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ORIGINAL ARTICLE

An interval neutrosophic linguistic multi-criteria group decision-making method and its application in selecting medical treatment options

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Abstract Selecting medical treatments is a critical activity in medical decision-making. Usually, medical treatments are selected by doctors, patients, and their families based on various criteria. Due to the subjectivity of decision-making and the large volume of information available, accurately and comprehensively evaluating information with traditional fuzzy sets is impractical. Interval neutrosophic linguistic numbers (INLNs) can be effectively used to evaluate information during the medical treatment selection process. In this study, a medical treatment selection method based on prioritized harmonic mean operators in an interval neutrosophic linguistic environment, in which criteria and decision-makers are assigned different levels of priority, is developed. First, the rectified linguistic scale functions of linguistic variables, new INLN operations, and an INLN comparison method are developed in order to prevent data loss and distortion during the aggregation process. Next, a generalized interval neutrosophic linguistic prioritized weighted harmonic mean operator and a generalized interval neutrosophic linguistic prioritized hybrid harmonic mean operator are developed in order to aggregate the interval neutrosophic linguistic information. Then, these operators are used to develop an interval neutrosophic linguistic multi-criteria group decision-making method. In addition, the proposed method is applied to a practical treatment selection method. Furthermore, the ranking results are compared to those obtained using a traditional approach in order to confirm the practicality and accuracy of the proposed method.

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1 Introduction

Selecting medical treatments is a critical activity in medical decision-making. Usually, medical treatments are selected collaboratively by doctors, patients, and their families in order to promote compliance and reduce medical risks. However, due to various factors, such as the probability that a treatment will cure the patient, the cost of that treatment, and the severity of its side effects, selecting an appropriate treatment can be difficult. Multi-criteria decision-making (MCDM) methods can be effectively applied to medical treatment selection problems [1]. In fact, many traditional MCDM methods have been used to select medical treatments [2–6]. Information regarding treatment options can be described with fuzzy sets (FSs) [7] using membership functions, intuitionistic fuzzy sets (IFSs) [8] using membership and non-membership functions, or hesitant fuzzy sets (HFSs) [9] using one or several degrees of membership. However, these sets are incapable of managing the indeterminate and inconsistent information frequently associated with medical decision-making problems. For example, when asked to assess whether a particular treatment would be "good" for a certain patient based on its probability of curing that patient, a doctor may deduce that the probability of truth is 0.5, the probability of falsity is 0.6, and the probability of indeterminacy is 0.2. Generalized IFSs [8], or neutrosophic sets (NSs) [10, 11], are powerful tools that can be used to describe uncertain, incomplete, indeterminate, and inconsistent information with truth-membership, indeterminacy-membership, and

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falsity-membership functions. The doctor's deduction in the aforementioned example would be expressed as $\langle \text{good}, (0.5, 0.2, 0.6) \rangle$ in an NS.

NSs have been successfully applied to many problems, such as medical diagnostic problems [12–15], investment selection problems [16, 17], image processing [18, 19], and supplier selection problems [20]. However, since NSs are based on philosophical thinking, they cannot be effectively applied to science and engineering problems without the addition of specific definitions. Numerous concepts, such as single-valued neutrosophic sets (SVNSs) [21], interval neutrosophic sets (INSs) [22], multi-valued neutrosophic sets (MVNSs) [23], normal neutrosophic sets (NNSs) [24], neutrosophic soft matrices (NSM) [25], and trapezoidal neutrosophic sets (TNSs) [26], have been developed in order to improve the applicability of NSs.

SVNSs were first introduced by Wang et al. [21], who used crisp numbers to describe the probability of truth, falsity, and indeterminacy of selection problems. Several extensions of these early SVNSs, such as single-valued neutrosophic hesitant fuzzy sets [27], have been developed. Research concerning the application of SVNSs to MCDM problems has primarily involved the development of aggregation operators [28–31], similarity measurements [13, 32], correlation coefficients [33], and distance measurements [16, 34–36]. Unlike SVNSs, INSs use membership intervals, non-membership intervals, and indeterminate intervals rather than real numbers to describe indeterminate and inconsistent information [37]. Moreover, several extensions of INSs, such as interval neutrosophic hesitant sets [38] and interval neutrosophic soft rough sets [39, 40], have been developed. Similar to SVNSs, INSs use aggregation operators [41–43], similarity measurements [44], correlation coefficients [45], cross-entropy measurements [46], and distance measurements [47] to solve MCDM problems.

Due to the complexity of objects and subjectivity of human thinking, obtaining accurate assessment values of problems too complex or ill-defined to be solved with quantitative information is difficult. In these cases, linguistic variables can be effectively used to enhance the reliability and flexibility of traditional decision-making models [48–53]. Several neutrosophic linguistic sets, including single-valued neutrosophic linguistic numbers (SVNLNs) [54], interval neutrosophic linguistic numbers (INLNs) [55], single-valued neutrosophic trapezoid linguistic numbers [56], and interval neutrosophic uncertain linguistic variables (INUNVs) [17], have been developed in order to improve the efficacy and practicality of NSs.

Aggregation operators, which can be used to effectively compile information, have been widely applied to MCDM problems in neutrosophic conditions. For example, Liu et al. [28] developed a family of generalized neutrosophic number Hamacher weighted averaging operators. Liu and Wang [29] proposed a single-valued neutrosophic normalized weighted Bonferroni mean operator based on the SVNSs. In addition, Peng et al. [31] developed a serial of simplified neutrosophic number weighted averaging operators. Sun et al. [41] introduced an interval neutrosophic number Choquet integral operator based on INSs, and Ye [42] introduced an interval neutrosophic number ordered weighted averaging operator and interval neutrosophic number ordered weighted geometric operator. Furthermore, Peng et al. [23] developed a multi-valued neutrosophic power weighted average operator and multi-valued neutrosophic power weighted geometric operator.

However, these aggregation operators are only capable of managing neutrosophic information expressed by crisp numbers or fuzzy numbers. Thus, they cannot be applied to the linguistic information prevalent in complex decision-making problems. Tian et al. [57] developed a simplified neutrosophic linguistic Bonferroni mean operator and simplified neutrosophic linguistic normalized weighted Bonferroni mean operator based on single-valued neutrosophic linguistic numbers. Because the degrees of truth-membership, indeterminate-membership, and falsity-membership of the linguistic values in a simplified neutrosophic linguistic number are described by three single numbers, these degrees of membership cannot be easily condensed into single numbers for the use of decision-makers. However, decision-makers can easily express information using interval numbers. Ye [55] developed an interval neutrosophic linguistic weighted arithmetic average (INLWAA) operator and interval neutrosophic linguistic weighted geometric average (INLWGA) operator based on INLNs. Although interval neutrosophic linguistic operators can be used to manage linguistic neutrosophic information, the linguistic terms of the neutrosophic linguistic information are operated upon based on their subscripts in linear functions. Furthermore, the correlations among linguistic terms and these three degrees of membership are neglected and the indeterminate degree of membership is assumed to be equal to the falsity degree of membership, resulting in information loss and distortion.

INLSs can effectively describe the information presented in complex decision-making problems. However, the operators of existing INLNs assume that criteria and decision-makers in a decision-making problem share the same level of priority. However, in medical treatment selection problems, doctors, patients, and family members all evaluate treatment options based on various criteria and with varying levels of priority. Thus, this evaluation information cannot be managed using the aforementioned operators. In this paper, a generalized interval neutrosophic linguistic prioritized weighted harmonic mean (GINLPWHM) operator and a generalized interval neutrosophic linguistic prioritized hybrid harmonic mean

(GINLPHHM) operator were developed based on the prioritized aggregation (PA) operators developed by Yager [58] to model the priority levels of criteria based on their weights as well as the harmonic mean (HM) operator, which has been widely used to aggregate central tendency data. The proposed operators accounted for the prioritization relationships among various criteria and decision-makers, utilized a harmonic mean operator to aggregate neutrosophic information, and prevented the complications associated with existing INLN operators. The proposed operators were then used to develop an interval neutrosophic linguistic MCGDM method of medical treatment selection.

The remainder of this paper is organized as follows. In Sect. 2, some basic concepts related to INLNs are introduced. In Sect. 3, the linguistic scale functions of the linguistic variables, new INLN operations, and an INLN comparison method are introduced. In Sect. 4, a GINLPWHM operator and GINLPHHM operator are developed based on the proposed INLN operations. In Sect. 5, an MCGDM method is developed based on the proposed GINLPWHM and GINLPHHM operators. In addition, the developed approach is demonstrated using a treatment selection problem in an interval neutrosophic linguistic environment, and a comparative analysis is conducted in order to verify the validity and feasibility of the proposed approach. The conclusions are presented in Sect. 6.

2 Preliminaries

In this section, some basic concepts and definitions related to interval neutrosophic linguistic numbers (INLNs) and the proposed aggregation operators utilized in the subsequent analysis, including neutrosophic sets (NSs), interval neutrosophic sets (INSs), linguistic term sets, interval neutrosophic linguistic sets (INLSs), prioritized aggregation (PA), and harmonic mean (HM) operators, are introduced.

2.1 NSs and INSs

Definition 1 [22] Let X be a space of points (objects) with a generic element in X denoted by x. Then an NS A in X is characterized by three membership functions, including a truth-membership function $T_A(x)$, indeterminacy-membership function $I_A(x)$, and falsity-membership function $F_A(x)$, and is defined as

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle | x \in X\},$$

where $T_A(x)$, $I_A(x)$, and $F_A(x)$ are real standard or nonstandard subsets of $]^-0, 1^+[$, i.e. $T_A(x) : X \to]^-0, 1^+[$, $I_A(x) : X \to]^-0, 1^+[$ and $F_A(x) : X \to]^-0, 1^+[$. The sum of $T_A(x)$, $I_A(x)$, and $F_A(x)$ is unrestricted, and $^-0 \le T_A(x) + I_A(x) + F_A(x) \le 3^+$.

NSs must be specifically defined; otherwise, they cannot be easily applied to science and engineering problems. Wang et al. [22] developed an INS that can act as an NS in order to improve the ease of computations during the operation process and improve the applicability of NSs.

Definition 2 [22] Let X be a space of points (objects). Then an INS A in X can be expressed as

$$A = \{x, \langle [\inf T_A(x), \sup T_A(x)], [\inf I_A(x), \sup I_A(x)], [\inf F_A(x), \sup F_A(x)]\rangle | x \in X\},$$

where $[\inf T_A(x), \sup T_A(x)] \subseteq [0, 1]$, $[\inf I_A(x), \sup I_A(x)] \subseteq [0, 1]$, and $[\inf F_A(x), \sup F_A(x)] \subseteq [0, 1]$.

Thus, the sum of $\sup T_A(x)$, $\sup I_A(x)$, and $\sup F_A(x)$ satisfies $0 \le \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \le 3$. When the inferior and superior limits of $T_A(x)$, $I_A(x)$, and $F_A(x)$ in an INS are equal, the INS is reduced to a single-valued neutrosophic set (SVNS).

2.2 Linguistic term sets

Definition 3 [59] Let $S = \{s_i | i = 1, 2, \ldots, 2t + 1, t \in N^*\}$ be a linguistic term set, where N^* is a set of positive integers, and s_i represents the value of a linguistic variable. Then the set S satisfies the following properties:

1. The linguistic term set is ordered: $i > j \Leftrightarrow s_i > s_j$, and
2. A negation operator exists: $\mathrm{Neg}(s_i) = s_j$, where $i + j = 2t + 1$.

In order to preserve information during the decision-making process, Xu [60, 61] expanded the discrete linguistic term set S into a continuous linguistic term set $\tilde{S} = \{s_i | i \in [1, l]\}$, where $s_i > s_j (i > j)$, and $l (l > 2t + 1)$ is a sufficiently large positive integer. If $s_i \in S$, the linguistic term is denoted as the original linguistic term; otherwise, s_i is denoted as a virtual linguistic term. In general, decision-makers use original linguistic terms to evaluate alternatives, and virtual linguistic terms are only included in operations to prevent information loss and enhance the decision-making process [60].

2.3 INLSs and INLNs

Due to the accuracy and practicality of linguistic variables and INSs, Ye [55] combined two concepts to develop INLSs.

Definition 4 [55] Let U be a space of points (objects). Then an INLS $\bar{A}$ in X can be defined as

$$\bar{A} = \{x, \langle s_{\theta(x)}, ([\inf T_{\bar{A}}(x), \sup T_{\bar{A}}(x)], [\inf I_{\bar{A}}(x), \sup I_{\bar{A}}(x)], [\inf F_{\bar{A}}(x), \sup F_{\bar{A}}(x)])\rangle | x \in X\},$$

where $[\inf T_{\bar{A}}(x), \sup T_{\bar{A}}(x)] \subseteq [0, 1]$, $[\inf I_{\bar{A}}(x), \sup I_{\bar{A}}(x)] \subseteq [0, 1]$, and $[\inf F_{\bar{A}}(x), \sup F_{\bar{A}}(x)] \subseteq [0, 1]$ represent the degrees of truth-membership, indeterminacy-membership, and falsity-membership of x in X to the linguistic term $s_{\theta(x)}$, and $s_{\theta(x)} \in S$.

Thus, the INLNs, which are elements of $\bar{A}$, can be expressed as

$$\langle s_{\theta(x)}, ([\inf T_{\bar{A}}(x), \sup T_{\bar{A}}(x)], [\inf I_{\bar{A}}(x), \sup I_{\bar{A}}(x)], [\inf F_{\bar{A}}(x), \sup F_{\bar{A}}(x)])\rangle.$$

2.4 PA and HM operators

Definition 5 [58] Let $C = \{C_1, C_2, \ldots, C_n\}$ be a set of criteria that satisfies the linear ordering prioritization $C_1 \succ C_2 \succ \cdots \succ C_n$, where the priority of C_k is higher than that of C_j if $k < j$. Then the value of $C_j(x_i)$ represents the performance of the alternative x_i under criterion C_j. Thus, the PA operator can be expressed as

$$\mathrm{PA}(C(x_i)) = \sum_{j=1}^{n} \left(w_j C_j(x_i)\right),$$

where $C_j(x_i) \in [0, 1]$, $w_j = T_j / \sum_{i=1}^{n} T_i$, $T_1 = 1$, $T_j = \prod_{k=1}^{j-1} C_k(x_i)$, and $j = 2, 3, \ldots, n$.

