Florentin Smarandache
(author and editor)

Collected Papers

(on Physics, Artificial Intelligence, Health Issues, Decision Making, Economics, Statistics)

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Florentin Smarandache is professor of mathematics at the University of New Mexico and he published over 1,000 articles and books. He coined the words "neutrosophy" [(French neutre < Latin neuter, neutral, and Greek sophia, skill/wisdom) means knowledge of neutral thought] and its derivatives: neutrosophic, neutrosophication, neutrosophicator, deneutrosophication, deneutrosophicator, etc. He is the founder and developer of neutrosophic set / logic / probability / statistics etc. In 2006 he introduced the degree of dependence/independence between the neutrosophic components T, I, F.

In 2007 he extended the neutrosophic set to Neutrosophic Overset (when some neutrosophic component is > 1), and to Neutrosophic Underset (when some neutrosophic component is < 0), and to Neutrosophic Offset (when some neutrosophic components are off the interval [0, 1], i.e. some neutrosophic component > 1 and some neutrosophic component < 0). Then, similar extensions to respectively Neutrosophic Over/Under/Off Logic, Measure, Probability, Statistics etc.
Then, introduced the Neutrosophic Tripolar Set and Neutrosophic Multipolar Set, also the Neutrosophic Tripolar Graph and Neutrosophic Multipolar Graph.

He then generalized the Neutrosophic Logic/Set/Probability to Refined Neutrosophic Logic/Set/Probability [2013], where T can be split into subcomponents T1, T2, ..., Tp, and I into I1, I2, ..., Ir, and F into F1, F2, ..., Fs, where p+r+s = n ≥ 1. Even more: T, I, and/or F (or any of their subcomponents Tj,Ik, and/or Fl) could be countable or uncountable infinite sets.

In 2015 he refined the indeterminacy "I", within the neutrosophic algebraic structures, into different types of indeterminacies (depending on the problem to solve), such as I1, I2,, Ip with integer p ≥ 1, and obtained the refined neutrosophic numbers of the form Np = a+b1I1+b2I2+ +bpIp where a, b1, b2,, bp are real or complex numbers, and a is called the determinate part of Np, while for each k in {1, 2,, p} Ik is called the k-th indeterminate part of Np.

Then consequently he extended the neutrosophic algebraic structures to Refined Neutrosophic Algebraic Structures [or Refined Neutrosophic I-Algebraic Structures] (2015), which are algebraic structures based on sets of the refined neutrosophic numbers a+b1I1+b2I2+ +bpIp.

He introduced the (T, I, F)-Neutrosophic Structures [2015]. In any field of knowledge, each structure is composed from two parts: a space, and a set of axioms (or laws) acting (governing) on it. If the space, or at least one of its axioms (laws), has some indeterminacy, that structure is a (T, I, F)-Neutrosophic Structure. And he extended them to the (T, I, F)-Neutrosophic I-Algebraic Structures [2015], i.e. algebraic structures based on neutrosophic numbers of the form a+bI, but also having indeterminacy related to the structure space (elements which only partially belong to the space, or elements we know nothing if they belong to the space or not) or indeterminacy related to at least an axiom (or law) acting on the structure space. Then he extended them to Refined (T, I, F)-Neutrosophic Refined I-Algebraic Structures.

Also, he proposed an extension of the classical probability and the imprecise probability to the 'neutrosophic probability' [1995], that he defined as a tridimensional vector whose components are real subsets of the non-standard interval [-0, 1+] introduced the neutrosophic measure and neutrosophic integral: http://fs.gallup.unm.edu/NeutrosophicMeasureIntegralProbability.pdf and also extended the classical statistics to neutrosophic statistics: http://fs.gallup.unm.edu/NeutrosophicStatistics.pdf

Introductory Note

This eleventh volume of *Collected Papers* includes 90 papers comprising 988 pages on Physics, Artificial Intelligence, Health Issues, Decision Making, Economics, Statistics, written between 2001-2022 by the author alone or in collaboration with the following 84 co-authors (alphabetically ordered) from 19 countries: Abhijit Saha, Abu Sufian, Jack Allen, Shahbaz Ali, Ali Safaa Sadiq, Aliya Fahmi, Atiqa Fakhar, Atiqa Firdous, Sukanto Bhattacharya, Robert N. Boyd, Victor Chang, Victor Christianto, V. Christy, Dao The Son, Debjit Dutta, Azeddine Elhassouny, Fazal Ghani, Fazli Amin, Anirudha Ghosha, Nasruddin Hassan, Hoang Viet Long, Jhulaneswar Baidya, Jin Kim, Jun Ye, Darjan Karabašević, Vasilios N. Katsikis, Ieva Meidutė-Kavaliauskienė, F. Kaymarm, Nour Eldeen M. Khalifa, Madad Khan, Qaisar Khan, M. Khoshnevisan, Kifayat Ullah,, Volodymyr Krasnoholovets, Mukesh Kumar, Le Hoang Son, Luong Thi Hong Lan, Tahir Mahmood, Mahmoud Ismail, Mohamed Abdel-Basset, Siti Nurul Fitriah Mohamad, Mohamed Loey, Mai Mohamed, K. Mohana, Kalyan Mondal, Muhammad Gulfam, Muhammad Khalid Mahmood, Muhammad Jamil, Muhammad Yaqub Khan, Muhammad Riaz, Nguyen Dinh Hoa, Cu Nguyen Giap, Nguyen Tho Thong, Peide Liu, Pham Huy Thong, Gabrijela Popović, Surapati Pramanik, Dmitri Rabounski, Roslan Hasni, Rumi Roy, Tapan Kumar Roy, Said Broumi, Saleem Abdullah, Muzafer Saračević, Ganeshsree Selvachandran, Shariful Alam, Shyamal Dalapati, Housila P. Singh, R. Singh, Rajesh Singh, Predrag S. Stanimirović, Kasan Susilo, Dragiša Stanujkić, Alexandra Şandru, Ovidiu Ilie Şandru, Zenonas Turskis, Yunita Umniyati, Alptekin Ulutaș, Maikel Yelandi Leyva Vázquez, Binyamin Yusoff, Edmundas Kazimieras Zavadskas, Zhao Loon Wang.

Keywords

Physics: Speed Barrier, Entangled Particles, Unmatter, Quantum Chromodynamics, Magnetic Resonance, Time Traveling, Superluminal Physics, Instantaneous Physics, Subluminal Physics, Euclidean Geometry, Smarandache Geometry, non-Euclidean Geometry, Syntropy, Precognition, Maxwell Equation, Advanced Wave Solution, Solitary Wave, Cosmological KdV Equation, Nonlinear Universe, Cellular Automata, Hirsch Theory, London Equations, Hydrodynamics, Proca Equations, Electrodynamics, Superconductor, Chiral Medium, Chiral Gravitation Theory, Non-locality.

Artificial Intelligence: Fusion Theories, Smart mobile application.

Health Issues: Wireless Technologies, Ho'oponopono, Immune System, Vitamin D3, Corona Virus, COVID-19, Deep Learning, Edge Computing, Pandemic, Transfer Learning.

Decision Making: α-Discounting MCDM-Method, Non-Linear Decision Making Problems, Interval Neutrosophic Linguistic, Multiple Attribute Decision Making, Maximizing Deviation Method, Neutrosophic Set, Rough Neutrosophic Set, Rough Variation Coefficient Similarity Measure, n-Wise Criteria Comparisons, AHP, Consistency, Inconsistency, Generalized Dynamic Interval-Valued Neutrosophic Set; Hesitant Fuzzy Set; Dynamic Neutrosophic Environment; Dynamic TOPSIS Method; Neutrosophic Data Analytics.

Economics: Anghel Rugina, Social Justice, Fuzzy logic, Investment, Management.

Statistics: Population, Averaging, Information, Measurement, Error Estimation.

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PHYSICS

There Is No Speed Barrier for a Wave Phase Nor for Entangled Particles

Florentin Smarandache

Florentin Smarandache (2005). There Is No Speed Barrier for a Wave Phase nor for Entangled Particles. *Progress in Physics* 1, 85-86

In this short paper, as an extension and consequence of Einstein-Podolski-Rosen paradox and Bell's inequality, one promotes the hypothesis (it has been called the Smarandache Hypothesis [1, 2, 3]) that: There is no speed barrier in the Universe and one can construct arbitrary speeds, and also one asks if it is possible to have an infinite speed (instantaneous transmission)? Future research: to study the composition of faster-than-light velocities and what happens with the laws of physics at faster-than-light velocities?

This is the new version of an early article. That early version, based on a 1972 paper [4], was presented at the Universidad de Blumenau, Brazil, May–June 1993, in the Conference on "Paradoxism in Literature and Science"; and at the University of Kishinev, in December 1994. See that early version in [5].

1 Introduction

What is new in science (physics)?

According to researchers from the common group of the University of Innsbruck in Austria and US National Institute of Standards and Technology (starting from December 1997, Rainer Blatt, David Wineland et al.):

- Photon is a bit of light, the quantum of electromagnetic radiation (quantum is the smallest amount of energy that a system can gain or lose);
- Polarization refers to the direction and characteristics of the light wave vibration;
- If one uses the entanglement phenomenon, in order to transfer the polarization between two photons, then: whatever happens to one is the opposite of what happens to the other; hence, their polarizations are opposite of each other;
- In quantum mechanics, objects such as subatomic particles do not have specific, fixed characteristic at any given instant in time until they are measured;
- Suppose a certain physical process produces a pair of entangled particles A and B (having opposite or complementary characteristics), which fly off into space in the opposite direction and, when they are billions of miles apart, one measures particle A; because B is the opposite, the act of measuring A instantaneously tells B what to be; therefore those instructions would somehow have to travel between A and B faster than the speed of light; hence, one can extend the Einstein-Podolsky-Rosen paradox and Bell's inequality and assert that the light speed is not a speed barrier in the Universe.

Such results were also obtained by: Nicolas Gisin at the University of Geneva, Switzerland, who successfully teleported quantum bits, or qubits, between two labs over 2 km of coiled cable. But the actual distance between the two labs was about 55 m; researchers from the University of Vienna and the Austrian Academy of Science (Rupert Ursin et al. have carried out successful teleportation with particles of light over a distance of 600 m across the River Danube in Austria); researchers from Australia National University and many others [6, 7, 8].

2 Scientific hypothesis

We even promote the hypothesis that:

> There is no speed barrier in the Universe, which would theoretically be proved by increasing, in the previous example, the distance between particles A and B as much as the Universe allows it, and then measuring particle A.

It has been called the *Smarandache Hypotesis* [1, 2, 3].

3 An open question now

If the space is infinite, is the maximum speed infinite?

"This Smarandache hypothesis is controversially interpreted by scientists. Some say that it violates the theory of relativity and the principle of causality, others support the ideas that this hypothesis works for particles with no mass or imaginary mass, in non-locality, through tunneling effect, or in other (extra-) dimension(s)." Kamla John, [9].

Scott Owens' answer [10] to Hans Gunter in an e-mail from January 22, 2001 (the last one forwarded it to the author): "It appears that the only things the Smarandache hypothesis can be applied to are entities that do not have real mass or energy or information. The best example I can come up with is the difference between the wavefront velocity of a photon and the phase velocity. It is common for the phase velocity to exceed the wavefront velocity c, but that does not mean that any real energy is traveling faster than c. So, while it is possible to construct arbitrary speeds from zero in infinite, the superluminal speeds can only apply to purely imaginary entities or components."

Would be possible to accelerate a photon (or another particle traveling at, say, $0.99c$ and thus to get speed greater than c (where c is the speed of light)?

4 Future possible research

It would be interesting to study the composition of two velocities v and u in the cases when:

$$v < c \text{ and } u = c;$$
$$v = c \text{ and } u = c;$$
$$v > c \text{ and } u = c;$$
$$v > c \text{ and } u > c;$$
$$v < c \text{ and } u = \infty;$$
$$v = c \text{ and } u = \infty;$$
$$v > c \text{ and } u = \infty;$$
$$v = \infty \text{ and } u = \infty.$$

What happens with the laws of physics in each of these cases?

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Verifying Unmatter by Experiments, More Types of Unmatter, and a Quantum Chromodynamics Formula

Florentin Smarandache

Florentin Smarandache (2006). Verifying Unmatter by Experiments, More Types of Unmatter, and a Quantum Chromodynamics Formula. *Infinite Energy* 67, 4

Abstract
As shown, experiments registered unmatter - a new kind of matter whose atoms include both nucleons and antinucle-ons, with a very short life span of no more than 10^{-20} sec. Stable states of unmatter can be built on quarks and anti-quarks: applying the unmatter principle, obtained is a quan-tum chromodynamics formula that gives many combina-tions of unmatter built on quarks and antiquarks.

Since the publication of my articles defining "matter, anti-matter, and unmatter,"[1,2] and Dr. S. Chubb's pertinent com-ment[3] on unmatter, new development has been made on the unmatter topic in the sense that experiments verifying unmatter have been found.

1. Definition of Unmatter

In short, unmatter is formed by matter and antimatter that bind together.[1,2] The building blocks (most elementary particles known today) are six quarks and six leptons; their twelve antiparticles also exist. Then unmatter will be formed by at least a building block and at least an anti-building block which can bind together.

2. Exotic Atom

If in an atom we substitute one or more particles with other particles of the same charge (constituents) we obtain an exotic atom whose particles are held together due to the electric charge. For example, we can substitute in an ordinary atom one or more electrons with other negative particles (say, p^-, anti-Rho meson, D^-, D_s^-, muon, tau, Ω^-, D^-, etc., generally clusters of quarks and antiquarks whose total charge is negative), or the positively charged nucleus replaced by other positive particles (say, clusters of quarks and antiquarks whose total charge is positive, etc.).

3. Unmatter Atom

It is possible to define unmatter in a more general way, using the exotic atom. The classical unmatter atoms were formed by particles like (a) electrons, protons, and antineutrons, or (b) antielectrons, antiprotons, and neutrons.

In a more general definition, an unmatter atom is a system of particles as above, or such that one or more particles are replaced by other particles of the same charge.

Other categories would be (c) a matter atom where one or more (but not all) of the electrons and/or protons are replaced by antimatter particles of the same corresponding charges, and (d) an antimatter atom such that one or more (but not all) of the antielectrons and/or antiprotons are replaced by matter particles of the same corresponding charges.

In a more composed system we can substitute a particle with an unmatter particle and form an unmatter atom.

Of course, not all of these combinations are stable, semi-stable, or quasi-stable, especially when their time to bind together might be longer than their lifespan.

4. Examples of Unmatter

From 1970 to 1975, numerous pure experimental verifications were obtained proving that "atom-like" systems built on nucleons (protons and neutrons) and antinucleons (antiprotons and antineutrons) are real. Such "atoms," where nucleon and antinucleon are moving at the opposite sides of the same orbit around the common center of mass, are very unstable; their lifespan is no more than 10^{-20} sec. Then nucleon and antinucleon annihilate into gamma-quanta and more light particles (pions) which can not be connected with one another (see References 6-8). The experiments were done mainly in Brookhaven National Laboratory (USA) and partially at CERN (Switzerland), where "proton—antiproton" and "antiproton—neutron" atoms were observed, called $\bar{p}p$ and $\bar{p}n$ respectively (see Figures 1 and 2).

After the experiments were done, the lifespan of such "atoms" was calculated theoretically in Chapiro's works.[9-11] His main idea was that nuclear forces, acting between nucleon and antinucleon, can keep them far way from each other, hindering their annihilation. For instance, a proton and antiproton are located at opposite sides in the same orbit and they are moved around the orbit center. If the diameter of their orbit is much more than the diameter of "annihilation area," they can be kept out of annihilation (see Figure 3). But because the orbit, according to quantum mechanics, is an actual cloud spreading far around the average radius, at any radius between the proton and the antiproton there is a probability that they can meet one another at the annihilation distance. Therefore nucleon—antinucleon system annihilates in any case, this system is unstable by definition, having a lifespan no more than 10^{-20} sec.

Unfortunately, the researchers limited the research to the consideration of $\bar{p}p$ and $\bar{p}n$ nuclei only. The reason was that they, in the absence of a theory, considered $\bar{p}p$ and $\bar{p}n$ "atoms" as only a rare exception, which gives no classes of matter.

The unmatter does exist—for example, some mesons and anti-mesons, through for a trifling of a second lifetime—so the pions are unmatter [which have the composition u^d and ud^, where by u^ we mean anti-up quark, d = down quark, and analogously u = up quark and d^ = anti-down quark, while by ^ means anti], the kaon K^+ (us^), K^- (u^s), Phi (ss^), D^+ (cd^), D^0 (cu^), D_s^+ (cs^), J/Psi (cc^), B^- (bu^), B^0 (db^), B_s^0 (sb^), Upsilon (bb^) [where c = charm quark, s = strange quark, b = bottom quark], etc. are unmatter too.

Also, the pentaquark Theta-plus (Q$^+$), of charge $^+$1, uudds^ (*i.e.*, two quarks up, two quarks down, and one anti-strange quark), at a mass of 1.54 geV and a narrow width of 22 MeV, is unmatter, observed in 2003 at the Jefferson Lab in Newport News, Virginia, in the experiments that involved multi-GeV photons impacting a deuterium target.

Similar pentaquark evidence was obtained by Takashi Nakano of Osaka University in 2002, by researchers at the ELSA accelerator in Bonn in 1997-1998, and by researchers at ITEP in Moscow in 1986.

Besides Theta-plus, evidence has been found in one experiment[4] for other pentaquarks, X_5^{--} (ddssu^) and X_5^{+} (uussd^). D.S. Carman[5] has reviewed the positive and null evidence for these pentaquarks and their existence is still under investigation.

In order for the paper to be self-contained let's recall that the *pionium* is formed by p$^+$ and p$^-$ mesons, the *positronium* is formed by an antielectron (positron) and an electron in a semi-stable arrangement, the *protonium* is formed by a proton and an antiproton which are also semi-stable, the *antiprotonic helium* is formed by an antiproton and electron together with the helium nucleus (semi-stable), and *muonium* is formed by a positive muon and an electron.

Also, the *mesonic atom* is an ordinary atom with one or more of its electrons replaced by negative mesons. The *strange matter* is an ultra-dense matter formed by a large number of strange quarks bound together with an electron atmosphere (this strange matter is hypothetical).

From the exotic atom, the pionium, positronium, protonium, antiprotonic helium, and muonium are unmatter. The mesonic atom is unmatter if the electron(s) are replaced by negatively-charged anti-mesons.

Also we can define a mesonic anti-atom as an ordinary anti-atomic nucleus with one or more of its antielectrons replaced by positively-charged mesons. Hence, this mesonic anti-atom is unmatter if the antielectron(s) are replaced by positively-charged mesons.

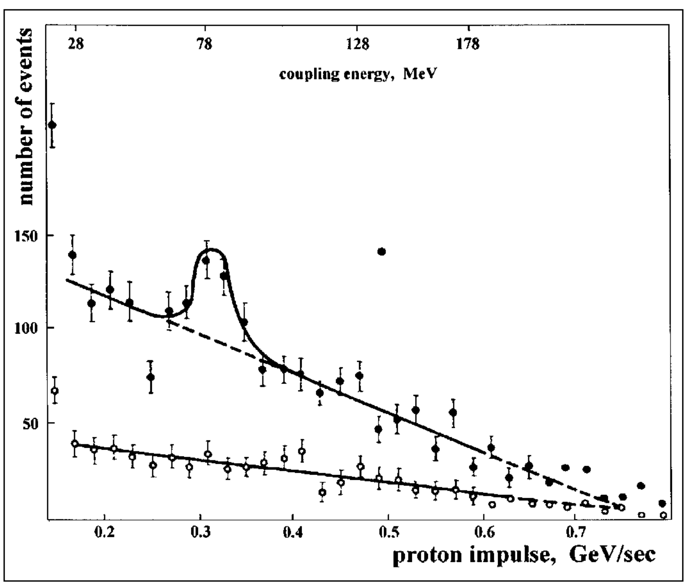

Figure 1. Spectra of proton impulses in the reaction p̄ + d fi (p̄n) + p. The upper arc is the annihilation of p̄n into even number of pions, the lower arc its annihilation into odd number of pions. The observed maximum points out that there is a connected system p̄n. Abscissa axis represents the proton impulse in GeV/sec (and the connection energy of the system p̄n). Ordinate axis is the number of events.

The strange matter can be unmatter if there exists at least an antiquark together with so many quarks in the nucleus. Also, we can define the strange antimatter as formed by a large number of antiquarks bound together with an antielectron around them. Similarly, the strange antimatter can be unmatter if there exists at least one quark together with so many antiquarks in its nucleus.

The bosons and antibosons help in the decay of unmatter. There are 13+1 (Higgs boson) known bosons and 14 antibosons in present.

5. Quantum Chromodynamics Formula

In order to save the colorless combinations prevailed in the theory of quantum chromodynamics (QCD) of quarks and antiquarks in their combinations when binding, we devise the following formula:

$$Q - A \;\text{¸}\; \pm M3 \qquad (1)$$

where M3 means multiple of three, *i.e.* ±M3 = {3·k | k ¸ Z} = {…, -12, -9, -6, -3, 0, 3, 6, 9, 12, …}, and Q = number of quarks, A = number of antiquarks.

But (1) is equivalent to:

$$Q \;\text{"}\; A \pmod 3 \qquad (2)$$

(Q is congruent to A modulo 3).

To justify this formula we mention that three quarks form a colorless combination, and any multiple of three (M3) combination of quarks too, *i.e.* 6, 9, 12, etc. quarks. In a similar way, three antiquarks form a colorless combination, and any multiple of three (M3) combination of antiquarks too, *i.e.* 6, 9, 12, etc. antiquarks. Hence, when we have hybrid

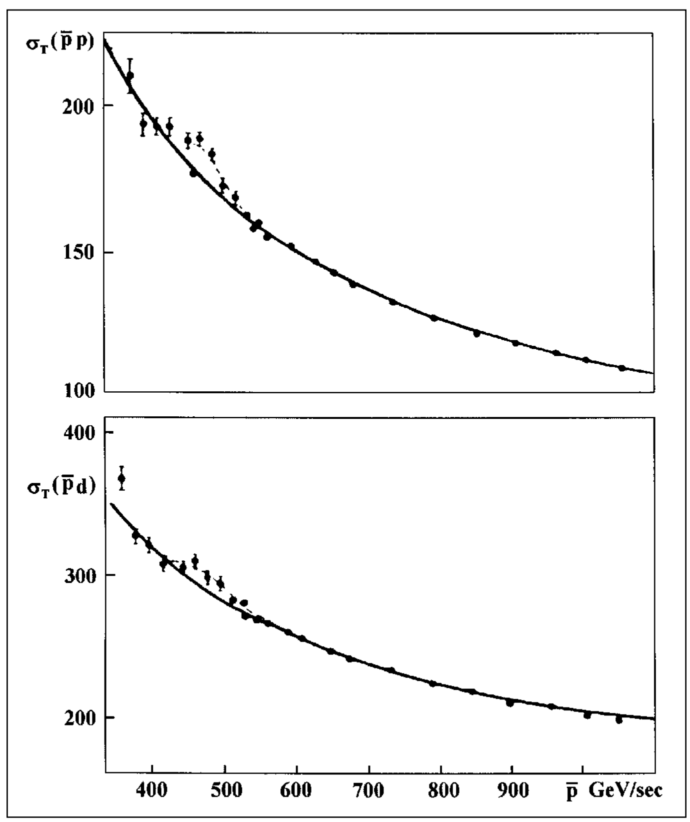

Figure 2. Probability s of interaction between p̄, p, and deutrons d (cited from [7]). The presence of maximum points out the existence of the resonance state of "nucleon–anti-nucleon."

combinations of quarks and antiquarks, a quark and an antiquark will annihilate their colors and, therefore, what's left should be a multiple of three number of quarks (in the case when the number of quarks is bigger, and the difference in the formula is positive), or a multiple of three number of antiquarks (in the case when the number of antiquarks is bigger, and the difference in the formula is negative).

6. Quark-Antiquark Combinations

Let's note by q = quark ˛ {Up, Down, Top, Bottom, Strange, Charm}, and by a = antiquark ˛ {Up^, Down^, Top^, Bottom^, Strange^, Charm^}. Hence, for combinations of n quarks and antiquarks, n ‡ 2, prevailing the colorless, we have the following possibilities:

- if n = 2, we have: qa (biquark, for example the mesons and anti-mesons);
- if n = 3, we have qqq, aaa (triquark, for example the baryons and anti-baryons);
- if n = 4, we have qqaa (tetraquark);
- if n = 5, we have qqqqa, aaaaq (pentaquark);
- if n = 6, we have qqqaaa, qqqqqq, aaaaaa (hexaquark);
- if n = 7, we have qqqqqaa, qqaaaaa (septiquark);
- if n = 8, we have qqqqaaaa, qqqqqqaa, qqaaaaaa (octoquark);
- if n = 9, we have qqqqqqqqq, qqqqqqaaa, qqqaaaaaa, aaaaaaaaa (nonaquark);
- if n = 10, we have qqqqqaaaaa, qqqqqqqqaa, qqaaaaaaaa (decaquark); etc.

7. Unmatter Combinations

From the above general case we extract the unmatter combinations:

— For combinations of two we have: qa (unmatter biquark), [mesons and anti-mesons]; the number of all possible unmatter combinations will be $6 \cdot 6 = 36$, but not all of them will bind together. It is possible to combine an entity with its mirror opposite and still bound them, such as: uu^, dd^, ss^, cc^, bb^, which form mesons. It is possible to combine, unmatter + unmatter = unmatter, as in ud^ + us^ = uud^s^ (if they bind together).

— For combinations of three (unmatter triquark) we can not form unmatter since the colorless can not hold.

— For combinations of four we have: qqaa (unmatter tetraquark); the number of all possible unmatter combinations will be $6^2 \cdot 6^2 = 1{,}296$, but not all of them will bind together.

— For combinations of five we have: qqqqa, or aaaaq (unmatter pentaquarks); the number of all possible unmatter combinations will be $6^4 \cdot 6 + 6^4 \cdot 6 = 15{,}552$, but not all of them will bind together.

— For combinations of six we have: qqqaaa (unmatter hexaquarks); the number of all possible unmatter combinations will be $6^3 \cdot 6^3 = 46{,}656$, but not all of them will bind together.

— For combinations of seven we have: qqqqqaa, qqaaaaa (unmatter septiquarks); the number of all possible unmatter combinations will be $6^5 \cdot 6^2 + 6^2 \cdot 6^5 = 559{,}872$, but not all of them will bind together.

— For combinations of eight we have: qqqqaaaa, qqqqqqqa, qaaaaaaa (unmatter octoquarks); the number of all possible unmatter combinations will be $6^4 \cdot 6^4 + 6^7 \cdot 6^1 + 6^1 \cdot 6^7 = 5{,}038{,}848$, but not all of them will bind together.

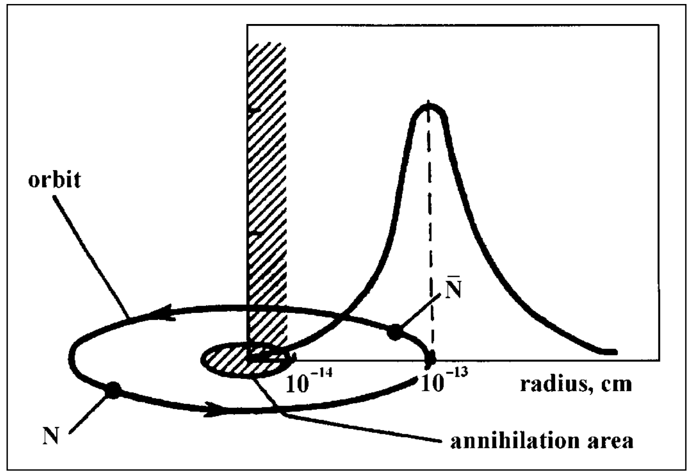

Figure 3. Annihilation area and the probability arc in "nucleon–anti-nucleon" system (cited from [11]).

— For combinations of nine we have: qqqqqqaaa, qqqaaaaaa (unmatter nonaquarks); the number of all possible unmatter combinations will be $6^6 \cdot 6^3 + 6^3 \cdot 6^6 = 2 \cdot 6^9 = 20{,}155{,}392$, but not all of them will bind together.

- For combinations of ten we have: qqqqqqqqaa, qqqqqaaaaa, qqaaaaaaaa (unmatter decaquarks); the number of all possible unmatter combinations will be $3 \cdot 6^{10} = 181{,}398{,}528$, but not all of them will bind together.

It may be possible to make infinite combinations of quarks/antiquarks and leptons/antileptons.

Unmatter can combine with matter and/or antimatter and the result may be any of these three. Some unmatter could be in the strong force, hence part of hadrons.

8. Unmatter Charge

The charge of unmatter may be positive, as in the pentaquark Theta-plus, 0 (as in positronium), or negative as in anti-Rho meson (u^d) [M. Jordan].

9. Containment

I think for the containment of antimatter and unmatter it would be possible to use electromagnetic fields (a container whose walls are electromagnetic fields). But its duration is unknown.

10. Further Research

Let's start from neutrosophy,[18] which is a generalization of dialectics, *i.e.* not only the opposites are combined but also the neutralities. Why? Because when an idea is launched, a category of people will accept it, others will reject it, and a third group will ignore it (won't care). But the dynamics between these three categories changes, so somebody accepting it might later reject or ignore it, and so on. Similarly, the dynamicity of <A>, <antiA>, <neutA>, where <neutA> means neither <A> nor <antiA>, but in between (neutral).

Neutrosophy considers not di-alectics but *tri-alectics* (based on three components: <A>, <antiA>, <neutA>). Hence unmatter is a kind of neutrality (not referring to the charge) between matter and antimatter, *i.e.* neither one, nor the other. Upon the model of unmatter we may look at ungravity, unforce, unenergy, etc.

Ungravity would be a mixture of gravity and antigravity (for example, attracting and rejecting simultaneously or alternatively; or a magnet which changes the positive and negative poles frequently).

Unforce: We may consider positive force (in the direction we want) and negative force (repulsive, opposed to the previous). There could be a combination of both positive and negative forces at the same time, or alternating positive and negative, etc.

Unenergy would similarly be a combination of positive and negative energies (as the alternating current (a.c.), which periodically reverses its direction in a circuit and whose frequency, f, is independent of the circuit's constants). Would it be possible to construct an alternating-energy generator?

In conclusion, according to the universal dialectic, unity is manifested in duality and duality in unity. "Thus, Unmatter (unity) is experienced as duality (matter vs. antimatter). Ungravity (unity) as duality (gravity vs. antigravity). Unenergy (unity) as duality (positive energy vs. negative energy). And thus also. . .between duality of being (existence) vs. nothingness (anti-existence) must be 'unexistence' (or pure unity)."[12]

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A Note on Computer Solution of Wireless Energy Transmit via Magnetic Resonance

Victor Christianto, Florentin Smarandache

Victor Christianto, Florentin Smarandache (2008). A Note on Computer Solution of Wireless Energy Transmit via Magnetic Resonance. *Progress in Physics* 1, 81-82

In the present article we argue that it is possible to find numerical solution of coupled magnetic resonance equation for describing wireless energy transmit, as discussed recently by Karalis (2006) and Kurs *et al.* (2007). The proposed approach may be found useful in order to understand the phenomena of magnetic resonance. Further observation is of course recommended in order to refute or verify this proposition.

1 Introduction

In recent years there were some new interests in methods to transmit energy without wire. While it has been known for quite a long-time that this method is possible theoretically (since Maxwell and Hertz), until recently only a few researchers consider this method seriously.

For instance, Karalis et al [1] and also Kurs et al. [2] have presented these experiments and reported that efficiency of this method remains low. A plausible way to solve this problem is by better understanding of the mechanism of magnetic resonance [3].

In the present article we argue that it is possible to find numerical solution of coupled magnetic resonance equation for describing wireless energy transmit, as discussed recently by Karalis (2006) and Kurs *et al.* (2007). The proposed approach may be found useful in order to understand the phenomena of magnetic resonance.

Nonetheless, further observation is of course recommended in order to refute or verify this proposition.

2 Numerical solution of coupled-magnetic resonance equation

Recently, Kurs *et al.* [2] argue that it is possible to represent the physical system behind wireless energy transmit using coupled-mode theory, as follows:

$$a_m(t) = (i\omega_m - \Gamma_m)\, a_m(t) + \\ + \sum_{n \neq m} i\kappa_{nm} a_n(t) - F_m(t)\,. \tag{1}$$

The simplified version of equation (1) for the system of two resonant objects is given by Karalis *et al.* [1, p. 2]:

$$\frac{da_1}{dt} = -\,i\,(\omega_1 - i\Gamma_1)\,a_1 + i\kappa a_2\,, \tag{2}$$

and

$$\frac{da_2}{dt} = -\,i\,(\omega_2 - i\Gamma_2)\,a_2 + i\kappa a_1\,. \tag{3}$$

These equations can be expressed as linear 1st order ODE as follows:

$$f'(t) = -\,i\,\alpha\,f(t) + i\,\kappa\,g(t) \tag{4}$$

and

$$g'(t) = -\,i\beta\,g(t) + i\,\kappa\,f(t)\,, \tag{5}$$

where

$$\alpha = (\omega_1 - i\,\Gamma_1) \tag{6}$$

and

$$\beta = (\omega_2 - i\Gamma_2) \tag{7}$$

Numerical solution of these coupled-ODE equations can be found using Maxima [4] as follows. First we find test when parameters (6) and (7) are set up to be 1. The solution is:

```
(%i5) 'diff(f(x),x)+%i*f=%i*b*g(x);
(%o5) 'diff(f(x),x,1)+%i*f=%i*b*g(x)
(%i6) 'diff(g(x),x)+%i*g=%i*b*f(x);
(%o6) 'diff(g(x),x,1)+%i*g=%i*b*f(x)
(%i7) desolve([%o5,%o6],[f(x),g(x)]);
```

The solutions for $f(x)$ and $g(x)$ are:

$$f(x) = \frac{\left[i\,g(0)\,b - i\,f(x)\right]\,\sin(bx)}{b} - \\ - \frac{\left[g(x) - f(0)\,b\right]\,\cos(bx)}{b} + \frac{g(x)}{b}\,, \tag{8}$$

$$g(x) = \frac{\left[i\,f(0)\,b - i\,g(x)\right]\,\sin(bx)}{b} - \\ - \frac{\left[f(x) - g(0)\,b\right]\,\cos(bx)}{b} + \frac{f(x)}{b}\,. \tag{9}$$

Translated back to our equations (2) and (3), the solutions for $\alpha = \beta = 1$ are given by:

$$a_1(t) = \frac{\left[i\,a_2(0)\kappa - i\,a_1\right]\,\sin(\kappa t)}{\kappa} - \\ - \frac{\left[a_2 - a_1(0)\,\kappa\right]\,\cos(\kappa t)}{\kappa} + \frac{a_2}{\kappa} \tag{10}$$

$$f(x) = e^{-(ic-ia)t/2} \left[\frac{\left[2i\,f(0)\,c + 2i\,g(0)\,b - f(0)(ic - i\,a)\right] \sin\left(\frac{\sqrt{c^2-2ac+4b^2+a^2}}{2}\,t\right)}{\sqrt{c^2 - 2ac + 4b^2 + a^2}} + \right.$$
$$\left. + \frac{f(0) \cos\left(\frac{\sqrt{c^2-2ac+4b^2+a^2}}{2}\,t\right)}{\sqrt{c^2 - 2ac + 4b^2 + a^2}} \right] \tag{13}$$

$$g(x) = e^{-(ic-ia)t/2} \left[\frac{\left[2i\,f(0)\,c + 2i\,g(0)\,a - g(0)(ic - i\,a)\right] \sin\left(\frac{\sqrt{c^2-2ac+4b^2+a^2}}{2}\,t\right)}{\sqrt{c^2 - 2ac + 4b^2 + a^2}} + \right.$$
$$\left. + \frac{g(0) \cos\left(\frac{\sqrt{c^2-2ac+4b^2+a^2}}{2}\,t\right)}{\sqrt{c^2 - 2ac + 4b^2 + a^2}} \right] \tag{14}$$

$$a_1(t) = e^{-(i\beta-i\,\alpha)t/2} \left(\frac{\left[2i\,a_1(0)\,\beta + 2i\,a_2(0)\,\kappa - (i\beta - i\alpha)\,a_1\right] \sin\left(\frac{\xi}{2}\,t\right)}{\xi} - \frac{a_1(0) \cos\left(\frac{\xi}{2}\,t\right)}{\xi} \right) \tag{15}$$

$$a_2(t) = e^{-(i\beta-i\alpha)t/2} \left(\frac{\left[2i\,a_2(0)\,\beta + 2i\,a_1(0)\,\kappa - (i\beta - i\,\alpha)\,a_2\right] \sin\left(\frac{\xi}{2}\,t\right)}{\xi} - \frac{a_2(0) \cos\left(\frac{\xi}{2}\,t\right)}{\xi} \right) \tag{16}$$

and

$$a_2(t) = \frac{\left[i\,a_1(0)\kappa - i\,a_2\right] \sin(\kappa t)}{\kappa} - $$
$$- \frac{\left[a_1 - a_2(0)\,\kappa\right] \cos(\kappa t)}{\kappa} + \frac{a_1}{\kappa}\,. \tag{11}$$

Now we will find numerical solution of equations (4) and (5) when $\alpha \neq \beta \neq 1$. Using Maxima [4], we find:

(%i12) 'diff(f(t),t)+%i*a*f(t)=%i*b*g(t);

(%o12) 'diff(f(t),t,1)+%i*a*f(t)=%i*b*g(t)

(%i13) 'diff(g(t),t)+%i*c*g(t)=%i*b*f(t);

(%o13) 'diff(g(t),t,1)+%i*c*g(t)=%i*b*f(t)

(%i14) desolve([%o12,%o13],[f(t),g(t)]);

and the solution is found to be quite complicated: these are formulae (13) and (14).

Translated back these results into our equations (2) and (3), the solutions are given by (15) and (16), where we can define a new "ratio":

$$\xi = \sqrt{\beta^2 - 2\alpha\beta + 4\kappa^2 + \alpha^2}\,. \tag{12}$$

It is perhaps quite interesting to remark here that there is no "distance" effect in these equations.

Nonetheless, further observation is of course recommended in order to refute or verify this proposition.

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Interpretation of Solution of Radial Biquaternion Klein-Gordon Equation and Comparison with EQPET/TSC Model

Victor Christianto, Florentin Smarandache

Victor Christianto, Florentin Smarandache (2008). Interpretation of Solution of Radial Biquaternion Klein-Gordon Equation and Comparison with EQPET/TSC Model. *Infinite Energy* 79, 2

Abstract

In a previous publication,[1] we argued that the biquaternionic extension of the Klein-Gordon equation has numerical solution with sinusoidal form, which differs appreciably from conventional Yukawa potential. In the present article we interpret and compare this result from the viewpoint of the EQPET/TSC (Electronic Quasi-Particle Expansion Theory/Tetrahedral Symmetric Condensate) model described by Takahashi.[2] Further observation is of course recommended in order to refute or verify this proposition.

Introduction

In the preceding article[1] we argued that biquaternionic extension of the radial Klein-Gordon equation (radialBQKGE) has numerical solution with sinusoidal form, which differs appreciably from conventional Yukawa potential. We also argued that this biquaternionic extension of KGE may be useful in particular to explore new effects in the context of low-energy nuclear reaction (LENR).[3]

Interestingly, Takahashi[2] has discussed key experimental results in condensed matter nuclear effects in light of EQPET/TSC. We argue here that the potential model used in his paper, STTBA (Sudden Tall Thin Barrier Approximation), may be comparable to our derived sinusoidal potential from radial biquaternion KGE.[1] While we can't yet offer numerical prediction, our qualitative comparison may be useful in verifying further experiments.

Solution of Radial Biquaternionic KGE (radial BQKGE)

In our previous paper,[1] we argued that it is possible to write the biquaternionic extension of the Klein-Gordon equation as follows:

$$\left(\diamond\bar{\diamond} + m^2\right)\varphi(x,t) = 0 \tag{1}$$

Provided we use this definition:[1,3]

$$\begin{aligned}\diamond = \nabla^q + i\nabla^q = &\left(-i\frac{\partial}{\partial t} + e_1\frac{\partial}{\partial x} + e_2\frac{\partial}{\partial y} + e_3\frac{\partial}{\partial z}\right) \\ &+ i\left(-i\frac{\partial}{\partial T} + e_1\frac{\partial}{\partial X} + e_2\frac{\partial}{\partial Y} + e_3\frac{\partial}{\partial Z}\right)\end{aligned} \tag{2}$$

where e_1, e_2, e_3 are *quaternion imaginary units* obeying (with ordinary quaternion symbols $e_1=i$, $e_2=j$, $e_3=k$):[3,4]

$$\begin{aligned} i^2 = j^2 = k^2 = -1,\ ij = -ji = k\ , \\ jk = -kj = i\ \ ,\ \ ki = -ik = j\ . \end{aligned} \tag{3}$$

And quaternion Nabla operator is defined as:[2]

$$\nabla^q = -i\frac{\partial}{\partial t} + e_1\frac{\partial}{\partial x} + e_2\frac{\partial}{\partial y} + e_3\frac{\partial}{\partial z} \tag{4}$$

By using polar coordinates transformation,[5] we get this for the one-dimensional situation:

$$\left(\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}\right) - i\frac{\partial}{\partial r}\left(\frac{\partial}{\partial r}\right) + m^2\right)\varphi(x,t) = 0 \tag{5}$$

The solution is given by:[1]

$$y = k_1 \cdot \sin\left(\frac{|m|r}{\sqrt{-i-1}}\right) + k_2 \cdot \cos\left(\frac{|m|r}{\sqrt{-i-1}}\right) \tag{6}$$

Therefore, we may conclude that numerical solution of radial biquaternionic extension of Klein-Gordon equation yields different potential compared to the well-known Yukawa potential:[1]

$$u(r) = -\frac{g^2}{r}e^{-mr} \tag{7}$$

In the next section we will discuss an interpretation of this new potential (6) compared to the findings discussed by Takahashi[2] from condensed matter nuclear experiments.

Comparison with Takahashi's EQPET/TSC/STTBA model

Takahashi[2] reported some findings from condensed matter nuclear experiments, including intense production of helium-4 (^{4}He) atoms by electrolysis and laser irradiation experiments.

Takahashi analyzed those experimental results using EQPET formation of TSC were modelled with numerical estimations by STTBA. This STTBA model includes strong interaction with negative potential near the center (where $r \to 0$). See Figure 1.

Takahashi described that Gamow integral of STTBA is given by:

$$\Gamma_n = 0.218\,(m^{1/2})\int_{r_0}^{b}(V_b - E_d)^{1/2}\,dr \tag{8}$$

Using b=5.6 fm and r_0=5 fm, he obtained:

$$P_{4d} = 0.77 \tag{9}$$

and

$$V_B = 0.257\ \text{MeV} \tag{10}$$

While his EQPET model gave significant underestimation for 4D fusion rate when rigid constraint of motion in 3D space was attained, introducing different values of λ_{4d} can improve the result.[2] Therefore we may conclude that STTBA can offer good approximation of condensed matter nuclear reactions.[5]

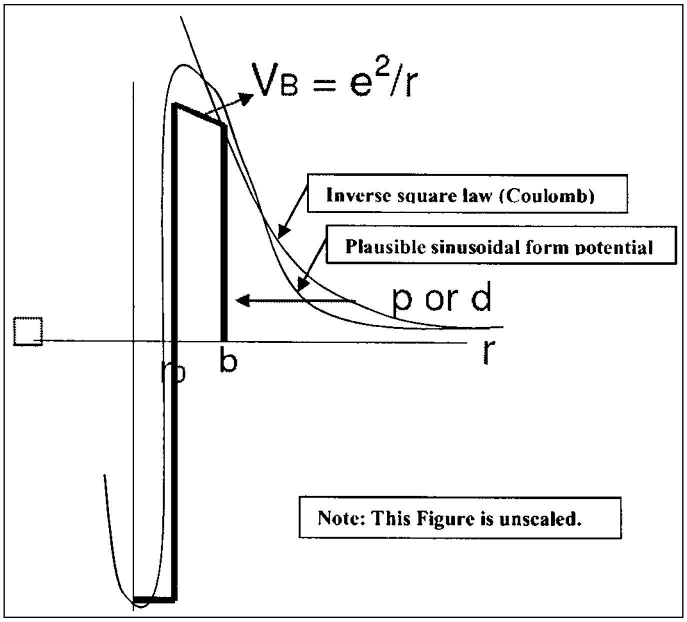

Figure 1. Potential for Coulomb barrier reversal for STTBA, from Takahashi.[2]

Interestingly, the STTBA lacks sufficient theoretical basis, therefore one can expect that a sinusoidal form (or combined sinusoidal waves such as in Fourier method) may offer better result which agrees with experiments. This will be pursued in a later paper.

Nonetheless, we recommend further observation in order to refute or verify this proposition of a new type of potential derived from the biquaternion radial Klein-Gordon equation.

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Five Paradoxes and a General Question on Time Traveling

Florentin Smarandache

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These are five paradoxes on time traveling, which come from Neutrosophy and Neutro-sophic Logics applied to the theory of relativity.

1 Traveling to the past

Joe40, who is 40 years old, travels 10 years back to the past when he was 30 years old. He meets himself when he was 30 years old, let's call this Joe30.

Joe40 kills Joe30.

If so, we mean if Joe died at age 30 (because Joe30 was killed), how could he live up to age 40?

2 Traveling to the future

Joe30, who is 30 years old, travels 10 years in the future and meats himself when he will be 40 years old, let's call him Joe40.

Joe40 kills Joe30.

At what age did Joe die, at 30 or 40?

If Joe30 died, then Joe40 would not exist.

3 Traveling pregnant woman

a) A 3-month pregnant woman, Jane3, travels 6 months to the future where she gives birth to a child Johnny3.
b) Then she returns with the child back, and after 1 month she travels 5 months to the future exactly at the same time as before.

Then how is it possible to have at exactly the same time two different situations: first only the pregnant woman, and second the pregnant woman and her child?

4 Traveling in the past before birth

Joe30, who is 30 years old, travels 40 years in the past, therefore 10 years before he was born.

How is it possible for him to be in the time when he did not exist?

5 Traveling in the future after death

Joe30, who is 30 years old, travels 40 years in the future, 10 years after his death. He has died when he was 60 years old, as Joe60.

How is it possible for him to be in the time when he did not exist any longer?

6 A general question about time traveling

When traveling say 50 years in the past [let's say from year 2010 to year 1960] or 50 years in the future [respectively from year 2010 to year 2060], how long does the traveling itself last?

If it's an instantaneous traveling in the past, is the time traveler jumping from year 2010 directly to year 1960, or is he continuously passing through all years in between 2010 and 1960? Similar question for traveling in the future.

If the traveling lasts longer say, a few units (seconds, minutes, etc.) of time, where will be the traveler at the second unit or third unit of time? I mean, suppose it takes 5 seconds to travel from year 2010 back to year 1960; then in the 1st second is he in year 2000, in the 2nd second in year 1990, in the 3rd second in year 1980, in the 4st second in year 1970, and in the 5st second in year 1960? So, his speed is 10 years per second?

Similar question for traveling in the future.

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Quantum Causality Threshold and Paradoxes

Florentin Smarandache

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Abstract

In this paper we consider two entangled particles and study all the possibilities: when both are immobile, or one of them is immobile, or both are moving in different directions, or one of them is moving in a different direction. Then we study the causality between them and the paradoxes, which are generated. We define the Causality Threshold of a particle A with respect to another particle B.

Keywords: Entangled particles, Causality, Causality threshold, Quantum paradoxes

1. Perfect simultaneousness

Let's consider two entangled particles A and B. {Schrödinger introduced the notion "entangled" in order to describe the non-separable states [Belavkin (2002)]}.

At the beginning, both are immobile, in the same space S(A,B) and time t (simultaneously), and none of them is in the causality cone of the other.

According to Einstein's Theory of Relativity, when a particle is moving with respect to the other, its time and space axes appear inclined from the perspective of the other particle, modifying what for this other particle is "before" or "after", but their causality cones remain the same. And, if both particles are moving with respect to each other, the appearance of the inclined time and space axes is reciprocal from the perspective of each other.

Let's define the Quantum Causality Threshold of the particle A with respect to the particle B, noted by $\tau_{A,B}$, to be the space-time when neither A nor B is a cause for the other on the B space-time axis (i.e. when the position-time vector vertex $t_A \equiv B$).

To change the causality of a particle A with respect to another particle B one has to pass through non-causality, i.e. one has to pass through their threshold.

Generally, $\tau_{A,B} \neq \tau_{B,A}$, because one can have $t_A \equiv B$ but $t_B \neq A$, or reciprocally [see, for example, Figure 1.1.1].

a) When $\tau_{A,B} = \tau_{B,A}$ there is no causality between A and B (and therefore there is no quantum causality paradox).

b) If one particle attains its threshold with respect to the other, and the other one does not, then there is a causality and a non-causality simultaneously (and thus a quantum causality paradox) [see Figures 1.1.1, 1.1.2, 1.2.1].

c) If no particle attains its threshold with respect to the other, one has two sub-cases: either opposite causalities (and thus, again, a quantum causality paradox) [see Figures 1.1.3, 1.1.4], or compatible causalities (and, consequently, there is no quantum causality paradox) [see Figures 1.2.2 (for t together with t\` time axes), Figure 1.2.3 (for t together with t\`\`\` time axes)].

1.1 Moving particle(s) keeping the same direction

1.1.1 Particle B is moving away from particle A

< Figure 1.1.1>

- S(A,B) is the space (represented here by a plane) of both entangled particles A and B.
- The left red vertical (t) continuous line represents the time axis of the particle A.
- Similarly, the green (t) continuous line represents the time axis of the particle B.

- On the left side one has the double cone of causality of the particle A: the cone beneath S(A,B) contains the events that are the cause for A (i.e. events that influenced A), and the cone above S(A,B) contains the events that A is a cause for (i.e. events influenced by A).
- Similarly, the right double cone represents the cone of causality of the particle B.
- Beneath S(A,B) it is the past time ("before A"), lying on the S(A,B) is the present time ("simultaneously with A"), and above S(A,B) it is the future time ("after A").
- Similarly, because the particles A and B are in the same space, S(A,B) separates the past, present, and future times for the particle B.

Relative to the same referential system, the particle A remains immobile, while the particle B starts moving in the opposite direction relative to A. [Figure 1.1.1]

Therefore, from the perspective of B, the entangled particles A and B are simultaneous, and none of them is the cause of the other ($t_A \equiv B$ on B's time axis); while from the perspective of A, the particle A is a cause for the particle B (i.e. $A < t_B$ on A's time axis).

Hence, it appears this quantum causality paradox: non-causality or causality simultaneously?

1.1.2 Particle B is moving closer to particle A

<Figure 1.1.2>

Relative to the same referential system, the particle A remains immobile, while the particle B starts moving in a direction towards A. [Figure 1.1.2]

Therefore, from the perspective of the particle B, the entangled particles A and B are simultaneous, and none of them is the cause of the other ($t_A \equiv B$ on B's time axis); while from the perspective of the particle A, the particle B is a cause for the particle A (i.e. tB < A on A's time axis).

Hence, again, it appears a similar quantum causality paradox: non-causality or causality simultaneously?

1.1.3 Both entangled particles are moving closer to each other

<Figure 1.1.3>

With respect to the same referential system, both particles A and B start moving towards each other. [Figure 1.1.3]

Therefore, from the perspective of the particle A, the particle B is a cause of the particle A (i.e. $t_B < A$ on A's time axis), and reciprocally: from the perspective of the particle B, the particle A is a cause of the particle B (i.e. $t_A < B$ on B's time axis). Thus one obtains the following:

Quantum Causality Paradox: How is it possible that simultaneously A is a cause of B, and B is a cause of A?

1.1.4 Both entangled particles are moving away from each other

<Figure 1.1.4>

With respect to the same referential system, both particles A and B start moving in opposite directions from each other. [Figure 1.1.4]

Therefore, from the perspective of A, the particle A is a cause of the particle B (i.e. $A < t_B$ on A's time axis), and reciprocally: from the perspective of B, the particle B is a cause of the particle A (i.e. $B < t_A$ on B's time axis). Thus, one obtains the following same statement:

Quantum Causality Paradox: How is it possible that simultaneously A is a cause of B, and B is a cause of A?

This theoretical case is similar to the 2002 Suarez Experiment [1], the only difference being that in Suarez's experiment there is not a perfect simultaneousness between the particles A and B.

1.2 Moving particle(s) changing the direction

1.2.1 With respect to the same referential system, the particle A is immobile; while the particle B is moving at the beginning in a direction towards A, and later B changes the direction moving away from A.

a) Then, from the perspective of A: The particle B is a cause for A (i.e. $t`_B < A$ on A's time axis). Then B changes its movement in a direction away from A, consequently B attains its quantum threshold $\tau_{B,A}$, i.e. $t``_B \equiv A$ on A's time axis (now there is no causality between A and B). B keeps moving further from A and crosses its quantum threshold, then A becomes a causality for B because $t``_B > A$ on A's time axis.

b) While, from the perspective of B, there is no causality between A and B, since B ≡ t_A on all B's three time axes t`, t``, t```. [Figure 1.2.1.]. Hence, this quantum causality paradox appears: simultaneously B is cause for A, and non-causality, and A is cause for B?

< Figure 1.2.1>

1.2.2 Relative to the same referential system, the particle A is moving away from B; while the particle B is moving at the beginning in a direction towards A, and later B changes the direction moving away from A.

a) Then from the perspective of A: B is a cause for A (i.e. $t`_B$ < A on A's time axis). Then B changes its movement in a direction away from A, consequently B attains its quantum threshold $\tau_{B,A}$, i.e. $t``_B$ ≡ A on A's time axis (now there is no causality among A and B). B keeps moving further from A and crosses its quantum threshold, then A becomes a causality for B because $t``_B$ > A on A's time axis.

b) While from the perspective of B, the particle B is always a cause for A, since B < t_A on all B's time axes t`, t``, and t```. [Figure 1.2.2]. Hence, this quantum causality paradox appears: simultaneously B is cause for A, and non-causality, and A is cause for B?

< Figure 1.2.2>

1.2.3 With respect to the same referential system, the particle A is moving closer to B; while the particle B is moving at the beginning in a direction towards A, and later B changes the direction moving away from A.

a) Then from the perspective of A: B is a cause for A (i.e. $t`_B$ < A on A's time axis). Then B changes its movement in a direction away from A, consequently B attains its quantum threshold $\tau_{B,A}$, i.e. $t``_B$ ≡ A on A's time axis (now there is no causality among A and B). B keeps moving further from A and crosses its quantum threshold, then A becomes a cause for B, because $t``_B$ > A on A's time axis.

b) While from the perspective of B, the particle A is always a cause for B, since t_A < B on all B's time axes t`, t``, and t```. [Figure 1.2.2]. Hence, this quantum causality paradox appears: simultaneously B is cause for A, and non-causality, and A is cause for B?

< Figure 1.2.3>

2. Let's consider the non-simultaneousness, when the particles A and B are in the separate spaces, S(A) and S(B) respectively, and different time axes, t and t` respectively

2.1 Moving particle(s) keeping the same direction

2.1.1 With respect to the same referential system, both particles A and B are moving in the same direction but with different high speeds. [Figure 2.1.1]

Therefore, from both perspectives, of A and of B, the particle B is cause for A.

< Figure 2.1.1>

2.1.2 With respect to the same referential system, both particles A and B are moving in the same direction and with the same high speeds. [Figure 2.1.2]

Therefore, from both perspectives, of A and of B, neither one is the causality of the other.

< Figure 2.1.2>

2.1.3 With respect to the same referential system, both particles A and B are moving closer to each other and with different high speeds [Figure 2.1.3]. Therefore, from the perspective of A the particle B is a cause of A, and reciprocally, thus again one gets a quantum causality paradox.

< Figure 2.1.3>

2.2 Moving particle(s) changing the direction

2.2.1 With respect to the same referential system, the particle A is moving towards B; while the particle B is moving at the beginning in a direction towards A, and later B changes the direction moving away from A.

a) Then from the perspective of A: B is a cause for A (i.e. $t`_B$ < A on A's time axis). Then B changes its movement in a direction away from A, consequently B attains its quantum threshold $\tau_{B,A}$, i.e. $t``_B$ ≡ A on A's time axis (now there is no causality among A and B). B keeps moving further from A and crosses its quantum threshold, then A becomes a cause for B because $t``_B$ > A on A's time axis.

b) While from the perspective of B, the particle A is always a cause for B, since t_A < B on all B's time axes t`, t``, and t```. [Figure 2.2.1]. Hence, this quantum causality paradox appears: simultaneously B is cause for A, and non-causality, and A is cause for B?

< Figure 2.2.1>

2.2.2 Relative to the same referential system, both particles are moving towards each other, and then both change the movement in the opposite directions.

Similarly, from both perspectives, of A and of B, there are normal causalities (corresponding to t1 and t` time axes), non-causalities (corresponding to t2 and t`` time axes), and opposite causalities (corresponding to t3 and t``` time axes) [Figure 2.2.2].

Hence, one again, one arrives at quantum causality paradoxes.

< Figure 2.2.2>

Acknowledgement

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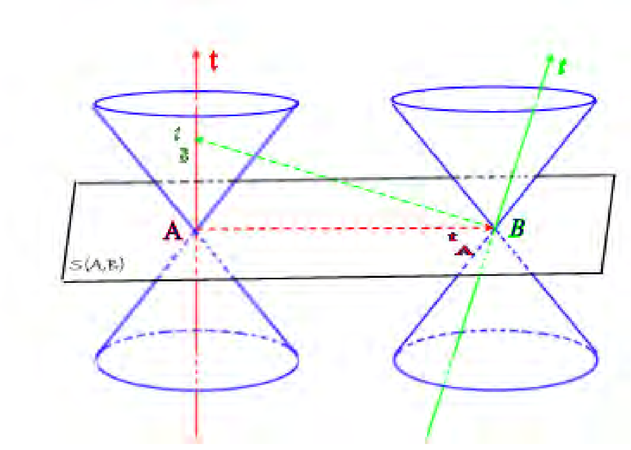

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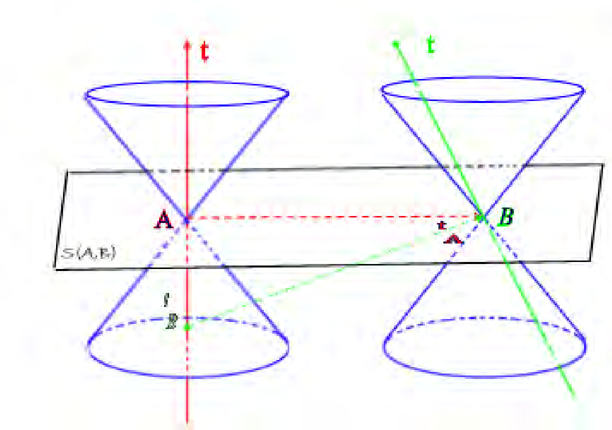

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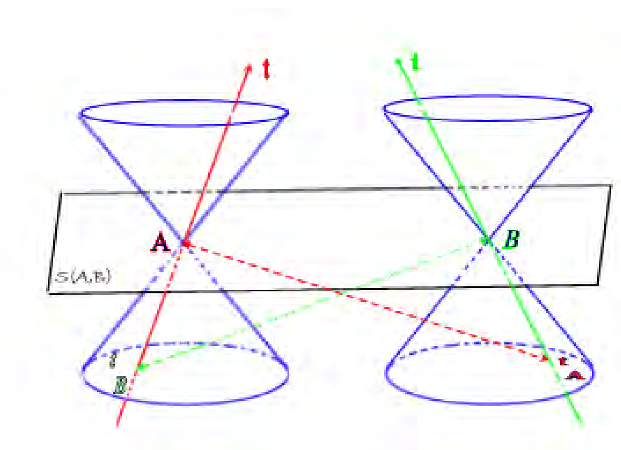

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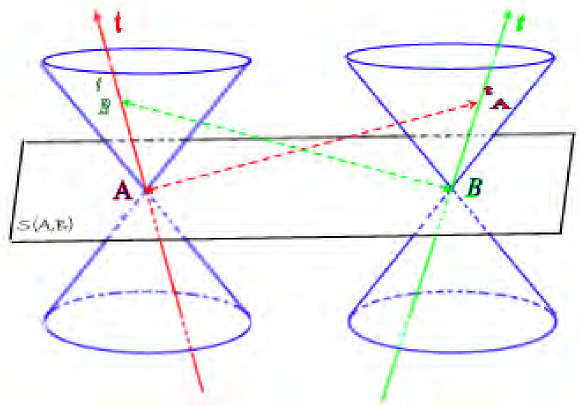

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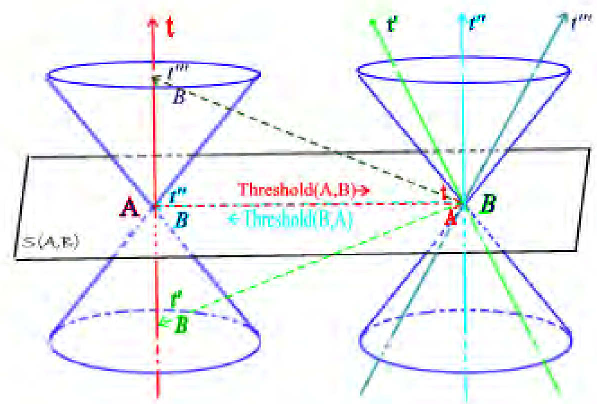

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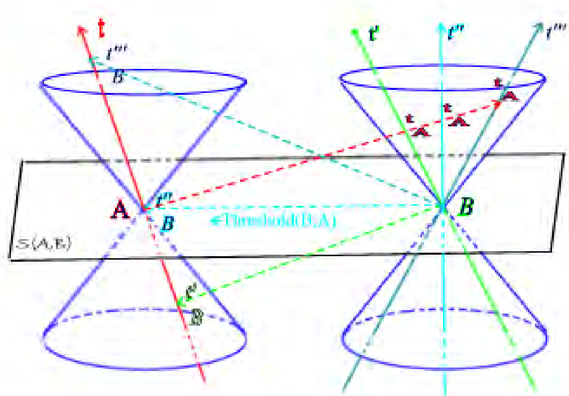

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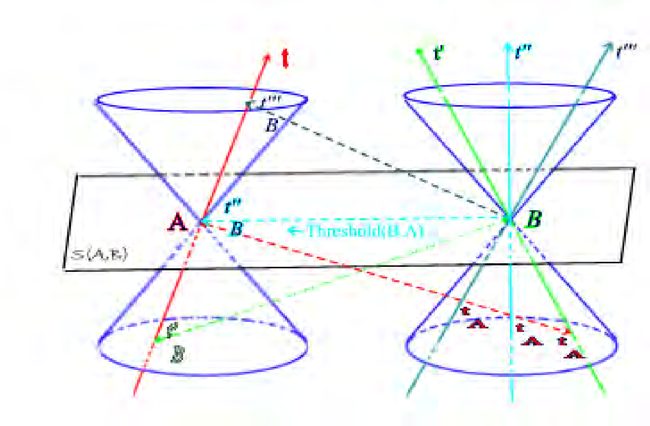

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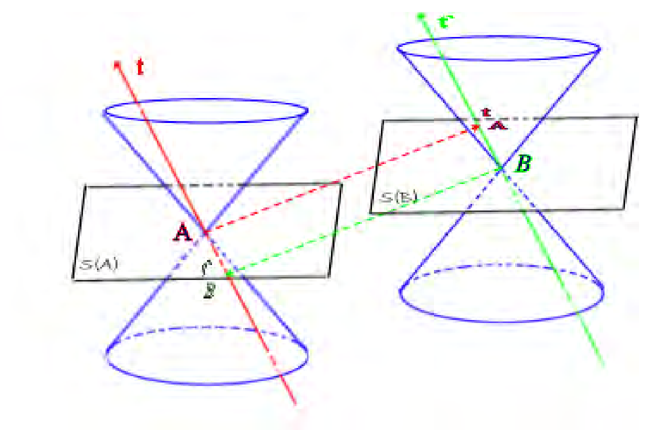

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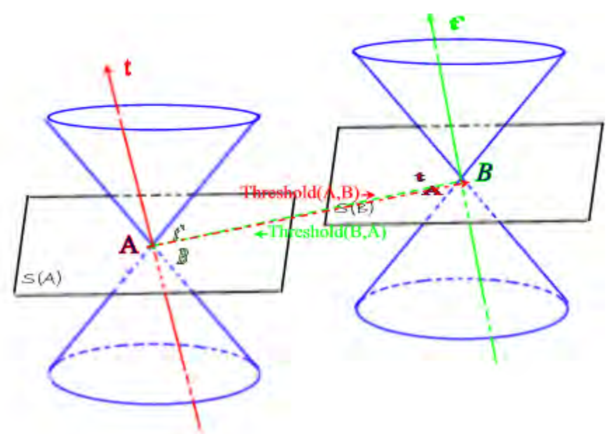

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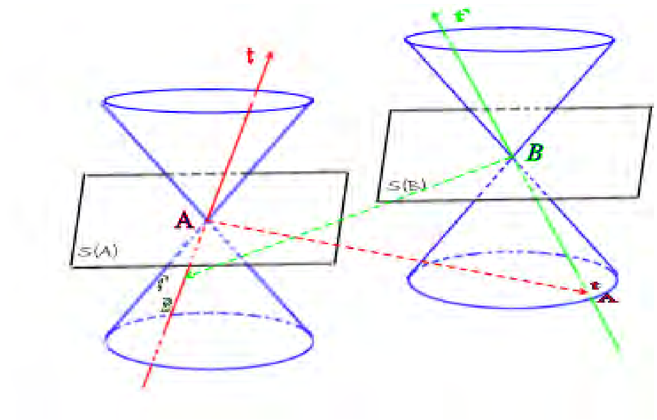

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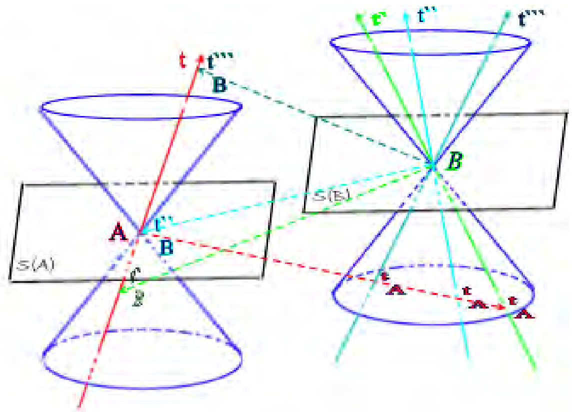

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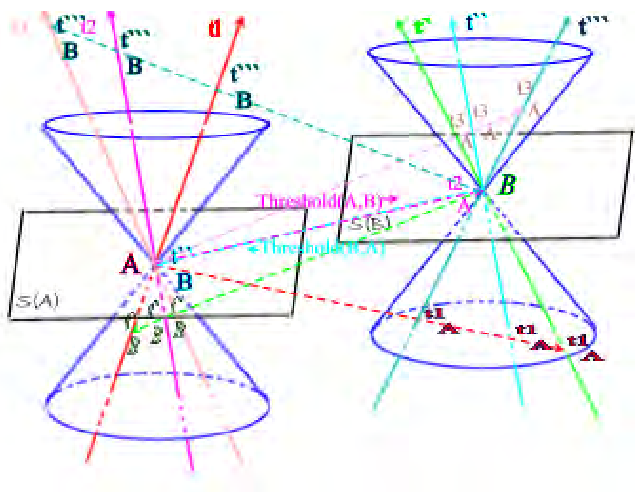

Figure 2.2.2

Superluminal Physics and Instantaneous Physics as New Trends in Research

Florentin Smarandache

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In a similar way as passing from Euclidean Geometry to Non-Euclidean Geometry, we can pass from Subluminal Physics to Superluminal Physics, and further to Instantaneous Physics. In the lights of two consecutive successful CERN experiments with superluminal particles in the Fall of 2011, we believe that these two new fields of research should begin developing.

1 Introduction

Let's start by recalling the history of geometry in order to connect it with the history of physics.

Then we present the way of S-denying a law (or theory) and building a spectrum of spaces where the same physical law (or theory) has different forms, then we mention the S-multispace with its multistructure that may be used to the Unified Field Theory by employing a *multifield*.

It is believed that the S-multispace with its multistructure is the best candidate for 21st century *Theory of Everything* in any domain.

2 Geometry's history

As in Non-Euclidean Geometry, there are models that validate the hyperbolic geometric and of course invalidate the Euclidean geometry, or models that validate the elliptic geometry and in consequence they invalidate the Euclidean geometry and the hyperbolic geometry.

Now, we can mix these geometries and construct a model in which an axiom is partially validated and partially invalidated, or the axiom is only invalidated but in multiple different ways [1]. This operation produces a degree of negation of an axiom, and such geometries are hybrid. We can in general talk about the *degree of negation of a scientific entity* P, where P can be a theorem, lemma, property, theory, law, etc.

3 S-denying of a theory

Let's consider a physical space S endowed with a set of physical laws L, noted by (S, L), such that all physical laws L are valid in this space S.

Then, we construct another physical space (or model) S_1 where a given law has a different form, afterwards another space S_2 where the same law has another form, and so on until getting a spectrum of spaces where this law is different.

We thus investigate spaces where anomalies occur [2].

4 Multispace theory

In any domain of knowledge, multispace (or S-multispace) with its multistructure is a finite or infinite (countable or uncountable) union of many spaces that have various structures. The spaces may overlap [3].

The notions of multispace (also spelt multi-space) and multistructure (also spelt multi-structure) were introduced by the author in 1969 under his idea of hybrid science: combining different fields into a unifying field (in particular combinations of different geometric spaces such that at least one geometric axiom behaves differently in each such space), which is closer to our real life world since we live in a heterogeneous multispace. Today, this idea is accepted by the world of sciences. S-multispace is a qualitative notion, since it is too large and includes both metric and non-metric spaces.

