Florentin Smarandache

Advances
of Standard and Nonstandard
Neutrosophic
Theories
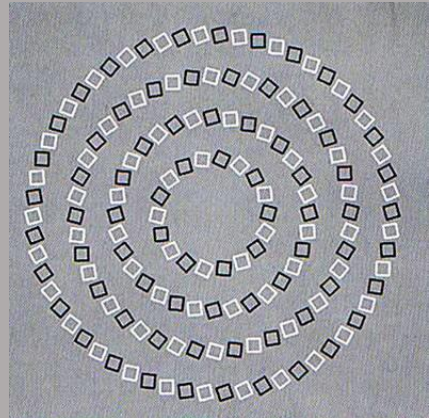
Florentin Smarandache

Advances of Standard and Nonstandard Neutrosophic Theories

Florentin Smarandache

Advances
of Standard
and
Nonstandard
Neutrosophic
Theories

CONTENTS

5

CHAPTER 2

REFINED NEUTROSOPHY & LATTICES VS. PAIR STRUCTURES & YINYANG BIPOLAR FUZZY SET 96

CHAPTER 3

ABOUT NONSTANDARD NEUTROSOPHIC LOGIC (ANSWERS TO IMAMURA'S "NOTE ON THE DEFINITION OF NEUTROSOPHIC LOGIC") 129

CHAPTER 4

EXTENDED NONSTANDARD NEUTROSOPHIC LOGIC, SET, AND PROBABILITY BASED ON EXTENDED NONSTANDARD ANALYSIS 153

CHAPTER 6

INTRODUCTION TO NEUTROALGEBRAIC STRUCTURES AND ANTIALGEBRAIC STRUCTURES (REVISITED)........................240

CHAPTER 7

NEW DEVELOPMENTS IN NEUTROSOPHIC THEORIES AND APPLICATIONS

PREFACE

In this book, we approach different topics related to neutrosophics, such as: *Neutrosophic Set, Intuitionistic Fuzzy Set, Inconsistent Intuitionistic Fuzzy Set, Picture Fuzzy Set, Ternary Fuzzy Set, Pythagorean Fuzzy Set, Atanassov's Intuitionistic Fuzzy Set of second type, Spherical Fuzzy Set, n-HyperSpherical Neutrosophic Set, q-Rung Orthopair Fuzzy Set, truth-membership, indeterminacy-membership, falsehood-nonmembership, Regret Theory, Grey System Theory, Three-Ways Decision, n-Ways Decision, Neutrosophy, Neutrosophication, Neutrosophic Probability, Refined Neutrosophy, Refined Neutrosophication, Nonstandard Analysis; Extended Nonstandard Analysis; Open and Closed Monads to the Left/Right; Pierced and Unpierced Binads; MoBiNad Set; infinitesimals; infinities; nonstandard reals; standard reals; Nonstandard Neutrosophic Lattices of First Type (as poset) and Second Type (as algebraic structure); Nonstandard Neutrosophic Logic; Extended Nonstandard Neutrosophic Logic; Nonstandard Arithmetic Operations; Nonstandard Unit Interval; Nonstandard Neutrosophic Infimum; Nonstandard Neutrosophic Supremum, Plithogeny; Plithogenic Set; Neutrosophic Set; Plithogenic Operators, Neutrosophic Triplets, (Axiom, NeutroAxiom, AntiAxiom), (Law, NeutroLaw, AntiLaw), (Associativity, NeutroAssociaticity, AntiAssociativity), (Commutativity, NeutroCommutativity, Anti-Commutativity), (WellDefined, NeutroDefined, AntiDefined), (Semigroup, NeutroSemigroup, AntiSemigroup), (Group, NeutroGroup, AntiGroup), (Ring, NeutroRing, AntiRing), (Algebraic Structures, NeutroAlgebraic Structures, AntiAlgebraic Structures), (Structure, NeutroStructure, AntiStructure), (Theory, NeutroTheory, AntiTheory), S-denying an Axiom, Multispace with Multistructure*, and so on.

In the first chapter (**Neutrosophic Set is a Generalization of Intuitionistic Fuzzy Set, Inconsistent Intuitionistic Fuzzy Set (Picture Fuzzy Set, Ternary Fuzzy Set), Pythagorean Fuzzy Set (Atanassov's Intuitionistic Fuzzy Set of second type), q-Rung Orthopair Fuzzy Set,**

Spherical Fuzzy Set, and n-HyperSpherical Fuzzy Set, while Neutrosophication is a Generalization of Regret Theory, Grey System Theory, and Three-Ways Decision - revisited), we prove that Neutrosophic Set (NS) is an extension of Intuitionistic Fuzzy Set (IFS) no matter if the sum of single-valued neutrosophic components is < 1, or > 1, or = 1. For the case when the sum of components is 1 (as in IFS), after applying the neutrosophic aggregation operators one gets a different result from that of applying the intuitionistic fuzzy operators, since the intuitionistic fuzzy operators ignore the indeterminacy, while the neutrosophic aggregation operators take into consideration the indeterminacy at the same level as truth-membership and falsehood-nonmembership are taken. NS is also more flexible and effective because it handles, besides independent components, also partially independent and partially dependent components, while IFS cannot deal with these. Since there are many types of indeterminacies in our world, we can construct different approaches to various neutrosophic concepts.

Neutrosophic Set (NS) is also a generalization of Inconsistent Intuitionistic Fuzzy Set (IIFS) { which is equivalent to the Picture Fuzzy Set (PFS) and Ternary Fuzzy Set (TFS) }, Pythagorean Fuzzy Set (PyFS) {Atanassov's Intuitionistic Fuzzy Set of second type}, Spherical Fuzzy Set (SFS), n-HyperSpherical Fuzzy Set (n-HSFS), and q-Rung Orthopair Fuzzy Set (q-ROFS). And Refined Neutrosophic Set (RNS) is an extension of Neutrosophic Set. And all these sets are more general than Intuitionistic Fuzzy Set.

We prove that Atanassov's Intuitionistic Fuzzy Set of second type (AIFS2), and Spherical Fuzzy Set (SFS) do not have independent components. And we show that n-HyperSphericalFuzzy Set that we now introduce for the first time, Spherical Neutrosophic Set (SNS) and n-HyperSpherical Neutrosophic Set (n-HSNS) {the last one also introduced now for the first time} are generalizations of IFS2 and SFS.

The main distinction between Neutrosophic Set (NS) and all previous set theories are: a) the *independence* of all three neutrosophic components {truth-membership (T), indeterminacy-membership (I),

falsehood-nonmembership (F)} with respect to each other in NS – while in the previous set theories their components are dependent of each other; and b) the importance of *indeterminacy* in NS - while in previous set theories indeterminacy is completely or partially ignored.

Neutrosophy is a particular case of Refined Neutrosophy, and consequently Neutrosophication is a particular case of Refined Neutrosophication. Also, Regret Theory, Grey System Theory, and Three-Ways Decision are particular cases of Neutrosophication and of Neutrosophic Probability. We have extended the Three-Ways Decision to n-Ways Decision, which is a particular case of Refined Neutrosophy.

In 2016 Smarandache defined for the first time the Refined Fuzzy Set (RFS) and Refined Fuzzy Intuitionistic Fuzzy Set (RIFS). We now, further on, define for the first time: Refined Inconsistent Intuitionistic Fuzzy Set (RIIFS){Refined Picture Fuzzy Set (RPFS), Refined Ternary Fuzzy Set (RTFS)}, Refined Pythagorean Fuzzy Set (RPyFS) {Refined Atanassov's Intuitionistic Fuzzy Set of type 2 (RAIFS2)}, Refined Spherical Fuzzy Set (RSFS), Refined n-HyperSpherical Fuzzy Set (R-n-HSFS), and Refined q-Rung Orthopair Fuzzy Set (R-q-ROFS).

In the second chapter (**Refined Neutrosophy & Lattices vs. Pair Structures & YinYang Bipolar Fuzzy Set**), we present the lattice structures of neutrosophic theories, we prove that Zhang-Zhang's YinYang Bipolar Fuzzy Set is a subclass of Single-Valued Bipolar Neutrosophic Set. Then we show that the Pair Structure is a particular case of Refined Neutrosophy, and the number of types of neutralities (sub-indeterminacies) may be any finite or infinite number.

The third chapter (**About Nonstandard Neutrosophic Logic - Answers to Imamura's "Note on the Definition of Neutrosophic Logic"**) intends to answer Imamura's criticism that we found benefic in better understanding the nonstandard neutrosophic logic – although the nonstandard neutrosophic logic was never used in practical applications.

In order to more accurately situate and fit the neutrosophic logic into the framework of nonstandard analysis, we present the neutrosophic inequalities, neutrosophic equality, neutrosophic infimum and supremum,

neutrosophic standard intervals, including the cases when the neutrosophic logic standard and nonstandard components T, I, F get values outside of the classical unit interval [0, 1], and a brief evolution of neutrosophic operators.

In the fourth chapter (**Extended Nonstandard Neutrosophic Logic, Set, and Probability based on Extended Nonstandard Analysis**), we extend for the second time the Nonstandard Analysis by adding the left monad closed to the right, and right monad closed to the left, while besides the pierced binad (we introduced in 1998) we add now the unpierced binad - all these in order to close the newly extended nonstandard space under nonstandard addition, nonstandard subtraction, nonstandard multiplication, nonstandard division, and nonstandard power operations. Then, we extend the Nonstandard Neutrosophic Logic, Nonstandard Neutrosophic Set, and Nonstandard Probability on this Extended Nonstandard Analysis space - that we prove it is a nonstandard neutrosophic lattice of first type (endowed with a nonstandard neutrosophic partial order) as well as a nonstandard neutrosophic lattice of second type (as algebraic structure, endowed with two binary neutrosophic laws, infN and supN). Many theorems, new terms introduced, better notations for monads and binads, and examples of nonstandard neutrosophic operations are given.

The fifth chapter (**Plithogenic Set and Hypersoft Set**) has two parts. The first part (**Plithogenic Set, an Extension of Crisp, Fuzzy, Intuitionistic Fuzzy, and Neutrosophic Sets - revisited**) introduces the plithogenic set (as generalization of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets), which is a set whose elements are characterized by many attributes' values. An attribute value v has a corresponding (fuzzy, intuitionistic fuzzy, neutrosophic or other types of sets) degree of appurtenance d(x,v) of the element x, to the set P, with respect to some given criteria. In order to obtain a better accuracy for the plithogenic aggregation operators in the plithogenic set, and for a more exact inclusion (partial order), a (fuzzy, intuitionistic fuzzy, or neutrosophic) contradiction (dissimilarity) degree is defined between each attribute value and the dominant (most important) attribute value. The plithogenic

intersection and union are linear combinations of the fuzzy operators tnorm and tconorm, while the plithogenic complement, inclusion (inequality), equality are influenced by the attribute values contradiction (dissimilarity) degrees. This article offers some examples and applications of these new concepts in our everyday life. The second part (**Extension of Soft Set to Hypersoft Set, and then to Plithogenic Hypersoft Set**) generalizes the soft set to the hypersoft set by transforming the function F into a multi-attribute function. Then we introduce the hybrids of Crisp, Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Plithogenic Hypersoft Set. Plithogeny (generalization of Dialectics and Neutrosophy), and Plithogenic Set/Logic/Probability/Statistics (generalization of fuzzy, intuitionistic fuzzy, neutrosophic set/logic/probability/statistics) were introduced by Smarandache in 2017.

In the sixth chapter (**Introduction to NeutroAlgebraic Structures and AntiAlgebraic Structures - revisited**), we opened for the first time [in 2019] new fields of research called NeutroStructures and AntiStructures respectively.

In all classical algebraic structures, the Laws of Compositions on a given set are well-defined. But this is a restrictive case, because there are many more situations in science and in any domain of knowledge when a law of composition defined on a set may be only partially-defined (or partially true) and partially-undefined (or partially false), that we call NeutroDefined, or totally undefined (totally false) that we call AntiDefined.

Again, in all classical algebraic structures, the Axioms (Associativity, Commutativity, etc.) defined on a set are totally true, but it is again a restrictive case, because similarly there are numerous situations in science and in any domain of knowledge when an Axiom defined on a set may be only partially-true (and partially-false), that we call NeutroAxiom, or totally false that we call AntiAxiom.

Finally, the seventh chapter (**New Developments in Neutrosophic Theories and Applications**) presents suggestions for future research in the area of neutrosophics, e.g. *neutrality and indeterminacy, types of indeterminacies, completeness or incompleteness in neutrosophy,*

dependence and independence of sources providing information, geometric representation of neutrosophic cubic set, nonstandard neutrosophic algebraic structure, three-ways model, division of quadruple neutrosophic numbers, neutrosophic quaternions, neutrosophic physics laws, neutrosophic physical constants, neutrosophic sorites paradox, determinate and indeterminate parts of a sky cloud, example of bipolar neutrosophic set, neutrosophic triplet hypertopology, plithogenic set in combination with all previous set-types, plithogenic graph, neutrosophic dynamic system: easier to break from inside, than from outside, degree of democracy, degree of indeterminate-democracy, and degree of antidemocracy, neutrosophic example in military, neutrosophic random variable, neutrosophic risk, neutrosophic satisfiability & neutrosophic randomness, neutrosophication vs. regret theory, expert systems vs. neutrosophic implications, neutrosophic applications in literature, arts, criminal justice, philosophy, and history, neutrosophy in arts and letters, and so on.

CHAPTER 1

Neutrosophic Set is a Generalization of Intuitionistic Fuzzy Set, Inconsistent Intuitionistic Fuzzy Set (Picture Fuzzy Set, Ternary Fuzzy Set), Pythagorean Fuzzy Set (Atanassov's Intuitionistic Fuzzy Set of second type), q-Rung Orthopair Fuzzy Set, Spherical Fuzzy Set, and n-HyperSpherical Fuzzy Set, while Neutrosophication is a Generalization of Regret Theory, Grey System Theory, and Three-Ways Decision (revisited)

Abstract

In this paper we prove that Neutrosophic Set (NS) is an extension of Intuitionistic Fuzzy Set (IFS) no matter if the sum of single-valued neutrosophic components is < 1, or > 1, or $= 1$. For the case when the sum of components is 1 (as in IFS), after applying the neutrosophic aggregation operators one gets a different result from that of applying the intuitionistic fuzzy operators, since the intuitionistic fuzzy operators ignore the indeterminacy, while the neutrosophic aggregation operators take into consideration the indeterminacy at the same level as truth-membership and falsehood-nonmembership are taken. NS is also more flexible and effective because it handles, besides independent components, also partially independent and partially dependent components, while IFS cannot deal with these. Since there are many types of indeterminacies in our world, we can construct different approaches to various neutrosophic concepts.

Neutrosophic Set (NS) is also a generalization of Inconsistent Intuitionistic Fuzzy Set (IIFS) { which is equivalent to the Picture Fuzzy Set (PFS) and Ternary Fuzzy Set (TFS) }, Pythagorean Fuzzy Set (PyFS) {Atanassov's Intuitionistic Fuzzy Set of second type}, Spherical Fuzzy Set (SFS), n-HyperSpherical Fuzzy Set (n-HSFS), and q-Rung Orthopair Fuzzy Set (q-ROFS). And Refined Neutrosophic Set (RNS) is an extension of Neutrosophic Set. And all these sets are more general than Intuitionistic Fuzzy Set.

We prove that Atanassov's Intuitionistic Fuzzy Set of second type (AIFS2), and Spherical Fuzzy Set (SFS) do not have independent components. And we show that n-HyperSphericalFuzzy Set that we now introduce for the first time, Spherical Neutrosophic Set (SNS) and n-HyperSpherical Neutrosophic Set (n-HSNS) {the last one also introduced now for the first time} are generalizations of IFS2 and SFS.

The main distinction between Neutrosophic Set (NS) and all previous set theories are: a) the *independence* of all three neutrosophic components {truth-membership (T), indeterminacy-membership (I), falsehood-nonmembership (F)} with respect to each other in NS – while in the previous set theories their components are dependent of each other; and b) the importance of *indeterminacy* in NS - while in previous set theories indeterminacy is completely or partially ignored.

Neutrosophy is a particular case of Refined Neutrosophy, and consequently Neutrosophication is a particular case of Refined Neutrosophication. Also, Regret Theory, Grey System Theory, and Three-Ways Decision are particular cases of Neutrosophication and of Neutrosophic Probability. We have extended the Three-Ways Decision to n-Ways Decision, which is a particular case of Refined Neutrosophy.

In 2016 Smarandache defined for the first time the Refined Fuzzy Set (RFS) and Refined Fuzzy Intuitionistic Fuzzy Set (RIFS). We now, further on, define for the first time: Refined Inconsistent Intuitionistic Fuzzy Set (RIIFS){Refined Picture Fuzzy Set (RPFS), Refined Ternary Fuzzy Set (RTFS)}, Refined Pythagorean Fuzzy Set (RPyFS) {Refined Atanassov's Intuitionistic Fuzzy Set of type 2 (RAIFS2)}, Refined Spherical Fuzzy Set (RSFS), Refined n-HyperSpherical Fuzzy Set (R-n-HSFS), and Refined q-Rung Orthopair Fuzzy Set (R-q-ROFS).

Keywords

Neutrosophic Set, Intuitionistic Fuzzy Set, Inconsistent Intuitionistic Fuzzy Set, Picture Fuzzy Set, Ternary Fuzzy Set, Pythagorean Fuzzy Set, Atanassov's Intuitionistic Fuzzy Set of second type, Spherical Fuzzy Set, n-HyperSpherical Neutrosophic Set, q-Rung Orthopair Fuzzy Set, truth-membership, indeterminacy-membership, falsehood-nonmembership, Regret Theory, Grey System Theory, Three-Ways Decision, n-Ways Decision, Neutrosophy, Neutrosophication, Neutrosophic Probability, Refined Neutrosophy, Refined Neutrosophication.

1.1. Introduction

This paper recalls ideas about the distinctions between neutrosophic set and intuitionistic fuzzy set presented in previous versions of this paper [1, 2, 3, 4, 5].

Mostly, in this paper we respond to Atanassov and Vassiliev's paper [6] about the fact that neutrosophic set is a generalization of intuitionistic fuzzy set.

We use the notations employed in the neutrosophic environment [1, 2, 3, 4, 5] since they are better descriptive than the Greek letters used in intuitionistic fuzzy environment, i.e.:

truth-membership (T), indeterminacy-membership (I), and falsehood-nonmembership (F).

We also use the triplet components in this order: (T, I, F).

Neutrosophic "Fuzzy" Set (as named by Atanassov and Vassiliev [6]) is commonly called "Single-Valued" Neutrosophic Set (i.e. the neutrosophic components are single-valued numbers) by the neutrosophic community that now riches about 1,000 researchers, from 60 countries around the world, which have produced about 2,000 publications (papers, conference presentations, book chapters, books, MSc theses, and PhD dissertations).

The NS is more complex and more general than previous (crisp / fuzzy / intuitionistic fuzzy / picture fuzzy / ternary fuzzy set / Pythagorean fuzzy / Atanassov's intuitionistic fuzzy set of second type / spherical fuzzy / q-Rung orthopair fuzzy) sets, because:

– A new branch of philosophy was born, called *Neutrosophy* [7], which is a generalization of Dialectics (and of YinYang Chinese philosophy), where not only the dynamics of opposites are studied, but the dynamics of opposites together with their neutrals as well, i.e. (<A>, <neutA>, <antiA>), where <A> is an item, <antiA> its opposite, and <neutA> their neutral (indeterminacy between them).

– Neutrosophy show the significance of neutrality / indeterminacy (<neutA>) that gave birth to neutrosophic set / logic / probability / statistics / measure / integral and so on, that have many practical applications in various fields.

– The sum of the Single-Valued Neutrosophic Set/Logic components was allowed to be up to 3 (this shows the importance of independence of the neutrosophic components among themselves), which permitted the characterization of paraconsistent/conflictual sets/propositions (by letting the sum of components > 1), and of paradoxical sets/propositions, represented by the neutrosophic triplet (1, 1, 1).

– NS can distinguish between *absolute* truth /indeterminacy /falsehood and *relative* truth/indeterminacy/falsehood using nonstandard analysis, which generated the Nonstandard Neutrosophic Set (NNS).

– Each neutrosophic component was allowed to take values outside of the interval [0, 1], that culminated with the introduction of the neutrosophic overset/underset/offset [8].

– NS was enlarged by Smarandache to Refined Neutrosophic Set (RNS), where each neutrosophic component was refined / split into sub-components [9]., i.e. T was refined/split into $T_1, T_2, \ldots, T_p$; I was refined / split into $I_1, I_2, \ldots, I_p$; and F was refined split into $F_1, F_2, \ldots, F_s$; where p, r, s $\geq$ 1 are integers and $p + r + s \geq 4$; all T_j, I_k, F_l are subsets of [0, 1] with no other restriction.

– RNS permitted the extension of the Law of Included Middle to the neutrosophic Law of Included Multiple-Middle [10].

– Classical Probability and Imprecise Probability were extended to Neutrosophic Probability [11], where for each event E one has: the chance that event E occurs (ch(E)), indeterminate-chance that event E occurs or not (ch(neutE)), and the chance that the event E does not occur (ch(antiE)), with: *0 ≤ sup{ch(E)} + sup{ch(neutE)} + sup{ch(antiE)} ≤ 3*.

– Classical Statistics was extended to Neutrosophic Statistics [12] that deals with indeterminate / incomplete / inconsistent / vague data regarding samples and populations.

And so on (see below more details). Several definitions are recalled for paper's self-containment.

1.2. Refinements of Fuzzy Types Sets

In 2016 Smarandache [8] introduced for the first time the Refined Fuzzy Set (RFS) and Refined Fuzzy Intuitionistic Fuzzy Set (RIFS).

Let $\mathcal{U}$ be a universe of discourse, and let $A \subset \mathcal{U}$ be a subset.

We give general definitions, meaning that the components may be any subsets of [0, 1]. In particular cases, the components may be single numbers, hesitant sets, intervals and so on included in [0, 1].

1.3. Fuzzy Set (FS)

A_{FS} = { $x(T_A(x))$, $x \in \mathcal{U}$ }, where T_A: U $\longrightarrow$ P([0, 1]) is the membership degree of the generic element x with respect to the set A, and P([0, 1]) is the powerset of [0, 1], is called a Fuzzy Set.

1.4. Refined Fuzzy Set (RFS)

We have split/refined the membership degree $T_A(x)$ into sub-membership degrees. Then:

$A_{RFS} = \{x(T_A^1(x), T_A^2(x),...,T_A^p(x)), p \geq 2, x \in U\}$, where $T_A^1(x)$ is a sub-membership degree of type 1 of the element x with respect to the set A, $T_A^2(x)$ is a sub-membership degree of type 2 of the element x with respect to the set A, ..., $T_A^p(x)$ is a sub-membership degree of type p of the element x with respect to the set A, and $T_A^j(x) \subseteq [0,1]$ for $1 \leq j \leq p$, and $\sum_{j=1}^{p} sup T_x^j \leq 1$ for all $x \in U$.

1.5. Intuitionistic Fuzzy Set (IFS)

Let $\mathcal{U}$ be a universe of discourse, and let $A \subset \mathcal{U}$ be a subset. Then:

A_{IFS} = { $x(T_A(x), F_A(x))$, $x \in \mathcal{U}$ }, where $T_A(x)$, $F_A(x)$: U $\longrightarrow$ P ([0, 1]) are the membership degree respectively the nonmembership of the generic element x with respect to the set A, and P ([0, 1]) is the powerset of [0, 1], and $sup T_A(x) + sup F_A(x) \leq 1$ for all $x \in U$, is called an Intuitionistic Fuzzy Set.

1.6. Refined Intuitionistic Fuzzy Set (RIFS)

We have split/refined the membership degree $T_A(x)$ into sub-membership degrees, and the nonmembership degree $F_A(x)$. Then:

$$A_{RIFS} = \{x(T_A^1(x), T_A^2(x), ..., T_A^p(x); F_A^1(x), F_A^2(x), ..., F_A^s(x)), p + s \geq 3, x \in U\},$$

with p, s positive nonzero integers, $\sum_{j=1}^{p} supT_x^j + \sum_{l=1}^{s} supF_x^l \leq 1$, and $T_A^j(x), F_A^l(x) \subseteq [0,1]$ for $1 \leq j \leq p$ and $1 \leq l \leq s$.

Where $T_A^1(x)$ is a sub-membership degree of type 1 of the element x with respect to the set A, $T_A^2(x)$ is a sub-membership degree of type 2 of the element x with respect to the set A, ..., $T_A^p(x)$ is a sub-membership degree of type p of the element x with respect to the set A.

And $F_A^1(x)$ is a sub-nonmembership degree of type 1 of the element x with respect to the set A, $F_A^2(x)$ is a sub-nonmembership degree of type 2 of the element x with respect to the set A, ..., $F_A^s(x)$ is a sub-nonmembership degree of type s of the element x with respect to the set A.

1.7. Inconsistent Intuitionistic Fuzzy Set (IIFS) { Picture Fuzzy Set (PFS), Ternary Fuzzy Set (TFS) }

Are defined as below:

$$A_{IIFS} = A_{PFS} = A_{TFS} = \{\langle x, T_A(x), I_A(x), F_A(x) \rangle | x \in \mathcal{U}\},$$

where $T_A(x), I_A(x), F_A(x) \in$ P([0,1]) and the sum $0 \leq supT_A(x) + supI_A(x) + supF_A(x) \leq 1$, for all $x \in \mathcal{U}$.

In these sets, the denominations are:

$T_A(x)$ is called *degree of membership* (or validity, or positive membership);

$I_A(x)$ is called *degree of neutral membership*;

$F_A(x)$ is called *degree of nonmembership* (or nonvalidity, or negative membership).

The *refusal degree* is: $R_A(x) = 1 - T_A(x) - I_A(x) - F_A(x) \in [0, 1]$, for all $x \in \mathcal{U}$.

1.8. Refined Inconsistent Intuitionistic Fuzzy Set (RIIFS) { Refined Picture Fuzzy Set (RPFS), Refined Ternary Fuzzy Set (RTFS) }

$$A_{RIIFS} = A_{RPFS} = A_{RTFS} = \{x(T_A^1(x), T_A^2(x), ..., T_A^p(x); I_A^1(x), I_A^2(x), ..., I_A^r(x);$$
$$F_A^1(x), F_A^2(x), ..., F_A^s(x)), p + r + s \geq 4, x \in U\},$$

with *p, r, s* positive nonzero integers, and:

$$T_A^j(x), I_A^k(x), F_A^l(x) \subseteq [0,1], \text{ for } 1 \leq j \leq p, \ 1 \leq k \leq r, \text{ and } 1 \leq l \leq s,$$

$$0 \leq \sum_1^p \sup T_A^j(x) + \sum_1^r \sup I_A^k(x) + \sum_1^s \sup F_A^l(x) \leq 1.$$

$T_A^j(x)$ is called *degree of sub-membership* (or sub-validity, or positive sub-membership) of type *j* of the element *x* with respect to the set *A*;

$I_A^k(x)$ is called *degree of sub-neutral membership* of type *k* of the element *x* with respect to the set *A*;

$F_A^l(x)$ is called *degree of sub-nonmembership* (or sub-nonvalidity, or negative sub-membership) of type *l* of the element *x* with respect to the set *A*;

and the *refusal degree* is:

$$R_A(x) = [1,1] - \sum_1^p T_A^j(x) - \sum_1^r I_A^k(x) - \sum_1^s F_A^l(x) \subseteq [0,1], \text{ for all } x \in \mathcal{U}.$$

1.9. Definition of single-valued Neutrosophic Set (NS)

Introduced by Smarandache [13, 14, 15] in 1998. Let U be a universe of discourse, and a set $A_{NS} \subseteq U$.

Then $A_{NS} = \{<x, T_A(x), I_A(x), F_A(x)> \mid x \in U\}$, where $T_A(x)$, $I_A(x)$, $F_A(x) : U \to [0, 1]$ represent the degree of truth-membership, degree of indeterminacy-membership, and degree of false-nonmembership respectively, with $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

The neutrosophic components $T_A(x)$, $I_A(x)$, $F_A(x)$ are independent with respect to each other.

1.10. Definition of single-valued Refined Neutrosophic Set (RNS)

Introduced by Smarandache [9] in 2013. Let U be a universe of discourse, and a set $A_{RNS} \subseteq U$. Then

$A_{RNS} = \{<x, T_{1A}(x), T_{2A}(x), \ldots, T_{pA}(x); I_{1A}(x), I_{2A}(x), \ldots, I_{rA}(x); F_{1A}(x), F_{2A}(x), \ldots, F_{sA}(x)> \mid x \in U\}$, where all $T_{jA}(x)$, $1 \leq j \leq p$, $I_{kA}(x)$, $1 \leq k \leq r$, $F_{lA}(x)$, $1 \leq l \leq s$, $: U \to [0, 1]$, and

$T_{jA}(x)$ represents the *j*-th sub-membership degree,

$I_{kA}(x)$ represents the *k*-th sub-indeterminacy degree,

$F_{lA}(x)$ represents the *l*-th sub-nonmembership degree,

with $p, r, s \geq 1$ integers, where $p + r + s = n \geq 4$, and:

$$0 \leq \sum_{j=1}^{p} T_{jA}(x) + \sum_{k=1}^{r} I_{kA}(x) + \sum_{l=1}^{s} T_{jA}(x) \leq n.$$

All neutrosophic sub-components $T_{jA}(x)$, $I_{kA}(x)$, $F_{lA}(x)$ are independent with respect to each other.

Refined Neutrosophic Set is a generalization of Neutrosophic Set.

1.11. Definition of single-valued Intuitionistic Fuzzy Set (IFS)

Introduced by Atanassov [16, 17, 18] in 1983. Let U be a universe of discourse, and a set $A_{IFS} \subseteq U$. Then $A_{IFS} = \{<x, T_A(x), F_A(x)> \mid x \in U\}$, where $T_A(x)$, $F_A(x) : U \to [0, 1]$ represent the degree of membership and

degree of nonmembership respectively, with $T_A(x) + F_A(x) \leq 1$, and $I_A(x)$ = 1 - $T_A(x)$ - $F_A(x)$ represents degree of indeterminacy (in previous publications it was called degree of hesitancy).

The intuitioinistic fuzzy components $T_A(x)$, $I_A(x)$, $F_A(x)$ are <u>dependent</u> with respect to each other.

1.12. Definition of single-valued Inconsistent Intuitionistic Fuzzy Set (equivalent to single-valued Picture Fuzzy Set, and with single-valued Ternary Fuzzy Set)

The single-valued *Inconsistent Intuitionistic Fuzzy Set* (IIFS), introduced by Hindde and Patching [19] in 2008, and the single-valued *Picture Fuzzy Set* (PFS), introduced by Cuong [20] in 2013, indeed coincide, as Atanassov and Vassiliev have observed; also we add that single-valued *Ternary Fuzzy Set*, introduced by Wang, Ha and Liu [21] in 2015 also coincide with them. All these three notions are defined as follows.

Let $\mathcal{U}$ be a universe of discourse, and let's consider a subset $A \subseteq \mathcal{U}$.

Then $A_{IIFS} = A_{PFS} = A_{TFS} = \{\langle x, T_A(x), I_A(x), F_A(x) \rangle | x \in \mathcal{U}\}$,

where $T_A(x), I_A(x), F_A(x) \in [0,1]$, and the sum $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 1$, for all $x \in \mathcal{U}$.

In these sets, the denominations are:

$T_A(x)$ is called *degree of membership* (or validity, or positive membership);

$I_A(x)$ is called *degree of neutral membership*;

$F_A(x)$ is called *degree of nonmembership* (or nonvalidity, or negative membership).

The *refusal degree* is: $R_A(x) = 1 - T_A(x) - I_A(x) - F_A(x) \in [0,1]$, for all $x \in \mathcal{U}$.

The IIFS (PFS, TFS) components $T_A(x), I_A(x), F_A(x), R_A(x)$ are <u>dependent</u> with respect to each other.

Wang, Ha and Liu's [21] assertion that "neutrosophic set theory is difficult to handle the voting problem, as the sum of the three components is greater than 1" is not true, since the sum of the three neutrosophic components is not necessarily greater than 1, but it can be less than or equal to any number between 0 and 3, i.e. $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$, so for example the sum of the three neutrosophic components can be less than 1, or equal to 1, or greater than 1 depending on each application.

1.13. Inconsistent Intuitionistic Fuzzy Set and the Picture Fuzzy Set and Ternary Fuzzy Set are particular cases of the Neutrosophic Set

The *Inconsistent Intuitionistic Fuzzy Set* and *the Picture Fuzzy Set* and *Ternary Fuzzy Set* are particular cases of the Neutrosophic Set (NS). Because, in neutrosophic set, similarly taking single-valued components $T_A(x), I_A(x), F_A(x) \in [0,1]$, one has the sum $T_A(x) + I_A(x) + F_A(x) \leq 3$, which means that $T_A(x) + I_A(x) + F_A(x)$ can be equal to or less than any number between 0 and 3.

Therefore, in the particular case when choosing the sum equal to $1 \in [0,3]$ and getting $T_A(x) + I_A(x) + F_A(x) \leq 1$, one obtains IIFS and PFS and TFS.

1.14. Single-valued Intuitionistic Fuzzy Set is a particular case of single-valued Neutrosophic Set

Single-valued Intuitionistic Fuzzy Set is a particular case of single-valued Neutrosophic Set, because we can simply choose the sum to be equal to 1:

$$T_A(x) + I_A(x) + F_A(x) = 1.$$

1.15. Inconsistent Intuitionistic Fuzzy Set and Picture Fuzzy Set and Ternary Fuzzy Set are also particular cases of single-valued Refined Neutrosophic Set

The *Inconsistent Intuitionistic Fuzzy Set* (IIFS), *Picture Fuzzy Set* (PFS), and *Ternary Fuzzy Set* (TFS), that coincide with each other, are in

addition particular case(s) of *Single-Valued Refined Neutrosophic Set* (RNS).

We may define:

$$A_{IIFS} \equiv A_{PFS} = A_{TFS} = \{x, T_A(x), I_{1_A}(x), I_{2_A}(x), F_A(x) | x \in \mathcal{U}\},$$

with $T_A(x), I_{1_A}(x), I_{2_A}(x), F_A(x) \in [0, 1]$,

and the sum $T_A(x) + I_{1_A}(x) + I_{2_A}(x) + F_A(x) = 1$, for all $x \in \mathcal{U}$;

where:

$T_A(x)$ is the degree of positive membership (validity, etc.);

I_{1_A} is the degree of neutral membership;

$I_{2_A}(x)$ is the refusal degree;

$F_A(x)$ is the degree of negative membership (non-validity, etc.).

$n = 4$, and as a particular case of the sum $T_A(x) + I_{1_A}(x) + I_{2_A}(x) + F_A(x) \leq 4$, where the sum can be any positive number up to 4, we take the positive number 1 for the sum:

$$T_A(x) + I_{1_A}(x) + I_{2_A}(x) + F_A(x) = 1.$$

1.16. Independence of Neutrosophic Components vs. Dependence of Intuitionistic Fuzzy Components

Section 4, equations (46) - (51) in Atanassov's and Vassiliev's paper [6] is reproduced below:

> *"4. Interval valued intuitionistic fuzzy sets, intuitionistic fuzzy sets, and neutrosophic fuzzy sets*
>
> (…) the concept of a Neutrosophic Fuzzy Set (NFS) is introduced, as follows:
>
> $$A^n = \{x, \mu_A^n(x), v_A^n(x), \pi_A^n(x) | x \in E\}, \qquad (46)$$
>
> where $\mu_A^n(x), v_A^n(x), \pi_A^n(x) \in [0, 1]$, and have the same sense as IFS.

Let

$$\sup_{y \in E} \mu_A^n(y) + \sup_{y \in E} v_A^n(y) + \sup_{y \in E} \pi_A^n(y) \neq 0. \quad (47)$$

Then we define:

$$\mu_A^i(x) = \frac{\mu_A^n(x)}{\sup\limits_{y \in E} \mu_A^n(y) + \sup\limits_{y \in E} v_A^n(y) + \sup\limits_{y \in E} \pi_A^n(y)}; \quad (48)$$

$$v_A^i(x) = \frac{v_A^n(x)}{\sup\limits_{y \in E} \mu_A^n(y) + \sup\limits_{y \in E} v_A^n(y) + \sup\limits_{y \in E} \pi_A^n(y)}; \quad (49)$$

$$\pi_A^i(x) = \frac{\pi_A^n(x)}{\sup\limits_{y \in E} \mu_A^n(y) + \sup\limits_{y \in E} v_A^n(y) + \sup\limits_{y \in E} \pi_A^n(y)}; \quad (50)$$

$$\iota_A^i(x) = 1 - \mu_A^i(y) - v_A^i(y) - \pi_A^i(y). \quad (51)"$$

Using the neutrosophic component common notations, $T_A(x) \equiv \mu_A^n(x)$, $I_A(x) \equiv \pi_A^n(x)$, and $F_A(x) \equiv v_A^n(x)$, the refusal degree $R_A(x)$, and $A_N \equiv A^n$ for the neutrosophic set, and considering the triplet's order (T, I, F), with the universe of discourse $\mathcal{U} \equiv E$, we can re-write the above formulas as follows:

$$A_N = \{\langle x_1, T_A(x), I_A(x), T_A(x) \rangle | x \in \mathcal{U}\} \quad (46)'$$

where $T_A(x), I_A(x), F_A(x) \in [0, 1]$, for all $x \in \mathcal{U}$.

Neutrosophic *Fuzzy* Set is commonly named *Single-Valued* Neutrosophic Set (SVNS), i.e. the components are single-valued numbers.

The authors, Atanassov and Vassiliev, assert that $T_A(x), I_A(x), F_A(x)$ "have the same sense as IFS" (Intuitionistic Fuzzy Set).

But this is untrue, since in IFS one has $T_A(x) + I_A(x) + F_A(x) \leq 1$, therefore the IFS components $T_A(x), I_A(x), T_A(x)$ are <u>dependent</u>, while in

SVNS (Single-Valued Neutrosophic Set), one has $T_A(x) + I_A(x) + F_A(x) \le 3$, what the authors omit to mention, therefore the SVNS components $T_A(x), I_A(x), F_A(x)$ are <u>independent</u>, and this makes a big difference, as we'll see below.

In general, for the *dependent components*, if one component's value changes, the other components values also change (in order for their total sum to keep being up to 1). While for the *independent components*, if one component changes, the other components do not need to change since their total sum is always up to 3.

Let's re-write the equations (47) - (51) from authors' paper:

Assume

$$\sup_{y \in \mathcal{U}} T_A(y) + \sup_{y \in \mathcal{U}} I_A(y) + \sup_{y \in \mathcal{U}} F_A(x) \ne 0 \cdot \qquad (47)'$$

The authors have defined:

$$T_A^{IIFS}(x) = \frac{T_A(x)}{\sup\limits_{y \in U} T_A(y) + \sup\limits_{y \in U} I_A(y) + \sup\limits_{y \in U} F_A(y)}; \qquad (48)'$$

$$I_A^{IIFS}(x) = \frac{I_A(x)}{\sup\limits_{y \in U} T_A(y) + \sup\limits_{y \in U} I_A(y) + \sup\limits_{y \in U} F_A(y)}; \qquad (50)'$$

$$F_A^{IIFS}(x) = \frac{F_A(x)}{\sup\limits_{y \in U} T_A(y) + \sup\limits_{y \in U} I_A(y) + \sup\limits_{y \in U} F_A(y)}. \qquad (49)'$$

These mathematical transfigurations, which transform [<u>change in form</u>] the neutrosophic components

$$T_A(x), I_A(x), F_A(x) \in [0, 1],$$

whose sum

$T_A(x) + I_A(x) + +F_A(x) \le 3$, into inconsistent intuitionistic fuzzy components:

$$T_A^{IIFS}(x), I_A^{IIFS}(x), F_A^{IIFS}(x) \in [0, 1],$$

whose sum

$$T_A^{IIFS}(x) + I_A^{IIFS}(x) + F_A^{IIFS}(x) \le 1,$$

and the refusal degree

$$R_A^{IIFS}(x) = 1 - T_A^{IIFS}(x) - I_A^{IIFS}(x) - F_A^{IIFS}(x) \in [0,1], \quad (51)'$$

distort the original application, i.e. the original neutrosophic application and its intuitioinistic fuzzy transformed application are not equivalent, see below.

This is because, in this case, *the change in form brings a change in content*.

1.17. By Transforming the Neutrosophic Components into Intuitionistic Fuzzy Components the Independence of the Neutrosophic Components is Lost

In reference paper [6], Section 4, Atanassov and Vassilev convert the neutrosophic components into intuitionistic fuzzy components.

But, converting a single-valued neutrosophic triplet (T_1, I_1, F_1), with $T_1, I_1, F_1 \in [0, 1]$ and

$T_1 + I_1 + F_1 \le 3$ that occurs into a neutrosophic application α_N, to a single-valued intuitionistic triplet (T_2, I_2, F_2), with $T_2, I_2, F_2 \in [0, 1]$ and $T_2 + F_2 \le 1$ (or $T_2 + I_2 + F_2 = 1$) that would occur into an intuitionistic fuzzy application α_{IF}, is just a *mathematical artifact*, and there could be constructed many such mathematical operators [the authors present four of them], even more: it is possible to convert from the sum $T_1 + I_1 + F_1 \le 3$ to the sum

$T_2 + I_2 + F_2$ equals to any positive number – but they are just *abstract transformations*.

The neutrosophic application α_N will not be equivalent to the resulting intuitionistic fuzzy application α_{IF}, since while in α_N the neutrosophic components T_1, I_1, F_1 are <u>independent</u> (because their sum is up to 3), in α_{IF} the intuitionistic fuzzy components T_2, I_2, F_2 are <u>dependent</u> (because their sum is 1). Therefore, the independence of components is lost.

And the *independence* of the neutrosophic components is the main distinction between neutrosophic set vs. intuitionistic fuzzy set.

Therefore, the resulted intuitionistic fuzzy application α_{IF} after the mathematical transformation is just a subapplication (particular case) of the original neutrosophic application α_N.

1.18. Degree of Dependence/Independence between the Components

The degree of dependence/independence between components was introduced by Smarandache [22] in 2006.

In general, the sum of two components *x* and *y* that vary in the unitary interval [0, 1] is:

$0 \leq x+y \leq 2\text{-}d(x,y)$, where *$d(x,y)$* is the degree of dependence between *x* and *y*, while *$1\text{-}d(x,y)$* is the degree of independence between x and y.

NS is also flexible because it handles, besides independent components, also partially independent and partially dependent components, while IFS cannot deal with these.

For example, if *T* and *F* are totally dependent, then *$0 \leq T + F \leq 1$,* while if component *I* is independent from them, thus *$0 \leq I \leq 1$,* then *$0 \leq T + I + F \leq 2$.* Therefore the components *T, I, F* in general are partially dependent and partially independent.

1.19. Intuitionistic Fuzzy Operators ignore the Indeterminacy, while Neutrosophic Operators give Indeterminacy the same weight as to Truth-Membership and Falsehood-Nonmembership

Indeterminacy in intuitioniostic fuzzy set is ignored by the intuitionistic fuzzy aggregation operators, while the neutrosophic aggregation operators treats the indeterminacy at the same weight as the other two neutrosophic components (truth-membership and falsehood-membership).

Thus, even if we have two single-valued triplets, with the sum of each three components equal to 1 { therefore triplets that may be treated both as *intuitionistic fuzzy triplet*, and *neutrosophic triplet* in the same time (since in neutrosophic environment the sum of the neutrosophic components can be any number between 0 and 3, whence in particular we may take the sum 1) }, after applying the intuitionistic fuzzy aggregation operators we get a different result from that obtained after applying the neutrosophic aggregation operators.

1.20. Intuitionistic Fuzzy Operators and Neutrosophic Operators

Let the intuitionistic fuzzy operators be denoted as: negation ($\neg_{IF}$), intersection ($\wedge_{IF}$), union ($\vee_{IF}$), and implication ($\rightarrow_{IF}$), and the neutrosophic operators [complement, intersection, union, and implication respectively] be denoted as: negation ($\neg_N$), intersection ($\wedge_N$), union ($\vee_N$), and implication ($\rightarrow_N$).

Let $A_1 = (a_1, b_1, c_1)$ and $A_2 = (a_2, b_2, c_2)$ be two triplets such that $a_1, b_1, c_1, a_2, b_2, c_2 \in [0, 1]$ and

$$a_1 + b_1 + c_1 = a_2 + b_2 + c_2 = 1.$$

The intuitionistic fuzzy operators and neutrosophic operators are based on *fuzzy t-norm* ($\wedge_F$) and *fuzzy t-conorm* ($\vee_F$). We'll take for this article the simplest ones:

$a_1 \wedge_F a_2 = \min\{a_1, a_2\}$ and $a_1 \vee_F a_2 = \max\{a_1, a_2\}$,

where $\wedge_F$ is the fuzzy intersection (t-norm) and $\vee_F$ is the fuzzy union (t-conorm).

For the intuitionistic fuzzy implication and neutrosophic implication, we extend the classical implication:

$A_1 \rightarrow A_2$ that is classically equivalent to $\neg A_1 \vee A_2$,

where $\rightarrow$ is the classical implication, $\neg$ the classical negation (complement),

and $\vee$ the classical union,

to the intuitionistic fuzzy environment and respectively to the neutrosophic environment.

But taking other fuzzy t-norm and fuzzy t-conorm, the conclusion will be the same, i.e. the results of intuitionistic fuzzy aggregation operators are different from the results of neutrosophic aggregation operators applied on the same triplets.

Intuitionistic Fuzzy Aggregation Operators { the simplest used intuitionistic fuzzy operations }:

Intuitionistic Fuzzy Negation:

$\neg_{IF} (a_1, b_1, c_1) = (c_1, b_1, a_1)$

Intuitionistic Fuzzy Intersection:

$(a_1, b_1, c_1) \wedge_{IF} (a_2, b_2, c_2) = (\min\{a_1, a_2\}, 1 - \min\{a_1, a_2\} - \max\{c_1, c_2\}, \max\{c_1, c_2\})$

Intuitionistic Fuzzy Union:

$(a_1, b_1, c_1) \vee_{IF} (a_2, b_2, c_2) = (\max\{a_1, a_2\}, 1 - \max\{a_1, a_2\} - \min\{c_1, c_2\}, \min\{c_1, c_2\})$

Intuitionistic Fuzzy Implication:

$(a_1, b_1, c_1) \rightarrow_{IF} (a_2, b_2, c_2)$ is intuitionistically fuzzy equivalent to $\neg_{IF}(a_1, b_1, c_1) \vee_{IF} (a_2, b_2, c_2)$

Neutrosophic Aggregation Operators { the simplest used neutrosophic operations }:

Neutrosophic Negation:

$\neg_N \ (a_1, b_1, c_1) = (c_1, \ 1\text{-}b_1, \ a_1)$

Neutrosophic Intersection:

$$(a_1, b_1, c_1) \wedge_N (a_2, b_2, c_2) = (\min\{a_1, a_2\}, \max\{b_1, b_2\}, \max\{c_1, c_2\})$$

Neutrosophic Union:

$$(a_1, b_1, c_1) \vee_N (a_2, b_2, c_2) = (\max\{a_1, a_2\}, \min\{b_1, b_2\}, \min\{c_1, c_2\})$$

Neutrosophic Implication:

$(a_1, b_1, c_1) \rightarrow_N (a_2, b_2, c_2)$ is neutrosophically equivalent to $\neg_N (a_1, b_1, c_1) \vee_N (a_2, b_2, c_2)$

1.21. Numerical Example of Triplet Components whose Summation is 1

Let $A_1 = (0.3, \ 0.6, \ 0.1)$ and $A_2 = (0.4, \ 0.1, \ 0.5)$ be two triplets, each having the sum:

$0.3 + 0.6 + 0.1 = 0.4 + 0.1 + 0.5 = 1.$

Therefore, they can both be treated as neutrosophic triplets and as intuitionistic fuzzy triplets simultaneously. We apply both, the intuitionistic fuzzy operators and then the neutrosophic operators and we prove that we get different results, especially with respect with Indeterminacy component that is ignored by the intuitionistic fuzzy operators.

1.21.1 Complement/Negation

Intuitionistic Fuzzy:

$\neg_{IF} (0.3, \ 0.6, \ 0.1) = (0.1, \ 0.6, \ 0.3),$

and $\neg_{IF}$ *(0.4, 0.1, 0.5) = (0.5, 0.1, 0.4).*

Neutrosophic:

$$\neg_N(0.3,\ 0.6,\ 0.1) = (0.1, 1 - 0.6, 0.3) = (0.1, 0.4, 0.3) \neq (0.1, 0.6, 0.3),$$

and

$$\neg_N(0.4,\ 0.1,\ 0.5) = (0.5, 1 - 0.1, 0.4) = (0.5, 0.9, 0.4) \neq (0.5, 0.1, 0.4).$$

1.21.2 Intersection

Intuitionistic Fuzzy

$$(0.3, 0.6, 0.1) \wedge_{IF} (0.4, 0.1, 0.5) = (\min\{0.3, 0.4\}, 1 - \min\{0.3, 0.4\} - \max\{0.1, 0.5\}, \max\{0.1, 0.5\}) = (0.3, 0.2, 0.5)$$

As we see, the indeterminacies 0.6 of A_1 and 0.1 of A_2 were completely ignored into the above calculations, which is unfair. Herein, the resulting indeterminacy from intersection is just what is left from truth-membership and falsehood-nonmembership { 1 - 0.3 - 0.5 = 0.2 }.

Neutrosophic

$$(0.3, 0.6, 0.1) \wedge_N (0.4, 0.1, 0.5) = (\min\{0.3, 0.4\}, \max\{0.6, 0.1\}, \max\{0.1, 0.5\}) = (0.3, 0.6, 0.5) \neq (0.3, 0.2, 0.5)$$

In the neutrosophic environment the indeterminacies 0.6 of A_1 and 0.1 of A_2 are given full consideration in calculating the resulting intersection's indeterminacy: $\max\{0.6, 0.1\} = 0.6$.

1.21.3 Union:

Intuitionistic Fuzzy:

$$(0.3, 0.6, 0.1) \vee_{IF} (0.4, 0.1, 0.5) = (\max\{0.3, 0.4\}, 1 - \max\{0.3, 0.4\} - \min\{0.1, 0.5\}, \max\{0.1, 0.5\}) = (0.4, 0.5, 0.1)$$

Again, the indeterminacies 0.6 of A_1 and 0.1 of A_2 were completely ignored into the above calculations, which is not fair. Herein, the resulting indeterminacy from the union is just what is left from truth-membership and falsehood-nonmembership { 1 - 0.4 - 0.1 = 0.5 }.

Neutrosophic:

$$(0.3, 0.6, 0.1) \vee_N (0.4, 0.1, 0.5) = (\max\{0.3, 0.4\}, \min\{0.6, 0.1\}, \min\{0.1, 0.5\}) = (0.4, 0.1, 0.1) \neq (0.4, 0.5, 0.1)$$

Similarly, in the neutrosophic environment the indeterminacies 0.6 of A_1 and 0.1 of A_2 are given full consideration in calculating the resulting union's indeterminacy: $\min\{0.6, 0.1\} = 0.1$.

1.21.4 Implication

Intuitionistic Fuzzy

$$(0.3,0.6,0.1) \rightarrow_{IF} (0.4,0.1,0.5) = \neg_{IF}(0.3,0.6,0.1) \vee_{IF} (0.4,0.1,0.5) = (0.1,0.6,0.3) \vee_{IF} (0.4,0.1,0.5) = (0.4,0.3,0.3)$$

Similarly, indeterminacies of A_1 and A_2 are completely ignored.

Neutrosophic

$$(0.3,0.6,0.1) \rightarrow_N (0.4,0.1,0.5) = \neg_N(0.3,0.6,0.1) \vee_N (0.4,0.1,0.5)$$
$$= (0.1,0.4,0.3) \vee_N (0.4,0.1,0.5)$$
$$= (0.4,0.1,0.3) \neq (0.4,0.3,0.3)$$

While in the neutrosophic environment the indeterminacies of A_1 and A_2 are taken into calculations.

1.21.5. Remark

We have proven that even when the sum of the triplet components is equal to 1, as demanded by intuitionistic fuzzy environment, the results of the intuitionistic fuzzy operators are different from those of the neutrosophic operators – because the indeterminacy is ignored into the intuitionistic fuzzy operators.

1.22. Simple Counterexample 1, Showing Different Results between Neutrosophic Operators and Intuitionistic Fuzzy Operators Applied on the Same Sets (with component sums > 1 or < 1)

Let the universe of discourse $\mathcal{U} = \{x_1, x_2\}$, and two neutrosophic sets included in $\mathcal{U}$:

$A_N = \{x_1(0.8, 0.3, 0.5), x_2(0.9, 0.2, 0.6)\}$, and

$B_N = \{x_1(0.2, 0.1, 0.3), x_2(0.6, 0.2, 0.1)\}$.

Whence, for A_N one has, after using Atanassov and Vassiliev's transformations (48)' - (51)':

$$T_A^{IIFS}(x_1) = \frac{0.8}{0.9+0.3+0.6} = \frac{0.8}{1.8} \approx 0.44;$$

$$I_A^{IIFS}(x_1) = \frac{0.3}{1.8} \approx 0.17;$$

$$F_A^{IIFS}(x_1) = \frac{0.5}{1.8} \approx 0.28.$$

The refusal degree for x_1 with respect to A_N is:

$$R_A^{IIFS}(x_1) = 1 - 0.44 - 0.17 - 0.28 = 0.11.$$

Then:

$$T_A^{IIFS}(x_2) = \frac{0.9}{1.8} = 0.50;$$

$$I_A^{IIFS}(x_2) = \frac{0.2}{1.8} \approx 0.11;$$

$$F_A^{IIFS}(x_2) = \frac{0.6}{1.8} \approx 0.33.$$

The refusal degree for x_2 with respect to A_N is:

$$R_A^{IIFS}(x_2) = 1 - 0.50 - 0.11 - 0.33 = 0.06.$$

Then:

$$A_{IIFS} = \{x_1(0.44, 0.17, 0.28), x_2(0.50, 0.11, 0.33)\}.$$

For B_N one has:

$$T_B^{IIFS}(x_1) = \mu_B^i(x_1) = \frac{0.2}{0.6+0.2+0.3} = \frac{0.2}{1.1} \approx 0.18;$$

$$I_B^{IIFS}(x_1) = v_B^i(x_1) = \frac{0.1}{1.1} \approx 0.09;$$

$$F_B^{IIFS}(x_1) = \pi_B^i(x_1) = \frac{0.3}{1.1} \approx 0.27.$$

The refusal degree for x_1 with respect to B_N is:

$$R_B^{IIFS}(x_1) = 1 - 0.18 - 0.09 - 0.27 = 0.46.$$

$$T_B^{IIFS}(x_2) = \frac{0.6}{1.1} \approx 0.55;$$

$$I_B^{IIFS}(x_2) = \frac{0.2}{1.1} \approx 0.18;$$

$$F_B^{IIFS}(x_2) = \frac{0.1}{1.1} \approx 0.09.$$

The refusal degree for x_2 with respect to the set B_N is:

$$R_B^{IIFS}(x_2) = 1 - 0.55 - 0.18 - 0.09 = 0.18.$$

Therefore:

$$B_{IIFS} = \{x_1, (0.18, 0.09, 0.27), x_2(0.55, 0.18, 0.09)\}.$$

Therefore, the neutrosophic sets:

$$A_N = \{x_1(0.8, 0.3, 0.5), x_2(0.9, 0.2, 0.6)\} \text{ and}$$

$$B_N = \{x_1(0.2, 0.1, 0.3), x_2(0.6, 0.2, 0.1)\},$$

where transformed (restricted), using Atanassov and Vassiliev's transformations (48)-(51), into inconsistent intuitionistic fuzzy sets respectively as follows:

$$A_{IIFS}^{(t)} = \{x_1(0.44, 0.17, 0.28), x_2(0.50, 0.11, 0.33)\} \text{ and}$$

$$B_{IIFS}^{(t)} = \{x_1(0.18, 0.09, 0.27), x_2(0.55, 0.18, 0.09)\},$$

where the upper script (t) means "after Atanassov and Vassiliev's transformations".

We shall remark that the set B_N, as neutrosophic set (where the sum of the components is allowed to also be strictly less than 1 as well), happens to be in the same time an inconsistent intuitionistic fuzzy set, or

$$B_N \equiv B_{IIFS}.$$

Therefore, B_N transformed into $B_{IIFS}^{(t)}$ was a distortion of B_N, since we got different IIFS components:

$$x_1^{B_N}(0.2, 0.1, 0.3) \equiv x_1^{B_{IIFS}}(0.2, 0.1, 0.3) \neq x_1^{B_{IIFS}^{(t)}}(0.18, 0.09, 0.27)$$

Similarly:

$$x_2^{B_N}(0.6, 0.2, 0.1) \equiv x_2^{B_{IIFS}}(0.6, 0.2, 0.1) \neq x_2^{B_{IIFS}^{(t)}}(0.55, 0.18, 0.09)$$

Further on, we show that the NS operators and IIFS operators, applied on these sets, give *different results*. For each individual set operation (intersection, union, complement/negation, inclusion/implication, and equality/equivalence) there exist classes of operators, not a single one. We choose the simplest one in each case, which is based on min / max (fuzzy t-norm / fuzzy t-conorm).

1.22.1 Intersection

Neutrosophic Sets (min / max / max)

$$x_1^A \wedge_N x_1^B = (0.8, 0.3, 0.5) \wedge_N (0.2, 0.1, 0.3)$$
$$= (\min\{0.8, 0.2\}, \max\{0.3, 0.1\}, \max\{0.5, 0.3\}) = (0.2, 0.3, 0.5).$$

$$x_2^A \wedge_N x_2^B = (0.9, 0.2, 0.6) \wedge_N (0.6, 0.2, 0.1) = (0.6, 0.2, 0.6).$$

Therefore:

$$A_N \wedge_N B_N = \{x_1(0.2, 0.3, 0.5), x_2(0.6, 0.2, 0.6)\} \stackrel{\text{def}}{=} C_N.$$

Inconsistent Intuitionistic Fuzzy Set (min / max / max)

$$x_1^A \wedge_{IIFS} x_1^B = (0.44, 0.17, 0.28) \wedge_{IIFS} (0.18, 0.09, 0.27) =$$
$$(\min\{0.44, 0.18\}, \max\{0.17, 0.09\}, \max\{0.28, 0.27\}) =$$
$$(0.18, 0.17, 0.28)$$

.

$$x_2^A \wedge_{IIFS} x_2^B = (0.50, 0.11, 0.33) \wedge_{IIFS} (0.55, 0.18, 0.09)$$
$$= (0.50, 0.18, 0.33).$$

Since in IIFS the sum of components is not allowed to surpass 1, we normalize:

$$\left(\frac{0.50}{1.01}, \frac{0.11}{1.01}, \frac{0.33}{1.01}\right) \approx (0.495, 0.109, 0.326).$$

Therefore:

$$A_{IIFS} \wedge_{IIFS} B_{IIFS} = \{x_1(0.18, 0.17, 0.28), x_2(0.495, 0.109, 0.326)\} \stackrel{\text{def}}{=} C_{IIFS}$$

Also:

$$T_{A_N \wedge_N B_N}(x_1) = 0.2 < 0.3 = I_{A_N \wedge_N B_N}(x_1),$$

while

$$T_{A_{IIFS} \wedge_{IIFS} B_{IIFS}}(x_1) = 0.18 > 0.17 = I_{A_{IIFS} \wedge_{IIFS} B_{IIFS}}(x_1),$$

and other discrepancies can be seen.

Inconsistent Intuitionistic Fuzzy Set (with min / min / max, as used by Cuong [20] in order to avoid the sum of components surpassing 1; but this is in discrepancy with the IIFS/PFS union that uses max / min / min, not max /max / min):

$$x_1^A \wedge_{IIFS2} x_1^B = (0.44, 0.17, 0.28) \wedge_{IIFS2} (0.18, 0.09, 0.27) =$$
$$(\min\{0.44, 0.18\}, \min\{0.17, 0.09\}, \max\{0.28, 0.27\}) =$$
$$(0.18, 0.09, 0.28)$$

$$x_2^A \wedge_{IIFS2} x_2^B = (0.50, 0.11, 0.33) \wedge_{IIFS2} (0.55, 0.18, 0.09)$$
$$= (0.50, 0.11, 0.33).$$

Therefore:

$$A_{IIFS} \wedge_{IIFS2} B_{IIFS} = \{x_1(0.18, 0.09, 0.28), x_2(0.50, 0.11, 0.33)\} \stackrel{\text{def}}{=} C_{IIFS2}$$

We see that:

$$A_N \wedge_N B_N \neq A_{IIFS} \wedge_{IIFS} B_{IIFS}, \text{ or } C_N \neq C_{IIFS};$$

and $A_N \wedge_N B_N \neq A_{IIFS} \wedge_{IIFS2} B_{IIFS}$, $C_N \neq C_{IIFS2}$. Also C_{IIFS} $\neq C_{IIFS2}$.

Let's transform the above neutrosophic set C_N, resulted from the application of the neutrosophic intersection operator,

$C_N = \{x_1(0.2, 0.3, 0.5), x_2(0.6, 0.2, 0.6)\}$,

into an inconsistent intuitionistic fuzzy set, employing the same equations (48) – (50) of transformations [denoted by (t)], provided by Atanassov and Vassiliev, which are equivalent {using (T, I, F)-notations} to (48)'-(50)'

$$(t)T_C^{IIFS}(x_1) = \frac{0.2}{0.6+0.3+0.6} = \frac{0.2}{1.5} \simeq 0.13;$$

$$(t)I_C^{IIFS}(x_1) = \frac{0.3}{1.5} = 0.20;$$

$$(t)F_C^{IIFS}(x_1) = \frac{0.5}{1.5} \simeq 0.33.$$

$$(t)T_C^{IIFS}(x_2) = \frac{0.6}{1.5} \simeq 0.40;$$

$$(t)I_C^{IIFS}(x_2) = \frac{0.2}{1.5} \simeq 0.13;$$

$$(t)F_C^{IIFS}(x_2) = \frac{0.6}{1.5} \simeq 0.40.$$

Whence the results of neutrosophic and IIFS/PFS are totally different:

$C_{IIFS}^{(t)} = \{x_1(0.13, 0.20, 0.33), x_2(0.40, 0.13, 0.40)\} \neq$

$\{x_1(0.18, 0.17, 0.28), x_2(0.495, 0.109, 0.326)\} \equiv C_{IIFS}$,

and

$C_{IIFS}^{(t)} \neq \{x_1(0.18, 0.09, 0.28), x_2(0.50, 0.11, 0.33)\} = C_{IIFS2}$.

1.22.2 Union

Neutrosophic Sets (max / min / min)

$$x_1^A \vee_N x_1^B = (0.8, 0.3, 0.5) \vee_N (0.2, 0.1, 0.3)$$

$$= (\max\{0.8, 0.2\}, \min\{0.3, 0.1\}, \min\{0.5, 0.3\}) = (0.8, 0.1, 0.3).$$

$$x_2^A \vee_N x_2^B = (0.9, 0.2, 0.6) \vee_N (0.6, 0.2, 0.1) = (0.9, 0.2, 0.1).$$

Therefore:

$$A_N \vee_N B_N = \{x_1(0.8, 0.1, 0.3), x_2(0.9, 0.2, 0.1)\} \stackrel{\text{def}}{=} D_N.$$

Inconsistent Intuitionistic Fuzzy Sets (max / min / min [3])

$$x_1^A \vee_{IIFS} x_1^B = (0.44, 0.17, 0.28) \vee_{IIFS} (0.18, 0.09, 0.27)$$

$$= (\max\{0.44, 0.18\}, \min\{0.17, 0.09\}, \min\{0.28, 0.27\})$$

$$= (0.44, 0.09, 0.27).$$

$$x_2^A \vee_{IIFS} x_2^B = (0.50, 0.11, 0.33) \vee_{IIFS} (0.55, 0.18, 0.09)$$

$$= (0.55, 0.11, 0.09).$$

Therefore:

$$A_{IIFS} \vee_{IIFS} B_{IIFS} = \{x_1(0.44, 0.09, 0.27), x_2(0.55, 0.11, 0.09)\} \stackrel{\text{def}}{=} D_{IIFS}$$

a) We see that the results are totally different:

$$A_N \vee_N B_N \neq A_{IIFS} \vee_{IIFS} B_{IIFS}, \text{ or } D_N \neq D_{IIFS}.$$

b) Let's transform the above neutrosophic set, D_N, resulted from the application of neutrosophic union operator,

$$D_N = \{x_1(0.8, 0.1, 0.3), x_2(0.9, 0.2, 0.1)\},$$

into an inconsistent intuitionistic fuzzy set, employing the same equations (48) -(50) of transformation [denoted by (*t*)], provided by

Atanassov and Vassiliev, which are equivalent [using (T, I, F) notations] to (48)'-(50)':

$$(t)T_D^{IIFS}(x_1) = \frac{0.8}{0.9+0.2+0.3} = \frac{0.8}{1.4} \simeq 0.57;$$

$$(t)I_D^{IIFS}(x_1) = \frac{0.1}{1.4} \simeq 0.07;$$

$$(t)I_D^{IIFS}(x_1) = \frac{0.3}{1.4} \simeq 0.21.$$

$$(t)T_D^{IIFS}(x_2) = \frac{0.9}{1.4} \simeq 0.64;$$

$$(t)I_D^{IIFS}(x_2) = \frac{0.2}{1.4} \simeq 0.14;$$

$$(t)F_D^{IIFS}(x_2) = \frac{0.1}{1.4} \simeq 0.07.$$

Whence:

$$D_{IIFS}^{(t)} = \{x_1(0.57, 0.07, 0.21), x_2(0.64, 0.14, 0.07)\}$$
$$\neq \{x_1(0.44, 0.09, 0.27), x_2(0.55, 0.11, 0.09)\} \equiv D_{IIFS}.$$

The results again are totally different.

1.22.3 Corollary

Therefore, no matter if we first transform the neutrosophic components into inconsistent intuitionistic fuzzy components (as suggested by Atanassov and Vassiliev) and then apply the IIFS operators, or we first apply the neutrosophic operators on neutrosophic components, and then later transform the result into IIFS components, in both ways the obtained results in the neutrosophic environment are totally different from the results obtained in the IIFS environment.

