A New Measure of Divergence with its Application to Multi-Criteria Decision Making under Fuzzy Environment

Shikha Maheshwari, Rajkumar Verma, Amit Srivastava
[1,3]Department of Mathematics, Jaypee Institute of Information Technology
Noida-201307, Uttar Pradesh, India
[2]Department of Applied Sciences, HMR Institute of Technology and Management, Guru Gobind Singh Indraprastha University, Hamidpur, Delhi-110036, India
maheshwari.shikha23@gmail.com, rkver83@gmail.com

Abstract: Divergence measure is an important tool for determining the amount of discrimination between two probability distributions. Since the introduction of fuzzy sets, divergence measures between two fuzzy sets have gained attention for their applications in various fields. Exponential entropy measure has some advantages over Shannon's entropy. In this paper, we used the idea of Jensen Shannon divergence to define a new divergence measure called '*fuzzy Jensen-exponential divergence* (FJSD)' for measuring the discrimination/difference between two fuzzy sets. The measure is demonstrated to satisfy some very elegant properties, which shows its strength for applications in multi-criteria decision making problems. Further, we develop a method to solve multi-criteria decision making problems under fuzzy phenomenon by utilizing the proposed measure and demonstrate by a numerical example.

Keywords - entropy, fuzzy sets, fuzzy divergence, Jensen-Shannon divergence, multi-criteria decision making

1. Introduction

In the last six decades, divergence measures have been extensively used to measure the difference between two probability distributions [4, 5, 15, 16 and 27] and widely applied in various fields.

The notion of divergence, introduced by Kullback and Leibler [16] in 1951, provides a measure of discrimination between two probability distributions. After its introduction, various generalized measures of divergence have been proposed by researchers [28 and 31] and studied their properties and application in details. In 1991, Lin [18] defined a new divergence measure named as Jensen-Shannon divergence, which has been gained quite some attention from researchers/practitioners and successfully applied in variety of disciplines [1, 7 – 9, 11, 19 and 21-24].

Parallel to the concept of probability theory, the notion of fuzzy sets (FSs) introduced by Zadeh [35] in 1965 to deal with vagueness. Since then, the theory of fuzzy set has become a vigorous area of research in different disciplines such as engineering, artificial intelligence, medical science, signal processing, and expert systems. In 1992, Bhandari and Pal [3] forwarded the concept of divergence measure from probabilistic to fuzzy phenomena and defined a divergence measure between two fuzzy sets. Fuzzy divergence measure gives fuzzy information measure for discrimination of a fuzzy set A relative to some other fuzzy set B. This fuzzy divergence measure has wide applications in many areas such as pattern recognition, fuzzy clustering, signal and image processing etc. Some generalized measures of fuzzy divergence have been studied by Hooda [12], Bajaj and Hooda [2], Shang and Jiang [29].

As mentioned above Jensen-Shannon divergence is an important measure from application point of view. This divergence measure is based on Shannon's entropy function [30]. In 1989, Pal and Pal [25, 26] critically analyze the Shannon's function and discussed its some limitations. To imbue these limitations, Pal and Pal [25, 26] defined a new measure of entropy based on exponential function. Futher, Kvålseth [17] and Verma and Sharma [34] defined some generalized version of exponential fuzzy entropy.

In 1999, Fan and Xie [6] introduced the fuzzy divergence measure based on exponential function and studied its relation with divergence measure introduced in [3]. Ghosh et al. [7] developed some applications of fuzzy divergence measure in the area of automated leukocyte recognition.

In this paper, we propose a new measure of divergence called '*fuzzy Jenson-exponential divergence (FJED)*' between two fuzzy sets. The new divergence measure has elegant properties, which are stated and proved in the paper to enhance the employability of this measure. The strength of this extension has been demonstrated by an example of multi-criteria decision making.

The paper is organized as follows: In Section 2 some basic definitions related to probability and fuzzy set theory are briefly given. In Section 3, the fuzzy Jenson-exponential divergence measure is proposed. In Section 4 some properties of the proposed fuzzy divergence measure are stated and proved. In Section 5 a fuzzy multi-criteria decision making method is proposed with the help of fuzzy Jenson-exponential divergence measure. A numerical example is also given to illustrate the solution process and our conclusions are presented in Section 6.

2. Preliminaries

Let $\Omega_n = \left\{ \left(p_1, p_2, ..., p_n \right) : p_j \geq 0, \ \sum_{j=1}^{n} p_j = 1 \right\}; n \geq 2$ denote the set of all complete finite discrete probability distributions.

For any probability distribution $P = \left(p_1, p_2, ..., p_n \right) \in \Omega_n$, Shannon [30] introduced the entropy to measure the uncertainty associated with probability distribution P as follows

$$H(P) = -\sum_{j=1}^{n} p_j \log p_j . \tag{1}$$

Pal and Pal [26] analyzed the classical Shannon information entropy and introduced a new probabilistic entropy called exponential entropy given by

$$_eH(P) = \sum_{j=1}^{n} p_j \left(e^{1-p_j} - 1 \right). \tag{2}$$

These authors point out that, the exponential entropy has an advantage over Shannon's entropy. For the uniform probability distribution $P = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n} \right)$, exponential entropy has a fixed upper bound

$$\lim {_eH}\left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n} \right) = e - 1 \qquad as\ n \to \infty . \tag{3}$$

which is not the case for Shannon's entropy.

Kullback and Leibler [16] measure of divergence of a probability distribution $Q=(q_1,q_2,\ldots,q_n)\in\Omega_n$ to probability distribution $P=\left(p_1,p_2,...,p_n\right)\in\Omega_n$, is given by

$$D(P\,|\,Q)=\sum_{j=1}^{n}p_j\log\frac{p_j}{q_j}. \tag{4}$$

In 1991, Lin [18] introduced the Jensen Shannon divergence between the two probability distributions $P=\left(p_1,p_2,\ldots,p_n\right)$ and $Q=\left(q_1,q_2,\ldots,q_n\right)$ as

$$JSD(P;Q)=H\left(\frac{P+Q}{2}\right)-\frac{H(P)+H(Q)}{2}, \tag{5}$$

where $H(P)$ is the Shannon entropy. Since $H(P)$ is a concave function, according to Jensen's inequality [10], $JSD(P;Q)$ is nonnegative and vanishes when $P=Q$. The JSD can also be represents in terms of KL divergence as

$$JSD(P;Q)=\frac{1}{2}\left(D\left(P\,|\,\frac{P+Q}{2}\right)+D\left(Q\,|\,\frac{P+Q}{2}\right)\right). \tag{6}$$

Definition 1. ***Fuzzy set*** [35]**:** A fuzzy set A defined in a finite universe of discourse $X=\{x_1,x_2,...,x_n\}$ is mathematically represented as

$$A=\left\{\langle x,\ \mu_A(x)\rangle\,\middle|\,x\in X\right\}, \tag{7}$$

where $\mu_A(x)\colon X\to[0,1]$ is measure of belongingness or degree of membership of an element x in A.

Definition 2 ***Set Operations on FSs*** [35]**:** Let $FS(X)$ denote the family of all FSs in the universe X, assume $A,B\in FS(X)$ given as

$$A=\{\langle x,\mu_A(x)\rangle\,|\,x\in X\},$$
$$B=\{\langle x,\mu_B(x)\rangle\,|\,x\in X\}.$$

Then some set operations defined as follows:

(i) $A\subseteq B$ iff $\mu_A(x)\le\mu_B(x)\quad\forall\ x\in X$;

(ii) $A=B$ iff $A\subseteq B$ and $B\subseteq A$;

(iii) $A^C=\left\{\langle x,\,1-\mu_A(x)\rangle\,\middle|\,x\in X\right\}$;

(iv) $A\cap B=\{\langle x,\mu_A(x)\wedge\mu_B(x)\rangle\,|\,x\in X\}$;

(v) $A\cup B=\{\langle x,\mu_A(x)\vee\mu_B(x)\rangle\,|\,x\in X\}$;

where $\vee,\wedge$ stand for max and min operators, respectively.

