Florentin Smarandache[1]

[1]Department of Mathematics, The University of New Mexico,
200 College Road, Gallup, NM 87301, U.S.A.

An In-Depth Look at Quantitative Information Fusion Rules

Published in:
Florentin Smarandache & Jean Dezert (Editors)
Advances and Applications of DSmT for Information Fusion
(Collected works), Vol. II
American Research Press, Rehoboth, 2006
ISBN: 1-59973-000-6
Chapter VIII, pp. 205 - 236

Abstract: *This chapter may look like a glossary of the fusion rules and we also introduce new ones presenting their formulas and examples: Conjunctive, Disjunctive, Exclusive Disjunctive, Mixed Conjunctive-Disjunctive rules, Conditional rule, Dempster's, Yager's, Smets' TBM rule, Dubois-Prade's, Dezert-Smarandache classical and hybrid rules, Murphy's average rule, Inagaki-Lefevre-Colot-Vannoorenberghe Unified Combination rules [and, as particular cases: Iganaki's parameterized rule, Weighted Average Operator, minC (M. Daniel), and newly Proportional Conflict Redistribution rules (Smarandache-Dezert) among which PCR5 is the most exact way of redistribution of the conflicting mass to non-empty sets following the path of the conjunctive rule], Zhang's Center Combination rule, Convolutive x-Averaging, Consensus Operator (Jøsang), Cautious Rule (Smets), α-junctions rules (Smets), etc. and three new T-norm & T-conorm rules adjusted from fuzzy and neutrosophic sets to information fusion (Tchamova-Smarandache). Introducing the degree of union and degree of inclusion with respect to the cardinal of sets not with the fuzzy set point of view, besides that of intersection, many fusion rules can be improved. There are corner cases where each rule might have difficulties working or may not get an expected result. As a conclusion, since no theory neither rule fully satisfy all needed applications, the author proposes a Unification of Fusion Theories extending the power and hyper-power sets from previous theories to a Boolean algebra obtained by the closures of the frame of discernment under union, intersection, and complement of sets (for non-exclusive elements one considers a fuzzy or neutrosophic complement). And, at each application, one selects the most appropriate model, rule, and algorithm of implementation.*

The material of this chapter has been presented at NASA Langley Research Center, Hampton, Virginia, on November 5, 2004.

8.1 Introduction

Let's consider the frame of discernment $\Theta = \{\theta_1, \theta_2, \ldots, \theta_n\}$, with $n \geq 2$, and two sources of information:

$$m_1(\cdot), m_2(\cdot) : S^\Theta \to [0, 1].$$

For the simplest frame $\Theta = \{\theta_1, \theta_2\}$ one can define a mass matrix as follows:

	θ_1	θ_2	$\theta_1 \cup \theta_2$	$\theta_1 \cap \theta_2$	$\mathcal{C}\theta_1$	$\mathcal{C}\theta_2$	$\mathcal{C}(\theta_1 \cap \theta_2)$	$\emptyset$
$m_1(\cdot)$	m_{11}	m_{12}	m_{13}	m_{14}	m_{15}	m_{16}	m_{17}	m_{18}
$m_2(\cdot)$	m_{21}	m_{22}	m_{23}	m_{24}	m_{25}	m_{26}	m_{27}	m_{28}

In calculations we take into account only the focal elements, i.e. those for which $m_1(\cdot)$ or $m_2(\cdot) > 0$. In the Shafer's model one only has the first three columns of the mass matrix, corresponding to θ_1, θ_2, $\theta_1 \cup \theta_2$, while in the Dezert-Smarandache free model only the first four columns corresponding to θ_1, θ_2, $\theta_1 \cup \theta_2$, $\theta_1 \cap \theta_2$. But here we took the general case in order to include the possible complements (negations) as well.

We note the combination of these bba's, using any of the below rule "r", by

$$m_r = m_1 \otimes_r m_2.$$

All the rules below are extended from their power set $2^\Theta = (\Theta, \cup) = \{\emptyset, \theta_1, \theta_2, \theta_1 \cup \theta_2\}$, which is a set closed under union, or hyper-power set $D^\Theta = (\Theta, \cup, \cap) = \{\emptyset, \theta_1, \theta_2, \theta_1 \cup \theta_2, \theta_1 \cap \theta_2\}$ which is a distributive lattice called hyper-power set, to the super-power set $S^\Theta = (\Theta, \cup, \cap, \mathcal{C}) = \{\emptyset, \theta_1, \theta_2, \theta_1 \cup \theta_2, \theta_1 \cap \theta_2, C\theta_1, \mathcal{C}\theta_2, \mathcal{C}(\theta_1 \cap \theta_2)\}$, which is a Boolean algebra with respect to the union, intersection, and complement ($\mathcal{C}$ is the complement).

Of course, all of these can be generalized for Θ of dimension $n \geq 2$ and for any number of sources $s \geq 2$.

Similarly one defines the mass matrix, power-set, hyper-power set, and super-power set for the general frame of discernment.

A list of the main rules we have collected from various sources available in the open literature is given in the next sections.

8.2 Conjunctive Rule

If both sources of information are telling the truth, then we apply the conjunctive rule, which means consensus between them (or their common part):

$$\forall A \in S^\Theta, \text{ one has } m_1(A) = \sum_{\substack{X_1, X_2 \in S^\Theta \\ X_1 \cap X_2 = A}} m_1(X_1) m_2(X_2),$$

where the *Total Conflicting Mass* is:

$$k_{12} = \sum_{\substack{X_1, X_2 \in S^\Theta \\ X_1 \cap X_2 = \emptyset}} m_1(X_1) m_2(X_2).$$

8.3 Disjunctive Rule

If at least one source of information is telling the truth, we use the optimistic disjunctive rule proposed by Dubois and Prade in [7]:

$$m_{\cup}(\emptyset) = 0, \text{ and } \forall A \in S^{\Theta} \setminus \emptyset, \text{ one has } m_{\cup}(A) = \sum_{\substack{X_1,X_2 \in S^{\Theta} \\ X_1 \cup X_2 = A}} m_1(X_1)m_2(X_2).$$

8.4 Exclusive Disjunctive Rule

If only one source of information is telling the truth, but we don't know which one, then one uses the exclusive disjunctive rule [7] based on the fact that $X_1 \veebar X_2$ means either X_1 is true, or X_2 is true, but not both in the same time (in set theory let's use $X_1 \veebar X_2$ for exclusive disjunctive):

$$m_{\veebar}(\emptyset) = 0, \text{ and } \forall A \in S^{\Theta} \setminus \emptyset, \text{ one has } m_{\veebar}(A) = \sum_{\substack{X_1,X_2 \in S^{\Theta} \\ X_1 \veebar X_2 = A}} m_1(X_1)m_2(X_2).$$

8.5 Mixed Conjunctive-Disjunctive Rule

This is a mixture of the previous three rules in any possible way [7]. As an example, suppose we have four sources of information and we know that: either the first two are telling the truth or the third, or the fourth is telling the truth. The mixed formula becomes:

$$m_{\cap\cup}(\emptyset) = 0,$$

and

$$\forall A \in S^{\Theta} \setminus \emptyset, \text{ one has } m_{\cap\cup}(A) = \sum_{\substack{X_1,X_2,X_3,X_4 \in S^{\Theta} \\ ((X_1 \cap X_2) \cup X_3) \veebar X_4 = A}} m_1(X_1)m_2(X_2)m_3(X_3)m_4(X_4).$$

8.6 Conditioning Rule

This classical *conditioning* rule proposed by Glenn Shafer in Dempster-Shafer Theory [26] looks like the conditional probability (when dealing with Bayesian belief functions) but it is different. Shafer's conditioning rule is commonly used when there exists a bba, say $m_S(\cdot)$, such that for an hypothesis, say A, one has $m_S(A) = 1$ (i.e. when the subjective certainty of an hypothesis to occur is given by an expert). Shafer's conditioning rule consists in combining $m_S(\cdot)$ directly with another given bba for belief revision using Dempster's rule of combination. We point out that this conditioning rule could be used also whatever rule of combination is chosen in any other fusion theory dealing with belief functions. After fusioning $m_1(.)$ with $m_S(A) = 1$, the conflicting mass is transferred to non-empty sets using Dempster's rule in DST, or DSmH or PCR5 in DSmT, etc. Another family of belief conditioning rules (BCR) is proposed as a new alternative in chapter 9 of this book.

8.7 Dempster's Rule

This is the most used fusion rule in applications and this rule influenced the development of other rules. Shafer has developed the Dempster-Shafer Theory of Evidence [26] based on the model that all hypotheses in the frame of discernment are exclusive and the frame is exhaustive. Dempster's rule for two *independent* sources is given by [26]

$$m_D(\emptyset) = 0,$$

and

$$\forall A \in S^\Theta \setminus \emptyset, \text{ one has } m_D(A) = \frac{1}{1-k_{12}} \cdot \sum_{\substack{X_1,X_2 \in S^\Theta \\ X_1 \cap X_2 = A}} m_1(X_1)m_2(X_2).$$

8.8 Modified Dempster-Shafer rule (MDS)

MDS rule was introduced by Dale Fixsen and Ronald P. S. Mahler in 1997 [11] for identifying objects in a finite universe $\mathcal{U}$ containing N elements, and it merges Bayesian and Dempster-Shafer theories.

Let B and C be two bodies of evidence:$B = \{(S_1, m_1), (S_2, m_2), \ldots, (S_b, m_b)\}$ and $C = \{(T_1, n_1), (T_2, n_2), \ldots (T_c, n_c)\}$ where S_i, $1 \leq i \leq b$, T_j , $1 \leq j \leq c$, are subsets of the universe $\mathcal{U}$,and (S_i, m_i) represents for the source B the hypothesis *object is in* S_i with a belief (mass assignment) of m_i with of course $\sum_i m_i = 1$. Similarly for (T_j, n_j) for each j.