Definition 6 [62] Let $h_i (i = 1, 2, \ldots, n)$ be a collection of positive real numbers and $w = (w_1, w_2, \ldots, w_n)$ be the weight vector of $h_i (i = 1, 2, \ldots, n)$. Then the weight harmonic mean can be expressed as

$$\mathrm{WHM}(h_1, h_2, \ldots, h_n) = \frac{1}{\sum_{i=1}^{n} (w_i / h_i)},$$

where $w_i \in [0, 1]$, and $\sum_{i=1}^{n} w_i = 1$.

If $w_i = 1$, $w_j \neq 1$, and $i \neq j$, then $\mathrm{WHM}(h_1, h_2, \ldots, h_n) = h_i$; if $w = (1/n, 1/n, \ldots, 1/n)$, then the WHM operator is reduced to the harmonic mean (HM) operator as

$$\mathrm{HM}(h_1, h_2, \ldots, h_n) = \frac{n}{\sum_{i=1}^{n} (1/h_i)}.$$

3 Comparison of the INLNs and their operations

In this section, rectified linguistic scale functions are introduced in order to allow for a higher degree of flexibility when modelling the linguistic information. New operations and an INLN comparison method are also developed in order to prevent information loss and distortion during the aggregation process [55].

3.1 Rectified linguistic scale functions

Linguistic scale functions play an active role in the conversion of linguistic arguments to real numbers belonging to [0, 1]. However, the smallest linguistic value s_0 is always converted to 0. Thus, if s_0 is involved in multiplicative operations, inaccurate results could be obtained.

Example 1 Let $a_1 = \langle s_0, ([0.3, 0.4], [0.2, 0.4], [0.3, 0.4])\rangle$ and $a_2 = \langle s_0, ([0.1, 0.2], [0.1, 0.2], [0.7, 0.8])\rangle$ be two INLNs. Then according to the score function, $E(a) = s_{\frac{1}{6}(4 + \inf T(a) - \inf I(a) - \inf F(a) + \sup T(a) - \sup I(a) - \sup F(a))\theta(a)}$, accuracy function $H(a) = s_{\frac{1}{2}(\inf T(a) - \inf F(a) + \sup T(a) - \sup F(a))\theta(a)}$, and certainty function $C(a) = s_{\frac{1}{2}(\inf T(a) + \sup T(a))\theta(a)}$ introduced by Ye [55], where $\theta(a)$ denotes the subscripts of the linguistic values,

$$E(a_1) = E(a_2) = H(a_1) = H(a_2) = C(a_1) = C(a_2) = s_0.$$

These two INLNs cannot be compared using the above functions. However, a_1 is known to be superior to a_2. In order to overcome these limitations and improve their applicability, the linguistic scale functions introduced in [49, 63, 64] were modified. These modifications allowed for more efficient and flexible linguistic assessment information use through the situation-dependent conversion of various linguistic assessment arguments to real numbers.

Definition 7 [49, 64] Let $S = \{s_i | i = 1, 2, \ldots, 2t + 1\}$ be a linguistic term set. Then the linguistic scale function φ can be expressed as

$$\varphi : S \rightarrow \theta_i (i = 1, 2, \ldots, 2t + 1),$$

where $0 < \theta_1 < \theta_2 < \cdots < \theta_{2t+1} \leq 1$.

The function φ monotonically increases with respect to subscript i. $\theta_i (i = 1, 2, \ldots, 2t + 1)$ is used to reflect the preferences of decision-makers, while assessment arguments are described in linguistic terms of $s_i \in S$. Therefore, these functions and values can identify differences in semantics.

Three rectified linguistic scale functions, which would be preferable in practice since they could yield more deterministic results when faced with differences in semantics, are outlined below.

1. The rectified average linguistic scale function can be expressed as $\varphi_1 = \varphi(s_i) = \frac{i}{2(t+1)}$, $i = 1, 2, \ldots, 2t + 1$.

The evaluation scale of the above linguistic information is usually averaged. Although this function is simple and frequently used, it lacks a reasonable theoretical basis [65].

This function also usually results in information loss and distortion during the aggregation process.

2. The rectified composite linguistic scale function can be expressed as

$$\varphi_2 = \varphi(s_i) = \begin{cases} (c^{t+1} - c^{t+1-i})/(2c^{t+1} - 2), & i = 1,2,\ldots,t+1 \\ (c^{t+1} + c^{i-t-1} - 2)/(2c^{t+1} - 2), & i = t+2, t+3, \ldots, 2t+1 \end{cases},$$

where c is a constant, and usually $c \in [1.36, 1.4]$ [66, 67].

When $c \in [1.36, 1.4]$, as the middle linguistic subscripts expand on both sides of the equation, the absolute deviation between each pair of adjacent linguistic terms also increases.

3. The rectified developed linguistic scale function can be expressed as

$$\varphi_3 = \varphi(s_i) = \begin{cases} [(t+1)^p - (t+1-i)^p]/2(t+1)^p, & i = 1,2,\ldots,t+1 \\ [(t+1)^q + (i-t-1)^q]/2(t+1)^q, & i = t+2, t+3, \ldots, 2t+1 \end{cases},$$

where $p, q \in [0, 1]$.

As the middle linguistic subscripts expand on both sides of the equation, the absolute deviation between each pair of adjacent linguistic terms decreases. p and q represent the curvatures of the subjective value functions of gains and losses, respectively [68]. Risky decision-makers usually select large p and q values. In contrast, conservative decision-makers usually select small p and q values.

Similarly, in order to preserve all of the assessment arguments and facilitate aggregation, the linguistic term set $S = \{s_i | i = 1, 2, \ldots, 2t+1\}$ can be expanded to the continuous linguistic term set $\tilde{S} = \{s_i | i \in [1, l]\}$, where $s_i > s_j (i > j)$ and $l\,(l > 2t+1)$ are sufficiently large positive integers. In addition, the function φ can be expanded to $\varphi^* : \tilde{S} \to R^+$ $(R^+ = \{r | r \geq 0, r \in R\})$, which is compatible with the above functions. Because $\varphi*$ increases monotonically and continuously, the inverse function of $\varphi*$, denoted as φ^{*-1}, exists.

3.2 New INLN operations

Ye [55] defined the operations of INLNs in order to describe the aggregation process during the decision-making problems.

Definition 8 [55] Let $a_1 = \langle s_{\theta(a_1)}, ([\inf T(a_1), \sup T(a_1)], [\inf I(a_1), \sup I(a_1)], [\inf F(a_1), \sup F(a_1)]) \rangle$ and $a_2 = \langle s_{\theta(a_2)}, ([\inf T(a_2), \sup T(a_2)], [\inf I(a_2), \sup I(a_2)], [\inf F(a_2), \sup F(a_2)]) \rangle$ be two INLNs and $\lambda \geq 0$. Then the operations of the INLNs can be expressed as

1. $\lambda a_1 = \left\langle s_{\lambda \times \theta(a_1)}, \left(\left[1 - (1 - \inf T(a_1))^\lambda, \quad 1 - (1 - \sup T(a_1))^\lambda\right], \quad [\inf I^\lambda(a_1), \sup I^\lambda(a_1)], \quad [\inf F^\lambda(a_1), \sup F^\lambda\ (a_1)]\right)\right\rangle$;
2. $a_1 \oplus a_2 = \langle s_{\theta(a_1)+\theta(a_2)}, ([\inf T(a_1) + \inf T(a_2) - \inf T(a_1) \times \inf T(a_2), \sup T(a_1) + \sup T(a_2) - \sup T(a_1) \times \sup T(a_2)], [\inf I(a_1) \times \inf I(a_2), \sup I(a_1) \times \sup I(a_2)], [\inf F(a_1) \times \inf F(a_2),\ \sup F(a_1) \times\ \sup F(a_2)]) \rangle$;
3. $a_1 \otimes a_2 = \langle s_{\theta(a_1) \times \theta(a_2)}, ([\inf T(a_1) \times \inf T(a_2),\ \sup T(a_1) \times \sup T(a_2)],\ [\inf I(a_1) + \inf I(a_2)\ - \inf I(a_1) \times \inf I(a_2), \sup I(a_1)\ + sup I(a_2)\ - \sup I(a_1)\ \times sup I(a_2)],\ [\inf F(a_1) + \inf F(a_2) - \inf F(a_1) \times \inf F(a_2), \sup F(a_1) +\ \sup F(a_2) - \sup F\ (a_1)\ \times\ \sup F(a_2)]) \rangle$;
4. $a_1^\lambda = \left\langle s_{\theta^\lambda(a_1)}, \quad ([\inf T^\lambda(a_1), \quad \sup T^\lambda(a_1)], \quad [1 - (1 - \inf I(a_1))^\lambda, 1 - (1 - \sup I(a_1))^\lambda], [1 - (1 - \inf F(a_1))^\lambda, 1 - (1 - \sup F(a_1))^\lambda]) \right\rangle$.

However, some limitations of Definition 8 exist.

1. The linguistic terms and three degrees of membership of the INLNs are assumed to be separate, neglecting any possible interrelationships.

Example 2 Let $a_1 = \langle s_2, ([0,0], [0,0], [1,1]) \rangle$ and $a_2 = \langle s_3, ([1,1], [0,0], [0,0]) \rangle$ be two INLNs.

$$\begin{aligned} a_1 \oplus a_2 &= \langle s_2, ([0,0], [0,0], [1,1]) \rangle \\ &\quad \oplus \langle s_3, ([1,1], [0,0], [0,0]) \rangle \\ &= \langle s_5, ([1,1], [0,0], [0,0]) \rangle. \end{aligned}$$

This result is inaccurate because the degree of falsity-membership of a_1, the correlations among the linguistic values, and the three degrees of membership of a_1 and a_2 were not considered. Thus, these operations would be impractical.

2. The linguistic terms are directly operated upon according to their subscripts, and the absolute deviations of any two pairs of adjacent linguistic terms are assumed to be equal. Thus, these operations would not reflect differences in semantics.

In order to overcome the limitations presented by the operations proposed by Ye [55], new INLN operations based on the linguistic scale function were defined, as shown below.

Definition 9 Let $a_1 = \langle s_{\theta(a_1)}, ([\inf T(a_1), \sup T(a_1)], [\inf I(a_1), \sup I(a_1)], [\inf F(a_1), \sup F(a_1)]) \rangle$ and $a_2 = \langle s_{\theta(a_2)}, ([\inf T(a_2), \sup T(a_2)], [\inf I(a_2), \sup I(a_2)], [\inf F(a_2), \sup F(a_2)]) \rangle$ be two INLNs and $\lambda \geq 0$. Then the modified operations of two INLNs can be expressed as

1. $\text{neg}(a_1) = \left\langle \varphi^{*-1}\left(\varphi^*(s_{2t+1}) - \varphi^*(s_{\theta(a_1)})\right), \quad ([\inf T(a_1), \sup T(a_1)], [\inf I(a_1), \sup I(a_1)], [\inf F(a_1), \sup F(a_1)]) \right\rangle$;
2. $\lambda a_1 = \left\langle \varphi^{*-1}\left(\lambda\varphi^*(s_{\theta(a_1)})\right), ([\inf T(a_1), \sup T(a_1)], [\inf I(a_1), \sup I(a_1)], [\inf F(a_1), \sup F(a_1)]) \right\rangle$;
3. $a_1 \oplus a_2 = \Big\langle \varphi^{*-1}\left(\varphi^*(s_{\theta(a_1)}) + \varphi^*(s_{\theta(a_2)})\right), \Big(\Big[\frac{\varphi^*(s_{\theta(a_1)}) \inf T(a_1) + \varphi^*(s_{\theta(a_2)}) \inf T(a_2)}{\varphi^*(s_{\theta(a_1)}) + \varphi^*(s_{\theta(a_2)})}, \frac{\varphi^*(s_{\theta(a_1)}) \sup T(a_1) + \varphi^*(s_{\theta(a_2)}) \sup T(a_2)}{\varphi^*(s_{\theta(a_1)}) + \varphi^*(s_{\theta(a_2)})}\Big], \Big[\frac{\varphi^*(s_{\theta(a_1)}) \inf I(a_1) + \varphi^*(s_{\theta(a_2)}) \inf I(a_2)}{\varphi^*(s_{\theta(a_1)}) + \varphi^*(s_{\theta(a_2)})}, \frac{\varphi^*(s_{\theta(a_1)}) \sup I(a_1) + \varphi^*(s_{\theta(a_2)}) \sup I(a_2)}{\varphi^*(s_{\theta(a_1)}) + \varphi^*(s_{\theta(a_2)})}\Big], \Big[\frac{\varphi^*(s_{\theta(a_1)}) \inf F(a_1) + \varphi^*(s_{\theta(a_2)}) \inf F(a_2)}{\varphi^*(s_{\theta(a_1)}) + \varphi^*(s_{\theta(a_2)})}, \frac{\varphi^*(s_{\theta(a_1)}) \sup F(a_1) + \varphi^*(s_{\theta(a_2)}) \sup F(a_2)}{\varphi^*(s_{\theta(a_1)}) + \varphi^*(s_{\theta(a_2)})}\Big]\Big)\Big\rangle$;
4. $a_1 \otimes a_2 = \left\langle \varphi^{*-1}\left(\varphi^*(s_{\theta(a_1)})\varphi^*(s_{\theta(a_2)})\right), ([\inf T(a_1) \times \inf T(a_2), \sup T(a_1) \times \sup T(a_2)], [\inf I(a_1) + \inf I(a_2) - \inf I(a_1) \times \inf I(a_2), \sup I(a_1) + supI(a_2) - \sup I(a_1) \times supI(a_2)], [\inf F(a_1) + \inf F(a_2) - \inf F(a_1) \times \inf F(a_2), \sup F(a_1) + \sup F(a_2) - \sup F(a_1) \times \sup F(a_2)]) \right\rangle$;
5. $a_1^{\lambda} = \left\langle \varphi^{*-1}\left(\left(\varphi^*(s_{\theta(a_1)})\right)^{\lambda}\right), \left([\inf T^{\lambda}(a_1), \ \sup T^{\lambda}(a_1)], \left[1 - (1 - \inf I(a_1))^{\lambda}, \ 1 - (1 - \sup I(a_1))^{\lambda}\right], \ [1 - (1 - \inf \ F(a_1))^{\lambda}, 1 - [1 - \sup F(a_1)]^{\lambda}]\right)\right\rangle$;
6. $1/a_1 = \left\langle \varphi^{*-1}\left(1/\varphi^*(s_{\theta(a_1)})\right), ([\inf T(a_1), \sup T(a_1)], [\inf I(a_1), \sup I(a_1)], [\inf F(a_1), \sup F(a_1)]) \right\rangle$.