In 1972, De Luca and Termini [20] defined fuzzy entropy for a fuzzy set A corresponding to Shannon entropy [20], given by

$$H(A)=-\frac{1}{n}\sum_{j=1}^{n}\left[\mu_A(x_j)\log\mu_A(x_j)+(1-\mu_A(x_j))\log(1-\mu_A(x_j))\right]. \tag{8}$$

Fuzzy exponential entropy for fuzzy set A corresponding to (2) has also been introduced by Pal and Pal [26] as

$$ {}_{e}H(A)=\frac{1}{n\left(\sqrt{e}-1\right)}\sum_{j=1}^{n}\left[\mu_A\left(x_j\right)e^{1-\mu_A(x_j)}+\left(1-\mu_A\left(x_j\right)\right)e^{\mu_A(x_j)}-1\right]. \quad (9) $$

Let A and B be two fuzzy sets defined in X . Corresponding to the Kullback and Leibler [16] divergence measure, Bhandari and Pal [3] defined the fuzzy divergence measure of a fuzzy set B with respect to fuzzy set A as

$$ D(A|B)=\frac{1}{n}\sum_{j=1}^{n}\left[\mu_A\left(x_j\right)\log\frac{\mu_A\left(x_j\right)}{\mu_B\left(x_j\right)}+\left(1-\mu_A\left(x_j\right)\right)\log\frac{\left(1-\mu_A\left(x_j\right)\right)}{\left(1-\mu_B\left(x_j\right)\right)}\right]. \quad (10) $$

Fan and Xie [6] proposed the fuzzy divergence measure based on exponential function as

$$ {}_{e}D(A|B)=\sum_{j=1}^{n}\left(1-\left(1-\mu_A\left(x_j\right)\right)e^{\mu_A(x_j)-\mu_B(x_j)}-\mu_A\left(x_j\right)e^{\mu_B(x_j)-\mu_A(x_j)}\right). \quad (11) $$

In the next section, motivated by the idea of Jensen-Shannon divergence, we propose a new measure called '*fuzzy Jensen-exponential divergence'* (FJED) to measure the difference between two fuzzy sets. Some properties of the proposed FJED are also studied here.

3. Fuzzy Jensen-Exponential Divergence (FJED)

Definition 3.1. Single Element Universe set: Let $A=\{\langle x,\mu_A(x)\rangle \mid x\in X\}$ and $B=\{\langle x,\mu_B(x)\rangle \mid x\in X\}$ be the two fuzzy sets defined in a single element universe of discourse $X=\{x\}$. We define the fuzzy Jensen-exponential divergence measure between two FSs A and B , as

$$ \begin{aligned} FJED^{*}\left(A\|B\right) &= {}_{e}H\left(\frac{A+B}{2}\right)-\left(\frac{{}_{e}H(A)+{}_{e}H(B)}{2}\right) \\ &= \frac{1}{\left(\sqrt{e}-1\right)}\left[\begin{array}{l} \left(\begin{array}{l}\left(\dfrac{\mu_A(x)+\mu_B(x)}{2}\right)\exp\left(\dfrac{2-\mu_A(x)-\mu_B(x)}{2}\right) \\ \quad +\left(\dfrac{2-\mu_A(x)-\mu_B(x)}{2}\right)\exp\left(\dfrac{\mu_A(x)+\mu_B(x)}{2}\right)\end{array}\right) \\ -\left(\dfrac{\begin{array}{l}\left(\mu_A(x)\exp\left(1-\mu_A(x)\right)+\left(1-\mu_A(x)\right)\exp\left(\mu_A(x)\right)\right) \\ \quad +\left(\mu_B(x)\exp\left(1-\mu_B(x)\right)+\left(1-\mu_B(x)\right)\exp\left(\mu_B(x)\right)\right)\end{array}}{2}\right)\end{array}\right]. \end{aligned} \quad (12) $$

Theorem 3.1: For all $A,B\in FS(X)$, the fuzzy Jenson-exponential divergence measure $FJED^{*}(A\|B)$ in (12) satisfies the following properties:

(i) $FJED^{*}(A\|B)\geq 0$,

(ii) $FJED^{*}(A\|B)=0$ if and only if $A=B$,

(iii) $0\leq FJED^{*}(A\|B)\leq 1$.

Proof: (i) It follows from Jensen inequality [10].

(ii) Let $A = B$, then it is obvious $FJED^{*}(A\|B) = 0$.
Conversely, let

$$FJED^{*}(A\|B) = 0$$

$${}_{e}H\left(\frac{A+B}{2}\right) - \left(\frac{{}_{e}H(A) + {}_{e}H(B)}{2}\right) = 0 \tag{13}$$

Since ${}_{e}H(*)$ is a concave function [26], then from Jensen inequality [10] expression (13) holds if and only if $A = B$.

(iii) $FJED^{*}(A\|B)$ attains the highest value for the following cases:

$$A = (1,0), B = (0,1) \text{ or } A = (0,1), B = (1,0),$$

which gives the required results, *i.e.*, $0 \le FJED^{*}(A\|B) \le 1$.

This completes the proof. □

In the above definition, we assumed single element universe set. Now, here we extended this notion to finite universe of set.

Definition 3.2: Let $A = \{\langle x, \mu_A(x_j)\rangle \mid x_j \in X\}$ and $B = \{\langle x, \mu_B(x_j)\rangle \mid x_j \in X\}$ be two fuzzy sets in the universe of discourse $X = \{x_1, x_2, ..., x_n\}$, then the fuzzy Jenson-exponential divergence measure between *A* and *B*, is given by

$$FJED(A\|B) = \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{j=1}^{n} \left[\begin{array}{l} \left(\left(\frac{\mu_A(x_j)+\mu_B(x_j)}{2}\right) \exp\left(\frac{2-\mu_A(x_j)-\mu_B(x_j)}{2}\right) + \left(\frac{2-\mu_A(x_j)-\mu_B(x_j)}{2}\right) \exp\left(\frac{\mu_A(x_j)+\mu_B(x_j)}{2}\right) \right) \\ - \left(\frac{\left(\mu_A(x_j)\exp\left(1-\mu_A(x_j)\right) + \left(1-\mu_A(x_j)\right)\exp\left(\mu_A(x_j)\right)\right) + \left(\mu_B(x_j)\exp\left(1-\mu_B(x_j)\right) + \left(1-\mu_B(x_j)\right)\exp\left(\mu_B(x_j)\right)\right)}{2} \right) \end{array} \right]. \tag{14}$$

In the next section, we prove some elegant properties of the proposed measure given by (14). For proving the properties, we will assume that the finite universe of discourse X is partitioned into two disjoint sets X_1 and X_2, such that

$$X_1 = \{x \mid x \in X,\ \mu_A(x_j) \ge \mu_B(x_j)\} \tag{15}$$

$$X_2 = \{x \mid x \in X,\ \mu_A(x_j) < \mu_B(x_j)\}. \tag{16}$$

4. Properties of Fuzzy Jensen Exponential Divergence

The measure given by (10) has the following important properties:

Theorem 4.1: For all $A, B \in FS(X)$,

(i) $FJED(A \| A \cup B) = FJED(B \| A \cap B)$,

(ii) $FJED(A \| A \cap B) = FJED(B \| A \cup B)$,

(iii) $FJED(A \cup B \| A \cap B) = FJED(A \| B)$,

(iv) $FJED(A \| A \cup B) + FJED(A \| A \cap B) = FJED(A \| B)$,

(v) $FJED(B \| A \cup B) + FJED(B \| A \cap B) = FJED(A \| B)$.