A such multispace can be used for example in physics for the Unified Field Theory that tries to unite the gravitational, electromagnetic, weak and strong interactions by constructing a *multifield* formed by a gravitational field united with an electromagnetic field united with a weak-interactions field and united with a strong-interactions field.

Or in the parallel quantum computing and in the mu-bit theory, in multi-entangled states or particles and up to multi-entangles objects.

We also mention: the algebraic multispaces (multi-groups, multi-rings, multi-vector spaces, multi-operation systems and multi-manifolds, also multi-voltage graphs, multi-embedding of a graph in an n-manifold, etc.) or structures included in other structures, geometric multispaces (combinations of Euclidean and Non-Euclidean geometries into one space as in S-geometries), theoretical physics, including the Relativity Theory [4], the M-theory and the cosmology, then multi-space models for p-branes and cosmology, etc.

The multispace is an extension of the neutrosophic logic and set, which derived from neutrosophy. Neutrosophy (1995) is a generalization of dialectics in philosophy, and takes into consideration not only an entity <A> and its opposite <antiA> as dialectics does, but also the neutralities <neutA> in between. Neutrosophy combines all these three <A>, <antiA>, and <neutA> together. Neutrosophy is a metaphilosophy.

Neutrosophic logic (1995), neutrosophic set (1995), and neutrosophic probability (1995) have, behind the classical values of truth and falsehood, a third component called indeterminacy (or neutrality, which is neither true nor false, or is both true and false simultaneously — again a combination of opposites: true and false in indeterminacy).

Neutrosophy and its derivatives are generalizations of the paradoxism (1980), which is a vanguard in literature, arts, and science, based on finding common things to opposite ideas (i.e. combination of contradictory fields).

5 Physics history and the future

a) With respect to the size of space there are: *Quantum Physics* which is referring to the subatomic space, the *Classical Physics* to our intuitive living space, while *Cosmology* to the giant universe;

b) With respect to the direct influence: the *Locality*, when an object is directly influenced by its immediate surroundings only, and the *Nonlocality*, when an object is directly influenced by another distant object without any interaction mediator;

c) With respect to the speed: the *Newtonian Physics* is referred to low speeds, the *Theory of Relativity* to subluminal speeds near to the speed of light, while *Superluminal Physics* will be referred to speeds greater than c, and *Instantaneous Physics* to instantaneous motions (infinite speeds).

A physical law has a form in Newtonian physics, another form in Relativity Theory, and different form at Superluminal theory, or at Infinite (Instantaneous) speeds — as above in the S-Denying Theory spectrum.

We get new physics at superluminal speeds and other physics at a very very big speed ($v \gg c$) speeds or at instantaneous (infinite) traveling.

At the beginning we have to extend physical laws and formulas to superluminal traveling and afterwards to instantaneous traveling.

For example, what/how would be Doppler effect if the motion of an emitting source relative to an observer is greater than c, or $v \gg c$ (much greater than c), or even at instantaneous speed?

Also, what addition rule should be used for superluminal speeds?

Then little by little we should extend existing classical physical theories from subluminal to superluminal and instantaneous traveling.

For example: if possible how would the Theory of Relativity be adjusted to superluminal speeds?

Lately we need to found a general theory that unites all theories at: law speeds, relativistic speeds, superluminal speeds, and instantaneous speeds — as in the S-Multispace Theory.

6 Conclusion

Today, with many contradictory theories, we can reconcile them by using the S-Multispace Theory.

We also propose investigating new research trends such as Superluminal Physics and Instantaneous Physics. Papers in these new fields of research should be e-mailed to the author by July 01, 2012, to be published in a collective volume.

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n-Valued Refined Neutrosophic Logic and Its Applications to Physics

Florentin Smarandache

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In this paper we present a short history of logics: from particular cases of 2-symbol or numerical valued logic to the general case of n-symbol or numerical valued logic. We show generalizations of 2-valued Boolean logic to fuzzy logic, also from the Kleene's and Lukasiewicz' 3-symbol valued logics or Belnap's *4*-symbol valued logic to the most general *n-symbol or numerical valued refined neutrosophic logic*. Two classes of neutrosophic norm (*n-norm*) and neutrosophic conorm (*n-conorm*) are defined. Examples of applications of neutrosophic logic to physics are listed in the last section. Similar generalizations can be done for *n-Valued Refined Neutrosophic Set*, and respectively *n-Valued Refined Neutrosopjhic Probability*.

1 Two-Valued Logic

1.1 The Two Symbol-Valued Logic

It is the Chinese philosophy: *Yin and Yang* (or Femininity and Masculinity) as contraries:

Fig. 1: Ying and Yang

It is also the Classical or *Boolean Logic*, which has two symbol-values: truth T and falsity F.

1.2 The Two Numerical-Valued Logic

It is also the Classical or *Boolean Logic*, which has two numerical-values: truth *1* and falsity *0*. More general it is the *Fuzzy Logic*, where the truth (T) and the falsity (F) can be any numbers in *[0,1]* such that $T + F = 1$.

Even more general, T and F can be subsets of *[0,1]*.

2 Three-Valued Logic

2.1 The Three Symbol-Valued Logics

1. *Lukasiewicz 's Logic*: True, False, and Possible.
2. *Kleene's Logic*: True, False, Unknown (or Undefined).
3. Chinese philosophy extended to: *Yin, Yang*, and *Neuter* (or Femininity, Masculinity, and Neutrality) - as in Neutrosophy. Neutrosophy philosophy was born from neutrality between various philosophies. *Connected with Extenics* (Prof. Cai Wen, 1983), and Paradoxism (F. Smarandache, 1980). *Neutrosophy* is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as <nonA>. Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only). According to this theory every idea <A> tends to be neutralized and balanced by <antiA> and <nonA> ideas - as a state of equilibrium. In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course) have common parts two by two, or even all three of them as well. *Such contradictions involves Extenics*. Neutrosophy is the base of all neutrosophics and it is used in engineering applications (especially for software and information fusion), medicine, military, airspace, cybernetics, physics.

2.2 The Three Numerical-Valued Logic

1. *Kleene's Logic*: True (*1*), False (*0*), Unknown (or Undefined) (*1/2*), and uses "min" for $\wedge$, "max" for $\vee$, and "1-" for negation.
2. More general is the *Neutrosophic Logic* [Smarandache, 1995], where the truth (T) and the falsity (F) and the indeterminacy (I) can be any numbers in *[0, 1]*, then $0 \leq T + I + F \leq 3$. More general: Truth (T), Falsity (F), and Indeterminacy (I) are standard or nonstandard subsets of the nonstandard interval $]^{-}0, 1^{+}[$.

3 Four-Valued Logic

3.1 The Four Symbol-Valued Logic

1. It is *Belnap's Logic*: True (T), False (F), Unknown (U), and Contradiction (C), where *T, F, U, C* are symbols,

not numbers. Below is the Belnap's conjunction operator table:

$\cap$	F	U	C	T
F	F	F	F	F
U	F	U	F	U
C	F	F	C	C
T	F	U	C	T

Restricted to *T,F,U,* and to *T,F,C,* the Belnap connectives coincide with the connectives in Kleene's logic.

2. Let G = Ignorance. We can also propose the following two 4-Symbol Valued Logics: *(T, F, U, G)*, and *(T, F, C, G)*.

3. *Absolute-Relative 2-, 3-, 4-, 5-, or 6-Symbol Valued Logics* [Smarandache, 1995]. Let T_A be truth in all possible worlds (according to Leibniz's definition); T_R be truth in at last one world but not in all worlds; and similarly let I_A be indeterminacy in all possible worlds; I_R be indeterminacy in at last one world but not in all worlds; also let F_A be falsity in all possible worlds; F_R be falsity in at last one world but not in all worlds; Then we can form several Absolute-Relative 2-, 3-, 4-, 5-, or 6-Symbol Valued Logics just taking combinations of the symbols T_A, T_R, I_A, I_R, F_A, and F_R. As particular cases, very interesting would be to study the Absolute-Relative 4-Symbol Valued Logic (T_A, T_R, F_A, F_R), as well as the Absolute-Relative 6-Symbol Valued Logic (T_A, T_R, I_A, I_R, F_A, F_R).

3.2 Four Numerical-Valued Neutrosophic Logic

Indeterminacy I is refined (split) as U = Unknown, and C = contradiction. T, F, U, C are subsets of [0, 1], instead of symbols; This logic generalizes Belnap's logic since one gets a degree of truth, a degree of falsity, a degree of unknown, and a degree of contradiction. Since $C = T \wedge F$, this logic involves the Extenics.

4 Five-Valued Logic

1. Five Symbol-Valued Neutrosophic Logic [Smarandache, 1995]: Indeterminacy I is refined (split) as U = Unknown, C = contradiction, and G = ignorance; where the symbols represent:
   T = truth;
   F = falsity;
   U = neither T nor F (undefined);
   $C = T \wedge F$, which involves the Extenics;
   $G = T \vee F$.

2. If *T, F, U, C,* G are subsets of *[0, 1]* then we get:a *Five Numerical-Valued Neutrosophic Logic*.

5 Seven-Valued Logic

1. *Seven Symbol-Valued Neutrosophic Logic* [Smarandache, 1995]:
   I is refined (split) as *U, C, G,* but *T* also is refined as T_A = absolute truth and T_R = relative truth, and *F* is refined as F_A = absolute falsity and F_R = relative falsity. Where: U = neither (T_A or T_R) nor (F_A or F_R) (i.e. undefined); $C = (T_A or T_R) \wedge (F_A or F_R)$ (i.e. Contradiction), which involves the Extenics;
   $G = (T_A or T_R) \vee (F_A or F_R)$ (i.e. Ignorance). All are symbols.

2. But if T_A, T_R, F_A, F_R, *U, C, G* are subsets of *[0, 1]*, then we get a *Seven Numerical-Valued Neutrosophic Logic*.

6 n-Valued Logic

1. *The n-Symbol-Valued Refined Neutrosophic Logic* [Smarandache, 1995]. In general:
   T can be split into many types of truths: $T_1, T_2, ..., T_p$, and *I* into many types of indeterminacies: $I_1, I_2, ..., I_r$, and *F* into many types of falsities: $F_1, F_2, ..., F_s$, where all $p, r, s \geq 1$ are integers, and $p + r + s = n$.
   All subcomponents T_j, I_k, F_l are symbols for $j \in \{1, 2..., p\}$, $k \in \{1, 2..., r\}$, and $l \in \{1, 2..., s\}$.
   If at least one $I_k = T_j \wedge F_l$ =contradiction, we get again the Extenics.

2. *The n-Numerical-Valued Refined Neutrosophic Logic.* In the same way, but all subcomponents T_j, I_k, F_l are not symbols, but subsets of *[0,1]*, for all $j \in \{1, 2..., p\}$, all $k \in \{1, 2..., r\}$, and all $l \in \{1, 2..., s\}$. If all sources of information that separately provide neutrosophic values for a specific subcomponent are independent sources, then in the general case we consider that each of the subcomponents T_j, I_k, F_l is independent with respect to the others and it is in the non-standard set $]^-0, 1^+[$. Therefore per total we have for crisp neutrosophic value subcomponents T_j, I_k, F_l that:

$$^-0 \leq \sum_{j=1}^{p} T_j + \sum_{k=1}^{r} I_k + \sum_{l=1}^{s} F_l \leq n^+ \qquad (1)$$

   where of course $n = p + r + s$ as above. If there are some dependent sources (or respectively some dependent subcomponents), we can treat those dependent subcomponents together. For example, if T_2 and I_3 are dependent, we put them together as$^-0 \leq T_2 + I_3 \leq 1^+$. The non-standard unit interval $]^-0, 1^+[$, used to make a distinction between absolute and relative truth/ indeterminacy /falsehood in philosophical applications, is replace for simplicity with the standard (classical) unit interval $[0, 1]$ for technical applications.
   For at least one $I_k = T_j \wedge F_l$ = contradiction, we get again the Extenics.

7 n-Valued Neutrosophic Logic Connectors

1. **n-Norm and n-Conorm defined on combinations of t-Norm and t-Conorm**

   The n-norm is actually the neutrosophic conjunction operator, NEUTROSOPHIC AND ($\wedge_n$); while the n-conorm is the neutrosophic disjunction operator, NEUTROSOPHIC OR ($\vee_n$).

   One can use the t-norm and t-conorm operators from the fuzzy logic in order to define the **n-norm** and respectively **n-conorm** in neutrosophic logic:

$$\begin{aligned} &n-norm((T_j)_{j=\{1,2,...,p\}},\\ &(I_k)_{k=\{1,2,...,r\}},(F_l)_{l=\{1,2,...,s\}})\\ &=([t-norm(T_j)]_{j=\{1,2,...,p\}},\\ &\quad [t-conorm(I_k)]_{k=\{1,2,...,r\}},\\ &\quad [t-conorm(F_l)]_{l=\{1,2,...,s\}}) \end{aligned} \tag{2}$$

   and

$$\begin{aligned} &n-conorm((T_j)_{j=\{1,2,...,p\}},(I_k)_{k=\{1,2,...,r\}},\\ &(F_l)_{l=\{1,2,...,s\}})\\ &=([t-conorm(T_j)]_{j=\{1,2,...,p\}},\\ &\quad [t-norm(I_k)]_{k=1,2,...,r},\\ &\quad [t-norm(F_l)]_{l=1,2,...,s}) \end{aligned} \tag{3}$$

   and then one normalizes if needed.

   Since the n-norms/n-conorms, alike t-norms/t-conorms, can only approximate the inter-connectivity between two n-Valued Neutrosophic Propositions, there are many versions of these approximations.

   For example, for the n-norm: the indeterminate (sub)components I_k alone can be combined with the t-conorm in a pessimistic way [i.e. lower bound], or with the t-norm in an optimistic way [upper bound]; while for the n-conorm: the indeterminate (sub)components I_k alone can be combined with the t-norm in a pessimistic way [i.e. lower bound], or with the t-conorm in an optimistic way [upper bound].

   In general, if one uses in defining an n-norm/n-conorm for example the t-norm $min\,\{x,y\}$ then it is indicated that the corresponding t-conorm used be $max\,\{x,y\}$; or if the t-norm used is the product $x\dot{y}$ then the corresponding t-conorm should be $x+y-x\dot{y}$; and similarly if the t-norm used is $max\,\{0,\,x+y-1\}$ then the corresponding t-conorm should be $min\,\{x+y,\,1\}$; and so on.

   Yet, it is still possible to define the n-norm and n-conorm using different types of t-norms and t-conorms.

2. **N-norm and n-conorm based on priorities**

   For the *n-norm* we can consider the priority: T<I<F, where the subcomponents are supposed to conform with similar priorities, i.e.

$$\begin{aligned} T_1 < T_2 < ... < T_p < I_1 < I_2 < ...\\ < I_r < F_1 < F_2 < ... < F_s. \end{aligned} \tag{4}$$

   While for the **n-conorm** one has the opposite priorities: T>I>F, or for the refined case:

$$\begin{aligned} T_1 > T_2 > ... > T_p > I_1 > I_2 > ...\\ > I_r > F_1 > F_2 > ... > F_s. \end{aligned} \tag{5}$$

By definition A<B means that all products between A and B go to B (the bigger).

Let's say, one has two neutrosophic values in simple (non-refined case):

$$(T_x, I_x, F_x) \tag{6}$$

and

$$(T_y, I_y, F_y) \tag{7}$$

Applying the n-norm to both of them, with priorities T < I < F, we get:

$$\begin{aligned} &(T_x, I_x, F_x) \wedge_n (T_y, I_y, F_y)\\ &= (T_xT_y, T_xI_y + T_yI_x + I_xI_y,\\ &\quad T_xF_y + T_yF_x + I_xF_y + I_yF_x + F_xF_y). \end{aligned} \tag{8}$$

Applying the n-conorm to both of them, with priorities T > I > F, we get:

$$\begin{aligned} &(T_x, I_x, F_x) \vee_n (T_y, I_y, F_y)\\ &= (T_xT_y + T_xI_y + T_yI_x + T_xF_y + T_yF_x,\\ &\quad I_xI_y + I_xF_y + I_yF_x, F_xF_y). \end{aligned} \tag{9}$$

In a lower bound (pessimistic) n-norm one considers the priorities T < I < F, while in an upper bound (optimistic) n-norm one considers the priorities I < T < F.

Whereas, in an upper bound (optimistic) n-conorm one considers T > I > F, while in a lower bound (pessimistic) n-conorm one considers the priorities T>F>I.

Various priorities can be employed by other researchers depending on each particular application.

8 Particular Cases

If in *6 a)* and *b)* one has all $I_k = 0, k = \{1, 2, ..., r\}$, we get the **n-Valued Refined Fuzzy Logic**.

If in *6 a)* and *b)* one has only one type of indeterminacy, i.e. k=1, hence $I_1 = I > 0$, we get the **n-Valued Refined Intuitionistic Fuzzy Logic**.

9 Distinction between Neutrosophic Physics and Paradoxist Physics

Firstly, we make a distinction between Neutrosophic Physics and Paradoxist Physics.

1. **Neutrosophic Physics**

   Let <A> be a physical entity (i.e. concept, notion, object, space, field, idea, law, property, state, attribute, theorem, theory, etc.), <antiA> be the opposite of <A>, and <neutA> be their neutral (i.e. neither <A> nor <antiA>, but in between).

Neutrosophic Physics is a mixture of two or three of these entities <A>, <antiA>, and <neutA> that hold together.

Therefore, we can have neutrosophic fields, and neutrosophic objects, neutrosophic states, etc.

2. **Paradoxist Physics**
   Neutrosophic Physics is an extension of Paradoxist Physics, since Paradoxist Physics is a combination of physical contradictories <A> and <antiA> only that hold together, without referring to their neutrality <neutA>. Paradoxist Physics describes collections of objects or states that are individually characterized by contradictory properties, or are characterized neither by a property nor by the opposite of that property, or are composed of contradictory sub-elements. Such objects or states are called paradoxist entities.

   These domains of research were set up in the *1995* within the frame of neutrosophy, neutrosophic logic/set/probability/statistics.

10 n-Valued Refined Neutrosophic Logic Applied to Physics

There are many cases in the scientific (and also in humanistic) fields that two or three of these items <A>, <antiA>, and <neutA> simultaneously coexist.
Several **Examples** of paradoxist and neutrosophic entities:

- anions in two spatial dimensions are arbitrary spin particles that are neither bosons (integer spin) nor fermions (half integer spin);
- among possible Dark Matter candidates there may be exotic particles that are neither Dirac nor Majorana fermions;
- mercury (Hg) is a state that is neither liquid nor solid under normal conditions at room temperature;
- non-magnetic materials are neither ferromagnetic nor anti-ferromagnetic;
- quark gluon plasma (QGP) is a phase formed by quasi-free quarks and gluons that behaves neither like a conventional plasma nor as an ordinary liquid;
- unmatter, which is formed by matter and antimatter that bind together (F. Smarandache, 2004);
- neutral kaon, which is a pion and anti-pion composite (R. M. Santilli, 1978) and thus a form of unmatter;
- neutrosophic methods in General Relativity (D. Rabounski, F. Smarandache, L. Borissova, 2005);
- neutrosophic cosmological model (D. Rabounski, L. Borissova, 2011);
- neutrosophic gravitation (D. Rabounski);
- qubit and generally quantum superposition of states;
- semiconductors are neither conductors nor isolators;
- semi-transparent optical components are neither opaque nor perfectly transparent to light;
- quantum states are metastable (neither perfectly stable, nor unstable);
- neutrino-photon doublet (E. Goldfain);
- the "multiplet" of elementary particles is a kind of "neutrosophic field" with two or more values (E. Goldfain, 2011);
- A "neutrosophic field" can be generalized to that of operators whose action is selective. The effect of the neutrosophic field is somehow equivalent with the "tunneling" from the solid physics, or with the "spontaneous symmetry breaking" (SSB) where there is an internal symmetry which is broken by a particular selection of the vacuum state (E. Goldfain). Etc.

Many types of logics have been presented above. For the most general logic, the n-valued refined neutrosophic logic, we presented two classes of neutrosophic operators to be used in combinations of neutrosophic valued propositions in physics.

Similar generalizations are done for **n-Valued Refined Neutrosophic Set, and respectively n-Valued Refined Neutrosophic Probability**.

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Oblique-Length Contraction Factor in the Special Theory of Relativity

Florentin Smarandache

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In this paper one generalizes the Lorentz Contraction Factor for the case when the lengths are moving at an oblique angle with respect to the motion direction. One shows that the angles of the moving relativistic objects are distorted.

1 Introduction

According to the Special Theory of Relativity, the Lorentz Contraction Factor is referred to the lengths moving along the motion direction. The lengths which are perpendicular on the direction motion do not contract at all [1].

In this paper one investigates the lengths that are oblique to the motion direction and one finds their Oblique-Length Contraction Factor [3], which is a generalization of the Lorentz Contraction Factor (for $\theta = 0$) and of the perpendicular lengths (for $\theta = \pi/2$). We also calculate the distorted angles of lengths of the moving object.

2 Length-Contraction Factor

Length-Contraction Factor $C(v)$ is just Lorentz Factor:

$$C(v) = \sqrt{1 - \frac{v^2}{c^2}} \in [0, 1] \;\; for\; v \in [0, 1] \tag{1}$$

$$L = L' \cdot C(v) \tag{2}$$

where L = non-proper length (length contracted), L' = proper length. $C(0) = 1$, meaning no space contraction [as in Absolute Theory of Relativity (ATR)].

$C(c) = 0$, which means according to the Special Theory of Relativity (STR) that if the rocket moves at speed 'c' then the rocket length and laying down astronaut shrink to zero! This is unrealistic.

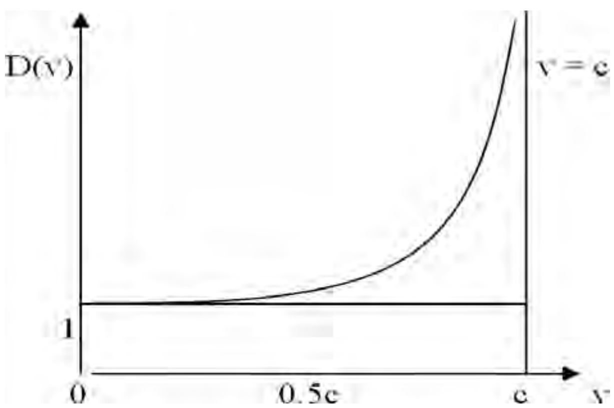

Fig. 1: The graph of the Time-Dilation Factor

3 Time-Dilation Factor

Time-Dilation Factor D(v) is the inverse of Lorentz Factor:

$$D(v) = \frac{1}{\sqrt{1 - \frac{v^2}{c^2}}} \in [1, +\infty] \;\; for\; v \in [0, c] \tag{3}$$

$$\Delta t = \Delta t' \cdot D(v) \tag{4}$$

where Δt = non-proper time and, $\Delta t'$ = proper time. $D(0) = 1$, meaning no time dilation [as in Absolute Theory of Relativity (ATR)]; $D(c) = \lim_{v \to c} D(v) = +\infty$, which means according to the Special Theory of Relativity (STR) that if the rocket moves at speed 'c' then the observer on earth measures the elapsed non-proper time as infinite, which is unrealistic. $v = c$ is the equation of the vertical asymptote to the curve of $D(v)$.

4 Oblique-Length Contraction Factor

The Special Theory of Relativity asserts that all lengths in the direction of motion are contracted, while the lengths at right angles to the motion are unaffected. But it didn't say anything about lengths at oblique angle to the motion (i.e. neither perpendicular to, nor along the motion direction), how would they behave? This is a generalization of Galilean Relativity, i.e. we consider the oblique lengths. The length contraction factor in the motion direction is:

$$C(v) = \sqrt{1 - \frac{v^2}{c^2}}. \tag{5}$$

Suppose we have a rectangular object with width W and length L that travels at a constant speed v with respect to an observer on Earth.

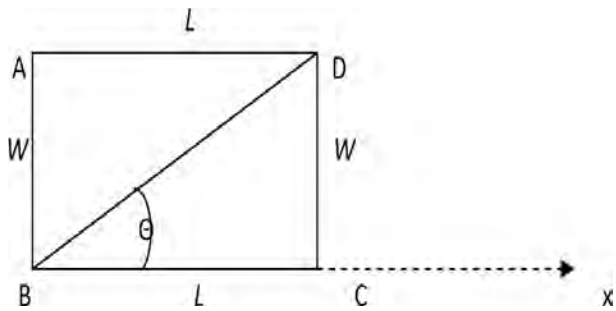

Fig. 2: A rectangular object moving along the x-axis

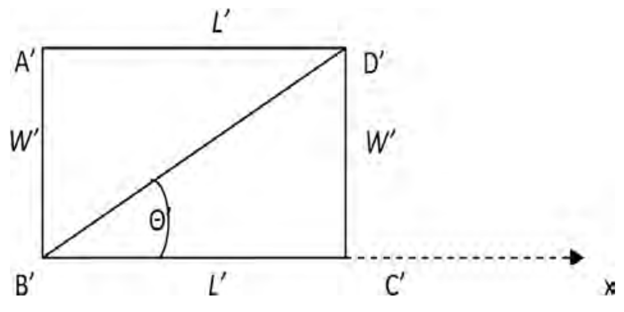

Fig. 3: Contracted lengths of the rectangular object moving along the x-axis

Then its lengths contract and its new dimensions will be L' and W': where $L' = L \cdot C(v)$ and $W' = W$. The initial diagonal of the rectangle ABCD is:

$$\begin{aligned} \delta &= |AC| = |BD| = \sqrt{L^2 + W^2} \\ &= \sqrt{L^2 + L^2 \tan^2\theta} = L\sqrt{1+\tan^2\theta} \end{aligned} \tag{6}$$

while the contracted diagonal of the rectangle $A'B'C'D'$ is:

$$\begin{aligned} \delta' &= |A'C'| = |B'D'| \\ &= \sqrt{(L')^2 + (W')^2} = \sqrt{L^2 \cdot C(v)^2 + W^2} \\ &= \sqrt{L^2 C(v)^2 + L^2\tan^2\theta} = L\sqrt{C(v)^2 + \tan^2\theta}. \end{aligned} \tag{7}$$

Therefore the lengths at oblique angle to the motion are contracted with the oblique factor

$$\begin{aligned} OC(v,\theta) &= \frac{\delta'}{\delta} = \frac{L\sqrt{C(v)^2+\tan^2\theta}}{L\sqrt{1+\tan^2\theta}} \\ &= \sqrt{\frac{C(v)^2+\tan^2\theta}{1+\tan^2\theta}} = \sqrt{C(v)^2\cos^2\theta + \sin^2\theta} \end{aligned} \tag{8}$$

which is different from C(v).

$$\delta' = \delta \cdot OC(v,\theta) \tag{9}$$

where $0 \le OC(v,\theta) \le 1$.

For unchanged constant speed v, the greater is θ in $\left(0, \frac{\pi}{2}\right)$ the larger gets the oblique-length contradiction factor, and reciprocally. By oblique length contraction, the angle

$$\theta \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right) \tag{10}$$

is not conserved.

In Fig. 4 the horizontal axis represents the angle θ, while the vertical axis represents the values of the Oblique-Length Contraction Factor $OC(v,\theta)$ for a fixed speed v. Hence $C(v)$ is thus a constant in this graph. The graph, for v fixed, is

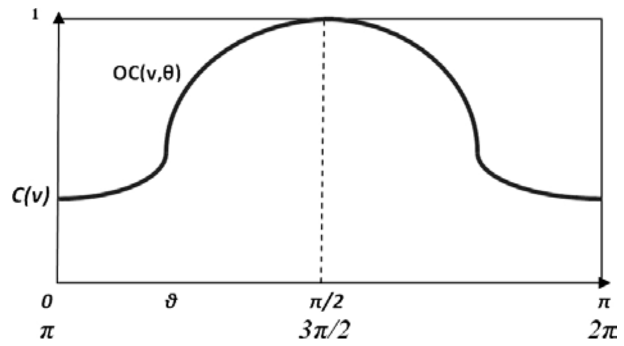

Fig. 4: The graph of the Oblique-Length Contraction Factor $OC(v,\theta)$

periodic of period π, since:

$$\begin{aligned} OC(v,\pi+\theta) &= \sqrt{C(v)^2\cos^2(\pi+\theta) + \sin^2(\pi+\theta)} \\ &= \sqrt{C(v)^2[-\cos\theta]^2 + [-\sin\theta]^2} \\ &= \sqrt{C(v)^2\cos^2\theta + \sin^2\theta} \\ &= OC(v,\theta). \end{aligned} \tag{11}$$

More exactly about the $OC(v,\theta)$ range:

$$OC(v,\theta) \in [C(v), 1] \tag{12}$$

but since $C(v) \in [0,1]$, one has:

$$OC(v,\theta) \in [0,1]. \tag{13}$$

The Oblique-Length Contractor

$$OC(v,\theta) = \sqrt{C(v)^2\cos^2\theta + \sin^2\theta} \tag{14}$$

is a generalization of Lorentz Contractor $C(v)$, because: when $\theta = 0$ or the length is moving along the motion direction, then $OC(v,0) = C(v)$. Similarly

$$OC(v,\pi) = OC(v,2\pi) = C(v). \tag{15}$$

Also, if $\theta = \frac{\pi}{2}$, or the length is perpendicular on the motion direction, then $OC(v,\pi/2) = 1$, i.e. no contraction occurs. Similarly $OC(v, \frac{3\pi}{2}) = 1$.

5 Angle Distortion

Except for the right angles $(\pi/2, 3\pi/2)$ and for the 0, π, and 2π, all other angles are distorted by the Lorentz transform.

Let's consider an object of triangular form moving in the direction of its bottom base (on the x-axis), with speed v, as in Fig. 5:

$$\theta \in \left(0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right) \tag{16}$$

is not conserved.

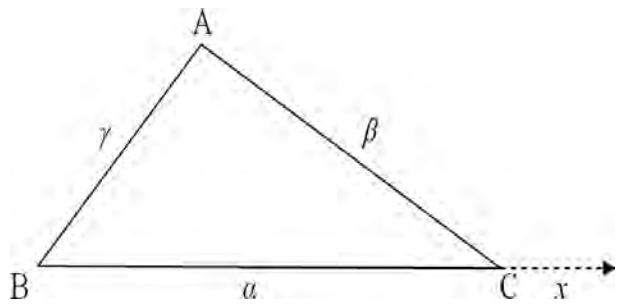

Fig. 5:

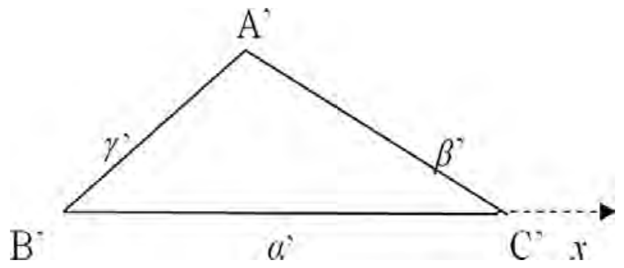

Fig. 6:

The side $|BC| = \alpha$ is contracted with the contraction factor $C(v)$ since BC is moving along the motion direction, therefore $|B'C'| = \alpha \cdot C(v)$. But the oblique sides AB and CA are contracted respectively with the oblique-contraction factors $OC(v, \measuredangle B)$ and $OC(v, \measuredangle\pi - C)$, where $\measuredangle B$ means angle B:

$$\left|A'B'\right| = \gamma \cdot OC(v, \measuredangle B) \tag{17}$$

and

$$\left|C'A'\right| = \beta \cdot OC(v, \measuredangle\pi - C) = \beta \cdot OC(v, \measuredangle A + B) \tag{18}$$

since

$$\measuredangle A + \measuredangle B + \measuredangle C = \pi. \tag{19}$$

Triangle ABC is shrunk and distorted to $A'B'C'$ as in Fig. 6.

Hence one gets:

$$\begin{aligned} \alpha' &= \alpha \cdot C(v) \\ \beta' &= \beta \cdot OC(v, \measuredangle A + B) \\ \gamma' &= \gamma \cdot OC(v, \measuredangle B) \end{aligned} \tag{20}$$

In the resulting triangle $A'B'C'$, since one knows all its side lengths, one applies the Law of Cosine in order to find each angle $\measuredangle A'$, $\measuredangle B'$, and $\measuredangle C'$. Therefore:

$$\measuredangle A' = \arccos \frac{-\alpha^2 \cdot C(v)^2 + \beta^2 \cdot OC(v, \measuredangle A + B)^2 + \gamma^2 \cdot OC(v, \measuredangle B)^2}{2\beta \cdot \gamma \cdot OC(v, \measuredangle B) \cdot OC(v, \measuredangle A + B)}$$

$$\measuredangle B' = \arccos \frac{\alpha^2 \cdot C(v)^2 - \beta^2 \cdot OC(v, \measuredangle A + B)^2 + \gamma^2 \cdot OC(v, \measuredangle B)^2}{2\alpha \cdot \gamma \cdot OC(v) \cdot OC(v, \measuredangle B)}$$

$$\measuredangle C' = \arccos \frac{\alpha^2 \cdot C(v)^2 + \beta^2 \cdot OC(v, \measuredangle A + B)^2 - \gamma^2 \cdot OC(v, \measuredangle B)^2}{2\alpha \cdot \beta \cdot OC(v) \cdot OC(v, \measuredangle A + B)}.$$

As we can see, the angles $\measuredangle A'$, $\measuredangle B'$, and $\measuredangle C'$ are, in general, different from the original angles A, B, and C respectively.

The distortion of an angle is, in general, different from the distortion of another angle.

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Superluminal Physics & Instantaneous Physics as New Trends in Research

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ABSTRACT

In a similar way as passing from Euclidean Geometry to Non-Euclidean Geometry, we can pass from Subluminal Physics to Superluminal Physics, and further to Instantaneous Physics. In the lights of two consecutive successful CERN experiments with superluminal particles in the Fall of 2011, we believe that these two new fields of research should begin developing.

Keywords: Subluminal physics, superluminal physics, Euclidean geometry, Smarandache geometry, non-Euclidean geometry

INTRODUCTION.

Let's start by recalling the history of geometry in order to connect it with the history of physics.

Then we present the way of S-denying a law (or theory) and building a spectrum of spaces where the same physical law (or theory) has different forms, then we mention the S-multispace with its multistructure that may be used to the Unified Field Theory by employing a ***multifield***.

It is believed that the S-multispace with its multistructure is the best candidate for 21st century ***Theory of Everything*** in any domain.

1. Geometry's History.

As in Non-Euclidean Geometry, there are models that validate the hyperbolic geometric and of course invalidate the Euclidean geometry, or models that validate the elliptic geometry and in consequence they invalidate the Euclidean geometry and the hyperbolic geometry.

Now, we can mix these geometries and construct a model in which an axiom is partially validated and partially invalidated, or the axiom is only invalidated but in multiple different ways [1]. This operation produces a degree of negation of an axiom, and such geometries are hybrid. We can in general talk about the ***degree of negation of a scientific entity P***, where P can be a theorem, lemma, property, theory, law, etc.

2. S-Denying of a Theory.

Let's consider a physical space S endowed with a set of physical laws ***L***, noted by (S, ***L***), such that all physical laws ***L*** are valid in this space S.

Then, we construct another physical space (or model) S_1 where a given law has a different form, afterwards another space S_2 where the same law has another form, and so on until getting a spectrum of spaces where this law is different.

We thus investigate spaces where anomalies occur [2].

3. Multispace Theory.

In any domain of knowledge, multispace (or S-multispace) with its multistructure is a finite or infinite (countable or uncountable) union of many spaces that have various structures. The spaces may overlap [3].

The notions of multispace (also spelt multi-space) and multistructure (also spelt multi-structure) were introduced by the author in 1969 under his idea of hybrid science: combining different fields into a unifying field (in particular combinations of different geometric spaces such that at least one geometric axiom behaves differently in each such space), which is closer to our real life world since we live in a heterogeneous multispace. Today, this idea is accepted by the world of sciences. S-multispace is a qualitative notion, since it is too large and includes both metric and non-metric spaces.

A such multispace can be used for example in physics for the Unified Field Theory that tries to unite the gravitational, electromagnetic, weak and strong interactions by constructing a ***multifield*** formed by a gravitational field united with an electromagnetic field united with a weak-interactions field and united with a strong-interactions field.

Or in the parallel quantum computing and in the mu-bit theory, in multi-entangled states or particles and up to multi-entangles objects.

We also mention: the algebraic multispaces (multi-groups, multi-rings, multi-vector spaces, multi-operation systems and multi-manifolds, also multi-voltage graphs, multi-embedding of a graph in an n-manifold, etc.) or structures included in other structures, geometric multispaces (combinations of Euclidean and Non-Euclidean geometries into one space as in S-geometries), theoretical physics, including the Relativity Theory [4], the M-theory and the cosmology, then multi-space models for p-branes and cosmology, etc.

The multispace is an extension of the neutrosophic logic and set, which derived from neutrosophy. Neutrosophy (1995) is a generalization of dialectics in philosophy, and takes into consideration not only an entity <A> and its opposite <antiA> as dialectics does, but also the neutralities <neutA> in between. Neutrosophy combines all these three <A>, <antiA>, and <neutA> together. Neutrosophy is a metaphilosophy.

Neutrosophic logic (1995), neutrosophic set (1995), and neutrosophic probability (1995) have, behind the classical values of truth and falsehood, a third component called indeterminacy (or neutrality, which is neither true nor false, or is both true and false simultaneously - again a combination of opposites: true and false in indeterminacy).

Neutrosophy and its derivatives are generalizations of the paradoxism (1980), which is a vanguard in literature, arts, and science, based on finding common things to opposite ideas [i.e. combination of contradictory fields].

4. Physics History and Future.

a) With respect to the <u>size of space</u> there are: ***Quantum Physics*** which is referring to the subatomic space, the ***Classical Physics*** to our intuitive living space, while ***Cosmology*** to the giant universe.

b) With respect to the <u>direct influence</u>: the ***Locality***, when an object is directly influenced by its immediate surroundings only, and the ***Nonlocality***, when an object is directly influenced by another distant object without any interaction mediator

c) With respect to the <u>speed</u>: the ***Newtonian Physics*** is referred to low speeds, the ***Theory of Relativity*** to subluminal speeds near to the speed of light, while ***Superluminal Physics*** will be referred to speeds greater than c, and ***Instantaneous Physics*** to instantaneous motions (infinite speeds).

   A physical law has a form in Newtonian physics, another form in Relativity Theory, and different form at Superluminal theory, or at Infinite (Instantaneous) speeds – as above in the S-Denying Theory spectrum.

   We get new physics at superluminal speeds and other physics at very very big speed (v >> c) speeds or at instantaneous (infinite) traveling.

   At the beginning we have to extend physical laws and formulas to superluminal traveling and afterwards to instantaneous traveling.

For example, what/how would be Doppler effect if the motion of an emitting source relative to an observer is greater than c, or v >> c (much greater than c), or even at instantaneous speed?

Also, what addition rule should be used for superluminal speeds?

Then little by little we should extend existing classical physical theories from subluminal to superluminal and instantaneous traveling.

For example: if possible how would the Theory of Relativity be adjusted to superluminal speeds?

Lately we need to found a general theory that unites all theories at: law speeds, relativistic speeds, superluminal speeds, and instantaneous speeds – as in the S-Multispace Theory.

CONCLUSION

Today, with many contradictory theories, we can reconcile them by using the S-Multispace Theory.

We also propose investigating new research trends such as Superluminal Physics and Instantaneous Physics. Papers in these new fields of research should be e-mailed to the author by July 1st, 2012, to be published in a collective volume.

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Unmatter Plasma Discovered

Florentin Smarandache

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The electron-positron beam plasma was generated in the laboratory in the beginning of 2015. This experimental fact shows that unmatter, a new form of matter that is formed by matter and antimatter bind together (mathematically predicted a decade ago) really exists. That is the electron-positron plasma experiment of 2015 is the experimentum crucis verifying the mathematically predicted unmatter.

Unmmatter Plasma is a novel form of plasma, exclusively made of matter and its antimatter counterpart. It was first generated in the 2015 experiment [1, 2] based on the 2004 considerations [3].

There are four fundamental states of matter: solid, liquid, gas, and plasma. Plasma consists of positive ions and free electrons (negative particles), typically at low pressures, and it is overall almost neutral. Plasma is an ionized gas (as in fluorescent neon, in lightning, in stars, in nuclear reactors). An ion is a positive or negative charged particle. A positive ion is called cation, while a negative ion is called anion. If the ion is an atom, then it may contain less electrons than needed for being neutrally charged (hence one has a cation), or more electrons than needed for being neutrally charged (hence one has an anion). Similarly if the ion is a molecule or a group (of atoms or molecules). The process of forming ions is called ionization. The degree of ionization depends on the proportion of atoms that have lost or gained electrons. By applying a strong electromagnetic field to a gas, or by heating a gas, one obtains plasma.

Unmatter as theoretically predicted in the framework of the neutrosophic logic and statistics [4–6] is considered as a combination of matter and antimatter that bound together, or a long-range mixture of matter and antimatter forming a weakly-coupled phase. For example, the electron-positron pair is a type of unmatter. We coined the word *unmatter* that means neither matter nor antimatter, but something in between. Besides matter and antimatter there may exist unmatter (as a new form of matter) in accordance with the neutrosophy theory that between an entity and its opposite there exist intermediate entities.

The 2015 experiment [1, 2] on matter-antimatter plasma (*unmatter plasma*, in terms of the neutrosophic logic and statistics) was recently successful in the Astra Gemini laser facility of the Rutherford Appleton Laboratory in Oxford, United Kingdom. The 2015 experiment has produced electron-positron plasma. The positron is the antimatter of the electron, having an opposite charge of the electron, but the other properties are the same.

Also, the meson is a clear example of unmatter whose configuration includes a pair quark-antiquark. Unmatter is mostly expected to emerge in exotic states outside the boundaries of the Standard Model for particle physics (for example in the Dark Matter sector) and in the regime of high-energy astrophysical objects [7].

"It is definitely a jet of unmatter, because a plasma consisting of the electrons and the positrons is neither matter nor antimatter in the same time. This experiment is the truly verification of unmatter as the theoretical achievements of neutrosophic logic and statistics. This experiment is a milestone of both experimental physics and pure mathematics" [8].

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On Syntropy & Precognitive Interdiction Based on Wheeler-Feynman's Absorber Theory

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ABSTRACT

It has been known for long time that intuition plays significant role in many professions and human life, including in entrepreneurship, government, and also in detective or law enforcement activities. Women are known to possess better intuitive feelings or "hunch" compared to men. Despite these examples, such a precognitive interdiction is hardly accepted in established science. In this letter, we discuss briefly the advanced solutions of Maxwell equations, and then explore plausible connection between syntropy and precognition.

Keywords: Intuition, syntropy, precognition, Maxwell equation, advanced wave solution.

1. Introduction

It has been known for long time that intuition plays significant role in many professions and other aspects of human life, including in entrepreneurship, government, and also in detective or law enforcement activities. Even women are known to possess better intuitive feelings or "*hunch*" compared to men[1]. Despite these examples, such a precognitive interdiction is hardly accepted in established science.

In this paper, we discuss briefly the advanced solutions of Maxwell equations in the context of Wheeler-Feynman-Cramer's absorber theory, and then make connection between *syntropy* and precognition from classical perspective. This may be regarded as first step to describe such precognition activities which are usually considered belong to quantum realm.

It is our hope that the new proposed interpretation can be verified with experimental data. Nonetheless, we admit that our model is still in its infancy, more researches are needed to fill all the missing details.

2. John Cramer's take on Wheeler-Feynman's absorber theory

The Wheeler-Feynman's paper on absorber theory has been discussed and generalized by John Cramer. He discussed among other things on the physical interpretation of advanced and retarded solutions of Maxwell equations and also Klein-Gordon equation.
Our discussion starts from the fundamental Maxwell's equations that unify electromagnetism[1]:

$$\begin{aligned}
&\nabla \cdot B = 0 (MagneticGauss),\\
&\nabla \cdot D = \rho_f (Gauss),\\
&\nabla \times E + \partial_t B = 0 (Faraday),\\
&\nabla \times H - \partial_t D = J_f (Amperecircuitallaw),
\end{aligned} \tag{1}$$

It is known that electromagnetic wave equation corresponding to (1) admits advanced wave solution.

Of course, here we do not have to accept all transactional QM interpretation by Cramer[1][2], but we can keep our discussion straightly within the scope of classical electromagnetic theory.
The electromagnetic wave equation for source-free space can be written in the form:

$$c^2 \nabla^2 \vec{F} = \frac{d^2 \vec{F}}{dt^2}, \tag{2}$$

where c represents the speed of light, and F represents either the electric field vector E or the magnetic field vector B of the wave [1].

Since this differential equation is second order in both time and space, it has two independent time solutions and two independent space solutions. Let us restrict our consideration to one dimension by requiring that the wave motion described by equation (2) moves along with x axis and that the E vector of the wave is along the y axis.

Then two independent time solutions of equation (2) might have the form [1]:

$$\vec{E}_{\pm}(x,t) = \hat{y} E_0 \sin\left[2\pi\left(\frac{x}{\lambda} \pm ft\right)\right], \tag{3}$$

and

$$\vec{B}_{\pm}(x,t) = \hat{y}B_0 \sin\left[2\pi\left(\frac{x}{\lambda} \pm ft\right)\right], \tag{4}$$

Quoting from Cramer's notes on the solutions of equations (3) and (4) [1] :

> Thus, wave $E_+(x,t)$ is a *negative-energy* (and negative-frequency) solution of Eq. (1). As mentioned above, it will arive at a point a distance x from the source at a time $t = x/c$ *before* the instant of emission. For this reason, it is called an *advanced* wave. Solution $E_-(x,t)$, on the other hand, is the more familiar positive-energy solution of Eq. (1). It arrives at x a time $t = x/c$ after the instant of emission and is called the *retarded* solution.

It should be clear, therefore, that advanced wave solution is inherent in the classical electromagnetic wave equations, without having to resort to Cramer's transactional interpretation of QM.

Next, we are going to discuss physical interpretation of such an advanced wave solution.

3. Interpretation of Advanced Wave Solution: Syntropy and Precognition

The above analysis by Cramer which seems to suggest that EPR paradox just disappears when considering the advanced waves to be real physical entities, has been suggested by other physicists too, notably: Costa de Beauregard and also Luigi Fantappie. While working on quantum mechanics and special relativity equations, Luigi noted that that retarded waves (retarded potentials) are governed by the law of entropy, while the advanced waves are governed by a symmetrical law that he named "*syntropy*"[3].

Therefore, some psychologists who work in this area began to make connection between the notion of syntropy and precognitive interdiction. And recently, a new journal by title *Syntropy* has been started to facilitate such a discussion.

But again let us emphasize here that equation (3) and (4) indicate that the advanced wave solutions have purely classical origin. Therefore, we do not discuss yet their connection with other alleged QM terminologies such as collapsing wave function which are hardly possible to prove experimentally, despite Bohr and Heisenberg insisted such a phenomenon is real. This is our departure to QM's inspired syntropy discussions in [3]-[6].

Our knowledge in this area is very limited, but we can expect that research in this direction of precognitive interdiction will flourish in the near future, once we can accept that it is purely classical origin, so we do not have to invoke complicated QM arguments.
As a last remark for experimenters, it may be advisable to verify this syntropy effect in women, especially those who have already proved themselves as good '*precogniters*."

4. Conclusion

It has been known for long time that intuition plays significant role in many professions and various aspects of human life, including in entrepreneurship, government, and also in detective or law enforcement activities. Even women are known to possess better intuitive feelings or "*hunch*" compared to men.

Despite these examples, such a precognitive interdiction or hunch (gut feeling) is hardly accepted in established science. In this paper, we discuss briefly the advanced solutions of Maxwell equations, and then make connection between syntropy and precognition from classical perspective. This may be regarded as first step to describe such precognition activities which are usually considered belong to quantum realm.

But we admit that our model is still in its infancy, more researches are needed to fill all the missing details. Further observations and experiments are recommended to verify the above propositions.

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How a synthesizer works

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ABSTRACT

In their classic book, the art of thinking, Robert Bramson and Allen Harrison describe five thinking styles that one tends to adopt: pragmatist, analyst, realist, synthesist, and idealist. In this paper, we tell our story on thinking modes we often use, and sometimes we took a synthesizer mode, i.e. to combine three or four of the above thinking styles. We present some examples too. We hope this retelling may be useful for young scientists and mathematicians in developing new theories either in theoretical physics and cosmology.

INTRODUCTION

Review of 5 thinking modes

Scientists in all fields need to adopt certain thinking modes, and analytical way is not necessarily to be the only approach he/she can adopt. In this regard, it seems worth to see 5 thinking styles of Bramson & Harrison. In this introductory section, allow us to quote in full an article by Carol Krucoff in Washington Post (1):

Over the past five years, Robert Bramson has asked several thousand people what seems like a simple question: "How do you think about things?"

"Most people find this extremely difficult to answer," says the 56-year-old organizational psychologist. "The typical response is a surprised stare, a blank look and words like, 'What do you mean, how do I think? I just think, that's all, as anybody else does."

What most people don't realize, Bramson says, is that "in our Western world there are five distinct approaches to thinking: Synthesist, Idealist, Pragmatist, Analyst and Realist. Each is useful in a given situation, but can be catastrophic if overused or used inappropriately. Yet almost all of us learn only one or two sets of strategies, and we go through life using them no matter what the situation.

"All around us we see people achieving success using strategies very different from our own. But despite the evidence, we persist in the ways that we believe work for us. We impose our own limitations, and we find it hard to understand those who persist in their own peculiar methods."

Psychologists call this human tendency "assuming similarity."

"In the absence of evidence to the contrary," says Bramson, "most people, most of the time assume others are just like them--only a little defective. Or, if their self-esteem is low, they think others are just like them only a little superior."

Bramson began researching styles of thinking in 1975 while trying to discover "why intelligent managers make terrible decisions." He and colleagues at their Berkeley, Calif., management-consulting firm uncovered two major studies relevant to the "problem-solving" issue: Philosopher C. West Churchman had identified five "inquiry modes" used by scientists, and Harvard professor Jerome Bruner had described four "conceptual strategies."

From this and other research (including Aristotle's description of the four different approaches to arguing) they isolated five styles of thinking and developed a test to determine thinking-style preference. In five years of conducting workshops and testing several thousand people--mostly white-collar professionals--they have isolated these characteristics of each style:

Idealists

Receptive and inquiring. Tend to focus on similarities among people and try to assimilate disparate views into a solution that will have something for everyone. Ethical, future-oriented and concerned with social values and goals. Excel in articulating goals and seeing the broad picture, but may try too hard for "perfect" solutions and screen out hard data and details.

Under stress, idealists often look hurt.

Analysts

Detail-oriented. Approach problems in a careful and methodical way. Gather as much information as possible before making a decision and look for the "one best way" to proceed. View themselves as factual, down-to-earth, practical people and view the world as logical, ordered and predictable. May screen out values and subjective factors and can appear inflexible and overly cautious.

Cool, studious and often hard to read, analysts under stress often withdraw.

Pragmatists

Flexible and adaptive. Focus on the shortest route to the payoff and excel at finding new ways of doing things with materials at hand. Believe the world is neither predictable nor understandable and are interested in "whatever works." May seem unpredictable, but tend to have well-developed social skills and are often well-liked.

Under stress, pragmatists may look bored.

Synthesists

Like to rearrange seemingly disparate things into new, creative combinations. Habitually question people's basic assumptions about things and enjoy philosophical arguments. Not likely to be interested in compromise or consensus. Best in controversial, conflict-laden situations. May be labeled as "troublemakers."

Under stress, synthesists tend to poke fun.

Realists

View "reality" as whatever they can feel, smell, see, hear or experience. Believe that any two-intelligent people ought to agree on the facts, and if something is wrong, want to fix it. Have a need to achieve and be in control. Pride themselves on incisiveness and can become impatient easily. Good at simplifying a problem and providing drive and momentum, but may try too hard for consensus and rush to over-simplified solutions.

Realists under stress become agitated.

The most popular style of thinking in this country, says Bramson, is Idealist with 37 percent of those tested showing that preference. Other styles, in order: Analyst (35 percent), Realist (24 percent), Pragmatist (18 percent) and Synthesist (11 percent).

"In the workplace we glorify the realists and analysts," he says, "and stomp out the synthesists. In the '60s there was a resurgence of interest in the synthesist style of thinking--which often comes up with new, fresh ideas--but today we tend to see them as troublemakers.

"In other cultures, style preferences may differ. That's something we're interested in testing. I believe there's likely a genetic bias toward one or two styles, which may be amplified or contradicted by early learning."

Sex is not a factor, Bramson claims, in the way people think. "We were surprised that we didn't find a difference in the style preference between men and women."

Occupations are, however, linked to style preferences.

"What we found," he says, "unfortunately supports the stereotypes. Social workers, for example, peak in idealist and are low in analyst, while budget officers are the exact opposite. . . Which makes it clear why the two groups often have trouble communicating. That can lead to poor use of funding."

Based on his study, Bramson believes that about half the population tends to rely on a single style of thinking and about 35 percent favors a combination of two styles.

Albert Einstein, he says, was probably an Analyst/Synthesist: "He had a vision, then backed it up with data." Thomas Jefferson was likely a "Synthesist/Idealist" who continually upset and confused "Analyst/Realist" Alexander Hamilton.

Ronald Reagan's style, he says, "is difficult to determine since he's so good at presenting himself . . . but he exemplifies the politician's profile: Realist/Pragmatist."

There is, stresses Bramson, no "best" style. "This is not a measure of ability, but of how you use your intellect. Each individual must stop wishing they were different, learn to be more skillful with the strengths they have and acknowledge their liabilities--which are usually simply the overuse or inappropriate use of their strengths."

Someone who learns to recognize the errors their preferred style of thinking may lead to, he says, can compensate for blind spots. The best way to broaden a style repertoire, he says, is to "link up with someone who is high in the areas you are low in and listen to them (2,3).

"My wife, who is also my partner, is a Pragmatist/Realist, low in Synthesist, while I'm a Synthesist/Realist, low in Pragmatist. She values the ideas I have as a Synthesist, but she can bring me down to earth when I've got my head in the clouds. We're sensitive to the ways we differ, try to listen and respect one another and value that different style of thinking."

How to be a Synthesizer

As Bramson argued, that each mode of thinking can be useful in certain context, and may be not so useful in other situations, therefore we also adopted a mode that you may call "Synthesizer" mode. A synthesizer (ˈsinTHəˌsīzər/) can be defined as follows:

1. an electronic musical instrument, typically operated by a keyboard, producing a wide variety of sounds by generating and combining signals of different frequencies.
2. any of various electronic, sometimes portable consoles or modules, usually computerized, for creating, modifying, and combining tones or reproducing the sounds of musical instruments by controlling voltage patterns, operated by means of keyboards, joysticks, sliders, or knobs.

In a similar way, we sometimes adopted 2-3 thinking styles altogether such as: Analytical/Realist/Synthesist like in Cantorian Navier-Stokes Cosmology. And sometimes Idealist/Synthesist/Analyst etc. like in Smarandache's Neutrosophic Logic. This mode is called as 'Synthesizer's way."

In the following section we will describe a few examples how to be a Synthesizer in physical sciences and mathematics fields.

Our story

A. *Neutrosophic logic:* One of us developed a new theory called Neutrosophic Logic as an extension of Intuitionistic Fuzzy Logic (4,5) Instead of working on Zadeh's Fuzzy Logic, he developed a novel way, to unify the whole logic and probability theory, with implications range far into new fields such as AI, Information Fusion, Dezer-Smarandache Theory (DSmT) etc.

Below is a summary of Neutrosophic Logic: (4)

Neutrosophic Logic is a general framework for unification of many existing logics, such as fuzzy logic (especially intuitionistic fuzzy logic), paraconsistent logic, intuitionistic logic, etc. The main idea of NL is to characterize each logical statement in a 3D Neutrosophic Space, where each dimension of the space represents respectively the truth (T), the falsehood (F), and the indeterminacy (I) of the statement under consideration, where T, I, F are standard or non-standard real subsets of]-0, 1+[with not necessarily any connection between them.

For software engineering proposals the classical unit interval [0,1] is used.

For single valued Neutrosophic logic, the sum of the components is:

$0 \le t+i+f \le 3$ when all three components are independent;

$0 \le t+i+f \le 2$ when two components are dependent, while the third one is independent from them;

$0 \le t+i+f \le 1$ when all three components are dependent.

When three or two of the components T, I, F are independent, one leaves room for incomplete information (sum <1), paraconsistent and contradictory information (sum >1), or complete information (sum =1).

If all three components T, I, F are dependent, then similarly one leaves room for incomplete information (sum <1), or complete information (sum =1).

B. *Cantorian navier-stokes cosmology:* Around mid of 2002 one of us tried to rekindle the superfluid interstellar medium in astrophysics, but after studying some existing papers, he ended up in superfluid quantized vortices model of the Solar System (5). He argued that the Universe can be modelled by Navier-Stokes equations, which reduce to superfluid quantised vortices. It was quite rare at the time to come up with a whole new idea in astrophysics, connecting NS equations and superfluidity, but now the use of NS into superfluidity context becomes more common (6,7).

Among our result there was a prediction of 3 new planetoids beyond Pluto orbit, which then the three new orbits have been found to be inhabited by new planetoids, like Sedna.

C. *Retro-classical physics:* This is a new term we argued in a paper discussing how we can work out new theories beyond the subjective-idealism tendency of Relativity Theory and Quantum Mechanics. Interested readers are advised to see our paper (8,9).

CONCLUSION

In the last 10-11 years, we have published more than ten books in this area of quantized astrophysics and also Neutrosophic Logic. Although there were different thinking modes between us (as mathematician and as a nocturnal physicist), we chose to publish rather than perish.

We hope this short article may inspire younger generation of physicists and mathematicians to rethink and furnish their approaches to Nature, and perhaps it may also help to generate new theories which will be useful for a better future of mankind.

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A Newtonian-vortex cosmology model from solar system to galaxy to large scale structures: Navier-stokes-inspired cosmography

Victor Christianto, Florentin Smarandache

Victor Christianto, Florentin Smarandache (2017). A Newtonian-vortex cosmology model from solar system to galaxy to large scale structures: Navier-Stokes-inspired cosmography. *J Mod Appl Phys.* 1(1); 10-13

ABSTRACT

Some years ago, Matt Visser asked the following interesting questions: How much of modern cosmology is really cosmography? How much of modern cosmology is independent of the Einstein equations? (Independent of the Friedmann equations?) These questions are becoming increasingly germane-as the model's cosmologists use for the stress-energy content of the universe become increasingly baroque. Therefore, in this paper we will discuss a novel Newtonian cosmology model with vortex, which offers wide implications from solar system, galaxy modeling up to large scale structures of the Universe. The basic starting point is very simple: It has been known for long time that most of the existing cosmology models have singularity problem. Cosmological singularity has been a consequence of excessive symmetry of flow, such as "Hubble's law." More realistic one is suggested, based on Newtonian cosmology model but here we include the vortical-rotational effect of the whole Universe. We review an Ermakov-type equation obtained by Nurgaliev, and solve the equation numerically with Mathematica. A potential application is also considered, namely for understanding tornado dynamics using 3D Navier-Stokes equations. It is our hope that the new proposed method can be verified with observations, in order to open new possibilities of more realistic nonlinear cosmology models.

Key Words: *Ermakov-type equation; Nonlinear cosmology; Newtonian cosmology; Vortex dynamics, turbulence; Navier-Stokes equations; Spiral galaxy*

INTRODUCTION

Some years ago, Matt Visser asked the following interesting questions: How much of modern cosmology is really cosmography? How much of modern cosmology is independent of the Einstein equations? (Independent of the Friedmann equations?) These questions are becoming increasingly germane-as the model's cosmologists use for the stress-energy content of the universe become increasingly baroque (1).

In this regard, academician Isaak Khalatnikov mentioned at the 13th Marcel Grossman Conference, that Lev Landau suggesting that something is too symmetric in the models yielding singularities, and that this problem is one of the three most important problems of modern physics. The aim of this report is to show that singularities are, indeed, consequences of such an overly "symmetrical approach" in building non-robust (i.e. without structural stability) toy models with singularities. Such models typically apply a synchronous system of reference and "Hubble's law", neglecting not-to-be-averaged-out quadratic terms of perturbations (specifically, differentially rotational velocities, vortexes) (2).

Only by accounting the overlooked factors instead of Einstein's ad hoc introduction of a new entity, which was later declared by him as his "biggest blunder", can we correctly interpret accelerated cosmological expansion, as well as provide possibility of static solution. The common perception of the observed accelerated expansion is that there is need either in modifying the General Relativity or discover new particles with unusual properties. Both ways are possible depending on what kind of system of reference and corresponding interpretation are chosen, a decision which is usually made depending on the level of "geometrization" (2).

Local rotations (vortices) play a role in radical stabilization of the cosmological singularity in the retrospective extrapolation, making possible a static or steady-state (on the average) Universe or local region. Therefore, Einstein could "permit" the galaxies to rotate instead of postulating a cosmological constant ad hoc in his general-relativistic consideration of a static Universe. Though, it does not necessarily mean that thecosmological constant is not necessary for other arguments (3).

In this paper, more realistic one is suggested, based on Newtonian cosmology model but here we include the vortical-rotational effect of the whole Universe. We review an Ermakov-type equation obtained by Nurgaliev (2,3), and solve the equation numerically with Mathematica 11.

In this paper we will also discuss a novel Newtonian cosmology model with vortex, which offers wide implications from solar system, galaxy modeling up to large scale structures of the Universe.

It is our hope that the new proposed method can be verified with observation data.

A FEW HISTORICAL NOTES

Since long time ago, there were numerous models of the Universe, dating back to Ptolemaic geocentric model, which was subsequently replaced by Nicolas Copernicus discovery. Copernicus model then was brought into fame after Isaac Newton published his book. But other than Newton, there was a model of Universe as a turbulent fluid (hurricane) brought by a French philosopher and mathematician, R. Descartes. But, this model was almost forgotten. Many physicists rejected Descartes' model because it stood against Newtonian model, but the truth is turbulence model can be expressed in Navier-Stokes equations, and Navier-Stokes equations can be considered as the rigorous formulation of Newtonian laws, especially for fluid dynamics. In other words, we can say that Newtonian turbulence Universe is not in direct contradiction with Newtonian laws. Therefore, in this paper we submit wholeheartedly a proposal that the Universe can be modelled as Newtonian-Vortex based on 3D Navier-Stokes equations. We shall show some implications of this new model in the following sections.

SOLAR SYSTEM MODEL

In this section, we will review the work which was carried out by VC and FS during the past ten years or so. The basic assumption here is that the Solar System's planetary orbits are quantized. But how do their orbits behave? Do they follow Titius-Bode's law? Our answer can be summarized as follows: (4-6).

Navier-Stokes equations→superfluid quantized vortices→Bohr's quantization

Our predictive model based on that scheme has yielded some interesting results which may be comparable with the observed orbits of planetoids beyond Pluto, including what is dubbed as Sedna (7). And it seems that the proposed model is slightly better compared to Nottale-Schumacher's gravitational Schrödinger model and also Titius-Bode's empirical law.

SPIRAL GALAXY MODEL

In this section, we discuss a simple model of galaxies based on a postulate of turbulence vortices which govern the galaxy dynamics. Abstract of Vatistas' paper told clearly (8).

Expanding our previous work on turbulent whirls (1) we have uncovered a similarity within the similarity shared by intense vortices. Using the new information we compress the tangential velocity profiles of a diverse set of vertices into one and thus identify those that belong to the same genus. Examining the Laser Doppler Anemometer (LDA) results of mechanically produced vortices and radar data of several tropical cyclones, we find that the uplift and flattening effect of tangential velocity is a consequence of turbulence. Reasoning by analogy we conclude that turbulence in the interstellar medium could indeed introduce a flattening effect in the galactic rotation curves.

The result of his model equation can yield prediction which is close to observation (without invoking dark matter hypothesis), as shown in the following diagram (Figure 1):

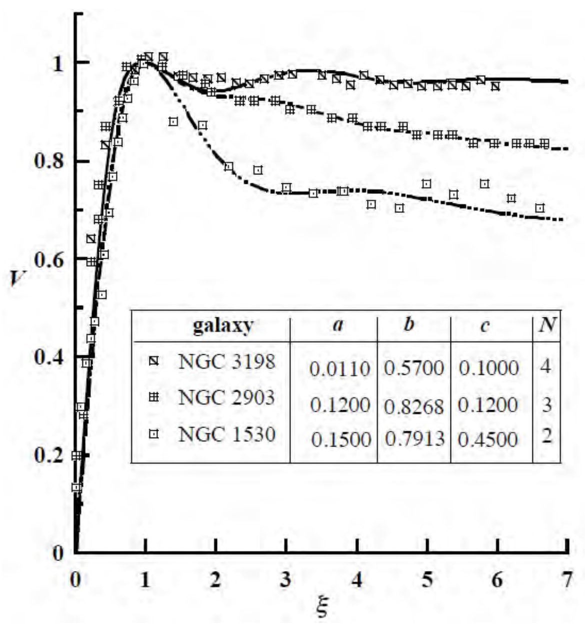


Figure 1) *From Vatistas (8)*

Therefore, it appears possible to model galaxies without invoking numerous ad hoc assumptions, once we accept the existence of turbulent interstellar medium. The model is also governed by Navier-Stokes equations (8).

DERIVING ERMAKOV-TYPE EQUATION FOR NEWTONIAN UNIVERSE WITH VORTEX

It has been known for long time that most of the existing cosmology models have singularity problem. Cosmological singularity has been a consequence of excessive symmetry of flow, such as "Hubble's law". A more realistic one is suggested, based on Newtonian cosmology model but here we include the vortical-rotational effect of the whole Universe.

In this section, we will derive an Ermakov-type equation following Nurgaliev (2). Then we will solve it numerically using Mathematica 11.

After he proceeds with some initial assumptions, Nurgaliev obtained a new simple local cosmological equation: (1)

$$\dot{H} + H^2 = \omega^2 + \frac{4\pi G}{3}\rho,$$

Where Here, $\dot{H} = dH/dt$. stand for Hubble constant, Newtonian gravitational constant, angular speed, and density, respectively.

The angular momentum conservation law ωR^2=const=K and the mass conservation law $(4\pi/3)\,\rho R^3$=const=M makes equation (1) solvable: (2)

$$\dot{\mathrm{H}} + \mathrm{H}^2 = \frac{K^2}{R^4} - \frac{\mathrm{GM}}{\mathrm{R}^3},$$

or

$$\ddot{R} + \frac{GM}{R^2} = \frac{K^2}{R^3}.$$

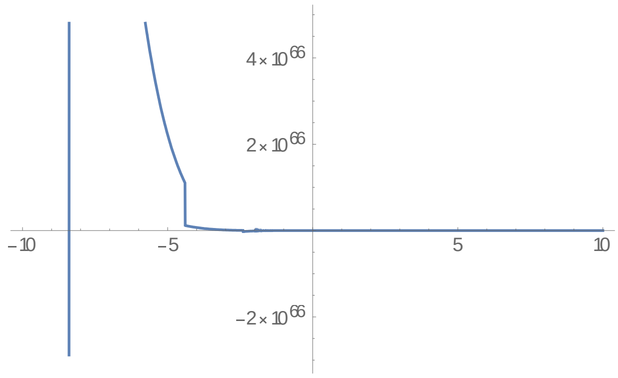
Equation (3) may be written as Ermakov-type nonlinear equation as follows;

$$\ddot{R} + \frac{GM}{R^2} = \frac{K^2}{R^3}.$$

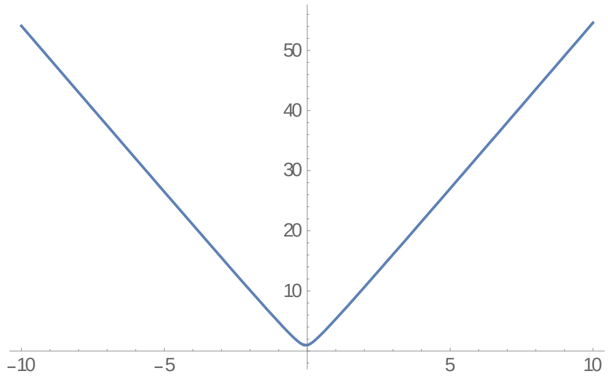
Nurgaliev tried to integrate equation (3), but now we will solve the above equation with Mathematica 11. First, we will rewrite this equation by replacing GM=A, K^2=B, so we get:

$$\ddot{R} + \frac{A}{R^2} = \frac{B}{R^3}.$$

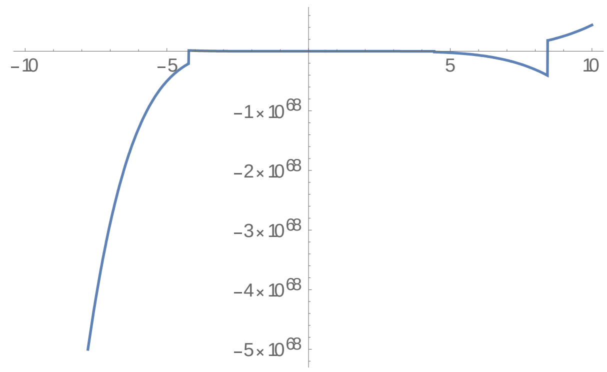
As with what Nurgaliev did in (1,2), we also tried different sets of A and B values, as follows:



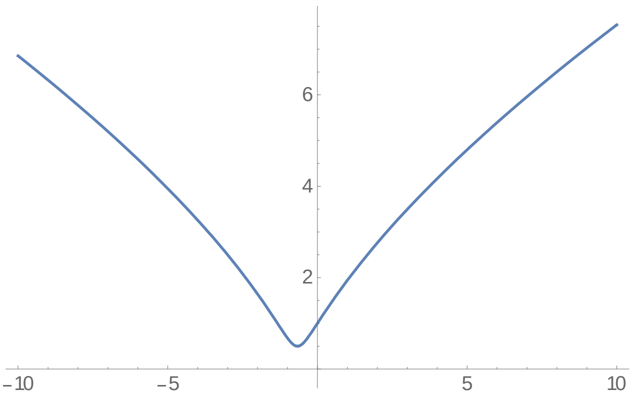
(A) A and B < 0; A=-10; B=-10; ODE=x''[t]+A/x[t]^2-B/x[t]^3==0; sol=NDSolve[{ODE,x[0]==1,x'[0]==1},x[t],{t,-10,10}] Plot[x[t]/.sol,{t,-10,10}].