However, λa_1, $a_1 \oplus a_2$, $a_1 \otimes a_2$, a_1^{λ}, and $1/a_1$ do not appear separately in actual applications due to the negligibility of their values. Thus, the values of $a_1 \oplus a_2$, λa_1, and $1/a_1$ are only combined during the aggregation process.

Example 3 Let $S = \{s_1, s_2 s_3, s_4, s_5, s_6, s_7\}$ be a linguistic term set and $a_1 = \langle s_3, ([0.5, 0.6], [0.1, 0.3], [0.1, 0.3]) \rangle$, $a_2 = \langle s_5, ([0.4, 0.5], [0.2, 0.3], [0.3, 0.4]) \rangle$ be two INLNs, where $\lambda = 2$.

If $\varphi^*(s_i) = \varphi_3$, $p = 0.8$, and $q = 0.7$, then

1. $\text{neg}(a_1) = \langle s_{4.26}, ([0.5, 0.6], [0.1, 0.3], [0.1,0.3]) \rangle$;
2. $2a_1 = \langle s_{4.86}, ([0.5, 0.6], [0.1, 0.3], [0.1, \ 0.3]) \rangle$;
3. $a_1 \oplus a_2 = \langle s_{8.28}, ([0.43, 0.53], [0.17, 0.30], [0.24, 0.37]) \rangle$;
4. $a_1 \otimes a_2 = \langle s_{2.16}, ([0.20, 0.30], [0.28, 0.51], [0.37, 0.58]) \rangle$;
5. $a_1^{\lambda} = \langle s_{1.10}, ([0.25, 0.36], [0.19, 0.51], [0.19, 0.51]) \rangle$; and
6. $1/a_1 = \langle s_{43.51}, ([0.5, 0.6], [0.1, 0.3], [0.1,0.3]) \rangle$.

These new INLN operations, which reflect differences in semantics and account for the interrelationships among the linguistic terms and three degrees of membership of INLNs, could be used to overcome the limitations of the operations proposed by Ye [55].

The following theorem can be proven in terms of the corresponding INLN operations.

Theorem 1 *Let* α_1, α_2, *and* α_3 *be three INLNs and* $\tau \geq 0$. *Then the following equations must be true*:

1. $\alpha_1 \oplus \alpha_2 = \alpha_2 \oplus \alpha_1$;
2. $(\alpha_1 \oplus \alpha_2) \oplus \alpha_3 = \alpha_1 \oplus (\alpha_2 \oplus \alpha_3)$;
3. $\alpha_1 \otimes \alpha_2 = \alpha_2 \otimes \alpha_1$;
4. $(\alpha_1 \otimes \alpha_2) \otimes \alpha_3 = \alpha_1 \otimes (\alpha_2 \otimes \alpha_3)$;
5. $\tau\alpha_1 \oplus \tau\alpha_2 = \tau(\alpha_2 \oplus \alpha_1)$; and
6. $(\alpha_2 \otimes \alpha_1)^{\tau} = \alpha_1^{\tau} \otimes \alpha_2^{\tau}$.

3.3 Method of INLN comparison

In this subsection, a new score function and method of INLN comparison are presented.

Definition 10 [55] Let $a = \langle s_{\theta(a)}, ([\inf T(a), \sup T(a)], [\inf I(a), \sup I(a)], \ [\inf F(a), \sup F(a)]) \rangle$ be an INLN. Then the score function $E(a)$ of a can be expressed as

$$E(a) = S_{\frac{1}{6}(4 + \inf T(a) - \inf I(a) - \inf F(a) + \sup T(a) - \sup I(a) - \sup F(a))\theta(a)}.$$

However, the score function is operated upon according to the subscripts of the linguistic terms and degrees of membership, and the absolute deviations of any two pairs of adjacent linguistic terms are assumed to be equal. Thus, this score function does not reflect differences in semantics, resulting in aggregation bias. Furthermore, this function equates the indeterminacy degree of membership to the falsity degree of membership, neglecting the preferences of the decision-makers. These limitations yield unreliable and inaccurate results.

Example 4 Let $a_1 = \langle s_2, ([0.3, 0.4], [0.5, 0.7], [0, 0]) \rangle$ and $a_2 = \langle s_2, ([0.3, 0.4], [0, 0], [0.5, 0.7]) \rangle$ be two INLNs.

Then the values of $E(a_1) = s_{\frac{1}{6} \times (4 + 0.3 - 0.5 - 0 + 0.4 - 0.7 - 0)} \times 2 = s_{1.67}$ and $E(a_2) = s_{1.67}$ can be calculated using Definition 10.

The expression $E(a_1) = E(a_2)$ denotes that $a_1 = a_2$. However, a_1 is obviously superior to a_2.

In order to overcome the limitations presented by Definition 10, a new score function was developed in order to reflect the levels of optimism, compromise, and pessimism expressed by decision-makers.

Definition 11 Let $a = \langle s_{\theta(a)}, ([\inf T(a), \sup T(a)], [\inf I(a), \sup I(a)], [\inf F(a), \sup F(a)])\rangle$ be an INLN. Then the score function of a can be expressed as

$$S(a) = \alpha\varphi^*(s_{\theta(a)})[0.5(\sup T(a) + 1 - \inf F(a)) + \alpha \sup I(a)] + (1-\alpha)\varphi^*(s_{\theta(a)})[0.5(\inf T(a) + 1 - \sup F(a)) + \alpha \inf I(a)],$$

where the values of $\alpha \in [0,1]$ reflect the attitudes of the decision-makers, and $\alpha > 0.5$, $\alpha = 0.5$, and $\alpha < 0.5$ denote the levels of optimism, compromise, and pessimism expressed by the decision-makers. In addition, different score functions can be obtained by applying different linguistic scale functions.

Definition 12 Let a and b be two INLNs. Then the INLN comparison method can be expressed by the following statements:

1. if $S(a) > S(b)$, then $a \succ b$, i.e. a is superior to b;
2. if $S(a) = S(b)$, then, $a \sim b$, i.e. a is equal to b; and
3. if $S(a) < S(b)$, then $a \prec b$, i.e. b is superior to a.

Example 5 Let $S_1 = \{s_1, s_2, \ldots, s_7\}$ be a linguistic term set and $\alpha = 0.7$. Using the data presented in Example 4, the following can be obtained.

If $\varphi^*(s_i) = \varphi_3$, $p = 0.8$ and $q = 0.7$, then the values of a_1 and a_2 can be calculated as

$$S(a_1) = 0.7 \times \left[(4^{0.8} - 2^{0.8})/(2 \times 4^{0.8})\right] \times [0.5 \times (0.4 + 1 - 0) + 0.7 \times 0.7] + 0.3 \times \left[(4^{0.8} - 2^{0.8})/(2 \times 4^{0.8})\right] \times [0.5 \times (0.3 + 1 - 0) + 0.7 \times 0.5] = 0.241;$$

$$S(a_2) = 0.7 \times \left[(4^{0.8} - 2^{0.8})/(2 \times 4^{0.8})\right] \times [0.5 \times (0.4 + 1 - 0.5) + 0.7 \times 0] + 0.3 \times \left[(4^{0.8} - 2^{0.8})/(2 \times 4^{0.8})\right] \times [0.5 \times (0.3 + 1 - 0.7) + 0.7 \times 0] = 0.086.$$

Similarly, if $\varphi^*(s_i) = \varphi_1$, then $S(a_1) = 0.324$ and $S(a_2) = 0.116$, and if $\varphi^*(s_i) = \varphi_2$, then $S(a_1) = 0.375$ and $S(a_2) = 0.134$.

Thus, the different rectified linguistic scale functions yielded similar results ($a_1 \succ a_2$). Because the proposed score function compensates for the limitations presented by the score function given in [55] by reflecting various semantic situations and calculating the indeterminacy degree of membership by accounting for the preferences of decision-makers, the results obtained using the proposed score function were closer to the expected results than those obtained using the score function presented in [55].

4 Generalized interval neutrosophic linguistic prioritized harmonic operators

In this section, GINLPWHM and GINLPWHHM operators based on the PA and HM operators are proposed. The new INLN operations are applied due to their flexibility and accuracy.

Definition 13 Let $a_i = \langle s_{\theta(a_i)}, ([\inf T(a_i), \sup T(a_i)], [\inf I(a_i), \sup I(a_i)], [\inf F(a_i), \sup F(a_i)])\rangle (i = 1, 2, \cdots, n)$ be a collection of INLNs. Then the generalized interval neutrosophic linguistic prioritized weighted harmonic mean (GINLPWHM) operator can be expressed as

$$\text{GINLPWHM}(a_1, a_2, \ldots, a_n) = \frac{1}{\left(\frac{T_1}{\sum_{i=1}^n T_i}(1/a_1)^\lambda \oplus \frac{T_2}{\sum_{i=1}^n T_i}(1/a_2)^\lambda \oplus \cdots \oplus \frac{T_n}{\sum_{i=1}^n T_i}(1/a_n)^\lambda\right)^{\frac{1}{\lambda}}} = \frac{1}{\left(\oplus_{i=1}^n \left(\frac{T_i(1/a_i)^\lambda}{\sum_{i=1}^n T_i}\right)\right)^{\frac{1}{\lambda}}},$$

where $\lambda > 0$, $T_1 = 1$, $T_i = \prod_{j=1}^{i-1} S(a_j)$ $(i = 2, 3, \cdots, n)$, and $S(a_j)$ is the score function of a_j.

Theorem 2 *Let* $a_i = \langle s_{\theta(a_i)}, ([\inf T(a_i), \sup T(a_i)], [\inf I(a_i), \sup I(a_i)], [\inf F(a_i), \sup F(a_i)])\rangle (i = 1, 2, \ldots, n)$ *be a collection of INLNs. Then the aggregated result obtained* via *the GINLPWHM operator is also an INLN, and*

$$\text{GINLPWHM}(a_1, a_2, \ldots, a_n) = \frac{1}{\left(\frac{T_1}{\sum_{i=1}^n T_i}(1/a_1)^\lambda \oplus \frac{T_2}{\sum_{i=1}^n T_i}(1/a_2)^\lambda \oplus \cdots \oplus \frac{T_n}{\sum_{i=1}^n T_i}(1/a_n)^\lambda\right)^{\frac{1}{\lambda}}}$$
$$= \left\langle \varphi^{*-1}\left(\left(\frac{1}{\sum_{i=1}^n \Upsilon_{a_i}}\right)^{\frac{1}{\lambda}}\right), \left(\left[\left(\frac{\sum_{i=1}^n \left(\Upsilon_{a_i}(\inf T(a_i))^\lambda\right)}{\sum_{i=1}^n \Upsilon_{a_i}}\right)^{\frac{1}{\lambda}}, \left(\frac{\sum_{i=1}^n \left(\Upsilon_{a_i}(\sup T(a_i))^\lambda\right)}{\sum_{i=1}^n \Upsilon_{a_i}}\right)^{\frac{1}{\lambda}}\right],\right.\right.$$
$$\left[1 - \left(1 - \frac{\sum_{i=1}^n \left(\Upsilon_{a_i}\left(1-(1-\inf I(a_i))^\lambda\right)\right)}{\sum_{i=1}^n \Upsilon_{a_i}}\right)^{\frac{1}{\lambda}}, 1 - \left(1 - \frac{\sum_{i=1}^n \left(\Upsilon_{a_i}\left(1-(1-\sup I(a_i))^\lambda\right)\right)}{\sum_{i=1}^n \Upsilon_{a_i}}\right)^{\frac{1}{\lambda}}\right],$$
$$\left.\left.\left[1 - \left(1 - \frac{\sum_{i=1}^n \left(\Upsilon_{a_i}\left(1-(1-\inf F(a_i))^\lambda\right)\right)}{\sum_{i=1}^n \Upsilon_{a_i}}\right)^{\frac{1}{\lambda}}, 1 - \left(1 - \frac{\sum_{i=1}^n \left(\Upsilon_{a_i}\left(1-(1-\sup F(a_i))^\lambda\right)\right)}{\sum_{i=1}^n \Upsilon_{a_i}}\right)^{\frac{1}{\lambda}}\right]\right)\right\rangle,$$

where $\lambda > 0$, $\Upsilon_{a_i} = \frac{T_i}{\sum_{i=1}^n T_i} \Big/ \left(\varphi^*(s_{\theta(a_i)})\right)^\lambda$, $T_1 = 1$, $T_i = \prod_{j=1}^{i-1} S(a_j)\ (i = 2, 3, \ldots, n)$, *and* $S(a_j)$ *is the score function of* a_j.

Proof Theorem 2 can be proven using Definition 9 via the mathematical induction of n.