Proof: (i) Considering measure given by (14), we have

$$FJED(A \| A \cup B) = \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{j=1}^{n} \left[\left\{ \begin{array}{l} \left(\frac{\mu_A(x_j) + \mu_{A\cup B}(x_j)}{2} \right) \exp\left(\frac{2 - \mu_A(x_j) - \mu_{A\cup B}(x_j)}{2} \right) \\ \quad + \left(\frac{2 - \mu_A(x_j) - \mu_{A\cup B}(x_j)}{2} \right) \exp\left(\frac{\mu_A(x_j) + \mu_{A\cup B}(x_j)}{2} \right) \end{array} \right\} - \left\{ \frac{ \begin{array}{l} \left(\mu_A(x_j) \exp\left(1 - \mu_A(x_j)\right) + \left(1 - \mu_A(x_j)\right) \exp\left(\mu_A(x_j)\right) \right) \\ \quad + \left(\mu_{A\cup B}(x_j) \exp\left(1 - \mu_{A\cup B}(x_j)\right) + \left(1 - \mu_{A\cup B}(x_j)\right) \exp\left(\mu_{A\cup B}(x_j)\right) \right) \end{array} }{2} \right\} \right]$$

$$= \frac{1}{n\left(\sqrt{e}-1\right)} \left[\begin{array}{l} \sum_{x_j \in X_1} \left\{ \left\{ \begin{array}{l} \left(\frac{\mu_A(x_j) + \mu_A(x_j)}{2} \right) \exp\left(\frac{2 - \mu_A(x_j) - \mu_A(x_j)}{2} \right) \\ \quad + \left(\frac{2 - \mu_A(x_j) - \mu_A(x_j)}{2} \right) \exp\left(\frac{\mu_A(x_j) + \mu_A(x_j)}{2} \right) \end{array} \right\} - \left\{ \frac{ \begin{array}{l} \left(\mu_A(x_j) \exp\left(1 - \mu_A(x_j)\right) + \left(1 - \mu_A(x_j)\right) \exp\left(\mu_A(x_j)\right) \right) \\ \quad + \left(\mu_A(x_j) \exp\left(1 - \mu_A(x_j)\right) + \left(1 - \mu_A(x_j)\right) \exp\left(\mu_A(x_j)\right) \right) \end{array} }{2} \right\} \right\} \\ + \sum_{x_j \in X_2} \left\{ \left\{ \begin{array}{l} \left(\frac{\mu_A(x_j) + \mu_B(x_j)}{2} \right) \exp\left(\frac{2 - \mu_A(x_j) - \mu_B(x_j)}{2} \right) \\ \quad + \left(\frac{2 - \mu_A(x_j) - \mu_B(x_j)}{2} \right) \exp\left(\frac{\mu_A(x_j) + \mu_B(x_j)}{2} \right) \end{array} \right\} - \left\{ \frac{ \begin{array}{l} \left(\mu_A(x_j) \exp\left(1 - \mu_A(x_j)\right) + \left(1 - \mu_A(x_j)\right) \exp\left(\mu_A(x_j)\right) \right) \\ \quad + \left(\mu_B(x_j) \exp\left(1 - \mu_B(x_j)\right) + \left(1 - \mu_B(x_j)\right) \exp\left(\mu_B(x_j)\right) \right) \end{array} }{2} \right\} \right\} \end{array} \right]$$

$$
= \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{x_j \in X_2} \left[\left\{ \begin{array}{l} \left(\dfrac{\mu_A(x_j)+\mu_B(x_j)}{2}\right)\exp\left(\dfrac{2-\mu_A(x_j)-\mu_B(x_j)}{2}\right) \\ \quad +\left(\dfrac{2-\mu_A(x_j)-\mu_B(x_j)}{2}\right)\exp\left(\dfrac{\mu_A(x_j)+\mu_B(x_j)}{2}\right) \end{array} \right\} - \left\{ \frac{\begin{array}{l} \left(\mu_A(x_j)\exp\left(1-\mu_A(x_j)\right)+\left(1-\mu_A(x_j)\right)\exp\left(\mu_A(x_j)\right)\right) \\ \quad +\left(\mu_B(x_j)\exp\left(\mu_B(x_j)\right)+\left(1-\mu_B(x_j)\right)\exp\left(\mu_B(x_j)\right)\right) \end{array}}{2} \right\} \right]. \tag{17}
$$

and

$$
FJED\left(B \| A\cap B\right)
$$

$$
= \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{j=1}^{n} \left[\left\{ \begin{array}{l} \left(\dfrac{\mu_B(x_j)+\mu_{A\cap B}(x_j)}{2}\right)\exp\left(\dfrac{2-\mu_B(x_j)-\mu_{A\cap B}(x_j)}{2}\right) \\ \quad +\left(\dfrac{2-\mu_B(x_j)-\mu_{A\cap B}(x_j)}{2}\right)\exp\left(\dfrac{\mu_B(x_j)+\mu_{A\cap B}(x_j)}{2}\right) \end{array} \right\} \right.
$$

$$
\left. - \left\{ \frac{\begin{array}{l} \left(\mu_B(x_j)\exp\left(1-\mu_B(x_j)\right)+\left(1-\mu_B(x_j)\right)\exp\left(\mu_B(x_j)\right)\right) \\ \quad +\left(\mu_{A\cap B}(x_j)\exp\left(1-\mu_{A\cap B}(x_j)\right)+\left(1-\mu_{A\cap B}(x_j)\right)\exp\left(\mu_{A\cap B}(x_j)\right)\right) \end{array}}{2} \right\} \right]
$$

$$
= \frac{1}{n\left(\sqrt{e}-1\right)} \left[\sum_{x_j \in X_1} \left\{ \left\{ \begin{array}{l} \left(\dfrac{\mu_B(x_j)+\mu_B(x_j)}{2}\right)\exp\left(\dfrac{2-\mu_B(x_j)-\mu_B(x_j)}{2}\right) \\ \quad +\left(\dfrac{2-\mu_B(x_j)-\mu_B(x_j)}{2}\right)\exp\left(\dfrac{\mu_B(x_j)+\mu_B(x_j)}{2}\right) \end{array} \right\} - \left\{ \frac{\begin{array}{l} \left(\mu_B(x_j)\exp\left(1-\mu_B(x_j)\right)+\left(1-\mu_B(x_j)\right)\exp\left(\mu_B(x_j)\right)\right) \\ \quad +\left(\mu_B(x_j)\exp\left(1-\mu_B(x_j)\right)+\left(1-\mu_B(x_j)\right)\exp\left(\mu_B(x_j)\right)\right) \end{array}}{2} \right\} \right\} \right.
$$

$$\left.-\sum_{x_j\in X_2}\left\{\left[\left\{\begin{array}{l}\left(\dfrac{\mu_B(x_j)+\mu_A(x_j)}{2}\right)\exp\left(\dfrac{2-\mu_B(x_j)-\mu_A(x_j)}{2}\right)\\ \qquad+\left(\dfrac{2-\mu_B(x_j)-\mu_A(x_j)}{2}\right)\exp\left(\dfrac{\mu_B(x_j)+\mu_A(x_j)}{2}\right)\end{array}\right\}-\left\{\dfrac{\begin{array}{l}\left(\mu_B(x_j)\exp\left(1-\mu_B(x_j)\right)+\left(1-\mu_B(x_j)\right)\exp\left(\mu_B(x_j)\right)\right)\\ \quad+\left(\mu_A(x_j)\exp\left(1-\mu_A(x_j)\right)+\left(1-\mu_A(x_j)\right)\exp\left(\mu_A(x_j)\right)\right)\end{array}}{2}\right\}\right]\right\}\right]$$

$$=\frac{1}{n\left(\sqrt{e}-1\right)}\sum_{x_j\in X_2}\left[\left\{\begin{array}{l}\left(\dfrac{\mu_B(x_j)+\mu_A(x_j)}{2}\right)\exp\left(\dfrac{2-\mu_B(x_j)-\mu_A(x_j)}{2}\right)\\ \qquad+\left(\dfrac{2-\mu_B(x_j)-\mu_A(x_j)}{2}\right)\exp\left(\dfrac{\mu_B(x_j)+\mu_A(x_j)}{2}\right)\end{array}\right\}-\left\{\dfrac{\begin{array}{l}\left(\mu_B(x_j)\exp\left(1-\mu_B(x_j)\right)+\left(1-\mu_B(x_j)\right)\exp\left(\mu_B(x_j)\right)\right)\\ \quad+\left(\mu_A(x_j)\exp\left(1-\mu_A(x_j)\right)+\left(1-\mu_A(x_j)\right)\exp\left(\mu_A(x_j)\right)\right)\end{array}}{2}\right\}\right]. \tag{18}$$

From (17) and (18), we get

$$FJED\left(A\,\|\,A\cup B\right)=FJED\left(B\,\|\,A\cap B\right).$$

(ii) The proof follows on similar lines as part (i).