Then B and C can be fused just following Dempster's rule and one gets a new body of evidence $B \star C$. The elements of $B \star C$ are intersections of $S_i \cap T_j$ for all $i = 1, \ldots, b$ and $j = 1, \ldots, c$, giving the following masses:

$$r_{ij} = m_i n_j \frac{\alpha_{DS}(S_i, T_j)}{\alpha_{DS}(B, C)}$$

if $\alpha_{DS}(B, C) \neq 0$ (it is zero only in the total degenerate case).
The *Dempster-Shafer agreement* $\alpha_{DS}(.,.)$ is defined by [11]:

$$\alpha_{DS}(B, C) = \sum_{i=1}^{b} \sum_{j=1}^{c} m_i n_j \alpha_{DS}(S_i, T_j) \qquad \text{with} \qquad \alpha_{DS}(S, T) = \frac{\rho(S \cap T)}{\rho(S)\rho(T)}$$

where the set function $\rho(S) = 1$ if $S \neq \emptyset$ and $\rho(\emptyset) = 0$; $\alpha_{DS}(S, T) = 1$ if $S \cap T \neq \emptyset$ and zero otherwise.

The agreement between bodies of evidence is just $1-k$, where k is the conflict from Dempster-Shafer Theory. In 1986, J. Yen had proposed a similar rule, but his probability model was different from Fixsen-Mahler MDS's (see [50] for details).

8.9 Murphy's Statistical Average Rule

If we consider that the bba's are important from a statistical point of view, then one averages them as proposed by Murphy in [24]:

$$\forall A \in S^\Theta, \text{ one has } m_M(A) = \frac{1}{2}[m_1(A) + m_2(A)].$$

Or, more general, $m_{\text{mixing}}(A) = \frac{1}{2}[w_1 m_1(A) + w_2 m_2(A)]$, where w_1, w_2 are weights reflecting the reliability of sources.

8.10 Dezert-Smarandache Classic Rule (DSmC)

DSmC rule [31] is a generalization of the conjunctive rule from the power set to the hyper-power set.

$$\forall A \in S^{\Theta},\ \text{one has } m_{\text{DSmC}}(A) = \sum_{\substack{X_1, X_2 \in S^{\Theta} \\ X_1 \cap X_2 = A}} m_1(X_1) m_2(X_2).$$

It can also be extended on the Boolean algebra $(\Theta, \cup, \cap, \mathcal{C})$ in order to include the complements (or negations) of elements.

8.11 Dezert-Smarandache Hybrid Rule (DSmH)

DSmH rule [31] is an extension of the Dubois-Prade rule for the dynamic fusion. The middle sum in the below formula does not occur in Dubois-Prade's rule, and it helps in the transfer of the masses of empty sets — whose disjunctive forms are also empty — to the total ignorance.

$$m_{\text{DSmH}}(\emptyset) = 0,$$

and

$$\begin{aligned}
&\forall A \in S^{\Theta} \setminus \emptyset \text{ one has} \\
m_{\text{DSmH}}(A) = &\sum_{\substack{X_1, X_2 \in S^{\Theta} \\ X_1 \cap X_2 = A}} m_1(X_1) m_2(X_2) + \sum_{\substack{X_1, X_2 \in \emptyset \\ (A=U) \vee \{U \in \emptyset \wedge A = I\}}} m_1(X_1) m_2(X_2) \\
&+ \sum_{\substack{X_1, X_2 \in S^{\Theta} \\ X_1 \cup X_2 = A \\ X_1 \cap X_2 \in \emptyset}} m_1(X_1) m_2(X_2)
\end{aligned}$$

where all sets are in canonical form (i.e. for example the set $(A \cap B) \cap (A \cup B \cup C))$ will be replaced by its canonical form $A \cap B$), and U is the disjunctive form of $X_1 \cap X_2$ and is defined as follows:

$$\begin{aligned}
U(X) &= X \text{ if } X \text{ is a singleton,} \\
U(X_1 \cap X_2) &= U(X_1) \cup U(X_2), \text{ and} \\
U(X_1 \cup X_2) &= U(X_1) \cup U(X_2);
\end{aligned}$$

while $I = \theta_1 \cup \theta_2 \cup \cdots \cup \theta_n$ is the total ignorance.

Formally the canonical form has the properties:

i) $c(\emptyset) = \emptyset$;

ii) if A is a singleton, then $c(A) = A$;

iii) if $A \subseteq B$, then $c(A \cap B) = A$ and $c(A \cup B) = B$;

iv) the second and third properties apply for any number of sets.

8.12 Smets' TBM Rule

In the TBM (Transferable Belief model) approach, Philippe Smets [36] does not transfer the conflicting mass, but keeps it on the empty set, meaning that $m(\emptyset) > 0$ signifies that there might exist other hypotheses we don't know of in the frame of discernment (this is called an *open world*).

$$m_S(\emptyset) = k_{12} = \sum_{\substack{X_1, X_2 \in S^\Theta \\ X_1 \cap X_2 = \emptyset}} m_1(X_1) m_2(X_2).$$

and

$$\forall A \in S^\Theta \setminus \emptyset, \text{ one has } m_S(A) = \sum_{\substack{X_1, X_2 \in S^\Theta \\ X_1 \cap X_2 = A}} m_1(X_1) m_2(X_2).$$

8.13 Yager's Rule

R. Yager transfers the total conflicting mass to the total ignorance [44], i.e.

$$m_Y(\emptyset) = 0, \qquad m_Y(I) = m_1(I) m_2(I) + \sum_{\substack{X_1, X_2 \in S^\Theta \\ X_1 \cap X_2 = \emptyset}} m_1(X_1) m_2(X_2)$$

where $I =$ total ignorance, and

$$\forall A \in S^\Theta \setminus \{\emptyset, I\}, \text{ one has } m_Y(A) = \sum_{\substack{X_1, X_2 \in S^\Theta \\ X_1 \cap X_2 = A}} m_1(X_1) m_2(X_2).$$

8.14 Dubois-Prade's Rule

This rule [8] is based on the principle that if two sources are in conflict, then at least one is true, and thus transfers the conflicting mass $m(A \cap B) > 0$ to $A \cup B$.

$$m_{DP}(\emptyset) = 0,$$

and

$$\forall A \in S^\Theta \setminus \emptyset \text{ one has}$$

$$m_{DP}(A) = \sum_{\substack{X_1, X_2 \in S^\Theta \\ X_1 \cap X_2 = A}} m_1(X_1) m_2(X_2) + \sum_{\substack{X_1, X_2 \in S^\Theta \\ X_1 \cup X_2 = A \\ X_1 \cap X_2 = \emptyset}} m_1(X_1) m_2(X_2).$$

8.15 Weighted Operator (Unification of the Rules)

The Weighted Operator (WO) proposed by T. Inagaki in [15] and later by Lefevre-Colot-Vannoorenberghe in [19] is defined as follows:

$$m_{WO}(\emptyset) = w_m(\emptyset) \cdot k_{12},$$

and

$$\forall A \in S^{\Theta} \setminus \emptyset, \text{ one has } m_{WO}(A) = \sum_{\substack{X_1,X_2 \in S^{\Theta} \\ X_1 \cap X_2 = A}} m_1(X_1)m_2(X_2) + w_m(A) \cdot k_{12}$$

where $w_m(A) \in [0,1]$ for any $A \in S^{\Theta}$ and $\sum_{X \in S^{\Theta}} w_m(X) = 1$ and $w_m(A)$ are called weighting factors.

8.16 Inagaki's Unified Parameterized Combination Rule

Inagaki's Unified Parameterized Combination Rule [15] is defined by

$$\forall A \in S^{\Theta} \setminus \{\emptyset, I\}, \text{ one has } m_p^U(A) = [1 + p \cdot k_{12}] \sum_{\substack{X_1,X_2 \in S^{\Theta} \\ X_1 \cap X_2 = A}} m_1(X_1)m_2(X_2),$$

and

$$m_p^U(\emptyset) = 0, m_p^U(I) = [1 + p \cdot k_{12}] \sum_{\substack{X_1,X_2 \in S^{\Theta} \\ X_1 \cap X_2 = I}} m_1(X_1)m_2(X_2) + [1 + p \cdot k_{12} - p]k_{12}$$

where the parameter $0 \leq p \leq 1/[1 - k_{12} - m_{\cap}(I)]$, and k_{12} is the conflict.

The determination of parameter p, used for normalization, is not well justified in the literature, but may be found through experimental data, simulations, expectations [39]. The greater is the parameter p, the greater is the change to the evidence.

8.17 The Adaptive Combination Rule (ACR)

Mihai Cristian Florea, Anne-Laure Jousselme, Dominic Grenier and Eloi Bossé propose a new class of combination rules for the evidence theory as a mixing between the disjunctive (p) and conjunctive (q) rules [12, 13]. The adaptive combination rule (ACR) between m_1 and m_2 is defined by $(m_1 \diamond m_2)(\emptyset) = 0$ and :

$$(m_1 \diamond m_2)(A) = \alpha(k)p(A) + \beta(k)q(A) \ , \qquad \forall A \subseteq \Theta, A \neq \emptyset \tag{8.1}$$

Here, α and β are functions of the conflict $k = q(\varnothing)$ from $[0,1]$ to $[0,+\infty[$. The ACR may be expressed according only to the function β (because of the condition $\sum_{A \subseteq \Theta}(m_1 \diamond m_2)(A) = 1$) as follows:

$$(m_1 \diamond m_2)(A) = [1 - (1-k)\beta(k)]p(A) + \beta(k)q(A) \ , \qquad \forall A \subseteq \Theta, A \neq \emptyset \tag{8.2}$$

and $(m_1 \diamond m_2)(\varnothing) = 0$ where β is any function such that $\beta : [0,1] \to [0,+\infty[$.

In the general case α and β are functions of k with no particular constraint. However, a desirable behaviour of the ACR is that it should act more like the disjunctive rule p whenever k is close to 1 (i.e. at least one source is unreliable), while it should act more like the conjunctive rule q, if k is close to 0 (i.e. both sources are reliable). This amounts to add three conditions on the general formulation:

(C1) α is an increasing function with $\alpha(0) = 0$ and $\alpha(1) = 1$;

(C2) β is a decreasing function with $\beta(0) = 1$ and $\beta(1) = 0$.

(C3) $\alpha(k) = 1 - (1-k)\beta(k)$

In particular, when $k = 0$ the sources are in total agreement and $(m_1 \diamond m_2)(A) = p(A), \forall A \subseteq \Theta$, the conjunction represents the combined evidence, while when $k = 1$ the sources are in total conflict and $(m_1 \diamond m_2)(A) = q(A), \forall A \subseteq \Theta$, the disjunction is the best choice considering that one of them is wrong.

