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Adjustable soft discernibility matrix based on picture fuzzy soft sets and its applications in decision making

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Abstract. In this work, we first define picture fuzzy soft sets and study some of their relevant properties, especially, a sufficient and necessary condition is presented to ensure that the dual laws are true in picture fuzzy soft theory. We then introduce an algorithm based on adjustable soft discernibility matrix by using level soft set of a picture fuzzy soft set to solve decision making problems, which can find an order relation of all the objects. Finally, an illustrative example is employed to show that they can be successfully applied to problems that contain uncertainties.

Keywords: Soft set, Picture fuzzy set, picture fuzzy soft set, soft discernibility matrix, decision making

1. Introduction

The real world is full of uncertainty, imprecision and vagueness. Actually most of the concepts we meet in everyday life are vague than precise. Dealing with uncertainties is a major problem in many areas such as economics, engineering, environmental science, medical science and social science. So many authors have become interested in modeling vagueness recently. Classical theories like probability theory, fuzzy set theory, rough set theory [1], vague set theory [2] and interval mathematics [3] are well known and play important roles in modeling uncertainty. However, with the rapidly increasing quantity and type of uncertainties, these theories show their inherent difficulties as pointed out by Molodtsov in [4]. In 1999, Molodtsov initiated soft set theory as a completely new mathematical tool for dealing with uncertainties that is free from the difficulties affecting existing methods [4]. This theory has shown wide application prospects in many different fields like decision making [5–14], data analysis [15] and forecasting [16].

Soft set theory has received much attention since its introduction by Molodtsov. The concept and basic properties of soft set theory are further studied in [4, 17, 18]. Up to the present, research on combinations of the soft set theory and other mathematical models is very active and many important results have been achieved in the theoretical aspect. Maji et al. presented the concept of fuzzy soft set [19] which is based on a combination of the fuzzy set and soft set models. Later, the same authors amalgamated the intuitionistic fuzzy set and soft set models and initiated the intuitionistic fuzzy soft set [20]. Then the concepts of the interval-valued fuzzy soft set [21], the soft rough set [22, 23], the vague soft set [24], the generalized fuzzy soft set [25], the trapezoidal fuzzy soft set [26], the multi-fuzzy soft set [27], the neutrosophic soft set [28] and the intuitionistic neutrosophic soft set [29] were introduced as some further extensions of soft sets. These models have been successfully applied in decision making problems under imprecise circumstances.

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In [6] Cagman and Enginoglu introduced soft matrix in soft sets and researched its operations. Then a method based on soft matrix was presented to solve the problems of decision making. With this method, the optimal choice object(s) can be found. Recently, the notion of soft discernibility matrix was firstly proposed by Feng and Zhou [30], and a new technique is shown to solve decision making problems based on soft discernibility matrix. It can find not only the optimal object(s), but also an order relation of all the objects easily by scanning the soft discernibility matrix at most one time.

In 2013, authors [31, 32] proposed the new concept of picture fuzzy sets, which are directly extensions of fuzzy sets (Zadeh) and of intuitionistic fuzzy sets (Atanassov), its membership function is a triple. Classical fuzzy sets give one degree (the degree of membership) of an element in a given set, intuitionistic fuzzy sets give two degrees (a degree of membership and a degree of non-membership) of an element, however, picture fuzzy sets give three degrees (a degree of positive membership, a degree of neutral membership and a degree of negative membership, respectively) of an element. The notion of picture fuzzy sets provides a new method to represent some problems which are difficult to explain in other extensions of fuzzy set theory, such as human opinions involving four answers of type: yes, abstain, no and refusal.

The purpose of this paper is to combine the picture fuzzy set and soft set, from which we can obtain a new soft set model named picture fuzzy soft set. This new model makes descriptions of the objective world more realistic, practical and accurate in some cases. Meanwhile, we aim to solve decision making problems by using adjustable soft discernibility matrix. To facilitate our discussion, we first review some background on soft set, soft discernibility matrix, fuzzy soft set, intuitionistic fuzzy soft set and picture fuzzy set in Section 2. In Section 3, the concept of picture fuzzy soft set with its operation rules are presented. In Section 4, the picture fuzzy soft set is used to analyze a decision making problem and an adjustable algorithm is proposed. Finally, some conclusions are pointed out in Section 5.

2. Preliminaries

In this section, we will briefly recall the basic concepts of soft sets, soft discernibility matrices, fuzzy soft sets, intuitionistic fuzzy soft sets and picture fuzzy sets. See especially [4, 17–20, 30–32] for further details and background.

The soft set is defined by Molodtsov [4] in the following way.

Let U be an initial universe set of objects and E the set of parameters in relation to objects in U. Parameters are often attributes, characteristics or properties of objects. Let $P(U)$ denote the power set of U and $A \subseteq E$.

Definition 1. ([4]) A pair $\langle F, A\rangle$ is called a soft set over U, where F is a mapping given by $F : A \to P(U)$.

In other words, a soft set over U is a mapping from parameters to the power set of U, and it is not a kind of set in ordinary sense, but a parameterized family of subsets of U. For any parameter $\varepsilon \in A$, $F(\varepsilon)$ may be considered as the set of $\varepsilon-$approximate elements of the soft set $\langle F, A\rangle$.

Definition 2. ([30]) Let $\langle F, A\rangle$ be a soft set over U. Partition $U|IND\langle F, A\rangle = \{C_i : i \leq |U|\}$ is determined by F. The soft discernibility matrix is defined as $\mathcal{D} = (D(C_i, C_j))_{i,j\leq|U|}$, where

$$D(C_i, C_j) = \{E^i \cup E^j : i, j \leq |U|\}$$

is called the set of soft discernibility parameters between C_i and C_j. In which

$$E^i = \{e_l^i : F(h_i, e_l) = 1 \text{ and } F(h_j, e_l) = 0, \forall h_i \in C_i, \forall h_j \in C_j, e_l \in A\}$$

and

$$E^j = \{e_l^j : F(h_j, e_l) = 1 \text{ and } F(h_i, e_l) = 0, \forall h_i \in C_i, \forall h_j \in C_j, e_l \in A\}$$

the symbol e_l^i (or e_l^j) represents the objects in C_i (or C_j) have the value 1 at parameter e_l, that is, $F(h_i, e_l) = 1, h_i \in C_i$ (or $F(h_j, e_l) = 1, h_j \in C_j$).

By introducing the concepts of fuzzy sets and intuitionistic fuzzy sets into the theory of soft sets, Maji et al. [19, 20] proposed the notions of fuzzy soft sets and intuitionistic fuzzy soft sets as follows.

Definition 3. ([19]) Let $\mathcal{F}(U)$ be the set of all fuzzy subsets of U. Let E be a set of parameters and $A \subseteq E$. A pair $\langle F, A\rangle$ is called a fuzzy soft set over U, where F is a mapping given by $F : A \to \mathcal{F}(U)$.

Generally speaking, for $\forall\varepsilon \in A$, $F(\varepsilon)$ is a fuzzy subset of U and it is called fuzzy value set of parameter ε. It is easy to see that every soft set may be considered as a fuzzy soft set. If $\forall\varepsilon \in A$, $F(\varepsilon)$ is a crisp subset of U, then $\langle F, A\rangle$ is degenerated to be a standard soft set.

Definition 4. ([20]) Let U be a universe set and E be a set of parameters. $\mathcal{IF}(U)$ denotes the set of all intuitionistic fuzzy sets of U. Let $A \subseteq E$. A pair $\langle F, A\rangle$ is called an intuitionistic fuzzy soft set over U, where F is a mapping given by $F : A \to \mathcal{IF}(U)$.

For any parameter $\varepsilon \in A$, $F(\varepsilon)$ is an intuitionistic fuzzy subset of U and it is called intuitionistic fuzzy value set of parameter ε. Obviously, a fuzzy soft set can be considered as a special intuitionistic fuzzy soft set.

Recently, Cuong and Kreinovich [31, 32] introduced a new concept of picture fuzzy sets, which can be seen as directly extensions of fuzzy sets and of intuitionistic fuzzy sets. They proposed the concept as follows.