(B) A < 0, B > 0; A=-10; B=10; ODE=x''[t]+A/x[t]^2-B/x[t]^3==0; sol=NDSolve[{ODE,x[0]==1,x'[0]==1},x[t],{t,-10,10}] Plot[x[t]/.sol,{t,-10,10}]



(C) A > 0, B < 0; A=1; B=-10; ODE=x''[t]+A/x[t]^2-B/x[t]^3==0; sol=NDSolve[{ODE,x[0]==1,x'[0]==1},x[t],{t,-10,10}] Plot[x[t]/.sol,{t,-10,10}]



(D) A >0, B > 0; A=1; B=1; ODE=x"[t]+A/x[t]^2-B/x[t]^3==0; sol=NDSolve[{ODE,x[0]==1,x'[0]==1},x[t],{t,-10,10}] Plot[x[t]/.sol,{t,-10,10}]

From the above numerical experiments, we conclude that the evolution of the Universe depends on the constants involved, especially on the rotational-vortex structure of the Universe. This needs to be investigated in more detailed for sure.

ENGINEERING APPLICATION: HURRICANE DYNAMICS AND SOLUTION OF 3D NAVIER-STOKES

Various methods to describe hurricane dynamics have been proposed in the literature, but most of them are based on 3D Navier-Stokes. Some existing models of tornado dynamics can be found in (9,10).

Now, we will discuss a simplified numerical solution of 3D Navier-Stokes equations based on Sergey Erhskov's papers (11,12).

In fluid mechanics, there is an essential deficiency of the analytical solutions of Navier–Stokes equations for 3D case of non-stationary flow. The Navier-Stokes system of equations for incompressible flow of Newtonian fluids should be presented in the Cartesian coordinates as below (under the proper initial conditions): (11)

$$\nabla.\vec{u} = 0,$$

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u}\cdot\nabla)\vec{u} = -\frac{\nabla p}{\rho} + \nu\cdot\nabla^2\vec{u} + \vec{F},$$

where u is the flow velocity, a vector field; ρ is the fluid density, p is the pressure, v is the kinematic viscosity, and F represents external force (per unit mass of volume) acting on the fluid (11).

In (11), Ershkov explores the ansatz of derivation of non-stationary solution for the Navier–Stokes equations in the case of incompressible flow, which was suggested earlier. In general case, such a solution should be obtained from the mixed system of 2 coupled Riccati ordinary differential equations (in regard to the time-parameter t). But instead of solving the problem analytically, we will try to find a numerical solution (12-15).

The coupled Riccati ODEs read as follows: (11)

$$a' = \frac{w_y}{2}\cdot a^2 - (w_x\cdot b)\cdot a - \frac{w_y}{2}(b^2-1) + w_z\cdot b$$

$$b' = -\frac{w_x}{2}\cdot b^2 - (w_y\cdot a)\cdot b - \frac{w_x}{2}(a^2-1) + w_z\cdot a.$$

First, equations (8) and (9) can be rewritten in the form as follows:

$$x(t)' = \frac{v}{2}\cdot x(t)^2 - (u\cdot y(t))\cdot x(t) - \frac{v}{2}(y(t)^2-1) + w\cdot y(t),$$

$$y(t)' = -\frac{u}{2}\cdot y(t)^2 - (v\cdot x(t))\cdot y(t) - \frac{u}{2}(x(t)^2-1) + w\cdot x(t).$$

Then we can put the above equations into Mathematica expression: (2)

v=1; u=1; w=1; {xans6[t_],yans6[t_]}={x[t],y[t]}/.Flatten[NDSolve[{x'[t]==(v/2)*x[t]^2-(u*y[t])*x[t]-(v/2)*(y[t]^2-1)+w*y[t], y'[t]==-(u/2)*y[t]^2-(v*x[t])*y[t]-(u/2)*(x[t]^2-1)+w*x[t], x[0]==1,y[0]==0}, {x[t],y[t]},{t,0,10}]] graphx6=Plot[xans6[t],{t,0,10}, AxesLabel->{"t","x"},PlotStyle->Dashing[{0.02,0.02}]];Show[graphx6,graphx6]

The result is as shown below Figure 2.

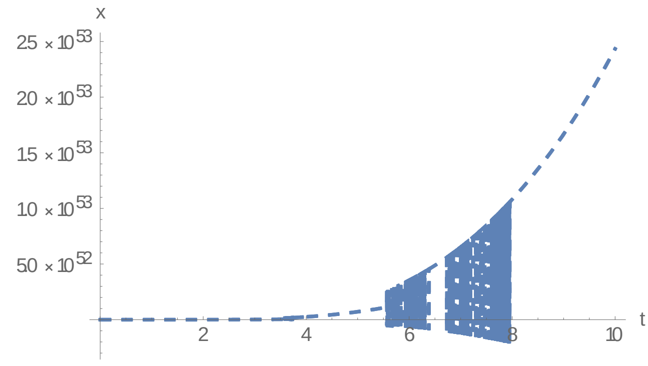


Figure 2) *Graphical plot of solution for case v=u=w=1 (2)*

It is our hope that the above numerical solution of 3D Navier-Stokes equations can be found useful for engineering purposes, such as controlling large tornadoes which happen quite often in various regions each year.

CONCLUSION

It has been known for long time that most of the existing cosmology models have singularity problem. Cosmological singularity has been a consequence of excessive symmetry of flow, such as "Hubble's law". More realistic one is suggested, based on Newtonian cosmology model but here we include the vortical-rotational effect of the whole Universe. We discuss a plausible model for describing planetary quantization in Solar system and also flattening velocity observed in numerous galaxies. We also review a Riccati-type equation obtained by Nurgaliev, and solve the equation numerically with Mathematica 11.

We also discuss an engineering application of this model, i.e. how to solve 3D Navier-Stokes equations numerically. It is our hope that the above numerical solution of 3D Navier-Stokes equations can be found useful for engineering purposes, such as controlling large hurricanes which happen quite often in various regions each year (16-18).

The solutions obtained here opens up new ways to interpret existing solutions of known 3D Navier-Stokes problem in physics, astrophysics, cosmology and engineering fields, especially those associated with nonlinear hydrodynamics and turbulence modelling (19-20).

It is our hope that the new proposed Cosmology model with vortex can be verified with more extensive observation data.

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A Short Introduction of Cellular Automaton Universe *via* Cosmological KdV Equation

Victor Christianto, Florentin Smarandache, Yunita Umnniyati

Victor Christianto, Florentin Smarandache, Yunita Umniyati (2017). A Short Introduction of Cellular Automaton Universe via Cosmological KdV Equation. *Prespacetime Journal* 8(5), 500-503

ABSTRACT

It has been long known that the cosmic sound wave was there since the early epoch of the Universe. Signatures of its existence are abound. However, such a sound wave model of cosmology is rarely developed fully into a complete framework. This paper can be considered as our *second* attempt towards such a complete description of the Universe based on soliton wave solution of cosmological KdV equation. We submit that Robert Kurucz's hypothesis that *Big Bang should be replaced with a finite cellular automaton universe with no expansion*. Our model is preliminary but close in spirit to what Konrad Zuse envisaged long time ago. It is our hope that the new model can be verified with observation data.

Keywords: Solitary wave, cosmological KdV equation, nonlinear universe, cellular automata, PDE, Konrad Zuse.

1. Introduction

Konrad Zuse is probably the first scholar who imagine a *Computing Universe*. In recent years, there are a few researchers who suggest similar vision in terms of cellular automata. For example, Stephen Wolfram, Gerard 't Hooft, and Robert Kurucz. Nonetheless, it seems that there is no existing model which can be connected with a nonlinear PDE of the Universe. Thus, we try to offer a working model of Computing Universe which can be modelled and solved using computer algebra packages such as Mathematica.

Korteweg-de Vries (KdV) equation is a non-linear wave equation plays a fundamental role in diverse branches of mathematical and theoretical physics. Its significance to cosmology has been discussed by a number of authors, such as Rosu and recently Lidsey [3, 7]. It is suggested that the KdV equation arises in a number of important scenarios, including inflationary cosmology etc. Analogies can be drawn between cosmic dynamics and the propagation of the solitonic wave solution to the equation, whereby quantities such as the speed and amplitude profile of the wave can be identified with cosmological parameters such as the spectral index of the density perturbation spectrum and the energy density of the universe.

Then we advance further this KdV equation by virtue to Cellular Automaton method to solve the PDEs. We submit wholeheartedly Kurucz's hypothesis that *Big Bang should be replaced with a finite cellular automaton universe with no expansion*.[4][5]

Nonetheless, we are fully aware that our model is far from being complete, but perhaps the proposed cellular automaton model of the Universe is very close in spirit to what Konrad Zuse envisaged long time ago. However, we do not exercise possible link between our model and Cellular automaton model of Gerard 't Hooft; that is beyond the scope of this paper.[14]
It is our hope that the new proposed equation can be verified with observation data both at lab scale and also at large scale astronomy data. But we admit that our model is still in its infancy, more researches are needed to fill all the missing details.

2. Cosmological KdV Equation

The Korteweg-de Vries (KdV) equation is the completely integrable, third-order, non-linear partial differential equation (PDE):[3]

$$\partial_t u + \partial_x^3 u + \frac{3}{u_0} u \partial_x u = 0, \tag{1}$$

where $u = u(x, t)$, $\partial_t = \partial/\partial t$, $\partial_x^3 = \partial^3/\partial x^3$, etc., u_0 is a constant and (x, t) represent space and time coordinates, respectively. This equation was originally derived within the context of small-amplitude, non-linear water wave theory and it is well known that it admits a solitonic wave solution of the form

$$u = u_0 \lambda^2 \sec h^2 \left[\lambda\left(x - \lambda^2 t\right)/2\right], \tag{2}$$

where the constant $\lambda/2$ represents the wavenumber of the soliton. The KdV soliton is characterized by the property that its speed and amplitude are proportional to the square of the wavenumber.

Rosu [7] and also Lidsey [3] both have considered some cosmological applications of KdV equation. We will consider here one application in inflationary universe model.
It can be shown that Friedmann equation after some steps which have been discussed in [3], yields to an equation which takes the form of (2), as follows:

$$H^2(\phi) = H_0^2 \lambda^2 \sec h^2 \left[\lambda A/2\right], \tag{3}$$

Where:

$$A = \frac{\sqrt{8\pi}}{m_p} \phi. \tag{4}$$

Therefore, it appears quite reasonable to consider this equation as originated from certain cosmological KdV physics.

3. Towards a Cellular Automaton Universe

Based on a paper by Steeb & Hardy [11], KdV equation can be written as a conservation law:

$$\frac{\partial u}{\partial t}+\frac{\partial}{\partial x}\left(-\frac{u^2}{2}-\frac{\partial^2 u}{\partial x^2}\right)=0, \tag{5}$$

It follows that, after the simplest discretization, we obtain the cellular automata:

$$u_j(t+1)=u_j(t)(u_{j+1}(t)-u_j(t))+u_{j+2}(t)-u_{j+1}(t)-u_{j-1}(t). \tag{6}$$

Thus $\sum_{j=0}^{N-1}u_j(t)$ is not an invariant.

In other words, it appears possible to discuss a Cellular Automaton-KdV Universe. But further analysis is required to study its potential applications.

4. Discussion

It has been long known that the cosmic sound wave was there since the early epoch of the Universe. Signatures of its existence are abound. However, such a sound wave model of cosmology is rarely developed fully into a complete framework. This paper can be considered as our *second* attempt towards such a complete description of the Universe based on soliton wave solution of cosmological KdV equation. We submit that Robert Kurucz's hypothesis that *Big Bang should be replaced with a finite cellular automaton universe with no expansion*. Our model is preliminary but close in spirit to what Konrad Zuse envisaged long time ago. It is our hope that the new model can be verified with observation data.

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Discovered “Angel Particle”, which is Both Matter and Antimatter, as a New Experimental Proof of Unmatter

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“Angel particle” bearing properties of both particles and anti-particles, which was re-cently discovered by the Stanford team of experimental physicists, is usually associated with Majorana fermions (predicted in 1937 by Ettore Majorana). In this message we point out that particles bearing properties of both matter and anti-matter were as well predicted without any connexion with particle physics, but on the basis of pure mathe-matics, namely — neutrosophic logic which is a generalization of fuzzy and intuition-istic fuzzy logics in mathematics.

Recently, a group of experimental physicists conducted by Prof. Shoucheng Zhang, in Stanford University, claimed about discovery of the particles that bear properties of both particles and anti-particles. The press-release [1] was issued on July 20, one day before the official publication [2].

Shoucheng Zhang told [1, 2] that the idea itself rose up from Ettore Majorana who in 1937 suggested that within the class of fermions a particle may exist which bear properties of particle and anti-particle in the same time. Such hypothetic particles are now know as “Majorana fermions”.

In their experiment, the Stanford team used the following experimental setup. Two stacked films — the top film made of superconductor and the bottom film made of magnetic insulator — were stored together in a cooled down vacuum box. And an electrical current was sent through this “sandwich”. Using a magnet mounted over the stackled films, the speed of the electrons in the film was able to be modifying. Varying the magnet’s properties, the experimentalists registered Majorana particles which appeared in pairs in the electron flow but deviated from the electrons (so they were able to be registered separately). The experimentalists referred to the supposed new particle as “Angel particle” (meaning that, as well as angels are neither male nor female, the supposed particle is neither matter nor anti-matter).

Shoucheng Zhang also declared the importance of this discovery because, he thinks, the particles bearing properties of matter and anti-matter in the same time shows a fantastic perspective for computer industry and machinery.

In this background, we should note that particles bearing properties of matter and anti-matter were as well theoretically predicted being non-connected with particle physics, but only on the basis of pure mathematics. This is a series of works [3–8] based on neutrosophic logic (one of the multi-valued modern logics, a part of mathematics) authored by Florentin Smarandache.

So, following the neutrosophic logics, “between an entity <A> and its opposite <AntiA> there exist intermediate entities <NeutA> which are neither <A> nor <AntiA> [. . .] Thus, between “matter” and “antimatter” there must exist something which is neither matter nor antimatter, let’s call it UNMATTER” [3]. Expanding this theory, a new type of matter — “unmatter” — was predicted.

Now, this theoretical study based on pure mathematics, elucidates that was discovered by the Stanford team conducted by Shoucheng Zhang. This fact shows that not only particle physics but also pure mathematics can make essential predictions that may change the wirld of science and techniques.

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From Acoustic Analog of Space to Acoustic Sachs-Wolfe Theorem: A Model of the Universe as a Guitar

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ABSTRACT

It has been known for long time that the cosmic sound wave was there since the early epoch of the Universe. Signatures of its existence are abundant. However, such an acoustic model of cosmology is rarely developed fully into a complete framework from the notion of space up to the sky. This paper may be the first attempt towards such a complete description of the Universe based on classical wave equation of sound. It is argued that one can arrive at a consistent description of space, elementary particles, Sachs-Wolfe acoustic theorem, starting from this simple classical wave equation of sound. We also discuss a plausible extension of Acoustic Sachs-Wolfe theorem based on its analogue with Klein-Gordon equation to a new equation. It is our hope that the new proposed equation can be verified with observation data. But we admit that our model is still in its infancy, more researches are needed to fill all the missing details.

Keywords: Acoustic metric, acoustic analogue of space, acoustic cosmology, Sachs-Wolfe theorem.

1. Introduction

In one of his papers, the late C.K. Thornhill wrote [1]:

> Relativists and cosmologists regularly refer to space-time without specifying precisely what they mean by this term. Here the two different forms of spacetime, real and imaginary, are introduced and contrasted. It is shown that, in real space-time (x, y, z, ct), Maxwell's equations have the same wave surfaces as those for sound waves in any uniform fluid at rest, and thus that Maxwell's equations are not general and invariant but, like the standard wave equation, only hold in one unique frame of reference. In other words, Maxwell's equations only apply to electromagnetic waves in a uniform ether at rest. But both Maxwell's equations and the standard wave equation, and their identical wave surfaces, transform quite properly, by Galilean transformation, into a

general invariant form which applies to waves in any uniform medium moving at any constant velocity relative to the reference-frame. It was the mistaken idea, that Maxwell's equations and the standard wave equation should be invariant, which led, by a mathematical freak, to the Lorentz transform (which demands the non-ether concept and a universally constant wave-speed) and to special relativity. The mistake was further compounded by misinterpreting the differential equation for the wave hypercone through any point as the quadratic differential form of a Riemannian metric in imaginary space-time (x, y, z, ict). Further complications ensued when this imaginary space-time was generalised to encompass gravitation in general relativity.

In this paper, we will start with a simple premise that the space itself has an acoustic origin and it relates to Maxwell equations. Maxwell equations can be expressed in terms of vortex sound equation. So it will indicate a new interpretation of aether in acoustic terminology.

It is argued that, starting from this simple classical wave equation of sound, one can arrive at a consistent description of space, elementary particles and Sachs-Wolfe acoustic theorem. We also discuss a plausible extension of Acoustic Sachs-Wolfe theorem to a new equation based on its analogue with Klein-Gordon equation.

It is our hope that the proposed new equation can be verified with observation data. It should be noted that this model is still in its infancy.

2. Acoustic Analogue of Space

In this section, we borrow some important ideas from C.K. Thornhill and also Tsutomu Kambe. According to Thornhill, real space-time is a four dimensional space consisting of three-dimensional space plus a fourth length dimension obtained by multiplying time by a constant speed. (This is usually taken as the constant wave-speed c of electromagnetic waves). If the four lengths, which define a four-dimensional metric (x, y, z, ict), are thought of as measured in directions mutually at right-angles, then the quadratic differential form of this metric is [1]:

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 - \bar{c}^2 (dt)^2 \qquad (1)$$

When the non-differential terms are removed from Maxwell's equations, i.e. when there is no charge distribution or current density, it can easily be shown that the components (E1 ,E2 ,E3) of the electrical field-strength and the components (H1 ,H2 ,H3) of the magnetic field-strength all satisfy the standard wave equation:[1]

$$\nabla \phi = \left(\frac{1}{\bar{c}^2} \right) \frac{\partial^2 \phi}{\partial t^2} \qquad (2)$$

It follows immediately, therefore, that the wave surfaces of Maxwell's equations are exactly the same as those for sound waves in any uniform fluid at rest, and that Maxwell's equations can only hold in one unique reference-frame and should not remain invariant when transformed into any other reference-frame. In particular, the equation for the envelope of all wave surfaces which pass through any point at any time is, for equation (2), and therefore also for Maxwell's equations,[1]

$$(dx)^2 + (dy)^2 + (dz)^2 = \bar{c}^2 (dt)^2 \tag{3}$$

or

$$\frac{(dx)^2}{(dt)^2} + \frac{(dy)^2}{(dt)^2} + \frac{(dz)^2}{(dt)^2} = \bar{c}^2 \tag{4}$$

It is by no means trivial, but it is, nevertheless, not very difficult to show, by elementary standard methods, that the general integral of the differential equation (4), which passes through (x1, y1, z1) at time t1, is the right spherical hypercone [1]:

$$(x - x_1)^2 + (y - y\ )^2 + (z - z_1)^2 = \bar{c}^2 (t - t_1)^2 \tag{5}$$

In other words, both Maxwell equations and space itself has the sound wave origin. We shall see later that this interpretation of Thornhill's work is consistent with the so-called acoustic Sachs-Wolfe theorem which is known in cosmology setting.

It is also interesting to remark here that Maxwell equations can be cast in the language of vortex sound theory, as follows.

T. Kambe from University of Tokyo has made a connection between the equation of vortex sound and fluid Maxwell equations. He wrote that it would be no exaggeration to say that any vortex motion excites *acoustic* waves. He considers the equation of vortex sound of the form: [2]:

$$\frac{1}{c^2}\partial_t^2 p - \nabla^2 p = \rho_0 \nabla . L = \rho_0 div(\omega \times v) \tag{6}$$

He also wrote that dipolar emission by the vortex-body interaction is:[3]

$$p_F(x,t) = -\frac{P_0}{4\pi c}\dot{\Pi}_i(t - \frac{x}{c})\frac{x_c}{x^2} \tag{7}$$

Then he obtained an expression of fluid Maxwell equations as follows [4]:

$$\begin{aligned} &\nabla \cdot H = 0 \\ &\nabla \cdot E = q \\ &\nabla \times E + \partial_t H = 0 \\ &a_0^2 \nabla \times H - \partial_t E = J \end{aligned} \tag{8}$$

where [4] a_0 denotes the sound speed, and

$$\begin{aligned} &q = -\partial_t (\nabla \cdot \upsilon) - \nabla \hbar, \\ &J = \partial_t^2 \nu + \nabla \partial_t h + a_o^2 \nabla \times (\nabla \times \upsilon) \end{aligned} \tag{9}$$

In our opinion, this new expression of fluid Maxwell equations suggests that there is a deep connection between vortex sound and electromagnetic fields. However, it should be noted that the above expressions based on fluid dynamics need to be verified with experiments. We should note also that in (8) and (9), the speed of sound a_0 is analogous of the speed of light in Maxwell equations, whereas in equation (6), the speed of sound is designated "c" (as analogous to the light speed in EM wave equation). For alternative hydrodynamics expression of electromagnetic fields, see [7].

The above interpretation of fluid Maxwell equations from vortex sound theory has been discussed in our recent paper, to appear in forthcoming issue of JCMNS [5].

3. Comparison between Schrödinger equation and Classical wave equation of sound

In the initial variant, the Schrodinger equation (SE) has the following form [8]:

$$\Delta \Psi + \frac{2m}{\hbar^2}\left(W + \frac{e^2}{4\pi \varepsilon_o r} \right)\Psi = 0 \tag{10}$$

The wave function satisfying the wave equation (10) is represented as:

$$\Psi = R(r)\Theta(\theta)\Phi(\varphi)T(t) = \psi(r,\theta,\varphi)T(t) \tag{11}$$

where $\psi(r,\theta,\varphi) = R(r)\Theta(\theta)\Phi(\varphi)$ is the complex amplitude of the wave function, because

$$\Phi_m(\varphi) = C_m e^{\pm im\varphi} \tag{12}$$

For standard method of separation of variables to solve spherical SE, see for example [11-13].

The Φ, Θ and T equations were known in the theory of wave fields. Hence these equations presented nothing new. Only the R was new. Its solution turned out to be *divergent*. However, Schrödinger together with H. Weyl (1885-1955), contrary to the logic of and all experience of theoretical physics, artificially cut off the divergent power series of the radial function R(r) at a κ-th term. This allowed them to obtain the radial solutions, which, as a result of the cut off operation, actually were the fictitious solutions [8].

Furthermore, it can be shown that the time-independent SE [9][10]:

$$\nabla\Psi + \frac{2m}{\hbar^2}(E-V)\Psi = 0, \tag{13}$$

can be written in the form of standard wave equation [8]:

$$\nabla\Psi + k^2\Psi = 0, \tag{14}$$

where

$$k = \pm\sqrt{\frac{2m}{\hbar^2}(E-V)}\,. \tag{15}$$

or if we compare (14) and (10), then we have [8]:

$$k = \pm\sqrt{\frac{2m}{\hbar^2}\left(W + \frac{e^2}{4\pi\varepsilon_o r}\right)}\,. \tag{16}$$

This means that the wave number k in Schrödinger's radial wave equation is a quantity that varies continuously in the radial direction. Is it possible to imagine a field where the wave number, and hence the frequency, change from one point to another in the space of the field? Of course, it is not possible. Such wave objects do not exist in Nature.

The unphysical nature of Schrödinger wave-function has created all confusing debates throughout 90 years. But it is rarely discussed in QM textbooks, on how he arrived at his equation. It is known that Schrodinger began with Einstein's mass-energy relation then he proceeded with Hamilton-Jacobian equation. At first he came to a similar fashion of Klein-Gordon equation, but then he arrived to a new equation which is non-relativistic. Logically speaking, he began with a relativistic assumption and he came to a nonrelativistic expression, and until now physicists remain debating on how to relativize Schrodinger equation. That is logically inconsistent and therefore unacceptable, and Schrodinger himself never knew where the problem lies. Until now people remain debating the problem of the meaning of his wavefunction, but it starts with unphysical nature of his equation. This is a common attitude of many young physicists who tend to neglect the process and logical implication of QM derivation, and they never asked about whether Schrodinger equation has deep logical inconsistency or not.

Moreover, there are some limitations in applying Schrödinger equation to experiments, although many textbooks on QM usually overlook existing problems on how to compare 3D spherical solution of Schrodinger equation with experimental data. The contradiction between QM and experiments are never discussed publicly, and this is why the most modern physicists hold the assertion that QM describes accurately "ALL" physical experiments; that is an unfounded assumption. George Shpenkov began with classical wave equation and he is able to derive a periodic table of elements which is very close to Mendeleev's table. And this is a remarkable achievement which cannot be done with standard wave mechanics.[1]

Nonetheless, equation (14) and (15) which suggests analogy between wave mechanics and sound wave equation has been discussed briefly by Hilbert & Batelaan [14]. And it seems worthy to explore further in experiments.

4. Derivation of Klein-Gordon equation from the Classical Wave equation

It is also possible to find theoretical correspondence between classical electromagnetic wave equation and Klein-Gordon equation. Such a correspondence has been discussed by David Ward & Sabine Volkmer [15]. They give a simple derivation of the KGE, which requires only knowledge of the electromagnetic wave equation and the basics of Einstein's special theory of relativity.

They begin with electromagnetic wave equation in one dimensional case:

$$\frac{\partial^2 E}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2 E}{\partial t^2}=0\,. \tag{17}$$

This equation is satisfied by plane wave solution:

$$E(x,t)=E_0 e^{i(kx-\omega t)}\,, \tag{18}$$

Where $k=\frac{2\pi}{\lambda}$ and $\omega=2\pi\nu$ are the spatial and temporal frequencies, respectively. Substituting equation (18) into (17), then we obtain:

[1]For further discussion, it is advisable to check the website of Dr. George Shpenkov, at http://shpenkov.janmax.com. See especially Shpenkov, George P. 2013. *Dialectical View of the World: The Wave Model (Selected Lectures).* Volume I: Philosophical and Mathematical Background. URL: http://shpenkov.janmax.com/Vol.1.Dialectics.pdf

$$\left(\frac{\partial^2}{\partial x^2}-\frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)E_0e^{i(kx-\omega t)}=0 \tag{19}$$

or

$$\left(k^2-\frac{\omega^2}{c^2}\right)E_0e^{i(kx-\omega t)}=0 \tag{20}$$

Solving the wave vector, we arrive at dispersion relation for light in free space: $k=\frac{\omega}{c}$. Note that this is similar to wave number k in equation (14).

Then, recall from Einstein and Compton that the energy of a photon is $\varepsilon=h\nu=\hbar\omega$ and the momentum of a photon is $p=\frac{h}{\lambda}=\hbar k$. We can rewrite equation (18) using these relations:

$$E(x,t)=E_0e^{\frac{i}{\hbar}(px-\varepsilon t)}, \tag{21}$$

Substituting this equation into (17) we find:

$$-\frac{1}{\hbar^2}\left(p^2-\frac{\varepsilon^2}{c^2}\right)E_0e^{\frac{i}{\hbar}(px-\varepsilon t)}=0 \tag{22}$$

Then we get an expression of relativistic total energy for a particle with zero rest mass:

$$\varepsilon^2=p^2c^2. \tag{23}$$

We now assume with de Broglie that frequency and energy, and wavelength and momentum, are related in the same way for classical particles as for photons, and consider a wave equation for non-zero rest mass particles. So we want to end up with:

$$\varepsilon^2=p^2c^2+m^2c^4. \tag{24}$$

Inserting this equation (24) into equation (22), it is straightforward from (19), that we get:

$$\left(\nabla^2-\frac{m^2c^2}{\hbar^2}\right)\Psi=\frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} \tag{25}$$

which is the Klein-Gordon equation for a free particle [15].

Having derived KGE from classical electromagnetic wave equation, now we are ready to discuss its implication in description of elementary particles. This will be discussed in the next section.

Interestingly, it can be shown that by using KGE one can describe hydrogen atom including electron spin without having to resort to the complicated Dirac equation [16]. It also appears worthnoting here that Meessen workout a description of elementary particles from excitation of spacetime, by starting from KGE and a novel assumption of quantized spacetime dx=n.a.[17] However, we will not discuss Ducharme's and Meessen's approach here, instead we will put more attention on how to extend Acoustic Sachs-Wolfe theorem by virtue of KGE.

5. Acoustic Sachs-Wolfe theorem and its plausible extension

According to Czaja, Golda, and Woszczyna [19], if one considers the acoustic field propagating in the radiation-dominated ($p=\varepsilon/3$) universe of arbitrary space curvature ($K=0,\pm1$), then the field equations are reduced to the d'Alembert equation in an auxiliary static Robertson-Walker spacetime. This is related to the so-called *Sachs-Wolfe acoustic* theorem, which can be found useful in the observation and analysis of Cosmic Microwave Background anisotropies.

In the meantime, there are papers suggesting that the integrated Sachs-Wolfe theorem may be useful to study dark energy, but we do not enter in such a discussion. See [22] for instance.
The Sachs–Wolfe acoustic theorem refers to the spatially flat ($K=0$), hot ($p=\varepsilon/3$) Friedmann–Robertson–Walker universe and the scalar perturbation propagating in it. The theorem states that with the appropriate choice of the perturbation variable, one can express the propagation equation in the form of d'Alembert's equation in Minkowski spacetime. Scalar perturbations in the flat, early universe propagate like electromagnetic or gravitational waves ([18], p. 79).
On the other hand, the wave equation for the scalar field of the dust ($p=0$) cosmological model can be transformed into the d'Alembert equation in the static Robertson–Walker spacetime, regardless of the universe's space curvature (see [18]). Therefore, we can suppose that the flatness assumption in the Sachs–Wolfe theorem is not needed and that the theorem is true in the general case. The proof of this fact, formulated as a symbolic computation, is presented in the first section of this paper.

In accordance with Czaja, Golda, and Woszczyna [19], we begin with Robertson–Walker metrics in spherical coordinates $x^{\sigma}=\{\eta,\chi,\vartheta,\phi\}$:

$$g_{(\mathcal{RW})} = a^2(\eta)\begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \dfrac{\sin^2(\sqrt{K}\chi)}{K} & 0 \\ 0 & 0 & 0 & \dfrac{\sin^2(\sqrt{K}\chi)\sin^2(\vartheta)}{K} \end{bmatrix} \tag{26}$$

with the scale factor a(□) appropriate for the equation of state p=□/3,

$$a(\eta) = \frac{\sin(\sqrt{K}\chi)}{\sqrt{K}}. \tag{27}$$

Let us define a new perturbation variable □ with the help of the second-order differential transformation of the density contrast □,

$$\Psi(x^\sigma) = \frac{1}{\cos(\sqrt{K}\chi)}\frac{\partial}{\partial\eta}\left(\frac{K}{\tan^2(\sqrt{K}\chi)}\frac{\partial}{\partial\eta}\left(\frac{\tan^2(\sqrt{K}\chi)}{K}\cos(\sqrt{K}\chi)\delta(x^\sigma)\right)\right). \tag{28}$$

The function $\Psi(x^{\sigma})$ is the solution of the d'Alembert equation:

$$\frac{\partial^2}{\partial\eta^2}\Psi(x^\sigma) - \frac{1}{3}\Delta\Psi(x^\sigma) = 0, \tag{29}$$

with the Beltrami–Laplace operator Δ acting in this space,

$$^{(3)}g = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{\sin^2(\sqrt{K}\chi)}{K} & 0 \\ 0 & 0 & \dfrac{\sin^2(\sqrt{K}\chi)\sin^2(\vartheta)}{K} \end{bmatrix}. \tag{30}$$

The Beltrami–Laplace operator Δ is defined as follow

$$\Delta = {}^{(3)}g_{mn}\nabla^m\nabla^n. \tag{31}$$

And it can be considered as an extension of Laplace operator for curved space.

Now let us discuss a basic question: what is Laplace-Beltrami operator? In differential geometry, the Laplace operator can be generalized to operate on functions defined on surfaces in Euclidean space and, more generally, on Riemannian and pseudo-Riemannian manifolds. This more general operator goes by the name Laplace-Beltrami operator, after Pierre-Simon Laplace and Eugenio Beltrami. Like the Laplacian, the Laplace-Beltrami operator is defined as the divergence of the gradient, and is a linear operator taking functions into functions. The operator can be extended to operate on tensors as the divergence of the covariant derivative. Alternatively, the operator can be generalized to operate on differential forms using the divergence and exterior derivative. The resulting operator is called the Laplace-de Rham operator (named after Georges de Rham).

Now, considering the formal equivalence between the form of (29) with KGE (25), minus the mass term, then it seems reasonable to include the mass term into (29). Then the extended version of equation (29) may be written as:

$$\frac{\partial^2}{\partial\eta^2}\Psi\left(x^\sigma\right)-\frac{1}{3}\Delta\Psi\left(x^\sigma\right)=-I\frac{m^2c^2}{\hbar^2}\Psi, \tag{32}$$

where I is identity matrix as follows:

$$I=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}. \tag{33}$$

The above equations (32) and (33) can be considered as a plausible extension of Acoustic Sachs-Wolfe theorem based on its analogue with Klein-Gordon equation to become Acoustic Sachs-Wolfe-Christianto-Smarandache-Umniyati (ASWoCSU) equation. Its usefulness remains to be verified with observation data.

6. Discussion and Concluding Remarks

It has been known for long time that the cosmic sound wave was there since the early epoch of the Universe. Signatures of its existence are abound.[24] However, such an acoustic model of cosmology is rarely developed fully into a complete framework from the notion of space, cancer therapy up to the sky. This paper may be the first attempt towards such a complete description of the Universe based on classical wave equation of sound.

We have discussed how the very definition of Newtonian space can be related to sound wave and also Maxwell equations, and also how fluid Maxwell equations can be formulated based on vortex sound theory.

We have also discussed the inadequacies of Schrodinger equation as a description of elementary particles, instead we established connection from classical electromagnetic wave equation to Klein-Gordon equation.

Then we discuss Acoustic Sachs-Wolfe theorem which is worthy to investigate further in the context of cosmology. We also propose an extension of Acoustic Sachs-Wolfe to become a new equation. In other words, it appears very reasonable to model the Universe and Cosmos in general in terms of sound wave equation.

To summarize, in this paper we tried our best to offer a novel picture of the Universe as a guitar. Further observation and experiments are recommended to verify the above propositions.

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A Derivation of Fluidic Maxwell-Proca Equations for Electrodynamics of Superconductors & Its Implication to Chiral Cosmology Model

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Abstract

Mario Liu described a hydrodynamic Maxwell equations [3] and, also discussed potential implications of these new approaches to superconductors which were made after Tajmar's paper [4]. In this paper, we present for the first time a derivation of *fluidic* Maxwell-Proca equations. The name of *fluidic* Maxwell-Proca is proposed because the equations were based on modifying Maxwell-Proca and Hirsch's theory of electrodynamics of superconductor. It is hoped that this paper may stimulate further investigations and experiments in superconductor. It may be expected to have some impact to cosmology modeling too, for instance we consider a hypothetical argument that photon mass can be origin of gravitation. Then, after combining with the so-called chiral modification of Maxwell equations (after Spröessig), then we consider chiral Maxwell-Proca equations as possible alternative of gravitation theory. Such a hypothesis has never considered in literature to the best of our knowledge.

Keywords: Hirsch theory, London equations, hydrodynamics Maxwell equations, Proca equations, electrodynamics, superconductor, chiral medium, chiral gravitation theory.

1. Introduction

According to J.E. Hirsch, from the outset of superconductivity research it was assumed that no electrostatic fields could exist inside superconductors and this assumption was incorporated into conventional London electrodynamics [2]. Hirsch suggests that there are difficulties with the two London equations. To summarize, London's equations together with Maxwell's equations lead to *unphysical* predictions [1]. Hirsch also proposes a new model for electrodynamics for superconductors [1-2].

In this regard, in a rather old paper, Mario Liu described a hydrodynamic Maxwell equations [3]. While he also discussed potential implications of these new approaches to superconductors, such a discussion of electrodynamics of superconductors is made only after Tajmar's paper.

Therefore, in this paper we present for the first time a derivation of *fluidic* Maxwell-Proca equations. The name of Maxwell-Proca is proposed because the equations were based on modifying Maxwell-Proca and Hirsch's theory of electrodynamics of superconductor.

Therefore, the aim of the present paper is to propose a version of *fluidic* Maxwell-Proca model for electrodynamics of superconductor. It is hoped that this paper may stimulate further investigations and experiments in particular for fractal superconductor. It may be expected to have some impact to cosmology modeling too, which will be discussed in the last section.

2. Hirsch's proposed revision of London's equations

According to J.E. Hirsch, from the outset of superconductivity research it was assumed that no electrostatic fields could exist inside superconductors and this assumption was incorporated into conventional London electrodynamics.[2] Hirsch suggests that there are difficulties with the two London equations. Therefore he concludes that London's equations together with Maxwell's equations lead to *unphysical* predictions.[1] However he still uses four- vectors J and A according to Maxwell's equations:

$$\Box^2 A = -\frac{4\pi}{c} J, \tag{1}$$

and

$$J - J_0 = -\frac{c}{4\pi\lambda_L^2}(A - A_0). \tag{2}$$

Therefore, Hirsch proposes a new fundamental equation for electrodynamics for superconductors as follows: [1]

$$\Box^2 (A - A_0) = \frac{1}{\lambda_L^2}(A - A_0), \tag{3a}$$

where

- London penetration depth λ_L is defined as follows:[2]

$$\frac{1}{\lambda_L^2} = \frac{4\pi n_s e^2}{m_e c^2}, \tag{3b}$$

- And d'Alembertian operator is defined as: [1]
-

$$\Box^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial_t^2} \quad . \tag{3c}$$

Then he proposes the following equations: [1]

$$\Box^2(F - F_0) = \frac{1}{\lambda_L^2}(F - F_0), \tag{4}$$

and

$$\Box^2(J - J_0) = \frac{1}{\lambda_L^2}(J - J_0), \tag{5}$$

where F is the usual electromagnetic field tensor and F_0 is the field tensor with entries $\vec{E}_0$ and 0 from $\vec{E}$ and $\vec{B}$ respectively when expressed in the reference frame at rest with respect to the ions.

Proca equations can be considered as an extension of Maxwell equations, and they have been derived in various ways, see for instance [4, 6, 7]. It can be shown that Proca equations can be derived from first principles [6], and also that Proca equations may have link with Klein-Gordon equation [7]. However, in this paper we will not attempt to re-derive Proca equations. Instead, I will use Proca equations as described in [6].

In the meantime, it is known that Proca equations can also be used to describe electrodynamics of superconductors, see [4]-[8]. The difference between Proca and Maxwell equations is that Maxwell equations and Lagrangian are based on the hypothesis that the photon has zero mass, but the Proca's Lagrangian is obtained by adding mass term to Maxwell's Lagrangian.[17] Therefore, the Proca equation can be written as follows:[17]

$$\partial_\mu F^{\mu\nu} + m_\gamma{}^2 A_\nu = \frac{4\pi}{c} J^\nu, \tag{6a}$$

where $m_\gamma = \frac{\omega}{c}$ is the inverse of the Compton wavelength associated with photon mass. [17] In terms of the vector potentials, equation (6a) can be written as [17]:

$$(\Box + m_\gamma) A_\mu = \frac{4\pi}{c} J_\mu. \tag{6b}$$

Similarly, according to Kruglov [15] the Proca equation for a free particle processing the mass m can be written as follows:

$$\partial_\nu \varphi_{\mu\nu}(x) + m^2 \varphi_\mu(x) = 0, \tag{7}$$

Now, the similarity between equations (1) and (6b) are remarkable with exception that equation (1) is in quadratic form. Therefore we propose to consider a modified form of Hirsch's model as follows:

$$(\Box^2 - m_\gamma^2)(F - F_0) = \frac{1}{\lambda_L^2}(F - F_0), \tag{8a}$$

and

$$(\square^2 - m_\gamma^2)(J - J_0) = \frac{1}{\lambda_L^2}(J - J_0). \tag{8b}$$

The relevance of the proposed new equations in lieu of (4)-(5) should be verified by experiments with superconductors [16]. For convenience, the equations (8a)-(8b) can be given a name: Maxwell-Proca-Hirsch equations.

3. Fluidic Maxwell-Proca Equations

In this regard, in a rather old paper, Mario Liu described a hydrodynamic Maxwell equations.[3] While he also discussed potential implications of these new approaches to superconductors, such a discussion of electrodynamics of superconductors is made only after Tajmar's paper. Therefore, in this section we present for the first time a derivation of *fluidic* Maxwell-Proca equations.

According to Blackledge, Proca equations can be written as follows [7]:

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} - \kappa^2 \phi, \tag{9}$$

$$\nabla \cdot \vec{B} = 0, \tag{10}$$

$$\nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}, \tag{11}$$

$$\nabla \times \vec{B} = \mu_0 j + \varepsilon_0 \mu_0 \frac{\partial \vec{E}}{\partial t} + \kappa^2 \vec{A}, \tag{12}$$

where:

$$\nabla \phi = -\frac{\partial \vec{A}}{\partial t} - \vec{E}, \tag{13}$$

$$\vec{B} = \nabla \times \vec{A}, \tag{14}$$

$$\kappa = \frac{m c_0}{\hbar}. \tag{15}$$

Therefore, by using the definitions in equations (9)-(12), and by comparing with hydrodynamic Maxwell equations of Liu [3, eq. 2], now we can arrive at *fluidic* Maxwell-Proca equations, as follows:

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0} - \kappa^2 \phi, \tag{16}$$

$$\nabla \cdot \vec{B} = 0, \tag{17}$$

$$\dot{B} = -\nabla \times E - \nabla \times \left(\hat{\beta} \nabla \times H_0\right), \tag{18}$$

$$\varepsilon_0 \mu_0 \dot{E} = \nabla \times B - \mu_0 j - \kappa^2 A - \left(\hat{\sigma} E_0 + \rho_e v + \hat{\gamma} \nabla T\right) - \nabla \times \left(\hat{a} \nabla \times E_0\right), \tag{19}$$

where:

$$\nabla \phi = -\frac{\partial \vec{A}}{\partial t} - \vec{E}, \tag{20}$$

$$\vec{B} = \nabla \times \vec{A}, \tag{21}$$

$$\kappa = \frac{m c_0}{\hbar}. \tag{22}$$

Since according to Blackledge, the Proca equations can be viewed as a *unified wavefield* model of electromagnetic phenomena [7], therefore we can also regard the fluidic Maxwell-Proca equations as a *unified wavefield* model for electrodynamics of superconductor.

Now, having defined fluidic Maxwell-Proca equations, we are ready to write down fluidic Maxwell-Proca-Hirsch equations using the same definition, as follows:

$$(\square_\alpha^2 - \kappa^2)(F - F_0) = \frac{1}{\lambda_L^2}(F - F_0), \tag{23}$$

And

$$(\square_\alpha^2 - \kappa^2)(J - J_0) = \frac{1}{\lambda_L^2}(J - J_0), \tag{24}$$

where

$$\square_\alpha^2 = \nabla^{\alpha 2} - \frac{1}{c^2} \frac{\partial^{\alpha 2}}{\partial_t^{\alpha 2}}. \tag{25}$$

As far as we know, the above fluidic Maxwell-Proca equations have never been presented elsewhere before. Provided the above equations can be verified with experiments, they can be used to describe electrodynamics of superconductors.

As a last note, it seems interesting to remark here that Kruglov [15] has derived a square-root of Proca equations as a possible model for hadron mass spectrum, therefore perhaps equations (23)-(25) may be *factorized* too to find out a model for hadron masses. Nonetheless, we leave this problem for future investigations.

4. Towards Chiral Cosmology model

The Maxwell-Proca electrodynamics corresponding to a finite photon mass causes a substantial change of the Maxwell stress tensor and, under certain circumstances, may cause the electromagnetic stresses to act effectively as "*negative pressure*." In a recent paper, Ryutov, Budker, Flambaum suggest that such a negative pressure imitates gravitational pull, and may produce effect similar to gravitation. In the meantime, there are other papers by Longo, Shamir etc. discussing observations indicating handedness of spiral galaxies, which seem to suggest chiral medium at large scale. However, so far there is no derivation of Maxwell-Proca equations in chiral medium.

In a recent paper, Ryutov, Budker, Flambaum suggest that Maxwell-Proca equations may induce a negative pressure imitates gravitational pull, and may produce effect similar to gravitation.[18]

In the meantime, there are other papers by Longo, Shamir etc. discussing observations indicating handedness of spiral galaxies, which seem to suggest chiral medium at large scale. As Shamir reported:

> "*A morphological feature of spiral galaxies that can be easily identified by the human eye is the handedness—some spiral galaxies spin clockwise, while other spiral galaxies rotate counterclockwise. Previous studies suggest large-scale asymmetry between the number of galaxies that rotate clockwise and the number of galaxies that rotate counterclockwise, and a large-scale correlation between the galaxy handedness and other characteristics can indicate an asymmetry at a cosmological scale*."[24]

However, so far there is no derivation of Maxwell-Proca equations in chiral medium. Therefore, inspired by Ryutov et al.'s paper, in this paper, we present for the first time a possibility to extend Maxwell-Proca-type equations to chiral medium, which may be able to explain origin of handedness of spiral galaxies as reported by M. Longo et al.[23. 24. 24a]

The present paper is intended to be a follow-up paper of our preceding paper, reviewing Shpenkov's interpretation of classical wave equation and its role to explain periodic table of elements and other phenomena [11][13][22].

5. Maxwell-Proca Equations in Chiral Medium

It shall be noted, that the relations between flux densities and the electric and magnetic fields depend on the material. It is well-known that for instance all organic materials contain carbon and realize in this way some kind of optical activity. Therefore, Lord Kelvin introduced the notion of the chirality measure of a medium. This coefficient expresses the optical activity of the underlying material. The correspondent constitutive laws are the following:[19]

$$D = \varepsilon E + \varepsilon\beta \text{ rot } E \text{ (Drude-Born-Feodorov laws)}, \tag{26}$$

$$B = \mu H + \mu\beta \text{ rot } H \tag{27}$$

where e = E(t, x) is the electric permittivity, j = p(t, x) is the magnetic permeability and the coefficient β describes the *chirality measure* of the material.[2]

Now, since we want to obtain Maxwell-Proca equations in chiral medium, then eq. (12) should be replaced with eq. (26). But such a hypothetical assertion should be investigated in more precise way.

Since according to Blackledge, the Proca equations can be viewed as a unified wavefield model of electromagnetic phenomena [7], then we can also regard the Maxwell-Proca equations in chiral medium as a further generalization of his *unified wavefield picture* into the realm of superconductors and may be also in cosmology modeling too.

6. Conclusion

One of our aims with the present paper is to propose a combined version of London-Proca-Hirsch model for electrodynamics of superconductor. Considering that Proca equations may be used to explain electrodynamics in superconductor, the proposed fluidic London-Proca equations may be able to describe electromagnetic of superconductors. It is hoped that this paper may stimulate further investigations and experiments in particular for superconductors. It may be expected to have some impact to cosmology modeling too.

Another purpose is to submit a new model of gravitation based on a recent paper by Ryutov, Budker, Flambaum, who suggest that Maxwell-Proca equations may induce a negative pressure imitates gravitational pull, and may produce effect similar to gravitation. In the meantime, there are other papers by Longo, Shamir etc. discussing observations indicating handedness of spiral galaxies, which seem to suggest chiral medium at large scale (as we know, there are recent findings on carbon molecules from the outer space or interstellar medium too).
.
However, so far there is no derivation of Maxwell-Proca equations in chiral medium. In this paper, we propose Maxwell-Proca-type equations in chiral medium, which may also explain (albeit hypothetically) origin of handedness of spiral galaxies as reported by M. Longo et al.

It may be expected that one can describe handedness of spiral galaxies by chiral Maxwell-Proca equations. This would need more investigations, both theoretically and empirically.

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Remark on vacuum fluctuation as the cause of Universe creation: Or How Neutrosophic Logic and Material Point Method may Resolve Dispute on the Origin of the Universe through re-reading Gen. 1:1-2

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Victor Christianto, Florentin Smarandache (2018). Remark on vacuum fluctuation as the cause of Universe creation: Or How Neutrosophic Logic and Material Point Method may Resolve Dispute on the Origin of the Universe through re-reading Gen. 1:1-2. *Asia Mathematika* 2(2), 1-11

Abstract: Questions regarding the formation of the Universe and what was there before the existence of Early Universe have been great interest to mankind of all times. In recent decades, the Big Bang as described by the Lambda CDM-Standard Model Cosmology has become widely accepted by majority of physics and cosmology communities. Among other things, we can cite A.A. Grib Pavlov who pointed out some problems of heavy particles creation out of vacuum and also other proposal of Creatio ex nihilo theory (CET). But the philosophical problems remain, as Vaas pointed out: Did the universe have a beginning or does it exist forever, i.e. is it eternal at least in relation to the past? This fundamental question was a main topic in ancient philosophy of nature and the Middle Ages, and still has its revival in modern physical cosmology both in the controversy between the big bang and steady state models some decades ago and in the contemporary attempts to explain the big bang within a quantum cosmological (vacuum fluctuation) framework. In this paper we argue that Neutrosophic Logic offers a resolution to the long standing disputes between beginning and eternity of the Universe. In other words, in this respect we agree with Vaas, i.e. it can be shown: "how a conceptual and perhaps physical solution of the temporal aspect of Immanuel Kant's „first antinomy of pure reason" is possible, i.e. how our universe in some respect could have both a beginning and an eternal existence. Therefore, paradoxically, there might have been a time before time or a beginning of time in time." By the help of computational simulation, we also show how a model of early Universe with rotation can fit this new picture. Further observations are recommended.

Key words: Big Bang, Steady state, rotating universe, fluid, singularity-free, cosmology model, early Universe, the book of Genesis, Spirit, Creation.

1. Introduction

Questions regarding the formation of the Universe and what was there before the existence of Early Universe have been great interest to mankind of all times. In recent decades, the Big Bang as described by the Lambda CDM-Standard Model Cosmology has become widely accepted by majority of physics and cosmology communities. Among other things, we can cite A.A. Grib Pavlov who pointed possible heavy particles creation out of vacuum and also other proposal such as Creatio Ex-Nihilo theory (CET).[36-37]

But the philosophical problems remain, as Vaas pointed out: Did the universe have a beginning or does it exist forever, i.e. is it eternal at least in relation to the past? This fundamental question was a main topic in ancient philosophy of nature and the Middle Ages. Philosophically it was more or less banished then by Immanuel Kant's Critique of Pure Reason. But it used to have and still has its revival in modern physical cosmology both in the controversy between the big bang and steady state models some decades ago and in the contemporary attempts to explain the big bang within a quantum cosmological framework.

Interestingly, Vaas also noted that Immanuel Kant, in his Critique of Pure Reason (1781/1787), argued that it is possible to prove both that the world has a beginning and that it is eternal (first antinomy of pure reason, A426f/B454f). As Kant believed he could overcome this Widerspruch "self-contradiction of reason" ("der Vernunft mit ihr selbst", A740) by what he called „transcendental idealism", the question whether the cosmos exists forever or not has almost vanished in philosophical discussions. [3]

Further problems also remain with the Big Bang theories, such as: a) what force was responsible to trigger the first creation of heavy particles out of vacuum?, b) if we take the statistical approach, i.e. the vacuum fluctuation alone was responsible for first creation, then one can ask how much is probability of such statistical chance to create all regularities as we observe of the Universe? (such as Tifft's quantized redshift data.)

In this paper we will take a closer look at Genesis 1:2 to see whether the widely-accepted notion of creatio ex-nihilo is supported by Hebrew Bible or not. It turns out that Neutrosophic Logic is in agreement with Kant and Vaas's position, it offers a resolution to the long standing disputes between beginning and eternity of the Universe. In other words, in this respect we agree with Vaas: "how a conceptual and perhaps physical solution of the temporal aspect of Immanuel Kant's „first antinomy of pure reason" is possible, i.e. how our universe in some respect could have both a beginning and an eternal existence. Therefore, paradoxically, there might have been a time before time or a beginning of time in time."[3] In the subsequent chapter we will discuss how to answer this question by the lens of hermeneutics of Sherlock Holmes. This is a tool of mind which we think to be a better way compared to critical hermeneutics.

1.1. What is Hermeneutics of Sherlock Holmes?

One article suggests: Holmes: "I have no data yet. It is a capital mistake to theorize before one has data. Insensibly one begins to twist facts to suit theories, instead of theories to suit facts." Far too often students of the Bible (and cosmology folks as well) twist verses to suit interpretations instead of formulating interpretations to suit what the verses say. Guide: Don't approach your passage assuming you know what it means. Rather, use the data in the passage – the words that are used and how they fit together – to point you toward the correct interpretation.

2. A closer look at Genesis 1:1-2 implications

One of the biggest mysteries in cosmogony and cosmology studies is perhaps: How to interpret properly Genesis chapter 1:2. Traditionally, philosophers proposed that God created the Universe out of nothingness (from reading "empty and formless" and "bara" words; this contention is called "creatio ex nihilo."). Understandably, such a model can lead to various interpretations, including the notorious "cosmic egg" (primeval atom) model as suggested by Georges Lemaitre, which then led to Big Bang model.[18-20] Subsequently, many cosmologists accept it without asking, that Big Bang stands as the most faithful and nearest theory to Biblical account of creation. But we can ask: Is that primeval atom model the true and faithful reading of Genesis 1:2? Let us start our discussion with examining key biblical words of Hebrew Bible, especially Genesis 1:1-2. It can be shown that the widely accepted creation ex nihilo is a post-biblical invention, rather than as faithful reading of the verses. To quote Ian Barbour: "Creation out of nothing is not a biblical concept."[4] Let us consider some biblical passages:

2.1. bereishit

The literal meaning of Gen. 1:1, "bareishit bara Elohim." This very first statement of the book of Genesis literally reads: 'first' and 'beginning' are reasonable alternatives for the Hebrew noun, reishit. Also note that in Hebrew, subjects and verbs are usually ordered verb-first (unlike English in which the subject is written first). If the verb and subject of this verse are reordered according to natural English grammar we read: [1] In, When first, beginning Elohim created. . . reishit: The noun, reishit, has as its root the letters, (Resh -Aleph-Shin). Words derived from this root often carry the meaning of 'primary', 'chief', 'begin', 'first' or "first-in-line", "head of", and so forth. Harris's Theological Wordbook of the Old Testament (TWOT) is more specific, namely, reishit means[1] ". . . first, beginning, choicest, first or best of a group. [Reishit is] a feminine noun derived from the root [Resh-Aleph-Shin], it appears fifty times in nearly all parts of the [Old Testament]. [Its] primary meaning is "first" or "beginning" of a series." Accordingly, we can now retranslate breishit bara Elohim as "When first created Elohim", or as we would render in English,[1] "When Elohim first created. . . "

2.2. Gen. 1:2

Gen. 1:2, "And the earth had been." In English this is easily handled by the past perfect tense (also called the pluperfect or the "flashback" tense). Likewise, if haytah in v 1:2 is translated as a past perfect verb, then verses 1:1-2 would read,[1] When Elohim first created the heavens and the earth, the earth had been . . . In this translation the universe, in some form or other, was already in existence when God executed His first creative act, the creation of light.

2.3. Re-reading Gen. 2:7 with Hermeneutics of Sherlock Holmes?

If we glance at Gen. 2: 7, we see at a glance that man is made up of the dust of the ground (adamah) which is breathed by the breath of life by God (nephesh). Here we can ask, does this text really support the Cartesian dualism view? We do not think so, because the Hebrew concept of man and life is integral. The bottom line: it is not the spirit trapped in the body (Platonic), but the body is flowing in the ocean of spirit. (check also Eric McKiddie's article: https://www.thegospelcoalition.org/blogs/trevin-wax/10-tips-on-solving-mysterious-bible-passages-from-sherlock-holmes/)

2.4. Gen. 1:2, formless and void

Gen. 1: 2, "The earth is without form and void, darkness over the deep, and the Spirit of God hovering over the waters." Patterns such as Adam's creation can also be encountered in the creation story of the universe. Earth and the oceans already exist (similar to adamah), but still empty and formless. Then the Spirit of God hovered over it, in the original text "ruach" can be interpreted as a strong wind (storm). So we can imagine there is wind/ hurricane, then in the storm that God said, and there was the creation of the universe. See also Amos Yong [6], also Hildebrandt [15]. From a scientific point of view, it is well known in aerodynamics that turbulence can cause sound (turbulence-generated sound). And primordial sound waves are indeed observed by astronomers.

2.5. Psalm 107

Ps. 107: 25, "He said, he raised up a storm that lifted up his waves." The relation between the word (sound) and the storm (turbulence) is interactive. Which one can cause other. That is, God can speak and then storms, or the Spirit of God causes a storm. Then came the voice.

2.6. Ezekiel 37

Ezekiel. 37: 7, "Then I prophesy as I am commanded, and as soon as I prophesy, it sounds, indeed, a crackling sound, and the bones meet with one another." In Ezekiel it appears that the story of the creation of Adam is repeated, that the Spirit of God is blowing (storm), then the sound of the dead bones arises.

2.7. Conclusion to re-reading the above three verses

The conclusion of the three verses above seems to be that man is made up of adamah which is animated by the breath or Spirit of God. He is not matter, more accurately referred to as spirit in matter. In other words, a close reading of Hebrew Bible seems to suggest that creatio ex-nihilo is a post-biblical invention. Other scholars have suggested an alternative concept, called creatio ex-materia, but many orthodox Christian scholars have raised objection to this notion, partly because the term seems to undermine God's ultimate power and control of the Universe. Besides, the notion of creatio ex-materia has been advocated by Mormon preachers. To overcome this problem, and based on what we learned recently, allow us now to come up with a new term: creatio ex-rotatione (rotatione is a Latin word for rotation). As we shall see in the next chapter, it is possible to come up with a physical model of early Universe with rotation, where the raw materials have been existed for long period of time, but suddenly it burst out into creation. And it seems to fit with Kant's idea to resolve the dichotomy between finite past or eternal Universe. Furthermore, it can be shown that the model naturally leads to accelerated expansion, without having to invoke ad hoc assumption like dark energy or cosmological constant.

3. A physical model of turbulence-generated sound for early Universe

Our discussion starts from the fundamental question: how can we include the rotation in early Universe model? After answering that question, we will discuss how "turbulence-generated sound" can be put into a mathematical model for the early Universe. We are aware that the notion of turbulence-generated sound is not new term at all especially in aerodynamics, but the term is rarely used in cosmology until now. We shall show that 3D Navier-Stokes will lead to non-linear acoustics models, which means that a turbulence/storm can generate sound wave.

3.1. How can we include rotation in early Universe model?

It has been known for long time that most of the existing cosmology models have singularity problem. Cosmological singularity has been a consequence of excessive symmetry of flow, such as "Hubble's law". More realistic one is suggested, based on Newtonian cosmology model but here we include the vortical-rotational effect of the whole Universe. In this section, we will derive an Ermakov-type equation following Nurgaliev [8]. Then we will solve it numerically using Mathematica 11. After he proceeds with some initial assumptions, Nurgaliev obtained a new simple local cosmological equation:[8][9]

$$\dot{H} + H^2 = \Omega + (4.\pi.G.\rho)/3$$

where

$$\dot{H} = dH/dt.$$

The angular momentum conservation law $\Omega.R^2 = const = K and the mass conservation law makes equation solvable:$ [9]

$$\dot{H} + H^2 = (K^2/R^4) - (GM)/R^3$$

or

$$\ddot{R} = (K^2/R^3) - (GM)/R^2$$

Equation above may be written as Ermakov-type nonlinear equation as follows:

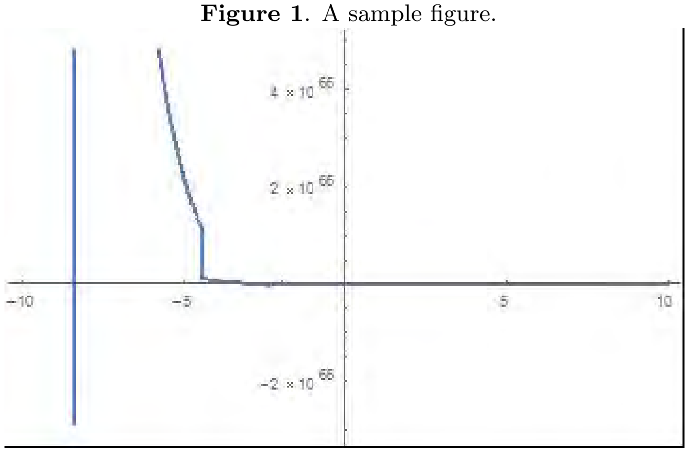
$$\ddot{R} + (GM)/R^2 = (K^2/R^3)$$

Nurgaliev tried to integrate equation (3), but now we will solve the above equation with Mathematica 11. First, we will rewrite this equation by replacing GM=A, K $^2 = B, soweget:$

$$\ddot{R} + A/R^2 = B/R^3$$

As with what Nurgaliev did in [8][9], we also tried different sets of A and B values, as follows: a. A and B $\leqslant 0 A = -10; B = -10; ODE = x''[t]{+}A/x[t]^2{-}B/x[t]^3 == 0; sol = NDSolve[ODE, x[0] == 1, x'[0] == 1, x[t], t, -10, 10] Plot[x[$

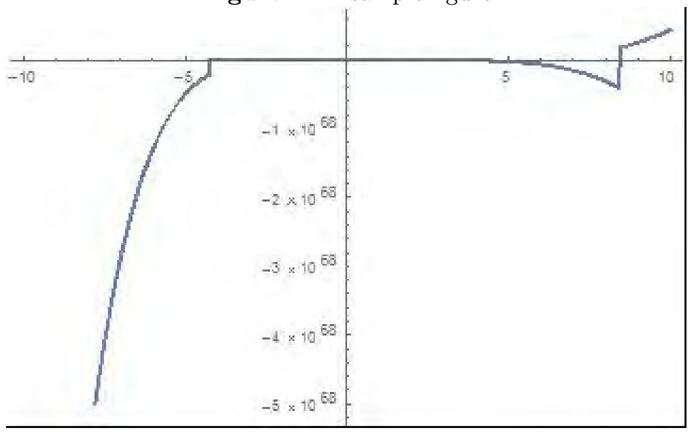
Figure 1. A sample figure.



b. A $\geqslant 0, B \leqslant 0 A = 0 B = -10; ODE = x''[t]{+}A/x[t]^2{-}B/x[t]^3 == 0; sol = NDSolve[ODE, x[0] == 1, x'[0] == 1, x[t],$

From the above numerical experiments, we conclude that the evolution of the Universe depends on the constants involved, especially on the rotational-vortex structure of the Universe. This needs to be investigated in more detailed for sure. One conclusion that we may derive especially from Figure 2, is that our computational simulation suggests that it is possible to consider that the Universe has existed for long time in prolonged stagnation period, then suddenly it burst out from empty and formless (Gen. 1:2), to take its current shape with accelerated expansion. As an implication, we may arrive at a precise model of flattening velocity of galaxies without having to invoke ad-hoc assumptions such as dark matter. Therefore, it is perhaps noteworthy to discuss briefly a simple model of galaxies based on a postulate of turbulence vortices which govern the galaxy dynamics. The result of Vatistas' model equation can yield prediction which is close to observation [14]. Therefore it appears possible to model galaxies without invoking numerous ad hoc assumptions such as dark matter, once we accept the existence of turbulent interstellar medium. The Vatistas model is also governed by Navier-Stokes equations, see for instance [14].

Figure 2. A sample figure.



3.2. How "turbulence-generated sound" can be put into a mathematical model for the early Universe

We are aware that the notion of turbulence-generated sound is not new term at all especially in aerodynamics, but the term is rarely used in cosmology until now. We will consider some papers where it can be shown that 3D Navier-Stokes will lead to non-linear acoustics models, which means that a turbulence/storm can generate sound wave. In this section we consider only two approaches:

3.2.1. Shugaev-Cherkasov-Solenaya's model

They investigate acoustic radiation emitted by three-dimensional (3D) vortex rings in air on the basis of the unsteady Navier–Stokes equations. Power series expansions of the unknown functions with respect to the initial vorticity which is supposed to be small are used. In such a manner the system of the Navier–Stokes equations is reduced to a parabolic system with constant coefficients at high derivatives. [16]

3.2.2. Rozanova-Pierrat's Kuznetsov equation

she analysed the existing derivation of the models of non-linear acoustics such as the Kuznetsov equation, the NPE equation and the KZK equation. The technique of introducing a corrector in the derivation ansatz allows to consider the solutions of these equations as approximations of the solution of the initial system (a compressible Navier-Stokes/Euler system). The direct derivation shows that the Kuznetzov equation is the first order approximation of the Navier-Stokes system, the KZK and NPE equations are the first order approximations of the Kuznetzov equation and the second order approximations of the Navier-Stokes system. [17]

4. Neutrosophic Logic perspective and implications

In the previous sections we have discussed how closer look at Gen. 1:1-2 leads to different scenario than the widely accepted creation ex-nihilo. This new scenario is quite in agreement with Kant's idea that it is possible that the Universe has both finite history in the past and also eternal background. We also discussed how such a mixed view can be modelled by introducing rotation in the early universe. Now there is an immediate question: Is this new look at the origin of Universe justifiable logically, or is it merely a compromised solution? So, in this chapter we will review Neutrosophic Logic, a new theory developed in recent decades by one of these authors

(FS). Vern Poythress argues that sometimes we need a modification of basic philosophy of mathematics, in order to re-define the redeemed mathematics; see [21]. In this context, allow us to argue in favor of Neutrosophic logic as one basic postutale, in lieu of the Aristotle logic which creates many problems in real world. In Neutrosophy, we can connect an idea with its opposite idea and with its neutral idea and get common parts, i.e. ¡A¿ / ¡nonA¿ = nonempty set. The common part of the uncommon things! It is true/real... paradox. From neutrosophy, all started: neutrosophic logic, neutrosophic set, neutrosophic probability, neutrosophic statistics, neutrosophic measure, neutrosophic physics, neutrosophic algebraic structures etc. It is true in restricted case, i.e. the Hegelian dialectics considers only the dynamics of opposites (¡A¿ and ¡antiA¿), but in our everyday life, not only the opposites interact, but the neutrals ¡neutA¿ between them too. For example: you fight with a man (so you both are the opposites). But neutral people around both of you (especially the police) interfere to reconcile both of you. Neutrosophy considers the dynamics of opposites and their neutrals. So, neutrosophy means that: ¡A¿, ¡antiA¿ (the opposite of ¡A¿), and ¡neutA¿ (the neutrals between ¡A¿ and ¡antiA¿) interact among themselves. A neutrosophic set is characterized by a truth-membership function (T), an indeterminacy-membership function (I), and a falsity-membership function (F), where T, I, F are subsets of the unit interval [0, 1]. As particular cases we have: single-valued neutrosophic set when T, I, F are crisp numbers in [0, 1], and interval-valued neutrosophic set when T, I, F are intervals included in [0, 1]. Neutrosophic Set is a powerful structure in expressing indeterminate, vague, incomplete and inconsistent information. See also [22]-[24]. To summarize, Neutrosophic Logic study the dynamics of neutralities. And from this viewpoint, we can understand that it is indeed a real possibility that the Universe has both initial start (creation) but with eternal background. This is exactly the picture we got after our closer look at Gen. 1:1-2 as discussed in the above section. In other words, our proposed term of "creatio ex-rotatione" has sufficient logical background.

5. Advantages of "creatio ex-rotatione" concept

In the preceding section, we have discussed on how our proposed term of "creatio ex-rotatione" has sufficient logical background. Now, allow us to discuss some advantages of the proposed "creatio ex-rotatione" cosmology view over the Lemaitre's primeval atom (which is the basis of Standard Model Cosmology).

5.1. Avoid inflationary scheme

It is known that inflationary models were proposed by Alan Guth et al. (see [25][26]), in order to explain certain difficulties in the Big Bang scenario. But some cosmology experts such as Hollands Wald has raised some difficulties with inflationary model, as follows: "We argue that the explanations provided by inflation for the homogeneity, isotropy, and flatness of our universe are not satisfactory, and that a proper explanation of these features will require a much deeper understanding of the initial state of our universe."[27] In our diagram plot above, it is clear that an early rotation model can explain why the Universe can burst out into creation in a very short period, without invoking ad hoc postulate such as inflation model.

5.2. Explain the observed late accelerated expansion.

As far as we know, one of the earliest models which gave prediction of accelerated expanding Universe is Carmeli's Cosmological General Relativity.[29] But it has been shown by Green Wald that for the large scale structures of the Universe, Newtonian model can give similar results compared to general relativity picture.[28] Furthermore, it seems that there is no quite clear arguments why we should accept Carmeli use of 5D metric model (space-time-velocity metric). In the meantime, in our rotating Universe model, we do not invoke ad hoc dimension into the metric.