1.23. Normalization

Further on, the authors propose the normalization of the neutrosophic components, where Atanassov and Vassiliev's [6] equations (57) – (59) are equivalent, using neutrosophic notations, to the following.

Let $\mathcal{U}$ be a universe of discourse, a set $A \subseteq \mathcal{U}$, and a generic element $x \in \mathcal{U}$, with the neutrosophic components:

$x\big(T_A(x), I_A(x), F_A(x)\big)$, where

$T_A(x), I_A(x), F_A(x) \in [0, 1]$, and

$T_A(x) + I_A(x) + F_A(x) \leq 3$, for all $x \in U$.

Suppose $T_A(x) + I_A(x) + F_A(x) \neq 0$, for all $x \in U$.

Then, by the below normalization of neutrosophic components, Atanassov and Vassiliev obtain the following intuitionistic fuzzy components $(T_A^{IFS}, I_A^{IFS}, F_A^{IFS})$:

$$T_A^{IFS}(x) = \frac{T_A(x)}{T_A(x)+I_A(x)+F_A(x)} \in [0, 1] \qquad (57)'$$

$$I_A^{IFS}(x) = \frac{I_A(x)}{T_A(x)+I_A(x)+F_A(x)} \in [0, 1] \qquad (58)'$$

$$F_A^{IFS}(x) = \frac{F_A(x)}{T_A(x)+I_A(x)+F_A(x)} \in [0, 1] \qquad (59)'$$

and

$T_A^{IFS}(x) + I_A^{IFS}(x) + F_A^{IFS}(x) = 1$, for all $x \in U$.

1.23.1 Counterexample 2

Let's come back to the previous *Counterexample 1*.

$\mathcal{U} = \{x_1, x_2\}$ be a universe of discourse, and let two neutrosophic

sets included in $\mathcal{U}$:

$A_N = \{x_1(0.8, 0.3, 0.5), x_2(0.9, 0.2, 0.6)\}$, and

$B_N = \{x_1(0.2, 0.1, 0.3), x_2(0.6, 0.2, 0.1)\}$.

Let's normalize their neutrosophic components, as proposed by Atanassov and Vassiliev, in order to restrain them to intuitionistic fuzzy components:

$$A_{IFS} = \left\{x_1\left(\frac{0.8}{0.8 + 0.3 + 0.5}, \frac{0.3}{1.6}, \frac{0.5}{1.6}\right), x_2\left(\frac{0.9}{1.7}, \frac{0.2}{1.7}, \frac{0.6}{1.7}\right)\right\}$$

$$\approx \{x_1(0.50, 0.19, 0.31), x_2(0.53, 0.12, 0.35)\}$$

$$\equiv \{x_1(0.50, 0.31), x_2(0.53, 0.35)\},$$

since the indeterminacy (called *hesitant degree* in IFS) is neglected.

$$B_{IFS} = \left\{x_1\left(\frac{0.2}{0.6}, \frac{0.1}{0.6}, \frac{0.3}{0.6}\right), x_2\left(\frac{0.6}{0.9}, \frac{0.2}{0.9}, \frac{0.1}{0.9}\right)\right\}$$

$$\approx \{x_1(0.33, 0.17, 0.50), x_2(0.67, 0.22, 0.11)\}$$

$$\equiv \{x_1(0.33, 0.50), x_2(0.67, 0.11)\},$$

since the indeterminacy (hesitance degree) is again neglected.

The intuitionistic fuzzy operators are applied only on truth-membership and false-nonmembership (but not on indeterminacy).

1.23.2 Intersection

Intuitionistic Fuzzy Intersection (min / max)

$$x_1^A \wedge_{IFS} x_1^B = (0.50, 0.31) \wedge_{IFS} (0.33, 0.50) =$$

$$(min\{0.50, 0.33\}, max\{0.31, 0.50\}) = (0.33, 0.50) \equiv (0.33, 0.17, 0.50)$$

after adding the indeterminacy which is what's left up to 1, i.e.
$1 - 0.33 - 0.50 = 0.17 \cdot$

$$x_2^A \wedge_{IFS} x_2^B = (0.53, 0.35) \wedge_{IFS} (0.67, 0.11)$$

$$= (min\{0.53, 0.63\}, max\{0.35, 0.11\}) = (0.53, 0.35)$$

$$\equiv (0.53, 0.12, 0.35),$$

after adding the indeterminacy.

The results of NS and IFS intersections are clearly very different:

$$A_N \wedge_N B_N = \{x_1(0.2, 0.3, 0.5), x_2(0.6, 0.2, 0.6)\}$$

$$\neq \{x_1(0.33, 0.17, 0.50), x_2(0.53, 0.12, 0.35)\} = A_{IFS} \wedge_{IFS} B_{IFS}$$

Even more distinction, between the NS intersection and IFS intersection of the same elements (whose sums of components equal 1) $x_1^A = (0.50, 0.19, 0.31)$ and $x_1^B = (0.33, 0.17, 0.50)$ one obtains

unequal results, using the (min / max / max) operator:

$$x_1^A \wedge_N x_1^B = (0.50, 0.19, 0.31) \wedge_N (0.33, 0.17, 0.50) = (0.33, 0.19, 0.50)$$

while

$$x_1^A \wedge_{IFS} x_1^B = (0.50, 0.19, 0.31) \wedge_{IFS} (0.33, 0.17, 0.50)$$

$$\equiv (0.50, 0.31) \wedge_{IFS} (0.33, 0.50) \qquad \{after \quad ignoring \quad the$$

indeterminacy in IFS}

$$= (0.33, 0.50) \equiv (0.33, 0.17, 0.50) \neq (0.33, 0.19, 0.50).$$

1.23.3 Union

Intuitionistic Fuzzy Union (max / min / min)

$$x_1^A \vee_{IFS} x_1^B = (0.50, 0.31) \vee_{IFS} (0.33, 0.50)$$

$$= (\max\{0.50, 0.33\}, \min\{0.31, 0.50\}) = (0.50, 0.31)$$

$$\equiv (0.50, 0.19, 0.31),$$

after adding the indeterminacy.

$$x_2^A \vee_{IFS} x_2^B = (0.53, 0.35) \vee_{IF} (0.67, 0.11)$$

$$= (\max\{0.53, 0.67\}, \min\{0.35, 0.11\}) = (0.67, 0.11)$$

$$\equiv (0.67, 0.22, 0.11),$$

after adding the indeterminacy.

The results of NS and IFS unions are clearly very different:

$$A_N \vee_N B_N = \{x_1(0.8, 0.1, 0.3), x_2(0.9, 0.2, 0.1)\}$$

$$\neq \{x_1(0.50, 0.19, 0.31), x_2(0.67, 0.22, 0.11)\} = A_{IFS} \vee_{IF} B_{IFS}.$$

Even more distinction, for the NS and IFS union of the same elements:

$$x_1^A \vee_N x_1^B = (0.50, 0.19, 0.31) \vee_N (0.33, 0.17, 0.50) = (0.50, 0.17, 0.31)$$

while

$$x_1^A \vee_{IFS} x_1^B = (0.50, 0.19, 0.31) \vee_{IFS} (0.33, 0.17, 0.50) \equiv$$

$$(0.50, 0.31) \vee_{IFS} (0.33, 0.50)$$

$$= (0.50, 0.31) \equiv (0.50, 0.19, 0.31) \ \{\text{after adding indeterminacy}\}$$

$$\neq (0.50, 0.17, 0.31).$$

1.24. Indeterminacy Makes a Big Difference between NS and IFS

The authors [6] assert that,

"Therefore, the NFS can be *represented* by an IFS" (page 5),

but this is not correct, since it should be:

The NFS (neutrosophic fuzzy set ≡single-valued neutrosophic set) can be *restrained* (degraded) to an IFS (intuitionistic fuzzy set), yet the <u>independence of components</u> is lost and the results of the aggregation operators are totally different between the neutrosophic environment and intuitionistic fuzzy environment, since <u>Indeterminacy is ignored by IFS operators</u>.

Since in single-valued neutrosophic set the neutrosophic components are *independent* (their sum can be up to 3, and if a component increases or decreases, it does not change the others), while in intuitionistic fuzzy set the components are *dependent* (in general if one changes, one or both the other components change in order to keep their sum equal to 1). Also, applying the neutrosophic operators is a better aggregation since the indeterminacy (I) is involved into all neutrosophic (complement/negation, intersection, union, inclusion / inequality / implication, equality / equivalence) operators while all intuitionistic fuzzy operators *ignore* (do not take into calculation) the indeterminacy.

That is why the results after applying the neutrosophic operators and intuitionistic fuzzy operators on the same sets are *different* as proven above.

1.25. Paradoxes cannot be Represented by the Intuitionistic Fuzzy Logic

No previous set/logic theories, including IFS or Intuitionistic Fuzzy Logic (IFL), since the sum of components was not allowed above 1, could characterize a paradox, which is a proposition that is true ($T = 1$) and false ($F = 1$) simultaneously, therefore the paradox is 100% indeterminate ($I = 1$). In Neutrosophic Logic (NL) a paradoxical proposition P_{NL} is represented as: $P_{NL}(1, 1, 1)$.

If one uses Atanassov and Vassiliev's transformations (for example the normalization) [6], we get $P_{IFL}(1/3, 1/3, 1/3)$, but this one cannot

represent a paradox, since a paradox is 100% true and 100% false, not 33% true and 33% false.

1.26. Single-Valued Atanassov's Intuitionistic Fuzzy Set of second type, also called Single-Valued Pythagorean Fuzzy Set

Single-Valued Atanassov's Intuitionistic Fuzzy Sets of second type (AIFS2) [23], also called Single-Valued Pythagorean Fuzzy Set (PyFS) [24], is defined as follows (using T, I, F notations for the components):

Definition of IFS2 (PyFS)

It is a set $A_{AIFS2} \equiv A_{PyFS}$ from the universe of discourse U such that:

$A_{AIFS2} \equiv A_{PyFS} = \{<x, T_A(x), F_A(x)> \mid x \in U\}$,

where, for all x ∈ U, the functions $T_A(x), F_A(x) : U \rightarrow [0, 1]$, represent the degree of membership (truth) and degree on nonmembership (falsity) respectively, that satisfy the conditions:

$$0 \le T_A^2(x) + F_A^2(x) \le 1$$
,

whence the hesitancy degree is:

$$I_A(x) = \sqrt{1 - T_A^2(x) - F_A^2(x)} \in [0,1]$$
.

1.27. Single-Valued Refined Pythagorean Fuzzy Set (RPyFS)

We propose now for the first time the Single-Valued Refined Pythagorean Fuzzy Set (RPyFS):

$$A_{RAIFS2} = A_{RPyFS} = \{x(T_A^1(x), T_A^2(x), ..., T_A^p(x); F_A^1(x), F_A^2(x), ..., F_A^s(x)), p+s \ge 3, x \in U\}$$

where p and s are positive nonzero integers, and for all $x \in U$, the functions $T_A^1(x), T_A^2(x), ..., T_A^p(x), F_A^1(x), F_A^2(x), ..., F_A^s(x) : U \rightarrow [0, 1]$, represent the degrees of sub-membership (sub-truth) of types *1, 2, ..., p,* and degrees on sub-nonmembership (sub-falsity) of types *1, 2, ..., s* respectively, that satisfy the condition:

$$0 \le \sum_{1}^{p} (T_A^j)^2 + \sum_{1}^{s} (F_A^l)^2 \le 1,$$

whence the refined hesitancy degree is:

$$I_A(x) = \sqrt{1 - \sum_{1}^{p} (T_A^j)^2 - \sum_{1}^{s} (F_A^l)^2} \in [0,1].$$

The Single-Valued Refined Pythagorean Fuzzy Set is a particular case of the Single-Valued Refined Neutrosophic Set.

1.28. The components of Atanassov's Intuitionistic Fuzzy Set of second type (Pythagorean Fuzzy Set) are not Independent

Princy R and Mohana K assert in [23] that:

> "the truth and falsity values and hesitancy value can be independently considered as membership and non-membership and hesitancy degrees respectively".

But this is untrue, since in IFS2 (PyFS) the components are not independent, because they are connected (dependent on each other) through this inequality:

$$T_A^2(x) + F_A^2(x) \le 1$$
.

1.29. Counterexample 3

If T = 0.9, then $T^2 = 0.9^2 = 0.81$, whence $F^2 \le 1 - T^2 = 1 - 0.81 = 0.19$, or $F \le \sqrt{0.19} \approx 0.44$.

Therefore, if T = 0.9, then F is restricted to be less than equal to $\sqrt{0.19}$.

While in NS if T = 0.9, F can be equal to any number in [0, 1], F can be even equal to 1.

Also, hesitancy degree clearly depends on T and F, because the formula of hesitancy degree is an equation depending on T and F, as below:

$$I_A(x) = \sqrt{1 - T_A^2(x) - F_A^2(x)} \in [0,1]$$

.

If T = 0.9 and F = 0.2, then hesitancy

$$I = \sqrt{1 - 0.9^2 - 0.2^2} = \sqrt{0.15} \approx 0.39$$

.

Again, in NS if T = 0.9 and F = 0.2, I can be equal to any number in [0, 1], not only to $\sqrt{0.15}$.

1.30. Neutrosophic Set is a Generalization of Pythagorean Fuzzy Set

In the definition of PyFS, one has $T_A(x), F_A(x) \in [0, 1]$, which involves that

$T_A(x)^2, F_A(x)^2 \in [0, 1]$ too;

we denote $T_A^{NS}(x) = T_A(x)^2, F_A^{NS}(x) = F_A(x)^2$, and $I_A^{NS}(x) = I_A(x)^2 = 1 - T_A(x)^2 - F_A(x)^2 \in [0,1]$, where "NS" stands for Neutrosophic Set.

Therefore, one gets: $T_A^{NS}(x) + I_A^{NS}(x) + F_A^{NS}(x) = 1$,

which is a particular case of the neutrosophic set, since in NS the sum of the components can be any number between 0 and 3, hence into PyFS has been chosen the sum of the components be equal to 1.

1.31. Spherical Fuzzy Set (SFS)

Definition of Spherical Fuzzy Set

A Single-Valued Spherical Fuzzy Set (SFS) [25, 26], of the universe of discourse U, is defined as follows:

A_{SFS} = {<x, $T_A(x)$, $I_A(x)$, $F_A(x)$> | x ∈ U},

where, for all x ∈ U, the functions $T_A(x)$, $I_A(x)$, $F_A(x)$: U → [0, 1], represent the degree of membership (truth), the degree of hesitancy, and degree on nonmembership (falsity) respectively, that satisfy the conditions:

$$0 \leq T_A^2(x) + I_A^2(x) + F_A^2(x) \leq 1$$,

whence the refusal degree is:

$$R_A(x) = \sqrt{1 - T_A^2(x) - I_A^2(x) - F_A^2(x)} \in [0,1]$$

1.32. Single-Valued n-HyperSpherical Fuzzy Set (n-HSFS)

Smarandache (2019) generalized for the first time the *spherical fuzzy set* to *n-hyperspherical fuzzy set*.

Definition of n-HyperSpherical Fuzzy Set.

A Single-Valued n-HyperSpherical Fuzzy Set (n-HSFS), of the universe of discourse U, is defined as follows:

$A_{n\text{-HSFS}}$ = {<x, T_A(x), I_A(x), F_A(x)> | x ∈ U},

where, for all x ∈ U, the functions T_A(x), I_A(x), F_A(x) : U → [0, 1], represent the degree of membership (truth), the degree of hesitancy, and degree on nonmembership (falsity) respectively, that satisfy the conditions:

$$0 \leq T_A^n(x) + I_A^n(x) + F_A^n(x) \leq 1$$, for *n ≥ 1*,

whence the refusal degree is:

$$R_A(x) = \sqrt{1 - T_A^n(x) - I_A^n(x) - F_A^n(x)} \in [0,1]$$.

It is clear that 2-HyperSpherical Fuzzy Set (i.e. when *n = 2*) is a spherical fuzzy set.

1.33. The n-HyperSpherical Fuzzy Set is a particular case of the Neutrosophic Set

Because, T_A(x), I_A(x), F_A(x) ∈ [0, 1] implies that, for *n ≥ 1* one has $T_A^n(x), I_A^n(x), F_A^n(x) \in [0,1]$ too, so they are neutrosophic components as well; therefore each n-HSFS is a NS. But the reciprocal is not true, since if at least one component is 1 and from the other two components

at least one is > 0, for example $T_A(x) = 1$, and $I_A(x) > 0$, $F_A(x) \in [0, 1]$, then $T_A^n(x) + I_A^n(x) + F_A^n(x) > 1$ for $n \geq 1$. Therefore, there are infinitely many triplets T, I, F that are NS components, but they are not n-HSFS components.

1.34. The components of the Spherical Fuzzy Set are not Independent

Princy R and Mohana K assert in [23] that:

> "In spherical fuzzy sets, while the squared sum of membership, non-membership and hesitancy parameters can be between 0 and 1, each of them can be defined between 0 and 1 independently."

But this is again untrue, the above parameters cannot be defined independently.

1.35. Counterexample 4

If *T = 0.9* then *F* cannot be for example equal to *0.8,*

since $0.9^2 + 0.8^2 = 1.45 > 1$,

but the sum of the squares of components is not allowed to be greater than 1.

So *F* depends on *T* in this example.

Two components are independent if no matter what value gets one component will not affect the other component's value.

1.36. Neutrosophic Set is a generalization of the Spherical Fuzzy Set

In [25] Gündoğlu and Kahraman assert about:

> "superiority of SFS [i.e. Spherical Fuzzy Set] with respect to Pythagorean, intuitionistic fuzzy and neutrosophic sets";
>
> also:

> "SFSs are a generalization of Pythagorean Fuzzy Sets (PFS) and neutrosophic sets".

While it is *true* that the spherical fuzzy set is a generalizations of Pythagorean fuzzy set and of intuitionistic fuzzy set, it is *false* that spherical fuzzy set is a generalization of neutrosophic set.

Actually it's the opposite: neutrosophic set is a generalization of spherical fuzzy set. We prove it bellow.

Proof

In the definition of the spherical fuzzy set one has:

$T_A(x)$, $I_A(x)$, $F_A(x) \in [0, 1]$, which involves that $T_A(x)^2$, $I_A(x)^2$, $F_A(x)^2$ $\in [0, 1]$ too.

Let's denote:

$$T_A^{NS}(x) = T_A(x)^2, I_A^{NS}(x) = I_A(x)^2, F_A^{NS}(x) = F_A(x)^2,$$

where "NS" stands for neutrosophic set, whence we obtain, using SFS definition:

$$0 \leq T_A^{NS}(x) + I_A^{NS}(x) + F_A^{NS}(x) \leq 1,$$

which is a particular case of the single-valued neutrosophic set, where the sum of the components T, I, F can be any number between 0 and 3. Now we can choose the sum up to 1.

1.37. Counterexample 5

If we take $T_A(x) = 0.9$, $I_A(x) = 0.4$, $F_A(x) = 0.5$, for some given element *x*, which are neutrosophic components, they are not spherical fuzzy set components because *$0.9^2 + 0.4^2 + 0.5^2 = 1.22 > 1$.*

There are infinitely many values for *$T_A(x)$, $I_A(x)$, $F_A(x)$* in *[0, 1]* whose sum of squares is strictly greater than 1, therefore they are not spherical fuzzy set components, but they are neutrosophic components.

The elements of a spherical fuzzy set form a 1/8 of a sphere of radius 1, centred into the origin O(0,0,0) of the Cartesian system of coordinates, on the positive *Ox (T), Oy (I), Oz (F)* axes.

While the standard neutrosophic set is a cube of side 1, that has the vertexes: *(0,0,0), (1,0,0), (0,1,0), (0,0,1), (1,1,0), (1,0,1), (0,1,1), (1,1,1)*.

The neutrosophic cube strictly includes the 1/8 fuzzy sphere.

1.38. Single-Valued Refined Spherical Fuzzy Set (RSFS)

We introduce now for the first time the Single-Valued Refined Spherical Fuzzy Set.

$$A_{RSFS} = \{x(T_A^1(x), T_A^2(x), ..., T_A^p(x); I_A^1(x), I_A^2(x), ..., I_A^r(x);$$
$$F_A^1(x), F_A^2(x), ..., F_A^s(x)), p + r + s \geq 4, x \in U\},$$

where *p, r, s* are nonzero positive integers, and for all $x \in U$, the functions

$$T_A^1(x), T_A^2(x), ..., T_A^p(x), I_A^1(x), I_A^2(x), ..., I_A^r(x), F_A^1(x), F_A^2(x), ..., F_A^s(x)$$

$: U \rightarrow [0, 1]$, represent the degrees of sub-membership (sub-truth) of types *1, 2, ..., p*, the degrees of sub-hesitancy of types *1, 2, ..., r*, and degrees on sub-nonmembership (sub-falsity) of types *1, 2, ..., s* respectively, that satisfy the condition:

$$0 \leq \sum_1^p (T_A^j)^2 + \sum_1^s (I_A^k)^2 + \sum_1^s (F_A^l)^2 \leq 1$$
,

whence the refined refusal degree is:

$$R_A(x) = \sqrt{1 - \sum_1^p (T_A^j)^2 - \sum_1^s (I_A^k)^2 - \sum_1^s (F_A^l)^2} \in [0,1]$$
.

The Single-Valued Refined Spherical Fuzzy Set is a particular case of the Single-Valued Refined Neutrosophic Set.

1.39. Single-Valued Spherical Neutrosophic Set

Spherical Neutrosophic Set (SNS) was introduced by Smarandache [27] in 2017.

A Single-Valued Spherical Neutrosophic Set (SNS), of the universe of discourse U, is defined as follows:

$A_{SNS} = \{<x, T_A(x), I_A(x), F_A(x)> | x \in U\}$,

where, for all $x \in U$, the functions $T_A(x), I_A(x), F_A(x) : U \to [0, \sqrt{3}]$, represent the degree of membership (truth), the degree of indeterminacy, and degree on nonmembership (falsity) respectively, that satisfy the conditions:

$$0 \leq T_A^2(x) + I_A^2(x) + F_A^2(x) \leq 3$$

.

The Spherical Neutrosophic Set is a generalization of Spherical Fuzzy Set, because we may restrain the SNS's components to the unit interval $T_A(x), I_A(x), F_A(x) \in [0, 1]$,

and the sum of the squared components to 1, i.e. $0 \leq T_A^2(x) + I_A^2(x) + F_A^2(x) \leq 1$.

Further on, if replacing $I_A(x) = 0$ into the Spherical Fuzzy Set, we obtain as particular case the Pythagorean Fuzzy Set.

1.40. Single-Valued n-HyperSpherical Neutrosophic Set (n-HSNS)

Definition of n-HyperSpherical Neutrosophic Set (Smarandache, 2019)

We introduce now for the first time the Single-Valued n-HyperSpherical Neutrosophic Set (n-HSNS), which is a generalization of the Spherical Neutrosophic Set and of n-HyperSpherical Fuzzy Set, of the universe of discourse U, for $n \geq 1$, is defined as follows:

$A_{n\text{-HNS}} = \{<x, T_A(x), I_A(x), F_A(x)> | x \in U\}$,

where, for all $x \in U$, the functions $T_A(x), I_A(x), F_A(x) : U \to [0, \sqrt[n]{3}]$, represent the degree of membership (truth), the degree of indeterminacy,

and degree on nonmembership (falsity) respectively, that satisfy the conditions:

$$0 \le T_A^n(x) + I_A^n(x) + F_A^n(x) \le 3.$$

1.41. Single-Valued Refined Refined n-HyperSpherical Neutrosophic Set (R-n-HSNS)

We introduce now for the first time the Single-Valued Refined n-HyperSpherical Neutrosophic Set (R-n-HSNS), which is a generalization of the n-HyperSpherical Neutrosophic Set and of Refined n-HyperSpherical Fuzzy Set.

On the universe of discourse U, for $n \ge 1$, we define it as:

$$A_{R-n-HSNS} = \{x(T_A^1(x), T_A^2(x), ..., T_A^p(x); I_A^1(x), I_A^2(x), ..., I_A^r(x);$$
$$F_A^1(x), F_A^2(x), ..., F_A^s(x)), p + r + s \ge 4, x \in U\},$$

where *p, r, s* are nonzero positive integers, and for all $x \in U$, the functions

$T_A^1(x), T_A^2(x), ..., T_A^p(x), I_A^1(x), I_A^2(x), ..., I_A^r(x), F_A^1(x), F_A^2(x), ..., F_A^s(x)$: $U \rightarrow [0, \; m^{1/n} \;]$,

represent the degrees of sub-membership (sub-truth) of types *1, 2, ..., p,* the degrees of sub-indeterminacy of types *1, 2, ..., r,* and degrees on sub-nonmembership (sub-falsity) of types *1, 2, ..., s* respectively, that satisfy the condition:

$$0 \le \sum_1^p (T_A^j)^n + \sum_1^r (I_A^k)^n + \sum_1^s (F_A^l)^n \le m, \text{ where } p + r + s = m.$$

1.42. Neutrosophic Set is a Generalization of q-Rung Orthopair Fuzzy Set (q-ROFS)

Definition of q-Rung Orthopair Fuzzy Set.

Using the same *T, I, F* notations one has as follows.

A *Single-Valued q-Rung Orthopair Fuzzy Set (q-ROFS)* [28], of the universe of discourse U, for a given real number $q \geq 1$, is defined as follows:

$A_{q\text{-ROFS}} = \{<x, T_A(x), F_A(x)> \mid x \in U\}$,

where, for all $x \in U$, the functions $T_A(x), F_A(x) : U \rightarrow [0, 1]$, represent the degree of membership (truth), and degree on nonmembership (falsity) respectively, that satisfy the conditions:

$$0 \leq T_A(x)^q + F_A(x)^q \leq 1$$.

Since $T_A(x), F_A(x) \in [0, 1]$, then for any real number $q \geq 1$ one has $T_A(x)^q, F_A(x)^q \in [0,1]$ too.

Let's denote: $T_A^{NS}(x) = T_A(x)^q, F_A^{NS}(x) = F_A(x)^q$, whence it results that:

$$0 \leq T_A^{NS}(x) + F_A^{NS}(x) \leq 1$$, where what's left may be Indeterminacy.

But this is a particular case of the neutrosophic set, where the sum of components T, I, F can be any number between 0 and 3, and for q-ROFS is it taken to be up to 1. Therefore, any Single-Valued q-Rung Orthopair Fuzzy Set is also a Neutrosophic Set, but the reciprocal is not true. See the next counterexample.

1.43. Counterexample 6.

Let's consider a real number $1 \leq q < \infty$, and a set of single-valued triplets of the form

(T, I, F), with *T, I, F ∈ [0, 1]* that represent the components of the elements of a given set.

The components of the form *(1, F)*, with *F > 0*, and of the form *(T, 1)*, with *T > 0*, constitute NS components as follows: *(1, I, F)*, with *F > 0* and any *I ∈ [0, 1]*, and respectively

(T, I, 1), with *T > 0* and any *I ∈ [0, 1]*, since the sum of the components is allowed to be greater than *1*, i.e. *1 + I + F > 1* and respectively *T + I + 1 > 1*.

But they cannot be components of the elements of a q-ROFS set, since:

$1^q + F^q = 1 + F^q > 1$, because *F > 0* and *$1 \leq q < \infty$*; but in q-ROFS the sum has to be *≤ 1*.

Similarly, *$T^q + 1^q = T^q + 1 > 1$*, because *T > 0* and *$1 \leq q < \infty$*; but in q-ROFS the sum has to be *≤ 1*.

1.44. Refined q-Rung Orthopair Fuzzy Set (R-q-ROFS)

We propose now for the first time the Single-Valued Refined q-Rung Orthopair Fuzzy Set (R-q-ROFS):

$$A_{R-q-ROFS} = \{x(T_A^1(x), T_A^2(x),...,T_A^p(x), F_A^1(x), F_A^2(x),...,F_A^s(x)), p+s \geq 3, x \in U\},$$

where *p* and *s* are positive nonzero integers, and for all *x ∈ U*, the functions $T_A^1(x), T_A^2(x),...,T_A^p(x), F_A^1(x), F_A^2(x),...,F_A^s(x): U \to [0, 1]$, represent the degrees of sub-membership (sub-truth) of types *1, 2, ..., p,* and degrees on sub-nonmembership (sub-falsity) of types *1, 2, ..., s* respectively, that satisfy the condition:

$$0 \leq \sum_1^p (T_A^j)^q + \sum_1^s (F_A^l)^q \leq 1, \text{ for } q \geq 1,$$

whence the refined hesitancy degree is:

$$I_A(x) = [1 - \sum_1^p (T_A^j)^q - \sum_1^s (F_A^l)^q]^{1/q} \in [0,1].$$

The Single-Valued Refined q-Rung Fuzzy Set is a particular case of the Single-Valued Refined Neutrosophic Set.

1.45. Regret Theory is a Neutrosophication Model

Regret Theory (2010) [29] is actually a Neutrosophication (1998) Model, when the decision making area is split into three parts, the

opposite ones (upper approximation area, and lower approximation area) and the neutral one (border area, in between the upper and lower area).

1.46. Grey System Theory as a Neutrosophication

A Grey System [30] is referring to a *grey area* (as <neutA> in neutrosophy), between extremes (as <A> and <antiA> in neutrosophy).

According to the Grey System Theory, a system with perfect information (<A>) may have a unique solution, while a system with no information (<antiA>) has no solution. In the middle (<neutA>), or grey area, of these opposite systems, there may be many available solutions (with partial information known and partial information unknown) from which an approximate solution can be extracted.

1.47. Three-Ways Decision as particular cases of Neutrosophication and of Neutrosophic Probability [31, 32, 33, 34, 35, 36]

1.47.1 Neutrosophication

Let *<A>* be an attribute value, *<antiA>* the opposite of this attribute value, and <neutA> the neutral (or indeterminate) attribute value between the opposites *<A>* and *<antiA>*.

For examples: <A> = *big*, then <antiA> = *small*, and <neutA> = *medium*; we may rewrite:

(<A>, <neutA>, <antiA>) = (big, medium, small);

or (<A>, <neutA>, <antiA>) = *(truth (*denoted as *T), indeterminacy (*denoted as *I), falsehood (*denoted as *F))* as in Neutrosophic Logic,

or (<A>, <neutA>, <antiA>) = (membership, indeterminate-membership, monmembership) as in Neutrosophic Set,

or (<A>, <neutA>, <antiA>) = (chance that an event occurs, indeterminate-chance that the event occurs or not, chance that the event does not occur) as in Neutrosophic Probability,

and so on.

And let's by "Concept" to mean: an item, object, idea, theory, region, universe, set, notion etc. that is characterized by this attribute.

The process of **neutrosophication** {Smarandache, 2019, [37]} means:

a) converting a Classical Concept

{ denoted as $(1_{<A>}, 0_{<neutA>}, 0_{<antiA>})$-*ClassicalConcept*, or *ClassicalConcept*$(1_{<A>}, 0_{<neutA>}, 0_{<antiA>})$ }, which means that the concept is, with respect to the above attribute,

$$100\% <A>, 0\% <neutA>, \text{ and } 0\% <antiA>,$$

into a *Neutrosophic Concept*

{ denoted as $(T_{<A>}, I_{<neutA>}, F_{<antiA>})$-*NeutrosophicConcept*, or *NeutrosophicConcept*$(T_{<A>}, I_{<neutA>}, F_{<antiA>})$ }, which means that the concept is, with respect to the above attribute,

$$T\% <A>, I\% <neutA>, \text{ and } F\% <antiA>,$$

which more accurately reflects our imperfect, non-idealistic reality,

where all *T, I, F* are subsets of *[0, 1]* with no other restriction;

- using Triangular / Pentagonal / Polygonal (etc. other function) Numbers.

- employing *neutrosophic IF-THEN rules* into reasoning.

b) or converting a Fuzzy Concept, or Intuitionistic Fuzzy Concept into a Neutrosophic Concept;

c) or converting other Concepts such as Inconsistent Intuitionistic Fuzzy (Picture Fuzzy, Ternary Fuzzy) Concept, or Pythagorean Fuzzy Concept, or Spherical Fuzzy Concept, or q-Rung Orthopair Fuzzy etc.

into a Neutrosophic Concept or into a Refined Neutrosophic Concept (i.e. $T_1\% <A_1>$, $T_2\% <A_2>$, …; $I_1\% <neutA_1>$, $I_2\% <neutA_2>$, …; and $F_1\% <antiA_1>$, $F_2\% <antiA_2>$, …),

where all $T_1, T_2, ...; I_1, I_2, ...; F_1, F_2, ...$ are subsets of *[0, 1]* with no other restriction.

d) or converting a *crisp real number* (*r*) into a *neutrosophic real number* of the form

$r = a + bI$, where "*I*" means (literal or numerical) indeterminacy, *a* and *b* are real numbers, and "*a*" represents the determinate part of the crisp real number *r*, while *bI* the indeterminate part of *r*;

e) or converting a *crisp complex number* (*c*) into a *neutrosophic complex number* of the form $c = a_1 + b_1i + (a_2 + b_2i)I = a_1 + a_2I + (b_1 + b_2I)i$, where "*I*" means (literal or numerical) indeterminacy, $i = \sqrt{-1}$, with a_1, a_2, b_1, b_2 real numbers, and "$a_1 + b_1i$" represents the determinate part of the complex real number *c*, while $a_2 + b_2i$ the indeterminate part of *c*;

(we may also interpret that as: a_1 is the determinate part of the real-part of *c*, and b_1 is the determinate part of the imaginary-part of *c*; while a_2 is the indeterminate part of the real-part of *c*, and b_2 is the indeterminate part of the imaginary-part of *c*);

f) converting a crisp, fuzzy, or intuitionistic fuzzy, or inconsistent intuitionistic fuzzy (picture fuzzy, ternary fuzzy set), or Pythagorean fuzzy, or spherical fuzzy, or q-rung orthopair fuzzy number and other numbers into a quadruple neutrosophic number of the form $a + bT + cI + dF$, where a, b, c, d are real or complex numbers, while T, I, F are the neutrosophic components.

g) splitting a set (or a region) into three parts (two opposites <A> and <antiA>, and one neutral <neutA> in between them), with respect to a given attribute.

h) splitting a set (or a region) into $n \geq 4$ parts (one group of parts <A_1>, <A_2>, … opposite to another group of parts <$antiA_1$>, <$antiA_1$>, …, and a third group of parts <$neutA_1$>, <$neutA_2$>, … as a neutral group of parts in between the opposite groups), with respect to a given attribute.

While the process of **deneutrosophication** means going backwards with respect to any of the above processes of neutrosophication.

Example 1:

Let the attribute <A> = cold temperature, then <antiA> = hot temperature, and <neutA> = medium temperature.

Let the concept be a country M, such that its northern part (30% of country's territory) is cold, its southern part is hot (50%), and in the middle there is a buffer zone with medium temperature (20%). We write:

$$M(\ 0.3_{cold\ temperature},\ 0.2_{medium\ temperature},\ 0.5_{hot\ temperature}\ )$$

where we took single-valued numbers for the neutrosophic components $T_M = 0.3$, $I_M = 0.2$, $F_M = 0.5$, and the neutrosophic components are considered dependent; their sum is equal to 1.

1.47.2 Three-Ways Decision is a particular case of Neutrosophication

Neutrosophy (based on <A>, <neutA>, <antiA>) was proposed by Smarandache [1] in 1998, and Three-Ways Decision by Yao [31] in 2009.

In Three-Ways Decision, the universe set is split into three different distinct areas, in regard to the decision process, representing:

Acceptance, Noncommitment, and Rejection respectively.

In this case, the decision attribute value <A> = Acceptance, whence <neutA> = Noncommitment, and <antiA> = Rejection.

The classical concept = *UniverseSet*.

Therefore, we got the *NeutrosophicConcept* ($T_{<A>}$, $I_{<neutA>}$, $F_{<antiA>}$), denoted as:

$$UniverseSet(\ T_{Acceptance},\ I_{Noncommitment},\ F_{Rejection}\ ),$$

where $T_{Acceptance}$ = *universe set's zone of acceptance*, $I_{Noncommitment}$ = *universe set's zone of noncomitment (indeterminacy)*, $F_{Rejection}$ = *universe set's zone of rejection*.

1.47.3 Three-Ways Decision as a particular case of Neutrosophic Probability

Let's consider the event, taking a decision on a universe set.

According to Neutrosophic Probability (NP) [1, 11] one has:

NP(decision) = (the universe set's elements for which the chance of the decision may be accept;   the universe set's elements for which there may be an indeterminate-chance of the decision;   the universe set's elements for which the chance of the decision may be reject).

1.47.4 Refined Neutrosophy

Refined Neutrosophy was introduced by Smarandache [9] in 2013 and it is described as follows:

<A> is refined (split) into subcomponents *<A$_1$>*, *<A$_2$>*, ..., *<A$_p$>*;

<neutA> is refined (split) into subcomponents <neutA$_1$>, <neutA$_2$>, ..., <neutA$_r$>;

and *<antiA>* is refined (split) into subcomponents *<antiA$_1$>*, *<antiA$_2$>*, ..., *<antiA$_s$>*;

where p, r, s $\geq$ 1 are integers, and p + r + s $\geq$ 4.

Refined Neutrosophy is a generalization of *Neutrosophy*.

Example 2:

If <A> = voting in country M, them <A$_1$> = voting in Region 1 of country M for a given candidate, <A$_2$> = voting in Region 2 of country M for a given candidate, and so on.

Similarly, <neutA$_1$> = not voting (or casting a white or a black vote) in Region 1 of country M, <A$_2$> = not voting in Region 2 of country M, and so on.

And <antiA$_1$> = voting in Region 1 of country M against the given candidate, <A$_2$> = voting in Region 2 of country M against the given candidate, and so on.

1.47.5 Extension of Three-Ways Decision to n-Ways Decision

n-Way Decision was introduced by Smarandache [37] in 2019.

In n-Ways Decision, the universe set is split into $n \geq 4$ different distinct areas, in regard to the decision process, representing:

Levels of Acceptance, Levels of Noncommitment, and *Levels of Rejection* respectively.

Levels of Acceptance may be: Very High Level of Acceptance *($<A_1>$)*, High Level of Acceptance *($<A_2>$),* Medium Level of Acceptance *($<A_3>$),* etc.

Similarly, Levels of Noncommitment may be: Very High Level of Noncommitment *($<neutA_1>$)*, High Level of Noncommitment *($<neutA_2>$),* Medium Level of Noncommitment *($<neutA_3>$),* etc.

And Levels of Rejection may be: Very High Level of Rejection *($<antiA_1>$)*, High Level of Rejection *($<antiA_2>$),* Medium Level of Rejection *($<antiA_3>$),* etc.

Then the *Refined Neutrosophic Concept*

{ denoted as $(T1_{<A1>}, T2_{<A2>}, \ldots, Tp_{<Ap>}; \quad I1_{<neutA1>}, I2_{<neutA2>}, \ldots, Ir_{<neutAr>};$

$F1_{<antiA1>}, F2_{<antiA2>}, Fs_{<antiAs>})$-*RefinedNeutrosophicConcept*,

or *RefinedNeutrosophicConcept*$(T1_{<A1>}, T2_{<A2>}, \ldots, Tp_{<Ap>}; I1_{<neutA1>}, I2_{<neutA2>}, \ldots, Ir_{<neutAr>}; F1_{<antiA1>}, F2_{<antiA2>}, Fs_{<antiAs>})$},

which means that the concept is, with respect to the above attribute value levels,

$$T1\% <A1>, T2\% <A2>, \ldots, Tp\% <Ap>;$$
$$I1\% <neutA1>, I2\% <neutA2>, \ldots, Ir\% <neutAr>;$$
$$F1\% <antiA1>, F2\% <antiA2>, Fs\% <antiAs>;$$

which more accurately reflects our imperfect, non-idealistic reality,

with where *p, r, s $\geq$ 1* are integers, and *p + r + s $\geq$ 4,*

where all T1, T2, …, Tp, I1, I2, …, Ir, F1, F2, …, Fs are subsets of [0, 1] with no other restriction.

1.48. Many More Distinctions between Neutrosophic Set (NS) and Intuitionistic Fuzzy Set (IFS) and other type sets

1.48.1 Neutrosophic Set can distinguish between absolute and relative

- *absolute membership* (i.e. membership in all possible worlds; we have extended Leibniz's absolute truth to absolute membership), and

- *relative membership* (membership in at least one world, but not in all), because

$$\text{NS (absolute membership element)} = 1^+$$

while

- $\text{NS (relative membership element)} = 1.$

This has application in philosophy (see the *neutrosophy*). That's why the unitary standard interval $[0, 1]$ used in IFS has been extended to the unitary non-standard interval $]^-0, 1^+[$ in NS.

Similar distinctions for *absolute* or *relative non-membership*, and *absolute* or *relative indeterminate appurtenance* are allowed in NS.

While IFS cannot distinguish the absoluteness from relativeness of the components.

1.48.2 In NS, there is no restriction on T, I, F other than they be subsets of $]^-0, 1^+[$, thus:

$$^-0 \leq \inf T + \inf I + \inf F \leq \sup T + \sup I + \sup F \leq 3^+.$$

The inequalities (2.1) and (2.4) [17] of IFS are relaxed in NS.

This non-restriction allows paraconsistent, dialetheist, and incomplete information to be characterized in NS {i.e. the sum of all three components if they are defined as points, or sum of superior limits of all

three components if they are defined as subsets can be >1 (for paraconsistent information coming from different sources), or < 1 for incomplete information}, while that information cannot be described in IFS because in IFS the components T (membership), I (indeterminacy), F (non-membership) are restricted either to t + i + f = 1 or to $t^2 + f^2 \leq 1$, if T, I, F are all reduced to the points (single-valued numbers) t, i, f respectively, or to sup T + sup I + sup F = 1 if T, I, F are subsets of [0, 1]. Of course, there are cases when paraconsistent and incomplete informations can be normalized to 1, but this procedure is not always suitable.

In IFS paraconsistent, dialetheist, and incomplete information cannot be characterized. This most important distinction between IFS and NS is showed in the below **Neutrosophic Cube** A'B'C'D'E'F'G'H' introduced by J. Dezert [38] in 2002.

Because in technical applications only the classical interval $[0,1]$ is used as range for the neutrosophic parameters t, i, f , we call the cube $ABCDEDGH$ the **technical neutrosophic cube** and its extension $A'B'C'D'E'D'G'H'$ the **neutrosophic cube** (or **nonstandard neutrosophic cube**), used in the fields where we need to differentiate between *absolute* and *relative* (as in philosophy) notions.
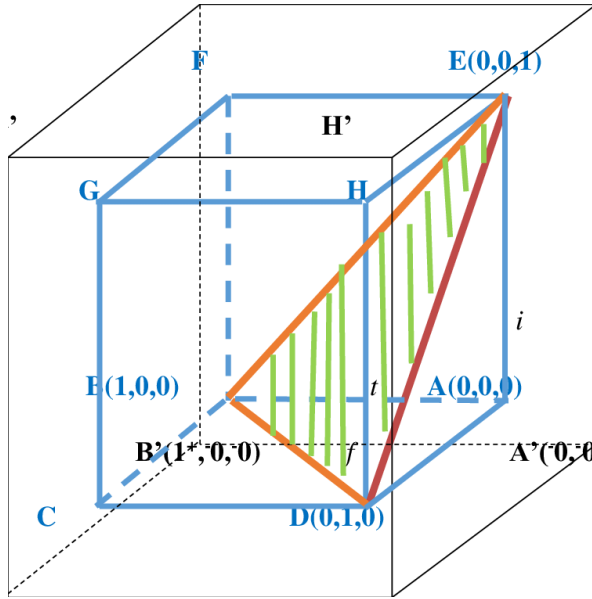
Fig. 1. Neutrosophic Cube

Let's consider a 3D Cartesian system of coordinates, where t is the truth axis with value range in $\left]^-0,1^+\right[$, f is the false axis with value range in $\left]^-0,1^+\right[$, and similarly i is the indeterminate axis with value range in $\left]^-0,1^+\right[$.

We now divide the technical neutrosophic cube $ABCDEDGH$ into three disjoint regions:

a) The shaded equilateral triangle BDE, whose sides are equal to $\sqrt{2}$, which represents the geometrical locus of the points whose sum of the coordinates is 1.

If a point Q is situated on the sides or inside of the triangle BDE, then $t_Q + i_Q + f_Q = 1$ as in Atanassov-intuitionistic fuzzy set $(A - IFS)$.

It is clear that IFS triangle is a restriction of (strictly included in) the NS cube.

b) The pyramid $EABD$ {situated in the right side of the $\triangle EBD$, including its faces $\triangle ABD$ (base), $\triangle EBA$, and $\triangle EDA$ (lateral faces), but excluding its face $\triangle BDE$ } is the locus of the points whose sum of coordinates is less than 1.

If $P \in EABD$ then $t_P + i_P + f_P < 1$ as in inconsistent intuitionistic fuzzy set (with incomplete information).

c) In the left side of $\triangle BDE$ in the cube there is the solid $EFGCDEBD$ (excluding $\triangle BDE$) which is the locus of points whose sum of their coordinates is greater than 1 as in the paraconsistent set.

If a point $R \in EFGCDEBD$, then $t_R + i_R + f_R > 1$.

It is possible to get the **sum of coordinates strictly less than 1 or strictly greater than 1**. For example having three independent sources of information:

- We have a source which is capable to find only the degree of membership of an element; but it is unable to find the degree of non-membership;

- Another source which is capable to find only the degree of non-membership of an element;

- Or a source which only computes the indeterminacy.

Thus, when we put the results together of these sources, it is possible that their sum is not 1, but smaller or greater.

Also, in information fusion, when dealing with indeterminate models (i.e. elements of the fusion space which are indeterminate/unknown, such as intersections we don't know if they are empty or not since we don't have enough information, similarly for complements of indeterminate elements, etc.): if we compute the believe in that element (truth), the disbelieve in that element (falsehood), and the indeterminacy part of that

element, then the sum of these three components is strictly less than 1 (the difference to 1 is the missing information).

1.48.3 Relation (2.3) from interval-valued intuitionistic fuzzy set is relaxed in NS, i.e. the intervals do not necessarily belong to Int[0,1] but to [0,1], even more general to $]^-0, 1^+[$.

1.48.4 In NS the components T, I, F can also be ***nonstandard subsets*** included in the unitary nonstandard interval $]^-0, 1^+[$, not only *standard* subsets included in the unitary standard interval [0, 1] as in IFS.

1.48.5 NS, like dialetheism, can describe **paradoxist elements**, NS(paradoxist element) = (1, 1, 1), while IFL cannot describe a paradox because the sum of components should be 1 in IFS.

1.48.6 The connectors/operators in IFS are defined with respect to T and F only, i.e. membership and nonmembership only (hence the Indeterminacy is what's left from 1), while in NS they can be defined with respect to any of them (no restriction).

But, for interval-valued intuitionistic fuzzy set one cannot find any left indeterminacy.

1.48.7 Component "*I*", indeterminacy, can be split into more subcomponents in order to better catch the vague information we work with, and such, for example, one can get more accurate answers to the *Question-Answering Systems* initiated by Zadeh (2003).

{In Belnap's four-valued logic (1977) indeterminacy is split into Uncertainty (*U*) and Contradiction (*C*), but they were interrelated.}

Even more, one can split "I" into Contradiction, Uncertainty, and Unknown, and we get a five-valued logic.

In a general Refined Neutrosophic Logic, T can be split into subcomponents $T_1, T_2, ..., T_p$, and I into $I_1, I_2, ..., I_r$, and F into $F_1, F_2, ...,F_s$, where p, r, s $\geq$ 1 and p + r + s = n $\geq$ 3. Even more: T, I, and/or F (or any of their subcomponents T_j, I_k, and/or F_l) can be countable or uncountable infinite sets.

1.48.8 Indeterminacy is independent from membership / truth and non-membership/falsehood in NS/Nl, while in IFS/IFL it is not.

In neutrosophics there are two types of indeterminacies:

a) *Numerical Indeterminacy (*or *Degree of Indeterminacy)*, which has the form *(t, i, f)* $\neq$ *(1, 0, 0),* where *t, i, f* are numbers, intervals, or subsets included in the unit interval *[0, 1],* and it is the base for the *(t, i, f)-* Neutrosophic Structures.

b) *Non-numerical Indeterminacy (or Literal Indeterminacy)*, which is the letter "*I*" standing for unknown (non-determinate), such that $I^2 = I$, and used in the composition of the neutrosophic number $N = a + bI$, where *a* and *b* are real or complex numbers, and *a* is the determinate part of number *N*, while *bI* is the indeterminate part of *N*. The neutrosophic numbers are the base for the *I*-Neutrosophic Structures.

1.48.9 NS has a better and clear terminology (name) as "neutrosophic" (which means the neutral part: i.e. neither true/membership nor false/nonmembership), while IFS's name "intuitionistic" produces confusion with Intuitionistic Logic, which is something different (see the article by Didier Dubois et al. [39], 2005).

1.48.10 The Neutrosophic Set was extended [Smarandache, 2007] to **Neutrosophic Overset** (when some neutrosophic component is > 1), and to **Neutrosophic Underset** (when some neutrosophic component is < 0), and to and to **Neutrosophic Offset** (when some neutrosophic components are off the interval [0, 1], i.e. some neutrosophic component > 1 and some neutrosophic component < 0). In IFS the degree of a component is not allowed to be outside of the classical interval [0, 1].

This is no surprise with respect to the classical fuzzy set/logic, intuitionistic fuzzy set/logic, or classical and imprecise probability where the values are not allowed outside the interval [0, 1], since our real-world has numerous examples and applications of over/under/off neutrosophic components.

Example:

In a given company a full-time employer works 40 hours per week. Let's consider the last week period.

Helen worked part-time, only 30 hours, and the other 10 hours she was absent without payment; hence, her membership degree was $30/40 = 0.75 < 1$.

John worked full-time, 40 hours, so he had the membership degree $40/40 = 1$, with respect to this company.

But George worked overtime 5 hours, so his membership degree was $(40+5)/40 = 45/40 = 1.125 > 1$. Thus, we need to make distinction between employees who work overtime, and those who work full-time or part-time. That's why we need to associate a degree of membership greater than 1 to the overtime workers.

Now, another employee, Jane, was absent without pay for the whole week, so her degree of membership was $0/40 = 0$.

Yet, Richard, who was also hired as a full-time, not only didn't come to work last week at all (0 worked hours), but he produced, by accidentally starting a devastating fire, much damage to the company, which was estimated at a value half of his salary (i.e. as he would have gotten for working 20 hours). Therefore, his membership degree has to be less that Jane's (since Jane produced no damage). Whence, Richard's degree of membership with respect to this company was $- 20/40 = - 0.50 < 0$.

Therefore, the membership degrees > 1 and < 0 are real in our world, so we have to take them into consideration.

Then, similarly, the Neutrosophic Logic / Measure / Probability / Statistics etc. were extended to respectively Neutrosophic Over/Under/Off Logic, Measure, Probability, Statistics etc. {Smarandache, 2007 [8]}.

1.48.11 Neutrosophic Tripolar (and in general **Multipolar**) **Set** and **Logic** {Smarandache, 2007 [8]} of the form:

$(<T^+_1, T^+_2, …, T^+_n; T^0; T^-_{-n}, …, T^{—}_{-2}, T^-_{-1} >, <I^+_1, I^+_2, …, I^+_n; I^0; I^-_{-n}, …, I^{—}_{-2}, I^-_{-1} >, <F^+_1, F^+_2, …, F^+_n; F^0; F^-_{-n}, …, F^{—}_{-2}, F^-_{-1} >)$

where we have multiple positive/neutral/negative degrees of T, I, and F respectively.

1.48.12 The **Neutrosophic Numbers** have been introduced by W.B. Vasantha Kandasamy and F. Smarandache [40] in 2003, which are numbers of the form $N = a + bI$, where *a, b* are real or complex numbers, while "I" is the indeterminacy part of the neutrosophic number N, such that $I^2 = I$ and $\alpha I + \beta I = (\alpha+\beta)I$.

Of course, indeterminacy "I" is different from the imaginary unit $i = \sqrt{-1}$.

In general one has $I^n = I$ if n > 0, and it is undefined if n ≤ 0.

1.48.13 Also, **Neutrosophic Refined Numbers** were introduced (Smarandache [31], 2015) as:

$a + b_1 I_1 + b_2 I_2 + … + b_m I_m$, where *a, b₁, b₂, …, bₘ* are real or complex numbers, while the $I_1, I_2, …, I_m$ are types of sub-indeterminacies, for $m \geq 1$.

1.48.14 The algebraic structures using neutrosophic numbers gave birth to the ***I*-Neutrosophic Algebraic Structures** [see for example "neutrosophic groups", "neutrosophic rings", "neutrosophic vector space", "neutrosophic matrices, bimatrices, …, n-matrices", etc.], introduced by W.B. Vasantha Kandasamy, Ilanthenral K., F. Smarandache [41] et al. since 2003.

Example of Neutrosophic Matrix: $\begin{bmatrix} 1 & 2+I & -5 \\ 0 & 1/3 & I \\ -1+4I & 6 & 5I \end{bmatrix}$.

Example of Neutrosophic Ring: ($\{a+bI,$ with a, b ϵ R$\}$, +, ·), where of course $(a+bI)+(c+dI) = (a+c)+(b+d)I$, and $(a+bI) \cdot (c+dI) = (ac) + (ad+bc+bd)I$.

1.48.15 Also, to **Refined *I*-Neutrosophic Algebraic Structures**, which are structures using sets of refined neutrosophic numbers [41].

1.48.16 Types of Neutrosophic Graphs (and Trees)

a-c) Indeterminacy "I" led to the definition of the **Neutrosophic Graphs** (graphs which have: either at least one indeterminate edge, or at least one indeterminate vertex, or both some indeterminate edge and some indeterminate vertex), and **Neutrosophic Trees** (trees which have: either at least one indeterminate edge, or at least one indeterminate vertex, or both some indeterminate edge and some indeterminate vertex), which have many applications in social sciences.

Another type of neutrosophic graph is when at least one edge has a neutrosophic (t, i, f) truth-value.

As a consequence, the Neutrosophic Cognitive Maps (Vasantha & Smarandache, 2003]) and Neutrosophic Relational Maps (Vasantha & Smarandache, 2004) are generalizations of fuzzy cognitive maps and respectively fuzzy relational maps, Neutrosophic Relational Equations (Vasantha & Smarandache, 2004), Neutrosophic Relational Data (Wang, Smarandache, Sunderraman, Rogatko - 2008), etc.

A Neutrosophic Cognitive Map is a neutrosophic directed graph with concepts like policies, events etc. as vertices, and causalities or indeterminates as edges.

It represents the causal relationship between concepts.

An edge is said indeterminate if we don't know if it is any relationship between the vertices it connects, or for a directed graph we don't know if it is a directly or inversely proportional relationship.

We may write for such edge that (t, i, f) = (0, 1, 0).

A vertex is indeterminate if we don't know what kind of vertex it is since we have incomplete information. We may write for such vertex that (t, i, f) = (0, 1, 0).

Example of Neutrosophic Graph (edges V_1V_3, V_1V_5, V_2V_3 are indeterminate and they are drawn as dotted):
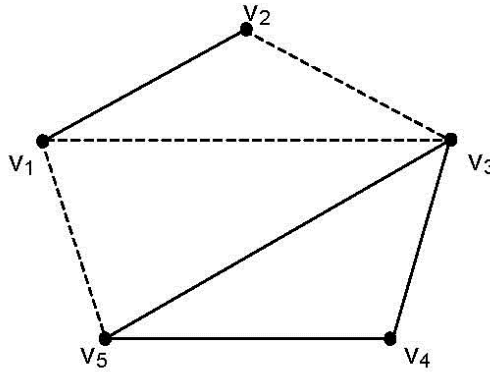
Fig. 2. Neutrosophic Graph { with I (indeterminate) edges }

and its neutrosophic adjacency matrix is:

$$
\begin{bmatrix}
0 & 1 & I & 0 & I \\
1 & 0 & I & 0 & 0 \\
I & I & 0 & 1 & 1 \\
0 & 0 & 1 & 0 & 1 \\
I & 0 & 1 & 1 & 0
\end{bmatrix}
$$

Fig. 3. Neutrosophic Adjacency Matrix of the Neutrosophic Graph

The edges mean: 0 = no connection between vertices, 1 = connection between vertices, I = indeterminate connection (not known if it is, or if it is not).

Such notions are not used in the fuzzy theory.

Example of Neutrosophic Cognitive Map (NCM), which is a generalization of the Fuzzy Cognitive Maps.

Let's have the following vertices:

C_1 - Child Labor

C_2 - Political Leaders

C_3 - Good Teachers

C_4 - Poverty

C_5 - Industrialists

C_6 - Public practicing/encouraging Child Labor

C_7 - Good Non-Governmental Organizations (NGOs)
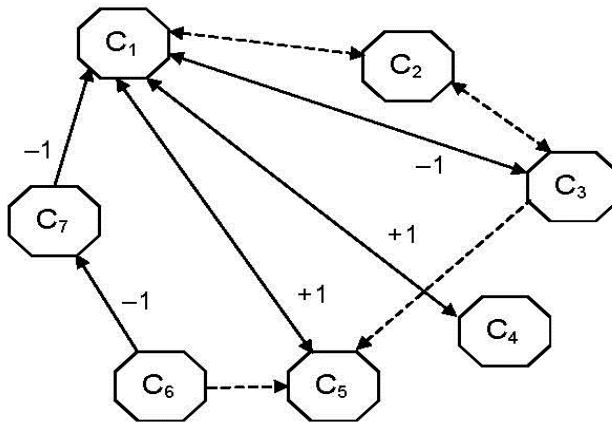
Fig. 4. Neutrosophic Cognitive Map

The corresponding neutrosophic adjacency matrix related to this neutrosophic cognitive map is:

$$
\begin{bmatrix}
0 & I & -1 & 1 & 1 & 0 & 0 \\
I & 0 & I & 0 & 0 & 0 & 0 \\
-1 & I & 0 & 0 & I & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & I & 0 & -1 \\
-1 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
$$

Fig. 4. Neutrosophic Adjacency Matrix
of the Neutrosophic Cognitive Map

The edges mean: 0 = no connection between vertices, 1 = directly proportional connection,

-1 = inversely proportionally connection, and I = indeterminate connection (not knowing what kind of relationship is between the vertices that the edge connects).

Such **literal indeterminacy** (letter **I**) does not occur in previous set theories, including intuitionistic fuzzy set; they had only *numerical indeterminacy*.

d) Another type of neutrosophic graphs (and trees) [Sma-randache, 2015, [41]]:

An edge of a graph, let's say from A to B (i.e. how A influences B), may have a neutrosophic value *(t, i, f)*,

where *t* means the positive influence of A on B,

   i means the indeterminate influence of A on B,

 and *f* means the negative influence of A on B.

Then, if we have, let's say: A->B->C such that A->B has the neutrosophic value (t_1, i_1, f_1) and B->C has the neutrosophic value (t_2, i_2, f_2), then A->C has the neutrosophic value $(t_1, i_1, f_1) \wedge (t_2, i_2, f_2)$, where $\wedge$ is the AND neutrosophic operator.

e) Also, again a different type of graph: we can consider a vertex A as: t% belonging/membership to the graph, i% indeterminate membership to the graph, and f% nonmembership to the graph.

f) Any of the previous types of graphs (or trees) put together.

g) **Tripolar (**and **Multipolar) Graph**, which is a graph whose vertexes or edges have the form $(<T^+, T^0, T^->, <I^+, I^0, I^->, <F^+, F^0, F^->)$ and respectively: $(<T^+_j, T^0, T^-_j>, <I^+_j, I^0, I^-_j>, <F^+_j, F^0, F^-_j>)$.

1.48.17 The **Neutrosophic Probability** (NP), introduced in 1995, was extended and developed as a generalization of the classical and imprecise probabilities {Smarandache, 2013 [11]}. NP of an event E is the chance that event E occurs, the chance that event E doesn't occur, and the chance of indeterminacy (not knowing if the event E occurs or not).

In classical probability $n_{sup} \leq 1$, while in neutrosophic probability $n_{sup} \leq 3^+$.

In imprecise probability: the probability of an event is a subset T in [0, 1], not a number p in   [0, 1], what's left is supposed to be the opposite, subset F (also from the unit interval [0, 1]); there is no indeterminate subset I in imprecise probability.

In neutrosophic probability one has, besides randomness, indeterminacy due to construction materials and shapes of the probability elements and space.

In consequence, neutrosophic probability deals with two types of variables: random variables and indeterminacy variables, and two types of processes: stochastic process and respectively indeterminate process.

1.48.18 And consequently the **Neutrosophic Statistics**, introduced in 1995 and developed in {Smarandache, 2014, [12]}, which is the analysis of the neutrosophic events.

Neutrosophic Statistics means statistical analysis of population or sample that has indeterminate (imprecise, ambiguous, vague, incomplete, unknown) data. For example, the population or sample size might not be exactly determinate because of some individuals that partially belong to the population or sample, and partially they do not belong, or individuals whose appurtenance is completely unknown. Also, there are population or sample individuals whose data could be indeterminate. It is possible to define the neutrosophic statistics in many ways, because there are various types of indeterminacies, depending on the problem to solve.

Neutrosophic statistics deals with neutrosophic numbers, neutrosophic probability distribution, neutrosophic estimation, neutrosophic regression.

The function that models the neutrosophic probability of a random variable x is called *neutrosophic distribution*: NP(x) = (T(x), I(x), F(x)), where T(x) represents the probability that value x occurs, F(x) represents the probability that value x does not occur, and I(x) represents the indeterminate / unknown probability of value x.

1.48.19 Also, **Neutrosophic Measure** and **Neutrosophic Integral** were introduced {Smarandache, 2013, [11]}.

1.48.20 Neutrosophy {Smarandache, 1995, [1, 2, 3, 4, 5, 7]} opened a new field in philosophy.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <Anti-A> and the spectrum of "neutralities" <Neut-A> (i.e. notions or ideas located between the two extremes, supporting neither <A> nor <Anti-A>). The <Neut-A> and <Anti-A> ideas together are referred to as <Non-A>.

According to this theory every idea <A> tends to be neutralized and balanced by <Anti-A> and <Non-A> ideas - as a state of equilibrium.

In a classical way <A>, <Neut-A>, <Anti-A> are disjoint two by two.

But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <Neut-A>, <Anti-A> (and <Non-A> of course) have common parts two by two as well.

Neutrosophy is the base of neutrosophic logic, neutrosophic set, neutrosophic probability and statistics used in engineering applications (especially for software and information fusion), medicine, military, cybernetics, physics.

We have extended dialectics (based on the opposites <A> and <antiA>) to neutrosophy (based on <A>, <antiA> and <neutA>.

1.48.21 In consequence, we extended the thesis-antithesis-synthesis to thesis-antithesis-neutrothesis-neutrosynthesis {Smarandache, 2015 [41]}.

1.48.22 Neutrosophy extended the Law of Included Middle to the **Law of Included Multiple-Middle** {Smarandache, 2014 [10]} in accordance with the n-valued refined neutrosophic logic.

1.48.23 Smarandache (2015 [41]) introduced the Neutrosophic Axiomatic System and Neutrosophic Deducibility.

1.48.24 Then he introduced the **(t, i, f)-Neutrosophic Structure** (2015 [41]), which is a structure whose space, or at least one of its axioms (laws), has some indeterminacy of the form (t, i, f) $\neq$ (1, 0, 0).

Also, we defined the combined *(t, i, f)-I-Neutrosophic Algebraic Structures*, i.e. algebraic structures based on neutrosophic numbers of the form $a + bI$, but also having some indeterminacy [of the form (t, i, f) $\neq$ (1, 0, 0)] related to the structure space (i.e. elements which only partially belong to the space, or elements we know nothing if they belong to the space or not) or indeterminacy   [of the form (t, i, f) $\neq$ (1, 0, 0)] related to at least one axiom (or law) acting on the structure space) .

Even more, we generalized them to Refined (t, i, f)- Refined I-Neutrosophic Algebraic Structures, or (t_j, i_k, f_l)-i_s-Neutrosophic Algebraic Structures; where t_j means that t has been refined to j subcomponents t_1, t_2, …, t_j; similarly for i_k, f_l and respectively i_s.

1.48.25 Smarandache and Ali [2014-2016] introduced the *Neutrosophic Triplet Structures* [42, 43, 44].

A ***Neutrosophic Triplet***, is a triplet of the form:

$$< a,\ neut(a),\ anti(a) >,$$

where *neut(a)* is the neutral of a, i.e. an element (different from the identity element of the operation *) such that *a*neut(a) = neut(a)*a = a,*

while anti(a) is the opposite of a, i.e. an element such that *a*anti(a) = anti(a)*a = neut(a).*

Neutrosophy means not only indeterminacy, but also neutral (i.e. neither true nor false). For example we can have neutrosophic triplet semigroups, neutrosophic triplet loops, etc.

Further on Smaradnache extended the neutrosophic triplet *< a, neut(a), anti(a) >* to a

m-valued refined neutrosophic triplet,

in a similar way as it was done for T_1, T_2, ...; I_1, I_2, ...; F_1, F_2, ... (i.e. the refinement of neutrosophic components).

It will work in some cases, depending on the composition law *. It depends on each * how many neutrals and anti's there is for each element "a".

We may have an m-tuple with respect to the element "a" in the following way:

(a; $neut_1(a)$, $neut_2(a)$, ..., $neut_p(a)$; $anti_1(a)$, $anti_2(a)$, ..., $anti_p(a)$),

where m = 1+2p,

such that:

- all $neut_1(a)$, $neut_2(a)$, ..., $neut_p(a)$ are distinct two by two, and each one is different from the unitary element with respect to the composition law *;

- also:

$$a*neut_1(a) \quad = \quad neut_1(a)*a \quad = \quad a$$
$$a*neut_2(a) \quad = \quad neut_2(a)*a \quad = \quad a$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$a*neut_p(a) = neut_p(a)*a = a;$$
$$- \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad and$$
$$a*anti_1(a) \quad = \quad anti_1(a)*a \quad = \quad neut_1(a)$$
$$a*anti_2(a) \quad = \quad anti_2(a)*a \quad = \quad neut_2(a)$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$
$$a*anti_p(a) = anti_p(a)*a = neut_p(a);$$

- where all $anti_1(a)$, $anti_2(a)$, ..., $anti_p(a)$ are distinct two by two, and in case when there are duplicates, the duplicates are discarded.