1. If $n = 2$, since

$$\frac{T_1}{\sum_{i=1}^n T_i}(1/a_1)^\lambda = \left\langle \varphi^{*-1}(\Upsilon_{a_1}), \left(\left[(\inf T(a_1))^\lambda, (\sup T(a_1))^\lambda\right], \left[1-(1-\inf I(a_1))^\lambda, 1-(1-\sup I(a_1))^\lambda\right], \left[1-(1-\inf F(a_1))^\lambda, 1-(1-\sup F(a_1))^\lambda\right]\right)\right\rangle,$$

$$\frac{T_1}{\sum_{i=1}^n T_i}(1/a_2)^\lambda = \left\langle \varphi^{*-1}(\Upsilon_{a_2}), \left(\left[(\inf T(a_2))^\lambda, (\sup T(a_2))^\lambda\right], \left[1-(1-\inf I(a_2))^\lambda, 1-(1-\sup I(a_2))^\lambda\right], \left[1-(1-\inf F(a_2))^\lambda, 1-(1-\sup F(a_2))^\lambda\right]\right)\right\rangle;$$

then

$$\text{GINLPWHM}(a_1, a_2) = \frac{1}{\left(\frac{T_1}{\sum_{i=1}^n T_i}(1/a_1)^\lambda \oplus \frac{T_2}{\sum_{i=1}^n T_i}(1/a_2)^\lambda\right)^{\frac{1}{\lambda}}}$$
$$= \left\langle \varphi^{*-1}\left(\frac{1}{\Upsilon_{a_1}+\Upsilon_{a_2}}\right), \left(\left[\frac{\Upsilon_{a_1}(\inf T(a_1))^\lambda + \Upsilon_{a_2}(\inf T(a_2))^\lambda}{\Upsilon_{a_1}+\Upsilon_{a_2}}, \frac{\Upsilon_{a_1}(\sup T(a_1))^\lambda + \Upsilon_{a_2}(\sup T(a_2))^\lambda}{\Upsilon_{a_1}+\Upsilon_{a_2}}\right],\right.\right.$$
$$\left[\frac{\Upsilon_{a_1}\left(1-(1-\inf I(a_1))^\lambda\right) + \Upsilon_{a_2}\left(1-(1-\inf I(a_2))^\lambda\right)}{\Upsilon_{a_1}+\Upsilon_{a_2}}, \frac{\Upsilon_{a_1}\left(1-(1-\sup I(a_1))^\lambda\right) + \Upsilon_{a_2}\left(1-(1-\sup I(a_2))^\lambda\right)}{\Upsilon_{a_1}+\Upsilon_{a_2}}\right],$$
$$\left.\left.\left[\frac{\Upsilon_{a_1}\left(1-(1-\inf F(a_1))^\lambda\right) + \Upsilon_{a_2}\left(1-(1-\inf F(a_2))^\lambda\right)}{\Upsilon_{a_1}+\Upsilon_{a_2}}, \frac{\Upsilon_{a_1}\left(1-(1-\sup F(a_1))^\lambda\right) + \Upsilon_{a_2}\left(1-(1-\sup F(a_2))^\lambda\right)}{\Upsilon_{a_1}+\Upsilon_{a_2}}\right]\right)\right\rangle.$$

2. If $n = k$, then

$$\mathrm{GINLPWHM}(a_1, a_2, \ldots, a_k) = \frac{1}{\left(\frac{T_1}{\sum_{i=1}^{k} T_i}(1/a_1)^{\lambda} \oplus \frac{T_2}{\sum_{i=1}^{k} T_i}(1/a_2)^{\lambda} \oplus \cdots \oplus \frac{T_k}{\sum_{i=1}^{k} T_i}(1/a_k)^{\lambda}\right)^{\frac{1}{\lambda}}}$$

$$= \left\langle \varphi^{*-1}\left(\frac{1}{\sum_{i=1}^{k} \Upsilon_{a_i}}\right), \left(\left[\frac{\sum_{i=1}^{k}\left(\Upsilon_{a_i}(\inf T(a_i))^{\lambda}\right)}{\sum_{i=1}^{k}\Upsilon_{a_i}}, \frac{\sum_{i=1}^{k}\left(\Upsilon_{a_i}(\sup T(a_i))^{\lambda}\right)}{\sum_{i=1}^{k}\Upsilon_{a_i}}\right], \left[\frac{\sum_{i=1}^{k}\left(\Upsilon_{a_i}\left(1-(1-\inf I(a_i))^{\lambda}\right)\right)}{\sum_{i=1}^{k}\Upsilon_{a_i}}, \frac{\sum_{i=1}^{k}\left(\Upsilon_{a_i}\left(1-(1-\sup I(a_i))^{\lambda}\right)\right)}{\sum_{i=1}^{k}\Upsilon_{a_i}}\right], \left[\frac{\sum_{i=1}^{k}\left(\Upsilon_{a_i}\left(1-(1-\inf F(a_i))^{\lambda}\right)\right)}{\sum_{i=1}^{k}\Upsilon_{a_i}}, \frac{\sum_{i=1}^{k}\left(\Upsilon_{a_i}\left(1-(1-\sup F(a_i))^{\lambda}\right)\right)}{\sum_{i=1}^{k}\Upsilon_{a_i}}\right]\right)\right\rangle.$$

When $n = k + 1$, according to the operations in Definition 9,

$$\mathrm{GINLPWHM}(a_1, a_2, \ldots, a_{k+1}) = \frac{1}{\left(\oplus_{i=1}^{k}\left(\frac{T_i}{\sum_{j=1}^{n} T_j}(1/a_i)^{\lambda}\right) \oplus \frac{T_{k+1}}{\sum_{j=1}^{n} T_j}(1/a_{k+1})^{\lambda}\right)^{\frac{1}{\lambda}}}$$

$$= \left\langle \varphi^{*-1}\left(\frac{1}{\sum_{i=1}^{k+1} \Upsilon_{a_i}}\right), \left(\left[\frac{\sum_{i=1}^{k+1}\left(\Upsilon_{a_i}(\inf T(a_i))^{\lambda}\right)}{\sum_{i=1}^{k}\Upsilon_{a_i}}, \frac{\sum_{i=1}^{k+1}\left(\Upsilon_{a_i}(\sup T(a_i))^{\lambda}\right)}{\sum_{i=1}^{k}\Upsilon_{a_i}}\right], \left[\frac{\sum_{i=1}^{k+1}\left(\Upsilon_{a_i}\left(1-(1-\inf I(a_i))^{\lambda}\right)\right)}{\sum_{i=1}^{k}\Upsilon_{a_i}}, \frac{\sum_{i=1}^{k+1}\left(\Upsilon_{a_i}\left(1-(1-\sup I(a_i))^{\lambda}\right)\right)}{\sum_{i=1}^{k}\Upsilon_{a_i}}\right], \left[\frac{\sum_{i=1}^{k+1}\left(\Upsilon_{a_i}\left(1-(1-\inf F(a_i))^{\lambda}\right)\right)}{\sum_{i=1}^{k}\Upsilon_{a_i}}, \frac{\sum_{i=1}^{k+1}\left(\Upsilon_{a_i}\left(1-(1-\sup F(a_i))^{\lambda}\right)\right)}{\sum_{i=1}^{k}\Upsilon_{a_i}}\right]\right)\right\rangle.$$

Thus, $n = k + 1$, and Theorem 2 is true.

According to (1) and (2), Theorem 2 holds for any value of n.

Theorem 3 *Let* $a_i = \langle s_{\theta(a_i)}, ([\inf T(a_i), \sup T(a_i)], [\inf I(a_i), \sup I(a_i)], [\inf F(a_i), \sup F(a_i)])\rangle (i = 1, 2, \cdots, n)$ *be a collection of INLNs. Then* $T_1 = 1$, $T_i = \prod_{j=1}^{i-1} S(a_j)\, (i = 2, 3, \cdots, n)$, *and* $S(a_j)$ *is the score function of* a_j. *If* $\upsilon > 0$, *then*

$$\mathrm{GINLPWHM}(\upsilon a_1, \upsilon a_2, \ldots, \upsilon a_n) = \upsilon \mathrm{GINLPWHM}(a_1, a_2, \ldots, a_n).$$

Proof According to Definition 9 and Theorem 2,

$$\mathrm{GINLPWHM}(\upsilon a_1, \upsilon a_2, \ldots, \upsilon a_n) = \frac{1}{\left(\frac{T_1}{\sum_{i=1}^{n} T_i}\left(\frac{1}{\upsilon a_1}\right)^{\lambda} \oplus \frac{T_2}{\sum_{i=1}^{n} T_i}\left(\frac{1}{\upsilon a_2}\right)^{\lambda} \oplus \cdots \oplus \frac{T_n}{\sum_{i=1}^{n} T_i}\left(\frac{1}{\upsilon a_n}\right)^{\lambda}\right)^{\frac{1}{\lambda}}}$$

$$= \left\langle \varphi^{*-1}\left(\left(\frac{1}{\sum_{i=1}^{n}(\Upsilon_{a_i}/\upsilon^{\lambda})}\right)^{\frac{1}{\lambda}}\right), \left(\left[\left(\frac{\sum_{i=1}^{n}\left((\Upsilon_{a_i}/\upsilon^{\lambda})(\inf T(a_i))^{\lambda}\right)}{\sum_{i=1}^{n}(\Upsilon_{a_i}/\upsilon^{\lambda})}\right)^{\frac{1}{\lambda}}, \left(\frac{\sum_{i=1}^{n}\left((\Upsilon_{a_i}/\upsilon^{\lambda})(\sup T(a_i))^{\lambda}\right)}{\sum_{i=1}^{n}(\Upsilon_{a_i}/\upsilon^{\lambda})}\right)^{\frac{1}{\lambda}}\right],\right.\right.$$

$$\left[\frac{\sum_{i=1}^{n}\left((\Upsilon_{a_i}/\upsilon^{\lambda})\left(1-(1-\inf I(a_i))^{\lambda}\right)\right)}{\sum_{i=1}^{n}(\Upsilon_{a_i}/\upsilon^{\lambda})}, \frac{\sum_{i=1}^{n}\left((\Upsilon_{a_i}/\upsilon^{\lambda})\left(1-(1-\sup I(a_i))^{\lambda}\right)\right)}{\sum_{i=1}^{n}(\Upsilon_{a_i}/\upsilon^{\lambda})}\right]$$

$$\left.\left.\left[\frac{\sum_{i=1}^{n}\left((\Upsilon_{a_i}/\upsilon^{\lambda})\left(1-(1-\inf F(a_i))^{\lambda}\right)\right)}{\sum_{i=1}^{n}(\Upsilon_{a_i}/\upsilon^{\lambda})}, \frac{\sum_{i=1}^{n}\left((\Upsilon_{a_i}/\upsilon^{\lambda})\left(1-(1-\sup F(a_i))^{\lambda}\right)\right)}{\sum_{i=1}^{n}(\Upsilon_{a_i}/\upsilon^{\lambda})}\right]\right)\right\rangle$$

$$= \left\langle \varphi^{*-1}\left(\upsilon\left(\frac{1}{\sum_{i=1}^{n}\Upsilon_{a_i}}\right)^{\frac{1}{\lambda}}\right), \left(\left[\left(\frac{\sum_{i=1}^{n}\left(\Upsilon_{a_i}(\inf T(a_i))^{\lambda}\right)}{\sum_{i=1}^{n}\Upsilon_{a_i}}\right)^{\frac{1}{\lambda}}, \left(\frac{\sum_{i=1}^{n}\left(\Upsilon_{a_i}(\sup T(a_i))^{\lambda}\right)}{\sum_{i=1}^{n}\Upsilon_{a_i}}\right)^{\frac{1}{\lambda}}\right],\right.\right.$$

$$\left[\frac{\sum_{i=1}^{n}\left(\Upsilon_{a_i}\left(1-(1-\inf I(a_i))^{\lambda}\right)\right)}{\sum_{i=1}^{n}\Upsilon_{a_i}}, \frac{\sum_{i=1}^{n}\left(\Upsilon_{a_i}\left(1-(1-\sup I(a_i))^{\lambda}\right)\right)}{\sum_{i=1}^{n}\Upsilon_{a_i}}\right],$$

$$\left.\left.\left[\frac{\sum_{i=1}^{n}\left(\Upsilon_{a_i}\left(1-(1-\inf F(a_i))^{\lambda}\right)\right)}{\sum_{i=1}^{n}\Upsilon_{a_i}}, \frac{\sum_{i=1}^{n}\left(\Upsilon_{a_i}\left(1-(1-\sup F(a_i))^{\lambda}\right)\right)}{\sum_{i=1}^{n}\Upsilon_{a_i}}\right]\right)\right\rangle$$

$$= \upsilon\mathrm{GINLPWHM}(a_1, a_2, \ldots, a_n).$$

Thus, $\mathrm{GINLPWHM}(\upsilon a_1, \upsilon a_2, \ldots, \upsilon a_n) = \upsilon\mathrm{GINLPWHM}(a_1, a_2, \ldots, a_n)$.

Theorem 4 *Let* $a_i = \langle s_{\theta(a_i)}, ([\inf T(a_i), \sup T(a_i)], [\inf I(a_i), \sup I(a_i)], [\inf F(a_i), \sup F(a_i)])\rangle (i = 1, 2, \ldots, n)$ *be a collection of INLNs. If* $a_i = a\,(i = 1, 2, \ldots, n)$, *then*

$\mathrm{GINLPWHM}(a_1, a_2, \ldots, a_n) = a$.

