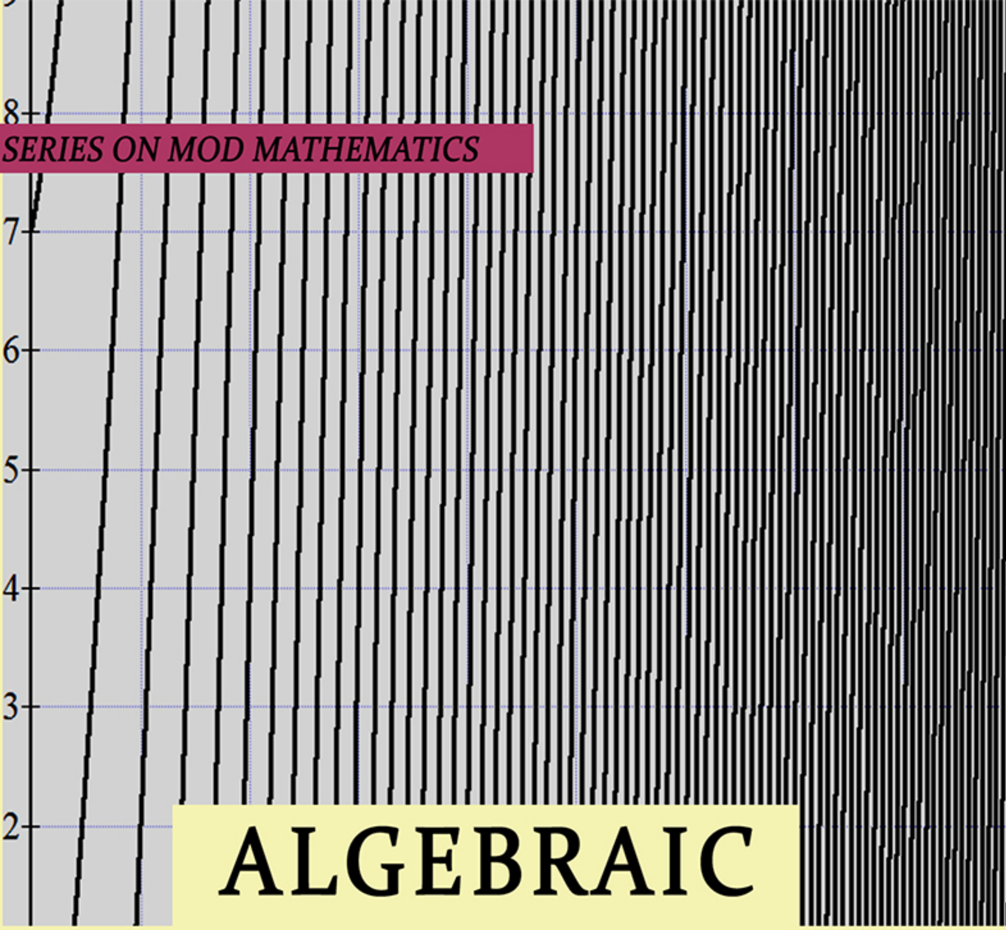

SERIES ON MOD MATHEMATICS

ALGEBRAIC STRUCTURES ON MOD PLANES

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CONTENTS

PREFACE

Study of MOD planes happens to a very recent one. Authors have studied several properties of MOD real planes $R_n(m)$; $2 \leq m \leq \infty$. In fact unlike the real plane $R \times R$ which is unique MOD real planes are infinite in number. Likewise MOD complex planes $C_n(m)$; $2 \leq m \leq \infty$, are infinitely many.

The MOD neutrosophic planes $R_n^I(m)$; $2 \leq m \leq \infty$ are infinite in number where as we have only one neutrosophic plane $R(I) = \langle R \cup I \rangle = \{a + bI \mid I^2 = I;\ a, b \in R\}$. Further three other new types of MOD planes constructed using dual numbers, special dual number like numbers and special quasi dual numbers are introduced.

$R_n^g(m) = \{a + bg \mid g^2 = 0,\ a, b \in [0, m)\}$ is the MOD dual number plane.

$R_n^h(m) = \{a + bh \mid h^2 = h; a, b \in [0, m)\}$ is the MOD special dual like number plane and $R_n^k(m) = \{a + bk \mid k^2 = (m - 1)k; a, b \in [0, m)\}$ is the MOD special quasi dual number plane.

Finally we have only one MOD fuzzy plane $R_n(1)$; which is unique.

Systematically algebraic structures on MOD planes like, MOD semigroups, MOD groups and MOD rings of different types are defined and studied.

Such study is very new and innovative for a large four quadrant planes are made into a small MOD planes. Several distinct features enjoyed by these MOD planes are defined, developed and described in this book.

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Chapter One

INTRODUCTION

MOD planes of different types was introduced in [24].

Now in this book authors give algebraic structures on MOD planes or small planes. We aim here to study one single associative binary operation on MOD planes. We use the term only MOD planes in this book.

The main notations used are as in [24].

However to make easy for the reader we just recall some essential notations. $R_n(5)$ denotes the MOD real plane built using the semi open interval $[0,5)$; $R_n(5) = \{(a, b) / a, b \in [0,5)\}$.

Thus in general $R_n(m)$ denotes the small real plane or MOD real plane associated with the MOD interval $[0, m)$, where m is a positive integer.

Now $R_n^I(9) = \{a + bI \mid a, b \in [0, 9); I^2 = I\}$ denotes the MOD neutrosophic plane or small neutrosophic plane associated with the interval $[0, 9)$.

Hence in general $R_n^I(m) = \{a + bI \mid a, b \in [0, m), I^2 = I\}$ denotes the MOD neutrosophic plane associated with the interval $[0, m)$.

We call $C_n(10) = \{a + bi_F \mid a, b \in [0, 10), i_F^2 = 9\}$ to be the MOD complex modulo integer plane associated with the interval [0, 10).

Thus $C_n(m) = \{a, + bi_F \mid a, b \in [0, m); i_F^2 = m - 1\}$ denotes the MOD complex modulo integer plane associated with the interval [0, m).

$R_n(8)g = \{a + bg \mid a, b \in [0, 8); g^2 = 0\}$ is the MOD dual number plane.

So in general $R_n(m)(g) = \{a + bg \mid a, b \in [0, m), g^2 = 0\}$ is the MOD dual number plane associated with the interval [0, m).

Let $R_n(10)(g) = \{a + bg \mid a, b \in [0, 10), g^2 = g\}$ be the MOD special dual like number plane associated with the interval [0, 10).

Hence $R_m(m)(g) = \{a + bg \mid a, b \in [0, m), g^2 = g\}$ denotes the special dual like number plane associated with the interval [0, m).

$R_n(15)(g) = \{a + bg \mid a, b \in [0, 15), g^2 = 14g\}$ denotes the MOD special quasi dual number plane associated with the interval [0, 15).

Thus $R_n(m)(g) = \{a + bg \mid a, b \in [0, m), g^2 = (m - 1)g\}$ is the MOD special quasi dual number plane associated with the interval [0, m).

Finally we define the MOD fuzzy plane which is only one plane given by $R_n(1) = \{(a, b) \mid a, b \in [0, 1)\}$. This plane will replace the real plane appropriately.

Thus for more about these concepts refer [24].

In this book groups and semigroups of various types are defined, described and developed.

Chapter Two

ALGEBRAIC STRUCTURES USING MOD SUBSETS OF THE MOD PLANES

We have in [24] introduced the concept of MOD planes like, real MOD plane $R_n(m)$, complex MOD plane $C_n(m)$, dual number MOD plane $R_n(m)(g)$, neutrosophic MOD plane, fuzzy MOD plane and so on.

Here we use the subsets of MOD planes and build algebraic structures like semigroup and group.

Such study will later be used in the construction of MOD linear algebras and MOD subset topological spaces of special types.

DEFINITION 2.1: *Let $S = S(R_n(m))$ = {Collection of all subsets of the MOD real plane}. S is defined as the MOD real subset collection of $R_n(m)$.*

Example 2.1: Let $S(R_n(9)) = \{\{(0, 0), (1, 7), (0.001, 0)\}, \{(0, 1)\}, \{(1, 0)\}, \{(0.1, 0.00052), (0, 0), (1, 0.2), (0.5, 0.1)\}$ and so on$\}$ be the MOD real subsets of $R_n(9)$.

Example 2.2: Let $S(R_n(11)) = \{\{(0, 10), (10, 0)\}, \{(0, 0), (1, 0), (1, 10), (10, 0.1), (0.7, 0.5)\}, \{(0.7, 0), (0.8, 0.4), (0.5, 0.7)\}, \ldots$ and so on$\}$ be the MOD real subsets of $R_n(11)$.

Example 2.3: Let $S(R_n(12)) = \{\{(0, 11), (11, 11), (3, 3), (2, 3)\}, \{(10, 10), (3, 5), (5, 8), (8, 3)\}, \{(3, 8), (3, 3), (2, 2), (1, 1)\}\ldots$ and so on$\}$ be the MOD real subsets of $R_n(12)$.

DEFINITION 2.2: *$S(C_n(m))$ = {Collection of all subsets of $C_n(m)$ with $i_F^2 = (m-1)$} is defined as the MOD complex subsets of $C_n(m)$.*

We will illustrate this situation by some examples.

Example 2.4: Let $S(C_n(12)) = \{\{9 + 10i_F\}, \{5 + 3.3i_F, 2.1, 5.6i_F\}, \{5.31 + 0.3i_F\}, \{0.72 + 3.2i_F, 11.3 + 4.1i_F\}\ldots$; and so on $i_F^2 = 11\}$ be the MOD complex subsets of $C_n(12)$.

Example 2.5: Let $S(C_n(17))$ = {Collection of all subsets of $C_n(17)$, the MOD complex plane} = $\{\{(0, 0), (1, 0.1), (0.7, 0.8)\}, \{(1, 0), (1, 0.8), (0, 0.101), (0, 0)\}, \ldots$ and so on$\}$ be the MOD complex subsets of $C_n(17)$.

DEFINITION 2.3: *Let $S(R_n^I(m))$ = {Collection of all subsets of the MOD neutrosophic plane $R_n^I(m)$}. $S(R_n^I(m), I^2 = I)$ is defined as the MOD neutrosophic subset of the MOD neutrosophic plane $R_n^I(m)$.*

We will illustrate this situation by some examples.

Example 2.6: Let $S(R_n^I(10)) = \{\{5 + 0.3I, 0 + 4I, 3.2 + 0.11I\}, \{0\}, \{0, 1, I, 4I, 0.0004I\}$ and so on$\}$ be the subsets of MOD neutrosophic plane.

Example 2.7: Let $S(R_n^I(17))$ = {Collection of all subsets of the neutrosophic MOD plane $R_n^I(17)$, $I^2 = I$}

$= \{\{0, I, 0.7I, 0.2 + 4I\}, \{0.4I, 0.7 + 0.2I, 0.6I\}\ldots$ and so on$\}$ be the MOD subsets of the neutrosophic MOD plane.

DEFINITION 2.4: *Let $S(R_n(1))$ = {Collection of all subsets of the MOD fuzzy set plane}. $S(R_n(1))$ is defined as the MOD subset of fuzzy MOD plane $R_n(1)$.*

Example 2.8: Let $S(R_n(1)) = \{\{(0, 0), (0, 0.5), (0.4, 0.1), (0.8, 0)\}, \{(0, 0), (0, 0.1), (0.4, 0.2)\}, \{(0.001, 0.001), (0.01, 0.01), (0.4, 0.4)\},\ldots$ and so on} be the MOD subsets of MOD fuzzy plane.

We have only one plane and only one MOD fuzzy subset collection.

Next we proceed on to define the notion of MOD subsets of MOD dual number plane.

DEFINITION 2.5: *Let $S(R_n(m)(g))$ = {Collection of all subsets from the MOD dual number plane $R_n(m)(g)$; $g^2 = 0$}. $S(R_n(m)(g))$ is defined as the MOD subsets of the MOD dual number plane.*

We will give an example or two.

Example 2.9: Let $S(R_n(20)(g)) = \{\{0.9 + 0.8g\}, \{0\}, \{g\}, \{0.7g, 0.9 + 1.2g\}, \ldots$ and so on} be the collection of all MOD subsets of the MOD dual number plane $R_n(20)(g)$.

Example 2.10: Let $S(R_n(11)(g)) = \{\{g\}, \{10g\}, \{10\}, \{g, 0\}, \{10g, 0.1 + 0.48g\},\ldots$ and so on} be the collection of all MOD subsets of the MOD dual number plane $R_n(11)(g)$.

Likewise we can define MOD subsets of special dual like number MOD planes and MOD subsets of special quasi dual number MOD planes.

We will illustrate this situation by some examples.

Example 2.11: $S(R_n(m)(g))$ = {Collection of all subsets of the MOD special dual like number plane $R_n(m)(g)$ with $g^2 = g$} is the MOD subset special dual like number plane.

Example 2.12: Let $S(R_n(12)(g))$ = {Collection of all subsets of the MOD special dual like number plane $g^2 = g$} = {{0, g}, {0.3 + 4g, 0.8 + 0.93g}, {1, 0.3g, 1.4, 1.5 + 2.7g}, … and so on} be the MOD subset special dual like number plane.

Example 2.13: Let $S(R_n(13)(g))$ = {Collection of all subsets of the MOD special quasi dual number plane, where $g^2 = 12g$} = {{0, 12g}, {g, 0.7 + 0.5g, 10g + 8}, {10 + 10g, 5 + 5g, g + 1, 2g + 2, 3 + 3g, 3, 5 1, 7}… and so on} be the MOD subsets of MOD special quasi dual number plane.

Now with these notations we will define operations such as '+' and '×' on them and illustrate these situations by some examples.

Example 2.14: Let $S(R_n(6))$ = {{0, 0}, {(3, 2.1), (1, 1.5)}, {(1, 1), (2, 2), (3, 3), (4, 4), (5, 5)} {(1, 0.1), (2, 0.2), (0.2, 2), (3, 3)} and so on}.

Consider A = {(0.1, 2), (2, 0.3), (4, 2), (5, 0.001)} and B = {(0, 0), (1, 2), (4, 5), (0.1, 0.2), (0.4, 0.5), (1, 1), (2, 2)} $\in S(R_n(6))$.

We find A + B = {(0.1, 2), (2, 0.3), (4, 2), (5, 0.001)} + {(0, 0), (1, 2), (4, 5), (0.1, 0.2), (0.4, 0.5), (1, 1), (2, 2)}

= {(0.1, 2), (2, 0.3), (4, 2), (5, 0.001), (1.1, 4), (3, 2.3), (5, 4), (0, 2.001), (4.1, 1), (0, 5.3), (2, 1), (3, 5.001), (0.2, 2.2), (2.1, 0.5), (4.1, 2.2), (5.1, 0.201), (0.5, 2.5), (2.4, 0.8), (4.4, 2.5), (5.4, 0.501), (1.1, 3), (3, 1.3), (5, 3), (0, 1.001), (2.1, 4), (4, 2.3), (0, 4), (1, 2.001)} $\in S(R_n(6))$.

This is the way + operation is performed on the MOD subsets of the MOD real plane $R_n(6)$.

It is easily verified {(0, 0)} $\in S(R_n(6))$ acts as the additive identity for every

A $\in S(R_n(6))$; A + {(0, 0)} = {(0, 0)} + A = A.

Thus $\{S(R_n(m)), +\}$ is defined as the MOD real subset semigroup or subset pseudo group of the MOD real plane $R_n(m)$. Clearly $S(R_n(6))$ is not a group under +.

We see the cardinality of $S(R_n(m))$ is infinite and the MOD additive MOD real subset pseudo group or semigroup is always commutative.

We see for every A in general we will not be in a position to find a B such that $A + B = \{(0, 0)\}$.

Thus only a semigroup structure can be given.

Example 2.15: Let $\{S(R_n(15)) + \} = G$ be the MOD additive subset pseudo real group (or a MOD subset semigroup).

Let $P = \{(0, 7), (8, 2), (4, 1.2), (1.5, 0.1)\} \in G$.

$$P + P = \{(0, 14), (8, 9), (1, 4), (3, 0.2), (4, 8.2), (8, 2.4), (1.5, 7.1), (12, 3.2), (9.5, 2.1), (5.5, 1.3)\} \in S(R_n(15)).$$

We can find addition

Let $B = \{(0.3, 0.2)\} \in G$.

$B + B = \{(0.6, 0.4)\}$,

$B + B + B = \{(0.9, 0.6)\}$ and so on.

This is the way addition is performed on the subset MOD pseudo real group or MOD real semigroup.

Thus finding MOD pseudo real subgroups or MOD real subsemigroups happens to be a difficult task.

Example 2.16: Let $\{S(R_n(19)), +\} = B$ be the MOD real pseudo subset group (MOD real subset semigroup).

For the MOD subset A = {(0, 1), (0, 0), (0, 2), …, (0, 17), (0, 18)} in B is a such A + A = A.

But we take subsets of A say S_1 = {(0, 0)}, S_2 = {(0, 1), (0, 0)}, S_3 = {(0, 18), (0, 14)} and so on we see S_2 has no additive inverse. Similarly S_3 has no additive inverse and so on.

$$\begin{aligned} -S_3 &= \{(0, 1), (0, 5)\} \text{ but } S_3 + (-S_3) \\ &= \{(0, 18), (0, 14)\} + \{(0, 1), (0, 5)\} \\ &= \{(0, 0), (0, 15), (0, 4)\}. \end{aligned}$$

Thus $S_3 + (-S_3) \neq \{(0, 0)\}$.

$$\begin{aligned} S_3 + S_3 &= \{(0, 18), (0, 14)\} + \{(0, 18), (0, 14)\} \\ &= \{(0, 17), (0, 13), (0, 9)\}. \end{aligned}$$

$\{(0, 0)\} \in B$ is such that for every $A \in B$,

$$A + \{(0, 0)\} = \{(0, 0)\} + A = A.$$

Hence B is a MOD real subset monoid.

Next we give examples of MOD subset complex pseudo group (semigroup) using $C_n(m)$.

Example 2.17: Let $S(C_n(10))$ = {Collection of all subsets of $C_n(10)$ with $i_F^2 = 9$} be the MOD subset complex modulo integer interval pseudo group (or semigroup) undet the operation +.

We will just show how + operation is performed and $\{S(C_n(10)), +\}$ is not a group as for A the MOD subset of $S(C_n(10))$, –A does not exist such that A + (–A) = {0}.

For take $A = \{3 + 0.4i_F, 7i_F, 0.8, 4.8 + 5.2i_F\} \in S(C_n(10))$. Clearly we cannot find a '–A' such that A + (–A) = 0. However for every A there is a unique –A but A + (–A) ≠ {(0, 0)} in general.

We have for every singleton MOD subset $B = \{5 + 2.7i_F\}$ a unique $-B = \{5 + 7.3i_F\}$ such that B + (–B) = {0}.

But not even a single MOD subset of cardinality two can have inverse. That is why we call $S(C_n(10))$ as only a MOD subset complex pseudo group.

Example 2.18: Let M = $\{S(C_n(7)), +\}$ be the MOD subset complex modulo integer pseudo group.

We see A = $\{0.7 + 2i_F, 5 + 4i_F, 0.071 + 3.002i_F, 4 + 0.3i_F\}$ and B = $\{2i_F, 5, 3 + 4.3i_F\} \in$ M.

We find A + B = $\{0.7 + 4i_F, 5 + 6i_F, 0.071 + 5.002i_F, 4 + 2.3i_F, 5.7 + 2i_F, 3 + 4i_F, 5.071 + 3.002i_F, 2 + 0.3i_F, 3.7 + 6.3i_F, 1 + 1.3i_F, 3.071 + 0.302i_F, 0 + 4.06i_F\} \in$ M.

This is the way addition operation is performed on M.

A + A = $\{5.7 + 6i_F, 0.771 + 5.002i_F, 4.7 + 2.3i_F, 1.4 + 4i_F, 3 + i_F, 5.071 + 0.002i_F, 2 + 4.3i_F, 0.142 + 6.004i_F, 4.071 + 3.302i_F, 1 + 0.6i_F\} \in$ M.

This is the way addition is performed.

Example 2.19: Let B = $\{S(C_n(12)), +\}$ be the MOD subset complex modulo integer pseudo group or semigroup. B has subgroups of both finite and infinite order. B also has pseudo subgroups or subsemigroups of infinite order.

Now having seen examples of MOD or small complex subset semigroups (pseudo group) we will develop and describe by examples MOD neutrosophic subset pseudo groups (semigroups) in the following using MOD neutrosophic plane.

Example 2.20: Let B = $\{S(R_n^I(20)), +\}$ be the MOD neutrosophic subset pseudo group. Clearly as in the case of $S(C_n(m))$ and $S(R_n(m))$. B is also not a group only a pseudo group or just a semigroup.

In fact B has subgroups. But B is also a monoid for {(0, 0)} acts as the additive identity.

Example 2.21: Let M = {S(R_n^I(12)), +} be the MOD or small subset MOD neutrosophic pseudo group.

Let P = {10 + I, 2 + 0.3I, 0.7 + 4I, 6 + 0.4I, 8 + 2I} and

Q = {5 + 0.2I, 7 + 3I, 11 + 0.7I, 0.8 + 11I} ∈ M.

P + Q = {3 + 1.2I, 7 + 0.5I, 5.7 + 4.6I, 11 + 0.6I, 1 + 2.2I, 5 + 4I, 9 + 3.3I, 7.7 + 7I, 1 + 3.4I, 3 + 5I, 9 + 1.7I, 1 + I, 11.7 + 4.7I, 5 + 1.1I, 7 + 2.7I, 10.8 + 0, 2.8 + 11.3I, 1.5 + 3I, 6.8 + 11.4I, 8.8 + I}∈ M.

This is the way addition is performed on M.

We can see {(0, 0)} acts as the additive identity, however we cannot guarantee the existence of inverse for every A ∈ M.

Example 2.22: Let {S(C_n(11)), +} = B be the MOD subset complex number pseudo group (semigroup).

We see B has subgroups of finite and infinite order.

Let P = {0.7 + 4.2i_F, 3 + 7i_F, 10.1 + 4.53i_F, 3i_F, 8i_F, 7, 0, 4.21 + 0.3i_F} ∈ B.

P + P = {1.4 + 8.4i_F, 6 + 3i_F, 9.2 + 9.06i_F, 6i_F, 5i_F, 3, 0, 8.42 + 0.6i_F, 3.7 + 0.2i_F, 10.8 + 8.73i_F, 0.7 + 7.2i_F, 0.7 + 1.2i_F, 7.7 + 4.2i_F, 10 + 7i_F, 10.1 + 4.53i_F, 7 + 3i_F, 7 + 8i_F, 7, 0.7 + 4.2i_F, 3 + 7i_F, 0.21 + 0.3i_F, 4.21 + 0.3i_F, 4.81 + 4.5i_F, 7.21 + 7.3i_F, 3.31 + 4.83i_F, 4.21 + 8.3i_F, 0.21 + 0.3i_F}∈ B.

This is the way '+' operation is performed on S(C_n(11)).

Example 2.23: Let {S(C_n(15)), +} = B be the MOD complex subset pseudo group (semigroup).

We see for any subset X, Y$\in$ B; X + Y $\in$ B.

Next we proceed on to describe once again the MOD neutrosophic subset pseudo group by some more examples.

Example 2.24: Let M = $\{S(R_n^I(12)), +\}$ be the MOD neutrosophic subset pseudo group (semigroup).

Let A = {10 + 4I, 3 + 0.8I, 10I, 2 + 5.3I, 8} and
B = {6 + 8I, 4 + 9.2I, 6I, 8 + 4I} $\in$ M.

A + B = {4 + 0I, 9 + 8.8I, 6 + 6I, 8 + 1.3I, 2 + 8I, 2 + 1.2I, 7 + 10I, 4 + 7.3I, 6 + 2.5I, 0 + 9.2I, 10 + 10I, 3 + 6.8I, 4I, 2 + 11.3I, 8 + 6I, 6 + 8I, 11 + 4.8I, 8 + 2I, 10 + 9.3I, 4 + 4I} $\in$ M.

This is the way + operation is performed on M.

Example 2.25: Let P = $\{R_n^I(4), +\}$ be the MOD subset neutrosophic pseudo group.

For A = {3 + 2.5I, 0.5, 2I, 0.58, 2I + 0.7} and
B = {2 + 3.5I, 2.4I, 3.8, 3.19, 0.9 + 2I} $\in$ P.

We find A + B = {1 + 2I, 2.5 + 3.5I, 2 + 1.5I, 2.58 + 3.5I, 2.7 + 1.5I, 3 + 0.9I, 0.5 + 2.4I, 0.4I, 2.4I + 0.58, 0.4I + 0.7, 2.8 + 2.5I, 0.3, 2I + 3.8, 0.38, 3.77, 2I + 3.89, 3.9 + 0.5I, 1.4 + 2I, 0.9, 1.48 + 2I, 0 + 0.7} $\in$ P. This is the way + operation is performed on P.

We now proceed on to give examples of MOD fuzzy subset pseudo groups under +.

Example 2.26: Let M = $\{S(R_n^I(1)), +\}$ be the MOD neutrosophic subset semigroup (pseudo group).

Let X = {0.3 + 0.8I, 0, 0.3I, 0.71, 0.3I + 0.2, 0.711I} and Y = {0.5 + 0.2I, 0.4, 0.31I, 0.8 + 0.4I, 0.7I} $\in$ M.

We now show how X + Y is calculated X + Y = {0.8 + 0I, 0.5 + 0.2I, 0.5 + 0.5I, 0.21 + 0.2I, 0.5I + 0.7, 0.5 + 0.911I, 0.7 + 0.8I, 0.4, 0.4 + 0.3I, 0.11, 0.3I + 0.6, 0.4 + 0.711I, 0.3 + 0.11I, 0.31I, 0.61I, 0.71 + 0.31I, 0.61I + 0.2, 0.011I, 0.1 + 0.2I, 0.8 + 0.4I, 0.8 + 0.7I, 0.51 + 0.4I, 0.7I, 0.8 + 0.111I, 0.3 + 0.5I, 0, 0.71 + 0.71I, 0.2, 0.411I} $\in$ M.

That is each element of Y is added with every element of X.

Example 2.27: Let B = {S(R_n(1)), +} be the MOD fuzzy subset pseudo group. We have one and only one MOD fuzzy subset pseudo group.

Let M = {{0}, {0.1}, {0.2}, {0.3}, {0.4}, {0.5}, {0.6}, {0.7}, {0.8}, {0.9}} $\subseteq$ B is a subgroup of order 10.

In fact this MOD fuzzy subset semigroup (pseudo group) has subgroups.

In view of all these examples we have the following theorem.

THEOREM 2.1: *Let {S(R_n(m)), +}, {S(C_n(m)), +}, {R_n(1), +} and {R_n^I(m), +} be the MOD subset real pseudo group, MOD subset complex pseudo group, MOD fuzzy pseudo group and MOD subset neutrosophic pseudo group respectively.*

Then all the four MOD subset pseudo groups (semigroups) are Smarandache pseudo groups (Smarandache semigroups).

The proof is direct and hence left as an exercise to the reader.

Next we proceed on to give examples of MOD or small subset dual number semigroup (pseudo group).

Example 2.28: Let B = {S(R_n(9)(g)), +, g^2 = 0} be the MOD subset dual number semigroup (pseudo group).

Let X = {6 + 2g, 3 + 4g, 5 + 6g, 8 + 8g, g + 2, 4 + 7g} and Y = {0, 8.5, 8g, 4.4 + g, 5g + 2, 6g + 6, 7 + 8g} $\in$ B.

X + Y = {6 + 2g, 3 + 4g, 5 + 6g, 8 + 8g, g + 2, 4 + 7g, 5.5 + 2g, 2.5 + 4g, 4.5 + 6g, 7.5 + 8g, 1.5 + g, 3.5 + 7g, 6 + g, 3 + 3g, 5 + 5g, 8 + 7g, 2, 6g + 4, 1.4 + 3g, 7.4 + 5g, 0.4 + 7g, 3.4, 2g + 6.4, 8.4 + 8g, 8 + 7g, 5 + 0, 7 + 2g, 1 + 4g, 6g + 4, 6 + 3g, 3 + 8g, g, 2 + 3g, 5 + 5g, 7g + 8, 1 + 4g, 2 + 6g, 5 + g, 1 + 3g, 3 + 5g, 6 + 7g, 0, 2 + 6g} $\in$ B.

This is the way + operation is performed on the MOD subset dual numbers semigroup.

Example 2.29: Let S = {S(R_n(7)(g)), $g^2 = 0$, +} be the MOD dual number subset pseudo group.

For A = {6g, 3, 3.5 + 2.8g, 5g + 3.8, 0} and
B = {0, 0.8, 6.9g, g, 5 + 4g, 6 + 6.8g, 1} $\in$ S.

We find A + B = {6g, 3, 3.5 + 2.8g, 5g + 3.8, 0, 0.8 + 6g, 3.8, 4.3 + 2.8g, 4.6 + 5g, 0.8, 5.9g, 3 + 6.9g, 3.5 + 2.7g, 3.8 + 4.9g, 6.9g, 3 + g, 3.5 + 3.8g, 6g + 3.8, g, 5 + 3g, 1 + 4g, 1.5 + 6.8g, 2g + 1.8, 5 + 4g, 6 + 5.8g, 2 + 6.8g, 2.5 + 2.6g, 2.8 + 4.8g, 6 + 6.8g, 1 + 6g, 4, 4.5 + 2.8g, 5g + 4.8, 1} $\in$ S.

Thus addition is performed for every element in A is added with every element of B. Clearly S is a S-semigroup or S-pseudo group.

Now we proceed on to describe with examples the notion of MOD subset special dual like numbers pseudo group (semigroup).

Example 2.30: Let B = {S(R_n(3)(g)) where g^2 = g, +} be the MOD subset special dual like number pseudo group.

Let X = {2 + g, 0, 0.3g, 0.4g, 2, g, 1, 0.2g} and
Y = {2 + 2g, 1, g, 2g, g + 1, 0.5, 0.8g} $\in$ B.

X + Y = {1, 2 + 2g, 2 + 2.3g, 2 + 2.4g, 1 + 2g, 2, 2g, 2 + 2.2g, g, 1, 1 + 0.3g, 1 + 0.4g, 0, 1 + g, 1 + 0.2g, g, 1.3g, 1.4g, 2 + g, 1.2g, 2.3g, 2.4g, 2g + 1, 2.2g, 1 + 1.3g, 1 + 1.4g, 1 + 1.2g, 2.5 + g, 0.5, 0.5 + 0.3g, 0.5 + 0.4g, 2.5, 0.5 + g, 1.5, 0.5 + 0.2g, 2 + 1.8g, 1.1g, 0.8g, 1.2g, 2 + 0.8g, 1.8g, 1 + 0.8g} $\in$ B.

This is the way + operation is performed.

Example 2.31: Let M = {S(R_n(10)g), g^2 = g, +} be the MOD subset special dual like semigroup (pseudo group).

Let X = {5, 5g, 8g, 7g, 2 + 0.5g, 0.5 + 2g, 0.4g, 0.6, 0.3 + 0.7g} and

Y = {5 + 5g, g, 0, 2g, 0.7g, 9g, 9, 9 + 7g} $\in$ M.

X + Y = {5g, 5, 5 + 3g, 5 + 2g, 7 + 5.5g, 5.5 + 7g, 5 + 5.4g, 5.6 + 5g, 5.3 + 5.7g, 5 + g, 6g, 9g, 8g, 2 + 1.5g, 0.5 + 3g, 1.4g, 0.6 + g, 0.3 + 1.7g, 7g, 2 + 0.5g, 0.5 + 2g, 0.4g, 0.6, 0.3 + 0.7g, 0, 2 + 2.5g, 0.5 + 4g, 2.4g, 0.6 + 2g, 0.3 + 2.7g, 5 + 0.7g, 5.7g, 8.7g, 7.7g, 2 + 1.2g, 0.5 + 2.7g, 1.1g, 0.6 + 0.7g, 0.3 + 1.4g, 5 + 9g, 4g, 2 + 9.5g, 0.5 + g, 9.4g, 0.6 + 9g, 0.3 + 9.7g, 9 + 5g, 9 + 8g, 9 + 7g, 1 + 0.5g, 9.5 + 2g, 9 + 0.4g, 9.6, 9.3 + 0.7g, 4 + 7g, 9 + 2g, 9 + 5g, 9 + 4g, 1 + 7.5g, 9.5 + 9g, 9 + 7.4g, 9.6 + 7g, 9.3 + 7.7g} $\in$ M.

Next we proceed on to give examples of MOD special quasi dual number pseudo group.

Example 2.32: Let G = {S(R_n(8)(g)), g^2 = 7g, +} be the MOD special subset quasi dual number pseudo group (semigroup).

It is important to keep in record that only the special quasi dual number is dependent on [0, 8) that is the MOD interval over which the MOD subsets are built.

Let A = {7.1 + 2g, 3 + 5g, 0.1 + 0.7g, 4.5 + 0.91g} and

B = {5 + 2g, 1, 0, 0.3 + 4g, 3g, 7, 6g} $\in$ G.

A + B = {4.1 + 4g, 7g, 5.1 + 2.7g, 1.5 + 2.91g, 0.1 + 2g, 4 + 5g, 1.1 + 0.7g, 5.5 + 0.91g, 7.1 + 2g, 3 + 5g, 0.1 + 0.7g, 4.5 + 0.91g, 7.4 + 6g, 3.3 + g, 0.4 + 4.7g, 4.8 + 4.91g, 7.1 + 5g, 3, 0.1 + 3.7g, 4.5 + 3.91g, 6.1 + 2g, 2 + 5g, 7.1 + 0.7g, 3.5 + 0.91g, 7.1, 3 + 3g, 0.1 + 6.7g, 4.5 + 6.91g} $\in$ G.

This is the way addition is performed on G.

Example 2.33: Let B = $\{S(R_n(5)g) \mid g^2 = 4g, +\}$ be the MOD subset special quasi dual pseudo number group.

Let X = {2 + 3g, 0, g, 1, 4g, 3.5, 1.5 + 2.7g} and

Y = {2g, 3g, 4, 2, 3.7 + 3.9g, 4.8 + 2.7g, 0, 4 + 4g, 1 + g} $\in$ B.

We find X + Y = {2, 2g, 3g, 1 + 2g, g, 3.5 + 2g, 1.5 + 4.7g, 2 + g, 4g, 1 + 3g, 3.5 + 3g, 1.5 + 0.7g, 1 + 3g, 4, 4 + g, 0, 4 + 4g, 2.5, 0.5 + 2.7g, 4 + 3g, 0.7 + 1.9g, 3.7 + 3.9g, 3.7 + 4.9g, 4.7 + 3.9g, 3.7 + 2.9g, 2.2 + 3.9g, 0.2 + 1.6g, 1.8 + 0.7g, 4.8 + 2.7g, 4.8 + 3.7g, 0.8 + 2.7g, 4.8 + 1.7g, 3.3 + 2.7g, 1.3 + 0.4g, 2 + 3g, 0, g, 1, 4g, 3.5, 1.5 + 2.7, 1 + 2g, 4 + 4g, 4, 4g, 4 + 3g, 2.5 + 4g, 0.5 + 1.7g, 3 + 4g, 1 + g, 1 + 2g, 2 + g, 1, 4.5 + g, 2.5 + 3.7g} $\in$ B.

This is the way addition operation is performed on B.

Now we define for every X in MOD collection of subsets the cardinality of X to be the number of distinct elements in X and is denoted by |X|.

|X| = n; n < ∞ or n = ∞. We see if |X| = m and |Y| = n then |X| + |Y| ≠ m + n. |X| + |Y| < m + n or greater than m + n.

We have only inequality in general.

We will show this by an example or two.

Example 2.34: Let G = $\{S_n(10)g;\ g^2 = 0, +\}$ be the MOD dual number pseudo subset group.

Let X = {5 + 5g, 2 + 2g, 0} and Y = {5 + 5g, 3 + 3g} $\in$ G.

X + Y = {0, 7 + 7g, 5 + 5g, 3 + 3g, 8 + 8g} $\in$ G.

We |X| = 3 and |Y| = 2 and |X| + |Y| = 5.

Let X = {2g, 3, 3g, 4, 5, 7g, 8g, 9} and Y = {5, 7g, 4} $\in$ G.