1.48.26 As latest minute development, the crisp, fuzzy, intuitionistic fuzzy, inconsistent intuitionistic fuzzy (picture fuzzy, ternary fuzzy), and neutrosophic sets were extended by Smarandache [45] in 2017 to **plithogenic set**, which is:

A set P whose elements are characterized by many attributes' values. An attribute value v has a corresponding (fuzzy, intuitionistic fuzzy, picture fuzzy, neutrosophic, or other types of sets) degree of appurtenance $d(x,v)$ of the element x, to the set P, with respect to some given criteria. In order to obtain a better accuracy for the *plithogenic aggregation operators* in the plithogenic set, and for a more exact inclusion (partial order), a (fuzzy, intuitionistic fuzzy, picture fuzzy, or neutrosophic) *contradiction (dissimilarity) degree* is defined between each attribute value and the dominant (most important) attribute value. The plithogenic intersection and union are *linear combinations of the fuzzy operators t-norm and t-conorm*, while the plithogenic complement (negation), inclusion (inequality), equality (equivalence) are influenced by the attribute values contradiction (dissimilarity) degrees.

1.49. Conclusion

In this paper we proved that neutrosophic set is a generalization of intuitionistic fuzzy set and inconsistent intuitionistic fuzzy set (picture fuzzy set, ternary fuzzy set).

By transforming (restraining) the neutrosophic components into intuitionistic fuzzy components, as Atanassov and Vassiliev proposed, the *independence of the components is lost* and the *indeterminacy is ignored by the intuitionistic fuzzy aggregation operators*. Also, the result after applying the neutrosophic operators is different from the result obtained after applying the intuitionistic fuzzy operators (with respect to the same problem to solve).

We presented many distinctions between neutrosophic set and intuitionistic fuzzy set, and we showed that neutrosophic set is more general and more flexible than previous set theories. Neutrosophy's applications in various fields such neutrosophic probability, neutrosophic statistics, neutrosophic algebraic structures and so on were also listed {see also [46]}.

Neutrosophic Set (NS) is also a generalization of Inconsistent Intuitionistic Fuzzy Set (IIFS) { which is equivalent to the Picture Fuzzy Set (PFS) and Ternary Fuzzy Set (TFS) }, Pythagorean Fuzzy Set (PyFS) {Atanassov's Intuitionistic Fuzzy Set of second type}, Spherical Fuzzy Set (SFS), n-HyperSpherical Fuzzy Set (n-HSFS), and q-Rung Orthopair Fuzzy Set (q-ROFS). And Refined Neutrosophic Set (RNS) is an extension of Neutrosophic Set. And all these sets are more general than Intuitionistic Fuzzy Set.

Neutrosophy is a particular case of Refined Neutrosophy, and consequently Neutrosophication is a particular case of Refined Neutrosophication. Also, Regret Theory, Grey System Theory, and Three-Ways Decision are particular cases of Neutrosophication and of Neutrosophic Probability. We have extended the Three-Ways Decision to n-Ways Decision, which is a particular case of Refined Neutrosophy.

References

[1] Florentin Smarandache, *Definition of Neutrosophic Logic – A Generalization of the Intuitionistic Fuzzy Logic*, Proceedings of the Third Conference of the European Society for Fuzzy Logic and Technology, EUSFLAT 2003, September 10-12, 2003, Zittau, Germany; University of Applied Sciences at Zittau/Goerlitz, 141-146.

[2-3-4] Florentin Smarandache, *Neutrosophic Set, A Generalization of the Intuitionistic Fuzzy Set*

  a. in International Journal of Pure and Applied Mathematics, Vol. 24, No. 3, 287-297, 2005;
  b. also in Proceedings of 2006 IEEE International Conference on Granular Computing, edited by Yan-Qing Zhang and Tsau Young Lin, Georgia State University, Atlanta, pp. 38-42, 2006;
  c. second version in Journal of Defense Resources Management, Brasov, Romania, No. 1, 107-116, 2010.

[5] Florentin Smarandache, *A Geometric Interpretation of the Neutrosophic Set – A Generalization of the Intuitionistic Fuzzy Set*, 2011 IEEE International Conference on Granular Computing, edited by Tzung-Pei Hong, Yasuo Kudo, Mineichi Kudo, Tsau-Young Lin, Been-Chian Chien, Shyue-Liang Wang, Masahiro Inuiguchi, GuiLong Liu, IEEE Computer Society, National University of Kaohsiung, Taiwan, 602-606, 8-10 November 2011, http://fs.unm.edu/IFS-generalized.pdf

[6] Krassimir Atanassov and Peter Vassilev, *Intuitionistic fuzzy sets and other fuzzy sets extensions representable by them*, Journal of Intelligent & Fuzzy Systems, xx (20xx) x–xx, 2019, IOS Press, DOI: 10.3233/JIFS-179426 (under press).

[7] Florentin Smarandache, *Neutrosophy, A New Branch of Philosophy*, Multiple-Valued Logic / An International Journal, Vol. 8, No. 3, 297-384, 2002. This whole issue of this journal is dedicated to Neutrosophy and Neutrosophic Logic, http://fs.unm.edu/Neutrosophy-A-New-Branch-of-Philosophy.pdf

[8] Florentin Smarandache, *Neutrosophic Overset, Neutrosophic Underset, and Neutrosophic Offset. Similarly for Neutrosophic Over-/Under-/Off- Logic, Probability, and Statistics*, 168 p., Pons Editions, Bruxelles, Belgique, 2016; https://arxiv.org/ftp/arxiv/papers/1607/1607.00249.pdf

[9] Florentin Smarandache, *n-Valued Refined Neutrosophic Logic and Its Applications in Physics*, Progress in Physics, 143-146, Vol. 4, 2013; https://arxiv.org/ftp/arxiv/papers/1407/1407.1041.pdf

[10] Florentin Smarandache, *Law of Included Multiple-Middle & Principle of Dynamic Neutrosophic Opposition*, by Florentin Smarandache, EuropaNova & Educational, Brussels-Columbus (Belgium-USA), 136 p., 2014; http://fs.unm.edu/LawIncludedMultiple-Middle.pdf

[11] Florentin Smarandache, *Introduction to Neutrosophic Measure, Neutrosophic Integral and Neutrosophic Probability*, Sitech, 2013, http://fs.gallup.unm.edu/NeutrosophicMeasureIntegralProbability.pdf

[12] Florentin Smarandache, *Introduction to Neutrosophic Statistics*, Sitech, 2014, http://fs.gallup.unm.edu/NeutrosophicStatistics.pdf.

[13] Florentin Smarandache, *Neutrosophy. Neutrosophic Probability, Set, and Logic*. ProQuest Information and Learning, Ann Arbor, Michigan, USA, 105 p., 1998, 2000, 2002, 2005, 2006.

[14] Florentin Smarandache, *A Unifying Field in Logics: Neutrosophic Logic*, Multiple-Valued Logic / An International Journal, Vol. 8, No. 3, 385-438, 2002, http://fs.unm.edu /eBook-Neutrosophics6.pdf

[15] Florentin Smarandache, editor, *Proceedings of the First International Conference on Neutrosophy, Neutrosophic Logic, Neutrosophic Set, Neutrosophic Probability and Statistics*, University of New Mexico, Gallup Campus, Xiquan, Phoenix, 147 p., 2002, http://fs.unm.edu /NeutrosophicProceedings.pdf.

[16] Krassimir Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, 20: 87-96, 1986.

[17] Krassimir T. Atanassov, *Intuitionistic Fuzzy Sets*, Physica-Verlag, Heidelberg, N.Y., 1999.

[18] Krassimir Atanassov, *Intuitionistic fuzzy sets*, VII ITKR's Session, Sofia, June 1983 (Deposed in Central Sci. - Techn. Library of Bulg. Acad. of Sci., 1697/84) (in Bulgarian); Reprinted: Int J Bioautomation **20**(S1) (2016), pp. S1-S6.

[19] C. Hinde and R. Patching, *Inconsistent intuitionistic fuzzy sets. Developments in Fuzzy Sets, Intuitionistic Fuzzy Sets*, Generalized Nets and Related Topics 1 (2008), 133–153.

[20] B.C. Cuong and V. Kreinovich, *Picture fuzzy sets - a new concept for computational intelligence problems*, Proceedings of the Third World Congress on Information and Communication Technologies WICT'2013, Hanoi, Vietnam, December *15-18*, 2013, pp. 1-6.

[21] Chao Wang, Minghu Ha and Xiaqowei Liu, *A mathematical model of ternary fuzzy set for votin*g, Journal of Intelligent & Fuzzy Systems, 29 (2015), 2381-2386.

[22] Florentin Smarandache, *Degree of Dependence and Independence of the (Sub)Components of Fuzzy Set and Neutrosophic Set*, Neutrosophic Sets and Systems, vol. 11, 2016, pp. 95-97, doi.org/10.5281/zenodo.571359,
http://fs.unm.edu/NSS/DegreeOfDependenceAndIndependence.pdf

[23] Princy R, Mohana K, *Spherical Bipolar Fuzzy Sets and its Application in Multi Criteria Decision Making Problem*, Journal of New Theory, 2019 (under press).

[24] R. R. Yager, *Pythagorean fuzzy subsets*, Joint IFSA World Congress and NAFIPS Annual Meeting (IFSA/NAFIPS), Edmonton, AB, Canada, pp. 57-61.

[25] Fatma Kutlu Gündoğdu, Cengiz Kahraman, *A novel spherical fuzzy QFD method and its application to the linear delta robot technology development*, Engineering Applications of Artificial Intelligence, 87 (2020) 103348.

[26] Abhishek Guleria, Rakesh Kumar Bajaj, *T-Spherical Fuzzy Graphs: Operations and Applications in Various Selection Processes*, Arabian Journal for Science and Engineering, https://doi.org/10.1007/s13369-019-04107-y

[27] Florentin Smarandache, *Neutrosophic Perspectives: Triplets, Duplets, Multisets, Hybrid Operators, Modal Logic, Hedge Algebras. And Applications*, Second extended and improved edition, Pons Publishing House Brussels, 2017,
http://fs.unm.edu/NeutrosophicPerspectives-ed2.pdf

[28] Ronald R. Yager, *Generalized orthopair fuzzy sets*, IEEE Trans. Fuzzy Syst., vol. 25, no. 5, pp. 1222-1230, Oct. 2017.

[29] H. Bleichrodt, A. Cillo, E. Diecidue, *A quantitative measurement of regret theory*, Manage. Sci. 56(1) (2010) 161-175.

[30] J. L. Deng, *Introduction to Grey System Theory*, The Journal of Grey System, 1(1): 1-24, 1989.

[31] Y. Yao, *Three-way decision: an interpretation of rules in rough set theory*, in Proceeding of 4th International Conference on Rough Sets and Knowledge Technology, LNAI, Vol. 5589, Springer Berlin Heidelberg, 2009, pp. 642–649.

[32] Florentin Smarandache, Three-Ways Decision is a particular case of Neutrosophication, in volume Nidus Idearum. Scilogs, VII: superluminal physics, vol. vii, Pons Ed., Brussels, pp. 97 - 102, 2019.

[33] Prem Kumar Singh, *Three-way fuzzy concept lattice representation using neutrosophic set*, International Journal of Machine Learning and Cybernetics, 2017, Vol 8, Issue 1, pp. 69-79.

[34] Prem Kumar Singh, *Three-way n-valued neutrosophic concept lattice at different granulation*, International Journal of Machine Learning and Cybernetics, November 2018, Vol 9, Issue 11, pp. 1839-1855.

[35] Prem Kumar Singh, *Complex neutrosophic concept lattice and its applications to Air quality analysis*, Chaos, Solitons and Fractals, Elsevier, 2018, Vol 109, pp. 206-213.

[36] Prem Kumar Singh, *Interval-valued neutrosophic graph representation of concept lattice and its (α, β, γ)-decomposition*, Arabian Journal for Science and Engineering, Springer, 2018, Vol. 43, Issue 2, pp. 723-740.

[37] Florentin Smarandache, *Extension of Three-Ways Decision to n-Ways Decision*, in NIDUS IDEARUM. scilogs, VII: superluminal physics Brussels, 2019, http://fs.unm.edu/NidusIdearum7.pdf

[38] Jean Dezert, *Open Questions to Neutrosophic Inferences,* Multiple-Valued Logic / An International Journal, Vol. 8, No. 3, 439-472, June 2002.

[39] Didier Dubois, S. Gottwald, P. Hajek, Henry Prade, *Terminological difficulties in fuzzy set theory – The case of intuitionistic fuzzy sets*, Fuzzy Sets and Systems, 156 (2005), 585-491.

[40] W. B. Vasantha Kandasamy, Florentin Smarandache, *Fuzzy Cognitive Maps and Neutrosophic Cognitive Maps*, ProQuest Information & Learning, Ann Arbor, Michigan, USA, 2003.

[41] Florentin Smarandache, *Symbolic Neutrosophic Theory*, Europa Nova, Bruxelles, 194 p., 2015;
https://arxiv.org/ftp/arxiv/papers/1512/1512.00047.pdf

[42] Florentin Smarandache, Mumtaz Ali, *The Neutrosophic Triplet Group and its Application to Physics*, presented by F. S. to Universidad Nacional de Quilmes, Department of Science and Technology, Bernal, Buenos Aires, Argentina, 02 June 2014.

[43] Florentin Smarandache and Mumtaz Ali, *Neutrosophic Triplet Group*, Neural Computing and Applications, Springer, 1-7, 2016, DOI: 10.1007/s00521-016-2535-x,
http://fs.unm.edu/NeutrosophicTriplets.htm

[44] F. Smarandache, M. Ali, *Neutrosophic triplet as extension of matter plasma, unmatter plasma, and antimatter plasma*, 69th annual gaseous electronics conference, Bochum, Germany, Veranstaltungszentrum & Audimax, Ruhr-Universitat, 10–14 Oct. 2016, http://meetings.aps.org/Meeting/GEC16/Session/HT6.111

[45] Florentin Smarandache, *Plithogeny, Plithogenic Set, Logic, Probability, and Statistics*, Infinite Study Publ. Hse., GoogleLLC, Mountain View, California, USA, 2017,
https://arxiv.org/ftp/arxiv/papers/1808/1808.03948.pdf
Harvard SAO/NASA ADS:
http://adsabs.harvard.edu/cgi-bin/bib_query?arXiv:1808.03948

[46] Florentin Smarandache, Neutrosophic Set as Generalization of Intuitioonistic Fuzzy Set, Picture Fuzzy Set and Spherical Fuzzy Set, and its Physical Applications, 2019 Joint Fall Meeting of the Texas Sections of American Physical Society (APS), AAPT and Zone 13 of the SPS, Friday–Saturday, October 25–26, 2019; Lubbock, Texas, USA, http://meetings.aps.org/Meeting/TSF19/scheduling?ukey=1480464-TSF19-otUQvu

CHAPTER 2

Refined Neutrosophy & Lattices vs. Pair Structures & YinYang Bipolar Fuzzy Set

Abstract

In this paper, we present the lattice structures of neutrosophic theories, we prove that Zhang-Zhang's YinYang Bipolar Fuzzy Set is a subclass of Single-Valued Bipolar Neutrosophic Set. Then we show that the Pair Structure is a particular case of Refined Neutrosophy, and the number of types of neutralities (sub-indeterminacies) may be any finite or infinite number.

2.1. Introduction

First, we prove that Klement and Mesiar's lattices do not fit the general definition of neutrosophic set, and we construct the appropriate nonstandard neutrosophic lattices of first type (as neutrosophically ordered set) [23], and of second type (as neutrosophic algebraic structure, endowed with two binary neutrosophic laws, $\inf_N$ and $\sup_N$) [23].

We also present the novelties that neutrosophy, neutrosophic logic, set, and probability and statistics, brought in with respect to the previous classical and multi-valued logics and sets, and with the classical and imprecise probability and statistics respectively.

Second, we prove that Zhang-Zhang's YinYang Bipolar Fuzzy Set is not equivalent with, but a subclass of Single-Valued Bipolar Neutrosophic Set.

Third, we show that Montero, Bustince, Franco, Rodríguez, Gómez, Pagola, Fernández, and Barrenechea's paired structure of knowledge representation model is a particular case of Refined Neutrosophy (a branch of philosophy that generalized dialectics) and of Refined Neutrosophic Set. We disprove again the claim that Bipolar Fuzzy Set (renamed as YinYang Bipolar Fuzzy Set) is the same of neutrosophic set as asserted by Montero et al.

About the three types of neutralities presented by Montero et al., we show, by examples, and formally, that there may be any finite number or an infinite number of types of neutralities n, or that indeterminacy (I), as neutrosophic component, can be refined (split) into $1 \leq n \leq \infty$ number

of sub-indeterminacies (not only 3 as Montero et al. said) as needed to each application to solve.

Also, we show, besides numerous neutrosophic applications, many innovatory contributions to science were brought on by the neutrosophic theories, such as: generalization of Yin Yang Chinese philosophy and dialectics to neutrosophy, a new branch of philosophy that is based on the dynamics of opposites and their neutralities; sum of the neutrosophic components *T, I, F* up to *3*; degrees of dependence / independence between the neutrosophic components; distinction between absolute truth and relative truth in neutrosophic logic; introduction of nonstandard neutrosophic logic, set, and probability after we have extended the nonstandard analysis; refinement of neutrosophic components into subcomponents; ability to express incomplete information, complete information, paraconsistent (conflicting) information; and extending the included middle principle to multiple-included middle principle, and so on.

2.2. Notations

$\leq_{nN}^{nonS}$ means nonstandard n-tuple neutrosophic inequality;

$\leq_{nN}$ means standard (real) *n*-tuple inequality;

$\leq_{N}^{nonS}$ means nonstandard unary neutrosophic inequality;

$\leq_{N}$ mean standard (real) unary neutrosophic inequality;

$=_{N}$ means neutrosophic equality;

$\neg_{N}$ means neutrosophic negation;

$\cup_{N}$ means neutrosophic union;

$=$ means classical equality;

$<, >, \leq, \geq$ mean classical inequalities.

2.3. Answers to Erich Peter Klement & Radko Mesiar

I have read the authors' paper [1], published in August 2018, and now it is my duty to publicly respond.

2.3.1. Oversimplification of Neutrosophic Set

At page 10 (Section 3.3) in their paper, related to neutrosophic sets, they wrote:

> As a straightforward generalization of the product lattice $\left(\mathbb{I} \times \mathbb{I}, \leq_{comp} \right)$, for each n ∈ N the *n-dimensional unit cube* $\left(\mathbb{I}^n, \leq_{comp} \right)$, i.e., the n-dimensional product of the lattice $(\mathbb{I}, \leq_{comp})$, can be defined by means of (1) and (2). The so-called "neutrosophic" sets introduced by F. Smarandache [93] (see also [94–97] are based on the bounded lattices $\left(\mathbb{I}^3, \leq_{\mathbb{I}^3} \right)$ and $\left(\mathbb{I}^3, \leq^{\mathbb{I}^3} \right)$, where the orders $\leq_{I^3}$ and $\leq_{I^3}$ on the unit cube I^3 are defined by
>
> $(x_1, x_2, x_3) \leq_{I^3} (y_1, y_2, y_3) \Leftrightarrow x_1 \leq y_1$ AND $x_2 \leq y_2$ AND $x_3 \geq y_3$ (13)
>
> $(x_1, x_2, x_3) \leq^{I^3} (y_1, y_2, y_3) \Leftrightarrow x_1 \leq y_1$ AND $x_2 \geq y_2$ AND $x_3 \geq y_3$ (14)

The authors have defined (1) and (2) as follows:

$$\left(\prod_{i=1}^{n} L_i, \leq_{comp} \right), \text{ where } \left(L_i, \leq_{L_i} \right) \text{ are fuzzy lattices, for all } 1 \leq i \leq n, \tag{1}$$

and

$$(x_1, x_2, ..., x_n) \leq_{comp} (y_1, y_2, ..., y_n) \Leftrightarrow x_1 \leq y_1 \text{ AND } x_2 \leq y_2 \text{ AND } ... \text{ AND } x_n \leq y_n \tag{2}$$

The authors did not specify what type of lattices they employ: of first type (lattice, as a partially ordered set), or second type (lattice, as an

algebraic structure). Since their lattices are endowed with some inequality (referring to the neutrosophic case) we assume it is as first type.

The authors have used the notations:

$\mathbb{I} = [0, 1]$,

$\mathbb{I}^2 = [0, 1]^2$,

$\mathbb{I}^3 = [0, 1]^3$,

and the order relationship $\leq_{comp}$ on $\mathbb{I}^3$ define as:

$(x_1, x_2, x_3) \leq_{comp} (y_1, y_2, y_3) \Longleftrightarrow x_1 \leq y_1$ and $x_2 \leq y_2$ and $x_3 \leq y_3$.

The three lattices they constructed, that I denote by KL_1, KL_2, KL_3 respectively

$KL_1 = \left(\mathbb{I}^3, \leq_{comp}\right), KL_2 = (\mathbb{I}^3, \leq_{I^3}), KL_3 = \left(\mathbb{I}^3, \leq^{I^3}\right)$,

contain ONLY the very particular case of standard single-valued neutrosophic set, i.e. when the neutrosophic components T (truth-membership), I (indeterminacy-membership), and F (false-membership) of the generic element $x(T, I, F)$, of a neutrosophic set N, are single-valued (crisp) numbers from the unit interval $[0, 1]$.

The authors have *oversimplified* the neutrosophic set. Neutrosophic is much more complex. Their lattices do not characterize the **initial definition of the neutrosophic set** ([15], 1998): a set whose elements have the degrees of appurtenance *T, I, F,* where *T, I, F* are standard or nonstandard subsets of the nonstandard unit interval: $]^-0, 1^+[$, where $]^-0, 1^+[$ overpasses the classical real unit interval $[0, 1]$ to the left and to the right.

2.3.2. Neutrosophic Cube vs. Unit Cube

Clearly, their $\mathbb{I}^3 = [0,1]^3 \subsetneqq]^-0, 1^+[^3$, where $^-0 = \mu(^-0)$ is the left nonstandard monad of number 0, and $1^+ = \mu(1^+)$ is the right nonstandard monad of number 1.

Fig. 1. Neutrosophic Cube
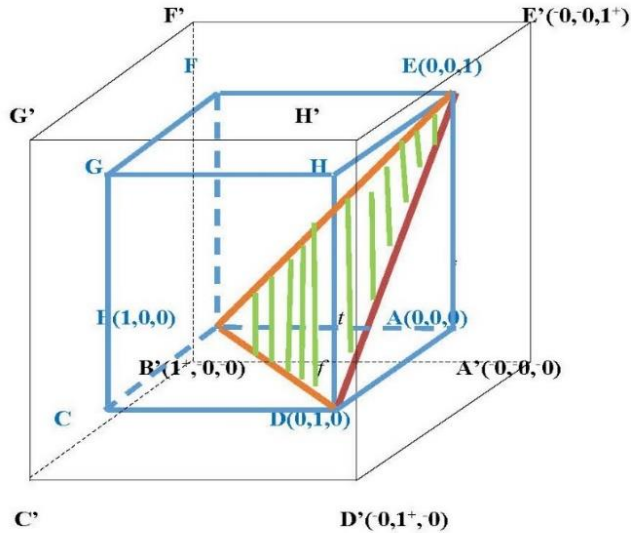
The unit cube $\mathbb{I}^3$ used by the authors does not equal the above neutrosophic cube. The **Neutrosophic Cube** *A'B'C'D'E'F'G'H'* was introduced by J. Dezert [5] in 2002.

2.3.3. The most general Neutrosophic Lattices

The authors' lattices are far from catching the **most general definition of neutrosophic set**.

Let $\mathcal{U}$ be a universe of discourse, and $M \subset \mathcal{U}$ be a set. Then an element $x(T(x), I(x), F(x)) \in M$, where $T(x), I(x), F(x)$ are standard or nonstandard subsets of nonstandard interval: $]^-\Omega, \Psi^+[$, where $\Omega \leq 0 < 1 \leq \Psi$, with $\Omega, \Psi \in \mathbb{R}$, whose values Ω and Ψ depend on each application, and:

$$]^-\Omega, \Psi^+[=_N \{\varepsilon, a, a^-, a^{-0}, a^+, a^{+0}, a^{-+}, a^{-0+} | \varepsilon, a \in [\Omega, \Psi], \varepsilon \text{ is infinitesimal}\},$$

where $\overset{m}{a}, m \in \{^-, ^{-0}, ^+, ^{+0}, ^{-+}, ^{-0+}\}$ are monads or binads [6, 7].

It follows that the nonstandard neutrosophic mobinad real offset lattices

$\left(]^-\Omega, \Psi^+[, \leq_N^{nonS}\right)$ and $(]^-\Omega, \Psi^+[, \inf_N, \sup_N, {}^-\Omega, \Psi^+)$ of first type and respectively second type are the most general (non-refined) neutrosophic lattices.

While the most general refined neutrosophic lattices of first type is:

$\left(]^-\Omega, \Psi^+[, \leq_{nN}^{nonS}\right)$, where $\leq_{nN}^{nonS}$ is the n-tuple nonstandard neutrosophic inequality dealing with nonstandard subsets, defined as:

$(T_1(x), T_2(x), \ldots, T_p(x); I_1(x), I_2(x), \ldots, I_r(x); F_1(x), F_2(x), \ldots, F_s(x)) \leq_{nN}^{nonS} (T_1(y), T_2(y), \ldots, T_p(y); I_1(y), I_2(y), \ldots, I_r(y); F_1(y), F_2(y), \ldots, F_s(y))$ iff

$T_1(x) \leq_{nN}^{nonS} T_1(y), T_2(x) \leq_{nN}^{nonS} T_2(y), \ldots, T_p(x) \leq_{nN}^{nonS} T_p(y);$

$I_1(x) \geq_{nN}^{nonS} I_1(y), I_2(x) \geq_{nN}^{nonS} I_2(y), \ldots, I_r(x) \geq_{nN}^{nonS} I_r(y);$

$F_1(x) \geq_{nN}^{nonS} F_1(y), F_2(x) \geq_{nN}^{nonS} F_2(y), \ldots, F_s(x) \geq_{nN}^{nonS} F_s(y).$

2.3.4. Distinction between Absolute Truth and Relative Truth

The authors' lattices are incapable of making distinctions between absolute truth (when $T = 1^+ >_N 1$) and relative truth (when $T = 1$) in the sense of Leibniz, which is the essence of nonstandard neutrosophic logic.

2.3.5. Neutrosophic Standard Subset Lattices

Their three lattices are not even able to deal with **standard subsets** [including intervals [8], and hesitant (discrete finite) subsets] $T, I, F \subseteq [0, 1]$, since they have defined the 3D-inequalities with respect to single-valued (crisp) numbers:

$x_1, x_2, x_3 \in [0, 1]$ and $y_1, y_2, y_3 \in [0, 1]$.

In order to deal with standard subsets, they should use *inf / sup*, i.e.

$$(T_1, I_1, F_1) \leq (T_2, I_2, F_2) \Longleftrightarrow$$

$\inf T_1 \leq \inf T_2$ and $\sup T_1 \leq \sup T_2,$

$\inf I_1 \geq \inf I_2$ and $\sup I_1 \geq \sup I_2$,

and $\inf F_1 \geq \inf F_2$ and $\sup F_1 \geq \sup F_2$.

[I have displayed the most used 3D-inequality by the neutrosophic community.]

2.3.6. Nonstandard and Standard Refined Neutrosophic Lattices

The **Nonstandard Refined Neutrosophic Set** ([9], 2013; [7], 2018; 23, 2019), defined on $]^-0, 1^+[^n$, strictly includes their n-dimensional unit cube ($\mathbb{I}^n$), and we use a <u>nonstandard</u> neutrosophic inequality, not the classical inequalities, to deal with inequalities of monads and binads, such as $\leq_{nN}^{nonS}$ and $\leq_N^{nonS}$.

Not even the Standard Refined Single-Valued Neutrosophic Set [9] (2013) may be characterized with KL_1, KL_2, KL_3 nor with $\left(\mathbb{I}^n, \leq_{\text{comp}}\right)$, since the n-D neutrosophic inequality is different from n-D $\leq_{\text{comp}}$, and from n-D extensions of $\leq_{I_3}$ or $\leq^{I_3}$ respectively, as follows.

Let T be refined into $T_1, T_2, \ldots, T_p$;

$\qquad\quad I$ be refined into $I_1, I_2, \ldots, I_r$;

$\quad$ and F be refined into $F_1, F_2, \ldots, F_s$;

with $p, r, s \geq 1$ are integers, and $p + r + s = n \geq 4$, produced the following n-D neutrosophic inequality.

Let $x\left(T_1^x, T_2^x, \ldots, T_p^x; I_1^x, I_2^x, \ldots, I_r^x; F_1^x, F_2^x, \ldots, F_s^x\right)$,

and $y\left(T_1^y, T_2^y, \ldots, T_p^y; I_1^y, I_2^y, \ldots, I_r^y; F_1^y, F_2^y, \ldots, F_s^y\right)$.

Then:

$$x \leq_N y \Longleftrightarrow \begin{pmatrix} T_1^x \leq T_1^y, T_2^x \leq T_2^y, \ldots, T_p^x \leq T_p^y; \\ I_1^x \geq I_1^y, I_2^x \geq I_2^y, \ldots, I_r^x \geq I_r^y; \\ F_1^x \geq F_1^y, F_2^x \geq F, \ldots, F_s^x \geq F_s^y. \end{pmatrix}$$

2.3.7. Neutrosophic Standard Overset/Underset/Offset Lattice

Their three lattice KL_1, KL_2, KL_3 are no match for neutrosophic overset (when the neutrosophic components $T, I, F > 1$), nor for neutrosophic underset (when the neutrosophic components $T, I, F < 0$), and in general no match for the neutrosophic offset (when the neutrosophic components T, I, F take values outside the unit interval $[0, 1]$ as needed in real life applications [10-14] (2006-2018):

$[\Omega, \Psi]$ with $\Omega \leq 0 < 1 \leq \Psi$.

So, a lattice may similarly be built on the **non-unitary neutrosophic cube** $[\varphi, \psi]^3$.

2.3.8. Sum of Neutrosophic Components up to 3

The authors say nothing on the novelty of neutrosophic theories that the sum of single-valued neutrosophic components $T + I + F \leq 3$, extended up to 3, and similarly the corresponding inequality when T, I, F are subsets of $[0, 1]$:

$\sup T + \sup I + \sup F \leq 3$,

for neutrosophic set, neutrosophic logic, and neutrosophic probability never done before in the previous classic logic and multiple-valued logics and set theories, nor in classical or imprecise probabilities.

This makes a big difference, since for a single-valued neutrosophic set S all unit cube $[0, 1]^3$ is fulfilled with points, each point $P(a, b, c)$ into the unit cube may represent the neutrosophic coordinates (a, b, c) of an element $x(a, b, c) \in S$, which was not the case for previous logics, sets, and probabilities.

Which is not the case for the Picture Fuzzy Set (B.C. Cuong, [25], 2013) whose domain is $\frac{1}{6}$ of the unit cube (a cube corner):

$\mathbb{D}^* = \{(x_1, x_2, x_3) \in \mathbb{I}^3 | x_1 + x_2 + x_3 \leq 1\}$,

while for Intuitionistic Fuzzy Set (K. Atanassov, [24], 1986)

$\mathbb{D}_A = \{(x_1, x_2, x_3) \in \mathbb{I}^3 | x_1 + x_2 + x_3 = 1\}$,

where x_1 = membership degree, x_2 = hesitant degree, and x_3 = nonmembership degree, whose domain is the main cubic diagonal triangle that connects the vertices: $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, i.e. triangle BDE (its sides and its interior) in previous Fig. 1.

2.3.9. Etymology of Neutrosophy and Neutrosophic

The authors write ironically twice, in between quotations, "neutrosophic", because they did not read the etymology [15] of the word published into my first book (1998), etymology which also appears into Denis Howe's 1999 *The Free Online Dictionary of Computing* [16], and afterwards repeated by many researchers from the neutrosophic community in their published papers:

> *Neutrosophy [16]:*
> *<philosophy> (From Latin "neuter" - neutral, Greek "sophia" - skill/wisdom) A branch of philosophy, introduced by Florentin Smarandache in 1980, which studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.*
> *Neutrosophy considers a proposition, theory, event, concept, or entity, "A" in relation to its opposite, "Anti-A" and that which is not A, "Non-A", and that which is neither "A" nor "Anti-A", denoted by "Neut-A".*
> *Neutrosophy is the basis of neutrosophic logic, neutrosophic probability, neutrosophic set, and neutrosophic statistics.*

While **neutrosophic** means what is derived/resulted from *neutrosophy*.

Unlike the "intuitionistic|" and "picture fuzzy" notions, the notion of *neutrosophic* was carefully and meaningfully chosen, coming from **neutral** (or indeterminate; denoted by <neutA>) between two opposites, $\langle A \rangle$ and $\langle \text{anti}A \rangle$, which made the main distinction between neutrosophic logic / set / probability, and the previous fuzzy, intuitionistic fuzzy logics and sets, i.e.:

- for neutrosophic logic neither true nor false, but neutral (or indeterminate) in between them;

- similarly for neutrosophic set: neither membership nor non-membership, but in between (neutral, or indeterminate);

- and analogously for neutrosophic probability: chance that an event E occurs, chance that the event E does not occur, and indeterminate (neutral) chance of the event E of occurring or not occuring.

Their irony is malicious and ungrounded.

2.3.10. Neutrosophy as Extension of Dialectics

Let $\langle A \rangle$ be a concept, notion, idea, theory or so on.

Then $\langle \text{anti}A \rangle$ is the opposite of $\langle A \rangle$, while $\langle \text{neut}A \rangle$ is the neutral (or indeterminate) part between them.

While in philosophy Dialectics is the dynamics of opposites ($\langle A \rangle$ and $\langle \text{anti}A \rangle$), Neutrosophy is an extension of dialectics, in other words neutrosophy is the dynamics of opposites and their neutrals ($\langle A \rangle$, $\langle \text{anti}A \rangle$, $\langle \text{neut}A \rangle$), because the neutrals play an important role in our world, interfering in one side or the other of the opposites.

2.3.11. Refined Neutrosophic Set and Lattice

At page 11, Klement and Mesiar ([1], 2018) assert that:

> *"Considering, for $n > 3$, lattices which are isomorphic to $\left(L_n(\mathbb{I}), \leq_{comp} \right)$, further generalizations of "neutrosophic" sets can be introduced.*

The authors are uninformed that a generalization was done long before, in 2013 when we have published a paper [9] that introduced for the first time the refined neutrosophic set / logic / probability, where *T, I, F* were refined into n neutrosophic subcomponents:

$T_1, T_2, \ldots, T_p; I_1, I_2, \ldots, I_r; F_1, F_2, \ldots, F_s,$

with $p, r, s \geq 1$ are integers and $p + r + s = n \geq 4$.

But in our lattice $(\mathbb{I}^n, \leq_{nN})$, the neutrosophic inequality is adjusted to the categories of sub-truths, sub-indeterminacies, and sub-falsehood respectively.

$(T_1(x), T_2(x), \ldots, T_p(x); I_1(x), I_2(x), \ldots, I_r(x); F_1(x), F_2(x), \ldots,$
$F_s(x)) \leq_{nN} (T_1(y), T_2(y), \ldots, T_p(y); I_1(y), I_2(y), \ldots, I_r(y); F_1(y),$
$F_2(y), \ldots, F_s(y))$ iff

$T_1(x) \leq T_1(y), T_2(x) \leq T_2(y), \ldots, T_p(x) \leq T_p(y);$

$I_1(x) \geq I_1(y), I_2(x) \geq I_2(y), \ldots, I_r(x) \geq I_r(y);$

$F_1(x) \geq F_1(y), F_2(x) \geq F_2(y), \ldots, F_s(x) \geq F_s(y).$

Therefore, $\leq_{nN}$ is different from the n-D inequalitties $\leq_{\text{comp}}$, and from $\leq_{\mathbb{I}^n}$ and $\leq^{\mathbb{I}^n}$ (extending from authors inequalities $\leq_{\mathbb{I}^3}$ and $\leq^{\mathbb{I}^3}$ respectively).

2.3.12. Nonstandard Refined Neutrosophic Set and Lattice

Even more, Nonstandard Refined Neutrosophic Set / Logic / Probability (which include infinitesimals, monads, and closed monads, binads and closed binads) has no connection and no isomorphism whatsoever with any of the authors' lattices or extensions of their lattices for *2D* and *3D* to *nD*.

2.3.13. Nonstandard Neutrosophic Mobinad Real Lattice

We have built (2018) a more complex Nonstandard Neutrosophic Mobinad Real Lattice, on the nonstandard mobinad unit interval $]^-0, 1^+[$ defined as:

$$]^-0, 1^+[= \{\varepsilon, a, a^-, a^{-0}, a^+, a^{+0}, a^{-+}, a^{-0+}| \text{ with } 0 \leq a \leq 1, a$$
$$\in \mathbb{R}, \text{and } \varepsilon > 0, \varepsilon \text{ infinitesimal}, \varepsilon \in \mathbb{R}^*\}$$

which is both **nonstandard neutrosophic lattice of first type** (as partially ordered set, under neutrosophic inequality $\leq_N$) and lattice of

second type (as algebraic structure, endowed with two binary nonstandard neutrosophic laws: $\inf_N$ and $\sup_N$).

Now, $]^-0, 1^+[^3$ is a nonstandard unit cube, with much higher density than $[0, 1]^3$ and which comprise not only real numbers $a \in [0, 1]$ but also infinitesimals $\varepsilon > 0$ and monads and binads neutrosophically included in $]^-0, 1^+[$.

2.3.14. New ideas brought by the neutrosophic theories, never done before

— The sum of the neutrosophic components is up to 3 (previously the sum was up to 1);

— Degree of independence and dependence between the neutrosophic components T, I, F, making their sum $T + I + F$ to vary between 0 and 3.

For example, when T, I, F are totally dependent with each other, then $T + I + F \leq 1$, so we obtain the particular cases of intuitionistic fuzzy set (when $T + I + F = 1$) and picture set when $T + I + F \leq 1$.

— Nonstandard analysis used in order to be able to distinguish between absolute and relative (truth, membership, chance).

— Refinement of the components into sub-components:

$$\left(T_1, T_2, \ldots, T_p;\ I_1, I_2, \ldots, I_r;\ F_1, F_2, \ldots, F_s\right),$$

with the newly introduced the Refined Neutrosophic Logic / Set / Probability.

— Ability to express *incomplete information* ($T + I + F < 1$) and paraconsistent (conflicting) and subjective information ($T + I + F > 1$).

— Law of Included Middle explicitly/independently expressed as $\langle \text{neut}A \rangle$ (indeterminacy, neutral).

— Law of Included Middle expanded to the Law of Included Multiple-Middles within the refined neutrosophic set and logic and probability.

- A large array of applications [21, 22] in a variety of fields, after two decades from their foundation ([15], 1998), such as: Artificial Intelligence, Information Systems, Computer Science, Cybernetics,

Theory Methods, Mathematical Algebraic Structures, Applied Mathematics, Automation, Control Systems, Communication, Big Data, Engineering, Electrical, Electronic, Philosophy, Social Science, Psychology, Biology, Biomedical, Engineering, Medical Informatics, Operational Research, Management Science, Imaging Science, Photographic Technology, Instruments, Instrumentation, Physics, Optics, Economics, Mechanics, Neurosciences, Radiology Nuclear, Medicine, Medical Imaging, Interdisciplinary Applications, Multidisciplinary Sciences etc.

As we can see, Klement's and Mesiar's claim that neutrosophic set (I do not talk herein about intuitionistic fuzzy set, picture fuzzy set, and Pythagorean fuzzy set that they also criticized) is not a new result… is far from the truth.

2.4. Neutrosophy vs. Yin Yang Philosophy

Ying Han, Zhengu Lu, Zhenguang Du, Gi Luo, and Sheng Chen [3] have defined the so-called "YinYang bipolar fuzzy set" (2018).

But "YinYang bipolar" is already a pleonasm, because in Taoist Chinese philosophy, from the 6th century BC, Yin and Yang is already a bipolarity, between: negative (Yin) / positive (Yang), or feminine (Yin) / masculine (Yang).

Dialectics was derived, much later in time, from Yin Yang.

Neutrosophy, as the dynamicity and harmony between opposites (Yin <A> and Yang (antiA>) together with their neutralities (things which are neither Yin nor Yang, or things which are blends of both: <neutA>) is an extension of Yin Yang Chinese philosophy. Neutrosophy came naturally, since in our world into the dynamicity and conflict and cooperation and even ignorance between opposites the neutrals are attracted and play an important role.

2.4.1. Yin Yang Bipolar Fuzzy Set is The Bipolar Fuzzy Set

The authors sincerely recognize that:

> *"In the existing papers, YinYang bipolar fuzzy set also was called bipolar fuzzy set [5] and bipolar-valued fuzzy set [13, 16]."*

See these papers cited at our References below as [17, 18, 19].

We prove that the YinYang bipolar fuzzy set is not equivalent with neutrosophic set, but a particular case of the bipolar neutrosophic set.

The authors *[3]* say that:

"Denote $I^P = [0 , 1]$ and $I^N = [-1 , 0]$, and

$$L = \{\tilde{\alpha} = (\tilde{\alpha}^P, \tilde{\alpha}^N) \mid \tilde{\alpha}^P \in I^P, \tilde{\alpha}^N \in I^N\},$$

then $\tilde{\alpha}$ is called YinYang bipolar fuzzy number.

(YinYang bipolar fuzzy set) $X = \{x_1 , \cdots, x_n\}$ represents the finite discourse. YinYang bipolar fuzzy set in X is defined by the mapping

$$\tilde{A} : X \to L, x \to \left(\tilde{A}^P(x), \tilde{A}^N(x) \right), \forall x \in X.$$

Where the functions $\tilde{A}^P : X \to I^P, x \to \tilde{A}^P(x) \in I^P$ and $\tilde{A}^N : X \to I^N, x \to \tilde{A}^N(x) \in I^N$ define the satisfaction degree of the element $x \in X$ to the property, and the implicit counter-property to the YinYang bipolar fuzzy set $\tilde{A}$ in X , respectively" (see [3], page 2).

With simpler notations, the above set L is equivalent to:
$L = \{(a, b), \text{ with } a \in [0, 1], b \in [-1, 0]\}$, and the authors denote *(a, b)* as YinYang bipolar fuzzy number.

Further on, again with simpler notations, the so-called *YinYang bipolar fuzzy set* in

$X = \{x_1, \dots, x_n\}$ is equivalent to:

$X = \{x_1(a_1, b_1), \dots, x_n(a_n, b_n)\}$, where all $a_1, \dots, a_n \in [0, 1]$, and

all $b_1, \dots, b_n \in [-1, 0]\}$.

Clearly, thus is the bipolar fuzzy set; no need to baptize it "YinYang bipolar fuzzy set".

The authors added that: "Montero et al. pointed that neutrosophic set is equivalent to the YinYang bipolar fuzzy set in syntax".

But, the bipolar fuzzy set is not equivalent to the neutrosophic set at all. The bipolar fuzzy set is actually a particular case of the bipolar neutrosophic set, defined as (keeping the previous notations):

$X = \{x_1((a_1, b_1), (c_1, d_1), (e_1, f_1)), \dots, x_n((a_n, b_n), (c_n, d_n), (e_n, f_n))\}$, where

all $a_1, \dots, a_n, c_1, \dots, c_n, e_1, \dots, e_n \in [0, 1]$, and all $b_1, \dots, b_n, d_1, \dots, d_n, f_1, \dots, f_n \in [-1, 0]\}$;

for a generic $x_j((a_j, b_j), (c_j, d_j), (e_j, f_j)) \in X$, $1 \leq j \leq n$,

a_i = positive membership degree of x_i, and b_i = negative membership degree of x_i;

c_i = positive indeterminate-membership degree of x_i, and d_i = negative indeterminate membership degree of x_i;

e_i = positive nonmembership degree of x_i, and f_i = negative nonmembership degree of x_i.

Using notations adequate to the neutrosophic environment, one has:

Let $\mathcal{U}$ be a universe of discourse, and $M \subset \mathcal{U}$ be a set. M is a **single-valued bipolar fuzzy set** (that authors call *YinYang bipolar fuzzy set*) if for any element $x\left(T_{(x)}^+, T_{(x)}^-\right) \in M$, $T_{(x)}^+ \in [0, 1]$, and $T_{(x)}^- \in [-1, 0]$, where $T_{(x)}^+$ is the positive membership of x, and $T_{(x)}^-$ is the negative membership of x.　　(BFS)

The authors write that:

> *"Montero et al. pointed that neutrosophic set [22] is equivalent to the YinYang bipolar fuzzy set in syntax [17]".*

Montero et al.'s paper is cited below at References as [4].

If somebody says something, it doesn't mean it is true, they have to verify. Actually, it is *untrue*, since the neutrosophic set is totally different from the so-called YinYang bipolar fuzzy set.

Let $\mathcal{U}$ be a universe of discourse, and $M \subset \mathcal{U}$ be a set, if for any element $x\big(T(x), I(x), F(x)\big) \in M$.

$T(x), I(x), F(x)$ are standard or nonstandard real subsets of the nonstandard real subsets of the nonstandard real unit interval $]^{-}0, 1^{+}[$. (NS)

Clearly, the definitions (BFS) and (NS) are totally different. In the so-called YinYang bipolar fuzzy set there is no indeterminacy $I(x)$, no nonstandard analysis involved, and the neutrosophic components may be subsets as well.

2.4.2. Single-Valued Bipolar Fuzzy Set as particular case of Single-Valued Bipolar Neutrosophic Set

The Single-Valued Bipolar Fuzzy Set (alias YinYang Bipolar Fuzzy Set) is a particular case of the Single-Valued Bipolar Neutrosophic Set, employed by the neutrosophic community, and defined as follows:

Let $\mathcal{U}$ be a universe of discourse, and $M \subset \mathcal{U}$ be a set. M is a single-valued bipolar neutrosophic set, if for any element:

$x\big(T_{(x)}^{+}, T_{(x)}^{-}; I_{(x)}^{+}, I_{(x)}^{-}; F_{(x)}^{+}, F_{(x)}^{-}\big) \in M,$

$T_{(x)}^{+}, I_{(x)}^{+}, F_{(x)}^{+} \in [0, 1],$

and $T_{(x)}^{-}, I_{(x)}^{-}, F_{(x)}^{-} \in [-1, 0].$

2.4.3. Dependent Indeterminacy vs. Independent Indeterminacy

The authors say:

> *"Attanassov's intuitionistic fuzzy set [4] perfectly reflects indeteminacy but not bipolarity".*

We disagree, since Atanassov's intuitionistic fuzzy set [24] perfectly reflects **hesitancy** between membership and nonmembership not **indeterminacy**, since **hesitancy is __dependent__** on membership and nonmembership: $H = 1 - T - F$, where H = hesitancy, T = membership, and F = nonmembership.

It is the single-valued neutrosophic set that "perfectly reflects indeterminacy", since indeterminacy (I) in the neutrosophic set is **independent** from membership (T) and from nonmembership (F).

On the other hand, neutrosophic set perfectly reflects the bipolarity membership/nonmembership as well, since the membership (T) and nonmembership (F) are independent of each other.

2.4.4. Dependent Bipolarity vs. Independent Bipolarity

The **bipolarity** in single-valued fuzzy set and intuitionistic fuzzy set is **dependent** (restrictive), in the sense that if the truth-membership is T, then it involves that the falsehood-nonmembership $F \leq 1 - T$ while the **bipolarity** in single-valued neutrosophic set is independent (nonrestrictive): if the truth-membership $T \in [0,1]$, the falsehood-nonmebership is not influenced at all, so also $F \in [0,1]$.

2.4.5. Equilibriums and Neutralities

Again:

> "While, in semantics, the YinYang bipolar fuzzy set suggests *equilibrium*, and neutrosophic set suggests a general *neutrality*. Al- though, neutrosophic set has been successfully applied to medical diagnosis [9, 27], from above analysis and the conclusion in [31], we see that YinYang bipolar fuzzy set is obvious the suitable model to bipolar disorder diagnosis and will be adopted in this paper."

I'd like to add that single-valued bipolar neutrosophic set suggests:

—three types of equilibrium, between: $T_{(x)}^+$ and $T_{(x)}^-$, $I_{(x)}^+$ and $I_{(x)}^-$, and $F_{(x)}^+$ and $F_{(x)}^-$;

— and two types of neutralities (indeterminacies) between $T_{(x)}^+$ and $F_{(x)}^+$, and between $T_{(x)}^-$ and $F_{(x)}^-$.

Therefore, the single-valued bipolar neutrosophic set is $3 \times 2 = 6$ times more complex and more flexible than the YinYang bipolar fuzzy set. Due to higher complexity, flexibility and capability of catching more details (such as falsehood-nonmembership, and indeterminacy) the single-valued bipolr neutrosophic set is more suitable than YinYang bipolar fuzzy set to be used in bipolar disorder diagnosis.

2.4.6. Zhang-Zhang's Bipolar Model is not equivalent with the Neutrosophic Set

Montero et al. [4] wrote:

> *"Zhang-Zhang's bipolar model is therefore equivalent to the neutrosophic sets proposed by Smarandache [70]" (p. 56).*

This sentence is false, and we proved previously that what Zhang & Zhang proposed in 2004 is a subclass of the single-valued bipolar neutrosophic set.

2.4.7. Tripolar and Multipolar Neutrosophic Sets

Not talking on the fact that in 2016 we have extended our bipolar neutrosophic set to tripolar and even multipolar neutrosophic sets [12], so more general than bipolar fuzzy model.

2.4.8. Neutrosophic Overset / Underset / Offset

Not talking that the unit interval $[0, 1]$ was extended in 2006 below 0 and above 1 into the neutrosophic overset/underset/offset: $[\Omega, \Psi]$ with $\Omega \leq 0 < 1 \leq \Psi$ (as explained above).

2.4.9. Neutrosophic Algebraic Structures

The Montero et al. continue:

> *"Notice also that none of these two equivalent models include any formal structure, as claimed in [48]".*

First, we have proved previously that these two models (Zhang-Zhang's bipolar fuzzy set, and neutrosophic logic) **are not equivalent** at all. Zhang-Zhang's bipolar fuzzy set is a subclass of a particular type of neutrosophic set, called single-valued bipolar neutrosophic set.

Second, since 2013, Kandasamy and Smarandache have developed various algebraic structures (such as neutrosophic semigroup, neutrosophic group, neutrosophic ring, neutrosophic field, neutrosophic vector space, etc.) [20] on the set of neutrosophic numbers:

$S_R = \{a + bI|, \text{where } a, b \in \mathbb{R}, \text{and } I = \text{indeterminacy}, I^2 = I\}$, where $\mathbb{R}$ is the set of real numbers.

And extended on:

$S_C = \{a + bI|, \text{where } a, b \in C, \text{and } I = \text{indeterminacy}, I^2 = I\}$, where C is the set of complex numbers.

But until 2016 [year of Montero et al.'s published paper], I did not developed a formal structure on neutrosophic set, Montero et al. are right.

Yet, in 2018 and consequently at the beginning of 2019, we developed, then generalized and proved that the neutrosophic set has a structure of lattice of first type (as neutrosophically partially ordered set):

$(]^-0, 1^+[, \leq_N)$, where $]^-0, 1^+[$ is the nonstandard neutrosophic mobinad (monads & binads) real unit interval, and $\leq_N$ is the nonstandard neutrosophic inequality.

Moreover,

$(]^-0, 1^+[, \inf_N, \sup_N, {}^-0, 1^+)$,

has also the structure of bound lattice of second type (as algebraic structure), under two binary laws $\inf_N$ (nonstandard neutrosophic infimum) and $\sup_N$ (nontandard neutrosophic supremum).

2.4.10. Neutrality (<neutA>)

Montero et al. continue:

> *"…the selected denominations within each model might suggest different underlying structures: while the model proposed by Zhang and Zhang suggests conflict between categories (a specific type of neutrality different from Atanassov's indeterminacy), Smarandache suggests a general neutrality that should perhaps jointly cover some of the specific types of neutrality considered in our paired approach."*

In neutrosophy and neutrosophic set / logic / probability, the neutrality <neutA> means everything in between <A> and <antiA>, everything which is neither <A> nor <antiA>, or everything which is a blending of <A> and <antiA>.

Further on, in Refined Neutrosophy and Refined Neutrosophic Set / Logic / Probability [9], the neutrality <neutA> was split (refined) in 2013 into sub-neutralities (or sub-indeterminacies), such as: <neutA$_1$>, <neutA$_2$>, …, <neutA$_n$> whose number could be finite or infinite – depending on each application to solve.

Thus, the paired structure becomes a particular case of refined neutrosophy (see next).

2.5. The Pair Structure as a particular case of Refined Neutrosophy

J. Montero, H. Bustince, C. Franco, J. T. Rodriguez, D. Gomez, M. Pagola, J. Fernandez, E. Barrenchea [4] in 2016 have defined a **paired structure**:

> *"composed by a pair of opposite concepts and three types of neutrality as primary valuations":*
>
> L = *{concept, opposite, indeterminacy, ambivalence, conflict}."*

Therefore, each element $x \in X$, where X is a universe of discourse, is characterized by a degree function, with respect to each attribute value from L:

$\mu: X \to [0,1]^5,$

$\mu(x) = \big(\mu_1(x), \mu_2(x), \mu_3(x), \mu_4(x), \mu_5(x)\big),$

where $\mu_1(x)$ represents the degree of x with respect to the concept;

$\mu_2(x)$ represents the degree of x with respect to the opposite (of the concept);

$\mu_3(x)$ represents the degree of x with respect to 'indeterminacy';

$\mu_4(x)$ represents the degree of x with respect to 'ambivalence';

$\mu_5(x)$ represents the degree of x with respect to 'conflict'.

But this paired structure is a particular case of Refined Neutrosophy.

2.5.1. Antonym vs Negation

Firstly, *Dialectics* is the dynamics of opposites, let's denote them by $\langle A \rangle$ and $\langle antiA \rangle$, where $\langle A \rangle$ may be an item, a concept, attribute, idea, theory, and so on; while $\langle antiA \rangle$ is the opposite of $\langle A \rangle$.

Secondly, Neutrosophy ([15], 1998), as a generalization of Dialectics, and new branch of philosophy, is the dynamics of opposites and their neutralities (denoted by $\langle neutA \rangle$). Therefore, Neutrosophy is the dynamics of $\langle A \rangle$, $\langle antiA \rangle$, and $\langle neutA \rangle$.

$\langle neutA \rangle$ means everything which is neither $\langle A \rangle$ nor $\langle antiA \rangle$, or which is a mixture of them, or which is indeterminate, vague, unknown.

The **antonym** of $\langle A \rangle$ is $\langle \text{anti}A \rangle$.

The **negation** of $\langle A \rangle$ (which we denote by $\langle \text{non}A \rangle$) is what is not $\langle A \rangle$, therefore:

$$\neg_N \langle A \rangle = \langle nonA \rangle =_N \langle neutA \rangle \cup_N \langle antiA \rangle.$$

We preferred to use the lower index $_N$ (neutrosophic) because we deal with items, concepts, attributes, ideas, theories etc., such as $\langle A \rangle$ and in consequence its derivates $\langle \text{anti}A \rangle$, $\langle \text{neut}A \rangle$, $\langle \text{non}A \rangle$, whose borders are ambiguous, vague, not clearly delimited.

2.5.2. Refined Neutrosophy as extension of Neutrosophy

Thirdly, *Refined Neutrosophy* ([9], 2013), as an extension of Neutrosophy, and refined branch of philosophy, is the dynamics of refined opposites:

$\langle A_1 \rangle$, $\langle A_2 \rangle$, …, $\langle A_p \rangle$ with $\langle \text{anti}A_1 \rangle$, $\langle \text{anti}A_2 \rangle$, …, $\langle \text{anti}A_s \rangle$, and their refined neutralities: $\langle \text{neut}A_1 \rangle$, $\langle \text{neut}A_2 \rangle$, …, $\langle \text{neut}A_r \rangle$, for integers p, r, $s \geq 1$,

and $p + r + s = n \geq 4$.

Therefore, the item $\langle A \rangle$ has been split into sub-items $\langle A_j \rangle$, $1 \leq j \leq p$, the $\langle \text{anti}A \rangle$ into sub-(anti-items) $\langle \text{anti}A_k \rangle$, $1 \leq l \leq s$, and the $\langle \text{neut}A \rangle$ into sub-(neutral-items) $\langle \text{neut}A_l \rangle$, $1 \leq k \leq r$.

2.5.3. Qualitative Scale as particular case of Refined Neutrosophy

Montero et al.'s *qualitative scale* is a particular case of Refined Neutrosophy, when the neutralities are split into three parts:

L = {concept, opposite, indeterminacy, ambivalence, conflict}

  = {<A>, <antiA>, <neutA₁>, <neutA₂>, <neutA₃>},

where of course:

> *<A> = concept, <antiA> = opposite, <neutA1> = indeterminacy,*
>
> *<neutA2> = ambivalence, <neutA3> = conflict.*

Yin Yang, Dialectics, Neutrosophy, and Refined Neutrosophy (the last one having only ⟨neut$A$⟩ as refined component), are bipolar: ⟨A⟩ and ⟨anti$A$⟩ are the poles.

Montero et al.'s qualitative scale is bipolar ('concept', and its 'opposite').

2.5.4. Multi-Subpolar Refined Neutrosophy

But the Refined Neutrosophy, whose at least one of ⟨A⟩ or ⟨anti$A$⟩ is refined, is *multi-subpolar*.

2.5.5. Multidimensional Fuzzy Set as particular case of Refined Neutrosophic Set

Montero et al. defined the *Multidimensional Fuzzy Set A_L* as:

$$A_t = \{< x; (\mu_s(x))_{s \in L} > | \, x \in X \},$$

where X is the universe of discourse, L = the previous qualitative scale, and $\mu_s(x) \in$ S, where S is a valuation scale (in most cases S = [0, 1]),

$\mu_s(x)$ is the degree of x with respect to s ∈ L.

A *Single-Valued Neutrosophic Set* is defined as follows. Let $\mathcal{U}$ be a universe of discourse, and $M \subset \mathcal{U}$ a set. For each element

$$x\big(T(x), I(x), F(x)\big) \in M,$$

$T(x) \in [0, 1]$ is the degree of truth-membership of element x with respect to the set M;

$I(x) \in [0, 1]$ is the degree of indeterminacy-membership of element x with respect to the set M;

and $F(x) \in [0, 1]$ is the degree of falsehood-nonmembership of element x with respect to the set M.

Let's refine $I(x)$ as $I_1(x)$, $I_2(x)$, $I_3(x) \in [0, 1]$ sub-indeterminacies. Whence we get a single-valued refined neutrosophic set.

$\mu_{concept}(x) = T(x)$ (truth-membership),

$\mu_{opposite}(x) = F(x)$ (falsehood-nonmembership),

$\mu_{indeterminacy}(x) = I_1(x)$ (first sub-indeterminacy),

$\mu_{ambivalence}(x) = I_2(x)$ (second sub-indeterminacy),

$\mu_{conflict}(x) = I_3(x)$ (third sub-indeterminacy).

The *Single-Valued Refined Neutrosophic Set* is defined as follows. Let $\mathcal{U}$ be a universe of discourse, and $M \subset \mathcal{U}$ a set. For each element

$$x\left(T_1(x), T_2(x), \ldots, T_p(x);\ I_1(x), I_2(x), \ldots, I_r(x);\ F_1(x), F_2(x), \ldots, F_s(x)\right)$$
$$\in M$$

$T_j(x)$, $1 \le j \le p$, are degrees of subtruth-submembership of element x with respect to the set M;

$I_k(x)$, $1 \le k \le r$, are degrees of subindeterminacy-membership of element x with respect to the set M;

and $F_l(x)$, $1 \le l \le s$, are degrees of subfalsehood-subnonmembership of element x with respect to the set M,

where integers p, r, $s \ge 1$, and $p + r + s = n \ge 4$.

Therefore, Montero et al.'s **multidimensional fuzzy set** is a particular case of the **refined neutrosophic set**, when $p = 1$, $r = 3$, and $s = 1$, whence $n = 1 + 3 + 1 = 5$.

2.5.6. Plithogeny and Plithogenic Set

Fourthly, in 2017 and 2018 [26 - 29], the Neutrosophy was extended to Plithogeny, which is multipolar, being the dynamics and hermeneutics [methodological study and interpretation] of many opposites and/or their neutrals, together with non-opposites:

⟨A⟩, ⟨neutA⟩, ⟨antiA⟩;

⟨B⟩, ⟨neutB⟩, ⟨antiB⟩; etc.

$\langle C \rangle$, $\langle D \rangle$, etc.

And consequently the **Plithogenic Set** was introduced, as a generalization of **Crisp**, **Fuzzy**, **Intuitionistic Fuzzy**, and **Neutrosophic Sets**.

Unlike previous sets defined, whose elements were characterized by the attribute 'appurtenance' (to the set), which has only one (membership), or two (membership, nonmembership), or three (membership, nonmembership, indeterminacy) attribute values respectively, for the Plithogenic Set each element may be characterized by a multi-attribute, with any number of attribute values.

2.5.7. Refined Neutrosophic Set as Unifying View of Opposite Concepts

Montero et al.'s statement from their paper *Abstract*:

> *"we propose a consistent and unifying view to all those basic knowledge representation models that are based on the existence of two somehow opposite fuzzy concepts".*

With respect to "unifying" claim, their statement is not true, since, as we proved before, their **paired structure** together with three types on neutralities (**indeterminacy**, **ambivalence**, **conflict**) is a simple particular case of the refined neutrosophic set.

The real unifying view nowadays is the **Refined Neutrosophic Set**.

{I was notified about this paired structure article by Dr. Said Broumi, who forwarded it to me.}

2.5.8. Counter-Example to Paired Structure

As a counter-example to the paired structure, it cannot catch a simple voting scenario.

The election for the United States President from 2016: Donald Trump vs. Hillary Clinton. USA has 50 states and since in the country there is used an **Electoral vote**, not a **Popular vote**, it is required to know the winner of each state.

There were two opposite candidates.

The candidate that receives in a state more votes than the other candidate, gets all the points of that state.

As in neutrosophic set, there are three possibilities:

T = percentage of USA people voting for Mr. Trump;

I = percentage of USA people not voting, or voting but giving either a blank vote (not selecting any candidate) or a black vote (cutting all candidates);

F = percentage of USA people voting against Mr. Trump.

The opposite concepts, using Montero et al.'s knowledge representation, are T (voting for, or truth-membership) and F (voting against, or false-membership). But $T > F$, or $T = F$, or $T < F$, that the Paired Structure can catch, mean only the Popular vote, which does not count in the United States.

Actually, it happened that $T < F$ in the US 2016 presidential election, or Mr. Trump lost the Popular vote, but he won the Presidency at the **Electoral vote**!

The paired structure is not capable to refine the opposite concepts (T and F), while the indeterminate (I) could be refined by the paired structure **only** in three parts.

Therefore, the paired structure is not a unifying view of all basic knowledge that use opposite fuzzy concepts. But the refined neutrosophic set / logic / probability do.

Using the refined neutrosophic set and logic, and splits (refines) T, I, and F as:

T_j = percentage of American state S_j people voting for Mr. Trump;

I_j = percentage of American state S_j people not voting, or casting a blank vote or a black vote;

F_j = percentage of American state S_j people voting against Mr. Trump,

with $T_j, I_j, F_j \in [0, 1]$ and $T_j + I_j + F_j = 1$, for all $j \in \{1, 2, \dots, 50\}$.

Therefore, one has:

$$(T_1, T_2, \dots, T_{50};\ I_1, I_2, \dots, I_{50};\ F_1, F_2, \dots, F_{50}).$$

On the other hand, due to the fact that the sub-indeterminacies I_1, I_2, ..., I_{50} did not count towards the winner or looser (only maybe for indeterminate voting statistics), it is not mandatory to refine I. We could simply refine as:

$$(T_1, T_2, \dots, T_{50};\ I;\ F_1, F_2, \dots, F_{50}).$$

2.5.9. Finite Number and Infinite Number of Neutralities

Montero et al.:

> „(...) we emphasize the key role of certain neutralities in our knowledge representation models, as pointed out by Atanassov [4] , Smarandache [70] and others. But notice that our notion of neutrality should not be confused with the neutral value in a traditional sense (see [22,–24,36,54] , among others). Instead, we will stress the existence of different kinds of neutrality that emerge (in the sense of [11]) from the semantic relation between two opposite concepts (and notice also that we refer to a neutral category that does not entail linearity between opposites)."

In neutrosophy, and consequently in neutrosophic set, logic, and probability, between the opposite items (concepts, attributes, ideas, etc.) ⟨A⟩ and ⟨anti$A$⟩ there may be a large number of neutralities/indeterminacies (all together denoted by ⟨neut$A$⟩ even an infinite spectrum – depending on the application to solve.

We agree with different kinds of neutralities and indeterminacies (vague, ambiguous, unknown, incomplete, contradictory, linear and non-linear information, and so on), but the authors display only three neutralities.

In our everyday life and in practical applications there are more neutralities, indeterminacies, even infinitely many.

Let's see another example (besides the previous about Electoral voting), where there may be any number of subindeterminacies / subneutralities).

The opposite concepts attributes are: $\langle A \rangle$ = white, $\langle \text{anti}A \rangle$ = black, while neutral concepts in between may be:

$\langle \text{neut}A_1 \rangle$ = yellow, $\langle \text{neut}A_2 \rangle$ = orange, $\langle \text{neut}A_3 \rangle$ = red, $\langle \text{neut}A_4 \rangle$ = violet, $\langle \text{neut}A_5 \rangle$ = green, $\langle \text{neut}A_6 \rangle$ = blue.

Therefore we have six neutralities.

Example with infinitely many neutralities:

— the opposite concepts: $\langle A \rangle$ = white, $\langle \text{anti}A \rangle$ = black;

— the neutralities: $\langle \text{neut}A_{1,2,\ldots,\infty} \rangle$ = the whole light spectrum between white and black, measured in nanometers (*nn*) [a nanometer is a billionth part of a meter].