Proof According to Theorem 2,

$$\begin{aligned}&\mathrm{GINLPWHM}(a_1,a_2,\ldots,a_n)\\&=\left\langle\varphi^{*-1}\left(\left(\frac{1}{\sum_{i=1}^{n}\Upsilon_{a_i}}\right)^{\frac{1}{\lambda}}\right),\left(\left[\left(\frac{\sum_{i=1}^{n}\left(\Upsilon_{a_i}(\inf T(a_i))^{\lambda}\right)}{\sum_{i=1}^{n}\Upsilon_{a_i}}\right)^{\frac{1}{\lambda}},\left(\frac{\sum_{i=1}^{n}\left(\Upsilon_{a_i}(\sup T(a_i))^{\lambda}\right)}{\sum_{i=1}^{n}\Upsilon_{a_i}}\right)^{\frac{1}{\lambda}}\right],\right.\right.\\&\left[1-\left(1-\frac{\sum_{i=1}^{n}\left(\Upsilon_{a_i}\left(1-(1-\inf I(a_i))^{\lambda}\right)\right)}{\sum_{i=1}^{n}\Upsilon_{a_i}}\right)^{\frac{1}{\lambda}},1-\left(1-\frac{\sum_{i=1}^{n}\left(\Upsilon_{a_i}\left(1-(1-\sup I(a_i))^{\lambda}\right)\right)}{\sum_{i=1}^{n}\Upsilon_{a_i}}\right)^{\frac{1}{\lambda}}\right],\\&\left.\left.\left[1-\left(1-\frac{\sum_{i=1}^{n}\left(\Upsilon_{a_i}\left(1-(1-\inf F(a_i))^{\lambda}\right)\right)}{\sum_{i=1}^{n}\Upsilon_{a_i}}\right)^{\frac{1}{\lambda}},1-\left(1-\frac{\sum_{i=1}^{n}\left(\Upsilon_{a_i}\left(1-(1-\sup F(a_i))^{\lambda}\right)\right)}{\sum_{i=1}^{n}\Upsilon_{a_i}}\right)^{\frac{1}{\lambda}}\right]\right)\right\rangle.\end{aligned}$$

Since $a_i = a$ for $\forall i$, then $\sum_{i=1}^{n}\Upsilon_{a_i} = \sum_{i=1}^{n}\left(\frac{T_i/\sum_{i=1}^{n}T_i}{\left(\varphi^*(s_{\theta(a_i)})\right)^{\lambda}}\right) = \sum_{i=1}^{n}\left(\frac{T_i/\sum_{i=1}^{n}T_i}{\left(\varphi^*(s_{\theta(a)})\right)^{\lambda}}\right) = \frac{1}{\left(\varphi^*(s_{\theta(a)})\right)^{\lambda}}$, and

$$\begin{aligned}&\mathrm{GINLPWHM}(a_1,a_2,\ldots,a_n)\\&=\left\langle\varphi^{*-1}\left(\left(\left(\varphi^*(s_{\theta(a)})\right)^{\lambda}\right)^{\frac{1}{\lambda}}\right),\left(\left[\left(\frac{\sum_{i=1}^{n}\left(\Upsilon_{a_i}(\inf T(a))^{\lambda}\right)}{\sum_{i=1}^{n}\Upsilon_{a_i}}\right)^{\frac{1}{\lambda}},\left(\frac{\sum_{i=1}^{n}\left(\Upsilon_{a_i}(\sup T(a))^{\lambda}\right)}{\sum_{i=1}^{n}\Upsilon_{a_i}}\right)^{\frac{1}{\lambda}}\right],\right.\right.\\&\left[1-\left(1-\frac{\sum_{i=1}^{n}\left(\Upsilon_{a_i}\left(1-(1-\sup I(a))^{\lambda}\right)\right)}{\sum_{i=1}^{n}\Upsilon_{a_i}}\right)^{\frac{1}{\lambda}},1-\left(1-\frac{\sum_{i=1}^{n}\left(\Upsilon_{a_i}\left(1-(1-\sup I(a))^{\lambda}\right)\right)}{\sum_{i=1}^{n}\Upsilon_{a_i}}\right)^{\frac{1}{\lambda}}\right],\\&\left.\left.\left[1-\left(1-\frac{\sum_{i=1}^{n}\left(\Upsilon_{a_i}\left(1-(1-\sup F(a))^{\lambda}\right)\right)}{\sum_{i=1}^{n}\Upsilon_{a_i}}\right)^{\frac{1}{\lambda}},1-\left(1-\frac{\sum_{i=1}^{n}\left(\Upsilon_{a_i}\left(1-(1-\sup F(a))^{\lambda}\right)\right)}{\sum_{i=1}^{n}\Upsilon_{a_i}}\right)^{\frac{1}{\lambda}}\right]\right)\right\rangle\\&=\left\langle s_{\theta(a)},([\inf T(a),\sup T(a)],[\inf I(a),\sup I(a)],[\inf F(a),\sup F(a)])\right\rangle\\&=a.\end{aligned}$$

Some special cases of the GINLPWHM operator are as follows.

1. If $\lambda = 1$, the GINLPWHM operator becomes the interval neutrosophic linguistic prioritized weight harmonic mean (INLPWHM) operator:

$$\text{INLPWHM}(a_1, a_2, \ldots, a_n) = 1\Big/\left(\oplus_{i=1}^{n}\left(\frac{T_i(1/a_i)}{\sum_{i=1}^{n} T_i}\right)\right) = \left\langle \varphi^{*-1}\left(\frac{1}{\sum_{i=1}^{n}\Upsilon_{a_i}}\right), \left(\left[\frac{\sum_{i=1}^{n}(\Upsilon_{a_i}\inf T(a_i))}{\sum_{i=1}^{n}\Upsilon_{a_i}}, \frac{\sum_{i=1}^{n}(\Upsilon_{a_i}\sup T(a_i))}{\sum_{i=1}^{n}\Upsilon_{a_i}}\right], \left[\frac{\sum_{i=1}^{n}(\Upsilon_{a_i}\inf I(a_i))}{\sum_{i=1}^{n}\Upsilon_{a_i}}, \frac{\sum_{i=1}^{n}(\Upsilon_{a_i}\sup I(a_i))}{\sum_{i=1}^{n}\Upsilon_{a_i}}\right], \left[\frac{\sum_{i=1}^{n}(\Upsilon_{a_i}\inf F(a_i))}{\sum_{i=1}^{n}\Upsilon_{a_i}}, \frac{\sum_{i=1}^{n}(\Upsilon_{a_i}\sup F(a_i))}{\sum_{i=1}^{n}\Upsilon_{a_i}}\right]\right)\right\rangle.$$

2. If $\lambda \to 0$, the GINLPWHM operator becomes the interval neutrosophic linguistic prioritized weight harmonic geometric (INLPWHG) operator:

$$\text{INLPWHG}(a_1,a_2,\ldots,a_n) = \prod_{i=1}^{n}(a_i)^{(T_i/\sum_{i=1}^{n}T_i)} = \left\langle \varphi^{*-1}\left(\prod_{i=1}^{n}\left(1\Big/\left((1/\varphi^{*}(s_{\theta(a_i)}))^{(T_i/\sum_{i=1}^{n}T_i)}\right)\right)\right), \left(\left[\prod_{i=1}^{n}\left((\inf T(a_i))^{(T_i/\sum_{i=1}^{n}T_i)}\right), \prod_{i=1}^{n}\left((\sup T(a_i))^{(T_i/\sum_{i=1}^{n}T_i)}\right)\right], \left[1-\prod_{i=1}^{n}\left((1-\inf I(a_i))^{(T_i/\sum_{i=1}^{n}T_i)}\right), 1-\prod_{i=1}^{n}\left((1-\sup I(a_i))^{(T_i/\sum_{i=1}^{n}T_i)}\right)\right], \left[1-\prod_{i=1}^{n}\left((1-\inf F(a_i))^{(T_i/\sum_{i=1}^{n}T_i)}\right), 1-\prod_{i=1}^{n}\left((1-\sup F(a_i))^{(T_i/\sum_{i=1}^{n}T_i)}\right)\right]\right)\right\rangle.$$

3. If $\lambda = 2$, the GINLPWHM operator becomes the interval neutrosophic linguistic prioritized weight harmonic quadratic mean (INLNPWHQM) operator:

$$\text{GINLPWHQM}(a_1, a_2, \ldots, a_n) = 1\Big/\left(\oplus_{i=1}^{n}\left(\frac{T_i(1/a_i)^2}{\sum_{i=1}^{n}T_i}\right)\right)^{\frac{1}{2}} = \left\langle \varphi^{*-1}\left(\left(\frac{1}{\sum_{i=1}^{n}\Upsilon_{a_i}}\right)^{\frac{1}{2}}\right), \left(\left[\left(\frac{\sum_{i=1}^{n}\left(\Upsilon_{a_i}(\inf T(a_i))^2\right)}{\sum_{i=1}^{n}\Upsilon_{a_i}}\right)^{\frac{1}{2}}, \left(\frac{\sum_{i=1}^{n}\left(\Upsilon_{a_i}(\sup T(a_i))^2\right)}{\sum_{i=1}^{n}\Upsilon_{a_i}}\right)^{\frac{1}{2}}\right], \left[1-\left(1-\frac{\sum_{i=1}^{n}\left(\Upsilon_{a_i}\left(1-(1-\inf I(a_i))^2\right)\right)}{\sum_{i=1}^{n}\Upsilon_{a_i}}\right)^{\frac{1}{2}}, 1-\left(1-\frac{\sum_{i=1}^{n}\left(\Upsilon_{a_i}\left(1-(1-\sup I(a_i))^2\right)\right)}{\sum_{i=1}^{n}\Upsilon_{a_i}}\right)^{\frac{1}{2}}\right], \left[1-\left(1-\frac{\sum_{i=1}^{n}\left(\Upsilon_{a_i}\left(1-(1-\inf F(a_i))^2\right)\right)}{\sum_{i=1}^{n}\Upsilon_{a_i}}\right)^{\frac{1}{2}}, 1-\left(1-\frac{\sum_{i=1}^{n}\left(\Upsilon_{a_i}\left(1-(1-\sup F(a_i))^2\right)\right)}{\sum_{i=1}^{n}\Upsilon_{a_i}}\right)^{\frac{1}{2}}\right]\right)\right\rangle.$$

Definition 14 Let $a_i = \langle s_{\theta(a_i)}, ([\inf T(a_i), \sup T(a_i)], [\inf I(a_i), \sup I(a_i)], [\inf F(a_i), \sup F(a_i)])\rangle (i = 1, 2, \ldots, n)$ be a collection of INLNs. Then the generalized interval neutrosophic linguistic prioritized hybrid harmonic mean (GINLPHHM) operator with an associated vector of $w = (w_1, w_2, \ldots, w_n)$ can be expressed as

$$\text{GINLPHHM}(a_1, a_2, \ldots, a_n) = \frac{1}{\left(w_1\left(r_{\sigma(1)}\right)^{\lambda} \oplus w_2\left(r_{\sigma(2)}\right)^{\lambda} \oplus \cdots \oplus w_n\left(r_{\sigma(n)}\right)^{\lambda}\right)^{\frac{1}{\lambda}}} = \frac{1}{\left(\oplus_{i=1}^{n}\left(w_i\left(r_{\sigma(i)}\right)^{\lambda}\right)\right)^{\frac{1}{\lambda}}},$$

where $\lambda > 0$, $w_i \geq 0\,(i = 1, 2, \cdots, n)$, $\sum_{i=1}^{n} w_i = 1$, $r_{\sigma(i)}$ is the ith largest of the prioritized weighted INLNs $r_i\left(r_i = n\frac{T_i}{\sum_{i=1}^{n}T_i}\frac{1}{a_i}, i = 1, 2, \ldots, n\right)$, $T_1 = 1$, $T_i = \prod_{j=1}^{i-1} S(a_j)$ $(i = 2, 3, \ldots, n)$, $S(a_j)$ is the score function of a_j, and n is the balancing coefficient of $r_i = n\frac{T_i}{\sum_{i=1}^{n}T_i}\frac{1}{a_i}$. The value of w_i can be determined using the method introduced in [69–71].

Theorem 5 *Let* $a_i = \langle s_{\theta(a_i)}, ([\inf T(a_i), \sup T(a_i)], [\inf I(a_i), \sup I(a_i)], [\inf F(a_i), \sup F(a_i)])\rangle (i = 1, 2, \cdots, n)$ *be a collection of INLNs with an associated vector of* $w = (w_1, w_2, \cdots, w_n)$. *The aggregated result, which is obtained* via *the GINLPHHM operator, is also an INLN, and*

$$\mathrm{GINLPHHM}(a_1, a_2, \ldots, a_n) = \frac{1}{\left(w_1\left(r_{\sigma(1)}\right)^{\lambda} \oplus w_2\left(r_{\sigma(2)}\right)^{\lambda} \oplus \cdots \oplus w_n\left(r_{\sigma(n)}\right)^{\lambda}\right)^{\frac{1}{\lambda}}}$$

$$= \left\langle \varphi^{*-1}\left(\left(\frac{1}{\sum_{i=1}^{n} \Upsilon_{r_i}}\right)^{\frac{1}{\lambda}}\right), \left(\left[\left(\frac{\sum_{i=1}^{n}\left(\Upsilon_{r_i}\left(\inf T(r_{\sigma(i)})\right)^{\lambda}\right)}{\sum_{i=1}^{n} \Upsilon_{r_i}}\right)^{\frac{1}{\lambda}}, \left(\frac{\sum_{i=1}^{n}\left(\Upsilon_{r_i}\left(\sup T(r_{\sigma(i)})\right)^{\lambda}\right)}{\sum_{i=1}^{n} \Upsilon_{r_i}}\right)^{\frac{1}{\lambda}}\right],\right.\right.$$

$$\left[1-\left(1-\frac{\sum_{i=1}^{n}\left(\Upsilon_{r_i}\left(1-\left(1-\inf I(r_{\sigma(i)})\right)^{\lambda}\right)\right)}{\sum_{i=1}^{n} \Upsilon_{r_i}}\right)^{\frac{1}{\lambda}}, 1-\left(\frac{\sum_{i=1}^{n}\left(\Upsilon_{r_i}\left(1-\left(1-\sup I(r_{\sigma(i)})\right)^{\lambda}\right)\right)}{\sum_{i=1}^{n} \Upsilon_{r_i}}\right)^{\frac{1}{\lambda}}\right],$$

$$\left.\left.\left[1-\left(1-\frac{\sum_{i=1}^{n}\left(\Upsilon_{r_i}\left(1-\left(1-\inf F(r_{\sigma(i)})\right)^{\lambda}\right)\right)}{\sum_{i=1}^{n} \Upsilon_{r_i}}\right)^{\frac{1}{\lambda}}, 1-\left(1-\frac{\sum_{i=1}^{n}\left(\Upsilon_{r_i}\left(1-\left(1-\sup F(r_{\sigma(i)})\right)^{\lambda}\right)\right)}{\sum_{i=1}^{n} \Upsilon_{r_i}}\right)^{\frac{1}{\lambda}}\right]\right)\right\rangle,$$

where $\lambda > 0$, $\Upsilon_{r_i} = w_i\left(\varphi^*(s_{\theta(r_{\sigma(i)})})\right)^{\lambda}$, $w_i \geq 0$, $\sum_{i=1}^{n} w_i = 1$, $r_{\sigma(i)}$ is the ith largest prioritized weighted INLN $r_i\left(r_i = n\frac{T_i}{\sum_{i=1}^{n} T_i}\frac{1}{a_i}, i = 1, 2, \ldots, n\right)$, $T_1 = 1$, $T_i = \prod_{j=1}^{i-1} S(a_j)\ (i = 2, 3, \ldots, n)$, $S(a_j)$ is the score function of a_j, and n is the balancing coefficient of $r_i = n\frac{T_i}{\sum_{i=1}^{n} T_i}\frac{1}{a_i}$.