X + Y = {5 + 2g, 8, 3g + 5, 9, 0, 5 + 7g, 5 + 8g, 4, 9g, 3 + 7g, 7g + 4, 4g, 5g, 9 + 7g, 4 + 2g, 7, 4 + 3g, 7g + 4, 4 + 8g, 3}
= {5 + 2g, 8, 3g + 5, 9, 0, 3, 4 + 8g, 4 + 3g, 5 + 7g, 5 + 8g, 4, 9g, 3 + 7g, 7, 7g + 4, 4 + 7g, 4g, 5g, 9 + 7g, 4 + 2g}.

$$|X| + |Y| = 20 > 8 + 3.$$

Thus we see

$$|X| + |Y| = |X + Y|.$$

$$|X| + |Y| > |X| + |Y|$$

Now let us consider

X = {5, 5g, 5 + 5g} and Y = {5 + 5g}

X + Y = {5g, 5, 0}.

$|X + Y| < |X| + |Y| = 3 + 1 = 4.$

Thus we can get the equality or inequality.

In view of this we have the following theorem.

THEOREM 2.2: *Let G = {S($R_n(m)$) or S($C_n(m)$) or S(R_n^I (m)) or S($R_n(m)g$); with g^2 = 0 or S($R_n(1)$) or S($R_n(m)g$) with g^2 = g or S($R_n(m)g$) with g^2 = (m – 1)g, +} be the MOD subset real or complex or neutrosophic or dual number or fuzzy or special dual like number or special quasi dual number respectively pseudo group (semigroup).*

For any X, Y ∈ G with $|X| = m$ and $|Y| = n$ we have

$$|X + Y| \underset{<}{\overset{>}{=}} |X| + |Y|.$$

The proof is direct and hence left as an exercise to the reader.

Now we proceed on to describe define and develop the notion of MOD subset real semigroup, MOD subset complex semigroup, MOD subset neutrosophic semigroup and so on and give examples of them and enumerate the special properties enjoyed by them.

DEFINITION 2.6: *G = {S($R_n(m)$), ×} is defined as the MOD subset real semigroup.*

G is commutative and is of infinite order.

We will give examples of them and describe their properties.

Example 2.35: Let S = {S($R_n(12)$), ×} be the MOD real subset semigroup.

o(S) = ∞ and is commutative.

S has infinite number of zero divisors only finite number of units and idempotents that too those X ∈ S are such that $|X| = 1$.

We will illustrate this situation.

Let

A = {(0, 4), (0, 2.001), (0, 5.3), (0, 9.2), (0, 1.32), (0, 0.3), (0, 0.005)} and

B = {(1, 0), (2, 0), (3, 0), (4, 0), (5, 0), (6.2, 0), (0.7, 0), (9.33, 0), (1.008, 0), (1.2, 0)}$\in$ S;

$$A \times B = \{(0, 0)\}.$$

Thus we see we can find infinite number of pairs of subsets for which the product is zero.

A = {(4, 8), (8, 4), (4, 4), (8, 8)} and

B = {(0, 6), (3, 3), (6, 0), (3, 6), (6, 3), (6, 6), (3, 0), (0, 3)} $\in$ S.

We see $A \times B = \{(0, 0)\}$.

Thus A and B are zero divisors of S.

We see X = {(9, 9)} $\in$ S is such that $X^2 = X$.

Y = {(4, 4)}$\in$ S is such that $Y^2 = Y$.

S has finite number of idempotents P = {(4, 0)} is an idempotent in S as P^2= P, T = {(9, 4)} is an idempotent as $T^2 = T$ and so on.

We have MOD subset subsemigroups of S.

Example 2.36: Let B = {S(R_n(11)), ×} be the MOD subset real semigroup.

B has infinite number of subset zero divisors, subset idempotents of the special form

$A = \{(1, 1), (1, 0), (0, 1), (0, 0)\}$ or

$D = \{(1, 1), (0, 0)\}$ or

$C = \{(0, 1), (1, 0), (0, 0)\}$ in B are such that

$A^2 = A$, $D^2 = D$ and $C^2 = C$.

Clearly all subset units X of B are such that $|X| = 1$, $T_1 = \{(1, 1)\}$, $T_2 = \{(2, 1)\}$, $T_3 = \{(1, 2)\}$, $T_4 = \{(2, 2)\}$, $T_5 = \{(1, 3)\}$, $T_6 = \{(3, 1)\}$, $T_7 = \{(3, 3)\}$, $T_8 = \{(1, 4)\}$, $T_9 = \{(4, 1)\}$, $T_{10} = \{(4, 4)\}$, $T_{11} = \{(1, 5)\}$, $T_{12} = \{(5, 1)\}$, $T_{13} = \{(5, 5)\}$, $T_{14} = \{(1, 6)\}$, $T_{15} = \{(6, 1)\}$, $T_{16} = \{(6, 6)\}$, $T_{17} = \{(1, 7)\}$, $T_{18} = \{(7, 1)\}$, $T_{19} = \{(7, 7)\}$, $T_{20} = \{(8, 1)\}$, $T_{21} = \{(8, 8)\}$, $T_{22} = \{(0, 8)\}$, … $T_p = \{(10, 8)\}$ are units.

For instance for $T_p = \{(10, 8)\}$ we have $T_q = \{(10, 7)\} \in B$ such that $T_p \times T_q = \{(1, 1)\}$.

However we have only finite number of subset units of B.

Let $A = \{(0.5, 8), (10, 4), (3, 2), (3.1, 4.7), (8.11, 0.11), (4.05, 7.1)\}$ and

$F = \{(0.2, 2), (5, 0.3), (3, 0.4), (0.8, 10), (9, 0.6)\} \in B$.

$A \times F = \{(0.1, 5), (2, 8), (0.6, 4), (0.62, 9.4), (1.622, 0.22), (0.810, 3.2), (2.5, 2.4), (6, 1.2), (4, 0.6), (4.5, 1.41), (4.055, 0.033), (9.25, 2.13), (1.5, 3.2), (8, 1.6), (9, 0.8), (9.3, 1.88), (2.433, 0.044), (1.15, 2.84), (0.4, 3), (8, 7), (2.4, 9), (2.48, 3), (6.488, 1.1), (3.240, 5), (4.5, 4.8), (2, 2.4), (5, 1.2), (5.9, 2.82), (6.99, 0.066), (3.45, 4.26)\} \in B$.

This is the way product is made. We see if both A and F are finite so is $A \times F$.

If one of them A or F or both is infinite and none of A or F is $\{(0, 0)\}$ then $A \times F$ is infinite.

Example 2.37: Let V = $\{S(R_n(20)), \times\}$ be the MOD subset real semigroup.

Let A = $\{(0, 10), (10, 0), (10, 10)\} \in V$.

We see $A^2 = \{(0, 0)\}$.

Thus A is a subset nilpotent element of order two.

Let B = $\{(1, 5), (1,1), (0, 5), (0, 0), (0, 1), (1, 0)\} \in V$.

We see $B \times B = B$ is a MOD subset idempotent of V.

V has subset zero divisors and subset idempotents.

We can define MOD complex subset semigroup $\{C_n(m), \times; i_F^2 = m - 1\}$ in a natural way as that of the MOD real subset semigroup.

This is considered as a matter of routine.

We will illustrate this situation by some examples.

Example 2.38: Let W = $\{S(C_n(6)), i_F^2 = 5, \times\}$ be the MOD complex subset semigroup.

Let A = $\{3 + 2.5i_F, 4.2 + 3.5i_F, 3i_F, 4.5i_F, 0.5i_F, 2.6, 1.4i_F\}$

and B = $\{5, 5i_F, 2 + 2i_F, 1 + 5i_F, 4 + 2i_F, 3i_F\} \in W$.

We can find A × B = $\{(3 + 0.5i_F), (3.0 + 5.5i_F), 3i_F, 4.5i_F, 2.5i_F, 1.0, 7.0i_F, 3i_F + 2.5, 3i_F + 5.5, 3, 4.5, 0.5, i_F, 5, 5 + 5i_F, 1.4 + 3.4i_F, 0, 3 + 3i_F, 5 + i_F, 5.2 + 5.2i_F, 2 + 2.8i_F, 5.5 + 5.5i_F, 1.7 + 0.5i_F, 4.5i_F + 4.5, 0.5 + 0.5i_F, 2.6 + 3i_F, 1.4 + i_F, 1 + 4.5i_F, 2.2 + 5.5i_F, 4, 4 + 0.5i_F, 4 + 2.5i_F, 0.6 + 2i_F, 4 + 3.4i_F, 3 + 5.5i_F, 4.2 + 0.5i_F, 0, 1.5i_F, 3.5i_F, 2.6 + 3i_F, 4.4i_F\} \in W$.

This is the way × is performed on W.

Let $X = \{i_F, 0.3i_F, 2.4i_F, 4i_F, 5.2i_F\}$ and
$Y = \{0, 2i_F, 4i_F, i_F, 5i_F, 3i_F, 0.5i_F\} \in W$.

$X \times Y = \{0, 4, 3, 2, 5, 1.5, 1, 4.5, 2.5, 0.75\} \in W$.

X and Y are pure complex numbers and $X \times Y$ are real.

Example 2.39: Let $P = \{C_n(5),\ i_F^2 = 4, \times\}$ be the MOD subset complex semigroup.

Let $B = \{2.5i_F, 0, 2.5, 2.5 + 2.5i_F\}$ and

$A = \{4, 2, 0, 4 + 2i_F, 4i_F + 2, 2i_F, 4i_F\} \in P$.

We see $A \times B = \{0\}$.

Thus P has MOD subset zero divisors.

We see $S(R_n(5)) \subseteq S(C_n(5))$ is a MOD subset subsemigroup and not an ideal.

M = {All subsets of Z_5 under $\times$} is a finite subset subsemigroup. Thus P has both finite and infinite subset subsemigroups.

Now we will see examples of MOD subset neutrosophic semigroups.

Example 2.40: Let
$S = \{S(R_n^I(9)),\ I^2 = I, \times\}$ be the MOD subset neutrosophic semigroup.

Let $A = \{3 + 5I, 2, 7, 0.8 + 0.2I, I, 5I, 3.1I\}$ and
$B = \{8I, 4 + 4I, 6I, 3 + 4I\} \in S$.

We find $A \times B = \{I, 7I, 2I, 8I, 4I, 6.8I, 8 + 8I, 3 + 7I, 1 + I, 5 + 4.8I, 6.8I, 3I, 6I, 0.6I, 6 + 8I, 3 + I, 3.7I, 2.4 + 4.6I\} \in S$.

This is the way operations are performed on MOD subset neutrosophic semigroups.

Example 2.41: Let S = {$R_n^I(11)$, $I^2 = I$, ×} be the MOD subset neutrosophic semigroup.

Let A = {5 + 5I, 0.7 + 3.1I, 6.8I, 4.3I, 2 + 0.9I, 10 + 5.2I, 3.5I + 4.7} and
B = {8, 8I, 7, 7I, 4, 0.2I, 0.5I, 0.9, 0.2, I, 1, 0} ∈ S.

We find A × B = {7 + 7I, 5.6 + 2.2I, 10.4I, 1.4I, 5 + 7.2I, 3 + 8.6I, 6I + 4.6, 3I, 7.8I, 10.4I, 1.4I, 1.2I, 0.6I, 10.6I, 2 + 2I, 4.9 + 10.7I, 3.6I, 8.1I, 3 + 6.3I, 6 + 3.4I, 10.9 + 2.5I, 4I, 4.6I, 3.6I, 8.1I, 9.3I, 9.4I, 8 + 8I, 2.8 + 1.4I, 5.2I, 6.2I, 8 + 3.6I, 7 + 9.8I, 3I + 3.8, 0, 0.1 + 0.1I, 0.14 + 0.62I, 1.36I, 0.86I, 0.4 + 0.18I, 2 + I, 0.7I + 0.94, 5 + 5I, 0.7 + 3.1I, 6.8I, 4.3I, 2 + 0.9I, 10 + 5.2I, 3.5I + 4.7, 8.2I, 10I, 3.8I, 6.8I, 4.3I, 2.9I, 4.2I, 1 + I, 0.14 + 0.61I, 1.36I, 0.86I, 0.4 + 0.18I, 2 + 1.04I, 0.7I + 0.94, 2.15I, 3.4I, 1.45I, 5I, 1.9I, 1.45I, 6.6I, 4.1I, 4.5 + 4.5I, 0.63 + 1.86I, 6.12I, 3.87I, 1.8 + 0.81I, 9 + 4.68I, 3.15 + 4.23}∈ S.

This is the way product operation is performed on the neutrosophic MOD subset semigroup.

Clearly these semigroups also has ideals, zero divisors, units and subsemigroups.

Now we proceed on to give examples of MOD fuzzy subset semigroups.

We see we have only one MOD fuzzy subset semigroup.

Example 2.42: Let F = {S(R_n(1)), ×} be the MOD fuzzy subset semigroup.

Let A = {(0, 0.1), (0.8, 0.5), (0.231, 0), (0.11, 0.52), (0.1114, 0.881), (0, 0), (0.75, 0.01)} and

B = {(0, 0.4), (0.2, 0.5), (0.7, 0.2), (0.9, 0.6), (0.7, 0.01), (0.02, 0.03)} $\in$ F.

We find A $\times$ B = {(0, 0.04), (0, 0.2), (0, 0), (0, 0.208), (0, 0.3524), (0, 0.004), (0, 0.05), (0.16, 0.25), (0.0462, 0), (0.022, 0.260), (0.02228, 0.4405), (0.15, 0.005), (0, 0.02), (0.56, 0.1), (0.1617, 0), (0.077, 0.104), (0.07798, 0.1762), (0.525, 0.002), (0, 0.06)., (0.72, 0.3), (0.2079, 0), (0.099, 0.312), (0.13326, 0), (0.099, 0.312), (0.10026, 0.5286), (0.675, 0.006), (0, 0.001), (0.56, 0.005), (0.1617, 0), (0.077, 0.0052), (0.07798, 0.00881), (0.525, 0.0001), (0, 0.003), (0.016, 0.015), (0.0462, 0), (0.022, 0.0156), (0.002228, 0.2643), (0.015, 0.0003)}$\in$ F.

It has infinite number of zero divisors given by

A = {(0, 0.34), (0, 0.0001), (0, 0.004321), (0, 0.0001123), (0, 0.015)} and

B = {(a, 0) | a $\in$ [0, 1)} $\in$ F is such that

A $\times$ B = {(0, 0)}.

It is important to keep on record that F is the one and only MOD subset fuzzy semigroup.

However MOD subset real semigroups, MOD subset neutrosophic semigroups and MOD subset complex semigroups are infinite in number.

Now we proceed on to give examples of MOD subset dual number semigroups.

Example 2.43: Let B = {S(R_n(7)g) | g^2 = 0} be the MOD subset dual number semigroup.

Let X = {3 + g, 0.4 + 0.5g, 0.8 + 0.9g, 0.06g + 4, 0.7 + 0.3g, 2.1 + 3.4g} and

Y = {6g, 3g, 0.2g, 0.7g, 0.9, 0.5, 6, 2g, 5g, 0.g} $\in$ B.

We find X × Y = {4g, 2.4g, 4.8g, 3g, 4.2g, 5.6g, 2g, 1.2g, 6.3g, 2.4g, 5g, 2.1g, 0.6g, 0.08g, 0.16g, 0.8g, 0.14g, 0.42g, 2.1g, 0.28g, 0.56g, 5.6g, 0.49g, 1.47g, 2.7 + 0.9g, 0.36 + 0.45g, 0.72 + 0.81g, 3.6 + 0.054g, 0.63 + 0.27g, 1.89 + 3.06g, 1.5 + 0.5g, 0.2 + 0.25g, 0.4 + 0.45g, 2 + 0.03g, 0.35 + 0.15g, 1.05 + 1.7g, 4 + 6g, 2.4 + 3g, 4.8 + 5.4g, 3 + 0.36g, 4.2 + 1.8g, 5.6 + 6.4g, 6g, 0.8g, 1.6g, g, 1.4g, 4.2g, g, 2g, 4g, 6g, 3.5g, 3g, 0.3g, 0.04g, 0.08g, 0.4g, 0.07g, 0.21g}∈ B.

This is the way product operation is performed on B has infinite number of MOD subset zero divisors.

Let D = {0.7g, 2g, 4g, 5g, 0.052g, 0.374g} and

E = {g, 2g, 4g, 5g, 0.224g, 6.735g, 4.332g} ∈ B.

Clearly D × E = {0}.

Several interesting properties about these MOD subset dual number semigroups can be got.

Example 2.44: Let S = {S(R_n(10)(g)), $g^2 = 0$, ×} be the MOD subset dual number semigroup.

Take X = {9g + 8.5, 2 + 4g, 5 + 0.8g, 8.1 + 4.5g, 7.5 + 8.2g, 8g, 6g, 4 + 4g, 2 + 2g} and

Y = {0.7g, 4g, 5g, 3g, 0.8g, 0.2g, g, 1, 2, 5, 6, 0.5, 0.2, 0.7} ∈ S.

X × Y = {5.95g, 1.4g, 5.67g, 5.25g, 1.4g, 2.8g, 0, 8g, 2.4g, 2g, 2.5g, 5g, 0.5g, 7.5g, 5.5g, 6g, 4.3g, 2.5g, 2g, 6g, 6.8g, 1.6g, 4g, 6.48g, 3.2g, 1.7g, 0.4g, 0.g, 1.62g, 1.5g, 0.8g, 0.4g, 8.5g, 2g, 5g, 8.1g, 7.5g, 4g, 2g, 9g + 8.5, 2 + 4g, 5 + 0.8g, 8.1 + 4.5g, 7.5 + 8.2g, 8g, 6g, 4 + 4g, 2 + 2g, 8g + 7, 4 + 8g, 1.6g, 6.2 + 9g, 5 + 6.4g, 8 + 8g, 5 + 2.5g, 5 + 4g, 0.5 + 2.5g, 7.5 + g, 4g + 1, 2 + 4g, 4.8g, 8.6 + 7g, 5 + 9.2g, 4.5g + 4.25, 1 + 2g, 2.5 + 0.4g, 4.05 + 2.25g, 3.75 + 4.1g, 4g, 3g, 1 + g, 1.8g + 1.7, 0.4 + 0.8g, 1 + 0.16g, 1.62 + 0.9g, 1.5 + 1.64g, 1.6g, 1.2g, 0.8 + 0.8g, 0.4 +

0.4g, 6.3 + 4.55g, 1.4 + 2.8g, 3.5 + 0.56g, 5.67 + 3.15g, 5.25 + 5.74g, 5.6g, 4.2g, 2.8 + 2.8g, 1.4 + 1.4g} $\in$ S.

S has infinite number of zero divisors, few units and idempotents.

Example 2.45: Let M = {S(R_n(13)g where $g^2 = 0$, $\times$} be the MOD subset dual number semigroup.

We see M has infinite number of zero divisors, only finite number of units and idempotents.

Let X = {0.7 + 0.6g, g + 1, 2 + 2g, 4 + 5g, 0.5 + 0.8g, 4g + 2, 10 + 5g} and

Y = {5g, 2g, 7g, 8g, g, 3g, 2, 5, 10, 7, 0.1, 0.5, 0.4} $\in$ M.

X $\times$ Y = {3.5g, 5g, 10g, 7g, 2.5g, 11g, 1.4g, 2g, 4g, 8g, g, 4g, 7g, 4.9g, 9g, 5.6g, 3g, 5g, 6g, 0.7g, 0.5g, 10g, 2.1g, 11g, 1.5g, 1.4 + 1.2g, 2 + 2g, 4 + 4g, 8 + 10g, 1 + 1.6g, 8g + 4, 7 + 10g, 3.5 + 3g, 5 + 5g, 10 + 10g, 7 + 12g, 2.5 + 4g, 7g + 10, 8 + 12g, 7 + 6g, 7 + 7g, 1 + 11g, 5 + 8g, g + 7, 9 + 8g, 4.9 + 4.2g, 1 + g, 2 + 9g, 3.5 + 5.6g, 0.1 + 0.1g, 2g + 1, 5 + 9g, 0.07 + 0.06g, 0.2 + 0.2g, 0.4 + 0.5g, 0.05 + 0.08g, 0.2 + 0.4g, 1 + 0.5g, 0.35 + 0.3g, 0.5 + 0.5g, 1 + g, 2 + 2.5g, 0.25 + 0.4g, 2g + 1, 5 + 2.5g, 0.28 + 0.24g, 0.4 + 0.4g, 0.8 + 0.8g, 1.6 + 2g, 2 + 3.2g, 1.6g + 0.8, 4 + 2g} $\in$ M.

This is the way product operation is performed on M.

We see M has infinite number of zero divisors.

For A = {0.2g, 0.5g, 8g, 9g, 0.4g} and
X = {g, 8g, 0, 5g, 1.4g, 3.11g, 2.109g, 10.009g, 11.21g, 12.45g} $\in$ M is such that A $\times$ X = {0}.

Thus we can get infinite collection of zero divisors in M.

Example 2.46: Let S = $\{S(R_n(27)g),\ g^2 = 0, \times\}$ be the MOD subset dual number semigroup. S has infinite number of zero divisors.

$S(R_n(27)) \subseteq S$ is a MOD subset real semigroup of infinite order. Clearly $S(R_n(27))$ is not an subset ideal of S.

Now we proceed on to describe with examples the notion of MOD subset special dual like number semigroup.

Example 2.47: Let S = $\{S(R_n(5)(g)) \mid g^2 = g, \times\}$ be the MOD subset special dual like number semigroup.

S is of infinite order and is commutative.

Let A = {3 + 4g, 2 + 2.2g, 4g, 2.5g, 0.5 + 0.7g, 2 + 4g, 3.2 + 4.1g} and

B = {3g, 2g, g, 1, 4g, 0.5g, 0.2g, 0.8g} $\in$ S.

A $\times$ B = {g, 2.6g, 2g, 2.5g, 3.6g, 3g, 1.9g, 4g, 3.4g, 0, 2.4g, 4.6g, 2g, 4.2g, 4g, 2.5g, 1.2g, g, 2.3g, 3 + 4g, 2 + 2.2g, 4g, 2.5g, 0.5 + 0.7g, 2 + 4g, 3.2 + 4.1g, 1.8g, 3.5g, 2g, 4.8g, 4.2g, 3.2g, 0.6g, 1.5g, 3.65g, 1.4g, 0.84g, 3.36g, 0.8g, 0.24g, 1.2g, 1.46g, 0.96g, 0.48g, 0.84g} $\in$ S.

Example 2.48: Let B = $\{S(R_n(12)(g)),\ g^2 = g, \times\}$ be the MOD subset special dual like number semigroup.

Let X = {0.8 + 4g, 5g + 10, 0.7g + 0.4, 6.5 + 3.5g, 2.1 + 5.2g, 10.01 + 7.2g, 6g, 0.4g} and

Y = {0.8g, 0.5g, 0.6g, g, 1, 0, 2, 4, 10} $\in$ B.

We now show how X $\times$ Y is found.

X $\times$ Y = {3.84g, 0, 0.88g, 8g, 5.84g, 1.768g, 0.32g, 2.4g, 7.5g, 0.55g, 5g, 3.65g, 8.605g, 3g, 0.2g, 2.88g, 9g, 0.66g, 6g, 4.38g, 10.326, 3.6g, 0.24g, 4.8g, 3g, 1.1g, 10g, 7.3g, 5.21g, 6g, 0.4g, 0.8 + 4g, 5g + 10, 0.7g + 0.4, 6.5 + 3.5g, 2.1 + 5.2g, 10.01 + 7.2g, 1.6 + 8g, 10g + 8, 1.4g + 0.8, 1 + 7g, 4.2 + 10.4g, 8.01 + 2.4g, 0.8g, 3.2 + 4g, 8g + 4, 2.8g + 1.6, 2 + 2g, 8.4 + 8.8g, 4.01 + 4.8g, 1.6g, 8 + 4g, 2g + 4, 7g + 4, 5 + 11g, 9 + 4g, 4.1, 4g}$\in$ B.

This is the way product operation is performed on the MOD subset special dual like number semigroups. In fact B has subset zero divisors.

Let X = {2g, 4g + 2, 2g + 4, 8g, 4g, 8 + 2g, 2 + 8g, 4 + 8g, 8 + 4g} and

Y = {6, 6g, 6 + 6g, 0} $\in$ B.

Clearly X $\times$ Y = {0}.

Example 2.49: Let S = $\{S(R_n(64)g),\ g^2 = g, \times\}$ be the MOD subset special dual like numbers semigroup. S has zero divisors. S has MOD subset subsemigroups which are not ideals.

Next we briefly describe semilattices on $S(R_n(m))$, $S(C_n(m))$, $S(R_n^I(m))$, $\{S(R_n(m)(g))$ with $g^2 = 0\}$, $\{S(R_n(m)(g))$ with $g^2 = g\}$ and $\{S(R_n(m)(g)) \mid g^2 = (m-1)g\}$.

DEFINITION 2.7: *Let $S(R_n(m))$ be the MOD subset of the MOD real plane. Define '$\cup$' on $S(R_n(m))$; $\{S(R_n(m)), \cup\}$ is a semilattice or an infinite idempotent semigroup. The same is true if $S(R_n(m))$ is replaced by $S(R_n^I(m))$ or $S(C_n(m))$ or $\{S(R_n(m)(g))$ with $g^2 = 0\}$ or $\{S(R_n(m)(g))$ with $g^2 = g\}$ or $\{S(R_n(m)(g))$ with $g^2 = (m-1)g\}$.*

We will only illustrate all these situations.

Example 2.50: Let $\{S(R_n(12)), \cup\}$ be MOD subset real semilattice or idempotent semigroup.

For any X = {(3, 8), (2, 0), (3.114, 0.532), (0, 8.1102), (0.11156, 6.7205)} and

Y = {(0, 0.732), (6.32, 1.207), (1.31, 5.38), (3, 8), (0.0710325, 8.31), (6, 0.36), (0.8, 0.841)} $\in$ $S(R_n(12))$;

$X \cup Y$ = {(3, 8), (2, 0), (3.114, 0.532), (0, 8.1102), (0.11156, 6.7205), (0, 0.732), (6.32, 1.207), (1.31, 5.38), (0.0710325, 8.31), (6, 0.36), (0.8, 0.841)} $\in S(R_n(12))$.

This is the way $\cup$ operation is performed on $S(R_n(12))$.

Example 2.51: Let M = $\{S(R_n^I(5)), \cup\}$ be the MOD subset neutrosophic idempotent semigroup.

Let X = {3 + 2I, 2.001I, 0.3211, 0.53 + I, 3.01I, 0.1304 + 0.63251I} and

Y = {I + 1, 0.4321I, 0.73 + 0.85I, 3 + 2I, 2.001I, 0.7 + 0.4I, 4.2 + 4.3I}$\in$ M.

$X \cup Y$ = {3 + 2I, 2.001I, 0.3211, 0.53 + I, 3.01I, 4.2 + 4.3I, 0.1304 + 0.6325I, 1 + I, 0.4321I, 0.73 + 0.85I, 0.7 + 0.4I}$\in$ M.

This is the way operation is performed.

Example 2.52: Let P = $\{S(C_n(8)), \cup\}$ be the MOD subset of complex modulo integer semilattice (idempotent semigroup). We see P is commutative and is of infinite order.

For every $X \in P$; $X \cup X = X$.

Let

X = {$0.73 + 7i_F$, $1.8 + 3i_F$, $6.5 + 6i_F$, $0.78i_F$, $0.42 + 0.8i_F$, $4 + 4i_F$} and

$Y = \{1, 0, 4, 4 + i_F, 0.0001, 0.8i_F, 0.78i_F, 3 + 3i_F\} \in P$.

We find $X \cup Y = \{0.73 + 7i_F, 1.8 + 3i_F, 6.5 + 6i_F, 0.78i_F, 0.42 + 0.8i_F, 4 + 4i_F, 1, 0, 4, 0.0001, 0.8i_F, 3 + 3i_F\} \in P$.

Next we give examples of MOD subset dual numbers semilattices.

Example 2.53: Let $S = \{S(R_n(12)(g))\ g^2 = 0, \cup\}$ be the MOD subset dual number idempotent semigroup (semilattice).

Let $X = \{3 + 4.8g, 0, g, 4 + 0.3g, 0.04 + 6.03g, 3.0g, 4.2, 7g\}$

and $Y = \{0, g, 3g, 0.7g, 5g, 2 + 4g, 10g, 4 + 5g\} \in S$.

$X \cup Y = \{3 + 4.8g, 0, g, 4 + 0.3g, 0.04 + 6.03g, 3.01g, 4.2, 7g, 0, g, 3g, 0.7g, 5g, 10g, 4 + 5g, 2 + 4g\} \in S$.

Example 2.54: Let $M = \{S(R_n(11)(g))$ where $g^2 = g, \cup\}$ be the MOD subset special dual like number semilattice (or idempotent semigroup).

Let $X = \{9 + 2g, 0.75 + 4.5g, 0.71 + 0.425g, 4g, 6, 7 + g, 0.001, 0.00004g\}$ and

$Y = \{g, 2g, 3g, 4 + g, 2 + 0.3g, 0.74g, 0.21g, 0, 1, 0.224g, 1.0059g, 9.58\} \in M$;

$X \cup Y = \{9 + 2g, 0.75 + 4.5g, 0.71 + 0.425g, 4g, 6, 7 + g, 0.001, 0.00004g, g, 2g, 3g, 4 + g, 2 + 0.3g, 0.74g, 0.21g, 0, 1, 0.224g, 1.0059g, 9.58\} \in M$.

This is the way operation $\cup$ is performed on M.

Example 2.55: Let $W = \{S(R_n(7)g)$ where $g^2 = 6g, \cup\}$ be the MOD subset special quasi dual number semigroup. W has subsemigroups.

In fact every subset X is a subsemigroup as $X \cup X = X$. Thus W has every singleton MOD subset to be subsemigroup.

Let X, Y $\in$ W, $X \cup Y \in$ W if and only if $X \subseteq Y$ or $Y \subseteq X$.

Let L = {X = {6 + g, 3g, 4g, 0.33g, 0.889g, 0.0052} and

Y = {3, 4g, 4 + 4g, 5g, 0.7778, 0.2225g, 0.7 + 4g}} $\subseteq$ W.

We see $X \cup X = X$ and $Y \cup Y = Y$ but $X \cup Y$ = {6 + g, 3g, 4g, 0.33g, 0.889g, 0.0052, 3, 4 + 4g, 5g, 0.7778, 0.2225g, 0.7 + 4g} $\notin$ L.

Thus L is only a subset of W but is not a subsemigroup.

However {X, Y, $X \cup Y$} is a subsemigroup which we define as the completion of the subset L and denote it by L^c and $L^c = L \cup \{X \cup Y\}$.

THEOREM 2.3: *Let M = {S($R_n(m)$), $\cup$} be the MOD subset real semigroup (semilattice) (or S($R_n^I(m)$)), S($C_n(m)$), {S($R_n(m)(g)$) / $g^2 = 0$}, {S($R_n(m)(g)$), $g^2 = g$}, {S($R_n(m)(g)$) / $g^2 = (m - 1)g$} be the MOD subset neutrosophic semigroup or MOD subset complex semigroup or MOD subset dual number semigroup, MOD subset special dual like number semigroup or MOD subset special quasi dual number semigroup.*

Every subset collection P of M can be made into a subsemigroup.

Proof is direct and simple hence left as an exercise to the reader.

We will illustrate this situation by an example or two.

Example 2.56: Let S = {S($R_n(5)(g)$) | $g^2 = g$, $\cup$} be the MOD subset special dual like semilattice.

Let P = {0.3g, 0.9, 2 + 4g, 0.38g + 4.02} $\in$ S.

We see P $\cup$P = P so P is a MOD subset subsemigroup of order 1.

In fact S has infinite number of MOD subset subsemigroups.

Let P = {0.13g, 2g + 0.8g, 2.4, 4 + 0.23g} and

Q = {2, 2g + 1, 0.21g + 3} $\in$ S.

P $\cup$ P = P, Q $\cup$Q = Q but
P $\cup$ Q = {0.13g, 2g + 0.8g, 2.4, 4 + 0.23g, 2, 2g + 1, 0.21g + 3} $\notin$ {P, Q}.

But B = {P, Q, P $\cup$ Q} is subsemigroup of order three in S.

In fact all subsets of order two in S which are not subsemigroups can be completed to form a MOD subset subsemigroups of order three.

Further S has infinite number of subsemigroups of order three.

Next if T = {A, B} where
A = {3 + 4g, 2g, 0.3371, 0, 2.00056g, 0.8101g + 2.0114}
and B = {3 + 4g, 2g, 0, 0.3371}.
Clearly A$\cup$A = A, B $\cup$B = B, A $\cup$ B = A so T is a MOD subset subsemigroup of S.

In fact S has subset subsemigroups of all finite order and also subsemigroups of infinite order.

Now in view of this we give the following theorem the proof of which is direct.

THEOREM 2.4: *Let S = {S($R_n(m)$), (or S($C_n(m)$) or S(R_n^I (m))), or {S($R_n(m)(g)$) g^2 = 0}, {S($R_n(m)(g)$) g^2 = g} or {S($R_n(m)(g)$) /*

$g^2 = (m - 1)g\}\}$ be the MOD subset semigroup under product operation .

i. S has MOD subset subsemigroups of all orders.

ii. If $P = \{A, B\}$ is a MOD subset of S of order 2. P is a MOD subset subsemigroup if and only if $A \subseteq B$ or $B \subseteq A$.

Next we proceed on to define MOD or small subset semigroups under $\cap$. In first place for us to have closure property under $\cap$ we need to adjoin ϕ the empty set with $S(R_n(m))$.

Now $S = \{\{S(R_n(m)) \cup \phi\}, \cap\}$ is a MOD subset real semigroup defined also as the MOD subset real semilattice.

Clearly $o(S) = \infty$ and every element in S is an idempotent hence every singleton element in S is a subsemigroup of S of order one.

We will illustrate all these situations by some examples.

Example 2.57: Let $S = \{S(R_n(24)(g)) \cup \phi, \cap\}$ be the MOD subset semilattice under the operation $\cap$. S is of infinite order.

Let $A = \{12g, 10 + 3g, 0.004g, 6.0025, 10.324g + 10.006, 0\}$ and

$B = \{5g, 4g, 2g, 20g, 21 + 0.88g, 0.44 + 23g\} \in S$.

Clearly $A \cap A = A$ and $B \cap B = B$.

Thus as in case of union '$\cup$'. We see also in case of '$\cap$'; we see every singleton subset of S is a subsemigroup of S.

Example 2.58: Let $S = \{S(R_n(24)) \cup \phi, \cap\}$. Every subsets of order two of S can be made into a subsemigroup under $\cap$ if that two element set is not a subsemigroup. However the completion will lead to a three element set.

Let A = {(10, 8), (0.2, 0.74), (10, 16), (19, 23)} and

B = {(5, 2), (0, 0), (1, 21), (4, 7)} $\in$ S.

$A \cap A = A$, $B \cap B = B$ and $A \cap B = \phi$.

Clearly {A, B, ϕ} = L is a MOD subset subsemigroup of order three.

However {A, B} is not a subsemigroup of S.

Let M = {A = {(0, 5), (8, 0.3), (0.4, 9.82), (23, 1)} and

B = {(0, 5), (23, 1), (5.772, 8.3), (11.2, 13.4)}} $\subseteq$ S;

But $A \cap B$ = {(0, 5), (23, 1)} $\notin$ M. So M is not a MOD subsemigroup only a subset.

But {A, B, $A \cap B$} $\subseteq$ S is a MOD subsemigroup called the completion of the subset M denoted by M^c.

Infact a MOD subset of order three which is not a subsemigroup only a subset can be completed to form a subsemigroup. In fact the same is true for any subset collection of any order.

Let N = {A = {(0, 3), (7, 5), (9, 2)},

B = {(0.337, 6.004), (7.331, 0), (4, 0.3), (2, 4), (3, 7.23)},

C = {(4, 0.3), (0, 3), (8.3, 0.3)},

D = {(0, 3), (2, 4), (4, 0.3)}} $\subseteq$ S be the MOD subset.

Clearly N is not a MODsubsemigroup.

We can obtain N^c which will be a subsemigroup.

$N^c = \{A, B, C, D, A \cap B = \phi, A \cap C = \{(0, 3)\}, A \cap D = \{(0, 3)\},$

$B \cap C = \{(4, 0.3)\}, B \cap D = \{(2, 4), (4, 0.3)\}$ and $C \cap D = \{(0, 3), (4, 0.3)\}\}$ is the completed MOD subsemigroup of the subset N.