Note that the three conditions above are dependent and (C1) can be removed, since it is a consequence of the (C2) and (C3). The particular case of the adaptive combination rule can be stated as Equation (8.2), $\forall A \subseteq \Theta, A \neq \varnothing$ and $m(\varnothing) = 0$, where β is any decreasing function such that $\beta : [0,1] \to [0,1]$ and $\beta(0) = 1$ and $\beta(1) = 0$.

8.18 The Weighted Average Operator (WAO)

The Weighted Average Operator (WAO) for two sources proposed in [17] consists in first, applying the conjunctive rule to the bba's $m_1(\cdot)$ and $m_2(\cdot)$ and second, redistribute the total conflicting mass k_{12} to all nonempty sets in S^Θ proportionally with their mass averages, i.e. for the set, say A, proportionally with the weighting factor:

$$w_{JDV}(A, m_1, m_2) = \frac{1}{2}(m_1(A) + m_2(A)).$$

The authors do not give an analytical formula for it. WAO does not work in degenerate cases as shown in chapter 1.

8.19 The Ordered Weighted Average operator (OWA)

It was introduced by Ronald R. Yager [46, 48]. The OWA of dimension n is defined as $F : R^n \mapsto R$ such that

$$F(a_1, a_2, \dots, a_n) = \sum_{j=1}^{n} w_j b_j$$

where $b_1 \geq b_2 \dots \geq b_n$ and the weights $w_j \in [0,1]$ with $\sum_{j=1}^{n} w_j = 1$.

OWA satisfies the following properties:
a) Symmetry: For any permutation Π, one has $F(a_{\Pi(1)}, a_{\Pi(2),\dots,a_{\Pi(n)}}) = F(a_1, a_2, \dots, a_n)$.
b) Monotonicity: If $\forall j$, $a_j \geq d_j$ then $F(a_1, a_2, \dots, a_n) \geq F(d_1, d_2, \dots, d_n)$.
c) Boundedness: $\min_j(a_j) \leq F(a_1, a_2, \dots, a_n) \leq \max_j(a_j)$.
d) Idempotency: If $\forall j$, $a_j = a$, then $F(a_1, a_2, \dots, a_n) = F(a, a, \dots, a) = a$.

A measure associated with this operator with weighting vector $W = (w_1, w_2, \dots, w_n)$ is the Attitude-Character (AC) defined as: $AC(W) = \sum_{j=1}^{n} w_j \frac{n-j}{n-1}$.

8.20 The Power Average Operator (PAO)

It was introduced by Ronald Yager in [47] in order to allow values being aggregated to support and reinforce each other. This operator is defined by:

$$PAO(a_1, a_2, \dots, a_n) = \frac{\sum_{i=1}^{n}(1 + T(a_i)a_i)}{\sum_{i=1}^{n}(1 + T(a-i))}$$

where $T(a_i) = \sum_{j=1 j\neq i}^{n} \sup(a_i, a_j)$ and $\sup(a, b)$ denotes the support for a from b which satisfies the properties:
a) $\sup(a, b) \in [0, 1]$;
b) $\sup(a, b) = \sup(b, a)$;
c) $\sup(a, b) \geq \sup(x, y)$ if $|a - b| < |x - y|$.

8.21 Proportional Conflict Redistribution Rules (PCR)

8.21.1 PCR1 Fusion rule

In 2004, F. Smarandache and J. Dezert independently developed a *Proportional Conflict Redistribution Rule (PCR1)*, which similarly consists in first, applying the conjunctive rule to the bba's $m_1(\cdot)$ and $m_2(\cdot)$ and second, redistribute the total conflicting mass k_{12} to all nonempty sets in S^Θ proportionally with their nonzero mass sum, i.e. for the set, say A, proportionally with the weighting factor:

$$w_{SD}(A, m_1, m_2) = m_1(A) + m_2(A) \neq 0.$$

The analytical formula for PCR1, non-degenerate and degenerate cases, is:

$$m_{PCR1}(\emptyset) = 0,$$

and

$$\forall A \in S^\Theta \setminus \emptyset, \text{ one has } m_{PCR1}(A) = \sum_{\substack{X_1, X_2 \in S^\Theta \\ X_1 \cap X_2 = A}} m_1(X_1) m_2(X_2) + \frac{c_{12}(A)}{d_{12}} \cdot k_{12},$$

where $c_{12}(A)$ is the sum of masses corresponding to the set A, i.e. $c_{12}(A) = m_1(A) + m_2(A) \neq 0$, d_{12} is the sum of nonzero masses of all nonempty sets in S^Θ assigned by the sources $m_1(\cdot)$ and $m_2(\cdot)$ [in many cases $d_{12} = 2$, but in degenerate cases it can be less], and k_{12} is the total conflicting mass.

Philippe Smets pointed out that PCR1 gives the same result as the WAO for non-degenerate cases, but PCR1 extends actually WAO, since PCR1 works also for the degenerate cases when all column sums of all non-empty sets are zero because in such cases, the conflicting mass is transferred to the non-empty disjunctive form of all non-empty sets together; when this disjunctive form happens to be empty, then one can consider an open world (i.e. the frame of discernment might contain new hypotheses) and thus all conflicting mass is transferred to the empty set.

For the cases of the combination of only one non-vacuous belief assignment $m_1(\cdot)$ with the vacuous belief assignment[1] $m_v(\cdot)$ where $m_1(\cdot)$ has mass assigned to an empty element, say $m_1(\cdot) > 0$ as in Smets' TBM, or as in DSmT dynamic fusion where one finds out that a previous non-empty element A, whose mass $m_1(A) > 0$, becomes empty after a certain time, then this mass of an empty set has to be transferred to other elements using PCR1, but for such case $[m_1 \otimes m_v](\cdot)$ is different from $m_1(\cdot)$. This severe draw-back of WAO and PCR1 forces us to develop more sophisticated PCR rules satisfying the neutrality property of VBA with better redistributions of the conflicting information.

[1]The VBA (vacuous belief assignment) is the bba m_v(total ignorance) $= 1$.

8.21.2 PCR2-PCR4 Fusion rules

F. Smarandache and J. Dezert then developed more improved versions of *Proportional Conflict Redistribution Rule (PCR2-4)*. A detailed presentation of these rules can be found in Chapter 1 of this book.

In the *PCR2* fusion rule, the total conflicting mass k_{12} is redistributed only to the non-empty sets involved in the conflict (not to all non-empty sets as in WAO and PCR1) proportionally with respect to their corresponding non-empty column sum in the mass matrix. The redistribution is then more exact (accurate) than in PCR1 and WAO. A nice feature of PCR2 is the preservation of the neutral impact of the VBA and of course its ability to deal with all cases/models.

$$m_{PCR2}(\emptyset) = 0,$$

and $\forall A \in S^\Theta \setminus \emptyset$ and A involved in the conflict, one has

$$m_{PCR2}(A) = \sum_{\substack{X_1,X_2 \in S^\Theta \\ X_1 \cap X_2 = A}} m_1(X_1)m_2(X_2) + \frac{c_{12}(A)}{e_{12}} \cdot k_{12},$$

while for a set $B \in S^\Theta \setminus \emptyset$ not involved in the conflict one has:

$$m_{PCR2}(B) = \sum_{\substack{X_1,X_2 \in S^\Theta \\ X_1 \cap X_2 = B}} m_1(X_1)m_2(X_2),$$

where $c_{12}(A)$ is the non-zero sum of the column of X in the mass matrix, i.e. $c_{12}(A) = m_1(A) + m_2(A) \neq 0$, k_{12} is the total conflicting mass, and e_{12} is the sum of all non-zero column sums of all non-empty sets only involved in the conflict (in many cases $e_{12} = 2$, but in some degenerate cases it can be less). In the degenerate case when all column sums of all non-empty sets involved in the conflict are zero, then the conflicting mass is transferred to the non-empty disjunctive form of all sets together which were involved in the conflict. But if this disjunctive form happens to be empty, then one considers an open world (i.e. the frame of discernment might contain new hypotheses) and thus all conflicting mass is transferred to the empty set.

A non-empty set $X \in S^\Theta$ is considered involved in the conflict if there exists another set $Y \in S^\Theta$ such that $X \cap Y = 0$ and $m_{12}(X \cap Y) > 0$. This definition can be generalized for $s \geq 2$ sources.

PCR3 transfers partial conflicting masses, instead of the total conflicting mass. If an intersection is empty, say $A \cap B = \emptyset$, then the mass $m(A \cap B) > 0$ of the partial conflict is transferred to the non-empty sets A and B proportionally with respect to the non-zero sum of masses assigned to A and respectively to B by the bba's $m_1(\cdot)$ and $m_2(\cdot)$. The PCR3 rule works if at least one set between A and B is non-empty and its column sum is non-zero. When both sets A and B are empty, or both corresponding column sums of the mass matrix are zero, or only one set is non-empty and its column sum is zero, then the mass $m(A \cap B)$ is transferred to the non-empty disjunctive form $u(A) \cup u(B)$ [which is defined as follows: $u(A) = A$ if A is a singleton, $u(A \cap B) = u(A \cup B) = u(A) \cup u(B)$]; if this disjunctive form is empty then $m(A \cap B)$ is transferred to the non-empty total ignorance in a closed world approach or to the empty set if one prefers to adopt the Smets' open world approach; but if even the total ignorance is empty (a completely degenerate case) then one considers an open world (i.e. new hypotheses might be in the frame of discernment) and the conflicting mass is transferred to the empty set, which means that the original problem has no solution in the close world initially chosen for the problem.

$$m_{PCR3}(\emptyset) = 0,$$

and $\forall A \in S^\Theta \setminus \emptyset$, one has

$$\begin{aligned} m_{PCR3}(A) = &\sum_{\substack{X_1,X_2\in S^\Theta\\ X_1\cap X_2=A}} m_1(X_1)m_2(X_2) + c_{12}(A) \\ &\cdot \sum_{\substack{X\in S^\Theta\setminus A\\ X\cap A=\emptyset}} \frac{m_1(A)m_2(X)+m_2(A)m_1(X)}{c_{12}(A)+c_{12}(X)} \\ &+ \sum_{\substack{X_1,X_2\in S^\Theta\setminus A\\ X_1\cap X_2=\emptyset\\ u(X_1)\cup u(X_2)=A}} [m_1(X_1)m_2(X_2)+m_1(X_2)m_2(X_1)] \\ &+ \Psi_\Theta(A)\cdot \sum_{\substack{X_1,X_2\in S^\Theta\setminus A\\ X_1\cap X_2=\emptyset\\ u(X_1)=u(X_2)=A}} [m_1(X_1)m_2(X_2)+m_2(X_1)m_1(X_2)] \end{aligned}$$

where $c_{12}(A)$ is the non-zero sum of the mass matrix column corresponding to the set A, and the total ignorance characteristic function $\Psi_\Theta(A) = 1$ if A is the total ignorance, and 0 otherwise.