Definition 5. ([31]) A picture fuzzy set $\mathcal{A}$ on a universe U is an object of the form

$$\mathcal{A} = \{(x, \mu_{\mathcal{A}}(x), \eta_{\mathcal{A}}(x), \nu_{\mathcal{A}}(x))|x \in U\},$$

where $\mu_{\mathcal{A}}(x) \in [0, 1]$ is called the degree of positive membership of x in $\mathcal{A}$, $\eta_{\mathcal{A}}(x) \in [0, 1]$ is called the degree of neutral membership of x in $\mathcal{A}$ and $\nu_{\mathcal{A}}(x) \in [0, 1]$ is called the degree of negative membership of x in $\mathcal{A}$, and $\mu_{\mathcal{A}}$, $\eta_{\mathcal{A}}$ and $\nu_{\mathcal{A}}$ satisfy the following condition:

$$0 \le \mu_{\mathcal{A}}(x) + \eta_{\mathcal{A}}(x) + \nu_{\mathcal{A}}(x) \le 1, \ \forall x \in U.$$

Then for $x \in U$, $r_{\mathcal{A}}(x) = 1 - (\mu_{\mathcal{A}}(x) + \eta_{\mathcal{A}}(x) + \nu_{\mathcal{A}}(x))$ could be called the degree of refusal membership of x in $\mathcal{A}$.

Clearly, if $\forall x \in U$, $r_{\mathcal{A}}(x) = 0$, then $\mathcal{A}$ will be degenerated to be a standard intuitionistic fuzzy set. If $\forall x \in U, \eta_{\mathcal{A}}(x) = 0$ and $r_{\mathcal{A}}(x) = 0$, then $\mathcal{A}$ will be degenerated to be a classical fuzzy set. Let $PF(U)$ denote the set of all the picture fuzzy sets of U.

Basically, the model of picture fuzzy set may be adequate in situations when we face human opinions involving more answers of type: yes, abstain, no, refusal. Voting can be a good example of such a situation as the human voters may be divided into four groups of those who: vote for, abstain, vote against, refusal of the voting.

In [31, 32], the authors defined the following relations and operations on picture fuzzy sets and studied some basic properties.

Definition 6. ([31]) For $\mathcal{A}, \mathcal{B} \in PF(U)$, define

(1) $\mathcal{A} \subseteq \mathcal{B}$ iff $\mu_{\mathcal{A}}(x) \le \mu_{\mathcal{B}}(x)$, $\eta_{\mathcal{A}}(x) \le \eta_{\mathcal{B}}(x)$ and $\nu_{\mathcal{A}}(x) \ge \nu_{\mathcal{B}}(x), \forall x \in U$.
(2) $\mathcal{A} = \mathcal{B}$ iff $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{A}$.
(3) $\mathcal{A} \cup \mathcal{B} = \{(x, \mu_{\mathcal{A}}(x) \vee \mu_{\mathcal{B}}(x), \eta_{\mathcal{A}}(x) \wedge \eta_{\mathcal{B}}(x), \nu_{\mathcal{A}}(x) \wedge \nu_{\mathcal{B}}(x))|x \in U\}$.
(4) $\mathcal{A} \cap \mathcal{B} = \{(x, \mu_{\mathcal{A}}(x) \wedge \mu_{\mathcal{B}}(x), \eta_{\mathcal{A}}(x) \wedge \eta_{\mathcal{B}}(x), \nu_{\mathcal{A}}(x) \vee \nu_{\mathcal{B}}(x))|x \in U\}$.
(5) $\mathcal{A}^c = \{(x, \nu_{\mathcal{A}}(x), \eta_{\mathcal{A}}(x), \mu_{\mathcal{A}}(x))|x \in U\}$.

Proposition 1. *Let $\mathcal{A}, \mathcal{B}, \mathcal{C} \in PF(U)$. Then*

(1) If $\mathcal{A} \subseteq \mathcal{B}$ and $\mathcal{B} \subseteq \mathcal{C}$, then $\mathcal{A} \subseteq \mathcal{C}$.
(2) $(\mathcal{A}^c)^c = \mathcal{A}$.
(3) *Operations $\cap$ and $\cup$ are commutative, associative and distributive.*
(4) *Operations $\cap$ and $\cup$ satisfy DeMorgan's laws.*

3. Picture fuzzy soft sets

In this section, we introduce the concept of a picture fuzzy soft set and define some operations on a picture fuzzy soft set, namely subset, complement, AND, OR, union, intersection and so on.

Now we propose the definition of a picture fuzzy soft set and we give an illustrative example of it.

Definition 7. Let U be an initial universe set and E a set of parameters. By a picture fuzzy soft set (PFSS) over U we mean a pair $\langle F, A\rangle$, where $A \subseteq E$ and F is a mapping given by $F : A \to PF(U)$.

Clearly, a PFSS is a mapping from parameters to $PF(U)$. It is not a set, but it is a parameterized family of picture fuzzy subsets of U. For any $\varepsilon \in A$, $F(\varepsilon)$ is a picture fuzzy subset of U. Clearly, $F(\varepsilon)$ can be written as a picture fuzzy set such that $F(\varepsilon) = \{(x, \mu_{F(\varepsilon)}(x), \eta_{F(\varepsilon)}(x), \nu_{F(\varepsilon)}(x))|x \in U\}$, where $\mu_{F(\varepsilon)}(x)$, $\eta_{F(\varepsilon)}(x)$ and $\nu_{F(\varepsilon)}(x)$ are the positive membership, neutral membership and negative membership functions, respectively. If $\forall x \in U$, $\mu_{F(\varepsilon)}(x) + \eta_{F(\varepsilon)}(x) + \nu_{F(\varepsilon)}(x) = 1$, then $F(\varepsilon)$ will be degenerated to be a traditional intuitionistic fuzzy set and $\langle F, A\rangle$ will be degenerated to be an intuitionistic fuzzy soft set. If $\forall x \in U$, $\mu_{F(\varepsilon)}(x) + \nu_{F(\varepsilon)}(x) = 1$, then $F(\varepsilon)$ will be degenerated to be a classical fuzzy set and $\langle F, A\rangle$ will be degenerated to be a fuzzy soft set.

Example 1. Consider a PFSS $\langle F, A\rangle$ over U, where $U = \{h_1, h_2, h_3, h_4\}$ is the set of four houses under consideration of a decision maker to purchase, and $A = \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$ is the set of parameters, where ε_1 stands for the parameter 'cheap', ε_2 stands for the parameter 'beautiful' and ε_3 stands for the parameter 'in the good location'. The PFSS $\langle F, A\rangle$ describes the "attractiveness of the houses" to this decision maker. Suppose that

Table 1
Tabular representation of the PFSS $\langle F, A\rangle$.

U	ε_1	ε_2	ε_3
h_1	(0.3, 0.2, 0.4)	(0.5, 0.2, 0.1)	(0.6, 0.1, 0.2)
h_2	(0.5, 0.1, 0.3)	(0.8, 0.1, 0.0)	(0.1, 0.2, 0.5)
h_3	(0.1, 0.4, 0.2)	(0.3, 0.4, 0.2)	(0.5, 0.2, 0.2)
h_4	(0.4, 0.0, 0.5)	(0.2, 0.6, 0.1)	(0.2, 0.5, 0.1)

$F(\varepsilon_1)$ =cheap houses
$= \{(0.3, 0.2, 0.4)/h_1, (0.5, 0.1, 0.3)/h_2, (0.1, 0.4, 0.2)/h_3, (0.4, 0.0, 0.5)/h_4\}$,
$F(\varepsilon_2)$ =beautiful houses
$= \{(0.5, 0.2, 0.1)/h_1, (0.8, 0.1, 0.0)/h_2, (0.3, 0.4, 0.2)/h_3, (0.2, 0.6, 0.1)/h_4\}$,
$F(\varepsilon_3)$ =houses in the good location
$= \{(0.6, 0.1, 0.2)/h_1, (0.1, 0.2, 0.5)/h_2, (0.5, 0.2, 0.2)/h_3, (0.2, 0.5, 0.1)/h_4\}$.

The PFSS $\langle F, A\rangle$ is a parameterized family $\{F(\varepsilon_i), i = 1, 2, 3\}$ of picture fuzzy sets on U, and $\langle F, A\rangle = \{$cheap houses $= \{(0.3, 0.2, 0.4)/h_1, (0.5, 0.1, 0.3)/h_2, (0.1, 0.4, 0.2)/h_3, (0.4, 0.0, 0.5)/h_4\}$, beautiful houses$= \{(0.5, 0.2, 0.1)/h_1, (0.8, 0.1, 0.0)/h_2, (0.3, 0.4, 0.2)/h_3, (0.2, 0.6, 0.1)/h_4\}$, houses in the good location$= \{(0.6, 0.1, 0.2)/h_1, (0.1, 0.2, 0.5)/h_2, (0.5, 0.2, 0.2)/h_3, (0.2, 0.5, 0.1)/h_4\}\}$, where each approximation has two parts: (i) a predicate p, and (ii) an approximate value set v. Thus, each PFSS $\langle F, A\rangle$ can be viewed as a collection of approximation like $\langle F, A\rangle = \{p_i = v_i | i = 1, 2, \cdots, |A|\}$.