$N^c = \{A, B, C, D, \phi, \{(0, 3)\}, \{(4, 0.3)\}, \{(2, 4), (4, 0.3)\}, \{(0, 3), (4, 0.3)\}\} \subseteq S.$

$|N| = 4$ and $|N^c| = 9$.

It is left as an open problem to study the following.

Problem 2.1: Is it possible for one to say if $|P| = n$ and $P \subseteq S$, where S is a MOD subset semigroup and P a subset of S. What is $|P^C|$?

(a) If $P = \{X_1, \ldots, X_n\}$ with $X_i \cap X_j = \phi$ for $i \leq i, j \leq n$.

(b) If $X_i \cap X_j \neq \phi$ for only one i and j and $i \neq j$.

(c) If $X_i \cap X_j \neq \phi$ for a set of 3 elements in P.

(d) If $X_i \cap X_j \neq \phi$ for only a set of 4 subsets in P.

(e) Generalize for $X_i \cap X_j \neq \phi$ for a m set of elements and $m < n$.

Example 2.59: Let $S = \{S(C_n(10)) \cup \{\phi\}, \cap\}$ be the MOD subset complex modulo integer semigroup.

Every singleton set is a MOD subset subsemigroup.

Let $A = \{3, 2.5, 7.5, 8, 0\} \in S, A \cap A = A.$

Thus A is a MOD subset subsemigroup.

Let B = $\{3i_F, 2i_F, 0.77i_F, 6.8051i_F, 2.15i_F\} \in S$;

$B \cap B = B$ so B is also a MOD subset subsemigroup.

Let A = $\{3 + 2i_F, 0.7i_F, 0.3321 + 9.2105i_F\}$ and

B = $\{0.7i_F, 0.8, 0.7389i_F + 4.8, 5 + 5i_F, 6 + 8i_F\} \in S$.

Let M = {A, B}; we see $A \cap B = \{0.7i_F\} \notin M$.

Thus $M^c = \{A, B, A \cap B\} \subseteq S$ is a MOD subset subsemigroup of order 3. M^c is called the completion of M.

We can always complete any subset collection which is not a subsemigroup to a subsemigroup.

In view of all these we mention the following theorem.

THEOREM 2.5: *Let S = $\{\{S(R_n(m)) \cup \phi\}$, (or $\{S(C_n(m)) \cup \phi\}$ or $\{S(R_n^I(m)) \cup \phi\}$), or $\{S(R_n^I(m)(g)) \cup \phi, g^2 = 0\}$ or $\{S(R_n^I(m)(g)) \cup \phi, g^2 = g\}$ or $\{S(R_n(m)(g)) \cup \phi$ with $g^2 = (m-1)g\}\}$ be the MOD subset real semigroup (semilattice) (or MOD subset complex semigroup and so on).*

Then

i. *Every singleton subset is a subsemigroup.*
ii. *S has subsemigroups of order two, three and so on.*
iii. *Every subset of S can be completed to a subsemigroup of S.*

Proof is direct and hence left as an exercise to the reader.

Example 2.60: Let S = $\{\{S(R_n(11)(g)), \cup \phi\}, g^2 = 10g, \cap\}$ be the MOD subset special quasi dual like number semigroup.

Let A = {3 + 4g, 10g + 3.33, 0.9 + 0.32g} and

B = {g + 0.72, 6.848 + 10.03g, 10g + 3.33, 0.999g, 2.22g, 2.001} $\in$ S we see

M = {A, B} is only a subset and not a subsemigroup.

M^c = {A, B, A $\cap$ B} $\subseteq$ S is the completed subsemigroup of the subset M.

Every subset in S can be completed to a subsemigroup.

Example 2.61: Let S = {{S(C_n(24))$\cup\phi$, $\cap$}} be the MOD subset complex modulo integer semigroup.

Let M = {Collection of all subsets from the set S(R_n(24)) $\cup\phi$} $\subseteq$ S is a subsemigroup of infinite order.

M is not an ideal of S only a subsemigroup.

M will be a S-subsemigroup if M has a subgroup under $\cap$ which is not possible.

THEOREM 2.6: *Let S = {{S(R_n(m))}, $\cup$} and R ={{S(R_n(m))$\cup\phi$}, $\cap$} be the MOD subset semigroups. S and R are never Smarandache subsemigroups.*

Proof is direct and hence left as an exercise to the reader.

THEOREM 2.7: *Let M = {S(R_n(m)), $\times$} be the MOD subset semigroup. M is a S-semigroup if and only if Z_m is S-semigroup.*

Proof is direct and hence left as an exercise to the reader.

THEOREM 2.8: *Let T = {S(R_n^I (m)), $\times$} (or S(C_n(m)) or S(R_n(m)(g)) with g^2 = 0 or S(R_n(m)g) with g^2 = g or S(R_n(m)(g)) with g^2 = (m – 1)g) be the MOD subset semigroup. T is a S-subsemigroup if and only if Z_m is a S-semigroup.*

The proof is left as an exercise to the reader.

All these MOD subset semigroups have finite subsemigroups as well as infinite subsemigroups which are not ideals. However all ideals are of infinite order.

Some of these claims are not true in case of the MOD subset semigroups which are semilattices.

In fact we have every singleton element of MOD subset semigroups which are semilattices to be a subsemigroup of order one. These semilattices have subsemigroups of all order. Further any finite subset of the MOD semilattice can always be completed to be a subsemigroup. This is the striking difference between MOD semigroups and MOD semilattices.

Study in this direction is interesting and innovative. Finally we are not in a position to give group structure to any MOD subset collection. Even under addition MOD subset is only a semigroup which has always groups.

The following problems are suggested.

Problems

1. List out some special features enjoyed by MOD subset real pseudo groups.

2. Let $\{S(R_n(m)), +\} = G$ be MOD subset real pseudo group.

   (i) Prove $o(G) = \infty$.
   (ii) Show G is commutative.
   (iii) Can G have subgroups which are not pseudo?
   (iv) Prove every subset of cardinality greater than or equal to two cannot have inverse with respect to +.
   (v) Show G has subgroups of finite order.
   (vi) Obtain any other special feature enjoyed by G.

3. Let $G_1 = \{S(R_n(15)), +\}$ be the MOD subset real pseudo group.

   Study questions i to vi of problem two for this G_1.

4. Let $H = \{S(R_n(19)), +\}$ be the MOD subset real pseudo group.

   Study questions i to vi of problem two for this H.

5. Obtain the special features enjoyed by MOD subset complex pseudo groups.

6. Let $\{S(C_n(16)),\ i_F^2 = 15, +\} = B$ be the MOD subset complex number pseudo group.

   (i) Show B is commutative and is of infinite order.
   (ii) Find some special features enjoyed by B.
   (iii) Show S has both finite and infinite subset groups which are not pseudo.
   (iv) Find all infinite pseudo subgroups of B.

7. Let $M = \{S(C_n(43)), +,\ i_F^2 = 42\}$ be the MOD subset complex number pseudo group.

   Study questions i to iv of problem 6 for this M.

8. Let $S = \{S(R_n^I(m)),\ I^2 = I, +\}$ be the MOD subset neutrosophic pseudo group.

   (i) Obtain the special features enjoyed by S.
   (ii) Compare a MOD subset real pseudo group with the MOD subset neutrosophic pseudo group.
   (iii) Find subgroups of S which are not pseudo.
   (iv) Can S have a subgroup of finite order which is pseudo?

9. Prove for all $S = \{(S(R_n(m)), +\}$ or $S(R_n^I(m))$, $S(C_n(m))$, $S(R_n(m)(g))$ with $g^2 = 0$, $S(R_n(m)(g))$, $g^2 = g$, $S(R_n(m)(g))$ with $g^2 = (m - 1)g\}$ the MOD pseudo subset group (or the MOD subset semigroup) is always a S-MOD subset semigroup.

10. Let $M = \{S(R_n(m)(g))$, '+' $g^2 = 0\}$ be the MOD subset dual number pseudo group (semigroup).

    (i) Find all finite MOD subset subsemigroups.
    (ii) Can M have infinite pseudo groups which are not ideals?
    (iii) Can M have ideals?
    (iv) Prove in general a subset X of M has no inverse.
    (v) Prove $\{0\}$ is the additive identity of M.
    (vi) Prove $B = \{$all subsets from ag with $a \in [0, m)\}$ is only a subsemigroup of M but not an ideal.

11. Let $B_1 = \{S(R_n(9)(g)), +; g^2 = g\}$ be the MOD subset special dual like number semigroup.

    Study questions i to vi of problem 10 for this B_1.

12. Let $T_1 = \{S(R_n(17)(g)), +; g^2 = g\}$ be the MOD subset special dual like number semigroup.

    Study questions i to vi of problem 10 for this T_1.

13. Let $W_1 = \{S(R_n(93)(g)), +; g^2 = 92g\}$ be the MOD subset special quasi dual number semigroup (pseudo group).

    Study questions i to vi of problem 10 for this W_1.

14. Let $M = \{S(R_n^I(18)(g)), +\}$ be the MOD subset neutrosophic semigroup.

    Study questions i to vi of problem 10 for this M.

15. Let W = $\{S(R_n^I(29)(g)), +\}$ be the MOD subset neutrosophic semigroup.

    Study questions i to vi of problem 10 for this W.

16. Let T = $\{\{S(R_n(20)) \cup \phi\}, \cap\}$ be the MOD subset real semilattice (semigroup).

    (i) Prove T has subsemigroups of order 1, 2, 3, … so on.
    (ii) Can every subset be completed to form a subsemigroup?.
    (iii) Obtain some interesting properties about T.

17. Let P = $\{\{S(R_n(43)) \cup \phi\}, \cap\}$ be the MOD subset real semilattice.

    Study questions i to iii of problem 16 for this P.

18. Let M = $\{\{S(R_n^I(24)) \cup \phi\}, \cap\}$ be the MOD subset neutrosophic semilattice.

    Study questions i to iii of problem 16 for this M.

19. Let S = $\{\{S(C_n(92)) \cup \phi\}, \cap\}$ be the complex modulo integer semilattice.

    Study questions i to iii of problem 16 for this S.

20. Let V = $\{\{S(R_n(24)(g)) \cup \phi\}, \cap, g^2 = 0\}$ be the MOD subset dual number semilattice.

    Study questions i to iii of problem 16 for this V.

21. Let W = $\{\{S(R_n(20)(g)) \cup \phi\}, \cap, g^2 = g\}$ be the MOD subset special dual like number semilattice.

    Study questions i to iii of problem 16 for this W.

22. Let M = $\{\{S(R_n(29), \cup\}$ be the MOD subset real semilattice.

    Study questions i to iii of problem 16 for this M.

23. Let X = $\{\{S(R_n^I(35)), \cup\}$ be the MOD subset neutrosophic semilattice.

    Study questions i to iii of problem 16 for this X.

24. Let Z = $\{\{S(C_n(20)), \cup\}$ be the MOD subset complex modulo integer semilattice.

    Study questions i to iii of problem 16 for this Z.

25. Let B = $\{\{S(R_n(19)(g)), \cup, g^2 = 0\}$ be the MOD subset dual number semilattice.

    Study questions i to iii of problem 16 for this B.

26. Let S = $\{\{S(R_n(224)(g)), \cup, g^2 = 0\}$ be the MOD subset dual number semilattice.

    Study questions i to iii of problem 16 for this S.

27. Let B = $\{\{S(R_n(16)(g)), \cup, g^2 = g\}$ be the MOD subset dual like number semilattice.

    Study questions i to iii of problem 16 for this B.

28. Let S = $\{\{S(R_n(29)(g)), \cup, g^2 = 28g\}$ be the MOD subset special quasi dual number semilattice.

    Study questions i to iii of problem 16 for this S.

29. Let M = $\{S(R_n^I(43)), \cup, I^2 = I\}$ be the MOD subset neutrosophic semilattice.

    Study questions i to iii of problem 16 for this M.

30. Let S = {{S(R_n^I(24)), ∪ ϕ}, ∩, I^2 = I} be the MOD subset neutrosophic semilattice.

    Study questions i to iii of problem 16 for this S.

31. Let P = {{S(C_n(43)), ∪ϕ}, ∩, i_F^2 = 42} be the MOD subset complex modular integer semilattice.

    Study questions i to iii of problem 16 for this P.

32. Can P = {{S(R_n(48)) ∪ϕ}, ∩} be a S-semigroup? Justify your claim.

33. Can B ={{S(C_n(47)) ∪ϕ}, ∩} be a S-semigroup? Justify your claim.

34. Can P and B in problems 32 and 33 have subsemigroups which are Smarandache?

35. Obtain any special and interesting feature enjoyed by MOD subset neutrosophic semilattices.

36. Obtain some special properties related with MOD subset complex modulo number semilattices.

37. Let M = {S(R_n(24)), ×} be the MOD subset real semigroup.

    (i) Prove M is commutative.
    (ii) Can M have ideals of finite order?
    (iii) Can M have subsemigroups of finite order?
    (iv) Can M have zero divisors?
    (v) Can M have S-ideals?
    (vi) Is M a S-semigroup?
    (vii) Can M have S-units?
    (viii) Is it possible for M to have infinite number of idempotents?

(ix) Can the concept of S-idempotent be possible in M?

38. Let S = $\{S(R_n^I(43)), \times, I^2 = I\}$ be the MOD subset neutrosophic semigroup.

    Study questions i to ix of problem 37 for this S.

39. Let B = $\{S(C_n(24)), \times, i_F^2 = 23\}$ be the MOD subset complex modulo integer semigroup.

    Study questions i to ix of problem 37 for this B.

40. Let T = $\{S(R_n(17)), \times\}$ be the MOD subset real semigroup.

    Study questions i to ix of problem 37 for this T.

41. Let W = $\{S(C_n(53)), \times, i_F^2 = 52\}$ be the MOD subset complex modulo integer semigroup.

    Study questions i to ix of problem 37 for this W.

42. Let N = $\{S(R_n^I(12)), \times, I^2 = I\}$ be the MOD subset neutrosophic semigroup.

    Study questions i to ix of problem 37 for this N.

43. Let A = $\{S(R_n(40)(g)), g^2 = 0, \times\}$ be the MOD subset dual number semigroup.

    Study questions i to ix of problem 37 for this A.

44. Let B = $\{S(R_n(37)(g)), g^2 = g, \times\}$ be the MOD subset special dual like number semigroup.

    Study questions i to ix of problem 37 for this B.

45. Let $C = \{S(R_n(42)(g)),\ g^2 = 41g,\ \times\}$ be the MOD special quasi dual number semigroup.

    Study questions i to ix of problem 37 for this C.

46. Obtain some special features enjoyed by $B = \{S(R_n(m)g);\ g^2 = (m-1)g, \times\}$.

47. What are the special features associated with $S = \{S(C_n(m));\ i_F^2 = m-1, \times\}$?

Chapter Three

MATRICES AND POLYNOMIALS USING MOD SUBSET SEMIGROUPS

In this chapter we for the first time build matrices and polynomials using MOD planes. However MOD polynomials have been dealt with in [25-6].

Here we build matrices and polynomials using MOD subsets and study algebraic structures enjoyed by them.

To this end we will define all types of MOD real matrices and they can be easily extended to the case of MOD complex planes etc.

The notations given in chapter II will be used in this chapter also.

DEFINITION 3.1: *Let* $M = \{(a_1, \ldots, a_p) \,/ a_i \in R_n(m);\ 1 \leq i \leq p\} = \{((x_1, y_1), (x_2, y_2), \ldots, (x_p, y_p)) \,/ a_i = (x_i, y_i);\ x_i, y_i \in [0, m);\ 1 \leq i \leq$

p} be defined as the MOD real row matrix with entries from the MOD plane $R_n(m)$.

We will illustrate this by a few examples.

Example 3.1: Let
P = ((0, 3), (0.771, 2), (5.1107, 8.01), (2.07, 11.37)) be a MOD real 1×4 row matrix with entries from the MOD real plane $R_n(12)$.

Example 3.2: Let X = ((0.001, 0.2), (0.112, 0.75), (6.113, 0.211), (0, 0), (1, 1), (5, 5), (2.00007, 0.8912)) be the MOD real 1×7 row matrix with entries from the MOD real plane $R_n(7)$.

On similar lines we can define MOD complex row matrix, MOD neutrosophic row matrix, MOD dual number matrix, MOD special dual like number matrix and MOD special quasi dual like number matrix. This is a matter of routine so left as an exercise to the reader.

We will now supply examples of them.

Example 3.3: Let
M = ($3 + 5.1i_F$, $2 + 0.3i_F$, $0.32i_F$, 7.021, $0.3 + 0.772i_F$) is a 1×5 MOD complex modulo integer row matrix with entries from $C_n(10)$.

Example 3.4: Let P = {($0.4i_F$, $2 + 0.37i_F$, 0.337, $0.812 + 4i_F$, $6.33312 + 5.035721i_F$, 0, 1, i_F)} be the 1×8 MOD complex modulo integer matrix with entries from $C_n(7)$.

Example 3.5: Let
B = (I + 0.31, 1, 1 + I, 0.334 + 0.888I, 4 + 5I, 3.814 + 6.9321I) be the 1×6 MOD neutrosophic row matrix with entries from $R_n^I(7)$.

Example 3.6: Let
M = (0.7, 8I, 0.2, 7I + 0.54, 0, I, 1.32 + I, 1, 3, 4I + 0.5) be a 1×10 MOD neutrosophic row matrix with entries from $R_n^I(10)$.

Example 3.7: Let
W = (3 + 0.5g, 2g, 7, 0, g, 0.81 + 0.71g, 5g + 0.77) be a 1×7 MOD dual number row matrix with entries from $R_n(12)(g)$ with $g^2 = 0$.

Example 3.8: Let
M = (0.3, 11.2g, 9, 12 + 0.475g, 0, 1, 4g, 5.007 + 2.18g) be a 1×8 MOD dual number row matrix with entries from $R_n(13)(g)$ with $g^2 = 0$.

Example 3.9: Let B = (0.921, 0.7741g, 4, 5g, 8g + 0.779, 0.012g, 9 + 0.6642g, 2 + 5g, 10 + 9g) be a 1×9 MOD special dual like number row matrix with entries from $R_n(11)(g)$ where $g^2 = g$.

Example 3.10: Let
N = (10 + 2g, 4 + 8g, 0.2111g, 0.31115, 0.2114g + 9, 8 + 0.51231g, 0.332 + 0.12g) be a 1×7 MOD special dual like number row matrix with entries from $R_n(15)(g)$ with $g^2 = g$.

Example 3.11: Let M = (0, 1, 5g, 3, 2g, 0.11g, 0.15g, 7 + 8g, 10 + 0.3315g, 0.712 + 8g, 0.57312 + 0.63215g) be a 1×11 row matrix of MOD special quasi dual numbers with entries from $R_n(11)g$ with $g^2 = 10g$.

Example 3.12: Let S = (0.331g, 0, 4, 2g, 6.331, 0.4895, 0.7 + 0.5g, 3g + 8, 0.831 + 7g, 6 + 0.4123g) be a 1×10 MOD special quasi dual number row matrix with entries from $R_n(10)(g)$ with $g^2 = 9g$.

Now we proceed to give various operations on MOD row matrices of all types.

DEFINITION 3.2: *$B = \{(a_1, \ldots, a_p) \mid a_i \in R_n(m)$ (or $R_n^I(m)$ or $C_n(m)$ or $R_n(m)(g)$ with $g^2=0$ or $R_n(m)(g)$ with $g^2 = g$ or $R_n(m)g$ with $g^2 = (m-1)g$, $1 \leq i \leq p, +\}$ is defined to be a MOD real row matrix group or MOD neutrosophic row matrix group or MOD complex modulo integer row matrix group or MOD dual number row matrix group or MOD special dual like row matrix group or MOD special quasi dual number group respectively.*

We will illustrate these by some examples.

Example 3.13: Let
$B = \{(a_1, a_2, a_3, a_4)$ where $a_i = (x_i, y_i) \in R_n(10)$, $1 \leq i \leq 4, +\}$ be the MOD real 1×4 row matrix group under +.

We see ((0, 0), (0, 0), (0, 0), (0, 0)) acts as the additive identity.

Let x = ((3, 1), (2, 0.112), (0.1, 0), (0.72, 0.45)) and

y = ((0, 0.7), (0.1, 0.1121), (0.571, 2), (0.5, 0.31)) $\in$ B.

x + y = ((3, 1.7), (2.1, 0.2241), (0.671, 2), (1.22, 0.76)) $\in$ B. This is the way '+' operation is performed on B.

To every x $\in$ B there exists a unique y $\in$ B with x + y = ((0, 0), (0, 0), (0, 0), (0, 0)).

Let x = ((4, 0.33), (6.32, 6.75), (2.3, 6.2), (9.1, 2.7)) $\in$ B.

We have a unique y = ((6, 9.67), (3.68, 3.25), (7.7, 3.8), (0.9, 7.3)) $\in$ B such that x + y = ((0, 0), (0, 0), (0, 0), (0, 0)).

We call y the inverse of x and x is the inverse of y. Thus B is a MOD real row matrix group of infinite order.

Example 3.14: Let $B = \{(a_1, a_2, a_3) \mid a_i \in C_n(12), +, 1 \leq i \leq 3\}$ be the MOD row matrix complex modulo integer group.

B has subgroups of both finite and infinite order.

Example 3.15: Let
M = $\{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in R_n^I(17), 1 \leq i \leq 5, +\}$ be the MOD neutrosophic row matrix group.

Example 3.16: Let
T = $\{(a_1, \ldots, a_{10}) \mid a_i \in R_n(15)(g); 1 \leq i \leq 10, g^2 = 0, +\}$ be the MOD row matrix of dual numbers group.

Example 3.17: Let
S = $\{(a_1, a_2, \ldots, a_{15}) \mid a_i \in R_n(9)(g); 1 \leq i \leq 15, +, g^2 = g\}$ be the MOD special dual like number row matrix group of infinite order.

Example 3.18: Let
M = $\{(a_1, a_2, \ldots, a_{16}) \mid a_i \in R_n(15)g, 1 \leq i \leq 16; +, g^2 = 14g\}$ be the MOD special quasi dual number row matrix group of infinite order.

Example 3.19: Let
T = $\{(a_1, a_2, \ldots, a_{20}) \mid a_i \in R_n(19)(g); g^2 = 18g, 1 \leq i \leq 20, +\}$ be the MOD special quasi dual number group.

Example 3.20: Let M = $\{(a_1, a_2, a_3) \mid a_i \in R_n(5)(g); g^2 = g, 1 \leq i \leq 3, +\}$ be the MOD special dual like number row matrix group.

Let x = (0.9 + 4.2g, 2.1 + 0.3g, g) and
y = (4.1 + 0.8g, 2.9 + 4.7g, 4g) $\in$ M;

$$x + y = (0, 0, 0).$$

Example 3.21: Let
B = $\{(a_1, a_2, a_3, a_4, a_5, a_6) \mid a_i \in C_n(10); 1 \leq i \leq 6, +, i_F^2 = 9\}$ be the MOD complex modulo integer row matrix group.

Let x = $(9.2 + 4i_F, 0.3i_F, 2 + 3i_F, 0, 0.5 + 0.8i_F, 2.37) \in B$.

We see $y = (0.8 + 6i_F, 9.7i_F, 8 + 7i_F, 0, 9.5 + 9.2i_F, 7.63) \in B$ is such that $x + y = (0, 0, 0, 0, 0, 0)$.

Thus y is the inverse of x and x is the inverse of y.

Next we prove $\times$ can be defined on MOD row matrix collection and they form a semigroup under $\times$.

We will give examples of them.

Example 3.22: Let
$M = \{(x_1, x_2, x_3, x_4) \mid x_i \in R_n(9)(g), g^2 = 0, 1 \leq i \leq 4; \times\}$ be the MOD real dual number row matrix semigroup under product. M has zero divisors. M also has subsemigroups as well as ideals.

$x = (5 + 6.2g, 0.32 + 4.3g, 0, g)$ we find

$x \times x = (7 + 8g, 0.1024 + 2.752g, 0, 0) \in M$.

This is the way product is performed on M.

Let $x = (8.923 + 4.632g, 0, 0, 0)$ and
$y = (0, 2.71 + 3.25g, 0, 0) \in M$.

We see $x \times y = (0, 0, 0, 0) \in M$.

Thus M has infact infinite number of zero divisors.

Let $x = (1, 5, 2, 1) \in M$.

We see $y = (1, 2, 5, 1) \in M$ is such that
$x \times y = y \times x = (1, 1, 1, 1)$. So M has units.

Let $x = (0.7g, 0, g, 0.334g)$ and
$y = (ag, bg, cg, dg)$ where $a, b, c, d \in [0, 9)$ is such that $x \times y = (0, 0, 0, 0)$ as $g^2 = 0$.

Example 3.23: Let
$B = \{(a_1, a_2, a_3, a_4, a_5, a_6) \mid a_i \in R_n(20); 1 \leq i \leq 6, \times\}$ be the MOD real row matrix semigroup. B has infinite number of zero divisors but only finite number of units and idempotents. In fact B has atleast ${}_6C_1 + {}_6C_2 + {}_6C_3 + {}_6C_4 + {}_6C_5$ number of subsemigroups which are ideals of B.

Example 3.24: Let
$S = \{(a_1, \ldots, a_7) \mid a_i \in C_n(7); 1 \leq i \leq 7, i_F^2 = 6, \times\}$ be the MOD complex modulo integer row matrix semigroup. S has units, zero divisors and ideals.

Let $x = (2i_F, 0, 3i_F, 2.1i_F, 1, i_F, 3)$ and
$y = (2.34 + 2.1i_F, 0.7 + 2.14i_F, 6.12 + 3.4i_F, 4, 0.3771 + 0.523i_F, 6.02134 + 4.0002i_F, 5.2103 + 2.01i_F) \in S$.

We find $x \times y = (4.68 + 4.2i_F, 0, 0.56i_F, 1.4i_F, 0.3771 + 0.523i_F, 6.02134i_F + 3.0012, 1.6309 + 6.03i_F) \in S$.

This is the way product operation is performed on S.

Example 3.25: Let
$B = \{(a_1, a_2, a_3) \mid a_i \in R_n^I(20), I^2 = I, 1 \leq i \leq 3, \times\}$ be the MOD neutrosophic row matrix semigroup.

Let $x = (3 + 14.1I, 19 + 12.3I, 4.31I + 2.758)$ and
$y = (4 + 5I, 6 + 8I, 10 + 10I) \in B$.

We find $x \times y = (3 + 14.1I, 19 + 12.3I, 4.31I + 2.758) \times (4 + 5I, 6 + 8I, 10 + 10I)$

$= (12 + 56.4I + 15I + 70.5I, 114 + 73.8I + 152I + 98.4I, 43.1I + 27.58 + 43.1I + 27.58I)$

$= (12 + 1.9I, 14 + 4.2I, 7.58 + 13.78I) \in B$. This is the way product is performed on B.

Example 3.26: Let
M = $\{(a_1, a_2, a_3, a_4) \mid a_i \in R_n(10)(g), g^2 = 9g, 1 \leq i \leq 4, \times\}$ be the MOD special quasi dual number semigroup.

Let x = (5 + 2g, 7 + 5g, 3g, 5) and y = (0.32 + 8g, 4.2g, 8.3g, 2.5g) $\in$ M.

x $\times$ y = (5 + 2g, 7 + 5g, 3g, 5) $\times$ (0.32 + 8g, 4.2g, 8.3g, 2.5g) = (1.60 + 0.64g + 40g + 16 $\times$ 9g, 29.4g + 31.0 $\times$ 9g, 24.9 $\times$ 9g, 12.5g)

= (1.6 + 4g, 18.4g, 4.1g, 12.5g) $\in$ M.

This is the way product operation is performed on M.

Now we proceed on to describe MOD column matrix groups and semigroups.

DEFINITION 3.3:

$$B = \{\text{Collection of all column matrices } \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_p \end{bmatrix} \mid a_i \in R_n(m),$$

(or $R_n^I(m)$, $C_n(m)$, $R_n(m)(g)$ with $g^2 = 0$, $R_n(m)(g)$ with $g^2 = g$, or $R_n(m)(g)$ with $g^2 = (m - 1)g)$, $1 \leq i \leq p\}$ is defined as MOD real column matrices (or MODneutrosophic column matrix or MOD complex modulo integer column matrix or MOD dual number column matrix or MOD special dual like numbers column matrix or MOD special quasi dual numbers column matrix).

We will illustrate these situations by some examples.

Example 3.27: Let

$$P = \begin{bmatrix} (3,2) \\ (0.31,0.75) \\ 0,0.7771) \\ (1,6.1102) \end{bmatrix}$$ be a 4×1 MOD real column matrix with

entries from $R_n(7)$.

Example 3.28: Let

$$T = \begin{bmatrix} 3.01+0.75i_F \\ 0.851+7.15i_F \\ 0+i_F \\ 3+0 \\ 2.007+4.003i_F \\ 1.213+0.52i_F \\ 4.021+3.2i_F \end{bmatrix}$$ be the 7×1 MOD complex modulo

integer column matrix with entries from T.

Example 3.29: Let

$$B = \begin{bmatrix} 0.21 + 0.5I \\ 3.01 + 6.004I \\ 0.52 + 0.7I \\ 8 + 3I \\ 0.74 + 5I \\ 7 + 0.331I \\ 8.331 + 6.3102I \end{bmatrix}$$ be the MOD neutrosophic column

matrix from the MOD neutrosophic plane $R_n^I(1)$.

Example 3.30: Let

$$Y = \begin{bmatrix} 7 + 3.5g \\ 0.00152g \\ 0.111371 \\ 0.58 + 0.62g \\ 5.7 + 4g \end{bmatrix}$$ be the MOD dual number column matrix

from the MOD dual number plane $R_n(12)$; $g^2 = 0$.

Example 3.31: Let

$$Y = \begin{bmatrix} 0.3g \\ 0.0073 \\ 1+0.789g \\ 0.452+3g \\ 4+5g \\ 5.002+3.11g \\ 0 \end{bmatrix}$$ be the MOD special dual like number

column matrix with entries from $R_n(16)$ (g) $g^2 = g$.

Example 3.32: Let

$$B = \begin{bmatrix} 0 \\ 1+4g \\ 0.335+2.7g \\ 6 \\ 2.11+0.38g \\ 5g \\ 3.52+7.3g \end{bmatrix}$$ be the MOD special quasi dual number

column matrix with entries from $R_n(1)$ (g) with $g^2 = 9g$.

Now having seen the concept of MOD column matrices we proceed on to define two types of operations on the collection of all $t \times 1$ MOD column matrices which will be illustrated by some examples.

Example 3.33: Let

$$B = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| \; a_i \in R_n(12); 1 \leq i \leq 6, + \right\}$$

be the MOD real 6×1 column matrix group.

Clearly B is of infinite order and is commutative.

$$D = \begin{bmatrix} (0,0) \\ (0,0) \\ (0,0) \\ (0,0) \\ (0,0) \\ (0,0) \end{bmatrix}$$ is the additive identity of the group B.

Further if $A = \begin{bmatrix} (0,3) \\ (2,0.112) \\ (3,1.1) \\ (0,0) \\ (4,4) \\ (7.01,2.52) \end{bmatrix}$ then $-A = \begin{bmatrix} (0,9) \\ (10,11.888) \\ (9,10.9) \\ (0,0) \\ (8,8) \\ (4.99,9.48) \end{bmatrix} \in B$ is

$$\text{such that } A + (-A) = \begin{bmatrix} (0,0) \\ (0,0) \\ (0,0) \\ (0,0) \\ (0,0) \\ (0,0) \end{bmatrix}.$$

Example 3.34: Let

$$V = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| \; a_i \in R_n(19);\ 1 \leq i \leq 5, \right\}$$

be the MOD subset real column matrix group under +. V has subgroups of infinite order.

$$M_1 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| \; a_i \in R_n(19), + \right\} \subseteq V,$$

$$M_2 = \left\{ \begin{bmatrix} 0 \\ a_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| \; a_2 \in R_n(19), + \right\} \subseteq V,$$

$$M_3 = \left\{\begin{bmatrix} 0 \\ 0 \\ a_3 \\ 0 \\ 0 \end{bmatrix} \middle| \; a_3 \in R_n(19), +\right\} \subseteq V;$$

$$M_4 = \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ a_4 \\ 0 \end{bmatrix} \middle| \; a_4 \in R_n(19), +\right\} \subseteq V \text{ and}$$

$$M_5 = \left\{\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_5 \end{bmatrix} \middle| \; a_5 \in R_n(19), +\right\} \subseteq V$$

are 5 subgroups of infinite order of V.

Infact V also has subgroups of finite order.

Example 3.35: Let

$$P = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| \; a_i \in R_n(12)\,(g);\ 1 \leq i \leq 6,\ +\ g^2 = 0 \right\}$$

be the MOD dual like number 6 × 1 column matrix group. P has subgroups of both finite and infinite order.

$$\text{Let } x = \begin{bmatrix} 8+3g \\ 2+0.5g \\ g \\ 3.2+2g \\ 4 \\ 4.11+3.13g \end{bmatrix} \text{ and } y = \begin{bmatrix} 3.11+4.72g \\ 3.2g \\ 7.32+4.37g \\ 2.15+7g \\ 0.2g+11 \\ 0.371+2.3312g \end{bmatrix} \in P$$

$$x + y = \begin{bmatrix} 11.11+7.72\,g \\ 2+2.5g \\ 7.32+5.37g \\ 5.36+9g \\ 0.2g+3 \\ 4.481+5.4612g \end{bmatrix}.$$

This is the way operations are made on P.

Further P has atleast ${}_6C_1 + {}_6C_2 + \ldots + {}_6C_5$ number of infinite order subgroups.

Example 3.36: Let

$$W = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \middle| \; a_i \in R_n(6)\,(g);\ 1 \leq i \leq 4,\ g^2 = g, + \right\}$$

be the MOD special dual like number group.

We see $V_1 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| \; a_1 \in Z_6, + \right\}$ is a subgroup of W.

$V_2 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| \; a_1 \in \{0, 3\} + \right\}$ is a subgroup of W.

$V_3 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| \; a_1 \in \{0,2,4\}, + \right\}$ is a subgroup of W.

$$V_4 = \left\{\begin{bmatrix} 0 \\ a_2 \\ 0 \\ 0 \end{bmatrix} \middle| \; a_1 \in Z_6, +\right\} \text{ is a subgroup of W.}$$

$$V_5 = \left\{\begin{bmatrix} 0 \\ a_2 \\ 0 \\ 0 \end{bmatrix} \middle| \; a_1 \in \{0,3\} +\right\} \text{ is a subgroup of W.}$$

$$V_6 = \left\{\begin{bmatrix} 0 \\ a_2 \\ 0 \\ 0 \end{bmatrix} \middle| \; a_2 \in \{0, 2,4\} +\right\} \text{ is a subgroup of W.}$$

$$V_7 = \left\{\begin{bmatrix} 0 \\ 0 \\ a_3 \\ 0 \end{bmatrix} \middle| \; a_3 \in Z_6, +\right\},$$

$$V_8 = \left\{\begin{bmatrix} 0 \\ 0 \\ a_3 \\ 0 \end{bmatrix} \middle| \; a_3 \in \{0,3\}, +\right\},$$

$$V_9 = \left\{ \begin{bmatrix} 0 \\ 0 \\ a_3 \\ 0 \end{bmatrix} \middle| \; a_3 \in \{0,2,4\}, + \right\} \text{ are subgroups of W.}$$

$$V_{10} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_4 \end{bmatrix} \middle| \; a_4 \in Z_6, + \right\},$$

$$V_{11} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_4 \end{bmatrix} \middle| \; a_4 \in \{0,3\}, + \right\} \text{ and}$$

$$V_{12} = \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ a_4 \end{bmatrix} \middle| \; a_4 \in \{0,2,4\} + \right\} \text{ are subgroup of W.}$$

$$V_{13} = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ 0 \\ 0 \end{bmatrix} \middle| \; a_1, a_2 \in Z_6, + \right\},$$

$$V_{14} = \left\{\begin{bmatrix} a_1 \\ a_2 \\ 0 \\ 0 \end{bmatrix} \middle|\; a_1 \in Z_6,\ a_2 \in \{0,3\},+\right\},$$

$$V_{15} = \left\{\begin{bmatrix} a_1 \\ a_2 \\ 0 \\ 0 \end{bmatrix} \middle|\; a_1 \in Z_6,\ a_2 \in \{0,2,4\},\ +\right\},$$

$$V_{16} = \left\{\begin{bmatrix} a_1 \\ a_2 \\ 0 \\ 0 \end{bmatrix} \middle|\; a_2 \in Z_6,\ \ a_1 \in \{0,3\},+\right\},$$

$$V_{17} = \left\{\begin{bmatrix} a_1 \\ a_2 \\ 0 \\ 0 \end{bmatrix} \middle|\; a_2 \in Z_6, \text{ and } a_1 \in \{0,2,4\},\ +\right\},$$

$$V_{18} = \left\{\begin{bmatrix} a_1 \\ a_2 \\ 0 \\ 0 \end{bmatrix} \middle|\; a_1, a_2 \in \{0,3\},\ +\right\},$$

$$V_{19} = \left\{\begin{bmatrix} a_1 \\ a_2 \\ 0 \\ 0 \end{bmatrix} \middle|\ a_1, a_2 \in \{0,2,4\}, +\right\},$$

$$V_{20} = \left\{\begin{bmatrix} a_1 \\ a_2 \\ 0 \\ 0 \end{bmatrix} \middle|\ a_1 \in \{0,3\} \text{ and } a_2 \in \{0, 2, 3\}, +\right\},$$

$$V_{21} = \left\{\begin{bmatrix} a_1 \\ a_2 \\ 0 \\ 0 \end{bmatrix} \middle|\ a_1 \in \{0,2,4\}\right.$$

and $a_2 \in \{0, 3\}, +\}$ are all subgroups of W of finite order.