5.3. Explain inhomogeneity, breeding galaxies etc.

Astronomers have known for long time, that the Universe is not homogeneous and isotropic as in the usual model. It contains of inhomogeneity, irregularity, clumpiness, voids, filaments etc, which indicate complex structures. Such inhomogeneous structures may be better modelled in terms of turbulence model such as Navier-Stokes equations, see also our early papers [11][12]. Furthermore, observations clearly suggest that matter ejected continuously in galaxy centers, which view is difficult to reconcile with Big Bang scenario of galaxy creation. From our point of view, the Hubble's law indicates that galaxies move fast from each other were because of continuous matter ejection, which effect seems to be a direct refutation of singularity point in the beginning of the Universe. This is in opposite to mainstream view that Hubble's law support Big Bang theory. It is also interesting remark that Edwin Hubble himself remained rather refrained from attributing his "law" to support the expanding Universe hypothesis. Remark on his seminal paper: "The features, however, include the phenomena of red-shifts whose significance is still uncertain. Alternative interpretations are possible, and, while they introduce only minor differences in the picture of the observable region, they lead to totally different conceptions of the universe itself. One conception, at the moment, seems less plausible than the other, but this dubious world, the expanding universe of relativistic cosmology, is derived from the more likely of the two interpretations of red-shifts. Thus the discussion ends in a dilemma, and the resolution must await improved observations or improved theory or both." See [34][35]. Added note: Moreover, for years, some researchers have developed a novel theory of gravity based on an old theory of Le Sage/Laplace (it is known as Le Sage's gravitation theory). An interesting remark on impetus to Le Sage gravitation theory can be found in article by the late Prof. Halton Arp on his work with Narlikar: "Nevertheless the ball had started rolling down hill so to speak and in 1991, with Narlikar's help, I outlined in Apeiron the way in which particle masses growing with time would account for the array of accumulated extragalactic paradoxes. Later Narlikar and Arp (1993) published in the Astrophysical Journal Narlikar's original, 1977 solution of the basic dynamical equations along with the Apeiron applications to the quasar/galaxy observations. The first insight came when I realized that the Friedmann solution of 1922 was based on the assumption that the masses of elementary particles were always and forever constant, m = const. He had made an approximation in a differential equation and then solved it. This is an error in mathematical procedure. What Narlikar had done was solve the equations for m = f(x,t). This a more general solution, what Tom Phipps calls a covering theory. ... But Narlikar had overwhelmed me with the beauty of the variable mass solution by showing how the local dynamics could be recovered by the simple conformal transformation from t time (universal) to what we called time (our galaxy) time. The advertisement here was that our solution inherited all the physics triumphs much heralded in general relativity but also accounted for the non-local phenomena like quasar and extragalactic redshifts."[2] Summarizing, it is very significant to consider matter creation process in nature. For instance, one can begin by considering the correct presentation of Newton's third law is not F=ma, but F=d(mv)/dt=v(dm/dt) + m(dv/dt). In other words, it is possible of matter creation (dm/dt), and this is consistent with Narlikar's work. We will explore this effect of receding Moon from Earth, in calculations to be presented in a forthcoming paper.

CONCLUSION

In this paper we argue that Neutrosophic Logic offers a resolution to the long standing disputes between beginning and eternity of the Universe. In other words, in this respect we agree with Vaas, i.e. it can be shown: “how a conceptual and perhaps physical solution of the temporal aspect of Immanuel Kant’s first antinomy ” of pure reason“ is possible, i.e. how our universe in some respect could have both a beginning and an eternal existence. Therefore, paradoxically, there might have been a time before time or a beginning of time in time.” We argue that a re-reading of Genesis 1:2 will lead us to another viable story, albeit the alternative has not been developed rigorously as LCDM theories. It took around three years before now we have been thinking this problem out loud, and here our answer can be summarized as follows: “The relic sound wave in early creation is a faithful interpretation of John 1:1, but we can come up with a more complete picture if we combine it with Gen. 1:2, that is the Holy Spirit came to hovering over the primordial fluid, then a kind of hurricane/storm started which created perfect medium where God spoke (Logos).” And one conclusion that we may derive especially from Figure 2, is that our computational simulation suggests that it is possible to consider that the Universe has existed for long time in prolonged stagnation period, then suddenly it burst out from empty and formless (Gen. 1:2), to take its current shape which is accelerating. Such a possibility has never been considered before in cosmology literature. It is our hope that our exploration will lead to nonlinear cosmology theories which are better in terms of observations, and also more faithful to Biblical account of creation.

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Modeling virus as elastic sphere in newtonian fluid based on 3d Navier-Stokes equations

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Christianto V, Smarandache F. Modeling virus as elastic sphere in newtonian fluid based on 3d Navier-Stokes equations. J Nanosci Nanomed.. March-2018;2(1):1-2.

Although virus is widely known to significantly affect many biological form of life, its physical model is quite rare. In this regard, experiments on the acoustic vibrations of elastic nanostructures in fluid media have been used to study the mechanical properties of materials, as well as for mechanical and biological sensing. The medium surrounding the nanostructure is typically modeled as a Newtonian fluid. In this paper, we present a mathematical model of virus as an elastic sphere in a Newtonian fluid, i.e. via 3D Navier-Stokes equations. We also obtain a numerical solution by the help of Mathematica 11.
KEYWORDS: Newtonian fluid; virus model; nonlinear physics; 3D Navier-Stokes equations; computational physics

INTRODUCTION

Although virus is widely known to significantly affect many biological form of life, its physical model is quite rare. In a paper, L.H. Ford wrote:

"Two simple models for the particle are treated, a liquid drop model and an elastic sphere model. Some estimates for the lowest vibrational frequency are given for each model. It is concluded that this frequency is likely to be of the order of a few GHz for particles with a radius of the order of 50nm" [1-4].

Such an investigation on acoustic vibration of virus particles may resonate with other reports by Prof. Luc Montagnier [8,9] and also our own hypothesis [10,11] on wave character of biological entities such as DNA, virus, water, etc.

In this regard, there are studies on the mechanical properties of (biology) materials based on experiments on the acoustic vibrations of elastic nanostructures in fluid media, where the medium surrounding the nanostructure is typically modeled as a Newtonian fluid.

In this paper we will also discuss a Newtonian fluid, i.e. 3D Navier-Stokes equations.

It is our hope that the new proposed method can be verified with experiments.

A MODEL OF LINEARIZED NAVIER-STOKES EQUATIONS

In 2015, Vahe Galstyan, On Shun Pak and Howard A. Stone published a paper where they discuss breathing mode of an elastic sphere in Newtonian and complex fluids [2]. They consider the radial vibration of an elastic sphere in a compressible viscous fluid, where the displacement field of the elastic fluid medium is governed by the Navier equation in elasticity. This spherically symmetric motion is also called the breathing mode.

They use a linearized version of Navier-Stokes equations, as follows: [2]

$$\rho_v \frac{\partial v}{\partial t} = -\nabla p + \eta \nabla^2 v + \left(\kappa + \frac{\eta}{3}\right)\nabla, \qquad (1)$$

where ρv is the density of the fluid, η is the shear viscosity, κ is the bulk viscosity, and p is the thermodynamic pressure.

There are other authors who work on linearized NS problem; here we mention a few of them: Foias and Saut [12], Thomann and Guenther [13], A. Leonard [14].

NUMERICAL SOLUTION OF 3D NAVIER-STOKES EQUATIONS

In fluid mechanics, there is an essential deficiency of the analytical solutions of non-stationary 3D Navier–Stokes equations. Now, instead of using linearized NS equations as above, we will discuss a numerical solution of 3D Navier-Stokes equations based on Sergey Erhskov's papers [4,5].

The Navier-Stokes system of equations for incompressible flow of Newtonian fluids can be written in the Cartesian coordinates as below (under the proper initial conditions): [4]

$$\nabla . \vec{u} = 0, \qquad (2)$$

$$\frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \nabla\right)\vec{u} = -\frac{\nabla p}{\rho} + \nu \cdot \nabla^2 \vec{u} + \vec{F}. \qquad (3)$$

Where u is the flow velocity, a vector field, ρ is the fluid density, p is the pressure, v is the kinematic viscosity, and F represents external force (per unit mass of volume) acting on the fluid. [4]

In ref. [4], Ershkov explores new ansatz of derivation of non-stationary solution for the Navier–Stokes equations in the case of incompressible flow, where his results can be written in general case as a mixed system of 2 coupled-Riccati ODEs (in regard to the time-parameter t). But instead of solving the problem analytically, we will try to find a numerical solution with the help of computer algebra package of Mathematica 11 (Figure 1).

The coupled Riccati ODEs read as follows: [4]

$$a' = \frac{w_y}{2} \cdot a^2 - (w_x \cdot b) \cdot a - \frac{w_y}{2}(b^2 - 1) + w_z \cdot b. \qquad (4)$$

$$b' = -\frac{w_x}{2} \cdot b^2 + (w_y \cdot a) \cdot b + \frac{w_x}{2}(a^2 - 1) - w_z \cdot a. \qquad (5)$$

First, equations (4) and (5) can be rewritten in the form as follows:

$$x(t)' = \frac{v}{2} \cdot x(t)^2 - (u \cdot y(t)) \cdot x(t) - \frac{v}{2}(y(t)^2 - 1) + w \cdot y(t). \quad (6)$$

$$y(t)' = -\frac{u}{2} \cdot y(t)^2 + (v \cdot x(t)) \cdot y(t) + \frac{u}{2}(x(t)^2 - 1) - w \cdot x(t). \quad (7)$$

Then we can put the above equations into Mathematica expression: [3]

v=1;

u=1;

w=1;

{xans6[t_],vans6[t_]}={x[t],y[t]}/.Flatten[NDSolve[{x'[t]==(v/2)*x[t]^2-(u*y[t])*x[t]-v/2)*(y[t]^2-1)+w*y[t], y'[t]==-(u/2)*y[t]^2+(v*x[t])*y[t]+(u/2)*(x[t]^2-1)-w*x[t], x[0]==1,y[0]==0}, {x[t],y[t]},{t,0,10}]]

graphx6 = Plot [xans6[t],{t,0,10}, Axes Label->{"t","x"}, Plot Style->Dashing[{0.02,0.02}]];

Show [graphx6,graphx6]

The result is as shown below: [3]

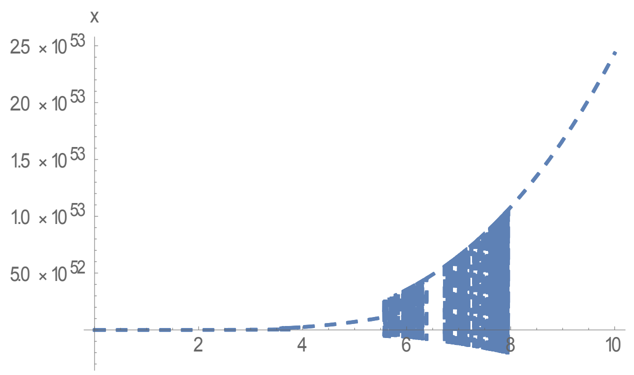

Figure 1: Graphical plot of solution for case v=u=w=1. See [3]

CONCLUDING REMARKS

In this paper we review 3D Navier-stokes equations obtained by Ershkov, as a model of virus as an elastic sphere in Newtonian fluid, and we solve the equations numerically with the help of Mathematica 11.

It is our hope that the above numerical solution of 3D Navier-Stokes equations can be found useful in mathematical modeling of virus [6-9].

All in all, here we would like to emphasize that such an investigation on acoustic vibration of virus particles may resonate with other reports by Prof. Luc Montagnier and also our own hypothesis, on wave character of biological entities such as DNA, virus, water, etc. [8-10].

We are quite optimistic that this novel approach can lead to new kinds of nanomedicine of virus based on acoustic vibration in Newtonian fluid medium [11-14].

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An outline of cellular automaton universe via cosmological KdV equation

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Abstract. It has been known for long time that the cosmic sound wave was there since the early epoch of the Universe. Signatures of its existence are abound. However, such a sound wave model of cosmology is rarely developed fully into a complete framework. This paper can be considered as our *second* attempt towards such a complete description of the Universe based on soliton wave solution of cosmological KdV equation. Then we advance further this KdV equation by virtue of Cellular Automaton method to solve the PDEs. We submit wholeheartedly Robert Kuruczs hypothesis that *Big Bang should be replaced with a finite cellular automaton universe with no expansion* [4][5]. Nonetheless, we are fully aware that our model is far from being complete, but it appears the proposed cellular automaton model of the Universe is very close in spirit to what Konrad Zuse envisaged long time ago. It is our hope that the new proposed method can be verified with observation data. But we admit that our model is still in its infancy, more researches are needed to fill all the missing details.

1. Introduction

Konrad Zuse is probably the first scholar who imagine a *Computing Universe*. In recent years, there are a few researchers who suggest similar vision in terms of cellular automata, for example Stephen Wolfram, Gerardus t Hooft, and Robert Kurucz from Harvard Smithsonian of Astrophysics. Nonetheless, it seems that there is no existing model which can be connected with a nonlinear PDE of the Universe. In this paper, we try to offer some working CA models based on the KdV equation, which can be modelled and solved using computer algebra packages such as Mathematica.

Meanwhile, Korteweg-de Vries (KdV) equation is a non-linear wave equation plays a fundamental role in diverse branches of mathematical and theoretical physics. Its significance to cosmology has been discussed by a number of authors, such as Rosu and recently Lidsey [3][7]. It is suggested that the KdV equation arises in a number of important scenarios, including inflationary cosmology etc. Analogies can be drawn between cosmic dynamics and the propagation of the solitonic wave solution to the equation, whereby quantities such as the speed and amplitude profile of the wave can be identified with cosmological parameters such as the spectral index of the density perturbation spectrum and the energy density of the universe.

2. Cosmological KdV equation

The Korteweg-de Vries (KdV) equation is the completely integrable, third-order, non-linear partial differential equation (PDE) [3]:

$$\partial_t u + \partial_x^3 u + \frac{3}{u_0} u \partial_x u = 0, \tag{1}$$

where $u = u(x,t), \partial_t = \frac{\partial}{\partial t}, \partial_x^3 = \frac{\partial^3}{\partial x^3}$, etc., u_0 is a constant and (x,t) represent space and time coordinates, respectively. This equation was originally derived within the context of small-amplitude, non-linear water wave theory and it is well known that it admits a solitonic wave solution of the form

$$u = u_0 \lambda^2 \sec h^2 \left[\frac{\lambda(x - \lambda^2 t)}{2} \right], \tag{2}$$

where the constant $\frac{\lambda}{2}$ represents the wavenumber of the soliton. The KdV soliton is characterized by the property that its speed and amplitude are proportional to the square of the wavenumber.

Rosu [7] and also Lidsey [3] both have considered some cosmological applications of KdV equation. We will consider here one application in inflationary universe model. It can be shown that Friedmann equation after some steps which have been discussed in [3], yields to an equation which takes the form of (2), as follows:

$$H^2(\phi) = H_0^2 \lambda^2 \sec h^2 \left[\frac{\lambda A}{2} \right], \tag{3}$$

where

$$A = \frac{\sqrt{8\pi}}{m_p} \phi. \tag{4}$$

Therefore, it appears quite reasonable to consider this equation as originated from certain cosmological KdV physics.

3. Cellular automata model of KdV equation: Towards cellular automaton universe

There are several methods to consider discretization of KdV equation into cellular automata models. Here we briefly discuss only few methods:

(i) Based on paper by [11], KdV equation can be written as a conservation law:

$$\frac{\partial u}{\partial t} + \frac{\partial}{\partial x} \left(-\frac{u^2}{2} - \frac{\partial^2 u}{\partial x^2} \right) = 0. \tag{5}$$

It follows that, after the simplest discretization, we obtain the cellular automata:

$$u_j(t+1) = u_j(t)[u_{j+1}(t) - u_j(t)] + u_{j+2}(t) - u_{j+1}(t) - u_{j-1}(t). \tag{6}$$

Thus $\Sigma_{j=0}^{N-1} u_j(t)$ is not an invariant.

(ii) The discrete analogue of the KdV equation is known thanks to the pioneering work of Hirota. It has the form [15]:

$$\frac{1}{u_{l+1}^{t+1}} - \frac{1}{u_l^t} = \delta(u_{l+1}^t - u_l^{t+1}) \tag{7}$$

(iii) Another model was proposed by Tokihiro et al around twenty years ago. They suggested that an integrable discretization (differential-difference equation) of the KdV equation is the Lotka-Volterra equation [14]:

$$\frac{d}{dt} b_j t = b_j(t)[b_{j+t}(t) - b_{j-1}(t)] \tag{8}$$

In other words, it appears possible at least in theory to consider a Cellular Automaton-KdV Universe, based on discretization of the original KdV equation. Nonetheless, further analysis is required to study its potential applications.

4. Discussion

We advance further KdV equation by virtue to Cellular Automaton method to solve the PDEs. Here, we consider some mathematical methods to discretize the original KdV equation in order to be transformed into cellular automata cosmology models. As far as our knowledge, this approach is different from existing cellular cosmology, such as by Conrad Ranzan (www.cellularuniverse.com).

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On Some Metaphysical Problems of Many Worlds Interpretation of Quantum Mechanics

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Abstract

Despite its enormous practical success, many physicists and philosophers alike agree that the quantum theory is full of contradictions and paradoxes which are difficult to solve consistently. Even after 90 years, the experts themselves still do not all agree what to make of it. The area of disagreement centers primarily around the problem of describing observations. Formally, the so-called quantum *measurement problem* can be defined as follows: the result of a measurement is a superposition of vectors, each representing the quantity being observed as having one of its possible values. The question that has to be answered is : how this superposition can be reconciled with the fact that in practice we only observe one value. *How is the measuring instrument prodded into making up its mind which value it has observed?* Among some alternatives to resolve the above QM measurement problem, a very counterintuitive one was suggested by Hugh Everett in his 1955 Princeton dissertation, which was subsequently called the Many-Worlds Interpretation of QM (MWI). In this paper, we will not discuss all possible scenarios to solve the measurement problem, but we will only shortly discuss Everett's MWI, because it has led to heated debates on possibility of multiverses, beyond the Universe we live in. We also discuss two alternatives against MWI proposal: (a) the so-called *scale symmetry theory*, and (b) the Maxwell-Dirac isomorphism. In last section, we also discuss shortly MWI hypothesis from philosophical perspective.

Keywords: Quantum measurement problem, many-worlds interpretation, quantum metaphysics, multiverse, realism interpretation, scale symmetry, Maxwell-Dirac isomorphism.

1. Introduction

In its simplest form the quantum theory of measurement considers a world composed of just two dynamical entities, a system and an apparatus. According to the Copenhagen interpretation of QM, at the point of time when an observer operates the apparatus to observe the system, the system's wave function collapse. But the exact mechanism of wave function collapse is unknown. Furthermore, it is difficult to model the correlation between a *macroscopic* observer and apparatus (governed by classical physics) with the *microscopic* system in question, which is supposed to be governed the Schrödinger's wave function. This is known as quantum measurement problem, which baffled many physicists since the early years of QM development.

To quote De Witt's paper in *Physics Today* [7]:

> At this point Bohr entered the picture and deflected Heisenberg somewhat from his original program. Bohr convinced Heisenberg and most other physicists that quantum mechanics has no meaning in the absence of a classical realm capable of unambiguously recording the results of observations. The *mixture of metaphysics with physics*, which this notion entailed, led to the almost universal belief that the chief issues of interpretation are epistemological rather than ontological: The quantum realm must be viewed as a kind of ghostly world whose symbols, such as the wave function, represent potentiality rather than reality.

Apparently, Everett also realized that Copenhagen interpretation is largely incomplete. In his 1955 PhD thesis, Everett essentially proposed a resolution from measurement problem by assuming a multitude of possibilities, which is why his hypothesis is called Many Worlds Interpretation.

In De Witt's words:[7]

> … it forces us to believe in the reality of all the simultaneous worlds represented in the superposition described by equation 5, in each of which the measurement has yielded a different outcome. Nevertheless, this is precisely what EWG would have us believe. According to them the real universe is faithfully represented by a state vector similar to that in equation 5 but of vastly greater complexity. This universe is constantly splitting into a stupendous number of branches, all resulting from the measurement like interactions between its myriads of components. Moreover, every quantum transition taking place on every star, in every galaxy, in every remote comer of the universe is splitting our local world on earth into myriads of copies of itself.

In other words, Everett's hypothesis called for a different picture of reality, and obviously this requires a careful consideration of the distinction and boundary between physics theories and metaphysics. In the next section we will discuss several objections and critics to MWI.

2. Some Critics to Many Worlds Interpretation

Since publication of his dissertation, Everett's MWI has caused debates especially on philosophical problems related to his proposal. Such a proposition leads some physicists to argue that MWI actually moves the measurement problem into wild metaphysical speculation of branching universes. Barrett has reviewed earlier discussions on this topics.[6]

Despite acceptance of MWI by some theoretical physicists, and even Barrau [9] argued in favor of possible experimental vindication of MWI, there are also those who raise serious criticisms on such a wild hypothesis.

One critics came from Adrian Kent from Princeton University, from the same department where Everett obtained his PhD. In essence, Kent's objection on MWI is because:

> The relevance of frequency operators to MWI is examined; it is argued that frequency operator theorems of Hartle and Farhi-Goldstone-Gutmann do not in themselves provide a probability interpretation for quantum mechanics, and thus neither support existing MWI nor would be useful in constructing new MWI. [5]

Furthermore, he argues:

> Firstly, the very failure of MWI proponents to axiomatize their proposals seems to have left the actual complexity of realistic MWI widely unappreciated. It may thus possibly be tempting for MWI advocates to assume that there is no real problem; that Everett's detractors either have not understood the motivation for, or merely have rather weak aesthetic objections to, his program. (Hence perhaps the otherwise inexplicable claim by one commentator that "Avoiding this [prediction of multiple co-existing consciousnesses for a single observer] is their [Everett's opponents'] motivation for opposing Everett in the first place.")
>
> Secondly, MWI seem to offer the attractive prospect of using quantum theory to make cosmological predictions. The trouble here is that *if MWI is ultimately incoherent and ill-founded, it is not clear why one should pay attention to any quantum cosmological* calculations based on it. [5, p. 27]

In answering frequent question of what are the alternatives to MWI hypothesis, Kent outlined a number of ideas, including subquantum physics.

Another critics came from Steven Weinberg. For example, in 2005 interview with Dan Falk, Steven Weinberg still has objection on multiverse hypothesis. Meanwhile, he agrees that positivism or constructivism may be no longer valid in physics sciences, but he also admits that he still tries to figure out an alternative interpretation of QM:

> SW: And sometimes, as with the example of positivism, the work of professional philosophers actually stands in the way of progress. That's also the case with the approach known as constructivism — the idea that every society's scientific theories are a social construct, like its political institutions, and have to be understood as coming out of a particular cultural milieu. I don't know whether you'd call it a philosophical theory or a historical theory, but at any rate, I think that view is wrong, and I also think it could impede the work of science, because it takes away one of science's great motivations, which is to discover something that, in an absolute sense, divorced from any cultural milieu, is actually true.

> Dan Falk: You're 81. Many people would be thinking about retirement, but you're very active. What are you working on now?
>
> SW: There's something I've been working on for more than a year — maybe it's just an old man's obsession, but I'm trying to find an approach to quantum mechanics that makes more sense than existing approaches. I've just finished editing the second edition of my book, *Lectures on Quantum Mechanics*, in which I think I strengthen the argument that none of the existing interpretations of quantum mechanics are entirely satisfactory.

Weinberg himself has proposed his own theoretical physical interpretation of QM, albeit his theory is non-ontological in nature. He wrote:[14]

> ψ is theoretically physical and describes the probabilistic possibilities, as the Copenhagen interpretation implies. It has physical units (see Eq. (3)). ψ is also, naively, a function of a real spacetime coordinate argument solving a partial differential equation, such as the time-dependent Schrödinger equation, for example, with spacetime partial derivatives. All this argues for theoretically physical formalism (*not ontology*), applicable predictably prior to 'Copenhagen observation.'

Therefore, in the same spirit with Weinberg's reserved position against MWI hypothesis, in the following section we will discuss two simpler alternatives which seem quite worthy for further considerations: (a) scale symmetry theory, (b) a more realistic interpretation of quantum wave function based on Maxwell-Dirac isomorphism.

3. Resolution to the problem based on scale symmetry theory

In a semi-popular article in *Quanta Magazine*[10], Wolchover describes how some theoretical physicists who feel unhappy with multiverse metaphysical problem, have come up with new theories where mass and length are no longer fundamental entities. In a scale-symmetry theory, advocated earlier by Bardeen around 1995, the origin of mass can be derived without invoking Higgs mechanism.[11]

Proponents of scale symmetry theory argue that this approach has clear prospect to prove that multiverse hypothesis (MWI) is an *excess baggage*. In essence, they believe that the key to the correct answer to the measurement problem is not by pondering metaphysical problems such as the existence of multiple realities and multiple histories, but by examining our assumptions on mass, length, and scales. See also Hashino et al. [12].

Resolution to the problem based on realistic Maxwell-Dirac isomorphism

Actually, there is a simpler resolution to the aforementioned QM measurement problem, although it is not quite popular yet, i.e. by admitting that (a) Schrodinger's wave function is unphysical therefore it has no value for realistic physical systems, (b) because of such an unrealistic wave function, the measurement problem is caused by confusion on the physical meaning of quantum wave function, (c) it is required to reconcile physical wave function obtained from QM and from classical electrodynamics theory.

Once we accept these, then we should find out the correct physical meaning of wave function, by formal connection between QM and classical electrodynamics. In other words, contradictions and confusions can be removed once we reconcile quantum picture with classical electrodynamics picture of wave function, instead of crafting unfounded assumption of many-worlds which only creates metaphysical excess baggage.

There are some papers in literature which concerned with the formal connection between classical electrodynamics and wave mechanics, especially there are some existing proofs on Maxwell-Dirac isomorphism. Here the author will review two derivations of Maxwell-Dirac isomorphism i.e. by Hans Sallhofer and Volodimir Simulik. In the last section we will also discuss a third option, i.e. by exploring Maxwell-Dirac isomorphism through quaternionic language.

a. Sallhofer's method

Summing up from one of Sallhofer's papers[1], he says that under the sufficiently general assumption of periodic time dependence the following connection exists between source-free electrodynamics and wave mechanics:

$$\sigma \cdot \begin{bmatrix} rotE + \frac{\mu}{c}\frac{\partial}{\partial t}H = 0 \\ rotH - \frac{\varepsilon}{c}\frac{\partial}{\partial t}H = 0 \\ div\varepsilon E = 0 \\ div\mu H = 0 \end{bmatrix}_{divE=0} \equiv \left[(\gamma \cdot \nabla + \gamma^{(4)}\partial_4)\Psi = 0\right] \tag{1}$$

In words: Multiplication of source-free electrodynamics by the Pauli-vector yields wave mechanics.[1]

In simple terms, this result can be written as follows:

$$P \cdot M = D, \tag{2}$$

Where:

P = Pauli vector,

M = Maxwell equations,

D = Dirac equations.

We can also say: *Wave mechanics is a solution-transform of electrodynamics.* Here one has to bear in mind that the well-known circulatory structure of the wave functions, manifest in Dirac's hydrogen solution, is not introduced just by the Pauli-vector.[1]

b. Simulik's method

Simulik described another derivation of Maxwell-Dirac isomorphism. In one of his papers[2], he wrote a theorem suggesting that the Maxwell equations of source-free electrodynamics which can be written as follows:

$$\begin{gathered} rotE+\frac{\mu}{c}\frac{\partial}{\partial t}H=0 \\ rotH-\frac{\varepsilon}{c}\frac{\partial}{\partial t}H=0 \\ divE=0 \\ divH=0 \end{gathered} \tag{3}$$

Are equivalent to the Dirac-like equation [2]:

$$\left[\gamma.\nabla-\begin{pmatrix}\varepsilon 1 & 0 \\ 0 & \mu 1\end{pmatrix}\frac{1}{c}\frac{\partial}{\partial t}\right]\Psi^{c1}=1, \tag{4}$$

Where in the usual representation

$$\gamma=\begin{pmatrix}0 & \sigma \\ \sigma & 0\end{pmatrix}, \tag{5}$$

And σ are the well-known Pauli matrices.

c. Maxwell-Dirac isomorphism through Quaternionic language

Recognizing that the Maxwell's equations were originally formulated in terms of quaternionic language, some authors investigate formal correspondence between Maxwell and Dirac equations. To name a few who worked on this problem: Kravchenko and Arbab. These authors have arrived to a similar conclusion, although with a different procedures based on Gersten decomposition of Dirac equation.[4]

It seems that the above arguments of Maxwell-Dirac isomorphism can be an alternative to the problematic MWI hypothesis. This MD isomorphism can also be extended further to classical description of boson mass which was usually called Higgs boson[3], so it may offer a simpler route to describe the origin of mass compared to scale symmetry theory.

4. Philosophical viewpoint

In our opinion, the essence of problem with MWI is captured in De Witt's remark as quoted above: "The *mixture of metaphysics with physics*." Formally speaking, Everett's many worlds interpretation of QM can be viewed as large scale implication if one accepts Feynman's *sum over history* interpretation of QM. But, it is known that Feynman famously declared that nobody understood completely Quantum Mechanics. Therefore, one should be very careful before generalize his sum over history interpretation of QM toward Universe.

Nonetheless, Everett's Multiverse found numerous followers, especially science fiction fans all over the world. And some people also relates his Multiverse as a realization of one of Borges's story: "*Garden of the forking paths*."

While we shall admit that such a Multiverse hypothesis is a nice material for science fiction novels or movies, now is the right time to ask: *Is it possible that God created Multiverse?*

Such a philosophical implication of cosmology development has been emphasized by Bernard Carr:

> By emphasizing the scientific legitimacy of anthropic and multiverse reasoning, I do not intend to deny the relevance of these issues to the science– religion debate [32]. The existence of a multiverse would have obvious religious implications [33], so contributions from theologians are important. More generally, cosmology addresses fundamental questions about the origin of matter and mind, which are clearly relevant to religion, so theologians need to be aware of the answers it provides.

Rodney identifies several problems related to multiverse hypothesis:[17]

> Among the problems identified with the hypothesis are:
>
> (1) the existence of infinitely many universes depends critically on parameter choice;
> (2) the probability that any universe in an ensemble is fine-tuned for life is zero;
> (3) the physical realization of any ensemble will exclude an infinity of possibilities;
> (4) the hypothesis is untestable and unscientific; and
> (5) the hypothesis is not consistent with the amount of order found in this universe, nor with the persistence of order.

> If these factors are taken into consideration the conclusion of the last chapter will be much stronger, because the prior probability of many universes will be further reduced and because the 'likelihood' entering Bayes's theorem will also be reduced.”

It seems worth noting here to quote George Ellis’s remark in his Emmanuel College lecture:[19]

> The very nature of the scientific enterprise is at stake in the multiverse debate: the multiverse proponents are proposing weakening the nature of scientific proof in order to claim that multiverses provide a scientific explanation. This is a dangerous tactic.
>
> …
>
> The often claimed existence of physically existing infinities (of universes, and of spatial sections in each universe) in the multiverse context (e.g.Vilenkin: Many Worlds in One: The Search for Other Universes) is dubious.
>
> …
>
> Here one must distinguish between explanation and prediction. Successful scientific theories make predictions, which can then be tested. The multiverse theory can’t make any predictions because it can explain anything at all.

Finally, Ellis warned his fellow cosmologists:[18]

> I suggest that cosmologists should be very careful not make methodological proposals that erode the essential nature of science in their enthusiasm to support specific theories as being scientific, for if they do so, there will very likely be unintended consequences in other areas where the boundaries of science are in dispute. It is dangerous to weaken the grounds of scientific proof in order to include multiverses under the mantle of ‘tested science’ for there are many other theories standing in the wings that would also like to claim that mantle.
>
> *It is a retrograde step towards the claim that we can establish the nature of the universe by pure thought*, and don’t then have to confirm our theories by observational or experimental tests: it abandons the key principle that has led to the extraordinary success of science.
>
> In fact we can’t establish definitively either the existence or the nature of expanding universe domains that are out of sight and indeed out of causal contact with us.

5. Conclusion

Despite its enormous practical success, many physicists and philosophers alike agree that the quantum theory is full of contradictions and paradoxes which are difficult to solve consistently. Even after 90 years, the experts themselves still do not all agree what to make of it. In this paper, we review QM measurement problem which paved a way to Many-Worlds Interpretation of QM. Nonetheless, it is clear that Everett’s hypothesis called for a different picture of reality, and obviously this requires a very careful consideration of the distinction between physics theories and metaphysics.

In the meantime, the problem of the formal connection between electrodynamics and wave mechanics has attracted the attention of a number of authors, especially there are some existing proofs on Maxwell-Dirac isomorphism. Here the authors review three derivations of Maxwell-Dirac isomorphism i.e. by Hans Sallhofer and Volodimir Simulik and also quaternion language.

In our opinion the above arguments of Maxwell-Dirac isomorphism can be a *simpler* alternative compared to the metaphysically problematic MWI hypothesis. (Allow us to recall Ockham's razor: the simpler explanation is more likely to be the correct answer.) This MD isomorphism can also be extended further to classical description of boson mass which was usually called Higgs boson [3], so it may be a simpler option compared to scale symmetry theory.

It is our hope that discussions presented in this paper have made clear that the entire Many Worlds Interpretation of QM is not required, once we begin to ask what is the physical meaning of wave function, instead of accepted blindly the macroscale implication of path integral interpretation of QM.

This paper was inspired by an old question: *Is there a consistent and realistic description of wave function, both classically and quantum mechanically?*

It can be expected that the above discussions will shed some lights on that old problem especially in the context of physical meaning of quantum wave function. This is reserved for further investigations.

To conclude this paper, allow us to repeat Ellis's warning to his over-enthusiastic fellow cosmologists:

> I suggest that cosmologists should be very careful not make methodological proposals that erode the essential nature of science in their enthusiasm to support specific theories as being scientific, for if they do so, there will very likely be unintended consequences in other areas where the boundaries of science are in dispute. It is dangerous to weaken the grounds of scientific proof in order to include multiverses under the mantle of 'tested science'

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A Plausible Resolution to Hilbert's Failed Attempt to Unify Gravitation & Electromagnetism

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Abstract

In this paper, we explore the reasons why Hilbert's axiomatic program to unify gravitation theory and electromagnetism failed and outline a plausible resolution of this problem. The latter is based on Gödel's incompleteness theorem and Newton's aether stream model.

Keywords: Unification, gravitation, electromagnetism, Hilbert, resolution.

Introduction

Hilbert and Einstein were in race at 1915 to develop a new gravitation theory based on covariance principle [1]. While Einstein seemed to win the race at the time, Hilbert produced two communications which show that he was ahead of Einstein in term of unification of gravitation theory and electromagnetic theory. Hilbert started with Mie's electromagnetic theory. However, as Mie theory became completely failed, so was the Hilbert's *axiomatic* program to unify those two theories [1]. Einstein might be learning from such an early failure of Hilbert to unify those theories, and years later returned to Mie theory [1].

What we would say here is that Hilbert's axiomatic failure can be explained by virtue of Gödel's incompleteness theorem: which says essentially that any attempt to build a consistent theory based on axiomatic foundations can be shown to be inconsistent. Nonetheless only few physicists seem to grasp this result.

What can we learn from that story?

First of all, it leads us back to Newton's aether stream model as will be discussed in the following sections. Moreover, it may be not only that it is an elusive dream to unify gravitation and electromagnetic theories from pure thoughts, but it clearly shows that we ought to return to the old days of Maxwell and also Heaviside who have given hints on how to come up with a more realistic unification of gravitation and electromagnetic theories.

To us, it also shows that we may need to re-read Maxwell's original papers, perhaps we should find out how he thought about cogwheel, molecular vortices etc…and they may lead us to a correct theory of gravitation (and also how to connect it with classical electrodynamics). In the meantime, it is worth noting here that Tesla and other experimenters have tried to come up with a simpler version of such unification theories, although most of them were not as familiar to many physicists unlike General Relativity theory.

Arthur Eddington's work

The modern era of cosmology began with the publication of Einstein's general theory of relativity in 1915. The first experimental test of this theory was Eddington's famous expedition to measure the bending of light at a total solar eclipse in 1919 [3].

According to Peter Coles's book [3]:

> Eddington was impressed by the beauty of Einstein's work, and immediately began to promote it. In a report to the Royal Astronomical Society in early 1917, he particularly stressed the importance of testing the theory using measurements of light bending. A few weeks later, the Astronomer Royal, Sir Frank Watson Dyson, realised that the eclipse of 29 May 1919 was especially propitious for this task. Although the path of totality ran across the Atlantic ocean from Brazil to West Africa, the position of the Sun at the time would be right in front of a prominent grouping of stars known as the Hyades. When totality occurred, the sky behind the Sun would be glittering with bright stars whose positions could be measured. Dyson began immediately to investigate possible observing sites. It was decided to send not one, but two expeditions. One, led by Eddington, was to travel to the island of Principe off the coast of Spanish Guinea in West Africa, and the other, led by Andrew Crommelin (an astronomer at the Royal Greenwich Observatory), would travel to Sobral in northern Brazil. An application was made to the Government Grant Committee to fund the expeditions, £100 for instruments and £1000 for travel and other costs. Preparations began, but immediately ran into problems. Although Britain and Germany had been at war since 1914, conscription into the armed forces was not introduced in England until 1917. At the age of 34, Eddington was eligible for the draft, but as a Quaker he let it be known that he would refuse to serve. …
>
> There were other problems too. The light deflection expected was quite small: less than two seconds of arc. But other things could cause a shifting of the stars' position on a photographic plate. For one thing, photographic plates can expand and contract with changes in temperature. The emulsion used might not be particularly uniform. The eclipse plates might have been exposed under different conditions from the reference plates, and so on. The Sobral team in particular realised that, having risen during the morning, the temperature fell noticeably during totality, with the probable result that the photographic plates would shrink. The refractive properties of the atmosphere also change during an eclipse, leading to a false distortion of the images. And perhaps most critically of all, Eddington's expedition was hampered by bad luck even after the eclipse. Because of an imminent strike of the local steamship operators, his team was in danger of being completely stranded. He was therefore forced to leave early, before taking any

reference plates of the same region of the sky with the same equipment. Instead he relied on one check plate made at Principe and others taken previously at Oxford. These were better than nothing, but made it impossible to check fully for systematic errors and laid his results open to considerable criticism. All these problems had to be allowed for, and corrected if possible in the final stage of data analysis. Scientific observations are always subject to errors and uncertainty of this kind. The level of this uncertainty in any experimental result is usually communicated in the technical literature by giving not just one number as the answer, but attaching to it another number called the ’standard error’, an estimate of the range of possible errors that could influence the result. If the light deflection measured was, say, 1 arc second, then this measurement would be totally unreliable if the standard error were as large as the measurement itself, 1 arc second. Such a resultwould bepresented as ’1±1’ arc second, and nobodywould believe it because the measured deflection could well be produced entirely by instrumental errors. In fact, as a rule of thumb, physicists never usually believe anything unless the measured number is larger than two standard errors. The expedition teams analysed their data, with Eddington playing the leading role, cross-checked with the reference plates, checked and double-checked their standard errors. Finally, they were ready. …

A special joint meeting of the Royal Astronomical Society and the Royal Society of London was convened on 6 November 1919. Dyson presented the main results, and was followed by contributions from Crommelin and Eddington. The results from Sobral, with measurements of seven stars in good visibility, gave the deflection as 1.98±0.16 arc seconds. Principe was less convincing. Only five stars were included, and the conditions there led to a much larger error. Nevertheless, the value obtained by Eddington was 1.61±0.40. Both were within two standard errors of the Einstein value of 1.74 and more than two standard errors away from either zero or the Newtonian value of 0.87. The reaction from scientists at this special meeting was ambivalent. Some questioned the reliability of statistical evidence from such a small number of stars. This skepticism seems in retrospect to be entirely justified. Although the results from Sobral were consistent with Einstein’s prediction, Eddington had been careful to remove from the analysis all measurements taken with the main equipment, the astrographic telescope and used only the results from the 4-inch. As I have explained, there were good grounds for this because of problems with the focus of the larger instrument. On the other hand, these plates yielded a value for the deflection of 0.93 seconds of arc, very close to the Newtonian prediction. Some suspected Eddington of cooking the books by leaving these measurements out.

Gödel’s incompleteness theorem

Gödel’s ground-breaking results were obtained against the backdrop of the foundational debate of the 1920s. In 1921, reacting in part to calls for a “revolution” in mathematics by the intuitionist L. E. J. Brouwer and his own student Hermann Weyl, Hilbert had proposed a program for a new foundation of mathematics. The program called for (i) a formalization of all of mathematics in an axiomatic systems followed by (ii) a demonstration that this formalization is consistent, i.e., that no contradiction can be derived from the axioms of mathematics. Partial progress had been made by Wilhelm Ackermann and John von Neumann, and Hilbert in 1928 claimed that consistency proofs had been established for first-order number theory. Gödel's results would later show that this assessment was too optimistic; but he had himself set out to with the aim of contributing to this program.[5[

According to Devlin's book [4]:

> Gödel's Theorem says that in any axiomatic mathematical system that is sufficiently rich to do elementary arithmetic, there will be some statements that are true but cannot be proved (from the axioms). In technical terminology, the axiom system must be incomplete. At the time Gödel proved this theorem, it was widely believed that, with sufficient effort, mathematicians would eventually be able to formulate axioms to support all of mathematics. The Incompleteness Theorem flew in the face of this expectation, and many took it to imply that there is a limit to the mathematical knowledge we may acquire. Few mathematicians think that way now, however. The change in our conception of mathematical truth that Godel's theorem brought about was so complete, that today most of us view the result itself as merely a technical observation about the limitations of axiom systems.

To summarize: "Kurt Gödel's Incompleteness Theorem changed the concept of mathematical truth and showed the limitations of axiom-based systems." In other words, Godel effectively put Hilbert's axiomatic program into ruins. And so was Hilber's approach to unify gravitation and electromagnetic theory.

Now the hard question: is it possible to find a door outside such a Godel's spider web?

A plausible resolution of the above problems

a. *Why do we need a new approach?*

Karl Popper's epistemology suggests that when the theory is refuted by observation, then it is time to look for a set of new approaches. Now, it is clear that Hilbert's axiomatic program has failed not only by experiment (Mie theory does not agree with experiment) but also in terms of logic (Godel theorem). Therefore we set out a new approach, starting from an old theory of Isaac Newton.

b. *Recalling Newton's aether stream model*

Newton brought up his aether stream model in a letter to Robert Boyle, 1678. For interested readers, complete letter of Isaac Newton to Boyle can be found in Appendix section. Comments on Newton aether stream model by DeMeo go as follows:

> The letter clearly shows the young Newton, who wrote this in 1679 when he was 37 years old, had a firm belief and working grasp of the ether of space as a thing of substance and

"ponderability", something which participated in the movement and ordering of the planets and universe, as a working force in optics, chemistry and gravitation. In this, Newton was continuing the conceptual ideas of Galileo, which had been such an irritant to the Vatican Bishops, who would tolerate no possibility of a motional force in nature other than God. The idea that ether and god might be identical descriptions for the "prime-mover" was equally intolerable, as while one could scientifically know and measure the ether, one could not by definition measure or know "the divine". The young Newton was not bothered by such conceptual difficulties as which bothered the Bishops of Rome, however, but the older Newton increasingly became preoccupied with theological matters, to the point that nearly all his biographers would agree he had become as much of a theologian as scientist in his last decades. Even only 20 years after penning this Letter to Boyle, he writes in the last query of his *Optics,* the following:

"Now by the help of these principles, all material things seem to have been composed of the hard and solid particles, above-mentioned, variously associated in the first creation by the counsel of an intelligent agent. For it became him who created them to set them in order. And if he did so, it's unphilosophical to seek for any other origin of the world, or to pretend that it might arise out of a chaos by the mere laws of nature; though being once formed, it may continue by those laws for many ages..." (quoted in Sullivan, p.125-126)

During those later periods, Newton would drop ideas such as a ponderable and moving cosmic ether in favor of more abstract concepts, such as the divine "prime mover" or deified "absolute space", which was foundational for most later astrophysical investigations into the nature of the cosmos. The most obvious result of this shift was, that in the original Michelson-Morley experiment for testing of ether-drift, everyone anticipated a very large ether-drift effect, based upon the assumption the Earth was racing through an intangible and substance-less static and immobile cosmic ether at very high speeds. No such intangible static ether has ever been demonstrated, nor could it be. But a material and substantive entrained ether, moving more slowly at lower altitudes and close to the speed of the earth itself, something quite similar to that proposed by the young Isaac Newton, was detected repeatedly.."[6]

c. Remark on Aether stream (by third author)

The higher the energy, the higher the velocity of the aether entities in the given place and time, and the lower the density. The phase states can exhibit turbulence, which is more marked at the higher densities, the way I am looking at this right now. The Kolmogorov Limit of 10e -58 meters plays a part here. Entities smaller than that will not exhibit much turbulence, primarily because they tend to be superluminal, so any turbulence will be hard to see.

The following figure is on Mishin's Aether phase states:

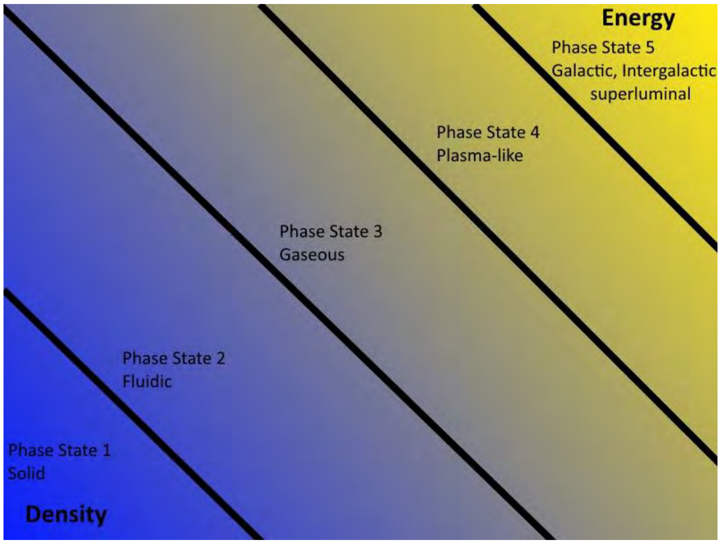

Figure 1. Aether phase states (Mishin)

There is an illustration of the process of aether particles being slowed by existing matter and eventually forming electron vortices as the local aether density and turbulence increases, while the energy drops due to interactions with existing matter, or aether in a denser phase state.

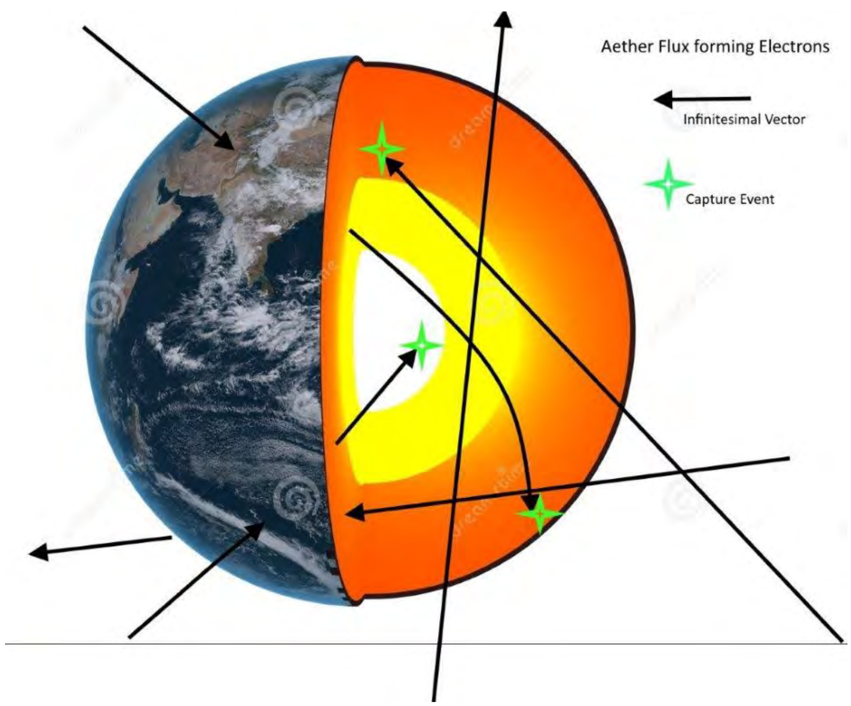

Figure 2. Illustration on how matter creation can take place in inner core of Earth (Source: https://www.dreamstime.com/royalty-free-stock-photography-earth-core-image1890727)

The process of matter creation can be attributed to electron vortex capture event.

This illustration shows stellar and interstellar aether flows interacting with electron vortices. In some cases the stellar flux is diverted by the electron vortex. In other cases, the flux entity misses entirely, similar to a neutrino. In some unusual cases the flux is captured by an electron vortex and participates in it for a while.

Each electron which already exists, acts as a large rock in a moving stream, causing deflections of the normal aether flow, slowing down the flow-rate, and producing eddy currents and turbulence in the ambient aether near the given electron. When the turbulence becomes large enough, additional electrons form in the media, which act to choke off the interstellar aether flow even more and impede its normally unencumbered motion. This is similar to adding more and more rocks into the channel of a stream of water, so that the flow rate of the water slows down, as more and more rocks are added.

This process was discovered by Nikola Tesla during his experiments at his Colorado Springs laboratory, where my grandfather was employed by Tesla, during those days. It is a good thing this happens, or aether avalanches produced by Tesla's 100,000,000 volt explosive electrical discharge events could have burned away the very air we live in.

Tesla was relieved to find out the discharges were choked off, accompanied by vast numbers of newly created electrons. Tesla found the excess electricity resulting from the excess electrons to be a nuisance to his other experiments, so he dumped the excess electrical power into the earth's crust.

Helmholtz electron vortices can be destroyed by aether shock fronts resulting from high dv/dt electrical discharges which are approaching the ideal of a Dirac delta function. In that situation, the Helmholtz vortex is disintegrated. The aether which originally formed the particle vortex, becomes part of the shock front and is carried along with the aether shock wave at velocities similar to the shock front, until the shock front dissipates. At that point, all that remains is a propagating aether stream, diverging at the rate of 1/r, relative to the source.

Everything is made of aether infinitesimals. Their group streaming motions precede the known forces, in the form of vector potentials. All matter is made from accumulations of infinitesimals. And all matter can be dissipated back into its constituent infinitesimals. See also figure below:

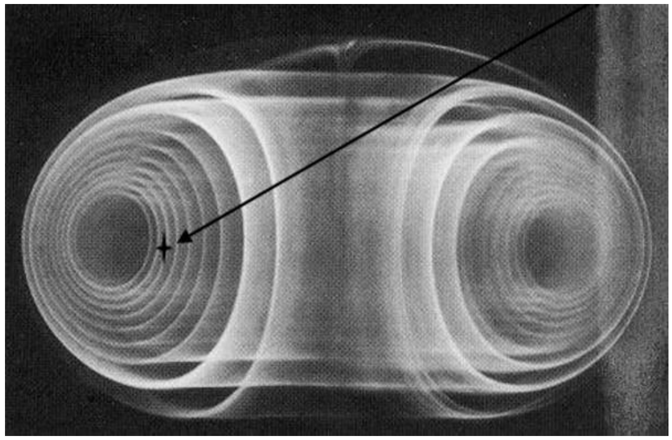

Figure 3. electron vortex capture event – Helmholtz electron vortex is *nearly indestructible* (after R.N. Boyd)

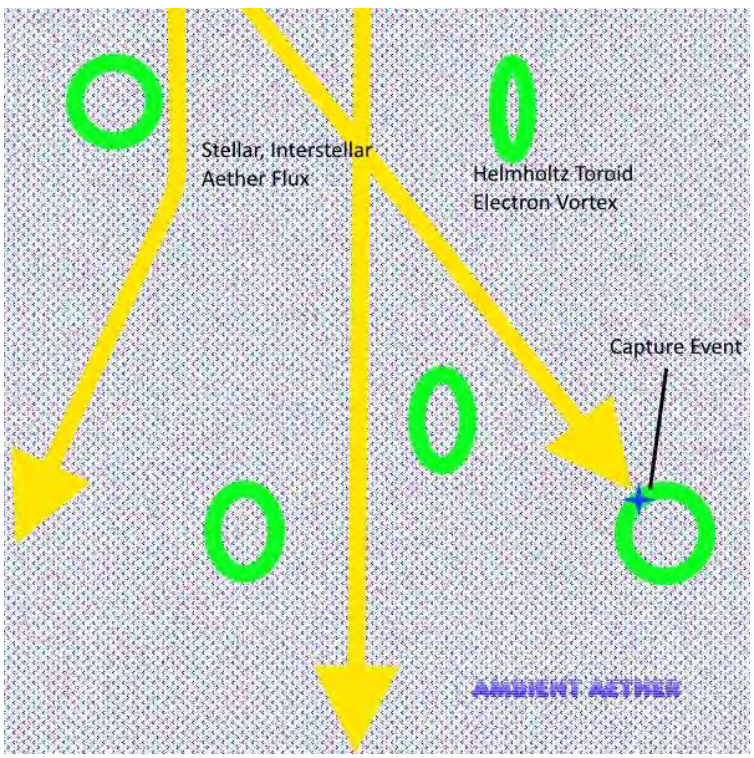

Figure 3a. electron vortex capture event(after R.N. Boyd)

The Helmholtz vortex model of the electron as illustrated in the photo of a Helmholtz vortex (Fig. 3), is a toroid made of nested concentric toroidal flows of smaller particles. Lines of constant flow are given by

$$r = \mathbf{a} \sin {}'\Omega = \mathbf{a} \sin {}'\Omega t,$$

where **a** is a constant. The velocity components are

$$dr/dt = \mathbf{a}\,\Omega \text{ cosine } \Omega\, t$$

and

$$r \ \ d\theta/dt = \mathbf{a}\,\Omega \sin \Omega\, t$$

The Ωt implies that a characteristic wave function is associated with the vortex, but we haven't worked on it yet. This may be an indication of origin of the de Broglie wave of the electron, or it may have something to do with the Compton radius of the electron, or both.

The constant **a** may represent the outer limit of the vortex-particle, if the internal circulation velocity of smaller particles does not exceed light speed. If the circulation velocity is larger than c, at the outer shells of the nested vortex, there may be a species of sub-particles which is always being removed from the nested toroidal form, which must be replenished to the vortex which is living in an "atmosphere" made larger circulations of sub-particles. This is due to considering the electron as having a fixed mass, a fixed extent, and a fixed charge (which may not be the case for all time and in all circumstances).

There should be some set of equations which shows vortex sub-particle replacement activities from the ambient aether, but we haven't worked on it either.

The first equation is a circle tangent to the z axis at the origin, with a center located in the X Y plane at the distance

$$\mathbf{a}/2 = p$$

where p is the potential of the electron, and is independent of the orientation of the electron vortex.

Then the electron can be viewed as a toroid, with a volume

$$V = 2\,\pi\, r \text{ times } \pi\, r\,\hat{}2 = 2\,\pi\hat{}2\, r\hat{}3$$

Three potentials are indicated here: Static potential, Spin potential, and a Dipole potential. Since the electron vortex has mass (which may change from its present value, according to the parameters of the ambient aether in the vicinity of the electron at the given place and time), a total of six potentials are implied.

d. *Introducing acoustic model of space*

With regards to spacetime metric which is conventionally attributed to Special Relativity, Thornhill has argued in favour of *acoustic nature of space* which conforms reality, instead of relativity with its notorious denial view on the existence of Aether stream. The following argument is derived from Thornhill.

In one of his remarkable papers, the late C.K. Thornhill wrote as follows:

> “Relativists and cosmologists regularly refer to space-time without specifying precisely what they mean by this term. Here the two different forms of spacetime, real and imaginary, are introduced and contrasted. It is shown that, in real space-time (x, y, z, ct), Maxwell’s equations have the same wave surfaces as those for sound waves in any uniform fluid at rest, and thus that Maxwell’s equations are not general and invariant but, like the standard wave equation, only hold in one unique frame of reference. In other words, Maxwell’s equations only apply to electromagnetic waves in a uniform ether at rest. But both Maxwell’s equations and the standard wave equation, and their identical wave surfaces, transform quite properly, by Galilean transformation, into a general invariant form which applies to waves in any uniform medium moving at any constant velocity relative to the reference-frame. It was the mistaken idea, that Maxwell’s equations and the standard wave equation should be invariant, which led, by a mathematical freak, to the Lorentz transform (which demands the non-ether concept and a universally constant wave-speed) and to special relativity. The mistake was further compounded by misinterpreting the differential equation for the wave hypercone through any point as the quadratic differential form of a Riemannian metric in imaginary space-time (x, y, z, ict). Further complications ensued when this imaginary space-time was generalised to encompass gravitation in general relativity.”[9]

Acoustic Analogue of Space

In this section, we borrow some important ideas from C.K. Thornhill and also Tsutomu Kambe. According to Thornhill, real space-time is a four dimensional space consisting of three-dimensional space plus a fourth length dimension obtained by multiplying time by a constant speed. (This is usually taken as the constant wave-speed c of electromagnetic waves). If the four lengths, which define a four-dimensional metric (x, y, z, ict), are thought of as measured in directions mutually at right-angles, then the quadratic differential form of this metric is: [9]

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 - \bar{c}^2 (dt)^2 \quad (1)$$

When the non-differential terms are removed from Maxwell’s equations, i.e. when there is no charge distribution or current density, it can easily be shown that the components (E1 ,E2 ,E3) of the electrical field-strength and the components (H1 ,H2 ,H3) of the magnetic field-strength all satisfy the <u>standard wave equation: </u>[9]

$$\nabla \phi = \left(\frac{1}{\bar{c}^2} \right) \frac{\partial^2 \phi}{\partial t^2} \quad (2)$$

It follows immediately, therefore, that the wave surfaces of Maxwell's equations are exactly the same as those for sound waves in any uniform fluid at rest, and that Maxwell's equations can only hold in one unique reference-frame and should not remain invariant when transformed into any other reference-frame. In particular, the equation for the envelope of all wave surfaces which pass through any point at any time is, for equation (2), and therefore also for Maxwell's equations [9],

$$(dx)^2 + (dy)^2 + (dz)^2 = \bar{c}^2 (dt)^2 , \tag{3}$$

or

$$\frac{(dx)^2}{(dt)^2} + \frac{(dy)^2}{(dt)^2} + \frac{(dz)^2}{(dt)^2} = \bar{c}^2 \tag{4}$$

It is by no means trivial, but it is, nevertheless, not very difficult to show, by elementary standard methods, that the general integral of the differential equation (4), which passes through (x_1, y_1, z_1) at time t_1, is the right spherical hypercone [9]

$$(x - x_1)^2 + (y - y \ )^2 + (z - z_1)^2 = \bar{c}^2 (t - t_1)^2 \tag{5}$$

In other words, both *Maxwell equations and space itself has the sound wave origin.*

It is also interesting to remark here that Maxwell equations can be cast in the language of vortex sound theory, as follows.
Prof. T. Kambe from University of Tokyo has made a connection between the equation of vortex sound and fluid Maxwell equations. He wrote that it would be no exaggeration to say that any vortex motion excites *acoustic* waves. He considers the equation of vortex sound of the form: [10]

$$\frac{1}{c^2}\partial_t^2 p - \nabla^2 p = \rho_0 \nabla . L = \rho_0 div(\omega \times v) \tag{6}$$

He also wrote that dipolar emission by the vortex-body interaction is [11]:

$$p_F(x,t) = -\frac{P_0}{4\pi c}\dot{\Pi}_i (t - \frac{x}{c})\frac{x_c}{x^2} \tag{7}$$

Then he obtained an expression of fluid Maxwell equations as follows [12]:

$$\begin{aligned} &\nabla \cdot H = 0 \\ &\nabla \cdot E = q \\ &\nabla \times E + \partial_t H = 0 \\ &a_0^2 \nabla \times H - \partial_t E = J \end{aligned} \tag{8}$$

Where [12] a_0 denotes the sound speed, and

$$\begin{aligned} &q = -\partial_t (\nabla \cdot \upsilon) - \nabla \hbar, \\ &J = \partial_t^2 v + \nabla \partial_t h + a_o^2 \nabla \times (\nabla \times \upsilon) \end{aligned} \tag{9}$$

In our opinion, this new expression of fluid Maxwell equations suggests that there is a deep connection between vortex sound and electromagnetic fields. However, it should be noted that the above expressions based on fluid dynamics need to be verified with experiments. We should note also that in (8) and (9), the speed of sound a_0 is analogous of the speed of light in Maxwell equations, whereas in equation (6), the speed of sound is designated "c" (as analogous to the light speed in EM wave equation). For alternative hydrodynamics expression of electromagnetic fields, see [14-15].

e. More proof: *Calculating matter creation in Earth and its effect*

One of us has performed a calculation to show that the observed receding Moon from Earth, should be properly attributed to increasing size of the Earth. The latter phenomenon could be attributed to “matter creation” as effect of aether stream (vortex). We will discuss this in a separate report.

f. More proof: *Dayton Miller’s experiment*

DeMeo remark on Dayton Miller’s experiment:

> The history of science records the 1887 ether-drift experiment of Albert Michelson and Edward Morley as a pivotal turning point, where the energetic *ether of space* was discarded by mainstream physics. Thereafter, the postulate of "empty space" was embraced, along with related concepts which demanded constancy in light-speed, such as Albert Einstein's relativity theory. The now famous *Michelson-Morley experiment* is widely cited, in nearly every physics textbook, for its claimed "null" or "negative" results. Less known, however, is the far more significant and detailed work of Dayton Miller.
>
> Dayton Miller's 1933 paper in *Reviews of Modern Physics* details the positive results from over 20 years of experimental research into the question of ether-drift, and remains *the most definitive body of work on the subject of light-beam interferometry.* Other positive ether-detection experiments have been undertaken, such as the work of Sagnac (1913) and Michelson and Gale (1925), documenting the existence in light-speed variations (c+v > c-v), but these were not adequately constructed for detection of a larger

cosmological *ether-drift,* of the Earth and Solar System moving through the background of space. Dayton Miller's work on ether-drift was so constructed, however, and yielded *consistently positive results.*

Miller's work, which ran from 1906 through the mid-1930s, most strongly supports the idea of an ether-drift, of the Earth moving through a cosmological medium, with calculations made of the actual direction and magnitude of drift. By 1933, Miller concluded that the Earth was drifting at a speed of 208 km/sec. towards an apex in the Southern Celestial Hemisphere, towards Dorado, the swordfish, right ascension 4 hrs 54 min., declination of -70° 33', in the middle of the Great Magellanic Cloud and 7° from the southern pole of the ecliptic. (Miller 1933, p.234)"[8]

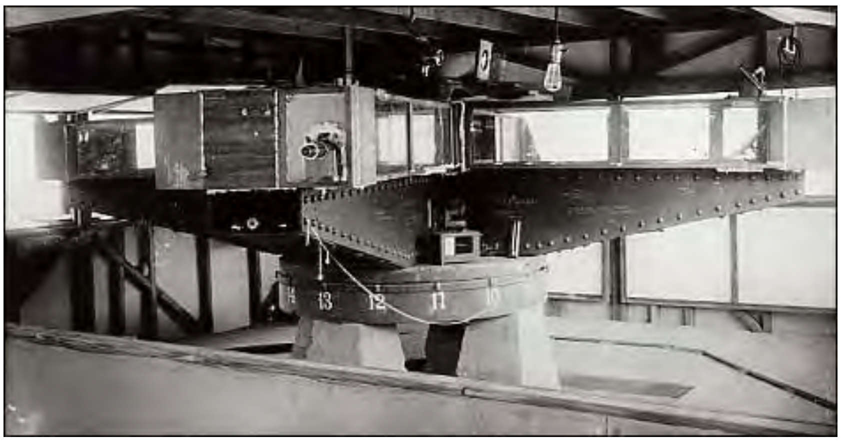

Figure 4. **Dayton Miller's light-beam interferometer,** at 4.3 meters across, was the largest and most sensitive of this type of apparatus ever constructed, with a mirror-reflected round-trip light-beam path of 64 meters. It was used in a definitive set of ether-drift experiments on Mt. Wilson, 1925-1926. Protective insulation is removed in this photograph, and windows were present all around the shelter at the level of the interferometer light-path. [8]

That Dayton Miller's experiment seems quite consistent with other experiments such as Michelson-Morley non-null result, which indicates solar system in motion. [21-22].

g. More proof: *preferred direction and Milky Way moving to The Great Attractor*

Another type of observations seems to suggest that there is preferred direction in the Universe at large scale, and especially that the Milky Way is moving at large speed toward the Great Attractor.[18-20] While this effect may be not detected in the Miller's days, two things are for sure: (a) no general relativity based theories can explain this effect, and (b) it makes Copernican Principle on question. This effect is seemingly consistent with Tifft's finding of rest background frame.[17]

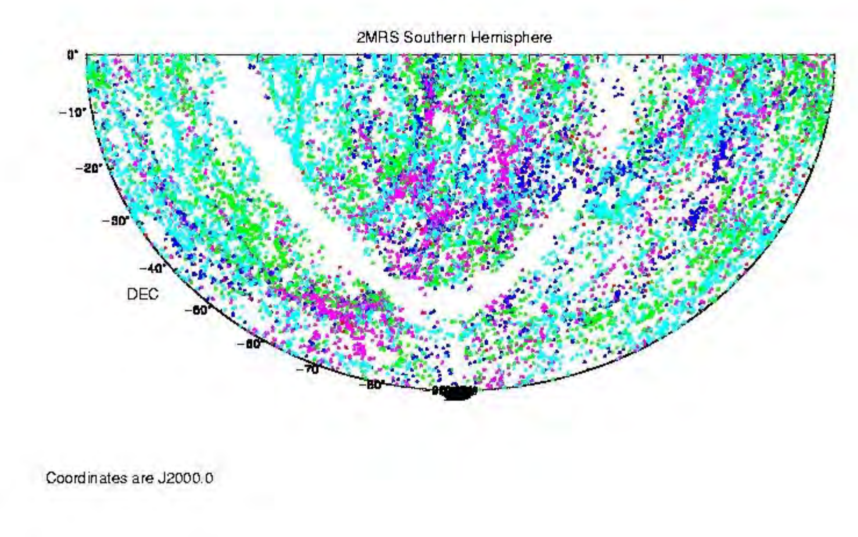

Figure 5. The Great Attractor from Southern Hemisphere

Figure 6. Shapley Supercluster

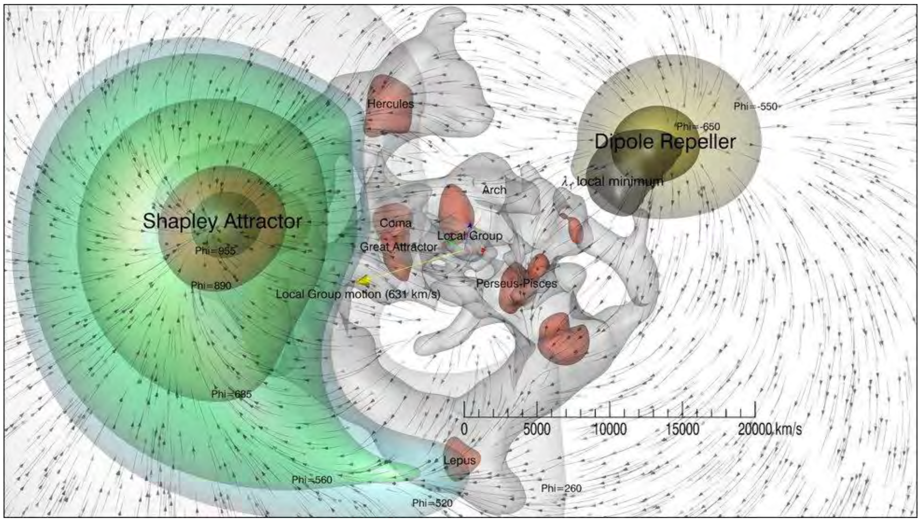

Figure 7. Shapley Supercluster

Conclusions

We begin with Hilbert's axiomatic program to unify electromagnetic and gravitation theory, and we remark that Godel finding effectively put Hilbert program into ruins. We also mentioned Eddington's observation.

In summary, it is very significant to consider matter creation process in nature. For instance, one can begin by considering the correct presentation of Newton's third law is not F=ma, but F=d(mv)/dt=v(dm/dt) + m(dv/dt). In other words, it is possible of matter creation (dm/dt), and this is consistent with Narlikar's work. This seems to be the essence of Le Sage gravity theory.

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Non-locality, Precognition & Spirit from the Physics Point of View

Victor Christianto, Florentin Smarandache

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Abstract

There are various supernatural phenomena which can hardly be explained by the existing mainstream science, for instance non-local interactions (e.g. ESP) and also precognitive interdictions. And there are other problems such as how to include the Spirit in the framework of physics. For example, it has been known for long time that intuition plays significant role in many professions and human life, including entrepreneurship, government, and also in detective or law enforcement activities. Despite these examples, such a precognitive interdiction is hardly accepted in mainstream science. In this paper, we discuss non-local interactions and advanced solutions of Maxwell equations, and argue in favor of precognitive interdiction from classical physics perspective. We also briefly discuss how "spirit" may be included in medicine, although we also warn against the danger of "*spiritism*."

Keywords: Non-locality, intuition, precognition, Maxwell equations, advanced wave solution, spirit.

1. Introduction

There are various supernatural phenomena which hardly can be explained by the mainstream science, for instance non-local interactions (e.g., ESP) and also precognitive interdictions. And there are other problems such as how to include the Spirit in our consciousness.

For example, it has been known for long time that intuition plays significant role in many professions and other aspects of human life, including in entrepreneurship, government, and also in detective or law enforcement activities. Even women are known to possess better intuitive feelings or "*hunch*" compared to men. Despite these examples, such a precognitive interdiction is hardly accepted in established science.