By analogy with soft sets, one easily sees that each PFSS can be viewed as an information system and be represented by a data table for the purpose of storing a PFSS in a computer. Table 1 is the tabular representation of the PFSS $\langle F, A\rangle$. For any $i = 1, 2, 3, 4$, $j = 1, 2, 3$, $h_{ij} = (\mu_{F(\varepsilon_j)}(h_i), \eta_{F(\varepsilon_j)}(h_i), \nu_{F(\varepsilon_j)}(h_i))$, where h_{ij} are the entries corresponding to the house h_i and the parameter ε_j.

For convenience of explanation, we can also represent the PFSS $\langle F, A\rangle$ which is described in the above in matrix form as follows: $\langle F, A\rangle =$

$$\begin{pmatrix} & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \\ h_1 & (0.3, 0.2, 0.4) & (0.5, 0.2, 0.1) & (0.6, 0.1, 0.2) \\ h_2 & (0.5, 0.1, 0.3) & (0.8, 0.1, 0.0) & (0.1, 0.2, 0.5) \\ h_3 & (0.1, 0.4, 0.2) & (0.3, 0.4, 0.2) & (0.5, 0.2, 0.2) \\ h_4 & (0.4, 0.0, 0.5) & (0.2, 0.6, 0.1) & (0.2, 0.5, 0.1) \end{pmatrix}.$$

Definition 8. Let $\langle F, A\rangle$ and $\langle G, B\rangle$ be two PFSSs over U. Then $\langle F, A\rangle$ is called a picture fuzzy soft subset of $\langle G, B\rangle$, denoted by $\langle F, A\rangle \tilde{\subseteq} \langle G, B\rangle$, if

(1) $A \subseteq B$, and
(2) $\forall \varepsilon \in A$, $F(\varepsilon) \subseteq G(\varepsilon)$.

Example 2. Let $U = \{x_1, x_2, x_3, x_4, x_5\}$ and $E = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5, \varepsilon_6\}$. Suppose that $\langle F, A\rangle$ and $\langle G, B\rangle$ are two PFSSs over U given by
$A = \{\varepsilon_1, \varepsilon_2, \varepsilon_3\}$, and $B = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_5\}$,
$\langle F, A\rangle =$

$$\begin{pmatrix} & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \\ x_1 & (0.2, 0.2, 0.5) & (0.1, 0.2, 0.6) & (0.6, 0.1, 0.3) \\ x_2 & (0.5, 0.1, 0.3) & (0.2, 0.1, 0.4) & (0.1, 0.2, 0.5) \\ x_3 & (0.1, 0.3, 0.2) & (0.1, 0.4, 0.2) & (0.5, 0.0, 0.2) \\ x_4 & (0.6, 0.0, 0.3) & (0.2, 0.0, 0.7) & (0.2, 0.4, 0.0) \\ x_5 & (0.2, 0.2, 0.6) & (0.2, 0.3, 0.1) & (0.2, 0.1, 0.5) \end{pmatrix}$$

$\langle G, B\rangle =$

$$\begin{pmatrix} & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_5 \\ x_1 & (0.4, 0.2, 0.3) & (0.1, 0.4, 0.2) & (0.7, 0.1, 0.2) & (0.1, 0.5, 0.3) \\ x_2 & (0.6, 0.2, 0.2) & (0.3, 0.1, 0.2) & (0.5, 0.3, 0.1) & (0.4, 0.2, 0.3) \\ x_3 & (0.3, 0.4, 0.1) & (0.2, 0.5, 0.2) & (0.8, 0.0, 0.1) & (0.1, 0.1, 0.1) \\ x_4 & (0.7, 0.1, 0.1) & (0.9, 0.0, 0.1) & (0.2, 0.5, 0.1) & (0.0, 0.7, 0.2) \\ x_5 & (0.2, 0.3, 0.5) & (0.4, 0.4, 0.1) & (0.3, 0.3, 0.2) & (0.8, 0.0, 0.0) \end{pmatrix}$$

Then $\langle F, A\rangle$ is a picture fuzzy soft subset of $\langle G, B\rangle$.

Definition 9. Two PFSSs $\langle F, A\rangle$ and $\langle G, B\rangle$ over U are called to be picture fuzzy soft equal, if and only if $\langle F, A\rangle$ is a picture fuzzy soft subset of $\langle G, B\rangle$ and $\langle G, B\rangle$ is a picture fuzzy soft subset of $\langle F, A\rangle$.

Definition 10. Let $\langle F, A\rangle$ be a PFSS over U. The complement of $\langle F, A\rangle$, denoted by $\langle F, A\rangle^c$, is defined by $\langle F, A\rangle^c = \langle F^c, A\rangle$, where $F^c : A \to PF(U)$ is a mapping given by $F^c(\varepsilon) = (F(\varepsilon))^c$ for all $\varepsilon \in A$.

It is worth noting that in the above definition, the parameter set of the complement $(\widetilde{\mathcal{F}}, A)^c$ is still the original parameter set A, instead of $\neg A$. Clearly, $(F^c)^c$ is the same as F and so $(\langle F, A\rangle^c)^c = \langle F, A\rangle$.

Example 3. Consider the PFSS $\langle F, A\rangle$ in Example 1. Then the complement of $\langle F, A\rangle$ is represented as follows:
$\langle F, A\rangle^c = \langle F^c, A\rangle =$

$$\begin{pmatrix} & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 \\ h_1 & (0.4, 0.2, 0.3) & (0.1, 0.2, 0.5) & (0.2, 0.1, 0.6) \\ h_2 & (0.3, 0.1, 0.5) & (0.0, 0.1, 0.8) & (0.5, 0.2, 0.1) \\ h_3 & (0.2, 0.4, 0.1) & (0.2, 0.4, 0.3) & (0.2, 0.2, 0.5) \\ h_4 & (0.5, 0.0, 0.4) & (0.1, 0.6, 0.2) & (0.1, 0.5, 0.2) \end{pmatrix}.$$

By the suggestions given by molodtsov in [4], we define the AND and OR operations on two PFSSs as follows.

Definition 11. If $\langle F, A\rangle$ and $\langle G, B\rangle$ are two PFSSs over U, the "$\langle F, A\rangle$ AND $\langle G, B\rangle$", denoted by $\langle F, A\rangle \wedge \langle G, B\rangle$, is defined by $\langle F, A\rangle \wedge \langle G, B\rangle = \langle H, A \times B\rangle$, where $H(\alpha, \beta) = F(\alpha) \cap G(\beta)$, for all $(\alpha, \beta) \in A \times B$.

Definition 12. If $\langle F, A\rangle$ and $\langle G, B\rangle$ are two PFSSs over U, the "$\langle F, A\rangle$ OR $\langle G, B\rangle$", denoted by $\langle F, A\rangle \vee \langle G, B\rangle$, is defined by $\langle F, A\rangle \vee \langle G, B\rangle = \langle \mathcal{O}, A \times B\rangle$, where $\mathcal{O}(\alpha, \beta) = F(\alpha) \cup G(\beta)$, for all $(\alpha, \beta) \in A \times B$.