Likewise we get $9 \times {}_4C_2$ number of subgroup of finite order.

We can also consider.

$$V_{22} = \left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ 0 \end{bmatrix} \middle|\ a_1, a_2, a_3 \in Z_6, + \right\},$$

$$V_{23} = \left\{\left.\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ 0 \end{bmatrix}\right| a_1, a_2, a_3 \in \{0,3\} +\right\},$$

$$V_{24} = \left\{\left.\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ 0 \end{bmatrix}\right| a_1, a_2, a_3 \in \{0, 2, 4\} +\right\},$$

$$V_{25} = \left\{\left.\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ 0 \end{bmatrix}\right| a_1 \in Z_6\ a_2, a_3 \in \{0,3\} +\right\},$$

$$V_{26} = \left\{\left.\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ 0 \end{bmatrix}\right| a_1 \in Z_6, a_2, a_3\ \{0, 2, 4\} +\right\},$$

$$V_{27} = \left\{\left.\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ 0 \end{bmatrix}\right| a_1 \in Z_6\ a_2 \in \{0,3\}\ a_3 \in \{0, 2, 3\}, +\right\},$$

$$V_{28} = \left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ 0 \end{bmatrix} \middle| a_1 \in Z_6, a_2 \in \{0, 2, 4\} \text{ and } a_3 \in \{0, 3\}\right\}$$

are subgroups of finite order.

We have $4 \times 3 \times 7 = 84$ such subgroups of finite order.

Thus we have got a big collection of finite order subgroups for W.

In view of all these we leave the following conjecture open.

Problem 3.1: Let $B = \{(a_1, \ldots, a_p) \mid a_i \in (R_n(m)), +;\ m = p_1\, p_2 \ldots p_t$ (each p_i is a prime different from other primes)$\}$ be the MOD real row matrix group.

Find the number of subgroups of finite order in B.

Problem 3.2: Let

$$C = \left\{\begin{bmatrix} a_1 \\ \vdots \\ a_p \end{bmatrix} \middle| a_1 \in R_n(m) \text{ m as in problem}, +\right\}$$

Find the number of finite subgroups and show both B and C has same number of finite subgroups.

Problem 3.3: Find the number of finite subgroups if; $m = p_1^{\alpha_1}\, p_2^{\alpha_2} \ldots p_t^{\alpha_t}$; $\alpha_i \geq 1$, $1 \leq i \leq t$ in $R_n(m)$.

Example 3.37: Let

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \middle| \; a_i \in R_n(6)(g);\ g^2 = 0,\ 1 \leq i \leq 6, + \right\}$$

be the MOD dual number column matrix group.

This has more number of finite subgroups than that of MOD real column matrix group.

$$\text{Let } V_1 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| \; a_1 \in \langle Z_6 \cup g \rangle, + \right\}$$

$$V_2 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| \; a_1 \in Z_6, + \right\}$$

$$V_3 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| \; a_1 \in \{0,3\}, + \right\}$$

$$V_4 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| a_1 \in \{0, 2, 4\}, + \right\},$$

$$V_5 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| a_1 \in \{0, 3 + 3g, 3g, 3\}, + \right\},$$

$$V_6 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| a_1 \in \{0, 2, 4, 2g, 4g, 2 + 4g, 2 + 2g, 4 + 2g, 4 + 4g\}, + \right\}$$

and so on.

Thus a MOD dual number group has more finite subgroups than MOD real group.

Example 3.38: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \\ a_8 \end{bmatrix} \middle| a_i \in R_n^I(12); 1 \leq i \leq 8, + \right\}$$

be the MOD neutrosophic column matrix group.

M also has several subgroups of finite order.

$$B_1 = \left\{ \left. \begin{bmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right| a_1 \in Z_{12}, +\right\},$$

$$B_2 = \left\{ \left. \begin{bmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right| a_1 \in \{0,6\}, +\right\},$$

$$B_3 = \left\{ \left. \begin{bmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right| a_1 \in \{0, 2, \ldots, 10\}, +\right\},$$

$$B_4 = \left\{ \left. \begin{bmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right| a_1 \in \{0, 3, 6, 9\}, +\right\},$$

$$B_5 = \left\{ \left. \begin{bmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right| a_1 \in \{0, 4, 8\}, +\right\},$$

$$B_6 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \middle| \; a_1 \in \langle Z_{12} \cup I \rangle, + \right\},$$

$$B_7 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \middle| \; a_1 \in \{0,6,6I, 6 + 6I\}, + \right\},$$

$$B_8 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \middle| \; a_1 \in \{0, 4, 8, 4I, 8I, 4 + 8I, 8 + 4I, 4 + 4I, 8 + 8I\}, + \right\}$$

$$B_9 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \middle| \; a_1 \in \{0, 2, 4, 6, 8, 10, 2I, 4I, 6I, 8I, 10I, 2 + 2I, 2 + 4I, 2 + 6I, 2 + 8I, 2 + 10I \ldots 10 + 10I\}, + \right\}$$

and so on are all subgroups of M of finite order.

M also has infinite order subgroups.

Example 3.39: Let

$$M = \left\{ \left. \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_9 \end{bmatrix} \right| a_i \in C_n\ (20);\ 1 \leq i \leq 9, +\right\}$$

be the MOD finite complex modulo integer 9 × 1 column group. We see M has several MOD subgroups of finite order.

Now we proceed onto define the notion of product of column matrix. Of course we can have only have natural product of matrices.

We see under natural product, M is only a semigroup.

We will illustrate this situation by some examples.

Example 3.40: Let

$$B = \left\{ \left. \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \right| a_i \in\ R_n\ (10),\ 1 \leq i \leq 7, \times_n\right\}$$

be the MOD real column matrix semigroup under natural product $\times_n$.

B has zero divisors and is commutative B has subsemigroups of finite order.

Example 3.41: Let

$$L = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \middle| a_1 \in R_n(10);\ 1 \leq i \leq 4, \times_n \right\}$$

be the MOD real column matrix semigroup.

$$\text{Let } x = \begin{bmatrix} (0, 0.21) \\ (1, 6.52) \\ (1.22, 3.52) \\ (3.125, 1.112) \end{bmatrix} \text{ and } y \begin{bmatrix} (0.771, 0) \\ (0.971, 0.2) \\ (0.7, 5) \\ (0.8, 2) \end{bmatrix} \in L;$$

$$\text{we find } x \times_n y = \begin{bmatrix} (0,0) \\ (0.971, 1.304) \\ (0.854, 7.60) \\ (2.5000, 2.224) \end{bmatrix} \in L.$$

This is the way natural product is performed on L.

Example 3.42: Let

$$B = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| \, a_i \in R_n^I(6); 1 \leq i \leq 7, \times_n \right\}$$

be the MOD neutrosophic column matrix semigroup.

B has infinite number of zero divisors but only finite number of idempotents and units. B has finite number of finite order subsemigroups.

Example 3.43: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{bmatrix} \middle| \, a_i \in C_n(24); 1 \leq i \leq 7, \times_n \right\}$$

be the MOD complex modulo integer column matrix semigroup.

M has several zero divisors but only finite number of units and idempotents.

Infact M has only finite number of finite order subsemigroups however all ideals of M are of infinite order.

Example 3.44: Let

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{12} \end{bmatrix} \middle| \; a_i \in R_n(12)(g), g^2 = 0, 1 \leq i \leq 2, \times_n \right\}$$

be the MOD dual number column matrix semigroup.

S has subsemigroups of finite order. All ideals of S are of infinite order.

Example 3.45: Let

$$T = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| \; a_i \in R_n(19)(g), g^2 = g, 1 \leq i \leq 5, \times_n \right\}$$

be the MOD special dual like number column matrix semigroup.

$$P_1 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \middle| \; a_i \in R_n(19); \times_n \right\}$$

is an infinite subsemigroup which is not an ideal.

T has infinite number of zero divisors.

Example 3.46: Let

$$S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{bmatrix} \middle| \; a_i \in R_n(29)(g) \text{ with } 1 \leq i \leq 10, g^2 = 28g, \times_n \right\}$$

be the MOD special quasi dual like number column matrix semigroup. S has atleast ${}_{10}C_1 + {}_{10}C_2 + \ldots + {}_{10}C_9$ number of ideals.

Also S has $3({}_{10}C_1 + \ldots + {}_{10}C_9 + {}_{10}C_{10})$ number infinite subsemigroups which are not ideals.

Next we proceed onto give examples of MOD matrix groups and semigroups of matrices which are not column or row matrices.

Example 3.47: Let

$$B = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in R_n(12); 1 \leq i \leq 9, + \right\}$$

be the MOD real square matrix group.

B has subgroups of both finite and infinite order.

Example 3.48: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{30} \\ a_{31} & a_{32} & \dots & a_{40} \end{bmatrix} \middle| \; a_i \in R_n^I(43); 1 \leq i \leq 40, \times_n \right\}$$

be the MOD neutrosophic 4×40 matrix group of infinite order.

Example 3.49: Let

$$T = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{40} & a_{41} & a_{42} \end{bmatrix} \middle| \; a_i \in C_n(27); 1 \leq i \leq 42, i_F^2 = 26, + \right\}$$

be the MOD complex modulo integer matrix group. T has subgroups of both finite and infinite order.

Example 3.50: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_6 \\ a_7 & a_8 & \dots & a_{12} \\ a_{13} & a_{14} & \dots & a_{18} \\ a_{19} & a_{20} & \dots & a_{24} \end{bmatrix} \middle| \; a_i \in R_n(43)(g); \; g^2 = 0, 1 \leq i \leq 24, + \right\}$$

be the MOD dual number matrix group.

M has subgroups of both finite and infinite order.

Example 3.51: Let

$$W = \left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ \vdots & \vdots & \vdots & \vdots \\ a_{37} & a_{38} & a_{39} & a_{40} \end{bmatrix} \middle| \, a_i \in R_n(18)(g);\ g^2 = g,\ 1 \le i \le 40, +\right\}$$

be the MOD special quasi dual like number group.

W has infinite number of subgroups.

W has atleast $5({}_{40}C_1 + {}_{40}C_2 + \ldots + {}_{40}C_{39} + {}_{40}C_{40})$ number of subgroups of finite order.

W also subgroups of infinite order.

Example 3.52: Let

$$T = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \\ a_{13} & a_{14} \end{bmatrix} \middle| \, a_i \in R_n(24)(g);\ g^2 = g,\ 1 \le i \le 14, +\right\}$$

be the MOD special dual like number group of infinite order.

T has several subgroups of finite and infinite order.

T has atleast $7({}_{14}C_1 + \ldots + {}_{14}C_{14})$ number of finite subgroups.

Example 3.53: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ \vdots & \vdots & \vdots & \vdots \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix} \middle| \; a_i \in R_n(12)(g);\ g^2 = g, \; 1 \leq i \leq 24, + \right\}$$

is the MOD special dual like number group.

M has atleast $2({}_{24}C_1 + {}_{24}C_2 + \ldots + {}_{24}C_{24})$ number of subgroups of infinite order.

M has atleast $5({}_{24}C_1 + {}_{24}C_2 + \ldots + {}_{24}C_{23} + {}_{24}C_{24})$ number of subgroups of finite order.

Now we proceed onto define and describe (MOD matrix semigroup under $\times$.

Let $R_n(m)$ (or $R_n^I(m)$, $C_n(m)$, $R_n(m)(g)$ with $g^2 = 0$, $R_n(m)(g)$ with $g^2 = g$ or $R_n(m)(g)$ with $g^2 = (m - 1)g$) be the MOD real plane (or MOD neutrosophic plane or MOD complex modulo integer plane or MOD dual number plane, MOD special dual like number plane or MOD special quasi dual number plane).

We built matrices using these planes and give algebraic structure of product on them.

This is illustrated by examples.

Example 3.54: Let

$$B = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| \; a_i \in R_n(23);\ 1 \leq i \leq 12, \times_n \right\}$$

be the MOD real matrix semigroup under the natural product of matrices.

Let

$$x = \{\begin{bmatrix} (0,3) & (2.1115,2) & (0.3312,0) & (1,4) \\ (0.311,1) & (5.221,3.101) & (14,20.01) & (3,0.31) \\ (0.431,5) & (6,7) & (0,0) & (2.110,3) \end{bmatrix}$$

and

$$y = \begin{bmatrix} (3.1115,2) & (0,5) & (0.2,8.1119) & (1.772,0) \\ (6,5.213) & (1,0) & (2,10) & (10,0.2) \\ (0,1.32) & (10.0.77) & (4.331,15.371) & (2,0.7) \end{bmatrix}$$

$\in$ B.

We find $x \times_n y$

$$= \begin{bmatrix} (0,6) & (0,10) & (0.6624,0) & (1.772,0) \\ (1.866,5.213) & (5.221,0) & (5,2.001) & (7,0.0622) \\ (0,66) & (17,5.39) & (0,0) & (4.220,2.1) \end{bmatrix}$$

$\in$ B.

This is the way product is defined on B. Infact B has infinite number of zero divisors and only a finite number of units.

Example 3.55: Let

$$M = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \, a_i \in R_n(6); 1 \leq i \leq 4, \times_n\right\}$$

be the MOD real square matrix semigroup under the natural product $\times_n$.

$$\text{Let } x = \begin{bmatrix} (0,3) & (0.3312, 0.4423) \\ (4.2227, 2.115) & (3,5) \end{bmatrix} \text{ and}$$

$$y = \begin{bmatrix} (5.17835, 1.831) & (0.3, 2) \\ (4,5) & (5.1113, 4.235) \end{bmatrix} \in M.$$

$$\text{We find } x \times_n y = \begin{bmatrix} (0, 5.493) & (0.09936, 0.8846) \\ (4.9908, 4.575) & (3.3339, 3.175) \end{bmatrix} \in M.$$

This is the way natural product operation is performed on M.

When we replace the natural product $\times_n$ by $\times$; M which is a commutative semigroup under $\times_n$ is a non commutative semigroup under $\times$. We will show $x \times y \neq y \times x$ in general.

$$\text{Let } A = \begin{bmatrix} (0.377, 2) & (1.5444, 3.2) \\ (4.31, 0) & (0, 5.21) \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} (3, 2.1) & (0, 5) \\ (0, 4) & (1, 4) \end{bmatrix} \in M$$

$$A \times B = \begin{bmatrix} (1.131, 4.2) + (0, 0.8) & (0, 4) + (1.544, 0.8) \\ (0.93, 0) + (0, 2.84) & (0, 0) + (0, 2.84) \end{bmatrix}$$

$$= \begin{bmatrix} (1.131, 5) & (1.544, 4.8) \\ (0.93, 2.84) & (0, 2.84) \end{bmatrix} \quad \text{- I}$$

Now we find

$$B \times A = \begin{bmatrix} (1.131, 4.2) + (0, 0) & (4.632, 0.72) + (0, 2.05) \\ (0, 2) + (4.31, 0) & (0, 0.8) + (0, 2.84) \end{bmatrix}$$

$$= \begin{bmatrix} (1.131, 4.2) & (4.632, 2.77) \\ (4.31, 2) & (0, 3.64) \end{bmatrix} \text{- II}$$

I and II are different hence the MOD real square matrix semigroup under usual product is a non commutative semigroup.

Example 3.56: Let

$$W = \left\{\left.\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \end{bmatrix}\right| a_i \in C_n(10),\ i_F^2 = 9,\ 1 \leq i \leq 8, \times_n\right\}$$

be the MOD complex modulo integer semigroup under the natural product $\times_n$.

$$\text{Let } x = \begin{bmatrix} 2+3i_F & 0.337i_F \\ 1+i_F & 4.7772 \\ 4.1732i_F & 2+3i_F \\ 0 & 1 \end{bmatrix} \text{ and}$$

$$y = \begin{bmatrix} 9.221+3.88i_F & 8+2i_F \\ 0.7784 & 5+5i_F \\ 2 & 9.2 \\ 0.7793+6.11192i_F & 4.7321+8.331i_F \end{bmatrix} \in W.$$

$x \times_n y =$

$$\begin{bmatrix} 8.442 + 7.663i_F + 7.76i_F + 1.64 \times 9 & 2.696i_F + 0.674 \times 9 \\ 0.7784 + 0.7784i_F & 3.8860 + 3.8860i_F \\ 0.3464i_F & 8.4 + 7.6i_F \\ 0 & 4.7321 + 8.331i_F \end{bmatrix}$$

$\in$ W.

This is the way the natural product is carried out on W.

Further W is commutative under $\times_n$.

Example 3.57: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & \ldots & \ldots & \ldots & a_{10} \\ a_{11} & \ldots & \ldots & \ldots & a_{15} \end{bmatrix} \middle| \ a_i \in R_n(6)(g); \right.$$

$$1 \leq i \leq 15,\ g^2 = 0, \times_n \}$$

be the MOD dual number semigroup matrix.

S has infinite number of zero divisors.

$$\text{Let } x = \begin{bmatrix} 2+3g & g & 0 & 0.2g & 1 \\ 0 & 1+g & 2 & 3g & 2+0.4g \\ 1 & 2g & 0.1+0.5g & 0 & 0.3g \end{bmatrix} \text{ and}$$

$$y = \begin{bmatrix} 0.2g & 0.1g & 4+0.21g & 0.3g & 0.5+3g \\ 4+0.2g & 0.7g & 0.32+0.11g & 0.12g & 4g \\ 0.315+2.17g & 0.7114g & 0.52g & 1.1117+2.32g & 4.1g \end{bmatrix}$$

$\in$ S.

We find

$x \times_n y =$

$$\begin{bmatrix} 0.4g & 0 & 0 & 0 & 0.05+3g \\ 0 & 0.7g & 0.64+0.22g & 0 & 2g \\ 0.315+2.17g & 0 & 0.052g & 0 & 0 \end{bmatrix}$$

$\in$ S.

This is the way we find the product on S.

Further $\{S, \times_n\}$ is a commutative semigroup.

Infact S has only finite number of units and idempotents.

The identity element of S is $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}$.

Example 3.58: Let

$$B = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \end{bmatrix} \middle| \, a_i \in R_n(10)\, g,\, g^2 = g, \times_n \right\}$$

be the MOD special dual like number semigroup. B has zero divisors units and idempotents.

$$\text{Let } x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \\ 3g & 2.11g \end{bmatrix} \text{ and}$$

$$y = \begin{bmatrix} 9.3321 + 0.33g & 0 \\ 0 & 7.2152 + 5.33g \\ 0 & 0 \end{bmatrix} \in B.$$

$$\text{Clearly } x \times_n y = \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}. \text{ Let } x = \begin{bmatrix} 5g & g \\ 5 & 6 \\ 6g & 1 \end{bmatrix} \in B;$$

$$\text{we see } x^2 = \begin{bmatrix} 5g & g \\ 5 & 6 \\ 6g & 1 \end{bmatrix} \in B \text{ is an idempotent in B.}$$

We have several idempotents in x but they are only finite in number.

$$P_1 = \left\{ \begin{bmatrix} a_1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \middle| \; a_1 \in R_n(10)(g) \right\}$$

is a MOD subsemigroup which is also an ideal of B.

$$P_2 = \left\{\left.\begin{bmatrix} a_1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\right| a_1 \in R_n(10)\right\}$$

is a MOD subsemigroup of B but P_2 is not an ideal of B and is of infinite order.

$$P_3 = \left\{\left.\begin{bmatrix} a_1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\right| a_1 \in [0,10)\right\}$$

is a subsemigroup of infinite order and is not an ideal.

$$P_4 = \left\{\left.\begin{bmatrix} a_1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\right| a_1 \in Z_{10}\right\}$$

is only a subsemigroup of finite order and is not an ideal.

We have atleast ${}_6C_1 + {}_6C_2 + \ldots + {}_6C_5$ number of MOD subsemigroups which are ideals of B.

$$T = \begin{bmatrix} 1 & 9 \\ 1 & 1 \\ 1 & 9 \end{bmatrix} \in \text{ B is such that}$$

$$T^2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix} \text{ is a unit in B.}$$

Example 3.59: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \end{bmatrix} \middle| \; a_i \in R_n(7)\ g;\ g^2 = 6g, \times_n \right\}$$

be the MOD special quasi dual number semigroup matrix under natural product $\times_n$.

$$\text{Let } x = \begin{bmatrix} 0 & 3+2g & g \\ 1+g & 5g & 2+g \end{bmatrix} \text{ and}$$

$$y = \begin{bmatrix} 0.3312+6.52g & 2g & 5g \\ 3+0.41g & 2+0.3g & 0.5+0.7g \end{bmatrix} \in M.$$

$$x \times_n y = \begin{bmatrix} 0 & 6g+4\times 6g & 5\times 6g \\ 3+0.41g+ & 3g+1.5\times 6g & 1+1.4g \\ 3g+0.41\times 6g & & +0.5g+0.7\times 6g \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 2g & 2g \\ 3+6.87g & 5g & 1+6.1g \end{bmatrix};$$

clearly the MOD semigroup is commutative and is of infinite order.

Now we proceed onto build MOD subset matrix groups and MOD subset matrix semigroups using MOD planes.

We will define for one type and the rest of the work is considered as a matter of routine so is left as an exercise to the reader.

DEFINITION 3.4: *Let*
$B = \{(a_{ij})_{s\times t} \mid a_{ij} \in S(R_n(m)); 1 \leq i \leq s, 1 \leq j \leq t\}$ be the collection of all subset $s\times t$ matrices. B is defined as the MOD subset real $s \times t$ matrices.

Likewise we can define using (subsets of the MOD plane $S(R_n^I(m))$, $S(C_n(m))$, $S(R_n(m)(g))$ with $g^2 = 0$, $S(R_n(m)(g))$ with $g^2 = g$ and $S(R_n(m)(g))$ with $g^2 = (m-1)g$).

Now we will illustrate first all these concepts by some examples.

Example 3.60: Let
$B = \{(x_1, x_2, x_3, x_4) \mid x_i \in S(R_n(12)); 1 \leq i \leq 4\}$ is the MOD subset real row matrix B has infinite number of elements.

$X = (\{(0.8), (4.113, 0.88), (2.31, 4.26), (0, 1)\}, \{(0, 0),$
$(1, 0.2314), (1.521, 1)\}, \{(0.321,0.2), (0.441, 6.732)\},$
$\{(6.312, 6.3331), (7.51, 9) (8, 4)\}) \in B.$

Observe the elements in the row matrix are subsets of $S(R_n(12))$.

Example 3.61: Let

$$M = \left\{\begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{bmatrix} \middle| \; y_i \in S(R_n^I(5)); 1 \leq i \leq 4\right\}$$

be the MOD subset column neutrosophic matrix.

$$P = \begin{bmatrix} \{3+0.7I, 0.532+4.21I, 0.711, 2.052I\} \\ \{0.3, 2.7I, 0.33I, 0.2105\} \\ \{0,1,4,3.2137I+4.7321\} \\ \{2+3.00062I, 3+4.02157I, 0.76215I, 1.22012\} \end{bmatrix} \in M;$$

we observe elements of P are subsets.

Example 3.62: Let

$$B = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \middle|\ a_i \in S(C_n(10)),\ 1 \leq i \leq 12,\ i_F^2 = 9 \right\}$$

be the MOD subset complex modulo integer matrix.

T =

$$\begin{bmatrix} \{3+4i_F, 2+3i_F\} & \{0, 0.33i_F\} & \{1, 0.221, 4.3i_F\} \\ \{4.33i_F, 3+6.33i_F\} & \{3i_F, 0.34i_F\} & \{1,2,0\} \\ \{3,4,5,7i_F\} & \{4.331+8.538i_F\} & \{7,2,0,0.3i_F\} \\ \begin{matrix}\{4.1+2i_F, 3+4i_F \\ 0.337i_F+8.222\}\end{matrix} & \begin{matrix}\{6.1257+7.43i_F, 3+4.0007i_F, \\ 8.01457+7i_F\}\end{matrix} & \{0\} \end{bmatrix}$$

$$\in B.$$

The entries of every matrix in B are subsets of $S(C_n(10))$.

Example 3.63: Let

$$W = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle| \; a_i \in S\,(R_n(12)\,(g));\, g^2 = 0, \right.$$

$$1 \leq i \leq 16\}$$

be the MOD subset dual number square matrix.

$$P = \begin{bmatrix} \{0, 3+g, 0.7g\} & \{0.42+7.2g, 0.11g & \{0.4g, 0.5, 0.11g\} & \{2.8g\} \\ \{4g, 0.7771g\} & 0,1,0.55228g & \{0.7, 0.9, 8g\} & \{4.2g\} \\ \{5\} & 0.777782g & \{0.111152\} & \{6.2221\} \\ \{9+51231+0.8g & 11.21g, 2.112+3g & \{0\} & \{4.2+3.77g\} \end{bmatrix} \in W.$$

This is the way the elements of W look like.

Example 3.64: Let
S = {(a_1, a_2, a_3, a_4) | $a_i \in S(R_n(6)g)$; $g^2 = g$, $1 \leq i \leq 4$} be the MOD subset special dual like number row matrix.

P = ({2g + 0.331, 1 + 0.5871g, 2 + 3g, 4 + 0.5g}, {g, 2, 2 + 5g, 0.7771g}, {1 + 2.378g, 4.317 + 0.3315g} {0.7 + 2.31g, 0.77g, 0.523}) is an elements of S.

Entries are subsets form $S(R_n(6)g)$.

Example 3.65: Let

$$P = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \; a_i \in S(R_n(15)g) \text{ with } g^2 = 4g\right\}$$

be the MOD subset special quasi dual number square matrix.

$$x = \begin{bmatrix} \{0.8, 4g, 0.2g\} & \{0.5g, 2+g\} \\ \\ \{0, 1+0.3g\} & \{1, 4+2g\} \end{bmatrix} \in P.$$

The entries are subsets. Now are can define operations on these MOD subset matrices.

They are as follows.

In this case we can define 4 types of distinct operations on them viz. +, ×, ∪ and ∩.

We see under all the four operations they are only semigroups.

They are not groups under + so only we also call these semigrousp as subset pseudo groups.

We see in general any matrix may not have inverse with respect to + that is why we call them as pseudo group.

Example 3.66: Let
M = {(a_1, a_2, a_3) | $a_i \in$ S(R_n(12)), $1 \leq i \leq 3$, +} is the MOD subset real row matrix pseudo group or semigroup. We see ({0}, {0}, {0}) acts as the additive identity of M.

But for every x ∈ M, we may not in general have an inverse with respect to addition.

For take x = ({(10, 3), (2.11, 3.6), (0, 0.2)}, {(4, 3.11), (2.001, 1)}, {(0,5.1), (2,1)}) be the element in M.

Clearly x has no additive inverse.

We show how we find x + x.

x + x = ({(10,3), (2.11, 3.6), (0, 0.02)}, {(4, 3.11), (2.001, 1)}, {(0, 5.1), (2, 1)}) + {(10,3), (2.11, 3.6), (0, 0.02)}, {(4, 3.11), (2.001, 1)}, {(0, 5.1), (2,1)})

= ({(8,6), (0.11, 6.6), (10, 3.02), (4.22, 7.2), (2.11, 3.62), (0, 0.04)}, {(8, 6.22), (6.001, 4.11), (4.002, 2)}, {(0, 10.2), (4.2), (2, 6.1)}) is in M.

Clearly we cannot find a – x in M such that

$$x + (-x) = (\{0\}, \{0\}, \{0\}).$$

Example 3.67: Let

$$B = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \middle| \; a_i \in R_n^I(10), 1 \leq i \leq 3, + \right\}$$

be the MOD subset neutrosophic column matrix pseudo group (semigroup).

$$\text{Let } x = \begin{bmatrix} \{3I, 2+4I, 0.7I\} \\ \\ \{0.3338I + 6.521, 0\} \\ \\ \{0.43 + 7I, 2 + 3.01I\} \end{bmatrix} \text{ and}$$

$$y = \begin{bmatrix} \{2 + 0.2I, 7 + 5.3I\} \\ \\ \{7 + 0.3I, 4I + 0.2\} \\ \\ \{0.81 + 4I\} \end{bmatrix} \in B$$

$$x + y = \begin{bmatrix} \{2 + 3.2I, 4 + 4.2I, \\ 2 + 0.9I, 7 + 8.3I, \\ 9 + 9.3I, 7 + 6I\} \\ \\ \{3.521 + 6.38I, 7 + 0.3I \\ 4.3338I + 6.721, \\ 4I + 0.2\} \\ \\ \{11I + 1.24, \\ 2.81 + 7.01I\} \end{bmatrix} \in B.$$

This is the way sum is got in B.

Further the additive identity is $\begin{bmatrix} \{0\} \\ \{0\} \\ \{0\} \end{bmatrix}$.

Example 3.68: Let

$$S = \left\{ \begin{bmatrix} a_1 & \dots & a_6 \\ a_7 & \dots & a_{12} \\ a_{13} & \dots & a_{18} \\ a_{19} & \dots & a_{24} \end{bmatrix} \middle| \; a_i \in S(R_n(10)\,(g)),\; g^2 = 0, \; 1 \le i \le 24 \right\}$$

be the MOD subset dual number matrix pseudo group (semigroup).

S is commutative and is of infinite order.

Example 3.69: Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{28} & a_{29} & a_{30} \end{bmatrix} \middle| \; a_i \in C_n(20);\; i_F^2 = 19,\; 1 \le i \le 30, + \right\}$$

be the MOD subset complex modulo integer semigroup.

This has subgroups of both finite and infinite order.

This has subsemigroups which are not groups are only of infinite order.

Example 3.70: Let

$$W = \left\{ \left[\begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \\ a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{array} \right] \right| a_i \in S(R_n(9)(g)\};$$

$$1 \leq i \leq 24, g^2 = g, +\}$$

be the MOD subset special dual like number semigroup.

Thus has atleast $2({}_{24}C_1 + {}_{24}C_2 + \ldots + {}_{24}C_{24})$ number of subsemigroups which enjoy group structure under +.

Example 3.71: Let

$$M = \left\{ \left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{array} \right] \right| a_i \in S(R_n)(g)); g^2 = 12g, 1 \leq i \leq 9, +\}$$

be the MOD quasi special dual like number subset matrix semigroup.

M is commutative and is of infinite order.

Next we proceed onto give examples of MOD subset matrix semigroups under product.

Example 3.72: Let $B = \{(a_1, a_2, a_3) \mid a_i \in S(R_n(4)), 1 \leq i \leq 3, \times\}$ be the MOD subset real row matrix semigroup under $\times$.

B has zero divisors.

$\{0\} = ((0, 0), (0, 0), (0, 0))$ is the zero of B.

Let x = ({(0, 3), (0.732, 0.0014), (2.312, 3.1119)}, {(2.112, 0), (1, 2), (2, 0.7775)}, {(0,3) (0, 0.42)}} and

y = ({(0,3), (0.2, 1)}, {(0, 0), (1, 0.23)}, {(0.7, 0.1), (1,1)}) $\in$ B.

We find x $\times$ y = ({(0,1), (0, 0.0052), (0, 1.3357), (0, 3), (0.1464, 0.0014), (0.4624, 3.1119)}, {(0,0), (2.112,0), (1, 0.46), (2, 0.178825)), {(0,0.3), (0,0.042), (0, 3), (0, 0.42)}}) $\in$ B.

This is the way product operation is performed on B.

Let x = ({(0,0)}, {(0.72, 0.65), (2.001, 0.725), (3.002, 0.412)}, {(0,0.0013), (0.44432, 0.9231), (0, 2), (2, 0.0012)}})

and y = ({(3, 2.115), (1.2, 3.2225), (1.432, 0.7352), (3.2114, 0.353)} {(0,0)}, {(0,0)}}) $\in$ B.

Clearly x $\times$ y = ({(0,0)}, {(0, 0)}, {(0,0)}}) $\in$ B.

Infact B has infinite number of zero divisors.

Example 3.73: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \middle| \; a_i \in S(C_n(10)); 1 \leq i \leq 5, \times_n \right\}$$

be the MOD subset complex modulo column matrix semigroup.

$$\text{Let } x = \begin{bmatrix} \{3+0.2i_F, 0.72+4.3i_F, 4.311+5i_F, \\ \quad 6i_F, 7.3\} \\ \\ \{0.721+6.352i_F, 4.379+6.003i_F, \\ \quad 1.325+9.003i_F\} \\ \\ \{0.39+7.2i_F, 4.333i_F, \\ \quad 5.8997, 5.32+0.7i_F\} \\ \\ \{2+7.312i_F, 3.7214+5i_F \\ \quad 3i_F, 8, 0\} \end{bmatrix}$$

$$y = \begin{bmatrix} \{0,1,2\} \\ \\ \{i_F, 0\} \\ \\ \{3i_F, 5\} \\ \\ 4i_F, 0.3\} \end{bmatrix} \in M.$$

We find the natural product $\times_n$ on M.

$$x \times_n y = \begin{bmatrix} \{0, 3+0.2i_F, 0.72+4.3i_F, \\ 4.3111+5i_F, 6i_F, 7.3, 6+0.4i_F \\ 1.44+8.6i_F, 8.6222, 2i_F, 4.6\} \\ \\ \{0, 0.721i_F+7.168, 4.379i_F+4.027, \\ 1.325i_F+1.0027\} \\ \\ \{7.6991i_F, 5.96i_F+8.9, 1.17i_F+8.4, 6.991 \\ 1.95+6i_F, 5.96i_F+8.9 \\ 1.95+6i_F, 1.665i_F, 4.985, \\ 6.60+3.5i_F\} \\ \\ \{0, 6+1.936i_F, 9i_F 1.1642+5i_F, 4 \\ 2i_F 4.8856, 8i_F+3.322\} \end{bmatrix}$$

This is the way the natural product $\times_n$ is performed on M we see M has infinite number of zero divisors.

M also has finite order subsemigroups.

However ideals of M are of infinite order.

Example 3.74: Let

$$B = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \middle| \; a_i \in R_n^I(16);\ 1 \leq i \leq 15, \times_n \right\}$$

be the MOD subset neutrosophic matrix semigroup under the natural product $\times_n$.

We see B has only finite number of units, infinite number of zero divisors.

All ideals of B are of infinite order but B has subsemigroups of finite order.

Example 3.75: Let

$$V = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \; a_i \in R_n(12)\ (g),\ g^2 = 0,\ 1 \le i \le 4, \times\right\}$$

be the MOD subset dual number matrix semigroup.