The PCR4 fusion rule improves Milan Daniel's minC rule [3–5]. After applying the conjunctive rule, Daniel uses the proportionalization with respect to the results of the conjunctive rule, and not with respect to the masses assigned to each nonempty set by the sources of information as done in PCR1-3 or the next PCR5. PCR4 also uses the proportionalization with respect to the results of the conjunctive rule, but with PCR4 the conflicting mass $m_{12}(A\cap B) > 0$ when $A\cap B = \emptyset$ is distributed to A and B only because only A and B were involved in the conflict $\{A\cup B$ was not involved in the conflict since $m_{12}(A\cap B) = m_1(A)m_2(B)+m_2(A)m_1(B)\}$, while minC [both its versions a) and b)] redistributes the conflicting mass $m_{12}(A\cap B)$ to A, B, and $A\cup B$. Also, for the mixed sets such as $C\cap(A\cup B) = \emptyset$ the conflicting mass $m_{12}(C\cap(A\cup B)) > 0$ is distributed to C and $A\cup B$ because only them were involved in the conflict by PCR4, while minC version a) redistributes $m_{12}(C\cap(A\cup B))$ to C, $A\cup B$, $C\cup A\cup B$ and minC version b) redistributes $m_{12}(C\cap(A\cup B))$ even worse to A, B, C, $A\cup B$, $A\cup C$, $B\cup C$, $A\cup B\cup C$. The PCR5 formula for the fusion of two sources is given by

$$m_{PCR4}(\emptyset) = 0,$$

and $\forall A \in S^\Theta \setminus \emptyset$, one has

$$m_{PCR4}(A) = m_{12}(A) + \sum_{\substack{X\in S^\Theta\setminus A\\ X\cap A=\emptyset}} m_{12}(X)\frac{m_{12}(A\cap X)}{m_{12}(A)+m_{12}(X)},$$

where $m_{12}(\cdot)$ is the conjunctive rule, and all denominators $m_{12}(A)+m_{12}(X) \neq 0$; (if a denominator corresponding to some X is zero, the fraction it belongs to is discarded and the mass $m_{12}(A\cap X)$ is transferred to A and X using PCR3.

8.21.3 PCR5 Fusion Rule

PCR5 fusion rule is the most mathematically exact form of redistribution of the conflicting mass to non-empty sets which follows backwards the tracks of the conjunctive rule formula. But it is the most difficult to implement. In order to better understand it, let's start with some examples:

- **Example 1**:

	A	B	$A \cup B$
m_1	0.6	0	0.4
m_2	0	0.3	0.7

The conjunctive rule yields:

m_{12}	0.42	0.12	0.28

and the conflicting mass $k_{12} = 0.18$.

Only A and B were involved in the conflict,

$$k_{12} = m_{12}(A \cap B) = m_1(A)m_2(B) + m_2(A)m_1(B) = m_1(A)m_2(B) = 0.6 \cdot 0.3 = 0.18.$$

Therefore, 0.18 should be distributed to A and B proportionally with respect to 0.6 and 0.3 {i.e. the masses assigned to A and B by the sources $m_1(\cdot)$ and $m_2(\cdot)$} respectively. Let x be the conflicting mass to be redistributed to A and y the conflicting mass to be redistributed to B (out of 0.18), then:

$$\frac{x}{0.6} = \frac{y}{0.3} = \frac{x+y}{0.6+0.3} = \frac{0.18}{0.9} = 0.2,$$

whence $x = 0.6 \cdot 0.2 = 0.12$, $y = 0.3 \cdot 0.2 = 0.06$, which is normal since 0.6 is twice bigger than 0.3. Thus:

$$\begin{aligned} m_{PCR4}(A) &= 0.42 + 0.12 = 0.54, \\ m_{PCR4}(B) &= 0.12 + 0.06 = 0.18, \\ m_{PCR4}(A \cup B) &= 0.28 + 0 = 0.28. \end{aligned}$$

This result is the same as PCR2-3.

- **Example 2**:

Let's modify a little the previous example and have the mass matrix

	A	B	$A \cup B$
m_1	0.6	0	0.4
m_2	0.2	0.3	0.5

The conjunctive rule yields:

$$m_{12} \quad 0.50 \quad 0.12 \quad 0.20$$

and the conflicting mass $k_{12} = 0.18$.

The conflict k_{12} is the same as in previous example, which means that $m_2(A) = 0.2$ did not have any impact on the conflict; why?, because $m_1(B) = 0$.

A and B were involved in the conflict, $A \cup B$ is not, hence only A and B deserve a part of the conflict, $A \cup B$ does not deserve.

With PCR5 one redistributes the conflicting mass 0.18 to A and B proportionally with the masses $m_1(A)$ and $m_2(B)$ respectively, i.e. identically as above. The mass $m_2(A) = 0.2$ is not considered to the weighting factors of redistribution since it did not increase or decrease the conflicting mass. One obtains $x = 0.12$ and $y = 0.06$, which added to the previous masses yields:

$$\begin{aligned}
m_{PCR4}(A) &= 0.50 + 0.12 = 0.62,\\
m_{PCR4}(B) &= 0.12 + 0.06 = 0.18,\\
m_{PCR4}(A \cup B) &= 0.20.
\end{aligned}$$

This result is different from all PCR1-4.

- **Example 3**:

  Let's modify a little the previous example and have the mass matrix

	A	B	$A \cup B$
m_1	0.6	0.3	0.1
m_2	0.2	0.3	0.5

  The conjunctive rule yields:

$$m_{12} \quad 0.44 \quad 0.27 \quad 0.05$$

  and the conflicting mass

$$k_{12} = m_{12}(A \cap B) = m_1(A)m_2(B) + m_2(A)m_1(B) = 0.6 \cdot 0.3 + 0.2 \cdot 0.3 = 0.18 + 0.06 = 0.24.$$

  Now the conflict is different from the previous two examples, because $m_2(A)$ and $m_1(B)$ are both non-null. Then the partial conflict 0.18 should be redistributed to A and B proportionally to 0.6 and 0.3 respectively (as done in previous examples, and we got $x1 = 0.12$ and $y1 = 0.06$), while 0.06 should be redistributed to A and B proportionally to 0.2 and 0.3 respectively.

  For the second redistribution one similarly calculate the proportions:

$$\frac{x2}{0.2} = \frac{y2}{0.3} = \frac{x2 + y2}{0.2 + 0.3} = \frac{0.06}{0.5} = 0.12,$$

  whence $x = 0.2 \cdot 0.12 = 0.024$, $y = 0.3 \cdot 0.12 = 0.036$. Thus:

$$\begin{aligned}
m_{PCR4}(A) &= 0.44 + 0.12 + 0.024 = 0.584,\\
m_{PCR4}(B) &= 0.27 + 0.06 + 0.036 = 0.366,\\
m_{PCR4}(A \cup B) &= 0.05 + 0 = 0.050.
\end{aligned}$$

  This result is different from PCR1-4.

The formula of the PCR5 fusion rule for two sources is given by [35]:

$$m_{PCR5}(\emptyset) = 0,$$

and $\forall A \in S^\Theta \setminus \emptyset$, one has

$$m_{PCR5}(A) = m_{12}(A) + \sum_{\substack{X \in S^\Theta \setminus \{A\} \\ X \cap A = \emptyset}} \left[\frac{m_1(A)^2 \cdot m_2(X)}{m_1(A) + m_2(X)} + \frac{m_2(A)^2 \cdot m_1(X)}{m_2(A) + m_1(X)} \right],$$

where $m_{12}(\cdot)$ is the conjunctive rule, and all denominators are different from zero; if a denominator is zero, the fraction it belongs to is discarded.

The general PCR5 formula for $s \geq 2$ sources is given by (see Chapter 1)

$$m_{PCR5}(\emptyset) = 0,$$

and $\forall A \in S^\Theta \setminus \emptyset$ by

$$m_{PCR5}(A) = m_{12\ldots s}(A) + \sum_{\substack{2 \leq t \leq s \\ 1 \leq r_1, \ldots, r_t \leq s \\ 1 \leq r_1 < r_2 < \ldots < r_{t-1} < (r_t = s)}} \sum_{\substack{X_{j_2}, \ldots, X_{j_t} \in S^\Theta \setminus \{A\} \\ \{j_2, \ldots, j_t\} \in \mathcal{P}^{t-1}(\{1, \ldots, n\}) \\ A \cap X_{j_2} \cap \ldots \cap X_{j_s} = \emptyset \\ \{i_1, \ldots, i_s\} \in \mathcal{P}^s(\{1, \ldots, s\})}} \frac{(\prod_{k_1=1}^{r_1} m_{i_{k_1}}(A)^2) \cdot [\prod_{l=2}^{t} (\prod_{k_l = r_{l-1}+1}^{r_l} m_{i_{k_l}}(X_{j_l})]}{(\prod_{k_1=1}^{r_1} m_{i_{k_1}}(A)) + [\sum_{l=2}^{t} (\prod_{k_l = r_{l-1}+1}^{r_l} m_{i_{k_l}}(X_{j_l})]},$$

where i, j, k, r, s and t are integers. $m_{12\ldots s}(A)$ corresponds to the conjunctive consensus on A between s sources and where all denominators are different from zero. If a denominator is zero, that fraction is discarded; $\mathcal{P}^k(\{1, 2, \ldots, n\})$ is the set of all subsets of k elements from $\{1, 2, \ldots, n\}$ (permutations of n elements taken by k), the order of elements doesn't count.