(iii) Using measure given by (10), shown as

$FJED\left(A\cup B\,\|\,A\cap B\right)$

$$=\frac{1}{n\left(\sqrt{e}-1\right)}\sum_{j=1}^{n}\left[\left\{\begin{array}{l}\left(\dfrac{\mu_{A\cup B}(x_j)+\mu_{A\cap B}(x_j)}{2}\right)\exp\left(\dfrac{2-\mu_{A\cup B}(x_j)-\mu_{A\cap B}(x_j)}{2}\right)\\ \qquad+\left(\dfrac{2-\mu_{A\cup B}(x_j)-\mu_{A\cap B}(x_j)}{2}\right)\exp\left(\dfrac{\mu_{A\cup B}(x_j)+\mu_{A\cap B}(x_j)}{2}\right)\end{array}\right\}\right.$$
$$\left.-\left\{\dfrac{\begin{array}{l}\left(\mu_{A\cup B}(x_j)\exp\left(1-\mu_{A\cup B}(x_j)\right)+\left(1-\mu_{A\cup B}(x_j)\right)\exp\left(\mu_{A\cup B}(x_j)\right)\right)\\ \quad+\left(\mu_{A\cap B}(x_j)\exp\left(1-\mu_{A\cap B}(x_j)\right)+\left(1-\mu_{A\cap B}(x_j)\right)\exp\left(\mu_{A\cap B}(x_j)\right)\right)\end{array}}{2}\right\}\right]$$

$$
= \frac{1}{n\left(\sqrt{e-1}\right)} \left[\sum_{x_j \in X_1} \left\{ \begin{array}{l} \left\{ \begin{array}{l} \left(\dfrac{\mu_A(x_j)+\mu_B(x_j)}{2}\right)\exp\left(\dfrac{2-\mu_A(x_j)-\mu_B(x_j)}{2}\right) \\ \quad +\left(\dfrac{2-\mu_A(x_j)-\mu_B(x_j)}{2}\right)\exp\left(\dfrac{\mu_A(x_j)+\mu_B(x_j)}{2}\right) \end{array} \right\} \\ -\left\{ \dfrac{\begin{array}{l} \left(\mu_A(x_j)\exp\left(1-\mu_A(x_j)\right)+\left(1-\mu_A(x_j)\right)\exp\left(\mu_A(x_j)\right)\right) \\ \quad +\left(\mu_B(x_j)\exp\left(1-\mu_B(x_j)\right)+\left(1-\mu_B(x_j)\right)\exp\left(\mu_B(x_j)\right)\right) \end{array}}{2} \right\} \end{array} \right\} \right.
$$

$$
\left. + \sum_{x_j \in X_2} \left\{ \begin{array}{l} \left\{ \begin{array}{l} \left(\dfrac{\mu_B(x_j)+\mu_A(x_j)}{2}\right)\exp\left(\dfrac{2-\mu_B(x_j)-\mu_A(x_j)}{2}\right) \\ \quad +\left(\dfrac{2-\mu_B(x_j)-\mu_A(x_j)}{2}\right)\log\left(\dfrac{\mu_B(x_j)+\mu_A(x_j)}{2}\right) \end{array} \right\} \\ -\dfrac{\left\{ \begin{array}{l} \left(\mu_B(x_j)\exp\left(1-\mu_B(x_j)\right)+\left(1-\mu_B(x_j)\right)\exp\left(\mu_B(x_j)\right)\right) \\ \quad +\left(\mu_A(x_j)\exp\left(1-\mu_A(x_j)\right)+\left(1-\mu_A(x_j)\right)\exp\left(\mu_A(x_j)\right)\right) \end{array} \right\}}{2} \end{array} \right\} \right]
$$

$$
= FJED\left(A \| B\right).
$$

(iv) Considering (10), we have

$FJED\left(A \| A \cup B\right)$

$$
= \frac{1}{n\left(\sqrt{e-1}\right)} \sum_{j=1}^{n} \left[\begin{array}{l} \left\{ \begin{array}{l} \left(\dfrac{\mu_A(x_j)+\mu_{A\cup B}(x_j)}{2}\right)\exp\left(\dfrac{2-\mu_A(x_j)-\mu_{A\cup B}(x_j)}{2}\right) \\ \quad +\left(\dfrac{2-\mu_A(x_j)-\mu_{A\cup B}(x_j)}{2}\right)\exp\left(\dfrac{\mu_A(x_j)+\mu_{A\cup B}(x_j)}{2}\right) \end{array} \right\} \\ -\left\{ \dfrac{\begin{array}{l} \left(\mu_A(x_j)\exp\left(1-\mu_A(x_j)\right)+\left(1-\mu_A(x_j)\right)\exp\left(\mu_A(x_j)\right)\right) \\ \quad +\left(\mu_{A\cup B}(x_j)\exp\left(1-\mu_{A\cup B}(x_j)\right)+\left(1-\mu_{A\cup B}(x_j)\right)\exp\left(\mu_{A\cup B}(x_j)\right)\right) \end{array}}{2} \right\} \end{array} \right]
$$

$$
= \frac{1}{n\left(\sqrt{e-1}\right)}\left[\sum_{x_j \in X_1}\left\{\begin{array}{l}\left\{\begin{array}{l}\left(\dfrac{\mu_A(x_j)+\mu_A(x_j)}{2}\right)\exp\left(\dfrac{2-\mu_A(x_j)-\mu_A(x_j)}{2}\right)\\ \quad +\left(\dfrac{2-\mu_A(x_j)-\mu_A(x_j)}{2}\right)\exp\left(\dfrac{\mu_A(x_j)+\mu_A(x_j)}{2}\right)\end{array}\right\}\\ -\left\{\dfrac{\begin{array}{l}\left(\mu_A(x_j)\exp\left(1-\mu_A(x_j)\right)+\left(1-\mu_A(x_j)\right)\exp\left(\mu_A(x_j)\right)\right)\\ \quad +\left(\mu_A(x_j)\exp\left(1-\mu_B(x_j)\right)+\left(1-\mu_A(x_j)\right)\exp\left(\mu_A(x_j)\right)\right)\end{array}}{2}\right\}\end{array}\right\}\right.
$$

$$
\left. + \sum_{x_j \in X_2}\left\{\begin{array}{l}\left\{\begin{array}{l}\left(\dfrac{\mu_A(x_j)+\mu_B(x_j)}{2}\right)\exp\left(\dfrac{2-\mu_A(x_j)-\mu_B(x_j)}{2}\right)\\ \quad +\left(\dfrac{2-\mu_A(x_j)-\mu_B(x_j)}{2}\right)\exp\left(\dfrac{\mu_A(x_j)+\mu_B(x_j)}{2}\right)\end{array}\right\}\\ -\left\{\dfrac{\begin{array}{l}\left(\mu_A(x_j)\exp\left(1-\mu_A(x_j)\right)+\left(1-\mu_A(x_j)\right)\exp\left(\mu_A(x_j)\right)\right)\\ \quad +\left(\mu_B(x_j)\exp\left(1-\mu_B(x_j)\right)+\left(1-\mu_B(x_j)\right)\exp\left(\mu_B(x_j)\right)\right)\end{array}}{2}\right\}\end{array}\right\}\right] \quad (19)
$$