In this paper, we discuss non-locality interactions in classical electromagnetic theory, and also the advanced solutions of Maxwell equations in the context of Wheeler-Feynman-Cramer's absorber theory, and then make connection between *syntropy* and precognition from classical perspective. This may be regarded as a first step to describe such precognition activities which are usually considered belong to quantum realm.

In the last section, we will discuss on how to include spirit in medicine, although we shall also warn against "*spiritism*." It is our hope that what we discuss here can be verified with experimental data.

2. Electromagnetic origin of non-local interactions

There is a widely-held belief among physicists that non-local interactions can only be explained as an effect of Quantum Mechanics. But what is surprising to reveal here is that non-local interactions can be explained from pure classical electromagnetic theory.

A recent paper by Butler and Gresnigt tried to elucidate this issue:

> "A fields-only formulation of EM interactions that does not invoke charge explicitly is presented. The EM interaction ceases to be the result of an asymmetric action of a field on a point charge locally, but instead is the result of applying Hamilton's principle of virtual work to the symmetric but non-local interaction of space-filling EM fields themselves. The fields themselves are therefore the only fundamental entities.[8]
>
> ….
>
> The pure-field force law presented here is both Lorentz invariant and symmetrical with respect to all sources. However, it is not local. The fields are the mediators of force, but not through the interaction of the fields with the test charge at a single point in space, but rather through the dispersed interaction of the fields from all charges throughout all space. The approach taken here has traded locality for symmetry…. Although the derivation of the previous section is classical, the dispersed interaction is reminiscent of the interaction of QM states." [8]

Therefore, from theoretical viewpoint, non-local interactions can be explained from classical electromagnetic theory itself, especially when we consider *knotted* solutions of Maxwell equations.

Butler and Gresnigt also remarked:

> "Likewise, the motion resulting from EM interaction of a multiple particle system is the result of each particle's EM field's contribution to the quadratic energy density ... This overlapping structure between EM and QM has also been highlighted by van der Mark who showed that the QM probability current arises as the EM 4-current from topological EM fields." [8]

3. John Cramer's take on Wheeler-Feynman's absorber theory

The Wheeler-Feynman's paper on absorber theory has been discussed and generalized by John Cramer. He discussed among other things on the physical interpretation of advanced and retarded solutions of Maxwell equations and also Klein-Gordon equation.

Our discussion starts from the fundamental Maxwell's equations that unify electromagnetism [1]:

$$\begin{aligned}
&\nabla \cdot B = 0 (MagneticGauss),\\
&\nabla \cdot D = \rho_f (Gauss),\\
&\nabla \times E + \partial_t B = 0 (Faraday),\\
&\nabla \times H - \partial_t D = J_f (Amperecircuitallaw),
\end{aligned} \tag{1}$$

It is known that electromagnetic wave equation corresponding to (1) admits advanced wave solution.

Of course, here we do not have to accept all transactional QM interpretation by Cramer [1][2], but we can keep our discussion straightly within the scope of classical electromagnetic theory.

The electromagnetic wave equation for source-free space can be written in the form:

$$c^2 \nabla^2 \vec{F} = \frac{d^2 \vec{F}}{dt^2}, \tag{2}$$

where c represents the speed of light, and F represents either the electric field vector E or the magnetic field vector B of the wave.[1]

Since this differential equation is second order in both time and space, it has two independent time solutions and two independent space solutions. Let us restrict our consideration to one dimension by requiring that the wave motion described by equation (2) moves along with x axis and that the E vector of the wave is along the y axis.

Then two independent time solutions of equation (2) might have the form [1]:

$$\vec{E}_{\pm}(x,t) = \hat{y} E_0 \sin\left[2\pi\left(\frac{x}{\lambda} \pm ft\right)\right], \tag{3}$$

and

$$\vec{B}_{\pm}(x,t) = \hat{y} B_0 \sin\left[2\pi\left(\frac{x}{\lambda} \pm ft\right)\right], \tag{4}$$

Quoting from Cramer's notes on the solutions of equations (3) and (4):[1]

Thus, wave $E_+(x,t)$ is a ***negative-energy*** (and negative-frequency) solution of Eq. (1). As mentioned above, it will arive at a point a distance x from the source at a time $t = x/c$.***before*** the instant of emission. For this reason, it is called an ***advanced*** wave. Solution $E_-(x,t)$, on the other hand, is the more familiar positive-energy solution of Eq. (1). It arrives at x a time $t = x/c$ after the instant of emission and is called the ***retarded*** solution.

It should be clear, therefore, that advanced wave solution is inherent in the classical electromagnetic wave equations, without having to resort to Cramer's transactional interpretation of QM.

Next, we are going to discuss physical interpretation of such an advanced wave solution.

4. Interpretation of Advanced Wave Solution: Precognitive Interdiction

The above analysis by Cramer which seems to suggest that EPR paradox disappears when considering the advanced waves to be real physical entities, has been suggested by other physicists too, notably, Costa de Beauregard and also Luigi Fantappie [1a]. While working on quantum mechanics and special relativity equations, Fantappie noted that that retarded waves (retarded potentials) are governed by the law of entropy, while the advanced waves are governed by a symmetrical law that he named "*syntropy*."[3]

Therefore, some psychologists who work in this area began to make connection between the notion of syntropy and precognitive interdiction. And recently, a new journal by title *Syntropy* has been started to facilitate such a discussion.[1]

But again let us emphasize here that equation (3) and (4) indicate that the advanced wave solutions have *purely classical* origin. Therefore, we do not discuss yet their connection with other alleged QM phenomena such as collapsing wave function which is hardly possible to prove experimentally, despite Bohr and Heisenberg insisted that such a phenomenon is real. This is our departure to QM's inspired syntropy discussions in [3-6].

Our knowledge in this area is very limited, but we can expect that research in this direction of precognitive interdiction will flourish in the near future, once we can accept that it is purely classical origin, so we do not have to invoke complicated QM arguments.

[1] url: http://www.syntropy.org/journal-english

5. Problem with Western medicine & a post-colonial reading of Gen. 2:7

Realizing that there are many things we still do not know and many stones remain to be lifted, now allow us to discuss shortly on deep problems with the so-called Western Medicine.

There are several scientific authors who describe fundamental problems with modern (Western) medicine. The fundamental problem is commonly expressed with a *mechanistic* worldview or Cartesian *dualism* philosophy.[9] Sheldrake revealed that such a mechanistic view is actually derived from *Neo-Platonic* philosophy, so it is not based on biblical teaching.[9][13]

A similar argument was developed by Fritjof Capra in his famous book, *The Turning Point*.[11] Similarly, a Christian philosopher Alvin Plantinga has written a paper criticizing *materialism*.[14]

Unfortunately, however, the thinking of scientists from such disciplines often fails in the midst of massive dis-information (and advertising) that modern (Western) medicine has managed to address almost all human health problems. Is that true?

Let's take a look at the colonial post-reading of Gen. 2: 7 and some other texts.

If we read closely Gen. 2: 7, we see at a glance that man is made up of the dust of the ground (*adamah*) then it was made to live by the breath of life by God (*nephesh*). Here we can ask, does this text really support the Cartesian dualism view?

We do not think so, because the Hebrew concept of man and life is integral. The bottom line: it is not the spirit that is trapped in the body (Platonic), but the body is flowing in the ocean of spirit.[10]

This means that we must think as an open possibility for developing an integral treatment approach (Ken Wilber), or perhaps it can be more properly called "**spirit-filled medicine**."[12]

Let's look at three more texts:

a. Gen. 1: 2: "*The earth is without form and void, darkness over the deep, and the Spirit of God hovering over the waters.*" In Hebrew:[2] וְהָאָרֶץ, הָיְתָה תֹהוּ וָבֹהוּ, וְחֹשֶׁךְ, עַל-פְּנֵי תְהוֹם; וְרוּחַ אֱלֹהִים, מְרַחֶפֶת עַל-פְּנֵי הַמָּיִם.

   Patterns such as Adam's creation can also be encountered in the creation story of the universe. Earth and the oceans already existed (similar to adamah), but still empty and formless. Then the Spirit of God hovered over it; in the original text "*ruach*" can be interpreted as a strong wind (storm). So we can imagine there was strong

[2] https://www.mechon-mamre.org/p/pt/pt0101.htm

wind/hurricane, then in the storm God said, and there was the beginning of creation process of the universe. From a scientific point of view, it is well known in aerodynamics that turbulence can cause sound (turbulence-generated sound). And primordial sound waves are indeed observed by astronomers.

b. Ps. 107: 25, "*He said, he raised up a storm that lifted up his waves*." The relation between the word (sound) and the storm (turbulence) is interactive. Which one can cause other. That is, God can speak and then storms, or the Spirit of God causes a storm. Then came the voice.[3]

c. Ezekiel. 37: 7, "*Then I prophesy as I am commanded, and as soon as I prophesy, it sounds, indeed, a crackling sound, and the bones meet with one another."* In Ezekiel it appears that the story of the creation of Adam is repeated, that the Spirit of God is blowing (storm), then the sound of the dead bones arises.

The conclusion of the three verses above seems to be that man is made up of adamah which is animated by the breath or Spirit of God. He is not matter, more accurately referred to as spirit in matter. Like a popular song around 80s goes: "*We are spirits in the material world*." See also Amos Yong [10]. Therefore, it is *inappropriate* to develop only materialistic or Cartesian dualism treatment. We should develop a more integral approach, based on integral view of anthropology.[9]

The integral view of humanity and spirituality, instead of *two-tiered* Western view of the world, appears to be more in line with majority of people in underdeveloping countries, especially in Asia and Africa. See for instance the work by Paul Hiebert. [16][17]

Among recent studies supporting such an integral approach is the view that *cells are waves*, see the paper from Prof. Luc Montagnier.[15][15a][18] Interested readers may also see our paper on the wave nature of matter, as well as the possibility of developing a wave-based (cancer) treatment. See our papers on this topic.[19-20]

6. A few cautious remarks on the danger of spiritism

While we argue in favor of returning the “spirit” into modern science and medicine, we also wish to make a few cautious remarks on the danger of “spiritism.”

But first of all, allow us to quote an interesting discussion on the problem of modern theology discourse:

> *Theologia* as a term which means 'reasoned discourse about God' or 'the doctrine of God' was probably invented by Plato and has been adopted into Christianity for the systematic study and

[3] Our idea of creation process from great turbulence has been reported in this journal over the last several years, see for instance: V. Christianto & F. Smarandache, Thinking Out Loud on Primeval Atom, Big Bang & Biblical Creation, *SciGod J.* vol. 1 no. 9 (2018), url: http://scigod.com/index.php/sgj/article/view/603

> presentation of topics relating to God. But in its wider connotations 'theology' is the systematic and scientific study of religion generally… It has been fashionable of late for influential theologians like R. Bultmann and R. H. Fuller to disavow the existence and influence of the evil spirits spoken of in the New Testament. This is supposedly *because of their modern 'scientific' or positivistic outlook, which asserts that only that which is scientifically verifiable by any of the five senses may be said to exist*. Evil spirits do not belong to this category, therefore they do not exist."[23]

So, we hope the readers begin to realize where the problem began: it started from positive philosophy influence to theology fields, which ultimately result in reluctance or skepticism to accept the reality of evil spirits. But in the post-modern era, such a reality of evil spirit has been accepted again along with critics by missiology experts like Paul Hiebert, who called such a Cartesian reductionistic mind-body dualism: "*the flaw of excluded middle*."[16][17]

However, we heard that "spiritism" is still widely practiced in many regions in Africa, Latin America, and also Asia.[4] While Christian believers in those regions should understand that reality (may be an old practice inherited from their ancestors), it does not mean they can invite those spiritism practices into their Christian life, otherwise there may be conflicts between their Christian faith and various forms of spiritism rituals (occultism). Nonetheless, Christian believers are called to encounter with those evil spirits when the situation calls them to do so.

Apart from such a theologian viewpoint, there were extensive experiments on physical mediumism, spiritism etc. by scientists in attempt to put this kind of research within domain of psychology and psychiatrists. For instance, researches in this area have been pioneered in Italy by Enrico Morselli, Tamburini *et al*.[24]

7. Conclusion

There are various supernatural phenomena which hardly can be explained by the existing electromagnetic science, for instance non-locality interactions (which may be associated with ESP phenomena etc.), and also precognitive interdictions. And there are other problems such as how to include the Spirit into our consciousness. See our recent papers where we discuss such a possibility of new consciousness model beyond Freudian mental model, which includes the "spirit."[25-26]

It has been known for long time that intuition plays significant role in many professions and various aspects of human life, including in entrepreneurship, government, and also in detective or law enforcement activities. Even women are known to possess better intuitive feelings or

[4] For an introduction to spiritism and other diabolical sects in Latin America etc., see for instance: Umberto Eco, *Foucault's Pendulum*. url: http://www.postmodernmystery.com/foucaults_pendulum.html

"hunch" compared to men. Despite these examples, such a precognitive interdiction (hunch) is hardly accepted in established science.

In this paper, we discuss briefly the advanced solutions of Maxwell equations, and then make connection between syntropy and precognition from classical physics perspective. This may be regarded as a first step to describe such precognition activities which are usually considered belong to quantum realm.

Further observations and experiments are recommended to verify the above propositions.

Acknowledgement: Discussions with Dr. Robert N. Boyd are noted and appreciated. One of us (VC) would extend his deepest gratitude to Jesus Christ and Holy Spirit who always guide him in all valleys of darkness. Jesus Christ is the *Good Shepherd.*

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Thinking Out Loud on Primeval Atom, Big Bang & Biblical Account of Creation

Victor Christianto, Florentin Smarandache

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Abstract

In recent years, the Big Bang as described by the Lambda CDM-Standard Model Cosmology has become widely accepted by majority of physics and cosmology communities. Some people even have concluded that it has no serious alternative in horizon. Is that true? First, as we argued elsewhere, Big Bang relies on singularity, so, when we are able to describe the observed data without invoking singularity, then Big Bang model is no longer required. In this paper, we explore a few alternatives other than Big Bang which most cosmologists believe is the closest to biblical account of creation. We argue that re-reading of Genesis 1:2 will lead one to another viable model, albeit it has not been developed rigorously as LCDM theories. We also briefly discuss a fluid Maxwell equations of Tsutomu Kambe based on vortex sound theory.

Keywords: Electromagnetic theory, Maxwell, fluid, singularity-free, cosmology model, vortex sound theory, early Universe, Genesis, Spirit, Creation.

1. Introduction

One of the biggest mysteries in cosmogony and cosmology studies is perhaps: *How to interpret properly Genesis chapter 1:2*. Traditionally, philosophers proposed that God created the Universe out of nothingness (from reading “empty and formless” and “*bara*” words; this contention is called “*creation ex nihilo*.”). Understandably, such a model can lead to various interpretations, including the notorious “cosmic egg” (primeval atom) model as suggested by Georges Lemaitre, which then led to Big Bang model.[18-20] Subsequently, many cosmologists accept it without asking, that Big Bang stands as the most faithful and nearest theory to Biblical account of creation. But we can ask: Is that cosmic egg model the true and faithful reading of Genesis 1:2?

In the subsequent chapter we will discuss how to answer this question by the lens of hermeneutics of Sherlock Holmes. This is a tool of mind which we think to be a better way compared to critical hermeneutics.

Now a word on the meaning of thinking out loud phrase. What we mean with this phrase is, according to a definition:

> **Thinking out loud** is the act of expressing in recoverable and external form new thoughts which you encourage your mind into exploring. Often these lead to new avenues of thought. When you **think out loud** you detect and explore ideas and concepts which are either unknown, or as yet unexplored.[1]

2. Several different interpretations of Genesis 1:2 & implications

Our discussion starts from the fundamental question that one of us (VC) has heard around three years ago (January 2015). At the time, he (VC) had a good time of conversation with a senior pastor who happens to be one of the most leading scholars from Jakarta Theology and Philosophy Seminary, i.e. Dr. Joas Adiprasetya (JA). VC tried to explain to him his idea on interpreting of Prolegomena of John Gospel as one of reliable biblical account of creation. In essence, VC told JA that it appears possible to interpret the Logos (in John 1:1) as the Sacred Voice of God, then from voice we can infer sound wave, then from sound wave we can infer frequency. Therefore, we can infer that there should be primordial/relic sound wave which emerged at the earliest time of creation. [10-13] And Prof. Wayne Hu & Martin White has written a paper about observation of such relic sound wave.[21]

But JA asked him (VC): okay, then where was the role of Holy Spirit in that creation story based on John 1:1? VC should admit that at the time he cannot come up with a convincing answer. He only said: "I do not think of that yet."

And it took around three years before now we have been thinking this problem out loud, and here our answer can be summarized as follows: "The relic sound wave in early creation is a faithful interpretation of John 1:1, but we can come up with a more complete picture if we combine it with Gen. 1:2, that is the Holy Spirit came to hovering over the primordial fluid, then a kind of hurricane/storm started which created perfect medium where God spoke (Logos)."

Let us consider some biblical passages:

- What is *Hermeneutics of Sherlock Holmes*?
  One article suggests:[2]
  Holmes: "I have no data yet. It is a capital mistake to theorize before one has data. Insensibly one begins to twist facts to suit theories, instead of theories to suit facts."

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[1] wiki.c2.com/?ThinkingOutLoud
[2] https://www.str.org/blog/learning-hermeneutics-from-holmes

Far too often students of the Bible (and cosmology folks as well) twist verses to suit interpretations instead of formulating interpretations to suit what the verses say.
Guide: Don't approach your passage assuming you know what it means. Rather, use the data in the passage – the words that are used and how they fit together – to point you toward the correct interpretation.

- A re-reading of Gen. 2:7 with Hermeneutics of Sherlock Holmes[3]

If we glance at Gen. 2: 7, we see at a glance that man is made up of the dust of the ground (*adamah*) which is breathed by the breath of life by God (*nephesh*). Here we can ask, does this text really support the Cartesian dualism view?

We do not think so, because the Hebrew concept of man and life is integral. The bottom line: it is not the spirit trapped in the body (Platonic), but the body is flowing in the ocean of spirit. [7]

Let's look at three more texts:

a. Gen. 1: 2, "*The earth is without form and void, darkness over the deep, and the Spirit of God hovering over the waters*." Patterns such as Adam's creation can also be encountered in the creation story of the universe. Earth and the oceans already exist (similar to *adamah*), but still empty and formless. Then the Spirit of God hovered over it, in the original text "*ruach*" can be interpreted as a strong wind (storm). So we can imagine there is wind/hurricane, then in the storm that God said, and there was the creation of the universe. See also Amos Yong [6], also Hildebrandt [15]. From a scientific point of view, it is well known in aerodynamics that turbulence can cause sound (*turbulence-generated sound*). And primordial sound waves are indeed observed by astronomers.

b. Ps. 107: 25, "*He said, he raised up a storm that lifted up his waves*." The relation between the word (sound) and the storm (turbulence) is interactive. Which one can cause other. That is, God can speak and then storms, or the Spirit of God causes a storm. Then came the voice.

c. Ezekiel. 37: 7, "*Then I prophesy as I am commanded, and as soon as I prophesy, it sounds, indeed, a crackling sound, and the bones meet with one another."* In Ezekiel it appears that the story of the creation of Adam is repeated, that the Spirit of God is blowing (storm), then the sound of the dead bones arises.

[3] Check Eric McKiddie's article: https://www.thegospelcoalition.org/blogs/trevin-wax/10-tips-on-solving-mysterious-bible-passages-from-sherlock-holmes/

The conclusion of the three verses above seems to be that man is made up of *adamah* which is animated by the breath or Spirit of God. He is not matter, more accurately referred to as spirit in matter. Like a popular song around 80s goes: "*We are spirits in the material world*."

3. A physical model of turbulence-generated sound for early Universe

Our discussion starts from the fundamental question: how can we include the rotation in early Universe model? After answering that question, we will discuss how "turbulence-generated sound" can be put into a mathematical model for the early Universe. We are aware that the notion of turbulence-generated sound is not new term at all especially in aerodynamics, but the term is rarely used in cosmology until now. We shall show that 3D Navier-Stokes will lead to non-linear acoustics models, which means that a turbulence/storm can generate sound wave.

a. <u>How can we include rotation in early Universe model?</u>

It has been known for long time that most of the existing cosmology models have singularity problem. Cosmological singularity has been a consequence of excessive symmetry of flow, such as "Hubble's law". More realistic one is suggested, based on Newtonian cosmology model but here we include the vortical-rotational effect of the whole Universe.

In this section, we will derive an Ermakov-type equation following Nurgaliev [8]. Then we will solve it numerically using Mathematica 11.

After he proceeds with some initial assumptions, Nurgaliev obtained a new simple local cosmological equation:[8][9]

$$\dot{H} + H^2 = \omega^2 + \frac{4\pi G}{3}\rho, \tag{1}$$

where $\dot{H} = dH/dt$.

The angular momentum conservation law ωR^2 = const = K and the mass conservation law $(4\pi/3)\rho R^3$ = const = M makes equation (5) solvable:[9]

$$\dot{\mathrm{H}} + \mathrm{H}^2 = \frac{K^2}{R^4} - \frac{\mathrm{GM}}{\mathrm{R}^3}, \tag{2}$$

or

$$\ddot{R} = \frac{K^2}{R^3} - \frac{GM}{R^2}. \tag{3}$$

Equation (3) may be written as Ermakov-type nonlinear equation as follows;

$$\ddot{R}+\frac{GM}{R^2}=\frac{K^2}{R^3}. \tag{4}$$

Nurgaliev tried to integrate equation (3), but now we will solve the above equation with Mathematica 11. First, we will rewrite this equation by replacing GM=A, K^2=B, so we get:

$$\ddot{R}+\frac{A}{R^2}=\frac{B}{R^3}. \tag{5}$$

As with what Nurgaliev did in [8][9], we also tried different sets of A and B values, as follows:

a. A and B < 0

```
A=-10;
B=-10;
ODE=x"[t]+A/x[t]^2-B/x[t]^3==0;
sol=NDSolve[{ODE,x[0]==1,x'[0]==1},x[t],{t,-10,10}]
Plot[x[t]/.sol,{t,-10,10}]
```

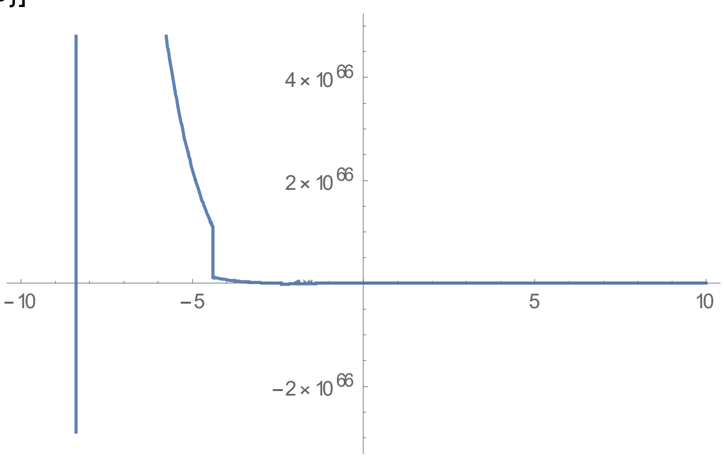

Figure 1. Plot of Ermakov-type solution for A=-10, B=-10

b. A > 0, B < 0

```
A=1;
B=-10;
ODE=x"[t]+A/x[t]^2-B/x[t]^3==0;
sol=NDSolve[{ODE,x[0]==1,x'[0]==1},x[t],{t,-10,10}]
Plot[x[t]/.sol,{t,-10,10}]
```

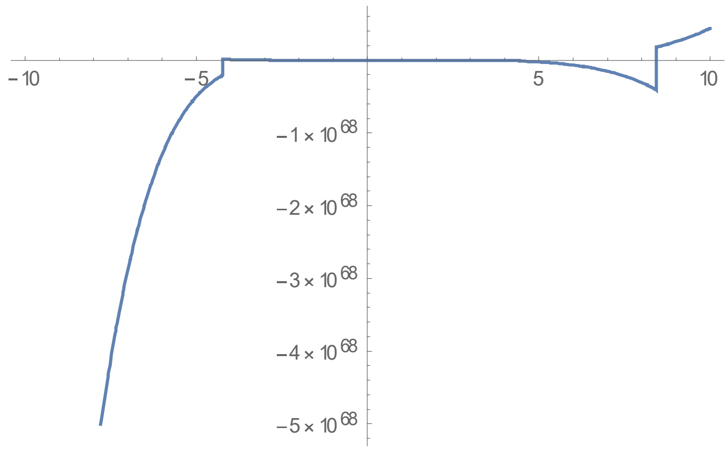

Figure 2. Plot of Ermakov-type solution for A=1, B=-10

From the above computational experiments, we conclude that the evolution of the Universe depends on the constants involved, especially on the rotational-vortex structure of the Universe. This needs to be investigated in more detailed for sure.

One conclusion that we may derive especially from Figure 2, is that our computational simulation suggests that it is possible to consider that the Universe has existed for long time in prolonged stagnation period, then suddenly it burst out from *empty and formless* (Gen. 1:2), to take its current shape with observed "accelerated expansion."

As an implication, we may arrive at a precise model of flattening velocity of galaxies without having to invoke *ad-hoc* assumptions such as dark matter.

Therefore, it is perhaps noteworthy to discuss briefly a simple model of galaxies based on a postulate of turbulence vortices which govern the galaxy dynamics. The result of Vatistas' model equation can yield prediction which is close to observation, as shown in the following diagram:[14]

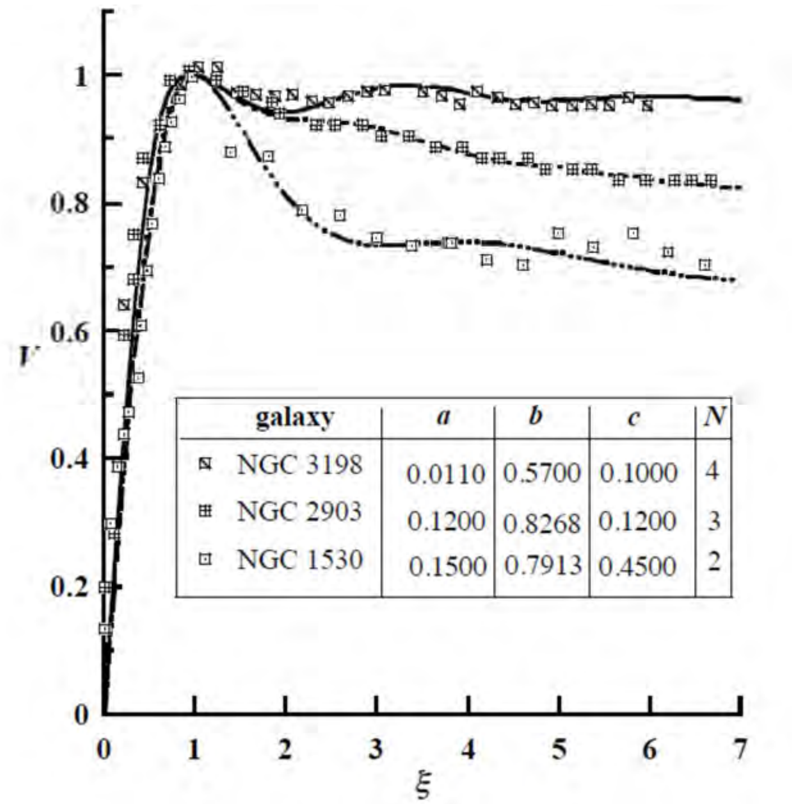

Figure 3. From Vatistas [14]

Therefore, it appears possible to model galaxies without invoking numerous *ad hoc* assumptions such as *dark matter*, once we accept the existence of turbulent interstellar medium. The Vatistas model is also governed by Navier-Stokes equations, see for instance [14].

b. <u>*How "turbulence-generated sound" can be put into a mathematical model for the early Universe*</u>

We are aware that the notion of turbulence-generated sound is not new term at all especially in aerodynamics, but the term is rarely used in cosmology until now. We will consider some papers where it can be shown that 3D Navier-Stokes will lead to non-linear acoustics models, which means that a turbulence/storm can generate sound wave.

In this section we consider only two approaches:

- Shugaev-Cherkasov-Solenaya's model: They investigate acoustic radiation emitted by three-dimensional (3D) vortex rings in air on the basis of the unsteady Navier–Stokes equations. Power series expansions of the unknown functions with respect to the initial vorticity which is supposed to be small are used. In such a manner the system of the Navier–Stokes equations is reduced to a parabolic system with constant coefficients at high derivatives. [16]

- Rozanova-Pierrat's Kuznetsov equation: she analysed the existing derivation of the models of non-linear acoustics such as the Kuznetsov equation, the NPE equation and the KZK equation. The technique of introducing a corrector in the derivation ansatz allows to consider the solutions of these equations as approximations of the solution of the initial system (a compressible Navier-Stokes/Euler system). The direct derivation shows that the Kuznetzov equation is the first order approximation of the Navier-Stokes system, the KZK and NPE equations are the first order approximations of the Kuznetzov equation and the second order approximations of the Navier-Stokes system. [17]

4. Vortex-sound theory and fluidic Maxwell equations

There are a number of proposals to revise Maxwell equations. But few has considered a fresh starting point with regards to the (sub-)structure of aether. It is very interesting to note that Prof. T. Kambe from University of Tokyo has made a connection between the equation of vortex-sound theory and its analogue fluid Maxwell equations. He wrote that it would be no exaggeration to say that any vortex motion excites *acoustic* waves. [2]

He considers the equation of vortex sound of the form: [2]

$$\frac{1}{c^2}\partial_t^2 p - \nabla^2 p = \rho_0 \nabla.L = \rho_0 div(\omega \times v) \tag{6}$$

He also wrote that dipolar emission by the vortex-body interaction is:[2]

$$p_F(x,t) = -\frac{P_0}{4\pi c}\dot{\Pi}_i(t - \frac{x}{c})\frac{x_c}{x^2} \tag{7}$$

Then he obtained an expression of fluid Maxwell equations as follows [2]:

$$\begin{aligned}
&\nabla \cdot H = 0 \\
&\nabla \cdot E = q \\
&\nabla \times E + \partial_t H = 0 \\
&a_0^2 \nabla \times H - \partial_t E = J
\end{aligned} \tag{8}$$

Where [2]:

a_0 denotes the sound speed, and

$$\begin{aligned}
&q = -\partial_t(\nabla \cdot \upsilon) - \nabla \hbar, \\
&J = \partial_t^2 v + \nabla \partial_t h + a_o^2 \nabla \times (\nabla \times \upsilon)
\end{aligned} \tag{9}$$

In our opinion, this new expression of fluid Maxwell equations suggests that there is a deep connection between vortex sound and electromagnetic fields.

However, it should be noted that the above expressions based on fluid dynamics need to be verified with experiments. We should note also that in (8) and (9), the speed of sound a_0 is analogous of the speed of light in Maxwell equations, whereas in equation (6), the speed of sound is designated "c" (as analogous to the light speed in EM wave equation).

As an added note, we can mention here that elsewhere Wang [5] was able to derive Coulomb law from the source-sink approach. We are wondering if it is also possible to re-derive Maxwell equations including displacement current from the same approach. If yes, then it may offer another fresh starting point to understand the physical meaning of displacement current.

5. Conclusions

In recent years, there is growing number of proposals to use a novel concept of singularity-free Cosmology models. It should be clear that if we are able to come up with such singularity-free models which agree well with observation data, then the Big Bang model is no longer required. Therefore, here we explore a few alternative stories other than Big Bang story, which most cosmologists believe it is the nearest to Biblical account of creation (as Fred Hoyle once remarked: the Big Bang is a fanatical religion).

We argue that a re-reading of Genesis 1:2 will lead us to another viable story, albeit the alternative has not been developed rigorously as LCDM theories.

It took around three years before now we have been thinking this problem out loud, and here our answer can be summarized as follows: "*The relic sound wave in early creation is a faithful interpretation of John 1:1, but we can come up with a more complete picture if we combine it with Gen. 1:2, that is the Holy Spirit came to hovering over the primordial fluid, then a kind of hurricane/storm started which created perfect medium where God spoke (Logos).*"

And one conclusion that we may derive especially from Figure 2, is that our computational simulation suggests that it is possible to consider that the Universe has existed for long time in prolonged stagnation period, then suddenly it burst out from *empty and formless (Gen. 1:2)*, to take its current shape which is accelerating. Such a possibility has never been considered before in cosmology literatures.

We also briefly discuss a plausible extension of Maxwell equations based on vortex sound theory of Tsutomu Kambe. It is our hope that our exploration will lead to nonlinear cosmology theories which are better in terms of observations, and also more faithful to Biblical account of creation.

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On Maxwell-Dirac Isomorphism

Victor Christianto, Florentin Smarandache

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Abstract

In this paper, we discuss Maxwell-Dirac isomorphism and quantum entanglement.

Keywords: Quantum entanglement, metaphysics, realism, Maxwell-Dirac isomorphism.

1. Introduction

In its simplest form, the features of quantum theory can be reduced to: (a) a wave function description of microscopic entities; and (b) entanglement. Entanglement is a key property that makes quantum information theory different from its classical counterpart [14].

But what is entanglement? Wootter gives one of clearest description [13]:

> In both classical mechanics and quantum mechanics, one can define a pure state to be a state that is as completely specified as the theory allows. In classical mechanics a pure state might be represented by a point in phase space. In quantum mechanics it is a vector in a complex vector space. Perhaps the most remarkable feature of quantum mechanics, a feature that clearly distinguishes it from classical physics, is this: for any composite system, there exist pure states of the system in which the parts of the system do not have pure states of their own. Such states are called entangled.

According to Scolarici and Solombrino [5]:

> The essential difference in the concept of state in classical and quantum mechanics is clearly pointed out by the phenomenon of entanglement, which may occur whenever the product states of a compound quantum system are superposed. Entangled states play a key role in all controversial features of QM; moreover, the recent developments in quantum information theory have shown that entanglement can be considered a concrete physical resource that it is important to identify, quantify and classify.

Nonetheless, they concluded that "our research has pointed out a puzzling situation, in which the same state of a physical system is entangled in CQM, while it seems to be separable in QQM."

While entanglement is usually considered as purely quantum effect, it by no means excludes the possibility to describe it in a classical way.

In this regard and from the history of QM, we learn that there were many efforts to describe QM features in a more or less classical picture. For example, in 1927 Einstein presented his version of the hidden variable theory of QM, starting from Schrödinger’s picture, which seems to influence his later insistence that “*God does not play dice*” [6][7].

Efforts have also been made to extend QM to QQM (quaternionic quantum mechanics), for instance by Stephen Adler from IAS [8].

In recent decades, however, another route began to appear, which may be called the Maxwell-Dirac isomorphism route, where it can be shown that there is close link between Maxwell’s equations of classical electromagnetism and the Dirac equation of the electron. Intuitively, this may suggest that there is a one-to-one correspondence between the electromagnetic wave and quantum wave function. But can it offer a classical description of entanglement? This problem will be explored in the next sections.

2. A Few Alternatives of a Realistic Maxwell-Dirac Isomorphism

There are some papers dealing with the formal connection between classical electrodynamics and wave mechanics, especially there are some existing proofs on the Maxwell-Dirac isomorphism. We will review here two derivations of the Maxwell-Dirac isomorphism, i.e., by Hans Sallhofer and Volodimir Simulik. In the last section, we will also discuss a third option, i.e., by exploring the Maxwell-Dirac isomorphism through quaternionic language.

a. Sallhofer’s method

Summing up one of Sallhofer's papers [1], he says that under the sufficiently general assumption of periodic time dependence, the following connection exists between source-free electro-dynamics and wave mechanics:

$$\sigma \cdot \left[\begin{array}{c} rotE + \frac{\mu}{c}\frac{\partial}{\partial t}H = 0 \\ rotH - \frac{\varepsilon}{c}\frac{\partial}{\partial t}H = 0 \\ div\varepsilon E = 0 \\ div\mu H = 0 \end{array} \right]_{divE=0} \equiv \left[(\gamma \cdot \nabla + \gamma^{(4)}\partial_4)\Psi = 0 \right] \quad (1)$$

That is, the multiplication of source-free electrodynamics by the Pauli-vector yields wave mechanics [1].

In simple terms, this result can be written as follows:

$$P \cdot M = D, \tag{2}$$

where P = Pauli vector, M = Maxwell's equations and D = the Dirac equation.

We can also say that wave mechanics is a solution-transform of electrodynamics. Here, one has to bear in mind that the well-known circulatory structure of the wave functions, manifest in Dirac's hydrogen solution, is not introduced just by the Pauli-vector [1].

b. Simulik's method

Simulik described another derivation of the Maxwell-Dirac isomorphism. In one of his papers [2], he wrote a theorem suggesting that Maxwell's equations of source-free electrodynamics which can be written as follows:

$$\begin{aligned} rotE + \frac{\mu}{c}\frac{\partial}{\partial t}H &= 0 \\ rotH - \frac{\varepsilon}{c}\frac{\partial}{\partial t}H &= 0 \\ divE &= 0 \\ divH &= 0 \end{aligned} \tag{3}$$

are equivalent to the Dirac-like equation [2]:

$$\left[\gamma.\nabla - \begin{pmatrix} \varepsilon 1 & 0 \\ 0 & \mu 1 \end{pmatrix} \frac{1}{c}\frac{\partial}{\partial t}\right]\Psi^{c1} = 1, \tag{4}$$

where in the usual representation

$$\gamma = \begin{pmatrix} 0 & \sigma \\ \sigma & 0 \end{pmatrix}, \tag{5}$$

and σ are the well-known Pauli matrices.

c. Maxwell-Dirac isomorphism through Quaternionic language

In text books, quantum theory is based on complex numbers of the form $a_0+a_1 i$, with i being the imaginary unit $i^2 = -1$. It has long been known that an alternative quantum mechanics can be based on the quaternion or hyper-complex numbers of the form $a_0 + a_1 i + a_2 j + a_3 k$, with i, j, k being three non-commuting imaginary units [8].

On the other hand, recognizing that Maxwell's equations were originally formulated in terms of quaternionic language, some authors investigated whether there could be a formal correspondence between Maxwell's and Dirac's equations. Kravchenko and Arbab are a few researchers who worked on this problem. And also the present authors arrived at a similar conclusion despite using different procedures based on the Gersten decomposition of the Dirac equation [4].

This MD isomorphism can also be extended further to the classical description of boson mass, which was usually called the Higgs boson [3], so it may be a simpler option compared to scale symmetry theory.

3. Quaternionic QM & Entanglement

Singh & Prabakaran are motivated to examine the geometry of a two-qubit quantum state using the formalism of the Hopf map. The "quaternions" again come in handy in studying the two-qubit state. [10]

In his exposition of Quaternionic Quantum Mechanics, Singh concluded that [9]:

> Having established the compatibility of the Hopf fibration representation with the conventional theory for unentangled states, let us, now, address the issue of measurability of entanglement in this formalism. In the context, "Wootters' Concurrence" and the related "Entanglement of Formation" constitute well accepted measures of entanglement, particularly so, for pure states. …It follows that any real linear combination of the "magic basis" would result in a fully entangled state with unit concurrence. Conversely, any completely entangled state can be written as a linear combination in the "magic basis" with real components, up to an overall phase factor. In fact, these properties are not unique to a state description in the "magic basis" and hold in any other basis that is obtained from the "magic basis" by an orthogonal transformation…

In a rather different way, Najarbashi et al. explored quaternionic Möbius transformations, which can be useful in theoretical physics in areas such as quaternionic quantum mechanics, quantum conformal field theory and quaternionic computations [11]. They found that "*as in the case of two-qubits, both octonionic stereographic projection and Möbius transformation are entanglement sensitive*."

5. Discussions & Conclusion

Despite its enormous practical success, many physicists and philosophers alike agree that quantum theory is full of contradictions and paradoxes that are difficult to solve consistently. Even after 90 years, experts still do not agree about what to make of it.

In the meantime, the problem of the formal connection between electrodynamics and wave mechanics has attracted the attention of a number of authors, especially there are some existing proofs on Maxwell-Dirac isomorphism. Here the author reviews two derivations of the Maxwell-Dirac isomorphism by Hans Sallhofer and Volodimir Simulik as well as quaternion language.

While this paper does not conclusively answer the question of whether the Maxwell-Dirac isomorphism and especially its quaternionic formulation can offer a classical description of entanglement, we have mentioned some recent discussions on this topic such the Hopf map and quaternionic Möbius transformations.

This paper was inspired by an old question: *Is there a consistent and realistic description of the wave function, both classically and quantum mechanically?* It can be expected that the above discussions will shed some light on such an old problem especially in the context of the physical meaning of the quantum wave function. This is reserved for further investigations.[1]

Acknowledgement: Special thanks to Prof. Thee Houw Liong for bringing up future science and technology issues in a recent RG forum.

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Lógica neutrosófica refinada n-valuada y sus aplicaciones a la física
n-Valued Refined Neutrosophic Logic and Its Applications to Physics

Florentin Smarandache

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Abstract: In this paper we present a short history of logics: from particular cases of 2-symbol or numerical valued logic to the general case of n-symbol or numerical valued logic. We show generalizations of 2-valued Boolean logic to fuzzy logic, also from the Kleene's and Lukasiewicz' 3-symbol valued logics or Belnap's *4*-symbol valued logic to the most general *n-symbol or numerical valued refined neutrosophic logic*. Examples of applications of neutrosophic logic to physics are listed in the last section. Similar generalizations can be done for *n-Valued Refined Neutrosophic Set*, and respectively

Keywords: n-symbol valued logic.Neutrosophic Logic, física neutrosófica, física paradoxista

1 Lógica de 2 valores

1.1 Lógica valuada en dos símbolos

En la filosofía china: Yin y Yang (o Feminidad y Masculinidad) se representan como contrarios::

Fig. 1: Ying y Yang

También en la lógica clásica o booleana, se tienen dos valores: verdad T y falsedad F.

1.2 Lógica de dos valores numéricos

También es la lógica clásica o booleana, se tiene dos valores numéricos: verdad 1 y falsedad 0. Más general es la lógica difusa, donde la verdad (T) y la falsedad (F) pueden ser cualquier número en [0, 1] tal que T + F = 1.
Aún más general, T y F pueden ser subconjuntos de [0,1].

2 Lógica de tres valores

2.1 Lógicas trivalente con tres símbolos

1. *Lógica de Lukasiewicz* : Verdadero, Falso, y Posible.
2. *Lógica de Kleene*: Verdadero, Falso, Desconocido (o Indefinido).
3. Filosofía China extendida a: *Yin, Yang,* y *Neutro* (o Feminidad, Masculinidad, y Neutralidad)- como en la neutrosofía.

4. La filosofía neutrosófica surgió de la neutralidad entre varias filosofías. Conectada con la exténica (Prof. Cai Wen, 1983), y el paradoxismo (F. Smarandache, 1980). La neutrosofía es una nueva rama de la filosofía que estudia el origen naturaleza y alcance de las neutralidades. Esta teoría considera cualquier noción o idea A junto a su opuesto o negación AntiA y el espectro de neutralidades neutA entre ellas (nociones o ideas que no soportan ni a A ni antiA) . NeutA y AntiA juntas se les conoce con noA . La neutrosofía es una generalización de la dialéctica de Hegel (esta solo se basa en A y antiA). De acuerdo a esta teoría toda idea A tiende a ser neutralizada y balanceada por antiA y noA como un estado de equilibrio. De una forma clásica A, neutA y antiA son disjuntos dos por dos. Sin embargo en la mayoría de los casos lo límites entre ellos resultan vagos e imprecisos. La neutrosofía es la base de todas las teorías neutrosóficas con múltiples aplicaciones a la ingeniería (especialmente en la ingeniería de software y la fusión de la información), medicina, militares, aeroespaciales, cibernética y física.

2.2 Lógica numericamente valudas de tres valores

1. *Lógica de Kleene*: Verdadero (*1*), Falso (*0*), Desconocido (o Indefinido) (*1/2*), y utiliza —min” para $\wedge$, —max” para $\vee$, y —1-” para la negación.
2. Más general resulta la lógica [Smarandache, 1995], donde la verdad (*T*) , la falsedad (*F*) y la indeterminación (*I*) pueden ser números en el intervalo *[0, 1]*, entonces : $0 \leq T + I + F \leq 3$.

3 Lógica de cuatro valores

3.1 Lógica de valuda en cuatro símboles Lógica

1. *Lógica de Belnap*: Verdadero (*T*), Falso (*F*), Desconocido(*U*) , y Contradicción (*C*), donde *T*, *F*, U, *C* son símbolo. A continuación la tabla del operador de conjunción de Belnap,

	F	U	C	T
F	F	F	F	F
U	F	U	F	U
C	F	F	C	C
T	F	U	C	T

Restrigda a *T,F,U*, y a *T,F,C*,los conectores d ela lógica de Belnap coincide con las conectivas lógicas de la lógica de Kleene.

2. Sea *G* = Ignorancia. Se puede proponer la siguiente lógica de cuatro símbolos: *(T, F, U, G)*, y *(T, F, C, G)*.
3. *Realidad Absoluta-Relativa 2-, 3-, 4-, 5-, Lógica Valuada en 6 Símbolos* [Smarandache, 1995]. Sea verdadero en todos los mundos posibles (de acuerdo a la definición de Leibniz), sea verdadero en al menos uno de los mundos posibles pero no en los otros, y de forma similar sea indeterminado en todos los mundos posibles , sea indeterminado en al menos uno de los mundos y no en otros; adicionalmente sea falso en todos los mundos posibles pero no todos los mundos, sea falso en al menos uno pero no en todos. los mundos posibles, entonces podemos formar varias lógicas Absolutas-Relativas 2-, 3-, 4-, 5-, o lógica valuada en 6 símbolos solo tomando combinaciones de estos símbolos. O A lógica valuada en 6 símbolos

3.2 Lógica de 4 Valores Numéricos

La indeterminación I se refina (divide) como U = Desconocida, y C = contradicción. T, F, U, C son subconjuntos de [0, 1], en lugar de símbolos; Esta lógica generaliza la lógica de Belnap ya que uno obtiene un grado de verdad, un grado de falsedad, un grado de desconocimiento y un grado de contradicción..

4 Lógica de 4 valores

Lógica neutrosófica valorada en cinco símbolos [Smarandache, 1995]: la indeterminación I se refina (divide) como U = Desconocido, C = contradicción y G = ignorancia; donde los símbolos representan:

T = verdad;

F = falsedad;

U = ni T ni F (indefinido);

C = T $\wedge$F , involucra la Exténica;

G = T $\vee$F

. Si T, F, U, C, G son subconjuntos de [0, 1], entonces obtenemos: una lógica neutrosófica de cinco valores numéricos.

5 Lógica de n valores

1. La lógica neutrosófica de n valores simbólicos [Smarandache, 1995]. En general:

T se puede dividir en muchos tipos de verdades: $T_1, T_2, \ldots, T_p$, I en muchos tipos de nes: $I_1, I_2, \ldots, I_r$ y F en muchos tipos de falsedades: d $F_1, F_2, \ldots, F_s$onde todos $p, r, s \geq 1$ son enteros y $p + r + s = n$

Todos los subcomponentes, son símbolos de T_j, I_k, F_l , para todos $j \in \{1,2, \ldots, p\}, k \in \{1,2, \ldots, r\}$ e $l \in \{1,2, \ldots, s\}$.

2. *La lógica neutrosófica refinada de n-valor numérico.* De la misma manera, pero todos los subcomponentes T_j, I_k, F_l, no son símbolos, sino subconjuntos de [0,1], para todos $j \in \{1,2, \ldots, p\}, k \in \{1,2, \ldots, r\}$ e $l \in \{1,2, \ldots, s\}$. Si todas las fuentes de información que proporcionan valores neutrosóficos por separado para un subcomponente específico fuentes independientes, entonces en el caso general consideramos que cada uno de los subcomponentes T_j, I_k, F_l, es independiente con respecto a los demás y está en el conjunto no estándar $]^{-}0, 1^{+}[$. Por lo tanto, tenemos un total para los subcomponentes T_j, I_k, F_l, que:

$^{-}0 \leq \sum_{j=1}^{p} T_j + \sum_{k=1}^{r} I_k + \sum_{l=1}^{s} F_s \leq n^{+}$

Donde $p + r + s = n$, por supuesto, como arriba. Si hay algunas fuentes dependientes (o, respectivamente, algunos subcomponentes dependientes), podemos tratar esos subcomponentes dependientes juntos.

6. Distinción entre física neutrosófica y física paradoxista

En primer lugar se realiza un distinción entre la física neutrosófica y al física paradoxista

1. **Física Nuetrosófica**

Sea A una entidad física (es decir, concepto, noción, objeto, espacio, campo, idea, ley, propiedad, estado, atributo,teorema, teoría, etc.), antiA sea lo opuesto a A, y neutA sea su neutral (es decir, ni A ni antiA, sino en el medio).

La Física Neutrosófica es una mezcla de dos o tres de estas entidades A, antiA y neutA que se mantienen juntas.

Por lo tanto, podemos tener campos neutrosóficos y objetos neutrosóficos, estados neutrosóficos, etc.2. **Paradoxist Physics**

La Física Neutrosófica es una extensión de la Física Paradoxista, ya que la Física Paradoxista es una combinación de contradictorios físicos A y antiA solo que se mantienen unidos, sin referirse a su neutralidad neutA. La física paradójica describe las colecciones de objetos o estados que se caracterizan individualmente por propiedades contradictorias, o que se caracterizan ni por una propiedad ni por el opuesto de esa propiedad, o están compuestos de subelementos contradictorios. Tales objetos o estados se llaman entidades paradojas.

Estos dominios de investigación se establecieron en 1995 en el marco de la neutrosofía, lógica / conjunto / probabilidad / estadística neutrosóficas.

10 Lógica Neutrosófica N-valuada Refinada Aplicada a la Física

Hay muchos casos en los campos científicos (y también humanísticos) en los que dos o tres de estos elementos A, antiA y neutA coexisten simultáneamente.

Varios ejemplos de entidades paradójicas y neutrosóficas:

• los aniones en dos dimensiones espaciales son partículas de spin arbitrarias que no son ni bosones (integerspin) ni fermions (giro de medio entero);

• entre los posibles candidatos de Dark Matter, puede haber partículas exóticas que no sean fermentos de Dirac ni de Majorana;

• mercurio (Hg) es un estado que no es líquido ni sólido en condiciones normales a temperatura ambiente;

• los materiales no magnéticos no son ni ferromagnéticos ni antiferromagnéticos;

• quark gluon plasma (QGP) es una fase formada por quarks casi libres y gluones que no se comporta como un plasma convencional ni como un líquido ordinario;
• no relacionado, que está formado por la materia y la antimateria que se unen (F. Smarandache, 2004);
• kaon neutral, que es un compuesto pión y anti-pión (R. M. Santilli, 1978) y por lo tanto una forma de desapego;
• Métodos neutrosóficos en general relatividad (D. Rabounski, F. Smarandache, L. Borissova, 2005);
• modelo cosmológico neutrosófico (D. Rabounski, L. Borissova, 2011);
• gravitación neutrosófica (D. Rabounski);
• superposición cuántica y en general cuántica de estados;
• los semiconductores no son conductores ni aisladores;
• los componentes ópticos semi-transparentes no son ni opacos ni perfectamente transparentes a la luz;
• los estados cuánticos son metaestables (ni perfectamente estables ni inestables);
• doblete de fotones de neutrinos (E. Goldfain);
• el "multiplete" de partículas elementales es una especie de "campo neutrosófico" con dos o más valores (E. Goldfain, 2011);
• Un "campo de neutrosofía" se puede generalizar al de los operadores cuya acción es selectiva. El efecto del campo neutrosophic es de alguna manera equivalente con el “túnel” de la física de los sólidos, o con la “ruptura espontánea de simetría” (SSB) en la que hay una simetría interna que se rompe por una selección particular del estado de vacío (E. Goldfain). Etc.

Conclusiones

Muchos tipos de lógicas se han presentado arriba. Para la lógica más general, la lógica neutrosófica refinada n-valorada.Se hacen generalizaciones similares para el conjunto neutrosófico refinado n-valorado y la probabilidad neutrosófica refinada n-valorada

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Electron Model Based on Helmholtz's Electron Vortex Theory & Kolmogorov's Theory of Turbulence

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Abstract

In this paper, we explore a new electron model based on Helmholtz's electron vortex and Kolmogorov theory of turbulence. We also discuss a new model of origination of charge and matter.

Keywords: Electron model, Helmholtz, electron vortex, Kolmogorov, turbulence.

Introduction

In a previous paper [1], we explored Hilbert's axiomatic program to unify electromagnetic and gravitation theory, and we remarked that Godel's finding effectively put Hilbert program into ruins. Summarizing, it is very significant to consider *matter creation* process in nature. For instance, one can begin by considering the correct presentation of Newton's second law is not F=ma, but F=d(mv)/dt=v(dm/dt) + m(dv/dt). In other words, it is possible of matter creation (dm/dt), and this seems quite consistent with *Narlikar's work*.

There are various models of electron which have been suggested, for instance see Chekh *et. al.* [10]. We seek a more realistic electron model which is able to describe to experiments conducted by Winston Bostick *et. al.* [9]. In our attempt to explain such experiments of electron creation in plasma, allow us to come up with a new model of electron, based on *Helmholtz's electron vortex* theory. In turn, we will discuss a plausible model of electron capture event inside Earth (matter creation), which may serve a basis to explain Le Sage/Laplace's push gravity.

The Helmholtz vortex model of the electron is a toroid made of nested concentric toroidal flows of smaller particles, perhaps the inertons of Krasnoholovets, or aggregate particles made from Bhutatmas.

The most salient part of the Kelvin-Helmholtz electron vortex form ("KH vortex"), at its outermost margins, is almost **spherical**, as well as toroidal, as can be seen from the diagrams and the photograph of KH vortices. Thus, due to laminar flows intersecting with existing spheres, vortex streets are caused to form into KH vortex rings, which are rotating in alternating opposite directions. Electrons and positrons also have equal and opposite "charge" and are considered to be "*anti-matter*" in relation to one another.

But at this point, readers may ask: what is "anti-matter" really, other than opposite directions of rotation of similar particles? And what is "charge" really, in terms of aether behaviors?

So, essentially, electron-positron pair formation is properly described and justified for the first time in the history of particle physics, as both electrons and positrons are KH vortices, rotating in opposite directions. Electron-positron pairs are, at least temporarily, linked by bridges of the same material particles which the *e-p* particle pairs are being formed in.

Pairs of electrons and positrons are required to make the larger particles, such as the proton, which is an agglomeration of an exact number of electrons and positrons, with one positron excess, to account for the positive charge produced by the proton.

What needs to be discovered here is: What property of the aether determines the exact numbers of electron-positron pairs, required to form protons and neutrons? Does this have to do with "packing" limitations, imposed by the media? Is this to do with the phi ratio inherent in the media?

Each electron which already exists, acts as a large rock in a moving stream, causing deflections of the normal aether flow, slowing down the flow-rate, and producing eddy currents and turbulence in the ambient aether near the given electron. When the turbulence becomes large enough, additional electrons form in the media, which act to choke off the interstellar aether flow even more and impede its normally unencumbered motion. This is similar to adding more and more rocks into the channel of a stream of water, so that the flow rate of the water slows down, as more and more rocks are added.

This process was discovered by Nikola Tesla during his experiments at his Colorado Springs laboratory. It is a good thing this happens, or aether avalanches produced by Tesla's 100,000,000 volt explosive electrical discharge events could have burned away the very air we live in.

Tesla was relieved to find out the discharges were choked off, accompanied by vast numbers of newly created electrons. Tesla found the excess electricity resulting from the excess electrons to be a nuisance to his other experiments, so he dumped the excess electrical power into the earth's crust.

Relation between Helmholtz's electron vortex model & turbulence theory

Solving the turbulence problem means finding (unknown) laws of the mixing of momentum and scalars, at asymptotically high Reynolds numbers. About hundred years ago, Osborne Reynolds and soon also Friedman & Keller thought that we can solve the problem by series expansions of the Navier-Stokes equations, a process which provides dynamic equations of motion for higher and higher (statistical) moments.

Unfortunately, such an expansion does not visibly converge. Certain closure assumptions are needed, such that this approach is not strict. With respect to theory, all subsequent research followed the paradigms of Reynolds, Friedman, and Keller, without any exact result.

The famous text by Landau & Lifshitz on fluid dynamics states that universal constants of turbulent motion, like von Karman's constant, can only be measured (rather than predicted by theoretical considerations).

Later, Kolmogorov realized the hopelessness of Reynolds-type paradigms and then he introduced an argument: *Similarity Analyses*, which immediately led to the scaling laws of turbulent spectra, e.g. the famous 5/3rd law, which is strict.

At an infinitely high Reynolds number, the physical properties of the specific fluid under study "vanish", due to vanishing viscosity. So the viscosity of the media at the given energy-density, is relevant, in aether considerations.

This sort of turbulence is consequently described by the (regularized) Euler equation, which represents an "inert geometry". By this, the turbulence problem rests on the Euler equation and its singular solutions, such as "vortex atoms", as first introduced by Lord Kelvin almost 200 years ago, based on von Helmholtz's vortex theorems. Such solutions can be treated as non-trivial three-dimensional particles, in motion.

In most cases these motions are extremely hard to predict are the focus of a special branch of mathematics – *topological hydrodynamics*. See also Kiehn [11-14].

There are two exceptions: Completely isolated vortices, and a "gas" of comprised of many vortices. The former case is trivial. In the latter case, one can do what has already been done by Maxwell in his kinetic theory of gases: Assume a chaotic (Brownian) motion of the entities involved. This paradigm, produces simple and comfortable equations of motion, of the advection-diffusion-reaction type, for the key variables of turbulence, turbulent kinetic energy, and r.m.s. vorticity.

This approach allows a theoretical prediction of von Karman's constant as 1/Sq Rt (2π) = 0.399 (The international standard value, based on measurements is 0.4).

This result is physically related to the Helmholtz vortex model of the electron. The correct aether turbulence model will produce electrons in the manner of a fluid flow producing turbulence.

The form of the Helmholtz vortex is circular at the surface, with toroidal shells made from the same smaller particles, circulating internally.

This allows the "substructure" requested by the "ring model." The ring model is constrained to behave according to Einstein's version of relativity, by extraneous artifices and excuses, all of which are wrong, from my point of view. There is nothing preventing **any** faster than light behaviors, other than Einstein's version of relativity, which is **completely non-physical**, and only functions internal to one's imagination.

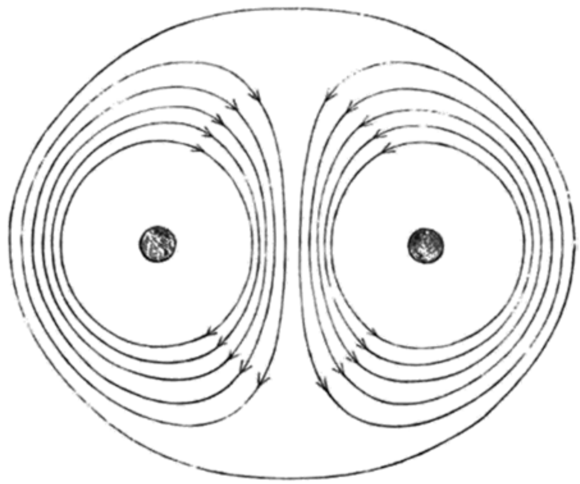

Figure 1. Helmholtz's atom model should be applied to electron vortex (after R.N. Boyd)

One of the hugest mistakes ever made in physics was Einstein's ill-advised attempts to constrain everything in existence to light speed, including time. This causes a conceptual wall to be erected in the mind, which prohibits superluminal behaviors of any kind, and makes interstellar travel and power without fuel, impossible, just because of a mathematical fantasy that cannot be proved as valid by any manner of physical experiment. There are vast numbers and types of experiments which refute every part and portion of the irrational arguments of Einstein's version of relativity.

It seems a good idea is to combine the "ring model" of the electron with the Helmholtz vortex model of the electron. The conclusions of the ring model which finds the Dirac and Schrodinger's equations invalid, are just a few of the mistakes in the development of the ring model that need to be corrected in the Helmholtz model which allows that *superluminal* behaviors of every kind may participate.

On the plus side, they have done most of the other physics requirements work already. Once we provide the corrective measures which exclude relativistic considerations, we will have a very compelling model for the electron, which is based on nested flows of SubQuantum particles, which comprise a toroid when considered as a unit whole.

Natural extensions of Kolmogorov's studies of turbulence, towards the infinitely small, have directly derived turbulence-generated vortices as small as 10^{-58} m, which we call Kolmogorov vortices. These are the smallest creatures which are still influenced by gravitation. Smaller creatures are the primary cause of gravitation, in this model, which is related to both the LaPlace and LeSage models of gravitation. Both these models are valid, depending on how one is looking at the situation, so we are combining them into one model. We also have reproducible experimental evidence and instrumented spacecraft observations, which physically support this model.

Fabriciuss suggested that multiple Kolmogorov vortices might form a geometric inter-relationship which would then comprise an electron.

The "Bhutatma" infinitesimal particle of Vedic lore is the ultimate building block of everything, being the smallest unit of matter, and at the same time, the smallest unit of Consciousness.

Once the errors are removed from the ring model, and we hope that soon we will be able to illustrate electron formation from Kolmogorov turbulence in a perfect fluid, then our Helmholtz vortex model will be excellent. An outline of such a model of electron creation will be discussed at the following section.

Turbulence origination of Kelvin-Helmholtz electron vortex

For a non-viscous fluid, pressure exerts a force of -grad p per unit volume. (There is also a gravitational aether force, ρg per unit volume.) The aether fluid obeys Newton's law of motion, so ρdv/dt = -grad p, as the equation of motion. (This is used to determine fluid pressure when the flow is known.)

A vorticity field is ω(x, y, z, t) in magnitude and direction, at any point. Lines drawn parallel to ω are called vortex lines, and their density can express the strength of the rotation, just as streamlines define the velocity field, and magnetic field lines define a magnetic field. (Such lines are not real, but greatly aid in visualization).

The line integral of the component of velocity, tangent to a closed curve, is called "circulation", and clearly measures the amount of rotation in the vortex. Let's take a small circle surrounding an area A = πr2 as the path of integration. If the angular velocity is ω, then the circulation will be 2πr x ωr = 2πωr2 = 2ωa. Thus, the circulation of the fluid, per unit area, is directly proportional to the angular velocity of rotation.

Stokes's Theorem states that the circulation of a vector about any curve C, is the surface integral of the curl (del cross) of the vector over the area enclosed by C. If this is applied to the present case, we find that curl v = 2ω, so that the rotation of the vortex is half the curl of the velocity. Since the divergence of the curl of a vector is identically zero, div ω = 0.

This means that if we consider a tube whose walls are parallel to ω, called a vortex tube, then this tube has the same "strength" (the product of the area and ω), at any point. This means that the vortex tube cannot end within the fluid, and must either close into a ring, or go to a boundary.

The Kelvin-Helmholtz theorem, states that the substantial derivative of the circulation about any curve C, in a fluid of zero viscosity, vanishes. This applies to any curve C on the walls of a vortex tube, or on any surface parallel to the vorticity, and implies that vortex lines are carried with the fluid, and that the "strength" at any point remains constant.

If the initial state of a fluid to which the KH theorem applies, has no rotation, that is, curl v = 0 everywhere, the fluid will remain irrotational as it moves. This also means that if rotation exists in the vortex, it will persist for all time.

The stream function in a fluid or gas is analogous to the use of the vector potential of the magnetic fields of electric currents. From this, the foundational basis of electromagnetism is actually a description of fluidic flows in the aether.

Consider a vector field A = kA(x, y). (A(x, y) may also vary with the time, but we will consider that later.) Suppose that v is derived from A by the rule v = curl A. Writing this out: $v = i(\partial A/\partial y) - j(\partial A/\partial x)$, so that $vx = \partial A/\partial y$ and $vy = -\partial A/\partial x$.

Now, writing out the continuity equation of div v = 0, it is automatically satisfied for any function A. To find the relationship between A and the vorticity, we write out the z-component of curl v, to find that $2\omega = \partial vy/\partial x - \partial vx/\partial y$ -div grad A.

In considering two-dimensional motions, the vorticity of the aether fluid can only be parallel to the z-axis, since the velocity must lie in the x y-plane and is independent of z. (The vector potential of a magnetic field satisfies the same equation, where the current takes the place of fluidic vorticity.) The above, is Helmholtz's equation. The one scalar function A, thus allows us to find two interrelated components of the fluid velocity.

If the aether flow is irrotational, then A will satisfy Laplace's equation, and solve the problem as well as the velocity potential φ . In fact, A and φ are conjugate functions. In two dimensions, they are the real and imaginary parts of a complex analytic function. The streamlines A = constant, are orthogonal to the equipotentials φ = constant, again pointing to the direct relation between fluidic aether flows and the Maxwell equations.

Vortex lines have been postulated to study fluid dynamics. A vortex line has a finite strength (vorticity times area), but zero area, similar to the understanding that a dipole has zero length. The resulting vortex lines tend to propagate at infinite velocity, unless the lines remain absolutely straight. This would be the 5^{th} aether phase state in *Mishin's 5-phase aetherdynamics*. Now *we are beginning to discover the origin of the various types of turbulences in the ambient aether flows which eventually manifest as KH electron vortices.* The aether flows around an already existing, but non-motional, electron vortex in a streaming aether fluid flow, sheds vortex pairs which are rotating in opposite directions, alternately from the two sides of the KH vortex, resulting in lines made of vortices, called a vortex "street" (also called a "*von Kármán street*"), behind it. These "streets" are seen on all scales, from flows in brooks, to the atmosphere, to the fluidic aether in which KH electron vortices eventually come into existence.

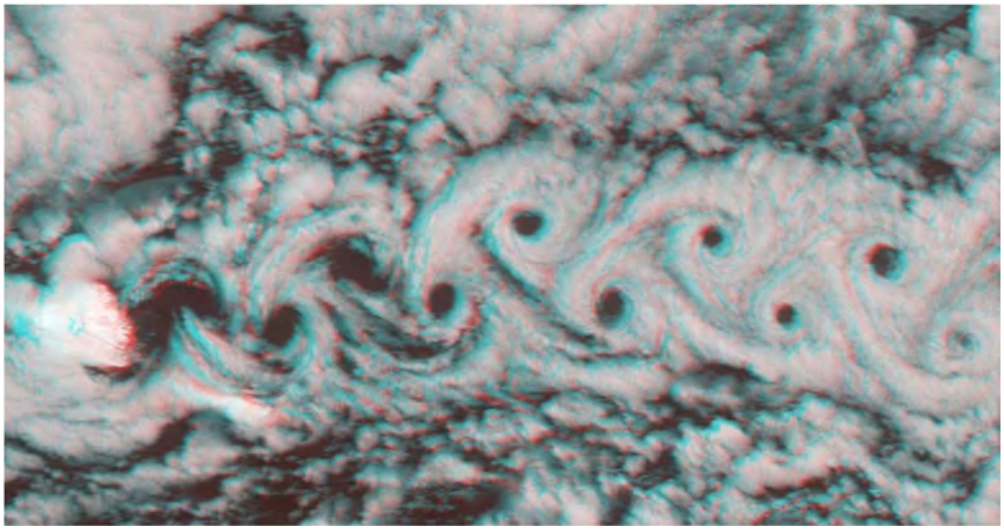

Figure 2. Illustration of von Karman street (source: [7], see also [8])

Alternating transverse forces can act on a cylinder, for example a telephone wire, which can make it vibrate. This is the reason why wires "sing" in the wind. The wire cylinder is stationary in a stream of moving media. Behind the cylinder is a turbulent wake of slowed air. Two vortex sheets are formed on each side of the wake, and their instability results in the vortex streets (streams of vortices). Vortices are formed in a Kelvin-Helmholtz instability in the same way. Analogous effects occur in aether flows which pass around an existing electron sphere, but in this situation the resulting "street" of vortices form into rings, which are exactly many newly formed KH vortices.

Vortex "shedding" produces resonances with the object that impeded the flow. In this case, the vortices are resonant with the existing electron. This means the positron could be viewed as an "anti-resonant" particle. Resonance at this level will constrain the vortices in the "street" to form duplicates that are the same as the original forms, in terms of "aether mass" (constrained aether forms). This also implies that positrons can be the basis for the formation of new electrons, in the parallel aether stream. See figure 3.

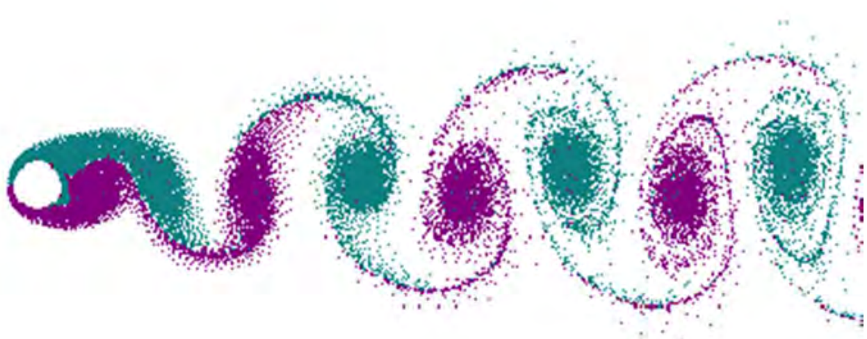

Figure 3. alternating electron-positron, alternating rotation directions (After RN Boyd)

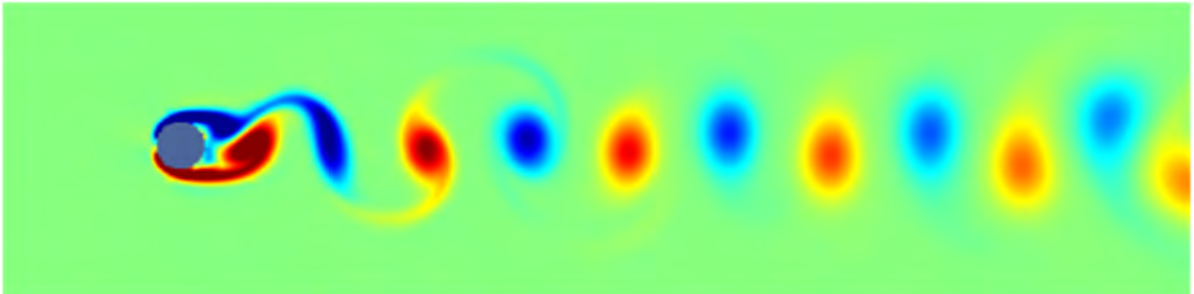

Figure 4. alternating electron-positron (After RN Boyd)

The above figure 4 is an alternative version of Figure 3. This raises a number of questions: Does this imply that both positive and negative charges already both exist, internal to the aether which comprises the aether winds? This implies that behaviors of obstructed aether flows are the origination of the **cause** of the distinct charges of electrons and positrons, and of electrons and protons.