The proof for the GINLPHHM operator can be found in Theorem 2.

Theorem 6 *Let* $a_i = \langle s_{\theta(a_i)}, ([\inf T(a_i), \sup T(a_i)], [\inf I(a_i), \sup I(a_i)], [\inf F(a_i), \sup F(a_i)])\rangle (i = 1, 2, \cdots, n)$ *be a collection of INLNs,* $w = (w_1, w_2, \cdots, w_n)$ *be the associated vector of the GINLPHHM operator, and* $r_{\sigma(i)}$ *be the ith largest of the prioritized weighted INLNs* $r_i\ (r_i = n\frac{T_i}{\sum_{i=1}^{n} T_i}\frac{1}{a_i}, i = 1, 2, \cdots, n)$. *In addition,* $T_1 = 1$, $T_i = \prod_{j=1}^{i-1} S(a_j)\ (i = 2, 3, \cdots, n)$, *and* $S(a_j)$ *is the score function of* a_j. *If* $\upsilon > 0$, *then*

$$\mathrm{GINLPHHM}(\upsilon a_1, \upsilon a_2, \ldots, \upsilon a_n) = \upsilon \mathrm{GINLPHHM}(a_1, a_2, \ldots, a_n).$$

The proof of this theorem can be found in Theorem 6.

Theorem 7 *Let* $a_i = \langle s_{\theta(a_i)}, ([\inf T(a_i), \sup T(a_i)], [\inf I(a_i), \sup I(a_i)], [\inf F(a_i), \sup F(a_i)])\rangle (i = 1, 2, \ldots, n)$ *be a collection of INLNs,* $r_{\sigma(i)}$ *be the ith largest of the prioritized weighted INLNs* $r_i\left(r_i = n\frac{T_i}{\sum_{i=1}^{n} T_i}\frac{1}{a_i}, i = 1, 2, \ldots, n\right)$, *and* $w = (w_1, w_2, \ldots, w_n)$ *be the associated vector of the GINLPHHM operator. In addition,* $T_1 = 1$, $T_i = \prod_{j=1}^{i-1} S(a_j)\ (i = 2, 3, \ldots, n)$, *and* $S(a_j)$ *is the score function of* a_j. *If* $a_i = a (i = 1, 2, \ldots, n)$, *then*

$$\mathrm{GINLPHHM}(a_1, a_2, \ldots, a_n) = a.$$

The proof of this theorem can be found in Theorem 4.

Theorem 8 *Let* $a_i = \langle s_{\theta(a_1)}, ([\inf T(a_i), \sup T(a_i)], [\inf I(a_i), \sup I(a_i)], [\inf F(a_i), \sup F(a_i)])\rangle (i = 1, 2, \ldots, n)$ *be a collection of INLNs and* $r_{\sigma(i)}$ *be the ith largest prioritized weighted INLN* $r_i\left(r_i = n\frac{T_i}{\sum_{i=1}^{n} T_i}\frac{1}{a_i}, i = 1, 2, \ldots, n\right)$. $T_1 = 1$, $T_i = \prod_{j=1}^{i-1} S(a_i)(1 = 2, 3, \ldots, n)$, *and* $S(a_j)$ *is the score function of* a_j. *If the associated vector of the GINLPHHM operator is* $w = (1/n, 1/n, \ldots, 1/n)$ and $\lambda = 1$, *then the GINLPHHM operator is reduced to the INLPWHM operator.*

Proof If $w = (1/n, 1/n, \ldots, 1/n)$ and $\lambda = 1$, then

$$\mathrm{GINLPHHM}(a_1, a_2, \ldots, a_n) = \frac{1}{(1/n)r_{\sigma(1)} \oplus (1/n)r_{\sigma(2)} \oplus \cdots \oplus (1/n)r_{\sigma(n)}}.$$

According to Theorem 5,

$$\mathrm{GINLPHHM}(a_1,a_2,\ldots,a_n)=\left\langle \varphi^{*-1}\left(\frac{1}{\sum_{i=1}^{n}\Upsilon_{r_i}}\right),\left(\left[\frac{\sum_{i=1}^{n}\left(\Upsilon_{r_i}\inf T(r_{\sigma(i)})\right)}{\sum_{i=1}^{n}\Upsilon_{r_i}},\frac{\sum_{i=1}^{n}\left(\Upsilon_{r_i}\sup T(r_{\sigma(i)})\right)}{\sum_{i=1}^{n}\Upsilon_{r_i}}\right],\left[\frac{\sum_{i=1}^{n}\left(\Upsilon_{r_i}\inf I(r_{\sigma(i)})\right)}{\sum_{i=1}^{n}\Upsilon_{r_i}},\frac{\sum_{i=1}^{n}\left(\Upsilon_{r_i}\sup I(r_{\sigma(i)})\right)}{\sum_{i=1}^{n}\Upsilon_{r_i}}\right],\left[\frac{\sum_{i=1}^{n}\left(\Upsilon_{r_i}\inf F(r_{\sigma(i)})\right)}{\sum_{i=1}^{n}\Upsilon_{r_i}},\frac{\sum_{i=1}^{n}\left(\Upsilon_{r_i}\sup F(r_{\sigma(i)})\right)}{\sum_{i=1}^{n}\Upsilon_{r_i}}\right]\right)\right\rangle,$$

where $\Upsilon_{r_i}=\left(\varphi^*\left(s_{\theta(r_{\sigma(i)})}\right)^{\lambda}\right)\Big/ n$.

Since $\sum_{i=1}^{n}\Upsilon_{r_i}=\sum_{i=1}^{n}\left(\frac{\varphi^*\left(s_{\theta(r_{\sigma(i)})}\right)}{n}\right)=\frac{1}{n}\sum_{i=1}^{n}\left(n\frac{T_i}{\sum_{i=1}^{n}T_i}\varphi^*\left(s_{\theta\left(\frac{1}{a_i}\right)}\right)\right)=\sum_{i=1}^{n}\left(\frac{T_i}{\sum_{i=1}^{n}T_i}\Big/\varphi^* s_{\theta(a_i)}\right)=\sum_{i=1}^{n}\Upsilon_{a_i}$,

$$\frac{\sum_{i-1}^{n}\left(\Upsilon_{r_i}\Omega(r_{\sigma(i)})\right)}{\sum_{i-1}^{n}\Upsilon_{r_i}}=\frac{\sum_{i=1}^{n}\left((1/n)\left(n\left(T_i/\sum_{i-1}^{n}T_i\right)/\varphi^*(s_{\theta(a_i)})\right)\Omega(a_i)\right)}{\sum_{i-1}^{n}\left((1/n)\left(n\left(T_i/\sum_{i=1}^{n}T_i\right)/\varphi^*\left(s_{\theta(a_i)}\right)\right)\right)}=\frac{\sum_{i-1}^{n}\left(\left(\left(T_i/\sum_{i=1}^{n}T_i\right)/\varphi^*\left(s_{\theta(a_i)}\right)\right)\Omega(a_i)\right)}{\sum_{i-1}^{n}\left(\left(\left(T_i/\sum_{i=1}^{n}T_i\right)/\varphi^*\left(s_{\theta(a_i)}\right)\right)\right)},$$

where Ω can represent any one character of the set $\{\inf T,\sup T,\inf I,\sup I,\inf F,\sup F\}$.

Thus,

Therefore, $\mathrm{GINLPHHM}(a_1a_2,\ldots,a_n)=\mathrm{INLPWHM}(a_1a_2,\ldots,a_n)$.

Some special cases of the GINLPHHM operator are as follows.

1. If $\lambda=1$, the GINLPHHM operator becomes the interval neutrosophic linguistic prioritized hybrid harmonic mean (INLPHHM) operator:

$$\mathrm{INLPHHM}(a_1,a_2,\ldots,a_n)=\frac{1}{\oplus_{i=1}^{n}\left(w_i r_{\sigma(i)}\right)}\left\langle\varphi^{*-1}\left(\frac{1}{\sum_{i=1}^{n}\Upsilon_{r_i}}\right),\left(\left[\frac{\sum_{i=1}^{n}\left(\Upsilon_{r_i}\inf T(r_{\sigma(i)})\right)}{\sum_{i=1}^{n}\Upsilon_{r_i}},\frac{\sum_{i=1}^{n}\left(\Upsilon_{r_i}\sup T(r_{\sigma(i)})\right)}{\sum_{i=1}^{n}\Upsilon_{r_i}}\right],\left[\frac{\sum_{i=1}^{n}\left(\Upsilon_{r_i}\inf I(r_{\sigma(i)})\right)}{\sum_{i=1}^{n}\Upsilon_{r_i}},\frac{\sum_{i=1}^{n}\left(\Upsilon_{r_i}\sup I(r_{\sigma(i)})\right)}{\sum_{i=1}^{n}\Upsilon_{r_i}}\right],\left[\frac{\sum_{i=1}^{n}\left(\Upsilon_{r_i}\inf F(r_{\sigma(i)})\right)}{\sum_{i=1}^{n}\Upsilon_{r_i}},\frac{\sum_{i=1}^{n}\left(\Upsilon_{r_i}\sup F(r_{\sigma(i)})\right)}{\sum_{i=1}^{n}\Upsilon_{r_i}}\right]\right)\right\rangle$$

2. If $\lambda\to 0$, the GINLPHHM operator becomes the interval neutrosophic linguistic prioritized hybrid harmonic geometric (INLPHHG) operator:

$$\begin{aligned}&\mathrm{GINLPHHM}(a_1,a_2,\ldots,a_n)\\&\left\langle\varphi^{*-1}\left(\frac{1}{\sum_{i=1}^{n}\gamma_{a_1}}\right),\left(\left[\frac{\sum_{i=1}^{n}\left(\left(\left(T_i/\sum_{i=1}^{n}T_i\right)/\varphi^*\left(s_{\theta(a_i)}\right)\right)\inf T(a_i)\right)}{\sum_{i=1}^{n}\left(\left(\left(T_i/\sum_{i=1}^{n}T_i\right)/\varphi^*\left(s_{\theta(a_i)}\right)\right)\right)},\frac{\sum_{i=1}^{n}\left(\left(\left(T_i/\sum_{i=1}^{n}T_i\right)/\varphi^*\left(s_{\theta(a_i)}\right)\right)\sup T(a_i)\right)}{\sum_{i=1}^{n}\left(\left(\left(T_i/\sum_{i=1}^{n}T_i\right)/\varphi^*\left(s_{\theta(a_i)}\right)\right)\right)}\right]\right.\right.\\&\left[\frac{\sum_{i=1}^{n}\left(\left(\left(T_i/\sum_{i=1}^{n}T_i\right)/\varphi^*\left(s_{\theta(a_i)}\right)\right)\inf I(a_i)\right)}{\sum_{i=1}^{n}\left(\left(\left(T_i/\sum_{i=1}^{n}T_i\right)/\varphi^*\left(s_{\theta(a_i)}\right)\right)\right)},\frac{\sum_{i=1}^{n}\left(\left(\left(T_i/\sum_{i=1}^{n}T_i\right)/\varphi^*\left(s_{\theta(a_i)}\right)\right)\sup I(a_i)\right)}{\sum_{i=1}^{n}\left(\left(\left(T_i/\sum_{i=1}^{n}T_i\right)/\varphi^*\left(s_{\theta(a_i)}\right)\right)\right)}\right],\\&\left.\left.\left[\frac{\sum_{i=1}^{n}\left(\left(\left(T_i/\sum_{i=1}^{n}T_i\right)/\varphi^*\left(s_{\theta(a_i)}\right)\right)\inf F(a_i)\right)}{\sum_{i=1}^{n}\left(\left(\left(T_i/\sum_{i=1}^{n}T_i\right)/\varphi^*\left(s_{\theta(a_i)}\right)\right)\right)},\frac{\sum_{i=1}^{n}\left(\left(\left(T_i/\sum_{i=1}^{n}T_i\right)/\varphi^*\left(s_{\theta(a_i)}\right)\right)\sup F(a_i)\right)}{\sum_{i=1}^{n}\left(\left(\left(T_i/\sum_{i=1}^{n}T_i\right)/\varphi^*\left(s_{\theta(a_i)}\right)\right)\right)}\right]\right)\right\rangle,\\&\left\langle\varphi^{*-1}\left(\frac{1}{\sum_{i=1}^{n}\Upsilon_{a_i}}\right),\left[\frac{\sum_{i=1}^{n}\left(\Upsilon_{a_i}\inf T(a_i)\right)}{\sum_{i=1}^{n}\Upsilon_{a_i}},\frac{\sum_{i=1}^{n}\left(\Upsilon_{a_i}\sup T(a_i)\right)}{\sum_{i=1}^{n}\Upsilon_{a_i}}\right],\left[\frac{\sum_{i=1}^{n}\left(\Upsilon_{a_i}\inf I(a_i)\right)}{\sum_{i=1}^{n}\Upsilon_{a_i}},\frac{\sum_{i=1}^{n}\left(\Upsilon_{a_i}\sup I(a_i)\right)}{\sum_{i=1}^{n}\Upsilon_{a_i}}\right],\right.\\&\left.\left[\frac{\sum_{i=1}^{n}\left(\Upsilon_{a_i}\inf F(a_i)\right)}{\sum_{i=1}^{n}\Upsilon_{a_i}},\frac{\sum_{i=1}^{n}\left(\Upsilon_{a_i}\sup F(a_i)\right)}{\sum_{i=1}^{n}\Upsilon_{a_i}}\right]\right)\right\rangle\\&=\mathrm{INLPWHM}(a_1,a_2,\ldots,a_n)\end{aligned}$$

$$\mathrm{INLPHHG}(a_1,a_2,\ldots,a_n)=1\Big/\left(\prod_{i=1}^{n}\left(r_{\sigma(i)}\right)^{w_i}\right)$$
$$=\Big\langle \varphi^{*-1}\left(1\Big/\left(\prod_{i=1}^{n}\left(\left(\varphi^{*}(s_{\theta(r_{\sigma(i)})})\right)^{w_i}\right)\right)\right),$$
$$\left(\left[\prod_{i=1}^{n}\left(\left(\inf T(r_{\sigma(i)})\right)^{w_i}\right),\prod_{i=1}^{n}\left(\left(\sup T(r_{\sigma(i)})\right)^{w_i}\right)\right],\right.$$
$$\left[1-\prod_{i=1}^{n}\left(\left(1-\inf I(r_{\sigma(i)})\right)^{w_i}\right),1-\prod_{i=1}^{n}\left(\left(1-\sup I(r_{\sigma(i)})\right)^{w_i}\right)\right],$$
$$\left.\left[1-\prod_{i=1}^{n}\left(\left(1-\inf F(r_{\sigma(i)})\right)^{w_i}\right),1-\prod_{i=1}^{n}\left(\left(1-\sup F(r_{\sigma(i)})\right)^{w_i}\right)\right]\right)\Big\rangle.$$