Example 4. Let $U = \{x_1, x_2, x_3, x_4\}$ and $E = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4, \varepsilon_5\}$. Take $A = \{\varepsilon_1, \varepsilon_2\}$ and $B = \{\varepsilon_1, \varepsilon_3, \varepsilon_5\}$, define

$$\langle F, A\rangle = \begin{pmatrix} & \varepsilon_1 & \varepsilon_2 \\ x_1 & (0.1, 0.2, 0.6) & (0.4, 0.2, 0.3) \\ x_2 & (0.2, 0.1, 0.1) & (0.3, 0.1, 0.6) \\ x_3 & (0.7, 0.3, 0.0) & (0.5, 0.2, 0.3) \\ x_4 & (0.4, 0.0, 0.3) & (0.8, 0.0, 0.1) \end{pmatrix}$$

and

$\langle G, B\rangle =$

$$\begin{pmatrix} & \varepsilon_1 & \varepsilon_3 & \varepsilon_5 \\ x_1 & (0.4, 0.3, 0.2) & (0.2, 0.7, 0.1) & (0.6, 0.1, 0.2) \\ x_2 & (0.7, 0.1, 0.1) & (0.8, 0.0, 0.2) & (0.3, 0.2, 0.4) \\ x_3 & (0.3, 0.5, 0.1) & (0.2, 0.4, 0.2) & (0.5, 0.3, 0.0) \\ x_4 & (0.6, 0.1, 0.2) & (0.4, 0.3, 0.1) & (0.1, 0.1, 0.6) \end{pmatrix}.$$

Then we have $\langle F, A\rangle \wedge \langle G, B\rangle$ and $\langle F, A\rangle \vee \langle G, B\rangle$ as follows:

$\langle F, A\rangle \wedge \langle G, B\rangle = \langle H, A \times B\rangle =$

$$\left(\begin{matrix} & (\varepsilon_1, \varepsilon_1) & (\varepsilon_1, \varepsilon_3) & (\varepsilon_1, \varepsilon_5) \\ x_1 & (0.1, 0.2, 0.6) & (0.1, 0.2, 0.6) & (0.1, 0.1, 0.6) \\ x_2 & (0.2, 0.1, 0.1) & (0.2, 0.0, 0.2) & (0.2, 0.1, 0.4) \\ x_3 & (0.3, 0.3, 0.1) & (0.2, 0.3, 0.2) & (0.5, 0.3, 0.0) \\ x_4 & (0.4, 0.0, 0.3) & (0.4, 0.0, 0.3) & (0.1, 0.0, 0.6) \end{matrix}\right.$$

$$\left.\begin{matrix} (\varepsilon_2, \varepsilon_1) & (\varepsilon_2, \varepsilon_3) & (\varepsilon_2, \varepsilon_5) \\ (0.4, 0.2, 0.3) & (0.2, 0.2, 0.3) & (0.4, 0.1, 0.3) \\ (0.3, 0.1, 0.6) & (0.3, 0.0, 0.6) & (0.3, 0.1, 0.6) \\ (0.3, 0.2, 0.3) & (0.2, 0.2, 0.3) & (0.5, 0.2, 0.3) \\ (0.6, 0.0, 0.2) & (0.4, 0.0, 0.1) & (0.1, 0.0, 0.6) \end{matrix}\right)$$

and

$\langle F, A\rangle \vee \langle G, B\rangle = \langle \mathcal{O}, A \times B\rangle =$

$$\left(\begin{matrix} & (\varepsilon_1, \varepsilon_1) & (\varepsilon_1, \varepsilon_3) & (\varepsilon_1, \varepsilon_5) \\ x_1 & (0.4, 0.2, 0.2) & (0.2, 0.2, 0.1) & (0.6, 0.1, 0.2) \\ x_2 & (0.7, 0.1, 0.1) & (0.8, 0.0, 0.1) & (0.3, 0.1, 0.1) \\ x_3 & (0.7, 0.3, 0.0) & (0.7, 0.3, 0.0) & (0.7, 0.3, 0.0) \\ x_4 & (0.6, 0.0, 0.2) & (0.4, 0.0, 0.1) & (0.4, 0.0, 0.3) \end{matrix}\right.$$

$$\left.\begin{matrix} (\varepsilon_2, \varepsilon_1) & (\varepsilon_2, \varepsilon_3) & (\varepsilon_2, \varepsilon_5) \\ (0.4, 0.2, 0.2) & (0.4, 0.2, 0.1) & (0.6, 0.1, 0.2) \\ (0.7, 0.1, 0.1) & (0.8, 0.0, 0.2) & (0.3, 0.1, 0.4) \\ (0.5, 0.2, 0.1) & (0.5, 0.2, 0.2) & (0.5, 0.2, 0.0) \\ (0.8, 0.0, 0.1) & (0.8, 0.0, 0.1) & (0.8, 0.0, 0.1) \end{matrix}\right)$$

Theorem 1. *Let $\langle F, A\rangle$ and $\langle G, B\rangle$ be two PFSSs over U. Then*

(1) $(\langle F, A\rangle \wedge \langle G, B\rangle)^c = \langle F, A\rangle^c \vee \langle G, B\rangle^c$.
(2) $(\langle F, A\rangle \vee \langle G, B\rangle)^c = \langle F, A\rangle^c \wedge \langle G, B\rangle^c$.

Proof. (1)Suppose that $\langle F, A\rangle \wedge \langle G, B\rangle = \langle H, A \times B\rangle$. Therefore $(\langle F, A\rangle \wedge \langle G, B\rangle)^c = \langle H, A \times B\rangle^c = \langle H^c, A \times B\rangle$. Now, $\langle F, A\rangle^c \vee \langle G, B\rangle^c = \langle F^c, A\rangle \vee \langle G^c, B\rangle = \langle \mathcal{O}, A \times B\rangle$. Take $(\alpha, \beta) \in A \times B$, therefore, $H^c(\alpha, \beta) = (H(\alpha, \beta))^c = (F(\alpha) \cap G(\beta))^c = F^c(\alpha) \cup G^c(\beta)$, again, $\mathcal{O}(\alpha, \beta) = F^c(\alpha) \cup G^c(\beta)$. Hence, $H^c(\alpha, \beta) = \mathcal{O}(\alpha, \beta)$. Proved.

(2) The result can be proved in the similar way.

Definition 13. Union of two PFSSs $\langle F, A\rangle$ and $\langle G, B\rangle$ over U can be defined as $\langle F, A\rangle \tilde{\cup} \langle G, B\rangle = \langle H, C\rangle$, where $C = A \cup B$, and $\forall \varepsilon \in C$,

$$H(\varepsilon)=\begin{cases}F(\varepsilon), & \text{if } \varepsilon\in A-B,\\ G(\varepsilon), & \text{if } \varepsilon\in B-A,\\ F(\varepsilon)\cup G(\varepsilon), & \text{if } \varepsilon\in A\cap B.\end{cases}$$

Theorem 2. *Union of two PFSSs $\langle F, A\rangle$ and $\langle G, B\rangle$ is a PFSS.*

Proof. In fact, according to Definition 13, $\forall\varepsilon\in C$, if $\varepsilon\in A-B$ or $\varepsilon\in B-A$, then $H(\varepsilon)=F(\varepsilon)$ or $H(\varepsilon)=G(\varepsilon)$. So, in either case, we have $H(\varepsilon)$ is a picture fuzzy set.

If $\varepsilon\in A\cap B$, for a fixed $x\in U$, without loss of generality, suppose $\mu_{F(\varepsilon)}(x)\le\mu_{G(\varepsilon)}(x)$, we have

$$\begin{aligned}&\mu_{H(\varepsilon)}(x)+\eta_{H(\varepsilon)}(x)+\nu_{H(\varepsilon)}(x)\\ =\;&\mu_{F(\varepsilon)}(x)\vee\mu_{G(\varepsilon)}(x)+\eta_{F(\varepsilon)}(x)\wedge\eta_{G(\varepsilon)}(x)+\nu_{F(\varepsilon)}(x)\wedge\nu_{G(\varepsilon)}(x)\\ =\;&\mu_{G(\varepsilon)}(x)+\eta_{F(\varepsilon)}(x)\wedge\eta_{G(\varepsilon)}(x)+\nu_{F(\varepsilon)}(x)\wedge\nu_{G(\varepsilon)}(x)\\ \le\;&\mu_{G(\varepsilon)}(x)+\eta_{G(\varepsilon)}(x)+\nu_{G(\varepsilon)}(x)\le 1.\end{aligned}$$

Therefore, $\langle H, C\rangle$ is a picture fuzzy soft set.