2.6. Conclusion

The neutrosophic community thank the authors for their critics and interest in the neutrosophic environment, and we wait for new comments and critics, since as Winston Churchill had said: *the eagles fly higher against the wind.*

References

[1] Erich Peter Klement and Radko Mesiar, L-Fuzzy Sets and Isomorphic Lattices: Are All the "New" Results Really New?, Mathematics 2018, 6, 146.

[2] Wen-Ran Zhang, Lulu Zhang, *YinYang bipolar logic and bipolar fuzzy logic*, Information Sciences 165 (2004) 265–287.

[3] Ying Han, Zhenyu Lu , Zhenguang Du, Qi Luo , Sheng Chen, *A YinYang bipolar fuzzy cognitive TOPSIS method to bipolar disorder diagnosis*, Computer Methods and Programs in Biomedicine, 158 (2018) 1-10.

[4] J. Montero, H. Bustince , C. Franco, J.T. Rodríguez, D. Gómez, M. Pagola, J. Fernández, E. Barrenechea, *Paired structures in knowledge representation*, Knowledge-Base d Systems, 100 (2016) 50-48.

[5] Jean Dezert, *Open Questions to Neutrosophic Inferences,* Multiple-Valued Logic / An International Journal, Vol. 8, No. 3, 439-472, June 2002.

[6] F. Smarandache, About Nonstandard Neutrosophic Logic (Answers to Imamura's 'Note on the Definition of Neutrosophic Logic'), Cornell University, New York City, USA ; (Submitted on 24 Nov 2018 (v1), last revised 13 Feb 2019 (this version, v2)),

Abstract: https://arxiv.org/abs/1812.02534v2

Full Paper: https://arxiv.org/ftp/arxiv/papers/1812/1812.02534.pdf

[7] F. Smarandache, *Extended Nonstandard Neutrosophic Logic, Set, and Probability based on Extended Nonstandard Analysis*, mss. 1-26, v1, Jan. 2019.

[8] H. Wang, F. Smarandache, Y.-Q. Zhang, R. Sunderraman, *Interval Neutrosophic Sets and Logic: Theory and Applications in Computing*, Hexis, 87 p., 2005, http://arxiv.org/pdf/cs/0505014.pdf

[9] F. Smarandache, *n-Valued Refined Neutrosophic Logic and Its Applications in Physics*, Progress in Physics, 143-146, Vol. 4, 2013; https://arxiv.org/ftp/arxiv/papers/1407/1407.1041.pdf

[10] Florentin Smarandache, *Neutrosophic Overset, Neutrosophic Underset, and Neutrosophic Offset. Similarly for Neutrosophic Over-/Under-/Off- Logic, Probability, and Statistics*, 168 p., Pons Editions, Bruxelles, Belgique, 2016; https://arxiv.org/ftp/arxiv/papers/1607/1607.00234.pdf;

https://hal.archives-ouvertes.fr/hal-01340830;

http://fs.unm.edu/NeutrosophicOversetUndersetOffset.pdf

[11] Florentin Smarandache, *Operadores con conjunto neutrosóficos de valor único Oversets, Undersets y Off-set*, Neutrosophic Computing and Machine Learning, Vol. 4, 2018, pp. 3-7. DOI: 10.5281/zenodo.2140877

[12] F. Smarandache, *Interval-Valued Neutrosophic Oversets, Neutrosophic Understes, and Neutrosophic Offsets*, International Journal of Science and Engineering Investigations, Vol. 5, Issue 54, 1-4, July 2016.

http://fs.unm.edu/IV-Neutrosophic-Overset-Underset-Offset.pdf

[13] F. Smarandache, *Operators on Single-Valued Neutrosophic Oversets, Neutrosophic Undersets, and Neutrosophic Offsets*, Journal of Mathematics and Informatics, Vol. 5, 63-67, 2016; https://hal.archives-ouvertes.fr/hal-01340833

http://fs.unm.edu/SVNeutrosophicOverset-JMI.pdf

[14] F. Smarandache, *Applications of Neutrosophic Sets in Image Identification, Medical Diagnosis, Fingerprints and Face Recognition* and *Neutrosophic verset/Underset/Offset*, COMSATS Institute of Information Technology, Abbottabad, Pakistan, December 26th, 2017.

[15] F. Smarandache, *Neutrosophy. / Neutrosophic Probability, Set, and Logic*, ProQuest Information & Learning, Ann Arbor, Michigan, USA, 105 p., 1998; http://fs.unm.edu/eBook-Neutrosophics6.pdf

[16] Denis Howe, *The Free Online Dictionary of Computing*, England, 1999;

[17] I. Bloch, *Geometry of spatial bipolar fuzzy sets based on bipolar fuzzy numbers and mathematical morphology, fuzzy logic and applications*, Lect. Notes Comput. Sci. 5571 (2009) 237–245 .

[18] Y. Han, P. Shi, S. Chen, *Bipolar-valued rough fuzzy set and its applications to decision information system*, IEEE Trans. Fuzzy Syst. 23 (2015) 2358–2370 .

[19] K.M. Lee, *Bipolar-valued fuzzy sets and their basic operations*, in: Proceedings of the International Conference, Bangkok, Thailand, 2010, pp. 307–317 .

[20] W. B. Vasantha Kandasamy, Florentin Smarandache, *Fuzzy Cognitive Maps and Neutrosophic Cognitive Maps*, Xiquan, Phoenix, 211 p., 2003; http://fs.unm.edu/NCMs.pdf

[21] Xindong Peng and Jingguo Dai, *A bibliometric analysis of neutrosophic set: two decades review from 1998 to 2017*, Artificial Intelligence Review, Springer, 18 August 2018; http://fs.unm.edu/BibliometricNeutrosophy.pdf

[22] Stefan Vladutescu, Florentin Smarandache, Daniela Gifu, Alina Tenescu (editors), *Topical Communication Uncertainties*, Sitech Publishing House (Craiova, Romania) and Zip Publishing (Columbus, Ohio, USA), 300 p., 2014.

[23] F. Smarandache, *Extended Nonstandard Neutrosophic Logic, Set, and Probability based on Extended Nonstandard Analysis*, 31 pages, arXiv, Cornell University, 11 March 2019,
Abstract: https://arxiv.org/abs/1903.04558,
Article: https://arxiv.org/ftp/arxiv/papers/1903/1903.04558.pdf

[24] K. T. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets Syst., 1986, 20, 87–96.

[25] Cuong, B.C.; Kreinovich, V. *Picture fuzzy sets—A new concept for computational intelligence problems.* In Proceedings of the Third World Congress on Information and Communication Technologies (WICT 2013), Hanoi, Vietnam, 15–18 December 2013; pp. 1–6.

[26] Florentin Smarandache, *Plithogeny, Plithogenic Set, Logic, Probability, and Statistics*, by Pons Publishing House, Brussels, Belgium, 141 p., 2017;
arXiv.org (Cornell University), Computer Science - Artificial Intelligence, 03Bxx:
book: https://arxiv.org/ftp/arxiv/papers/1808/1808.03948.pdf
abstract: https://arxiv.org/abs/1808.03948
Harvard SAO/NASA ADS:
http://adsabs.harvard.edu/cgi-bin/bib_query?arXiv:1808.03948

[27] F. Smarandache, *Extension of Soft Set to Hypersoft Set, and then to Plithogenic Hypersoft Set*, Neutrosophic Sets and Systems, vol. 22, 2018, pp. 168-170. DOI: 10.5281/zenodo.2159754

[28] Florentin Smarandache: *Plithogenic Set, an Extension of Crisp, Fuzzy, Intuitionistic Fuzzy, and Neutrosophic Sets – Revisited*,

Neutrosophic Sets and Systems, vol. 21, 2018, pp. 153-166; https://doi.org/10.5281/zenodo.1408740

[29] Florentin Smarandache, *Physical Plithogenic Set*, 71st Annual Gaseous Electronics Conference, American Physical Society (APS), Session LW1, Oregon Convention Center Room, Portland, Oregon, USA, November 5 –9, 2018; http://meetings.aps.org/Meeting/GEC18/Session/LW1.110

CHAPTER 3

About Nonstandard Neutrosophic Logic

(Answers to Imamura's "Note on the Definition of

Neutrosophic Logic")

Abstract

In order to more accurately situate and fit the neutrosophic logic into the framework of nonstandard analysis, we present the neutrosophic inequalities, neutrosophic equality, neutrosophic infimum and supremum, neutrosophic standard intervals, including the cases when the neutrosophic logic standard and nonstandard components *T, I, F* get values outside of the classical unit interval [0, 1], and a brief evolution of neutrosophic operators. The paper intends to answer Imamura's criticism that we found benefic in better understanding the nonstandard neutrosophic logic – although the nonstandard neutrosophic logic was never used in practical applications.

3.1 Uselessness of Nonstandard Analysis in Neutrosophic Logic, Set, Probability, et al.

Imamura's discussion [1] on the definition of neutrosphic logic is welcome, but it is useless, since from all neutrosophic papers and books published, from all conference presentations, and from all MSc and PhD theses defended around the world, etc. (more than one thousand) in the last two decades since the first neutrosophic research started (1998-2018), and from hundreds of neutrosophic researchers, not even a single one ever used the nonstandard form of neutrosophic logic, set, or probability and statistics in no occasion (extended researches or applications).

All researchers, with no exception, have used the *Standard Neutrosophic Set and Logic* [so no stance whatsoever of *Nonstandard Neutrosophic Set and Logic*], where the neutrosophic components *T, I, F* are subsets of the standard unit interval [0, 1].

Even more, for simplifying the calculations, the majority of researchers have utilized the *Single-Valued Neutrosophic Set and Logic* {when *T, I, F* are single numbers from [0, 1]}, on the second place was *Interval-Valued Neutrosophic Set and Logic* {when *T, I, F* are intervals included in [0, 1]}, and on the third one the *Hesitant Neutrosophic Set and Logic* {when *T, I, F* were discrete finite sets included in [0, 1]}.

In this direction, there have been published papers on single-valued "neutrosophic standard sets" [12, 13, 14], where the neutrosophic components are just *standard real numbers*, considering the particular case when $0 \le T + I + F \le 1$ (in the most general case $0 \le T + I + F \le 3$).

Actually, Imamura himself acknowledges on his paper [1], page 4, that:

> "neutrosophic logic does not depend on transfer, so the use of non-standard analysis is not essential for this logic, and can be eliminated from its definition".

Entire neutrosophic community has found out about this result and has ignored the non-standard analysis in the studies and applications of neutrosophic logic for two decades.

3.2 Applicability of Neutrosophic Logic et al. vs. Theoretical Nonstandard Analysis

Neutrosophic logic, set, measure, probability, statistics and so on were designed with the primordial goal of being applied in practical fields, such as:

> Artificial Intelligence, Information Systems, Computer Science, Cybernetics, Theory Methods, Mathematical Algebraic Structures, Applied Mathematics, Automation, Control Systems, Big Data, Engineering, Electrical, Electronic, Philosophy, Social Science, Psychology, Biology, Biomedical, Engineering, Medical Informatics, Operational Research, Management Science, Imaging Science, Photographic Technology, Instruments, Instrumentation, Physics, Optics, Economics, Mechanics, Neurosciences, Radiology Nuclear, Medicine, Medical Imaging, Interdisciplinary Applications, Multidisciplinary Sciences etc. [2],

while nonstandard analysis is mostly a pure mathematics.

Since 1990, when I emigrated from a political refugee camp in Turkey to America, working as a software engineer for Honeywell Inc., in Phoenix, Arizona State, I was advised by American co-workers to do theories that have *practical applications*, not pure-theories and abstractizations as "art pour art".

3.3 Theoretical Reason for the Nonstandard Form of Neutrosophic Logic

The only reason I have added the nonstandard form to neutrosophic logic (and similarly to neutrosophic set and probability) was in order to make a distinction between *Relative Truth* (which is truth in some Worlds, according to Leibniz) and *Absolute Truth* (which is truth in all possible Words, according to Leibniz as well) that occur in philosophy.

Another possible reason may be when the neutrosophic degrees of truth, indeterminacy, or falsehood are infinitesimally determined, for example a value infinitesimally bigger than 0.8 (or 0.8+), or infinitesimally smaller than 0.8 (or -0.8). But these can easily be overcome by roughly using interval neutrosophic values, for example (0.80, 0.81) and (0.79, 0.80) respectively.

I wanted to get the neutrosophic logic as general as possible [6], extending all previous logics (Boolean, fuzzy, intuitionistic fuzzy logic, intuitionistic logic, paraconsistent logic, dialethism), and to have it able to deal with all kind of logical propositions (including paradoxes, nonsensical propositions, etc.).

That's why in 2013 I extended the *Neutrosophic Logic* to *Refined Neutrosophic Logic* [from generalizations of *2-valued Boolean logic* to *fuzzy logic*, also from the *Kleene's* and *Lukasiewicz's* and *Bochvar's 3-symbol valued logics* or *Belnap's 4-symbol valued logic* to the most general *n-symbol* or *n-numerical valued refined neutrosophic logic*, for any integer $n \geq 1$], the largest ever so far, when some or all neutrosophic components *T, I, F* were respectively split/refined into neutrosophic

subcomponents: $T_1, T_2, \ldots; I_1, I_2, \ldots; F_1, F_2, \ldots$ which were deduced from our everyday life [3].

3.4 From Paradoxism movement to Neutrosophy branch of philosophy and then to Neutrosophic Logic

I started first from *Paradoxism* (that I founded in 1980's as a movement based on antitheses, antinomies, paradoxes, contradictions in literature, arts, and sciences), then I introduced the *Neutrosophy* (as generalization of Dialectics, neutrosophy is a branch of philosophy studying the dynamics of triads, inspired from our everyday life, triads that have the form:

<A>,

its opposite <antiA>,

and their neutrals <neutA>,   (3.1)

where <A> is any item or entity [4].

(Of course, we take into consideration only those triads that make sense in our real and scientific world.)

The Relative Truth neutrosophic value was marked as 1, while the Absolute Truth neutrosophic value was marked as 1^+ (a tinny bigger than the Relative Truth's value):

$1^+ >_N 1,$

where $>_N$ is a neutrosophic inequality, meaning 1^+ is neutrosophically bigger than 1.

Similarly for Relative Falsehood / Indeterminacy (which falsehood / indeterminacy in some Worlds), and Absolute Falsehood / Indeterminacy (which is falsehood / indeterminacy in all possible worlds).

3.5. Introduction to Nonstandard Analysis [15, 16]

An *infinitesimal number* (ε) is a number ε such that:

$$|\varepsilon| < 1/n,$$

for any non-null positive integer n. An infinitesimal is close to zero, and so small that it cannot be measured.

The infinitesimal is a number smaller, in absolute value, than anything positive nonzero.

Infinitesimals are used in calculus.

An *infinite number* (ω) is a number greater than anything:

$$1 + 1 + 1 + \ldots + 1 \text{ (for any finite number terms)} \qquad (3.2)$$

The infinites are reciprocals of infinitesimals.

The set of hyperreals (non-standard reals), denoted as R^*, is the extension of set of the real numbers, denoted as R, and it comprises the infinitesimals and the infinites, that may be represented on the hyperreal number line

$$1/\varepsilon = \omega/1. \qquad (3.3)$$

The set of hyperreals satisfies the *transfer principle*, which states that the statements of first order in R are valid in R^* as well.

A *monad* (*halo*) of an element $a \; \varepsilon \in R^*$, denoted by $\mu(a)$, is a subset of numbers infinitesimally close to a.

Let's denote by $R_+{}^*$ the set of positive nonzero hyperreal numbers.

We consider the left monad and right monad, and we have introduced the *binad* [5]:

Left Monad { that we denote, for simplicity, by (^-a) or only ^-a } is defined as:

$$\mu(^-a) = (^-a) = {}^-a =$$

$$= \{a - x, x \in R_+{}^* \mid x \text{ is infinitesimal}\}. \qquad (3.4)$$

Right Monad { that we denote, for simplicity, by (a^+) or only by a^+ } is defined as:

$$\mu(a^+) = (a^+) = a^+ =$$

$$= \{a + x, x \in R_+{}^* \mid x \text{ is infinitesimal}\}. \qquad (3.5)$$

Bimonad { that we denote, for simplicity, by $(^-a^+)$ or only $^-a^+$ } is defined as:

$$\mu(^-a^+) = (^-a^+) = {}^-a^+ = \{a - x, \; x \in R+^* \mid x \text{ is infinitesimal}\}$$

$$\cup \{a + x, \; x \in R+^* \mid x \text{ is infinitesimal}\}$$

$$= \{ a \pm x, \; x \in R+^* \mid x \text{ is infinitesimal}\}. \qquad (3.6)$$

The left monad, right monad, and the bimonad are subsets of R^*.

3.6. Neutrosophic Strict Inequalities

We recall the neutrosophic inequality which is needed for the inequalities of nonstandard numbers.

Let α, β be elements in a partially ordered set M.

We have defined the neutrosophic strict inequality

$$\alpha >_N \beta \qquad (3.7)$$

and read as

"*α is neutrosophically greater than β*" $\qquad (3.8)$

if   *α in general is greater than β,*

or *α is approximately greater than β,*

or *subject to some indeterminacy* (unknown or unclear ordering relationship between α and β) *or subject to some contradiction (*situation when α is smaller than or equal to β*) α is greater than β.*

It means that in most of the cases, on the set M, *α is greater than β.*

And similarly for the opposite neutrosophic strict inequality *α* $<_N$ *β.*

3.7. Neutrosophic Equality

We have defined the *neutrosophic inequality*

$$\alpha =_N \beta \qquad (3.9)$$

and read as

"*α is neutrosophically equal to β*" $\qquad (3.10)$

if   *α in general is equal to β*,

or *α is approximately equal to β*,

or *subject to some indeterminacy* (unknown or unclear ordering relationship between α and β)

*or subject to some contradiction (*situation when α is not equal to β*) α is equal to β.*

It means that in most of the cases, on the set *M*,

α is equal to β.

3.8. Neutrosophic (Non-Strict) Inequalities

Combining the neutrosophic strict inequalities with neutrosophic equality, we get the ≥*N* and ≤*N* neutrosophic inequalities.

Let *α, β* be elements in a partially ordered set *M*.

The *neutrosophic (non-strict) inequality*

$$\alpha \geq_N \beta \tag{3.11}$$

and read as

"*α is neutrosophically greater than or equal to β*"   (3.12)

if

α in general is greater than or equal to β,

or *α is approximately greater than or equal to β*,

or *subject to some indeterminacy* (unknown or unclear ordering relationship between α and β) or *subject to some contradiction* (situation when α is smaller than β) *α is greater than or equal to β.*

It means that in most of the cases, on the set *M*, *α is greater than or equal to β.*

And similarly for the opposite neutrosophic (non-strict) inequality *α ≤N β.*

3.9. Neutrosophically Ordered Set

Let M be a set. $(M, <_N)$ is called a neutrosophically ordered set if:

$\forall\ \alpha, \beta \in M$, one has:

either $\alpha <_N \beta$, or $\alpha =_N \beta$, or $\alpha >_N \beta$.   (3.13)

3.10. Neutrosophic Nonstandard Inequalities

Let $\mathscr{P}(R^*)$ be the power-set of R^*. Let's endow $(\mathscr{P}(R^*), <_N)$ with a neutrosophic inequality   Let $a, b \in R$, where R is the set of (standard) real numbers.

And let $(^-a), (a^+), (^-a^+) \in \mathscr{P}(R^*)$, and $(^-b), (b^+), (^-b^+) \in \mathscr{P}(R^*)$, be the left monads, right monads, and the bimonads of the elements (standard real numbers) a and b respectively. Since all monads are subsets, we may treat the single real numbers $a = [a, a]$ and $b = [b, b]$ as subsets too.

$P(R^*)$ is a set of subsets, and thus we deal with neutrosophic inequalities between subsets.

i) If the subset α has many of its elements above all elements of the subset β,

then $\alpha >_N \beta$ (partially).

ii) If the subset α has many of its elements below all elements of the subset β,

then $\alpha <_N \beta$ (partially).

iii) If the subset α has many of its elements equal with elements of the subset β,

then $\alpha =_N \beta$ (partially).

If the subset α verifies *i)* and *iii)* with respect to subset β, then $\alpha \geq_N \beta$. If the subset α verifies *ii)* and *iii)* with respect to subset β, then $\alpha \leq_N \beta$.

If the subset α verifies *i)* and *ii)* with respect to subset β,

then there is no neutrosophic order (inequality) between α and β.

{ For example, between $(^-a^+)$ and a there is no neutrosophic order. }

Similarly, if the subset α verifies *i), ii)* and *iii)* with respect to subset β, then there is no neutrosophic order (inequality) between α and β.

3.11. Open Neutrosophic Research

The quantity or measure of "many of its elements" of the above *i), ii),* and *iii)* conditions depends on each *neutrosophic application* and on its *neutrosophic experts*.

For the *neutrosophic nonstandard inequalities*, we propose based on the above three conditions the following:

$$(\text{-a}) <_N \ a \ <_N \ (a^+) \tag{3.14}$$

because $\forall x \in R^*, a - x < a < a + x$, where x is of course a (nonzero) positive infinitesimal (the above double neutrosophic inequality actually becomes a double classical standard real inequality for each fixed positive infinitesimal).

$$(^-a) \leq_N (^-a^+) \leq_N (a^+) \tag{3.15}$$

This double neutrosophic inequality may be justified due to $(^-a^+) = (^-a) \cup (a^+),$ so:

$$(^-a) \leq_N (^-a) \cup (a^+) \leq_N (a^+) \tag{3.16}$$

whence the left side of the inequality middle term coincides with the inequality first term, while the right side of the inequality middle term coincides with the third inequality term.

If $a > b,$ which is a (standard) classical real inequality, then we have the following neutrosophic nonstandard inequalities:

$$a >_N (^-b), \quad a >_N (b^+), \quad a >_N (^-b^+); \tag{3.17}$$

$$(^-a) >_N b, (^-a) >_N (^-b), (^-a) >_N (b^+), \quad (^-a) >_N (^-b^+); \tag{3.18}$$

$$(a^+) >_N b, \ (a^+) >_N (^-b),$$

$$(a^+) >_N (b^+), \quad (a^+) >_N (^-b^+); \tag{3.19}$$

$$(^-a^+) >_N b, \ (^-a^+) >_N (^-b),$$

$$(^-a^+) >_N (b^+), \quad (^-a^+) >_N (^-b^+). \tag{3.20}$$

If $a \geq b$, which is a (standard) classical real inequality, then we have the following neutrosophic nonstandard inequalities:

a $\geq_N$ (-b);   (3.21)

(-a) $\geq_N$ (-b);   (3.22)

(a+) $\geq_N$ (-b), (a+) $\geq_N$ b, a+) $\geq_N$ (b+),

(a+) $\geq_N$ (-b+);   (3.23)

($^-a^+$) $\geq_N$ (^-b),   ($^-a^+$) $\geq_N$ ($^-b^+$).   (3.24)

And similarly for $<_N$ and $\leq_N$ neutrosophic nonstandard inequalities.

3.12. Neutrosophic Nonstandard Equalities

Let a, b be standard real numbers; if $a = b$ that is a (classical) standard equality, then:

(^-a) $=_N$ (^-b), (a^+) $=_N$ (b^+), ($^-a^+$) $=_N$ ($^-b^+$).   (3.25)

3.13. Neutrosophic Infimum and Neutrosophic Supremum

As an extension of the classical infimum and classical supremum, and using the neutrosophic inequalities and neutrosophic equalities, we define the neutrosophic infimum (denoted as *infN*) and the neutrosophic supremum (denoted as *supN*).

3.13.1 Neutrosophic Infimum.

Let $(S, <_N)$ be a set that is neutrosophically partially ordered, and M a subset of S.

The neutrosophic infimum of M, denoted as *infN(M)* is the neutrosophically greatest element in S that is neutrosophically less than or equal to all elements of M.

3.13.2 Neutrosophic Supremum.

Let $(S, <_N)$ be a set that is neutrosophically partially ordered, and M a subset of S.

The neutrosophic supremum of *M*, denoted as *sup$_N$(M)* is the neutrosophically smallest element   in *S* that is neutrosophically greater than or equal to all elements of *M*.

3.14. Classical Infimum and Supremum vs. Neutrosophic Infimum and Supremum

Giving the definitions of neutrosophic components from my book [5]:

"Let *T, I, F* be standard or non-standard real subsets of *]$^-$0, 1$^+$[*,

with sup T = t_sup, inf T = t_inf, sup I = i_sup, inf I = i_inf,

sup F = f_sup, inf F = f_inf,

and n_sup = t_sup+i_sup+f_sup, n_inf = t_inf+i_inf+f_inf."

Imamura argues (page 3) that:

> "Subsets of R$_*$, even bounded, may have neither infima nor suprema, because the   transfer principle ensures the existences of infima and suprema only for internal sets."

This is true from a classical point of view, yet according to the definitions of the neutrosophic inequalities, the neutrosophic infimum and supremum do exist for the nonstandard intervals, for   example:

inf$_N$ ( ]$^-$a, b$^+$[ ) = $^-$a, and sup$_N$ ( ]$^-$a, b$^+$[ ) = b$^+$.   (3.26)

Indeed, into my definition above I had to clearly mention that we <u>talk neutrosophically</u> [*mea   culpa*] by inserting an "N" standing for neutrosophic (*inf$_N$* and *sup$_N$*):

Let *T, I, F* be standard or non-standard real subsets of *]$^-$0, 1$^+$[*,

with sup$_N$ T = t_sup, inf$_N$ T = t_inf, sup$_N$ I = i_sup, inf$_N$ I = i_inf,

sup$_N$ F = f_sup, inf$_N$ F = f_inf,

and n_sup = t_sup+i_sup+f_sup, n_inf = t_inf+i_inf+f_inf.

I was more prudent when I presented the sum of single valued standard neutrosophic components, saying:

Let T, I, F be single valued numbers, T, I, F ∈ [0, 1], such that $0 \leq T + I + F \leq 3$.

A friend alerted me: "If T, I, F are numbers in [0, 1], of course their sum is between 0 and 3."

"Yes, I responded, I afford this tautology, because if I did not mention that the sum is up to *3*, readers would take for granted that the sum *T + I + F* is bounded by *1*, since that is in all logics and in probability!"

3.15. Notations

Imamura is right when criticizing my confusion of notations between hyperreals (numbers) and monads (subsets). I was rather informal than formal at the beginning.

By ^-a and b^+ most of times I wanted to mean the subsets of left monad and right monad respectively. Taking an arbitrary positive infinitesimal ε, and writing $^-a = a\text{-}\varepsilon$ and $b^+ = b+\varepsilon$ was actually picking up a representative from each class (monad).

Similarly, representations of the monads by intervals were not quite accurate from a classical point of view:

$$(^-a) = (a\text{-}\varepsilon,\ a), \tag{3.27}$$
$$(b^+) = (b,\ b+\varepsilon), \tag{3.28}$$
$$(^-a^+) = (a\text{-}\varepsilon,\ a) \cup (b,\ b+\varepsilon), \tag{3.29}$$

but they were rather neutrosophic equalities (approximations):

$$(^-a) =_N (a\text{-}\varepsilon,\ a), \tag{3.30}$$
$$(b^+) =_N (b,\ b+\varepsilon), \tag{3.31}$$
$$(^-a^+) =_N (a\text{-}\varepsilon,\ a) \cup (b,\ b+\varepsilon). \tag{3.32}$$

3.16. Nonarchimedean Ordered Field

At pages 5-6 of note [1], Imamura proposed the following Nonarchimedean Ordered Field K:

"Let $x, y \in K$. x and y are said to be *infinitely close* (denoted by $a \approx b$) if $a - b$ is infinitesimal. We say that x is *roughly smaller than y* (and write $x^{<} y$) if $x < y$ or $x \approx y$."

An ordered field is called nonarchimedian field, if it has non-null infinitesimals.

While it is a beautiful definition to consider that x and y are *infinitely close* (denoted by $a \approx b$) if $a - b$ is infinitesimal, it produces confusions into the nonstandard neutrosophic logic. Why? Because one cannot distinguish any-longer between ^{-}a, a, and a^{+} (which is essential in, and the flavor of, nonstandard neutrosophic logic, in order to differentiate the *relative truth/indeterminacy/falsehood* from *absolute truth/indeterminacy/falsehood* respectively), since one gets that:

$$(^{-}a) \approx a \approx (a^{+}) \tag{3.33}$$

or with the simplest notations:

$$^{-}a \approx a \approx a^{+}. \tag{3.34}$$

Proof

$$\forall x \in R^{*}, a - (a - x) = x = \text{infinitesimal},$$

whence $a \approx (^{-}a)$ $\qquad\qquad$ (3.35)

and $\forall x \in R^{*}, (a + x) - a = x = \text{infinitesimal},$

whence $a^{+} \approx a.$ $\qquad\qquad$ (3.36)

For the definition of nonstandard interval $]^{-}a, b^{+}[$, Imamura proposes at page 6:

"For $a, b \in K$ the set $]^{-}a, b^{+}[K$ is defined as follows:

$$]^{-}a, b^{+}[K = \{x \in K \mid a^{<} x^{<} b\}. "$$

In nonstandard neutrosophic logic and set, we may have not only $]^{-}a, b^{+}[$, but various forms of nonstandard intervals:

$$m_1 \quad m_2$$
$$]a, b[\tag{3.37}$$

where m_1 and m_2 stand for: left monads ($^-$), right monads ($^+$), or bimonads ($^-{^+}$), in all possible combinations (in total $3 \times 3 = 9$ possibilities).

Yet, Imamura's definition cannot be adjusted for all above nonstandard intervals, for example the nonstandard intervals of the form $]a^+, {}^-b[$, because if one writes

$$]a^+, b^-[_K = \{x \in K \mid a^<_{,}x^<b\} \tag{3.38}$$

one arrives at proving that

$$]{}^-a, b^+[_K \subseteq]a^+, b^-[_K \tag{3.39}$$

which is obviously false, since: ^-a is below a and hence below a^+, and in the same way b^+ is above b and hence above ^-b {one gets a bigger nonstandard interval included in or equal to a smaller nonstandard interval}. This occurs because $^-a \approx a^+$ and $b^+ \approx b^-$ (in Imamura's notation).

3.17. Nonstandard Unit Interval

Imamura cites my work:

> "by "$-a$" one signifies a monad, i.e., a set of hyper-real numbers in non-standard analysis:
>
> $(-a) = \{ a - x \in R^* \mid x \text{ is infinitesimal} \}$,
>
> and similarly "b+" is a hyper monad:
>
> $(b+) = \{ b + x \in R^* \mid x \text{ is infinitesimal} \}$. ([5] p. 141; [6] p. 9)"

But these are inaccurate, because my exact definitions of monads, since my 1998 first world neutrosophic publication {see [5], page 9; and [6], pages 385 - 386}, were:

> "$(^-a) = \{ a - x: x \in R+^* \mid x \text{ is infinitesimal} \}$,
>
> and similarly "b+" is a hyper monad:

$(b^+) = \{ \ b + x: x \in R+^* \mid x \text{ is infinitesimal } \}$"

Imamura says that:

> "The correct definitions are the following:
>
> $(-a) = \{ \ a - x \in R^* \mid x \text{ is } positive \text{ infinitesimal } \}$,
>
> $(b+) = \{ \ b + x \in R^* \mid x \text{ is } positive \text{ infinitesimal } \}$."

I did not have a chance to see how my article was printed in *Proceedings of the 3rd Conference of the European Society for Fuzzy Logic and Technology* [7], that Imamura talks about, maybe there were some typos, but Imamura can check the *Multiple Valued Logic / An International Journal* [6], published in England in 2002 (ahead of the European Conference from 2003, that Imamura cites) by the prestigious Taylor & Francis Group Publishers, and clearly one sees that it is: $\boldsymbol{R}_+^*$ (so, *x* is a *positive* infinitesimal into the above formulas), therefore there is no error.

Then Imamura continues:

> "Ambiguity of the definition of the nonstandard unit interval. Smarandache did not give any explicit definition of the notation $]^-0, 1^+[$ in [5] (or the notation $\Vdash^-0, 1^+\dashv\mathrel{\|}$ in [6]). He only said:
>
> Then, we call $] \ ^-0, 1^+ [$ a non-standard unit interval. Obviously, 0 and 1, and analogously non-standard numbers infinitely small but less than 0 or infinitely small but greater than 1, belong to the non-standard unit interval. ([5] p. 141; [6] p. 9)."

Concerning the notations I used for the nonstandard intervals as $\Vdash \dashv\mathrel{\|}$ or $] \ [$, it was imperative to employ notations different from the classical *[ ]* or *( )* intervals, since the extremes of the nonstandard unit interval were unclear, vague. I thought it was easily understood that:

$$]^-0, 1^+[\ = (^-0) \cup [0, 1] \cup (1^+). \qquad (3.40)$$

Or, using the previous neutrosophic inequalities, we may write:

$]^-0, 1^+[= \{x \in R^*, {}^-0 \leq_N x \leq_N 1^+\}.$ (3.41)

Imamura says that:

> "Here $^-0$ and 1^+ are particular real numbers defined in the previous paragraph:
>
> $^-0 = 0-\varepsilon$ and $1^+ = 1+ \varepsilon$, where ε is a fixed non-negative infinitesimal."

This is untrue, I never said that "ε is a *fixed* non-negative infinitesimal", ε was not fixed, I said that for any real numbers *a* and *b* {see again [5], page 9; and [6], pages 385 - 386}:

> *"(^-a) = { a − x: x ∈ R+* | x is infinitesimal },*
>
> *(b^+) = { b + x: x ∈ R+* | x is infinitesimal }".*

Therefore, once we replace a = 0 and b = 1 we get:

($^-0$) = { 0 − x: x ∈ R$_+$* | x is infinitesimal },

(1^+) = { 1 + x: x ∈ R$_+$* | x is infinitesimal }.

Thinking out of box, inspired from the real world, was the first intent, i.e. allowing neutrosophic components (truth / indeterminacy / falsehood) values be outside of the classical (standard) unit real interval *[0, 1]* used in all previous (Boolean, multi-valued etc.) logics if needed in applications, so neutrosophic component values *< 0* and *> 1* had to occurs due to the Relative / Absolute stuff, with:

$^-0 <_N 0$ and $1^+ >_N 1.$ (3.42)

Later on, in 2007, I found plenty of cases and real applications in Standard Neutrosophic Logic and Set (therefore, not using the Nonstandard Neutrosophic Logic and Set), and it was thus possible the extension of the neutrosophic set *to Neutrosophic Overset (when some neutrosophic component is > 1), and to Neutrosophic Underset (when some neutrosophic component is < 0), and to Neutrosophic Offset (when some neutrosophic components are off the interval [0, 1], i.e. some neutrosophic component > 1 and some neutrosophic component < 0).*

Then, similar extensions to respectively *Neutrosophic Over/Under/Off Logic, Measure,*

Probability, Statistics etc. [8, 17, 18, 19], extending the unit interval [0, 1] to

$$[\Psi, \Omega], \text{ with } \Psi \leq 0 < 1 \leq \Omega, \tag{3.43}$$

where Ψ, Ω are standard real numbers.

Imamura says, ref. the definition of neutrosophic logic that:

"In this logic, each proposition takes a value of the form (T, I, F), where T, I, F are subsets of the nonstandard unit interval $]^-0, 1^+[$ and represent all possible values of Truthness, Indeterminacy and Falsity of the proposition, respectively."

Unfortunately, this is not exactly how I defined it.

In my first book {see [5], p. 12; or [6] pp. 386 – 387} it is stated:

"Let T, I, F be real standard or non-standard subsets of]-0, 1+[" meaning that T, I, F may also be "real standard" not only real non-standard.

In *The Free Online Dictionary of Computing*, 1999-07-29, edited by Denis Howe from England, it is written:

Neutrosophic Logic:

<logic> (Or "Smarandache logic") A generalization of fuzzy logic based on Neutrosophy. A proposition is t true, i indeterminate, and f false, where t, i, and f are real values from the ranges T, I, F, with no restriction on T, I, F, or the sum n=t+i+f. Neutrosophic logic thus generalizes:

- intuitionistic logic, which supports incomplete theories (for 0<n<100, 0<=t,i,f<=100);

- fuzzy logic (for n=100 and i=0, and 0<=t,i,f<=100);

- Boolean logic (for n=100 and i=0, with t,f either 0 or 100);

- multi-valued logic (for 0<=t,i,f<=100);

- paraconsistent logic (for n>100, with both t,f<100);

- dialetheism, which says that some contradictions are true (for t=f=100 and i=0; some paradoxes can be denoted this way).

Compared with all other logics, neutrosophic logic introduces a percentage of "indeterminacy" - due to unexpected parameters hidden in some propositions. It also allows each component t,i,f to "boil over" 100 or "freeze" under 0. For example, in some tautologies t>100, called "overtrue". ["Neutrosophy / Neutrosophic probability, set, and logic", F. Smarandache, American Research Press, 1998].

As Denis Howe said in 1999, the neutrosophic components *t, i, f* are "*real values* from the ranges *T, I, F*", not nonstandard values or nonstandard intervals. And this was because nonstandard ones were not important for the neutrosophic logic (the Relative/Absolute plaid no role in technological and scientific applications and future theories).

3.18. The Logical Connectives ∧, ∨, →

Imamura's critics of my first definition of the neutrosophic operators is history for long ago.

All fuzzy, intuitionistic fuzzy, and neutrosophic logic operators are inferential approximations, not written in stone. They are improved from application to application.

Let's denote:

∧F, ∧N, ∧P representing respectively the fuzzy conjunction, neutrosophic conjunction, and plithogenic conjunction;

Similarly

∨F, ∨N, ∨P representing respectively the fuzzy disjunction, neutrosophic disjunction, and plithogenic disjunction,

and

→F, →N, →P representing respectively the fuzzy implication, neutrosophic implication, and plithogenic implication.

I agree that my beginning neutrosophic operators (when I applied the same fuzzy t-norm, or the same fuzzy t-conorm, to all neutrosophic components T, I, F) were less accurate than others developed later by the neutrosophic community researchers. This was pointed out since 2002 by Ashbacher [9] and confirmed in 2008 by Rivieccio [10]. They observed that if on T1 and T2 one applies a fuzzy t-norm, on their opposites F1 and F2 one needs to apply the fuzzy t-conorm (the opposite of fuzzy t-norm), and reciprocally.

About inferring I1 and I2, some researchers combined them in the same directions as T1 and T2.

Then:

$$(T_1, I_1, F_1) \wedge_N (T_2, I_2, F_2) =$$
$$= (T_1 \wedge_F T_2, I_1 \wedge_F I_2, F_1 \vee_F F_2), \qquad (3.44)$$
$$(T_1, I_1, F_1) \vee_N (T_2, I_2, F_2) =$$
$$= (T_1 \vee_F T_2, I_1 \vee_F I_2, F_1 \wedge_F F_2), \qquad (3.45)$$
$$(T_1, I_1, F_1) \rightarrow_N (T_2, I_2, F_2) = (F_1, I_1, T_1) \vee_N (T_2, I_2, F_2) =$$
$$= (F_1 \vee_F T_2, I_1 \vee_F I_2, T_1 \wedge_F F_2); \qquad (3.46)$$

others combined I_1 and I_2 in the same direction as F_1 and F_2 (since both I and F are negatively qualitative neutrosophic components), the most used one:

$$(T_1, I_1, F_1) \wedge_N (T_2, I_2, F_2) =$$
$$= (T_1 \wedge_F T_2, I_1 \vee_F I_2, F_1 \vee_F F_2), \qquad (3.47)$$
$$(T_1, I_1, F_1) \vee_N (T_2, I_2, F_2) =$$
$$= (T_1 \vee_F T_2, I_1 \wedge_F I_2, F_1 \wedge_F F_2), \qquad (3.48)$$
$$(T_1, I_1, F_1) \rightarrow_N (T_2, I_2, F_2) = (F_1, I_1, T_1) \vee_N (T_2, I_2, F_2) =$$
$$= (F_1 \vee_F T_2, I_1 \wedge_F I_2, T_1 \wedge_F F_2). \qquad (3.49)$$

Now, applying the neutrosophic conjunction suggested by Imamura:

"This causes some counterintuitive phenomena. Let A be a (true) proposition with value ({ 1 } , { 0 } , { 0 }) and let B be a (false) proposition with value ({ 0 } , { 0 } , { 1 }).

Usually we expect that the falsity of the conjunction A ∧ B is { 1 }. However, its actual falsity is { 0 }."

we get:

$$(1, 0, 0) \wedge_N (0, 0, 1) = (0, 0, 1), \tag{3.50}$$

which is correct (so the falsity is *1*).

Even more, recently, in an extension of neutrosophic set to *plithogenic set* [11] (which is a set whose each element is characterized by many attribute values), the *degrees of contradiction c(,)* between the neutrosophic components *T, I, F* have been defined (in order to facilitate the design of the aggregation operators), as follows:

*c(T, F) = 1 (*or *100%,* because they are totally opposite*), c(T, I) = c(F, I) = 0.5 (*or *50%,* because they are only half opposite), then:

$$(T_1, I_1, F_1) \wedge_P (T_2, I_2, F_2) =$$

$$= (T_1 \wedge_F T_2, 0.5(I_1 \wedge_F I_2) + 0.5(I_1 \vee_F I_2), F_1 \vee_F F_2), \tag{3.51}$$

$$(T_1, I_1, F_1) \vee_P (T_2, I_2, F_2) =$$

$$= (T_1 \vee_F T_2, 0.5(I_1 \vee_F I_2) + 0.5(I_1 \wedge_F I_2), F_1 \wedge_F F_2). \tag{3.52}$$

$$(T_1, I_1, F_1) \rightarrow_N (T_2, I_2, F_2) = (F_1, I_1, T_1) \vee_N (T_2, I_2, F_2)$$

$$= (F_1 \vee_F T_2, 0.5(I_1 \vee_F I_2) + 0.5(I_1 \wedge_F I_2), T_1 \wedge_F F_2). \tag{3.53}$$

Conclusion

We thank very much Dr. Takura Imamura for his interest and critics of *Nonstandard Neutrosophic Logic*, which eventually helped in improving it. {In the history of mathematics, critics on nonstandard analysis, in general, have been made by Paul Halmos, Errett Bishop, Alain Connes and others.} We hope we'll have more dialogues on the subject in the future.

References

[1] Takura Imamura, *Note on the Definition of Neutrosophic Logic*, arxiv.org, 7 Nov. 2018.

[2] Xindong Peng and Jingguo Dai, *A bibliometric analysis of neutrosophic set: two decades review from 1998 to 2017*, Artificial Intelligence Review, Springer, 18 August 2018; http://fs.unm.edu/BibliometricNeutrosophy.pdf   [3]   Florentin Smarandache, *n-Valued Refined Neutrosophic Logic and Its Applications in Physics*, Progress in Physics, 143-146, Vol. 4, 2013;
http://fs.unm.edu/n-ValuedNeutrosophicLogic-PiP.pdf

[4] F. Smarandache, Neutrosophy, *A New Branch of Philosophy*, <Multiple Valued Logic / An International Journal>, USA, ISSN 1023-6627, Vol. 8, No. 3, pp. 297-384, 2002.

[5] Florentin Smarandache, *Neutrosophy. / Neutrosophic Probability, Set, and Logic*, ProQuest Information & Learning, Ann Arbor, Michigan, USA, 105 p., 1998; http://fs.unm.edu/eBook-Neutroosphics6.pdf.

[6] F. Smarandache, *A Unifying Field in Logics: Neutrosophic Logic*, <Multiple Valued Logic / An International Journal>, USA, ISSN 1023-6627, Vol. 8, No. 3, pp. 385-438, 2002. {The whole issue of this journal is dedicated to Neutrosophy and Neutrosophic Logic.}

[7] Florentin Smarandache, *Definition of neutrosophic logic — a generalization of the intuitionistic fuzzy logic*, Proceedings of the 3rd Conference of the European Society for Fuzzy Logic and Technology, 2003, pp. 141–146.

[8] Florentin Smarandache, *Neutrosophic Overset, Neutrosophic Underset, and Neutrosophic Offset. Similarly for Neutrosophic Over-/Under-/Off- Logic, Probability, and Statistics*, 168 p., Pons Editions, Bruxelles, Belgique, 2016;
https://arxiv.org/ftp/arxiv/papers/1607/1607.00234.pdf

[9] Charles Ashbacher, *Introduction to Neutrosophic Logic*, ProQuest Information & Learning, Ann Arbor, 2002,
http://fs.unm.edu/IntrodNeutLogic.pdf

[10] Umberto Rivieccio, *Neutrosophic logics: Prospects and problems*, Fuzzy Sets and Systems, v. 159, issue 14, 1860–1868, 2008.

[11] Florentin Smarandache: *Plithogeny, Plithogenic Set, Logic, Probability, and Statistics*, Pons Publishing House, Brussels, Belgium, 141 p., 2017; arXiv.org (Cornell University), Computer Science - Artificial Intelligence, 03Bxx:

https://arxiv.org/ftp/arxiv/papers/1808/1808.03948.pdf

[12] Nguyen Xuan Thao, Florentin Smarandache, *(I, T)-Standard neutrosophic rough set and its topologies properties*, Neutrosophic Sets and Systems, Vol. 14, 2016, pp. 65-70; doi.org/10.5281/zenodo.570892

http://fs.unm.edu/NSS/RoughStandardNeutrosophicSets.pdf

[13] Nguyen Xuan Thao, Bui Cong Cuong, Florentin Smarandache, *Rough Standard Neutrosophic Sets: An Application on Standard Neutrosophic Information Systems*, Neutrosophic Sets and Systems, Vol. 14, 2016, pp. 80-92; doi.org/10.5281/zenodo.570890

http://fs.unm.edu/NSS/RoughStandardNeutrosophicSets.pdf

[14] Bui Cong Cuong, Pham Hong Phong, Florentin Smarandache, *Standard Neutrosophic Soft Theory - Some First Results*, Neutrosophic Sets and Systems, Vol. 12, 2016, pp. 80-91; doi.org/10.5281/zenodo.571149

http://fs.unm.edu/NSS/StandardNeutrosophicSoftTheory.pdf

[15] Insall, Matt and Weisstein, Eric W. *Nonstandard Analysis*. From MathWorld--A Wolfram Web Resource.

http://mathworld.wolfram.com/NonstandardAnalysis.html

[16] Insall, Matt. *Transfer Principle*. From MathWorld--A Wolfram Web Resource, created by Eric W. Weisstein.

http://mathworld.wolfram.com/TransferPrinciple.html

[17] F. Smarandache, *Applications of Neutrosophic Sets in Image Identification, Medical Diagnosis, Fingerprints and Face Recognition and Neutrosophic Overset/Underset/Offset*, COMSATS Institute of Information Technology, Abbottabad, Pakistan, December 26th, 2017;

[18] F. Smarandache, *Interval-Valued Neutrosophic Oversets, Neutrosophic Understes, and Neutrosophic Offsets*, International Journal of Science and Engineering Investigations, Vol. 5, Issue 54, 1-4, July 2016.

[19] F. Smarandache, *Operators on Single-Valued Neutrosophic Oversets, Neutrosophic Undersets, and Neutrosophic Offsets*, Journal of Mathematics and Informatics, Vol. 5, 63-67, 2016.

CHAPTER 4

Extended Nonstandard Neutrosophic Logic, Set, and Probability based on Extended Nonstandard Analysis

Abstract

We extend for the second time the Nonstandard Analysis by adding the left monad closed to the right, and right monad closed to the left, while besides the pierced binad (we introduced in 1998) we add now the unpierced binad - all these in order to close the newly extended nonstandard space under nonstandard addition, nonstandard subtraction, nonstandard multiplication, nonstandard division, and nonstandard power operations. Then, we extend the Nonstandard Neutrosophic Logic, Nonstandard Neutrosophic Set, and Nonstandard Probability on this Extended Nonstandard Analysis space - that we prove it is a nonstandard neutrosophic lattice of first type (endowed with a nonstandard neutrosophic partial order) as well as a nonstandard neutrosophic lattice of second type (as algebraic structure, endowed with two binary neutrosophic laws, infN and supN). Many theorems, new terms introduced, better notations for monads and binads, and examples of nonstandard neutrosophic operations are given.

4.1. Short Introduction

In order to more accurately situate and fit the neutrosophic logic into the framework of extended nonstandard analysis, we present the nonstandard neutrosophic inequalities, nonstandard neutrosophic equality, nonstandard neutrosophic infimum and supremum, nonstandard neutrosophic intervals, including the cases when the neutrosophic logic standard and nonstandard components *T, I, F* get values outside of the

classical unit interval *[0, 1],* and a brief evolution of neutrosophic operators.

4.2. Theoretical Reason for the Nonstandard Form of Neutrosophic Logic

The only reason we have added the nonstandard form to neutrosophic logic (and similarly to neutrosophic set and probability) was in order to make a distinction between *Relative Truth* (which is truth in some Worlds, according to Leibniz) and *Absolute Truth* (which is truth in all possible Words, according to Leibniz as well) that occur in philosophy.

Another possible reason may be when the neutrosophic degrees of truth, indeterminacy, or falsehood are infinitesimally determined, for example a value infinitesimally bigger than *0.8 (*or *0.8⁺),* or infinitesimally smaller than *0.8 (*or *⁻0.8).* But these can easily be overcome by roughly using interval neutrosophic values, for example (0.80, 0.81) and (0.79, 0.80) respectively.

4.3. Why the Sum of Neutrosophic Components is up to 3

I was more prudent when I presented the sum of single valued standard neutrosophic components, saying:

Let T, I, F be single valued numbers, T, I, F ∈ [0, 1], such that:

$$0 \leq T + I + F \leq 3. \tag{4.1}$$

The sum of the single-valued neutrosophic components, $T + I + F$ is up to *3* since they are considered completely (*100%*) independent of each other [28]. But if, let's say, two components *T* and *F* are completely (*100%*) dependent, then $T + F \leq 1$ (as in fuzzy and intuitionistic fuzzy logics), and let's assume the neutrosophic middle component *I* is completely (*100%*) independent from *T* and *F,* then $I \leq 1,$ whence *T + F + I ≤ 1 + 1 = 2.*

But the degree of dependence/independence between *T, I, F* all together, or taken two by two, may be, in general, any number between *0* and *1*.

4.4. Neutrosophic Components Outside the Unit Interval [0, 1]

Thinking out of box, inspired from the real world, was the first intent, i.e. allowing neutrosophic components (truth / indeterminacy / falsehood) values be outside of the classical (standard) unit real interval *[0, 1]* used in all previous (Boolean, multi-valued etc.) logics if needed in applications, so neutrosophic component values < *0* and > *1* had to occurs due to the Relative / Absolute stuff, with:

$$^-0 <_N 0 \quad \text{and} \quad 1^+ >_N 1. \tag{4.2}$$

Later on, in 2007, I found plenty of cases and real applications in Standard Neutrosophic Logic and Set (therefore, not using the Nonstandard Neutrosophic Logic and Set), and it was thus possible the extension of the neutrosophic set *to Neutrosophic Overset (when some neutrosophic component is > 1), and to Neutrosophic Underset (when some neutrosophic component is < 0), and to Neutrosophic Offset (when some neutrosophic components are off the interval [0, 1], i.e. some neutrosophic component > 1 and some neutrosophic component < 0). Then, similar extensions to respectively Neutrosophic Over/Under/Off Logic, Measure, Probability, Statistics etc.* [8, 17, 18, 19], extending the unit interval [0, 1] to: [Ψ, Ω], with $\Psi \leq 0 < 1 \leq \Omega$, (4.3)

where *Ψ, Ω* are standard real numbers.

4.5. Refined Neutrosophic Logic, Set, and Probability

We wanted to get the neutrosophic logic as general as possible [6], extending all previous logics (Boolean, fuzzy, intuitionistic fuzzy logic, intuitionistic logic, paraconsistent logic, dialethism), and to have it able to deal with all kind of logical propositions (including paradoxes, nonsensical propositions, etc.).

That's why in 2013 we extended the Neutrosophic Logic to Refined Neutrosophic Logic [from generalizations of 2-valued Boolean logic to fuzzy logic, also from the Kleene's and Lukasiewicz's and Bochvar's 3-symbol valued logics or Belnap's 4-symbol valued logic to the most general n-symbol or n-numerical valued refined neutrosophic logic, for any integer $n \geq 1$], the largest ever so far, when some or all neutrosophic components T, I, F were respectively split/refined into neutrosophic subcomponents: $T_1, T_2, \ldots; I_1, I_2, \ldots; F_1, F_2, \ldots$ which were deduced from our everyday life [3].

4.6. From Paradoxism movement to Neutrosophy branch of philosophy and then to Neutrosophic Logic

We started first from *Paradoxism* (that I founded in 1980's as a movement based on antitheses, antinomies, paradoxes, contradictions in literature, arts, and sciences), then we introduced the *Neutrosophy* (as generalization of Dialectics of Hegel and Marx, which is actually the ancient YinYang Chinese philosophy), neutrosophy is a branch of philosophy studying the dynamics of triads, inspired from our everyday life, triads that have the form:

<A>, its opposite *<antiA>*, and their neutrals *<neutA>*, (4.4)

where *<A>* is any item or entity [4].

(Of course, we take into consideration only those triads that make sense in our real and scientific world.)

The Relative Truth neutrosophic value was marked as *1*, while the Absolute Truth neutrosophic value was marked as 1^+ (a tinny bigger than the Relative Truth's value):

$1^+ >_N 1$, where $>_N$ is a neutrosophic inequality, meaning 1^+ is neutrosophically bigger than *1*.

Similarly for Relative Falsehood / Indeterminacy (which falsehood / indeterminacy in some Worlds), and Absolute Falsehood / Indeterminacy (which is falsehood / indeterminacy in all possible worlds).

4.7. Introduction to Nonstandard Analysis [25, 15, 16]

An *infinitesimal* [or infinitesimal number] (ε) is a number ε such that $|\varepsilon| < 1/n$, for any non-null positive integer n. An infinitesimal is close to zero, and so small that it cannot be measured.

The infinitesimal is a number smaller, in absolute value, than anything positive nonzero.

Infinitesimals are used in calculus.

An *infinite* [or infinite number] *(ω)* is a number greater than anything:

1 + 1 + 1 + ... + 1 (for any finite number terms)   (4.5)

The infinites are reciprocals of infinitesimals.

The set of *hyperreals* (or *non-standard reals*), denoted as R^*, is the extension of set of the real numbers, denoted as R, and it comprises the infinitesimals and the infinites, that may be represented on the *hyperreal number line*

1/ε = ω/1.   (4.6)

The set of hyperreals satisfies the *transfer principle*, which states that the statements of first order in R are valid in R^* as well.

A *monad* (*halo*) of an element $a \in R^*$, denoted by $\mu(a)$, is a subset of numbers infinitesimally close to a.

4.8. First Extension of Nonstandard Analysis

Let's denote by $R_+^{\;*}$ the set of positive nonzero hyperreal numbers.

We consider the left monad and right monad, and the (*pierced*) *binad* that we have introduced as extension in 1998 [5]:

Left Monad { that we denote, for simplicity, by (^-a) or only ^-a } is defined as:

$\mu(^-a) = (^-a) = {}^-a = \bar{a} = \{a - x,\ x \in R_+^{\;*} \mid x \text{ is infinitesimal}\}.$ (4.7)

Right Monad { that we denote, for simplicity, by (a^+) or only by a^+ } is defined as:

$$\mu(a^+) = (a^+) = a^+ = \overset{+}{a} = \{a + x, \ x \in R_+^* \mid x \text{ is infinitesimal}\}. \qquad (4.8)$$

Pierced Binad { that we denote, for simplicity, by $(\bar{\ }a^+)$ or only $\bar{\ }a^+$ } is defined as:

$$\mu(\bar{\ }a^+) = (\bar{\ }a^+) = \bar{\ }a^+ = \overset{-+}{a} =$$

$$= \{a - x, \ x \in R_+^* \mid x \text{ is infinitesimal}\} \cup \{a + x, \ x \in R_+^* \mid x \text{ is infinitesimal}\}$$

$$= \{\ a \pm x \ , \ x \in R_+^* \mid x \text{ is infinitesimal}\}. \qquad (4.9)$$

The left monad, right monad, and the pierced binad are subsets of R^*.

4.9. Second Extension of Nonstandard Analysis

For necessity of doing calculations that will be used in nonstandard neutrosophic logic in order to calculate the nonstandard neutrosophic logic operators (conjunction, disjunction, negation, implication, equivalence) and in order to have the Nonstandard Real MoBiNad Set closed under arithmetic operations, we extend now for the time: the left monad to the Left Monad Closed to the Right, the right monad to the Right Monad Closed to the Left; and the Pierced Binad to the Unpierced Binad, defined as follows:

Left Monad Closed to the Right

$$\mu\left(\overset{-0}{a}\right) = \left(\overset{-0}{a}\right) = \overset{-0}{a} = \{a - x \mid x = 0, \text{ or } x \in R_+^*$$

and x is infinitesimal$\} = \mu(\bar{\ }a) \cup \{a\} = (\bar{\ }a) \cup \{a\}$

$= \bar{\ }a \cup \{a\}. \qquad (4.10)$

Right Monad Closed to the Left

$$\mu\left(\overset{0+}{a}\right) = \left(\overset{0+}{a}\right) = \overset{0+}{a} = \{a + x \mid x = 0, \text{ or } x \in R_+^*$$

and x is infinitesimal$\} = \mu(a^+) \cup \{a\} = (a^+) \cup \{a\}$

$= a^+ \cup \{a\}.$ (4.11)

Unpierced Binad

$$\mu\left(\overset{-0+}{a}\right) = \left(\overset{-0+}{a}\right) = \overset{-0+}{a} = \{a - x \mid x \in R_+^* \text{ and x is infinitesimal}\}$$

$\cup \{a + x \mid x \in R_+^* \text{ and x is infinitesimal}\} \cup \{a\} =$

$= \{\, a \pm x \mid x = 0, \text{ or } x \in R_+^* \text{ and } \text{x is infinitesimal}\,\}$

$= \mu(^-a^+) \cup \{a\} = (^-a^+) \cup \{a\} = {}^-a^+ \cup \{a\}$ (4.12)

The element *{a}* has been included into the left monad, right monad, and pierced binad respectively.

4.10. Nonstandard Neutrosophic Function

In order to be able to define equalities and inequalities in the sets of monads, and in the sets of binads, we construct a *nonstandard neutrosophic function* that approximates the monads and binads to tiny open (or half open and half closed respectively) standard real intervals as below. It is called 'neutrosophic' since it deals with indeterminacy: unclear, vague monads and binads, and the function approximates them with some tiny real subsets.

Taking an arbitrary infinitesimal:

$\varepsilon_1 > 0,$

and writing $^-a = a-\varepsilon_1$, $a^+ = a+\varepsilon_1$,

and $^-a^+ = a \pm \varepsilon_1,$ (4.13)

or taking an arbitrary infinitesimal $\varepsilon_2 \geq 0,$

and writing

$$\overset{-0}{a} = (a - \varepsilon_2, a], \overset{0+}{a} = [a, a + \varepsilon_2), \overset{-0+}{a} = (a - \varepsilon_2, a + \varepsilon_2)$$ (4.14)

we meant actually picking up a representative from each class of the monads and of the binads respectively.

Representations of the monads and binads by intervals is not quite accurate from a classical point of view, but it is an approximation that helps in finding a partial order and computing nonstandard arithmetic operations on the elements of the nonstandard set NR_{MB}.

Let ε be a generic positive infinitesimal, while a be a generic standard real number.

Let $P(R)$ be the power set of the real number set R.

$$\mu_N : NR_{MB} \rightarrow P(R) \tag{4.15}$$

For any $a \in R$, the set of real numbers, one has:

$$\mu_N(\,(^-a)\,) =_N (a - \varepsilon, a), \tag{4.16}$$

$$\mu_N(\,(a^+)\,) =_N (a, a + \varepsilon), \tag{4.17}$$

$$\mu_N(\,(^-a^+)\,) =_N (a - \varepsilon, a) \cup (a, a + \varepsilon) \tag{4.18}$$

$$\mu_N\left(\left(\begin{smallmatrix} -0 \\ a \end{smallmatrix}\right)\right) =_N (a - \varepsilon, a],$$

$$\mu_N\left(\left(\begin{smallmatrix} 0+ \\ a \end{smallmatrix}\right)\right) =_N [a, a + \varepsilon),$$

$$\mu_N\left(\left(\begin{smallmatrix} -0+ \\ a \end{smallmatrix}\right)\right) =_N (a - \varepsilon, a + \varepsilon),$$

$$\text{(4.19-4.20-4.21)}$$

$$\mu_N\left(\left(\begin{smallmatrix} 0 \\ a \end{smallmatrix}\right)\right) =_N \mu_N(a) =_N a = [a, a],$$

$$\text{(4.22)}$$

in order to set it as real interval too.

4.11. General Notations for Monads and Binads

Let $a \in R$ be a standard real number. We use the following general notation for monads and binads:

$$\overset{m}{a} \in \{\overset{-}{a}, \overset{-0}{a}, \overset{+}{a}, \overset{0+}{a}, \overset{-+}{a}, \overset{-0+}{a}, \overset{\ }{a}\} \text{ and by convention } \overset{0}{a} = a; \tag{4.23}$$

$$\text{or } m \in \{\ , \overset{-}{\ }, \overset{-0}{\ }, \overset{+}{\ }, \overset{+0}{\ }, \overset{-+}{\ }, \overset{-0+}{\ }\} = \{\overset{0}{\ }, \overset{-}{\ }, \overset{-0}{\ }, \overset{+}{\ }, \overset{+0}{\ }, \overset{-+}{\ }, \overset{-0+}{\ }\}; \tag{4.24}$$

therefore "*m*" above a standard real number "*a*" may mean anything: a standard real number (^{0}, or nothing above), a left monad ($^-$), a left monad closed to the right ($^{-0}$), a right monad ($^+$), a right monad closed to the left ($^{0+}$), a pierced binad ($^{-+}$), or a unpierced binad ($^{-0+}$) respectively.

The notations of monad's and binad's diacritics above (not laterally) the number *a* as

$$\overset{-}{a}, \overset{-0}{a}, \overset{+}{a}, \overset{0+}{a}, \overset{-+}{a}, \overset{-0+}{a} \qquad (4.25)$$

are the best, since they also are designed to avoid confusion for the case when the real number *a* is negative.

For example, if $a = {}^-2$, then the corresponding monads and binads are respectively represented as:

$$\overset{-}{{}^-2}, \overset{-0}{{}^-2}, \overset{+}{{}^-2}, \overset{0+}{{}^-2}, \overset{-+}{{}^-2}, \overset{-0+}{{}^-2} \qquad (4.26)$$

4.12. Classical and Neutrosophic Notations

Classical notations on the set of real numbers:

$$<, \leq, >, \geq, \wedge, \vee, \rightarrow, \leftrightarrow, \cap, \cup, \subset, \supset, \subseteq, \supseteq, =, \in,$$

$$+, -, \times, \div, \wedge, * \qquad (4.27)$$

Operations with real subsets: $\circledast$ \qquad (4.28)

Neutrosophic notations on nonstandard sets (that involve indeterminacies, approximations, vague boundaries):

$$<_N, \leq_N, >_N, \geq_N, \wedge_N, \vee_N, \rightarrow_N, \leftrightarrow_N, \cap_N, \cup_N, \subset_N, \supset_N, \subseteq_N, \supseteq_N, =_N, \in_N$$

$$+_N, -_N, \times_N, \div_N, \wedge_N, *_N \qquad (4.29)$$

4.13. Neutrosophic Strict Inequalities

We recall the neutrosophic strict inequality which is needed for the inequalities of nonstandard numbers.

Let *α, β* be elements in a partially ordered set *M*.

We have defined the *neutrosophic strict inequality*

$$\alpha >_N \beta \qquad\qquad\qquad (4.30)$$

and read as

"α is neutrosophically greater than β"

if α in general is *greater than* β,

or α is *approximately greater* than β,

or *subject to some indeterminacy* (unknown or unclear ordering relationship between α and β) *or subject to some contradiction (*situation when α is smaller than or equal to β*) α is greater than β.*

It means that in most of the cases, on the set *M*, *α is greater than β.*

And similarly for the opposite neutrosophic strict inequality $\alpha <_N \beta$.

$$(4.31)$$

4.14. Neutrosophic Equality

We have defined the neutrosophic inequality

$$\alpha =_N \beta \qquad\qquad\qquad (4.32)$$

and read as

"α is neutrosophically equal to β"

if α in general is *equal* to β,

or α is *approximately equal* to β,

or *subject to some indeterminacy* (unknown or unclear ordering relationship between α and β) *or subject to some contradiction (*situation when α is not equal to β*) α is equal to β.*

It means that in most of the cases, on the set *M*, *α is equal to β.*

4.15. Neutrosophic (Non-Strict) Inequalities

Combining the neutrosophic strict inequalities with neutrosophic equality, we get the $\geq_N$ and $\leq_N$ neutrosophic inequalities.

Let *α, β* be elements in a partially ordered set *M*.

The *neutrosophic (non-strict) inequality*

$$\alpha \geq_N \beta \tag{4.33}$$

and read as

"**α is neutrosophically greater than or equal to β**"

if

α in general is *greater than* or *equal* to β,

or α is *approximately greater than* or *equal* to β,

or *subject to some indeterminacy* (unknown or unclear ordering relationship between α and β) or subject to some *contradiction* (situation when α is smaller than β) α is greater than or equal to β.

It means that in most of the cases, on the set *M*, *α is greater than or equal to β.*

And similarly for the opposite neutrosophic (non-strict) inequality α $\leq_N$ β. (4.34)

4.16. Neutrosophically Ordered Set

Let *M* be a set. (*M*, $<_N$) is called a neutrosophically ordered set if:

$$\forall \ \alpha, \beta \in M, \text{ one has: either } \alpha <_N \beta, \text{ or } \alpha =_N \beta, \text{ or } \alpha >_N \beta. \tag{4.35}$$

4.17. Neutrosophic Infimum and Neutrosophic Supremum

As an extension of the classical infimum and classical supremum, and using the neutrosophic inequalities and neutrosophic equalities, we define the neutrosophic infimum (denoted as inf_N) and the neutrosophic supremum (denoted as sup_N).