V is a non commutative semigroup under the usual product.

$$\text{Let } x = \begin{bmatrix} \{0.7, 0.2g, 4g, 0.5\} & \{0\} \\ \{0\} & \{3+2g, 0.2g, 4, 0.5\} \end{bmatrix} \text{ and}$$

$$y = \begin{bmatrix} \{0.5, 5g, 3g, 2+g\} & \{1\} \\ \{2\} & \{4, 2g+5\} \end{bmatrix} \in V.$$

We find x × y =

$$\begin{bmatrix} \begin{matrix}\{0.35, 0.1g, 0, 2g, 0.25, 3.5 \\ 2.5g, 2.1g, 1.5g, 1.4+0.7g \\ 0.4g, 8g, 1+0.5\}+\{0\}\end{matrix} & \{0.7, 0.2g, 4g, 0.5\} \\ \{6+4g, 0.4g\ 8,\ 1\} & \begin{matrix}\{8g, 0.8g, 4, 2, g, \\ 8g+8\ g+2.5\ 3+4g\}\end{matrix} \end{bmatrix} \quad \text{I}$$

Consider y × x =

$$\begin{bmatrix} \begin{matrix}\{0.35, 0.1g, 0, 2g, 0.25, 3.5 \\ 2.5g, 2.1g, 1.5g, 1.4+0.7g \\ 0.4g, 8g, 1+0.5g\}\end{matrix} & \{3+2g, 0.2g, 4, 0.5\} \\ \{1.4, 0.4g, 8g, 11\} & \begin{matrix}\{8g, 0.8g, 4, 2, g, \\ 8g+8g+2.5, 3+4g\}\end{matrix} \end{bmatrix} \quad \text{II}$$

Clearly I and II are distinct, thus V is not a commutative semigroup.

It is interesting to find subsemigroups, right ideals and left ideals of V.

This task is left as an exercise to the reader.

However the MOD semigroup V under natural product $\times_n$ is commutative.

$$\text{For } x \times_n y = \begin{bmatrix} \begin{matrix}\{0.35, 0.1g, 0, 2g, 0.25, 3.5 \\ 2.5g, 2.1g, 1.5g, 1.4+0.7g \\ 0.4g, 8g, 1+0.5g\}\end{matrix} & \{0\} \\ \{0\} & \begin{matrix}\{8g, 0.8g, 4, 2, g, \\ 3+4g, g+2.5\}\end{matrix} \end{bmatrix}$$

Clearly $x \times_n y = y \times_n x$.

Further $x \times_n y \neq x \times y$.

Thus if natural product is defined on V, V becomes a commutative MOD subset dual number matrix semigroup.

Example 3.76: Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \middle| \; a_i \in S\,(R_n^I\,(15)) : 1 \leq i \leq 15, \times_n \right\}$$

be the MOD subset neutrosophic matrix semigroup under natural product $\times_n$.

Example 3.77: Let

$$P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \end{bmatrix} \middle| \; a_i \in S\,(R_n\,(10)\,(g)) : g^2 = g, \times_n, 1 \leq i \leq 10 \right\}$$

be the MOD subset special dual like number semigroup under natural product $\times_n$.

P is commutative and of infinite order.

P has at least $4({}_{10}C_1 + {}_{10}C_2 + \ldots + {}_{10}(C_{10})$ number of finite order subsemigroup.

P has atleast ${}_{10}C_1 + {}_{10}C_2 + \ldots + {}_{10}C_9$ number of distinct infinite order subsemigroups which are ideals of P.

Example 3.78: Let

$$B = \left\{\left.\begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \cdots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{30} \\ a_{31} & a_{32} & \ldots & a_{40} \\ a_{41} & a_{42} & \cdots & a_{50} \end{bmatrix}\right| a_i \in S\,(R_n(13)\,g) : g^2 = 12g,\right.$$

$$1 \leq i \leq 50, \times_n\}$$

be the MOD subset special quasi dual number matrix semigroup under natural product $\times_n$.

B is of infinite order and is commutative B has infinite number of zero divisors.

B has subsemigroups of finite order and B also has subsemigroups of infinite order which are not ideals.

Next we proceed onto define MOD subset semigroups using $\cup$ and $\cap$.

Example 3.79: Let
$M = \{a_1, a_2, a_3, a_4) \mid a_i \in S(R_n(24)) : 1 \leq i \leq 4, \cup\}$ be the MOD subset real semigroup.

Infact M is a semilattice.

Every element in M is an idempotent.

Infact every singleton element is a subsemigroup of order one M has subsemigroups of order two order three and so on.

Example 3.80: Let

$$B = \left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| \; a_i \in t\, S(R_n^I(24)) : 1 \leq i \leq 5, \cup\right\}$$

be the MOD subset neutrosophic column matrix semigroup of infinite order under $\cup$.

B has every element to be subsemigroup.

Example 3.81: Let

$$W = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| \; a_i \in S(R_n(12)\, g)) ; g^2 = 0, \right.$$

$$\left. 1 \leq i \leq 12, \cup\right\}$$

be the MOD subset dual number matrix semigroup under $\cup$ which is also a semilattice as every element in W is an idempotent.

Example 3.82: Let

$$B = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \\ a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} \end{bmatrix} \middle| \; a_i \in S\,(C_n(17)) \; ; \; i_F^2 = 16; \right.$$

$$\left. 1 \leq i \leq 24, \cup \right\}$$

be the MOD subset complex modulo integer semigroup. B is of infinite order.

Example 3.83: Let

$$W = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle| \; a_i \in S\,(R_n(15)\,g) \; ; \right.$$

$$\left. 1 \leq i \leq 16, g^2 = g, \cup \right\}$$

be the MOD subset special dual like number semilattice of infinite order which has subsemigroups of all orders.

Example 3.84: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in S\,(R_n(14)\,g) \; ; \; 1 \leq i \leq 9, \right.$$

$$\left. g^2 = 13g; \cup \right\}$$

be the MOD subset special quasi dual like number semigroup of infinite order.

M is commutative.

Now we give examples of MOD subset semigroup under $\cap$. To have intersection we need to adjoin the empty set to $S(R_n(m)) \cup \{\phi\}$, $S(R_n^I(m)) \cup S(R_n(m)(g)) \cup \{\phi\}$, $S(R_n(m)(g)) \cup \{\phi\}$ and so on $S(C_n(m)) \cup \{\phi\}$.

Example 3.85: Let
$M = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in S(R_n(15)) \cup \{\phi\}; 1 \leq i \leq 5, \cap\}$ be the MOD subset real matrix semigroup under $\cap$.

Example 3.86: Let

$$B = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} \end{bmatrix} \middle| \; a_i \in S\,(C_n(24)\,g)\,;\, 1 \leq i \leq 21, \cap \right\}$$

be the MOD subset complex modulo integer matrix semigroup.

Every element in B is a subsemigroup.

Subsets can be completed to form a subsemigroup. Infact B has subsemigroups of all orders 1, 2, 3 and so on.

Example 3.87: Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \; a_i \in S\,(R_n(20)\,(g)) \cup \{\phi\}; \; g^2 = 0, 1 \leq i \leq 4, \cap \right\}$$

be the MOD subset dual number matrix semigroup.

$$\text{Let } x = \begin{bmatrix} \{2+g, 0.335g, 0.2g\} & \{4g, 0, 2\} \\ \\ \{0.555g, 6.72+0.8g\} & \{0, 1, 3g\} \end{bmatrix} \text{ and}$$

$$y = \begin{bmatrix} \{0, 2g, 0.2g, 0.55g\} & \{0, 2, 8g\} \\ 2+g, 5g\} & \\ \\ \{0.555g, 7g, 2g, 5g, & \{0, 1, 5g, \\ 0.112+8g & 5.311+6.2g\} \end{bmatrix} \in S$$

$$x \cap y = \begin{bmatrix} \{2+g, 0.2g\} & \{0, 2\} \\ \\ \{0.555g\} & \{0, 1\} \end{bmatrix}.$$

This is the way '$\cap$' operation is defined on S.

Now we see $P^c = \{x, y, x \cap y\}$ is a subsemigroup we call it as the completion of the subset P = {x, y}.

Example 3.88: Let

$$M = \{(a_1, a_2, a_3, a_4) \mid a_i \in S(R_n(10)g) \cup \{\phi\},$$
$$g^2 = 9, 1 \leq i \leq 4; \cap\}$$

be the MOD subset special quasi dual number matrix semigroup.

Let P = {x, y, z} where x = ({0, 2.31 + 0.4g, 3.011 + 5g}, {0, 0.3g, 0.4 + 0.7g}, {0, 1, g}, {0, 4 + 5g, 1 + g}),

y = ({1, 2 + 3g, 4, 5g}, {2, 5g}, {ϕ}, {4 + 0.3g, 0, 0.9 + 0.12g}) and

z = ({0, 1, 2g, 3.007 + 0.115g}, {2g, 4 + 0.007g}, {0.32g}, {1.5632, 7.3214, 6.32g, 9.3021g}) $\in$ M.

$x \cap y = (\{\phi\}, \{\phi\}, \{\phi\}, \{0\}) = u$

$x \cap z = (\{0\}, \{\phi\}, \{\phi\}, \{\phi\}, \{\phi\}) = v$

$z \cap y = (\{1\}, \{\phi\}, \{\phi\}, \{\phi\}) = w$

$u \cap v = (\{\phi\}, \{\phi\}, \{\phi\}, \{\phi\}) = t$ and

$u \cap w = (\{\phi\}, \{\phi\}, \{\phi\}, \{\phi\})$,

$v \cap w = (\{\phi\}, \{\phi\}, \{\phi\}, \{\phi\})$.

Thus P^c = {x, y, z, u, v, w, t} is a MOD subsemigroup which is the completed subsemigroup of the subset P of M.

Next we just mention the theorem the proof is left as an exercise to the reader.

Now we study the MOD subset polynomial MOD real polynomials, MOD complex modulo integer polynomials, MOD dual number polynomials, MOD special dual like number polynomials and dual special quasi dual number polynomials have been studied in Chapter II.

These are pseudo groups under + but only semigroups under $\times$. These semigroups have ideals and zero divisors. So this study is considered as a matter of routine and left as an exercise to the reader.

Let $S(R_n(m)[x])$ = {Collection of all MOD real subset polynomials}; that is any $A \in S(R_n(m)[x])$ is of the form $A = \{a_0 + a_1x + a_3x^3, b_0 + b_1x^2, c_0 + c_1x^7, dx^{20}$ where $a_0, a_1, a_3, b_0, b_1, c_0, c_1$ and $d \in R_n(m)\}$.

We will illustrate this situation by some examples.

Example 3.89: Let M = $\{S(R_n(6)[x])\}$ be the MOD real subset polynomial.

Let A = $\{(0.3, 4.2)x^3 + (0.7, 0)x^2 + (5.0221, 3.223), (0.3333, 0.6)x^4 + (0.818, 5.1)x^3 + (2.1, 0.5)x^2, (4, 3)x^7 + (0.33333, 0.4444)\} \in M$ is a MOD real subset polynomial.

However N = $\{S(R_n(6))[x]\}$ is also a MOD real polynomial subset collection.

Any X $\in$ N is of the form X = $\{\{(0.3, 4.2), (0, 0.7), (0.2, 2), (4, 3), (6.3, 2)\}x^3 + \{(0.1, 5.2), (3, 2), (4.5, 0.3)\}x^2 + \{(0.312, 0.111), (4.1, 2.5), (3.01, 0.72)\}x + \{(0, 0), (0, 3), (3, 0), (3, 3)\}\} \in N$.

Clearly X $\notin$ M and A $\notin$ M. Both are entirely different collections.

We can define 4 different types of operations on M as well on N.

We will illustrate this situation by some examples.

Example 3.90: Let S = $\{S(R_n(7)[x]), +\}$ be the MOD real subset polynomial semiring. S has finite subsemigroups also elements of M will be of the form

A = $\{(0, 0.7)x^8 + (0.31, 4)x^4 + (6.311, 5.321), (0.3, 0.3)x^4 + (0.771, 4.23)x^3 + (1.27, 6.3)\} \in M$.

Let B = $\{(0, 0.5)x^8 + (2.12, 6.1), (6.3, 0.7)x^4 + (1.22, 0.11)x^3 + (0.73, 5.2)\} \in M$.

We add the elements in M as follows:

A + B = $\{(0, 1.2)x^8 + (0.31, 4)x^4 + (1.43, 4.421); (0, 0.5)x^8 + (0.3, 0.3)x^4 + (0.771, 4.23)x^3 + (3.39, 5.4), (0, 0.7)x^8 + (6.61,$

4.7)x^4 + (1.22, 0.11)x^3 + (0.041, 6.021), (6.6, 1)x^4 + (1.991, 4.34)x^3 + (2, 4.5)}.

This is the way '+' operation is performed on $S(R_n(7)[x])$.

We find A × B = {(0, 0.35)x^{16}+ (0, 2)x^{12} + (0, 5.505)x^8 + (0, 4.27)x^8 + (0.6572, 3.4)x^4 + (6.3792, 2.4581), (0, 0.49)x^{12} + (1.953, 2.8)x^8 + (6.7593, 2.961)x^4 + (0, 0.77)x^{11} + (0.3682, 0.44)x^7 + (0.69942, 0.58531)x^3 + (0, 3.64)x^8 + (0.2263, 6.8)x^4 + (2.20703, 6.6692), (1.89, 0.21)x^8 + (5.2233, 2.994)x^7 + (0.94062, 0.4653)x^6 + (1.22, 3.45)x^4 + (2.11223, 1.689)x^3 + (0.9271, 4.76); (0, 0.15)x^{12} + (0, 2.115)x^{11} + (0, 3.15)x^8 + (0.636, 1.83)x^4 + (1.63452, 4.803)x^3 + (2.6924, 3.43)}∈ M.

Thus {$S(R_n(7)[x])$, ×} is the MOD real subset polynomial semigroup of infinite order. This has infinite number of zero divisors.

Now we consider A ∪ B = {(0, 0.7)x^8 + (0.31, 4)x^4 + (6.311, 5.321), (0.3, 0.3)x^4 + (0.771, 4.23)x^3 + (1.23, 6.3), (0, 0.5)x^8 + (2.12, 6.1), (6.3, 0.7)x^4 + (1.22, 0.11)x^3 + (0.73, 5.2)} ∈ M.

Thus ($S(R_n(7)[x])$, ∪) is a MOD real polynomial subset semigroup.

In fact it is a semilattice and every element in $S(R_n(7)[x])$ is a MOD real polynomial subset subsemigroup.

Next we consider A ∩ B = {ϕ}.

Thus {$S(R_n(7)[x])$ ∪ {ϕ}, ∩} is a MOD real polynomial subset semigroup which is a semilattice and every element in $S(R_n(7)[x])$ ∪ {ϕ} is an idempotent.

We see all the four semigroups are distinct.

On similar lines we can for $(S(R_n(m)[x]))$ the MOD real subset polynomial define all the four operations which is illustrated by the following example.

Example 3.91: Let B = $\{S(R_n(12))[x]\}$ be the collection of all polynomials with coefficients as subset from $R_n(12)$.

Let X = $\{(0, 3), (0.4, 8), (4, 4), (6, 3)\}x^4 + \{(0.2, 0.4), (0, 0), (0.3, 0.7)\}x^2 + \{(0.5, 0.2), (6, 8), (7, 0)\}$ and

Y = $\{(0.3, 0), (7, 1), (0.9, 1.6)\}x^2 + \{(0.7, 0), (0.8, 0), (0, 4)\} \in$ B.

Now we see one can define all the four operations +, $\times$, $\cup$ and $\cap$.

X + Y = $\{(0, 3), (0.4, 8), (4, 4), (6, 3)\}x^4 + \{(0.5, 0.4), (0.3, 0), (0.6, 0.7), (7.2, 1.4), (7, 1), (7.3, 1.7), (1.1, 2), (0.9, 1.6), (1.2, 2.3)\}x^2 + \{(1.2, 0.2), (6.7, 8), (7.7, 0), (1.3, 0.2), (6.8, 8), (7.8, 0), (7, 4), (6, 0), (0.5, 4.2)\} \in$ B.

Thus $((S(R_n(12))[x], +)$ is a MOD real polynomial subset semigroup. All other properties associated with (B, +) is a matter of routine.

X $\times$ Y = $\{(0, 0), (0.12, 0), (1.2, 0), (1.8, 0), (0, 3), (2.8, 8), (4, 4), (6, 3), (0, 4.8), (0.36, 0.8), (3.6, 6.4), (5.4, 4.8)\}x^6 + \{(0.06, 0), (0, 0), (0.09, 0), (1.4, 0.4), (2.1, 0.7), (0.18, 0.64), (0.27, 1.12)\}x^4 + \{(0.15, 0), (1.8, 0), (2.1, 0), (3.5, 0.2), (1, 0), (6, 8), (0.45, 0.32), (5.4, 0.8), (6.3, 0)\}x^2 + \{(0, 0), (0.28, 0), (2.8, 0), (4.2, 0), (0.32, 0), (3.2, 0), (4.8, 0), (0, 8)\}x^4 + \{(0.14, 0), (0, 0), (0.21, 0), (0.16, 0), (0.24, 0), (0, 1.6), (0, 2.8)\}x^2 + \{(0.35, 0), (4.9, 0), (0.63, 0), (0.4, 0), (4.8, 0), (5.6, 0), (0, 0.8), (0, 8), (0, 0)\} \in$ B.

Thus T = $\{S(R_n(12))[x], \times\}$ is the MOD subset real polynomial semigroup. T has infinite number of zero divisors.

All ideals of T are of infinite order. T has also finite subsemigroups.

Now we find

$$X \cap Y = \phi \in B.$$

Thus $P = \{SR_n(12)[x] \cup \{\phi\}, \cap\}$ is a MOD subset real polynomial semigroup which is also a semilattice.

Every singleton subset of P is a subsemigroup. P has subsemigroups of all order.

Further every proper subset of P can be completed to get a subsemigroup.

Next we find $X \cup Y = \{(0, 3), (0.4, 8), (4, 4), (6, 3)\}x^4 + \{(0.2, 0.4), (0, 0), (0.3, 0.7), (0.3, 0), (7, 1), (0.9, 1.6)\}x^2 + \{(0.5, 0.2), (6, 8), (7, 0), (0.7, 0), (0.8, 0), (0, 4)\} \in B$.

We see $W = \{S(R_n(12))[x] \cup\}$ is a MOD real subset polynomial semigroup which is an infinite lattice and every singleton set in W is a subsemigroup.

W has subsemigroups of all orders. Finally any subset of W can be completed into a subsemigroup.

Now we give examples of various type of MOD subset polynomial semigroups.

Example 3.92: Let $S(R_n^I(10))[x] = \left\{\sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in S(R_n^I(10))\right\}$ be the MOD neutrosophic subset polynomial set.

We can define all the four operations on $S(R_n^I(10))[x]$.

Let $X = \{3 + 2I, 0.5 + 0.2I, 8I, 5, 0.9 + 4I\}x^3 + \{0, 2I, 4 + 5I, 7I + 1\}$ and

Y = {2 + I, 0.6 + 4I, 5I, 8}x^2 + {2.03 + I, 2 + 3I, 6I + 0.8, 4I + 0.9} $\in$ S(R_n^I(10))[x].

X + Y = {3 + 2I, 0.5 + 0.2I, 8I, 5, 0.9 + 4I}x^3 + {2 + I, 0.6 + 4I, 5I, 8} x^2 + {2.03 + I, 2 + 3I, 6I + 0.8, 4I + 0.9, 2.03 + 3I, 2 + 6I, 8I + 0.8, 6I + 0.9, 6.03 + 6I, 6 + 8I, I + 4.8, 9I + 4.9, 8I + 3.03, 3, 3I + 1.8, I + 1.9}$\in$ S(R_n^I(10))[x].

Thus {S(R_n^I(10))[x], +} is a MOD subset neutrosophic polynomial semigroup.

Consider

X + Y = {6 + 9I, 1.8 + 1.2I, 5I, 0, 4.5I, 4 + 6I, 4 + 1.6I, 4I, 7.2 + 2I, 1 + 1.1I, 3, 1.8 + 2.9I, 3.5I, 0.3 + 4I, 6.8I, 2I}x^5 + {0, 6I, 9.2I, 8 + 9I, 2.4 + 9I, 5I, 2, 2 + 2I, 6.2I + 0.6, 8 + 6I}x^2 + {0.15 + 1.106I, 4, 9.06I + 6.09, 9I + 6, 2.4 + 1.6I, 2.7 + 1.8I, 1 + 2.5I, 0, 5I, 1.8 + 2.7I, 0.4 + 4.36I, 4.4I, 0.45, 0.45 + 2.98, 4.24I, 9.2I}x^2 + {0, 6.06I, 3.6I, 9.8I, 8.12I + 0.15, 8 + 7I, 7.2I + 3.2, 0.5I + 3.6, 2.2I + 2.03, 2 + 8I, 0.8 + 3.6I, 0.9 + 8.3I}.

Thus {S(R_n^I(10))[x], $\times$} = B is a MOD neutrosophic subset polynomial semigroup.

B has infinite number of zero divisors.

Consider X $\cap$ Y = {ϕ}. Thus P = {S(R_n^I(10))[x] $\cup$ {ϕ}, $\cap$} is the MOD subset neutrosophic polynomial semigroup which is also a semilattice.

Every singleton element is a subsemigroup. Infact P has subsemigroups of all orders.

Every subset of P can be completed to a subsemigroup.

Next we find X ∪ Y = {3 + 2I, 0.5 + 0.2I, 8I, 5, 0.9 + 4I}x^3 + {2 + I, 0.6 + 4I, 5I, 8}x^2 + {0, 2I, 4 + 5I, 7I + 1, 2.03 + I, 2 + 3I, 6I + 0.8, 4I + 0.9}.

This is the way the operation is performed on S(R_n^I(10))[x].

T = {S(R_n^I(10))[x], ∪} is a MOD subset neutrosophic polynomial semigroup.

Every singleton element is a subsemigroup T has subsemigroups of all orders. Infact every proper subset of T can be completed to form a subsemigroup.

Now we consider S = S(R_n^I(10)[x]) = {Collection of all MOD neutrosophic polynomial subsets}.

We can define all the four operations on S and S is different from S(R_n^I(10))[x].

We will illustrate this situation also.

Let X = {(4 + 3I)x^7 + (0.3 + 0.5I)x^2 + (5.3 + 6.2I), (0.4 + 0.2I)x^3 + (0.7 + 0.2I)} and

Y = {(0.3 + 7I)x^2 + (0.6 + 0.8I)x + (0.5 + 2I), 0, (6.8 + 2I)x^4 + (0.4 + 0.7I)} ∈ S.

X + Y = {(4 + 3I)x^7 + (0.3 + 0.5I)x^2 + (5.3 + 6.2I), (0.4 + 0.2I)x^3 + (0.7 + 0.2), (4 + 3I)x^7 + (0.6 + 7.5I)x^2 + (0.6 + 0.8I)x + (5.8 + 8.2I), (0.4 + 0.2I)x^3 + (6.8 + 2I)x^4 + (1.1 + 0.9I), (4 + 3I)x^7 + (0.3 + 0.5I)x^2 + (6.8 + 2I)x^4 + (5.7 + 6.9I), (0.4 + 0.2I)x^3 + (0.3 + 7I)x^2 + (0.6 + 0.8I)x + (1.2 + 2.2I)} ∈ S

{S(R_n^I(10)[x]),+} is a MOD neutrosophic polynomial subset semigroup.

We now find

$X \times Y = \{0, (1.2 + 9.9I)x^9 + (8.4I + 2.4)x^8 + (5.5I + 2)x^7 + (0.18 + 0.94)x^3 + (0.09 + 5.75I)x^4 + (1.59 + 2.36I)x^2 + (2.65 + 3.7I)x^2 + (0.15 + 1.85I)\} \in S$

Thus $\{S(R_n^I(10)[x]), \times\}$ is the MOD neutrosophic polynomial subset semigroup.

This semigroup is of infinite number of zero divisors.

All ideals of this semigroup are of infinite order.

Next we consider the MOD subset complex modulo integer polynomials and MOD complex modulo integer subset polynomials by the following examples.

Example 3.93: Let
$S(C_n(9)[x]) = \{$Collection of all subsets from $C_n(9)[x]\}$.

We can define all the four binary operations on $S(C_n(9)[x])$ in the following.

Let $A = \{(3 + 2i_F)x^7 + (2 + 0.5i_F), (1 + 0.5i_F)x^3 + (0.7 + 0.9i_F)x, (5 + 0.3i_F)x^4, (4i_F + 3)x^5\}$ and

$B = \{(5 + 0.3i_F)x^4, (2 + i_F)x^2 + (3 + 0.01i_F), (0.7 + 0.8i_F)\} \in S(C_n(9)[x])$.

$A + B = \{(3 + 2i_F)x^7 + (5 + 0.3i_F)x^4 + (2 + 0.5i_F), (1 + 0.5i_F)x^3 + (0.7 +0.9i_F)x + (5 + 0.3i_F)x^4, (1 + 0.6i_F)x^4, (4i_F + 3)x^5 + (5 + 0.3i_F)x^4, (3 + 2i_F)x^7 + (2 + i_F)x^2 + (5 + 0.51i_F), (1 + 0.5i_F)x^3 + (2 + i_F)x^2 + (0.7 + 0.9i_F)x + (3 + 0.01i_F), (5 + 0.3i_F)x^4 + (2 + i_F)x^2 + (3 + 0.01i_F), (4i_F + 3)x^5 + (2 + i_F)x^2 + (3 + 0.01i_F), (3 + 2i_F)x^7 + (2.7 + 1.3i_F), (1 + 0.5i_F)x^3 + (0.7 + 0.9i_F)x + (0.7 + 0.8i_F), (1 + 0.5i_F)x^3 + (0.7 + 0.9i_F)x + (0.7 + 0.8i_F), (5 + 0.3i_F)x^4 + (0.7 + 0.8i_F), (4i_F + 3)x^5 + (0.7 + 0.8i_F)\} \in S(C_n(9)[x])$.

Thus $\{S(C_n(9)[x]), +\}$ is a MOD complex modulo integer polynomial subset semigroup.

Now we proceed on to find $A \times B = \{(1.8 + 1.9i_F)x^{11} + (2.2 + 3.1i_F)x^4 + (7 + 2.8i_F)x^7 + (5.46 + 4.71i_F)^5, (7.81, 3i_F)x^8, \ldots\} \in S(C_n(9)[x])$.

Hence $\{S(C_n(9)[x]), \times\}$ is the MOD complex modulo integer polynomial subset semigroup.

We can find $A \cap B$ and $A \cup B$ and $\{S(C_n(9)[x]), \cup\}$ and $\{S(C_n(9)[x]) \cup \{\phi\}, \cap\}$ are both MOD semigroups which are MOD semilattices.

Next we just show

$$S(C_n(9))[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \,\middle|\, a_i \in S(C_n(9)) = \text{subsets of } C_n(9) \right\}.$$

We see $S(C_n(9))[x]$ is different from $S(C_n(9)[x])$.

We can define four operations on $S(C_n(9))[x]$ under which $S(C_n(9))[x]$ is a semigroup.

We will just give some examples to show how operations are performed on $S(C_n(9))[x]$.

Let $A = \{3 + 0.5i_F, 0.2 + 0.4i_F\}x^3 + \{0.8, 4i_F, 1 + i_F\}$ and

$B = \{4.2 + 0.4i_F, 3i_F\}x^3 + \{1.2 + i_F, 0.3\}x^2 + \{0.7, 4i_F + 1\} \in S(C_n(9))[x]$.

$A + B = \{7.2 + 0.9i_F, 4.4 + 0.8i_F, 3 + 3.5i_F, 0.2 + 3.4i_F\}x^3 + \{1.2 + i_F, 3i_F\}x^2 + \{1.5, 0.7 + 4i_F, 1.7 + i_F, 4i_F + 1.8, 8i_F + 1, 2 + 5i_F\} \in S(C_n(9))[x]$.

Thus $\{S(C_n(9))[x], +\}$ is a MOD subset complex modulo integer polynomial semigroup.

Now we find

$A \times B = \{\{3 + 0.5i_F, 0.2 + 0.4i_F\} \times \{4.2 + 0.4i_F, 3i_F\}x^6 + \{0.8, 4i_F, 1 + i_F\}\times \{4.2 + 0.4i_F, 3i_F\}x^3 + \{3 + 0.5i_F, 0.2 + 0.4i_F\} \times \{1.2 + i_F, 0.3\}x^5 + \{0.8, 4i_F, 1 + i_F\} \times \{1.2 + i_F, 0.3\} \times x^2 + \{3 + 0.5i_F, 0.2 + 0.4i_F\} \times \{0.7, 4i_F + 1\}x^3 + \{0.8, 4i_F, 1 + i_F\} \times \{0.7, 4i_F + 1\}\}\in S(C_n(9))[x]$.

Hence $\{S(C_n(9))[x], \times\}$ is a MOD complex modulo integer polynomial subset semigroup of infinite order.

We can find $A \cap B$ and $A \cup B$ also.

$A \cap B = (\{3 + 0.5i_F, 0.2 + 0.4i_F\} \cap \{4.2 + 0.4i_F, 3i_F\})x^3 + \{\phi\} + (\{0.8, 4i_F, 1 + i_F\} \cap \{0.7, 4i_F + 1\}) \in S(C_n(9))[x]$.

We see $\{S(C_n(9))[x] \cup \{\phi\}, \cap\}$ is a MOD complex modulo integer subset semigroup.

$A \cup B = (\{3 + 0.5i_F, 0.2 + 0.4i_F\}\cup \{4.2 + 0.4i_F, 3i_F\})x^3 + \{1.2 + i_F, 0.3\}x^2 + \{0.8, 4i_F, 1 + i_F\} \cup \{0.7, 4i_F + 1\} \in S(C_n(9))[x]$. So $\{S(C_n(9))[x], \cup\}$ is a MOD complex modulo integer subset polynomial semigroup.

The study of properties associated with them is considered as a matter of routine and hence left as an exercise to the reader.

Next we give examples of MOD dual number subset polynomials.

Example 3.94: Let $S = \{S(R_n(15)(g))[x]$ that is all polynomials in the variable x whose coefficients are from the MOD dual number subsets from $R_n(15)(g)\}$.

Let

$X = \{2g, 5g + 0.3, 9.2g + 4\}x^2 + \{14g + 0.1, 12g, 10, 5 + 10g\}$ and

$Y = \{5g + 0.3, 11g, 12, 11 + 5g\}x^2 + \{14g + 0.1, 10, 2g, 3g, 5g, 12\} \in S$.

$X + Y = \{7g + 0.3, 10g + 0.6, 14.2g + 4.3, 13g, g + 0.3, 5.2gg + 4, 2g + 12, 5g + 12.3, 9.2g + 1, 11 + 7g, g + 11.3, 14.2g, x^2 + 13g + 0.2, 11g + 0.1, 14g + 10.1, 9g + 5.1, 10.1 + 14g, 10 + 12g, 5, 10g, 14g, g + 0.1, 10 + 2g, 5 + 12g, 2g + 0.1, 0, 10 + 3g, 5 + 13g, 4g + 0.1, 2g, 10 + 5g, 12.1 + 14g, 12 + 12g, 7, 2 + 10g\} \in S$.

Hence {S, +} is the MOD dual number subset polynomial semigroup.

Consider

$X \times Y = \{0.6g, 3g + 0.09, 1.2 + 7.76g, 0, 3.3g, 14g, 10g, 1.5 + 13g, 5 + 11g\}x^4 + \{4.7g + 0.03, 3.6g, 5g + 3, 1.5 + 13g, g, 0, 1.1g, 10g, 9g, 1.2 + 13g, 1.1 + 4.5g, 12g, 5g + 5, 10\}x^2 + \{0.2g, 5g, 0, 9g, 3.6, 1.5g, 0.9g, 0.6g, 5g + 3, 0.03 + 4.7g, 2g + 10, 8g, 12g, 3 + 5.4g, 0.4 + 11.92g\}x^2 + \{(14g + 0.1)^2, 1.2g, 5g + 1, 0.5 + 11g, 10, 5 + 10g, 0.2g, 9g, 5g, 10g, 0.3g, 0, 0.5g, 1.2 + 3g\} \in S$.

Hence $\{S(R_n(15)(g))[x], \times\}$ is the MOD dual number polynomial subset semigroup. This semigroup has infinite number of zero divisors.

$X \cup Y = \{2g, 5g + 0.3, 9.2g + 4, 11g, 12, 11 + 5g\}x^2 + \{14g + 0.1, 10, 12g, 2g, 5 + 10g, 3g, 5g, 12\} \in S$.

Thus $\{S, \cup\}$ is the MOD dual number subset polynomial semigroup.

This is a semilattice and every singleton element is a subsemigroup.

Further every proper subset of S can be extended under the operation $\cup$ to form a subsemigroup. S has finite subsemigroups of all orders.

$$X \cap Y = \{5g + 0.3\}x^2 + \{14g + 0.1, 10\} \in S.$$

Thus $\{S \cup \{\phi\}, \cap\}$ is a MOD dual number polynomial subset semigroup. This is infact a semilattice.

Every subset can be completed into a subsemigroup. Finally this semilattice has subsemigroups of all orders 1, 2, 3, ….

Now consider B = $S(R_n(15)(g)[x])$ = {Collection of all MOD special dual like number polynomial subsets}.

On B we can define all the four operations +, ×, ∪, ∩ and under these operations.

B is a semigroup. We just show how the 4 operations are performed on B.

Let $X = \{(3 + 0.4g)x^2 + (0.2 + g), (7g + 0.8)x^3, (2 + 4g)x^4\}$ and

$Y = \{(4g + 0.5)x^2 + (0.4g + 2), (2 + 4g)x^4, (3 + 4g)x + (2 + g)\} \in B$.

We find $X + Y = \{(4.4g + 3.5)x^2 + (2.2 + 1.4g), (7g + 0.8)x^3 + (4g + 0.5)x^2 + (0.2 + g), (2 + 4g)x^4 + (4g + 0.5)x^2 + (0.4 + 2), (3 + 0.4g) x^2 + (0.2 + g) + (2 + 4g) x^4, (7g + 0.8) x^3 + (2 + 4g) x^4 (4 + 8g) x^4, (3 + 0.4g) x^2 + (3 + 4g) x + (2.2 + 2g), (7g + 0.8) x^3 + (3 + 4g) x + (2 + g), (2 + 4g) x^4 + (3 + 4g) x + (2 + g)\} \in B$.

Thus {B, +} is the MOD dual number subset polynomial semigroup.

Now $X \times Y = \{(1.5 + 13.5g) x^4 + (1.3g + 0.1) x^2 + (2g + 6) x^2 + (0.4 + 2.08g), (6.7g + 0.4) x^5 + (1.6 + 14.32g) x^3, (10g + 1) x^6 + (8.8g + 4)x^4, (6 + 12.8g) x^6 + (0.4 + 2.8g) x^4, (1.6 + 2.2g) x^7 (4 + g) x^8, (9 + 13.2g) x^3 + (0.6 + 3.8g) x + (6 + 3.8g) x^2 + (0.4 + 2.2g), (2.4 + 9.2g) x^4 + (1.6 + 0.8g) x^3, (6 + 5g)x^5 + (4 + 10g) x^4\} \in B$.

Hence {B, $\times$} is the MOD dual number subset polynomial group.

Consider X $\cup$ Y = {(3 + 0.4g) x^2 + (0.2 + g), (7g + 0.8) x^3, (2 + 4g) x^4, (2 + 4g) x^4 (4g + 0.5) x^2 + (0.4g + 2), (3 + 4g) x + (2 + g)} $\in$ B.

{B, $\cup$} is the MOD subset dual number polynomial semigrup. This is also a semilattice. Every singleton element is a subsemigroup.

Consider X $\cap$ Y = {(2 + 4g) x^4} $\in$ B, (B, $\cap$) is a MOD dual number subset polynomial semigroup.

Clearly S(R_n(15)g[x]) is different from S(R_n(15)(g))[x] in structure.

Likewise we can study MOD special dual like number polynomials.

Example 3.95: Let B = S(R_n(12)(g))[x] = {(collection of all polynomials with coefficients from S(R_n(12)(g)) subsets as from the MOD special dual like number plane $g^2 = g$}.

X = {3 + 9.4g, 2.5 + 7.2g, 10.3g 4, 5g} x^2 + {6 + 2g, 4, 7.1 + 0.2g) and

Y = {4, 4g, 3 +2g} + {1 + 5g, 2g} x $\in$ B.

X + Y = {3 + 9.4g, 2.5 + 7.2g, 10.3g, 4, 5g} x^2 + {10 + 2g, 11.1 + 0.2g, 6 + 6g, 4 + 4g, 7.1 + 4.2g, 9 + 4g, 7 + 2g, 10.1 + 2.2g} + {1 + 5g, 2g}x $\in$ P.

Hence {P, +} is the MOD special dual like number subset polynomial semigroup.