The KH vortex model of the electron is simultaneously a sphere, surrounding a nest of concentric smaller vortices, which have a vortex ring at the middle of the concentric aether flows which comprise the particle. So the ring model is only partially valid.

Kelvin-Helmholtz electron vortex & origination of charge and matter

Vortex lines have been postulated to study fluid dynamics. A vortex line has a finite "strength" (vorticity times area), but a zero area, similar to the understanding that a dipole has zero length. Vortex lines tend to propagate at infinite velocity, unless the lines remain absolutely straight. (This would be the 5^{th} aether phase state in Mishin's 5-phase aether dynamics. See diagram no. 5)

Importantly, the instant a vortex line departs from an absolutely straight line of propagation, **charge develops** in all the vortex lines that are bent. According to the direction of the bend, away from a perfectly straight line, a positive or a negative charge develops.

Parity (handedness) is directly involved in the development of charge. Parity determines the sign of the charge. The internal quantum numbers of electrons are opposite to those of positrons, which is just a restatement of the handedness (parity) of the internal aether circulation directions. The involvement of superluminal SQ infinitesimals in the formation of electrons and positrons, and superluminal internal circulations of the aether constituents of electrons and positrons, eliminates Lorentz "invariance" from consideration.

Lorentz "invariance" is only valid for a **single absolute value of c**, which value was experimentally proven to vary by as much as plus and minus 3000 meters per second, as recorded in the handwritten log-books associated with the hundreds of repetitions of the Michelson-Morely experiments during the last century. In addition, Lorentz "invariance" has nothing to do with electrons, positrons, and so on, due to the fact that "invariance" is only valid for exact specific-velocity photons, which are **not identical** to electrons, contrary to the expressions of Heisenberg in his first book on quantum theory.

Vortex lines circulating internal to electrons or positrons are **always** bent away from a straight line, so the vortex lines circulating internal to electrons and positrons are **always creating charge**. This is **the origination of charge** and the reason charge never ceases, as long as the charged particle exists.

In addition, the electron-positron pairs are forming in aether-connected chains, which chains are responsible for the creations of atoms, as well as protons and neutrons, in a manner which depends on how long is the “street” of connected electron-positron pairs, which in turn, become parts of the nucleus of the new atom, in terms of the atomic number of the nucleus of the atom, in an *e-p* pair model of the composition of, and the construction of, the protons and neutrons which comprise the nuclear particles of atoms.

If the parallel aether flows which are forming chains of *e-p* pairs are short-lived, we will only see hydrogen, or perhaps the occasional helium atom being generated. Longer *e-p* chains result in larger atoms. The local density of types of atoms and alignments of atoms, may give an indication of the frequency of aether wind streamlines, in that region. Proper instrumentation of vortex-line (SQ infinitesimals) resultant behaviors can be used to map astronomical space, comprising an infinite range observation capability, due to the fact that vortex lines propagate with infinite velocity.

Conclusion

There are various models of electron which have been suggested, for instance see Chekh et al. [10]. But we seek a model which is close to experiments conducted by Bostick et al. [9]. Our attempt to explain such experiments of electron creation in plasma allows us to come up with a new model of electron, based on Helmholtz’s electron vortex theory. In turn, we discussed a plausible model of electron capture event inside Earth (matter creation), which in turn could serve as a basis to explain Le Sage/Laplace’s push gravity.

We also discussed among other things how relevant is Kolmolgorov theory of turbulence, von Karman vortex street etc. to KH electron vortex. We further discuss a new model of origination of charge and matter.

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On the New Concept Creatio Ex-Rotatione

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Abstract

It is known that the Big Bang theory was based on the concept of creation ex nihilo, after ancient Greek philosophers. In this paper, we discuss the concept of *creatio ex nihilo*, as well as two other approaches - Intelligent Design and Emergence Theory. We argue that beside the above three approaches, a new concept called *creatio ex-rotatione* offers a resolution to the long standing disputes between beginning and eternity of the Universe. We agree with Vaas: *[h]ow a conceptual and perhaps physical solution of the temporal aspect of Immanuel Kant's "first antinomy of pure reason" is possible, i.e., how our universe in some respect could have both a beginning and an eternal existence. Therefore, paradoxically, there might have been a time before time or a beginning of time in time."* By computational simulation, we also show how a model of early Universe with rotation can fit this new picture.

Keywords: Big Bang, Steady state, rotating universe, fluid, singularity free, cosmology model, early Universe, Genesis, Spirit in Creation.

Introduction

Considering the Big Bang Theory promulgated by priest Georges Lemaître in 1927 and based on the Christian belief that the universe was created, the following questions will naturally arise:

a) where did this primeval atom come from? and
b) what was before this big bang?

The term "big bang" was derogatorily coined by Fred Hoyle in a BBC interview and it is supposed that the universe, according to this theory, was created between 10-20 billion years ago [33].

In this article we will discuss three approaches to the origin of the universe, all of them can be related to the notion of Big Bang (spontaneous creation). In the last section, we will explore a new proposed concept: *creatio ex-rotatione*, based on our investigation in the past few years.

Three Approaches on the Origin of the Universe

Creatio Ex Nihilo

In a paper to appear in European Journal of Science and Theology [31], Kalachanis, the authors state in the abstract:

> The Big Bang Theory considers that the Universe, space and time have a beginning. Similar is the position of the Christian writers of the early Christian Church, who support the ex nihilo - ἐκ μὴὴ ὄντος (ek me ontos = from the "non-being") creation of the world through the divine "energy", with the two theories converging to the fact that space and time have a beginning.

That the Big Bang concept has a beginning, that is true, but what kind of beginning that its originator had in mind is rather different from the concept that Christian writers had in mind - See , *e.g.*, [32].

The Big Bang hypothesis was formulated by Lemaitre based on the notion of primeval atom ("cosmic egg"). Although it is true that some Christian writers also mentioned "Creation from nothing", they were more likely to have different concepts compared to the primeval atom. Moreover, the notion of "creation from nothing" should be accepted as debatable, since it was mentioned in a few verses only in NT, and it can be traced back to the book of Maccabee in Deuterocanonica. So, in the next sections, we will take a look directly and closely at Hebrew version of the book of Genesis 1.

In short, we argue that: (a) while both the Big Bang originator and Christian writers shared similar concept of creatio ex nihilo, they have different views on "primeval atom;" and (b) even the idea of primeval atom itself seems in direct contradiction with "ex nihilo" term.

Intelligent Design

With regards to ID hypothesis, some philosophers began with Psalm 19 to argue in favor of *The Intelligent Creator*:

> The heavens declare the glory of God;
> And the firmament shows His handiwork.
> **2** Day unto day utters speech,
> And night unto night reveals knowledge.
> **3** *There is* no speech nor language
> *Where* their voice is not heard.

4 Their line[a] has gone out through all the earth,
And their words to the end of the world. (Psalm 19: 1-4, NKJV)[1]

We can note some proponents of ID, such as Michael Behe etc. While such attempt to link the old conception of Intelligent Design to Biblical account may sound interesting at first glance, one can note immediately that all ID proponents seem to avoid to point to God of Bible as the Intelligent Creator that they talk about.

Yes, ID theory is a nice hypothesis to talk about, but the end of the day, such a hesitation to speak about the Biblical God reflects their adherence (perhaps) to a number of theoretical possibilities which enable them to theorize around and around without daring to point at the Real Subject behind all Design in the Universe. And clearly, such a hesitation to point to God is not without implications, as Erkki Vesa Kojonen wrote in his dissertation in University of Helsinki [30]:

> ID's design arguments are quite minimalistic, not aspiring to prove the existence of God, but merely of an **unidentified** intelligent designer of cosmic and biological teleology.

At the price of giving too much "intellectual room," then we find in recent decades some scientists or pseudo-scientists come up with alternative hypothesis on who or what is behind the Design of the Universe.

In their book "*Grand Design*," Hawking and Mlodinow argue that in their TOE model based on certain variations of Superstring theories, that such TOE does need the role of God as Creator.[2] In other words, they seem to argue that physical laws exist eternally before the Universe exist, so by such physical laws themselves, there was Big Bang triggered by primordial vacuum fluctuations. But how did it happen…it seems many cosmologists remain silent on this vague hypothesis. This fact alone should alert us that Hawking and Mlodinow ask their readers to believe in a story based on a baseless-theory which does not conform to any experimental backup. See also article by Michael G. Strauss.[3] Moreover, other alerts may come from the fact that: It is worth noting, that calculation shows that Quantum Field theory predicts cosmological constant at astronomical error compared to observed value.[4] Even mathematicians like Peter Woit already wrote a book called "*Not even wrong*" to alert us on the fact that Superstring

[1] https://www.biblegateway.com/passage/?search=Psalm+19&version=NKJV
[2] https://www.reasonablefaith.org/writings/popular-writings/science-theology/the-grand-design-truth-or-fiction/
[3] http://www.michaelgstrauss.com/2017/08/the-grand-design-is-god-unnecessary.html
[4] Quote from J.R. Roldan: "The quantum field **theory prediction** of the **cosmological constant** is 120 orders of magnitude higher than the observed value. This is known as the **cosmological constant problem**." https://arxiv.org/abs/1011.5708

theories do not predict anything which can be measured.[5] See also his other book: "*String theory: an evaluation*."[6]

And much worse than Grand Design, some college students (and may be with support of their professors) have come up with a new god called "*Flying Spaghetti Monster*" (FSM religion). They even managed to push their case that FSM religion should be taught at high schools and colleges in the same way of ID/evolution theory.[7] Such a fancy FSM reminds us to the *golden cow* made by Aaron and the Israelites soon after Moses went to the mount.

Emergence Philosophy

According to Amos Yong, a professor in Fuller Seminary:

> To be clear, emergence is a philosophical or metaphysical hypothesis rather than a theological doctrine or scientific datum. Yet the theory of emergence, Clayton suggests, identifies patterns of developments in the natural history of the cosmos as understood through the findings of the various scientific disciplines. …" [6, p. 145]

In other words, emergence philosophy as proposed by Clayton[8] seems to be founded on certain metaphysical assumptions on how nature functions. We will not go into details of Emergence here, suffice it to say (with all respect to Amos Yong as a leading contemporary theological scholar from Fuller) that there is danger that we do **eisegesis** on biblical narratives, rather than doing a fair and faithful reading (exegesis) on Biblical account of Creation.

Therefore, in the next section, we shall show what we can infer from Biblical narratives, with minimal assumptions.

How *creatio ex-rotatione* may Resolve Dispute on the Origin of the Universe

In recent years, the Big Bang as described by the Lambda CDM-Standard Model Cosmology has become widely accepted by majority of physics and cosmology communities. But the philosophical problems remain, as Vaas pointed out: Did the universe have a beginning or does it exist forever, i.e. is it eternal at least in relation to the past? This fundamental question was a main topic in ancient philosophy of nature and the Middle Ages. Philosophically it was more or

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[5] http://www.math.columbia.edu/~woit/rutgers.pdf
[6] https://www.math.columbia.edu/~woit/strings.pdf
[7] https://www.telegraph.co.uk/news/worldnews/northamerica/usa/1498162/In-the-beginning-there-was-the-Flying-Spaghetti-Monster.html
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less banished then by Immanuel Kant's *Critique of Pure Reason*. But it used to have and still has its revival in modern physical cosmology both in the controversy between the big bang and steady state models some decades ago and in the contemporary attempts to explain the big bang within a quantum cosmological framework.

Interestingly, Vaas also noted that Immanuel Kant, in his *Critique of Pure Reason* (1781/1787), argued that it is possible to prove both that the world has a beginning and that it is eternal (first antinomy of pure reason, A426f/B454f). As Kant believed he could overcome this „self-contradiction of reason" („*Widerspruch der Vernunft mit ihr selbst*", A740) by what he called „*transcendental idealism*", the question whether the cosmos exists forever or not has almost vanished in philosophical discussions. [3]

In this paper, we will take a closer look at Genesis 1:2 to see whether the widely-accepted notion of *creation ex-nihilo* is supported by Hebrew Bible or not. It turns out that a new concept called **creatio ex-rotatione** is in agreement with Kant and Vaas's position, it offers a resolution to the long standing disputes between beginning and eternity of the Universe. In other words, in this respect we agree with Vaas: "how a conceptual and perhaps physical solution of the temporal aspect of Immanuel Kant's "*first antinomy of pure reason*" is possible, i.e. how our universe in some respect could have both a beginning and an eternal existence. Therefore, paradoxically, there might have been a time before time or a beginning of time in time."[3]

We will discuss how to answer this question by the lens of hermeneutics of Sherlock Holmes. This is a tool of mind which we think to be a better way compared to critical hermeneutics.

What is *Hermeneutics of Sherlock Holmes*?[9]

The following are 10 tips from Eric McKiddie to adapt Sherlock Holmes to interpreting biblical passages:[10]

Tip no 1:

> *Holmes: "I have no data yet. It is a capital mistake to theorize before one has data. Insensibly one begins to twist facts to suit theories, instead of theories to suit facts."*
> Far too often students of the Bible (and cosmology folks as well) twist verses to suit interpretations instead of formulating interpretations to suit what the verses say.
> Guide: Don't approach your passage assuming you know what it means. Rather, use the data in the passage – the words that are used and how they fit together – to point you toward the correct interpretation.

[9] https://www.str.org/blog/learning-hermeneutics-from-holmes
[10] https://www.thegospelcoalition.org/blogs/trevin-wax/10-tips-on-solving-mysterious-bible-passages-from-sherlock-holmes/

Tip no 2: **The kind of looking that solves mysteries**

- *Holmes: "You have frequently seen the steps which lead up from the hall to this room."*
- *Watson: "Hundreds of times."*
- *Holmes: "Then how many are there?"*
- *Watson: "How many? I don't know!"*
- *Holmes: "Quite so! You have not observed. And yet you have seen. That is just my point. Now, I know that there are seventeen steps, because I have both seen and observed."*
- There is a difference between reading a Bible verse and observing it. Observation is a way of collecting details contained in a passage. As you read and reread the verses, pull the words into your brain where you can think about them and figure them out.
- This habit will shed light on how you understand the text, even if the passage is as familiar as the stairs in your house.

Tip no 3: **Know what to look for**

- *Watson: "You appeared to [see] what was quite invisible to me."*
- *Holmes: "Not invisible but unnoticed, Watson. You did not know where to look, and so you missed all that was important."*
- Know where to look for clues that will illuminate your passage. Look for repeated words and phrases, bookends (where the beginning and end of the passage contain similarities), and clues in the context around your passage.
- Don't know what to look for? *Living by the Book* by Howard Hendricks and *How to Read the Bible for All Its Worth* by Gordon Fee and Douglas Stuart are great resources to start learning how to study the Bible.

Tip no 4: **Mundane details are important!**

- *Watson: "I had expected to see Sherlock Holmes impatient under this rambling and inconsequential narrative, but, on the contrary, he had listened with the greatest concentration of attention."*
- Don't ignore parts of the passage that seem insignificant to its meaning. Treat every word as if it contains clues to the interpretation of the passage.

Tip no 5: **Use solutions to little mysteries to solve bigger ones.**

- *Holmes: "The ideal reasoner would, when he had once been shown a single fact in all its bearings, deduce from it not only all the chain of events which led up to it but also all the results which would follow from it."*
- Once you understand the passage that baffled you, your work is not done!
- Now it's time to locate that passage in the grand narrative of the Bible. How do previous books and stories lead up to your passage? How does your passage anticipate the consummation of all things that results at Jesus' second coming?

Tip no 6: **The harder the mystery, the more evidence you need**

- *"This is a very deep business," Holmes said at last. "There are a thousand details which I should desire to know before I decide upon our course of action."*
- In grad school, one professor gave us an assignment requiring us students to make 75 observations on Acts 1:8. The verse does not even contain that many words!
- The professor's goal was to train us in compiling evidence. Harder Bible passages demand that we collect as much information as possible.

Tip no 7: **Break big mysteries down into little ones**

- *Watson: "Holmes walked slowly round and examined each and all of [the pieces of evidence] with the keenest interest."*
- Difficult passages can be overwhelming. Break chapters down into paragraphs, paragraphs into verses, and verses into clauses. Devote careful attention to each chunk of the passage individually. Then try to piece together the meaning they have when added up as a whole.

Tip no 8: **Don't be so committed to a solution that you ignore new evidence**

- *"I had," said Holmes, "come to an entirely erroneous conclusion which shows, my dear Watson, how dangerous it always is to reason from insufficient data…I can only claim the merit that I instantly reconsidered my position."*
- After you've put the hard work into grasping a mysterious passage, the case isn't necessarily closed. Often you'll run across other passages that shed new light on your passage. Or you'll hear someone preach those verses in a different way than how you interpreted it.
- Always be willing to consider new insights. This will at least help you nuance your understanding of the passage, if not take a different stance.

Tip no 9: **Simple solutions often provide answers to manifold mysteries**

- *Holmes: "The case has been an interesting one…because it serves to show very clearly how simple the explanation may be of an affair which at first sight seems to be almost inexplicable."*
- Many passages that seem mysterious at first end up not being so bad. Their bark is worse than their bite. For example, several passages in Revelation, intimidating to so many, have simple explanations. (Not all, but some!)

Tip no 10: **On the other hand, so-called simple passages may be more complicated than initially meets the eye.**

- *Holmes: "This matter really strikes very much deeper than either you or the police were at first inclined to think. It appeared to you to be a simple case; to me it seems exceedingly complex."*

- This is often true of coffee mug and bumper sticker verses. We think they are simple to understand because we see them all the time. But once you dig into them, you realize they are more mysterious than meets the eye.

A close reading at Genesis 1:1-2

One of the biggest mysteries in cosmogony and cosmology studies is perhaps: *How to interpret properly Genesis chapter 1:2*. Traditionally, philosophers proposed that God created the Universe out of nothingness (from reading "empty and formless" and "*bara*" words; this contention is called "*creatio ex nihilo*."). Understandably, such a model can lead to various interpretations, including the notorious "cosmic egg" (primeval atom) model as suggested by Georges Lemaitre, which then led to Big Bang model.[18-20] Subsequently, many cosmologists accept it without asking, that Big Bang stands as the most faithful and nearest theory to Biblical account of creation. But we can ask: Is that primeval atom model the true and faithful reading of Genesis 1:2?

Let us start our discussion with examining key biblical words of Hebrew Bible, especially Genesis 1:1-2. It can be shown that the widely accepted creation ex nihilo is a *post-biblical invention*, rather than as faithful reading of the verses. To quote Ian Barbour: "*Creation out of nothing is not a biblical concept*."[4]

Let us consider some biblical passages:

The literal meaning of Gen. 1:1, "bareishit bara Elohim." This very first statement of the book of Genesis literally reads: 'first' and 'beginning' are reasonable alternatives for the Hebrew noun, *reishit*. Also note that in Hebrew, subjects and verbs are usually ordered verb-first (unlike English in which the subject is written first). If the verb and subject of this verse are reordered according to natural English grammar we read: [1]

{In, When} {first, beginning} Elohim created…**reishit**: The noun, reishit, has as its root the letters, ראש (Resh -Aleph-Shin). Words derived from this root often carry the meaning of 'primary', 'chief', 'begin', 'first' or "first-in-line", "head of", and so forth. Harris's Theological Wordbook of the Old Testament (TWOT) is more specific, namely, reishit means[1]

> *…first, beginning, choicest, first or best of a group. [Reishit is] a feminine noun derived from the root [Resh-Aleph-Shin], it appears fifty times in nearly all parts of the [Old Testament]. [Its] primary meaning is "first" or "beginning" of a series.*

Accordingly, we can now retranslate *bəreishit bara Elohim* as "When first created Elohim", or as we would render in English,[1] *When Elohim first created…*

Gen. 1:2, "And the earth had been." In English this is easily handled by the past perfect tense (also called the pluperfect or the "flashback" tense). Likewise, if *haytah* in v 1:2 is translated as a past perfect verb, then verses 1:1-2 would read,[1]

When Elohim first created the heavens and the earth, the earth had been …

In this translation the universe, in some form or other, was already in existence when God executed His first creative act, the creation of light.

In other words, a close reading of Hebrew Bible seems to suggest that *creatio ex-nihilo* is a post-biblical invention. Other scholars have suggested an alternative concept, called *creatio ex-materia,* but many orthodox Christian scholars have raised objection to this notion, partly because the term seems to undermine God's ultimate power and control of the Universe. Besides, the notion of *creatio ex-materia* has been advocated by Mormon preachers.

To overcome this problem, and based on what we learned recently, allow us now to come up with a new term: **creatio ex-rotatione** (rotatione is a Latin word for "rotation)". As we shall see in the next chapter, it is possible to come up with a physical model of early Universe with rotation, where the raw materials have been existed for long period of time, but suddenly it burst out into creation. And it seems to fit with Kant's idea to resolve the dichotomy between finite past or eternal Universe. Furthermore, it can be shown that the model naturally leads to accelerated expansion, without having to invoke *ad hoc* assumptions like dark energy or cosmological constant.

A computational model of rotation in early Universe

Our discussion starts from the fundamental question: how can we include the rotation in early Universe model? After answering that question, we will discuss how "turbulence-generated sound" can be put into a mathematical model for the early Universe. We are aware that the notion of turbulence-generated sound is not new term at all especially in aerodynamics, but the term is rarely used in cosmology until now. We shall show that 3D Navier-Stokes will lead to non-linear acoustics models, which means that a turbulence/storm can generate sound wave.

It has been known for long time that most of the existing cosmology models have singularity problem. Cosmological singularity has been a consequence of excessive symmetry of flow, such as "Hubble's law". More realistic one is suggested, based on Newtonian cosmology model but here we include the vortical-rotational effect of the whole Universe.

In other paper, we obtained an Ermakov-type equation following Nurgaliev [8]. Then we solve it numerically using Mathematica 11. An interesting result from that simple computational simulation is shown in the following diagram:[9]

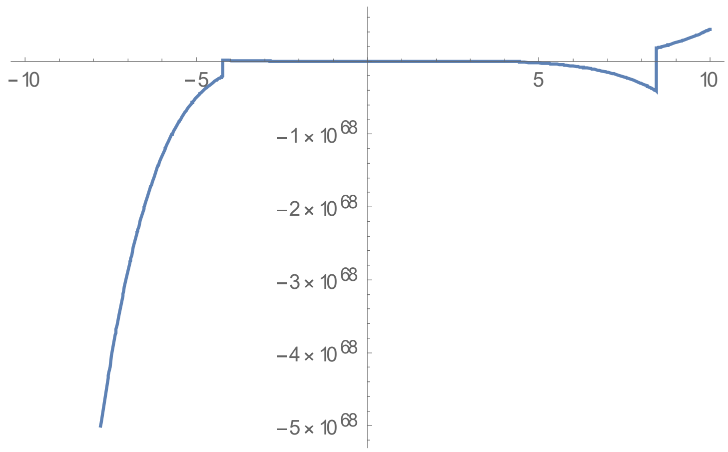

Diagram 1. Plot of Ermakov-type solution for A=1, B=-10 (from [9])

From the above computational experiment, we conclude that the evolution of the Universe depends on the constants involved, especially on the rotational-vortex structure of the Universe. This needs to be investigated in more detailed for sure.

One conclusion that we may derive especially from Diagram 1, is that our computational simulation suggests that it is possible to consider that the Universe has existed for long time in prolonged stagnation period, then suddenly it burst out from *empty and formless* (Gen. 1:2), to take its current shape with observed "accelerated expansion."

As an implication, we may arrive at a precise model of flattening velocity of galaxies without having to invoke *ad-hoc* assumptions such as dark matter.

Therefore, it is perhaps noteworthy to discuss briefly a simple model of galaxies based on a postulate of turbulence vortices which govern the galaxy dynamics. The result of Vatistas' model equation can yield prediction which is close to observation, as shown in the following diagram [14]:

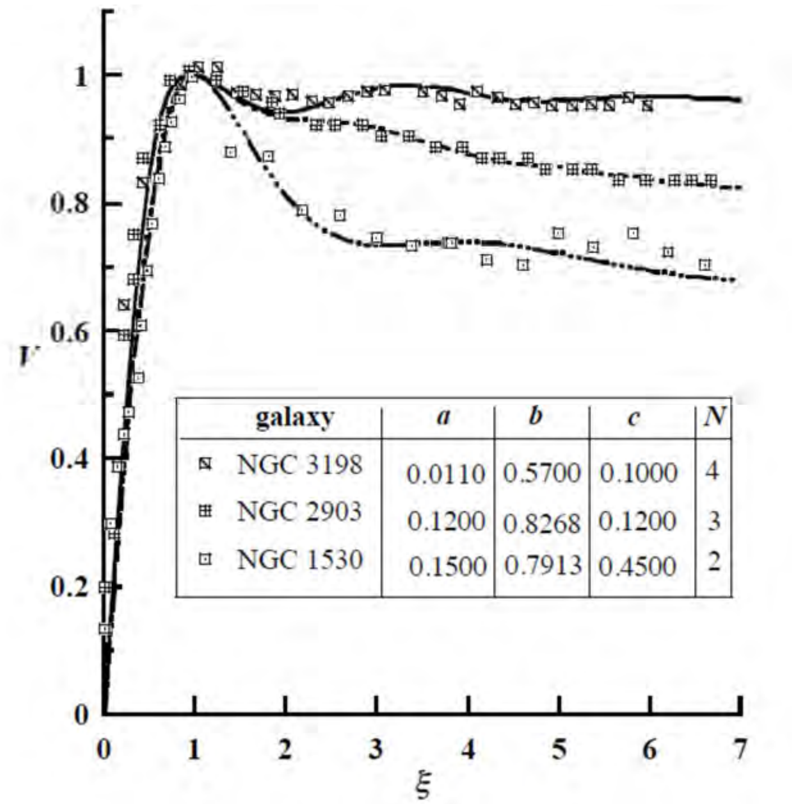

Figure 1. From Vatistas [14]

Therefore, it appears possible to model galaxies without invoking numerous *ad hoc* assumptions such as *dark matter*, once we accept the existence of turbulent interstellar medium. The Vatistas model is also governed by Navier-Stokes equations, see for instance [14].

Advantages of the "creatio ex-rotatione" concept

In the preceding section, we have discussed on how our proposed term of "*creatio ex-rotatione*" has sufficient logical background. Now, allow us to discuss some advantages of the proposed "*creatio ex-rotatione*" cosmology view over the Lemaitre's primeval atom (which is the basis of Standard Model Cosmology):

(a) Explain excess of handedness in spiral galaxies

As reported by Longo et al, there is an excess of left-handedness in spiral galaxies. According to Longo, the simplest explanation of such left-handedness is that there is net angular momentum of the Universe. This seems to suggest that our hypothesis of *creatio ex-rotatione* is closer to the truth with respect to origin of the Universe. [2]

(b) Avoid inflationary scheme

It is known that inflationary models were proposed by Alan Guth et al. (see [25][26]), in order to explain certain difficulties in the Big Bang scenario. But some cosmology experts such as Hollands & Wald has raised some difficulties with inflationary model, as follows:

> We argue that the explanations provided by inflation for the homogeneity, isotropy, and flatness of our universe are not satisfactory, and that a proper explanation of these features will require a much deeper understanding of the initial state of our universe [27]

In our diagram plot above, it is clear that an early rotation model can explain why the Universe can burst out into creation in a very short period, without invoking ad hoc postulate such as inflation model.

(c) Explain accelerated expansion:

As far as we know, one of the earliest models which gave prediction of accelerated expanding Universe is Carmeli's Cosmological General Relativity.[29] But it has been shown by Green & Wald that for the large scale structures of the Universe, Newtonian model can give similar results compared to general relativity picture.[28] Furthermore, it seems that there is no quite clear arguments why we should accept Carmeli use of 5D metric model (space-time-velocity metric). In the meantime, in our rotating Universe model, we do not invoke ad hoc dimension into the metric.

(d) Explain inhomogeneity, breeding galaxies etc.:

Astronomers have known for long time, that the Universe is not homogeneous and isotropic as in the usual model. It contains of inhomogeneity, irregularity, clumpiness, voids, filaments etc, which indicate complex structures. Such inhomogeneous structures may be better modelled in terms of turbulence model such as Navier-Stokes equations, see also our early papers [11][12]. Furthermore, observations clearly suggest that matter ejected continuously in galaxy centers, which view is difficult to reconcile with Big Bang scenario of galaxy creation.

Conclusions

In summary, we argue that: (a) while both the Big Bang originator and Christian writers shared similar concept of creatio ex nihilo, they have different views on "primeval atom," (b) even the idea of primeval atom itself seems in direct contradiction with "ex nihilo" term; (c) the proposed *creatio ex-rotatione* offers a resolution to the long standing disputes between beginning and eternity of the Universe. In other words, in this respect we agree with Vaas, i.e. it can be shown: "how a conceptual and perhaps physical solution of the temporal aspect of Immanuel Kant's „*first antinomy of pure reason*" is possible, i.e. how our universe in some respect could have both a beginning and an eternal existence. Therefore, paradoxically, there might have been a time before time or a beginning of time in time."

We argue that a close re-reading of Genesis 1:2 will lead us to another viable story which is different from Lemaitre's primeval atom model of early Universe, albeit this alternative has not been developed rigorously as LCDM theories.

It is our hope that our exploration will lead to more realistic nonlinear cosmology theories which are better in terms of observations, and also more faithful to Biblical account of creation.

We hope this short review may inspire younger generation of physicists and biologists to rethink and renew their approaches to Nature, and perhaps it may also help to generate new theories which will be useful for a better future of mankind.

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From Big Science to "Deep Science"

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Abstract

The Standard Model of particle physics has accomplished a great deal including the discovery of Higgs boson in 2012. However, since the supersymmetric extension of the Standard Model has not been successful so far, some physicists are asking what alternative deeper theory could be beyond the Standard Model? This article discusses the relationship between mathematics and physical reality and explores the ways to go from Big Science to "Deep Science".

Keywords: Particle physics, Standard Model, mathematics, physical reality, Big Science.

There are more things in heaven and earth, Horatio, Than are dreamt of in your philosophy. - Hamlet (1.5.167-8), Hamlet to Horatio

Introduction

It was said that young Einstein received a mathematics book from a medical student with the witty remark that "*Mathematics is like catching a mouse; you run after the mouse until it runs into a corner of the room. Then you get it*."

Einstein's mental process was probably shaped along this line of thought. Einstein wrote three important papers in Einstein's miraculous year of 1905, one of the papers being on light quanta and another being the special relativity theory. These three papers have been regarded as the cornerstone of modern physics. However, one may still ask the question: how effective is mathematics when describing physical reality?

Beyond Platonic World

The question of effectiveness of mathematics in physical sciences has been discussed by many physicists, notably Eugene Wigner [1]. Perhaps it is true that mathematics is "the art of catching a mouse". But many physics problems are so delicate that the situation may be more analogous to how to catch a black mouse in a dark room.

While most physicists believe that they can "catch the mouse" with the latest development in mathematics, there are others who think that there are problems with relying too much on mathematics.

Derek Abbott argues that [3]:

> Mathematical Platonism is an inaccurate view of reality. Instead, he argues for the opposing viewpoint, the non-Platonist notion that mathematics is a product of the human imagination that we tailor to describe reality.
> …
> So if mathematicians, engineers, and physicists can all manage to perform their work despite differences in opinion on this philosophical subject, why then does the true nature of mathematics and its relation to the physical world really matter?
>
> The reason, Abbott says, is that when you recognise that math is just an abstract mental construct—*just an approximation of reality that has its frailties and limitations, and that such abstractions will break down* at some point, because perfect mathematical forms do not exist in the physical universe—then you can see how ineffective math actually is.
>
> That is Abbott's main point (and the most controversial one): that mathematics is not exceptionally good at describing reality, and mathematics is definitely not the "miracle" that some scientists have marvelled at. Einstein, a mathematical non-Platonist, was one scientist who marvelled at the power of mathematics. He asked, "How can it be that mathematics, being after all, a product of human thought, which is independent of experience, is so admirably appropriate to the objects of reality?
> (*Emphasis added*)

The above quoted text gives us some clues regarding how we should think of mathematics. Yes, mathematics is useful for numerous problems, but as with human being himself, it is bound to limitations, contradictions etc. This problem of mathematical consistency has been discussed bluntly in Godel's famous incompleteness theorem[1].

In other words, we should continue being humble, provided we accept such a non-Platonic view of mathematics. Even if sometimes our theory gives out a series of correct predictions, it does not necessarily mean that we already hit the jackpot of physical reality. But the problem is more acute, because numerous theoretical physicists hold a position which may resemble the following joke:

> *An engineer considers that his/her equations approximate reality,*
> *A theoretical physicist considers that reality approximates his/her equations,*
> *A mathematician doesn't care about reality.*

[1] See for instance Mark Buchanan, Nature, https://www.nature.com/articles/nphys550

It seems that many physicists hold the mathematician's viewpoint of reality, in their stubborn adherence to mathematical models, as if the models themselves hold all the answers. This behaviour happens with the Standard Model of particle physics and with the standard model of cosmology. If there are new observational findings which contradict those standard models, they are discarded, or new words are used to describe the difficulties of the apparent mathematical situation such as "dark energy", "dark matter", and numerous "ghosts" here and there, as mental deflections. Such deflection has become quite the norm in "Big Science."

One scientist[2] once wrote an email to the first author stating that many physicists tend to forget that what they are working with are only models, *i.e.,* approximate and tentative descriptions of physical reality, such as Standard Model of Particle Physics and Standard Model of Cosmology and so on.

In retrospect, perhaps the root cause of such a strict adherence to those models could be traced back to blind acceptance of Einstein's photon model and his special relativity theory. A deeper look into those two theories will reveal that they may be problematic. For example, special relativity theory rejects the notion of ether.

Scaffold to the Moon

There is another philosophical question related to Platonic view: *Do we live in a mathematical universe?* Although this question appears simple, the answer is not. Beyond the Platonic and non-Platonic views, as we discussed above, there is a variety of other possible answers. To mention a few:

- Neo-Platonic view; for example the dodecahedron universe model by Luminet et al.
- number theoretic model of universe (from Pythagoras: "The world is a number".)
- set theory view
- geometric view
- string theory inspired models
- adhesion universe model
- Voronoi tessellatice model
- cellular model, e.g. Konrad Ranzan [9];
- soliton model, e.g. our recent paper on the cellular automaton KdV model (see Appendix);
- nonlinear cosmology based on Kolmogorov's turbulence[3] or Pfaffian turbulence theories [8];

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[2] cf. the late Prof. Robert M. Kiehn from Houston University.

[3] For instance, in few previous papers, we reported our exploration on an early Universe model with rotation. As per our summary report submitted to J. Mathematics (MDPI), we suggest among other things:
"Questions regarding the formation of the Universe and what was there before the existence of the Early Universe have been of great interest to mankind of all times. In recent decades, the Big Bang as described by the Lambda CDM-Standard Model Cosmology has become widely accepted by the majority of physics and cosmology communities. Among other things, we can cite A.A. Grib & Pavlov who pointed to problems with assumptions of heavy particles creation out of vacuum and also launched other proposal such as *Creatio Ex-Nihilo theory* (CET).

So, which one to choose? Our opinion is: you can start with a few assumptions which you find convenient with, and work them out seriously. But after your paper has been published, keep on being flexible, keeping in mind other possibilities.

There are a number of analogies which advise us to remain humble in our journey to decipher the hidden layers of physical reality, *e.g.*, the story of an elephant and five blind men, which you may have heard before. And there is also a Confucius saying: "*The wise man points his finger to the moon, but the fool only sees the finger, not the moon."*[4] The message here is striking: our given theory is only the pointing finger. Yes, the theory surely helps us to see the moon (the Universe), but we should not forget that the theory is not really the Universe.

In a reader doesn't like Zen koans, alternatively there is another analogy held by Murray Gell-Mann, *e.g.*, that his theory helps him like a scaffolding, in order to describe certain physical phenomena. After the work is done, the scaffolding may be not necessary anymore.

One can develop one's theories to be more and more accurate in order to explore the hidden realities of Nature. One may consider this approach as "a scaffold to the moon":

(a) When one's equations have been confirmed by observations and experiments, *Voila!* Yes, it is normal that the first thing that comes to your mind is to celebrate the confirmation of the equations with champagne or a bottle of vodka. An accumulation of such self-celebrating non-physical "experimental confirmations" of mathematical abstractions, leads to *Big Science*.

(b) But that is only the first in the iteration of steps up the scaffold. Perhaps one or two years later, one figures out that there were too many assumptions or there were logical flaws in one's equations, and one must find a better and simpler way to figure out the hidden structure of Nature. That is *Deep Science.* Feynman once remarked something like: "the faster we find out

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But the philosophical problems remain, as Vaas pointed out: Did the universe have a beginning or does it exist forever, i.e. is it eternal at least in relation to the past? This fundamental question was a main topic in ancient philosophy of nature and the Middle Ages. Philosophically it was more or less banished then by Immanuel Kant's *Critique of Pure Reason*. But it used to have and still has its revival in modern physical cosmology both in the controversy between the big bang and steady state models some decades ago and in the contemporary attempts to explain the big bang within a quantum cosmological framework.
Interestingly, Vaas also noted that Immanuel Kant, in his *Critique of Pure Reason* (1781/1787), argued that it is possible to prove both that the world has a beginning and that it is eternal (first antinomy of pure reason, A426f/B454f). As Kant believed he could overcome this „self-contradiction of reason" („*Widerspruch der Vernunft mit ihr selbst*", A740) by what he called „*transcendental idealism*", the question whether the cosmos exists forever or not has almost vanished in philosophical discussions.
It turns out that Neutrosophic Logic is in agreement with Kant and Vaas's position, it offers a resolution to the long standing disputes between beginning and eternity of the Universe. In other words, in this respect we agree with Vaas: "how a conceptual and perhaps physical solution of the temporal aspect of Immanuel Kant's "*first antinomy of pure reason*" is possible, i.e. how our universe in some respect could have both a beginning and an eternal existence. Therefore, paradoxically, *there might have been a time before time or a beginning of time in time*."
To summarize, Neutrosophic Logic study the dynamics of neutralities. And from this viewpoint, we can understand that it is indeed a real possibility that the Universe has both *an initial start (creation) but with an eternal background*."

4 "*When a wise man points at the moon the imbecile examines the finger."* — Confucius.

flaws in our theory, the better, because it will lead us to move one step forward."

Deep Science is quite comparable to Deep Learning, *i.e.*, the merging of Machine Intelligence and Big Data. This kind of merging becomes crucial in studying physical phenomena, because the amount of data involved in physical science is getting very large.[5] For example, one or two decades ago, a PhD student in astronomy may have needed to analyse a few Gigabytes of data, but these days the data requirements have reached Terabyte levels. In the same way, it is not enough to find the logical structures while studying cosmology. We should also learn physical patterns. After all, mathematics is not all about logic and proof building, but it is also about pattern recognition.

There are two things one should keep in mind: (1) Physics is more than an acrobatic juxtaposition of the latest trends of theoretical jargon; and (2) Mathematics is more than a semiotic game of symbols and operators. In other words, mathematical language is required, but that is not the goal. If one wants to find the light at the end of the tunnel, one should ask different questions and think differently.

A Cat Tied to the Pole

If one wants to find the Holy Grail of Nature, where shall one find the answers? The following story may be helpful:

A long time ago, a Zen teacher in a distant village was disturbed by voice of a cat in his house, so he ordered his students to tie the cat to a pole in the backyard, so he could pray, undisturbed. Decades later, long after the teacher had passed away, all his followers still hold on to their former Zen teachers' exhibited behaviours and have made it into a tradition of tying a cat to a pole in the backyard. They also publish many books discussing the spiritual advantages of praying beside a cat.

That is an old story in a book written by Father Anthony de Mello, a wise priest from India. The lesson is simple but it has deep message: A temporary solution for certain problem can become a cult, worshipped by future generations of ignorant followers.

One may laugh at this story, but let us see four examples to show that the same problem may plague many areas of our modern life:

a. Max Planck. In a desperate move, he used a partition function in order to solve the blackbody paradox. His artificial trick was hailed as quanta of energy by Einstein in his 1905 (photoelectric paper), a development that Planck himself remained sceptical of. The

[5] For interested readers, there are a number of new papers in arXiv which explain this new field of Deep Learning applied to physics, see for instance: J. Thierry-Mieg, https://arxiv.org/pdf/1811.00576.pdf; Dan Guest et al., Annu. Rev. Nucl. Part. Sci. 2018. 68:1–22. url: https://arxiv.org/pdf/1806.11484.pdf; Maziar Raissi, https://arxiv.org/pdf/1801.06637.pdf; Emmanuel de Bezenac et al., https://arxiv.org/pdf/1711.07970.pdf.

photon was then traditionally accepted as real entity by later generations of physicists. In recent years, other physicists prove that Planck's blackbody law can be re-derived by assuming monochromatic waves. Earlier, Timothy Boyer derived Planck's law by applications of stochastic electrodynamics and zero point radiation.[6] Now we ask this question: Does the photon really exist or is it really a "cat tied to a pole" a kind of tradition?

b. Albert Einstein. Einstein developed his general relativity theory with the help of a friend, Marcell Grossmann. He was fully aware that his castle constructions assumed many things, including assumptions of continuous structures. We can argue that it is actually possible to develop various new models of cosmology starting from a discrete space, instead of the "continuous space" assumption. Ask this question: Does space-time curvature physically exist or does it exist only in your mind?

c. Murray Gell-Mann. After spending a few years learning group theory, Gell-Mann developed further Sakata's model, in order to explain certain experimental results which had arisen at the time. He called his extension of Sakata's model, "quark theory". But Gell-Mann himself never considered "quarks" as real entities.[7] They were only his pet name for a mathematical construction. Unfortunately, from that point, Zweig and Yuval Ne'eman developed a theory which assumed "real quarks" in the "cat tied to a pole" tradition. Later on, experimenters realised that quarks cannot be isolated. This realisation points at the fact that "quarks" are merely fictional creatures, based on unquestioned assumptions regarding "the correctness of quarks" in the "tradition" of Gell-Mann. Indeed, there is still a whole subculture in particle physics devoted to the "quark confinement" problem. What is the point of the "quark confinement" problem, if all quarks are just mathematical artefacts in the "tradition" of misunderstanding the origins of the term "quark" as "a particle", rather than a personal term for a mathematical operation, as originally coined by Gell-Mann?

d. Abraham Maslow. He was the "father of humanist psychology" at the time, who was famous for his "hierarchy of needs". Probably you have read that you should fulfil basic human needs first (food, clothes etc.), then begin to meet education and health needs and relationships with others, then seek actualisation of your life. Millions of people followed Maslow's hierarchy advise [5]. The story goes that later in his life, Maslow regretted how his theory was used. Of course, if you think rather deeply, you will find out that if you follow Maslow's recipe, then by the age of 60 you will have no more energy to do actualisation, or to live meaningfully, let alone doing something good for your community. It is much better to do things the other way around: Begin to seek God's purpose in your life, find His Kingdom and His truth, and you will have a purpose in your life. Then gradually God will help you to fulfil all your needs. But unfortunately only a few people

[6] See for example, Timothy Boyer, url: https://arxiv.org/abs/physics/0206033; https://arxiv.org/pdf/physics/0605003.pdf; https://arxiv.org/abs/1809.09093; and also O.A. Senatchin, arXiv:physics/0101054.

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can see the way. Most people only follow Maslow's hierarchy blindly, and they forget that is just a hypothesis. That is "*the road less travelled*."

Conclusion: Another Day in Paradise

Perhaps the true purpose of doing mathematics, as well as logic, is to purify our mind. If our mind is like a mirror, then we have to remove all the dust. But minds have limitations too. As Hui Neng[8] put it: "*If there is no shining mirror. ...Where can dust collect?"* In other words, while it is true that it takes years to get mathematical mastery in a field, may be the right answer lies somewhere else. Therefore one needs to ask a variety of different questions. Set one's mind free, since "[i]deas are your only currency" [6].

One may learn somewhere that to become a good scientist, one should have great ambition to dominate the entire world. One should be fast like a jaguar, strong like a gorilla, and cruel like a shark. If one follows such an advice, no wonder one gradually become a beast. And that is what they urge one to become in numerous universities. If one is a professor, one is forced to publish 25 papers and maybe more each year. They call it the "*publish or perish*" policy.[9] Actually that is an unnatural draconian "Social Darwinism" policy, while Nature is based on Harmony, caring, and cooperation.

And the plan of Social Darwinism (competition and "survival of the fittest") is to make one's life miserable, like living in constant danger, in a jungle, a policy intended to curtail any real progress in the sciences. No wonder, many leading professors have no time anymore to give lectures, because they are too busy catching up their publication requirements schedule. If one follows this story, this is why scientific productivity in the USA tends to gradually decrease, as a recent article in the Economist magazine reports. See also two other articles cited below[10].

But the truth is, one can do science in entirely different way: Find the *inner peace*, feel the rhythm of your heart, know how much God love you, and begin to love your family and your neighbours. The truth is not out there, but it is inside you. Instead of living like an animal in a jungle, and doing cruel things to one's colleagues, and having even crueller things done to you, try to listen to a soft voice of the Old Friend: "*You can walk with Me in Paradise, right now.*"

Contrary to what most people tell you, "become a mean and cruel animal", or "dominate the whole

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8 Yeno (Hui-neng, 638-713), traditionally considered the Sixth Patriarch of the Zen teaching in China. url: https://www.zinzin.com/observations/2014/zen-in-action-no-tree-no-mirror-no-dust/

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10 https://www.focus-economics.com/blog/why-is-productivity-growth-so-low-23-economic-experts-weigh-in; see also https://www.vox.com/new-money/2016/10/24/13327014/productivity-paradox-innovation-growth

world", you can ask: "What is the benefit of dominating the world, if I lose my life? How can I find purpose and happiness?"

In the end, if you seek peace and happiness for you and your neighbours, you will find that you live not in any jungle. Instead, you will find that this day is just another day in Paradise. What we mean by *Deep Science* is:

Deep Learning,
Deep Meaning,
Deep Purpose,
Deep Life,
Deep Spirituality, and
Don't live a superficial life - Go Deep.

To err is human, but to generate original ideas is divine.

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Cellular Automata Representation of Submicroscopic Physics

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Abstract

Krasnoholovets theorized that the microworld is constituted as a *tessellation* of primary topological balls. The tessellattice becomes the origin of a submicrospic mechanics in which a quantum system is subdivided to two subsystems: the particle and its inerton cloud, which appears due to the interaction of the moving particle with oncoming cells of the tessellattice. The particle and its inerton cloud periodically change the momentum and hence move like a wave. The new approach allows us to correlate the Klein-Gordon equation with the deformation coat that is formed in the tessellatice around the particle. The submicroscopic approach shows that the source of any type of wave movements including the Klein-Gordon, Schrödinger, and classical wave equations is hidden in the tessellattice and its basic exciations – inertons, carriers of mass and inert properties of matter.

Keywords: Schrödinger, Klein-Gordon, classical wave equation, periodic table, molecule, cellular automata, submicroscopic.

1. Introduction

Elze [1] wrote about possible re-interpretation of quantum mechnics (QM) starting from classical automata principles. This is surely a fresh approach to QM, initiated by some authors including Gerard ‘t Hooft [3]. In the mean time, in a series of papers Shpenkov [2, 3-14] suggested that the spherical solution of Schrödinger’s equation says nothing about the structure of molecules. According to Shpenkov [2, 3-14], the classical wave equation is able to derive a periodic table of elements which is close to Mendeleyev’s periodic table and also other phenomena related to the structure of molecules.

However, the Schrödinger equation is a quantum equation that describes the motion of the appropriate particle-wave since all quantum objects manifest charecteristics of both particles and waves. Considering Shpenkov’s results, one can ask: *why do the particle’s characteristics dissapear and what exactly is the subject of purely wave behaviour in a quantum system?*

Recently, Krasnoholovets has developed a submicroscopic concept in which the motion of a canonical particle occurs in physical space constructed as a cellular structure named the tessellattice(see, e.g. Ref. [15]).

In this paper, we carry out studies of the Schrödinger equation and classical wave equation and show how they both are related to the idea of tessellatice. The Appendix contains a more detailed proof on how "space" has the form of acoustic/sound wave.

2. Correspondence betwwen Classical Wave & Quantum Mechanics

A connection between classical and quantum mechanics has been studied at least by several researchers (see e.g. Refs. [26-28]). Ward and Volkmer [29] discussed a relation between the classical electromagnetic wave equation and Schrödinger equation. They derived the Schrödinger equation based on the electromagnetic wave equation and Einstein's special theory of relativity. They began with electromagnetic wave equation in one-dimensional case:

$$\frac{\partial^2 E}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2 E}{\partial t^2} = 0. \tag{13}$$

This equation is satisfied by plane wave solution:

$$E(x,\ t) = E_0 e^{i(kx-\omega t)}, \tag{14}$$

Where $k = 2\pi/\lambda$ and $\omega = 2\pi\nu$ are the spatial and temporal frequencies, respectively. Substituting equation (14) into (13), then we obtain

$$\left(\frac{\partial^2}{\partial x^2} - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) E_0 e^{i(kx-\omega t)} = 0, \tag{15}$$

or

$$\left(k^2 - \frac{\omega^2}{c^2}\right) E_0 e^{i(kx-\omega t)} = 0, \tag{16}$$

which arrives us to a dispersion relationship for light in free space: $k = \omega/c$. This is similar to the wave number k in eq. (8).

Then, recalling from Einstein and Compton that the energy of a photon is $\varepsilon = h\nu = \hbar\omega$ and the momentum of a photon is $p = h/\lambda = \hbar k$, which allows us to rewrite eq. (14) using these relations:

$$E(x,\ t) = E_0\, e^{\frac{i}{\hbar}(px-\varepsilon t)}. \tag{17}$$

Substituting expression (17) into eq. (13) we find

$$-\frac{1}{\hbar^2}\left(p^2 - \frac{\varepsilon^2}{c^2}\right) E_0 e^{\frac{i}{\hbar}(px-\varepsilon t)} = 0, \tag{18}$$

which results in the relativistic total energy of a particle with zero rest mass

$$\varepsilon^2 = p^2c^2. \tag{19}$$

Following de Broglie, we may write the total relativistic energy for a particle with non-zero rest mass

$$\varepsilon^2 = p^2c^2 + m_0^2c^4. \tag{20}$$

Inserting expression (20) into eq. (18), it is straightforward from (15) that we get

$$\left(\nabla^2 - \frac{m_0^2c^2}{\hbar^2}\right)\Psi = \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2}, \tag{21}$$

which is the Klein-Gordon equation [30, 31] for a free particle [29]. Now we want to obtain Schrödinger equation, which is non-relativistic case of eq. (21). The first step is to approximate $\varepsilon^2 = p^2c^2 + m_0^2c^4$ as follows

$$\varepsilon = m_0c^2\sqrt{1+\frac{p^2}{m_0^2c^2}} \approx m_0c^2 + \frac{p^2}{2m_0} \approx m_0c^2 + \Im. \tag{22}$$

After some approximation steps, Ward and Volkmer [29] arrived at the Schrödinger equation

$$-\frac{\hbar^2}{2m}\nabla^2\phi = i\hbar\frac{\partial\phi}{\partial t}, \tag{23}$$

where the non-relativistic wave function ϕ is also constrained to the condition that it be normalisable to unit probability.

In the meantime, Hilbert and Batelaan [32] explored equivalence between the quantum and acoustic system. A simple physical system was discussed, which mirrored the quantum mechanical infinite square well with a central delta well potential. They find that the analytic solution to the quantum system exhibits level splitting, as does the acoustic system. They compare the acoustic resonances in a closed tube and the quantum mechanical eigen-frequencies of an infinite square well and showed that the acoustic displacement standing wave is

$$\xi(x) = \xi_{\max}\sin\left(n\pi x/(2a)\right) \tag{24}$$

for the n-th resonance. Eq. (24) has the same shape as the quantum mechanical wave function.
So we can conclude that there exists formal connection between the classical wave equation and Schrödinger equation, but it still requires some assumptions and approximations. Shpenkov's

interpretation of classical wave equation looks as more realistic for atomic and molecular modeling.

3. Cellular Automata Model of Classical Wave Equation

In the previous section, we have argued that Shpenkov's interpretation of classical wave equation looks as more realistic for atomic and molecular modeling. Now we shall outline a cellular automata model of classical wave equation.

But first of all, let us give a few remarks on cellular automata. The term cellular automata cellular automata is plural. Our code examples will simulate just one—a cellular automaton cellular automaton, singular. To simplify our lives, we'll also refer to cellular automata as "CA." Cellular automata make a great first step in building a system of many objects that have varying states over time:

> A cellular automaton is a model of a system of "cell" objects with the following characteristics.
>
> • The cells live on a grid grid. (We'll see examples in both one and two dimensions in this chapter, though a cellular automaton can exist in any finite number of dimensions.)
>
> • Each cell has a state. The number of state possibilities is typically finite. The simplest example has the two possibilities of 1 and 0 (otherwise referred to as "on" and "off" or "alive" and "dead").
>
> • Each cell has a neighborhood. This can be defined in any number of ways, but it is typically a list of adjacent cells.[36]

Now consider a set of simple rules that would allow that pattern to create copies of itself on that grid. This is essentially the process of a CA that exhibits behaviour similar to biological reproduction and evolution. (Incidentally, von Neumann's cells had twenty-nine possible states.) Von Neumann's work in self-replication and CA is conceptually similar to what is probably the most famous cellular automaton: the "Game of Life." Perhaps the most significant scientific (and lengthy) work studying cellular automata arrived in 2002: Stephen Wolfram's *A New Kind of Science* (http://www.wolframscience.com/nks/) [36].

A plausible method to describe cellular automata model of wave equation was described for instance by Yang and Young [33]. For the 1D linear wave equation, where c is the wave speed they presented a scheme:

$$\frac{u_i^{n+1} - 2u_i^n + u_i^{n-1}}{(\Delta t)^2} = c^2 \frac{u_{i+1}^n - 2u_i^n + u_{i-1}^n}{(\Delta x)^2}. \tag{25}$$

After some steps eq. (25) can be rewritten in a generic form (by choosing $\Delta t = \Delta x = 1,\ t = n$) as follows $u_i^{t+1} + u_i^{t-1} = g(u^t)$, which is reversible under certain conditions. This property comes from the reversibility of the wave equation because it is invariant under the transformation: $t \rightarrow -t$.

O'Reilly has shown that the coupled Maxwell-Dirac electrodynamic system can be implemented in an analog cellular-automaton operating within a 3D regular face-centered cubic lattice [34]. The result of this approach can be expressed in terms of a second order wave equation, namely: $s_i^{t+1} = s_i^t + \dot{s}_i^{t+1}$. He concludes that the second order wave equation is arguably one of the simplest possible continuous-valued cellular automata update equations that do anything physically interesting, though all of electrodynamics can be built of elaborations of this one fundamental interaction.

Thus, cellular approach allows one to construct equations that describe physical systems without using second order equations.

Correspondence with Konrad Zuse's work: from static space to calculating space

To trace the development of physical thoughts in this field, we would like to mention two books. In the late 1970s, Konrad Zuse conceived an essay entitled *Calculating Space*, in which he advocated that physical laws are discrete by nature and that the entire history of our universe is just the output of a giant deterministic CA.[37]

It shall be clear, that we should let go our assumption of static space (Newtonian), nor dynamical space (Einsteinian), toward calculating space (Zusian). In this new model, space itself has a kind of computing capability, hence intelligence, albeit perhaps not the same kind of human intelligence. If such a new proposition can be proved true, then it may open up an array of explanations on many puzzling cosmology questions, such as: why galaxies apparently grow and then move to other directions (for instance, it is known that our Milky Way is moving toward *The Great Attractor*). Such an observed dynamics is very difficult to comprehend in terms of classical picture based on static space (differential equations).

In closing, we would like to quote Zuse's perceptive predictions made forty years ago: "*Incorporation of the concepts of information and the automaton theory in physical observations will become even more critical, as even more use is made of whole numbers, discrete states and the like.*"[37]

Nonetheless, we shall also keep in mind Zuse's question: "Is nature digital, analog or hybrid?" It is clear that classical physics is built in analogue way, but it does not mean that Nature is perfectly working in accordance with that model. That question needs to be investigated in more precise manner. [37, p. 22]

One way to investigate such a discrete model of space is by assuming a tessellattice model of space, as will be discussed in the next section.

4. The Tessellattice as the Source for the Formalism of Conventional Quantum Mechanics

A detailed theory of real physical space was developed by Bounias and Krasnoholovets starting from pure mathematical principles (see e.g. Ref. [35]). A submicroscopic theory of physical processes occurring in real physical space was elaborated by Krasnoholovets in a series of works (see e.g., monograph [15]). Those studies show that our ordinary space is constructed as a mathematical lattice of primary topological balls, which was named a tessellattice.

In the tesselllattice, primary topological balls play the role of cells. This is a physical vacuum, or aether. Matter emerges at local deformations of the tessellattice when a cell (or some cells) changes its volume following a fractal law of transformations. Such a deformation in the tessellattice can be associated with the physical notion of mass.

The motion of a fractal-deformed cell, i.e. a mass particle, is occurred with the fractal decomposition of its mass owing to its interaction with ongoing cells of the tessellattice. This is a further development of Zuse's idea about calculating space because cells can exchange by fractals, which locally change properties of space.

The interaction of matter with space generates a cloud of a new kind of spatial excitations named 'inertons'. This means that "hidden variables" introduced in the past by Louis de Broglie, David Bohm and Jean-Pierre Vigier have acquired a sense of real quasiparticles of space.

Thus in monograph [15] it has been shown that inertons are carriers of a new physical field (the inerton field), which appears as a basic field of the universe. Inertons as quasi-particles of the inerton field are responsible for quantum mechanical, nuclear and gravitational interactions of matter. Inertons carry mass and also fractal properties of space, i.e. they are real carries of information.

A particle moving in the tessellattice is surrounded with its inerton cloud. The particle actualizes the real motion between ongoing cells, though its inertons emitted when the particle rubs again the tessellattice's cells, migrate as excitations hopping from cell to cell. Such sophisticated motion in which the particle is surrounded with its inerton cloud can easily be compared with the formalism of quantum mechanics because the particle wrapped with its inertons can be projected to the particle's wave ψ-function determined in an abstract phase space. In such a pattern, the overlapping of wave ψ-functions of nearest particles means that the particles' inerton clouds overlap and thus we obtain real carriers of the quantum mechanical interaction, which provide a short-range action between the particles studied.

The particle's de Broglie wavelength λ plays the role of a section in which the moving particle emits its inerton cloud (an odd section) and in the next even section λ these inertons come back to the particle passing the momentum on to it. Inertons emitted by the freely moving particle come back to the particle owing to the elasticity of the tessellattice as such.

How can we write the interaction of a moving particle with its inerton cloud? The interaction can be written between the particle and an ensemble of inertons, which accompany the particle. The ensemble is presented as one integral object, an inerton cloud. The speed υ_0 of the particle the particle satisfies the inequality $\upsilon_0 << c$. At such presentation, our study is significantly simplified and is reduced to the consideration of a system of two objects: the particle and its cloud of inertons, which the particle periodically emits and adsorbs when moving along its path. In this case the Lagrangian (2.1) is transformed to the following one written in two-dimensional Euclidean space

$$L = \tfrac{1}{2} m_0 \dot{x}^2 + \tfrac{1}{2}\mu_0 \cdot \left[(\dot{\chi}^{\parallel})^2 + (\dot{\chi}^{\perp})^2 \right] - \frac{2\pi}{T}\sqrt{m_0 \mu_0}\, x \dot{\chi}^{\perp}. \tag{26}$$

In the Lagrangian (26)(2.49) the first term describes the kinetic energy of the particle with the mass *m* and the velocity $\dot{x}$, which moves along the axis *X*; the second term depicts the kinetic energy of the whole inerton cloud whose mass is μ_0 and its center-of-mass has the coordinate $\chi^{\parallel}$ along the particle's path and $\chi^{\perp}$ is the transverse coordinate; the third term is the interaction energy between the particle and the inerton cloud where $1/T$ is the frequency of their collisions.

By using the substitution

$$\dot{x}^{\perp} = \dot{\tilde{\chi}} + 2\pi \sqrt{m_0 / \mu_0}\; x / T, \tag{27}$$

we carry out a kind of a canonical transformation that leads to the following Lagrangian

$$\tilde{L} = \tfrac{1}{2} m_0 \dot{x}^2 - \tfrac{1}{2}(2\pi / T)^2 m_0 x^2 + \tfrac{1}{2}\mu_0 \cdot \left(\dot{\tilde{\chi}}^2 + (\dot{\chi}^{\parallel})^2 \right). \tag{28}$$

We can see from the effective Lagrangian (28)(2.51) that in such a presentation the particle's behavior is described as a classical harmonic oscillator and the accompanying inerton cloud moves by its own hidden principle (though it does not disturb the particle).

The Hamiltonian function according to the definition

$$H = \sum_i \dot{Q}_i \partial L / \partial \dot{Q}_i - L.$$

In our case the Hamiltonian is

$$H = \dot{x}\partial L / \partial \dot{x} + \dot{\tilde{\chi}} \partial L / \partial \dot{\tilde{\chi}} - \tilde{L}. \tag{29}$$

The effective Hamiltonian based on the Lagrangian (28)(2.51) of the oscillating particle in the system of the center-of-mass of the particle and its inerton cloud in the explicit form becomes

$$H = p^2 / (2m_0) + m_0 (2\pi / T)^2 x^2 / 2. \tag{30}$$

Solutions of the equations of motion given by the Hamiltonian (30) are well known for different presentations. In particular, the function (30) allows one to derive the Hamilton-Jacobi equation

$$(\partial S_1 / \partial x)^2 / (2m_0) + m_0 (2\pi / T)^2 x^2 / 2 = E \tag{31}$$

from which we obtain the equation for a shortened action

$$S_1 = \int_{x_0}^{x} p\, dx = \int_{x_0}^{x} \sqrt{2m_0 [E - (2\pi / T)^2 x^2 / 2]}\, dx . \tag{32}$$

The function (32)(2.55) enables the solution x as a function of t in the form

$$x = \frac{\sqrt{2E / m_0}}{2\pi / T} \sin(2\pi t / T) . \tag{33}$$

Now we can calculate the increment ΔS_1 of the action (32)(2.55) of the particle during the period T; in terms of the action-angle variables

$$\begin{aligned} \Delta S_1 &= \oint p\, dx = \oint \sqrt{2m_0 \left(E - m_0 (2\pi / T)^2 x^2 \right)}\, dx \\ &= \oint \sqrt{2m_0 \left(E - E \sin^2(2\pi t / T) \right)} \sqrt{2E / m_0} \cos(2\pi t / T)\, dt \\ &= 2E \int_0^T \cos^2(2\pi t / T)\, dt = 2E \left(\frac{t}{2} + \frac{\sin(4\pi t / T)}{4(2\pi / T)} \right) \Bigg|_{t=0}^{t=T} = ET . \end{aligned} \tag{34}$$

The final result (34) can be rewritten as follows

$$\Delta S_1 = E \times T = E / \nu \tag{35}$$

where the notation $\nu = 1/T$ is entered.

Since the constant E is the initial energy of the particle, i.e., $E = \frac{1}{2} m_0 \upsilon_0^2$, the increment of action (35) can also be presented in the form

$$\Delta S_1 = \tfrac{1}{2} m_0 \upsilon_0^2 \times T = m_0 \upsilon_0 \times \tfrac{1}{2} \upsilon_0 T = m_0 \upsilon_0 \times \lambda \tag{36}$$

where the parameter λ is the spatial amplitude of oscillations of the particle along its path.

If we equate the increment of the action ΔS_1 to the Planck constant h, we immediately arrive at the two major relationships of quantum mechanics introduced by de Broglie for a particle:

$$E = h\nu , \qquad \lambda = h / (m_0 \upsilon_0) . \tag{37}$$

Thus the amplitude of special oscillation of a particle is exactly the particle's de Broglie wavelength.

Having obtained the relationships (37), we can present the complete action for a particle

$$S = S_1 - Et = \int^x p\,dx - Et \tag{38}$$

in two equivalent forms:

$$S = m_0 \upsilon_0 x - Et \tag{39}$$

and

$$S = h \times (x/\lambda - \nu t). \tag{40}$$

The relationships (39), (40) and (37) allow the derivation of the Schrödinger equation. If in a conventional wave equation

$$\Delta\psi - \frac{1}{(\upsilon_0/2)^2}\frac{\partial^2\psi}{\partial t^2} = 0 \tag{41}$$

(where $\frac{1}{2}\upsilon_0$ is the average velocity of the particle in the spatial period λ) we insert a wave function, whose phase is based on the action (40),

$$\psi = a\exp\left\{i2\pi[x/\lambda - \nu t]\right\}, \tag{42}$$

and set $\upsilon_0 = \lambda \times 2\nu$, we get the wave equation in the following presentation:

$$\Delta\psi + \left(2\pi/\lambda\right)^2\psi = 0. \tag{43}$$

Then putting $\lambda = h/p$ and extracting the momentum p from the function (32) (i.e., $p^2 = 2mE$) we finally obtain a conventional time-independent Schrödinger equation

$$\Delta\psi + \frac{2m_0 E}{\hbar^2}\psi = 0. \tag{44}$$

Thus, we can see that the moving system of a particle and its inerton cloud obeys the Schrödinger equation.

5. The Deformation Coat of Particle & the Klein-Gordon Equation

As we discussed above, Ward and Volkmer [29] demonstrated the derivation of the Klein-Gordon equation (21) for a mass particle starting from its total relativistic energy $\varepsilon^2 = p^2c^2 + m_0^2c^4$ (20). They also showed that a non-relativistic approximation of the same energy (20) results in the Schrödinger time-dependent equation (23).

Usually the Klein-Gordon equation [30, 31] is applied for the description of an abstract relativistic particle that does not possess spin. However, the submicroscopic concept of physics presented in monograph [15] makes *it possible to relate the Klein-Gordon equation to a real object,* namely, a deformation coat that is developed around the mass particle created in the tessellatice.

In fact the creation a particle means the appearance of a local deformation, i.e. a volumetric fractal deformation of the appropriate cell of the tesselllatice. The local deformation must induce a tension state in ambient cells, which may extend only to a definite radius R. So behind the radius R, the tessellattice does not have any distortion, it is found here in a degenerate state.

The study [15] shows that in the microworld such fundamental physical parameters as mass and charge vary at the motion. Namely, in a section (the even section) equal to the particle's de Broglie wavelength λ the mass m is transferred to a tension ξ and the charge e changes to the magnetic monopole g. In the odd section λ the mass and charge are restored. The same happened with cells that form the particle's deformation coat. When the particle is moving, it pulls its deformation coat as well, i.e. ambient cells adjust to state of the particle. In the deformation coat the state of cells oscillates between the tension ξ and mass m. A collective oscillating mode of the deformation coat is specified by the energy [15] $E=\hbar\omega$, which in turn equals the total energy of the particle mc^2.

The discussed oscillations can be described by a plane wave mode $E(x,t)=E_0 e^{i(kx-\varepsilon t)}$ (17). Then following the arguments (17) – (21), we immediately derive the Klein-Gordon equation (21). Note that in our case the particle that obeys the Klein-Gordon equation is the deformation coat that accompanies the moving particle. This deformation coat is specified with the radius equal to the particle's Compton wavelength [15] (see p. 57).

If the speed υ of a particle satisfies the inequality $\upsilon << c$, we following reasoning (22) and (23) will arrive at the Schrödinger equation (23).

6. Conclusion

We have reviewed a plausible cellular automaton molecular model for classical wave equation, as an alternative to Cellular automaton quantum mechanics (by Elze, Gerard 't Hooft etc).

Then we have considered the submicroscopic concept that allows one to easily derive the Schrödinger and Klein-Gordon equations starting from first submicroscopic principles. It is interesting that for the first time we now can identify the Klein-Gordon equation with a real object that is described by this equation – it is the particle's deformation coat that is induced in the tessellattice at around the appropriate created canonical particle.

The submicroscopic concept, which is based on space constituted as the tessellattice of primary topological balls, introduces a new physical field, namely the inerton field, which appears as a fundamental field of the universe. Inertons emerge at any motion of particles; in particular, they arise in atoms and around owing to uninterrupted motion of electrons, nuclei and nucleons.

Thus the motion of a quantum system is characterized by its separation to two joined subsystems: the particle itself and its inerton cloud. Their oscillation dynamics exhibits obvious features of the wave motion. Although the deformation coat that accompanies the moving particle behaves in a special way, it is described by the Klein-Gordon equation, which also manifests the wave properties.

Our analysis shows that oscillations of inertons are present in any movement of a material object. Inertons clearly demonstrate wave behavior. This means that inerton oscillations appear in atoms and molecules. Hence inerton oscillations justify Shpenkov's model [4–14], which applies a classical wave equation of sound to atoms and molecules: the wave function Ψ used by Shpenkov describes oscillations of an inerton field and the location of the corresponding nodes in the oscillating wave studied.