8.21.4 PCR6 Fusion Rule

PCR6 was developed by A. Martin and C. Osswald in 2006 (see Chapters [22] and [23] for more details and applications of this new rule) and it is an alternative of PCR5 for the general case when the number of sources to combine become greater than two (i.e. $s \geq 3$). PCR6 does not follow back on the track of conjunctive rule as PCR5 general formula does, but it gets better intuitive results. For $s = 2$ PCR5 and PCR6 coincide. The general formula for PCR6[2] when extended to super-power set S^Θ is:

$$m_{PCR6}(\emptyset) = 0,$$

and $\forall A \in S^\Theta \setminus \emptyset$

[2]Two extensions of PCR6 (i.e. PCR6f and PCR6g) are also proposed by A. Martin and C. Osswald in [22].

$$m_{\mathrm{PCR6}}(A) = m_{12\ldots s}(A) + \sum_{i=1}^{s} m_i(A)^2 \sum_{\substack{\bigcap_{k=1}^{s-1} Y_{\sigma_i(k)} \cap A \equiv \emptyset \\ (Y_{\sigma_i(1)},\ldots,Y_{\sigma_i(s-1)}) \in (S^\Theta)^{s-1}}} \left(\frac{\prod_{j=1}^{s-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)})}{m_i(A) + \sum_{j=1}^{s-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)})} \right),$$

with $m_i(A) + \sum_{j=1}^{s-1} m_{\sigma_i(j)}(Y_{\sigma_i(j)}) \neq 0$ and where $m_{12\ldots s}(.)$ is the conjunctive consensus rule and σ_i counts from 1 to s avoiding i, i.e.:

$$\begin{cases} \sigma_i(j) = j & \text{if } j < i, \\ \sigma_i(j) = j+1 & \text{if } j \geq i, \end{cases}$$

8.22 The minC Rule

The minC rule (minimum conflict rule) proposed by M. Daniel in [3–5] improves Dempster's rule since the distribution of the conflicting mass is done from each partial conflicting mass to the subsets of the sets involved in partial conflict proportionally with respect to the results of the conjunctive rule results for each such subset. It goes by types of conflicts. The author did not provide an analytical formula for this rule in his previous publications but only in Chapter 4 of this volume. minC rule is commutative, associative, and non-idempotent.

Let $m_{12}(X \cap Y) > 0$ be a conflicting mass, where $X \cap Y = \emptyset$, and X, Y may be singletons or mixed sets (i.e. unions or intersections of singletons).

minC has two versions, minC a) and minC b), which differs from the way the redistribution is done: either to the subsets X, Y, and $X \cup Y$ in version a), or to all subsets of $P(u(X) \cup u(Y))$ in version b).

One applies the conjunctive rule, and then the partial conflict, say $m_{12}(A \cap B)$, when $A \cap B = \emptyset$, is redistributed to A, B, $A \cup B$ proportionally to the masses $m_{12}(A)$, $m_{12}(B)$, and $m_{12}(A \cup B)$ respectively in both versions a) and b). PCR4 redistributes the conflicting mass to A and B since only them were involved in the conflict.

But for a mixed set, as shown above, say $C \cap (A \cup B) = \emptyset$, the conflicting mass $m_{12}(C \cap (A \cup B)) > 0$ is distributed by PCR4 to C and $A \cup B$ because only them were involved in the conflict, while the minC version a) redistributes $m_{12}(C \cap (A \cup B))$ to C, $A \cup B$, $C \cup A \cup B$, and minC version b) redistributes $m_{12}(C \cap (A \cup B))$ even worse to A, B, C, $A \cup B$, $A \cup C$, $B \cup C$, $A \cup B \cup C$.

Another example is that the mass $m_{12}(A \cap B \cap C)) > 0$, when $A \cap B \cap C = \emptyset$, is redistributed in both versions minC a) and minC b) to A, B, C, $A \cup B$, $A \cup C$, $B \cup C$, $A \cup B \cup C$.

When the conjunctive rule results are zero for all the nonempty sets that are redistributed conflicting masses, the conflicting mass is averaged to each such set.

8.23 The Consensus Operator

Consensus Operator (CO) proposed by A. Jøsang in [16] is defined only on binary frames of discernment. CO doesn't work on non-exclusive elements (i.e. on models with nonempty intersections of sets).

On the frame $\Theta = \{\theta_1, \theta_2\}$ of exclusive elements, θ_2 is considered the complement/negation of θ_1.

If the frame of discernment has more than two elements, then by a simple or normal coarsening it is possible to derive a binary frame containing any element A and its complement $\mathcal{C}(A)$. Let $m(\cdot)$ be a bba on a (coarsened) frame $\Theta = \{A, \mathcal{C}(A)\}$, then one defines an opinion resulted from this bba is:

$$w_A = (b_A, d_A, u_A, \alpha_A),$$

where $b_A = m(A)$ is the belief of A, $d_A = m(\mathcal{C}(A))$ is the disbelief of A, $u_A = m(A \cup \mathcal{C}(A))$ is the uncertainty of A, and α_A represents the atomicity of A. Of course $b_A + d_A + u_A = 1$, for $A \neq \emptyset$.

The relative atomicity expresses information about the relative size of the state space (i.e. the frame of discernment). For every operator, the relative atomicity of the output belief is computed as a function of the input belief operands. The relative atomicity of the input operands is determined by the state space circumstances, or by a previous operation in case that operation's output is used as input operand. The relative atomicity itself can also be uncertain, and that's what's called state space uncertainty. Possibly the state space uncertainty is a neglected problem in belief theory. It relates to Smets' open world, and to DSm paradoxical world. In fact, the open world breaks with the "exhaustive" assumption, and the paradoxical world breaks with the "exclusive" assumption of classic belief theory.

CO is commutative, associative, and non-idempotent.

Having two experts with opinions on the same element A,

$$\begin{aligned} w_{1A} &= (b_{1A}, d_{1A}, u_{1A}, \alpha_{1A}) \text{ and} \\ w_{2A} &= (b_{2A}, d_{2A}, u_{2A}, \alpha_{2A}), \text{ one first computes} \\ k &= u_{1A} + u_{2A} - u_{1A} \cdot u_{2A}. \end{aligned}$$

Let's note by $b_{12A} = (b_{12A}, d_{12A}, u_{12A}, \alpha_{12A})$ the consensus opinion between w_{1A} and w_{2A}. Then:

a) for $k \neq 0$ one has:

$$\begin{aligned} b_{12A} &= (b_{1A} \cdot u_{2A} + b_{2A} \cdot u_{1A})/k \\ d_{12A} &= (d_{1A} \cdot u_{2A} + d_{2A} \cdot u_{1A})/k \\ u_{12A} &= (u_{1A} \cdot u_{2A})/k \\ \alpha_{12A} &= \frac{\alpha_{1A} u_{2A} + \alpha_{2A} u_{1A} - (\alpha_{1A} + \alpha_{2A}) u_{1A} u_{2A}}{u_{1A} + u_{2A} - 2u_{1A} u_{2A}} \end{aligned}$$

b) for $k = 0$ one has:

$$\begin{aligned} b_{12A} &= (\gamma_{12A} \cdot b_{1A} + b_{2A})/(\gamma_{12A} + 1) \\ d_{12A} &= (\gamma_{12A} \cdot d_{1A} + d_{2A})/(\gamma_{12A} + 1) \\ u_{12A} &= 0 \\ \alpha_{12A} &= (\gamma_{12A} \cdot \alpha_{1A} + \alpha_{2A})/(\gamma_{12A} + 1) \end{aligned}$$

where $\gamma_{12A} = u_{2A}/u_{1A}$ represents the relative dogmatism between opinions b_{1A} and b_{2A}.

The formulas are not justified, and there is not a well-defined method for computing the relative atomicity of an element when a bba is known.

For frames of discernment of size greater than n, or with many sources, or in the open world it is hard to implement CO.

A bba $m(\cdot)$ is called *Bayesian* on the frame $\Theta = \{\theta_1, \theta_2\}$ of exclusive elements if $m(\theta_1 \cup \theta_2) = 0$, otherwise it is called *non Bayesian*.

If one bba is Bayesian, say $m_1(\cdot)$, and another is not, say $m_2(\cdot)$, then the non Bayesian bba is ignored! See below $m_{CO}(\cdot) = m_1(\cdot)$:

Example

	A	B	$A \cup B$
m_1	0.3	0.7	0.0
m_2	0.8	0.1	0.1
m_{CO}	0.3	0.7	0.0

Because

	b_A	d_A	u_A	α_A
m_{1A}	0.3	0.7	0.0	0.5
m_{2A}	0.8	0.1	0.1	0.5

$\alpha_{1A} = \alpha_{2A} = \frac{|A \cap \Theta|}{|\Theta|} = 0.5$, where $|X|$ means the cardinal of X, whence $\alpha_{12A} = 0.5$.

Similarly one computes the opinion on B, because:

	b_B	d_B	u_B	α_B
m_{1B}	0.7	0.3	0.0	0.5
m_{2B}	0.1	0.8	0.1	0.5

If both bba's are Bayseian, then one uses their arithmetic mean.

8.24 Zhang's Center Combination Rule

The Center Combination Rule proposed by L. Zhang in [54] is given by

$$\forall A \in S^\Theta, \text{ one has } m_Z(A) = k \cdot \sum_{\substack{X_1, X_2 \in S^\Theta \\ X_1 \cap X_2 = A}} \frac{|X_1 \cap X_2|}{|X_1| \cdot |X_2|} m_1(X_1) m_2(X_2).$$

where k is a renormalization factor, $|X|$ is the cardinal of the set X, and

$$r(X_1, X_2) = \frac{|X_1 \cap X_2|}{|X_1| \cdot |X_2|}$$

represents the degree (measure) of intersection of the sets X_1 and X_2.

In Dempster's approach the degree of intersection was assumed to be 1.

The degree of intersection could be defined in many ways, for example

$$r(X_1, X_2) = \frac{|X_1 \cap X_2|}{|X_1 \cup X_2|}$$

could be better defined this way since if the intersection is empty the degree of intersection is zero, while for the maximum intersection, i.e. when $X_1 = X_2$, the degree of intersection is 1.

One can attach the $r(X_1, X_2)$ to many fusion rules.