and

$FJED\left(A \| A \cap B\right)$

$$
= \frac{1}{n\left(\sqrt{e-1}\right)}\sum_{j=1}^{n}\left[\begin{array}{l}\left\{\begin{array}{l}\left(\dfrac{\mu_A(x_j)+\mu_{A\cap B}(x_j)}{2}\right)\log\left(\dfrac{2-\mu_A(x_j)-\mu_{A\cap B}(x_j)}{2}\right)\\ \quad +\left(\dfrac{2-\mu_A(x_j)-\mu_{A\cap B}(x_j)}{2}\right)\log\left(\dfrac{\mu_A(x_j)+\mu_{A\cap B}(x_j)}{2}\right)\end{array}\right\}\\ -\left\{\dfrac{\begin{array}{l}\left(\mu_A(x_j)\exp\left(1-\mu_A(x_j)\right)+\left(1-\mu_A(x_j)\right)\exp\left(\mu_A(x_j)\right)\right)\\ \quad +\left(\mu_{A\cap B}(x_j)\exp\left(1-\mu_{A\cap B}(x_j)\right)+\left(1-\mu_{A\cap B}(x_j)\right)\exp\left(\mu_{A\cap B}(x_j)\right)\right)\end{array}}{2}\right\}\end{array}\right]
$$

$$= \frac{1}{n\left(\sqrt{e}-1\right)}\left[\sum_{x_j\in X_1}\left\{\begin{array}{l}\left\{\begin{array}{l}\left(\dfrac{\mu_A(x_j)+\mu_A(x_j)}{2}\right)\exp\left(\dfrac{2-\mu_A(x_j)-\mu_A(x_j)}{2}\right)\\ \quad +\left(\dfrac{2-\mu_A(x_j)-\mu_A(x_j)}{2}\right)\exp\left(\dfrac{\mu_A(x_j)+\mu_A(x_j)}{2}\right)\end{array}\right\}\\ -\left\{\dfrac{\begin{array}{l}\left(\mu_A(x_j)\exp\left(1-\mu_A(x_j)\right)+\left(1-\mu_A(x_j)\right)\exp\left(\mu_A(x_j)\right)\right)\\ \quad +\left(\mu_A(x_j)\exp\left(1-\mu_A(x_j)\right)+\left(1-\mu_A(x_j)\right)\exp\left(\mu_A(x_j)\right)\right)\end{array}}{2}\right\}\end{array}\right\}\right.$$

$$\left. +\sum_{x_j\in X_2}\left\{\begin{array}{l}\left\{\begin{array}{l}\left(\dfrac{\mu_A(x_j)+\mu_B(x_j)}{2}\right)\log\left(\dfrac{2-\mu_A(x_j)-\mu_B(x_j)}{2}\right)\\ \quad +\left(\dfrac{2-\mu_A(x_j)-\mu_B(x_j)}{2}\right)\log\left(\dfrac{\mu_A(x_j)+\mu_B(x_j)}{2}\right)\end{array}\right\}\\ -\left\{\dfrac{\begin{array}{l}\left(\mu_A(x_j)\exp\left(1-\mu_A(x_j)\right)+\left(1-\mu_A(x_j)\right)\exp\left(\mu_A(x_j)\right)\right)\\ \quad +\left(\mu_B(x_j)\exp\left(1-\mu_B(x_j)\right)+\left(1-\mu_B(x_j)\right)\exp\left(\mu_B(x_j)\right)\right)\end{array}}{2}\right\}\end{array}\right\}\right]. \tag{20}$$

On adding (19) and (20), we get

$$FJED\left(A\,\|\,A\cup B\right)+FJED\left(A\,\|\,A\cap B\right)=FJED\left(A\,\|\,B\right).$$

(v) The proof is as similar as part (iv).

This completes the proof. □

Theorem 4.2: For all $A, B\in FS(X)$,

(i) $FJED\left(A\,\|\,B^C\right)=FJED\left(A^C\,\|\,B\right)$,

(ii) $FJSD\left(A\,\|\,B\right)+FJSD\left(A^C\,\|\,B\right)=FJSD\left(A^C\,\|\,B^C\right)+FJSD\left(A\,\|\,B^C\right)$.

where A^C and B^C are the complement of fuzzy set *A* and *B*, respectively.

Proof: Consider the expression

(i) $FJED\left(A\,\|\,B^C\right)-FJED\left(A^C\,\|\,B\right)$ (21)

$$
=\frac{1}{n\left(\sqrt{e}-1\right)}\sum_{j=1}^{n}\left[\left\{\begin{array}{l}\left(\frac{\mu_A(x_j)+\mu_{B^C}(x_j)}{2}\right)\exp\left(\frac{2-\mu_A(x_j)-\mu_{B^C}(x_j)}{2}\right)\\ \qquad +\left(\frac{2-\mu_A(x_j)-\mu_{B^C}(x_j)}{2}\right)\exp\left(\frac{\mu_A(x_j)+\mu_{B^C}(x_j)}{2}\right)\end{array}\right\}-\left\{\frac{\begin{array}{l}\left(\mu_A(x_j)\exp\left(\mu_A(x_j)\right)+\left(1-\mu_A(x_j)\right)\exp\left(1-\mu_A(x_j)\right)\right)\\ \qquad +\left(\mu_{B^C}(x_j)\exp\left(\mu_{B^C}(x_j)\right)+\left(1-\mu_{B^C}(x_j)\right)\exp\left(1-\mu_{B^C}(x_j)\right)\right)\end{array}}{2}\right\}\right]
$$

$$
-\frac{1}{n\left(\sqrt{e}-1\right)}\sum_{j=1}^{n}\left[\left\{\begin{array}{l}\left(\frac{\mu_{A^C}(x_j)+\mu_B(x_j)}{2}\right)\exp\left(\frac{2-\mu_{A^C}(x_j)-\mu_B(x_j)}{2}\right)\\ \qquad +\left(\frac{2-\mu_{A^C}(x_j)-\mu_B(x_j)}{2}\right)\exp\left(\frac{\mu_{A^C}(x_j)+\mu_B(x_j)}{2}\right)\end{array}\right\}-\left\{\frac{\begin{array}{l}\left(\mu_{A^C}(x_j)\exp\left(1-\mu_{A^c}(x_j)\right)+\left(1-\mu_{A^C}(x_j)\right)\exp\left(\mu_{A^C}(x_j)\right)\right)\\ \qquad +\left(\mu_B(x_j)\exp\left(1-\mu_B(x_j)\right)+\left(1-\mu_B(x_j)\right)\exp\left(\mu_B(x_j)\right)\right)\end{array}}{2}\right\}\right]
$$

$$
=\frac{1}{n\left(\sqrt{e}-1\right)}\sum_{j=1}^{n}\Bigg[\Bigg[\left\{\begin{array}{l}\left(\frac{\mu_A(x_j)+1-\mu_B(x_j)}{2}\right)\exp\left(\frac{1-\mu_A(x_j)+\mu_B(x_j)}{2}\right)\\ \qquad +\left(\frac{1-\mu_A(x_j)+\mu_B(x_j)}{2}\right)\exp\left(\frac{\mu_A(x_j)+1-\mu_B(x_j)}{2}\right)\end{array}\right\}-\left\{\frac{\begin{array}{l}\left(\mu_A(x_j)\exp\left(1-\mu_A(x_j)\right)+\left(1-\mu_A(x_j)\right)\exp\left(\mu_A(x_j)\right)\right)\\ \qquad +\left(\left(\mu_B(x_j)\right)\exp\left(1-\mu_B(x_j)\right)+\left(1-\mu_B(x_j)\right)\exp\left(\mu_B(x_j)\right)\right)\end{array}}{2}\right\}\Bigg]
$$

$$
\left. \left. - \left\{ \left(\frac{1-\mu_A(x_j)+\mu_B(x_j)}{2}\right)\exp\left(\frac{1+\mu_A(x_j)-\mu_B(x_j)}{2}\right) + \left(\frac{1+\mu_A(x_j)-\mu_B(x_j)}{2}\right)\exp\left(\frac{1-\mu_A(x_j)+\mu_B(x_j)}{2}\right) \right\} - \left\{ \frac{\left(\left(1-\mu_A(x_j)\right)\exp\left(\mu_A(x_j)\right)+\mu_A(x_j)\exp\left(1-\mu_A(x_j)\right)\right) + \left(\left(1-\mu_B(x_j)\right)\exp\left(\mu_B(x_j)\right)+\left(\mu_B(x_j)\right)\exp\left(1-\mu_B(x_j)\right)\right)}{2} \right\} \right] \right]
$$

$= 0$.