X × Y = {1.6g, 10 + 4.8g, 2.8g, 4, 4g, 8, 8g, 5.2g, 6.9 + 6.6g, 9 + 4g, 7.5 + 5g} x^2 + {0.8g, 7.4g, 8g, 10g, 8.6g, 4 + 2g, 6g, 1.8g, 3 + 11.4g, 2.5 + 7.7g} x^3 + {8g, 4, 4.4 + 0.8g, 4g, 5.2g, 6 + 10g, 9.3 + 3.2g} + {6 + 6h, 7.1 + 0.5g, 4g, 8g, 4 + 8g, 2.04g} x ∈ B.

Thus {B, ×} is a MOD special dual like number polynomial subset semigroup of infinite order. {B, ×} has no finite polynomial subsemigroups, only finite constant subsemigroups. Further {B, ×} has infinite number of zero divisors.

X ∩ Y = {4} and

X ∪ Y = {3 + 9.4g, 2.5 + 7.2g, 10.3g, 4, 5g} x^2 + {1 + 5g, 2g} x + {6 + 2g, 4, 7.1 + 0.2g, 4, 4g, 3 + 2g} ∈ B are MOD subset special dual like number polynomial semigroups which are also semilattices.

Every singleton element is a subsemigroup and every subset can be completed to form a semigroup.

Let S = {S(R_n(12)(g)[x]} be the collection of all MOD special dual like number polynomial subset}.

(S, +) {S, ×} (S, ∪} ({S ∪ ϕ}, ∩} are four distinct MOD special dual like number polynomial subsets semigroup. S is different from B.

Likewise we can build MOD special quasi dual like number polynomial subsets semigroups of two types using the special quasi dual like number MOD planes.

We will describe them very briefly by an example.

Example 3.96: Let M = $\{S(R_n(7)(g))[x], g^2 = 6g\}$ be the collection of all MOD special quasi dual number polynomials with subsets from $S(R_n(7)(g))$ $\{M +\}$ is a MOD special quasi dual number polynomial subset semigroup.

Likewise $\{S(R_n(7)(g)[x], \times\}$, $\{S(R_n(7)(g))[x], \cup\}$ and $\{S(R_n(7)\ (g))[x] \cup \{\phi\}, \cap\}$ are all MOD special quasi dual number subsets semigroup.

Any $X \in S(R_n(7)(g))[x]$ is of the form $X = \{0.77g + 6.53, 1.70001 + 0.77775g, 6.01g, 4.00001, 5.66661 + 4.77772g\}\ x^3 + \{2g, 3+5g, 0.00007g + 5.001, 6.0008g + 5.0062\}\ x^2 + \{g + 1, 0, 2 + 6.089213g, 1.342000 + 6g\}$.

Now we can also define MOD special quasi dual like number polynomial subset semigroups of four types using $S(R_n(7)(g)[x]$ = {collection of all polynomial subsets from $R_n(7)(g)[x]$}.

Any element Y of $S(R_n(7)\ (g)[x])$ is of the form

$Y = \{(0.37008 + 2g)\ x^7 + (4 + 3g)\ x^6 + (5.00092 + 2.0019)\ x^3 + (6.01 + 0.72g), (3.0012 + 4g)\ x^5 + (0.1123, 4.003g), (0.33337 + 0.8g) + (0.777776 + 0.999998g)x^3\}$.

Operations such that +, $\times$, $\cup$ and $\cap$ can be performed on $S\ (R_n(7)(g)[x])$.

Study in this direction is a matter of routine and is left as an exercise to the reader.

We suggest the following problems.

Problems:

1. Let $B = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in S\ (R_n(12);\ 1 \leq i \leq 5, +\}$ be the MOD real row matrix group.

i. Find all subgroups of B.
ii. Can B have subsemigroups of finite order?
iii. Can B have subgroups of infinite order?

2. Let $S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_9 \end{bmatrix} \middle| \; a_i \in R_n(17); 1 \leq i \leq 9, + \right\}$ be the MOD real column matrix group.

   Study questions (i) to (iii) of problem (1) for this S.

3. Let $N = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_5 \\ a_6 & a_7 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{15} \\ a_{16} & a_{17} & \ldots & a_{20} \end{bmatrix} \middle| \; a_i \in R_n(40); 1 \leq i \leq 20, + \right\}$ be the MOD real matrix group.

   Study questions (i) to (iii) of problem (1) for thus N.

4. Let $B = \{(a_1, a_2, \ldots, a_9) \mid a_i \in R_n^I(19); 1 \leq i \leq 9, +\}$ be the MOD neutrosophic row matrix group.

   Study questions (i) to (iii) of problem (1) for this B.

5. Let $W = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \\ a_{13} & a_{14} \end{bmatrix} \middle| a_i \in R_n^I(24); 1 \leq i \leq 14, +\right\}$ be the MOD neutrosophic matrix group.

   Study questions (i) to (iii) of problem (1) for this W.

6. Let $S = \{(a_1, a_2, \ldots, a_{18}) \mid a_i \in R_n(10)(g)$ where $g^2 = 0$, $1 \leq i \leq 18, +\}$ be the MOD row matrix dual number group.

   Study questions (i) to (iii) of problem (1) for this S.

7. Let $P = \left\{\begin{bmatrix} a_1 & a_2 & \ldots & a_7 \\ a_8 & a_9 & \ldots & a_{14} \\ a_{15} & a_{16} & \ldots & a_{21} \\ a_{22} & a_{23} & \ldots & a_{28} \\ a_{29} & a_{30} & \ldots & a_{35} \\ a_{36} & a_{37} & \ldots & a_{42} \end{bmatrix} \middle| a_i \in R_n(13)(g) \text{ where}\right.$ $g^2 = 0$, $1 \leq i \leq 42, +\}$ be the MOD dual number matrix group.

   Study questions (i) to (iii) of problem (1) for this P.

8. Let
$V = \{(a_1, a_2, \ldots, a_9) \mid a_i \in C_n(12)\ i_F^2 = 11, 1 \leq i \leq 0, +\}$ be the MOD complex modulo integer row matrix group.

Study questions (i) to (iii) of problem (1) for this V.

9. Let $P = \left\{\begin{bmatrix} a_1 & a_2 & \ldots & a_{18} \\ a_{19} & a_{20} & \ldots & a_{36} \\ a_{37} & a_{38} & \ldots & a_{54} \\ a_{55} & a_{56} & \ldots & a_{72} \end{bmatrix} \right| a_i \in C_n(19);\ i_F^2 = 18,$

$1 \leq i \leq 72, +\}$ be the MOD complex modulo integer matrix group.

Study questions (i) to (iii) of problem (1) for this P.

10. Let $M = \left\{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{15} \end{bmatrix} \right| a_i \in R_n(10)(g), g^2 = g, 1 \leq i \leq 15, +\}$ be

the MOD special dual like number column matrix group.

Study questions (i) to (iii) of problem (1) for this M.

11. Let $S = \left\{\begin{bmatrix} a_1 & a_2 & \ldots & a_{19} \\ a_{20} & a_{21} & \ldots & a_{38} \\ a_{39} & a_{40} & \ldots & a_{57} \end{bmatrix} \right| a_i \in R_n(9); (g),$

$g^2 = 8g,\ 1 \leq i \leq 57, +\}$ be the MOD special quasi dual like number matrix group.

Study questions (i) to (iii) of problem (1) for this S.

12. Let T = $\left\{\begin{bmatrix} a_1 & a_2 & \dots & a_8 \\ a_9 & a_{10} & \dots & a_{16} \\ a_{17} & a_{18} & \dots & a_{24} \\ a_{24} & a_{25} & \dots & a_{34} \\ \vdots & & \dots & \vdots \\ a_{57} & a_{58} & \dots & a_{64} \end{bmatrix}\right|$ $a_i \in R_n(14)(g)$,

$g^2 = 13g$, $1 \leq i \leq 64$, +} be the MOD special quasi dual number square matrix.

Study questions (i) to (iii) of problem (1)

13. Let B = $[(a_1, \ldots, a_9) \mid a_i \in R_n(6), 1 \leq i \leq 6, \times\}$ be the MOD real row matrix semigroup.

i) Is B a S-semigroup?

ii) Can B have infinite number of zero divisors?

iii) Can B have S-units?

iv) Can ideals of B be infinite.

v) Prove B can have finite subsemigroups.

vi) Can B have S-idempotents?

vii) What are the other special features enjoyed by B?

14. Let M = $\left\{\begin{bmatrix} a_1 & a_2 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{30} \\ a_{31} & a_{32} & \dots & a_{40} \\ a_{41} & a_{42} & \dots & a_{50} \end{bmatrix} \right|$ $a_i \in R_n(19)$; $1 \leq i \leq 50$,

$\times_n\}$ be the MOD real matrix semigroup.

Study questions (i) to (vii) of problem (13) for this M.

15. Let V = $\left\{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_9 \end{bmatrix} \right|$ $a_i \in R_n^I(10)$; $1 \leq i \leq 9$, $\times_n$, $I^2 = I\}$ be the

MOD neutrosophic column matrix semigroup.

Study questions (i) to (vii) of problem (13) for this V.

16. Let W= $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \right|$ $a_i \in R_n^I(23)$; $1 \leq i \leq 16$,

$\times$, $I^2 = I$, $\times_n\}$ be the MOD neutrosophic square matrix semigroups.

Study questions (i) to (vii) of problem (13) for this W.

17. Let S = $\{(a_1, \dots, a_9) \mid a_i \in C_n(20)$, $i_F^2 = 19$. $1 \leq i \leq 9$, $\times_n\}$ be the MOD complex modulo integer matrix semigroup.

Study questions (i) to (vii) of problem (13) for this S.

18. Let $P = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ \vdots & \vdots & \vdots \\ a_{28} & a_{29} & a_{30} \end{bmatrix} \middle| a_i \in C_n(31);\ i_F^2 = 30,\ 1 \le i \le 30, \right.$

$\times_n\}$ be the MOD complex modulo integer matrix semigroup.

Study questions (i) to (vii) of problem (13) for this P.

19. Obtain some special featurs enjoyed by MOD real matrix groups.

20. Study the nature of MOD complex modulo integer groups.

21. Let $B = \{(a_1, a_2, \ldots, a_9) \mid a_i \in R_n(5)(g)$ with $g^2 = 0$, $1 \le i \le 9, \times\}$ be the MOD dual number row matrix semigroup.

Study questions (i) to (vii) problem (13) for this B.

22. Let $M = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \\ a_{21} & a_{22} & \cdots & a_{30} \end{bmatrix} \middle| a_i \in R_n(14)\, g;\ g^2 = g, \right.$

$1 \le i \le 30, \times_n\}$ be the MOD dual number matrix semigroup.

Study questions (i) to (vii) of problem (13) for this M.

23. Let $P = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{bmatrix} \right|$ $a_i \in R_n(13)g;\ g^2 = g,\ 1 \leq i \leq 10, \times_n\}$ be the MOD special dual like number column matrix semigroup.

Study questions (i) to (vii) of problem (13) for this P.

24. Let

$$T = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \right| \ a_i \in R_n(10)(g), \times;\ g^2 = g,$$

$1 \leq i \leq 9\}$ be the MOD special dual like number matrix semigroup under usual product.

i) Prove T is non commutative.
ii) Prove T has right ideals which are not left ideals.
iii) Prove T has right zero divisors which are not left zero divisors.
iv) Study questions (i) to (vii) of problem (13) for this T.

24. Let
$B = \{(a_1, a_2, \ldots, a_{12}) \mid a_i \in R_n(10)(g),\ g^2 = 9g,\ 1 \leq i \leq 12, \times\}$ be the MOD special quasi dual number row matrix semigroups.

Study questions (i) to (vii) of problem (13) for this B.

25. Let $S = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_5 \\ a_6 & a_7 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{15} \\ a_{16} & a_{17} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{25} \end{bmatrix} \right|$ $a_i \in R_n(19)(g)$, with

$g^2 = 18g$, $1 \leq i \leq 25$, $\times_n\}$ be the MOD special quasi dual number matrix semigroup under usual product $\times$.

Study questions (i) to (iv) of problem (23) for this S.

26. Let V be the MOD special quasi dual like number semigroup given in problem 25 under $\times$ the usual operation is replaced by $\times_n$.

Study questions (i) to (iv) of problem (23) for this V.

Compare S of problem 25 with this V.

27. Obtain some special and distinct features enjoyed by S $(R_n(m))$ under the four operations $+$, $\times$, $\cap$ and $\cup$.

28. Let $V = \{(a_1, a_2, \ldots, a_5) \mid a_i \in S(R_n(20)); 1 \leq i \leq 5, +\}$ be a MOD real subset row matrix semigroup.

i) Can V have finite order subsemigroups.
ii) Is V a S-semigroup?
iii) Give atleast 5 subsemigroups of infinite order.

29. Let $W = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & \dots & \dots & \dots & a_{10} \\ a_{11} & \dots & \dots & \dots & a_{15} \\ a_{15} & \dots & \dots & \dots & a_{20} \\ a_{21} & \dots & \dots & \dots & a_{25} \end{bmatrix} \right| a_i \in S(R_n(19)),$

   $1 \leq i \leq 25, +\}$ be the MOD subset real matrix semigroup.

   Study questions (i) to (iii) of problem (28) for this W.

30. Let $B = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{15} \end{bmatrix} \right| a_i \in S(R_n^I(10)), 1 \leq i \leq 15, +\}$ be the

   MOD subset neutrosophic column matrix semigroup.

   Study questions (i) to (iii) of problem (28) for this B.

31. Let $W = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_7 \\ a_8 & a_9 & \dots & a_{14} \\ a_{15} & a_{16} & \dots & a_{21} \\ a_{22} & a_{23} & \dots & a_{28} \end{bmatrix} \right| a_i \in S(R_n^I(19));$

   $1 \leq i \leq 28, +\}$ be the MOD subset neutrosophic matrix semigroup.

   Study questions (i) to (iii) of problem (28) for this W.

32. Let M = $\left\{\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \end{pmatrix}\right| a_i \in$

$(R_n(10)g)$, $g^2 = 0$, $1 \leq i \leq 12$, +} be the MOD subset dual number row matrix semigroup.

Study questions (i) to (iii) of problem (28) for this M.

33. Let L = $\left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \\ a_{13} & a_{14} \end{bmatrix}\right| a_i \in S(R_n(13)(g)), g^2 = 0,$

$1 \leq i \leq 14$, +} be the subset dual number column matrix semigroup.

Study questions (i) to (iii) of problem (28) for this L.

34. Let S = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & \ldots & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & \ldots & a_{15} \\ a_{16} & a_{17} & \ldots & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & \ldots & a_{25} \end{bmatrix}\right| a_i \in S(R_n(16)(g))$

with $g^2 = g$, $1 \leq i \leq 25$, +} be the MOD subset special dual like number semigroups.

Study questions (i) to (iii) of problem (28) for this S.

35. Let $B = \left\{ \begin{pmatrix} a_1 & a_2 & \ldots & a_{12} \\ a_{13} & a_{14} & \ldots & a_{24} \\ a_{25} & a_{26} & \ldots & a_{36} \end{pmatrix} \middle| \ a_i \in S(R_n(19)(g));\right.$

$g^2 = g$; $1 \leq i \leq 36, +\}$ be the MOD subset special dual like number matrix semgroups.

Study questions (i) to (iii) of problem (28) for this B.

36. Let $M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \\ a_{17} & a_{18} & a_{19} & a_{20} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{25} & a_{26} & a_{27} & a_{28} \end{bmatrix} \middle| \ a_i \in S(R_n(12)g);\right.$

$g^2 = 11g$; $1 \leq i \leq 28, +\}$ be the MOD subset special quasi dual number matrix semigroup.
Study questions (i) to (iii) of problem (28) for this M.

37. Let $S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \end{bmatrix} \middle| \ a_i \in\right.$

$S(R_n(19)g)$, $g^2 = 18g, 1 \leq i \leq 18, +\}$ be the MOD subset special quasi dual like number matrix semigroup.

Study questions (i) to (iii) of problem (28) for this S.

38. Let T = {(a_1, a_2, …, a_{12}) | $a_i \in$ S(C_n(12)), i_F^2 = 11; 1 ≤ i ≤ 12, +} be the MOD subset complex modulo integer matrix semigroup.

    Study questions (i) to (iii) of problem (28) for this T.

39. Let B = {(a_1, a_2, a_3, a_4, a_5, a_6) | $a_i \in$ S(C_n(43)); 1 ≤ i ≤ 6, i_F^2 = 42, +} be the MOD subset complex modulo integer row matrix semigroup.

    Study questions (i) to (iii) of problem (28) for this B.

40. Let B = $\left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{18} \end{bmatrix} \right|$ $a_i \in$ S(R_n(9)), 1 ≤ i ≤ 18, $\times_n$} be the MOD subset real column matrix semigroup.

    i) Is B a S-semigroup?
    ii) Prove B has infinite number of zero divisors.
    iii) Can B have S-zero divisors?
    iv) Can B has S ideals of infinite order?
    v) How many subsemigroups of finite order does B contain?
    vi) Find subsemigroups of infinite order which are not ideals.
    vii) Can B have S-units?
    viii) Can B have idempotents?
    ix) Can B have S-idempotents?
    x) Show the number of idempotents in B are only finite in number.

41. Let $S = \left\{\begin{bmatrix} a_1 & a_2 & \ldots & a_6 \\ a_7 & a_8 & \ldots & a_{12} \\ a_{13} & a_{11} & \ldots & a_{18} \\ a_{19} & a_{20} & \ldots & a_{24} \\ a_{25} & a_{26} & \ldots & a_{30} \\ a_{31} & a_{32} & \ldots & a_{36} \end{bmatrix}\right| a_i \in S(R_n(19));$

$1 \leq i \leq 36, \times\}$ be the MOD real square matrix semigroup under usual product of matrices.

i) Prove S is non commutative.
ii) Study questions (i) to (x) of problem (40) for this S.

42. If in problem (41) the operation '×' by replaced by $\times_n$ show S with $\times_n$ commutative.

i) What are different properties enjoyed by $\{S, \times_n\}$?
ii) Compare ideals of $\{S, \times\}$ with $\{S, \times_n\}$.

iii) Can an ideal in $\{S, \times\}$ be an ideal in $\{S, \times_n\}$? Justify.

43. Let $M = \left\{\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_{10} \end{bmatrix}\right| a_i \in S\ (C_n(20)),\ i_F^2 = 19, \times_n,$

$1 \leq i \leq 10\}$ be the MOD subset complex modulo integer column matrix semigroup.

Study questions (i) to (x) of problem (40) for this M.

44. Let $B = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ a_8 & \dots & \dots & \dots & \dots & \dots & a_{14} \\ a_{15} & \dots & \dots & \dots & \dots & \dots & a_{21} \\ a_{22} & \dots & \dots & \dots & \dots & \dots & a_{28} \\ a_{29} & \dots & \dots & \dots & \dots & \dots & a_{35} \end{bmatrix} \right|$ $a_i \in$

$S(C_n(43))$, $1 \leq i \leq 35$, $i_F^2 = 42$; $\times_n\}$ be the MOD subset complex modulo integer matrix semigroup.

Study questions (i) to (x) of problem (40) for this B.

45. Let $P = \{(a_1, a_2, \ldots, a_9) \mid a_i \in S(R_n^I(42); 1 \leq i \leq 9, \times\}$ be the MOD subset neutrosophic matrix semigroup.

Study questions (i) to (x) of problem (40) for this P.

46. Let $W = \left\{ \begin{pmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{pmatrix} \right|$ $a_i \in S(R_n^I(19); 1 \leq i \leq 9, \times\}$

be the MOD subset neutrosophic square matrix semigroup under usual product.

i) Study questions (i) to (x) of problem (40) for this W.
ii) Show W has right ideals which are not left ideals and vice versa.
iii) Show W has left zero divisors which are not right zero divisors.

47. Let $W = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| \, a_i \in S(C_n(12));\ i_F^2 = 11;\ 1 \leq i \leq 6, \times_n \right\}$ be the MOD complex modulo integer column matrix semigroup under natural product $\times_n$.

Study questions (i) to (x) of problem (40) for this T.

48. In problem 46 '$\times$' operation is replaced by $\times_n$ show $\{W, \times_n\}$ is commutative semigroup.

Study questions (i) to (x) problem (40) for this $\{W, \times_n\}$.

49. Let $B = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & \ldots & \ldots & \ldots & \ldots & a_{12} \\ a_{13} & \ldots & \ldots & \ldots & \ldots & a_{18} \\ a_{19} & \ldots & \ldots & \ldots & \ldots & a_{24} \end{bmatrix} \middle| \, a_i \in S(C_n(29));\ i_F^2 = 28;\ 1 \leq i \leq 24, \times_n \right\}$ be the MOD subset complex modulo integer matrix semigroup.

Study questions (i) to (x) of problem (40) for this B.

50. Let $M = \{(a_1, \ldots, a_{12}) \mid a_i \in S(R_n(12)\ g);\ g^2 = 0,\ 1 \leq i \leq 12, \times\}$ be the MOD subset dual number matrix semigroup.

Study questions (i) to (x) of problem (40) for this M.

51. Let $B = \left\{\left[\begin{matrix} a_1 & a_2 & \dots & a_9 \\ a_{10} & a_{11} & \dots & a_{18} \\ a_{19} & a_{20} & \dots & a_{27} \\ a_{28} & a_{29} & \dots & a_{36} \\ a_{37} & a_{38} & \dots & a_{45} \end{matrix}\right] \right| a_i \in S(R_n(43)g);$

$g^2 = 0$; $1 \leq i \leq 45$, $\times_n\}$ be the MOD subset dual number matrix semigroup.

Study questions (i) to (x) of problem (40) for this B.

52. Let $V = \left\{\left(\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ a_8 & \dots & \dots & \dots & \dots & \dots & a_{14} \end{matrix}\right) \right| a_i \in$

$S(R_n(11)g)$; $g^2 = g$; $1 \leq i \leq 14$, $\times_n\}$ be the MOD subset special dual like number matrix semigroup.

Study questions (i) to (x) of problem (40) for this V.

53. Let $W = \left\{\left[\begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & \dots & \dots & \dots & \dots & a_{12} \\ a_{13} & \dots & \dots & \dots & \dots & a_{18} \\ a_{19} & \dots & \dots & \dots & \dots & a_{24} \\ a_{25} & \dots & \dots & \dots & \dots & a_{30} \\ a_{31} & \dots & \dots & \dots & \dots & a_{36} \end{matrix}\right] \right| a_i \in$

$S(R_n(24)g)$; $g^2 = g$; $1 \leq i \leq 36$, $\times\}$ be the MOD subset square matrix semigroup under the usual product of matrices.

Study questions (i) to (iii) of problem (46) for this W.

54. If in W '$\times$' is replaced by '$\times_n$' show W is commutative.

Enumerate the differences between the semigroup $\{W, \times\}$ and $\{W, \times_n\}$.

55. Let $S = \left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix}\right| a_i \in S(R_n(12)\ g));\ g^2 = 11g,\right.$

$1 \leq i \leq 15, \times_n\}$ be the MOD subset special quasi dual number matrix semigroup under the natural product $\times_n$.

Study questions (i) to (x) of problem (40) for this B.

56. Let $M = \left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix}\right| a_i \in S(R_n(43)(g));\right.$

$g^2 = 42g;\ 1 \leq i \leq 16, \times_n\}$ be the MOD subset special quasi dual like number matrix semigroup under usual product $\times$.

Study questions (i) to (iii) of problem (46) for this M.

57. If in M '$\times$' is replaced by the natural product $\times_n$.

Study questions (i) to (x) of problem 40.

Also compare (M, ×} with {M, $\times_n$}.

58. Let M = {(a_1, …, a_9) | $a_i \in$ S (R_n (12)); ∪} is a MOD subset real matrix semigroup (which is a semilattice)

   i) Study question (i) to (x) of problem 40.
   ii) What are the questions in problem (40) which are irrelevant?
   iii) Show every singleton is a subsemigroup.
   iv) Prove every subset can be completed into a subsemigroup.

59. Let V = $\left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \end{bmatrix} \middle|\ a_i \in S(R_n(13)); 1 \leq i \leq 12, \cup\right\}$ be

   the MOD subset real matrix semigroup.

   Study questions (i) to (iv) of problem (58) for this V.

60. Let B = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \middle|\ a_i \in S\,(R_n^I(16)); I^2 = I;\right.$

   $1 \leq i \leq 12, \cup\}$ be the MOD subset neutrosophic matrix semigroup.

Study questions (i) to (iv) of problem (58) for this B.

61. Let L = $\left\{\begin{pmatrix} a_1 & a_2 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{20} \end{pmatrix}\right|$ $a_i \in S(C_n(21))$;

$1 \leq i \leq 20$, $\cup\}$ be the MOD subset complex modulo integer matrix semigroup.

Study questions (i) to (iv) of problem (58) for this L.

62. Let B = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ a_7 & a_8 & \dots & \dots & \dots & a_{12} \\ a_{13} & a_{14} & \dots & \dots & \dots & a_{18} \\ a_{19} & a_{20} & \dots & \dots & \dots & a_{24} \end{bmatrix}\right|$

$a_i \in S(R_n(12)(g))$; $g^2 = 0$; $1 \leq i \leq 24$, $\cup\}$be the MOD subset dual number matrix semigroup.

Study questions (i) to (iv) of problem (58) for this B.

63. Let D = $\left\{\begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ a_8 & a_9 & \dots & \dots & \dots & \dots & a_{14} \end{pmatrix}\right|$ $a_i \in$

$S(R_n(14)$ $g^2 = g$, $1 \leq i \leq 14$, $\cup\}$ be the MOD subset special dual like number matrix semigroup.

Study questions (i) to (iv) of problem (58) for this D.

64. Let $E = \left\{\left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \end{array}\right]\right|$ $a_i \in S(R_n(11)g);\ g^2 = 10g,$

$1 \leq i \leq 18,\ \cup\}$ be the MOD subset special dual like number matrix semigroup.

Study questions (i) to (iv) of problem (58) for this E.

65. Let $S = \{(a_1, a_2, \ldots, a_{10}) \mid a_i \in S(R_n(24)) \cup \{\phi\}$ be the MOD subset real matrix semigroup.

Study questions (i) to (iv) of problem (58) for this S.

66. Let $B = \left\{\left[\begin{array}{ccc} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \end{array}\right]\right|$ $a_i \in S(R_n^I(10)) \cup \{\phi\};$

$1 \leq i \leq 18,\ I^2 = I,\ \cap\}$ be the MOD subset neutrosophic matrix semigroup.

Study questions (i) to (iv) of problem (58) for this B.

67. Let M = $\left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & a_8 & a_9 & a_{10} \\ a_{11} & a_{12} & a_{13} & a_{14} & a_{15} \end{bmatrix}\right|\, a_i \in S(R(20)g) \cup\right.$

$\{\phi\}$; $1 \leq i \leq 15, \cap\}$ be the MOD subset dual number matrix semigroup.

Study questions (i) to (iv) of problem (58) for this M.

68. Let T = $\left\{\left.\begin{pmatrix} a_1 & a_2 & \ldots & a_{11} \\ a_{12} & a_{13} & \ldots & a_{22} \\ a_{23} & a_{24} & \ldots & a_{33} \end{pmatrix}\right|\, a_i \in S(C_n(10)) \cup \{\phi\};\right.$

$1 \leq i \leq 33, \cap\}$ be the MOD subset complex modulo integer matrix semigroup.

Study questions (i) to (iv) of problem (58) for this T.

69. Let R = $\left\{\left.\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix}\right|\, a_i \in S\,(R_n(3)g) \cup \{\phi\};\ g^2 = 0;\ 1 \leq i \leq 6,\right.$

$\cap\}$ be the MOD subset dual number column matrix semigroup.

Study questions (i) to (iv) of problem (58) for this R.

70. Let $V = \left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & \dots & \dots & \dots & a_{10} \\ a_{11} & \dots & \dots & \dots & a_{15} \\ a_{16} & \dots & \dots & \dots & a_{20} \\ a_{21} & \dots & \dots & \dots & a_{25} \end{bmatrix} \right| a_i \in S(R_n(8)g) \cup$

$\{\phi\}$; $g^2 = g$, $1 \leq i \leq 25$, $\cap\}$ be the MOD subset special dual like number matrix semigroup.

Study questions (i) to (iii) of problem (58) for this V.

71. Let $B = \{(a_1, a_2, \ldots, a_{12}) \mid a_i \in S(R_n(15)\, g) \cup \{\phi\}$; $g^2 = 14g$, $1 \leq i \leq 12$, $\cap\}$ be the MOD subset special quasi dual number matrix semigroup.

Study questions (i) to (iv) of problem (58) for this W.

72. Obtain all special features enjoyed by MOD subset matrices.

73. Let $M = \{S(R_n(20))[x], +\}$ be the MOD subset real polynomial semigroup.

    i) Can M have finite subsemigroups?
    ii) Find atleast three infinite order subsemigroups.

74. Let $R = \{S(R_n(29))[x], +\}$ be the MOD subset real polynomial semigroup.

Study questions (i) and (ii) of problem (73) for this M.

75. Let T = {S(R_n^I (27))[x], +} be the MOD subset neutrosophic polynomial semigroup.

Study questions (i) and (ii) of problem (73) for this T.

76. Let S = {S(R_n(12)(g))[x] | g^2 = 0; +} be the MOD subset dual number polynomial semigroup.

Study questions (i) and (ii) of problem (73) for this W.

77. Let S = {S(R_n(13)(g))[x]; g^2 = g, +} be the MOD subset special dual like number polynomial semigroup.

Study questions (i) and (ii) of problem (73) for this S.

78. Let P = {S(C_n(15))[x], i_F^2 = 14, +} be the MOD subset complex modulo integer polynomial semigroup.

Study questions (i) and (ii) of problem (73) for this P.

79. Let W = {S(R_n(19)(g))[x], g^2 = 18g, +} be the MOD subset special quasi dual like number semigroup.

Study questions (i) and (ii) of problem (73) for this W.

80. Let B = {S(R_n(29)[x], ×} be the MOD real subset polynomial semigroup under ×.

i) Prove B has infinite number of zero divisors.
ii) Can B have finite subsemigroup?
iii) Can B have ideals of finite order?
iv) Can B have infinite subsemigroups which are not ideals?

81. Let G = $\{S(C_n(23))[x],\ i_F^2 = 22, \times\}$ be the subset MOD complex modulo integer polynomial semigroup.

    Study questions (i) to (iv) of problem (80) for this G.

82. Let W = $\{S(R_n^I(43))[x]\ I^2 = I, \times\}$ be the MOD subset neutrosophic polynomial semigroup.

    Study questions (i) to (iv) of problem (80) for this W.

83. Let V = $\{S(R_n(10)g)[x], g^2 = 0, \times\}$ be the MOD subset dual number polynomial semigroup.

    Study questions (i) to (iv) of problem (80) for this V.

84. Let M = $\{S(R_n(14)g, g^2 = g, \times\}$ be the MOD subset special dual like number polynomial semigroup.

    Study questions (i) to (iv) of problem (80) for this M.

85. Let X = $\{S(R_n(17)g)[x], g^2 = 16g, \times\}$ be the MOD subset polynomial semigroup of special quasi dual like numbers.

    Study questions (i) to (iv) of problem (80) for this ×.

86. Let Y = $\{S(R_n(10))[x], \cup\}$ be the MOD subset real polynomial semigroup.

    i) Study questions (i) to (iv) of problem (80) for this Y.
    ii) Prove Y has subsemigroups of order 1, 2, 3, …
    iii) Prove every subset of Y can be completed to a subsemigroup of Y.

87. Let $Z = \{S(R_n^I(15))[x];\ I^2 = I, \cup\}$ be the MOD subset neutrosophic polynomial semigroup.

    Study questions (i) to (iii) of problem (86) for this Z.

88. Let $P = \{S(R_n(16)(g))[x],\ g^2 = 0, \cup\}$ be the MOD subset dual number polynomial semigroup.

    Study questions (i) to (iii) of problem (86) for this P.

89. Let $Q = \{S(R_n(11)(g))[x],\ g^2 = g, \cup\}$ be the MOD subset special dual like number polynomial semigroup.

    Study questions (i) to (iii) of problem 86 for this Q.

90. Let $H = \{S(R_n(15)(g))[x],\ g^2 = 14g, \cup\}$ be the MOD subset special quasi dual like number polynomial semigroup.

    Study questions (i) to (iii) of problem (86) for this H.

91. Let $S = \{S\ (C_n(23))[x],\ i_F^2 = 22, \cup\}$ be the MOD subset complex modulo integer polynomial semigroup.

    Study questions (i) to (iii) of problem (86) for this S.

92. In problem (86), (87), (8), (89), (90) and (91) replace the operation $\cup$ by $\cap$ and add ϕ to the set so that the resultant is a semigroup.

    For those semigroups study questions (i) to (iii) of problem (86).

93. Enumerate any of the properties associated with $S(R_n(m))[x]$.

94. Let P = $\{S(R_n(10)[x]), +\}$ be the MOD real subset polynomial semigroup

    i) Is P a S-semigroup?
    ii) Can P have finite subsemigroups?
    iii) Find atleast 5 infinite subsemigroups of P.

95. Compare $S(R_n(10))[x]$ with $S(R_n(10)[x]$ as semigroups.

96. Let V = $\{S(R_n^I(13)[x]), +\}$ be the MOD neutrosophic subset polynomial semigroup.

    Study questions i to iii of problem 94 for this V.

97. Let M = $\{S(R_n^I(18)[x]), +\}$ be the MOD neutrosophic subset polynomial semigroup.

    Study questions i to iii of problem 94 for this M.

98. Let W = $\{S(C_n(23)[x]), +\}$ be the MOD complex modulo integer subset polynomial semigroup.

    i. Study questions i to iii of problem 94 for this W.
    ii. Compare $S(C_n(23))[x]$ with $S(C_n(23)[x])$ as semigroups.

99. Let B = $\{S((R_n(14)(g))[x], +), +\}$ be the MOD dual number polynomial subset semigroup.

    Study questions i to iii of problem 94 for this B.

100. Let D = $\{S(R_n(15)(g)[x]), +; g^2 = g\}$ be the MOD special dual like number semigroup.

Study questions i to iii of problem 94 for this D.

101. Let S = {S($R_n(24)[x]$), ×} be the MOD real subset polynomial semigroup.

   i. Can S have zero divisors?
   ii. Is S a S-semigroup?
   iii. Can S have S-idempotents?
   iv. Can S have S-ideals?
   v. Can S have finite subsemigroups?
   vi. Can S have finite ideals?
   vii. Compare S($R_n(24)[x]$) with S($R_n(24)$)[x].

102. Let B = {S($R_n^I(18)[x]$), ×} be the MOD neutrosophic polynomial subset semigroup.

   i. Study questions i to vi of problem 101 for this B.
   ii. Compare {S($R_n^I(18)[x]$), ×} with {S($R_n^I(18)$)[x], ×} as semigroups.

103. Let V = {S($C_n(23)[x]$); $i_F^2 = 22$, ×} be the MOD complex modulo integer polynomial subset semigroup.

   i. Study questions i to vii of problem 101 for this V.
   ii. Compare V with V_1 = {S($C_n(23)$)[x], ×} as semigroups.

104. Let M = {S($R_n(14)(g)[x]$), $g^2 = 0$, ×} be the MOD dual number subset polynomial semigroups.

   i. Study questions i to vii of problem 101 for this M.
   ii. Compare M with M_1 = {S($R_n(14)(g)$)[x], ×} as semigroups.

105. Let T = {S($R_n(24)(g)[x]$); $g^2 = 23g$, ×} be the MOD special quasi dual like number subset polynomial semigroup.

   i. Study questions i to vii of problem 101 for this T.

ii. Compare T with $T_1 = \{S(R_n(24)(g))[x], g^2 = 23g, \times\}$ as semigroups.

106. Let $B = \{S(R_n(15)[x]) \cup \{\phi\}, \cap\}$ be the MOD real subset polynomial semigroup.

i. Study questions i to vii of problem 101 for this B.
ii. Show every singleton element of B is a subsemigroup.
iii. Prove every subset can be completed to a subsemigroup in B.
iv. Prove B has subsemigroups of all orders 2, 3, ….
v. Compare B with the appropriate MOD real polynomials with subset coefficients.

107. Let $S = \{S(R_n^I(9)[x]) \cup \{\phi\}, I^2 = I, \cap\}$ be the MOD neutrosophic subset polynomial semigroup.

Study questions i to v of problem 106 for this S.

108. Let $T = \{S(C_n(15)[x]) \cup \{\phi\}, i_F^2 = 14, \cap\}$ be the MOD complex modulo integer subset polynomial semigroup.