8.25 The Convolutive x-Averaging

The Convolutive x-Averaging proposed by Ferson-Kreinovich in [10] is defined as

$$\forall A \in S^{\Theta}, \text{ one has } m_X(A) = \sum_{\substack{X_1, X_2 \in S^{\Theta} \\ (X_1+X_2)/2=A}} m_1(X_1)m_2(X_2)$$

This rule works for hypotheses defined as subsets of the set of real numbers.

8.26 The α-junctions Rules

The α-junctions rules [37] are generalizations of the above Conjunctive and Disjunctive Rules, and they are parameterized with respect to $\alpha \in [0, 1]$. Philippe Smets finds the rules for the elementary frame of discernment Θ with two hypotheses, using a matrix operator $\mathbf{K}_X$, for each $X \in \{\emptyset, A, B, A \cup B\}$ and shows that it is possible to extend them by iteration to larger frames of discernment. These rules are more theoretical and hard to apply.

8.27 The Cautious Rule

The Cautious Rule[3] has been proposed by Philippe Smets in 2000 and is just theoretical. Also, Smets does not provide a formula or a method for calculating this rule. He states [38] this *Theorem*:

Let m_1, m_2 be two bba's, and q_1, q_2 their corresponding commonality functions, and $SP(m_1)$, $SP(m_2)$ the set of specializations of m_1 and m_2 respectively. Then the hyper-cautious combination rule

$$m_{1\circledast 2} = \min\{m \mid m \in SP(m_1) \cap SP(m_2)\},$$

and the commonality of $m_{1\circledast 2}$ is q_{12} where $q_{12}(A) = \min\{q_1(A), q_2(A)\}$.

We recall that the *commonality function* of a bba $m(\cdot)$ is $q : S^{\Theta} \to [0, 1]$ such that:

$$q(A) = \sum_{\substack{X \in S^{\Theta} \\ X \supseteq A}} m(X) \text{ for all } A \in S^{\Theta}.$$

Now a few words about the least commitment and specialization.

a) *Least Commitment, or Minimum Principle*, means to assign a missing mass of a bba or to transfer a conflicting mass to the least specific element in the frame of discernment (in most of the cases to the partial ignorances or to the total ignorance). "The Principle of Minimal Commitment consists in selecting the least committed belief function in a set of equally justified belief functions. This selection procedure does not always lead to a unique solution in which case extra requirements are added. The principle formalizes the idea that one should never give more support than justified to any subset of Ω. It satisfies a form of skepticism, of a commitment, of conservatism in the allocation of our belief. In its spirit, it is not far from what the probabilists try to achieve with the maximum entropy principle." [Philippe Smets]

[3] More details about this rule can be found in [6].

b) About *specialization* [18]:

Suppose at time t_o one has the evidence $m_0(\cdot)$ which gives us the value of an hypothesis A as $m_0(A)$. When a new evidence $m_1(\cdot)$ comes in at time $t_1 > t_0$, then $m_0(A)$ might flow down to the subsets of A therefore towards a more specific information. The impact of a new bba might result in a redistribution of the initial mass of A, $m_0(A)$, towards its more specific subsets. Thus $m_1(\cdot)$ is called a specialization of $m_0(\cdot)$.

8.28 Other fusion rules

Yen's rule is related to fuzzy set, while the p-boxes method to upper and lower probabilities (neutrosophic probability is a generalization of upper and lower probability) - see Sandia Tech. Rep.

8.29 Fusion rules based on T-norm and T-conorm

These rules proposed by Tchamova, Dezert and Smarandache in [40] started from the T-norm and T-conorm respectively in fuzzy and neutrosophic logics, where the "and" logic operator $\wedge$ corresponds in fusion to the conjunctive rule, while the "or" logic operator $\vee$ corresponds to the disjunctive rule. While the logic operators deal with degrees of truth and degrees of falsehood, the fusion rules deal with degrees of belief and degrees of disbelief of hypotheses.

A **T-norm** is a function $T_n : [0,1]^2 \to [0,1]$, defined in fuzzy/neutrosophic set theory and fuzzy/neutrosophic logic to represent the "intersection" of two fuzzy/neutrosophic sets and the fuzzy/neutrosophic logical operator "and" respectively. Extended to the fusion theory the T-norm will be a substitute for the conjunctive rule.

The T-norm satisfies the conditions:

a) Boundary Conditions: $T_n(0,0) = 0$, $T_n(x,1) = x$.

b) Commutativity: $T_n(x,y) = T_n(y,x)$.

c) Monotonicity: If $x \leq u$ and $y \leq v$, then $T_n(x,y) \leq T_n(u,v)$.

d) Associativity: $T_n(T_n(x,y),z) = T_n(x,T_n(y,z))$.

There are many functions which satisfy the T-norm conditions. We present below the most known ones:

- The Algebraic Product T-norm: $T_{n-\text{algebraic}}(x,y) = x \cdot y$
- The Bounded T-norm: $T_{n-\text{bounded}}(x,y) = \max\{0, x+y-1\}$
- The Default (min) T-norm [21, 51]: $T_{n-\min}(x,y) = \min\{x,y\}$.

A **T-conorm** is a function $T_c : [0,1]^2 \to [0,1]$, defined in fuzzy/neutrosophic set theory and fuzzy/neutrosophic logic to represent the "union" of two fuzzy/neutrosophic sets and the fuzzy/neutrosophic logical operator "or" respectively. Extended to the fusion theory the T-conorm will be a substitute for the disjunctive rule.

The T-conorm satisfies the conditions:

a) Boundary Conditions: $T_c(1,1) = 1$, $T_c(x,0) = x$.

b) Commutativity: $T_c(x,y) = T_c(y,x)$.

c) Monotonicity: if $x \leq u$ and $y \leq v$, then $T_c(x,y) \leq T_c(u,v)$.

d) Associativity: $T_c(T_c(x,y),z) = T_c(x,T_c(y,z))$.

There are many functions which satisfy the T-conorm conditions. We present below the most known ones:

- The Algebraic Product T-conorm: $T_{c-\text{algebraic}}(x,y) = x + y - x \cdot y$
- The Bounded T-conorm: $T_{c-\text{bounded}}(x,y) = \min\{1, x+y\}$
- The Default (max) T-conorm [21, 51]: $T_{c-\max}(x,y) = \max\{x,y\}$.

Then, the T-norm Fusion rules are defined as follows:

$$m_{\cap 12}(A) = \sum_{\substack{X,Y \in S^\Theta \\ X \cap Y = A}} T_n(m_1(X), m_2(Y))$$

and the T-conorm Fusion rules are defined as follows:

$$m_{\cup 12}(A) = \sum_{\substack{X,Y \in S^\Theta \\ X \cup Y = A}} T_c(m_1(X), m_2(Y))$$

The min T-norm rule yields results, very closed to Conjunctive Rule. It satisfies the principle of neutrality of the vacuous bba, reflects the majority opinion, converges towards idempotence. It is simpler to apply, but needs normalization.

"What is missed it is a strong justification of the way of presenting the fusion process. But we think, the consideration between two sources of information as a vague relation, characterized with the particular way of association between focal elements, and corresponding degree of association (interaction) between them is reasonable." (Albena Tchamova)

"Min rule can be interpreted as an optimistic lower bound for combination of bba and the below Max rule as a prudent/pessimistic upper bound." (Jean Dezert)

The T-norm and T-conorm are commutative, associative, isotone, and have a neutral element.

8.30 Improvements of fusion rules

Degree of Intersection

The degree of intersection measures the percentage of overlapping region of two sets X_1, X_2 with respect to the whole reunited regions of the sets using the cardinal of sets not the fuzzy set point of view [27]:

$$d(X_1 \cap X_2) = \frac{|X_1 \cap X_2|}{|X_1 \cup X_2|},$$

where $|X|$ means cardinal of the set X.

This definition of the degree of intersection is different from Zhang's previous one. For the minimum intersection/overlapping, i.e. when $X_1 \cap X_2 = \emptyset$, the degree of intersection is 0, while for the maximum intersection/overlapping, i.e. when $X_1 = X_2$, the degree of intersection is 1.

Degree of Union

The degree of intersection measures the percentage of non-overlapping region of two sets X_1, X_2 with respect to the whole reunited regions of the sets using the cardinal of sets not the fuzzy set point of view [27]:

$$d(X_1 \cup X_2) = \frac{|X_1 \cup X_2| - |X_1 \cap X_2|}{|X_1 \cup X_2|}.$$

For the maximum non-overlapping, i.e. when $X_1 \cap X_2 = \emptyset$, the degree of union is 1, while for the minimum non-overlapping, i.e. when $X_1 = X_2$, the degree of union is 0.

The sum of degrees of intersection and union is 1 since they complement each other.

Degree of Inclusion

The degree of intersection measures the percentage of the included region X_1 with respect to the includant region X_2 [27]:

Let $X_1 \subseteq X_2$, then

$$d(X_1 \subseteq X_2) = \frac{|X_1|}{|X_2|}.$$

$d(\emptyset \subseteq X_2) = 0$ because nothing is included in X_2, while $d(X_2 \subseteq X_2) = 1$ because X_2 is fulfilled by inclusion. By definition $d(\emptyset \subseteq \emptyset) = 1$.

And we can generalize the above degree for $n \geq 2$ sets.

Improvements of Credibility, Plausibility and Communality Functions

Thus the $\mathrm{Bel}(\cdot)$, $\mathrm{Pl}(\cdot)$ and $\mathrm{Com}(\cdot)$ functions can incorporate in their formulas the above degrees of inclusion and intersection respectively:

- Credibility function improved:

$$\forall A \in S^{\Theta} \setminus \emptyset, \text{ one has } \mathrm{Bel}_d(A) = \sum_{\substack{X \in S^{\Theta} \\ X \subseteq A}} \frac{|X|}{|A|} m(X)$$

- Plausibility function improved:

$$\forall A \in S^{\Theta} \setminus \emptyset, \text{ one has } \mathrm{Pl}_d(A) = \sum_{\substack{X \in S^{\Theta} \\ X \cap A \neq \emptyset}} \frac{|X \cap A|}{|X \cup A|} m(X)$$

- Communality function improved:

$$\forall A \in S^{\Theta} \setminus \emptyset, \text{ one has } \mathrm{Com}_d(A) = \sum_{\substack{X \in S^{\Theta} \\ A \subseteq X}} \frac{|A|}{|X|} \cdot m(X)$$

Improvements of quantitative fusion rules

- Disjunctive rule improved:

$$\forall A \in S^{\Theta} \setminus \emptyset, \text{ one has } m_{\cup d}(A) = k_{\cup d} \cdot \sum_{\substack{X_1, X_2 \in S^{\Theta} \\ X_1 \cup X_2 = A}} \frac{|X_1 \cup X_2| - |X_1 \cap X_2|}{|X_1 \cup X_2|} m_1(X_1) m_2(X_2),$$

  where $k_{\cup d}$ is a constant of renormalization.