This proves (i).

(ii) The proof of (ii) is on similar lines as part (i).

Theorem 4.3: For all $A, B, C \in FS(X)$,

(i) $FJED(A \cup B \| C) \leq FJED(A \| C) + FJED(B \| C)$,

(ii) $FJED(A \cap B \| C) \leq FJED(A \| C) + FJED(B \| C)$,

(iii) $FJED(A \cup B \| C) + FJED(A \cap B \| C) = FJED(A \| C) + FJED(B \| C)$.

Proof: (i) Consider the expression

$$
FJSD(A \| C) + FJSD(B \| C) - FJSD(A \cup B \| C) \tag{22}
$$

$$
= \frac{1}{n\left(\sqrt{e}-1\right)} \sum_{j=1}^{n} \left[\left\{ \left(\frac{\mu_A(x_j)+\mu_C(x_j)}{2}\right)\exp\left(\frac{2-\mu_A(x_j)-\mu_C(x_j)}{2}\right) + \left(\frac{2-\mu_A(x_j)-\mu_C(x_j)}{2}\right)\exp\left(\frac{\mu_A(x_j)+\mu_C(x_j)}{2}\right) \right\} - \left\{ \frac{\left(\mu_A(x_j)\exp\left(1-\mu_A(x_j)\right)+\left(1-\mu_A(x_j)\right)\exp\left(\mu_A(x_j)\right)\right) + \left(\mu_C(x_j)\exp\left(1-\mu_C(x_j)\right)+\left(1-\mu_C(x_j)\right)\exp\left(\mu_C(x_j)\right)\right)}{2} \right\} \right]
$$

$$+\frac{1}{n\left(\sqrt{e}-1\right)}\sum_{j=1}^{n}\left[\begin{array}{l}\left\{\begin{array}{l}\left(\frac{\mu_B(x_j)+\mu_C(x_j)}{2}\right)\exp\left(\frac{2-\mu_B(x_j)-\mu_C(x_j)}{2}\right)\\ \quad+\left(\frac{2-\mu_B(x_j)-\mu_C(x_j)}{2}\right)\exp\left(\frac{\mu_B(x_j)+\mu_C(x_j)}{2}\right)\end{array}\right\}\\ -\left\{\frac{\begin{array}{l}\left(\mu_B(x_j)\exp\left(1-\mu_B(x_j)\right)+\left(1-\mu_B(x_j)\right)\exp\left(\mu_B(x_j)\right)\right)\\ \quad+\left(\mu_C(x_j)\exp\left(1-\mu_C(x_j)\right)+\left(1-\mu_C(x_j)\right)\exp\left(\mu_C(x_j)\right)\right)\end{array}}{2}\right\}\end{array}\right]$$

$$-\frac{1}{n\left(\sqrt{e}-1\right)}\sum_{j=1}^{n}\left[\begin{array}{l}\left\{\begin{array}{l}\left(\frac{\mu_{A\cup B}(x_j)+\mu_C(x_j)}{2}\right)\exp\left(\frac{2-\mu_{A\cup B}(x_j)-\mu_C(x_j)}{2}\right)\\ \quad+\left(\frac{2-\mu_{A\cup B}(x_j)-\mu_C(x_j)}{2}\right)\exp\left(\frac{\mu_{A\cup B}(x_j)+\mu_C(x_j)}{2}\right)\end{array}\right\}\\ -\left\{\frac{\begin{array}{l}\left(\mu_{A\cup B}(x_j)\exp\left(1-\mu_{A\cup B}(x_j)\right)+\left(1-\mu_{A\cup B}(x_j)\right)\log\left(\mu_{A\cup B}(x_j)\right)\right)\\ \quad+\left(\mu_C(x_j)\exp\left(1-\mu_C(x_j)\right)+\left(1-\mu_C(x_j)\right)\exp\left(\mu_C(x_j)\right)\right)\end{array}}{2}\right\}\end{array}\right]$$

$$=\frac{1}{n\left(\sqrt{e}-1\right)}\sum_{x_j\in X_1}\left[\left\{\begin{array}{l}\left\{\begin{array}{l}\left(\frac{\mu_B(x_j)+\mu_C(x_j)}{2}\right)\exp\left(\frac{2-\mu_B(x_j)-\mu_C(x_j)}{2}\right)\\ \quad+\left(\frac{2-\mu_B(x_j)-\mu_C(x_j)}{2}\right)\exp\left(\frac{\mu_B(x_j)+\mu_C(x_j)}{2}\right)\end{array}\right\}\\ -\left\{\frac{\begin{array}{l}\left(\mu_B(x_j)\exp\left(1-\mu_B(x_j)\right)+\left(1-\mu_B(x_j)\right)\exp\left(\mu_B(x_j)\right)\right)\\ \quad+\left(\mu_C(x_j)\exp\left(1-\mu_C(x_j)\right)+\left(1-\mu_C(x_j)\right)\exp\left(\mu_C(x_j)\right)\right)\end{array}}{2}\right\}\end{array}\right\}\right.$$

$$+\sum_{x_j\in X_2}\left\{\begin{array}{l}\left\{\begin{array}{l}\left(\dfrac{\mu_A(x_j)+\mu_C(x_j)}{2}\right)\log\left(\dfrac{2-\mu_A(x_j)-\mu_C(x_j)}{2}\right)\\ \quad+\left(\dfrac{2-\mu_A(x_j)-\mu_C(x_j)}{2}\right)\log\left(\dfrac{\mu_A(x_j)+\mu_C(x_j)}{2}\right)\end{array}\right\}\\ -\left\{\dfrac{\begin{array}{l}\left(\mu_A(x_j)\exp\left(1-\mu_A(x_j)\right)+\left(1-\mu_A(x_j)\right)\exp\left(\mu_A(x_j)\right)\right)\\ \quad+\left(\mu_C(x_j)\exp\left(1-\mu_C(x_j)\right)+\left(1-\mu_C(x_j)\right)\exp\left(\mu_C(x_j)\right)\right)\end{array}}{2}\right\}\end{array}\right\}\Bigg]\Bigg]$$

$\geq 0.$

This proves (i).

(ii) The proof is on similar lines as part (i).