Neutrosophic Infimum

Let (*S*, $<_N$) be a set that is neutrosophically partially ordered, and *M* a subset of *S*. The neutrosophic infimum of *M*, denoted as $inf_N(M)$ is the neutrosophically greatest element in *S* that is neutrosophically less than or equal to all elements of *M*.

Neutrosophic Supremum

Let $(S, <_N)$ be a set that is neutrosophically partially ordered, and M a subset of S. The neutrosophic supremum of M, denoted as $sup_N(M)$ is the neutrosophically smallest element in S that is neutrosophically greater than or equal to all elements of M.

4.18. Definition of Nonstandard Real MoBiNad Set

Let $\mathbb{R}$ be the set of standard real numbers, $\mathbb{R}^*$ the set of hyper-reals (or non-standard reals) which consists of infinitesimals and infinites.

The Nonstandard Real MoBiNad Set is now defined, for the first time, as follows:

$$NR_{MB} =_N \left\{ \begin{array}{c} \varepsilon, \omega, a, (^-a), (^-a^0), (a^+), (^0a^+), (^-a^+), (^-a^{0+}) \,\Big|\, \text{where } \varepsilon \text{ are infinitesimals,} \\ \text{with } \varepsilon \in \mathbb{R}^*; \ \omega = \frac{1}{\varepsilon} \text{are infinites,} \\ \text{with } \omega \in \mathbb{R}^*; \text{and } a \text{ are real numbers, with } a \in \mathbb{R} \end{array} \right\}$$

$$(4.36)$$

Therefore:

$$NR_{MB} =_N \mathbb{R}^* \cup \mathbb{R} \cup \mu(^-\mathbb{R}) \cup \mu(^-\mathbb{R}^0) \cup \mu(\mathbb{R}^+) \cup \mu(^0\mathbb{R}^+)$$

$$\cup \mu(^-\mathbb{R}^+) \cup \mu(^-\mathbb{R}^{0\,+}), \qquad\qquad (4.37)$$

where

$\mu(^-\mathbb{R})$ is the set of all real left monads,

$\mu(^-\mathbb{R}^0)$ is the set of all real left monads closed to the right,

$\mu(\mathbb{R}^+)$ is the set of all real right monads,

$\mu(^0\mathbb{R}^+)$ is the set of all real right monads closed to the left,

$\mu(^-\mathbb{R}^+)$ is the set of all real pierced binads,

and

$\mu(^-\mathbb{R}^{0\,+})$ is the set of all real unpierced binads.

Also,

$$NR_{MB} =_N \left\{ \varepsilon, \omega, \overset{m}{a} \; \middle| \; \begin{array}{l} \text{where } \varepsilon, \omega \in \mathbb{R}^*, \varepsilon \text{ are infinitesimals,} \\ \omega = \frac{1}{\varepsilon} \text{ are infinities; } a \in \mathbb{R}; \\ \text{and } m \in \{ \,^-, ^{-0}, ^+, ^{+0}, ^{-+}, ^{-0+} \} \end{array} \right\} \quad (4.38)$$

NR_{MB} is closed under addition, subtraction, multiplication, division

[except division by $\overset{m}{a}$, with $a = 0$ and $m \in \{ \,^-, ^{-0}, ^+, ^{+0}, ^{-+}, ^{-0+} \}$],

and power

$$\left\{ \left(\overset{m_1}{a} \right)^{\left(\overset{m_2}{b} \right)} \quad \text{with: either } a > 0, \text{ or } a = 0 \text{ and } b > 0, \text{ or } a < 0 \text{ but } b = \frac{p}{r} \right.$$

(irreducible fraction) and *p, r* are integers with *r* an odd positive number, $r \in \{1, 3, 5, ...\}$ \}.

These mobinad (nonstandard) above operations are reduced to set operations, using Set Analysis and Neutrosophic Analysis (both introduced by the author [24, page 11], which are generalizations of Interval Analysis), and they deal with sets that have indeterminacies.

4.19. Etymology of MoBiNad

MoBiNad comes from **mo***nad* + **bi***nad*, introduced now for the first time.

4.20. Definition of Nonstandard Complex MoBiNad Set

The **Nonstandard Complex MoBiNad Set**, introduced here for the first time, is defined as:

$$NC_{MB} =_N \{ \alpha + \beta i | \text{ where } i = \sqrt{-1}; \; \alpha, \beta \in NR_{MB} \}. \quad (4.39)$$

4.21. Definition of Nonstandard Neutrosophic Real MoBiNad Set

The **Nonstandard Neutrosophic Real MoBiNad Set**, introduced now for the first time, is defined as:

$$NNR_{MB} =_N \left\{ \begin{array}{l} \alpha + \beta I| \text{ where } I = \text{literal indeterminacy,} \\ \qquad\qquad I^2 = I; \; \alpha, \beta \in NR_{MB} \end{array} \right\}. \quad (4.40)$$

4.22. Definition of Nonstandard Neutrosophic Complex MoBiNad Set

The **Nonstandard Neutrosophic Complex MoBiNad Set**, introduced now for the first time, is defined as:

$$NNC_{MB} =_N \left\{ \begin{array}{l} \alpha + \beta I| \text{ where } I = \text{literal indeterminacy,} \\ \qquad\qquad I^2 = I; \; \alpha, \beta \in NC_{MB} \end{array} \right\}. \quad (4.41)$$

4.23. Properties of the Nonstandard Neutrosophic Real Mobinad Set

Since in nonstandard neutrosophic logic we use only the nonstandard neutrosophic real mobinad set, we study some properties of it.

Theorem 1

The nonstandard real mobinad set $(NR_{MB}, \leq_N)$, endowed with the nonstandard neutrosophic inequality is a lattice of first type [as partially ordered set (*poset*)].

Proof

The set NR_{MB} is partially ordered, because [except the two-element subsets of the form $\left\{ a, \overset{-+}{a} \right\}$, and $\left\{ a, \overset{-0+}{a} \right\}$, with $a \in \mathbb{R}$, beetwen which there is no order] all other elements are ordered:

If $a < b$, where $a, b \in \mathbb{R}$, then: $\overset{m_1}{a} <_N \overset{m_2}{b}$, for any monads or binads

$m_1, m_2 \in_N \{ ^-, ^{-0}, ^+, ^{0+}, ^{-+}, ^{-0+} \}$. $\qquad\qquad\qquad (4.42)$

If $a = b$, one has:

$\overset{-}{a} <_N a,$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.43)$

$a^- <_N a^+,$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.44)$

$a <_N a^+,$ $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (4.45)$

$$^-a \leq_N {}^-a^+, \tag{4.46}$$

$$^-a^+ \leq_N a^+, \tag{4.47}$$

and there is no neutrosophic ordering relationship between a and $^-a^+$,

nor between a and $\overset{-0+}{a}$ (that is why $\leq_N$ on NR_{MB} is a partial ordering

set). (4.48)

If $a > b$, then: $\overset{m_1}{a} >_N \overset{m_2}{b}$,

for any monads or binads m_1, m_2. (4.49)

Any two-element set $\{\alpha, \beta\} \subset_N NR_{MB}$ has a **neutrosophic nonstandard infimum** (**meet**, or **greatest lower bound**) that we denote by $\inf_N$, and a **neutrosophic nonstandard supremum** (**joint**, or **least upper bound**) that we denote by $\sup_N$, where both

$$\inf_N\{\alpha, \beta\} \text{ and } \sup_N\{\alpha, \beta\} \in NR_{MB}. \tag{4.50}$$

For the non-ordered elements a and $^-a^+$:

$$\inf_N\{a, {}^- a^+\} =_N {}^-a \in_N NR_{MB}, \tag{4.51}$$

$$\sup_N\{a, {}^- a^+\} =_N a^+ \in_N NR_{MB}. \tag{4.52}$$

And similarly for non-ordered elements a and $^-a^{0\,+}$:

$$\inf_N\{a, {}^- a^{0\,+}\} =_N {}^-a \in_N NR_{MB}, \tag{4.53}$$

$$\sup_N\{a, {}^- a^{0\,+}\} =_N a^+ \in_N NR_{MB}. \tag{4.54}$$

Dealing with monads and binads which neutrosophically are real subsets with indeterminate borders, and similarly $a = [a, a]$ can be treated as a subset, we may compute $\inf_N$ and $\sup_N$ of each of them.

$$\inf_N({}^-a) =_N {}^-a \text{ and } \sup_N({}^-a) =_N {}^-a; \tag{4.55}$$

$$\inf_N(a^+) =_N a^+ \text{ and } \sup_N(a^+) =_N a^+; \tag{4.56}$$

$$\inf_N({}^-a^+) =_N {}^-a \text{ and } \sup_N({}^-a^+) =_N a^+; \tag{4.57}$$

$$\inf_N({}^-a^{0\,+}) =_N {}^-a \text{ and } \sup_N({}^-a^{0\,+}) =_N a^+. \tag{4.58}$$

Also, $\inf_N(a) =_N a$ and $\sup_N(a) =_N a$. (4.59)

If $a < b$, then $\overset{m_1}{a} <_N \overset{m_2}{b}$, whence $\inf_N \left\{ \overset{m_1}{a}, \overset{m_2}{b} \right\} =_N \inf_N \left(\overset{m_1}{a} \right)$

and $\sup_N \left\{ \overset{m_1}{a}, \overset{m_2}{b} \right\} =_N \sup_N \left(\overset{m_2}{b} \right),$ \hfill (4.60)

which are computed as above.

Similarly, if $a > b$, with $\overset{m_1}{a} >_N \overset{m_2}{b}$. \hfill (4.61)

If $a = b$, then:

$\inf_N \left\{ \overset{m_1}{a}, \overset{m_2}{a} \right\} =_N$ the neutrosophically smallest ($<_N$) element among

$\inf_N \left\{ \overset{m_1}{a} \right\}$ and $\inf_N \left\{ \overset{m_2}{a} \right\}.$ \hfill (4.62)

While $\sup_N \left\{ \overset{m_1}{a}, \overset{m_2}{a} \right\} =_N$ the neutrosophically greatest ($>_N$) element among

$\sup_N \left\{ \overset{m_1}{a} \right\}$ and $\sup_N \left\{ \overset{m_2}{a} \right\}.$ \hfill (4.63)

Examples:

$\inf_N(^-a, a^+) =_N {}^-a$ and $\sup_N(^-a, a^+) =_N a^+;$ \hfill (4.64)

$\inf_N(^-a, {}^-a^+) =_N {}^-a$ and $\sup_N(^-a, {}^-a^+) =_N a^+;$ \hfill (4.65)

$\inf_N(^-a^+, a^+) =_N {}^-a$ and $\sup_N(^-a^+, a^+) =_N a^+.$ \hfill (4.66)

Therefore, $(NR_{MB}, \leq_N)$ is a nonstandard real mobinad lattice of first type (as partially ordered set).

Consequence

If we remove all pierced and unpierced binads from NR_{MB} and we denote the new set by

$$NR_M = \left\{ \begin{array}{c} \varepsilon, \omega, a, {}^- a, {}^- a^0, a^+, {}^0 a^+, \\ \text{where } \varepsilon \text{ are infinitesimals, } \omega \text{ are infinites, and } a \in \mathbb{R} \end{array} \right\} \text{ we}$$

obtain a totally neutrosophically ordered set.

Theorem 2

Any finite non-empty subset L of $(NR_{MB}, \leq_N)$ is also a sublattice of first type.

Proof

It is a consequence of any classical lattice of first order (as partially ordered set).

Theorem 3

$(NR_{MB}, \leq_N)$ is not bounded neither to the left nor to the right, since it does not have a **minimum** (bottom, or least element), nor a **maximum** (top, or greatest element).

Proof

Straightforward, since NR_{MB} includes the set of real number $R = (-\infty, +\infty)$ which is clearly unbounded to the left and right-hand sides.

Theorem 4

$(NR_{MB}, \inf_N, \sup_N)$, where $\inf_N$ and $\sup_N$ are two binary operations, dual to each other, defined before, is a **lattice of second type** (as an algebraic structure).

Proof

We have to show that the two laws $\inf_N$ and $\sup_N$ are commutative, associative, and verify the absorption laws.

Let $\alpha, \beta, \gamma \in NR_{MB}$ be two arbitrary elements.

Commutativity Laws

i) $\inf_N\{\alpha, \beta\} =_N \inf_N\{\beta, \alpha\}.$ (4.67)

ii) $\sup_N\{\alpha, \beta\} =_N \sup_N\{\beta, \alpha\}.$ (4.68)

Their proofs are straightforward.

Associativity Laws

$\inf_N\{\alpha, \inf_N\{\beta, \gamma\}\} =_N \inf_N\{\inf_N\{\alpha, \beta\}, \gamma\}.$ (4.69)

Proof

i) $\inf_N\{\alpha, \inf_N\{\beta, \gamma\}\} =_N \inf_N\{\alpha, \beta, \gamma\},$

(4.70)

and

$\inf_N\{\inf_N\{\alpha, \beta\}, \gamma\} =_N \inf_N\{\alpha, \beta, \gamma\},$ (4.71)

where we have extended the binary operation $\inf_N$ to a trinary operation $\inf_N$.

ii) $\sup_N\{\alpha, \sup_N\{\beta, \gamma\}\} =_N \sup_N\{\sup_N\{\alpha, \beta\}, \gamma\}$ (4.72)

Proof

$\sup_N\{\alpha, \sup_N\{\beta, \gamma\}\} =_N \sup_N\{\alpha, \beta, \gamma\},$ (4.73)

and

$\sup_N\{\sup_N\{\alpha, \beta\}, \gamma\} =_N \sup_N\{\alpha, \beta, \gamma\},$ (4.74)

where similarly we have extended the binary operation $\sup_N$ to a trinary operation $\sup_N$.

Absorption Laws (as peculiar axioms to the theory of lattice)

i) We need to prove that $\inf_N\{\alpha, \sup_N\{\alpha, \beta\}\} =_N \alpha.$ (4.75)

Let $\alpha \leq_N \beta$, then $\inf_N\{\alpha, \sup_N\{\alpha, \beta\}\} =_N \inf_N\{\alpha, \beta\} =_N \alpha.$ (4.76)

Let $\alpha >_N \beta$, then $\inf_N\{\alpha, \sup_N\{\alpha, \beta\}\} =_N \inf_N\{\alpha, \alpha\} =_N \alpha.$ (4.77)

ii) Now, we need to prove that $\sup_N\{\alpha, \inf_N\{\alpha, \beta\}\} =_N \alpha.$ (4.78)

Let $\alpha \leq_N \beta$, then $\sup_N\{\alpha, \inf_N\{\alpha, \beta\}\} =_N \sup_N\{\alpha, \alpha\} =_N \alpha.$ (4.79)

Let $\alpha >_N \beta$, then $\sup_N\{\alpha, \inf_N\{\alpha, \beta\}\} =_N \sup_N\{\alpha, \beta\} =_N \alpha.$ (4.80)

Consequence

The binary operations $\inf_N$ and $\sup_N$ also satisfy the idempotent laws:

$$\inf_N\{\alpha, \alpha\} =_N \alpha, \qquad (4.81)$$

$$\sup_N\{\alpha, \alpha\} =_N \alpha. \qquad (4.82)$$

Proof

The axioms of idempotency follow directly from the axioms of absorption proved above. Thus, we have proved that *(NR$_{MB}$, inf$_N$, sup$_N$)* is a **lattice of second type** (as algebraic structure).

4.24. Definition of General Nonstandard Real MoBiNad Interval

Let $a, b \in \mathbb{R}$, with $-\infty < a \leq b < \infty$,   (4.83)

$$]^-a, b^+[_{MB} = \{x \in NR_{MB}, {}^-a \leq_N x \leq_N b^+\}. \qquad (4.84)$$

As particular edge cases:

$]^-a, a^+[_{MB} =_N \{{}^-a, a, {}^-a^+, a^+\}$, a discrete nonstandard real set of cardinality 4.   (4.85)

$$]^-a, {}^-a[_{MB} =_N \{{}^-a\}; \qquad (4.86)$$

$$]a^+, a^+[_{MB} =_N \{a^+\}; \qquad (4.87)$$

$$]a, a^+[_{MB} =_N \{a, a^+\}; \qquad (4.88)$$

$$]^-a, a[_{MB} =_N \{{}^-a, a\}; \qquad (4.89)$$

$]^-a, {}^-a^+[_{MB} =_N \{{}^-a, {}^-a^+, a^+\}$, where $a \notin]^-a, {}^-a^+[_{MB}$ since $a \nleq_N {}^-a^+$ (there is no relation of order between a and ${}^-a^+$);   (4.90)

$$]^-a^+, a^+[_{MB} =_N \{{}^-a^+, a^+\}. \qquad (4.91)$$

Theorem 5

$(\,]^-a, b^+[, \leq_N)$ is a nonstandard real mobinad sublattice of first type (poset). (4.92)

Proof

Straightforward since $\,]^-a, b^+[$ is a sublattice of the lattice of first type NR_{MB}.

Theorem 6

$(\,]^-a, b^+[, \inf_N, \sup_N, {}^-a, b^+)$ is a nonstandard *bounded* real mobinad sublattice of second type (as algebraic structure).

(4.93)

Proofs

$]^-a, b^+[_{MB}$ as a nonstandard subset of NR_{MB} is also a poset, and for any two-element subset $\{\alpha, \beta\} \subset_N \,]^-0, 1^+[_{MB}$ (4.94)

one obviously has the triple neutrosophic nonstandard inequality:

$${}^-a \leq_N \inf_N\{\alpha, \beta\} \leq_N \sup_N\{\alpha, \beta\} \leq_N b^+, \qquad (4.95)$$

whence $(\,]^-a, b^+[_{MB} \leq_N)$ is a nonstandard real mobinad sublattice of first type (poset), or sublattice of NR_{MB}.

Further on, $]^-a, b^+[$, endowed with two binary operations $\inf_N$ and $\sup_N$, is also a sublattice of the lattice NR_{MB}, since the lattice axioms (Commutative Laws, Associative Laws, Absortion Laws, and Idempotent Laws) are clearly verified on $]^-a, b^+[$.

The nonstandard neutrosophic modinad **Identity Join Element** (Bottom) is ${}^-a$, and the nonstandard neutrosophic modinad **Identity Meet Element** (Top) is b^+,

or $\inf_N]^-a, b^+[=_N {}^-a$ and $\sup_N]^-a, b^+[=_N b^+$. (4.96)

The sublattice **Identity Laws** are verified below.

Let $\alpha \in_N]^-a, b^+[$, whence ${}^-a \leq_N \alpha \leq_N b^+$. (4.97)

Then:

$$\inf_N\{\alpha, b^+\} =_N \alpha, \text{ and } \sup_N\{\alpha, {}^-a\} =_N \alpha. \qquad (4.98)$$

4.25. Definition of Nonstandard Real MoBiNad Unit Interval

$$]^-0, 1^+[_{MB} =_N \{x \in NR_{MB}, {}^-0 \leq_N x \leq_N 1^+\} \qquad (4.99)$$

$$=_N \left\{ \varepsilon, a, \overset{-}{a}, \overset{-0}{a}, \overset{+}{a}, \overset{0+}{a}, \overset{-+}{a}, \overset{-0+}{a} \;\middle|\; \begin{array}{l} \text{where } \varepsilon \text{ are infinitesimals,} \\ \varepsilon \in \mathbb{R}^*, \text{ with } \varepsilon > 0, \text{ and } a \in [0, 1] \end{array} \right\}. \qquad (4.100)$$

This is an extension of the previous definition (1998) of nonstandard unit interval

$$]^-0, 1^+[=_N ({}^-0) \cup [0, 1] \cup (1^+) \qquad (4.101)$$

Associated to the first published definitions of neutrosophic set, logic, and probability was used.

One has: $]^-0, 1^+[\subset_N]^-0, 1^+[_{MB}, \qquad (4.102)$

where the index $_{MB}$ means: all monads and binads included in $]^-0, 1^+[$, for example:

$$({}^-0.2), ({}^-0.3^0), (0.5^+), ({}^-0.7^+), ({}^-0.8^{0+}) \text{ etc.} \qquad (4.103)$$

{or, using the top diacritics notation, respectively:

$$\overset{-}{0.2}, \overset{-0}{0.3}, \overset{+}{0.5}, \overset{-+}{0.7}, \overset{-0+}{0.8} \text{ etc.}\}. \qquad (4.104)$$

Theorem 7

The Nonstandard Real MoBiNad Unit Interval $]^-0, 1^+[_{MB}$ is a partially ordered set (poset) with respect to $\leq_N$, and any of its two elements have an $\inf_N$ and $\sup_N$ whence $]^-0, 1^+[_{MB}$ is a nonstandard neutrosophic lattice of first type (as poset).

Proof:

Straightforward.

Theorem 8

The Nonstandard Real MoBiNad Unit Interval $]^-0, 1^+[_{MB}$, endowed with two binary operations $\inf_N$ and $\sup_N$, is also a nonstandard neutrosophic lattice of second type (as an algebraic structure).

Proofs

Replace $a = 0$ and $b = 1$ into the general nonstandard real mobinad interval $]^-a, b^+[$.

4.26. Definition of Extended General Neutrosophic Logic

We extend and present in a clearer way our 1995 definition (published in 1998) of neutrosophic logic.

Let $\mathcal{U}$ be a universe of discourse of propositions, and $P \in \mathcal{U}$ a generic proposition.

A **General Neutrosophic Logic** is a multivalued logic in which each proposition P has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsehood (F), where T, I, F are standard or nonstandard real mobinad subsets of the nonstandard real mobinat unit interval $]^-0, 1^+[_{MB}$,

with $T, I, F \subseteq_N]^-0, 1^+[_{MB}$, $\hspace{2cm}$ (4.105)

where

$$^-0 \leq_N \inf_N T + \inf_N I + \inf_N F \leq_N \sup_N T + \sup_N I + \sup_N F \leq 3^+.$$
$$(4.106)$$

4.27. Definition of Standard Neutrosophic Logic

If in the above definition of general neutrosophic logic all neutrosophic components, T, I, F, are standard real subsets, included in or equal to the standard real unit interval, $T, I, F \subseteq [0, 1]$,

where $0 \leq \inf T + \inf I + \inf F \leq \sup T + \sup I + \sup F \leq 3$, (107)

we have a standard neutrosophic logic.

4.28. Definition of Extended Nonstandard Neutrosophic Logic

If in the above definition of general neutrosophic logic at least one of the neutrosophic components T, I, F is a nonstandard real mobinad subset, neutrosophically included in or equal to the nonstandard real mobinad unit interval $]^-0, 1^+[_{MB}$,

where

$$^-0 \leq_N \inf_N T + \inf_N I + \inf_N F \leq_N \sup_N T + \sup_N I + \sup_N F \leq 3^+,$$
(4.108)

we have an extended nonstandard neutrosophic logic.

Theorem 9

If M is a standard real set, $M \subset \mathbb{R}$,

then $\inf_N(M) = \inf(M)$ and $\sup_N(M) = \sup(M)$. (4.109)

Proof

The neutrosophic infimum and supremum coincide with the classical infimum and supremum since there is no indeterminacy on the set M, meaning M contains no nonstandard numbers.

4.29. Definition of Extended General Neutrosophic Set

We extend and present in a clearer way our 1995 definition of neutrosophic set.

Let $\mathcal{U}$ be a universe of discourse of elements, and $S \in \mathcal{U}$ a subset.

A **Neutrosophic Set** is a set such that each element x from S has a degree of membership (T), a degree of indeterminacy (I), and a degree of nonmembership (F), where T, I, F are standard or nonstandard real mobinad subsets, neutrosophically included in or equal to the nonstandard real mobinat unit interval $]^-0, 1^+[_{MB}$,

with $T, I, F \subseteq_N]^-0, 1^+[_{MB}$, (4.110)

where

$$^-0 \leq_N \inf_N T + \inf_N I + \inf_N F \leq_N \sup_N T + \sup_N I + \sup_N F \leq 3^+.$$
(4.111)

4.30. Definition of Standard Neutrosophic Set

If in the above general definition of neutrosophic set all neutrosophic components, T, I, F, are standard real subsets included in or equal to the classical real unit interval,

$$T, I, F \subseteq [0, 1],$$
where

$$0 \leq \inf T + \inf I + \inf F \leq \sup T + \sup I + \sup F \leq 3, \quad (4.112)$$
we have a standard neutrosophic set.

4.31. Definition of Extended Nonstandard Neutrosophic Set

If in the above general definition of neutrosophic set at least one of the neutrosophic components T, I, F is a nonstandard real mobinad subsets, neutrosophically included in or equal to $]^-0, 1^+[_{MB}$, where

$$^-0 \leq_N \inf_N T + \inf_N I + \inf_N F \leq_N \sup_N T + \sup_N I + \sup_N F \leq 3^+,$$
(4.113)

we have a nonstandard neutrosophic set.

4.32. Definition of Extended General Neutrosophic Probability

We extend and present in a clearer way our 1995 definition of neutrosophic probability.

Let $\mathcal{U}$ be a universe of discourse of events, and $E \in \mathcal{U}$ be an event.

A **Neutrosophic Probability** is a multivalued probability such that each event E has a chance of occuring (T), an indeterminate (unclear) chance of occuring or not occuring (I), and a chance of not occuring (F), where T , I , F are standard or nonstandard real mobinad subsets, neutrosophically included in or equal to the nonstandard real mobinat unit interval $]^-0, 1^+[_{MB}, T, I, F \subseteq_N]^-0, 1^+[_{MB},$

where

$$^-0 \leq_N \inf{}_N T + \inf{}_N I + \inf{}_N F \leq_N \sup{}_N T + \sup{}_N I + \sup{}_N F \leq 3^+.$$
(4.114)

4.33. Definition of Standard Neutrosophic Probability

If in the above general definition of neutrosophic probability all neutrosophic components, T, I, F, are standard real subsets, included in or equal to the standard unit interval,

$T, I, F \subseteq [0, 1],$

where

$$0 \leq \inf T + \inf I + \inf F \leq \sup T + \sup I + \sup F \leq 3, \qquad (4.115)$$

we have a standard neutrosophic probability.

4.34. Definition of Extended Nonstandard Neutrosophic Probability

If in the above general definition of neutrosophic probability at least one of the neutrosophic components T, I, F is a nonstandard real mobinad subsets, neutrosophically included in or equal to $]^-0, 1^+[_{MB}$,

where

$$^-0 \leq_N \inf{}_N T + \inf{}_N I + \inf{}_N F \leq_N \sup{}_N T + \sup{}_N I + \sup{}_N F \leq 3^+,$$
(4.116)

we have a nonstandard neutrosophic probability.

4.35. Classical Operations with Real Sets

Let $A, B \subseteq \mathbb{R}$ be two real subsets. Let $\circledast$ and $*$ denote any of the <u>real subset classical operations</u> and <u>real number classical operations</u> respectively: addition $(+)$, subtraction $(-)$, multiplication $(\times)$, division $(\div)$, and power $(\wedge)$.

Then, $A \circledast B = \{a * b, \text{where } a \in A \text{ and } b \in B\}.$ \qquad (4.117)

Thus:

$$A \oplus B = \{a + b \mid a \in A, b \in B\}$$
$$A \ominus B = \{a - b \mid a \in A, b \in B\}$$
$$A \otimes B = \{a \times b \mid a \in A, b \in B\}$$
$$A \oslash B = \{a \div b \mid a \in A, b \in B - \{0\}\}$$
$$A^B = \{a \wedge b \mid a \in A, a > 0; b \in B\}$$

(4.118-4.122)

For the division ($\div$), of course, we consider $b \neq 0$. While for the power ($\wedge$), we consider $a > 0$.

4.36. Operations on the Nonstandard Real MoBiNad Set (NR_{MB})

For all nonstandard (addition, subtraction, multiplication, division, and power) operations,

for $\alpha, \beta \in_N NR_{MB}$, $\alpha *_N \beta =_N \mu_N(\alpha) \circledast \mu_N(\beta)$ (4.123)

where $*_N$ is any neutrosophic arithmetic operations with neutrosophic numbers ($+_N$, $-_N$, $\times_N, \div_N, \wedge_N$), while the corresponding $\circledast$ is an arithmetic operation with real subsets.

So, we approximate the nonstandard operations by standard operations of real subsets.

We sink the nonstandard neutrosophic real mobinad operations into the standard real subset operations, then we resurface the last ones back to the nonstandard neutrosophic real mobinad set.

Let ε_1 and ε_2 be two non-null positive infinitesimals. We present below some particular cases, all others should be deduced analogously.

Nonstandard Addition

First Method

$(^-a) + (^-b) =_N (a - \varepsilon_1, a) + (b - \varepsilon_2, b) =_N (a + b - \varepsilon_1 - \varepsilon_2, a + b) =_N (a + b - \varepsilon, a + b) =_N {}^-(a + b),$ (4.124)

where we denoted $\varepsilon_1 + \varepsilon_2 = \varepsilon$

(the addition of two infinitesimals is also an infinitesimal)

Second Method

$$(^-a) + (^-b) =_N (a - \varepsilon_1) + (b - \varepsilon_2)$$
$$=_N (a + b - \varepsilon_1 - \varepsilon_2) =_N {}^-(a + b). \tag{4.125}$$

Adding two left monads, one also gets a left monad.

Nonstandard Subtraction

First Method

$$(^-a) - (^-b) =_N (a - \varepsilon_1, a) - (b - \varepsilon_2, b) =_N (a - \varepsilon_1 - b, a -$$
$$b + \varepsilon_2) =_N (a - b - \varepsilon_1, a - b + \varepsilon_2) =_N \left(\begin{smallmatrix} - & & + \\ & 0 & \\ & a - b & \end{smallmatrix} \right) \tag{4.126}$$

Second Method

$$(^-a) - (^-b) =_N (a - \varepsilon_1) - (b - \varepsilon_2) =_N a - b - \varepsilon_1 + \varepsilon_2, \tag{4.127}$$

since ε_1 and ε_2 may be any positive infinitesimals,

$$=_N \begin{cases} {}^-(a - b), \text{when } \varepsilon_1 > \varepsilon_2; \\ \left(\begin{smallmatrix} 0 \\ a - b \end{smallmatrix} \right), \text{when } \varepsilon_1 = \varepsilon_2 =_N \left(\begin{smallmatrix} 0 \\ a - b \end{smallmatrix} \right) =_N a - b; \\ (a - b)^+, \text{when } \varepsilon_1 < \varepsilon_2. \end{cases}$$
$$\text{(4.128-4.130)}$$

Subtracting two left monads, one obtains an unpierced binad (that's why the unpierced binad had to be introduced).

Nonstandard Division

Let a, b > 0.

$$(^-a) \div (^-b) =_N (a - \varepsilon_1, a) \div (b - \varepsilon_2, b) =_N \left(\frac{a - \varepsilon_1}{b}, \frac{a}{b - \varepsilon_2} \right). \tag{4.131}$$

Since $\varepsilon_1 > 0$ and $\varepsilon_2 > 0, \frac{a - \varepsilon_1}{b} < \frac{a}{b}$ and $\frac{a}{b - \varepsilon_2} > \frac{a}{b},$ \hfill (4.132)

while between $\frac{a-\varepsilon_1}{b}$ and $\frac{a}{b-\varepsilon_2}$ there is a continuum whence there are some infinitesimals ε_1^0 and ε_2^0 such that $\frac{a-\varepsilon_1^0}{b-\varepsilon_2^0} = \frac{a}{b}$, or $ab - b\varepsilon_1^0 = ab - a\varepsilon_2^0$, and for a given ε_1^0.

there exist an $\varepsilon_2^0 = \varepsilon_1^0 \cdot \frac{b}{a}$. $\qquad\qquad$ (4.133)

Whence $\frac{(^-a)}{(^-b)} =_N \left(\begin{array}{ccc} - & 0 & + \\ & \frac{a}{b} & \end{array} \right)$. $\qquad\qquad$ (4.134)

For a or/and b negative numbers, it's similar but it's needed to compute the inf_N and sup_N of the products of intervals.

Dividing two left monads, one obtains an unpierced binad.

Nonstandard Multiplication

Let $a, b \geq 0$.

$(^-a^0) \times (^-b^{0\,+}) =_N (a - \varepsilon_1, a] \times (b - \varepsilon_2, b + \varepsilon_2) =_N ((a - \varepsilon_1) \cdot (b - \varepsilon_2), a \cdot (b + \varepsilon_2)) =_N (^-ab^{0\,+})$ $\qquad$ (4.135)

since $(a - \varepsilon_1) \cdot (b - \varepsilon_2) < a \cdot b$ and $a \cdot (b + \varepsilon_2) > a \cdot b$. $\quad$ (4.136)

For a or/and b negative numbers, it's similar but it's needed to compute the inf_N and sup_N of the products of intervals.

Multiplying a positive left monad closed to the right, with a positive unpierced binad, one obtains an unpierced binad.

Nonstandard Power

Let $a, b > 1$.

$(^0a^+)^{(^-b^0)} =_N [a, a + \varepsilon_1)^{(b - \varepsilon_2, b]} =_N (a^{b-\varepsilon_2}, (a + \varepsilon_1)^b) =_N \left(\begin{array}{ccc} - & 0 & + \\ & a^b & \end{array} \right)$ $\qquad$ (4.137)

since $a^{b-\varepsilon_1} < a^b$ and $(a + \varepsilon_1)^b > a^b$. $\qquad\qquad$ (4.138)

Raising a right monad closed to the left to a power equal to a left monad closed to the right, for both monads above 1, the result is an unpierced binad.

Consequence

In general, when doing arithmetic operations on nonstandard real monads and binads, the result may be a different type of monad or binad.

That's why is was imperious to extend the monads to closed monads, and the pierced binad to unpierced binad, in order to have the whole nonstandard neutrosophic real mobinad set closed under arithmetic operations.

4.37. Conditions of Neutrosophic Nonstandard Inequalities

Let NR_{MB} be the Nonstandard Real MoBiNad. Let's endow $(NR_{MB}, <_N)$ with a neutrosophic inequality.

Let $\alpha, \beta \in NR_{MB}$, where α, β may be real numbers, monads, or binads.

And let

$$\left(\begin{smallmatrix}-\\a\end{smallmatrix}\right), \left(\begin{smallmatrix}-0\\a\end{smallmatrix}\right), \left(\begin{smallmatrix}+\\a\end{smallmatrix}\right), \left(\begin{smallmatrix}0+\\a\end{smallmatrix}\right), \left(\begin{smallmatrix}-+\\a\end{smallmatrix}\right), \left(\begin{smallmatrix}-0+\\a\end{smallmatrix}\right) \in NR_{MB},$$

and $$\left(\begin{smallmatrix}-\\b\end{smallmatrix}\right), \left(\begin{smallmatrix}-0\\b\end{smallmatrix}\right), \left(\begin{smallmatrix}+\\b\end{smallmatrix}\right), \left(\begin{smallmatrix}0+\\b\end{smallmatrix}\right), \left(\begin{smallmatrix}-+\\b\end{smallmatrix}\right), \left(\begin{smallmatrix}-0+\\b\end{smallmatrix}\right) \in NR_{MB},$$ 　　　　(4.139)

be the left monads, left monads closed to the right, right monads, right monads closed to the left, and binads, and binads nor prierced of the elements (standard real numbers) a and b respectively. Since all monads and binads are real subsets, we may treat the single real numbers

$a = [a, a]$ and $b = [b, b]$ as real subsets too. 　　　　(4.140)

NR_{MB} is a set of subsets, and thus we deal with neutrosophic inequalities between subsets.

i) If the subset α has many of its elements above all elements of the subset β, then $\alpha >_N \beta$ (partially).

ii) If the subset α has many of its elements below all elements of the subset β, then $\alpha <_N \beta$ (partially).

iii) If the subset α has many of its elements equal with elements of the subset β, then $\alpha =_N \beta$ (partially).

If the subset α verifies *i)* and *iii)* with respect to subset β, then $\alpha \geq_N \beta$.

If the subset α verifies *ii)* and *iii)* with respect to subset β, then $\alpha \leq_N \beta$.

If the subset α verifies *i)* and *ii)* with respect to subset β, then there is no neutrosophic order (inequality) between α and β.

{ For example, between a and $({}^-a^+)$ there is no neutrosophic order, similarly between a and ${}^{-0+}a$. }

Similarly, if the subset α verifies *i), ii)* and *iii)* with respect to subset β, then there is no neutrosophic order (inequality) between α and β.

4.38. Open Neutrosophic Research

The quantity or measure of "many of its elements" of the above *i), ii),* or *iii)* conditions depends on each *neutrosophic application* and on its *neutrosophic experts*.

An approach would be to employ the *Neutrosophic Measure* [21, 22], that handles indeterminacy, which may be adjusted and used in these cases.

In general, we do not try in purpose to validate or invalidate an existing scientific result, but to investigate how an existing scientific result behaves in a new environment (that may contain indeterminacy), or in a new application, or in a new interpretation.

4.39. Nonstandard Neutrosophic Inequalities

For the *neutrosophic nonstandard inequalities*, we propose based on the previous six neutrosophic equalities, the following:

$$(-a) <_N a <_N (a^+) \qquad\qquad (4.141)$$

since the standard real interval *(a - ε, a)* is below *a,* and *a* is below the standard real interval

(a, a + ε) by using the approximation provided by the nonstandard neutrosophic function *μ*,

$$\text{or because } \forall x \in R_{+}^{*}, a - x < a < a + x, \qquad (4.142)$$

where *x* is of course a (nonzero) positive infinitesimal (the above double neutrosophic inequality actually becomes a double classical standard real inequality for each fixed positive infinitesimal).

The converse double neutrosophic inequality is also neutrosophically true:

$$(a^{+}) >_{N} a >_{N} (^{-}a) \qquad (4.143)$$

Another nonstandard neutrosophic double inequality:

$$(^{-}a) \leq_{N} (^{-}a^{+}) \leq_{N} (a^{+}) \qquad (4.144)$$

This double neutrosophic inequality may be justified since $(^{-}a^{+}) = (^{-}a) \cup (a^{+})$ and, geometrically, on the Real Number Line, the number *a* is in between the subsets $^{-}a = (a\text{-}ε, a)$ and

$a^{+} = (a, a+ε)$, so:

$$(^{-}a) \leq_{N} (^{-}a) \cup (a^{+}) \leq_{N} (a^{+}) \qquad (4.145)$$

whence the left side of the inequality's middle term coincides with the inequality first term, while the right side of the inequality middle term coincides with the third inequality term.

Conversely, it is neutrosophically true as well:

$$(a^{+}) \geq_{N} (^{-}a) \cup (a^{+}) \geq_{N} (^{-}a) \qquad (4.146)$$

Also, $^{-}a \leq_{N} {}^{-0}a \leq_{N} {}^{0+}a \leq_{N} {}^{+}a$ and $^{-}a \leq_{N} {}^{-+}a \leq_{N} {}^{-0+}a \leq_{N} {}^{+}a$. (4.147)

Conversely, they are also neutrosophically true:

$$^{+}a \geq_{N} {}^{0+}a \geq_{N} {}^{-0}a \geq_{N} {}^{-}a$$

and $^{+}a \geq_{N} {}^{-0+}a \geq_{N} {}^{-+}a \geq_{N} {}^{-}a$ respectively. (4.148)

If $a > b$, which is a (standard) classical real inequality, then we have the following neutrosophic nonstandard inequalities:

$$a >_N (\bar{}b), \quad a >_N (b^+), \quad a >_N (\bar{}b^+), \quad a >_N \overset{-0}{b}, a >_N \overset{0+}{b}, a >_N \overset{-0+}{b};$$

(4.149)

$$(\bar{}a) >_N b, \quad (\bar{}a) >_N (\bar{}b), \quad (\bar{}a) >_N (b^+), \quad (\bar{}a) >_N (\bar{}b^+),$$
$$\overset{-}{a} >_N \overset{-0}{b}, \overset{-}{a} >_N \overset{0+}{b}, \overset{-}{a} >_N \overset{-0+}{b};$$ (4.150)

$$(a^+) >_N b, \quad (a^+) >_N (\bar{}b), \quad (a^+) >_N (b^+), \quad (a^+) >_N (\bar{}b^+),$$ (4.151)

$$(\bar{}a^+) >_N b, \quad (\bar{}a^+) >_N (\bar{}b), \quad (\bar{}a^+) >_N (b^+), \quad (\bar{}a^+) >_N (\bar{}b^+), \text{ etc.}$$ (4.152)

No Ordering Relationships

For any standard real number a, there is no relationship of order between the elements a and $(\bar{}a^+)$,

nor between the elements a and $\left(\overset{-0+}{a} \right)$. (4.153)

Therefore, NR_{MB} is a neutrosophically partially order set.

If one removes all binads from NR_{MB}, then $(NR_{MB}, \leq_N)$ is neutrosophically totally ordered. (4.154)

Theorem 10.

Using the nonstandard general notation one has:

If $a > b$, which is a (standard) classical real inequality, then

$$\overset{m_1}{a} >_N \overset{m_2}{b} \text{ for any } m_1, m_2 \in \{ \ , \ \bar{}, \ ^{-0}, \ ^+, \ ^{+0}, \ ^{-+}, \ ^{-0+}\}.$$ (4.155)

And conversely,

If $a < b$, which is a (standard) classical real inequality, then

$$\overset{m_1}{a} <_N \overset{m_2}{b} \text{ for any } m_1, m_2 \in \{ \ , \ \bar{}, \ ^{-0}, \ ^+, \ ^{+0}, \ ^{-+}, \ ^{-0+}\}.$$ (4.156)

4.40. Nonstandard Neutrosophic Equalities

Let a, b be standard real numbers; if $a = b$ that is a (classical) standard equality, then:

$$({}^-a) =_N ({}^-b), \ (a^+) =_N (b^+), \ ({}^-a^+) =_N ({}^-b^+), \tag{4.157}$$

$$\left(\overset{-0}{a} \right) =_N \left(\overset{-0}{b} \right), \left(\overset{0+}{a} \right) =_N \left(\overset{0+}{b} \right), \left(\overset{-0+}{a} \right) =_N \left(\overset{-0+}{b} \right). \tag{4.158}$$

4.41. Nonstandard Neutrosophic Belongingness

On the nonstandard real set NR_{MB}, we say that

$$\overset{m}{c} \in_N \ ]\overset{m_1}{a}, \overset{m_2}{b}[\quad \text{iff} \quad \overset{m_1}{a} \leq_N \overset{m}{c} \leq_N \overset{m_2}{b}, \tag{4.159}$$

where m_1, m_2, $m \in \{ \ , \ {}^-, \ {}^{-0}, \ {}^+, \ {}^{+0}, \ {}^{-+}, \ {}^{-0+} \}$. $\tag{4.160}$

{ We use the previous nonstandard neutrosophic inequalities. }

4.42. Nonstandard Hesitant Sets

Are sets of the form: $A = \{a_1, a_2, ..., a_n\}$, $2 \leq n < \infty$, $A \subset_N NR_{MB}$,

where at least one element $a_{i_0}, 1 \leq i_0 \leq n,$ is either an infinitesimal, or a monad or a binad (of any type);

while other elements may be standard real numbers, infinitesimals, or also monads or binads (of any type). $\tag{4.161}$

If the neutrosophic components T, I, F are nonstandard hesitant sets, then one has a

Nonstandard Hesitant Neutrosophic Logic / Set / Probability.

4.43. Nonstandard Neutrosophic Strict Interval Inclusion

On the nonstandard real set NR_{MB},

$$]\overset{m_1}{a}, \overset{m_2}{b}[\subset_N \ ]\overset{m_3}{c}, \overset{m_4}{d}[\ \text{iff} \tag{4.162}$$

$$\overset{m_3}{c} \leq_N \overset{m_1}{a} <_N \overset{m_2}{b} <_N \overset{m_4}{d}$$

or $\overset{m_3}{c} <_N \overset{m_1}{a} <_N \overset{m_2}{b} \leq_N \overset{m_4}{d}$ or $\overset{m_3}{c} <_N \overset{m_1}{a} <_N \overset{m_2}{b} <_N \overset{m_4}{d}$. (4.163)

4.44. Nonstandard Neutrosophic (Non-Strict) Interval Inclusion

On the nonstandard real set NR_{MB},

$$]\overset{m_1}{a}, \overset{m_2}{b}[\subseteq_N]\overset{m_3}{c}, \overset{m_4}{d}[\text{ iff} \tag{4.164}$$

$$\overset{m_3}{c} \leq_N \overset{m_1}{a} <_N \overset{m_2}{b} \leq_N \overset{m_4}{d} \tag{4.165}$$

4.45. Nonstandard Neutrosophic Strict Set Inclusion

The nonstandard set A is neutrosophically strictly included in the nonstandard set B, $A \subset_N B$, if:

$$\forall x \in_N A, x \in_N B \text{ , and } \exists y \in_N B : y \notin_N A. \tag{4.166}$$

4.46. Nonstandard Neutrosophic (Non-Strict) Set Inclusion

The nonstandard set A is neutrosophically not-strictly included in the nonstandard set B,

$$A \subseteq_N B \text{ , iff:} \tag{4.167}$$

$$\forall x \in_N A, x \in_N B . \tag{4.168}$$

4.47. Nonstandard Neutrosophic Set Equality

The nonstandard sets A and B are neutrosophically equal, $A =_N B$, iff:

(4.169)

$$A \subseteq_N B \text{ and } B \subseteq_N A . \tag{4.170}$$

4.48. The Fuzzy, Neutrosophic, and Plithogenic Logical Connectives ∧, ∨, →

All fuzzy, intuitionistic fuzzy, and neutrosophic logic operators are *inferential approximations*, not written in stone. They are improved from application to application.

Let's denote:

$\wedge_F, \wedge_N, \wedge_P$ representing respectively the fuzzy conjunction, neutrosophic conjunction, and plithogenic conjunction;   (4.171)

similarly

$\vee_F, \vee_N, \vee_P$ representing respectively the fuzzy disjunction, neutrosophic disjunction, and plithogenic disjunction,   (4.172)

and

$\rightarrow_F, \rightarrow_N, \rightarrow_P$ representing respectively the fuzzy implication, neutrosophic implication, and plithogenic implication.   (4.173)

I agree that my beginning neutrosophic operators (when I applied the same *fuzzy t-norm*, or the same *fuzzy t-conorm*, to all neutrosophic components T, I, F) were less accurate than others developed later by the neutrosophic community researchers. This was pointed out since 2002 by Ashbacher [9] and confirmed in 2008 by Rivieccio [10]. They observed that if on T_1 and T_2 one applies a fuzzy t-norm, on their opposites F_1 and F_2 one needs to apply the *fuzzy t-conorm* (the opposite of *fuzzy t-norm*), and reciprocally.

About inferring I_1 and I_2, some researchers combined them in the same directions as T_1 and T_2.

Then:

$(T_1, I_1, F_1) \wedge_N (T_2, I_2, F_2)$

$= (T_1 \wedge_F T_2, I_1 \wedge_F I_2, F_1 \vee_F F_2),$ (4.174)

$(T_1, I_1, F_1) \vee_N (T_2, I_2, F_2) = (T_1 \vee_F T_2, I_1 \vee_F I_2, F_1 \wedge_F F_2),$ (4.175)

$(T_1, I_1, F_1) \rightarrow_N (T_2, I_2, F_2) = (F_1, I_1, T_1) \vee_N (T_2, I_2, F_2) =$

$$= (F_1 \vee_F T_2, I_1 \vee_F I_2, T_1 \wedge_F F_2); \tag{4.176}$$

others combined I_1 and I_2 in the same direction as F_1 and F_2 (since both I and F are negatively qualitative neutrosophic components, while F is qualitatively positive neutrosophic component), the most used one:

$$(T_1, I_1, F_1) \wedge_N (T_2, I_2, F_2) = (T_1 \wedge_F T_2, I_1 \vee_F I_2, F_1 \vee_F F_2), \tag{4.177}$$

$$(T_1, I_1, F_1) \vee_N (T_2, I_2, F_2) = (T_1 \vee_F T_2, I_1 \wedge_F I_2, F_1 \wedge_F F_2), \tag{4.178}$$

$$(T_1, I_1, F_1) \rightarrow_N (T_2, I_2, F_2) = (F_1, I_1, T_1) \vee_N (T_2, I_2, F_2) =$$
$$= (F_1 \vee_F T_2, I_1 \wedge_F I_2, T_1 \wedge_F F_2). \tag{4.179}$$

Even more, recently, in an extension of neutrosophic set to *plithogenic set* [11] (which is a set whose each element is characterized by many attribute values), the *degrees of contradiction c(,)* between the neutrosophic components *T, I, F* have been defined (in order to facilitate the design of the aggregation operators), as follows:

*c(T, F) = 1 (*or *100%,* because they are totally opposite),

*c(T, I) = c(F, I) = 0.5 (*or *50%,* because they are only half opposite),
$$\tag{4.180}$$

then:

$$(T_1, I_1, F_1) \wedge_P (T_2, I_2, F_2) =$$
$$= (T_1 \wedge_F T_2, 0.5(I_1 \wedge_F I_2) + 0.5(I_1 \vee_F I_2), F_1 \vee_F F_2), \tag{4.181}$$

$$(T_1, I_1, F_1) \vee_P (T_2, I_2, F_2) =$$
$$= (T_1 \vee_F T_2, 0.5(I_1 \vee_F I_2) + 0.5(I_1 \wedge_F I_2), F_1 \wedge_F F_2). \tag{4.182}$$

$$(T_1, I_1, F_1) \rightarrow_N (T_2, I_2, F_2) = (F_1, I_1, T_1) \vee_N (T_2, I_2, F_2)$$
$$= (F_1 \vee_F T_2, 0.5(I_1 \vee_F I_2) +$$
$$+ 0.5(I_1 \wedge_F I_2), T_1 \wedge_F F_2). \tag{4.183}$$

4.49. Fuzzy t-norms and Fuzzy t-conorms

The most used $\wedge_F$ (Fuzzy t-norms), and $\vee_F$ (Fuzzy t-conorms) are:

Let $a, b \in [0, 1]$. $\tag{4.184}$

Fuzzy t-norms (fuzzy conjunctions, or fuzzy intersections):

$$a \wedge_F b = \min\{a, b\};$$ (4.185)

$$a \wedge_F b = ab;$$ (4.186)

$$a \wedge_F b = \max\{a + b - 1, 0\}.$$ (4.187)

Fuzzy t-conorms (fuzzy disjunctions, or fuzzy unions):

$$a \vee_F b = \max\{a, b\};$$ (4.188)

$$a \vee_F b = a + b - ab;$$ (4.189)

$$a \vee_F b = \min\{a + b, 1\}.$$ (4.190)

4.50. Nonstandard Neutrosophic Operators

Nonstandard Neutrosophic Conjunctions

$(T_1, I_1, F_1) \wedge_N (T_2, I_2, F_2) = (T_1 \wedge_F T_2, I_1 \vee_F I_2, F_1 \vee_F F_2) =$
$(\inf_N(T_1, T_2), \sup_N(I_1, I_2), \sup_N(F_1, F_2))$ (4.191)

$(T_1, I_1, F_1) \wedge_N (T_2, I_2, F_2) = (T_1 \wedge_F T_2, I_1 \vee_F I_2, F_1 \vee_F F_2) =$
$(T_1 \times_N T_2, I_1 +_N I_2 -_N I_1 \times_N I_2, F_1 +_N F_2 -_N F_1 \times_N F_2)$ (4.192)

Nonstandard Neutrosophic Disjunctions

$(T_1, I_1, F_1) \vee_N (T_2, I_2, F_2) = (T_1 \vee_F T_2, I_1 \wedge_F I_2, F_1 \wedge_F F_2) =$
$(\sup_N(T_1, T_2), \inf_N(I_1, I_2), \inf_N(F_1, F_2))$ (4.193)

$(T_1, I_1, F_1) \vee_N (T_2, I_2, F_2) = (T_1 \vee_F T_2, I_1 \wedge_F I_2, F_1 \wedge_F F_2) =$
$(T_1 +_N T_2 -_N T_1 \times_N T_2, I_1 \times_N I_2, F_1 \times_N F_2)$ (4.194)

Nonstandard Neutrosophic Negations

$$\neg(T_1, I_1, F_1) = (F_1, I_1, T_1)$$ (4.195)

$$\neg(T_1, I_1, F_1) = (F_1, (1^+) -_N I_1, T_1)$$ (4.196)

Nonstandard Neutrosophic Implications

$(T_1, I_1, F_1) \rightarrow_N (T_2, I_2, F_2) = (F_1, I_1, T_1) \vee_N (T_2, I_2, F_2) =$

$= (F_1 \vee_F T_2, I_1 \wedge_F I_2, T_1 \wedge_F F_2)$

$= (F_1 +_N T_2 -_N F_1 \times_N T_2, I_1 \times_N I_2, T_1 \times_N F_2)$ (4.197)

$(T_1, I_1, F_1) \rightarrow_N (T_2, I_2, F_2) = (F_1, (1^+) -_N I_1, T_1) \vee_N (T_2, I_2, F_2)$

$= (F_1 \vee_F T_2, ((1^+) -_N I_1) \wedge_F I_2, T_1 \wedge_F F_2) =$

$= (F_1 +_N T_2 -_N F_1 \times_N T_2, ((1^+) -_N I_1) \times_N I_2, T_1 \times_N F_2)$ \qquad (4.198)

Let $P_1(T_1, I_1, F_1)$ and $P_2(T_2, I_2, F_2)$ be two nonstandard neutrosophic logical propositions, whose nonstandard neutrosophic components are respectively:

$T_1, I_1, F_1, T_2, I_2, F_2 \in_N NR_{MB}.$ \qquad (4.199)

4.51. Numerical Examples of Nonstandard Neutrosophic Operators

Let's take a particular numeric example, where:

$P_1 =_N (\overset{0+}{0.3}, \overset{-+}{0.2}, 0.4), P_2 =_N (\overset{-0}{0.6}, \overset{-0+}{0.1}, \overset{+}{0.5})$ \qquad (4.200)

are two nonstandard neutrosophic logical propositions.

We use the nonstandard arithmetic operations previously defined

Numerical Example of Nonstandard Neutrosophic Conjunction

$\overset{0+}{0.3} \times_N \overset{-0}{0.6} =_N [0.3, 0.3 + \varepsilon_1) \times (0.6 - \varepsilon_2, 0.6] = (0.18 - 0.3\varepsilon_2, 0.18 + 0.6\varepsilon_1) =_N \overset{-0+}{0.18}$
\qquad (4.201)

$\overset{-+}{0.2} +_N \overset{-0+}{0.1} -_N \overset{-+}{0.2} \times_N \overset{-0+}{0.1} =_N [(0.2 - \varepsilon_1, 0.2) \cup (0.2, 0.2 + \varepsilon_1)] + (0.1 - \varepsilon_2, 0.1 + \varepsilon_2)$

$- [(0.2 - \varepsilon_1, 0.2) \cup (0.2, 0.2 + \varepsilon_1)] \times (0.1 - \varepsilon_2, 0.1 + \varepsilon_2)$

$= [(0.3 - \varepsilon_1 - \varepsilon_2, 0.3 + \varepsilon_2) \cup (0.3 - \varepsilon_2, 0.3 + \varepsilon_1 + \varepsilon_2)]$

$- [(0.2 - \varepsilon_1) \times (0.1 - \varepsilon_2), (0.02 + 0.2\varepsilon_2)] \cup [(0.02 - 0.2\varepsilon_2), (0.2 + \varepsilon_1) \times (0.1 + \varepsilon_2)]$

$= [\overset{-0+}{0.3} \cup \overset{-0+}{0.3}] - [\overset{-0+}{0.02} \cup \overset{-0+}{0.02}] = [\overset{-0+}{0.3}] - [\overset{-0+}{0.02}] = \overset{-0+}{0.3} - \overset{-0+}{0.02} =_N \overset{-0+}{0.28}$
\qquad (4.202)

$$0.4 +_N \overset{+}{0.5} =_N [0.4, 0.4] + (0.5, 0.5 + \varepsilon_1) - [0.4, 0.4] \times (0.5, 0.5 + \varepsilon_1)$$
$$= (0.4 + 0.5, 0.4 + 0.5 + \varepsilon_1) - (0.4 \times 0.5, 0.4 \times 0.5 + 0.4\varepsilon_1)$$
$$= (0.9, 0.9 + \varepsilon_1) - (0.2, 0.2 + 0.4\varepsilon_1)$$
$$= (0.9 - 0.2 - 0.4\varepsilon_1, 0.9 + \varepsilon_1 - 0.2) = (0.7 - 0.4\varepsilon_1, 0.7 + \varepsilon_1) =_N \overset{-0+}{0.70}$$

$$(4.203)$$

Whence

$$P_1 \wedge_N P_2 =_N (\overset{-0+}{0.18}, \overset{-0+}{0.28}, \overset{-0+}{0.70}) \tag{4.204}$$

Numerical Example of Nonstandard Neutrosophic Disjunction

$$\overset{0+}{0.3} +_N \overset{-0}{0.6} - \overset{0+}{0.3} \times_N \overset{-0}{0.6} =_N \{[0.3, 0.3 + \varepsilon_1) + (0.6 - \varepsilon_1, 0.6]\} - \{[0.3, 0.3 + \varepsilon_1) \times (0.6 - \varepsilon_1, 0.6]\}$$
$$= (0.9 - \varepsilon_1, 0.9 + \varepsilon_1) - (0.18 - 0.3\varepsilon_1, 0.18 + 0.6\varepsilon_1) = (0.72 - 1.6\varepsilon_1, 0.72 + 1.3\varepsilon_1) =_N \overset{-0+}{0.72}$$

$$(4.205)$$

$$\overset{-+}{0.2} \times_N \overset{-0+}{0.1} =_N \left(\overset{-0+}{0.2 \times 0.1} \right) =_N \overset{-0+}{0.02} \tag{4.206}$$

$$0.4 \times_N \overset{+}{0.5} =_N \left(\overset{+}{0.4 \times 0.5} \right) =_N \overset{+}{0.20} \tag{4.207}$$

Whence

$$P_1 \vee_N P_2 =_N (\overset{-0+}{0.72}, \overset{-0+}{0.28}, \overset{+}{0.20}) \tag{4.208}$$

Numerical Example of Nonstandard Neutrosophic Negation

$$\neg_N P_1 =_N \neg_N (\overset{0+}{0.3}, \overset{-+}{0.2}, 0.4) =_N (0.4, \overset{-+}{0.2}, \overset{0+}{0.3}) \tag{4.209}$$

Numerical Example of Nonstandard Neutrosophic Implication

$$(P_1 \rightarrow_N P_2) \Leftrightarrow_N (\neg_N P_1 \vee_N P_2) =_N (0.4, \overset{-+}{0.2}, \overset{0+}{0.3}) \vee_N (\overset{-0}{0.6}, \overset{-0+}{0.1}, \overset{+}{0.5}) \tag{4.210}$$

Afterwards,

$$0.4 +_N \overset{-0}{0.6} -_N 0.4 \times_N \overset{-0}{0.6} =_N \left(\overset{-0}{0.4 + 0.6} \right) -_N \left(\overset{-0}{0.4 \times 0.6} \right) =_N \overset{-0}{1.0} -_N \overset{-0}{0.24} =_N \overset{-0+}{0.76}$$

$$\overset{-+}{0.2} \times_N \overset{-0+}{0.1} =_N \overset{-0+}{0.02}$$

$$\overset{0+}{0.3} \times_N \overset{+}{0.5} =_N \overset{+}{0.15}$$

$$(4.211\text{-}4.213)$$

whence $\quad \neg_N P_1 =_N (\overset{-0+}{0.76}, \overset{-0+}{0.02}, \overset{+}{0.15})$

$$(4.214)$$

Therefore, we have showed above how to do nonstandard neutrosophic arithmetic operations on some concrete examples.

4.52. Conclusion

In the history of mathematics, critics on nonstandard analysis, in general, have been made by Paul Halmos, Errett Bishop, Alain Connes and others. That's why we have extended in 1998 for the first time the monads to pierced binad, and then in 2019 for the second time we extended the left monad to left monad closed to the right, the right monad to right monad closed to the left, and the pierced binad to unpierced binad.

These were necessary in order to construct a general nonstandard neutrosophic real mobinad space, which is closed under the nonstandard neutrosophic arithmetic operations (such as addition, subtraction, multiplication, division, and power) which are needed in order to be able to define the nonstandard neutrosophic operators (such as conjunction, disjunction, negation, implication, equivalence) on this space, andto transform the newly constructed nonstandard neutrosophic real mobinad space into a lattice of first order (as partially ordered nonstandard set, under the neutrosophic inequality) and a lattice of second type [as algebraic structure, endowed with two binary laws: neutrosophic infimum (infN) and neutrosophic supremum (supN)].

As a consequence of extending the nonstandard analysis, we also extended the nonstandard neutrosophic logic, set, and probability.

As future research it would be to find applications of extended nonstandard neutrosophic logic, set, and probability into calculus, since in calculus one deals with infinitesimals and their aggregation operators.

References

[1] Takura Imamura, *Note on the Definition of Neutrosophic Logic*, arxiv.org, 7 Nov. 2018.

[2] Xindong Peng and Jingguo Dai, *A bibliometric analysis of neutrosophic set: two decades review from 1998 to 2017*, Artificial Intelligence Review, Springer, 18 August 2018; http://fs.unm.edu/BibliometricNeutrosophy.pdf

[3] Florentin Smarandache, *n-Valued Refined Neutrosophic Logic and Its Applications in Physics*, Progress in Physics, 143-146, Vol. 4, 2013; http://fs.unm.edu/n-ValuedNeutrosophicLogic-PiP.pdf

[4] F. Smarandache, *Neutrosophy, A New Branch of Philosophy*, <Multiple Valued Logic / An International Journal>, USA, ISSN 1023-6627, Vol. 8, No. 3, pp. 297-384, 2002.

[5] Florentin Smarandache, *Neutrosophy. / Neutrosophic Probability, Set, and Logic*, ProQuest Information & Learning, Ann Arbor, Michigan, USA, 105 p., 1998; http://fs.unm.edu/eBook-Neutroosphics6.pdf.

[6] F. Smarandache, *A Unifying Field in Logics: Neutrosophic Logic*, <Multiple Valued Logic / An International Journal>, USA, ISSN 1023-6627, Vol. 8, No. 3, pp. 385-438, 2002. {The whole issue of this journal is dedicated to Neutrosophy and Neutrosophic Logic.}

[7] Florentin Smarandache, *Definition of neutrosophic logic — a generalization of the intuitionistic fuzzy logic*, Proceedings of the 3rd Conference of the European Society for Fuzzy Logic and Technology, 2003, pp. 141–146.

[8] Florentin Smarandache, Neutrosophic Overset, Neutrosophic Underset, and Neutrosophic Offset. Similarly for Neutrosophic Over-/Under-/Off- Logic, Probability, and Statistics, 168 p., Pons Editions, Bruxelles, Belgique, 2016; https://arxiv.org/ftp/arxiv/papers/1607/1607.00234.pdf

[9] Charles Ashbacher, Introduction to Neutrosophic Logic, ProQuest Information & Learning, Ann Arbor, 2002, http://fs.unm.edu/IntrodNeutLogic.pdf

[10] Umberto Rivieccio, Neutrosophic logics: Prospects and problems, Fuzzy Sets and Systems, v. 159, issue 14, 1860–1868, 2008.

[11] Plithogeny, Plithogenic Set, Logic, Probability, and Statistics, by Florentin Smarandache, Pons Publishing House, Brussels, Belgium, 141 p., 2017; arXiv.org (Cornell University), Computer Science - Artificial Intelligence, https://arxiv.org/ftp/arxiv/papers/1808/1808.03948.pdf

[12] Nguyen Xuan Thao, Florentin Smarandache, *(I, T)-Standard neutrosophic rough set and its topologies properties*, Neutrosophic Sets and Systems, Vol. 14, 2016, pp. 65-70; doi.org/10.5281/zenodo.570892; http://fs.unm.edu/NSS/RoughStandardNeutrosophicSets.pdf

[13] Nguyen Xuan Thao, Bui Cong Cuong, Florentin Smarandache, *Rough Standard Neutrosophic Sets: An Application on Standard Neutrosophic Information Systems*, Neutrosophic Sets and Systems, Vol. 14, 2016, pp. 80-92; doi.org/10.5281/zenodo.570890; http://fs.unm.edu/NSS/RoughStandardNeutrosophicSets.pdf

[14] Bui Cong Cuong, Pham Hong Phong, Florentin Smarandache, *Standard Neutrosophic Soft Theory - Some First Results*, Neutrosophic Sets and Systems, Vol. 12, 2016, pp. 80-91; doi.org/10.5281/zenodo.571149; http://fs.unm.edu/NSS/StandardNeutrosophicSoftTheory.pdf

[15] Insall, Matt and Weisstein, Eric W. "Nonstandard Analysis." From *MathWorld*--A Wolfram Web Resource. http://mathworld.wolfram.com/NonstandardAnalysis.html

[16] Insall, Matt. "Transfer Principle." From *MathWorld*--A Wolfram Web Resource, created by Eric W. Weisstein. http://mathworld.wolfram.com/TransferPrinciple.html

[17] F. Smarandache, *Applications of Neutrosophic Sets in Image Identification, Medical Diagnosis, Fingerprints and Face Recognition* and *Neutrosophic Overset/Underset/Offset*, COMSATS Institute of Information Technology, Abbottabad, Pakistan, December 26th, 2017;

[18] F. Smarandache, *Interval-Valued Neutrosophic Oversets, Neutrosophic Understes, and Neutrosophic Offsets*, International Journal of Science and Engineering Investigations, Vol. 5, Issue 54, 1-4, July 2016.

[19] F. Smarandache, *Operators on Single-Valued Neutrosophic Oversets, Neutrosophic Undersets, and Neutrosophic Offsets*, Journal of Mathematics and Informatics, Vol. 5, 63-67, 2016.