Thus, quantum mechanical models, cellular automata, and a cellular automaton molecular model that uses a wave equation can be covered by studies originated from the tessellattice and the submicroscopic behavior of quantum systems, which involves an inerton field that binds canonical particles with the tessellattice and between themselves.

Nonetheless, there remains many questions to ponder, for example: whether the notion of cellular automata corresponds neatly to Zuse's calculating space hypothesis [37], and whether the latter in turn leads to cellular intelligence (see for instance [37a]). Therefore, further investigations in this direction are recommended, which will shed light on the cornerstones of the microworld.

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Numerical Solution of an Equation Corresponding to Schumann Waves

Victor Christianto, Florentin Smarandache

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Abstract

We consider an one-dimensional Schumann wave equation proposed by Marciak-Kozlowska. Numerical solution of that equation was obtained with the help of Mathematica.

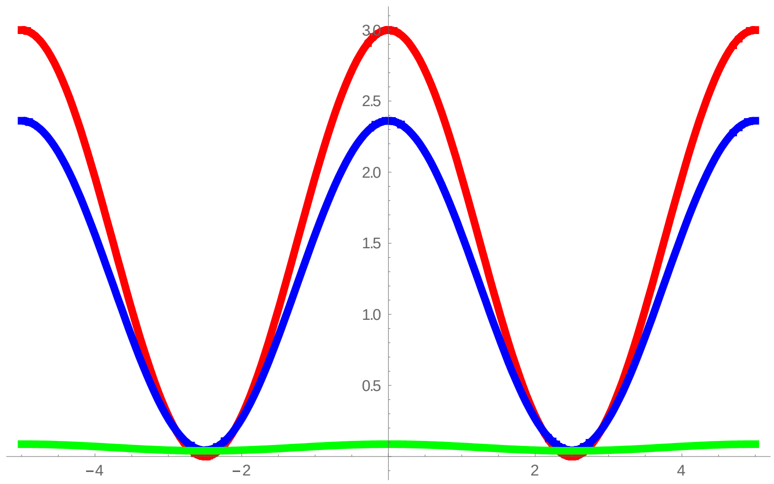
Keywords: Schumann wave, numerical solution, equation.

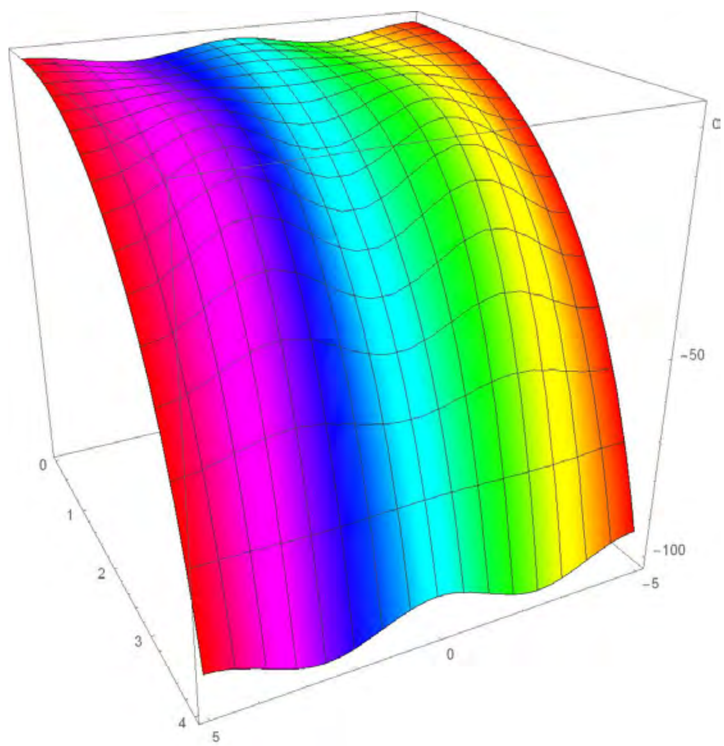
Introduction

The measured frequencies of Schuman and brainwaves are nearly the same [1]. It is worthwhile to point out that both calculated curves give a rather good description of the measured frequencies of Schuman and brain waves [2-3].

We consider an one-dimensional Schumann wave equation proposed by Marciak-Kozlowska. Numerical solution of that equation was obtained with the help of Mathematica.

A hyperbolic equation for Schuman wave phenomena was formulated [4-5] where m is the mass of the neuron, $\hbar$ is the Planck constant, V is potential and v is the velocity propagation of the Schumann wave in the brain.
Now we will obtain its numerical solution without having to recourse to Klein-Gordon equation as its approximation [4]. Instead, we will look for direct numerical solution and its plot using Mathematica 9 [6]:





Concluding remarks

A direct numerical solution of master equation corresponding to Schumann wave has been presented. We hope this result may be useful for further studies on the connection between Schuman resonance, brainwaves and other related topics.

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Appendix: Mathematica code

```
SetOptions[Plot,ImageSize->500,PlotRange->All,PlotPoints->nP*2,PlotStyle-
>{Blue,Thickness[0.01]}];{s=1/100,nP=100}
{nN=3,l=1,l1={Red,Blue,Green},l2={0,1/2,1}}
f[u_]:=2*b*a/c^2;f[u]
eKG=D[u[x,t],{t,2}]+a*D[u[x,t]/c,{t,1}]-D[u[x,t],{x,2}]+f[u]==0
fIC1[f1_]:=u[x,0]==f1;fIC2[f2_]:=(D[u[x,t],t]/.t->0)==f2;
fBC1[c_,f1_]:=(D[u[x,t],x]/.x->c)==f1;
fBC2[d_,f2_]:=(D[u[x,t],x]/.x->d)==f2;
{fIC1[f1],fIC2[f2],fBC1[c,f1],fBC2[d,f2]};
params5={a->1,b->1,c->-1,aN->1.5};{c5=-
5,d5=5,tF5=4,xI5=c5,xF5=d5,f15=aN*(1+Cos[2*Pi*x/d5]),f25=0,f35=0,f45=0,eKG5=N[eKG/.params5],ic5=N[{fI
C1[f15],fIC2[f25]}/.params5],bc5=N[{fBC1[c5,f35],fBC2[d5,f45]}/.params5]}
sol5=NDSolve[Flatten[{eKG5,ic5,bc5}],u,{x,xI5,xF5},{t,0,tF5},MaxStepSize->s,PrecisionGoal->2]
Do[g[i]=Plot[Evaluate[u[x,l2[[i]]]/.sol5],{x,xI5,xF5},PlotStyle-
>{l1[[i]],Thickness[0.01]}],{i,1,nN}];Show[Table[g[i],{i,1,nN}]]
Plot3D[Evaluate[u[x,t]/.sol5],{x,xI5,xF5},{t,0,tF5},ColorFunction->Function[{x,y},Hue[x]],BoxRatios-
>1,ViewPoint->{1,2,1},PlotRange->All,PlotPoints->{20,20},ImageSize->500]
Animate[Plot[Evaluate[u[x,t]/.sol5,{x,xI5,xF5}],PlotRange->{-3,3}],{t,0,tF5},AnimationRate->0.5]
```

One-Note-Samba Approach to Cosmology

Victor Christianto, Florentin Smarandache

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Abstract

Inspired by One Note Samba, a standard jazz repertoire, we present an outline of Bose-Einstein Condensate Cosmology. Although this approach seems awkward and a bit off the wall at first glance, it is not impossible to connect altogether BEC, Scalar Field Cosmology and Feshbach Resonance with Ermakov-Pinney equation. We also briefly discuss possible link with our previous paper which describes Newtonian Universe with Vortex in terms of Ermakov equation.

Keywords: Cosmology, Bose-Einstein condensate, scalar field, Feshbach resonance, Ermakov-Pinney equation, Newtonian universe, vortex.

Introduction

From time to time, it is often found useful to come up with a new approach in cosmology studies, in order to seek a new insight from where we can develop and take further step.

For instance, it is known that flat spacetime cosmology can explain many cosmology phenomena (see for instance Narlikar & Arp [30]). But then, if we know that there is quite high likelihood that our Universe can be modeled as flat spacetime, then what else to be done?

In this occasion, allow us to put forth an argument that our Universe has remarkable similarity with a macroscale Bose-Einstein condensate, especially on the grounds: a. the CMBR temperature is found to be as low as 2.73^0 Kelvin, therefore it may indicate a low temperature physics model of Universe (see G. Volovik [31]), and b. recent discovery of "black hole" seems remarkably similar to a vortex ring of BEC experiment (it shows a dark spot circled with a white ring).

All in all, although we admit that this approach seems awkward and a bit off the wall at first glance, it is not impossible to connect altogether BEC, Scalar Field Cosmology and Feshbach Resonance with Ermakov-Pinney equation. We also discuss shortly possible link with our previous paper, where we describe Newtonian Universe with Vortex in terms of Ermakov equation [5].

We submit to call this approach: "*one note Samba*," i.e. starting with a very simple premise (BEC) you arrive at a model of the entire Universe.

Lidsey's BEC-Cosmodynamics Correspondence

According to Hawkins & Lidsey [2], the dynamics of cosmologies sourced by a mixture of perfect fluids and self–interacting scalar fields are described by the non–linear, Ermakov–Pinney equation. The general solution of this equation can be expressed in terms of particular solutions to a related, linear differential equation. In general, an Ermakov system is a pair of coupled, second–order, non–linear ordinary differential equations (ODEs) and such systems often arise in studies of nonlinear optics [12], nonlinear elasticity, molecular structures.

They developed an analytical approach to models of this type by expressing the cosmological field equations in terms of an Ermakov system. In the one–dimensional case, the two equations decouple and the system reduces to a single equation known as the Ermakov–Pinney equation [2]:

$$\frac{d^2 b}{d\tau^2} + Q(\tau) b = \frac{\lambda}{b^3} \tag{1}$$

where Q is an arbitrary function of τ and λ is a constant. Equation (1) is sometimes referred to as the Milne–Pinney equation. They also showed that the field equations for a spatially flat, Friedmann–Robertson–Walker (FRW) universe with a scalar field and perfect fluid matter source reduce to Ermakov-Pinney equation. To summarize, the dynamics of a pure scalar field cosmology is determined by a one–dimensional oscillator equation with a time–dependent frequency. [2]

In his subsequent paper, Lidsey managed to show that there is dynamical correspondence between positively curved, isotropic, perfect fluid cosmologies and quasi-two-dimensional, harmonically trapped Bose-Einstein condensates by mapping the equations of motion for both systems onto the one-dimensional Ermakov system.[1]

He developed that connection based on *analogies between various condensed matter systems and different branches of gravitational physics which* have been developed in recent years. For example, the propagation of acoustic waves in an irrotational, inviscid, barotropic fluid is formally equivalent to that of a massless scalar field on a curved, Lorentzian spacetime. Furthermore, it is possible to model a black hole acoustically in terms of supersonic fluid flow and, in principle, quantum effects associated with black hole event horizons may then be studied within the context of condensed matter configurations.[1]

Identifying such a link is quite significant because if it can be shown that there exists dynamical correspondence between isotropic, four–dimensional cosmological models and harmonically trapped, quasi–two–dimensional Bose–Einstein condensates, then there is a big hope to do simulation or cosmology experiments in lab. The correspondence arises because the equations of motion for both systems can be mapped onto the one–dimensional Ermakov system.

In that paper, Lidsey showed that the dynamics of a positively curved (k > 0) FRW cosmology can be modeled in terms of a harmonically trapped Bose–Einstein condensate when cosmic time, τ is related to 'laboratory' time, t.[1]

To summarize our discussion thus far, it can be shown that positively curved, perfect fluid FRW cosmologies can be modeled dynamically in terms of quasi–two–dimensional Bose–Einstein condensates, where there exists a one–to–one correspondence between the type of matter in the universe and the functional form of the time–dependent trapping potential of the condensate. The physical properties of the wavefunction can be identified with the fundamental cosmological parameters.[1]

The Pinney equation corresponding to FRW cosmology can be written as follows [1]:

$$\frac{d^2 a}{dt^2} + \left(\frac{d\phi}{dt}\right)^2 a = \frac{k}{a^3} \tag{2}$$

A key assumption that was made in establishing the correspondence between the condensate and cosmological systems was that the dynamics of the condensate wavefunction can be described in terms of the Gross–Pitaevskii equation at each moment of time, i.e., that the configuration reacts instantaneously to changes in the trapping potential and scattering length of the atomic interactions. If this assumption is to remain valid, the majority of the atoms must remain in the condensate state (mean–field approximation) and the particle density and scattering length must be sufficiently small (dilute gas approximation) [1].

Nevertheless, one advantage of establishing correspondences between cosmology and condensed matter physics through Ermakov systems is that insight into the hidden symmetries of the two systems may be uncovered.

Subsequent work by Herring et al. revisit the topic of two-dimensional Bose-Einstein condensates under the influence of time-dependent magnetic confinement and time-dependent scattering length. A moment approach reduces the examination of moments of the wavefunction (in particular, of its width) to an Ermakov-Pinney (EP) ordinary differential equation (ODE). They discussed Feshbach resonance managed BEC and how EP equation connects with the case of anisotropic scalar field cosmologies [3].

There is also a more recent report by D'Ambroise and Williams showing that there is also dynamic correspondence not only to FLRW but also Bianchi I cosmologies and BEC system in arbitrary dimension, especially when a cosmological constant is present [4].

Comparison with Newtonian Dynamics Model

In a previous paper [5], we presented a numerical solution of Newtonian Universe with vortex; see also [6][7]. Now we will present a more detailed account of our model.

A physical model of turbulence-generated sound for early Universe

Our discussion starts from the fundamental question: how can we include the rotation in early Universe model? After answering that question, we will discuss how "turbulence-generated sound" can be put into a mathematical model for the early Universe. We are aware that the notion of turbulence-generated sound is not new term at all especially in aerodynamics, but the term is rarely used in cosmology until now. We shall show that 3D Navier-Stokes will lead to non-linear acoustics models, which means that a turbulence/storm can generate sound wave.

a. *How can we include rotation in early Universe model?*

It has been known for long time that most of the existing cosmology models have singularity problem. Cosmological singularity has been a consequence of excessive symmetry of flow, such as "Hubble's law". More realistic one is suggested, based on Newtonian cosmology model but here we include the vortical-rotational effect of the whole Universe.

As shown in previous paper, we derived an Ermakov-type equation following Nurgaliev [26][27].

After he proceeds with some initial assumptions, Nurgaliev obtained a new simple local cosmological equation:[8][9]

$$\dot{H} + H^2 = \omega^2 + \frac{4\pi G}{3}\rho, \tag{3}$$

where $\dot{H} = dH/dt$.

The angular momentum conservation law ωR^2 = const = K and the mass conservation law $(4\pi/3)\rho R^3$ = const = M makes equation (3) solvable:[26]

$$\dot{\mathrm{H}} + \mathrm{H}^2 = \frac{K^2}{R^4} - \frac{\mathrm{GM}}{\mathrm{R}^3}, \tag{4}$$

or

$$\ddot{R} = \frac{K^2}{R^3} - \frac{GM}{R^2}.$$

(5)

Equation (5) may be written as Ermakov-type nonlinear equation as follows;

$$\ddot{R} + \frac{GM}{R^2} = \frac{K^2}{R^3}. \tag{6}$$

Nurgaliev tried to integrate equation (5), but we solved the above equation numerically. The results are as follows: First, we rewrite this equation by replacing GM=A, K^2=B, so we get an expression of Ermakov equation:[26]

$$\ddot{R} + \frac{A}{R^2} = \frac{B}{R^3}. \tag{7}$$

As with what Nurgaliev did, we also tried different sets of A and B values, as follows:

a. A and B < 0

```
A
B=-10;
ODE=x"[t]+A/x[t]^2-B/x[t]^3==0;
sol=NDSolve[{ODE,x[0]==1,x'[0]==1},x[t],{t,-10,10}]
Plot[x[t]/.sol,{t,-10,10}]
```

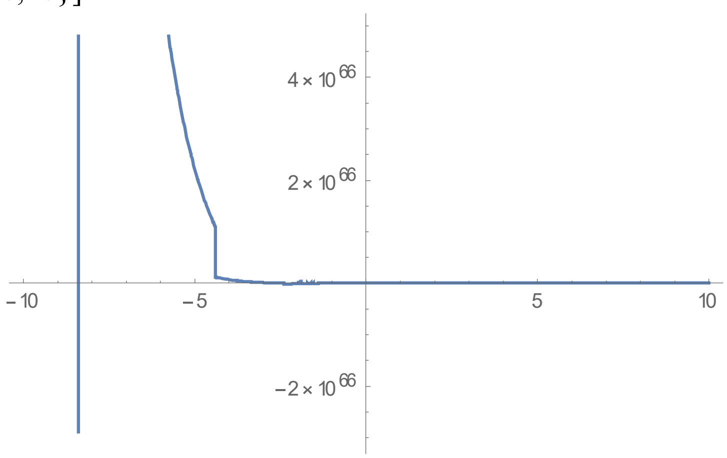

Figure 1. Plot of Ermakov-type solution for A=-10, B=-10 [5]

b. A > 0, B < 0

```
A=1;
B=-10;
ODE=x"[t]+A/x[t]^2-B/x[t]^3==0;
sol=NDSolve[{ODE,x[0]==1,x'[0]==1},x[t],{t,-10,10}]
Plot[x[t]/.sol,{t,-10,10}]
```

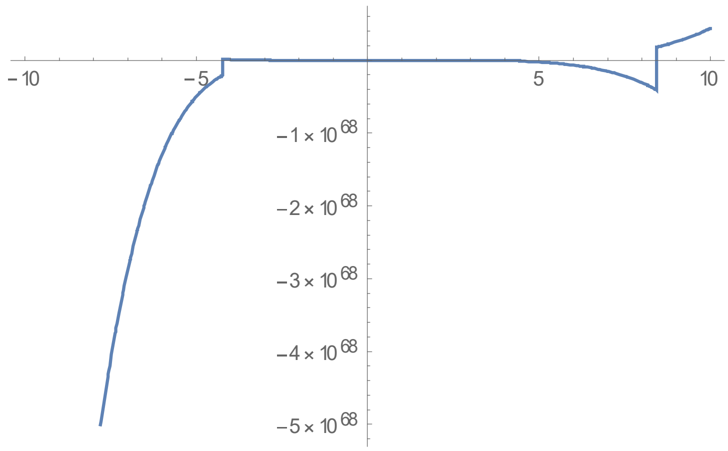

Figure 2. Plot of Ermakov-type solution for A=1, B=-10 [5]

From the above numerical experiments, we conclude that the evolution of the Universe depends on the constants involved, especially on the rotational-vortex structure of the Universe. This needs to be investigated in more detailed for sure.

One conclusion that we may derive especially from Figure 2, is that our computational simulation suggests that it is possible to consider that the Universe has existed for long time in prolonged stagnation period, then suddenly it burst out from *empty and formless* (Gen. 1:2), to take its current shape with accelerated expansion.

As an implication, we may arrive at a precise model of flattening velocity of galaxies without having to invoke *ad-hoc* assumptions such as dark matter.

Therefore, it is perhaps noteworthy to discuss briefly a simple model of galaxies based on a postulate of turbulence vortices which govern the galaxy dynamics. The result of Vatistas' model equation can yield prediction which is close to observation, as shown in the following diagram:[10]

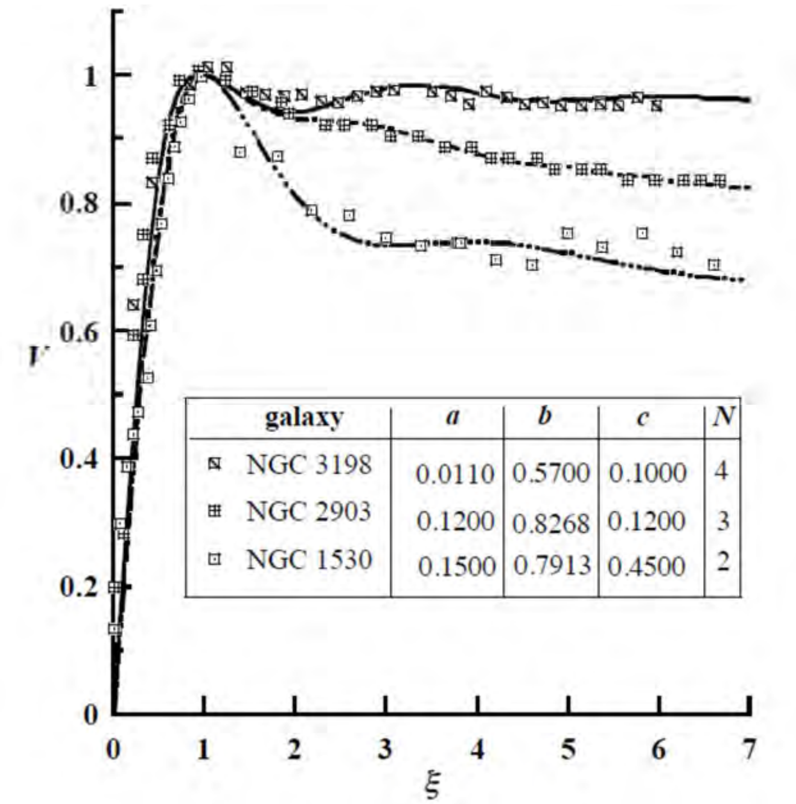

Figure 3. From Vatistas [10]

Therefore it appears possible to model galaxies without invoking numerous *ad hoc* assumptions such as *dark matter*, once we accept the existence of turbulent interstellar medium. The Vatistas model is also governed by Navier-Stokes equations, see for instance [10].

b. *How "turbulence-generated sound" can be put into a mathematical model for the early Universe*

We are aware that the notion of turbulence-generated sound is not new term at all especially in aerodynamics, but the term is rarely used in cosmology until now. We will consider some papers where it can be shown that 3D Navier-Stokes will lead to *non-linear acoustics* models, which means that a turbulence/storm can generate sound wave.

In this section we consider only two approaches:

- Shugaev-Cherkasov-Solenaya's model: They investigate acoustic radiation emitted by three-dimensional (3D) vortex rings in air on the basis of the unsteady Navier–Stokes equations. Power series expansions of the unknown functions with respect to the initial vorticity which is supposed to be small are used. In such a manner the system of the

Navier–Stokes equations is reduced to a parabolic system with constant coefficients at high derivatives. [11]

- Rozanova-Pierrat's Kuznetsov equation: she analysed the existing derivation of the models of non-linear acoustics such as the Kuznetsov equation, the NPE equation and the KZK equation. The technique of introducing a corrector in the derivation ansatz allows to consider the solutions of these equations as approximations of the solution of the initial system (a compressible Navier-Stokes/Euler system). The direct derivation shows that the Kuznetzov equation is the first order approximation of the Navier-Stokes system, the KZK and NPE equations are the first order approximations of the Kuznetzov equation and the second order approximations of the Navier-Stokes system. [12]

Remark on Neutrosophic Logic perspective and implications

It seems obvious, how this new scenario is quite in agreement with Kant's idea that it is possible that the Universe has both finite history in the past and also eternal background (our new term: "*time before time*"); see [6]. We also discussed how such a mixed view can be modelled by introducing rotation in the early universe; see in particular Fig. 2.

Now there is an immediate question: Is this new look at the origin of Universe justifiable logically, or is it merely a compromised solution?

So, in this chapter we will review *Neutrosophic Logic*, a new theory developed in recent decades by one of these authors (FS). In this context, allow us to argue in favor of Neutrosophic logic as one basic postutale, in lieu of the Aristotle logic which creates many problems in real world.

In Neutrosophy, we can connect an idea with its opposite idea and with its neutral idea and get common parts, i.e. <A> /\ <nonA> = nonempty set. The common part of the uncommon things! It is true/real... paradox. From neutrosophy, all started: neutrosophic logic, neutrosophic set, neutrosophic probability, neutrosophic statistics, neutrosophic measure, neutrosophic physics, neutrosophic algebraic structures etc.

It is true in restricted case, i.e. the Hegelian dialectics considers only the dynamics of opposites (<A> and <antiA>), but in our everyday life, not only the opposites interact, but the neutrals <neutA> between them too. For example: you fight with a man (so you both are the opposites). But neutral people around both of you (especially the police) interfere to reconcile both of you. Neutrosophy considers the dynamics of opposites and their neutrals.

So, neutrosophy means that: <A>, <antiA> (the opposite of <A>), and <neutA> (the neutrals between <A> and <antiA>) interact among themselves. A neutrosophic set is characterized by a truth-membership function (T), an indeterminacy-membership function (I), and a falsity-membership function (F), where T, I, F are subsets of the unit interval [0, 1].

As particular cases we have: single-valued neutrosophic set {when T, I, F are crisp numbers in [0, 1]}, and interval-valued neutrosophic set {when T, I, F are intervals included in [0, 1]}. Neutrosophic Set is a powerful structure in expressing indeterminate, vague, incomplete *and* inconsistent information. See [16]-[18].

To summarize, Neutrosophic Logic studies the dynamics of opposites and neutralities. And from this viewpoint, we can understand that it is indeed a real possibility that the Universe has both initial start (creation) but with eternal background. This is exactly the picture we got after our closer look at Gen. 1:1-2 as discussed in the above section.

In other words, our proposed term of Kantian "*time before time*" has sufficient logical background, especially in turbulence Universe model. This new interpretation of cosmic dynamics can be considered as Neutrosophic Logic application in cosmology studies, see also our previous article [32].

In the next section, we will consider some advantages of this new model of Universe.

Advantages of our Turbulence Universe model

Now, allow us to discuss some advantages of the proposed turbulence cosmology view over the Lemaitre's primeval atom (which is the basis of Standard Model Cosmology); see [13]-[15].

a. Avoid inflationary scheme.
   It is known that inflationary models were proposed by Alan Guth et al. (see [19][20]), in order to explain certain difficulties in the Big Bang scenario. But some cosmology experts such as Hollands & Wald has raised some difficulties with inflationary model, as follows:

   > "We argue that the explanations provided by inflation for the homogeneity, isotropy, and flatness of our universe are not satisfactory, and that a proper explanation of these features will require a much deeper understanding of the initial state of our universe."[21]

   In our diagram plot above, it is clear that an early rotation model can explain why the Universe can burst out into creation in a very short period, without invoking ad hoc postulate such as inflation model.

b. Explain the observed late accelerated expansion.
   As far as we know, one of the earliest models which gave prediction of accelerated expanding Universe is Carmeli's Cosmological General Relativity.[23]

But it has been shown by Green & Wald that for the large scale structures of the Universe, Newtonian model can give similar results compared to general relativity picture.[22]
Furthermore, it seems that there is no quite clear arguments why we should accept Carmeli use of 5D metric model (*space-time-velocity metric*). In the meantime, in our rotating Universe model, we do not invoke *ad hoc* dimension into the metric.

c. Explain inhomogeneity, breeding galaxies etc.
Astronomers have known for long time, that the Universe is not homogeneous and isotropic as in the usual model. It contains of inhomogeneity, irregularities, clumpiness, voids, filaments etc,, which indicate complex structures. Such inhomogeneous structures may be better modelled in terms of turbulence model such as Navier-Stokes equations, see also our early papers [7][8], also [10].

Conclusions

In this paper we start with reviewing Lidsey's work on connection between Ermakov-Pinney equation with cosmology schemes, then we also review his further work which attempts to establish the connection with BEC experiments in lab and cosmology setting.

Nonetheless, our additional note in this paper is pertaining to the use of similar Ermakov equation in Newtonian Universe with vortex, which indicates early universe with rotation can be modeled using Ermakov equation instead of trying to modify Friedmann equation for *rotating metric*.

In retrospect, noting similarity between EP equation in Lidsey's work and ours; it seems to indicate that it is possible to consider a turbulence Universe in terms of BEC experiments too. (It may be worth noting here, that superfluid vortex dynamics may be modelled in classical turbulence too, but it is beyond the scope of this paper).

That is why, we call this approach: "*one note samba*" approach to cosmology. Further investigation and experiment are recommended in this direction.

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A few calculations of receding Moon from spherical kinetic dynamics, receding planetary orbits, and the quantization of celestial motions

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Abstract

The present article discusses some interesting phenomena including the Lense-Thirring type anomalous precession, using a known spherical kinetic dynamics approach. Other implications include a plausible revised version of the celestial quantization equation described by Nottale and Rubcic & Rubcic. If the proposition described herein corresponds to the facts, then this kinetic dynamics interpretation of '*frame-dragging'* effect could be viewed as a step to unification between GTR-type phenomena and QM. Further observation to verify or refute this conjecture is recommended, plausibly using LAGEOS-type satellites.

Keywords: Lense-Thirring effect, celestial quantization, LAGEOS satellite, boson condensation, gravitation

Introduction

It is known, that the use of Bohr radius formula to predict celestial quantization has led to numerous verified observations [1]. This approach was based on Bohr-Sommerfeld quantization rules [2][3]. While this kind of approach is not widely accepted yet, this could be related to wave mechanics equation to describe large-scale structure of the Universe [4], and also a recent suggestion to reconsider Sommerfeld's conjectures in Quantum Mechanics [5]. Some implications of this quantum-like approach include exoplanet prediction, which becomes a rapidly developing subject in recent years [6][7].

Rubcic & Rubcic's approach [2] is particularly interesting in this regard, because they begin with a conjecture that Planck mass ($m_p = \sqrt{\hbar c / 2\pi G}$) is the basic entity of Nature, which apparently corresponds to Winterberg's assertion of superfluid Planckian aether comprised of phonon-roton pairs [8]. In each of these pairs, superfluid vortices can form with circulation quantized according to $\oint v_{+-}.dx = n\hbar / m_p$. This condition implies the Helmholtz vortex theorem, $d/dt \oint v_{+-}.dx = 0$. This relationship seems conceivable, at least from the viewpoint of likely neat linkage between cosmology phenomena and various low-temperature condensed matter physics [9][10][11]. In effect, celestial objects at various scales could be regarded as spinning Bose-Einstein condensate; which method has been used for neutron stars [32].

Despite these aforementioned advantages, it is also known that all of the existing celestial quantization methods [1][2][3] thus far have similarity that they assume a circular motion, while the actual celestial orbits (and also molecular orbits) are elliptical. Historically, this was the basis of Sommerfeld's argument in contrast to

Bohr's model, which also first suggested that any excess gravitational-type force would induce a precessed orbit. This is the starting premise of the present article, albeit for brevity we will not introduce elliptical effect yet [12].

Using a known spherical kinetic dynamics approach, some interesting phenomena are explained, including the receding Moon, the receding Earth from the Sun, and also anomalous precession of the first planet (Lense-Thirring effect). Despite some recent attempts to rule out the gravitational quadrupole moment (J_2) contribution to this effect [13][14][15][16][17], it seems that the role of spherical kinetic dynamics [12] to Lense-Thirring effect has not been taken into consideration thus far, at least to this author's knowledge.

After deriving prediction for some known observed phenomena, this article will also present a revised version of quantization equation of L. Nottale [1] in order to take into consideration this spherical kinetic dynamics effect. If the proposition as described here corresponds to the facts, then this approach could be viewed as a step to unification of GTR-type phenomena and Quantum Mechanics.

Spherical kinetic dynamics, Earth bulging effect

In this section we start with some basic equations that will be used throughout the present article. It is assumed that the solar nebula is disk-shaped and is in hydrostatic equilibrium in the vertical direction. Let suppose that the disk has approximately Keplerian rotation, ω; then the half-thickness of the disk is given by [4d; p.4-5]:

$$d = c_s / \omega \qquad (1)$$

and

$$c_s \approx \sqrt{kT/m} \qquad (1a)$$

where d and c_s represents half-thickness of the disk and sound velocity, respectively.

In order to find the spherical kinetic dynamics contribution to Lense-Thirring effect, we begin with the spinning dynamics of solid sphere with mass M. Using the known expression [12; p.6, p.8]:

$$E_{kinetic} = -I_{zz}\omega^2 / 2 \qquad (2)$$

$$I_{sphere} = 2MR^2 / 5 \qquad (2a)$$

where I_{zz}, ω, M, R represents angular momentum, angular velocity, spinning mass of the spherical body, and radius of the spherical body, respectively. Inserting equation (2a) into (2) yields:

$$E_{kinetic} = -MR^2\omega^2 / 5 \qquad (3)$$

This known equation is normally interpreted as the amount of energy required by a spherical body to do its axial rotation. But if instead we conjecture that '*galaxies get their angular momentum from the global rotation of the Universe due to the conservation of the angular momentum*' [34], and likewise the solar system rotates because of the corresponding galaxy rotates, then this equation implies that the rotation itself exhibits extra kinetic energy. Furthermore, it has been argued that the global rotation gives a natural explanation of the empirical relation between the angular momentum and mass of galaxies: $J \approx \alpha M^{5/3}$ [34]. This conjecture is also relevant in the context of Cartan torsion description of the Universe [18]. For reference purpose, it is worthnoting in this regard that sometime ago R. Forward has used an argument of non-Newtonian gravitation force of this kind, though in the framework of GTR (*Amer.J.Phys*. **31** No. 3, 166, 1963).

Let suppose this kind of extra kinetic energy could be transformed into mass using a known expression in condensed-matter physics [10b; p.4], with exception that c_s is used here instead of v to represent the sound velocity:

$$E_{kinetic}(n, p) = c_s.p = m_s.c_s^{\,2} \qquad (4)$$

where the sound velocity obeying [10b; p.4]:

$$c_s^{\,2}(n) = (n/m)(d^2 \in / dn^2) \qquad (4a)$$

Physical mechanism of this kind of mass-energy transformation is beyond the scope of the present article, albeit there are some recent articles suggesting that such a condensed-matter radiation is permitted [35]. Now inserting this equation (4) into (3), and by dividing both sides of equation (3) by Δt, then we get the incremental mass-energy equivalent relation of the spinning mass:

$$\Delta m_s / \Delta t = -\omega.(\Delta\omega / \Delta t).MR^2 / (5.c_s^{\,2}) \qquad (5)$$

By denoting $\dot{\omega} = \Delta\omega / \Delta t$, then this equation (5) can be rewritten as:

$$\Delta M / \Delta t = -\dot{\omega}.MR^2\omega / (5.c_s^{\ 2}) \qquad (5a)$$

For $\dot{\omega} = 0$ the equation (5a) shall equal to zero, therefore this equation (5a) essentially says that a linear change of angular velocity observed at the *surface* of the spinning mass corresponds to mass flux, albeit this effect is almost negligible in daily experience. But for celestial mechanics, this effect could be measurable.

If, for instance, we use the observed anomalous *decceleration rate* [30] of angular velocity of the Earth as noted by Kip Thorne [19]:

$$|\omega| / |\dot{\omega}| = 6x10^{11} \ years \qquad (6)$$

And using values as described in Table 1 for other parameters:

Table 1. Parameter values to compute kinetic expansion of the Earth

Parameter	Value	Unit
R_e	$6.38x10^6$	m
M_e	$5.98x10^{24}$	kg
T_e	$2.07x10^6$	sec
ω_e	$3.04x10^{-6}$	rad/s
c_s	0.14112	m/s

It is perhaps worth noting that the only free parameter here is c_s =0.14112 m/sec. This value is approximately within the range of Barcelo *et al.*'s estimate of sound velocity (at the order of cm/sec) for gravitational Bose-Einstein condensate [11], provided the Earth could be regarded as a spinning Bose-Einstein condensate. Alternatively, the sound velocity could be calculated using equation (1a), but this obviously introduces another kind of uncertainty in the form of determining temperature (T) inside the center of the Earth; therefore this method is not used here.

Then inserting these values from equation (6) and Table 1 into equation (5a) yields:

$$\Delta M / \Delta t \approx 3.76x10^{16} kg / year \qquad (7)$$

Perhaps this effect could be related to a recent Earth bulging data, which phenomenon lacks a coherent explanation thus far [36].

Prediction of the receding Moon and the receding planets from the Sun

Now let suppose this predicted value (7) is fully conserved to become inertial mass, and then we could rewrite Nottale's method of celestial quantization [1]. Alternatively, we could begin with the known Bohr-Sommerfeld quantization rule [3]:

$$\oint p_j.dq_j = n_j.2\pi.e^2 / (\alpha_e.c) \qquad (8a)$$

Then, supposing that the following substitution is plausible [3]:

$$e^2 / \alpha_e \to GMm / \alpha_g \qquad (8b)$$

where e,α_e,α_g represents electron charge, Sommerfeld's fine structure constant, and gravitational-analogue of fine structure constant, respectively. This corresponds to Nottale's basic equations $v_n = \alpha_g.c / n = v_o / n$ and v_o=144 km/sec [1]. And by introducing the gravitational potential energy [12]:

$$\Phi(r,\vartheta) = -GM / r.\left[1 - J_2.(a/r)^2.(3\cos^2\vartheta - 1)/2\right] \qquad (8c)$$

where ϑ is the polar angle (collatude) in spherical coordinate, M the total mass, and a the equatorial radius of the solid.

Neglecting higher order effects of the gravitational quadrupole moment J_2 [13][14][15][16][17], then we get the known Newtonian gravitational potential:

$$\Phi = -GM / r \qquad (8d)$$

Then it follows that the semi-major axes of the celestial orbits are given by [1][3]:

$$r_n = GMn^2 / v_o^{\ 2} \qquad (8e)$$ where n=1,2,….is the principal quantum number.

It could be shown, that equation (8a) also corresponds to the conjecture of quantization of circulation [4b].

By reexpressing equation (8e) for mass flux effect (5) by defining $M_{n+1} = M_n + \Delta M_n / \Delta t_n$, then the total equation of motion becomes:

$(M + \Delta M / \Delta t) = (r + \Delta r / \Delta t).v_0^{\ 2} /(G.n^2)$ (8f)

For $\Delta \to 0$, equation (8f) can be rewritten as:

$dM / dt - \chi.dr / dt + M - r.\chi = 0$ (8g)

where

$\chi = v_0^{\ 2} /(G.n^2)$ (8h)

Now inserting (5a) into equation (8g), and dividing both sides by χ, yields:

$dr / dt - M / \chi + r + \dot{\omega}.MR^2\omega /(\chi 5.c_s^{\ 2}) = 0$ (8i)

This equation (8i) can be rewritten in the form:

$\dot{r} + r + \varphi = 0$ (8j)

by denoting $\dot{r} = dr / dt$ and

$\varphi = -M / \chi.[1 - \dot{\omega}.R^2\omega /(5.c_s^{\ 2})]$ (8k)

if we suppose a linear decceleration at the surface of the spinning mass. Equation (8j) and (8k) is obviously a first-order linear ODE equation [26], which admits exponential solution. In effect, this implies that the revised equation for celestial quantization [1][2] takes the form of spiral motion. This could also be interpreted as a plausible solution of diffusion equation in *dissipative* medium [33], which perhaps may also correspond to the origin of spiral galaxies formation [28]. And if this corresponds to the fact, then it could be expected that the spiral galaxies and other gravitational clustering phenomena [22b] could also be modeled using the same quantization method [39], as described by Nottale [1] and Rubcic & Rubcic [2].

To this author's knowledge these equations (8j) and (8k) have not been presented before elsewhere, at least in the context of celestial quantization.

Inserting result in equation (7) into (8e) by using n=3 and v_o=23.71 km/sec for the Moon [2] yields a receding orbit radius of the Moon as large as 0.0401 m/year, which is very near to the observed value ~ 0.04 m/year [20]. The quantum number and specific velocity here are also free parameters, but they have less effect because these could be replaced by the actual Moon orbital velocity using $v_n = v_0 / n$ [1].

While this kind of receding Moon observation could be described alternatively using oscillation of gravitational potential [30], it seems the kinetic expansion explanation is more preferable particularly with regard to a known phenomenon of continental drift [29], and also perhaps a known periodic geological layering described by Alvarez *et al.* Furthermore, using this kind of approach it seems that we could also offer a plausible explanation for the kinetic origin of volcanoes eruption. Apparently, none of these effects could be explained using oscillation of gravitational field argument, because they are relentless effects.

In this regard, it is interesting to note that Sidharth has argued in favor of varying G [21]. From this starting point, he was able to explain –among other things-- anomalous precession (Lense-Thirring effect) of the first planet and also anomalous Pioneer acceleration. This will be discussed in the subsequent section. In principle, Sidharth's basic assertion is [21]:

$G = G_{\otimes}.(1 + t / t_{\otimes})$ (9)

It is worthnoting here that Barrow [40c] has also considered a somewhat similar argument in the context of varying constants:

$G = G_{\otimes}.t_{\otimes} /(t - c)$ (9a)

However, in this article we will use (9) instead of (9a), partly because it will lead to more consistent predictions with observation data. Alternatively, we could also hypothesize using Maclaurin formula:

$G = G_{\otimes}.e^{t / t_{\otimes}} = G_{\otimes}.(1 + t / t_{\otimes} + (t / t_{\otimes})^2 / 2! + (t / t_{\otimes})^3 / 3! + ...)$ (9b)

This expression is a bit more consistent with the exponential solution of equation (8j) and (8k). Therefore, from this viewpoint equation (9) could be viewed as first-order approximation of (9b), by neglecting second and higher orders in the series. It will be shown in subsequent sections, that equation (9) is more convenient for deriving predictions.

If we conjecture that instead of varying G, the spinning mass M varies, then it would result in the same effect as explained by Sidharth [21], because for Keplerian dynamics we could assert k=GM, where k represents the stiffness coefficient of the system. Accordingly, Gibson [22] has derived similar conjecture of exponential mass flux from Navier-Stokes gravitational equation, which can be rewritten in the form:

$M = G_{\otimes}.e^{t / t_{\otimes}} = M_{\otimes}.(1 + t / t_{\otimes} + (t / t_{\otimes})^2 / 2! + (t / t_{\otimes})^3 / 3! + ...)$ (9c)

provided we denote for consistency [22]:

$$t_{\otimes} = \tau_g / 2\pi \quad (9d)$$

Using the above argument of Maclaurin series, equation (9c) could be rewritten in the similar form with (9) by neglecting higher order effects:

$$M = M_{\otimes}.(1 + t / t_{\otimes}) \quad (10)$$

In a recent article Gibson & Schild [23] argue that their gravitational Navier-Stokes approach results in better explanation than what is offered by Jeans instability. Furthermore, R.M. Kiehn has also shown that the Navier-Stokes equation corresponds exactly to Schroedinger equation [27], which seems to support the idea of quantization of celestial motion [1][2][3]. A plausible extension of Euler equation and Jeans instability to describe gravitational clustering has been discussed in [22b], which corresponds to viscosity term and also turbulence phenomena [22c,22d] described by Gibson. Therefore, apparently equation (10) is more consistent with kinematical gravitational instability consideration than (9).

From equation (10) we could write for M at time difference $\Delta t = t_2 - t_1$:

$$M_2 = M_{\otimes}.(1 + t_2 / t_{\otimes}) \quad (11)$$

$$M_1 = M_{\otimes}.(1 + t_1 / t_{\otimes}) \quad (12)$$

from which we get:

$$\Delta M = (M_{\otimes} / t_{\otimes}).(t_2 - t_1) \quad (13)$$

Inserting our definition $\Delta t = t_2 - t_1$ yields:

$$\Delta M / \Delta t = (M_{\otimes} / t_{\otimes}) = k \quad (14)$$

For verification of this assertion, we could use equation (14) instead of (5a) to predict mass flux of the Earth. Inserting the present mass of the Earth from Table 1 and a known estimate of Earth epoch of $2.2\text{x}10^9$ years, we get k=$0.272\text{x}10^{16}$ kg/year, which is approximately at the same order of magnitude (ratio=13.83) with equation (7).

Inserting equation (14) into equation (5a), we get:

$$M_{\otimes} / t_{\otimes} \approx -\dot{\omega}.MR^2\omega / (5.c^2) \quad (15)$$

which is the basic conjecture of the present article. From this viewpoint we could rewrite equation (8j) and (8k):

$$dr / dt = M / \chi.[1 - \dot{\omega}.R^2\omega / (5.c_s{}^2)] - r \quad (15a)$$

and inserting equation (15), we get:

$$dr / dt = M / \chi.[1 + M_{\otimes} / (t_{\otimes}.M)] - r \quad (15b)$$

A plausible test of this conjecture could be made by inserting this result (14) into equation (8e) and using $M_{\otimes} = 1.98951x10^{33}\,g$ and $t_{\otimes} = 2x10^{10}\,year$ as the epoch of the solar system [21], and specific velocity v_o=144 km/sec [1], then from equation (15b) we get a receding orbit radius for Earth at the order of:

$$\Delta r_{Earth} / \Delta t = 6.03 m / year \quad (16)$$

Interestingly, there is an article [24] hypothesizing that there is a tad effect of *receding Earth orbit from the Sun at the order of 7.5 m/year*, supposing Earth orbit radius has been expanding as large as $93\text{x}10^6$ miles since the beginning of the solar epoch at $t_{\otimes} = 2x10^{10}\,year$ ago (in the quoted article, it was assumed that the epoch is $4.5\text{x}10^9$ years). Of course, it shall be noted that there is large uncertainty of the estimate of solar epoch, for instance Gibson prefers $4.6\text{x}10^{17}$ sec (or $1.46\text{x}10^{10}$year, see [22]). Therefore, it is suggested here to verify this assumption of solar epoch using the same tad effect for other planets. For observation purposes, some estimate values were presented in Table 2 using the same approach with (15b).

Table 2. Prediction of planetary orbit radii (r) increment

Celestial object	Quantum number (n)	Orbit increment (m/yr)
Mercury	3	2.17
Venus	4	3.86
Earth	5	6.03
Mars	6	8.68

Quantization of anomalous celestial precession

It is known that the Newtonian gravitation potential equation (8d) is only weak-field approximation, and that GTR makes a basic assertion that this equation is *exact*. And if the gravitation could be related to boson condensation phenomena [9][10][11], then it seems worth to quote a remark by Consoli [9b; p.2]: "*for weak gravitational fields, the classical tests of general relativity would be fulfilled in any theory that incorporates the Equivalence Principle*." And also [9b; p.18]: "Einstein had to start from the peculiar properties of Newtonian gravity to get the basic idea of transforming the classical effects of this type of interaction into a metric structure. For this reason, classical general relativity cannot be considered a *dynamical* explanation of the origin of gravitational forces." Furthermore, Consoli also argued that the classical GTR effects other than anomalous precession could be explained without introducing non-flat metric, as described by Schiff [9b; p.19], therefore it seems that the only remarkable observational 'proof' of GTR is anomalous precession of the first planet [37]. Therefore, it seems reasonable to expect that the anomalous precession effect could be predicted without invoking non-flat metric, which suggestion is particularly attributed to R. Feynman, who 'believed that the geometric interpretation of gravity beyond what is necessary for special relativity is not essential in physics.' [9d] It will be shown that a consistent approach with equation (10) will yield not only the anomalous celestial precession, but also a conjecture that such an anomalous precession is quantized.

By using the same method as described by Sidharth [21], except that we assert varying mass M instead of varying G – in accordance with Gibson's solution [22]--, and denoting the average angular velocity of the planet by

$$\dot{\Omega} \equiv 2\pi / T \tag{17}$$

and period T, according to Kepler's Third Law:

$$T = 2\pi . a^{3/2} / \sqrt{GM} \tag{18}$$

Then from equation (10), (17), (18) we get:

$$\dot{\Omega} - \dot{\Omega}_o = -\dot{\Omega}_0 . t / t_\otimes \tag{19}$$

Integrating equation (19) yields:

$$\varpi(t) = \Omega - \Omega_o = -(\pi / T).t^2 / t_\otimes \tag{20}$$

which is average precession at time 't'. Therefore the anomalous precession corresponds to the epoch of the corresponding system. For Mercury, with T=0.25 year, equation (20) yields the average precession per year at time 't':

$$\varpi(t)_{Mercury} = \Omega - \Omega_0 = -4\pi . t^2 / t_\otimes \tag{21}$$

Using again $t_\otimes = 2x10^{10}\ year$ as the epoch of the solar system and integrating for years n=1 … 100, equation (21) will result in total anomalous precession in a century:

$$\varpi(n) = \sum_{n=1}^{n=100} \varpi(n) = 43.86''\ percentury \tag{22}$$

It would be more interesting in this regard if we also get prediction of this effect for other planets using the same method (20), and then compare the results with GTR-Lense-Thirring prediction. Table 3 presents the result, in contrast with observation by Hall and also prediction by Newcomb, which are supposed to be the same [25].

Table 3. Comparison of prediction and observed anomalous precession

Celestial Object	Period, T	$\omega_{prediction}$	Hall/ Newcomb	Diff.	GTR/ Thirring	Diff.
	(year)	(arcsec/cy)	(arcsec/cy)	(%)	(arcsec/cy)	(%)
Mercury	0.25	43.86	43.00	2.03	42.99	-0.05
Venus	0.57	19.24	16.80	14.54	0.8	-95.2
Earth	1.00	10.96	10.40	5.46	3.84	-63.1
Mars	1.88	5.83	5.50	6.02	1.36	-76.0
Jupiter	4346.5	$2.52x10^{-3}$				
Saturn	10774.9	$1.02x10^{-3}$				
Uranus	30681.0	$3.57x10^{-4}$				
Neptune	60193.2	$1.82x10^{-4}$				
Pluto	90472.4	$1.21x10^{-4}$				

It is obvious from Table 3 above that the result of equation (20) appears near to GTR and observation by Hall for the first planet, but there is substantial difference between GTR and observation for other planets particularly Venus. In the mean time, average percentage of error from prediction using equation (20) and observation (Hall) is 7.01%. The numerical prediction for Jovian planets is negligible; though perhaps they could be observed provided there will be more sensitive observation methods in the near future.

It is perhaps also worth noting here, that if we use the expression of quantization of period [3]:

$$T = 2\pi.GM.n^3 / v_0^3 \quad (23)$$

where $v_0 = \alpha_g.c = 144 km/s$ in accordance with Nottale [1]. Inserting this equation (23) into (20), yields:

$$\varpi(t)_{precess} = \Omega - \Omega_0 = -(v_0^3 / 2GMn^3).t^2 / t_\otimes \quad (24)$$

or

$$T_{precess} = 2\pi / \varpi(t)_{precess} = -4\pi.t_\otimes GMn^3 / v_o^3 t^2 \quad (24a)$$

This equation (24) and (24a) imply that the anomalous precession of Lense-Thirring type is also *quantized*. Apparently no such an assertion has been made before in the literature. It would be interesting therefore, to verify this assertion for giant planets and exoplanets, but this is beyond the scope of the present article.

A plausible test using LAGEOS-type satellites

In this regard, one of the most obvious methods to observe those tad effects as described in this article is using LAGEOS-type satellites, which have already been used to verify Lense-Thirring effect of Earth. What is presented here is merely an approximation, neglecting higher order effects [12][16][31].

Using equation (8c) we could find the rotational effect to satellite orbiting the Earth. Supposed we want to measure the precessional period of the inclined orbit period. Then the best way to measure quadrupole moment (J_2) effect would be to measure the ϑ component of the gravity force (8c):

$$g = 1/r.\partial V / \partial\vartheta = -3GM.a^2 J_2.\sin\vartheta.\cos\vartheta / r^4 \quad (25)$$

This component of force will apply a torque to the orbital angular momentum and it should be averaged over the orbit. This yields a known equation, which is often used in satellite observation:

$$\omega_p / \omega_s = -3a^2 J_2.\cos i / 2r^2 \quad (26)$$

where i is the inclination of the satellite orbit with respect to the equatorial plane, a is Earth radius, r is orbit radius of the satellite, ω_s is the orbit frequency of the satellite, and ω_p is the precession frequency of the orbit plane in inertial space. Now using LAGEOS satellite data [31] as presented in Table 4:

Table 4. LAGEOS satellite parameters

Parameter	Value	Unit
R_{LAGEOS}	$12.265x10^6$	M
i_{LAGEOS}	109.8	o
T_{LAGEOS}	13673.4	sec
ω_s	$4.595x10^{-4}$	rad/s
J_2	$1.08x10^{-3}$	

Inserting this data into equation (26) yields a known value:

$$\omega_p = 0.337561^\circ / day \quad (27)$$

which is near enough to the observed LAGEOS precession = 0.343°/day.

Now let suppose we want to get an estimate of the effect of Earth kinetic expansion to LAGEOS precession. Assuming a solid sphere, we start with a known equation [34]:

$$M = 4\pi.\rho_{sphere}.r^3 / 3 \quad (28)$$

where ρ_{sphere} is the average density of the 'equivalent' solid sphere. For Earth data (Table 1), we get ρ_{sphere} $=5.50x10^6$ gr/m^3. Using the same method with equation (8f), equation (28) could be rewritten as:

$$M + \Delta M / \Delta t = 4\pi.\rho_{sphere}.(r + \Delta r / \Delta t)^3 / 3 \quad (29)$$

or

$$\Delta r / \Delta t = \sqrt[3]{(M + \Delta M / \Delta t).3 / (4\pi.\rho_{sphere})} - r \quad (30)$$

From equation (30) we get dr/dt=13.36 mm/year for Earth. Inserting this value (r+dr/dt) to compute back equation (26) yields:

$$\Delta\omega_p = \omega_{p,n+1} - \omega_{p,n} = 1.41x10^{-9\circ} / day = 2.558 arc\sec/ year \quad (31)$$

Therefore, provided the aforementioned propositions correspond to the facts, it could be expected to find a tad extra precession of LAGEOS-satellite around 2.558 arcsecond/year. To this author's knowledge this tad effect has not been presented before elsewhere. And also thus far there is no coherent explanation of those aforementioned phenomena altogether, except perhaps in [21] and [30].

As an alternative to this method, it could be expected to observe Earth gravitational acceleration change due to its radius increment. By using equation (28):

$$\ddot{r}(t) = GM / r^2 = 4\pi.G.\rho_{sphere}.r / 3 \quad (32)$$

From this equation, supposing there is linear radius increment, then we get an expression of the rate of change of the gravitational acceleration:

$$\dddot{r}(t) = \Delta\ddot{r} / \Delta t = 4\pi.G.\rho_{sphere}.(r + \Delta r / \Delta t) / 3 - \ddot{r}(t) \quad (33)$$

It would be interesting to find observation data to verify or refute this equation.

Constraint of varying α

In recent years there is suggestion that unification of the fundamental interactions requires cosmological solutions in which low-energy limits of fundamental physical constants vary with time, including α [40][41][42]. This assertion began with Dirac's remark: "*the constancy of the fundamental physical constants should be checked in an experiment*" [42, p.439]. While this has not been widely accepted yet, a plausible way to verify this proposition is using celestial quantization method: "offers a possibility to check the variability of the constants by studying, for example, lunar and Earth's secular accelerations, which has been done using satellite data, tidal records, and ancient eclipses" [42, p. 441].

In this regard, instead of using Nottale's celestial quantization method [1], alternatively we use Rubcic & Rubcic's assertion [2] that the celestial quantization equation could be related to Planck constant and Planck mass, by introducing:

$$H' / M = fA \quad (34)$$

$$r_n = (fA)^2.Mn^2 / G \quad (35)$$

where

$$A = \hbar / (\alpha.m_p^2) \quad (36)$$

therefore

$$r_n = (f\hbar / \alpha m_p{}^2)^2.Mn^2 / G \quad (37)$$

where f, $\hbar$, α, m_p=2.177x10^{-8} kg represents a specific ratio for given system [2], Planck constant, fine structure constant (~1/137), and Planck mass, respectively. This alternative expression is quite interesting, particularly if compared to Winterberg's argument of superfluid Planckian phonon-roton as the basic entity of Nature [8].

From equation (34) and (36) we get:

$$\alpha = f\hbar M / (H'.m_p{}^2) \quad (38)$$

Using the same method with equation (8f) we get:

$$\alpha + \dot{\alpha} = (M + \dot{M}).f\hbar / (H'.m_p{}^2) \quad (39)$$

Now dividing both sides of equation (39) with M and α, we get:

$$1 + \dot{\alpha} / \alpha = (1 + \dot{M} / M).[f\hbar M / (H'.m_p{}^2.\alpha)] \quad (40)$$

From equation (34) and (36) we know that components in the square bracket of the right side of equation (40) equals to unity, therefore we conclude by using equation (14):

$$\dot{\alpha} / \alpha \approx \dot{M} / M = (M_\otimes / t_\otimes) / M_\otimes = 1 / t_\otimes \quad (41)$$

From this viewpoint, we argue that α varies corresponding to inverse of the epoch of the system in question. Supposing the epoch of the Universe is 1.09 Tyr (larger than epoch of the solar system in the previous section, 2x10^{10}year), and then from equation (41) we get an estimate of varying α:

$$\dot{\alpha} / \alpha \approx 9.2x10^{-13}\, year^{-1} \quad (42)$$

For comparison, other values for varying α as proposed in the literature are presented in Table 5. Alternatively, we could use Sidharth's original assertion (9) and (14), and by using an equation described in [43]:

$$\dot{G} / G = t_\otimes^{-1} = 78.2x\dot{\alpha} / \alpha \quad (43)$$

and supposing epoch of the Universe =13.9 Gyr [40d], then we get estimate of $\dot{\alpha} / \alpha = 9.2x10^{-13}$ year^{-1}, which is near to Bahcall *et al*'s prediction [40b]. Of course, this subject of varying α is not conclusive yet, partly because of large uncertainty in determining the epoch of the Universe, but at least equation (42) could be viewed as an alternative constraint based on the celestial quantization method. However, it seems that this proposition of varying α from Rubcic & Rubcic's celestial quantization approach [2] is quite conceivable, particularly from the viewpoint of recent suggestion of the *invariance* and possible time variation of the Planck mass [2b][2c].

Table 5. Range of values of varying α

Ref.	$\dot{\alpha}/\alpha$	Unit
Prestage *et al.* [40, p.31]	3.7×10^{-14}	$year^{-1}$
Moffat [40d, p.5]	3.8×10^{-16}	$year^{-1}$
Ivanchik *et al.* [42, p.439]	1.9×10^{-14}	$year^{-1}$
Slyakhter [42a, p.2]	10^{-19}	$year^{-1}$
Bahcall *et al.* [40b,p.520]	2×10^{-13}	$year^{-1}$
Bahcall *et al.* [40b,p.530]	6×10^{-12}	$year^{-1}$
Nguyen [40e,p.1]	1.2×10^{-18}	$year^{-1}$

Discussion

Allow us to make some more remarks: one of the most relevant being the inconstancy of the "constants". Following Dirac's suggestion, the various constants should be validated at period intervals, with no assumption that the values will remain exactly the same as they were at the last measurement.

The fine structure "constant" ***a*** , is the most vulnerable to changes in other physical parameters, because it is relational. Instrumented observations of the *fine structure constant have recorded changes* in the value starting during the 1980s, and with divergences becoming larger than the original value at the time of its inception by Sommerfeld, as time passes.

The fine structure "constant" is a variable, G is a variable, c is a variable, and if *e* is variable, the normal understandings of the standard physics will be soon be demolished.

Variations in ***a*** are directly related to quantized red-shift, as first pointed to by astrophysicist *Halton Arp*. He also points at quantized changes in gravitation and mass, as related to quasars and active galaxies. It has been verified by astrophysical observations, that all galaxies and quasars exhibit quantized red shift.

But what causes quantized red shift?

At the core of every galaxy and quasar, there lives an enormous and very complex **plasmoid**, which exhibits periodic episodes of powerful radiations spanning the entire of the E/M spectrum. (There are no black holes, anywhere. Those figments are cartoon fantasies produced by Hollywood science to support ongoing systematic frauds.)

E/M sources are also aether sources, and act superluminally. The explosive events of galactic core plasmoids produce expanding shells of aether, with abnormal aether density. The various constants, and the laws of physics which rely on those "constants" are changed as each aether-density shell expands from the center of the galaxy, to the outer edges, changing the local "constants" along the way.

Where we live, the laws of physics **are changing** in a gradient manner, along with many of the "constants". This gradient of changing values will continue until the entire of the aether shell has passed through our solar system. At that point, the constants will remain stable in value, and the physics will change in a reliable manner, until the next episode of galactic core-plasma aether ejection occurs and passes through where we live.

The the entire process of changing "constants" will start again, and continue until the shell has passed through our location, when the values will once more stabilize and the local physics will again be reliable. Until the next galactic core-plasmoid event.

This is based on current observations of variations in the constants and on quantized red shift, seen in all galaxies, as correlated with the SQ originations of all things physical. We are reaching a crescendo of change, including changes of Consciousness and the appearance, ab initio, of new life forms, specially constructed to take full advantage of the new conditions which will comprise their environment. (The "*morphogenic field*" of Sheldrake is relevant here.)

Concluding note

If physical theories could be regarded as a *continuing search* to find systematic methods to reduce the entropy required to do calculations to minimum; then the fewer free parameters, the better is the method. Accordingly, it is shown in this article that some twelve phenomena can be explained using only few free parameters, including:

- The Moon is receding from the Earth [20];
- Earth's angular velocity decrease (K. Thorne) [19];
- Planets are receding from the Sun [24];
- Lense-Thirring effect for inner planets, corresponding to Hall/Newcomb's observation;
- Celestial orbit prediction in solar system [1][2][3];
- Exoplanets orbit prediction [1][3];
- Pioneer-type anomalous acceleration [21];
- A plausible origin of volcanoes eruption;
- A plausible origin of continental drift effect [29];
- A plausible origin of spiral motion in spiral nebulae [22];
- Prediction of extra precession of LAGEOS satellite [31];
- Prediction of angular velocity decrease of other planets.
-

As a plausible observation test of the propositions described here, it is recommended to measure the following phenomena:

- Lense-Thirring effect of inner planets, compared to spherical kinetic dynamics prediction derived herein;
- Annual extra precession of Earth-orbiting LAGEOS-type satellites;
- Receding planets from the Sun;
- Receding satellites from their planets, similar to receding Moon from the Earth;
- Angular velocity decrease of the planets;
- Angular velocity decrease of the Sun.

It appears that some existing spacecrafts are already available to do this kind of observation, for instance LAGEOS-type satellites [31]. Further refinement of the method as described here could be expected, including using ellipsoidal kinetic dynamics [12] or using analogy with neutron star dynamics [32]. Further extensions to cosmological scale could also be expected, for instance using some versions of Cartan-Newton theory [38]; or to find refinement in predictions related to varying constants.

All in all, the present article is not intended to rule out the existing methods in the literature to predict Lense-Thirring effect, but instead to argue that perhaps the notion of '*frame dragging*' in GTR [14][16] could be explained in terms of dynamical interpretation, through invoking the spherical kinetic dynamics. In this context, the dragging effect is induced by the spinning spherical mass to its nearby celestial objects.

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A Thousand Words: How Shannon Entropy Perspective Provides Link between Exponential Data Growth, Average Temperature of the Earth and Declining Earth Magnetic Field

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ABSTRACT

The sunspot data seems to indicate that the Sun is likely to enter Maunder Minimum, then it will mean that low Sun activity may cause low temperature in Earth. If this happens then it will cause a phenomenon which is called by some climatology experts as "The Little Ice Age" for the next 20-30 years, starting from the next few years. Therefore, the Earth climate in the coming years tend to be cooler than before. This phenomenon then causes us to ask: what can we do as human being in Earth to postpone or avoid the worsening situation in terms of Earth cooling temperature in the coming years? We think this is a more pressing problem for the real and present danger that we are facing in the Earth. What we are suggesting in this paper is that perhaps it is possible to model Sun-Earth interaction in terms of Shannon entropy. Since Shannon entropy can be expressed as bits of information, then it would mean that perhaps we can do something with Earth temperature by controlling the amount of information transfer and storage in the Earth. This proposal is somewhat in resemblance with message of a 2012 movie "A Thousand words" where we shall strive to love our neighbours and nature, instead of being absorbed in a culture of less-meaningful fast-talk (starred by Eddie Murphy).

1. INTRODUCTION

The historical recognition that the Sun warms the Earth has suggested a direct connection between the average global temperature and solar activity. Consequently, any significant changes in solar activity should result in equivalent changes in the Earth's global temperature. The literature on the solar influence on the Earth's temperature is quite extensive, indicating the importance of the problem [5].

In this regards, it is very important to note here that some reports made by climate experts have indicated that it is highly likely that the Sun will enter into a Maunder minimum in thenext couple years, which will last for 20-30 years to come. Since the Sun activity highly affects Earth temperature, then it can be expected that the Earth will experience cooling, which some climatologists refer to as the Little Ice Age. This global cooling can be observed in recent extreme climate conditions such as snow storms in some regions in USA in January-February 2014 and also during this winter (January 2015). Other indication includes the fact that the Arctic Ice has increased 29% in size from 2012-2013, which indicates the coming of "global cooling" [9].

Such a global cooling phenomenon has been related to low solar activity, as reported by Mr. John Casey (www.spaceandscience.net) and Dr. Dong Choi (www.ncgt.org). This phenomenon then causes us to ask concerning what we can do as human being in Earth to avoid the worsening situation in terms of Earth cooling temperature in the coming years.

It is well known that Shannon information entropy can reduce to the Boltzmann entropy, but we are not sure yet how temperature in thermodynamics sense can be related to the information entropy measures. Here we submit a viewpoint that it is possible to put temperature in thermodynamics sense in terms of information entropy. This result is quite new, and it is worth to be communicated to wider audience, since it affects temperature of the Earth. We expect that people start to be wiser and more efficient in using and sending information especially via online and electronic media.

2. BACKGROUND THEORY ON INFORMATION ENTROPY

Shannon information entropy is defined as follows [1, p.4]:

$$S = -k \sum_{i=1}^{w} \rho i \, ln \, \rho i \qquad (1)$$

For the uniform distribution, then the Shannon entropy takes on its maximum value and it reduces to be

Boltzmann entropy [1, p.5]:

$$S = k \, ln \mathrm{W} \qquad (2)$$

And then we conclude that both equations essentially correspond to the same process, i.e. the sending and receiving of information, provided we assume that the Earth is a large information retrieval system. Therefore we can accept that actually Boltzmann entropy is neatly related to information entropy, and therefore we can proceed further to accept that the thermodynamics temperature of the Earth corresponds neatly to the amount of information sent and received in the Earth. Actually Boltzmann himself did not realize the full implications of his thermodynamics equation, because he did not know beforehand how the Sun activity actually corresponds to the ambience temperature of the Earth.

The correspondence between the process of information retrieval and thermodynamics entropy can be expressed as follows [2, p.6]:

$$\left| \frac{\delta Q}{dS/(ln2)} \right| \geq kT.ln2 \qquad (3)$$

Where the principle is based on Clausius inequality and states that many-to-one operations like erasure of information requires the dissipation of energy. And the right hand side of the inequality is known as Landauer bound.

In other words, one should be very careful because sending and receiving useless information can affect temperature without one realizes it, although how precisely the mechanism that information can affect global temperature remains mystery. This increasing information content of the Earth has been discussed in a few papers, see for instance Hosoya-Buchert-Morita's paper [3], although they figure out the problem without connecting it with the increasing of temperature of the Earth. It is because they assume that the increasing information content is related to the Relative Information Entropy of a cosmological model containing dust matter [3]; but actually the increasing information content in the Universe corresponds strongly to the increasing use of online information in recent decades.

3. SHANNON ENTROPY AND GLOBAL TEMPERATURE

According to Nicola Scafetta and Bruce West [5], Earth's short-term temperature anomalies and the solar flare intermittency are linked, and the relation can be expressed in terms of Shannon entropy, S(t):

$$S(t) = -\int_{-\infty}^{\infty} \rho\ (x, t \ln[\ \rho(x,t)] = A + \delta \ln{(t)} \quad (4)$$

Where A is constant and δ is found to be 0.67 for global temperature data between 1860-2000. However, since 2000 the global temperature shows declining change significantly caused by low Sun activity.

It should be emphasized here that solar activity is not the only factor that affects Earth's temperature; other factors may include planetary synchronicity [6].

Moreover, it should be noted that there is a critique on the hypothesis that Solar activity affects global temperature, see for instance Gil-Alana et al. [8], nonetheless their arguments have been refuted by Scafetta in his recent paper [7].

So the conclusion is that there is nonlinear relationship between Sunspot number and Earth temperature. In the subsequent section, I will discuss a possible model in terms of Momentary Information Transfer as proposed by Runge et al.

4. MOMENTARY INFORMATION TRANSFER (MIT) AND SOURCE ENTROPY

In his dissertation, Jakob G.B. Runge describes some new notions [4]. The notion of momentary information is introduced in Section 3.1.3, and momentary information transfer is explained in Section 3.4.5. The basic approach is to measure causal coupling strength (see Section 3.4.5) based on source entropy (also termed entropy rate from Shannon, 1948). The goal is to quantify the interaction between two causally linked processes as well as along causal paths and between multiple processes such as the earth's surface temperature (cooling and heating), atmosphere, moon and sun.

Climatological analysis using MIT is introduced in Appendix B. Large MIT values indicate strong coupling between Earth's surface and upper tropospheric levels, as discussed in Appendix B.3.

As an example, following Runge et al. (2012b), we compare mutual information (MI), transfer entropy (TE), the CMI defining causal links (LINK), information transfer to Y (ITY) and from X (ITX), and momentary information transfer (MIT) on an analytically tractable model of a multivariate Gaussian process: [4, p. 93]

$$Z_t = C_{XZ} X_{t-1} + \eta t^z$$

$$X_t = a_X X_{t-1} + \eta t^z$$

$$Y_t = C_{XY} X_{t-1} + C_{WY} W_{t-1} \eta t^z$$

$$W_t = \eta t^w$$

We hope that in the near future, more exact physical models will be developed to describe how information exchange can affect Earth's ambient temperature.