Example 5. Consider Example 4. We have $\langle F, A\rangle\tilde{\cup}\langle G, B\rangle=\langle H, C\rangle$, where $C=A\cup B=\{\varepsilon_1,\varepsilon_2,\varepsilon_3,\varepsilon_5\}$, and $\langle F, A\rangle\tilde{\cup}\langle G, B\rangle=\langle H, C\rangle=$

$$\begin{pmatrix} & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_5\\ x_1 & (0.4, 0.2, 0.2) & (0.4, 0.2, 0.3) & (0.2, 0.7, 0.1) & (0.6, 0.1, 0.2)\\ x_2 & (0.7, 0.1, 0.1) & (0.3, 0.1, 0.6) & (0.8, 0.0, 0.2) & (0.3, 0.2, 0.4)\\ x_3 & (0.7, 0.3, 0.0) & (0.5, 0.2, 0.3) & (0.2, 0.4, 0.2) & (0.5, 0.3, 0.0)\\ x_4 & (0.6, 0.0, 0.2) & (0.8, 0.0, 0.1) & (0.4, 0.3, 0.1) & (0.1, 0.1, 0.6)\end{pmatrix}$$

Definition 14. Intersection of two PFSSs $\langle F, A\rangle$ and $\langle G, B\rangle$ with $A\cap B\neq\phi$ over U can be defined as $\langle F, A\rangle\tilde{\cap}\langle G, B\rangle=\langle H, C\rangle$, where $C=A\cap B$, and $\forall\varepsilon\in C$,

$$H(\varepsilon)=F(\varepsilon)\cap G(\varepsilon).$$

Theorem 3. *Intersection of two PFSSs $\langle F, A\rangle$ and $\langle G, B\rangle$ is a PFSS.*

Proof. According to Definition 14, $\forall\varepsilon\in A\cap B$, for a fixed $x\in U$, without loss of generality, suppose $\nu_{F(\varepsilon)}(x)\le\nu_{G(\varepsilon)}(x)$, we have

$$\begin{aligned}&\mu_{H(\varepsilon)}(x)+\eta_{H(\varepsilon)}(x)+\nu_{H(\varepsilon)}(x)\\ =\;&\mu_{F(\varepsilon)}(x)\wedge\mu_{G(\varepsilon)}(x)+\eta_{F(\varepsilon)}(x)\wedge\eta_{G(\varepsilon)}(x)+\nu_{F(\varepsilon)}(x)\vee\nu_{G(\varepsilon)}(x)\\ =\;&\mu_{F(\varepsilon)}(x)\wedge\mu_{G(\varepsilon)}(x)+\eta_{F(\varepsilon)}(x)\wedge\eta_{G(\varepsilon)}(x)+\nu_{G(\varepsilon)}(x)\\ \le\;&\mu_{G(\varepsilon)}(x)+\eta_{G(\varepsilon)}(x)+\nu_{G(\varepsilon)}(x)\le 1.\end{aligned}$$

Therefore, $\langle H, C\rangle$ is a picture fuzzy soft set.

Example 6. Reconsider Example 4. We have $\langle F, A\rangle\tilde{\cap}\langle G, B\rangle=\langle H, C\rangle$, where $C=A\cap B=\{\varepsilon_1\}$, and

$$\langle F, A\rangle\tilde{\cap}\langle G, B\rangle=\langle H, C\rangle=\begin{pmatrix} & \varepsilon_1\\ x_1 & (0.1, 0.2, 0.6)\\ x_2 & (0.2, 0.1, 0.1)\\ x_3 & (0.3, 0.3, 0.1)\\ x_4 & (0.4, 0.0, 0.3)\end{pmatrix}.$$

Theorem 4. *Let $\langle F, A\rangle$, $\langle G, B\rangle$ and $\langle H, C\rangle$ be PFSSs over U. Then*

(1) $\langle F, A\rangle\tilde{\cup}\langle F, A\rangle=\langle F, A\rangle$.
(2) $\langle F, A\rangle\tilde{\cap}\langle F, A\rangle=\langle F, A\rangle$.
(3) $\langle F, A\rangle\tilde{\cup}\langle G, B\rangle=\langle G, B\rangle\tilde{\cup}\langle F, A\rangle$.
(4) $\langle F, A\rangle\tilde{\cap}\langle G, B\rangle=\langle G, B\rangle\tilde{\cap}\langle F, A\rangle$.
(5) $(\langle F, A\rangle\tilde{\cup}\langle G, B\rangle)\tilde{\cup}\langle H, C\rangle=\langle F, A\rangle\tilde{\cup}(\langle G, B\rangle\tilde{\cup}\langle H, C\rangle)$.
(6) $(\langle F, A\rangle\tilde{\cap}\langle G, B\rangle)\tilde{\cap}\langle H, C\rangle=\langle F, A\rangle\tilde{\cap}(\langle G, B\rangle\tilde{\cap}\langle H, C\rangle)$.

Proof. The proofs are straightforward by using the definitions of union and intersection and Proposition 1.

Theorem 5. *Let $\langle F, A\rangle$, $\langle G, B\rangle$ and $\langle H, C\rangle$ be PFSSs over U. Then*

(1) $\langle F, A\rangle\tilde{\cup}(\langle G, B\rangle\tilde{\cap}\langle H, C\rangle)=(\langle F, A\rangle\tilde{\cup}\langle G, B\rangle)\tilde{\cap}(\langle F, A\rangle\tilde{\cup}\langle H, C\rangle)$.
(2) $\langle F, A\rangle\tilde{\cap}(\langle G, B\rangle\tilde{\cup}\langle H, C\rangle)=(\langle F, A\rangle\tilde{\cap}\langle G, B\rangle)\tilde{\cup}(\langle F, A\rangle\tilde{\cap}\langle H, C\rangle)$.

Proof. The proofs are straightforward by using the fact that the union and intersection of picture fuzzy sets are distributive in Proposition 1.

Theorem 6. *(the dual law) Let $\langle F, A\rangle$ and $\langle G, B\rangle$ be PFSSs over U. Then*

(1) $(\langle F, A\rangle \tilde{\cup} \langle G, B\rangle)^c = \langle F, A\rangle^c \tilde{\cap} \langle G, B\rangle^c$ iff $A = B$.
(2) $(\langle F, A\rangle \tilde{\cap} \langle G, B\rangle)^c = \langle F, A\rangle^c \tilde{\cup} \langle G, B\rangle^c$ iff $A = B$.

Proof. (1) If $A = B$, then we have $\langle F, A\rangle \tilde{\cup} \langle G, B\rangle = \langle F, A\rangle \tilde{\cup} \langle G, A\rangle = \langle H, A\rangle$. Now for $\forall \varepsilon \in A$, $H(\varepsilon) = F(\varepsilon) \cup G(\varepsilon)$. Hence $(\langle F, A\rangle \tilde{\cup} \langle G, B\rangle)^c = (\langle F, A\rangle \tilde{\cup} \langle G, A\rangle)^c = \langle H, A\rangle^c = \langle H^c, A\rangle$, and $H^c(\varepsilon) = (F(\varepsilon) \cup G(\varepsilon))^c = F^c(\varepsilon) \cap G^c(\varepsilon)$.

Again suppose that $\langle F, A\rangle^c \tilde{\cap} \langle G, B\rangle^c = \langle F, A\rangle^c \tilde{\cap} \langle G, A\rangle^c = \langle F^c, A\rangle \tilde{\cap} \langle G^c, A\rangle = \langle I, A\rangle$, and $\forall \varepsilon \in A$, $I(\varepsilon) = F^c(\varepsilon) \cap G^c(\varepsilon)$.

We see that $\forall \varepsilon \in A$, $I(\varepsilon) = H^c(\varepsilon)$. Therefore, the result is true.

Conversely, hypothesize $A \neq B$. Suppose that $\langle F, A\rangle \tilde{\cup} \langle G, B\rangle = \langle H, C\rangle$, where $C = A \cup B$, and $\forall \varepsilon \in C$,

$$H(\varepsilon) = \begin{cases} F(\varepsilon), & \text{if } \varepsilon \in A - B, \\ G(\varepsilon), & \text{if } \varepsilon \in B - A, \\ F(\varepsilon) \cup G(\varepsilon), & \text{if } \varepsilon \in A \cap B. \end{cases}$$

Thus $(\langle F, A\rangle \tilde{\cup} \langle G, B\rangle)^c = \langle H, C\rangle^c = \langle H^c, C\rangle$, and

$$H^c(\varepsilon) = \begin{cases} F^c(\varepsilon), & \text{if } \varepsilon \in A - B, \\ G^c(\varepsilon), & \text{if } \varepsilon \in B - A, \\ F^c(\varepsilon) \cap G^c(\varepsilon), & \text{if } \varepsilon \in A \cap B. \end{cases}$$

Again suppose that $\langle F, A\rangle^c \tilde{\cap} \langle G, B\rangle^c = \langle F^c, A\rangle \tilde{\cap} \langle G^c, B\rangle = \langle I, J\rangle$, where $J = A \cap B$, and $\forall \varepsilon \in J$, $I(\varepsilon) = F^c(\varepsilon) \cap G^c(\varepsilon)$.

Obviously, when $A \neq B$, we have $C = A \cup B \neq A \cap B = J$, so, $\langle H, C\rangle^c \neq \langle I, J\rangle$. This contradicts the condition $(\langle F, A\rangle \tilde{\cup} \langle G, B\rangle)^c = \langle F, A\rangle^c \tilde{\cap} \langle G, B\rangle^c$. Hence, $A = B$.

(2)The result can be proved in the similar way.