[20] Florentin Smarandache, *Answers to Imamura Note on the Definition of Neutrosophic Logic*, 24 November 2018, http://arxiv.org/abs/1812.02534, https://arxiv.org/ftp/arxiv/papers/1812/1812.02534.pdf

[21] Florentin Smarandache, *Introduction to Neutrosophic Measure, Neutrosophic Integral, and Neutrosophic Probability*, Sitech & Educational, Craiova, Columbus, 140 p., 2013; https://arxiv.org/ftp/arxiv/papers/1311/1311.7139.pdf

[22] Florentin Smarandache, An Introduction to Neutrosophic Measure, American Physical Society, April Meeting 2014, Volume 59, Number 5, Saturday-Tuesday, April 5-8, 2014; Savannah, Georgia, http://meetings.aps.org/Meeting/APR14/Session/T1.26

[23] F. Smarandache, A*bout Nonstandard Neutrosophic Logic (Answers to Imamura's 'Note on the Definition of Neutrosophic Logic')*, Cornell University, New York City, USA, (Submitted on 24 Nov 2018 (v1), last revised 13 Feb 2019 (this version, v2)), Abstract: https://arxiv.org/abs/1812.02534v2; Full Paper: https://arxiv.org/ftp/arxiv/papers/1812/1812.02534.pdf

[24] F. Smarandache, *Neutrosophic Precalculus and Neutrosophic Calculus*, EuropaNova, Brussels, Belgium, 154 p., 2015; https://arxiv.org/ftp/arxiv/papers/1509/1509.07723.pdf

[25] Abraham Robinson, *Non-Standard Analysis*, Princeton University Press, Princeton, NJ, 1996.

[26] Florentin Smarandache, *Extended Nonstandard Neutrosophic Logic, Set, and Probability based on Extended Nonstandard Analysis*, first version, 31 pages, arXiv, Cornell University, 11 March 2019; Abstract: https://arxiv.org/abs/1903.04558, Article: https://arxiv.org/ftp/arxiv/papers/1903/1903.04558.pdf

[27] Peter A. Loeb, (ed.); Manfred Wolff (ed.). *Nonstandard analysis for the working mathematician*. [B] Mathematics and its Applications (Dordrecht). 510. Dordrecht: Kluwer Academic Publishers. xiv, 311 p., 2000.

[28] F. Smarandache, *Degree of Dependence and Independence of the (Sub)Components of Fuzzy Set and Neutrosophic Set*, Neutrosophic Sets and Systems, Vol. 11, 95-97, 2016; http://fs.unm.edu/NSS/DegreeOfDependenceAndIndependence.pdf

CHAPTER 5

Plithogenic Set and Hypersoft Set

5.1 Plithogenic Set, an Extension of Crisp, Fuzzy, Intuitionistic Fuzzy, and Neutrosophic Sets (Revisited)

Abstract

In this paper, we introduce the plithogenic set (as generalization of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets), which is a set whose elements are characterized by many attributes' values. An attribute value v has a corresponding (fuzzy, intuitionistic fuzzy, neutrosophic, or other types of sets) degree of appurtenance d(x,v) of the element x, to the set P, with respect to some given criteria. In order to obtain a better accuracy for the plithogenic aggregation operators in the plithogenic set, and for a more exact inclusion (partial order), a (fuzzy, intuitionistic fuzzy, or neutrosophic) contradiction (dissimilarity) degree is defined between each attribute value and the dominant (most important) attribute value. The plithogenic intersection and union are linear combinations of the fuzzy operators tnorm and tconorm, while the plithogenic complement, inclusion (inequality), equality are influenced by the attribute values contradiction (dissimilarity) degrees. This article offers some examples and applications of these new concepts in our everyday life.

5.1.1. Informal Definition of Plithogenic Set

Plithogeny is the genesis or origination, creation, formation, development, and evolution of new entities from dynamics and organic fusions of contradictory and/or neutrals and/or non-contradictory multiple old entities.

While plithogenic means what is pertaining to plithogeny.

A plithogenic set P is a set whose elements are characterized by one or more attributes, and each attribute may have many values. Each

attribute's value v has a corresponding degree of appurtenance d(x,v) of the element x, to the set P, with respect to some given criteria.

In order to obtain a better accuracy for the plithogenic aggregation operators, a contradiction (dissimilarity) degree is defined between each attribute value and the dominant (most important) attribute value.

{However, there are cases when such dominant attribute value may not be taking into consideration or may not exist [therefore it is considered zero by default], or there may be many dominant attribute values. In such cases, either the contradiction degree function is suppressed, or another relationship function between attribute values should be established.}

The plithogenic aggregation operators (intersection, union, complement, inclusion, equality) are based on contradiction degrees between attributes' values, and the first two are linear combinations of the fuzzy operators' tnorm and tconorm.

Plithogenic set is a generalization of the crisp set, fuzzy set, intuitionistic fuzzy set, and neutrosophic set, since these four types of sets are characterized by a single attribute value (appurtenance): which has one value (membership) – for the crisp set and fuzzy set, two values (membership, and nonmembership) – for intuitionistic fuzzy set, or three values (membership, nonmembership, and indeterminacy) – for neutrosophic set.

5.1.2. Formal Definition of Single (Uni-Dimensional) Attribute Plithogenic Set

Let U be a universe of discourse, and P a non-empty set of elements, $P \subseteq U$.

5.1.2.1 Attribute Value Spectrum

Let A be a non-empty set of uni-dimensional attributes
A = $\{\alpha_1, \alpha_2, \ldots, \alpha_m\}$,

$m \geq 1$; and $\alpha \in A$ be a given attribute whose spectrum of all possible values (or states) is the non-empty set S, where S can be a finite discrete set, $S = \{s_1, s_2, ..., s_l\}$, $1 \leq l < \infty$, or infinitely countable set $S = \{s_1, s_2, ..., s_\infty\}$, or infinitely uncountable (continuum) set $S =]a, b[$, $a < b$, where $]...[$ is any open, semi-open, or closed interval from the set of real numbers or from other general set.

5.1.2.2 Attribute Value Range

Let V be a non-empty subset of S, where V is the *range of all attribute's values* needed by the experts for their application.

Each element $x \in P$ is characterized by all attribute's values in $V = \{v_1, v_2, ..., v_n\}$, for $n \geq 1$.

5.1.2.3 Dominant Attribute Value

Into the attribute's value set V, in general, there is a <u>dominant attribute value</u>, which is determined by the experts upon their application. Dominant attribute value means the most important attribute value that the experts are interested in.

{However, there are cases when such dominant attribute value may not be taking into consideration or not exist, or there may be many dominant (important) attribute values - when different approach should be employed.}.

5.1.2.4 Attribute Value Appurtenance Degree Function

Each attributes value $v \in V$ has a corresponding degree of appurtenance $d(x, v)$ of the element x, to the set P, with respect to some given criteria.

The degree of appurtenance may be: a fuzzy degree of appurtenance, or intuitionistic fuzzy degree of appurtenance, or neutrosophic degree of appurtenance to the plithogenic set.

Therefore, the attribute value appurtenance degree function is:

$$\forall x \in P, \; d: P \times V \rightarrow P([0, 1]^z), \tag{5.1.1}$$

so $d(x, v)$ is a subset of $[0, 1]^z$, where $P([0, 1]^z)$ is the power set of the $[0, 1]^z$, where $z = 1$ (for fuzzy degree of appurtenance), $z = 2$ (for intuitionistic fuzzy degree of appurtenance), or $z = 3$ (for neutrosophic degree de appurtenance).

5.1.2.5 Attribute Value Contradiction (Dissimilarity) Degree Function

Let the cardinal $|V| \geq 1$. Let $c: V \times V \to [0, 1]$ be the <u>attribute value contradiction degree function</u> (that we introduce now for the first time) between any two attribute values v_1 and v_2, denoted by

$c(v_1, v_2)$, and satisfying the following axioms:

$c(v_1, v_1) = 0$, the contradiction degree between the same attribute values is zero;

$c(v_1, v_2) = c(v_2, v_1)$, commutativity.

For simplicity, we use a <u>fuzzy attribute value contradiction degree function</u> (c as above, that we may denote by c_F in order to distinguish it from the next two), but an <u>intuitionistic attribute value contradiction function</u> ($c_{IF} : V \times V \to [0, 1]^2$), or more general a <u>neutrosophic attribute value contradiction function</u> ($c_N : V \times V \to [0, 1]^3$) may be utilized increasing the complexity of calculation but the accuracy as well.

We mostly compute the contradiction degree between *uni-dimensional attribute values*. For *multi-dimensional attribute values* we split them into corresponding uni-dimensional attribute values.

The attribute value contradiction degree function helps the plithogenic aggregation operators, and the plithogenic inclusion (partial order) relationship to obtain a more accurate result.

The attribute value contradiction degree function is designed in each field where plithogenic set is used in accordance with the application to solve. If it is ignored, the aggregations still work, but the result may lose accuracy.

Several examples will be provided into this paper.

Then (P, a, V, d, c) is called a plithogenic set:

● where "P" is a set, "a" is a (multi-dimensional in general) attribute, "V" is the range of the attribute's values, "d" is the degree of appurtenance of each element x's attribute value to the set P with respect to some given criteria ($x \in P$), and "d" stands for "d_F" or "d_{IF}" or "d_N", when dealing with fuzzy degree of appurtenance, intuitionistic fuzzy degree of appurtenance, or neutrosophic degree of <u>appurtenance</u> respectively of an element x to the plithogenic set P;

● and "c" stands for "c_F" or "c_{IF}" or "c_N", when dealing with fuzzy degree of contradiction, intuitionistic fuzzy degree of contradiction, or neutrosophic degree of <u>contradiction</u> between attribute values respectively.

The functions $d(\cdot,\cdot)$ and $c(\cdot,\cdot)$ are defined in accordance with the applications the experts need to solve.

One uses the notation:

$$x\big(d(x,V)\big),$$

where $d(x,V) = \{d(x,v), \text{for all } v \in V\}, \forall x \in P$.

5.1.2.6 About the Plithogenic Aggregation Set Operators

The attribute value contradiction degree is calculated between each attribute value with respect to the dominant attribute value (denoted v_D) in special, and with respect to other attribute values as well.

The attribute value contradiction degree function c between the attribute's values is used into the definition of <u>plithogenic aggregation operators</u> {*Intersection (AND), Union (OR), Implication ($\Rightarrow$), Equivalence ($\Leftrightarrow$), Inclusion Relationship (Partial Order),* and other plithogenic aggregation operators that combine two or more attribute value degrees - that t_{norm} and t_{conorm} act upon}.

Most of the plithogenic aggregation operators are linear combinations of the fuzzy t_{norm} *(denoted* $\wedge_F$ *),* and fuzzy t_{conorm} *(denoted* $\vee_F$*),* but non-linear combinations may as well be constructed.

If one applies the t_{norm} on dominant attribute value denoted by v_D, and the contradiction between v_D and v_2 is $c(v_D, v_2)$, then onto attribute value v_2 one applies:

$$[1 - c(v_D, v_2)] \cdot t_{norm}(v_D, v_2)$$
$$+ c(v_D, v_2) \cdot t_{conorm}(v_D, v_2), \tag{5.1.2}$$

Or, by using symbols:

$$[1 - c(v_D, v_2)] \cdot (v_D \wedge_F v_2)$$
$$+ c(v_D, v_2) \cdot (v_D \vee_F v_2). \tag{5.1.3}$$

Similarly, if one applies the t_{conorm} on dominant attribute value denoted by v_D, and the contradiction between v_D and v_2 is $c(v_D, v_2)$, then onto attribute value v_2 one applies:

$$[1 - c(v_D, v_2)] \cdot t_{conorm}(v_D, v_2)$$
$$+ c(v_D, v_2) \cdot t_{norm}(v_D, v_2), \tag{5.1.4}$$

Or, by using symbols:

$$[1 - c(v_D, v_2)] \cdot (v_D \vee_F v_2)$$
$$+ c(v_D, v_2) \cdot (v_D \wedge_F v_2). \tag{5.1.5}$$

5.1.3 Plithogenic Set as Generalization of other Sets

The plithogenic set is an extension of all: crisp set, fuzzy set, intuitionistic fuzzy set, and neutrosophic set.

For examples:

Let U be a universe of discourse, and a non-empty set $P \subseteq U$. Let $x \in P$ be a generic element.

5.1.3.1 Crisp (Classical) Set (CCS)

The attribute is $\alpha = $ "appurtenance";

the *set of attribute values V = {membership, nonmembership},* with cardinal $|V| = 2$;

the *dominant attribute value = membership*;

the *attribute value appurtenance degree function*:

$d: P \times V \rightarrow \{0, 1\},$ (5.1.6)

$d(x, membership) = 1, \ d(x, nonmembership) = 0,$

and the *attribute value contradiction degree function*:

$c: V \times V \rightarrow \{0, 1\},$ (5.1.7)

$c(membership, membership) = c(nonmembership, nonmembership) = 0,$

$c(membership, nonmembership) = 1.$

5.1.3.1.1. Crisp (Classical) Intersection

$a \wedge b \in \{0, 1\}$ (5.1.8)

5.1.3.1.2. Crisp (Classical) Union

$a \vee b \in \{0, 1\}$ (5.1.9)

5.1.3.1.3. Crisp (Classical) Complement (Negation)

$\neg a \in \{0, 1\}.$ (5.1.10)

5.1.3.2 Single-Valued Fuzzy Set (SVFS)

The *attribute* is $\alpha =$ "*appurtenance*";

the *set of attribute values V = {membership}*, whose cardinal $|V| = 1$;

the *dominant attribute value = membership;*

the *appurtenance attribute value degree function:*

$d: P \times V \rightarrow [0, 1],$ (5.1.11)

with $d(x, membership) \in [0, 1]$;

and the *attribute value contradiction degree function*:

$c: V \times V \rightarrow [0, 1],$ (5.1.12)

$c(membership, membership) = 0.$

5.1.3.2.1. Fuzzy Intersection

$a \wedge_F b \in [0, 1]$ (5.1.13)

5.1.3.2.2. Fuzzy Union

$a \vee_F b \in [0, 1]$ (5.1.14)

5.1.3.2.3. Fuzzy Complement (Negation)

$$\neg_F a = 1 - a \in [0, 1]. \tag{5.1.15}$$

5.1.3.3 Single-Valued Intuitionistic Fuzzy Set (SVIFS)

The *attribute* is α = "*appurtenance*";

the *set of attribute values V = {membership, nonmembership}*, whose cardinal $|V| = 2$;

the *dominant attribute value = membership;*

the *appurtenance attribute value degree function*:

$$d: P \times V \rightarrow [0, 1], \tag{5.1.16}$$

$d(x, membership) \in [0, 1]$,

$d(x, nonmembership) \in [0, 1]$,

with $d(x, membership) + d(x, nonmembership) \leq 1$,

and the *attribute value contradiction degree function*:

$$c: V \times V \rightarrow [0, 1], \tag{5.1.17}$$

$c(membership, membership) = c(nonmembership, nonmembership) = 0$,

$c(membership, nonmembership) = 1$,

which means that for SVIFS aggregation operators' intersection (AND) and union (OR), if one applies the t_{norm} on membership degree, then one has to apply the t_{conorm} on nonmembership degree – and reciprocally.

Therefore:

5.1.3.3.1 Intuitionistic Fuzzy Intersection

$$(a_1, a_2) \wedge_{\text{IFS}} (b_1, b_2) =$$
$$= (a_1 \wedge_F b_1, a_2 \vee_F b_2) \tag{5.1.18}$$

5.1.3.3.2 Intuitionistic Fuzzy Union

$$(a_1, a_2) \vee_{\text{IFS}} (b_1, b_2) =$$
$$= (a_1 \wedge_F b_1, a_2 \vee_F b_2), \tag{5.1.19}$$

and

5.1.3.3.3 Intuitionistic Fuzzy Complement (Negation)

$$\neg_{IFS} (a_1, a_2) = (a_2, a_1). \tag{5.1.20}$$

where $\wedge_F$ and $\vee_F$ are the fuzzy t_{norm} and fuzzy t_{conorm} respectively.

5.1.3.3.4 Intuitionistic Fuzzy Inclusions (Partial Orders)

5.1.3.3.4.1. **Simple Intuitionistic Fuzzy Inclusion** *(the most used by the intuitionistic fuzzy community):*

$$(a_1, a_2) \leq_{IFS} (b_1, b_2) \tag{5.1.21}$$

iff $a_1 \leq b_1$ and $a_2 \geq b_2$.

5.1.3.3.4.2. **Plithogenic (Complete) Intuitionistic Fuzzy Inclusion** *(that we now introduce for the first time):*

$$(a_1, a_2) \leq_P (b_1, b_2) \tag{5.1.22}$$

iff $a_1 \leq (1 - c_v) \cdot b_1, a_2 \geq (1 - c_v) \cdot b_2,$

where $c_v \in [0, 0.5)$ is the contradiction degree between the attribute dominant value and the attribute value v { the last one whose degree of appurtenance with respect to Expert A is *(a₁, a₂)*, while with respect to Expert B is *(b₁, b₂)* }. If c_v does not exist, we take it by default as equal to zero.

5.1.3.4 Single-Valued Neutrosophic Set (SVNS)

The *attribute* is α = "*appurtenance*";

the *set of attribute values V = {membership, indeterminacy, nonmembership}*, whose cardinal $|V| = 3$;

the *dominant attribute value = membership*;

the *attribute value appurtenance degree function*:

$$d: P \times V \to [0, 1], \tag{5.1.23}$$

d(x, membership) $\in [0, 1]$,

d(x, indeterminacy) $\in [0, 1]$,

d(x, nonmembership) $\in [0, 1]$,

with $0 \leq d(x, membership) + d(x, indeterminacy) + d(x, nonmembership) \leq 3$;

and the *attribute value contradiction degree function*:

$c: V \times V \rightarrow [0, 1]$,　　　　　　　　　　　　(5.1.24)

$c(membership, membership) = c(indeterminacy, indeterminacy) = c(nonmembership, nonmembership) = 0$,

$c(membership, nonmembership) = 1$,

$c(membership, indeterminacy) =$

$c(nonmembership, indeterminacy) = 0.5$,

which means that for the SVNS aggregation operators (Intersection, Union, Complement etc.), if one applies the t_{norm} on membership, then one has to apply the t_{conorm} on nonmembership {and reciprocally), while on indeterminacy one applies the average of t_{norm} and t_{conorm}, as follows:

5.1.3.4.1. Neutrosophic Intersection

5.1.3.4.1.1. Simple Neutrosophic Intersection *(the most used by the neutrosophic community):*

$$(a_1, a_2, a_3) \wedge_{NS} (b_1, b_2, b_3) =$$
$$\left(a_1 \wedge_F b_1, a_2 \vee_F b_2, a_3 \vee_F b_3 \right)$$　　　(5.1.25)

5.1.3.4.1.2. Plithogenic Neutrosophic Intersection

$$(a_1, a_2, a_3) \wedge_P (b_1, b_2, b_3) =$$
$$\left(a_1 \wedge_F b_1, \frac{1}{2}\left[\left(a_2 \wedge_F b_2 \right) + \left(a_2 \vee_F b_2 \right) \right], a_3 \vee_F b_3 \right)$$　　　(5.1.26)

5.1.3.4.2. Neutrosophic Union

5.1.3.4.2.1. Simple Neutrosophic Union (the most used by the neutrosophic community)

$$(a_1, a_2, a_3) \vee_{NS} (b_1, b_2, b_3) =$$
$$\left(a_1 \vee_F b_1, a_2 \wedge_F b_2, a_3 \wedge_F b_3 \right)$$　　　(5.1.27)

5.1.3.4.2.2. Plithogenic Neutrosophic Union

$$(a_1, a_2, a_3) \vee_P (b_1, b_2, b_3) =$$

$$= \begin{pmatrix} a_1 \vee_F b_1, \\ \frac{1}{2}[(a_2 \wedge_F b_2) + (a_2 \vee_F b_2)], \\ a_3 \wedge_F b_3 \end{pmatrix} \qquad (5.1.28)$$

In other way, with respect to what one applies on the membership, one applies the opposite on non-membership, while on indeterminacy one applies the average between them.

5.1.3.4.3. Neutrosophic Complement (Negation):

$$\neg_{NS} (a_1, a_2, a_3) = (a_3, a_2, a_1). \qquad (5.1.29)$$

5.1.3.4.4. Neutrosophic Inclusions (Partial-Orders)

5.1.3.4.4.1. Simple Neutrosophic Inclusion (the most used by the neutrosophic community):

$$(a_1, a_2, a_3) \leq_{NS} (b_1, b_2, b_3) \qquad (5.1.30)$$

iff $a_1 \leq b_1$ and $a_2 \geq b_2$, $a_3 \geq b_3$.

5.1.3.4.4.2. Plithogenic Neutrosophic Inclusion *(defined now for the first time):*

Since the degrees of contradiction are

$$c(a_1, a_2) = c(a_2, a_3) = c(b_1, b_2) = c(b_2, b_3) = 0.5, \qquad (5.1.31)$$

one applies:

$$a_2 \geq [1 - c(a_1, a_2)]b_2 \text{ or } a_2 \geq (1-0.5)b_2 \text{ or } a_2 \geq 0.5 \cdot b_2$$

while

$$c(a_1, a_3) = c(b_1, b_3) = 1 \qquad (5.1.32)$$

{having $a_1 \leq b_1$ one does the opposite for $a_3 \geq b_3$},

whence

$$(a_1, a_2, a_3) \leq_P (b_1, b_2, b_3) \qquad (5.1.33)$$

iff $a_1 \leq b_1$ and $a_2 \geq 0.5 \cdot b_2$, $a_3 \geq b_3$.

5.1.3.5 Single-Valued Refined Fuzzy Set (SVRFS)

For the first time the fuzzy set was refined by Smarandache [2] in 2016 as follows:

A SVRFS number has the form:

$$(T_1, T_2, \ldots, T_p),$$

where $p \geq 2$ is an integer, and all $T_j \in [0, 1]$, for $j \in \{1, 2, \ldots, p\}$.

The *attribute* $\alpha =$ "*appurtenance*";

the *set of attribute values* $V = \{m_1, m_2, \ldots, m_p\}$, where "m" means *submembership*;

the *dominant attribute values* $= m_1, m_2, \ldots, m_p$;

the *attribute value appurtenance degree function*:

$d: P \times V \rightarrow [0, 1]$, $\qquad\qquad\qquad\qquad$ (5.1.34)

$d(x, m_j) \in [0, 1]$, for all j,

and

$$\sum_{j=1}^{p} d_x(m_j) \leq 1; \qquad\qquad\qquad\qquad (5.1.35)$$

and the *attribute value contradiction degree function*:

$$c(m_{j_1}, m_{j_2}) = 0, \qquad\qquad\qquad\qquad (5.1.36)$$

for all $j_1, j_2 \in \{1, 2, \ldots, p\}$.

Aggregation operators on SVRFS:

Let $(a_j, 1 \leq j \leq p)$, with all $a_j \in [0, 1]$, be a SVRFS number, which means that the sub-truths $T_j = a_j$ for all $1 \leq j \leq p$.

5.1.3.5.1. Refined Fuzzy Intersection

$$(a_j, 1 \leq j \leq p) \wedge_{\text{RFS}} (b_j, 1 \leq j \leq p)$$
$$= \left(a_j \wedge_F b_j, 1 \leq j \leq p \right).$$
$\qquad\qquad\qquad\qquad$ (5.1.37)

5.1.3.5.2. Refined Fuzzy Union

$$\left(a_j, 1 \leq j \leq p\right) \vee_{\text{RFS}} \left(b_j, 1 \leq j \leq p\right)$$
$$= \left(a_j \vee_F b_j, 1 \leq j \leq p\right).$$

(5.1.38)

5.1.3.5.3. Refined Fuzzy Complement (Negation)

$$\neg_{RFS} \left(T_j = a_j, 1 \leq j \leq p\right) =$$
$$= \left(F_j = a_j, 1 \leq j \leq p\right),$$

(5.1.39)

where F_j are the sub-falsehoods, for all $1 \leq j \leq p$.

5.1.3.5.4. Refined Fuzzy Inclusion (Partial-Order)

$$\left(a_j, 1 \leq j \leq p\right) \leq_{\text{RFS}} \left(b_j, 1 \leq j \leq p\right)$$

(5.1.40)

iff $a_j \leq b_j$ for all $1 \leq j \leq p$.

5.1.3.6 Single-Valued Refined Intuitionistic Fuzzy Set (SVRIFS)

For the first time, the intuitionistic fuzzy set was refined by Smarandache [2] in 2016, as follows:

A SVRIFS number has the form:

$$(T_1, T_2, \ldots, T_p; F_1, F_2, \ldots, F_s),$$

where *p, r ≥ 1* are integers, and *p + r ≥ 3*, and all T_j, $F_l \in [0, 1]$, for *j* ∈{1, 2, …, *p*} and *l* ∈{1, 2, …, *s*}.

The *attribute* α = "*appurtenance*";

the *set of attribute values V = {m₁, m₂, …, mₚ; nm₁, nm₂, …, nmₚ}*, where "m" means *submembership*, and "nm" *subnonmembership*;

the *dominant attribute values = m₁, m₂, …, mₚ*;

the *attribute value appurtenance degree function*:

d: P×V→[0, 1],

(5.1.41)

d(x, mⱼ) ∈ [0, 1], for all *j*, and *d(x, nmₗ)* ∈ [0, 1], for all *l*, where

$$\sum_{j=1}^{p} d_x(m_j) + \sum_{l=1}^{s} d_x(nm_l) \leq 1;$$

(5.1.42)

and the *attribute value contradiction degree function*:

$$c\left(m_{j_1}, m_{j_2}\right) = c\left(nm_{l_1}, nm_{l_2}\right) = 0, \tag{5.1.43}$$

for all j_1, $j_2 \in \{1, 2, \ldots, p\}$, and l_1, $l_2 \in \{1, 2, \ldots, s\}$, while $c\left(m_j, nm_l\right) = 1$ for all j and l.

Aggregation operators on SVRIFS:

5.1.3.6.1. Refined Intuitionistic Set Intersection

$$\left(a_j, 1 \leq j \leq p; b_l, 1 \leq l \leq s\right) \wedge_{RIFS} \tag{5.1.44}$$

$$\left(c_j, 1 \leq j \leq p;\ d_l, 1 \leq l \leq s\right) =$$

$$\left(a_j \wedge_F c_j, 1 \leq j \leq p; b_l \vee_F d_l, 1 \leq l \leq s\right).$$

5.1.3.6.2. Refined Intuitionistic Set Union

$$\left(a_j, 1 \leq j \leq p; b_l, 1 \leq l \leq s\right) \vee_{RIFS} \tag{5.1.45}$$

$$\left(c_j, 1 \leq j \leq p;\ d_l, 1 \leq l \leq s\right)$$

$$= \left(a_j \vee_F c_j, 1 \leq j \leq p; b_l \wedge_F d_l, 1 \leq l \leq s\right).$$

5.1.3.6.3. Refined Intuitionistic Complement (Negation)

$$\neg_{RIFS}\left(T_j = a_j, 1 \leq j \leq p; F_j = b_l, 1 \leq l \leq s\right) =$$

$$\left(T_j = b_l, 1 \leq l \leq s;\ F_j = a_j, 1 \leq j \leq p\right). \tag{5.1.46}$$

5.1.3.6.4. Refined Intuitionistic Inclusions (Partial Orders)

5.1.3.6.4.1. Simple Refined Intuitionistic Inclusion

$$\left(a_j, 1 \leq j \leq p; b_l, 1 \leq l \leq s\right) \leq_{RIFS}$$

$$\left(u_j, 1 \leq j \leq p;\ w_l, 1 \leq l \leq s\right) \tag{5.1.47}$$

iff

$a_j \leq u_j$ for all $1 \leq j \leq p$ and $w_l \geq d_l$ for all $1 \leq l \leq s$.

5.1.3.6.4.2. Plithogenic Refined Intuitionistic Inclusion

$$\left(a_j, 1 \le j \le p; b_l, 1 \le l \le s\right) \le_P$$
$$\left(u_j, 1 \le j \le p; w_l, 1 \le l \le s\right) \tag{5.1.48}$$

iff $a_j \le (1 - c_v) \cdot u_j$ for all $1 \le j \le p$ and $b_l \ge (1 - c_v) \cdot w_l$ for all $1 \le l \le s$,

where similarly $c_v \in [0, 0.5)$ is the contradiction degree between the attribute dominant value and the attribute value v. If c_v does not exist, we take it by default as equal to zero.

5.1.3.7 Single-Valued Finitely Refined Neutrosophic Set (SVFRNS)

The Single-Valued Refined Neutrosophic Set and Logic were first defined by Smarandache [3] in 2013.

A SVFRNS number has the form:

$$(T_1, T_2, \ldots, T_p; I_1, I_2, \ldots, I_r; F_1, F_2, \ldots, F_s),$$

where $p, r, s \ge 1$ are integers, with $p + r + s \ge 4$,

and all $T_j, I_k, F_l \in [0, 1]$, for $j \in \{1, 2, \ldots, p\}$, $k \in \{1, 2, \ldots, r\}$, and $l \in \{1, 2, \ldots, s\}$.

The *attribute* $\alpha =$ "*appurtenance*";

the *set of attribute values* $V = \{m_1, m_2, \ldots, m_p; i_1, i_2, \ldots, i_r; f_1, f_2, \ldots, f_s\}$, where "*m*" means submembership, "*i*" subindeterminacy, and "*f*" sub-nonmembership;

the *dominant attribute values* $= m_1, m_2, \ldots, m_p$;

the *attribute value appurtenance degree function*:

$d: P \times V \rightarrow [0, 1],$ (5.1.49)

$d(x, m_j) \in [0, 1], d(x, i_k) \in [0, 1], d(x, f_l) \in [0, 1]$, for all j, k, l,

with

$$0 \le \sum_{j=1}^{p} d_x(m_j) + \sum_{k=1}^{r} d_x(i_k) +$$
$$+ \sum_{l=1}^{s} d_x(f_l) \le p + r + s; \tag{5.1.50}$$

and the *attribute value contradiction degree function*:

$$c\left(m_{j_1}, m_{j_2}\right) = \; c\left(i_{k_1}, i_{k_2}\right) = \; c\left(f_{l_1}, f_{l_2}\right) = 0, \tag{5.1.51}$$

for all j_1, $j_2 \in \{1, 2, ..., p\}$, k_1, $k_2 \in \{1, 2, ..., r\}$, and l_1, $l_2 \in \{1, 2, ..., s\}$;

$$c\left(m_j, f_l\right) = 1, \tag{5.1.52}$$

$$c\left(m_j, i_k\right) = c(f_l, i_k) = 0.5, \tag{5.1.53}$$

for all j, k, l.

Aggregation operators on SVFRNS:

5.1.3.7.1. Refined Neutrosophic Set Union

$$\left(a_j, 1 \le j \le p; b_k, 1 \le k \le r; \; g_l, 1 \le l \le s \;\right) \vee_{RNS}$$
$$\left(u_j, 1 \le j \le p; \; o_k, 1 \le k \le r; \; w_l, 1 \le l \le s\right) =$$

$$\left(\begin{array}{c} a_j \vee_F u_j, 1 \le j \le p; \dfrac{1}{2}\left[\left(b_k \wedge_F o_k\right) + \left(b_k \vee_F o_k\right)\right], \\ 1 \le k \le r; g_l \wedge_F w_l, 1 \le l \le s \end{array} \right). \tag{5.1.54}$$

5.1.3.7.2. Refined Neutrosophic Complement (Negation)

$$\neg_{NS} \left(\begin{array}{c} T_j = a_j, 1 \le j \le p; I_k = b_k, \\ 1 \le k \le r; F_l = \; g_l, 1 \le l \le s \end{array} \right) =$$
$$= \left(\begin{array}{c} T_j = g_l, 1 \le l \le s \; ; I_k = b_k, 1 \le k \le r; \\ g_l, 1 \le l \le s; \; F_l = a_j, 1 \le j \le p \end{array} \right), \tag{5.1.55}$$

where all T_j = sub-truths, all I_k = sub-indeterminacies, and all F_l = sub-falsehoods.

5.1.3.7.3. Refined Neutrosophic Inclusions (Partial-Orders)

5.1.3.7.3.1. Simple Refined Neutrosophic Inclusion

$$\left(a_j, 1 \le j \le p; b_k, 1 \le k \le r; \; g_l, 1 \le l \le s \;\right) \le_{RNS}$$
$$\left(u_j, 1 \le j \le p; \; o_k, 1 \le k \le r; \; w_l, 1 \le l \le s\right) \tag{5.1.56}$$

iff all $a_j \le u_j$, all $b_k \ge o_k$ and all $g_l \ge w_l$.

5.1.3.7.3.2. Plithogenic Refined Neutrosophic Inclusion

$$\left(a_j, 1 \le j \le p; b_k, 1 \le k \le r; g_l, 1 \le l \le s\right)$$
$$\le_P \left(u_j, 1 \le j \le p; o_k, 1 \le k \le r; w_l, 1 \le l \le s\right) \tag{5.1.57}$$

iff all $a_j \le (1-c_v) \cdot u_j$, all $b_k \ge (1-c_v) \cdot o_k$ and all $g_l \ge (1-c_v) \cdot w_l$,

where $c_v \in [0, 0.5)$ is the contradiction degree between the attribute dominant value and the attribute value v. If c_v does not exist, we take it by default as equal to zero.

5.1.4 One-Attribute-Value Plithogenic Single-Valued Set Operators

If onto the dominant attribute value v_D one applies the plithogenic t_{norm}, then on an attribute value v_1 whose contradiction degree with respect to v_D is 1, one applies the opposite, i.e. the plithogenic t_{conorm}.

While onto an attribute value v_2 whose contradiction degree with respect to v_D belongs to $(0, 1)$, one applies a linear combination of the t_{norm} and t_{conorm}:

$$\alpha \cdot t_{norm}[d_A(v_2), d_B(v_2)] +$$
$$\beta \cdot t_{conorm}[d_A(v_2), d_B(v_2)], \tag{5.1.58}$$

with $\alpha, \beta \in (0, 1)$, and $\alpha + \beta = 1$.

When doing a *plithogenic intersection*: the closer is $c(v_D, v_2)$ to 0, the larger is the percentage of t_{norm} added and the smaller is the percentage of t_{conorm} added.

And reciprocally, when doing a *plithogenic union*: the closer is $c(v_D, v_2)$ to 0, the smaller is the percentage of t_{norm} added and the bigger is the percentage of t_{conorm} added.

If $c(v_D, v_2) = \frac{1}{2}$, then the plithogenic intersection coincides with the plithogenic union:

$$d_A(v_2) \wedge_p d_B(v_2) =$$
$$\frac{1}{2} \cdot [d_A(v_2) \wedge_F d_B(v_2)] + \frac{1}{2} \cdot [d_A(v_2) \vee_F d_B(v_2)], \tag{5.1.59}$$

while

$$d_A(v_2) \vee_p d_B(v_2) =$$
$$\frac{1}{2} \cdot [d_A(v_2) \vee_F d_B(v_2)] +$$
$$+\frac{1}{2} \cdot [d_A(v_2) \wedge_F d_B(v_2)]. \tag{5.1.60}$$

If onto v_D one applies $\wedge_p$, then on all v's with $c(v_D, v) < 0.5$ one also applies $\wedge_p$, while on those v's with $c(v_D, v) \geq 0.5$ one applies the opposite ($\vee_p$).

And reciprocally: if on v_D one applies $\vee_p$, then on all v's with $c(v_D, v) < 0.5$ one also applies $\vee_p$, while on those v's with $c(v_D, v) \geq 0.5$ one applies the opposite ($\wedge_p$).

5.1.4.1 One-Attribute-Value Plithogenic Single-Valued Fuzzy Set Operators

Let U be a universe of discourse, and a subset of it P be a plithogenic set, and $x \in P$ an element. Let α be a uni-dimensional attribute that characterize x, and v an attribute value, $v \in V$, where V is set of all attribute's α values used into solving an application.

The degree of contradiction $c(v_D, v) = c_0 \in [0, 1]$ between the dominant attribute value v_D and the attribute value v.

Let's consider two experts, A and B, each evaluating the single-valued fuzzy degree of appurtenance of attribute value v of x to the set P with respect to some given criteria:

$$d_A^F(v) = a \in [0, 1], \text{ and}$$
$$d_B^F(v) = b \in [0, 1].$$

Let $\wedge_F$ and $\vee_F$ be a fuzzy t$_{norm}$ and respectively fuzzy t$_{conorm}$.

5.1.4.2 One-Attribute-Value Plithogenic Single-Valued Fuzzy Set Intersection

$$a \wedge_p b = (1 - c_0) \cdot [a \wedge_F b] + c_0 \cdot [a \vee_F b]. \tag{5.1.61}$$

If $c(v_D, v) = c_0 \in [0, 0.5)$ then more weight is assigned onto the $t_{norm}(a, b) = a \wedge_F b$ than onto $t_{conorm}(a,b) = a \vee_F b$; this is a proper plithogenic intersection.

If $c(v_D, v) = c_0 \in (0.5, 1]$ then less weight is assigned onto the $t_{norm}(a, b) = a \wedge_F b$ than onto $t_{conorm}(a,b) = a \vee_F b$; this becomes (rather) an improper plithogenic union.

If $c(v_D, v) = c_0 \in 0.5$ then the same weight $\{0.5\}$ is assigned onto the $t_{norm}(a, b) = a \wedge_F b$ and on $t_{conorm}(a,b) = a \vee_F b$.

5.1.4.3 One-Attribute-Value Plithogenic Single-Valued Fuzzy Set Union

$$a \vee_p b = (1 - c_0) \cdot [a \vee_F b] + c_0 \cdot [a \wedge_F b]. \qquad (5.1.62)$$

If $c(v_D, v) = c_0 \in [0, 0.5)$ then more weight is assigned onto the $t_{conorm}(a, b) = a \vee_F b$ than onto $t_{norm}(a,b) = a \wedge_F b$; this is a proper plithogenic union.

If $c(v_D, v) = c_0 \in (0.5, 1]$ then less weight is assigned onto the $t_{conorm}(a, b) = a \vee_F b$ than onto $t_{norm}(a,b) = a \wedge_F b$; this is (rather) an improper plithogenic intersection.

If $c(v_D, v) = c_0 \in 0.5$ then the same weight $\{0.5\}$ is assigned onto the

$$t_{conorm}(a, b) = a \wedge_F b \text{ and on } t_{norm}(a,b) = a \vee_F b.$$

5.1.4.4 One-Attribute-Value Plithogenic Single-Valued Fuzzy Set Complements (Negations)

5.1.4.4.1. Denying the Attribute Value

$$\neg_p(v) = anti(v), \qquad (5.1.63)$$

i.e. the opposite of v, where $anti(v) \in V$ or $anti(v) \in RefinedV$ (refined set of V).

So, we get:

$$d_A^F(anti(v)) = a. \qquad (5.1.64)$$

5.1.4.4.2. Denying the Attribute Value Degree

$$\neg_p(a) = 1 - a, \text{ or } \neg_p d_A^F(v) = 1 - a. \qquad (5.1.65)$$

$$\binom{v}{a} \xrightarrow{negation} \binom{anti(v)}{a} \text{ or } \binom{v}{1-a}. \qquad (5.1.66)$$

5.1.5 Singe-Valued Fuzzy Set Degrees of Appurtenance

According to Expert A:

$d_A: \{green, yellow, red; tall, medium\} \rightarrow [0, 1]$

One has:

$d_A(green) = 0.6,$

$d_A(yellow) = 0.2,$

$d_A(red) = 0.7;$

$d_A(tall) = 0.8,$

$d_A(medium) = 0.5.$

We summarize as follows:

According to Expert A:

Contradiction Degrees	0	$\frac{1}{3}$	$\frac{2}{3}$		0	$\frac{1}{2}$
Attributes' Values	green	yellow	red		tall	medium
Fuzzy Degrees	0.6	0.2	0.7		0.8	0.5

Table 1.

According to Expert B:

Contradiction Degrees	0	$\frac{1}{3}$	$\frac{2}{3}$		0	$\frac{1}{2}$
Attributes' Values	green	yellow	red		tall	medium
Fuzzy Degrees	0.7	0.4	0.6		0.6	0.4

Table 2.

The element

<p style="text-align:center">x{ (green, tall), (green, medium), (yellow, tall), (yellow,
medium), (red, tall), (red, medium) } ∈ P</p>

with respect to the two experts as above is represented as:

$$x_A\{(0.6, 0.8), (0.6, 0.5), (0.2, 0.8), (0.2, 0.5), (0.7, 0.8), (0.7, 0.5)\}$$

and

$$x_B\{(0.7, 0.6), (0.7, 0.4), (0.4, 0.6), (0.4, 0.4), (0.6, 0.6), (0.6, 0.4)\}.$$

In order to find the optimal representation of x, we need to intersect x_A and x_B, each having six duplets. Actually, we separately intersect the corresponding duplets.

In this example, we take the fuzzy t_{norm}: $a \wedge_F b = ab$ and the fuzzy t_{conorm}: $a \vee_F b = a + b - ab$.

5.1.5.1 Application of Uni-Attribute Value Plitho-genic Single-Valued Fuzzy Set Intersection

Let's compute $x_A \wedge_p x_B$.

$$\begin{matrix} 0 & \quad 0 & \quad 0 & \quad 0 & \{degrees\ of\ contradictions\} \end{matrix}$$
$$(0.6, 0.8) \wedge_p (0.7, 0.6) = (0.6 \wedge_p 0.7, 0.8 \wedge_p 0.6) = (0.6 \cdot 0.7, 0.8 \cdot 0.6) = (0.42, 0.48),$$

where above each duplet we wrote the degrees of contradictions of each attribute value with respect to their correspondent dominant attribute value. Since they were zero, $\wedge_p$ coincided with $\wedge_F$.

{*the first raw below 0 ½ and again 0 ½ represents the contradiction degrees*}

$$\begin{pmatrix} 0 & \frac{1}{2} \\ 0.6, & 2 \\ & 0.5 \end{pmatrix} \wedge_p \begin{pmatrix} 0 & \frac{1}{2} \\ 0.7, & 2 \\ & 0.4 \end{pmatrix} = (0.6 \wedge_p 0.7, 0.5 \wedge_p 0.4)$$
$$= (0.6 \cdot 0.7, (1 - 0.5) \cdot [0.5 \wedge_F 0.4] + 0.5 \cdot [0.5 \vee_F 0.4])$$
$$= (0.42, 0.5[0.2] + 0.5[0.5 + 0.4 - 0.5 \cdot 0.4])$$
$$= (0.42, 0.45).$$

$$\begin{pmatrix} \dfrac{1}{3}, & 0 \\ 0.2 & 0.8 \end{pmatrix} \wedge_p \begin{pmatrix} \dfrac{1}{3}, & 0 \\ 0.4 & 0.6 \end{pmatrix} = \left(0.2 \wedge_p 0.4, 0.8 \wedge_p 0.6\right)$$

$$= \left(\left\{1 - \dfrac{1}{3}\right\} \cdot [0.2 \wedge_F 0.4] + \left\{\dfrac{1}{3}\right\} \cdot [0.2 \vee_F 0.4], 0.8\right.$$
$$\left. \cdot\, 0.6\right) \approx (0.23, 0.48).$$

$$\begin{pmatrix} \dfrac{1}{3}, & \dfrac{1}{2} \\ 0.2 & 0.5 \end{pmatrix} \wedge_p \begin{pmatrix} \dfrac{1}{3}, & \dfrac{1}{2} \\ 0.4 & 0.4 \end{pmatrix} = \left(0.2 \wedge_p 0.4, 0.5 \wedge_p 0.4\right)$$

(they were computed above)

$$\approx (0.23, 0.45).$$

$$\begin{pmatrix} \dfrac{2}{3}, & 0 \\ 0.7 & 0.8 \end{pmatrix} \wedge_p \begin{pmatrix} \dfrac{2}{3}, & 0 \\ 0.6 & 0.6 \end{pmatrix} = \left(0.7 \wedge_p 0.8, 0.8 \wedge_p 0.6\right)$$

$$= \left(\left\{1 - \dfrac{2}{3}\right\} \cdot [0.7 \wedge_F 0.6] + \left\{\dfrac{2}{3}\right\} \cdot [0.7 \vee_F 0.6], 0.48\right)$$

(the second component was computed above)

$$= \left(\dfrac{1}{3}[0.7 \cdot 0.6] + \dfrac{2}{3}[0.7 + 0.6 - 0.7 \cdot 0.6], 0.48\right) \approx (0.73, 0.48).$$

$$\begin{pmatrix} \dfrac{2}{3}, & \dfrac{1}{2} \\ 0.7 & 0.5 \end{pmatrix} \wedge_p \begin{pmatrix} \dfrac{2}{3}, & \dfrac{1}{2} \\ 0.6 & 0.4 \end{pmatrix} = \left(0.7 \wedge_p 0.6, 0.5 \wedge_p 0.4\right) \approx (0.73, 0.45).$$

Finally:

$$x_A \wedge_p x_B \approx \left\{\begin{array}{c} (0.42, 0.48), (0.42, 0.45), (0.23, 0.48), (0.23, 0.45), \\ (0.73, 0.48), (0.73, 0.45) \end{array}\right\},$$

or, after the intersection of the experts' opinions $A \wedge_P B$, we summarize the result as:

Contradiction Degrees	0	$\dfrac{1}{3}$	$\dfrac{2}{3}$		0	$\dfrac{1}{2}$
Attributes' Values	green	yellow	red		tall	medium

Fuzzy Degrees of Expert A for x	0.6	0.2	0.7		0.8	0.5
Fuzzy Degrees of Expert B for x	0.7	0.4	0.6		0.6	0.4
Fuzzy Degrees of $x_A \wedge_p x_B$	0.42	0.23	0.73		0.48	0.45
Fuzzy Degrees of $x_A \vee_p x_B$	0.88	0.37	0.57		0.92	0.45

Table 3.

5.1.5.2 Application of Uni-Attribute Value Plithogenic Single-Valued Fuzzy Set Union

We separately compute for each single attribute value:

$$d_A^F(x, green) \vee_p d_B^F(x, green) = 0.6 \vee_p 0.7$$
$$= (1 - 0) \cdot [0.6 \vee_F 0.7] + 0 \cdot [0.6 \wedge_F 0.7]$$
$$= 1 \cdot [0.6 + 0.7 - 0.6 \cdot 0.7] + 0 = 0.88.$$

$$d_A^F(x, yellow) \vee_p d_B^F(x, yellow) = 0.2 \vee_p 0.4$$
$$= \left(1 - \frac{1}{3}\right) \cdot [0.2 \vee_F 0.4] + \frac{1}{3} \cdot [0.2 \wedge_F 0.4]$$
$$= \frac{2}{3} \cdot (0.2 + 0.4 - 0.2 \cdot 0.4) + \frac{1}{3}(0.2 \cdot 0.4) \approx 0.37.$$

$$d_A^F(x, red) \vee_p d_B^F(x, red) = 0.7 \vee_p 0.6$$
$$= \left\{1 - \frac{2}{3}\right\} \cdot [0.7 \vee_F 0.6] + \frac{2}{3} \cdot [0.7 \wedge_F 0.6]$$
$$= \frac{1}{3} \cdot (0.7 + 0.6 - 0.7 \cdot 0.6) + \frac{2}{3}(0.7 \cdot 0.6) \approx 0.57.$$

$$d_A^F(x, tall) \vee_p d_B^F(x, tall) = 0.8 \vee_p 0.6$$
$$= (1 - 0) \cdot (0.8 + 0.6 - 0.8 \cdot 0.6) + 0 \cdot (0.8 \cdot 0.6)$$
$$= 0.92.$$

$$d_A^F(x, medium) \vee_p d_B^F(x, medium) = 0.5 \vee_p 0.4$$
$$= \frac{1}{2}(0.5 + 0.4 - 0.5 \cdot 0.4) + \frac{1}{2} \cdot (0.5 \cdot 0.4) = 0.45.$$

5.1.5.3 Properties of Plithogenic Single-Valued Set Operators in Applications

1) When the attribute value contradiction degree with respect to the corresponding dominant attribute value is 0 (zero), one simply use the fuzzy intersection:

$$d_{A\wedge_p B}(x, green) = d_A(x, green) \wedge_F d_B(x, green) =$$
$$= 0.6 \cdot 0.7 = 0.42,$$
$$d_{A\wedge_p B}(x, tall) = d_A(x, tall) \wedge_F d_B(x, tall) = 0.8 \cdot 0.6 = 0.48.$$

2) But, if the attribute value contradiction degree with respect to the corresponding dominant attribute value is different from 0 and from 1, the result of the plithogenic intersection is between the results of fuzzy t_{norm} and fuzzy t_{conorm}. Examples:

$$d_A(x, yellow) \wedge_F d_B(x, yellow) = 0.2 \wedge_F 0.4 = 0.2 \cdot 0.4$$
$$= 0.08 \ (t_{norm}),$$
$$d_A(x, yellow) \vee_F d_B(x, yellow) = 0.2 \vee_F 0.4 = 0.2 + 0.4 - 0.2 \cdot 0.4$$
$$= 0.52 \ (t_{conorm});$$

while

$$d_A(x, yellow) \wedge_p d_B(x, yellow) =$$
$$= 0.23 \in [0.08, 0.52]$$

{or 0.23 ≈ 0.2266… = (2/3)×0.08 + (1/3)×0.52, i.e.

a linear combination of t_{norm} and t_{conorm}}.

Similarly:

$$d_A(x, red) \wedge_p d_B(x, red) = 0.7 \wedge_F 0.6 = 0.7 \cdot 0.6 = 0.42 \ (t_{norm}),$$
$$d_A(x, red) \vee_p d_B(x, red) = 0.7 \vee_F 0.6 = 0.7 + 0.6 - 0.7 \cdot 0.6 =$$
$$0.88 \ (t_{conorm});$$

while

$$d_A(x, red) \wedge_p d_B(x, red) = 0.57 \in [0.42, 0.88]$$

{linear combination of t_{norm} and t_{conorm}}.

And

$$d_A(x, medium) \wedge_F d_B(x, medium) = 0.5 \wedge_F 0.4 = 0.5 \cdot 0.4 = 0.20,$$
$$d_A(x, medium) \vee_F d_B(x, medium) = 0.5 \vee_F 0.4$$
$$= 0.5 + 0.4 - 0.5 \cdot 0.4 = 0.70,$$

while

$d_A(x, medium) \wedge_p d_B(x, medium) = 0.45$, which is just in the middle (because "medium" contradiction degree is $\frac{1}{2}$) of the interval $[0.20, 0.70]$.

5.1.6 Single-Valued Intuitionistic Fuzzy Set Degree of Appurtenance

5.1.6.1 One-Attribute Value Plithogenic Single-Valued Intuitionistic Fuzzy Set Intersection

{degrees of contradictions}

$$\overset{0}{} \qquad \overset{0}{}$$

$$d_A^{IF}(x, green) \wedge_p d_B^{IF}(x, green) = (0.4, 0.5) \wedge_p (0.6, 0.3)$$
$$= \left(0.4 \wedge_p 0.6, 0.5 \vee_p 0.3\right) =$$
$$= (1 \cdot [0.4 \cdot 0.6] + 0 \cdot [0.4 + 0.6 - 0.4 \cdot 0.6], 0$$
$$\cdot [0.5 \cdot 0.3] + 1 \cdot [0.5 + 0.3 - 0.5 \cdot 0.3])$$
$$= (0.24, 0.65).$$

$$\overset{1/3}{} \qquad \overset{1/3}{}$$

$$d_A^{IF}(x, yellow) \wedge_p d_B^{IF}(x, yellow) = (0.1, 0.2) \wedge_p (0.4, 0.3)$$
$$= \left(0.1 \wedge_p 0.4, 0.2 \vee_p 0.3\right)$$
$$= \left(\left\{1 - \frac{1}{3}\right\} \cdot [0.1 \wedge_F 0.4] + \left\{\frac{1}{3}\right\} \cdot [0.1 \vee_F 0.4], \left\{1 - \frac{1}{3}\right\} \cdot [0.2 \vee_F 0.3]\right)$$
$$+ \left\{\frac{1}{3}\right\} \cdot [0.2 \wedge_F 0.3]$$
$$= \left(\frac{2}{3} \cdot [0.1 \cdot 0.4] + \frac{1}{3} \cdot [0.1 + 0.4 - 0.1 \cdot 0.4],\right.$$
$$\left.\frac{2}{3} \cdot [0.2 + 0.3 - 0.2 \cdot 0.3]\right) + \frac{1}{3} \cdot [0.2 \cdot 0.3]$$
$$\approx (0.18, 0.31).$$

$$\overset{2/3}{} \qquad \overset{2/3}{}$$

$$d_A^{IF}(x, red) \wedge_p d_B^{IF}(x, red) = (0, 0.3) \wedge_p (0.2, 0.5)$$
$$= \left(0 \wedge_p 0.2, 0.3 \vee_p 0.5\right)$$
$$= \left(\left\{1 - \frac{2}{3}\right\} \cdot [0 \wedge_F 0.2] + \left\{\frac{2}{3}\right\} \cdot [0 \vee_F 0.2], \left\{1 - \frac{2}{3}\right\} \cdot [0.3 \vee_F 0.5] + \left\{\frac{2}{3}\right\}\right.$$
$$\left. \cdot [0.3 \wedge_F 0.5] + \right)$$
$$= \left(\frac{1}{3} \cdot [0 \cdot 0.2] + \frac{2}{3} \cdot [0 + 0.2 - 0 \cdot 0.2], \frac{1}{3}\right.$$
$$\left. \cdot [0.3 + 0.5 - 0.3 \cdot 0.5] + \frac{2}{3} \cdot [0.3 \cdot 0.5]\right)$$
$$\approx (0.13, 0.32).$$

$$\overset{0}{} \qquad \overset{0}{}$$

$$d_A^{IF}(x, tall) \wedge_p d_B^{IF}(x, tall) = (0.8, 0.2) \wedge_p (0.6, 0.1)$$
$$= \left(0.8 \wedge_p 0.6, 0.2 \vee_p 0.1\right)$$
$$= (\{1 - 0\} \cdot [0.8 \wedge_F 0.6] + \{0\} \cdot [0.8 \vee_F 0.6], \{1 - 0\}$$
$$\cdot [0.2 \vee_F 0.1] + \{0\} \cdot [0.2 \wedge_F 0.1])$$
$$= (1 \cdot [0.8 \cdot 0.6] + 0 \cdot [0.8 + 0.6 - 0.8 \cdot 0.6], 1$$
$$\cdot [0.2 + 0.1 - 0.2 \cdot 0.1] + 0 \cdot [0.2 \cdot 0.1])$$
$$= (0.48, 0.28).$$

$$\overset{½}{} \qquad \overset{½}{}$$

$$d_A^{IF}(x, medium) \wedge_p d_B^{IF}(x, medium) = (0.4, 0.5) \wedge_p (0.5, 0.3)$$
$$= \left(0.4 \wedge_p 0.5, 0.5 \vee_p 0.3\right)$$
$$= \left(\left\{1 - \frac{1}{2}\right\} \cdot [0.4 \wedge_F 0.5] + \left\{\frac{1}{2}\right\} \cdot [0.4 \vee_F 0.5], \left\{1 - \frac{1}{2}\right\}\right.$$
$$\left. \cdot [0.5 \vee_F 0.3] + \left\{\frac{1}{2}\right\} \cdot [0.5 \wedge_p 0.3]\right)$$

$$= \left(\frac{1}{2} \cdot [0.4 \cdot 0.5] + \frac{1}{2} \cdot [0.4 + 0.5 - 0.4 \cdot 0.5], \frac{1}{2} \cdot [0.5 + 0.3 - 0.5 \cdot 0.3] \right.$$
$$\left. + \frac{1}{2} \cdot [0.5 \cdot 0.3] \right) = (0.45, 0.40).$$

5.1.6.2 One-Attribute Value Plithogenic Single-Valued Intuitionistic Fuzzy Set Union

$$d_A^{IF}(x, green) \vee_p d_B^{IF}(x, green) = (0.4, 0.5) \vee_p (0.6, 0.3)$$
$$= \left(0.4 \vee_p 0.6, 0.5 \wedge_p 0.3 \right)$$
$$= (\{1 - 0\} \cdot [0.4 \vee_F 0.6] + \{0\} \cdot [0.4 \wedge_F 0.6], \{1 - 0\}$$
$$\cdot [0.5 \wedge_F 0.3] + \{0\} \cdot [0.5 \vee_F 0.3])$$
$$= (1 \cdot [0.4 + 0.6 - 0.4 \cdot 0.6] + 0 \cdot [0.4 \cdot 0.6], 0$$
$$\cdot [0.5 \cdot 0.3] + 1 \cdot [0.5 + 0.3 - 0.5 \cdot 0.3])$$
$$= (0.76, 0.15).$$

$$\overset{1/3}{} \quad \overset{1/3}{}$$
$$d_A^{IF}(x, yellow) \vee_p d_B^{IF}(x, yellow) = (0.1, 0.2) \vee_p (0.4, 0.3)$$
$$= \left(0.1 \vee_p 0.4, 0.2 \wedge_p 0.3 \right)$$
$$= \left(\left\{ 1 - \frac{1}{3} \right\} \cdot [0.1 \vee_F 0.4] + \left\{ \frac{1}{3} \right\} \cdot [0.1 \wedge_F 0.4], \left\{ 1 - \frac{1}{3} \right\} \right.$$
$$\left. \cdot [0.2 \wedge_F 0.3] + \left\{ \frac{1}{3} \right\} \cdot [0.2 \vee_F 0.3] \right)$$
$$= \left(\frac{2}{3} \cdot [0.1 + 0.4 - 0.1 \cdot 0.4] + \frac{1}{3} \cdot [0.1 \cdot 0.4], \frac{2}{3} \right.$$
$$\left. \cdot [0.2 \cdot 0.3] + \frac{1}{3} \cdot [0.2 + 0.3 - 0.2 \cdot 0.3] \right)$$
$$\approx (0.32, 0.19).$$

$$\overset{2/3}{} \quad \overset{2/3}{}$$
$$d_A^{IF}(x, red) \vee_p d_B^{IF}(x, red) = (0, 0.3) \vee_p (0.2, 0.5) =$$
$$= \left(0 \vee_p 0.2, 0.3 \wedge_p 0.5 \right) = \left(\left\{ 1 - \frac{2}{3} \right\} \cdot [0 \vee_p 0.2] + \left\{ \frac{2}{3} \right\} \cdot \right.$$
$$[0 \wedge_p 0.2], \left\{ 1 - \frac{2}{3} \right\} \cdot [0.3 \wedge_p 0.5] + \left\{ \frac{2}{3} \right\} \cdot [0.3 \vee_p 0.5] \right) = \left(\frac{1}{3} [0 + 0.2 - \right.$$
$$0 \cdot 0.2] + \frac{2}{3} [0 \cdot 0.2], \frac{1}{3} [0.3 \cdot 0.5] + \frac{2}{3} \cdot [0.3 + 0.5 - 0.3 \cdot 0.5] \right) \approx$$
$$(0.07, 0.48).$$

$$0 \qquad\qquad 0$$

$$d_A^{IF}(x, tall) \vee_p d_B^{IF}(x, tall) = (0.8, 0.2) \vee_p (0.6, 0.1)$$
$$= \left(0.8 \vee_p 0.6, 0.2 \wedge_p 0.1\right)$$
$$= (\{1 - 0\} \cdot [0.8 \vee_F 0.6] + \{0\} \cdot [0.8 \wedge_F 0.6], \{1 - 0\}$$
$$\cdot [0.2 \wedge_F 0.1] + \{0\} \cdot [0.2 \vee_F 0.1])$$
$$= (1 \cdot [0.8 + 0.6 - 0.8 \cdot 0.6] + 0 \cdot [0.8 \cdot 0.6], 1$$
$$\cdot [0.2 \cdot 0.1] + 0 \cdot [0.2 + 0.1 - 0.2 - 0.1])$$
$$= (0.92, 0.02).$$

$$1/2 \qquad\qquad 1/2$$

$$d_A^{IF}(x, medium) \vee_p d_B^{IF}(x, medium) = (0.4, 0.5) \vee_p (0.5, 0.3)$$
$$= \left(0.4 \vee_p 0.5, 0.5 \wedge_p 0.3\right) =$$
$$= \left(\left\{1 - \frac{1}{2}\right\} \cdot [0.4 \vee_p 0.5] + \left\{\frac{1}{2}\right\} \cdot [0.4 \wedge_F 0.5], \left\{1 - \frac{1}{2}\right\} \cdot [0.5 \wedge_F 0.3] + \left\{\frac{1}{2}\right\}\right.$$
$$\left. \cdot [0.5 \vee_F 0.3]\right)$$
$$= \left(\frac{1}{2} \cdot [0.4 + 0.5 - 0.4 \cdot 0.5] + \frac{1}{2} \cdot [0.4 \cdot 0.5], \frac{1}{2} \cdot [0.5 \cdot 0.3]\right.$$
$$\left. + \frac{1}{2} \cdot [0.5 + 0.3 - 0.5 - 0.3]\right) = (0.45, 0.40).$$

5.1.7 Single Valued Neutrosophic Set Degree of Appurtenance

5.1.7.1 One-Attribute Value Plithogenic Single-Valued Neutrosophic Set Intersection

$$d_A^N(x, green) \wedge_p d_B^N(x, green) = (0.4, 0.1, 0.5) \wedge_p (0.5, 0.2, 0.4)$$
$$= \left(0.4 \wedge_p 0.5, \left\{1 - \frac{1}{2}\right\} \cdot (0.1 \wedge_p 0.2) + \left\{\frac{1}{2}\right\}\right.$$
$$\left. \cdot (0.1 \vee_p 0.2), 0.5 \vee_p 0.4\right)$$
$$= \left(0.4 \wedge_p 0.5, \frac{1}{2} \cdot [0.1 \wedge_p 0.2] + \frac{1}{2}\right.$$
$$\left. \cdot [0.1 \vee_p 0.2], 0.4 \vee_p 0.5\right)$$

{ Using first the *interior neutrosophic contradiction degrees* (between the neutrosophic components *T, I,* and *F)*:

$$\left\{ \begin{matrix} 0 & \frac{1}{2} & 1 \\ T, & I, & F \end{matrix} \right\}$$

$$= \begin{pmatrix} \{1-0\} \cdot [0.4 \wedge_F 0.5] + \{0\} \cdot [0.4 \vee_F 0.5], \\ \frac{1}{2} \cdot [0.1 \wedge_p 0.2] + \frac{1}{2} \cdot [0.1 \vee_p 0.2], \\ \{1-0\} \cdot [0.5 \vee_F 0.4] + \{0\} \cdot [0.5 \wedge_F 0.4] \end{pmatrix} =$$

$$= \begin{pmatrix} 1 \cdot [0.4 \cdot 0.5] + 0 \cdot [0.4 + 0.5 - 0.4 \cdot 0.5], \\ \frac{1}{2} \cdot [0.1 \wedge_p 0.2] + \frac{1}{2} \cdot [0.1 \vee_p 0.2], \\ (1-0) \cdot [0.5 + 0.4 - 0.5 \cdot 0.4] + 0 \cdot [0.5 \cdot 0.4] \end{pmatrix} =$$

$$= \left(0.20, \frac{1}{2} \cdot [0.1 \wedge_F 0.2] + \frac{1}{2} \cdot [0.1 \vee_F 0.2], 0.70 \right)$$

$$= \left(0.20, \frac{1}{2}(0.1 \cdot 0.2) + \frac{1}{2} \right.$$

$$\left. \cdot [0.1 + 0.2 - 0.1 \cdot 0.2], 0.70 \right) = (0.20, 0.15, 0.70).$$

$$d_A^N(x, yellow) \wedge_p d_B^N(x, yellow)$$

$$= \begin{pmatrix} & \frac{1}{3} & \\ 0.3, & 0.6, & 0.2 \end{pmatrix} \wedge_p \begin{pmatrix} & \frac{1}{3} & \\ 0.4, & 0.1, & 0.3 \end{pmatrix}$$

$$= \left(0.3 \wedge_p 0.4, \frac{1}{2} \cdot [0.6 \wedge_p 0.1] + \frac{1}{2} \right.$$

$$\left. \cdot [0.6 \vee_p 0.1], 0.2 \vee_p 0.3 \right)$$

$$\left\{ \begin{matrix} one\ firstly\ used\ the\ interior \\ neutrosophic\ contradiction\ degrees: \\ c(T, I) = \frac{1}{2}, c(T, F) = 1. \end{matrix} \right\}$$

$$= \left(\left\{1 - \frac{1}{3}\right\} \cdot [0.3 \wedge_F 0.4] + \left\{\frac{1}{3}\right\} \cdot [0.3 \vee_F 0.4], \frac{1}{2} \cdot [0.6 \cdot 0.1] + \frac{1}{2}\right.$$

$$\cdot [0.6 + o.1 - 0.6 \cdot 0.1], \left\{1 - \frac{1}{3}\right\} \cdot [0.2 \vee_F 0.3] + \left\{\frac{1}{3}\right\}$$

$$\left. \cdot [0.2 \wedge_F 0.3]\right)$$

$$= \left(\frac{2}{3} \cdot [0.3 \cdot 0.4] + \frac{1}{3} \cdot [0.3 + 0.4 - 0.3 \cdot 0.4], 0.35, \frac{2}{3}\right.$$

$$\left. \cdot [0.2 + 0.3 - 0.2 \cdot 0.3] + \frac{1}{3} \cdot [0.2 \cdot 0.3]\right)$$

$$\approx (0.27, 0.35, 0.31).$$

$$\overset{2/3}{} \qquad \overset{2/3}{}$$

$$d_A^N(x, red) \wedge_p d_B^N(x, red)$$

$$= (0.2, 0.1, 0.4) \wedge_p (0.3, 0.4, 0.2)$$

$$= \left(0.2 \wedge_p 0.3, 0.1 \vee_p 0.4, 0.4 \vee_p 0.2\right)$$

$$= \left(\left\{1 - \frac{2}{3}\right\} \cdot [0.2 \wedge_F 0.3] + \left\{\frac{2}{3}\right\} \cdot [0.2 \vee_p 0.3]\right), \frac{1}{2}$$

$$\cdot [I_1 \wedge_F I_2 + I_1 \vee_F I_2], \{\text{according to Theorem 5}\}$$

$$\left\{1 - \frac{2}{3}\right\} \cdot [0.4 \vee_F 0.2] + \left\{\frac{2}{3}\right\} \cdot [0.4 \wedge_F 0.2])$$

$$= \left(\frac{1}{3} \cdot [0.2 \cdot 0.3] + \frac{2}{3} \cdot [0.2 + 0.3 - 0.2 \cdot 0.3], \frac{1}{2}\right.$$

$$\cdot [0.1 \cdot 0.4 + 0.1 + 0.4 - 0.1 \cdot 0.4], \frac{1}{3}$$

$$\left. \cdot [0.4 + 0.2 - 0.4 \cdot 0.2] + \frac{2}{3} \cdot [0.4 \cdot 0.2]\right)$$

$$\approx (0.31, 0.25, 0.23).$$

{The degree of contradiction is $2/3 > 0.5$.}

$$d_A^N(x, tall) \wedge_p d_B^N(x, tall) = (0.8, 0.3, 0.1) \wedge_p (0.7, 0.1, 0.6)$$

$$= \left(0.8 \wedge_p 0.7, 0.3 \vee_p 0.1, 0.1 \vee_p 0.6\right)$$

$$= \left(0.8 \wedge_F 0.7, \frac{1}{2}\right.$$

$$\left. \cdot (0.3 \wedge_F 0.1 + 0.3 \vee_F 0.1), 0.1 \vee_F 0.6\right)$$

(since the exterior degree of contradiction is zero)

$$= \left(0.8 \cdot 0.7, \frac{1}{2} \cdot (0.3 \cdot 0.1 + 0.3 + 0.1 - 0.3 \cdot 0.1), 0.1 + 0.6 - 0.1\right.$$
$$\left. \cdot 0.6\right) = (0.56, 0.20, 0.64).$$

$$d_A^N(x, medium) \wedge_p d_B^N(x, medium) = (0.6, 0.2, 0.3) \wedge_p (0.5, 0.1, 0.3)$$
$$= \left(0.6 \wedge_p 0.5, 0.2 \vee_p 0.1, 0.3 \vee_p 0.3\right)$$
$$= \begin{pmatrix} \frac{1}{2} \cdot [0.6 \cdot 0.5] + \frac{1}{2} \cdot [0.6 + 0.5 - 0.6 \cdot 0.5], \\ \frac{1}{2} \cdot [0.2 \cdot 0.1 + 0.2 + 0.1 - 0.2 \cdot 0.1], \\ \frac{1}{2} \cdot [0.3 \cdot 0.3] + \frac{1}{2} \cdot [0.3 + 0.3 - 0.3 \cdot 0.3] \end{pmatrix}$$
$$= (0.55, 0.15, 0.30).$$

{Since the degree of contradiction is 1/2. }

5.1.7.2 One-Attribute Value Plithogenic Single-Valued Neutrosophic Set Union

$$d_A^N(x, green) \vee_p d_B^N(x, green)$$
$$= \left(\begin{matrix} & 0 & \\ 0.4, & 0.1, & 0.5 \end{matrix}\right) \vee_p \left(\begin{matrix} & 0 & \\ 0.5, & 0.2, & 0.4 \end{matrix}\right)$$
$$= \left(0.4 \vee_p 0.5, 0.1 \wedge_p 0.2, 0.5 \wedge_p 0.4\right)$$

{since the degree of contradiction is zero}

$$= \left(0.4 + 0.5 - 0.4\right.$$
$$\left. \cdot 0.5, \frac{1}{2}(0.1 \cdot 0.2 + 0.1 + 0.2 - 0.1 \cdot 0.2), 0.5 \cdot 0.4\right)$$
$$= (0.70, 0.15, 0.20).$$

$d_A^N(x, yellow) \vee_p d_B^N(x, yellow)$

$$= \begin{pmatrix} \dfrac{1}{3} \\ 0.3, \quad 0.6, \quad 0.2 \end{pmatrix} \vee_p \begin{pmatrix} \dfrac{1}{3} \\ 0.4, \quad 0.1, \quad 0.3 \end{pmatrix}$$

$= \left(0.3 \vee_p 0.4, 0.6 \wedge_p 0.1, 0.2 \wedge_p 0.3\right)$

$= \left(\left\{1 - \dfrac{1}{3}\right\} \cdot [0.3 \vee_F 0.4] + \left\{\dfrac{1}{3}\right\} \cdot [0.3 \wedge_F 0.4], \dfrac{1}{2}\right.$

$\cdot [0.6 \wedge_F 0.1 + 0.6 \vee_F 0.1], \left\{1 - \dfrac{1}{3}\right\} \cdot [0.2 \wedge_F 0.3] + \left\{\dfrac{1}{3}\right\}$

$\left. \cdot [0.2 \vee_F 0.3]\right)$

$= \left(\dfrac{2}{3} \cdot [0.3 + 0.4 - 0.3 \cdot 0.4] + \dfrac{1}{3} \cdot [0.3 \cdot 0.4], \dfrac{1}{2}\right.$

$\cdot [0.6 \cdot 0.1 + 0.6 + 0.1 - 0.6 \cdot 0.1], \dfrac{2}{3} \cdot [0.2 \cdot 0.3] + \dfrac{1}{3}$

$\left. \cdot [0.2 + 0.3 - 0.2 \cdot 0.3]\right) \approx (0.43, 0.35, 0.19).$

$d_A^N(x, red) \vee_p d_B^N(x, red) = \begin{pmatrix} \dfrac{2}{3} \\ 0.2, \quad 0.1, \quad 0.4 \end{pmatrix} \vee_p \begin{pmatrix} \dfrac{2}{3} \\ 0.3, \quad 0.4, \quad 0.2 \end{pmatrix}$

$= \left(0.2 \vee_p 0.3, 0.1 \wedge_p 0.4, 0.4 \wedge_p 0.2\right)$

$= \left(\left\{1 - \dfrac{2}{3}\right\} \cdot [0.2 \vee_p 0.3] + \left\{\dfrac{2}{3}\right\} \cdot [0.2 \wedge_p 0.3]\right), \dfrac{1}{2}$

$\cdot [0.1 \wedge_F 0.4 + 0.1 \vee_F 0.4], \left\{1 - \dfrac{2}{3}\right\} \cdot [0.4 \wedge_F 0.2] + \left\{\dfrac{2}{3}\right\}$

$\cdot [0.4 \vee_F 0.2]$

$$= \begin{pmatrix} \dfrac{1}{3} \cdot [0.2 + 0.3 - 0.2 \cdot 0.3] + \dfrac{2}{3} \cdot [0.2 \cdot 0.3], \\ \dfrac{1}{2} \cdot [0.1 \cdot 0.4 + 0.4 + 0.1 - 0.1 \cdot 0.4], \\ \dfrac{1}{3} \cdot [0.4 \cdot 0.2] + \dfrac{2}{3} \cdot [0.4 + 0.2 - 0.4 \cdot 0.2] \end{pmatrix}$$

$\approx (0.19, 0.25, 0.37).$

{The degree of contradiction is $\dfrac{2}{3} > 0.5$.}.

$$d_A^N(x, tall) \vee_p d_B^N(x, tall) = \begin{pmatrix} 0 \\ 0.8, & 0.3, & 0.1 \end{pmatrix} \vee_p \begin{pmatrix} 0 \\ 0.7, & 0.1, & 0.6 \end{pmatrix}$$

$$= \left(0.8 \vee_p 0.7, 0.3 \wedge_p 0.1, 0.1 \wedge_p 0.6\right)$$

$$= \begin{pmatrix} 0.8 \vee_F 0.7, \\ \frac{1}{2}(0.3 \wedge_F 0.1 + 0.3 \vee_F 0.1), \\ 0.1 \wedge_F 0.6 \end{pmatrix}$$

$$= \begin{pmatrix} 0.8 + 0.7 - 0.8 \cdot 0.7, \\ \frac{1}{2}(0.3 \cdot 0.1 + 0.3 + 0.1 - 0.3 \cdot 0.1), \\ 0.1 \cdot 0.6 \end{pmatrix}$$

$$= (0.94, 0.20, 0.06).$$

$$d_A^N(x, medium) \vee_p d_B^N(x, medium)$$

$$= \begin{pmatrix} \frac{1}{2} \\ 0.6, & 0.2, & 0.3 \end{pmatrix} \vee_p \begin{pmatrix} \frac{1}{2} \\ 0.5, & 0.1, & 0.3 \end{pmatrix}$$

$$= \left(0.6 \vee_p 0.5, 0.2 \vee_p 0.1, 0.3 \vee_p 0.3\right)$$

$$= \left(\left\{1 - \frac{1}{2}\right\} \cdot [0.6 \vee_p 0.5] + \left\{\frac{1}{2}\right\} \cdot [0.6 \wedge_F 0.5]\right),$$

$$\frac{1}{2} \cdot [0.2 \wedge_F 0.1 + 0.2 \vee_p 0.1],$$

$$\left\{1 - \frac{1}{2}\right\} \cdot [0.3 \wedge_F 0.3] + \left\{\frac{1}{2}\right\} \cdot [0.3 \vee_p 0.3]$$

$$= \left(\frac{1}{2} \cdot [0.6 + 0.5 - 0.6 \cdot 0.5] + \frac{1}{2} \cdot [0.6 \cdot 0.5], \frac{1}{2}\right.$$

$$\cdot [0.2 \cdot 0.1 + 0.2 + 0.1 - 0.2 \cdot 0.1], \frac{1}{2} \cdot [0.3 \cdot 0.3] + \frac{1}{2}$$

$$\left. \cdot [0.3 + 0.3 - 0.3 \cdot 0.3]\right) = (0.55, 0.15, 0.30).$$

Conclusion & Future Research

As generalization of dialectics and neutrosophy, plithogeny will find more use in blending diverse philosophical, ideological, religious, political and social ideas.