(iii) Let us assume

$$FJED(A\cup B\,\|\,C)$$

$$=\frac{1}{n\left(\sqrt{e}-1\right)}\sum_{j=1}^{n}\left[\begin{array}{l}\left\{\begin{array}{l}\left(\dfrac{\mu_{A\cup B}(x_j)+\mu_C(x_j)}{2}\right)\exp\left(\dfrac{2-\mu_{A\cup B}(x_j)-\mu_C(x_j)}{2}\right)\\ \quad+\left(\dfrac{2-\mu_{A\cup B}(x_j)-\mu_C(x_j)}{2}\right)\exp\left(\dfrac{\mu_{A\cup B}(x_j)+\mu_C(x_j)}{2}\right)\end{array}\right\}\\ -\left\{\dfrac{\begin{array}{l}\left(\mu_{A\cup B}(x_j)\exp\left(1-\mu_{A\cup B}(x_j)\right)+\left(1-\mu_{A\cup B}(x_j)\right)\log\left(\mu_{A\cup B}(x_j)\right)\right)\\ \quad+\left(\mu_C(x_j)\exp\left(1-\mu_C(x_j)\right)+\left(1-\mu_C(x_j)\right)\exp\left(\mu_C(x_j)\right)\right)\end{array}}{2}\right\}\end{array}\right]$$

$$=\frac{1}{n\left(\sqrt{e}-1\right)}\left[\sum_{x_j\in X_1}\left\{\begin{array}{l}\left\{\begin{array}{l}\left(\dfrac{\mu_A(x_j)+\mu_C(x_j)}{2}\right)\exp\left(\dfrac{2-\mu_A(x_j)-\mu_C(x_j)}{2}\right)\\ +\left(\dfrac{2-\mu_A(x_j)-\mu_C(x_j)}{2}\right)\exp\left(\dfrac{\mu_A(x_j)+\mu_C(x_j)}{2}\right)\end{array}\right\}\\ -\left\{\dfrac{\begin{array}{l}\left(\mu_A(x_j)\exp\left(1-\mu_A(x_j)\right)+\left(1-\mu_A(x_j)\right)\exp\left(\mu_A(x_j)\right)\right)\\ \quad+\left(\mu_C(x_j)\exp\left(1-\mu_C(x_j)\right)+\left(1-\mu_C(x_j)\right)\exp\left(\mu_C(x_j)\right)\right)\end{array}}{2}\right\}\end{array}\right\}\right.$$

$$
\left.+\sum_{x_j\in X_2}\left\{\begin{array}{l}\left\{\begin{array}{l}\left(\dfrac{\mu_B(x_j)+\mu_C(x_j)}{2}\right)\exp\left(\dfrac{2-\mu_B(x_j)-\mu_C(x_j)}{2}\right)\\+\left(\dfrac{2-\mu_B(x_j)-\mu_C(x_j)}{2}\right)\exp\left(\dfrac{\mu_B(x_j)+\mu_C(x_j)}{2}\right)\end{array}\right\}\\-\left\{\dfrac{\begin{array}{c}\big(\mu_B(x_j)\exp\big(1-\mu_B(x_j)\big)+\big(1-\mu_B(x_j)\big)\exp\big(\mu_B(x_j)\big)\big)\\+\big(\mu_C(x_j)\exp\big(1-\mu_C(x_j)\big)+\big(1-\mu_C(x_j)\big)\exp\big(\mu_C(x_j)\big)\big)\end{array}}{2}\right\}\end{array}\right\}\right].
\tag{23}
$$

and

$FJED\left(A\cap B\,\|\,C\right)$

$$
=\frac{1}{n\left(\sqrt{e}-1\right)}\sum_{j=1}^{n}\left[\begin{array}{l}\left\{\begin{array}{l}\left(\dfrac{\mu_{A\cap B}(x_j)+\mu_C(x_j)}{2}\right)\exp\left(\dfrac{2-\mu_{A\cap B}(x_j)-\mu_C(x_j)}{2}\right)\\+\left(\dfrac{2-\mu_{A\cap B}(x_j)-\mu_C(x_j)}{2}\right)\exp\left(\dfrac{\mu_{A\cap B}(x_j)+\mu_C(x_j)}{2}\right)\end{array}\right\}\\-\left\{\dfrac{\begin{array}{c}\big(\mu_{A\cap B}(x_j)\exp\big(1-\mu_{A\cap B}(x_j)\big)+\big(1-\mu_{A\cap B}(x_j)\big)\exp\big(\mu_{A\cap B}(x_j)\big)\big)\\+\big(\mu_C(x_j)\exp\big(1-\mu_C(x_j)\big)+\big(1-\mu_C(x_j)\big)\exp\big(\mu_C(x_j)\big)\big)\end{array}}{2}\right\}\end{array}\right]
$$

$$
=\frac{1}{n\left(\sqrt{e}-1\right)}\sum_{x_j\in X_1}\left[\left\{\begin{array}{l}\left\{\begin{array}{l}\left(\dfrac{\mu_B(x_j)+\mu_C(x_j)}{2}\right)\exp\left(\dfrac{2-\mu_B(x_j)+\mu_C(x_j)}{2}\right)\\+\left(\dfrac{2-\mu_B(x_j)-\mu_C(x_j)}{2}\right)\exp\left(\dfrac{\mu_B(x_j)+\mu_C(x_j)}{2}\right)\end{array}\right\}\\-\left\{\dfrac{\begin{array}{c}\big(\mu_B(x_j)\exp\big(1-\mu_B(x_j)\big)+\big(1-\mu_B(x_j)\big)\exp\big(\mu_B(x_j)\big)\big)\\+\big(\mu_C(x_j)\exp\big(1-\mu_C(x_j)\big)+\big(1-\mu_C(x_j)\big)\exp\big(\mu_C(x_j)\big)\big)\end{array}}{2}\right\}\end{array}\right\}\right.
$$

$$+\sum_{x_j\in X_2}\left[\left\{\begin{array}{l}\left\{\begin{array}{l}\left(\dfrac{\mu_A(x_j)+\mu_C(x_j)}{2}\right)\exp\left(\dfrac{2-\mu_A(x_j)-\mu_C(x_j)}{2}\right)\\ +\left(\dfrac{2-\mu_A(x_j)-\mu_C(x_j)}{2}\right)\exp\left(\dfrac{\mu_A(x_j)+\mu_C(x_j)}{2}\right)\end{array}\right\}\\ -\left\{\dfrac{\begin{array}{c}\Big(\mu_A(x_j)\exp\big(1-\mu_A(x_j)\big)+\big(1-\mu_A(x_j)\big)\exp\big(\mu_A(x_j)\big)\Big)\\ +\Big(\mu_C(x_j)\exp\big(1-\mu_C(x_j)\big)+\big(1-\mu_C(x_j)\big)\exp\big(\mu_C(x_j)\big)\Big)\end{array}}{2}\right\}\end{array}\right\}\right]. \tag{24}$$

Adding (23) and (24), we obtain the required result.

This completes the proof. □

5. Application of FJED to Fuzzy Multi-Criteria Decision Making

5.1 *Fuzzy Multi-criteria Decision Making Problem*:

Decision making is a process of selecting the best option (or options) from a finite number of feasible options. It is a very common activity that usually occurs in our daily life and plays an important role in business, finance, management, economics, social and political science, engineering and computer science, biology and medicine etc.

Multi criteria decision making (MCDM) refers to select optimal option from a finite number of feasible options under several criteria. To find the most preferred option, the decision maker provides his/her preference information for the options.

In many real life decision making problems, the available information is vague or imprecise. To adequately solve decision problems with vague or imprecise information, fuzzy set theory has become powerful tool. In the literature, a number of multiple criteria decision making theories and methods under fuzzy environment have been proposed for effectively solving the multi criteria decision making problems and various applications have been cited in the literature [13, 32, 33].

Let $M=\{M_1,M_2,...,M_m\}$ be a set of options and let $C=\{C_1,C_2,...,C_n\}$ be a set of criteria. For decision making, the decision maker characterized each option in terms of FSs by assigning appropriate values to $\mu-$functions. Suppose that the characteristics of the option M_i in terms of the set of criteria C are presented by FS shown as follows:

$$FS-M_i=\left\{\left\langle C_j,\mu_{ij}\right\rangle \mid C_j\in C\right\},\qquad i=1,2,...,m \text{ and } j=1,2,...,n, \tag{25}$$

where μ_{ij} indicates the degree with which the option M_i satisfies the criterion C_j and $\mu_{ij}\in[0,1]$, $i=1,2,...,m$ and $j=1,2,...,n$.