From the above theory, we know that DeMorgan's laws are invalid for PFSSs with the different parameter sets, but they are true for PFSSs with the same parameter set.

4. Application of the picture fuzzy soft set model based on adjustable soft discernibility matrix

Combining the algorithm based on soft discernibility matrix [30] with the decision making methods regard to fuzzy soft sets, intuitionistic fuzzy soft sets and interval-valued intuitionistic fuzzy soft sets in [5, 10, 14], in this section, we present an adjustable approach to picture fuzzy soft set based decision making problems. Using this approach, we not only choose the optimal object, but also can obtain an order relation among all the objects. We first propose the concept of level soft sets of a PFSS.

Definition 15. Let $\varpi = \langle F, A\rangle$ be a PFSS over U. For a triple $(r, s, t) \in [0, 1]^3$, the $(r, s, t)-$level soft set of ϖ is a crisp soft set $L(\varpi; (r, s, t)) = \langle F_{(r,s,t)}, A\rangle$ defined by

$$F_{(r,s,t)}(\varepsilon) = L(F(\varepsilon); (r, s, t)) = \{x \in U | \mu_{F(\varepsilon)}(x) \geq r, \eta_{F(\varepsilon)}(x) \leq s \text{ and } \nu_{F(\varepsilon)}(x) \leq t\}, \text{ for all } \varepsilon \in A.$$

In the above definition, $r \in [0, 1]$ can be viewed as a given least threshold on the degree of positive membership, $s \in [0, 1]$ can be viewed as a given greatest threshold on the degree of neutral membership, and $t \in [0, 1]$ can be viewed as a given greatest threshold on the degree of negative membership. For practical applications, the thresholds are pre-established by decision makers and represent their requirements.

To illustrate this idea, let us consider the following example.

Example 7. Assume that Mr.X wants to buy a house, and describes the "attractiveness of houses" by a PFSS $\langle F, A\rangle$.

Suppose that there five houses $U = \{h_1, h_2, h_3, h_4, h_5\}$ under consideration and that $A = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ is a set of decision parameters. The $\varepsilon_i (i = 1, 2, 3, 4)$ stand for the parameters 'expensive', 'large', 'beautiful' and 'in the good location', respectively.

Suppose that $\varpi = \langle F, A\rangle =$

$$\begin{pmatrix} & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 \\ h_1 & (0.3, 0.4, 0.2) & (0.4, 0.2, 0.2) & (0.5, 0.1, 0.2) & (0.3, 0.2, 0.4) \\ h_2 & (0.7, 0.0, 0.1) & (0.5, 0.2, 0.1) & (0.1, 0.3, 0.2) & (0.5, 0.2, 0.1) \\ h_3 & (0.4, 0.3, 0.1) & (0.3, 0.4, 0.2) & (0.6, 0.2, 0.1) & (0.4, 0.2, 0.3) \\ h_4 & (0.2, 0.1, 0.3) & (0.6, 0.1, 0.2) & (0.8, 0.1, 0.1) & (0.6, 0.2, 0.2) \\ h_5 & (0.6, 0.2, 0.1) & (0.2, 0.3, 0.1) & (0.4, 0.0, 0.0) & (0.4, 0.0, 0.5) \end{pmatrix}$$

Now we take $(r, s, t) = (0.4, 0.2, 0.3)$, then we have the following results:

Table 2
Tabular representation of the $(0.4, 0.2, 0.3)-$level soft set $L(\varpi; (0.4, 0.2, 0.3))$

U	ε_1	ε_2	ε_3	ε_4
h_1	0	1	1	0
h_2	1	1	0	1
h_3	0	0	1	1
h_4	0	1	1	1
h_5	1	0	1	0

$L(F(\varepsilon_1); (0.4, 0.2, 0.3)) = \{h_2, h_5\}$,
$L(F(\varepsilon_2); (0.4, 0.2, 0.3)) = \{h_1, h_2, h_4\}$,
$L(F(\varepsilon_3); (0.4, 0.2, 0.3)) = \{h_1, h_3, h_4, h_5\}$,
$L(F(\varepsilon_4); (0.4, 0.2, 0.3)) = \{h_2, h_3, h_4\}$.

Hence, the $(0.4, 0.2, 0.3)-$level soft set of $\varpi = \langle F, A\rangle$ is a soft set $L(\varpi; (0.4, 0.2, 0.3))$. Table 2 gives the tabular representation of $L(\varpi; (0.4, 0.2, 0.3))$.

Sometimes, decision makers need to impose different thresholds on different parameters. To cope with such problems, we use a function to replace a constant value triple as the thresholds on positive membership value, neutral membership value and negative membership value, respectively.

Definition 16. Let $\varpi = \langle F, A\rangle$ be a PFSS over U. Let $\lambda : A \to [0, 1]^3$ be a function, i.e. $\forall\varepsilon \in A$, $\lambda(\varepsilon) = (r(\varepsilon), s(\varepsilon), t(\varepsilon))$, and $r(\varepsilon), s(\varepsilon), t(\varepsilon) \in [0, 1]$. The level soft set of ϖ with respect to λ is a crisp soft set $L(\varpi; \lambda) = \langle F_\lambda, A\rangle$ defined by

$F_\lambda(\varepsilon) = L(F(\varepsilon); \lambda(\varepsilon)) = \{x \in U | \mu_{F(\varepsilon)}(x) \geq r(\varepsilon), \eta_{F(\varepsilon)}(x) \leq s(\varepsilon) \text{ and } \nu_{F(\varepsilon)}(x) \leq t(\varepsilon)\}$,
for all $\varepsilon \in A$.

Remark 1. Here the function $\lambda : A \to [0, 1]^3$ is not restricted to be a picture fuzzy set, it is only a function, $r(\varepsilon)$ can be viewed as a given least threshold w.r.t. the parameter ε on the degree of positive membership, $s(\varepsilon)$ can be viewed as a given greatest threshold w.r.t. the parameter ε on the degree of neutral membership, and $t(\varepsilon)$ can be viewed as a given greatest threshold w.r.t. the parameter ε on the degree of negative membership. In Definition 5 in [10], λ is an intuitionistic fuzzy set in A which is called a threshold intuitionistic fuzzy set, in fact, the top-top-threshold is not an intuitionistic fuzzy set as shown in Example 3 in [10].

Let $\varpi = \langle F, A\rangle$ be a PFSS over U. The familiar four threshold functions are shown as follows.

i. the Mid-level threshold function mid_ϖ:

$\text{mid}_\varpi : A \to [0, 1]^3$, i.e. $\text{mid}_\varpi(\varepsilon) = (r_{\text{mid}_\varpi}(\varepsilon), s_{\text{mid}_\varpi}(\varepsilon), t_{\text{mid}_\varpi}(\varepsilon))$ for all $\varepsilon \in A$, where

$$r_{\text{mid}_\varpi}(\varepsilon) = \frac{1}{|U|}\sum_{x\in U}\mu_{F(\varepsilon)}(x),$$
$$s_{\text{mid}_\varpi}(\varepsilon) = \frac{1}{|U|}\sum_{x\in U}\eta_{F(\varepsilon)}(x),$$
$$t_{\text{mid}_\varpi}(\varepsilon) = \frac{1}{|U|}\sum_{x\in U}\nu_{F(\varepsilon)}(x).$$

The function mid_ϖ is called the mid-threshold of $\varpi = \langle F, A\rangle$, the level soft set w.r.t. mid_ϖ, namely $L(\varpi; \text{mid}_\varpi)$ is called the mid-level soft set of ϖ.

ii. the Top-bottom-bottom-level (simply Tbb-level) threshold function tbb_ϖ:

$\text{tbb}_\varpi : A \to [0, 1]^3$, i.e. $\text{tbb}_\varpi(\varepsilon) = (r_{\text{tbb}_\varpi}(\varepsilon), s_{\text{tbb}_\varpi}(\varepsilon), t_{\text{tbb}_\varpi}(\varepsilon))$ for all $\varepsilon \in A$, where

$$r_{\text{tbb}_\varpi}(\varepsilon) = \max_{x\in U}\mu_{F(\varepsilon)}(x),$$
$$s_{\text{tbb}_\varpi}(\varepsilon) = \min_{x\in U}\eta_{F(\varepsilon)}(x),$$
$$t_{\text{tbb}_\varpi}(\varepsilon) = \min_{x\in U}\nu_{F(\varepsilon)}(x).$$

The function tbb_ϖ is called the top-bottom-bottom-threshold of $\varpi = \langle F, A\rangle$, the level soft set w.r.t. tbb_ϖ, namely $L(\varpi; \text{tbb}_\varpi)$ is called the top-bottom-bottom-level soft set of ϖ.