Study questions i to v of problem 106 for this T.

109. Let $M = \{S(R_n(23)(g)[x]) \cup \{\phi\}, g^2 = 0, \cap\}$ be the MOD dual number subset polynomial semigroup.

Study questions i to v of problem 106 for this M.

110. Let $S = \{S(R_n(45)(g)[x]) \cup \{\phi\}, g^2 = g, \cap\}$ be the MOD special dual like number subset polynomial semigroup.

Study questions i to v of problem 106 for this S.

111. Let $M = \{S(R_n(11)(g)[x]) \cup \{\phi\}, g^2 = 10g, \cap\}$ be the MOD special quasi dual like number subset polynomial semigroup.

     Study questions i to v of problem 106 for this M.

112. In problems 106, 107, 108, 109, 110 and 111 replace '$\cap$' and '$\cup$' and study the problem.

113. Give any special applications of these new MOD algebraic structures.

114. Can we say MOD algebraic structures makes the real plane into a small plane depending on the number m which we choose?

115. Is it true by using MOD structure one can save time?

116. Can one say the –ve numbers which cannot be visualized can be overcome by MOD techniques?

117. Can replacing MOD complex planes by the complex plane C advantages and saves time working with it?

118. As these are ages of computer will MOD methods lessen the storage space of any appropriate document.

Chapter Four

ALGEBRAIC STRUCTURES USING MOD FUZZY PLANE $R_n(1)$

Unlike usual MOD real plane or MOD complex modulo integer plane or MOD dual number plane or MOD special dual like number plane or MOD special quasi dual number plane we see the MOD fuzzy plane is only one and rest of the planes are infinite in number.

So we using the MOD fuzzy plane $R_n(1) = \{(a, b) \mid a, b \in [0, 1)\}$ construct matrices and polynomials.

It is pertinent to keep on record that these MOD fuzzy planes will also find its applications in fuzzy models. Thus we are interested in this special MOD fuzzy plane.

DEFINITION 4.1: *Let $R_n(1)$ be the MOD fuzzy plane.*

Any $s \times t$ matrix $P = (a_{ij})$ where $a_{ij} \in R_n(1)$ will be defined as the MOD fuzzy $s \times t$ matrix

$$1 \leq i \leq s, \quad 1 \leq j \leq t, \; s \geq 1 \quad \text{and} \quad t \geq 1.$$

When s = 1 we call P as the MOD fuzzy row matrix.

It t = 1 we call P to be the MOD fuzzy column matrix.

It t = s we define P to be the MOD fuzzy square matrix.

We will give examples of different types of MOD fuzzy matrices.

Example 4.1: Let X = ((0.331, 0.211), (0.9, (0.3815), (0.3, 0.754), (0, 0.342), (0.67537102, 0)) be a MOD fuzzy 1×5 row matrix.

Example 4.2: Let

$$Y = \begin{bmatrix} (0.92, 0.1114) \\ (0.213, 0.375) \\ (0.15, 0.72) \\ (0.3, 0.93) \\ (0.04, 0.001) \\ (0.119, 0.2134) \\ (0.612, 0.314) \\ (0.751, 0.75) \end{bmatrix}$$ be the MOD fuzzy 8×1 column matrix.

Example 4.3: Let

$$A = \begin{bmatrix} (0.03) & (0.2,0) & (0.0) \\ (0.1,0.2) & (0.4,0.3) & (0.2,0) \\ (0.0) & (0.5,0) & (0,0.3) \\ (0.7,0.9) & (0.8,0.1) & (0.7,(0.7) \end{bmatrix}$$

be a MOD fuzzy 4×3 matrix.

Example 4.4: Let

$$P = \begin{bmatrix} (0,0.4) & (0.7,0) & (0,0) & (0.1,0) \\ (0.5,0.1) & (0,0) & (0.7,0.5) & (0,0.1) \\ (0.6,0.8) & (0.7,0) & (0,0.91) & (0,0) \\ (0,0) & (0.71,0.17) & (0.31,0.13) & (0,0.1) \end{bmatrix}$$

be a MOD fuzzy 4×4 square matrix.

Now we will give some operations on the collection of MOD fuzzy matrices of some order. The MOD fuzzy matrices under + is a group and under $\times$ it is only a semigroup.

DEFINITION 4.2: *Let $M = \{(a_1, \ldots a_n)\ a_i \in R_n(1);\ 1 \le i \le m, +\}$ under the operation of + is defined as the MOD fuzzy row matrix group.*

We will give one or two examples of them.

Example 4.5: Let $T = \{(a_1, a_2, a_3) \mid a_i \in R_n(1), 1 \le i \le 3, +\}$ be the MOD fuzzy row matrix group.

Let x $= ((0.7, 0.25), (0.31, 0.02), (0.74, 0)) \in T$

x + x $= ((0.4, 0.5), (0.62, 0.04), (0.58, 0)) \in T$.

We see $0 = ((0, 0), (0, 0), (0, 0))$ acts as the additive identity of T.

$$0 + x = x + 0 = x \text{ for all } x \in T.$$

We have a unique $-x \in T$ such that $x + (-x) = 0$.

For the given x

$-x = ((0.3, 0.75), (0.69, 0.98), (0.26, 0)) \in T$ is such that

$$x + (-x) = ((0, 0), (0, 0), (0,0)).$$

We see T has subgroups of both infinite and finite order.

P = ⟨((0, 0.5), (0, 0), (0, 0))⟩ is a subgroup of order two under addition as P + P = (0).

We also have subgroups of finite order.

Further T has subgroups of infinite order also.

Example 4.6: Let

$$B = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \middle| \; a_i \in R_n(1);\ 1 \leq i \leq 4, + \right\}$$

be the MOD fuzzy column matrix group.

$$B \text{ has } (0) = \begin{bmatrix} (0,0) \\ (0,0) \\ (0,0) \\ (0,0) \end{bmatrix} \text{ to be the additive identity.}$$

For every y ∈ B we have a unique y′ ∈ B such that y + y′ = (0).

$$\text{Take } y = \begin{bmatrix} (0, 0.75) \\ (0.011, 0.223) \\ (0.771, 0.305) \\ (0.9015, 0.1572) \end{bmatrix} \in B$$

$$y' = \begin{bmatrix} (0, 0.25) \\ \\ (0.989, 0.777) \\ \\ (0.229, 0.695) \\ \\ (0.0985, 0.8428) \end{bmatrix} \in B$$

is such that y + y′ = (0); B has subgroups of both finite and infinite order.

Example 4.7: Let

$$D = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in R_n(1);\ 1 \leq i \leq 9, + \right\}$$

be the MOD fuzzy square matrix group.

D has finite order subgroups as well infinite order subgroups.

$$\text{We see } (0) = \begin{bmatrix} (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) \\ (0,0) & (0,0) & (0,0) \end{bmatrix}$$

is the additive identity of B.

$$\text{We see if } x = \begin{bmatrix} (0.71, 0.2) & (0, 0.3) & (0.51, 0) \\ (0.13, 0.31) & (0.71, 0.17) & (0.5, 0.7) \\ (0.79, 0.34) & (0.26, 0.14) & (0.71, 0) \end{bmatrix} \in B$$

then the unique x′ ∈ B such that x + x′ = (0).

$$x' = \begin{bmatrix} 0.28, 0.8) & (0, 0.7) & (0.49, 0) \\ (0.87, 0.69) & (0.29, 0.83) & (0.5, 0.3) \\ (0.21, 0.66) & (0.74, 0.86) & (0.29, 0) \end{bmatrix} \in B$$

is the unique inverse of x with respect to +.

Example 4.8: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \middle| \; a_i \in R_n(1),\ 1 \leq i \leq 15, +\right\}$$

be the MOD fuzzy 5×3 matrix group.

M has subgroups of finite and infinite order.

Example 4.9: Let

$$A = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_9 \\ a_{10} & a_{11} & \ldots & a_{18} \\ a_{19} & a_{20} & \ldots & a_{27} \\ a_{28} & a_{29} & \ldots & a_{36} \\ a_{37} & a_{38} & \ldots & a_{45} \end{bmatrix} \middle| \; a_i \in R_n(1);\ 1 \leq i \leq 45, +\right\}$$

be the MOD fuzzy 5×9 matrix groups.

A has subgroups of both finite and infinite order.

$$X = \left\{ \begin{bmatrix} a_0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix} \middle| \; a_i \in R_n(1), + \right\} \subseteq A$$

is a subgroup of infinite order.

$$Y = \left\{ \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \begin{bmatrix} 0.5 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix} \right\} \subseteq A$$

is a subgroup of order two.

$$N = \left\{ \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \begin{bmatrix} 0.25 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} 0.5 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \begin{bmatrix} 0.75 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix} \right\} \subseteq A$$

is a subgroup of order four.

$$T = \left\{ \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \begin{bmatrix} 0.1 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} 0.2 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \dots, \begin{bmatrix} 0.9 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix} \right\} \subseteq A$$

is a subgroup of order 10.

Next our motivation is to define $\times_n$ the natural product on the MOD fuzzy matrix collection.

We will illustrate this situation by examples.

DEFINITION 4.3: *Let* $M = \{(a_{ij})_{m\times t} \mid a_{ij} \in R_n(1);\ 1 \leq i \leq m,\ 1 \leq j \leq t, \times_n\}$; *M is defined as the MOD fuzzy matrix semigroup. M is of infinite order and M is commutative.*

We give examples of such MOD fuzzy matrix semigroup.

Example 4.10: Let $S = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in R_n(1)\ ;\ 1 \leq i \leq 5, \times\}$ be the MOD fuzzy row matrix semigroup.

$(0) = ((0, 0), (0, 0), (0,0), (0, 0), (0,0))$ is the zero of S. For any $x \in S$; we see

$$x \times (0) = (0) \times x = (0).$$

Let x = ((0.3, 0), (0.81, 0), (0.53, 0), (0.3314, 0) (0.75, 0))

and y = ((0, 0.7) (0, 0.94) (0, 0.33), (0, 0.315) (0, 0.001)) $\in$ S

$x \times y = (0)$. Let $A = \{(a_1, 0, 0, 0, 0) \mid a_1 \in R_n(1), \times\} \subseteq S$; A is an ideal of S and this ideal is of infinite order.

Example 4.11: Let

$$N = \left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| a_i \in R_n(1), 1 \leq i \leq 5, \times_n\right\}$$

be the MOD fuzzy column matrix semigroup under natural product $\times_n$.

$$(0) \;=\; \begin{bmatrix} (0,0) \\ (0,0) \\ (0,0) \\ (0,0) \\ (0,0) \end{bmatrix} \text{ is the zero of N.}$$

For each x $\in$ N; $x \times_n (0) = (0) \times_n x = (0)$.

We see there exists for every $A = \begin{bmatrix} (a_1, a_2) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ of this form we

have several B $\in$ N with $A \times_n B = (0)$.

$$\text{For } P = \left\{ \begin{bmatrix} (a_1,0) \\ (a_2,0) \\ (a_3,0) \\ (a_4,0) \\ (a_5,0) \end{bmatrix} \right| (a_i, 0) \in R_n(1);\ 1 \leq i \leq 5, \times_n \} \subseteq N$$

we have a

$$Q = \left\{ \begin{bmatrix} (0,a_1) \\ (0,a_2) \\ (0,a_3) \\ (0,a_4) \\ (0,a_5) \end{bmatrix} \right| (0, a_i) \in R_n(1);\ 1 \leq i \leq 5, \times_n \} \subseteq N$$

such that $P \times_n Q = \{(0)\}$.

Thus N has infinite number of zero divisors.

.

$$\text{We see } P_1 = \left\{ \begin{bmatrix} a_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right| a_1 \in R_n(1), \times_n \} \subseteq N$$

is a subsemigroup which is an ideal of N.

$$P_2 = \left\{ \begin{bmatrix} 0 \\ a_2 \\ 0 \\ \vdots \\ 0 \end{bmatrix} \right| a_2 \in R_n(1), \times_n \} \subseteq N,$$

$$P_3 = \left\{\left.\begin{bmatrix} 0 \\ 0 \\ a_3 \\ 0 \\ 0 \end{bmatrix}\right| a_3 \in R_n(1), \times_n\right\} \subseteq N,$$

$$P_4 = \left\{\left.\begin{bmatrix} 0 \\ 0 \\ 0 \\ a_4 \\ 0 \end{bmatrix}\right| a_4 \in R_n(1), \times_n\right\} \subseteq N$$

$$\text{and } P_5 = \left\{\left.\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ a_5 \end{bmatrix}\right| a_5 \in R_n(1), \times_n\right\} \subseteq N$$

be the subsemigroups of N which are ideals of N.

N has atleast ${}_5C_1 + {}_5C_2 + {}_5C_3 + {}_5C_4$ number of ideals.

Example 4.12: Let

$$M = \left\{\left.\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \\ a_{17} & a_{18} & a_{19} & a_{20} \end{bmatrix}\right| a_i \in R_n(1); 1 \leq i \leq 20, \times_n\right\}$$

be the MOD fuzzy matrix semigroup of infinite order.

M has infinite number of zero divisors and has atleast ${}_{20}C_1 + {}_{20}C_2 + {}_{20}C_3 + \ldots {}_{20}C_{19}$ number of ideals.

Example 4.13: Let

$$N = \left\{ \left[\begin{array}{cccc} a_1 & a_2 & \ldots & a_5 \\ a_6 & a_7 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{15} \\ a_{16} & a_{17} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{25} \end{array} \right] \middle| \; a_i \in R_n(1), \times_n, 1 \leq i \leq 25 \right\}$$

be a MOD fuzzy square matrix semigroup.

We can have several interesting properties associated with it.

This study is considered as a matter of routine and hence left as an exercise to the reader.

Example 4.14: Let

$$B = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \; a_i \in R_n(1); 1 \leq i \leq 4, \times \right\}$$

be the MOD fuzzy square matrix semigroup under the usual matrix operation $\times$.

We see B is a non commutative semigroup of infinite order.

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ acts as the multiplicative identity.

$$\text{Let } x = \begin{bmatrix} (0.31, 0.25) & (0.72, 0.1) \\ (0.6, 0.5) & (0.3, 0.8) \end{bmatrix} \text{ and}$$

$$y = \begin{bmatrix} (0, 0.4) & (0, 0.7) \\ (0, 0.2) & (0, 0.9) \end{bmatrix} \in B.$$

Consider

$$x \times y = \begin{bmatrix} (0, 0.02) + (0, 0.1) & (0, 0.175) + (0, 0.09) \\ (0, 0.2) + (0, 0.16) & (0, 0.35) + (0, 0.72) \end{bmatrix}$$

$$= \begin{bmatrix} (0, 0.12) & (0, 0.265) \\ (0, 0.36) & (0, 0.07) \end{bmatrix} \quad \text{I}$$

$$y \times x = \begin{bmatrix} (0, 0.1) + (0, 0.35) & (0, 0.04) + (0, 0.56) \\ (0, 0.01) + (0, 0.45) & (0, 0.02) + (0, 0.72) \end{bmatrix}$$

$$= \begin{bmatrix} (0, 0.45) & (0, 0.6) \\ (0, 0.46) & (0, 0.74) \end{bmatrix} \quad \text{II}$$

Clearly I and II are distinct so the MOD fuzzy square semi group is non commutative.

This if we have MOD fuzzy square matrices M, M is a MOD commutative semigroup under the product $\times_n$ and non commutative under the usual $\times$.

Next we proceed onto define the concept of MOD subset fuzzy matrices.

DEFINITION 4.4: *Let*
$M = \{(a_{ij}) \mid a_{ij} \in S(R_n(1))\}$ $1 \leq i \leq s$, $1 \leq j \leq t\}$ be the collection of all $s \times t$ fuzzy matrices whose entries are subsets of $R_n(1)$.

We will give examples of them.

Example 4.15: Let $B = \{(a_1, a_2, a_3) / a_i \in S(R_n(1)); 1 \leq i \leq 3\}$ be the MOD fuzzy subset row matrix collection.

Let x = ({(0.33, 0.201), (0.00042, 0.01), (0.025, 0), (0, 0.0014)}, {(0.4, 0.1), (0.6, 0.93), (0.33, 0.101), (0.00001, 0.00072), (0.0015, 0.00152)}, {(0.54 0.589), (0.5, 0), (0, 0.5), (0.3, 0.3), (0.000123, 0.0012), (0.5, 0.00123456)}) $\in$ B is a typical element.

Clearly the entries of x are subsets from $S(R_n(1))$.

Example 4.16: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| \; a_i \in S(R_n(1)); 1 \leq i \leq 5 \right\}$$

be the MOD subset fuzzy column matrices.

$$y = \begin{bmatrix} \{(0,0.31),\ (0,0.1,0.7,0),(0,0.06),0.7,(0.3),(0,0.9) \\ (0.00035,0.2)\} \\ \\ \{(0.1,0.1),(0.2,0.3)(0.4,0.5),(0,0.5)\} \\ \\ \{(0.7,0.7)(0.32,0.25)(0,0.33372)\} \\ \\ \{(0.57,0)\ \ (0.37253,0.74),(0.331,0.2)\} \\ \\ \{(0.34,0.333333),(0.00005,0.243), \\ (0.842,0.752),(0.435,0.786) \\ (0.0.3)\ (0.1,0.9)\ (0.2,0.8)\} \end{bmatrix} \in M$$

is a MOD fuzzy subset column matrix.

Example 4.17: Let

$$W = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| \ a_i \in S(R_n(1));\ 1 \le i \le 12 \right\}$$

be the MOD fuzzy subset 4×3 matrix collection.

$$x = \begin{bmatrix} \{(0.3,0.2),(0,0.1),(0.7,0)\} & \{(0,0.6),(0.7,0.3)\} & \{(0,0.9)\} \\ \{(0.7,0.2),(0.4,0.6)\} & \{(0,0.8)(0.8,0)\} & \{(0,0.1),(0.2,0)\} \\ \begin{matrix}\{(0.5,0),(0.8,0.3),(0,0), \\ (0.75,0.83),(0.92,0)\}\end{matrix} & \begin{matrix}\{(0,0.41112), \\ (0.32227,0.999)\}\end{matrix} & \begin{matrix}\{(0.9,0.9), \\ (0.9999,0)\}\end{matrix} \\ \begin{matrix}\{(0.9999,0.090909), \\ (0.08989,0.191999), \\ (0.42199,0.39999)\}\end{matrix} & \begin{matrix}\{(0,0.9903), \\ (0.9909,0.309), \\ (0.311,0.889)\}\end{matrix} & \begin{matrix}\{(0,0.9),\ (0.99,0.9), \\ (0,0.999),(0.3,0.9)\}\end{matrix} \end{bmatrix} \in W$$

is an element W.

Example 4.18: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \end{bmatrix} \right| a_i \in s\,(R(1));\ 1 \leq i \leq 15\}$$

be the MOD fuzzy subset 5×3 matrix.

Let T =

$$\begin{bmatrix} \{(0,0.3),(0.5,0)\} & \{(0.43,0.72),(0,0)\} & \{(0,0),(0.3,0.4)\} \\ \{(0.333,0.9999)\} & \{(0.38999),(0.0001)\} & \{(0,0.1)\} \\ \{(0.12,0.99)\} & \begin{array}{l}\{(0.999,0.09), \\ (0.329,0.3399)\}\end{array} & \{(0,0),(0.3,0.1)\} \\ \begin{array}{l}\{(0.11999,0.9), \\ (0.99,0.99)\}\end{array} & \{(0.9,0.9),(0.99,0.99)\} & \{(0,0.9)\} \\ \begin{array}{l}\{(0,0.3),(0.3,0.6), \\ (0.6,0.9),(0.9,0)\}\end{array} & \begin{array}{l}\{(0.999999,0.9), \\ (0.99,0.999)\}\end{array} & \{(0.9,0),(0.3,0.9)\} \end{bmatrix}$$

$\in M$;

T is a MOD fuzzy subset 5×3 matrix.

Example 4.19: Let

$$S = \left\{\begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_3 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in S\,(R_n(1));\ 1 \le i \le 9\right\}$$

be the MOD fuzzy subset collection of square matrices.

$$A = \begin{bmatrix} \{(0.332, 0), (0.99, 0.9), (0.9, 0.00099)\} & \{(0,0)\} & \{(0,0)\} \\ \{(0,0)\} & \{(0.992, 0.45), (0.9992, 0.1111), (0.112, 0.9999)\} & \{(0,0)\} \\ \{(0,0)\} & \{(0,0)\} & \{(0, 0.9), (0.999, 0.9), (0.9, 0.999)\} \end{bmatrix} \in S$$

is a MOD fuzzy subset square matrix.

Now on the MOD fuzzy subset matrices of same order we can define four different operations viz. +, $\times$, $\cup$ and $\cap$ and under these four operations S is only a semigroup.

We will only illustrate this situation by some examples.

Example 4.20: Let

B = $\{(a_1, a_2, a_3, a_4) \mid a_i \in S(R_n(1));\ 1 \le i \le 4, +\}$ be the MOD fuzzy subset row matrix pseudo group (semigroup) under +.

Let x = ({(0, 0.2), (0.11, 0.3) (0.5210, 0)}, {(0.1, 0)}, {(0.312, 0.631), (0.9215, 0.1115)}, {(0, 0.72), (0.15, 0)}) and

y = ({(0, 0), (0.1, 0.2)}, {(0.3, 0.4), (0.5, 0.6)} {(0.31, 0), (0.62, 0.3), (0.71, 0.2)}, {(0.42, 0), (0.1, 0.5), (0.3, 0.7)}) $\in$ B.

x + y = ({(0, 0.2), (0.1, 0.3), (0.5210, 0), (0.1, 0.4), (0.21, 0.5), (0.6210, 0.2)}, {(0.4, 0.4), (0.6, 0.6)}, {(0.622, 0.631), (0.2315, 0.1115), (0.932, 0.931), (0.5415, 0.4115), (0.022, 0.831), (0.2315, 0.3115)}, {(0.42, 0.72), (0.57, 0), (0.1, 0.22), (0.25, 0.5), (0.3, 0.42), (0.45, 0.7)}) $\in$ B.

This is the way the operation + is performed on B.

It is clear for any x $\in$ B we do not have a – x in B such that

x + (– x) = ({0}, {0}, {0}, {0}) the additive identity of B.

B has several subsemigroups of both of infinite and finite order.

Example 4.21: Let

$$S = \left\{ \left. \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_9 \end{bmatrix} \right| a_i \in S(R_n(1));\ 1 \leq i \leq 9, + \right\}$$

be the MOD subset fuzzy column matrix semigroup. S is of infinite order.

$$(0) = \begin{bmatrix} \{0\} \\ \{0\} \\ \vdots \\ \{0\} \end{bmatrix} \text{ acts as the additive identity of S.}$$

Example 4.22: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in S\,(R_n(1));\ 1 \leq i \leq 9, + \right\}$$

be the MOD fuzzy subset square matrix semigroup of infinite order.

$$P_1 = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in \{0, 0.5\},\ 1 \leq i \leq 9 \right\} \subseteq M$$

is a subsemigroup of finite order.

Example 4.23: Let

$$B = \left\{ \begin{bmatrix} a_1 & a_2 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{30} \\ a_{31} & a_{32} & \dots & a_{40} \\ a_{41} & a_{42} & \dots & a_{50} \\ a_{51} & a_{52} & \dots & a_{60} \end{bmatrix} \middle| \; a_i \in S\,(R_n(1));\ 1 \leq i \leq 60, + \right\}$$

be the MOD fuzzy subset matrix semigroup.

B has several subsemigroups of both finite and infinite order.

Now we provide examples of MOD fuzzy subset matrix semigroups under $\times_n$ or $\times$ the natural product on matrices of the usual product.

Example 4.24: Let B = $\{(a_1, a_2, a_3) \mid a_i \in S(R_n(1)); 1 \leq i \leq 3, \times\}$ be the MOD fuzzy subset matrix semigroup of infinite order.

Let x = ({(0, 0.4), (0.3, 0.2), (0.7, 0.112), (0, 0.1)}, {(0, 0.4), (0, 0.3) (0, 0.6), (0.6, 0.7)}, {(0.1, 0.01), (0.002, 0.001), (0.006, 0.2)}) and

y = ({(0.12, 0.16), (0.25, 0.02)}, {(0.15,0.09), (0.8,0.12)}, {(0.3,0.5), (0.12, (0.16)}) $\in$ B;

x $\times$ y = ({(0, 0.064), (0.036, 0.032), (0.084, 0.01792), (0, 0.016), (0, 0.032), (0, 0.075, 0.004), (0.175, 0.00224), (0, 0.002)}, {(0, 0.036), (0, 0.027), (0, 0.054), (0.09, 0.063), (0, 0.048), (0, 0.036), (0, 0.072), (0.48, 0.0084)}, {(0.03, 0.005), (0.0006,0.0005), (0.0018,0.1), (0.012,0.0016), (0.00024, 0.00016), (0.00072,0.032)}) $\in$ B.

This is the way product operation is performed on B.

Infact ({0},{0}, {0}) = (0) acts as the zero of B.

We see B has infinite number of zero divisors.

(1) = ({1}, {1}, {1}) acts as the multiplicative identity of B. B has only finite number of idempotents.

Example 4.25: Let

$$M = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \middle| \, a_i \in S(R_n(1)); 1 \leq i \leq 4; \times_n \right\}$$

be the MOD fuzzy subset semigroup matrix of infinite order.

$$\text{Let } x = \begin{bmatrix} \{(0,0.32),(0,0.1)(0,0)\} \\ \\ \{(0.1,0.2)(0,0.8)\} \\ \\ \{(0.4,0.5),(0.7,0)\} \\ \\ \{(0.12,0)\ (0,0.13)\} \end{bmatrix} \text{ and}$$

$$y = \begin{bmatrix} \{(0,0.2)\ \ (0.1,0.01),(0,0.5)\} \\ \\ \{(0,0.7)(0.3,0.2)(0.5,0)\} \\ \\ \{(0,0),(0.1,0.2)\ (0.3,0.6)\} \\ \\ \{(0.1,0.2),(0.1,0.3)\} \end{bmatrix} \text{ be in M}$$

$$x \times_n y = \begin{bmatrix} \{(0,0.064),(0,0.02),(0,0), \\ (0,0.0032),(0,0.001), \\ (0,0.15)(0,0.05)\} \\ \\ \{(0,0.14),(0,0.56), \\ (0.03),(0.04),(0,0.16), \\ (0.05,0),(0,0)\} \\ \\ \{(00),(0.04,0.1),(0,0.7,0) \\ (0.12,0.3),(0.21,0)\} \\ \\ \{(0.012,0),(0,0.026), \\ (0.012,0),(0,0.039)\} \end{bmatrix}.$$

This is the way product operation $\times_n$ is performed on M. However M has infinite number of zero divisors.

M has atleast ${}_4C_1 + {}_4C_2 + {}_4C_3$ number of MOD subset fuzzy ideals.

Example 4.26: Let

$$B = \left\{ \begin{bmatrix} a_1 & a_2 & \ldots & a_6 \\ a_6 & a_7 & \ldots & a_{12} \\ a_{13} & a_{14} & \ldots & a_{18} \\ a_{19} & a_{20} & \ldots & a_{24} \end{bmatrix} \middle| \; a_i \in S(R_n(1)); \; 1 \leq i \leq 24, \times_n \right\}$$

be the MOD fuzzy subset matrix semigroup under the natural product $\times_n$. B is of infinite order.

B has subsemigroup of finite under B has ideals of infinite order B has atleast ${}_{24}C_1 + \ldots + {}_{24}C_{23}$ number of ideals.

Example 4.27: Let

$$B = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle| \; a_i \in S(R(1)); \; 1 \leq i \leq 16, \times_n \right\}$$

be the MOD fuzzy subset square matrix semigroup of infinite order and B is commutative.

However if $\times_n$ is replaced by $\times$ on B we see B is a MOD fuzzy subset square matrix semigroup which is non commutative.

To this end we give an example.

Example 4.28: Let

$$M = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle| \; a_i \in S(R_n(1));\ 1 \leq i \leq 4, \times \right\}$$

be the MOD subset fuzzy square matrix semigroup under the usual product × M has zero divisors and M is non commutative.

$$\text{Let } x = \begin{bmatrix} \{(0,0.3)\ (0.2,0.1)\ (0,4,0.8), \\ (0,0)\ (0.1,0.1)\} & \{(0,0)\} \\ \{(0,0)\} & \{(0,0.4(0,0.2) \\ & (0.3,0.1)\} \end{bmatrix} \text{ and}$$

$$y = \begin{bmatrix} \{(0,0.3), (0,0.2), \\ (0.1,0)\} & \{(0.3,0.1), (0.2,0.3)\} \\ \{(0,0.1), (0.2,0.3)\} & \{(0.2, 0.3),(0,0), (0.1,0.3)\} \end{bmatrix} \in M$$

$x \times y =$

$$\begin{bmatrix} \{(0,0.9)\ (0,0.03), & \{(0,0.3),(0,0.09), \\ (0,0.24)(0,0),(0,0.03), & (0.06, 0.01), \\ (0,0.06),(0.02,0), & (0.04,0.03),(0.12,0.08), \\ (0.04,0), (0.01,0)\} & (0.08,0.24),(0,0) \\ & (0.02,0.03),(0.03,0.01)\} \\ & \{(0,0),(0.06,0.03), \\ \{(0,0)\} & (0,0.06)(0,0), \\ & (0,0.12), (0.03,0.03)\} \end{bmatrix}$$

$y \times x =$

$$\begin{bmatrix} \begin{array}{l}\{(0,0.09),(0,0.03),(0.0),\\ (0,0.24),\ (0.06)(0,0.02),\\ (0,0.16),\ (0.02,0),(0.01,0),(0.04,0)\}\end{array} & \begin{array}{l}\{(0,0.04),(0,0.02),\\ (0.09),(0.01),(0,0.04)\\ (0,0.06),(0,0.6,0.03)\}\end{array} \\ \\ \begin{array}{l}\{(0,0.03),(0,0.01),(0,0.24),(0,0),\\ (0,0.06),(0,0.02),(0,0.16),(0.02,0),\\ (0.04,0),(0.01,0)\}\end{array} & \begin{array}{l}\{(0,0),(0,0.12),(0,0.06),\\ (0,0.03),(0,0.03,0.03)\}\end{array} \end{bmatrix}$$

Clearly $x \times y \neq y \times x$.

Thus the MOD subset fuzzy semigroup matrix is non commutative.

Example 4.29: Let
$M = \{(a_1, a_2, a_3, a_4) \mid a_i \in S(R_n(1));\ 1 \leq i \leq 4, \cup\}$ be the MOD subset fuzzy row matrix semigroup.

Clearly M is a semilattice. Every singleton element in M is a subsemigroup.

Example 4.30*:* Let

$$P = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \middle|\ a_i \in S(R_n(1);\ 1 \leq i \leq 3, \cap \right\}$$

be the MOD subset fuzzy column matrix semigroup.

$$\text{Let } x = \begin{bmatrix} \{(0,0.3),(0,2,0.1),(0,0),(0.5,0)\} \\ \\ \{(0.4,0),(0.3,0.2),(0.3,0),(0.7,0.9)\} \\ \\ \{(0,0),(0.1,0.3),(0.2,0.2)\} \\ \end{bmatrix}$$

and

$$y = \begin{bmatrix} \\ \{(0.0.3),(0.771,0.885),(0.5,0),(0.9912,0.7753)\} \\ \\ \{(0.4,0),(0.7,0.9),(0.331,0.4434), \\ (0.997,0.87031),(0.5006,0.7009)\} \\ \\ \{(0.778,0.112),(0.993,0.11214),(0.993,0.73), \\ (0.091,0),(0,1)\} \end{bmatrix}$$

$\in$ P.

$$\text{We see } x \cap y = \begin{bmatrix} \{(0,0.3),(0.5,0)\} \\ \\ \{(0.4,0),(0.7,0.9)\} \\ \\ \{(\phi,\phi)\} \\ \end{bmatrix}$$

Example 4.31: Let

$$A = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \end{bmatrix} \middle| \; a_i \in S(R_n(1));\ 1 \leq i \leq 9, \cup \right\}$$

be the MOD fuzzy subset matrix semigroup.

$$\text{Let } x = \begin{bmatrix} \{(0,0)(0.3,0)\} & \{(0,0.2)(0.2,0)\} & \{(0,0.31)\} \\ \{(0.4,0.324)\} & \{(0.321,0.21)\} & \{(0,0.4)\} \\ \{(0.1,0.3)\} & \{(0.4,0.5)\} & \{(0,0.7)\} \end{bmatrix}$$

$$\text{and } y = \begin{bmatrix} \{(0.3,0.2)\} & \{(0.1,0.5)\} & \{(0.31,0)\} \\ \{(0.2,0.31)\} & \{(0.11,0.21)\} & \{(0.1,0)(0,0)\} \\ \{(0,0)(0,0.2)\} & \begin{matrix}\{(0.4,0.5), \\ (0.1,0),(0,0.1)\}\end{matrix} & \{(0,0)\} \end{bmatrix}$$

$\in$ A.

x ∪ y =

$$\begin{bmatrix} \{(0,0),(0.3,0), & \{(0.02)(0.2,0) & \{(0,0.31) \\ (0.3,0.2)\} & (0.1,0.5)\} & (0.31,0)\} \\ & & \\ \{(0.4,0.324), & \{(0.321,0.21), & \{(0,0.4), \\ (0.2,0.31)\} & (0.11,0.21)\} & (0,0),(0.1,0)\} \\ & & \\ \{(0.1,0.3),(0.0) & \{(0.4,0.5),(0.1,0) & \{(0,0.7) \\ (0,0.2)\} & (0,0.1)\} & (0,0)\} \end{bmatrix}$$

$\in$ A.

This is the way '∪' operation is performed on A.

Example 4.32*:* Let

$$S = \left\{ \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \end{bmatrix} \middle| \; a_i \in S(R_n(1)) \cup 1 \leq i \leq 12, \cap \right\}$$

be the MOD subset fuzzy matrix semigroup S is a semilattice.

Infact every singleton element of S is a subsemigroup S has subsemigroups of all orders.

Example 4.33: Let

$$B = \left\{ \left. \begin{bmatrix} a_1 & a_2 & a_3 \\ a_4 & a_5 & a_6 \\ a_7 & a_8 & a_9 \\ a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} \\ a_{16} & a_{17} & a_{18} \\ a_{19} & a_{20} & a_{21} \\ a_{22} & a_{23} & a_{24} \end{bmatrix} \right| a_i \in S(R_n(1));\ 1 \leq i \leq 24, \cup \right\}$$

be a MOD fuzzy subset matrix semigroup. B has infinite number of subsemigroups.

Example 4.34: Let

$$M = \left\{ \left. \begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \right| a_i \in S(R_n(1)) \cup \{\phi\};\ 1 \leq i \leq 16, \cap \right\}$$

be the MOD fuzzy subset matrix semilattice (semigroup).

We will illustrate by an example that all the four operations '+' $\times$, $\cup$ and $\cap$ are distinct.

Example 4.35: Let B = $\{(a_1, a_2, a_3) \mid a_i \in S(R_n(1)); 1 \leq i \leq 3\}$ be the MOD fuzzy subset row matrix semigroup.

We will show for a pair of elements x, y in B $x \cup y$, $x \cap y$, $x \times y$ and $x + y$ are distinct.

Let x = ({(0, 0.31), (0.9, 0.09), (0.1, 0) (0.2, 0.9)}, {(0,0), (0.39, 0.9), (0.9, 0.29)},{(0,0.1), (0.1,0), (0.319, 0.29), (0.9, 0.39)}) and

y = ({(0.3,0.1), (0.9,0.09),(0.1, (0.2)}, {(0, 0.1) (0.2, 0.1), (0.3, 0.1)}, {(0.1,0.2), (0.1, 0.9)}) $\in$ B.