- Dezert-Smarandache classical rule improved:

$$\forall A \in S^{\Theta}, \text{ one has } m_{\text{DSmCd}}(A) = k_{\text{DSmCd}} \cdot \sum_{\substack{X_1, X_2 \in S^{\Theta} \\ X_1 \cap X_2 = A}} \frac{|X_1 \cap X_2|}{|X_1 \cup X_2|} m_1(X_1) m_2(X_2),$$

  where k_{DSmCd} is a constant of renormalization. It is similar with the Zhang's Center Combination rule extended on the Boolean algebra $(\Theta, \cup, \cap, \mathcal{C})$ and using another definition for the degree of intersection.

- Dezert-Smarandache hybrid rule improved:

$$\begin{aligned} &\forall A \in S^{\Theta} \setminus \emptyset \text{ one has} \\ m_{\text{DSmHd}}(A) = k_{\text{DSmHd}} &\cdot \Bigg\{ \sum_{\substack{X_1, X_2 \in S^{\Theta} \\ X_1 \cap X_2 = A}} \frac{|X_1 \cap X_2|}{|X_1 \cup X_2|} m_1(X_1) m_2(X_2) \\ &+ \sum_{\substack{X_1, X_2 \in \emptyset \\ (A=U) \vee \{U \in \emptyset \wedge A = I\}}} m_1(X_1) m_2(X_2) \\ &+ \sum_{\substack{X_1, X_2 \in S^{\Theta} \\ X_1 \cup X_2 = A \\ X_1 \cap X_2 = \emptyset}} \frac{|X_1 \cup X_2| - |X_1 \cap X_2|}{|X_1 \cup X_2|} m_1(X_1) m_2(X_2) \Bigg\} \end{aligned}$$

  where k_{DSmHd} is a constant of renormalization.

- Smets' rule improved:

$$m_S(\emptyset) = k_{12} = k_{Sd} \cdot \sum_{\substack{X_1, X_2 \in S^{\Theta} \\ X_1 \cap X_2 = \emptyset}} \frac{|X_1 \cap X_2|}{|X_1 \cup X_2|} m_1(X_1) m_2(X_2),$$

  and

$$\forall A \in S^{\Theta} \setminus \emptyset, \text{one has } m_S(A) = k_{Sd} \cdot \sum_{\substack{X_1, X_2 \in S^{\Theta} \\ X_1 \cap X_2 = A}} \frac{|X_1 \cap X_2|}{|X_1 \cup X_2|} m_1(X_1) m_2(X_2),$$

  where k_{Sd} is a constant of renormalization.

- Yager's rule improved:

$$m_Y(\emptyset) = 0, m_Y(I) = k_{Yd} \cdot \{m_1(I)m_2(I) + \sum_{\substack{X_1, X_2 \in S^\Theta \\ X_1 \cap X_2 = \emptyset}} \frac{|X_1 \cap X_2|}{|X_1 \cup X_2|} m_1(X_1)m_2(X_2)\}$$

and

$$\forall A \in S^\Theta \setminus \{\emptyset, I\}, \text{ one has } m_{Yd}(A) = k_{Yd} \cdot \sum_{\substack{X_1, X_2 \in S^\Theta \\ X_1 \cap X_2 = A}} \frac{|X_1 \cap X_2|}{|X_1 \cup X_2|} m_1(X_1)m_2(X_2).$$

where I = total ignorance and k_{Yd} is a constant of renormalization.

- Dubois-Prade's rule improved:

$$m_{DP}(\emptyset) = 0,$$

and

$$\forall A \in S^\Theta \setminus \emptyset \text{ one has}$$

$$m_{DP}(A) = k_{DPd} \cdot \left\{ \sum_{\substack{X_1, X_2 \in S^\Theta \\ X_1 \cap X_2 = A}} \frac{|X_1 \cap X_2|}{|X_1 \cup X_2|} m_1(X_1)m_2(X_2) + \sum_{\substack{X_1, X_2 \in S^\Theta \\ X_1 \cup X_2 = A \\ X_1 \cap X_2 = \emptyset}} \frac{|X_1 \cup X_2| - |X_1 \cap X_2|}{|X_1 \cup X_2|} m_1(X_1)m_2(X_2) \right\},$$

where k_{DPd} is a constant of renormalization.

8.31 Extension of bba on neutrosophic sets

- Let T, I, $F \subseteq [0, 1]$. An element $x(T, I, F)$ belongs to a *neutrosophic set* M as follows: its membership is T, its nonmembership is F, and its indeterminacy is I.
- Define a **neutrosophic basic belief assignment** (nbba):

$$n(\cdot) : S^\Theta \to [0, 1]^3, \quad n(A) = (T_A, I_A, F_A),$$

  where T_A = belief in A, F_A = disbelief in A, I_A = indeterminacy on A.
- *Admissibility condition*: For each $A \in S^\Theta$, there exist scalars $t_A \in T_A$, $i_A \in I_A$, $f_A \in F_A$ such that:

$$\sum_{A \in S^\Theta} (t_A + i_A + f_A) = 1.$$

- When $F_A = I_A = \phi$ nbba coincides with bba.

- Intuitionistic Fuzzy Set can not be applied since the sum of components for a single set < 1.
- N-norms/conorms [42, 43] use nbba's for information fusion.
- *All fusion rules and functions can be extended on nbba's.*

N-norms

$$N_n : \big([0,1]\times[0,1]\times[0,1]\big)^2 \to [0,1]\times[0,1]\times[0,1]$$

$$N_n\Big(\big(x_1,x_2,x_3\big),\big(y_1,y_2,y_3\big)\Big) = \Big(N_nT\big(x_1,y_1\big),N_nI\big(x_2,y_2\big),N_nF\big(x_3,y_3\big)\Big)$$

- For each component, $J \in \{T, I, F\}$, N_n satisfies the conditions:
  a) Boundary Conditions: $N_nJ(0,0) = 0$, $N_nJ(x,1) = x$.
  b) Commutativity: $N_nJ(x,y) = N_nJ(y,x)$.
  c) Monotonicity: If $x \leq u$ and $y \leq v$, then $N_nJ(x,y) \leq N_nJ(u,v)$.
  d) Associativity: $N_nJ(N_nJ(x,y),z) = N_nJ(x,N_nJ(y,z))$.

  N_n represents *intersection* in neutrosophic set theory, respectively the *and* operator in neutrosophic logic.
- Most known ones:
  - The Algebraic Product N-norm: $N_{n-\text{algebraic}}J(x,y) = x\cdot y$
  - The Bounded N-Norm: $N_{n-\text{bounded}}J(x,y) = \max\{0, x+y-1\}$
  - The Default (min) N-norm: $N_{n-\min}J(x,y) = \min\{x,y\}$.

N-conorms

$$N_c : \big([0,1]\times[0,1]\times[0,1]\big)^2 \to [0,1]\times[0,1]\times[0,1]$$

$$N_c\Big(\big(x_1,x_2,x_3\big),\big(y_1,y_2,y_3\big)\Big) = \Big(N_cT\big(x_1,y_1\big),N_cI\big(x_2,y_2\big),N_cF\big(x_3,y_3\big)\Big)$$

- For each component, $J \in \{T, I, F\}$, N_c satisfies the conditions:
  a) Boundary Conditions: $N_cJ(1,1) = 1$, $N_cJ(x,0) = x$.
  b) Commutativity: $N_cJ(x,y) = N_cJ(y,x)$.
  c) Monotonicity: if $x \leq u$ and $y \leq v$, then $N_cJ(x,y) \leq N_cJ(u,v)$.
  d) Associativity: $N_cJ(N_cJ(x,y),z) = N_cJ(x,N_cJ(y,z))$.

  N_c represents *union* in neutrosophic set theory, respectively the *or* operator in neutrosophic logic.
- Most known ones:
  - The Algebraic Product N-conorm: $N_{c-\text{algebraic}}J(x,y) = x+y-x\cdot y$
  - The Bounded N-conorm: $N_{c-\text{bounded}}J(x,y) = \min\{1, x+y\}$
  - The Default (max) N-conorm: $N_{c-\max}J(x,y) = \max\{x,y\}$.

N-norm and N-conorm based fusion rules

Let $n_1(\cdot)$ and $n_2(\cdot)$ be nbba's.

N-norm fusion rulesare defined as follows:

$$n_{\cap 12}(A) = \sum_{\substack{X,Y \in S \wedge \Theta \\ X \cap Y = A}} N_n(n_1(X), n_2(Y))$$

N-conorm fusion rules are defined as follows:

$$n_{\cup 12}(A) = \sum_{\substack{X,Y \in S \wedge \Theta \\ X \cup Y = A}} N_c(n_1(X), n_2(Y))$$

- they can replace the conjunctive respectively disjunctive rules (Smarandache 2004)
- need normalizations

Example of N-norm fusion rule

A first doctor's belief about a patient's disease A is 0.4, disbelief 0.2, while about the second disease B his belief is 0.1 with disbelief 0.2 and not sure 0.1. Second doctor is more confident in disease B with 0.6, his disbelief on A is 0.1 and 0.3 not sure on A.