iii. the Bottom-bottom-bottom-level (simply Bbb-level) threshold function bbb_ϖ:

$\text{bbb}_\varpi : A \to [0, 1]^3$, i.e. $\text{bbb}_\varpi(\varepsilon) = (r_{\text{bbb}_\varpi}(\varepsilon), s_{\text{bbb}_\varpi}(\varepsilon), t_{\text{bbb}_\varpi}(\varepsilon))$ for all $\varepsilon \in A$, where

$$r_{\text{bbb}_\varpi}(\varepsilon) = \min_{x\in U}\mu_{F(\varepsilon)}(x),$$
$$s_{\text{bbb}_\varpi}(\varepsilon) = \min_{x\in U}\eta_{F(\varepsilon)}(x),$$
$$t_{\text{bbb}_\varpi}(\varepsilon) = \min_{x\in U}\nu_{F(\varepsilon)}(x).$$

The function bbb_ϖ is called the bottom-bottom-bottom-threshold of $\varpi = \langle F, A\rangle$, the level soft set w.r.t. bbb_ϖ, namely $L(\varpi; \text{bbb}_\varpi)$ is called the bottom-bottom-bottom-level soft set of ϖ.

iv. the Med-level threshold function med_ϖ:

$\text{med}_\varpi : A \to [0, 1]^3$, i.e. $\text{med}_\varpi(\varepsilon) = (r_{\text{med}_\varpi}(\varepsilon), s_{\text{med}_\varpi}(\varepsilon), t_{\text{med}_\varpi}(\varepsilon))$ for all $\varepsilon \in A$,
where for $\forall\varepsilon \in A$, $r_{\text{med}_\varpi}(\varepsilon)$ is the median by ranking the degree of positive membership of all alternatives according to order from large to small (or from small to large), namely

$$r_{\text{med}_\varpi}(\varepsilon) = \begin{cases} \mu_{F(\varepsilon)}\left(x_{\left(\frac{|U|+1}{2}\right)}\right), & \text{if } |U| \text{ is odd,} \\ (\mu_{F(\varepsilon)}\left(x_{\left(\frac{|U|}{2}\right)}\right) + \\ \mu_{F(\varepsilon)}\left(x_{\left(\frac{|U|}{2}+1\right)}\right))/2, & \text{if } |U| \text{ is even.} \end{cases}$$

$s_{\text{med}_{\varpi}}(\varepsilon)$ is the median by ranking the degree of neutral membership of all alternatives according to order from large to small (or from small to large), namely

$$s_{\text{med}_{\varpi}}(\varepsilon)=\begin{cases}\eta_{F(\varepsilon)}(x_{\left(\frac{|U|+1}{2}\right)}), & \text{if } |U| \text{ is odd,}\\ (\eta_{F(\varepsilon)}(x_{\left(\frac{|U|}{2}\right)})+ & \\ \eta_{F(\varepsilon)}(x_{\left(\frac{|U|}{2}+1\right)}))/2, & \text{if } |U| \text{ is even.}\end{cases}$$

and $t_{\text{med}_{\varpi}}(\varepsilon)$ is the median by ranking the degree of negative membership of all alternatives according to order from large to small (or from small to large), namely

$$t_{\text{med}_{\varpi}}(\varepsilon)=\begin{cases}\nu_{F(\varepsilon)}(x_{\left(\frac{|U|+1}{2}\right)}), & \text{if } |U| \text{ is odd,}\\ (\nu_{F(\varepsilon)}(x_{\left(\frac{|U|}{2}\right)})+ & \\ \nu_{F(\varepsilon)}(x_{\left(\frac{|U|}{2}+1\right)}))/2, & \text{if } |U| \text{ is even.}\end{cases}$$

The function med_{ϖ} is called the med-threshold of $\varpi=\langle F, A\rangle$, the level soft set w.r.t. med_{ϖ}, namely $L(\varpi;\text{med}_{\varpi})$ is called the med-level soft set of ϖ.

Example 8. Let us reconsider the PFSS $\varpi=\langle F, A\rangle$ with its matrix form shown in Example 7. The above mentioned thresholds and the level soft sets with their tabular representations are given as follows.

i. $\text{mid}_{\varpi}=\{\langle\varepsilon_1,(0.44, 0.20, 0.16)\rangle, \langle\varepsilon_2,(0.40, 0.24, 0.16)\rangle, \langle\varepsilon_3,(0.48, 0.14, 0.12)\rangle, \langle\varepsilon_4,(0.44, 0.16, 0.30)\rangle\}$

ii. $\text{tbb}_{\varpi}=\{\langle\varepsilon_1,(0.7, 0.0, 0.1)\rangle, \langle\varepsilon_2,(0.6, 0.1, 0.1)\rangle, \langle\varepsilon_3,(0.8, 0.0, 0.0)\rangle, \langle\varepsilon_4,(0.6, 0.0, 0.1)\rangle\}$

iii. $\text{bbb}_{\varpi}=\{\langle\varepsilon_1,(0.2, 0.0, 0.1)\rangle, \langle\varepsilon_2,(0.2, 0.1, 0.1)\rangle, \langle\varepsilon_3,(0.1, 0.0, 0.0)\rangle, \langle\varepsilon_4,(0.3, 0.0, 0.1)\rangle\}$

iv. $\text{med}_{\varpi}=\{\langle\varepsilon_1,(0.4, 0.2, 0.1)\rangle, \langle\varepsilon_2,(0.40, 0.2, 0.2)\rangle, \langle\varepsilon_3,(0.5, 0.1, 0.1)\rangle, \langle\varepsilon_4,(0.4, 0.2, 0.3)\rangle\}$

Remark 2. The function mid_{ϖ} sets threshold triples by use of mean value of all positive membership, neutral membership and negative membership degrees under different attributes. But in reality, mean value is easily affected by extreme value. So decision makers may not choose the mid-level decision rule under extreme circumstances. To avoid adverse effect of extreme value, they often use other decision rule (usually the med-level decision rule) instead of the mid-level decision rule.

Table 3
Tabular representation of the mid-level soft set $L(\varpi;\text{mid}_{\varpi})$

U	ε_1	ε_2	ε_3	ε_4
h_1	0	0	0	0
h_2	1	1	0	0
h_3	0	0	0	0
h_4	0	0	1	0
h_5	1	0	0	0

Table 4
Tabular representation of the top-bottom-bottom-level soft set $L(\varpi;\text{tbb}_{\varpi})$

U	ε_1	ε_2	ε_3	ε_4
h_1	0	0	0	0
h_2	1	0	0	0
h_3	0	0	0	0
h_4	0	0	0	0
h_5	0	0	0	0

Table 5
Tabular representation of the bottom-bottom-bottom-level soft set $L(\varpi;\text{bbb}_{\varpi})$

U	ε_1	ε_2	ε_3	ε_4
h_1	0	0	0	0
h_2	1	0	0	0
h_3	0	0	0	0
h_4	0	0	0	0
h_5	0	0	1	0

Table 6
Tabular representation of the med-level soft set $L(\varpi;\text{med}_{\varpi})$

U	ε_1	ε_2	ε_3	ε_4
h_1	0	1	0	0
h_2	1	1	0	1
h_3	0	0	0	1
h_4	0	1	1	1
h_5	1	0	0	0

According to the discussions above, once the level soft set has been arrived at, an order relation of the objects can be easily obtained from the soft discernibility matrix as pointed out in [30]. Now a novel algorithm based on adjustable soft discernibility matrix for solving the problems of decision making will be given as follows.

Algorithm. Decision making based on adjustable soft discernibility matrix

1. Input the picture fuzzy soft set $\varpi=\langle F, A\rangle$.
2. Input a threshold function $\lambda: A\rightarrow[0,1]^3$ (or give a threshold triple $(r, s, t)\in[0,1]^3$; or choose the mid-level decision rule; or choose the top-bottom-bottom-level decision rule; or choose the bottom-bottom-bottom-level decision rule; or choose the med-level decision rule) for decision making.