$x \cap y$ = ({(0.9,0.09)}, {ϕ}, {ϕ}) I

$x \cup y$ = ({(0,0.31),(0.9,(0.09),(0.1,0), (0.2,0.9),(0.3,0.1), (0.1,0.2)} {(0,0), (0.39,0.9), (0.9,0.29, (0, 0.1), (0.2,0.1), (0.3, 0.1)}, {(0,0.1), (0.1,0) (0.319, 0.29), (0.9,0.39), (0.1, 0.2), (0.1, 0.9)}) II

x + y = ({(0.3,0.41), (0.2,0.19), (0.4,0.1) (0.5,0) (0.9,0.4), (0.8,0.18), (0,0.9), (0.1,0.99), (0.1, 0.51) (0, 0.29), (0.2, 0.2), (0.3, 0.1)} {(0, 0.1), (0.39, 0), (0.9, 0.39), (0.59,0), (0.2,0.1), (0.1,0.39), (0.3,0.1), (0.69,0), (0.2, 0.39)}, {(0.1,0.3), (0.2, 0.2), (0.419, 0.49), (0, 0.59), (0.1, 0), (0.2, 0.9), (0.419, 0.19), (0, 0.29)}) III

$x \times y$ = ({(0, 0.031), (0.27, 0.009), (0.03,0), (0.06,0.09), (0, 0.0279), (0.81, 0.0081), (0.09, 0), (0.18, 0.081), (0, 0.062), (0.09, 0.018), (0.01, 0) (0.02, 0.18)} {(0, 0), (0, 0.09), (0, 0.029), (0.078, 0.09), (0.18, 0.029), (0.117, 0.09), (0.27, 0.029)}, {(0, 0.02), (0.01, 0), (0.0319, 0.058), (0.09, 0.072), (0, 0.09), (0.01, 0), (0.0319, 0.261), (0.09, 0.321)}) IV

It is clearly seen all the four operations are distinct thus we get 4 distinct MOD fuzzy subset matrix semigroup.

We have MOD fuzzy polynomials as well as MOD fuzzy subset polynomials.

$$R_n(1)[x] = \left\{ \sum_{i=0}^{\infty} a_i x^i \;\middle|\; a_i \in R_n(1) \right\}.$$

We can define only two operations viz + and $\times$, where as on $S(R_n(1))[x]$ we can define four distinct operations, +, $\times$, $\cup$ and $\cap$.

We will briefly describe them by examples.

Example 4.36: Let $R_n(1)[x]$ be the MOD fuzzy polynomials any $p(x) = (0, 0.32)\, x^7 + (0.2, 0.51)\, x^6 + (0, 0)\, x^5 + (0, 0.1)\, x^4 + (0,0)\, x^3 + (0.32, 0)\, x^2 + (0,0.91)\, x + (0.3, 0.1)$

$$= (0,0.32)x^7 + (0.2, 0.51)x^6 + (0, 0.1)x^4 + (0.32, 0)x^2 + (0, 0.91)x + (0.3, 0.1).$$

Let $p(x) \in R_n(1)[x]$, we say degree of $p(x)$ to be n if n is the largest power of x in $p(x)$ whose coefficient is non zero.

We call elements of $R_n(1)\, [x]$ as MOD fuzzy polynomials.

We have only one MOD fuzzy polynomial collection denoted by $R_n(1)[x]$.

We can define two binary operations on $R_n(1)[x]$, + and $\times$; under $\times$ $R_n(1)[x]$ is only a commutative semigroup where as under + $R_n(1)[x]$ is a commutative group.

We will just illustrate how operastion are performed on $R_n(1)\, [x]$.

Let $p(x) = (0.3, 0.2)\, x^3 + (0.7, 0.4)\, x^2 + (0, 0.1)\, x + (0.3, 0)$ and

$q(x) = (0.712, 0.51)x^4 + (0.71, 0.92)x^3 + (0.4, 0.3)x^2 + (0.714,0.8)x + (0.923, 0.8) \in R_n(1)[x]$

We find

$$p(x) + q(x) = (0.712, 0.51)x^4 + (0.01, 0.12)x^3 + (0.1, 0.7)x^2 + (0.714, 0.9)x + (0.223, 0.8) \in R_n(1)[x].$$

This is the way + operation on MOD fuzzy polynomials are performed. In fact $p(x) + q(x) = q(x) + p(x)$.

We see $(0, 0) = (0,0)x^n + (0, 0)x^{n-1} + \ldots + (0, 0)x + (0, 0)$ is the zero polynomial in $R_n(1)[x]$.

Further $(0, 0) + p(x) = p(x) + (0, 0)$.

To every $p(x) \in R_n(1)[x]$ there exists a unique element denoted by $-p(x)$ such that $p(x) + (-p(x)) = (0, 0)$.

Let $p(x) = (0, 0.2)x^5 + (0.31, 0.8)x^4 + (0.34, 0.708)x^3 + (0.11, 0)x^2 + (0, 0.314)x + (0.2, 0.46)$ be MOD fuzzy polynomials in $R_n(1)[x]$.

The inverse of $p(x)$ is $-p(x) = (0, 0.8)x^5 + (0.69, 0.2)x^4 + (0.66, 0.292)x^3 + (0.89, 0)x^2 + (0, 0.686)x + (0.8, 0.54) \in R_n(1)[x]$ such that $p(x) + (-p(x)) = (0, 0)$.

Thus $R_n(1)[x]$ is the MOD fuzzy polynomial group of infinite order. $R_n(1)[x]$ is an additive abelian group.

Now we show how product of two MOD fuzzy polynomials are made.

Let $p(x) = (0, 0.2)x^4 + (0, 0.5)x^3 + (0.3, 0.7)x^2 + (0.4, 0.5)x + (0.7, 0.8)$ and

$q(x) = (0.4, 0.3)x^3 + (0.5, 0.7)x^2 + (0.16, 0)x + (0.11, 0.12) \in R_n(1)[x]$.

We find the product $p(x) \times q(x) = (0, 0.12)x^7 + (0, 0.15)x^6 + (0.12, 0.21)x^5 + (0.16, 0.15)x^4\ (0.28, 0.24)x^3 + (0, 0.14)x^6 + (0, 0.35)x^5 + (0.15, 0.49)x^4 + (0.2, 0.35)x^3 + (0.35, 0.56)x^2 + (0, 0)x^5 + (0, 0)x^4 + (0.48, 0)\ x^3 + (0.064, 0)x^2 + (0.112, 0)x + (0, 0.024)x^4 + (0, 0.06)x^3 + (0.033, 0.084)x^2 + (0.044, 0.06)x + (0.077, 0.096)$

$= (0, 0.12)x^7 + (0, 0.29)x^6 + (0.12, 0.56)x^5 + (0.31, 0.664)x^4 + (0.96, 0.65)x^3 + (0.447, 0.644)x^2 + (0.156, 0.06)x + (0.077, 0.096)$ is in $R_n(1)[x]$.

$R_n(1)[x]$ has zero divisors.

$p(x) = (0.3, 0)x^4 + (0.8, 0)x^3 + (0.92, 0)x^2 + (0.144, 0)$ and $q(x) = (0, 0.8)x^3 + (0, 0.9)x^2 + (0, 0.632) \in R_n(1)[x]$.

Clearly $p(x) \times q(x) = (0, 0)$, but $p(x) \neq (0, 0)$ and $q(x) \neq (0, 0)$.

In fact $\{R_n(1)[x], \times\}$ is a semigroup which has infinite number of zero divisors. Solving equations has no meaning as we have not get structured the MOD fuzzy polynomial with more than one operation.

We have seen $R_n(1)[x]$ as a group under addition and $R_n(1)[x]$ is a semigroup under $\times$.

Next we give operations on MOD fuzzy subset polynomials and describe them.

Let $S(R_n(1))[x]$ be the MOD fuzzy subset polynomial.

Elements of $S(R_n(1))[x]$ will be of the following form:

$p(x) = \{(0, 0.3), (0.4, 0.1), (0.8, 0)\}x^7 + \{(0, 0.5), (0.8, 0.4), (0, 0.1114), (0.5, 0.7), (0.9, 0.12)\}x^5 + \{(0.402, 0.11142), (0.1117, 0.1117), (0.0123, (0.131)x^3 + \{(0, 0), (0.1, 0.3), (0.8, 0.4), (0.5, 0.63), (0.31, 0.23)\} \in S(R_n(1))[x]$.

This is a MOD fuzzy subset polynomial of degree 7.

Clearly coefficients of $p(x)$ are in $S(R_n(1))$.

Further $\{(0,0)\}$ is the zero polynomial for
$\{(0,0)\} = \{(0,0)\}x^n + \{(0,0)\}x^{n-1} + \ldots + \{(0,0)\}x + \{(0,0)\}$.

We can on $S(R_n(1)[x])$ define four types of distinct operation and under all these operations $S(R_n(1)[x])$ is only a semigroup.

Let $p(x) = a_nx^n + a_{n-1}\, x^{n-1} + \ldots + a_0$ and

$q(x) = b_mx^m + b_{m-1}\, x^{m-1} + \ldots. + b_0$; $a_i, b_j \in S(R_n(1))$. $0 \leq i \leq n$ and $1 \leq j \leq m$.

Clearly $(S_n(R_n(1))\,[x], +)$ is a semigroup defined as the MOD subset fuzzy polynomial semigroup.

We will just show how the operation '+' is performed on $S(R_n(1))\,[x]$.

Let $p(x) = \{(0,0.3), (0.8, 0), (0, 0), (0.3, 0.2), (0.5, 0.7)\}x^3 + \{(0,0.4), (0.3, 0.2), (0.7, (0.4)\}x^2 + \{(0, 0.43), (0.31,0), (0.2, 0.4)\}$ and

$q(x) = \{(0, 0.4), 0.2, 0.3)\}x^2 + \{(0.4, 0.3), (0.7, 0.8)\} \in S(R_n(1))[x]$.

$p(x) + q(x) = \{(0, 0.3), (0.8, 0), (0,0), (0.3, 0.2), (0.5, 0.7)\}x^3 + \{(0, 0.8), (0.3,0.6), (0.7, 0.8), (0.2, 0.7), (0.5, 0.5), (0.9, 0.7)\}x^2 + \{(0.4, 0.73), (0.71, 0.3), (0.6, 0.7)), (0.7, 0.2), (0, 0), (0.4, 0.2)\} \in S(R_n(1))[x]$.

This is the way '+' operation on $S(R_n(1))[x]$ is performed. Clearly $S(R_n(1))[x]$ is not a group only a semigroup or in particular a monoid.

We see every $p(x) \in S(R_n(1))[x]$ need not in general have an inverse such that $p(x) + (-p(x)) = \{(0,0)\}$.

Interested reader is left with the task of studying the MOD subset fuzzy semigroup.

We see $S(R_n(1))[x]$ has subsemigroups of finite order under + also.

Now we briefly indicate how the product operation is performed on $S(R_n(1))[x]$.

Let p(x) = {(0, 0), (0.20) (0.7, 0.4) (0, 0.5) (0.8, 0.1)}x^2 + {(0, 0.3), (0.7, 0.8), (0.9, 0.2)}x + {(0, 0.7), (0.2, 0.4), (0.5, 0.9)} and

q(x) = {(0, 0.7), (0.3, 0.2), (0.5, 0)} + {(0, 0.4), (0.3, 0.2), (0.7, (0.8), (0.2, 0.9)} $x^2 \in S(R_n(1))[x]$.

We find p(x) × q(x) = {(0, 0), (0, 0.06), (0, 0.20), (0, 0.04), (0.06, 0), (0.21, 0.08), (0, 0.01), (0.24, (0.02), (0.14, 0), (0.49, 0.32), (0, 0.4), (0.56, 0.08), (0.04, 0), (0.14, 0.36), (0, 0.45), (0.16, 0.09)}x^4 + {(0, 0.12), (0.21, 0.16), (0, 0.32), (0, 0.08), (0, 0.6), (0.27, 0.04), (0, 0.24), (0.49, 0.64), (0.63, 0.16), (0, 0.27), (0.14, 0.72), (0.18, 0.18)}x^3 + {}x + {(0, 0), (0, 0.28), (0, 0.35), (0, 0.07), (0.06, 0), (0.21, 0.08), (0, 0.1), (0.24, 0.02), (0.1, 0), (0.35, 0), (0.4, 0)}x^2 + {(0, 0.49), (0, 0.28), (0, 0.63), (0, 0.14), (0.06, 0.08), (0.15, 0.18), (0, 0), (0.1, 0), (0.25, 0)} + {} x^2 …

= {(0, 0), (0, 0.6), (0, 0.2), (0, 0.04), (0.06, 0), (0.21, 0.08), (0, 0.1), (0.24, 0.02), (0.14, 0), (0.49, 0.32), (0, 0.4), (0.56, 0.08), (0.04, 0), (0.14, 0.36), (0, 0.45), (0.16, 0.09)}x^4 + {…} x^3 + {} x^2 + {} $\in S(R_n(1))[x]$.

This is the way product is performed on $S(R_n(1))$ [x].

Let p(x) = {(0, 0.03), (0.2, 0.3), (0.1, 0.3)} + {(0.8, 0.7), (0.1, 0.5)}x and

q(x) = {(0.3, 0.2), (0.8, 0.7), (0, 0.5)} x + {(0, 0), (0.8, 0.2)} $\in S(R_n(1))[x]$.

p(x) × q(x) = {(0, 0.06), (0.06, 0.06), (0.03, 0.06), (0, 0.21), (0.16, 0.21), (0.08, 0.21), (0, 0.15)}x + {(0.24, 0.14), (0.03, 0.1), (0.64, 0.49), (0.08, 0.35), (0, 0.35), (0, 0.25)}x^2 + {(0, 0), (0, 0.06), (0.16, 0.06), (0.08, 0.06)} + {(0, 0), (0.64, 0.14), (0.08, 0.1)}x.

= {(0.24, 0.14), (0.03, 0.1), (0.64, 0.49), (0.08, 0.35), (0, 0.35), (0, 0.25)}x^2 + {(0, 0.06), (0.06, 0.06), (0.03, 0.06), (0, 0.21), (0.16, 0.21), (0.08, 0.21), (0, 0.12), (0.6, 0.12), (0.03, 0.12), (0, 0.27), (0.16, 0.27), (0.08, 0.27), (0, 0.15), (0, 0.21), (0.16, 0.12), (0.22, 0.12), (0.19, 0.12), (0.16, 0.27), (0.32, 0.27), (0.24, 0.27), (0.16, 0.21), (0.08, 0.12), (0.14, 0.12), (0.11, 0.12), (0.08, 0.27), (0.24, 0.27), (0.16, 0.27), (0.08, 0.21)}x + {(0, 0), (0, 0.06), (0.16, 0.06), (0.08, 0.06)} $\in$ $S(R_n(1))[x]$.

This is the way product is performed on $S(R_n(1))$ [x].

We can define $\cup$ and $\cap$ on $S(R_n(1))$ [x], but it is a matter of routine so we just give some examples to explain this concept.

Let p(x) = {(0.3, 0.4), (0.2, 0.5), (0.1, 0.6)} + {(0.2, 0), (0, 0.7), (0.4, 0.2)}x + {(0, 0), (0, 0.2), (0.3, 0)}x^2 and

q(x) = {(0.3, 0), (0.2, 0)} + {(0, 0.5), (0.31, 0.82), (0.4, 0.7), (0.6, 0.9)}x^2.

p(x) $\cup$ q(x) = {(0.3, 0.4), (0.2, 0.5), (0.1, 0.6), (0.3, 0), (0.2)} {(0.2, 0), (0, 0.7), (0.4, 0.2)}x + {(0, 0), (0, 0.2), (0.3, 0), (0, 0.5), (0.31, 0.82), (0.4, 0.7), (0.6, 0.9)}x^2 $\in$ $S(R_n(1))[x]$.

This is the way '$\cup$' operation is performed on $S(R_n(1))[x]$. Next we proceed on to define '$\cap$' on $S(R_n(1))[x] \cup \{\phi\}$.

Let p(x) = {(0, 0.3), (0.2, 0), (0.7, 0)}x^2 + {(0.7, 0.2), (0.3, 0.7), (0.5, 0.03), (0.03, 0)}x + {(0, 0.21), (0.31, 0), (0, 0), (0, 0.35), (0.001, 0.04)} and

q(x) = {(0, 0.3), (0.2, 0), (0.711, 0), (0.5, 0.321), (0.312, 0.112), (0, 0), (0, 0.1)}x^2 + {(0.3, 0.7), (0.5, 0.03), (0.03, 0), (0.031, 0.215), (0.0012, 0.013)}x + {(0, 0.21), (0.31, 0), (0, 0.35), (0.005, 0.000072), (0.00012, 0.3), (0.312, 0.064)} ∈ $S(R_n(1))[x] \cup \{\phi\}$.

$p(x) \cap q(x)$ = {(0, 0.3), (0.2, 0)}x^2 + {(0.3, 0.7), (0.5, 0.03), (0.03,0)}x + {(0, 0.21), (0.31, 0), (0, 0.35)} ∈ $S(R_n(1))[x] \cup \{\phi\}$.

This is the way +, ×, ∪ and ∩ operations are performed on $S(R_n(1))$ [x].

We see {$S(R_n(1)[x]$, }, {$S(R_n(1)[x]$, ×}, {$S(R_n(1))[x]$, ∪} and {$S(R_n(1))[x]$, ∩} are all four distinct MOD subset fuzzy polynomial semigroup of infinite order and all the four groups are commutative.

Let p(x) = {(0, 0.3), (0.71, 0.2), (0.7, 0.2), (0, 0.5), (0.2, 0)}x^2 + {(0.3, 0.1), (0.21, 0), (0.33, 0.5), (0.12, 0.16), (0,0)}x + {(0.7, 0.5), (0.7, 0), (0.5, 0), (0.5, 0.2), (0.2,0.3)} and

q(x) = {(0, 0.3), (0.71, 0.2), (0.2, 0) (0.5, 0.5), (0.7, 0.3)}x^2 + {(0.3, 0.1), (0.21, 0), (0, 0.5) (0.2, 0)} x + {(0.7, 0.5), (0.7, 0), (0.5, 0), (0.001, 0.02) (0.04, 0.005)} ∈ $S(R_n(1))[x] \cup \{\phi\}$.

Now we find p(x) + q(x) = {(0.3, 0.4), (0.01,0.3), (0, 0.3), (0.3, 0.6), (0.5, 0.1), (0.21, 0.3), (0.92, 0.2), (0.91, 0.2), (0.21, 0.5) (0.04, 0.7) (0.41, 0), (0.33, 0.8), (0.03, 0.7), (0.33, 0), (0.53, 0.5), (0.12, 0.46), (0.83, 0.18), (0.7, 0.2) (0, 0.5), (0.19, 0.18), (0.12, 0.21), (0.2, 0), (0.14, 0.16), (0, 0.3), (0.71, 0.2)} + {(0.3, 0.1), (0.21, 0), (0, 0.5), (0.2, 0), (0.6, 0.2), (0.51, 0.1), (0.63, 0.6), (0.41, 0), (0.53,0.5), (0.15, 0.26), (0.42, 0), (0.5, 0.1) (0.32, 0.16), (0.54, 0.5), (0.33, 0.16), (0.3, 0.6), (0.21, 0.5), (0.33, 0), (0.12, 0.66)}x + {(0.4, 0), (0.4, 0.5), (0.2, 0.5) (0.2, 0.7), (0.9, 0.8), (0.2, 0), (0.2, 0.2), (0.9, 0.3), (0.2, 0.5), (0.2, 0), (0,0), (0, 0.2), (0.7, 0.3), (0.701, 0.52), (0.701, 0.02), (0.501, 0.02), (0.501, 0.22), (0.201, 0.32), (0.74, 0.505), (0.74, 0.005), (0.54, 0.005), (0.54, 0.205), (0.24, 0.305)} …. I

Thus p(x) + q(x) ∈ $S(R_n(1))[x]$.

Next we find

$p(x) \times q(x) = \{(0, 0.09), (0, 0.06), (0, 0.15), (0, 0), (0, 0.04), (0.497, 0.04), (0, 0.1), (0.142, 0) (0.142, 0), (0.14, 0), (0.04, 0), (0, 0.15), (0.355, 0.1), (0.35, 0.1) (0, 0.25), (0.497, 0.06), (0.49, 0.06), (0, 0.15), (0.14, 0)\}x^2 + \{(0.09, 0.01), (0.063, 0), (0.099, 0.05), (0.036, 0.016), (0, 0), (0.063, 0), (0.0441, 0), (0.0393, 0) (0.0252, 0), (0, 0.05), (0, 0.25), (0, 0.08), (0.06,0) (0.042, 0), (0.066, 0), (0.024, 0)\}x + \{(0.49, 0.25) (0.49, 0), (0.35, 0), (0.14, 0.15), (0.14, 0), (0.25, 0), (0.00033, 0.1), (0.00012, 0.0032), (0.012, (0.0005), (0.0084, 0), (0.0132, 0.00025), (0.0048, 0.0008)\}$ … II

That is $p(x) \times q(x) \in S(R_n(1))[x]$.

Clearly one and two are distinct

$p(x) \cap q(x) = \{(0, 0.3), (0.71, 0.2), (0.2, 0)x^2 + \{(0.3, 0.1), (0.21, 0)\}x + \{(0.7, 0.5), (0.7, 0), (0.5, 0)\}$ … III

$p(x) \cap q(x) \in S(R_n(1))[x]$.

Clearly all the three $p(x) + q(x)$, $p(x) \times q(x)$ and $p(x) \cap q(x)$ are different.

Now consider

$p(x) \cup q(x) = \{(0, 0.3), (0.71, 0.2), (0.7, 0.2), (0, 0.5), (0.2, 0), (0.5, 0.5), (0.7, 0.3)\}x^2 + \{(0.3, 0.1), (0.21, 0), (0.33, 0.5), (0.12, 0.16), (0, 0) (0, 0.5), (0.2, 0)\}x + \{(0.7, 0.5), (0.7, 0), (0.5, 0), (0.5, 0.2), (0.2, 0.3), (0.001, 0.02), (0.04, 0.005)\}$

… IV

Clearly $p(x) \cup q(x) \in S(R_n(1))[x]$.

IV is different from I, II and III. Thus all the four operations on $S(R_n(1))$ [x] are distinct from each other giving four different MOD subset fuzzy semigroups of infinite order.

Now we define MOD fuzzy polynomial subset as all subsets from the MOD fuzzy polynomial $R_n(1)[x]$ denoted by $S(R_n(1)[x])$.

Clearly $S(R_n(1)[x])$ is different from $S(R_n(1))[x]$ we will illustrate this situation by some examples and also define operations on them.

$p(x) = \{p(x), q(x), r(x), t(x)\} \in S(R_n(1)\ [x])$, to be more exact $p(x) = \{(0, 0.3)x^3 + (0.3, 0)x^2 + (0, 0.71), (0.32, 0.671)x^4 + (0.1114, 0.551), (0.621, 0.853)x^5 + (0.31, 0.61)x^3 + (0.421, 0.53)\} \in S(R_n(1)[x])$.

Elements of $S(R_n(1)[x])$ are sets of MOD fuzzy polynomials from $R_n(1)[x]$.

We will describe all t he four distinct binary operations $+, \times, \cap$ and $\cup$ under which, $S(R_n(1)[x])$ is a MOD polynomial subset semigroup.

Let $P(x) = \{(0.02)x^3 + (0.3, 0)x^2 + (0.1, 0.2), (0.2, 0.5)x^4 + (0.3, 0.4)\}$

$Q(x) = \{(0.2, 0.5)x^4 + (0.3, 0.4), (0.4, 0.8)x^2 + (0, 0.1)\} \in S(R_n(1)[x])$.

We show how the four operation $+, \times, \cap$ and $\cup$ are performed on $S(R_n(1)\ [x])$.

$P(x) + Q(x) = \{(0, 0.2)x^3 + (0.3, 0)x^2 + (0.4, 0.6) + (0.2, 0.5)x^4, (0.4, 0)\ x^4 + (0.6, 0.8), (0, 0.2)x^3 + (0.7, 0.8)x^2 + (0, 0.5)\} \in S(R_n(1)[x])$ … I

$S(R_n(1)[x])$ is a semigroup known as the MOD fuzzy polynomial subset semigroup under +

$P(x) + Q(x) = \{(0, 0.1)\ x^7 + (0.06, 0)x^6 + (0.02, 0.1)x^4 + (0, 0.08)x^3 + (0.09, 0)x^2 + (0.03, 0.08), (0.04, 0.25)x^8 + (0.06, 0.2)x^4 + (0.06, 0.2)x^4 + (0.9, 0.16); (0, 0.16)x^5 + (0.12, 0)x^4 +$

$(0.04, 0.16)x^2 + (0, 0.02)x^3 + (0, 0.02), (0.008, 0.4)x^6 + (0.12, 0.32)x^2 + (0, 0.05)x^4 + (0, 0.04)\} = \{(0, 0.1)x^7, (0.06, 0)x^6 + (0.02, 0.1)x^4 + (0, 0.08)x^3 + (0.09, 0)x^2 + (0, 0.3, 0.08), (0.04, 0.25)x^8 + (0.12, 0.4)x^4 + (0.9, 0.16); (0, 0.16)x^5 + (0.12, 0)x^4 + (0.04, 0.16)x^2 + (0, 0.02)x^3 + (0, 0.02), (0.08, 0.4)x^6 + (0.12, 0.32)x^2 + (0, 0.05)x^4 + (0, 0.04)\}$ … II

This is the way $\times$ operation is performed on $S(R_n(1)[x])$ and $\{S(R_n(1)[x]), \times\}$ is the MOD subset polynomial fuzzy semigroup.

Clearly I and II are distinct so they form two different semigroup.

Next we consider
$P(x) \cup Q(x) = \{(0, 0.2)x^3 + (0.3, 0)x^2 + (0.1, 0.2), (0.2, 0.5)x^4 + (0.3, 0.4), (0.4, 0.8)x^2 + (0, 0.1)\}$ … III

Clearly $\{S(R_n(1)\ [x]), \cup\}$ is a semigroup in which every element is a subsemigroup of order one. Further this semigroup is different from the other two semigroups.

Now consider
$P(x) \cap Q(x) = \{(0.2, 0.5)\ x^4 + (0.3, 0.4)\}$
$\in S(R_n(1)[x]) \cup \{\phi\}$.

We see $\{S(R_n(1)[x]) \cup \{\phi\}, \cap\}$ is a semigroup defined as the MOD fuzzy polynomial subset semigroup and is different from the rest of the three semigroups just described and developed.

All properties related with this semigroup can be studied by any interested reader as it is considered as a matte of routine.

Now we proceed on to give some problems for the reader.

Problems:

1. What are the special features enjoyed by the MOD fuzzy plane $R_n(1)$?

2. Obtain some important properties associated with the MOD fuzzy group $G = \{R_n(1), +\}$.

   (i) Can G have subgroups of finite order?
   (ii) Can G have subgroups of infinite order?
   (iii) Can G be isomorphic with the group $H = \{(a, b) \mid a, b \in R\}$?

3. Let $H = \{R_n(1), \times\}$ be the MOD fuzzy semigroup.

   (i) Prove H is commutative and is infinite order.
   (ii) Can H be a Smarandache semigroup?
   (iii) Can H have ideals of finite order?
   (iv) Can H have subsemigroups of finite order?
   (v) Can H have S-ideals?
   (vi) Can H have S-zero divisors?
   (vii) Can H have idempotents?
   (viii) Can H has units? (Justify)
   (ix) Obtain any other interesting feature associated with H.

4. Let $B = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in R_n(1); 1 \leq i \leq 5, +\}$ be the MOD fuzzy row matrix group.

   (i) Prove B has both subgroups of finite and infinite order.

   (ii) Find any interesting feature about B.

5. Let P = $\left\{\begin{bmatrix} a_1 & a_2 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{30} \\ a_{31} & a_{32} & \dots & a_{40} \end{bmatrix} \right|$ $a_i \in R_n(1)$; $1 \leq i \leq 40$,

+} be the MOD fuzzy matrix group.

Study questions (i) and (ii) of problem 4 for this P.

6. Let M = $\left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \\ a_5 & a_6 \\ a_7 & a_8 \\ a_9 & a_{10} \\ a_{11} & a_{12} \end{bmatrix} \right|$ $a_i \in R_n(1)$; $1 \leq i \leq 12$, +} be the

MOD fuzzy matrix group.

Study questions (i) and (ii) problem (4) for this M.

7. Let B = $\{(a_1, a_2, a_3, \ldots, a_4) \mid a_i \in R_n(1)$; $1 \leq i \leq 9, \times\}$ be the MOD fuzzy matrix semigroup.

Study questions (i) to (ix) of problem (3) for this B.

8. Let T = $\left\{\begin{bmatrix} a_1 & a_2 & \dots & a_9 \\ a_{10} & a_{11} & \dots & a_{18} \\ a_{19} & a_{20} & \dots & a_{27} \\ a_{28} & a_{29} & \dots & a_{36} \\ a_{37} & a_{38} & \dots & a_{45} \\ a_{46} & a_{47} & \dots & a_{56} \end{bmatrix}\right|$ $a_i \in R_n(1); 1 \leq i \leq 56,$

$\times_n\}$ be the MOD fuzzy matrix semigroup under the natural product.

(i) Study questions (i) to (ix) of problem (3) for this T.

(ii) Show all ideals of t are of infinite order.

9. Let L = $\left\{\begin{bmatrix} a_1 & a_2 & \dots & a_5 \\ a_6 & a_7 & \dots & a_{10} \\ a_{11} & a_{12} & \dots & a_{15} \\ a_{16} & a_{17} & \dots & a_{20} \\ a_{21} & a_{22} & \dots & a_{25} \end{bmatrix}\right|$ $a_i \in R_n(1); 1 \leq i \leq 25,$

$\times\}$ be the MOD fuzzy square matrix semigroup under the usual product.

(i) Prove L is a non commutative group.
(ii) Find atleast 2 right ideals in L which are not left Ideals.
(iii) Find 3 left ideals which are not right ideals.
(iv) Prove L has atleast ${}_{25}C_1 + {}_{25}C_2 \dots + {}_{25}C_{25}$ number of ideals which are both right and left.
(v) Can L has left zero divisors which are not right zero divisors?
(vi) Can L have S-ideals?
(vii) Can L have S-idempotents?
(viii) Can L have S-units?
(ix) Obtain any other special feature associated with L.

10. Let $M = \left\{\begin{bmatrix} a_1 \\ \vdots \\ a_{18} \end{bmatrix} \middle| \; a_i \in R_n(1);\ 1 \leq i \leq 18, \times_n\right\}$ be the MOD fuzzy column matrix semigroup.

    Study questions i to ix of problem 9.

11. Let $B = \left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 \\ a_5 & a_6 & a_7 & a_8 \\ a_9 & a_{10} & a_{11} & a_{12} \\ a_{13} & a_{14} & a_{15} & a_{16} \end{bmatrix} \middle| \; a_i \in R_n(1);\ 1 \leq i \leq 16, \times_n\right\}$ be the MOD fuzzy square matrix semigroup.

    Study questions i to ix of problem 9 for this B.

12. Obtain any important features enjoyed by MOD fuzzy matrix semigroups.

13. Show these concepts can be used in fuzzy models?

14. Find the special features enjoyed by MOD fuzzy subset $S(R_n(1))$?

15. Show $\{S(R_n(1)), +\}$ is only a MOD fuzzy subset semigroup and not a group.

16. What are the special features associated with $\{S(R_n(1)), +\}$?

17. Can $G = \{S(R_n(1)), +\}$ be a MOD fuzzy smarandache semigroup?

18. Can G have subsemigroups of finite order?

19. Let H = $\{S(R_n(1)), \times\}$ be the MOD subset fuzzy semigroup.

(i) Is H a S-semigroup?
(ii) Find ideals of H.
(iii) Can ideals of H be of finite order?
(iv) Does H have S-ideals?
(v) Can H have finite subsemigroups?
(vi) Can H have zero divisors?

20. Let B = $\{(a_1, a_2, \ldots, a_9) \mid a_i \in S(R_n(1)); 1 \leq i \leq 9, \times\}$ be the MOD subset row matrix semigroup.

Study questions i to vi of problem 19 for this B.

21. Let M = $\left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| a_i \in S(R_n(1))\ 1 \leq i \leq 6, \times_n\right\}$ be the MOD subset column matrix semigroup under natural product.

Study questions i to vi of problem 19 for this M.

22. Let V = $\left\{\begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{30} \\ a_{31} & a_{32} & \ldots & a_{40} \end{bmatrix} \middle| a_i \in S(R_n(1));\ 1 \leq i \leq 40, \times_n\right\}$ be the MOD subset matrix semigroup under natural product.

Study questions i to vi of problem 19 for this V.

23. Let M = $\left\{\begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{30} \end{bmatrix} \middle|\; a_i \in S(R_n(1));\right.$

$1 \leq i \leq 30, \times_n\}$ be the MOD subset matrix semigroup under natural product.

Study questions i to vi of problem 19 for this M.

24. Let V = $\left\{\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ a_6 & a_7 & \ldots & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & \ldots & a_{15} \\ a_{16} & a_{17} & \ldots & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & \ldots & a_{25} \end{bmatrix} \middle|\; a_i \in S(R_n(1));\right.$

$1 \leq i \leq 25, \times\}$ be the non commutative MOD fuzzy matrix semigroup.

(i) Find atleast 5 ideals of V.
(ii) Find atleast 3 right ideals which are not left ideals.

(iii) Find right zero divisors if any which are not left zero divisors.
(iv) Show V has infinite number of zero divisors.

25. Find any of the special features enjoyed by $R_n(1)[x]$.

26. Can $R_n(1)[x]$ have zero divisors?

27. Find ideals if any in $R_n(1)[x]$.

28. Study and analyse the MOD subset fuzzy polynomials $(SR_n(1))[x] = R$.

29. Can R have zero divisors?

30. Can R have S-ideals?

31. Let S = $\{S(R_n(1))[x], +\}$ be the MOD subset fuzzy polynomial semigroup.

    (i) Find all interesting properties associated with S.
    (ii) Compare S with B = $\{R_n(1)[x], +\}$.
    (iii) What are the special and distinct features enjoyed by B?

32. Let M = $\{S(R_n(1))[x], \times\}$ be the MOD subset fuzzy polynomial semigroup.

    (i) Find all interesting properties enjoyed by M.
    (ii) Compare $\{R_n(1)[x], \times\}$ with $\{S(R_n(1))[x], \times\}$.
    (iii) Prove M has infinite number of zero divisors.
    (iv) Can M have ideals of finite order?
    (v) Prove in general all subsemigroups of M are not ideals of M.
    (vi) Can M have S-idempotents?

33. Let S = $\{S(R_n(1))[x], \cup\}$ be the MOD subset fuzzy polynomial semigroup.

    (i) Study any other interesting properties associated with S.
    (ii) Prove every subset M of S can be completed into a subsemigroup.
    (iii) Can M have ideals?

34. Let M = $\{S(R_n(1))[x] \cup \{\phi\}, \cap\}$ be the MOD subset fuzzy polynomial semigroup.

    (i) Compare M with $\{R_n(1)[x], \times\}$
    (ii) Can M have ideals?

(iii) Prove every subset N of M can be made into (or completed into) a subsemigroup.
(iv) Obtain some interesting properties associated with M.

35. Let B = $\{S(R_n(1)[x])\}$ be the MOD fuzzy polynomial subsets of $R_n(1)[x]$.

(i) Obtain all the special features enjoyed by B.
(ii) Compare B with D = $S(R_n(1))[x]$.

36. Let S = $\{S(R_n(1)[x]), +\}$ be the MOD fuzzy polynomial subsets semigroup under +.

(i) Can S have finite order subsemigroup?
(ii) Can S be a S-subsemigroup?
(iii) Find atleast 2 subsemigroups of infinite order.

37. Let W = $\{S(R_n(1)[x]), \times\}$ be the MOD fuzzy polynomial subset semigroup.

(i) Can W have S-ideals?
(ii) Is W a S-semigroup?
(iii) Can W have S-zero divisors?
(iv) Prove W has infinite number of zero divisors.
(v) Can W have finite subsemigroups?
(vi) Can W have S-idempotents?
(vii) What are the special features enjoyed by W?

38. Let B = $\{S(R_n(1)[x]), \cup\}$ be the MOD fuzzy polynomial subset semigroup.

(i) Prove every element is a subsemigroup.

(ii) Prove every subset of B can be completed into a subsemigroup.
(iii) Find any special features enjoyed by B.

39. Let $D = \{S(R_n(1)[x]) \cup \{\phi\}, \cap\}$ be the MOD fuzzy polynomial subset semigroup.

    (i) Study the special features enjoyed by D.
    (ii) Prove every element in D is a subsemigroup.
    (iii) Prove every subset of D can be completed into a subsemigroup.

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INDEX

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In this book, the authors introduce the new notion of algebraic structures on the MOD planes. Clearly under the operation + the MOD intervals and MOD planes is an infinite abelian group.

However MOD planes are only semigroups under product but they are never rings as the distributive law is not true. They are only pseudo rings.

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