We introduce a four-step algorithm to solve above fuzzy multi-criteria decision making problem using the measure defined in (14).

5.2 *Algorithm*:

Step 1: Finding the positive ideal and negative ideal solutions denoted by M^{+} and M^{-}, respectively given by:

$$M^{+}=\left\{\left\langle \mu_{1+}\right\rangle,\left\langle \mu_{2+}\right\rangle,\ldots,\left\langle \mu_{n+}\right\rangle\right\}, \tag{25}$$

$$M^{-}=\left\{\left\langle \mu_{1-}\right\rangle,\left\langle \mu_{2-}\right\rangle,\ldots,\left\langle \mu_{n-}\right\rangle\right\}, \tag{26}$$

where

$$\left.\begin{array}{l}\left\langle \mu_{j_+}\right\rangle=\left\langle \max\limits_{i} \mu_{ij}\right\rangle \\ \left\langle \mu_{j_-}\right\rangle=\left\langle \min\limits_{i} \mu_{ij}\right\rangle\end{array}\right\}. \tag{27}$$

Step 2: Calculating the values of $FJSD(M^{+}\,|\,M_i)$ and $FJSD(M^{-}\,|\,M_i)$ by the following formulas respectively:

$$FJED(M^{+}\,|\,M_i)=\frac{1}{n}\sum_{j=1}^{n}\left[\left\{\left(\frac{\mu_{j+}+\mu_{ij}}{2}\right)\exp\left(\frac{2-\mu_{j_+}-\mu_{ij}}{2}\right)+\left(\frac{2-\mu_{j_+}-\mu_{ij}}{2}\right)\exp\left(\frac{\mu_{j_+}+\mu_{ij}}{2}\right)\right\}+\left\{\frac{\left(\mu_{j_+}\exp\left(1-\mu_{j_+}\right)+\left(1-\mu_{j_+}\right)\exp\left(\mu_{j_+}\right)\right)+\left(\mu_{ij}\exp\left(1-\mu_{ij}\right)+\left(1-\mu_{ij}\right)\exp\left(\mu_{ij}\right)\right)}{2}\right\}\right] \tag{28}$$

and

$$FJED(M^{-}\,|\,M_i)=\frac{1}{n}\sum_{j=1}^{n}\left[\left\{\left(\frac{\mu_{j-}+\mu_{ij}}{2}\right)\exp\left(\frac{2-\mu_{j_-}-\mu_{ij}}{2}\right)+\left(\frac{2-\mu_{j_-}-\mu_{ij}}{2}\right)\exp\left(\frac{\mu_{j_-}+\mu_{ij}}{2}\right)\right\}+\left\{\frac{\left(\mu_{j_-}\exp\left(1-\mu_{j_-}\right)+\left(1-\mu_{j_-}\right)\exp\left(\mu_{j_-}\right)\right)+\left(\mu_{ij}\exp\left(1-\mu_{ij}\right)+\left(1-\mu_{ij}\right)\exp\left(\mu_{ij}\right)\right)}{2}\right\}\right]. \tag{29}$$

Step 3: Calculate the relative fuzzy Jensen-exponential divergence $FJED\left(M_i\right)$ of options M_i with respect to M^{+} and M^{-}, where

$$FJED\left(M_i\right)=\frac{FJED(M^{+}\,\|\,M_i)}{FJED(M^{+}\,\|\,M_i)+FJED(M^{-}\,\|\,M_i))}. \tag{30}$$

Step 4: Rank the options M_i, $i = 1,2,...,m$, according to the values of $FJED(M_i)$, $i = 1,2,...,m$, in ascending order. The leading M_i, with smallest value of $FJED(M_i)$, is the best option.

In order to illustrate the applicability of the proposed algorithm, we consider below an example of investment company decision-making problem.

Example [33]**:** Consider an investment company problem. Assume that an investment company desires to invest a definite amount of money in the best option among five options: **(i)** a car company M_1, **(ii)** a food company M_2, **(iii)** a computer company M_3, **(iv)** an arms company M_4 and **(v)** a TV company M_5. The investment company wants to take a decision according to the following four criteria:

i. C_1, the risk analysis,
ii. C_2, the growth analysis,
iii. C_3, the social-political impact analysis,
iv. C_4, the environmental impact analysis.

After evaluation of options, the decision maker forms the following fuzzy sets, given by

$$M_1 = \{\langle C_1, 0.5\rangle, \langle C_2, 0.6\rangle, \langle C_3, 0.3\rangle, \langle C_4, 0.2\rangle\},$$
$$M_2 = \{\langle C_1, 0.7\rangle, \langle C_2, 0.7\rangle, \langle C_3, 0.7\rangle, \langle C_4, 0.4\rangle\},$$
$$M_3 = \{\langle C_1, 0.6\rangle, \langle C_2, 0.5\rangle, \langle C_3, 0.5\rangle, \langle C_4, 0.6\rangle\},$$
$$M_4 = \{\langle C_1, 0.8\rangle, \langle C_2, 0.6\rangle, \langle C_3, 0.3\rangle, \langle C_4, 0.2\rangle\},$$
$$M_5 = \{\langle C_1, 0.6\rangle, \langle C_2, 0.4\rangle, \langle C_3, 0.7\rangle, \langle C_4, 0.5\rangle\}.$$

We need to find a ranking of the feasible options, with a view to find the best option.

Step 1: Obtaining the M^{+} and M^{-} given by

$$M^{+} = \{\langle C_1, 0.8\rangle, \langle C_2, 0.7\rangle, \langle C_3, 0.7\rangle, \langle C_4, 0.6\rangle\},$$
$$M^{+} = \{\langle C_1, 0.5\rangle, \langle C_2, 0.4\rangle, \langle C_3, 0.3\rangle, \langle C_4, 0.2\rangle\}.$$

Step 2: Using (28) and (29) to calculate $FJSD(M^{+} \mid M_i)$ and $FJSD(M^{-} \mid M_i)$, respectively, we have the following tables:

Table 1:

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Table 2:

****

Step 3. Calculating the relative fuzzy Jensen-exponential divergence measure $FJED(M_i)$ of M_i's with respect to M^+ and M^- by using (30). We obtain the following table:

Table 3

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Based on Table 3, the ranking order of options is given by,

$$M_2 \succ M_5 \succ M_3 \succ M_4 \succ M_1 .$$

Therefore, M_2 is the most preferable option, which is in agreement as obtained in [33].

6. Conclusion

In this paper, we proposed a new information measure called '*fuzzy Jensen-exponential divergence*' in the setting of fuzzy set theory. A number of properties of the proposed measure have been stated and proved. Then, based on fuzzy Jensen-exponential divergence, we have developed an algorithm to deal the problems of multi-criteria decision making with fuzzy information. Further study of this measure will be reported separately.

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Table 1: Values of $FJSD(M^{+} \mid M_i)$

$FJSD(M^{+} \mid M_1)$	0.0403
$FJSD(M^{+} \mid M_2)$	0.0048
$FJSD(M^{+} \mid M_3)$	0.0116
$FJSD(M^{+} \mid M_4)$	0.0316
$FJSD(M^{+} \mid M_5)$	0.0135

Table 2: Values of $FJSD(M^{-} \mid M_i)$

$FJSD(M^{-} \mid M_1)$	0.0038
$FJSD(M^{-} \mid M_2)$	0.0315
$FJSD(M^{-} \mid M_3)$	0.0211
$FJSD(M^{-} \mid M_4)$	0.0125
$FJSD(M^{-} \mid M_5)$	0.0249

Table 3: Values of Relative $FJED\left(M_i\right)$

$FJED\left(M_1\right)$	0.9143
$FJED\left(M_2\right)$	0.1319

$FJED(M_3)$	0.3548
$FJED(M_4)$	0.7158
$FJED(M_5)$	0.3507