Hence $n_1(A) = (0.4, 0, 0.2)$, $n_1(B) = (0.1, 0.1, 0.2)$, and $n_2(A) = (0, 0.3, 0.1)$, $n_2(B) = (0.6, 0, 0)$ are nbba's and frame of discernment $\{A, B\}$. Using the Algebraic Product N-norm fusion rule we get:

$$\begin{gathered} n_{12}(A) = (0, 0, 0.02), n_{12}(B) = (0.06, 0, 0), \\ n_{12}(A \cap B) = (0.24, 0, 0) + (0, 0.03, 0.02) = (0.24, 0.03, 0.02). \end{gathered}$$

Transfer the conflicting mass $n_{12}(A \cap B)$ to A and B proportionally to their mass sums (following PCR3):

$$x_1/(0.4 + 0) = y_1/(0.1 + 0.6) = 0.24/1.1,$$

hence

$$x_1 = 0.4(0.24/1.1) = 0.087273, \quad y_1 = 0.7(0.24/1.1) = 0.152727;$$

$$x_2/0.3 = y_2/0.1 = 0.03/0.4,$$

hence

$$x_2 = 0.3(0.03/0.4) = 0.0225, \quad y_2 = 0.1(0.03/0.4) = 0.0075;$$

$$x_3/0.3 = y_3/0.2 = 0.02/0.5,$$

hence

$$x_3 = 0.3(0.02/0.5) = 0.012, y_3 = 0.2(0.02/0.5) = 0.008.$$

Summing them with the previous one gets: $n_{12tr}(A) = (0.087273, 0.0225,\ 0.032)$, $n_{12tr}(B) = (0.212727, 0.0075, 0.008)$ and renormalize (divide by their sum= 0.37):

$$n_{\text{12tr-norm}}(A) = (0.235873, 0.060811, 0.086486),$$
$$n_{\text{12tr-norm}}(B) = (0.574938, 0.02027, 0.021622).$$

Remark: If first done the normalization and second the transfer the result will be the same [20].

8.32 Unification of Fusion Rules (UFR)

If variable y is *directly proportional* with variable p, then $y = k \cdot p$, where k is a constant. If variable y is *inversely proportional* with variable q, then $y = k \cdot \frac{1}{q}$; we can also say that y is directly proportional with variable $\frac{1}{q}$. In a general way, we say that y is directly proportional with variables $p_1, p_2, \ldots, p_m$ and inversely proportionally with variables $q_1, q_2, \ldots, q_n$, where $m, n \geq 1$, then:

$$y = k \cdot \frac{p_1 \cdot p_2 \cdot \ldots \cdot p_m}{q_1 \cdot q_2 \cdot \ldots \cdot q_n} = k \cdot \frac{P}{Q},$$

where $P = \prod_{i=1}^{m} p_i$ and $Q = \prod_{j=1}^{n} q_j$.

Then a Unification of Fusion Rules (UFR) is given by : $m_{UFR}(\emptyset) = 0$ and $\forall A \in S^\Theta \setminus \emptyset$ one has

$$m_{UFR}(A) = \sum_{\substack{X_1, X_2 \in S^\Theta \\ X_1 \star X_2 = A}} d(X_1 \star X_2) R(X_1, X_2) + \frac{P(A)}{Q(A)} \cdot \sum_{\substack{X \in S^\Theta \setminus A \\ X \star A \in E}} d(X \star A) \cdot \frac{R(A, X)}{P(A)/Q(A) + P(X)/Q(X)}$$

where $\star$ means intersection or union of sets (depending on the application or problem to be solved);
$d(X \star Y)$ is the degree of intersection or union respectively;
$R(X, Y)$ is a T-norm/conorm (or N-norm/conorm in a more general case) fusion combination rule respectively (extension of conjunctive or disjunctive rules respectively to fuzzy or neutrosophic operators) or any other fusion rule; the T-norm and N-norm correspond to the intersection of sets, while the T-conorm and N-conorm to the disjunction of sets;
E is the ensemble of sets (in majority cases they are empty sets) whose masses must be transferred (in majority cases to non-empty sets, but there are exceptions for the open world);
$P(A)$ is the product of all parameters directly proportional with A;
while $Q(A)$ is the product of all parameters inversely proportional with A [in most of the cases $P(A)$ and $Q(A)$ are derived from the masses assigned to the set A by the sources].

8.33 Unification of Fusion Theories (UFT)

As a conclusion, since no theory neither rule fully satisfy all needed applications, the author proposes [27–30] a Unification of (Quantitative) Fusion Theories extending the power and hyper-power sets from previous theories to a Boolean algebra obtained by the closures of the frame of discernment under union, intersection, and complement of sets (for non-exclusive elements one considers a fuzzy or neutrosophic complement).

And, at each application, one selects the most appropriate model, rule, and algorithm of implementation.

Since everything depends on the application/problem to solve, this scenario looks like a logical chart designed by the programmer in order to write and implement a computer program, or even like a cooking recipe.

Here it is the scenario attempting for a unification and reconciliation of the fusion theories and rules:

1) If all sources of information are reliable, then apply the conjunctive rule, which means consensus between them (or their common part):

2) If some sources are reliable and others are not, but we don't know which ones are unreliable, apply the disjunctive rule as a cautious method (and no transfer or normalization is needed).

3) If only one source of information is reliable, but we don't know which one, then use the exclusive disjunctive rule based on the fact that $X_1 \veebar X_2 \veebar \cdots \veebar X_n$ means either X_1 is reliable, or X_2, or and so on, or X_n, but not two or more in the same time.

4) If a mixture of the previous three cases, in any possible way, use the mixed conjunctive-disjunctive rule.

5) If we know the sources which are unreliable, we discount them. But if all sources are fully unreliable (100%), then the fusion result becomes the vacuum bba (i.e. $m(\Theta) = 1$, and the problem is indeterminate. We need to get new sources which are reliable or at least they are not fully unreliable.

6) If all sources are reliable, or the unreliable sources have been discounted (in the *default case*), then use the DSm classic rule (which is commutative, associative, Markovian) on Boolean algebra $(\Theta, \cup, \cap, \mathcal{C})$, no matter what contradictions (or model) the problem has. I emphasize that the super-power set S^Θ generated by this Boolean algebra contains singletons, unions, intersections, and complements of sets.

7) If the sources are considered from a statistical point of view, use *Murphy's average rule* (and no transfer or normalization is needed).

8) In the case the model is not known (the *default case*), it is prudent/cautious to use the free model (i.e. all intersections between the elements of the frame of discernment are non-empty) and DSm classic rule on S^Θ, and later if the model is found out (i.e. the constraints of empty intersections become known), one can adjust the conflicting mass at any time/moment using the DSm hybrid rule.

9) Now suppose the model becomes known [i.e. we find out about the contradictions (= empty intersections) or consensus (= non-empty intersections) of the problem/application]. Then:

   9.1) If an intersection $A \cap B$ is not empty, we keep the mass $m(A \cap B)$ on $A \cap B$, which means consensus (common part) between the two hypotheses A and B (i.e. both hypotheses A and B are right) [here one gets *DSmT*].

   9.2) If the intersection $A \cap B = \emptyset$ is empty, meaning contradiction, we do the following :

      9.2.1) if one knows that between these two hypotheses A and B one is right and the other is false, but we don't know which one, then one transfers the mass $m(A \cap B)$ to $m(A \cup B)$, since $A \cup B$ means at least one is right [here one gets *Yager's* if $n = 2$, or *Dubois-Prade,* or *DSmT*];

      9.2.2) if one knows that between these two hypotheses A and B one is right and the other is false, and we know which one is right, say hypothesis A is right and B is false, then one transfers the whole mass $m(A \cap B)$ to hypothesis A (nothing is transferred to B);

      9.2.3) if we don't know much about them, but one has an optimistic view on hypotheses A and B, then one transfers the conflicting mass $m(A \cap B)$ to A and B (the nearest specific sets in the Specificity Chains) [using *Dempster's*, *PCR2–5*]

      9.2.4) if we don't know much about them, but one has a pessimistic view on hypotheses A and B, then one transfers the conflicting mass $m(A \cap B)$ to $A \cup B$ (the more pessimistic the further one gets in the Specificity Chains: $(A \cap B) \subset A \subset (A \cup B) \subset I$); this is also the *default case* [using *DP's*, *DSm hybrid rule*, *Yager's*]; if one has a very pessimistic view on hypotheses A and B then one transfers the conflicting mass $m(A \cap B)$ to the total ignorance in a closed world [*Yager's*, *DSmT*], or to the empty set in an open world [*TBM*];

      9.2.5.1) if one considers that no hypothesis between A and B is right, then one transfers the mass $m(A \cap B)$ to other non-empty sets (in the case more hypotheses do exist in the frame of discernment) — different from A, B, $A \cup B$ — for the reason that: if A and B are not right then there is a bigger chance that other hypotheses in the frame of discernment have a higher subjective probability to occur; we do this transfer in a *closed world* [DSm hybrid rule]; but, if it is an *open world*, we can transfer the mass $m(A \cap B)$ to the empty set leaving room for new possible hypotheses [here one gets *TBM*];

      9.2.5.2) if one considers that none of the hypotheses A, B is right and no other hypothesis exists in the frame of discernment (i.e. $n = 2$ is the size of the frame of discernment), then one considers the *open world* and one transfers the mass to the empty set [here *DSmT* and *TBM* converge to each other].

Of course, this procedure is extended for any intersections of two or more sets: $A \cap B \cap C$, etc. and even for mixed sets: $A \cup (B \cup C)$, etc.

If it is a dynamic fusion in a real time and associativity and/or Markovian process are needed, use an algorithm which transforms a rule (which is based on the conjunctive rule and the transfer of the conflicting mass) into an associative and Markovian rule by storing the previous result of the conjunctive rule and, depending of the rule, other data. Such rules are called quasi-associative and quasi-Markovian.

Some applications require the necessity of *decaying the old sources* because their information is considered to be worn out.

If some bba is not normalized (i.e. the sum of its components is < 1 as in incomplete information, or > 1 as in paraconsistent information) we can easily divide each component by the sum of the components and normalize it. But also it is possible to fusion incomplete and paraconsistent masses, and then normalize them after fusion. Or leave them unnormalized since they are incomplete or paraconsistent.

PCR5 does the most mathematically exact (in the fusion literature) redistribution of the conflicting mass to the elements involved in the conflict, redistribution which exactly follows the tracks of the conjunctive rule.

Acknowledgement

We want to thank Dr. Wu Li from NASA Langley Research Center, Prof. Philippe Smets from the Université Libre de Bruxelles, Belgium; Dr. Jean Dezert from ONERA in Châtillon, France; Dr. Albena Tchamova from the Bulgarian Academy of Sciences, Sofia; Dr. Milan Daniel from the Czech Academy of Sciences, Praha and Dr. Audun Jøsang from DSTC Australia for their comments.

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