After the extension of fuzzy set, intuitionistic fuzzy set, and neutrosophic set to the plithogenic set; the extension of classical logic,

fuzzy logic, intuitionistic fuzzy logic and neutrosophic logic to plithogenic logic; and the extension of classical probability, imprecise probability, and neutrosophic probability to plithogenic probability [12] – more applications of the plithogenic set/logic/probability/statistics in various fields should follow.

The classes of plithogenic implication operators and their corresponding sets of plithogenic rules are to be constructed in this direction.

Also, exploration of non-linear combinations of tnorm and tconorm, or of other norms and conorms, in constructing of more sophisticated plithogenic set, logic and probabilistic aggregation operators, for a better modeling of real life applications.

References

[1] L.A. Zadeh, *Fuzzy sets*, Inform. and Control 8 (1965), 338-353.

[2] K. Atanassov, *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems 20 (1986), 87-96.

[3] K. Atanassov, *Two variants of intuitionistic fuzzy propositional calculus*, Preprint IM-MFAIS-5-88, 1988.

[4] Florentin Smarandache, *Neutrosophic Overset, Neutrosophic Underset, and Neutrosophic Offset. Similarly for Neutrosophic Over-/Under-/Off- Logic, Probability, and Statistics*, 168 p., Pons Editions, Bruxelles, Belgique, 2016;

[5] Florentin Smarandache, *n-Valued Refined Neutrosophic Logic and Its Applications in Physics*, Progress in Physics, 143-146, Vol. 4, 2013;
https://arxiv.org/ftp/arxiv/papers/1407/1407.1041.pdf

[6] Florentin Smarandache, *Subtraction and Division of Neutrosophic Numbers*, Critical Review, Creighton University, The Society for Mathematics of Uncertainty (SMU), Vol. XIII, 2016, pp. 103-110.

[7] S. Broumi, S. & F. Smarandache, *Cosine similarity measure of interval neutrosophic sets*. Neutrosophic Sets and Systems, 5, 15-20, 2014.

[8] J. Ye, (2014). *Improved correlation coefficients of single valued neutrosophic sets and interval neutrosophic sets for multiple*

attribute decision making. Journal of Intelligent & Fuzzy Systems, 27, 2453-2462, 2014.

[9] F. Smarandache, *Neutrosophy, A New Branch of Philosophy*, <Multiple Valued Logic / An International Journal>, USA, ISSN 1023-6627, Vol. 8, No. 3, pp. 297-384, 2002.

[10] F. Smarandache, *A Unifying Field in Logics: Neutrosophic Logic*, <Multiple Valued Logic / An International Journal>, USA, ISSN 1023-6627, Vol. 8, No. 3, pp. 385-438, 2002. The whole issue of this journal is dedicated to Neutrosophy and Neutrosophic Logic.

[11] F. Smarandache, *Definitions Derived from Neutrosophics*, <Multiple Valued Logic / An International Journal>, USA, ISSN 1023-6627, Vol. 8, No. 5-6, pp. 591-604, 2002.

[12] F. Smarandache, *Plithogeny, Plithogenic Set, Logic, Probability, and Statistics*, Pons, Brussels, Belgium, 2017.

5.2 Extension of Soft Set to Hypersoft Set, and then to Plithogenic Hypersoft Set

Abstract

In this paper, we generalize the soft set to the hypersoft set by transforming the function F into a multi-attribute function. Then we introduce the hybrids of Crisp, Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Plithogenic Hypersoft Set.

5.2.1 Introduction

We generalize the soft set to the **hypersoft set** by transforming the function F into a **multi-argument function**.

Then we make the distinction between the types of Universes of Discourse: **crisp**, **fuzzy**, **intuitionistic fuzzy**, **neutrosophic**, and respectively **plithogenic**.

Similarly, we show that a hypersoft set can be **crisp**, **fuzzy**, **intuitionistic fuzzy**, **neutrosophic**, or **plithogenic**.

A detailed numerical example is presented for all types.

5.2.2 Definition of Soft Set [1]

Let $\mathcal{U}$ be a universe of discourse, $\mathcal{P}(\mathcal{U})$ the power set of $\mathcal{U}$, and A a set of attributes. Then, the pair $(F, \mathcal{U})$, where

$$F: A \longrightarrow \mathcal{P}(\mathcal{U}) \tag{5.2.1}$$

is called a **Soft Set** over $\mathcal{U}$.

5.2.3 Definition of Hypersoft Set

Let $\mathcal{U}$ be a universe of discourse, $\mathcal{P}(\mathcal{U})$ the power set of $\mathcal{U}$.

Let $a_1, a_2, \ldots, a_n$, for $n \geq 1$, be n distinct attributes, whose corresponding attribute values are respectively the sets $A_1, A_2, \ldots, A_n$, with $A_i \cap A_j = \emptyset$, for $i \neq j$, and $i, j \in \{1, 2, \ldots, n\}$.

Then the pair $(F, A_1 \times A_2 \times \ldots \times A_n)$, where:

$$F: A_1 \times A_2 \times \ldots \times A_n \longrightarrow \mathcal{P}(\mathcal{U}) \qquad (5.2.2)$$

is called a **Hypersoft Set** over $\mathcal{U}$.

5.2.4 Particular case

For $n = 2$, we obtain the Γ–Soft Set [2].

5.2.5 Types of Universes of Discourses

5.2.5.1 A **Universe of Discourse** $\mathcal{U}_C$ is called **Crisp** if $\forall x \in \mathcal{U}_C$, x belongs 100% to $\mathcal{U}_C$, or x's membership (T_x) with respect to $\mathcal{U}_C$ is 1. Let's denote it $x(1)$.

5.2.5.2 A **Universe of Discourse** $\mathcal{U}_F$ is called **Fuzzy** if $\forall x \in \mathcal{U}_c$, x partially belongs to $\mathcal{U}_F$, or $T_x \subseteq [0, 1]$, where T_x may be a subset, an interval, a hesitant set, a single-value, etc. Let's denote it by $x(T_x)$.

5.2.5.3 A **Universe of Discourse** $\mathcal{U}_{IF}$ is called **Intuitionistic Fuzzy** if $\forall x \in \mathcal{U}_{IF}$, x partially belongs (T_x) and partially doesn't belong (F_x) to $\mathcal{U}_{IF}$, or $T_x, F_x \subseteq [0, 1]$, where T_x and F_x may be subsets, intervals, hesitant sets, single-values, etc. Let's denote it by $x(T_x, F_x)$.

5.2.5.4 A **Universe of Discourse** $\mathcal{U}_N$ is called **Neutrosophic** if $\forall x \in \mathcal{U}_N$, x partially belongs (T_x), partially its membership is indeterminate (I_x), and partially it doesn't belong (F_x) to $\mathcal{U}_N$, where $T_x, I_x, F_x \subseteq [0, 1]$, may be subsets, intervals, hesitant sets, single-values, etc. Let's denote it by $x(T_x, I_x, F_x)$.

5.2.5.5 A **Universe of Discourse** $\mathcal{U}_P$ **over a set V of attributes' values**, where $V = \{v_1, v_2, \ldots, v_n\}, n \geq 1$, is called Plithogenic, if $\forall x \in \mathcal{U}_P$, x belongs to $\mathcal{U}_P$ in the degree $d_x^0(v_i)$ with respect to the attribute value v_i, for all $i \in \{1, 2, \ldots, n\}$. Since the degree of membership $d_x^0(v_i)$

may be crisp, fuzzy, intuitionistic fuzzy, or neutrosophic, the Plithogenic Universe of Discourse can be Crisp, Fuzzy, Intuitionistic Fuzzy, or respectively Neutrosophic.

Consequently, a Hypersoft Set over a Crisp / Fuzzy / Intuitionistic Fuzzy / Neutrosophic / or Plithogenic Universe of Discourse is respectively called **Crisp / Fuzzy / Intuitionistic Fuzzy / Neutrosophic / or Plithogenic Hypersoft Set**.

5.2.6 Numerical Example

Let $\mathcal{U} = \{x_1, x_2, x_3, x_4\}$ and a set $\mathcal{M} = \{x_1, x_3\} \subset \mathcal{U}$.

Let the attributes be: a_1 = size, a_1 = color, a_1 = gender, a_1 = nationality, and their attributes' values respectively:

Size = A_1 ={small, medium, tall},

Color = A_2 ={white, yellow, red, black},

Gender = A_3 ={male, female},

Nationality = A_4 ={American, French, Spanish, Italian, Chinese}.

Let the function be:

$$F: A_1 \times A_2 \times A_3 \times A_4 \longrightarrow \mathcal{P}(\mathcal{U}). \qquad (5.2.3)$$

Let's assume:

$F(\{\text{tall}, \text{white}, \text{female}, \text{Italian}\}) = \{x_1, x_3\}$.

With respect to the set $\mathcal{M}$, one has:

5.2.6.1 Crisp Hypersoft Set

$$F(\{\text{tall}, \text{white}, \text{female}, \text{Italian}\}) = \{x_1(1), x_3(1)\}, \qquad (5.2.4)$$

which means that, with respect to the attributes' values $\{\text{tall}, \text{white}, \text{female}, \text{Italian}\}$ *all together*, x_1 belongs 100% to the set $\mathcal{M}$; similarly x_3.

5.2.6.2 Fuzzy Hypersoft Set

$$F(\{\text{tall}, \text{white}, \text{female}, \text{Italian}\}) = \{x_1(0.6), x_3(0.7)\}, \qquad (5.2.5)$$

which means that, with respect to the attributes' values {tall, white, female, Italian} *all together*, x_1 belongs 60% to the set $\mathcal{M}$; similarly, x_3 belongs 70% to the set $\mathcal{M}$.

5.2.6.3 Intuitionistic Fuzzy Hypersoft Set

$$F(\{\text{tall, white, female, Italian}\}) =$$

$$\{x_1(0.6, 0.1), x_3(0.7, 0.2)\}, \tag{5.2.6}$$

which means that, with respect to the attributes' values {tall, white, female, Italian} *all together*, x_1 belongs 60% and 10% it does not belong to the set $\mathcal{M}$; similarly, x_3 belongs 70% and 20% it does not belong to the set $\mathcal{M}$.

5.2.6.4 Neutrosophic Hypersoft Set

$$F(\{\text{tall, white, female, Italian}\}) =$$

$$= \{x_1(0.6, 0.2, 0.1), x_3(0.7, 0.3, 0.2)\}, \tag{5.2.7}$$

which means that, with respect to the attributes' values {tall, white, female, Italian} all together, x_1 belongs 60% and its indeterminate-belongness is 20% and it doesn't belong 10% to the set $\mathcal{M}$; similarly, x_3 belongs 70% and its indeterminate-belongness is 30% and it doesn't belong 20%.

5.2.6.5 Plithogenic Hypersoft Set

$$F(\{\text{tall, white, female, Italian}\}) =$$

$$= \begin{cases} x_1\left(d_{x_1}^0(\text{tall}), d_{x_1}^0(\text{white}), d_{x_1}^0(\text{female}), d_{x_1}^0(\text{Italian})\right), \\ x_2\left(d_{x_2}^0(\text{tall}), d_{x_2}^0(\text{white}), d_{x_2}^0(\text{female}), d_{x_2}^0(\text{Italian})\right) \end{cases}, \tag{5.2.8}$$

where $d_{x_1}^0(\alpha)$ means the degree of appurtenance of element x_1 to the set $\mathcal{M}$ with respect to the attribute value α; and similarly $d_{x_2}^0(\alpha)$ means the degree of appurtenance of element x_2 to the set $\mathcal{M}$ with respect to the attribute value α; where $\alpha \in \{\text{tall, white, female, Italian}\}$.

Unlike the Crisp / Fuzzy / Intuitionistic Fuzzy / Neutrosophic Hypersoft Sets [where the degree of appurtenance of an element x to the set $\mathcal{M}$ is *with respect to all attribute values **tall, white, female, Italian** together* (as a whole), therefore a degree of appurtenance *with respect to a set of attribute values*], the Plithogenic Hypersoft Set is a **refinement** of Crisp / Fuzzy / Intuitionistic Fuzzy / Neutrosophic Hypersoft Sets [since the degree of appurtenance of an element x to the set $\mathcal{M}$ is with respect to each single attribute value].

But the Plithogenic Hypersoft Set is also **combined** with each of the above, since the degree of degree of appurtenance of an element x to the set $\mathcal{M}$ with respect to each single attribute value may be: crisp, fuzzy, intuitionistic fuzzy, or neutrosophic.

5.2.7 Classification of Plithogenic Hypersoft Sets

5.2.7.1 Plithogenic Crisp Hypersoft Set

It is a plithogenic hypersoft set, such that the degree of appurtenance of an element x to the set $\mathcal{M}$, with respect to each attribute value, is **crisp**:

$d_x^0(\alpha) = 0$ (nonappurtenance), or 1 (appurtenance).

In our example:

$F(\{\text{tall}, \text{white}, \text{female}, \text{Italian}\}) =$

$= \{x_1(1, 1, 1, 1), x_3(1, 1, 1, 1)\}.$ (5.2.9)

5.2.7.2 Plithogenic Fuzzy Hypersoft Set

It is a plithogenic hypersoft set, such that the degree of appurtenance of an element x to the set $\mathcal{M}$, with respect to each attribute value, is **fuzzy**:

$d_x^0(\alpha) \in \mathcal{P}([0, 1])$, power set of $[0, 1]$, where $d_x^0(\cdot)$ may be a subset, an interval, a hesitant set, a single-valued number, etc.

In our example, for a single-valued number:

$F(\{\text{tall}, \text{white}, \text{female}, \text{Italian}\}) =$

$= \{x_1(0.4, 0.7, 0.6, 0.5), x_3(0.8, 0.2, 0.7, 0.7)\}.$ (5.2.10)

5.2.7.3 Plithogenic Intuitionistic Fuzzy Hypersoft Set

It is a plithogenic hypersoft set, such that the degree of appurtenance of an element x to the set $\mathcal{M}$, with respect to each attribute value, is **intuitionistic fuzzy**:

$d_x^0(\alpha) \in \mathcal{P}([0,1]^2)$, power set of $[0,1]^2$, where similarly $d_x^0(\alpha)$ may be: a Cartesian product of subsets, of intervals, of hesitant sets, of single-valued numbers, etc.

In our example, for single-valued numbers:

$F(\{\text{tall}, \text{white}, \text{female}, \text{Italian}\}) =$

$$= \begin{Bmatrix} x_1(0.4,0.3)(0.7,0.2)(0.6,0.0)(0.5,0.1), \\ x_3(0.8,0.1)(0.2,0.5)(0.7,0.0)(0.7,0.4) \end{Bmatrix}. \tag{5.2.11}$$

5.2.7.4 Plithogenic Neutrosophic Hypersoft Set

It is a plithogenic hypersoft set, such that the degree of appurtenance of an element x to the set $\mathcal{M}$, with respect to each attribute value, is **neutrosophic**:

$d_x^0(\alpha) \in \mathcal{P}([0,1]^3)$, power set of $[0,1]^3$, where $d_x^0(\alpha)$ may be: a triple Cartesian product of subsets, of intervals, of hesitant sets, of single-valued numbers, etc.

In our example, for single-valued numbers:

$F(\{\text{tall}, \text{white}, \text{female}, \text{Italian}\}) =$

$$\begin{Bmatrix} x_1[(0.4,0.1,0.3)(0.7,0.0,0.2)(0.6,0.3,0.0)(0.5,0.2,0.1)], \\ x_3[(0.8,0.1,0.1)(0.2,0.4,0.5)(0.7,0.1,0.0)(0.7,0.5,0.4)] \end{Bmatrix} \tag{5.2.12}$$

5.2.8 Future Research

For all types of plithogenic hypersoft sets, the aggregation operators (union, intersection, complement, inclusion, equality) have to be defined and their properties found.

Applications in various engineering, technical, medical, social science, administrative, decision making and so on, fields of knowledge of these types of plithogenic hypersoft sets should be investigated.

References

[1] D. Molodtsov: *Soft Set Theory First Results*. *Computer Math. Applic.* 37, 1999, pp. 19-31.

[2] T. Srinivasa Rao, B. Srinivasa Kumar, S. Hanumanth Rao: *A Study on Neutrosophic Soft Set in Decision Making Problem*. *Journal of Engineering and Applied Sciences*, Asian Research Publishing Network (ARPN), vol. 13, no. 7, April 2018.

[3] Florentin Smarandache: *Plithogeny, Plithogenic Set, Logic, Probability, and Statistics*. Brussels: Pons Editions, 2017.

[4] Florentin Smarandache: *Plithogenic Set, an Extension of Crisp, Fuzzy, Intuitionistic Fuzzy, and Neutrosophic Sets – Revisited*. *Neutrosophic Sets and Systems*, vol. 21, 2018, pp. 153-166. https://doi.org/10.5281/zenodo.1408740.

CHAPTER 6

**Introduction to NeutroAlgebraic Structures
and AntiAlgebraic Structures (revisited)**

Abstract

In all classical algebraic structures, the *Laws of Compositions* on a given set are well-defined. But this is a restrictive case, because there are many more situations in science and in any domain of knowledge when a law of composition defined on a set may be only partially-defined (or partially true) and partially-undefined (or partially false), that we call *NeutroDefined*, or totally undefined (totally false) that we call *AntiDefined*.

Again, in all classical algebraic structures, the *Axioms* (Associativity, Commutativity, etc.) defined on a set are totally true, but it is again a restrictive case, because similarly there are numerous situations in science and in any domain of knowledge when an Axiom defined on a set may be only partially-true (and partially-false), that we call *NeutroAxiom*, or totally false that we call *AntiAxiom*.

Therefore we open for the first time in 2019 new fields of research called *NeutroStructures* and *AntiStructures* respectively.

Keywords

6.1. Introduction

For the necessity to more accurately reflect our reality, Smarandache [1] introduced for the first time in 2019 the NeutroDefined and AntiDefined Laws, as well as the NeutroAxiom and AntiAxiom, inspired from Neutrosophy ([2], 1995), giving birth to new fields of research called NeutroStructures and AntiStructures.

Let's consider a given classical algebraic Axiom. We define for the first time the neutrosophic triplet corresponding to this Axiom, which is the following: *(Axiom, NeutroAxiom, AntiAxiom)*; while the classical Axiom is 100% or totally true, the NeutroAxiom is partially true and partially false (the degrees of truth and falsehood are both > 0), while the AntiAxiom is 100% or totally false.

For the classical algebraic structures, on a non-empty set endowed with well-defined binary laws, we have properties (axioms) such as: associativity & non-associativity, commutativity & non-commutativity, distributivity & non-distributivity; the set may contain a neutral element with respect to a given law, or may not; and so on; each set element may have an inverse, or some set elements may not have an inverse; and so on.

Consequently, we construct for the first time the neutrosophic triplet corresponding to the Algebraic Structures, which is this: *(Algebraic Structure, NeutroAlgebraic Structure, AntiAlbegraic Structure)*.

Therefore, we now introduce for the first time the *NeutroAlgebraic Structures* & the *AntiAlgebraic Structures*.

A (classical) <u>Algebraic Structure</u> is an algebraic structure dealing only with (classical) Axioms (which are totally true).

Then a <u>NeutroAlgebraic Structure</u> is an algebraic structure that has at least one NeutroAxiom, and no AntiAxioms.

While an <u>AntiAlgebraic Structure</u> is an algebraic structure that has at least one AntiAxiom.

These definitions can straightforwardly be extended from Axiom/NeutroAxiom/AntiAxiom

to any

> Property/NeutroProperty/AntiProperty,
>
> Proposition/NeutroProposition/AntiProposition,
>
> Theorem/NeutroTheorem/AntiTheorem,
>
> Theory/NeutroTheory/AntiTheory, etc.

and from Algebraic Structures to other Structures in any field of knowledge.

6.2. Neutrosophy

We recall that in neutrosophy we have for an item *<A>*, its opposite *<antiA>*, and in between them their neutral *<neutA>*.

We denoted by *<nonA>* = *<neutA>* ∪ *<antiA>*, where ∪ means union, and *<nonA>* means what is not *<A>*.

Or *<nonA>* is refined/split into two parts: *<neutA>* and *<antiA>*.

The neutrosophic triplet of *<A>* is:

$(\langle A\rangle, \langle neutA\rangle, \langle antiA\rangle)$, with $\langle neutA\rangle \cup \langle antiA\rangle = \langle nonA\rangle$.

6.3. Definition of Neutrosophic Triplet Axioms

Let $\mathcal{U}$ be a universe of discourse, endowed with some well-defined laws, a non-empty set $\mathcal{S} \subseteq \mathcal{U}$, and an Axiom α, defined on S, using these laws. Then:

1) If all elements of $\mathcal{S}$ verify the axiom α, we have a *Classical Axiom*, or simply we say *Axiom*.

2) If some elements of $\mathcal{S}$ verify the axiom α and others do not, we have a *NeutroAxiom* (which is also called *NeutAxiom*).

3) If no elements of $\mathcal{S}$ verify the axiom α, then we have an *AntiAxiom*.

The Neutrosophic Triplet Axioms are:

(Axiom, NeutroAxiom, AntiAxiom)

with

NeutroAxiom ∪ AntiAxiom = NonAxiom,

and *NeutroAxiom ∩ AntiAxiom = φ* (empty set),

where ∩ means intersection.

Theorem 1

The Axiom is 100% true, the NeutroAxiom is partially true (its truth degree > 0) and partially false (its falsehood degree > 0), and the AntiAxiom is 100% false.

Proof is obvious.

Theorem 2

Let

$$d : \{Axiom, NeutroAxiom, AntiAxiom\} \rightarrow [0, 1]$$

represent the **degree of negation function**.

The NeutroAxiom represents a degree of partial negation { $d \in (0, 1)$ } of the Axiom, while the AntiAxiom represents a degree of total negation { $d = 1$ } of the Axiom.

Proof is also evident.

6.4. Neutrosophic Representation

We have:

⟨*A*⟩ = *Axiom;*

⟨*neutA*⟩ = *NeutroAxiom (or NeutAxiom);*

⟨*antiA*⟩ = *AntiAxiom;*

and ⟨*nonA*⟩ = *NonAxiom.*

Similarly as in *Neutrosophy, NonAxiom* is refined/split into two parts: *NeutroAxiom* and *AntiAxiom*.

6.5. Application of NeutroLaws in Soft Science

In *soft sciences* the laws are interpreted and re-interpreted; in social and political legislation the laws are flexible; the same law may be true from a point of view, and false from another point of view. Thus the law is partially true and partially false (it is a neutrosophic law).

For example, "gun control". There are people supporting it because of too many crimes and violence (and they are right), and people that oppose it because they want to be able to defend themselves and their houses (and they are right too).

We see two opposite propositions, both of them true, but from different points of view (from different criteria/parameters; plithogenic logic may better be used herein). How to solve this? Going to the middle, in between opposites (as in neutrosophy): <u>allow</u> military, police, security, registered hunters to bear arms; <u>prohibit</u> mentally ill, sociopaths, criminals, violent people from bearing arms; and <u>background check</u> on everybody that buys arms, etc. Definition of Classical Associativity

Let $\mathcal{U}$ be a universe of discourse, and a non-empty set $\mathcal{S} \subseteq \mathcal{U}$, endowed with a well-defined binary law $*$. The law $*$ is associative on the set $\mathcal{S}$, iff $\forall\, a, b, c\, \in \mathcal{S}, a * (b * c) = (a * b) * c$.

6.6. Definition of Classical NonAssociativity

Let $\mathcal{U}$ be a universe of discourse, and a non-empty set $\mathcal{S} \subseteq \mathcal{U}$, endowed with a well-defined binary law $*$. The law $*$ is non-associative on the set $\mathcal{S}$, iff $\exists a, b, c\, \in \mathcal{S}$, such that $a * (b * c) \neq (a * b) * c$.

So, it is sufficient to get a single triplet a, b, c (where a, b, c may even be all three equal, or only two of them equal) that doesn't satisfy the associativity axiom.

Yet, there may also exist some triplet $d, e, f \in \mathcal{S}$ that satisfies the associativity axiom: $d * (e * f) = (d * e) * f$.

The classical definition of NonAssociativity does not make a distinction between a set $(\mathcal{S}_1, *)$ whose all triplets $a, b, c \in \mathcal{S}_1$ verify the non-associativity inequality, and a set $(\mathcal{S}_2, *)$ whose some triplets verify the non-associativity inequality, while others don't.

6.7. NeutroAssociativity & AntiAssociativity

If $\langle A \rangle$ = (classical) Associativity, then $\langle nonA \rangle$ = (classical) NonAssociativity.

But we refine/split $\langle nonA \rangle$ into two parts, as above:

$\langle neutA \rangle$ = *NeutroAssociativity;*

$\langle antiA \rangle$ = *AntiAssociativity.*

Therefore,

NonAssociativity = NeutroAssociativity ∪ AntiAssociativity.

The Associativity's neutrosophic triplet is:

<Associativity, NeutroAssociativity, AntiAssociativity>.

6.8. Definition of NeutroAssociativity

Let $\mathcal{U}$ be a universe of discourse, endowed with a well-defined binary law $*$, and a non-empty set $\mathcal{S} \subseteq \mathcal{U}$.

The set $(\mathcal{S}, *)$ is NeutroAssociative if and only if:

there exists at least one triplet $a_1, b_1, c_1 \in \mathcal{S}$ such that:

$a_1 * (b_1 * c_1) = (a_1 * b_1) * c_1$;

and there exists at least one triplet $a_2, b_2, c_2 \in \mathcal{S}$ such that:

$a_2 * (b_2 * c_2) \neq (a_2 * b_2) * c_2$.

Therefore, some triplets verify the associativity axiom, and others do not.

6.9. Definition of AntiAssociativity

Let $\mathcal{U}$ be a universe of discourse, endowed with a well-defined binary law $*$, and a non-empty set $S \subseteq \mathcal{U}$.

The set $(S, *)$ is AntiAssociative if and only if:

for any triplet $a, b, c \in S$ one has $a * (b * c) \neq (a * b) * c$.

Therefore, none of the triplets verify the associativity axiom.

6.10. Example of Associativity

Let $N = \{0, 1, 2, ..., \infty\}$, the set of natural numbers, be the universe of discourse, and the set $S = \{0, 1, 2, ..., 9\} \subset N$, also the binary law $*$ be the classical addition modulo 10 defined on N.

Clearly the law * is well-defined on S, and associative since:

$a + (b + c) = (a + b) + c \pmod{10}$, for all $a, b, c \in S$.

The degree of negation is 0%.

6.11. Example of NeutroAssociativity

$S = \{0, 1, 2, ..., 9\}$, and the well-defined binary law $*$ constructed as below:

$a * b = 2a + b \pmod{10}$.

Let's check the associativity:

$a * (b * c) = 2a + (b * c) = 2a + 2b + c$

$(a * b) * c = 2(a * b) + c = 2(2a + b) + c = 4a + 2b + c$

The triplets that verify the associativity result from the below equality:

$2a + 2b + c = 4a + 2b + c$

or $2a = 4a \pmod{10}$

or $0 = 2a \pmod{10}$, whence $a \in \{0, 5\}$.

Hence, two general triplets of the form:

$\{(0, b, c), (5, b, c), \text{where } b, c \in S\}$

verify the associativity.

The degree of associativity is $\frac{2}{10} = 20\%$, corresponding to the two numbers $\{0, 5\}$ out of ten.

While the other general triplet:

$\{(a, b, c), \text{where } a \in S \setminus \{0, 5\}, \text{while } b, c \in S\}$

do not verify the associativity.

The degree of negation of associativity is $\frac{8}{10} = 80\%$.

6.12. Example of AntiAssociativity

$S = \{a, b\}$, and the binary law $*$ well-defined as in the below Cayley Table:

$*$	a	b
a	b	b
b	a	a

Theorem 1.

For any $x, y, z \in S$, $x * (y * z) \neq (x * y) * z$.

Proof.

We have $2^3 = 8$ possible triplets on S:

1) (a, a, a)

$a * (a * a) = a * b = b$

while $(a * a) * a = b * a = a \neq b$.

2) (a, a, b)

$a * (a * b) = a * b = b$

$(a * a) * b = b * b = a \neq b$.

3) (a, b, a)

$a * (b * a) = a * a = b$

$(a * b) * a = b * a = a \neq b.$

4) (b, a, a)

$b * (a * a) = b * b = a$

$(b * a) * a = a * a = b \neq a.$

5) (a, b, b)

$a * (b * b) = a * a = b$

$(a * b) * b = b * b = a \neq b.$

6) (b, a, b)

$b * (a * b) = b * b = a$

$(b * a) * b = a * b = b \neq a.$

7) (b, b, a)

$b * (b * a) = b * a = a$

$(b * b) * a = a * a = b \neq a.$

8) (b, b, b)

$b * (b * b) = b * a = a$

$(b * b) * b = a * b = b \neq a.$

Therefore, there is no possible triplet on $\mathcal{S}$ to satisfy the associativity. Whence the law is AntiAssociative. The degree of negation of associativity is $\frac{8}{8} = 100\%$.

6.13. Definition of Classical Commutativity

Let $\mathcal{U}$ be a universe of discourse endowed with a well-defined binary law $*$, and a non-empty set $\mathcal{S} \subseteq \mathcal{U}$. The law $*$ is Commutative on the set $\mathcal{S}$, iff $\forall\ a, b\ \in \mathcal{S}, a * b = b * a.$

6.14. Definition of Classical NonCommutativity

Let $\mathcal{U}$ be a universe of discourse, endowed with a well-defined binary law $*$, and a non-empty set $\mathcal{S} \subseteq \mathcal{U}$. The law $*$ is NonCommutative on the set $\mathcal{S}$, iff $\exists a, b \in \mathcal{S}$, such that $a * b \neq b * a$.

So, it is sufficient to get a single duplet $a, b \in \mathcal{S}$ that doesn't satisfy the commutativity axiom.

However, there may exist some duplet $c, d \in \mathcal{S}$ that satisfies the commutativity axiom: $c * d = d * c$.

The classical definition of NonCommutativity does not make a distinction between a set $(\mathcal{S}_1, *)$ whose all duplets $a, b \in \mathcal{S}_1$ verify the NonCommutativity inequality, and a set $(\mathcal{S}_2, *)$ whose some duplets verify the NonCommutativity inequality, while others don't.

That's why we refine/split the NonCommutativity into NeutroCommutativity and AntiCommutativity**.**

6.15. NeutroCommutativity & AntiCommutativity

Similarly to Associativity we do for the Commutativity:

If $\langle A \rangle$ = (classical) *Commutativity*, then $\langle nonA \rangle$ = (classical) *NonCommutativity*.

But we refine/split $\langle nonA \rangle$ into two parts, as above:

$\langle neutA \rangle$ = *NeutroCommutativity;*

$\langle antiA \rangle$ = *AntiCommutativity.*

Therefore,

NonCommutativity =

= NeutroCommutativity $\cup$ *AntiCommutativity.*

The Commutativity's neutrosophic triplet is:

<Commutativity, NeutroCommutativity, AntiCommutativity>.

In the same way, Commutativity means all elements of the set commute with respect to a given binary law, NeutroCommutativity means that some elements commute while others do not, while AntiCommutativity means that no elements commute.

6.16. Example of NeutroCommutativity

$S = \{a, b, c\}$, and the well-defined binary law $*$.

$*$	a	b	c
a	b	c	c
b	c	b	a
c	b	b	c

$a * b = b * a = c$ (commutative);

$\begin{cases} a * c = c \\ c * a = b \neq c \end{cases}$ (not commutative);

$\begin{cases} b * c = a \\ c * b = b \neq a \end{cases}$ (not commutative).

We conclude that $(S,*)$ is $\frac{1 \text{ pair}}{3 \text{ pairs}} \approx 33\%$ commutative, and $\frac{2 \text{ pair}}{3 \text{ pairs}} \approx 67\%$ not commutative.

Therefore, the degree of negation of the commutativity of $(S,*)$ is 67%.

6.17. Example of AntiCommutativity

$S = \{a, b\}$, and the below binary well-defined law $*$.

$*$	a	b
a	b	b
b	a	a

where $a * b = b$, $b * a = a \neq b$ (not commutative)

Other pair of different element does not exist, since we cannot take $a * a$ nor $b * b$. The degree of negation of commutativity of this $(S,*)$ is 100%.

6.18. Definition of Classical Unit Element

Let $\mathcal{U}$ be a universe of discourse endowed with a well-defined binary law $*$ and a non-empty set $S \subseteq \mathcal{U}$.

The set S has a classical unit element $e \in S$, iff e is unique, and for any $x \in S$ one has $x * e = e * x = x$.

6.19. Partially Negating the Definition of Classical Unit Element

It occurs when at least one of the below statements occurs:

1) There exists at least one element $a \in S$ that has no unit element.

2) There exists at least one element $b \in S$ that has at least two distinct unit elements $e_1, e_2 \in S$, $e_1 \neq e_2$, such that:

$b * e_1 = e_1 * b = b,$

$b * e_2 = e_2 * b = b.$

3) There exists at least two different elements $c, d \in S$, $c \neq d$, such that they have different unit elements $e_c, e_d \in S$, $e_c \neq e_d$, with $c * e_c = e_c * c = c$, and $d * e_d = e_d * d = d$.

6.20. Totally Negating the Definition of Classical Unit Element

The set $(S, *)$ has *AntiUnitElements*, if:

1) Each element $x \in S$ has either no unit element, or two or more unit elements (unicity of unit element is negated);

2) If some elements $x \in S$ have only one unit element each, then these unit elements are different two by two.

6.21. Definition of NeutroUnitElements

The set $(S, *)$ has *NeutroUnit Elements*, if:

1) [Degree of Truth] There exist at least an element that has a single unit-element.

2) [Degree of Falsehood] There exist at least one element that either has no unit-element, or has two or more unit-elements.

6.22. Definition of AntiUnit Elements

The set $(S, *)$ has *AntiUnit Elements*, if:

Each element $x \in S$ has either no unit-element, or two or more distinct unit-elements.

6.23. Example of NeutroUnit Elements

$S = \{a, b, c\}$, and the well-defined binary law $*$:

$*$	a	b	c
a	b	b	a
b	b	b	a
c	a	b	c

Since,

$a * c = c * a = a$

$c * c = c$

the common unit-element of a and c is c (two distinct elements a $\neq c$ have the same unit element c).

From

$b * a = a * b = b$

$b * b = b$

we see that the element b has two distinct unit-elements a and b.

Since only one element b does not verify the classical unit axiom (i.e. to have a unique unit), out of 3 elements, the degree of negation of unit element axiom is $\frac{1}{3} \approx 33\%$, while $\frac{2}{3} \approx 67\%$ is the degree of truth (validation) of the unit element axiom.

6.24. Example of AntiUnit Elements

$S = \{a, b, c\}$, endowed with the well-defined binary law $*$ as follows:

*	a	b	c
a	a	a	a
b	a	c	b
c	a	c	b

Element a has 3 unit elements: a, b, c, because:

$a * a = a$

$a * b = b * a = a$

and $a * c = c * a = a.$

Element b has no unit element, since:

$b * a = a \neq b$

$b * b = c \neq b$

and $b * c = b$, but $c * b \neq b.$

Element c has no unit element, since:

$c * a = a \neq c$

$c * b = c$, but $b * c = b \neq c,$

and $c * c = b \neq c.$

The degree of negation of the unit element axiom is $\frac{3}{3} = 100\%$.

6.25. Definition of Classical Inverse Element

Let $\mathcal{U}$ be a universe of discourse endowed with a well-defined binary law $*$ and a non $-$ empty set $S \subseteq \mathcal{U}$.

Let $e \in S$ be the classical unit element, which is unique.

For any element $x \in S$, there exists a unique element, named the inverse of x, denoted by x^{-1}, such that:

$x * x^{-1} = x^{-1} * x = e.$

6.26. Partially Negating the Definition of Classical Inverse Element

It occurs when at least one statement from below occurs:

1) There exist at least one element $a \in S$ that has no inverse with respect to no ad-hoc unit-elements;

or

2) There exist at least one element $b \in S$ that has two or more distinct inverses with respect to some ad-hoc unit-elements.

6.27. Totally Negating the Definition of Classical Inverse Element

Each element has either no inverse, or two or more distinct inverses with respect to some ad-hoc unit-elements respectively.

6.28. Definition of NeutroInverse Elements

The set $(S, *)$ has NeutroInverse Elements if:

1) [Degree of Truth] There exist at least an element that has an inverse with respect to some ad-hoc unit-element.

2) [Degree of Falsehood] There exists at least one element that does not have any inverse with respect to no ad-hoc unit-element, or has at least two or more distinct inverses with respect to some ad-hoc unit-elements.

6.29. Definition of AntiInverse Elements

The set $(S, *)$ has AntiInverse Elements, if: each element has either no inverse with respect to no ad-hoc unit-element, or two or more distinct inverses with respect to some ad-hoc unit-elements.

6.30. Example of NeutroInverse Elements

$S = \{a, b, c\}$, endowed with the binary well-defined law * as below:

*	a	b	c
a	a	b	c
b	b	a	a
c	b	b	b

Because $a * a = a$, hence its ad-hoc unit/neutral element $neut(a) = a$ and correspondingly its inverse element is $inv(a) = a$.

Because $b * a = a * b = b$, hence its ad-hoc inverse/neutral element $neut(b) = a$;

from $b * b = a$, we get $inv(b) = b$.

No $neut(c)$, hence no $inv(c)$.

Hence a and b have ad-hoc inverses, but c doesn't.

6.31. Example of AntiInverse Elements

Similarly, $S = \{a, b, c\}$, endowed with the binary well-defined law *
as below:

*	a	b	c
a	b	b	c
b	a	a	a
c	c	a	a

There is no *neut(a)* and no *neut(b)*, hence: no *inv(a)* and *no inv(b)*.

$c * a = a * c = c$, hence: $neut(c) = a$.

$c * b = b * c = a$, hence: $inv(c) = b$;

$c * c = c * c = a$, hence: $inv(c) = c$; whence we get two inverses of c.

6.32. Cases When Partial Negation (NeutroAxiom) Does Not Exist

Let's consider the classical geometric *Axiom*:

On a plane, through a point exterior to a given line it's possible to draw a single parallel to that line.

The *total negation* is the following *AntiAxiom*:

On a plane, through a point exterior to a given line it's possible to draw either no parallel, or two or more parallels to that line.

The *NeutroAxiom* does not exist since it is not possible to partially deny this classical axiom.

6.33. Connections between the neutrosophic triplet *(Axiom, NeutroAxiom, AntiAxiom)* and the *S-denying an Axiom*

The *S-denying of an Axiom* was first defined by Smarandache [3, 4] in 1969 when he constructed *hybrid geometries* (or *S-geometries*) [5 – 18].

6.34. Definition of S-denying an Axiom

An *Axiom* is said *S-denied* [3, 4] if in the same space the axiom behaves differently (i.e., validated and invalided; or only invalidated but in at least two distinct ways).

Therefore, we say that an axiom is partially negated (or there is a degree of negation of an axiom): http://fs.unm.edu/Geometries.htm.

6.35. Definition of S-geometries

A geometry is called *S-geometry* [5] if it has at least one S-denied axiom.

Therefore, the Euclidean, Lobachevsky-Bolyai-Gauss, and Riemannian geometries were united altogether for the first time, into the same space, by some S-geometries. These S-geometries could be partially Euclidean and partially Non-Euclidean, or only Non-Euclidean but in multiple ways.

The most important contribution of the S-geometries was the introduction of the *degree of negation of an axiom* (and more general the degree of negation of any theorem, lemma, scientific or humanistic proposition, theory, etc.).

Many geometries, such as pseudo-manifold geometries, Finsler geometry, combinatorial Finsler geometries, Riemann geometry, combinatorial Riemannian geometries, Weyl geometry, Kahler geometry are particular cases of S-geometries. (Linfan Mao)

6.36. Connection between S-denying an Axiom and NeutroAxiom / AntiAxiom

"Validated and invalidated" Axiom is equivalent to NeutroAxiom. While "only invalidated but in at least two distinct ways" Axiom is part of the AntiAxiom (depending on the application).

"Partially negated" (or $0 < d < 1$, where d is the degree of negation) is referred to NeutroAxiom. While "there is a degree of negation of an axiom" is referred to both NeutroAxiom (when $0 < d < 1$) and AntiAxiom (when $d = 1$).

6.37. Connection between NeutroAxiom and MultiSpace

In any domain of knowledge, a *S-multispace with its multistructure* is a finite or infinite (countable or uncountable) union of many spaces that have various structures (Smarandache, 1969, [19]). The multi-spaces with their multi-structures [20, 21] may be non-disjoint. The multispace with multistructure form together a *Theory of Everything*. It can be used, for example, in the Unified Field Theory that tries to unite the gravitational, electromagnetic, weak, and strong interactions in physics.

Therefore, a NeutroAxiom splits the set S, which it is defined upon, into two subspaces: one where the Axiom is true and another where the Axiom is false. Whence S becomes a BiSpace with BiStructure (which is a particular case of MultiSpace with MultiStructure).

6.38. (Classical) WellDefined Binary Law

Let $\mathcal{U}$ be a universe of discourse, a non-empty set $\mathcal{S} \subseteq \mathcal{U}$, and a binary law $*$ defined on U. For any $x, y \in \mathcal{S}$, one has $x * y \in \mathcal{S}$.

6.39. NeutroDefined Binary Law

There exist at least two elements (that could be equal) $a, b \in \mathcal{S}$ such that $a * b \in \mathcal{S}$. And there exist at least other two elements (that could be equal too) $c, d \in \mathcal{S}$ such that $c, d \notin \mathcal{S}$.

6.40. Example of NeutroDefined Binary Law

Let $U = \{a, b, c\}$ be a universe of discourse, and a subset $S = \{a, b\}$, endowed with the below NeutroDefined Binary Law $*$:

$*$	a	b
a	b	b
b	a	c

We see that: $a * b = b \in S, b * a = a \in S,$ but $b * b = c \notin S.$

6.41. AntiDefined Binary Law

For any $x, y \in S$ one has $x * y \notin S$.

6.42. Example of AntiDefined Binary Law

Let $U = \{a, b, c, d\}$ a universe of discourse, and a subset $S = \{a, b\}$, and the below binary well-defined law $*$.

$*$	a	b
a	c	d
b	d	c

where all combinations between a and b using the law $*$ give as output c or d who do not belong to S.

6.43. Theorem of the Degenerate Case

If a set is endowed with AntiDefined Laws, all its algebraic structures based on them will be AntiStructures.

6.44. WellDefined n-ary Law

Let $\mathcal{U}$ be a universe of discourse, a non-empty set $S \subseteq \mathcal{U}$, and a n-ary law, for n integer, $n \geq 1$, defined on $\mathcal{U}$.

$L: \mathcal{U}^n \to \mathcal{U}.$

For any $x_1, x_2, \ldots, x_n \in S$, one has $L(x_1, x_2, \ldots, x_n) \in S$.

6.45. NeutroDefined n-ary Law

There exists at least a n-plet $a_1, a_2, \ldots, a_n \in \mathcal{S}$ such that $L(a_1, a_2, \ldots, a_n) \in \mathcal{S}$. The elements $a_1, a_2, \ldots, a_n$ may be equal or not among themselves.

And there exists at least a n-plet $b_1, b_2, \ldots, b_n \in \mathcal{S}$ such that $L(b_1, b_2, \ldots, b_n) \notin \mathcal{S}$. The elements $b_1, b_2, \ldots, b_n$ may be equal or not among themselves.

6.46. AntiDefined n-ary Law

For any $x_1, x_2, \ldots, x_n \in \mathcal{S}$, one has $L(x_1, x_2, \ldots, x_n) \notin \mathcal{S}$.

6.47. WellDefined n-ary HyperLaw

Let $\mathcal{U}$ be a universe of discourse, a non-empty set $\mathcal{S} \subset_{\neq} \mathcal{U}$, and a n-ary hyperlaw, for n integer, $n \geq 1$:

$H: \mathcal{U}^n \to \mathcal{P}(\mathcal{U})$, where $\mathcal{P}(\mathcal{U})$ is the power set of $\mathcal{U}$.

For any $x_1, x_2, \ldots, x_n \in \mathcal{S}$, one has $H(x_1, x_2, \ldots, x_n) \in \mathcal{P}(\mathcal{S})$.

6.48. NeutroDefined n-ary HyperLaw

There exists at least a n-plet $a_1, a_2, \ldots, a_n \in \mathcal{S}$ such that $H(a_1, a_2, \ldots, a_n) \in \mathcal{P}(\mathcal{S})$. The elements $a_1, a_2, \ldots, a_n$ may be equal or not among themselves.

And there exists at least a n-plet $b_1, b_2, \ldots, b_n \in \mathcal{S}$ such that $H(b_1, b_2, \ldots, b_n) \notin \mathcal{P}(\mathcal{S})$. The elements $b_1, b_2, \ldots, b_n$ may be equal or not among themselves.

6.49. AntiDefined n-ary HyperLaw

For any $x_1, x_2, \ldots, x_n \in \mathcal{S}$, one has $H(x_1, x_2, \ldots, x_n) \notin \mathcal{P}(\mathcal{S})$.

*

The most interesting are the cases when the composition law(s) are well-defined (classical way) and neutro-defined (neutrosophic way).

6.50. WellDefined NeutroStructures

Are structures whose laws of compositions are well-defined, and at least one axiom is NeutroAxiom, and one has no AntiAxiom.

6.51. NeutroDefined NeutroStructures

Are structures whose at least one law of composition is NeutroDefined, and all other axioms are NeutroAxioms or Axioms.

6.52. Example of NeutroDefined NeutroGroup

Let U = {a, b, c, d} be a universe of discourse, and the subset

$S = \{a, b, c\}$, endowed with the binary law $*$:

$*$	a	b	c
a	a	c	b
b	c	a	c
c	a	c	d

NeutroDefined Law of Composition:

Because, for example: a*b = c ∈ S, but c*c = d ∉ S.

NeutroAssociativity:

Because, for example:

b*(c*b) = b*c = c and (b*c)*b = c*b = c;

while, for example:

a*(a*b) = a*c = b and (a*a)*b = a*b = c ≠ b.

NeutroCommutativity:

Because, for example:

a*b = b*a = c, but a*c = b while c*a = a ≠ c.

NeutroUnit Element:

There exists a unit element b for c, since $c*b = b*c = c$; and there is a unit element a for a, since $a*a = a$.

But there is no unit element for b, because $b*x = a$ or c, not b, for any $x \in S$ (according to the above Cayley Table)

NeutroInverse Element:

There exists an inverse element for a, which is a, because $a*a = a$.

But there is no inverse element for b, since b has no unit element.

Therefore $(S, *)$ is a NeutroDefined NeutroCommutative NeutroGroup.

6.53. WellDefined AntiStructures

Are structures whose laws of compositions are well-defined, and have at least one AntiAxiom.

6.54. NeutroDefined AntiStructures

Are structures whose at least one law of composition is NeutroDefined and no law of composition is AntiDefined, and has at least one AntiAxiom.

6.55. AntiDefined AntiStructures

Are structures whose at least one law of composition is AntiDefined, and has at least one AntiAxiom.

6.56. Conclusion

The neutrosophic triplet (<A>, <neutA>, <antiA>), where <A> may be an "Axiom", a "Structure", a "Theory" and so on, <antiA> the opposite of <A>, while <neutA> (or <neutroA>) their neutral in between, are studied in this paper.

The NeutroAlgebraic Structures and AntiAlgebraic Structures are introduced now for the first time, because they have been ignored by the classical algebraic structures. Since, in science and technology and

mostly in applications of our everyday life, the laws that characterize them are not necessarily well-defined or well-known, and the axioms / properties / theories etc. that govern their spaces may be only partially true and partially false (as *<neutA>* in neutrosophy, which may be a blending of truth and falsehood).

Mostly in idealistic or imaginary or abstract or perfect spaces we have rigid laws and rigid axioms that totally apply (that are 100% true). But the laws and the axioms should be more flexible in order to comply with our imperfect world.

References

[1] Smarandache, Florentin, *Advances of Standard and Nonstandard Neutrosophic Theories*, Pons Editions, Brussels, European Union, 2019.

[2] Smarandache, F., *Neutrosophy. / Neutrosophic Probability, Set, and Logic*, ProQuest Information & Learning, Ann Arbor, Michigan, USA, 105 p., 1998.

[3] Bhattacharya, S., *A Model to The Smarandache Geometries (S-denied, or smarandachely-denied)*, Journal of Recreational Mathematics, Vol. 33, No. 2, p. 66, 2004-2005; updated version in Octogon Mathematical Magazine, Vol. 14, No. 2, 690-692, October 2006.

[4] F. Smarandache, *S-Denying a Theory*, International J.Math. Combin. Vol. 2(2013), 01-07, https://zenodo.org/record/821509#.XjR5QTJKgs4.

[5] Kuciuk, L., Antholy M., *An Introduction to Smarandache Geometries (S-geometries)*, Mathematics Magazine, Aurora, Canada, Vol. 12, 2003,

and online:

http://www.mathematicsmagazine.com/1-2004/Sm_Geom_1_2004.htm;

also presented at New Zealand Mathematics Colloquium, Massey University, Palmerston North, New Zealand, December 3-6, 2001; also presented at the Intl. Congress of Mathematicians (ICM2002), Beijing, 20-28 August 2002, http://www.icm2002.org.cn/B/Schedule_ Section04.htm and in 'Abstracts of Short Communications to the International Congress of Mathematicians', Intl. Congress of Mathematicians, 20-28 August 2002, Beijing, China; and in JP Journal of Geometry and Topology, Allahabad, India, Vol. 5, No. 1, 77-82, 2005.

[6] Mao Linfan, *An introduction to Smarandache Geometries on Maps*, 2005 International Conference on Graph Theory and Combinatorics, Zhejiang Normal University, Jinhua, Zhejiang, P. R. China, June 25-30, 2005; also appeared in "Smarandache geometries & map theory with applications" (I), Chinese Branch Xiquan House, 2007.

[7] Ashbacher, C., *Smarandache Geometries*, Smarandache Notions Journal, Vol. 8, 212-215, No. 1-2-3, 1997.

[8] Chimienti, S. and Bencze, M., *Smarandache Paradoxist Geometry*, Bulletin of Pure and Applied Sciences, Delhi, India, Vol. 17E, No. 1, 123-1124, 1998; http://fs.unm.edu/prd-geo1.txt.

[9] Mao, Linfan, *An introduction to Smarandache geometries on maps*, 2005 International Conference on Graph Theory and Combinatorics, Zhejiang Normal University, Jinhua, Zhejiang, P. R. China, June 25-30, 2005.

[10] Mao, Linfan, *Automorphism Groups of Maps, Surfaces and Smarandache Geometries*, partially post-doctoral research, Chinese Academy of Science, Am. Res. Press, Rehoboth, 2005.

[11] Mao, Linfan, *Selected Papers on Mathematical Combinatorics (I),* World Academic Press, Liverpool, U.K., 2006.

[12] Iseri, H., *Partially Paradoxist Smarandache Geometries*, http://fs.unm.edu/Howard-Iseri-paper.pdf.

[13] Iseri, H., *Smarandache Manifolds*, Am. Res. Press, Rehoboth, 2002, http://fs.unm.edu/Iseri-book1.pdf

[14] Perez, M., *Scientific Sites*, in 'Journal of Recreational Mathematics', Amityville, NY, USA, Vol. 31, No. 1, 86, 2002-20003.

[15] Smarandache, F., *Paradoxist Mathematics*, in Collected Papers (Vol. II), Kishinev University Press, Kishinev, 5-28, 1997.

[16] Mao, Linfan, *Automorphism Groups of Maps, Surfaces and Smarandache Geometries* (Partially postdoctoral research for the Chinese Academy of Sciences), Beijing, 2005, http://fs.unm.edu/Geometries.htm.

[17] Smarandache, F., *Paradoxist Mathematics (1969)*, in Collected Papers (Vol. II), Kishinev University Press, Kishinev, 5-28, 1997.

[18] Smarandache, F., *Paradoxist Geometry*, State Archives from Valcea, Rm. Valcea, Romania, 1969.

[19] Smarandache, Florentin, *Neutrosophic Transdisciplinarity (MultiSpace & MultiStructure),* Arhivele Statului, Filiala Valcea, Romania, 1969; http://fs.unm.edu/NeutrosophicTransdisciplinarity.htm.

[20] Smarandache, F., *Multi-space and Multi-structure*, in "Neutrosophy. Neutrosophic Logic, Set, Probability and Statistics", ProQuest Information & Learning, Ann Arbor, Michigan, USA, 105 p., 1998, http://fs.unm.edu/Multispace.htm.

[21] Rabounski, D., *Smarandache Spaces as a new extension of the basic space-time of general relativity*, Progress in Physics, Vol. 2, L1-L2, 2010.

CHAPTER 7

New Developments in Neutrosophic Theories and Applications

7.1 Definition of <neutA>

<neutA> is everything which is in between the opposites <A> and <antiA>.

It was called *Neutrality*, because it was neither <A> nor <antiA>, but the neutral in between them.

And it was also called *Indeterminacy*, because it was an indeterminate part from <A> and <antiA>.

Since there are many types of opposite pairs (<A>, <antiA>), one has many types of intermediaries (denoted by <neutA>) in between them.

"<neutA>" is just a generic denomination (general term) used for everything which is in between two opposites. Not to be taken *ad litteram* (literally).

<neutA> is a class of concepts, not a single one, and depends on the pair of opposites that <neutA> is in between.

Depending on each (<A>, <antiA>) particular opposite pair, <neutA> may be:

> neutrality, indeterminacy, tie result, unknown, contradiction, uncertainty, vagueness, unclear, mixtures of <A> and <antiA>, etc.

7.2 Neutrality and Indeterminacy

In Neutrosophic Logic/Set/Probability, between opposites <A> and <antiA>, i.e. between T = Truth / Membership / Chance of An Event to Occur, and F = Falsehood / Nonmembership / Chance of the Event Not to Occur, it is used the concept I = Indeterminacy (also called Neutrality).

Indeterminacy (or Neutrality) is all between *Truth* and *Falsehood* (in *Neutrosophic Logic*);

> or all in between *Membership* and *Nonmembership* (in *Neutrosophic Set*);

or all in between *Chance of An Event to Occur*, and *Chance of the Event Not to Occur* (in *Neutrosophic Probability*).

In *Neutrosophic Statistics*, Indeterminacy is referred to the statistical data, that may be:

> incomplete, partially known/unknown data, unknown exact sample or population side, probability distribution functions with indetermination (unclear, vague, contradictory data).

It does not mean that Neutrality is the same as Indeterminacy, but some people call them those ways.

One should not take linguistic dictionaries to extract the definitions of Indeterminacy and of Neutrality. Indeterminacy (and rarely used Neutrality) is a *generic terminology*, meaning it has a large meaning, not a narrow one. Indeterminacy (Neutrality) = <neutA>.

An example:

In mathematical topology we have "open set". But this does not mean that it has something to do with, for example: "open door", "open account", "open person", etc. "Open set" is just a set that satisfies some mathematical axioms.

Similarly, "Indeterminacy" was defined in approximation theories (fuzzy, neutrosophy etc.) as everything which is in between Truth and Falsehood. Period.

According to Merriam-Webster Dictionary:

indeterminacy = the quality or state of being indeterminate.

Synonyms and Near Synonyms for indeterminate:

> general, indefinable, indefinite, indistinct, mushy, undefined, undetermined, unsettled, vague, approximate, approximative, ballpark, imprecise, inaccurate, inexact, loose, squishy, erroneous, false, incorrect, off, wrong, faulty, flawed, mistaken, specious, distorted, fallacious,

misleading, doubtful, dubious, questionable, uncertain, inconclusive, indecisive, debatable, disputable, invalidated, unconfirmed, unsubstantiated, unsupported.

According to Cambridge Dictionary:

indeterminacy = the state of not being measured, counted, or clearly known.

Synonyms and Near Synonyms for indeterminate:

arguable, be neither fish nor fowl (idiom), blurred, circumstantial, conflicted, debatable, definite maybe, disputable, dodgy, don't bet on it (idiom), doubt, doubtful, dubious, dubiously, elusively, fuzzily, fuzzy, gray area, iffy, inchoate, inconclusive, inconclusively, indecisive, indecisively, indefinite, indefinitely, insecurely, insecurity, it remains to be seen (idiom), kinda, knife edge, lack direction (idiom), limbo, maybe, mistily, mistiness, haziness, murky, nebulous, nebulousness, no-man's-land, not be set/carved in stone (idiom), on a razor edge (idiom), open-ended, parlous, possibly, provisionally, questionable, quite, shade, shades of grey (idiom), shakily, shakiness, shallow, shallowly, spec, sputter, squishy, stutter, swither, tell, tentative, tentatively, tentativeness, tenuous, tenuously, there's no knowing (idiom), touch-and-go, uncertain, uncertainly, uncertainty, unclear, unclearly, unconfirmed, undecided, unlikely, unofficial, unofficially, unsafe, unspecified, unsupported, unwritten, up in the air (idiom), vagaries, vague, vaguely, vagueness, wild card, wishy-washy, indescribable, indescribably, indestructibility, indestructible, indeterminate, indeterminism, index, index card.

We have listed all definitions of "indeterminacy" and "indeterminate, together with their synonyms and near-synonyms, provided by Merriam-Webster and Cambridge Dictionaries, in order to show that the concept *Indeterminacy* is capable of comprising all possible <neutA> versions between the opposites <A> and <antiA>.