3. If $\lambda = 2$, the GINLPHHM operator becomes the interval neutrosophic linguistic prioritized hybrid harmonic quadratic mean (INLNPHHQM) operator:

$$\mathrm{INLPHHQM}(a_1,a_2,\ldots,a_n)=1\Big/\left(\oplus_{i=1}^{n}\left(w_i\left(r_{\sigma(i)}\right)^2\right)\right)^{\frac{1}{2}}$$
$$=\left\langle \varphi^{*-1}\left(\left(\frac{1}{\sum_{i=1}^{n}\Upsilon_{r_i}}\right)^{\frac{1}{2}}\right),\left(\left[\left(\frac{\sum_{i=1}^{n}\left(\Upsilon_{r_i}\left(\inf T(r_{\sigma(i)})\right)^2\right)}{\sum_{i=1}^{n}\Upsilon_{r_i}}\right)^{\frac{1}{2}},\left(\frac{\sum_{i=1}^{n}\left(\Upsilon_{r_i}\left(\sup T(r_{\sigma(i)})\right)^2\right)}{\sum_{i=1}^{n}\Upsilon_{r_i}}\right)^{\frac{1}{2}}\right],\right.\right.$$
$$\left[1-\left(1-\frac{\sum_{i=1}^{n}\left(\Upsilon_{r_i}\left(1-\left(1-\inf I(r_{\sigma(i)})\right)^2\right)\right)}{\sum_{i=1}^{n}\Upsilon_{r_i}}\right)^{\frac{1}{2}},1-\left(\frac{\sum_{i=1}^{n}\left(\Upsilon_{r_i}\left(1-\left(1-\sup I(r_{\sigma(i)})\right)^2\right)\right)}{\sum_{i=1}^{n}\Upsilon_{r_i}}\right)^{\frac{1}{2}}\right],$$
$$\left.\left.\left[1-\left(1-\frac{\sum_{i=1}^{n}\left(\Upsilon_{r_i}\left(1-\left(1-\inf F(r_{\sigma(i)})\right)^2\right)\right)}{\sum_{i=1}^{n}\Upsilon_{r_i}}\right)^{\frac{1}{2}},1-\left(1-\frac{\sum_{i=1}^{n}\left(\Upsilon_{r_i}\left(1-\left(1-\sup F(r_{\sigma(i)})\right)^2\right)\right)}{\sum_{i=1}^{n}\Upsilon_{r_i}}\right)^{\frac{1}{2}}\right]\right)\right\rangle.$$

5 MCGDM method of selecting medical treatments in an interval neutrosophic linguistic environment

In this section, interval neutrosophic linguistic prioritized harmonic operators are used to select medical treatments based on interval neutrosophic linguistic information.

For a medical treatment selection problem with interval neutrosophic linguistic information, let $\bar{A}$ be a set of interval neutrosophic linguistic information, $S = \{s_i | i = 1, 2, \ldots, 2t + 1\}$ be the linguistic term set, and $\tilde{S} = \{s_i | i \in [1, l]\}$ be the extended linguistic term set, which satisfies $s_i > s_j\ (i > j)$ and $l\ (l > 2t + 1)$. Assume that $B = \{B_1, B_2, \ldots, B_m\}$ is a set of medical treatment options and $D = \{D_1, D_2, \ldots, D_t\}$ is a set of decision-makers who evaluate these treatment options according to the criteria $C = \{C_1, C_2, \ldots, C_n\}$. Prioritization relationships exist among the decision-makers, which satisfy $D_1 \succ D_2 \succ \cdots \succ D_t$, and the treatment option criteria, which satisfy $C_1 \succ C_2 \succ \cdots s \succ C_n$. Evaluation information $a_{ij}^{y}(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n; y = 1, 2, \ldots, t)$ is provided by the decision-makers $D_y(y = 1, 2, \ldots, t)$ as they assess the medical treatment options $B_i(i = 1, 2, \ldots, m)$ with respect to the criteria $C_j(j = 1, 2, \ldots, n)$, where $a_{ij}^{y} \in \bar{A}$. Then, the decision matrix $R_y = (a_{ij}^{y})_{m\times n}$ is obtained. The method used to determine the rankings of the treatment options and decision-making procedures is described in the following passages.

Step 1 Normalize the decision matrices.

First, the decision-making information a_{ij}^{y} in the matrix $R_y = (a_{ij}^{y})_{m\times n}$ must be normalized. The criteria can be classified into benefit-type and cost-type criteria. The evaluation information does not have to be changed for the benefit-type criteria; however, the negation operator must be used for the cost-type criteria.

The normalizations of the decision matrices can be expressed as

$$\begin{cases}\tilde{a}_{ij}^{y} = a_{ij}^{y}, & C_j \in B_T\\ \tilde{a}_{ij}^{y} = neg\left(a_{ij}^{y}\right), & C_j \in C_T\end{cases},$$

where B_T denotes the set of benefit-type criteria and C_T denotes the set of cost-type criteria.

The normalized decision matrices can be denoted as $\bar{R}_y = (\tilde{a}_{ij}^{y})_{m\times n}$.

Step 2 Aggregate all of the values of each treatment option based on each criterion.

When $\lambda \to 0$, the collective INLNs $a_i^{y}(a_i^{y} \in \bar{A})$ or $\tilde{a}_i^{y}\ (\tilde{a}_i^{y} \in \bar{A})$ can be obtained via the GINLPWHM discussed in Definition 13 or the INLPWHG operator discussed in Theorem 4 as

Table 1 Evaluative criteria used to select treatments

Steroid therapy (B_1)	Plasmapheresis (B_2)
(C_{11}) About a medium probability of a cure	(C_{21}) A high probability of a cure
(C_{12}) There are some uncertain side effects	(C_{22}) The possibility of a blood pressure drop
(C_{13}) High expense	(C_{23}) Medium expense
Albumin immune therapy (B_3)	
(C_{31}) A high probability of a cure	
(C_{32}) The possibility of a cold or weariness	
(C_{33}) Low expenses	

$$a_i^y = \text{GINLPWHM}\left(\tilde{a}_{i1}^y, \tilde{a}_{i2}^y, \ldots, \tilde{a}_{in}^y\right) \quad \text{or}$$
$$b_i^y = \text{INLPWHG}\left(\tilde{a}_{i1}^y, \tilde{a}_{i2}^y, \ldots, \tilde{a}_{in}^y\right).$$

Then, the collective preference matrix $P = (a_i^y)_{m\times y}$ or $\tilde{P} = (b_i^y)_{m\times y}$ can be obtained.

Step 3 Calculate the overall value of each treatment option B_i.

When $\lambda \to 0$, the overall value $a_i\ (a_i \in \bar{A})$ or $b_i\ (b_i \in \bar{A})$ of each treatment option B_i can be obtained using the GINLPHHM discussed in Definition 14 or the INLPHHG operator discussed in Theorem 8 as

$$a_i = \text{GINLNPHHM}\left(a_i^1, a_i^2, \ldots, a_i^t\right) \quad \text{or}$$
$$b_i = \text{INLPHHG}\left(b_i^1, b_i^2, \ldots, b_i^t\right).$$

Step 4 Calculate the score values of $a_i(i = 1, 2, \ldots, m)$ or $b_i(i = 1, 2, \ldots, m)$ using Definition 11.

Step 5 Rank the medical treatment options and select the optimum treatment.

Based on the results obtained in Step 4, the medical treatments are ranked, and the optimum treatment is selected.

5.1 Illustration of the proposed approach

In this section, a medical treatment selection problem is used to illustrate the validity and efficacy of the developed method.

The following case is adapted from [6].

The patient, a 48-year-old wealthy woman with a history of diabetes mellitus, was diagnosed with acute inflammatory demyelinating disease by her doctor. This disease, which is characterized by ascending paralysis manifesting as weakness beginning in the feet and hands and migrating towards the trunk, can affect the peripheral nervous system and cause life-threatening complications. Most patients can recover from this disease with appropriate treatment within a few months to a year, although minor by-effects, such as areflexia, may persist. Few patients with this disease recover from a severe disability, such as severe proximal motor dysfunction. The doctor selected three treatment options, including steroid therapy (B_1), plasmapheresis (B_2), and albumin immune therapy (B_3), based on her medical history and current physical conditions. In order to improve the patient and her family's understanding of the benefits and disadvantages of each treatment option, the hospital provided descriptions of the treatment options in the form of $B_i(i = 1, 2, 3)$ using three criteria, including the probability of a cure (C_1), severity of the side effects (C_2), and cost (C_3), based on a large number of cases, as summarized in Table 1. A prioritization relationship among the criteria $C_j(j = 1, 2, 3)$, which satisfies $C_1 \succ C_2 \succ C_3$, was determined according to the patient's preferences and current financial situation. In order to select the optimum treatment, the patient (D_1), doctor (D_2), and patient's family (D_3), with a prioritization relationship among the decision-makers $D_y(y = 1, 2, 3)$ satisfying $D_1 \succ D_2 \succ D_3$, evaluated the three treatment options based on these criteria using INLNs and the linguistic term set $S = \{s_1 = \text{extremely poor}(EP),\ s_2 = \text{very poor}(VP),\ s_3 = \text{poor}(P),\ s_4 = \text{medium}(M),\ s_5 = \text{good}(G),\ s_6 = \text{very good}(VG), s_7 = \text{extremely good}\ (EG)\}$, yielding the INLNs $a_{ij}^y(i = 1, 2, 3; j = 1, 2, 3; y = 1, 2, 3)$. The decision matrices are shown in R_1, R_2, and R_3.

$$R_1 = \begin{pmatrix} \langle s_3, ([0.4, 0.5], [0.2, 0.3], [0.3, 0.4]) \rangle & \langle s_5, ([0.5, 0.6], [0.1, 0.2], [0.2, 0.4]) \rangle & \langle s_5, ([0.5, 0.6], [0.1, 0.2], [0.2, 0.3]) \rangle \\ \langle s_4, ([0.5, 0.7], [0.1, 0.2], [0.2, 0.3]) \rangle & \langle s_3, ([0.6, 0.7], [0.1, 0.2], [0.2, 0.3]) \rangle & \langle s_3, ([0.4, 0.7], [0.2, 0.2], [0.1, 0.2]) \rangle \\ \langle s_5, ([0.5, 0.6], [0.1, 0.2], [0.3, 0.4]) \rangle & \langle s_6, ([0.5, 0.7], [0.1, 0.3], [0.3, 0.4]) \rangle & \langle s_1, ([0.5, 0.6], [0.1, 0.3], [0.1, 0.3]) \rangle \end{pmatrix},$$

$$R_2 = \begin{pmatrix} \langle s_4, ([0.5, 0.6], [0.3, 0.4], [0.1, 0.2]) \rangle & \langle s_4, ([0.6, 0.7], [0.2, 0.3], [0.3, 0.4]) \rangle & \langle s_4, ([0.5, 0.8], [0.2, 0.3], [0.2, 0.3]) \rangle \\ \langle s_7, ([0.6, 0.8], [0.1, 0.2], [0.1, 0.2]) \rangle & \langle s_3, ([0.7, 0.8], [0.1, 0.2], [0.1, 0.3]) \rangle & \langle s_2, ([0.5, 0.7], [0.3, 0.4], [0.1, 0.2]) \rangle \\ \langle s_6, ([0.5, 0.5], [0.3, 0.4], [0.2, 0.3]) \rangle & \langle s_5, ([0.6, 0.8], [0.1, 0.3], [0.2, 0.4]) \rangle & \langle s_1, ([0.4, 0.6], [0.2, 0.3], [0.1, 0.2]) \rangle \end{pmatrix},$$

$$R_3 = \begin{pmatrix} \langle s_4, ([0.4, 0.5], [0.4, 0.5], [0, 0.2]) \rangle & \langle s_4, ([0.5, 0.7], [0.3, 0.4], [0.3, 0.4]) \rangle & \langle s_4, ([0.5, 0.6], [0.3, 0.4], [0.1, 0.3]) \rangle \\ \langle s_6, ([0.5, 0.7], [0.2, 0.3], [0.1, 0.2]) \rangle & \langle s_3, ([0.6, 0.7], [0.2, 0.3], [0, 0.3]) \rangle & \langle s_3, ([0.7, 0.8], [0.2, 0.4], [0, 0.2]) \rangle \\ \langle s_5, ([0.3, 0.5], [0.3, 0.5], [0.1, 0.3]) \rangle & \langle s_6, ([0.5, 0.8], [0, 0.3], [0.2, 0.3]) \rangle & \langle s_2, ([0.5, 0.5], [0.3, 0.4], [0.1, 0.2]) \rangle \end{pmatrix}.$$

5.2 Interval neutrosophic linguistic MCGDM method

The proposed MCGDM method is used to rank the treatment options.

Without the loss of generality, let $\varphi^*(s_i) = \varphi_3$, $p = 0.8$, $q = 0.7$, $t = 3$, $l > 7$, and $\lambda = 1$.