3. Compute the level soft set $L(\varpi;\lambda)$ of $\varpi = \langle F, A\rangle$ with respect to the threshold function λ (or the $(r, s, t)-$level soft set $L(\varpi;(r, s, t))$; or the mid-level soft set $L(\varpi;\mathrm{mid}_{\varpi})$; or the top-bottom-bottom-level soft set $L(\varpi;\mathrm{tbb}_{\varpi})$; or the bottom-bottom-bottom-level soft set $L(\varpi;\mathrm{bbb}_{\varpi})$; or the med-level soft set $L(\varpi;\mathrm{med}_{\varpi})$).
4. Present the level soft set $L(\varpi;\lambda)$ (or $L(\varpi;(r, s, t))$; or $L(\varpi;\mathrm{mid}_{\varpi})$; or $L(\varpi;\mathrm{tbb}_{\varpi})$; or $L(\varpi;\mathrm{bbb}_{\varpi})$; or $L(\varpi;\mathrm{med}_{\varpi})$) in tabular form.
5. Compute the partition of U and the soft discernibility matrix $\mathcal{D} = D(C_i, C_j)$.
6. Select D_1 and D_2 from the soft discernibility matrix, respectively, where

   $D_1 = \{D(C_i, C_j) : |D(C_i, C_j)| = 2n, n \in N^+\}$ and $D_2 = \{D(C_i, C_j) : |D(C_i, C_j)| = 2n + 1, n \in N^+\}$.

7. For every element of D_1, if $|E^i| = |E^j|$, then the object(s) $h_i \in C_i$ and $h_j \in C_j$ are kept in the same class. Otherwise, there must exist an order relation between $h_i \in C_i$ and $h_j \in C_j$, that is, either h_i is superior to h_j, or h_j is superior to h_i.
8. Output the result of the step 7. If it is a global relation for all of the objects in U, then turn to the step 11, otherwise, turn to the next step.
9. Combine with the result of the step 8, find the corresponding elements in D_2 to compare the order relation.
10. Output the order relation among all the objects by combining the step 8 with the step 9.
11. Choose the optimal object(s) at the first place lined according to order relation from large to small. If the optimal object is more than one, then any one of them may be chosen.

Remark 3. In the last step of the above algorithm, one may go back to the second step and change the threshold (or decision rule) that one once used so as to adjust the final optimal decision, especially when there are too many "optimal choices" to be chosen.

The following example is utilized to illustrate the basic idea of Algorithm given above.

Example 9. Assume there is an investment company, which wants to invest a sum of money in the best option (adapted from [33]). Let us consider a PFSS $\varpi = \langle F, A\rangle$ which describes the "attractiveness of projects" that the investment company is considering for investment. Suppose there are six alternative projects under consideration, $U = \{p_1, p_2, p_3, p_4, p_5, p_6\}$. The investment company must take a decision according to criteria set $A = \{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$, where ε_1 stands for the risk analysis, ε_2 stands for the growth analysis, ε_3 stands for the social-political impact analysis, and ε_4 stands for the environment analysis. The company evaluates the alternatives p_i $(i = 1, 2, 3, 4, 5, 6)$ with respect to the criteria ε_j $(j = 1, 2, 3, 4)$ and constructs a PFSS $\varpi = \langle F, A\rangle$ given as follows.

$$\varpi = \langle F, A\rangle = \begin{pmatrix} & \varepsilon_1 & \varepsilon_2 & \varepsilon_3 & \varepsilon_4 \\ p_1 & (0.31, 0.22, 0.41) & (0.54, 0.21, 0.15) & (0.60, 0.14, 0.26) & (0.38, 0.21, 0.40) \\ p_2 & (0.12, 0.41, 0.33) & (0.81, 0.11, 0.02) & (0.26, 0.51, 0.20) & (0.65, 0.15, 0.18) \\ p_3 & (0.23, 0.52, 0.21) & (0.13, 0.48, 0.37) & (0.72, 0.15, 0.03) & (0.29, 0.58, 0.12) \\ p_4 & (0.45, 0.09, 0.36) & (0.23, 0.59, 0.18) & (0.32, 0.49, 0.15) & (0.14, 0.32, 0.45) \\ p_5 & (0.57, 0.30, 0.05) & (0.60, 0.23, 0.14) & (0.81, 0.11, 0.06) & (0.43, 0.18, 0.35) \\ p_6 & (0.44, 0.40, 0.13) & (0.42, 0.36, 0.22) & (0.43, 0.27, 0.13) & (0.35, 0.29, 0.34) \end{pmatrix}$$

Suppose the investment company would like to select the optimal and the suboptimal projects to invest. As an adjustable approach, one can use different rules (or the thresholds) in decision making problem. For example, if the company deals with this problem by med-level decision rule, it is clear that the med-threshold of $\varpi = \langle F, A\rangle$ is

$$\mathrm{med}_{\varpi} = \{\langle \varepsilon_1, (0.375, 0.35, 0.27)\rangle, \langle \varepsilon_2, (0.48, 0.295, 0.165)\rangle, \langle \varepsilon_3, (0.515, 0.21, 0.14)\rangle, \langle \varepsilon_4, (0.365, 0.25, 0.345)\rangle\}$$

and we shall obtain the med-level soft set $L(\varpi;\mathrm{med}_{\varpi})$ of ϖ with tabular representation as in Table 7.

From Table 7, we can obtain the partition of U is $\{C_1 = \{p_1\}, C_2 = \{p_2\}, C_3 = \{p_3\}, C_4 = \{p_4, p_6\}, C_5 = \{p_5\}\}$ and the constructed soft discernibility matrix is as Table 8.

Table 7
Tabular representation of the med-level soft set $L(\varpi; \mathrm{med}_{\varpi})$

U	ε_1	ε_2	ε_3	ε_4
p_1	0	1	0	0
p_2	0	1	0	1
p_3	0	0	1	0
p_4	0	0	0	0
p_5	1	1	1	0
p_6	0	0	0	0

Table 8
The soft discernibility matrix of the med-level soft set $L(\varpi; \mathrm{med}_{\varpi})$

	C_1	C_2	C_3	C_4	C_5
C_1	ϕ				
C_2	$\{\varepsilon_4^2\}$	ϕ			
C_3	$\{\varepsilon_2^1, \varepsilon_3^3\}$	$\{\varepsilon_2^2, \varepsilon_3^3, \varepsilon_4^2\}$	ϕ		
C_4	$\{\varepsilon_2^1\}$	$\{\varepsilon_2^2, \varepsilon_4^2\}$	$\{\varepsilon_3^3\}$	ϕ	
C_5	$\{\varepsilon_1^5, \varepsilon_3^5\}$	$\{\varepsilon_1^5, \varepsilon_3^5, \varepsilon_4^2\}$	$\{\varepsilon_1^5, \varepsilon_2^5\}$	$\{\varepsilon_1^5, \varepsilon_2^5, \varepsilon_3^5\}$	ϕ

From Table 8, we have
$D_1 = \{D(C_3, C_1), D(C_4, C_2), D(C_5, C_1), D(C_5, C_3)\}$
$D_2 = \{D(C_2, C_1), D(C_3, C_2), D(C_4, C_1), D(C_4, C_3), D(C_5, C_2), D(C_5, C_4)\}$.

In D_1, we note that $|E^1| = |E^3|$ in $D(C_3, C_1)$. So the objects in C_3 and C_1 are in the same decision class, that is, p_3 and p_1 are in the same decision class. Since $|E^2| > |E^3|$ in $D(C_3, C_2)$, the objects in C_2 are superior to the objects in C_3, that is, p_2 is superior to p_3. Thus we have $p_2 > \{p_1, p_3\}$. In D_2, we note that $|E^5| > |E^2|$ in $D(C_5, C_2)$ and $|E^1| > |E^4|$ in $D(C_4, C_1)$, so the objects in C_5 are superior to the objects in C_2 and the objects in C_1 are superior to the objects in C_4, that is, p_5 is superior to p_2 and p_1 is superior to p_4 and p_6. With the above discussions, we have $p_5 > p_2 > \{p_1, p_3\} > \{p_4, p_6\}$. So an order relation is obtained. And the optimal decision is to select p_5 and the suboptimal is to select p_2. Therefore, the company should select the project p_5 as the best project and project p_2 as the second-best choice to invest.

5. Conclusion

In this paper, we generalize the concept of soft sets. Concretely, we propose the concept of the picture fuzzy soft set, which is a combination of a picture fuzzy set and a soft set. We then define various operations on picture fuzzy soft sets and study their properties. Especially, we prove DeMorgan's laws in the theory of picture fuzzy soft sets. Finally, an illustrative example is used to show the validity of the picture fuzzy soft set by using adjustable soft discernibilit matrix in a decision making problem.

This new extension not only provides a significant mathematical model to deal with uncertainties, but also leads to potential areas of further field research and pertinent applications. We hope that our work would help enhancing this study on picture fuzzy soft sets for the researchers.

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