7.3 Many Types of Indeterminacies

Since there are many types of indeterminacies in our world, we can construct different approaches to various neutrosophic concepts.

Indeed, the neutrosophic dynamic system was approached from a classical perspective but taking into account the indeterminacies.

Having many types of indeterminacies, what neutrosophic science are studying on, there are many approaches for the same topic that deals with different indeterminacies.

7.4 Completeness or Incompleteness in Neutrosophy

I have defined from the beginning that $t + i + f = 3$ if the information is complete, but $t + i + f < 3$ if the information is incomplete.

In general, I wrote $t + i + f \leq 3$ (depending on the completeness or incompleteness of the information provided by sources)

7.5 A concept <A> has many opposites and many neutralities

For a concept <A>, with respect to an attribute α_1 that characterizes it, there is an opposite <antiA$_1$>, and a neutral <neutA$_1$> among the opponents.

But, for <A>, with respect to another attribute α_2 that characterizes it, there is another opposite <antiA$_2$>, and another neutral <neutA$_2$> among the opponents.

Etc.

In conclusion, for a concept <A>, there are many opposites {<antiA$_1$>, <antiA$_2$>, ...} and many neutrals {<neutA$_1$>, <neutA$_2$>, ...}.

This is the reality.

7.6 Dependence and Independence of Sources providing Information

— We have independence when we judge a Proposition / Event with respect to a parameter, then we respect to another parameter completely independent of the previous parameter.

For example: let's say there will be a soccer game between India and China.

First parameter P1: History of India-China games.

Suppose according to the statistics of the games between India and China, India won most of the time. Therefore, we may approximate/guess that T = 0.7 (70% that India will win).

Second parameter P2: Playing home or not.

The game will be played in China, where China has a bigger chance to win. Hence, we may say F = 0.6 (60% that China will win).

— We may have independence when there are multiple independent sources (that do not communicate with each other), that may be subjective and give information on T, I, F neutrosophic components separately.

The same game: India - China.

Somebody, Raj from India, being patriot, will say that India will win. Someone from China, Young, being patriot for his country, may say that China will win. A third neutral person may say that it is a big chance that the game will be tied.

For more on dependence and independence of neutrosophic components:

http://fs.unm.edu/NSS/DegreeOfDependenceAndIndependence.pdf

7.7 Geometric Representation of Neutrosophic Cubic Set

Internal Neutrosophic Cubic Set { [a_1, a_2] × [b_1, b_2] × [c_1, c_2], (λ_1 × λ_2 × λ_3) },

where [a_1, a_2] × [b_1, b_2] × [c_1, c_2] is included or equal to [0, 1]^3,

λ_1 belongs to [a_1, a_2]

λ_2 belongs to [b_1, b_2]

and λ_3 belongs to [c_1, c_2],

is represented into the **Standard Neutrosophic Cube** ([0, 1] × [0, 1] × [0, 1]) by small prisms included into the **Standard Neutrosophic Cube**,

where each small prism P has the sides:

lying between a_1 and a_2 on x-axis,

between b_1 and b_2 on the y-axis,

and between c_1 and c_2 on the z-axis,

and a point M of coordinates (λ_1, λ_2, λ_3) included into the small prism P.

Also, for **Single-Valued Neutrosophic Set**, each element x of neutrosophic components (T_x, I_x, F_x) is represented by a point of coordinates T_x on the T-axis, I_x on the I-axis, F_x on the F-axis into the **Neutrosophic Cube**.

Now, for **Interval-Valued Neutrosophic Set**, each element x of coordinates ([T_x^-, T_x^+], [I_x^-, I_x^+], [F_x^-, F_x^+]) is represented by a small prism determined by the intervals

[T_x^-, T_x^+] on the T-axis,

[I_x^-, I_x^+] on the I-axis,

[F_x^-, F_x^+] on the F-axis,

of inside the **Neutrosophic Cube**.

Then, an **Internal Cubic Neutrosophic Set** is represented by a small prism (as the **Interval-Valued Neutrosophic Set**) included into the

Neutrosophic Cube, and each small prism contains a point (as the **Single-Valued Neutrosophic Set**).

7.8 Uncertainty, Contradiction

In neutrosophic logic we have:

T = Truth, I = Indeterminacy, F = Falsehood.

Indeterminacy does not have the definition from the Larrouse or Webster dictionaries etc.

Indeterminacy means everything that is different from T and F.

Indeterminacy might be:

$T \vee F$ = *Uncertainty*, $T \wedge F$ = *Contradiction* (what Belnap said).

Both are part of the Indeterminacy. In this case the Indeterminacy has been refined in *Uncertainty* and *Contradiction*.

What *Indeterminacy* means is different from one application to another.

In a soccer game, we have I = Equality.

In a relationship (friend, neutral, enemy) we have I = neutral.

In another relation we can have I = unknown (if we have no information).

In the logic where we do not know if a proposition is true or false, we can have I = 40% true and 60% false (for example).

Indeterminacy is different from one application to another.

Indeterminacy can be *Neutrality* in an application (for example a country that does not mix in war), or *Indeterminacy* can be *Uncertainty* (but it depends on the application too), or *Contradiction,* or *Unknown,* or *partially true and partially false,* etc.

7.9 Nonstandard Neutrosophic Algebraic Structures

A Japanese has criticized me for the nonstandard form of neutrosophic set/logic, i.e. about the nonstandard interval $]^-0, 1^+[$. His name is Dr. Imamura T. [1]. I answered to him [1, 2] and he agreed.

When I answered back, I extended the **Nonstandard Analysis** by introducing **monads closed to one side**, and **bimonds**.

Dr. Vasantha & I did work on neutrosophic algebraic structures, based on sets of the form $a+bI$, where a, b are real or complex numbers, and $I^2 = I$ is indeterminacy.

But we can extend these to **Nonstandard Neutrosophic Algebraic Structures** (never done before), $a+bI$, where a, b are monads or bimonas. I defined the operations (not all) of them into the above paper, but algebraic structures were not developed.

References

[1] Florentin Smarandache: *About Nonstandard Neutrosophic Logic (Answers to Imamura 'Note on the Definition of Neutrosophic Logic')*, pp. 1-16, Cornell University, New York City, USA. Submitted on 24 Nov 2018 (v1), last revised 13 Feb 2019 (v2). Abstract: https://arxiv.org/abs/1812.02534v2. Full paper: https://arxiv.org/ftp/arxiv/papers/1812/1812.02534.pdf.

[2] Florentin Smarandache: *Extended Nonstandard Neutrosophic Logic, Set, and Probability Based on Extended Nonstandard Analysis*. Symmetry 2019, 11, 515. Full paper: https://www.mdpi.com/2073-8994/11/4/515.

7.10 Three-ways model

The "three-ways model", that means to split the universe of discourse in:

> *Acceptance* (<A>), *Rejection* (<antiA>), and *Neither* (neutral or indeterminacy) (<neutA>),

is just a process of neutrosophication of the universe of discourse, because in neutrosophy we deal with the triplet of the form <A>, <antiA>, <neutA>.

7.11 Three-Ways Decision is a particular case of Neutrosophication

7.11.1 Neutrosophication

Let *<A>* be an attribute value, *<antiA>* the opposite of this attribute value, and <neutA> the neutral (or indeterminate) attribute value between the opposites *<A> and <antiA>*.

For examples: <A> = *big*, then <antiA> = *small*, and <neutA> = *medium*; we may rewrite:

(<A>, <neutA>, <antiA>) = (big, medium, small);

or (<A>, <neutA>, <antiA>) = (*truth* - denoted as *T, indeterminacy* - denoted as *I, falsehood* - denoted as *F*) as in *Neutrosophic Logic*,

or (<A>, <neutA>, <antiA>) = (membership, indeterminate-membership, monmembership) as in *Neutrosophic Set*,

or (<A>, <neutA>, <antiA>) = (chance that an event occurs, indeterminate-chance that the event occurs or not, chance that the event does not occur) as in *Neutrosophic Probability*,

and so on.

And let by "concept" mean: an item, object, idea, theory, region, universe, set, notion etc. that is characterized by this attribute.

The process of **neutrosophication** means:

- converting a Classical Concept { denoted as (**1**$_{<A>}$, **0**$_{<neutA>}$, **0**$_{<antiA>}$)-**ClassicalConcept**,

or ***ClassicalConcept*(*1*$_{<A>}$, *0*$_{<neutA>}$, *0*$_{<antiA>}$)** }, which means that the concept is, with respect to the above attribute,

100% <A>, 0% <neutA>, and 0% <antiA>,

into a Neutrosophic Concept { denoted as ($T_{<A>}$, $I_{<neutA>}$, $F_{<antiA>}$)-NeutrosophicConcept,

or ***NeutrosophicConcept($T_{<A>}$, $I_{<neutA>}$, $F_{<antiA>}$)*** }, which means that the concept is, with respect to the above attribute,

T% <A>, I% <neutA>, and F% <antiA>,

which more accurately reflects our imperfect, non-idealistic reality,

where *T, I, F* are subsets of *[0, 1]* with no other restriction.

7.11.1.1 Example 1

Let the attribute <A> = cold temperature, then <antiA> = hot temperature, and <neutA> = medium temperature.

Let the concept be a country *M*, such that its northern part (30% of country's territory) is cold, its southern part is hot (50%), and in the middle there is a buffer zone with medium temperature (20%). We write:

M($0.3_{\text{cold temperature}}$, $0.2_{\text{medium temperature}}$, $0.5_{\text{hot temperature}}$)

where we took single-valued numbers for the neutrosophic components $T_M = 0.3$, $I_M = 0.2$, $F_M = 0.5$, and the neutrosophic components are considered dependent so their sum is equal to 1.

7.11.1.2 Example 2 (Three-Ways Decision is a particular case of Neutrosophication)

Neutrosophy (based on *<A>, <neutA>, <antiA>*) was proposed by Smarandache [1] in 1998, and Three-Ways Decision by Yao [2] in 2009.

In Three-Ways Decision, the universe set is split into three different distinct areas, in regard to the decision process, representing:

Acceptance, Noncommitment, and Rejection respectively.

In this case, the decision attribute value *<A>* = Acceptance, whence *<neutA>* = Noncommitment, and *<antiA>* = Rejection.

The classical concept = *UniverseSet*.

Therefore, we got the NeutrosophicConcept($T_{<A>}$, $I_{<neutA>}$, $F_{<antiA>}$), denoted as:

UniverseSet($T_{Acceptance}$, $I_{Noncommitment}$, $F_{Rejection}$),

where $T_{Acceptance}$ = universe set's zone of acceptance, $I_{Noncommitment}$ = universe set's zone of noncomitment (indeterminacy), $F_{Rejection}$= = universe set's zone of rejection.

7.11.2 Three-Ways Decision as a particular case of Neutrosophic Probability

Let's consider the event, taking a decision on a universe set.

According to Neutrosophic Probability (NP) [3] one has:

NP(decision) = (the universe set's elements for which the chance of the decision may be accept; the universe set's elements for which there may be an indeterminate-chance of the decision; the universe set's elements for which the chance of the decision may be reject).

7.11.3 Refined Neutrosophy

Refined Neutrosophy was introduced by Smarandache [4] in 2013 and it is described as follows:

<A> is refined (split) into subcomponents *<A₁>, <A₂>, …, <Aₚ>;*

<neutA> is refined (split) into subcomponents <neutA₁>, <neutA₂>, …, <neutAᵣ>;

and *<antiA>* is refined (split) into subcomponents *<antiA₁>, <antiA₂>, …, <antiAₛ>;*

where *p, r, s ≥ 1* are integers, and *p + r + s ≥ 4.*

7.11.3.1 Example 3

If <A> = voting in country M, them <A₁> = voting in Region 1 of country M for a given candidate, <A₂> = voting in Region 2 of country M for a given candidate, and so on.

Similarly, $\langle \text{neut}A_1 \rangle$ = not voting (or casting a white or a black vote) in Region 1 of country M, $\langle A_2 \rangle$ = not voting in Region 2 of country M, and so on.

And $\langle \text{anti}A_1 \rangle$ = voting in Region 1 of country M against the given candidate, $\langle A_2 \rangle$ = voting in Region 2 of country M against the given candidate, and so on.

7.11.4 Extension of Three-Ways Decision to n-Ways Decision

n-Way Decision was introduced by Smarandache in 2019.

In n-Ways Decision, the universe set is split into $n \geq 4$ different distinct areas, in regard to the decision process, representing:

Levels of Acceptance, Levels of Noncommitment, and Levels of Rejection respectively.

Levels of Acceptance may be: Very High Level of Acceptance *($\langle A_1 \rangle$)*, High Level of Acceptance *($\langle A_2 \rangle$),* Medium Level of Acceptance *($\langle A_3 \rangle$),* etc.

Similarly, Levels of Noncommitment may be: Very High Level of Noncommitment *($\langle \text{neut}A_1 \rangle$)*, High Level of Noncommitment *($\langle \text{neut}A_2 \rangle$),* Medium Level of Noncommitment *($\langle \text{neut}A_3 \rangle$),* etc.

And Levels of Rejection may be: Very High Level of Rejection *($\langle \text{anti}A_1 \rangle$)*, High Level of Rejection *($\langle \text{anti}A_2 \rangle$),* Medium Level of Rejection *($\langle \text{anti}A_3 \rangle$),* etc.

Then the Refined Neutrosophic Concept

{ denoted as $(T1_{\langle A1 \rangle}, T2_{\langle A2 \rangle}, \ldots, Tp_{\langle Ap \rangle};\ I1_{\langle \text{neut}A1 \rangle}, I2_{\langle \text{neut}A2 \rangle}, \ldots, Ir_{\langle \text{neut}Ar \rangle};$

$F1_{\langle \text{anti}A1 \rangle}, F2_{\langle \text{anti}A2 \rangle}, Fs_{\langle \text{anti}As \rangle})$-RefinedNeutrosophicConcept,

or RefinedNeutrosophicConcept$(T1_{\langle A1 \rangle}, T2_{\langle A2 \rangle}, \ldots, Tp_{\langle Ap \rangle};\ I1_{\langle \text{neut}A1 \rangle}, I2_{\langle \text{neut}A2 \rangle}, \ldots, Ir_{\langle \text{neut}Ar \rangle};\ F1_{\langle \text{anti}A1 \rangle}, F2_{\langle \text{anti}A2 \rangle}, Fs_{\langle \text{anti}As \rangle}))\}$,

which means that the concept is, with respect to the above attribute value levels,

T1% <A1>, T2% <A2>, …, Tp% <Ap>;

I1% <neutA1>, I2% <neutA2>, …, Ir% <neutAr>;

F1% <antiA1>, F2% <antiA2>, Fs% <antiAs>;

which more accurately reflects our imperfect, non-idealistic reality,

with where *p, r, s ≥ 1* are integers, and *p + r + s ≥ 4,*

where all T1, T2, …, Tp, I1, I2, …, Ir, F1, F2, …, Fs are subsets of [0, 1] with no other restriction.

References

[1] F. Smarandache, *Neutrosophy. Neutrosophic Probability, Set, and Logic*, ProQuest Information & Learning, Ann Arbor, Michigan, USA, 105 p., 1998; http://fs.unm.edu/eBook-Neutrosophics6.pdf (edition online).

[2] Y. Yao, *Three-way decision: an interpretation of rules in rough set theory*, in Proceeding of 4th International Conference on Rough Sets and Knowledge Technology, LNAI, Vol. 5589, Springer Berlin Heidelberg, 2009, pp. 642–649.

[3] F. Smarandache, *Introduction to Neutrosophic Measure, Neutrosophic Integral, and Neutrosophic Probability*, by Florentin Smarandache, Sitech & Educational, Craiova, Columbus, 140 p., 2013; https://arxiv.org/ftp/arxiv/papers/1311/1311.7139.pdf

[4] F. Smarandache, *n-Valued Refined Neutrosophic Logic and Its Applications in Physics*, Progress in Physics, 143-146, Vol. 4, 2013; https://arxiv.org/ftp/arxiv/papers/1407/1407.1041.pdf

7.11 Division of Quadruple Neutrosophic Numbers

We can define the division of Quadruple Neutrosophic Numbers, but it does not work all the time.

I though that we might extend to quadruple neutrosophic field:

$(a_1+b_1T+c_1I+d_1F)/(a_2+b_2T+c_2I+d_2F) \equiv x+yT+zI+wF$,

whence:

$a_1+b_1T+c_1I+d_1F \equiv$ (identical with)

$(a_2+b_2T+c_2I+d_2F)(x+yT+zI+wF)$

We normally multiply, and then we solve for x, y, z, w and we get an algebraic nonlinear system of four equations with four unknowns x, y, x, w. Surely, there are some exceptions when the division does not work, i.e. $a_2 = 0$, etc.

7.12 Neutrosophic Quaternions

I have extended the Classical Quaternions to Neutrosophic Quaternions, that have the form:

$(a_1+a_2I) + (b_1 + b_2I)i + (c_1 + c_2I)j + (d_1 + d_2I)k$, with $a_1, a_2, b_1, b_2, c_1, c_2, d_1, d_2$ real numbers,

and I = indeterminacy (which can be any real subset),

where i, j, k have the same properties as in classical quaternions,

but $A = a_1 + a_2I$ is a neutrosophic number, where a_1 is the determinate part of A, while a_2I is the indeterminate part of A;

for example: A = 3 + 2I, where I = [0.1, 0.2], so we get [3.2, 3.4].

Similarly for $B = b_1 + b_2I$, $C = c_1 + c_2I$, and $D = d_1 + d_2I$.

The Classical Quaternions have the following properties:

$i^2 = j^2 = k^2 = ijk = -1$

$ij = -ji, ij = k$

Quaternion multiplication				
×	**1**	*i*	*j*	*k*
1	1	*i*	*j*	*k*
i	*i*	−1	*k*	−*j*
j	*j*	−*k*	−1	*i*
k	*k*	*j*	−*i*	−1

And I want to see how to use them in physical law and equations.

Instead of extending to quaternions, let's extend to neutrosophic quaternion some physical laws and equations.

Many physical constants and even physical laws are not accurate, but varying/approximations, so we can called the neutrososphic physical constants and respectively neutrosophic physical laws.

I think we can develop algebraic structures on them as well.

7.13 Neutrosophic Physics Laws

There are real applications of neutrosophic statistics, probability, logic in classical physics.

Not only Hubble's Law is not linear, but many classical Physics Laws may be represented by Neutrosophic Physics Laws, i.e. their equations have neutrosophic constants, neutrosophic coefficients, neutrosophic derivatives, neutrosophic integrals...

Instead of crisp number we have neutrosophic numbers, and instead of simple curves we have thick curves...

Many classical physical laws and equations should be interpreted from a neutrosophic point of view, i.e. including indeterminacy and approximations into the variables and coefficients involved into classical physical laws and equations, since our world is imperfect, not idealistic as modelled by the modern physics. [Robert Neil Boyd, Victor Christianto, Florentin Smarandache]

7.14 Not-Exact Physical Laws

In my book

http://fs.unm.edu/NewRelativisticParadoxes.pdf,

I stated that:

> not all physical laws are the same in all inertial reference frames.

We can get the same physical law that behaves differently in one place than in another,

or in some conditions than in others...

We can do something on not-exact physical laws…

7.15 Neutrosophic Physical Constants

Neutrosophic Constant in physics means a value that is not exact, but varies upon different parameters, such us: physical law, space, conditions etc.

Any classical physical constant c is actually a neutrosophic constant, i.e.

$c \pm \alpha$, where α is a positive real number

or a classical physical constant c actually is not a constant, but a variable c in a given set:

$c \in S$

7.16 Neutrosophic Sorites Paradox

Let *(<A>, <antiA>)* be a duplet, where *<A>* is an item (concept, object, idea, etc.) and *<antiA>* is the opposite of *<A>*, and there is no neutral *<neutA>* between them. Therefore the duplet *(<A>, <antiA>)* is not part of a neutrosophic triplet of the form *(<A>, <neutA>, <antiA>)*. Then:

A *Neutrosophic Sorites Paradox* is referring to the fact there exist specific items *<A>* such that: between *<A>* and its opposite *<antiA>* there is no clear frontier.

7.17 Determinate and Indeterminate parts of a Sky Cloud

A cloud on the sky is formed by a determinate part and an indeterminate part.

It is like the neutrosophic number N = a + bI, where "a" is the determinate part of the number N, while "bI" is the indeterminate part of N.

We transform a real number "r" into a neutrosophic number

$$N = a + bI,$$

where a, b are real numbers and I = indeterminacy, "I" is a subset,

whence N becomes a subset itself that captures "r" inside.

A simple example:

Real number r = 5, that we are not very sure about, may be a neutrosophic number of the form

N1 = 4.8 + 2I, with I = [0.05, 0.15],

whence N1 = [4.9, 5.1] that captures/includes 5.

There are many ways to capture a real number, let's say:

N2 = 5.2 + 3I, where I = [-0.2, 0.2],

whence N2 = [4.6, 5.8]. Or

N3 = 5.2 + 3I, where I = {-0.1, -0.2/3, 0, 1/3},

whence N3 = {4.9, 5.0, 5.2, 6.2}.

The corresponding neutrosophic numbers depend on the applications and experts.

One can then also go further and consider REFINED neutrosophic numbers, if needed for applications:

In a general Refined Neutrosophic Set/Logic/Probability,

T can be split into subcomponents T1, T2, ..., Tp,

and I into I1, I2, ..., Ir,

and F into F1, F2, ...,Fs, where p+r+s = n ≥ 1.

Even more: T, I, and/or F (or any of their subcomponents Tj, Ik, and/or Fl) can be countable or uncountable infinite sets:

http://fs.unm.edu/n-ValuedNeutrosophicLogic-PiP.pdf.

7.18 n-ary Neutrosophic Triplet of Weaker Type

An *n-ary Neutrosophic Triplet of Weaker Type* on *M* is defined in the following way.

Let an element $x \in M$. If there exist some element $e \in M$ such that

$$\circ_n(\underbrace{x, e, \ldots, e}_{n-1}) = \circ_n(\underbrace{e, \ldots, e}_{n-1}, x) = x.$$

Then it is considered the neutral element of *x* and it is denoted as $e \equiv neut_n(x)$.

Further, if there exist some element $x^{-1} \in M$, such that

$$\circ_n(x, \underbrace{x^{-1}, \ldots, x^{-1}}_{n-1}) = \circ_n(\underbrace{x^{-1}, \ldots, x^{-1}}_{n-1}, x) = e$$

Then it is considered the inverse element of *x* and it is denoted as $x^{-1} \equiv anti_n(x)$.

Therefore, *(x, neut_n(x), anti_n(x))* is called an n-ary neutrosophic triplet.

7.18.1 Remark

For an element *x*, there may exist more n-ary neutrals $neut_n(x)$'s and more n-ary inverses $anti_n(x)$.

7.18.2 Definition of n-ary (strong) Neutrosophic Triplet Set

An *n-ary (strong) Neutrosophic Triplet Set*, is a set M such that for any $x \in M$ there exist at least one $neut_n(x) \in M$ and one $anti_n(x) \in M$.

7.18.3 Definition of n-ary (weak) Neutrosophic Triplet Set

An *n-ary (weak) Neutrosophic Triplet Set*, is a set *M* such that for any $x \in M$ there exist at least one n-ary neutrosophic triplet $(y, neut_n(y), anti_n(y))$ in *M*, such that $x = y$, or $x = neut_n(y)$, or $x = anti_n(y)$.

7.18.4 Definition of n-ary (strong) Neutrosophic Triplet Group

An *n-ary (strong) Neutrosophic Triplet Group* is an n-ary (strong) Neutrosophic Triplet Set whose n-ary law $\circ_n$ is associative.

7.18.5 Definition of n-ary (weak) Neutrosophic Triplet Group

An *n-ary (weak) Neutrosophic Triplet Group* is an n-ary (weak) Neutrosophic Triplet Set whose n-ary law $\circ_n$ is associative.

7.19 Example of Bipolar Neutrosophic Set

Let us see an example for single-valued Bipolar Neutrosophic Set, whose neutrosophic components have the form (T+, T-; I+, I-, F+, F-), which T+, I+, F+ ∈ [0, 1], and T-, I-, F- ∈ [-1, 0],

where $0 \leq (T+) + (I+) + (F+) \leq 3$, and $-3 \leq (T-) + (I-) + (F-) \leq 0$.

At a company each employee has to work 40 hours a week and produce pieces of good quality.

John works only 35 hours this week (so his positive membership T+ = 36/40 = +0.90), but unfortunately his work is of low quality and below the required standard (so his negative membership T- is estimated by his supervisor to be T- = -0.30).

John does not work 4 hours this week, therefore his positive nonmembership (what's left from the positive membership) F+ = 4/40 = +0.10;

and in addition for not working unfortunately he comes to the company and accidentally destroys some machinery, that his supervisor estimate as negative nonmembership F- = -0.20.

The supervisor is not sure, but he believes that John may have worked 2 hours extra-time in the weekend, therefore John's positive indeterminacy is I+ = 2/40 = +0.05 but again of the same low quality work, that is estimated as negative indeterminacy: I- = -0.01.

Whence, John's single-valued bipolar neutrosophic membership to his company is:

John(+0.90, -0.30; +0.05, -0.01; +0.10, -0.20).

7.20 Neutrosophic Triplet Hypertopology

A new evolution from Neutrosophic Triplet Structures to Neutrosophic Triplet HyperStructures has been created. Therefore, a Neutrosophic Triplet Hypertopology may be defined.

Reference

Xiaohong Zhang, Florentin Smarandache, and Yingcang Ma: *Symmetry in Hyperstructure: Neutrosophic Extended Triplet Semihypergroups and Regular Hypergroups*, *Symmetry* 2019, 11, 1217; doi:10.3390/sym11101217

7.21 Plithogenic Set in Combination with all Previous Set-Types

7.21.1 Definition of Plithogenic Set

A **plithogenic set** P is a set such that each of its elements is characterized by many attribute-values. Almost all sets in our everyday life are plithogenic sets, because each element into a set is characterized by some attribute-values.

The (P, a, V, d, c) is called a plithogenic set, where:

"P" is a non-empty set included into a given universe of discourse U;

"a" is a (multi-dimensional in general) attribute;

"V" is the range of the attribute's values;

"$d(x, v)$", where $x \in U$ and $v \in V$, is the function that represents the degree of appurtenance of the element x, with respect to its attribute-value v, to the set P; and $d(x, v)$ may be of any type (see below);

"$c(v_k, v_D)$", where v_k is an attribute-value and v_D is the dominant (most important) attribute-value, is the function that represents the degree of contradiction (or dissimilarity) between an attribute-value and the dominant attribute-value.

The functions $(\cdot, \cdot)$ and $c(\cdot, \cdot)$ are defined by experts in accordance with the applications they need to solve. One uses the notation: $(d(x, V))$, where

$$(x, V) = \{d(x, v), \text{ for all } v \in V\}, \forall x \in P\}.$$

The contradiction (dissimilarity) degree was defined in order to obtain a better accuracy for the plithogenic aggregation operators.

The plithogenic aggregation operators (intersection, union, complement, inclusion, equality) are based on contradiction degrees between attribute-values, and the first two are linear combinations of the fuzzy t-norm and fuzzy t-conorm operators.

The **degree of appurtenance $d(x, v)$,** of the element x, with respect to its attribute-value v, to the plithogenic set P, may be of any type:

Crisp {0 or 1}, Fuzzy, Intuitionistic Fuzzy, Inconsistent Intuitionistic Fuzzy (Picture Fuzzy, Ternary Fuzzy), Pythagorean Fuzzy (Atanassov's Intuitionistic Fuzzy Set of second type), Spherical Fuzzy, n-HyperSpherical Fuzzy, n-HyperSpherical Neutrosophic, q-Rung Orthopair Fuzzy, Neutrosophic, Refined Fuzzy, Refined Intuitionistic Fuzzy Set, Refined Inconsistent Intuitionistic Fuzzy, {Refined Picture Fuzzy), Refined Ternary Fuzzy}, Refined Pythagorean Fuzzy {Refined Atanassov's Intuitionistic Fuzzy of type 2}, Refined Spherical Fuzzy, Refined n-HyperSpherical Fuzzy, Refined q-Rung Orthopair Fuzzy, Refined Neutrosophic, etc.

Similarly, the **degree of contradiction (or dissimilarity)** $c(v_k, v_D)$ between an attribute-value and the dominant attribute-value can be of any type: crisp, fuzzy, etc. as above.

7.21.2 Remark

In my previous publications [1 – 8], I have considered as degree of appurtenance

$d(x, y)$ and degree of contradiction (dissimilarity) $c(v_k, v_D)$ only the degrees of the types: crisp, fuzzy, intuitionoistic fuzzy, and neutrosophic.

But now, I extend them to more types of degrees of appurtenance and degrees of contradiction (dissimilarity) as above.

7.21.3 Open Research

In the previous plithogenic publications { [1-8], from years 2017-2019 }, only the <u>fuzzy</u> contradiction (dissimilarity) degree function between an attribute-value and its dominant attribute-value has been considered, or 1D (one dimensional) function:

$$c : V \times V \rightarrow [0,1]$$,

whence the plithogenic operators were linear combinations of fuzzy t-norm and fuzzy t-conorm.

But for other types of contradiction (dissimilarity) degree functions, such as <u>intuitionistic fuzzy, neutrosophic, refined fuzzy and refined neutrosophic types, etc.</u> one has:

$$c_k : V \times V \rightarrow [0,1]^k$$,

where $k = 2$ (for intuitionistic fuzzy), 3 (for neutrosophic), and in general $n \geq 2$ (for n-valued refined fuzzy and refined neutrosophic),

building the k-D (k-dimensional) plithogenic operators has not yet been studied and applied. One hint may be to construct *bi-linear, tri-linear, …,* or *n-linear* respectively plithogenic operators.

The readers are welcome to try building such *n-ary* plithogenic operators.

References

1. Florentin Smarandache, *Plithogenic Set, an Extension of Crisp, Fuzzy, Intuitionistic Fuzzy, and Neutrosophic Sets – Revisited*, Neutrosophic Sets and Systems, vol. 21, 2018, pp. 153-166. https://doi.org/10.5281/zenodo.1408740
2. Florentin Smarandache, *Plithogeny, Plithogenic Set, Logic, Probability, and Statistics*, Pons Publishing House, Brussels, Belgium, 141 p., 2017;
    arXiv.org (Cornell University), Computer Science – Artificial Intelligence, 03Bxx:

book: https://arxiv.org/ftp/arxiv/papers/1808/1808.03948.pdf
abstract: https://arxiv.org/abs/1808.03948
   UNM Repository:
https://digitalrepository.unm.edu/math_fsp/20/
   Harvard SAO/NASA ADS:
http://adsabs.harvard.edu/cgi-bin/bib_query?arXiv:1808.03948

3. F. Smarandache, *Plithogeny, Plithogenic Set, Logic, Probability, and Statistics*, Infinite Study Publ. Hse., GoogleLLC, Mountain View, California, USA, 2017.

4. Shazia Rana, Madiha Qayyum, Muhammad Saeed, F. Smarandache, Bakhtawar Ali Khan: Plithogenic Fuzzy Whole Hypersoft Set, Construction of Operators and their Application in Frequency Matrix Multi Attribute Decision Making Technique, Neutrosophic Sets and Systems, vol. 28, 2019, pp. 34-50. DOI: 10.5281/zenodo.3382507

5. Mohamed Abdel-Basset, Rehab Mohamed, Abd El-Nasser H. Zaied, and Florentin Smarandache, *A Hybrid Plithogenic Decision-Making Approach with Quality Function Deployment for Selecting Supply Chain Sustainability Metrics*, Symmetry 2019, 11, 903, pp. 1-21; doi:10.3390/sym11070903; https://www.mdpi.com/2073-8994/11/7/903

6. Florentin Smarandache, Extension of Soft Set to Hypersoft Set, and then to Plithogenic Hypersoft Set. Neutrosophic Sets and Systems, Vol. 22, 2018, pp. 168-170. http://doi.org/10.5281/zenodo.2838716

7. F. Smarandache, *Extension of Soft Set to Hypersoft Set, and then to Plithogenic Hyeprsoft Set*, Octogon Mathematical Magazine, Vol. 27, No. 1, April 2019, pp. 413-418.

8. Florentin Smarandache, *Physical Plithogenic Set*, 71st Annual Gaseous Electronics Conference, Session LW1, Oregon Convention Center Room, Portland, Oregon, USA, November 5–9, 2018; http://meetings.aps.org/Meeting/GEC18/Session/LW1.110

7.22 Plithogenic Set as extension of Neutrosophic Set

While the crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets are sets whose elements x are characterized by a single attribute, called "appurtenance", whose attribute values are: "membership" (for crisp sets and fuzzy sets),

or "membership" and "nonmembership" (for intuitionistic fuzzy set),

or "membership" and "nonmembership" and "indeterminacy" (for neutrosophic set),

a plithogemic set is a set whose elements x are characterized by many attributes, and each attribute may have many attribute values.

Neutrosophic set was extended to *plithogenic set* by Smarandache in 2017.

A simple **example**:

Let's consider a set M = {x1, x2, x3}, such that each element is characterized by two attributes:

C = color, and S = size. Suppose the attribute values of C = {white (w), blue (b), green (g)} and of size are S = {small (s), medium (m)}.

Thus, each x element of M is characterized by the all five attribute values: white, blue, green, small, tall, i.e.

M = {x1(w, b, g; s, m), x2(w, b, g; s, m), x3(w, b, g; s, m)}.

Therefore, each element x belongs to the set M with a degree of white d(w), a degree of blue d(b), a degree of green d(g), a degree of small d(s), and a degree of medium d(m).

Thus, M = { x1(d1(w), d1(b), d1(g); d1(s), d1(m)), x2(d2(w), d2(b), d2(g); d2(s), d2(m)), x3(d3(w), d3(b), d3(g); d3(s), d3(m)) }

where d1(.), d2(.), and d3(.) are the degrees of appurtenance of x1, x2, and x3 respectively to the set M with respect to each of the five attribute values.

But the degree of appurtenance may be: classical degree { whose values are 0 or 1 }, fuzzy degree { whose values are in [0, 1] }, intuitionistic fuzzy degree { whose values are in $[0, 1]^2$ }, or neutrosophic degree { whose values are in $[0, 1]^3$ }.

Therefore, we may get:

- A Plithogenic Classical Set:

M = { x1(0, 1, 0; 0, 1), x2(1, 0, 0; 0, 0), x3(1, 0, 0; 1, 0) },

which means that:

x1 is not white, x1 is blue, x1 is not green, x1 is not small, x1 is medium; similarly for x2 and x3.

- A Plithogenic Fuzzy Set:

M = { x1(0.2, 0.7, 0.5; 0.8, 0.3), x2(0.5, 0.1, 0.0; 0.9, 0.2), x3(0.5, 1, 0.6; 0.4, 0.3) },

which means that:

x1 has the fuzzy degree of white equals to 0.2, x1 has the fuzzy degree of blue equals to 0.7,

x1 has the fuzzy degree of green equals to 0.1, x1 has the fuzzy degree of small size equals to 0.8,

and x1 has the fuzzy degree of medium size equals to 0.3;

similarly for x2 and x3.

- A Plithogenic Intuitionistic Fuzzy Set:

M = { x1((0.4,0.1), (0.2,0.7), (0.0,0.3); (0.8,0.5), (0.2,0.3)), x2((0.7,0.2), (0.2,0.6), (1.0,0.0); (0.6,0.4), (0.1,0.5)), x3((0.4,0.4), (0.5,0.6, (0.5,0.1); (0.5,0.6), (0.3,0.3));

which means that:

x1 has the truth-degree of white equals to 0.4 and the false-degree of white equals to 0.1;

x1 has the truth-degree of blue equals to 0.2 and the false-degree of blue equals to 0.7;

x1 has the truth-degree of green equals to 0.0 and the false-degree of green equals to 0.3;

x1 has the truth-degree of small size equals to 0.8 and the false-degree of small size equals to 0.5;

x1 has the truth-degree of medium size equals to 0.2 and the false-degree of white equals to 0.3;

similarly for x2 and x3.

- A Plithogenic Neutrosophic Set:

M = { x1((0.2,0.4,0.3), (0.5,0.2,0.7), (0.6,0.4,0.3); (0.9,0.6,0.5), (0.1,0.2,0.3)), x2((0.1,0.7,0.2), (0.3,0.2,0.7), (0.0,0.2,1.0); (0.6,0.6,0.1), (0.0,0.1,0.6)), x3((0.7,0.4,0.4), (0.5,0.6, (0.3,0.5,0.1); (0.0,0.5,0.6), (0.8,0.3,0.2));

which means that:

x1 has the truth-degree of white equals to 0.2, the indeterminacy-degree of white equals to 0.4, and the false-degree of white equals to 0.3;

x1 has the truth-degree of blue equals to 0.5, the indeterminacy-degree of blue equals to 0.2, and the false-degree of blue equals to 0.7;

x1 has the truth-degree of green equals to 0.6, the indeterminacy-degree of green equals to 0.4, and the false-degree of green equals to 0.3;

x1 has the truth-degree of small size equals to 0.9, the indeterminacy-degree of small size equals to 0.6, and the false-degree of small size equals to 0.5;

x1 has the truth-degree of medium size equals to 0.1, the indeterminacy-degree of minimum size equals to 0.2, and the false-degree of minimum size equals to 0.3;

similarly for x2 and x3.

Of course, we have considered the Single-Valued Plithogenic Set, i.e. when all degrees are single-valued (crip) numbers from [0, 1].

But similarly we may define:

Interval-Valued Plithogenic Set (when the degrees are intervals included into [0, 1]),

or Hesitant Plithogenic Set (when the degrees are discrete finite subsets included into [0, 1]),

or in the most general case Subset Plithogenic Set (when the degrees are any subsets included into [0, 1]).

Plithogenic References

1. F. Smarandache, *Plithogenic Set, an Extension of Crisp, Fuzzy, Intuitionistic Fuzzy, and Neutrosophic Sets – Revisited*, Neutrosophic Sets and Systems, vol. 21, 2018, pp. 153-166; http://fs.unm.edu/NSS/PlithogenicSetAnExtensionOfCrisp.pdf https://doi.org/10.5281/zenodo.1408740

2. Florentin Smarandache, *Plithogeny, Plithogenic Set, Logic, Probability, and Statistics*, Pons Publishing House, Brussels, Belgium, 141 p., 2017; arXiv.org (Cornell University), Computer Science - Artificial Intelligence, 03Bxx: book: https://arxiv.org/ftp/arxiv/papers/1808/1808.03948.pdf abstract: https://arxiv.org/abs/1808.03948 Harvard SAO/NASA ADS: http://adsabs.harvard.edu/cgi-bin/bib_query?arXiv:1808.03948

3. Florentin Smarandache, *Plithogeny, Plithogenic Set, Logic, Probability, and Statistics*, Infinite Study Publ. Hse., GoogleLLC, Mountain View, California, USA, 2017.

4. Florentin Smarandache, *Physical Plithogenic Set*, 71st Annual Gaseous Electronics Conference, Session LW1, Oregon Convention Center Room, Portland, Oregon, USA, November 5–9, 2018; http://meetings.aps.org/Meeting/GEC18/Session/LW1.110

5. Florentin Smarandache, Extension of Soft Set to Hypersoft Set, and then to Plithogenic Hypersoft Set. Neutrosophic Sets and Systems, Vol. 22, 2018, pp. 168-170; http://fs.unm.edu/NSS/ExtensionOfSoftSetToHypersoftSet.pdf http://doi.org/10.5281/zenodo.2838716

6. Mohamed Abdel-Basset, Rehab Mohamed, Abd El-Nasser H. Zaied, and Florentin Smarandache, *A Hybrid Plithogenic Decision-Making Approach with Quality Function Deployment for Selecting Supply Chain Sustainability Metrics,* Symmetry 2019, 11, 903, pp. 1-21; doi:10.3390/sym11070903; https://www.mdpi.com/2073-8994/11/7/903

7.23 Plithogenic Set

Plithogenic set is a set P whose elements x are characterized by many attribute values $v_1, v_2, ..., v_n$, and the generic element x belongs to the set P with respect to each attribute value with a fuzzy / intuitionistic fuzzy / or neutrosophic degree:

7.23.1 Plithogenic Fuzzy Set

$x(v_1(t_1), v_2(t_2), ..., v_n(t_n))$

7.23.2 Plithogenic Intuitionistic Fuzzy Set

$x(v_1(t_1, f_1), v_2(t_2, f_2), ..., v_n(t_n, f_n))$,

with $0 \leq t_j + f_j \leq 1$, for all $j \in \{1, 2, …, n\}$.

7.23.3 Plithogenic Picture Fuzzy Set

$x(v_1(t_1, e_1, f_1), v_2(t_2, e_2, f_2), ..., v_n(t_n, e_n, f_n))$,

with $0 \leq t_j + e_j + f_j \leq 1$, for all $j \in \{1, 2, …, n\}$.

7.23.4 Plithogenic Neutrosophic Set

$x(v_1(t_1, i_1, f_1), v_2(t_2, i_2, f_2), ..., v_n(t_n, i_n, f_n))$,

with $0 \leq t_j + i_j + f_j \leq 3$, for all $j \in \{1, 2, …, n\}$.

where $t_j, i_j, f_j \in [0, 1]$ are degrees of membership, indeterminacy, and nonmembership respectively.

Plithogenic Set is much used in Multi-Criteria Decision Making.

Plithogenic References

7. F. Smarandache, *Plithogenic Set, an Extension of Crisp, Fuzzy, Intuitionistic Fuzzy, and Neutrosophic Sets – Revisited*, Neutrosophic Sets and Systems, vol. 21, 2018, pp. 153-166; http://fs.unm.edu/NSS/PlithogenicSetAnExtensionOfCrisp.pdf
https://doi.org/10.5281/zenodo.1408740

8. Florentin Smarandache, *Plithogeny, Plithogenic Set, Logic, Probability, and Statistics*, Pons Publishing House, Brussels, Belgium, 141 p., 2017;
    arXiv.org (Cornell University), Computer Science - Artificial Intelligence, 03Bxx:
    book: https://arxiv.org/ftp/arxiv/papers/1808/1808.03948.pdf
    abstract: https://arxiv.org/abs/1808.03948
    Harvard SAO/NASA ADS: http://adsabs.harvard.edu/cgi-bin/bib_query?arXiv:1808.03948
9. Florentin Smarandache, *Plithogeny, Plithogenic Set, Logic, Probability, and Statistics*, Infinite Study Publ. Hse., GoogleLLC, Mountain View, California, USA, 2017.
10. Florentin Smarandache, *Physical Plithogenic Set*, 71st Annual Gaseous Electronics Conference, Session LW1, Oregon Convention Center Room, Portland, Oregon, USA, November 5–9, 2018;
    http://meetings.aps.org/Meeting/GEC18/Session/LW1.110
11. Florentin Smarandache, Extension of Soft Set to Hypersoft Set, and then to Plithogenic Hypersoft Set. Neutrosophic Sets and Systems, Vol. 22, 2018, pp. 168-170;
    http://fs.unm.edu/NSS/ExtensionOfSoftSetToHypersoftSet.pdf
    http://doi.org/10.5281/zenodo.2838716
12. Mohamed Abdel-Basset, Rehab Mohamed, Abd El-Nasser H. Zaied, and Florentin Smarandache, *A Hybrid Plithogenic Decision-Making Approach with Quality Function Deployment for Selecting Supply Chain Sustainability Metrics,* Symmetry 2019, 11, 903, pp. 1-21; doi:10.3390/sym11070903;
    https://www.mdpi.com/2073-8994/11/7/903

7.24 Neutrosophic / Plithogenic Entropies

I think in a similar way to Neutrosophic Entropies it is possible to define Plithogenic Entropies, making the distance between a plithogenic set P and its plithogenic complement P^C.

My question for the many definitions of neutrosophic / plithogenic entropies: which one is the best?

How to identify the most accurate neutrosophic / plithogenic entropy formula?

Does it depend on the application, on expert, or on other parameters?

7.25 Plithogenic Graph

We can define and study the plithogenic graph, never studied before, as a graph whose vertices are elements of a plithogenic set, while the graph's edges may be: crisp, fuzzy, intuitionistic fuzzy, neutrosophic, or plithogenic relationships between vertices.

For example, a vertex A of the plithogenic graph may be characterized by many attribute values; for example, the attribute "color" and its attribute values: white, blue, violet, green, black;

we write $A(d(\text{white}), d(\text{blue}), d(\text{violet}), d(\text{green}), d(\text{black}))$,

where $d(\text{white})$ means degree of white, etc.

Reference

F. Smarandache: *Plithogenic Set, an Extension of Crisp, Fuzzy, Intuitionistic Fuzzy, and Neutrosophic Sets – Revisited*, in *Neutrosophic Sets and Systems*, vol. 21, 2018, pp. 153-166;

http://fs.unm.edu/NSS/PlithogenicSetAnExtensionOfCrisp.pdf

7.26 Dialectics is Incomplete

We talk about Hegel's dialectics, but actually it is Chinese Yin-Yang philosophy (dynamic of the opposites) that appeared more than 2,500 years ahead of Hegel and Marx!

And, by the way, dialectics is incomplete, because in our world there is not only the dynamic of opposites, but also the dynamic of opposites and the neutrality between them [called neutrosophy].

When two countries go to war, other neutral countries help in one side or the other.

In the world of ideas, sometimes contradictory ideas are resolved (reconciled) by the middle (neutral) between them, which may be a mixture of opposites (a degree of an opposite, plus a degree of the other opposite).

So, the world is more complicated than black and white [opposites], but comprising also a grey area [neutrality or mixture of opposites] in between.

http://fs.unm.edu/Neutrosophy-A-New-Branch-of-Philosophy.pdf

7.27 A Neutrosophic Dynamic System: Easier to break from inside, than from outside

Smarandache and Vatuiu enounced the law that:

"It is easier to break from inside, than from outside." This is a *neutrosophic dynamic system*.

A neutrosophic dynamic system is a dynamic system that has some indeterminacy, but all real dynamic systems have some indeterminacy.

Only the idealistic and pure-science dynamic systems may be indeterminacy-free - in theory.

Now, countries are easily destroyed from inside (by spies, saboteurs etc.) than from outside - militarily.

7.28 Capitalism and Communism blended

Each society has a degree of communism and a degree of capitalism (as in neutrosophy, and as in life in general).

And the degrees fluctuate between extremes in each society, going either closer to capitalism, then moving back towards the communism, and so on.

A capitalist country has a higher degree of capitalism than the communism, and therefore inversely a communist country;

while a socialist country is somewhere near the middle in between communism and capitalism.

7.29 Degree of Democracy, Degree of Indeterminate-Democracy, and Degree of Antidemocracy

In all countries (61) that I visited I observed good, bad, and neutral/indeterminate things, each of them in some degree different from a country to another.

I observed some degree of democracy, another degree of antidemocracy, and a third degree of indeterminate-democracy (contradictory ideas/lays/behaviors etc.) present in each society.

Example of Indeterminate-Democracy degree: "abortion"; the religious people say it kills a life (which is true); the women say that they are masters of their bodies (which is true also, so they may have the right to abortion).

How one measures the democracy? I mean, why today is a better democracy than yesterday? What new democratic elements have been added in the meantime?

7.30 Neutrosophic Example in Military

A simple neutrosophic example in military is in Target Identification: a plane is detected on the sky, this may be: friendly, neutral, enemy. Then we have several sources that give information (t, i, f) about the nature of this plane. Then we use the neutrosophic conjunction to find an optimal estimation about the plane.

7.31 Neutrosophic Random Variable

In general, a neutrosophic random variable is a random variable that has some indeterminacy (with respect to its argument or/and with respect to its values): http://fs.unm.edu/NeutrosophicMeasureIntegralProbability.pdf http://fs.unm.edu/NeutrosophicStatistics.pdf.

7.32 Neutrosophic Risk

There is an optimist risk and there is a pessimist risk.

I think one can extend it to

Neutrosophic Risk = (Optimistic Risk, Pessimistic Risk, Neutral (or Indeterminate) Risk).

Let us give an example to prove the existence of neutral risk.

Playing the lottery. Suppose the ticket is \$10.00.

The Optimistic Risk (R_o) is to gain > \$10, therefore you gain money.

The Pessimistic Risk (R_p) is to gain between [0, 10), therefore you lose money.

The Neutral Risk (R_n) is to gain just \$10 (therefore you neither gain, nor loose).

In a neutrosophic set, a generic element x has the neutrosophic coordinates (t, i, f), whence one endows each of the components with the risk possibilities:

$$(t(R_o, R_n, R_p), i(R_o, R_n, R_p), f(R_o, R_n, R_p)).$$

Something similar may be done for the plithogenic set.

In a plithogenic set A, a generic element x is characterized by many attribute values $v_1, v_2, ..., v_n, n \geq 1$.

The element x has, with respect to each attribute value v, a degree of appurtenance t with respect to the set A. We write:

$$x(v_1(t_1), v_2(t_2), ..., v_n(t_n)).$$

We considered the easiest type of degree, the fuzzy degree (t), but we can also consider intuitionistic fuzzy degrees (t, f), neutrosophic degrees (t, i, f) etc.

Then each (fuzzy, intuitionistic fuzzy, neutrosophic, or other types of sets) degree has some degree of risk.

A plithogenic set, with fuzzy attribute value degree, and neutrosophic risk degree:

$$x(v_1(t_1(Ro_1, Rn_1, Rp_1)), \ v_2(t_2(Ro_2, Rn_2, Rp_2)), \ ...,$$

$$v_n(t_n(Ro_n, Rn_n, Rp_n))).$$

One can use Neutrosophic Probability, see:

http://fs.unm.edu/NeutrosophicMeasureIntegralProbability.pdf

i.e.

> chance of a risk factor to occur (*event E*),
>
> chance that the risk factor will not occur (*event antiE*),
>
> indeterminate chance - not sure if it will occur or not (*neutE*).

7.33 Neutrosophic Satisfiability & Neutrosophic Randomness

1) A *Boolean Formula* (or *Expression*) F_B is constructed from Boolean variables $(x_1, x_2, ..., x_n)$,

with $n \geq 1$, parentheses, and Boolean operators {*AND* (conjunction) $\wedge$, *OR* (disjunction) $\vee$, *NOT* (negation) $\neg$}. The Boolean formula is well-defined, i.e. it makes sense in the Boolean algebra.

We denote it by $F_B(x_1, x_2, ..., x_n)$.

Each variable may take the Boolean values: *0* (*False*), or *1* (*True*).

The *Boolean Formula F_B* is said to be *satisfiable,* if $F_B(x_1, x_2, ..., x_n) = 1$ for some values *0* or *1* assigned to each of its *n* variables.

Otherwise it is called *unsatisfiable*, i.e. when $F_B(x_1, x_2, ..., x_n) = 0$ for all 2^n possible assignments of values *0* or *1* to its variables.

The Boolean satisfiability problem (SAT) is used in artificial intelligence.

2) The *Degree of Boolean Randomness*, considered as the degree/measure of uncertainty in a random process (where the order of events is unpredictable), is:

$$\frac{2^n - m}{2^n}$$

where 2^n represents all possible values of the n-tuple $(x_1, x_2, ..., x_n)$, when each variable x_j may take the value 0 or 1, and m is the number of solutions (of n-tuples) of the equation $F_B(x_1, x_2, ..., x_n) = 1$.

3) A *Single-Valued Neutrosophic Formula* (or *Expression*) F_N is constructed from Neutrosophic Variables $(x_1(t_1, i_1, f_1), x_2(t_2, i_2, f_2), ..., x_n(t_n, i_n, f_n))$, with $n \geq 1$, parentheses, and Neutrosophic Operators $\{AND_N$ (neutrosophic conjunction) $\wedge_N$, OR_N (neutrosophic disjunction) $\vee_N$, NOT_N (neutrosophic negation) $\neg_N\}$.

The neutrosophic formula is considered well-defined, i.e. it makes sense in the neutrosophic environment.

We denote it by $F_N(x_1(t_1, i_1, f_1), x_2(t_2, i_2, f_2), ..., x_n(t_n, i_n, f_n))$.

Each variable x_k, $1 \leq k \leq n$, may take the neutrosophic values: $t_k, i_k, f_k \in [0, 1]$.

Let's consider a *neutrosophic tautological threshold* $\tau(t_\tau , i_\tau , f_\tau)$; then each neutrosophic proposition P whose neutrosophic truth value is equal to or above/greater than the neutrosophic truth value of this neutrosophic tautological threshold should be considered a **neutrosophic tautology**; while if it is below it should be a *neutrosophic non-tautology*; in addition there are neutrosophic propositions whose neutrosophic truth value is neither above nor below the neutrosophic tautological threshold; they are called *neutrosophically undecided propositions* [1]. How to establish such threshold? Of course, this should be handled by experts upon the application or problem they need to solve.

The neutrosophic inequality $\leq_N$ is defined as:

$(t_1, i_1, f_1) \leq_N (t_2, i_2, f_2)$ iff $t_1 \leq t_2, i_1 \geq i_2, f_1 \geq f_2$,

where $t_1, i_1, f_1, t_2, i_2, f_2 \in [0, 1]$, and $>, \geq , <, \leq$ are classical inequalities.

Then, a *Neutrosophic Formula F_N* is said to be *satisfiable,* if $F_B(x_1, x_2, ..., x_n) \geq_N \tau(t_\tau, i_\tau, f_\tau)$, for some values between *[0, 1]* assigned to all t_k, i_k, f_k neutrosophic components of its *n* variables.

It is called *unsatisfiable*, i.e. when $F_B(x_1, x_2, ..., x_n) \leq_N \tau(t_\tau, i_\tau, f_\tau)$, for all possible assignments of values between *[0, 1]* to its variables' neutrosophic components.

Or it is called *undecidable* [neither satisfiable nor unsatisfiable], if $F_B(x_1, x_2, ..., x_n)$ is neither $\leq_N \tau(t_\tau, i_\tau, f_\tau)$ nor $>_N \tau(t_\tau, i_\tau, f_\tau)$.

7.33.1 Open Question.

How the calculate the *Degree of Neutrosophic Randomness*, considered as the neutrosophic degree/measure of uncertainty in a neutrosophic random process (where the order of events is unpredictable)?

A neutrosophic random process is based on the process of many neutrosophic random variables, which are variables whose outputs contain indeterminacy.

Reference:

[1] Florentin Smarandache, *Neutrosophic Perspectives: Triplets, Duplets, Multisets, Hybrid Operators, Modal Logic, Hedge Algebras. And Applications*. Pons Editions, Bruxelles, second edition 346 p., September 2017; pages 176-179; http://fs.unm.edu/NeutrosophicPerspectives-ed2.pdf

7.34 Neutrosophic Statistics vs. Interval Statistics

– Interval Statistics uses *Interval Analysis*, i.e. intervals instead of crisp numbers in order to approximate/capture the data inside the interval.

Neutrosophic Statistics uses *Set Analysis*, i.e. any type of sets, non only intervals [either intervals as before, either hesitant sets (discrete finite sets) such as for example {0.2, 0.3, 1.2}, or any kind of sets such as for example [1, 3] ∨ {3.1, 3.2}, or (1, 2) ∨ {3, 4, 5} ∨ [6, 7), etc.] to approximate/capture the data inside.

Set Analysis is a generalization of Interval Analysis {see my book "Neutrosophic Precalculus and Neutrosophic Calculus" (pages 11-16)}: http://fs.unm.edu/NeutrosophicPrecalculusCalculus.pdf.

– Neutrosophic Statistics takes also into consideration *the degrees of appurtenance of the individuals to a sample or to a population* (for example, some individuals only partially belong and partially do not belong to a sample or to a population), while Classical Statistics and Interval Statistics do not.

In Classical Statistics and Interval Statistics all individuals are mutually supposed belonging 100% to the sample and to the population.

– In Neutrosophic Statistics, it is allowed to the sample size or population size to not be well known [as it happens in our everyday life], while in Classical Statistics and Interval Statistics they have to be well known.

– Neutrosophic Statistics takes into consideration the Indeterminacy, but Indeterminacy may be of many different types, whence different types of Neutrosophic Statistics.

We may have, for example, various indeterminate functions that may be neutrosophic distributions.

For example:

The function: $f(2) = 5$ or 6, which means that we do not know if $f(2) = 5$ or $f(2) = 6$.

Or the function: $g(3 \text{ or } 5) = 7$, which means that we do not know if $g(3) = 7$ or $g(5) = 7$.

– The $N = a + bI$ neutrosophic number, where a, b are real or complex numbers and I = indeterminacy, is in general different from an interval, because I can be any subset of R (set of all real numbers) or of C (set of complex numbers).

For example, if $a = 3$, $b = 2$ and $I = \{0.1, 4.0, 9.6\}$, then

$$N = a+bI = 3 + 2I = 3 + 2 \cdot \{0.1, 4.0, 9.6\} =$$

$$= 3 + \{0.2, 8.0, 19.2\} = \{3.2, 11, 22.2\},$$

which is a more accurate approach than taking *I = [0.1, 9.6]* as an interval, and then obtaining *[3.2, 22.2]* that is an interval containing infinitely many numbers that we need to choose from in Interval Analysis, instead of only three numbers that we need to choose from as in Set Analysis *{3.2, 11, 22.2}*.

– Based on Neutrosophic Probability, a Neutrosophic Probability Distribution may be constituted from three curves [not a single one as in Classical Statistics and Interval Statistics]: one curve that represents the degree or occurring, second curve representing the degree of indeterminate occurrence, and third which represent the degree of non-occurrence.

See my book "Introduction to Neutrosophic Measure, Neutrosophic Integral, and Neutrosophic Probability", 140 p., 2013, http://fs.unm.edu/NeutrosophicMeasureIntegralProbability.pdf.

So, Neutrosophic Probability Distribution is more detailed, gives more information on the distribution function.

Let us come up with an example.

Suppose a candidate *C* runs for the presidency of Pakistan.

> Curve 1: those who might vote for him.
>
> Curve 2: those who might not vote, or cast a blank or a black vote.
>
> Curve 3: those who might vote against him.

Make a graph with these three curves.

Take any distribution, let's say normal distribution with respect to an attribute value <A>.

Then the normal distribution with respect to the opposite of this attribute value (let's call it <antiA>), and afterwards the normal distribution with respect to their neutral attribute value (<neutA>).

And make a curve for each of them, all curves on the same graph.

7.35 Grey System Theory as a Neutrosophication

A Grey System is referring to a grey area (as <neutA> in neutrosophy), between extremes (as <A> and <antiA> in neutrosophy).

According to the Grey System Theory, a system with perfect information (<A>) may have a unique solution, while a system with no information (<antiA>) has no solution. In the middle (<neutA>), or grey area, of these opposite systems, there may be many available solutions (with partial information known and partial information unknown) from which an approximate solution can be extracted.

Reference

J. L. Deng: *Introduction to Grey System Theory*, in *The Journal of Grey System*, 1(1): 1-24, 1989.

7.36 Neutrosophication vs. Regret Theory

Regret Theory (2010) is actually a Neutrosophication (1998) model, when the decision making area is split into three parts, the opposite ones (upper approximation area, and lower approximation area) and the neutral one (border area, in between the upper and lower area).

References

H. Bleichrodt, A. Cillo, E. Diecidue: *A quantitative measurement of regret theory*, *Manage. Sci.* 56(1) (2010) 161-175.

F. Smarandache: *Neutrosophy. / Neutrosophic Probability, Set, and Logic*, ProQuest Information & Learning, Ann Arbor, Michigan, USA, 105 p., 1998.

7.37 Modern vs. Classic

I do not think that neutrosophic statistics, from which has been extracted the indeterminacy, should be exactly reduced to classical statistics. This is not strident to be true.

Many laws are altered when passing from classical field to modern field: for example the Law of Excluded Middle in classical logic was replaced by the Law of Included Middle in intuitionistic fuzzy logic and

in neutrosophic logic, while in refined neutrosophic logic it became Law of Included Multiple-Middle...

7.38 Refinement means Detailed Information

Refinement is needed when detailed information is needed.

For example in voting process in a given country:

T = percentage of people voting for the candidate;

I = percentage of people not voting or casting a black or a white vote;

F = percentage people voting against the candidate.

But the political analysts want to know in detail what happened in each region of the country in order to take care for future elections. This is refinement.

So:

T1 = percentage of people from Region 1 voting for the candidate;

I1 = percentage of people from Region 1 not voting or casting a black or a white vote;

F1 = percentage of people from Region 1 voting against the candidate.

T2 = percentage of people from Region 2 voting for the candidate;

I2 = percentage of people from Region 2 not voting or casting a black or a white vote;

F2 = percentage of people from Region 2 voting against the candidate.

Etc.

7.39 Expert Systems vs. Neutrosophic Implications

"Expert Systems (ES) are not necessarily based on exact rules but are often based on non-evaluated assumptions, and hence answers are produced as statistically fuzzy conclusions." [from ResearchGate.net]

Expert Systems are similar to fuzzy / intuitionistic fuzzy / and neutrosophic If-THEN rules, i.e.: "If A then B" or "A --> B", where A and B are fuzzy / intuitionistic fuzzy / neutrosophic propositions, but using the fuzzy / intuitionistic fuzzy / or neutrosophic implications.

These rules/implications are approximations, of course, since their premises are approximations too (i.e. not 100% true as in classical logic).

I extended Luckasiewicz' four-valued logic named VL4 to n-valued refined neutrosophic logic, symbolically and numerically; see this article:

http://fs.unm.edu/n-ValuedNeutrosophicLogic-PiP.pdf .

7.40 Neutrosophic Applications in Literature, Arts, Criminal Justice, Philosophy, and History

New applications: neutrosophic analysis of literature creature, neutrosophic analysis of the arts, neutrosophical criminal justice (i.e. laws that are contradictory, for example marijuana is prohibited for ordinary citizen, yet marijuana is allowed in medical treatment; etc.), in history (opposite historical events, etc.).

Mustapha Kachchouh had a good idea to consider three categories of people, as in neutrosophy (<A>, <neutA>, <antiA>):

people who believe in God,

people who partially believe and partially do not believe,

and people who do not believe in God.

7.41 Neutrosophy in Arts and Letters

It is possible to use the neutrosophy (based on opposites and neutrals, <A>, <antiA>, and <neutA>) in art critics and literature essays, for example in Comparative Literature, or Comparative Art, i.e. making comparisons between the study work with respect to opposite and neutral works.

Similarly as we did with Maikel Leyva Vazquez with neutrosophy used in study of Marti's poetics.

In the first chapter (**Neutrosophic Set is a Generalization of Intuitionistic Fuzzy Set, Inconsistent Intuitionistic Fuzzy Set (Picture Fuzzy Set, Ternary Fuzzy Set), Pythagorean Fuzzy Set (Atanassov's Intuitionistic Fuzzy Set of second type), q-Rung Orthopair Fuzzy Set, Spherical Fuzzy Set, and n-HyperSpherical Fuzzy Set, while Neutrosophication is a Generalization of Regret Theory, Grey System Theory, and Three-Ways Decision - revisited**), we prove that Neutrosophic Set (NS) is an extension of Intuitionistic Fuzzy Set (IFS) no matter if the sum of single-valued neutrosophic components is < 1, or > 1, or = 1. For the case when the sum of components is 1 (as in IFS), after applying the neutrosophic aggregation operators one gets a different result from that of applying the intuitionistic fuzzy operators, since the intuitionistic fuzzy operators ignore the indeterminacy, while the neutrosophic aggregation operators take into consideration the indeterminacy at the same level as truth-membership and falsehood-nonmembership are taken. NS is also more flexible and effective because it handles, besides independent components, also partially independent and partially dependent components, while IFS cannot deal with these.

In the second chapter (**Refined Neutrosophy & Lattices vs. Pair Structures & YinYang Bipolar Fuzzy Set**), we present the lattice structures of neutrosophic theories, we prove that Zhang-Zhang's YinYang Bipolar Fuzzy Set is a subclass of Single-Valued Bipolar Neutrosophic Set. Then we show that the Pair Structure is a particular case of Refined Neutrosophy, and the number of types of neutralities (sub-indeterminacies) may be any finite or infinite number.

The third chapter (**About Nonstandard Neutrosophic Logic - Answers to Imamura's "Note on the Definition of Neutrosophic Logic"**) intends to answer Imamura's criticism that we found benefic in better understanding the nonstandard neutrosophic logic – although the nonstandard neutrosophic logic was never used in practical applications.

In the fourth chapter (**Extended Nonstandard Neutrosophic Logic, Set, and Probability based on Extended Nonstandard Analysis**), we extend for the second time the Nonstandard Analysis by adding the left monad closed to the right, and right monad closed to the left, while besides the pierced binad (we introduced in 1998) we add now the unpierced binad - all these in order to close the newly extended nonstandard space under nonstandard addition, nonstandard subtraction, nonstandard multiplication, nonstandard division, and nonstandard power operations.

The fifth chapter (**Plithogenic Set and Hypersoft Set**) has two parts. The first part (**Plithogenic Set, an Extension of Crisp, Fuzzy, Intuitionistic Fuzzy, and Neutrosophic Sets - revisited**) introduces the plithogenic set (as generalization of crisp, fuzzy, intuitionistic fuzzy, and neutrosophic sets), which is a set whose elements are characterized by many attributes' values. An attribute value v has a corresponding (fuzzy, intuitionistic fuzzy, neutrosophic or other types of sets) degree of appurtenance d(x,v) of the element x, to the set P, with respect to some given criteria. This article offers some examples and applications of these new concepts in our everyday life. The second part (**Extension of Soft Set to Hypersoft Set, and then to Plithogenic Hypersoft Set**) generalizes the soft set to the hypersoft set by transforming the function F into a multi-attribute function. Then we introduce the hybrids of Crisp, Fuzzy, Intuitionistic Fuzzy, Neutrosophic, and Plithogenic Hypersoft Set.

In the sixth chapter (**Introduction to NeutroAlgebraic Structures and AntiAlgebraic Structures - revisited**), we open for the first time new fields of research called NeutroStructures and AntiStructures respectively. In all classical algebraic structures, the Laws of Compositions on a given set are well-defined. But this is a restrictive case, because there are many more situations in science and in any domain of knowledge when a law of composition defined on a set may be only partially-defined (or partially true) and partially-undefined (or partially false), that we call NeutroDefined, or totally undefined (totally false) that we call AntiDefined.

Finally, the seventh chapter (**New Developments in Neutrosophic Theories and Applications**) presents suggestions for future research in the area of neutrosophics.