Step 1 Normalize the decision matrices.

The probability of a cure (C_1) is considered a benefit-type criterion, while the severity of the side effects (C_2) and cost (C_3) are considered cost-type criteria. Therefore, the information a_{ij}^y in the decision matrices $R_y = (a_{ij}^y)_{4\times 3} (i = 1, 2, 3; j = 1, 2, 3; y = 1, 2, 3)$ is normalized using negation operators.

The normalized decision matrices can be expressed as $\bar{R}_1$, $\bar{R}_2$, and $\bar{R}_3$.

$$\bar{R}_1 = \begin{pmatrix} \langle s_3, ([0.4, 0.5], [0.2, 0.3], [0.3, 0.4]) \rangle & \langle s_{2.06}, ([0.5, 0.6], [0.1, 0.2], [0.2, 0.4]) \rangle & \langle s_{2.06}, ([0.5, 0.6], [0.1, 0.2], [0.2, 0.3]) \rangle \\ \langle s_4, ([0.5, 0.7], [0.1, 0.2], [0.2, 0.3]) \rangle & \langle s_{4.26}, ([0.6, 0.7], [0.1, 0.2], [0.2, 0.3]) \rangle & \langle s_{4.26}, ([0.4, 0.7], [0.2, 0.2], [0.1, 0.2]) \rangle \\ \langle s_5, ([0.5, 0.6], [0.1, 0.2], [0.3, 0.4]) \rangle & \langle s_{0.98}, ([0.5, 0.7], [0.1, 0.3], [0.3, 0.4]) \rangle & \langle s_{5.98}, ([0.5, 0.6], [0.1, 0.3], [0.1, 0.3]) \rangle \end{pmatrix},$$

$$\bar{R}_2 = \begin{pmatrix} \langle s_4, ([0.5, 0.6], [0.3, 0.4], [0.1, 0.2]) \rangle & \langle s_{3.52}, ([0.6, 0.7], [0.2, 0.3], [0.3, 0.4]) \rangle & \langle s_{3.52}, ([0.5, 0.8], [0.2, 0.3], [0.2, 0.3]) \rangle \\ \langle s_7, ([0.6, 0.8], [0.1, 0.2], [0.1, 0.2]) \rangle & \langle s_{4.26}, ([0.7, 0.8], [0.1, 0.2], [0.1, 0.3]) \rangle & \langle s_{5.05}, ([0.5, 0.7], [0.3, 0.4], [0.1, 0.2]) \rangle \\ \langle s_6, ([0.5, 0.5], [0.3, 0.4], [0.2, 0.3]) \rangle & \langle s_{2.06}, ([0.6, 0.8], [0.1, 0.3], [0.2, 0.4]) \rangle & \langle s_{5.98}, ([0.4, 0.6], [0.2, 0.3], [0.1, 0.2]) \rangle \end{pmatrix},$$

$$\bar{R}_3 = \begin{pmatrix} \langle s_4, ([0.4, 0.5], [0.4, 0.5], [0, 0.2]) \rangle & \langle s_{3.52}, ([0.5, 0.7], [0.3, 0.4], [0.3, 0.4]) \rangle & \langle s_{3.52}, ([0.5, 0.6], [0.3, 0.4], [0.1, 0.3]) \rangle \\ \langle s_6, ([0.5, 0.7], [0.2, 0.3], [0.1, 0.2]) \rangle & \langle s_{4.26}, ([0.6, 0.7], [0.2, 0.3], [0, 0.3]) \rangle & \langle s_{4.26}, ([0.7, 0.8], [0.2, 0.4], [0, 0.2]) \rangle \\ \langle s_5, ([0.3, 0.5], [0.3, 0.5], [0.1, 0.3]) \rangle & \langle s_{0.98}, ([0.5, 0.8], [0, 0.3], [0.2, 0.3]) \rangle & \langle s_{5.05}, ([0.5, 0.5], [0.3, 0.4], [0.1, 0.2]) \rangle \end{pmatrix}.$$

Step 2 Aggregate all of the values of each treatment option based on each criterion.

The GINLPWHM and INLPWHG operators are used to aggregate all of the assessment information of each treatment option based on each criterion. The collective INLNs are expressed as P and $\tilde{P}$.

$$P=\begin{pmatrix}\langle s_{2.72},([0.43,0.53],[0.17,0.27],[0.27,0.40])\rangle & \langle s_{3.83},([0.53,0.66],[0.26,0.36],[0.18,0.28])\rangle & \langle s_{3.82},([0.45,0.58],[0.35,0.45],[0.11,0.28])\rangle \\ \langle s_{4.06},([0.51,0.70],[0.11,0.20],[0.19,0.29])\rangle & \langle s_{5.18},([0.63,0.78],[0.14,0.24],[0.10,0.25])\rangle & \langle s_{4.79},([0.59,0.72],[0.20,0.32],[0.03,0.24])\rangle \\ \langle s_{2.21},([0.50,0.68],[0.10,0.28],[0.30,0.40])\rangle & \langle s_{3.44},([0.57,0.71],[0.16,0.33],[0.20,0.36])\rangle & \langle s_{2.01},([0.46,0.74],[0.06,0.34],[0.18,0.30])\rangle\end{pmatrix},$$

$$\tilde{P}=\begin{pmatrix}\langle s_{2.76},([0.42,0.52],[0.18,0.28],[0.28,0.40])\rangle & \langle s_{3.84},([0.53,0.65],[0.26,0.36],[0.17,0.28])\rangle & \langle s_{3.83},([0.44,0.57],[0.36,0.46],[0.11,0.28])\rangle \\ \langle s_{4.06},([0.51,0.70],[0.11,0.20],[0.19,0.29])\rangle & \langle s_{5.31},([0.61,0.78],[0.14,0.24],[0.10,0.24])\rangle & \langle s_{4.86},([0.57,0.72],[0.20,0.32],[0.05,0.24])\rangle \\ \langle s_{3.25},([0.50,0.63],[0.10,0.24],[0.30,0.40])\rangle & \langle s_{3.96},([0.53,0.61],[0.22,0.36],[0.19,0.33])\rangle & \langle s_{2.97},([0.37,0.60],[0.20,0.43],[0.14,0.30])\rangle\end{pmatrix}.$$

Step 3. Calculate the overall value of each treatment option

The GINLPHHM and INLPHHG operators, where $w=(0.243, 0.514, 0.243)$, the value of which was derived using the normal distribution method [71], are used to obtain the collective values.

$$U=\begin{pmatrix}\langle s_{3.43},([0.46,0.56],[0.20,0.30],[0.24,0.36])\rangle \\ \langle s_{4.73},([0.56,0.73],[0.13,0.23],[0.14,0.27])\rangle \\ \langle s_{2.81},([0.51,0.69],[0.11,0.29],[0.27,0.39])\rangle\end{pmatrix} \text{ and } \tilde{U}=\begin{pmatrix}\langle s_{4.97},([0.48,0.59],[0.27,0.37],[0.19,0.31])\rangle \\ \langle s_{5.76},([0.58,0.74],[0.15,0.25],[0.11,0.25])\rangle \\ \langle s_{4.50},([0.48,0.61],[0.19,0.35],[0.21,0.34])\rangle\end{pmatrix}.$$

Step 4 Calculate the score values of the treatment options.

$$Q=(0.32 \quad 0.58 \quad 0.26) \text{ and } \tilde{Q}=(0.62 \quad 0.72 \quad 0.54).$$

Step 5 Rank the treatment options and select the optimum treatment.

The treatment options are ranked as $B_2 \succ B_1 \succ B_3$. Therefore, the patient and her family opt for plasmapheresis (B_2).

The GINLPWHM and GINLPHHM operators and INLPWHG and INLPHHG operators yield the same treatment ranking setting when $\lambda = 1$. The treatment options' rankings for different values of λ are shown in Fig. 1. In general, high values of λ were associated with relatively pessimistic decision-makers, while low values of λ were associated with relatively conservative decision-makers. When the decision-makers did not indicate any preferences, the most commonly used value ($\lambda = 1$) was used.

5.3 Comparative analysis and discussion

In order to validate the accuracy of the proposed interval neutrosophic linguistic MCGDM method, a comparative study based on the illustrative example provided in this paper was conducted. The method developed in this paper was compared to the method proposed by Ye [55].

When applying the approach described in [55] to the above example, which involves the use of the interval neutrosophic linguistic weighted arithmetic average (INLWAA) and interval neutrosophic linguistic weighted geometric average (INLWGA) operators with known weights to comprehensively analyse treatment options, the weights of the criteria and decision-makers can be determined using the PA operator ($w_{ij} = T_{ij}\big/\sum_{j=1}^{n} T_{ij}, T_{ij} = \prod_{k=1}^{j-1} (E(a_{ik})/6$, where $E(a_{ij})$ [55] is the score function value of the INLN a_{ij}). The overall values of the treatment options are denoted as $\bar{U}$, and the score function values of all of the treatment options are denoted as $\bar{Q}$.

$$\bar{U}=\begin{pmatrix}\langle s_{2.38},([0.90,0.97],[0,0.02],[0,0.02])\rangle \\ \langle s_{3.48},([0.95,0.99],[0,0.01],[0,0.01])\rangle \\ \langle s_{3.32},([0.90,0.97],[0,0.02],[0,0.02])\rangle\end{pmatrix} \text{ and}$$

$$\bar{Q}=(2.16 \quad 2.91 \quad 3.06).$$

Thus, the treatment options are ranked as $B_3 \succ B_2 \succ B_1$, and B_3 is the optimum treatment. The ranking results are shown in Table 2.

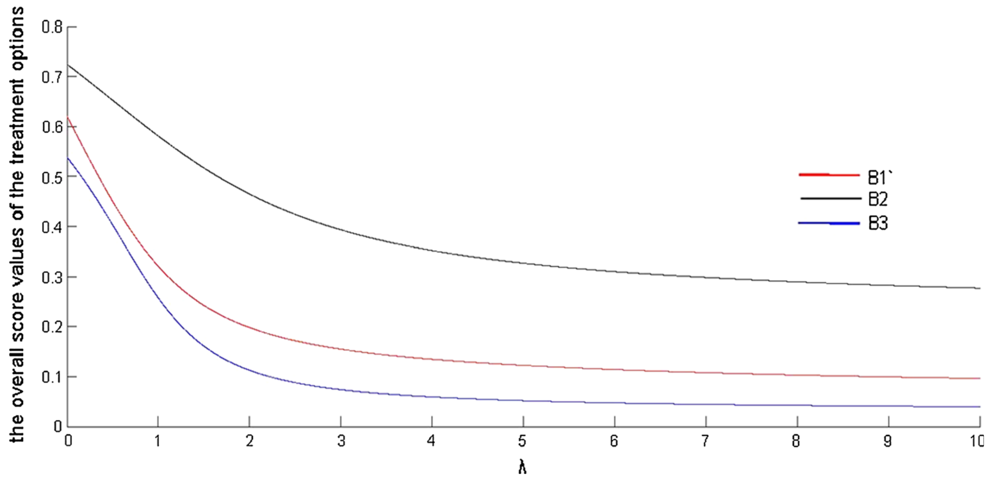

Fig. 1 Rankings of the various treatment options for different values of λ

Table 2 Ranking results obtained using the proposed method and method presented in [55]

Methods	Operators	Ranking of alternatives
Method presented in [55]	INLWAA	$B_3 \succ B_2 \succ B_1$
	INLWGA	$B_3 \succ B_1 \succ B_2$
Proposed method	GINLPWHM and GINLPHHM, $\lambda = 1$	$B_2 \succ B_1 \succ B_3$
	INLPWHG and INLPHHG	$B_2 \succ B_1 \succ B_3$

As shown in Table 1, the method developed in this paper and the method introduced in [55] yielded significantly different results. These differences were attributed to the following:

1. In the proposed approach, the linguistic terms were operated upon by the linguistic scale functions based on differences in semantics. Thus, the approach developed in this paper effectively reflected the semantics in the example. However, the linguistic terms in the approach developed in [55] were directly operated upon based on their subscripts, and the absolute deviations of any two pairs of adjacent linguistic terms were assumed to be equal, resulting in inaccurate aggregation results.
2. The new INLN operations defined in this paper accounted for the correlations among the linguistic terms and three degrees of membership of the INLNs. In addition, the new operations applied conservative and reliable principles, preventing information loss and distortion. However, the operations presented in [55] divided the linguistic terms and three degrees of membership of the INLNs into two parts and calculated their values separately, neglecting their interrelationships.
3. The weights of the criteria and decision-makers in the approach presented in [55] were expressed in real numbers, neglecting the priority rankings among the criteria and decision-makers that exist in practice. However, the weights of the criteria and decision-makers in this paper were calculated using PA operators according to their levels of priority. The method proposed in this paper also combined the advantages of PA and HM operators in order to obtain the overall INLNs of the alternatives. Thus, the method proposed in this paper yielded more objective and accurate results than the method developed in [55].

6 Conclusions

In this paper, the medical treatment option selection process was studied in an interval neutrosophic linguistic environment. In order to improve the applicability of methods based on interval neutrosophic linguistic aggregation operators and compensate for the limitations of existing operators, new interval neutrosophic linguistic aggregation operators were developed and applied to the medical treatment selection process. First, rectified linguistic scale functions, new operations, and an INLN comparison method were developed in order to prevent information loss and distortion during the aggregation process and comparative study. Then, GINLPWHM and GINLPHHM operators were developed based on these scale functions and operations. Furthermore, an interval neutrosophic linguistic MCGDM method based on these operators was developed and demonstrated using a practical example. Unlike the other methods, the proposed method effectively managed the preferential information expressed by the INLNs while considering the prioritization relationships that often exist among criteria and decision-makers in practical decision-making problems, preventing information loss and distortion. The proposed method was applied to a special case, in which the priority levels of the decision-makers and treatment option criteria varied. The results were compared to the results obtained by another operator-based method in order to demonstrate the practicality and efficacy of the proposed approach. In future research, the developed operator-based method will be applied to other domains, such as personnel selection and image processing.

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Compliance with ethical standards

Conflict of interest The authors declare that there is no conflict of interest regarding the publication of this paper.

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