5. FURTHER DISCUSSION

While our proposition here is somewhat simplified, here we discuss further how things are possibly linked:

Global data growth → Shannon entropy → global average temperature → Schumann resonance

For instance, some researchers have shown:

1. Global average temperature is linked to Schumann resonance

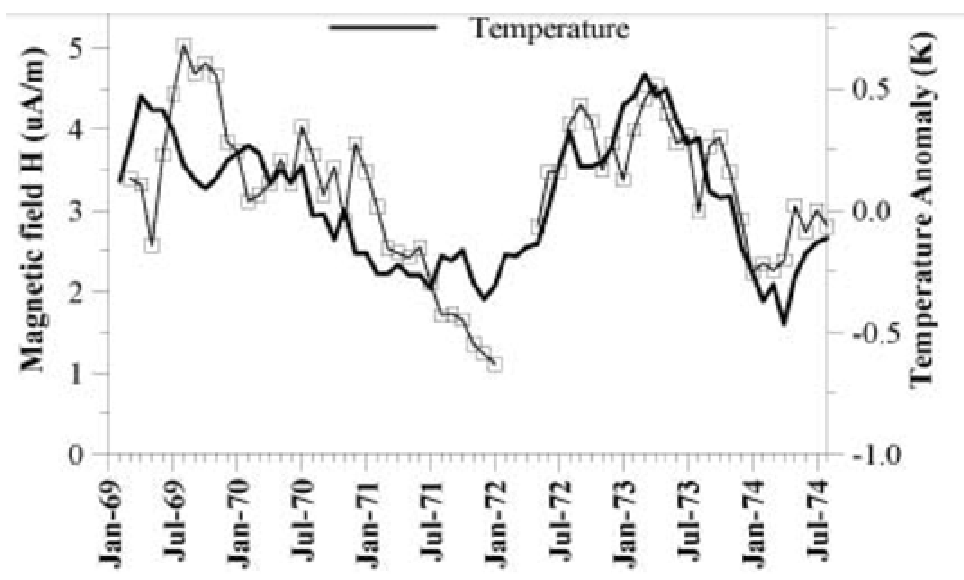

Figure 1: Correlation between global temperature and the intensity of Schumann resonance oscillations (adopted from Williams, 1992)
Source: ref. [12]

2. Global data is increasing exponentially, almost following Moore's law.

Figure 2:
Source: Cisco VNI Mobile, 2016. See [10][11]

3. Shannon entropy is also linked to variation of Earth magnetic field (using Kolmogorov or k-entropy). See ref. [13].
4. Declining Earth magnetic field is also linked to Earth climate, as emphasized by Campuzano et al. in a recent report in Plos ONE [14]:

"The debated question on the possible relation between the Earth's magnetic field and climate has been usually focused on direct correlations between different time series representing both systems. However, the physical mechanism able to potentially explain this connection is still an open issue. Finding hints about how this connection could work would suppose an important advance in the search of an adequate physical mechanism. Here, we propose an innovative information-theoretic tool, i.e. the transfer entropy, as a good candidate for this scope because is able to determine, not simply the possible existence of a connection, but even the direction in which the link is produced. We have applied this new methodology to two real time series, the South Atlantic Anomaly (SAA) area extent at the Earth's surface (representing the geomagnetic field system) and the Global Sea Level (GSL) rise (for the climate system) for the last 300 years, to measure the possible information flow and sense between them. This connection was previously suggested considering only the long-term trend while now we study this possibility also in shorter scales. The new results seem to support this hypothesis, with more information transferred from the SAA to the GSL time series, with about 90% of confidence level. This result provides new clues on the existence of a link between the geomagnetic field and the Earth's climate in the past and on the physical mechanism involved because, thanks to the application of the transfer entropy, we have determined that the sense of the connection seems to go from the system that produces geomagnetic field to the climate system. Of course, the connection does not mean that the geomagnetic field is fully responsible for the climate changes, rather that it is an important driving component to the variations of the climate."

6. URGENT RECOMMENDATION

Now we obtain that temperature of the Earth can be modeled by assuming that the Earth is a large information retrieval system, therefore Shannon information entropy can be used to represent the amount of information sent and received in the Earth. Therefore if many people send and receive information to the system without taking care to its effects to the temperature of the Earth, then the accumulative result can be dangerous to the entire system, including to the human population and environment of the Earth. Now we see that the use of online information is already increasing rapidly in recent years largely because of the Internet, and as a result it contributes to the declining temperature in this Earth.

Therefore, we urge that server administrators of the online information, including online email servers, to reduce the amount of information which are put ‘online’. This action shall include reducing the amount of emails which are put online, and reserve those emails into offline databases. But this action shall be made carefully and responsibly, otherwise it may cause Ice Age again in this Earth, and also disturbance of environment stability, because of rapid decreasing of temperature.

We wrote this article very shortly because we want to emphasize that information shall be sent and received more efficiently and more responsibly. The server administrators of the online information channels shall take care too on how much emails and other information shall be kept online in order to maintain the ambience temperature to remain within the acceptable range, i.e. between 25-27 degree Celsius. Therefore we urge that server administrators also monitor the effect of the already increasing amount of the online information and email messages in the past few days to the ambience temperature.

The effect of reducing the amount of online information can be observed and felt almost immediately, because of the entropy and temperature is transmitted immediately; it is because the Earth is intertwined to the Universe.

We recommend that all server administrators of online information channels to pray and ask for guidance from God, especially on how to maintain their online servers in a better and more effective way, in order to avoid further damage and destruction of this Earth because of rapidly increasing online information.

Furthermore, scientific journal Editors should maintain the published papers in the most efficient way possible, and do not upload too many large files if they can be kept as “optional” online. By keeping online communication at the most efficient, we can do the best to avoid the Earth magnetic field from declining further, in line with “A thousand words” spirit.

We hope this short article will be read in front of other physicists and also in front of all server administrators of online information channels, including Yahoo!, Google, Hotmail and other large email servers.

7. CONCLUDING REMARKS

The sunspot data seems to indicate that the Sun is likely to enter Maunder Minimum, then it will mean that low Sun activity may cause low temperature in Earth. If this happens then it will cause a phenomenon which is called by some climatology experts as “The Little Ice Age” for the next 20-30 years, starting from this year (2015). Therefore, the Earth climate in the coming years tend to be cooler than before. This phenomenon then causes us to ask: what can we do as human being in Earth to postpone or avoid the worsening situation in terms of Earth cooling temperature in the coming years? I think this is a more pressing problem for the real and present danger that we are facing in the Earth. What I am suggesting in this paper is that perhaps it is possible to model Sun-Earth interaction in terms of Shannon entropy. Since Shannon entropy can be expressed as bit of information, then it would mean that perhaps we can do something with Earth temperature by controlling the amount of information transfer and storage in the Earth.

Our proposal is somewhat in resemblance with message of a 2012 movie “A Thousand words” where we shall strive to love our neighbours and nature, instead of dwelving in a culture of fast-talk (starred by Eddie Murphy).

Since Shannon entropy can be expressed as bits of information, then it would mean that perhaps we can do something with Earth temperature by controlling the amount of information transfer and storage in the Earth. We hope that in the near future, more exact physical models will be developed to describe how information exchange can affect Earth's ambient temperature.

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There is No Constant in Physics: a Neutrosophic Explanation

Victor Christianto, Robert N. Boyd, Florentin Smarandache

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Abstract

In Neutrosophic Logic, a basic assertion is that there are variations of about everything that we can measure; the variations surround three parameters called T,I,F (truth, indeterminacy, falsehood) which can take a range of values. Similarly, in this paper we consider NL applications in physics constants. Those constants actually all have a window of plus and minus values, relative to the average value of the constant. For example, *speed of light, c, can vary in a window up to +/- 3000 m/s*. Therefore it should be written: 300000 km/s +/- 3 km/s. We also discuss some implications of this new perspective of physics constants, including in gravitation physics etc.

Keywords: Neutrosophic Logic, Physical Neutrosophy, gravitation, physics constants, Michelson-Morley experiment

1.Introduction

For majority of physicists, constants play a fundamental role. Like an anchor for a ship, they allow physicists build theories on the ground of those constants as basic "known" quantities. However, in real experiments, there are always variation of those constants. Moreover, from Neutrosophic Logic perspective, those constants always fluctuate depending on various circumstances.

In Neutrosophic Logic, a basic assertion is that there are variations of about everything that we can measure, the variations surround three parameters called T,I,F (truth, indeterminacy, falsehood) which can take a range of values. Similarly, in this paper we consider NL applications in physics constants. Those constants actually all have a window of plus and minus values, relative to the average value of the constant. For example, *speed of light, c, can vary in a window up to +/- 3000 m/s*.[1] Therefore it should be written: 300000 km/s +/- 3 km/s.

[1] Note by one of us (RNB): "The data from the experiment was recorded in the actual handwritten log books from the actual M-M experiments as **up to** plus and minus 3000 meters per second variation in the measured speed of light. I closely examined all the handwritten logs and lab notes personally. Most of the light speed excursions recorded in the actual log books were smaller than this. I recall calculating the average speed of light excursion to be in the vicinity of 300 meters per second. The apparatus was capable of measuring c to an accuracy of 0.00025 meters per second, as I recall.

We also discuss some implications of this new perspective of physics constants, including in gravitation physics etc.

It is our hope that this new perspective on physics constants will point to a more substantial and evidence-based approach to physics sciences.

2. Definition

Neutrosophic Logic, as developed by one of us (FS), is generalization of fuzzy logic based on **Neutrosophy**. A proposition is t true, i indeterminate, and f false, where t, i, and f are real values from the ranges T, I, F, with no restriction on T, I, F, or the sum n=t+i+f. Neutrosophic logic thus generalises:

- intuitionistic logic, which supports incomplete theories (for 0<n<100 and i=0, 0<=t,i,f<=100);

- fuzzy logic (for n=100 and i=0, and 0<=t,i,f<=100);

- Boolean logic (for n=100 and i=0, with t,f either 0 or 100);

- multi-valued logic (for 0<=t,i,f<=100);

- paraconsistent logic (for n>100 and i=0, with both t,f<100);

- dialetheism, which says that some contradictions are true (for t=f=100 and i=0; some paradoxes can be denoted this way).

Compared with all other logics, neutrosophic logic introduces a percentage of "indeterminacy" - due to unexpected parameters hidden in some propositions. It also allows each component t,i,f to "boil over" 100 or "freeze" under 0. For example, in some tautologies t>100, called "overtrue".[1]

Neutrosophic Logic allows one to develop new approaches in many fields of science, including a redefinition of physics constants, as will be discussed in the next section.

3. Neutrosophic reasoning: There is no Physics Constant

In accordance with Neutrosophic Logic, actually all physics constants have a window of plus and minus values, relative to the average value of the constant. For example, variation of c is approximately within the range of plus or minus 3000 meters/second.

There may be larger excursions, but we would not expect larger excursions to happen very often. Probability considerations are thus also involved in determining the average value and the statistical extremes for the given constant.

There are also curves which vary according to the materials involved, and the environment. For example, most recently and most importantly, it has been realized that h and h_bar cannot be used for any material other than carbon black (soot).

All other materials must have their thermal emissions curve instrumented. Then the h and h_bar for that material can be calculated. But the values calculated are subject to modifications by the local environment. Unless

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Both periodic and stochastic measurements of *speed of light variances* are recorded in the handwritten log books from the M-M experiments. Should be as listed: “variation of c is approximately within the range of plus or minus 3000 meters/second.” No larger excursions were recorded.” See also [10][11].

the aether environment can be considered and measured, the calculated values of h and h_bar for the given material will not be as reliable as we might prefer. (It depends on the specific application which requires instrumented measurements of the thermal emissions curve of the given material.)

So there should be a way to produce an accurate thermal emissions curve using a neutrosophic approach. Because all thermal emissions curves have extremes from absolute zero to very high heat values. Neutrosophic modifications of Kirchoff's law of "blackbody radiation", and Planck's "constant" would be very useful. (See for instance, a report by Robitaille and Crothers on the flaws of Kirchoff law, [2-4]). It is worth noting here, that from dynamical perspective, Shpenkov argues for a redefinition of Planck constant: “The Planck constant h is the quantity the value of which is equal to the orbital action of the electron on the Bohr first orbit in the hydrogen atom, namely to its orbital moment of momentum P_{orb} multiplied by 2π, or it can be rewritten as: $h=2\pi.\ P_{orb}$.” According to him, Planck constant also has acoustic origin. [5]

There are also physical situations where the variations of the value of one constant, directly alters the values of physically-related constants. The fine structure "constant" is an example of this kind of mutual influence. If the fine structure value changes, it changes the value of e, the charge of the electron. (Which informs us that the charge on the electron is an environmentally influenced Neutrosophic window.) Going the other way, if the value of e changes, it changes the value of the fine structure constant.

Another aspect of this to consider is that some constants-windows may not be perfectly symmetrical, but large on one side of the center value, and small on the other side, and exhibit dependence on the environment, such that under most conditions the value of the given "constant" would live inside the window, while there could be large asymmetrical extremes at other times, depending on the local and non-local environmental parameters of the aether, at the location where we are examining the measured value of the constant.

4. A few applications

At this point, some readers may ask: Can we get an example when a so-called constant has a value, while in another example the same so-called constant gets another value?

Answer: Gravitation is a good example. g changes depending on where and when it is measured. This is used in gravitational prospecting and by the GRACE experiment (NASA) which maps the gravitation variations of the Earth, over time. [6] In ref [6], they show many more data sets and graphical images showing gravitational variations on the Earth.

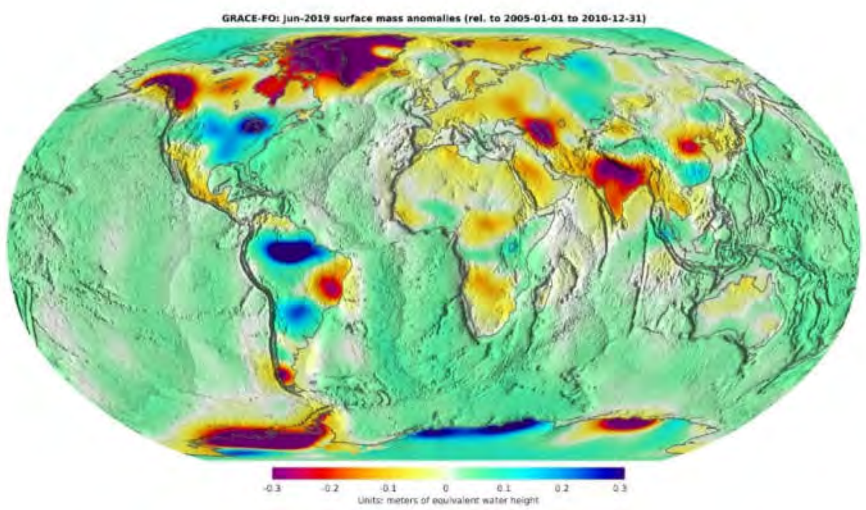

Figure 1. gravitational variations on the Earth. Source: [6]

Another article contains a good table of measurements of gravitation from 1798, until 2004 [7].

There is also a discussion of the increase in the force due to gravitation of the Earth, showing the dinosaurs would be crushed by their own weight if they were subjected to the gravitational force of the Earth today.

The *gravitational "constant"* is a good one to start with, since the variations can't be denied.

The next best one would be speed of light variations, although these days they refuse to allow one-way measurements of light velocity, because vast numbers of variations show up, depending on the time and place of the measurement. The mainstream insists that the speed of light can only be measured by round-trip measurements. This is because the light going back and forth along the same line results in many of the measured one-way variations in the velocity, being averaged out.

Typically, speed of light experiments cook their books and throw out any large deviations in measured light velocity. This tactic is similar to even more egregious cheating methods which are used by "global warming" and "climate change" advocates, paid for by the oil companies.

The next best one would be variations in Planck's "constant". h and h_bar are only valid for carbon black (soot). Every different material has a different thermal radiation when plotted on a thermal radiation curve. Some examples are displayed by Robitaille and Crothers in some of their presentations on the original "black body" thermal radiation constant known as Kirchoff's law, which was never measured by instrumented experiments, and was accepted as universally valid by Planck, who never did experiments to measure the thermal radiation curves of anything.[2-4]

5. Conclusions

In this article, we discussed how physics constants can vary in a wide range of values, in particular from Neutrosophic Logic perspective. We also discussed some examples, including variation in Earth gravitation measurements, speed of light measurement, and also Planck constant. It is our hope that this short discussion will be found as good impetus for a new direction in physics, more corresponding to experimental data, toward: "*evidence-based physics*." This new direction is in direct contrast to the unfortunate development of theoretical physics in the last 30-40 years with their overreliance on too much abstraction, oversophisticated mathematics, and other fantasies, which often have less and less to do with the actual physics as an empirical science. Two books can be mentioned here in relation to the present situation of physics science, see [8][9].

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A Review on Superluminal Physics and Superluminal Communication in light of the Neutrosophic Logic perspective

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Abstract

In a recent paper, we describe a model of quantum communication based on combining consciousness experiment and entanglement, which can serve as impetus to stop 5G-network-caused diseases. Therefore, in this paper we consider superluminal physics and superluminal communication as a bridge or intermediate way between subluminal physics and action-at-a-distance (AAAD) physics, especially from neutrosophic logic perspective. Although several ways have been proposed to bring such a superluminal communication into reality, such as Telluric wave or Telepathy analog of Horejev and Baburin, here we also review two possibilities: quaternion communication and also quantum communication based on quantum noise. Further research is recommended in the direction outlined herein. *Aim of this paper:* We discuss possibilities to go beyond 4G and 5G network, and avoid the unnecessary numerous health/diseases problems caused by massive 5G network. *Contribution*: We consider quaternionic communication and quantum communication based on quantum noise, which are largely unnoticed in literature. *Limitation:* We don't provide scheme for operationalization, except what we have provided in other paper.

Keywords: quantum entanglement, quantum communication, consciousness, superluminal communication, action at a distance.

Quote: "The Hertz wave theory of wireless transmission may be kept up for a while, but I do not hesitate to say that in a short time it will be recognized as one of the most remarkable and inexplicable aberrations of the scientific mind which has ever been recorded in history." —Nikola Tesla, *The True Wireless,* 1919.

1. Introduction

In line with the rapid development of new branch of foundational mathematics, i.e. Neutrosophic Logic, here we discuss potential application of NL theory in the field of telecommunication. See for recent papers on NL: [31-35]. It is known that nowadays telecommunication systems are predominated by RF systems, including numerous wireless systems, such as 4G Wi-Fi, 4G network etc. And the world is now in transition stage towards 5G network deployment.

A growing number of individuals are coming together in many countries - to attempt to block or stop the current telecoms roll-out of 5G electromagnetic microwave radiation which has proved to be extremely harmful to all sentient life forms, including plant life.[1]

It would prove invaluable if some methods could be found to render the 5G millimetre wave transmissions ineffective. In other words, to block or dissolve their ability to irradiate surrounding matter/life. Our concern is not to find a way to 'to protect the individual' but to prevent whole areas from being affected by microwaves via the tens of thousands of transmission bases that 5G requires - and from satellite sources.

In literature, there are known proposals or experiments which were purported to suggest possible ways to develop superluminal communication, to name a few: Telluric wave and also Telepathy analog way. For instance, in biofield site, it is written, which can be paraphrased as follows:

> "Torsion fields is one of the names given to the more unpretentious parts of the biofield. Torsion fields have additionally been alluded to as orgone, od, tachyon, aether, Tesla waves, scalar waves, the zero point field and then some. There is no settled upon logical agreement on these increasingly inconspicuous parts of the biofield. Torsion fields are guessed to the moderating mode for separation recuperating, which happens immediately and which research has been demonstrated that to be difficult to be transmitted through exemplary electromagnetic frequencies."[2]

See also Horejev and Baburin's paper [27]. Besides, there are also other suggestions of telepathic analog communications [28-30].

From Neutrosophic Logic perspective, we need to distinguish the subluminal communication from superluminal communication. In fact, Smarandache's Hypothesis states that there is no speed limit of anything, including light and "particles [16]. One of us (FS) also wrote in this regards:

> "In a similar way as passing from Euclidean Geometry to Non-Euclidean Geometry, we can pass from Subluminal Physics to Superluminal Physics, and further to Instantaneous Physics (instantaneous traveling). In the lights of two consecutive successful CERN experiments with superluminal particles in the Fall of 2011, we believe these two new fields of research should begin developing. A physical law has a form in Newtonian physics, another form in the Relativity Theory, and different form at Superluminal theory, or at Instantaneous (infinite) speeds – according to the S-Denying Theory spectrum. First, we extend physical laws and formulas

[1] A brief explanation here: https://www.5gspaceappeal.org/the-appeal
[2] https://www.biofieldlab.com/whatisthebiofield

to superluminal traveling and to instantaneous traveling. Afterwards, we should extend existing classical physical theories from subluminal to superluminal and instantaneous traveling."[3]

While the idea behind Smarandache hypothesis is quite simple and based on known hypothesis of quantum mechanics, called Einstein-Podolski-Rosen paradox, in reality such a *superluminal physics* seems still hard to accept by majority of physicists.

The background idea and our motivation for suggesting to go beyond RF/subluminal communication towards superluminal communication are two previous papers: (a) there will be more than 720! (factorial) types of new diseases which may arise, if the 5G network is massively implemented – and the present covid-19 pandemic may be just the beginning; (b) in a recent paper [14], we describe a model of quantum communication based on combining consciousness experiment and entanglement, which can serve as impetus to stop 5G-network-caused diseases, and (c) another recent paper that we presented in CTPNP 2019, where we discuss a realistic interpretation of wave mechanics, based on a derivation of Maxwell equations from quaternionic Dirac equation [36]. From these previous papers, we come to conclusion to superluminal communication is not only possible, but it is indeed embedded in quaternionic Maxwell equations, which are close to their original idea by Prof. James Clerk Maxwell.

Therefore, in this paper we consider superluminal physics and superluminal communication as a bridge or intermediate way between subluminal physics and action-at-a-distance (AAAD) physics, especially from *Neutrosophic Logic* perspective. Although several ways have been proposed to bring such a superluminal communication into reality, such as Telluric wave or Telepathy analog of Horejev and Baburin, here we also review two possibilities: quaternion communication and also quantum communication based on quantum noise.

These two choices will be discussed in section 4. But first of all we will discuss what is the difference between subluminal communication and superluminal communication.

2. What is the difference between subluminal communication and superluminal communication?

According to Belrose, which can be paraphrased as follows:

> "The extremely plausibility of remote correspondences was established on the investigations of James Clerk Maxwell, and his conditions structure the premise of computational electromagnetics. Their accuracy was built up by Heinrich Hertz, when in 1887 he found EM radiation at UHF frequencies as anticipated by Maxwell. Since the spearheading work of Maxwell starting in the center 1850s, and of his adherents, a little gathering that got known as Maxwellians, which incorporated UK's Poynting and Heaviside, Maxwell's conditions have been read for longer than a century, and have demonstrated to be one of the best speculations in the historical backdrop of radioscience. For instance, when Einstein saw that Newtonian elements had as changed to be good with his exceptional hypothesis of relativity, he found that Maxwell's conditions were at

[3] See Florentin Smarandache. url: http://fs.unm.edu/SuperluminalPhysics.htm

> that point relativistically right. EM field impacts are created by the increasing speed of charges, thus Maxwell had naturally incorporated relativity with his conditions." [25]

History also told us that around a century ago, there was proponents of wireless telegraphy, including Marconi, and opponents of it, including Tesla. And there were records of conflict between Tesla and Marconi. History also told us that around the same months after wireless telegraphy networks were installed everywhere, there was a flu pandemic, just like we observed now. It is also known, that wireless telegraphy was based on RF technology, which is actually a subluminal physics, while Tesla preferred a superluminal technology beyond Radio Frequency (sometimes he referred to as *non-Hertzian* waves). See also [18-26].

For instance, Paul Brenner wrote:

> "Marconi's work is based on copies of patents of many other inventors without their permission. His so called original "*two-circuit*" device, a spark-gap transmitter plus a coherer-receiver, was similar to those used by Oliver Lodge in a series of worldwide reported tests in 1894. Tesla disputed that Marconi was able to signal for greater distances than anyone else when using the spark-gap and coherer combination. In 1900 Alexander Stepanovich Popov declared to the Congress of Russian Electrical Engineers: "[...] *the emission and reception of signals by Marconi by means of electric oscillations [was] nothing new. In America, the famous engineer Nikola Tesla carried the same experiments in 1893*." [20]

It is also known from history books, that in the last century, the understanding of the nature of electromagnetic phenomena was proceeding with a constant rivalry between two concepts of interaction: namely, Newton instantaneous *action at a distance* (IAAAD) and Faraday-Maxwell *short-range interaction*. Finally, the discovery of Faraday's law of induction (explicit time dependence of electromagnetic phenomena) and the experimental observation of electromagnetic waves seemed to confirm the short-range interaction. Nevertheless, the idea of IAAAD still has many supporters. Among the physicists who have developed some theories based, in any case, on this concept, we can find names such as Tetrode and Fokker, Frenkel and Dirac, Wheeler and Feynman, and Hoyle and Narlikar. This interest in the concept of IAAAD is explained by the fact that classical theory of electromagnetism is an unsatisfactory theory all by itself, and so there have been many attempts to modify either the Maxwell equations or the principal ideas of electromagnetism.

As Augusto Garrido wrote in his review to Chubykalo et al's book:

> "On the other hand, the famous article "*Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?*" by Einstein, Rosen and Podolsky published in Physical Review in 1935 revived this discussion in a new panorama. In this article Einstein made public his position against the Copenhagen interpretation of the quantum mechanics. The controversy unleashed since then made this article a very popular one for its implications in our physical and philosophical understanding of the physical reality. The main objective of this article was to demonstrate that the quantum mechanics, the same way the Newtonian mechanics was for the relativistic mechanics, is an incomplete theory, and therefore, transitory of reality. For that reason Einstein made evident what is now known as the EPR paradox. According to EPR quantum mechanics is no local theory, that is to say, it permits action at a distance and, that is forbidden by the relativity theory, instantaneous action at a distance.
> Unfortunately for Einstein, and for common sense the experiment performed by Aspect seems to indicate that the IAAAD following from quantum mechanics exists. As a consequence of this confusion, physicists

> are divided in two big groups according their position about IAAAD. These disputants are the quantum physicists and the relativists, who, almost after a century, have not been able to answer the old question whether the subject of their studies is a complete and integrated Universe – a physical Universe in its own right – or simply a assemblage of locally interacting parts."[15]

Therefore, to summarize the above paragraph, it seems we can distinguish among few technologies for communication:

- Subluminal RF physics → subluminal wave → subluminal communication
- Close to light speed physics → relativistic wave
- Superluminal physics → superluminal wave → superluminal communication
- Action at a distance physics → AAAD/quantum communication

From *Neutrosophic Logic* perspective, we can see that superluminal physics and superluminal communication are an intermediate way between the subluminal physics and AAAD physics, because in NL theory there is always a possibility to find a third way or intermediate state.

Summarizing:

Standpoint A (subluminal) → Intermediate (*superluminal*) → Standpoint B (AAAD)

In the next sections, we will discuss shortly quantum entanglement and how it can be used in developing new telecommunication technologies.

3. What is quantum entanglement?

In its simplest form the quantum theory's features can be reduced to: (a) wave function description of microscopic entities and (b) entanglement. Entanglement is a key property that makes quantum information theory dierd ent from its classical counterpart.[14]

According to Scolarici and Solombrino [5]:

> The essential difference in the concept of state in classical and quantum mechanics is clearly pointed out by the phenomenon of entanglement, which may occur whenever the product states of a compound quantum system are superposed. Entangled states play a key role in all controversial features of QM; moreover, the recent developments in quantum information theory have shown that entanglement can be considered a concrete physical resource that it is important to identify, quantify and classify.

Nonetheless, they concluded: "our research has pointed out a puzzling situation, in which the same state of a physical system is entangled in CQM, while it seems to be separable in QQM."

While entanglement is usually considered as purely quantum effect, it by no means excludes possibility to describe it in a classical way.

In this regards, from history of QM we learn that there were many efforts to describe QM features in more or less classical picture. For example: Einstein in 1927 presented his version of hidden variable theory of QM, starting from Schrödinger's picture, which seems to influence his later insistence that "*God does not play dice*" philosophy.[6][7]

Efforts have also been made to extend QM to QQM (quaternionic Quantum Mechanics), for instance by Stephen Adler from IAS.[8]

But in recent decades, another route began to appear, what may be called as Maxwell-Dirac isomorphism route, where it can be shown that there is close link between Maxwell equations of classical electromagnetism and Dirac equations of electron. Intuitively, this may suggest that there is one-to-one correspondence between electromagnetic wave and quantum wave function.

4. Two possible ways of superluminal communications

4.a. Maxwell-Dirac isomorphism through Quaternion algebra

Textbook quantum theory is based on complex numbers of the form $a_0+a_1 i$, with i the imaginary unit $i^2 = -1$. It has long been known that an alternative quantum mechanics can be based on the quaternion or hyper-complex numbers of the form $a_0 + a_1 i + a_2 j + a_3 k$, with i,j,k three non-commuting imaginary units.[8]

On the other hand, recognizing that the Maxwell's equations were originally formulated in terms of quaternionic language, some authors investigate formal correspondence between Maxwell and Dirac equations. To name a few who worked on this problem: Kravchenko and Arbab. These authors have arrived to a similar conclusion, although with a different procedures based on Gersten decomposition of Dirac equation.[4]

This MD isomorphism can also be extended further to classical description of boson mass which was usually called Higgs boson[3], so it may be a simpler option compared to scale symmetry theory.

4.b. Quaternionic QM and Entanglement

Having convinced ourselves that Maxwell-Dirac isomorphism has sufficient reasoning to consider seriously in order to come up with realistic interpretation of quantum wave function, now let us discuss QQM and entanglement.

Singh & Prabakaran are motivated to examine the geometry of a two qubit quantum state using the formalism of the Hopf map. However, when addressing multiple qubit states, one needs to carefully consider the issue of quantum entanglement. The "quaternions" again come in handy in studying the two qubit state. [10]

In his exposition of Quaternionic Quantum Mechanics, J.P. Singh concluded that [9]:

> "Having established the compatibility of the Hopf fibration representation with the conventional theory for unentangled states, let us, now, address the issue of measurability of entanglement in this formalism. In the

> context, "Wootters' Concurrence" and the related "Entanglement of Formation" constitute well accepted measures of entanglement, particularly so, for pure states. …
>
> It follows that any real linear combination of the "magic basis" would result in a fully entangled state with unit concurrence. Conversely, any completely entangled state can be written as a linear combination in the "magic basis" with real components, up to an overall phase factor. In fact, these properties are not unique to a state description in the "magic basis" and hold in any other basis that is obtained from the "magic basis" by an orthogonal transformation…"

Singh & Prabakaran also suggest that this quaternionic QM may be useful for exploring quaternionic computing.[10]

Therefore, it shall be clear that entanglement and quantum communication have a sound theoretical basis.

4.c. Basic principles of quantum communication based on quantum noise

Our proposed communication method can provide an infinite number of infinite bandwidth communications channels for each user. See our recent paper describing one of plausible way to do quantum communication [14].

Communication using this method travels much faster than light. It does not use radio waves and does not need wires. It cannot be monitored nor tracked nor interfered with. It cannot be regulated due to the infinities involved, and due to the fact that it is unmonitorable. Each user benefits personally from the perfect information security provided by quantum communications.

Quantum communication does not harm any form of life, nor the environment, in any way, as quantum events are, and always have been, constantly a part of the Natural Environment. This method is not related to "Q-bits" nor "quantum teleportation" nor "quantum amplification" approaches, in any way. It is based on the Schrödinger equations of Quantum Mechanics. One of the features of the Schrödinger equations is a descriptive prediction of what is called "*quantum noise*". This is the constant "hiss" that one hears when using an FM radio, and setting the frequency selector in between active broadcast channels. The sound is called "quantum noise"

Quantum noise is observable at every location in the infinite volume universe. Quantum noise is the result of non-local Subquantum processes which cause apparently random quantum behaviors in physical systems, particularly those which involve electric, magnetic, or electromagnetic fields.

The situation is described by the quantum observable A of the system. This boils down to the fact that there is an expectation value in situations which involve quantum noise, which should normally appear as perfect randomity in the quantum system we are observing. Perfect randomity is called 3rd Order randomity and is completely unpredictable.

3rd Order randomity then represents the normal behavior of our quantum system as it interacts with Subquantum entities which are interacting with the system from up to infinity away and with up to an infinite velocity. 3rd Order randomity is the quantum expectation value of all Natural systems, in all locations and at all times.

There are ways to detect and predict quantum noise and the physical changes produced by quantum noise in quantum systems (These methods will not be discussed at this time). When we detect the quantum noise, for example, in the form of "white noise" between radio stations, we expect the quantum spectrum centered on the channel of our receiver to exhibit 3rd Order randomity in both electromagnetic frequency and magnitude domains, in our selected channel. However, environmental factors such as the presence of physical or non-physical forms of Consciousness can act on the 3rd Order randomity so as to bring predictability and order to the stream of random number which our E/M detector array passes on to our discriminator system.

Related to this, it was proved by instrumented experiments in the USA and in France during the 1990s that the Attention, Intentions, and Emotional State of operators of symplectic, complex, and standard electromagnetic transmission facilities, resulted in instantaneous changes in the radiation patterns of the transmission antennas.

All of the above mentioned facts can be useful for developing a working quantum communication device, see for further exposition of our method in [14].

5. Concluding remarks

Despite its enormous practical success, many physicists and philosophers alike agree that the quantum theory is so full of contradictions and paradoxes which are difficult to solve consistently. Even after 90 years, the experts themselves still do not all agree what to make of it. In this paper, we review the most puzzling feature of QM, i.e. entanglement.

In the meantime, the problem of the dangers of 5G creates a potential to develop new solutions of telecommunications, without having to use 5G/RF technologies. Therefore, in this paper we consider superluminal physics and superluminal communication as a bridge or intermediate way between subluminal physics and action-at-a-distance (AAAD) physics, especially from Neutrosophic Logic perspective. Although several ways have been proposed to bring such a superluminal communication into reality, such as Telluric wave or Telepathy analog of Horejev and Baburin, here we also review two possibilities: quaternion communication and also quantum communication based on quantum noise.

From *Neutrosophic Logic* perspective, we discuss on superluminal physics and superluminal communication as an intermediate way between the subluminal physics and AAAD physics, because in NL theory there is always a possibility to find a third way or intermediate state.

Summarizing:

Standpoint A (*subluminal*) → Intermediate (*superluminal*) → Standpoint B (*AAAD*)

This paper was inspired by an old question: Is there an alternate way to communication beyond RF method?

Further research is recommended for future implementation.

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An Expanded Model of Unmatter from Neutrosophic Logic perspective: Towards Matter-Spirit Unity View

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Abstract

In Neutrosophic Logic, a basic assertion is that there are variations of about everything that we can measure; the variations surround three parameters called T, I, F (truth, indeterminacy, falsehood) which can take a range of values. A previous paper in IJNS, 2020 shortly reviews the links among aether and matter creation from the perspective of Neutrosophic Logic. In any case, matter creation process in nature stays a major puzzle for physicists, scientists and other science analysts. To this issue neutrosophic logic offers an answer: "unmatter." This paper examines an extended model of unmatter, to incorporate issue soul solidarity. So, neutrosophic logic may demonstrate helpful in offering goal to long standing clashes.

Keywords: Neutrosophic Logic, Physical Neutrosophy, aether, matter creation, unmatter, unparticle

1.Introduction

In accordance with the quick improvement of new part of basic science, for example neutrosophic logic, here we talk about possible use of NL hypothesis in the field of media transmission. See for ongoing papers: [31-35].

It is known that *matter creation* processes in nature remains a big mystery for physicists, biologists and other science researchers. To this problem neutrosophic logic offers a solution, i.e. *unmatter and unparticle*. See also previous papers on unmatter [21-27].

To put it plainly, neutrosophic logic may demonstrate helpful in offering goal to long standing clashes. See likewise our past papers on this issue. [1-2].

2. Matter creation processes and Grusenick experiment

Physicists all through numerous hundreds of years have bantered over the physical presence of aether medium. Since its origin by Isaac Newton, many accepted that it is required in light of the fact that in any case it is highly unlikely to clarify communication a good ways off in a vacuum space. We need mechanism of connection, of which has been called by different names, for example, quantum vacuum, zero point field, and so forth.

The celebrated Michelson-Morley tests were thought to give invalid outcome to aether speculation, and truly it was the premise of Einstein's STR. In any case, more up to date conversations demonstrated that the proof was fairly equivocal, from MM information itself. Particularly after Dayton Miller examinations of aether float were accounted for, an ever increasing number of information came to help aether speculation, albeit numerous physicists would lean toward another terms, for example, physical vacuum or superfluid vacuum. See [9-13].

In this regards, an experiment which is worthy to mention here is by Grusenick. Actually, his method is quite similar to Michelson-Morley experiment, except that he puts the interferometer *vertically*, which makes him able to detect the vertical aether inflow perpendicularly toward the surface of the Earth. Because only few papers discuss his result, let us give him space to tell in his own words, which can be paraphrased as follows:

> "I have perused your information with much intrigue. Numerous individuals state that my development is precisely excessively flimsy, and that gravity impacts my contraption. So I assembled another. A man named Norbert Feist gave me better optical hardware to utilize. The new interferometer is just a steel plate with 189mm width and 8mm thick. The mirrors and the mirror holders are fabricated by Edmund, USA. Their shaft splitter anyway is precisely excessively insecure, so I utilized the one I made myself.
> The obstruction design is anticipated on a little bit of paper. During a 180° pivot with the new Interferometer, I can see on normal 1.5 impedance periphery shifts during the night and 2.0 during daytime. With the more established one, which you can find in the YouTube film, there are 11.0 movements around evening time and 11.5 if the trial is performed during daytime. In this way, the two Interferometers (the more seasoned and the more up to date one) show a distinction of 0.5 obstruction periphery shifts among day and night.
> I additionally might want to make reference to that a slight variety in the quality of the periphery design development happens during various days of the month. On Thursday 16.10.2009 at 24.00 o clock, I could see a full 3.0 obstruction periphery shifts per 180° pivot (with the new interferometer). The zero point, where a stop of the example development occurs, is for the two interferometers at a similar position. There are two zero focuses in a 360° turn of the two interferometers when the shaft splitter is situated evenly to the world's surface. To all individuals who state that the main impact on the interferometer is gravity, I have a straightforward inquiry. Why would that be no zero point or stop of the periphery design development when the shaft splitter is in the vertical position? In the pillar splitter's vertical position, the mirrors and the mirror holders are evenly pushed by gravity. In any case, there is no zero point."[19]

According to Paul LaViolette, Grusenick's experiment proves the existence of ether and also his G-ons theory:

> "Subquantum Kinetics requires that G etherons (G-ons) reliably diffuse into the Earth, driven by the incline in the Earth's 1/r gravitational conceivable field. The low G-on center inside the Earth, as differentiated and the Earth's space condition, develops considering the way that G-ons are foreseen to be conveyed at an all the more moderate rate in the unbiased issue inside the Earth as differentiated and enveloping space. … Later

he built up an improved adjustment of the interferometer, showed as follows, and found a total fringe move of 1.5 to 2.0 as the mechanical gathering was turned in the vertical bearing. This value comes closer to that of U.S. investigator Frank Pearce who has played out a type of the Grusenick break down using a 1 inch thick stone square, as opposed to an aluminum board, for mounting the interferometer mirrors and who found a move of just around one half to one outskirts when the mechanical get together was turned in the vertical bearing."[20]

Alternatively, let us assume that under certain conditions that aether can transform using Bose condensation process to become "*unmatter*", a transition phase of material, which then it sublimates into matter (solid, gas, liquid). Unmatter can also be considered as "*pre-physical matter*."

Summarizing our idea, it is depicted in the following block diagram [1]:

Aether → bose condensation → "unmatter" (pre-physical matter) → sublimation → ordinary matter/particle

Diagram 1. How aether becomes ordinary matter

Actually the term "unmatter" can be viewed as a solution from perspective of Neutrosophic Logic. A bit of history of unmatter term may be useful here:

> "*The word 'Unmatter' was instituted by one of us (F. Smarandache) and distributed in 2004 of every three papers regarding the matter. Unmatter is framed by mixes of issue and antimatter that bound together, or by long-extend blend of issue and antimatter shaping a pitifully coupled stage. The possibility of unparticle was first considered by F. Smarandache in 2004, 2005 and 2006, when he transferred a paper on CERN site and he distributed three papers about what he called 'unmatter', which is another type of issue framed by issue and antimatter that quandary together. Unmatter was presented with regards to 'neutrosophy' (Smarandache, 1995) and 'paradoxism' (Smarandache, 1980), which depend on blends of inverse substances 'An' and 'antiA' along with their neutralities 'neutA' that are in the middle."*[1] See also Christianto & Smarandache [17]. See also F. Smarandache et al.'s papers and books, [21-27].

In any case, in this paper, unmatter is considered as a progress state (pre-physical) from aether to get common particles, see also [1]. Moreover, superfluid model of dark matter has been discussed by some authors [6-7].

3. An expanded model of *unmatter*

In other side, it is known that astronomers find that only 1% of matter in the universe is observed, while 99% is undetected. That is why they call it the Hidden Universe.

[1] http://fs.unm.edu/unmatter.htm

Could it be that aether (may be in form of superfluid medium, a ka Mishin phase state) can be intermediate entity in neutrosophic sense?

In this line of thought, it is possible to come up with an expanded model of unmatter, as follows:

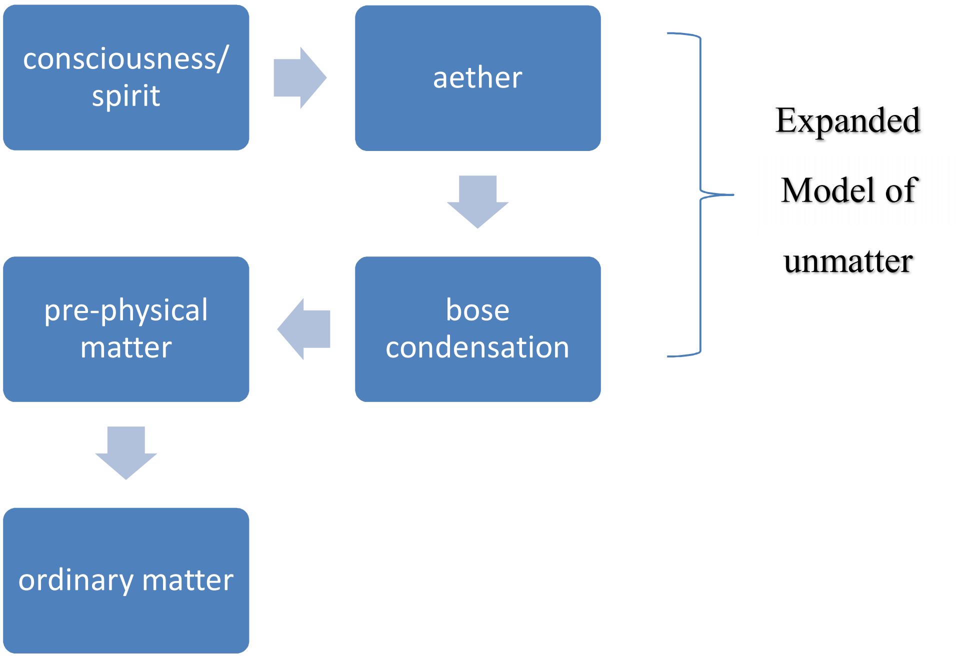

Diagram 2. An *expanded model of unmatter*

May be it is because the remaining entities are in the form of consciousness, aether and pre-physical matter. That is what can be called as "*expanded model of unmatter*."

4. Remark on grid cells, bhutatmas, and consciousness

May be it is possible to come up with a model of how spirit affect matter and vice versa, which reminds us to papers by Ervard Moser et al. on grid cells, space cells etc.
We can add some remark as follows:

> "Space cells and grid cells were first discussed by Alfven (Nobel Prize in Physics) regarding plasma behaviors. I brought it out that these cells have, and evoke, personality traits in all who occupy the given cell, over large spans of time. Which means each star and each cell in the spaces between stars, has a unique personality.

The Russians did a research project that covered most of Asia, and all of Europe, and determined that each cell contains life forms plants animals birds insects fish, and so on, that are unique to that specific cell, and that the people who are native to a given cell have similar personality traits and behaviors unlike the inhabitants of other cells.

The personality of the land of a given cell produces an ambient "personality field" and a unique "magic" that can be learned and used by the inhabitants as a source of benefits which are specific to that cell.

These personality cells are produced by the aether energy-information contents of the plasmas which originate the personality of the given cell. These cell have distinct boundaries and are directly involved with creating life-forms which are perfectly suited to life in the given cell. Some life forms are able to cross over into other cells without undue stresses. Others do not live long when they are removed from their native information-energy habitat.

For life forms which are able to transit and occupy various cells, if the given Being spends a large amount of time in a specific cell, they start to change physically and psychologically in alignment with the qualities of the personality of the land they are spending large amounts of time in.

The Bhutatmas are conveyed by plasmas and "stick to" every material form. Aether clings to matter at all scales, interpenetrating it and forming an atmosphere, similar to the photographs taken by Krasnoholovets of the "atmosphere" of "inertons" which surround electrons. Inertons are much larger than Bhutatmas, however. There are many layers of behaviors related to the smallness of the entities involved which form thresholds of altered physical behaviors, as seen in Pendry structures and other metamaterials.

Air currents, water currents, electrical flows, plasma flows, and all wave forms in all media, regardless of phase state, convey aether and information energy between end points and all along the lines of the flows. Aether circuits are always bi-directional between end-points, while plasma and electrical flows are one-directional. Gravitation and time aether flows also carry information-energy and can alter the given local energetic informational environment fairly rapidly, or over large spans of time.

Marjanovic's model does not cover any of this, as he has no attention for the physics of information-energy, Consciousness, nor studies of the activities of Divinity. Bhutatmas cover all the bases."

Moreover we can add…

"Personality cells are determinant in what kinds of matter are formed, and in where and when they are formed. Stars each have a unique personality, a unique chemistry, and a unique radiation spectrum, exactly because they are formed in different cells with different personalities, which personality cells act as environmental factors during star formation and planet formation.

This is related to the Telluric Intelligence (inhabiting aether rivers) which is endemic to and inherent in each star and each planet. Probably, each Telluric Intelligence is unique, as well as being involved with the unique star and the unique planets associated with the given star.

According to Wal Thornhill and Steven Smith, with whom we agree on this, planets are formed internal to stars by precipitation processes resulting from the creation of atomic elements in the outer-most layers of stars, due to charge separation in stellar plasmas creating enormous gradients in the stellar electric field, thus urging the aether involved with the given star to create new atoms, as put into evidence in the SAFIRE Project. The newly formed atoms tend to precipitate and drift towards the central regions of the given star. Each planet will be unique, but have some traits in common with its parent star.

Uniqueness is partly due to non-local influences being imposed on stellar systems by infinite velocity infinitesimals which carry and convey information to the given stellar system which influence the personality and material composition of the given star. This is a process due to the 5th phase state in Mishin's 5 phase-state aether. (Tesla talks about non-local influences imparting information and various forms of organization to local systems.) This is in addition to the Personality information inherent to the given aether-plasma space-cell, which can modify the local personality over time, and in response to superluminal activities of quantized red-shifts resulting in local variations in the laws of physics in the region, and local variations in the fine structure "constant", leaving leaving the galaxy-core aether-plasmoid in superluminal 3D shells, modifying the physics in the volume of the given out-bound shell."

Hopefully many more approaches can be developed in the direction as mentioned above.

5. Concluding remarks

In this paper, we discussed three possible applications of Neutrosophic Logic in the field of matter creation processes etc. For instance, a redefinition of term "unmatter" is proposed here, where under certain conditions, aether can transform using Bose condensation process to become "*unmatter*", a transition phase of material, which then it sublimates into matter (solid, gas, liquid). Unmatter can also be considered as "pre-physical matter." Moreover, we can extend it further to include consciousness/spirit, which may explain why the 99% of matter in this Universe is undetected. Further researches are recommended in the above directions.

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Towards Realism Interpretation of Wave Mechanics based on Maxwell Equations in Quaternion Space and some implications, including Smarandache's Hypothesis

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Abstract. From time to time, some eminent physicists commenced to ask: What is the reality behind quantum mechanical predictions? Is there a realism interpretation of Quantum Physics? This paper is intended to explore such a possibility of a realism interpretation of QM, based on a derivation of Maxwell equations in Quaternion Space. In this regards, we begin with Quaternion space and its respective Quaternion Relativity (it also may be called as Rotational Relativity) as it has been discussed in several papers including [1]. The purpose of the present paper is to review our previous derivation of Maxwell equations in Q-space [17], with discussion on some implications. First, we will review our previous results in deriving Maxwell equations using Dirac decomposition, introduced by Gersten (1999). Then we will shortly make a few remark on helical solutions of Maxwell equations, Smarandache's Hypothesis and possible cosmological entanglement. Further observations are of course recommended to refute or verify some implications of this proposition.

INTRODUCTION

From time to time, some eminent physicists began to ask: What is the reality behind quantum mechanical predictions? Is there a realism interpretation of Quantum Physics? For instance, in a recent paper Prof. G. 't Hooft asks questions which can be paraphrased as follows: "The creator has made his own analysis of the acknowledged facts, and got here to the conclusion that the Copenhagen doctrine, that is, the consensus reached by many of the world specialists at the beginning of the 20th century, partly for the duration of their numerous gatherings in the Danish capital, has it almost right: there is a wave function, or rather, something we name a quantum state, being a vector in Hilbert space, which obeys a Schrodinger equation. The absolute squares of the vector elements can also be used to describe possibilities every time we wish to predict or give an explanation for something. Powerful techniques had been developed, enabling one to bet the proper Schrodinger equation if one knows how things evolve classically, that is, in the ancient theories the place quantum mechanics had not but been incorporated. It all works magnificently well. According to Copenhagen, however, there is one question one no longer ask: "What does fact seem like of some thing moves round in our experimental settings?", or: what is actually going on? According to Copenhagen, such a question can never be addressed via capability of any experiment, so it has no reply within the set of logical statements we can make of the world. Period, schluss, fini. Those questions are senseless. It is this answer that we dispute. Even if this form of questions cannot be answered via experiments, we can nonetheless in idea try to construct credible models of reality." (see [16]) This paper is intended to explore such a possibility of realism interpretation of QM, especially based on a derivation of Maxwell equations in Quaternion Space. In this regards, we begin with Quaternion space and its respective Quaternion Relativity (it also may be called as Rotational Relativity) as it has been defined in a number of papers including [1], and it can be shown that this new theory is capable to describe relativistic motion in elegant and straightforward way. For instance, it can be shown that the Pioneer spacecraft's Doppler shift anomaly can be explained as a relativistic effect of Quaternion Space [11]. The Yang-Mills field also can be shown to be consistent with Quaternion Space [1]. Nonetheless there are subsequent theoretical developments which remains an open issue, for instance to derive Maxwell equations in Q-space [1].

Therefore the purpose of the present paper is to review our previous derivation of Maxwell equations in Q-space, also with a few remarks on helical solutions and the shape of electron. First, we will review a derivation of Maxwell equations using Dirac decomposition, introduced by Gersten (1999). Then we will shortly make a few remark on helical solutions of Maxwell equations and Smarandache's hypothesis. Further observations are of course recommended in order to refute or verify some implications of what. we discuss here.

Review of previous result: derivation of Maxwell equations in Quaternion Space by virtue of Dirac-Gersten decomposition.

In this section we review our previous derivation of Maxwell equations in Quaternion space based on Gersten's method to derive Maxwell equations from one photon equation by virtue of Dirac decomposition [4]. It can be noted here that there are other methods to derive such a 'quantum Maxwell equations' (i.e. to find link between photon equation and Maxwell equations), for instance by Asim Barut quite a long time ago (see ICTP preprint no. IC/91/255).

As a short review of our previous paper [17], we started with remark that Maxwell equations [9] will be obtained by substitution of E and p with the ordinary quantum operators (see for instance Bethe, *Field Theory*):

$$E \rightarrow i\hbar \frac{\partial}{\partial t} \quad , \tag{1}$$

and

$$p \rightarrow -ih\nabla, \tag{2}$$

and the wavefunction substitution:

$$\vec{\Psi} = \vec{E} - i\vec{B}, \tag{3}$$

where E and B are electric and magnetic fields, respectively. With the identity:

$$(\vec{p} \cdot \vec{S})\vec{\Psi} = \hbar\nabla \times \vec{\Psi}. \tag{4}$$

Then one will obtain:

$$i\frac{\hbar}{c}\frac{\partial(\vec{E} - i\vec{B})}{\partial t} = -\hbar\nabla \times (\vec{E} - i\vec{B}), \tag{5}$$

$$\nabla \cdot (\vec{E} - i\vec{B}) = 0, \tag{6}$$

Which are the Maxwell equations if the electric and magnetic fields are real [4,5].

We can remark here that the combination of E and B as introduced in (3) is quite well known in literature [6,7]. For instance, if we use positive signature, then it is known as Bateman's representation of Maxwell equations ($div\vec{\varepsilon} = 0; rot\vec{\varepsilon} = \frac{1}{c}\frac{\partial\vec{\varepsilon}}{\partial t}; \varepsilon = \vec{E} + i\vec{B}$). But the equation (3) with negative signature represents the *complex nature* of Electromagnetic fields [6], which indicates that these fields can also be represented in quaternion form.

Then after some steps, one will obtain *the Maxwell equations in Quaternion-space* as follows [17]:

$$i\frac{\hbar}{c}\frac{\partial\vec{\varepsilon}_q k}{\partial t} = -\hbar\nabla_k \times \vec{\varepsilon}_{qk} \quad , \tag{7}$$

$$\nabla_k \cdot \vec{\varepsilon}_{qk} = 0. \tag{8}$$

Now the remaining question is to define quaternion differential operators in the right hand side of (7) and left hand side of (8).

In this regards one can choose some definitions o f q uaternion d ifferential o perator, f or i nstance t he so-called 'Moisil-Theodoresco operator'[8].

As with cross product, we can note here that there could be more rigorous approach to define such a quaternionic curl operator.[7]

In the present paper we only discuss derivation of Maxwell equations in Quaternion Space using the decomposition method described by Gersten [4,5].

In the next section we will discuss three physical implications of this new derivation of Maxwell equations in Quaternion Space.

Implication 1: Helical solutions of Maxwell equations; one-to-one correspondence with Wave Mechanics; and limitations of this paper.

In the foregoing section we derived a consistent description of Maxwell equations in Q-Space by virtue of Dirac-Gersten's decomposition. Now we discuss some plausible implications of the new proposition.

First, in accordance with Gersten, we submit the viewpoint that the Maxwell equations yield wavefunctions which can be used as guideline for interpretation of quantum mechanics [4,5]. We shall emphasize here: "The one-to-one correspondence between classical and quantum wave interpretation actually can be expected not only in the context of Feynman's derivation of Maxwell equations from Lorentz force, but also from known exact correspondence between commutation relation and Poisson bracket." See also [3,4].

Secondly, the above expressions of Maxwelll equations in Q-space are still missing the A term. Provided we include A term as defined by R.W. Bass [2], as follows:

$$A = \nabla x(\psi u) + (1/\lambda)\nabla x(\nabla x(\psi u)) \tag{9}$$

then we got helical solutions of Maxwell eq., which are more consistent with many experimental results, instead of the well-known "sinusoidal" solutions.[2]

A few more remark deserves further attention, as follows: "This helical form of EM ties into the KH electron, during photon capture events. The captured photon causes an energetic imbalance in the desired and required stability of the electron, which causes the photon to be ejected in a short while. We're still trying to discover what is the cause of the form of the electron being as stable as it is, and why it has such a strong surface boundary." (KH=Kelvin-Helmholtz)

Further implications of this new proposition of helical solutions of Maxwell equations require further study, and therefore they are excluded from the present paper; see also Bass [2].

Before we close this paper, allow us to point out two limitations of the procedure as outlined above: (a) It begins with Gersten decomposition and Dirac equation, which assumes that mass-energy equivalence holds true. Whenever one can show that such an equivalence does not hold true, then our procedure should be revised. (b) Our model assumes orthogonality. Some authors have argued that orthogonality of Dirac equation prevents it to predict fractional states of hydrogen (hydrino, cf. R.L. Mills [15]). However, the majority of interactions and appearances in Nature are not orthogonal, but are comprised of acute and oblique angles, usually far from orthogonality. This fact is covered by dot products and cross products, as required by engineering applications of the Maxwell equations. Quaternion solutions may be not capable of doing dot products, to our knowledge (although it may be possible to define "quaternionic curl," as we mentioned in previous section). Perhaps it is better to use the graded Grassmann algebra as the foundational basis, since projective Grassmann solutions are easy to obtain. Furthermore, the graded Grassmann algebras do not require a metric, nor do they need to know which way is "up" as the Clifford algebras require. Thus, the Grassmann algebras are better suited for cosmological considerations, which the Clifford algebras are incapable of handling, due to the limitations and shortcomings of the Clifford algebras. But we reserve such an extension to Grassmann algebra for future investigation.

Implication 2: Smarandache's Hypothesis

As we discussed above, Maxwell equations in Quaternion Space are formally equivalent with Dirac equations, this result brings to a mathematical support of Smarandache's hypothesis. So in this section we will review shortly what Smarandache's hypothesis is. Smarandache's Hypothesis states that there is no speed limit of anything, including light and particles [18]. Eric Weisstein also wrote implications of Smarandache's Hypothesis [18a], which can be paraphrased as follows: "...the velocity of light c is no longer a maximum at which statistics can be transmitted and that arbitrary speeds of data or mass switch can occur. These assertions fly in the face of each idea and experiment, as they violate both Einstein's exceptional principle of relativity and causality and lack any experimental support. It is authentic that modern-day experiments have confirmed the existence of positive sorts of measurable superluminal phenomena. However, none of these experiments are in conflict with causality or distinct relativity, because no statistics or bodily object absolutely travels at speeds v large than c to produce the located phenomena." (see [18a]) While the idea is quite simple and based on known hypothesis of quantum mechanics, called Einstein-Podolski-Rosen paradox, in reality such a superluminal physics seems still hard to accept by majority of physicists. Since 2011, there was an apparent surprising result as announced by OPERA team. Nonetheless, few months later it was renounced, on the ground of errors in handling the measurement. The story was retold by Lukasz Glinka [20], which can be paraphrased as follows: "On September 22, 2011, the human world overloaded at some point of the various paradigms and dogmas had experienced without a doubt innovative excitations. Namely, on this day the OPERA Collaboration, an global experimental undertaking of the European Organization for Nuclear Research - CERN, announced that their results, which arose from the high-statistics experimental data, naturally display existence of the superluminal neutrinos... During the subsequent 5 months, the public opinion was once a witness of many a variety of speculations about faster-then-light motion, but already on February 22, 2012, OPERA pointed out the two issues, based totally on the technology of the Global Positioning System whose construction in itself makes use of the arguments of Special Relativity, which ought to at once have an effect on on the dimension process... The first trouble was once linked to the oscillator producing the occasions time-stamps in between the GPS synchronizations, whereas the 2d one used to be the cable of the optical fiber bringing the external GPS sign to the OPERA grasp clock. Both these probabilities probably could supply the anomaly regarded as an experimental error which led to registration of the faster-than-light neutrino. In February 2012, the 2nd cause was once regarded extra seriously than the first one, and the CERN experimentalists announced that the state of affairs will be demonstrated as soon as once more but in 2012. In March 2012, the ICARUS experiment, another CERN experimental collaboration initiated in 1977 via Carlo Rubbia, who shared the 1984 Nobel Prize in Physics for discovery of the weak gauge bosons W and Z, confirmed the absolutism of the pace of light in the dimension of the neutrino motion. Already in June 2012, the CERN Research Director Sergio Bertolucci, at the twenty fifth International Conference on Neutrino Physics and Astrophysics held in Kyoto, established the fallacious size due to the OPERA Collaboration… Moreover, it is worth stressing that the superluminal kingdom of affairs is regular in current astronomy when you consider that the early 1980s, when the faster-than-light movement had been advised in order to contradict the quasars having the cosmological distances. In the present-day situation, the experimental information exhibit that the superluminal travels are the phenomena which are very regularly met in radio galaxies, quasars and microquasars."

Allow us to make few comments on such an apparent failure to detect faster than light speed as follows: Despite those debates over OPERA results, we thought that a more convincing experiment has been done by Alain Aspect etc., who were able to show that quantum non-locality interaction is real. In 1980 Alain Aspect performed the first EPR experiment (Einstein-Podolski-Rosen) which proved the existence of space nonlocality (Aspect 1982). Alain Aspect and his team at Orsay, Paris, conducted three Bell tests using calcium cascade sources. The first and last used the CH74 inequality. The second was the first application of the CHSH inequality [24]. The third (and most famous) was arranged such that the choice between the two settings on each side was made during the flight of the photons (as originally suggested by John Bell). Some experimenters have repeated this experiment and prove similar result until distance of more than 90 km. So the notion of "spooky action at a distance" is a real physical phenomenon. Moreover, action at a distance was already mentioned in Newton's Principia Mathematica. Despite, apparently Einstein was trying to make all of Newton's expressions into nothing, our result suggests that the Maxwell equations in classical electrodynamics have "spooky interaction at a distance" type of interactions (as it has also been proven for Coulomb potential), which may be observed both at small scale experiments as well as in cosmological scale, as recent evidences show. The latter will be discussed in next section.

Implication 3: Possibility of Cosmological Entanglement

In the present paper, we argued in favor of mathematical correspondence which is more known as Dirac-Maxwell isomorphism. Its implications include Smarandache's Hypothesis and also quantum entanglement both as small scale experiments and also at cosmological scale. Interestingly, there is a recent report from MIT suggesting that ancient quasars support such quantum entanglement at large scale phenomena. In an article it is reported about possibility of cosmological entanglement [21], which can be paraphrased as follows: "In 2014, Kaiser and two individuals of the contemporary team, Jason Gallicchio and Andrew Friedman, proposed an test to produce entangled photons on Earth — a method that is pretty fashionable in research of quantum mechanics. They planned to shoot every member of the entangled pair in contrary directions, towards mild detectors that would additionally make a measurement of every photon the use of a polarizer. Researchers would measure the polarization, or orientation, of every incoming photon's electric powered field, with the aid of putting the polarizer at quite a number angles and watching whether or not the photons surpassed thru — an outcome for each photon that researchers should compare to decide whether the particles confirmed the hallmark correlations expected by using quantum mechanics. The team delivered a special step to the proposed experiment, which used to be to use mild from ancient, far away astronomical sources, such as stars and quasars, to decide the attitude at which to set each respective polarizer. As each entangled photon was once in flight, heading towards its detector at the velocity of light, researchers would use a telescope placed at every detector site to measure the wavelength of a quasar's incoming light. If that light used to be redder than some reference wavelength, the polarizer would tilt at a certain perspective to make a particular size of the incoming entangled photon — a size desire that was once determined by means of the quasar. If the quasar's mild was once bluer than the reference wavelength, the polarizer would tilt at a special angle, performing a one of a kind measurement of the entangled photon. In their preceding experiment, the team used small outdoor telescopes to measure the light from stars as shut as 600 light years away. In their new study, the researchers used a good deal larger, greater effective telescopes to seize the incoming mild from even greater ancient, far away astrophysical sources: quasars whose light has been travelling towards the Earth for at least 7.8 billion years — objects that are relatively a ways away and yet are so luminous that their mild can be located from Earth. On Jan. 11, 2018, "the clock had just ticked past nighttime neighborhood time," as Kaiser recalls, when about a dozen individuals of the crew gathered on a mountaintop in the Canary Islands and started amassing information from two large, 4-meter-wide telescopes: the William Herschel Telescope and the Telescopio Nazionale Galileo, both located on the equal mountain and separated via about a kilometer. One telescope focused on a particular quasar, whilst the different telescope appeared at every other quasar in a specific patch of the night time sky. Meanwhile, researchers at a station located between the two telescopes created pairs of entangled photons and beamed particles from each pair in contrary directions toward every telescope. In the fraction of a 2nd before each entangled photon reached its detector, the instrumentation determined whether or not a single photon arriving from the quasar used to be extra pink or blue, a dimension that then mechanically adjusted the angle of a polarizer that finally received and detected the incoming entangled photon." (see [21]) Therefore such a discovery has opened up a new way to look at the Universe: an entangled Cosmos.[22,23]

Concluding remarks

In the present paper we review our previous result on consistent derivation of Maxwell equations in Q-space. In accordance with Gersten, we submit the viewpoint that the Maxwell equations yield wavefunctions which can be used as guideline for interpretation of quantum mechanics. The one-to-one correspondence between classical and quantum wave interpretation here actually can be expected not only in the context of Feynman's derivation of Maxwell equations from Lorentz force, but also from known exact correspondence between commutation relation and Poisson bracket [2,4].

A somewhat unique implication obtained from the above results of Maxwell equations in Quaternion Space, is that it suggests that the helical solutions, especially if we consider the real physical meaning of A vector from R. Bass. Further implications, include possibility of faster than light entities, which seem to have been supported by a recent experiment of (quantum) cosmological entanglement.

In the present paper we only discuss our result on derivation of Maxwell equations in Quaternion Space using the decomposition method, and its implications [4,5]. Further extension to Proca equations in Quaternion Space seems possible too using the same method [5], but it will not be discussed here.

This proposition, however, deserves further theoretical considerations. Further observation is of course recommended in order to refute or verify some implications of this result.

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On the Possibility of Binary Companion of the Sun

Victor Christianto, Florentin Smarandache

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Abstract

In this article, we present some new arguments on the theoretical dwarf star thought to be a companion to our Sun, known as the Nemesis, which is postulated to explain a perceived cycle of mass extinctions in Earth's history. Some speculated that such a star could affect the orbit of objects in the far outer solar system, sending them on a collision course with Earth. While recent astronomical surveys failed to find any evidence that such a star exists, we outline in this article some theoretical findings including our own serendipity finding, suggesting that such a dwarf star companion of the Sun remains a plausibility. We also discuss possible theoretical framework to advance further this line of thought.

Keywords: Planet nine, dwarf star, binary star formation, solar system.

Introduction

The existence of a dwarf star as binary companion of the Sun has been debated for long time, although its existence has not been found yet. Some argues that no evidence at all supports such a dwarf star companion, sometimes dubbed as Nemesis. Nemesis is a theoretical dwarf star thought to be a companion to our sun. The theory was postulated to explain a perceived cycle of mass extinctions in Earth's history. Scientists speculated that such a star could affect the orbit of objects in the far outer solar system, sending them on a collision course with Earth. While recent astronomical surveys failed to find any evidence that such a star exists, a 2017 study suggests there could have been a "Nemesis" in the very ancient past [1].

Some arguments supporting plausible existence of Nemesis

Among arguments supporting possible existence of Nemesis, one of the most popular is the past extinction record. In the early 1980s, scientists noticed that extinctions on Earth seemed to fall in a cyclical pattern. Mass extinctions seem to occur more frequently every 27 million years. The long span of time caused them to turn to astronomical events for an explanation. In 1984, Richard Muller of the University of California Berkeley suggested that a red dwarf star 1.5 light-years away could be the cause of the mass extinctions. Later theories have suggested that Nemesis could be a brown or white dwarf, or a low-mass star only a few times as massive as Jupiter. All would cast dim light, making them difficult to spot.[1]

Scientists speculated that Nemesis may affect the Oort cloud, which is made up of icy rocks surrounding the sun beyond the range of Pluto. Many of these chunks travel around the sun in a long-term, elliptical orbit. As they draw closer to the star, their ice begins to melt and stream behind them, making them recognizable as comets. If Nemesis traveled through the Oort cloud every 27 million years, some argue, it could kick extra comets out of the sphere and send them hurling toward the inner solar system - and Earth. Impact rates would increase, and mass extinctions would be more common.

Another theory is there is a huge ice giant, nicknamed "Planet Nine," that is at the edge of our solar system. Researchers Konstantin Batygin and Mike Brown (both from the California Institute of Technology) suggested in 2016 that such a body might be stirring up smaller icy bodies in the Kuiper Belt.

More recent discoveries from the Chilean observatories of Sedna have only confirmed the existence of a massive “perturber” (to use the old language) or a “shepherd” (to use newer lingo). This article from *The Smithsonian’s Air & Space* publication makes it clear that the Caltech astronomers Mike Brown and Konstantin Batygin used a March 2014 issue of *Nature* to guide them to their discovery of Planet 9:

> When the two astronomers published the announcement, in the journal *Nature* in March 2014, they noted something odd about 2012VP, Sedna, and 10 other super-distant objects. They were all clustered in an unexpected way; their orbits all crossed the ecliptic plane - the conceptual flat disk around which the eight planets orbit the sun - very close to the spot where they came closest to the sun. So odd was the uniformity that the pair suspected something was causing it; they wrote, *‘[A]n unknown massive perturbing body may be shepherding these objects into these similar orbital configurations.’* [2]

A search for Planet Nine is ongoing. Notably, Brown was part of the research team that found Sedna and several other icy bodies in the Kuiper Belt, and he was one of the lead advocates for reclassifying Pluto (once considered a planet) to a dwarf planet in 2006 [1]. While the above two chief arguments: the past record of mass extinction and Planet Nine are quite interesting, they are not conclusive yet, because there are other possibilities to explain the data.

A third view, suggests that it is possible that all stars were formed in pairs, according to Berkeley news. The new assertion is based on a radio survey of a giant molecular cloud filled with recently formed stars in the constellation Perseus, and a mathematical model that can explain the Perseus observations only if all sunlike stars are born with a companion:

> We are saying, yes, there probably was a Nemesis, a long time ago," said co-author Steven Stahler, a UC Berkeley research astronomer. "We ran a series of statistical models to see if we could account for the relative populations of young single stars and binaries of all separations in the Perseus molecular cloud, and the only model that could reproduce the data was one in which all stars form initially as wide binaries. These systems then either shrink or break apart within a million years [4].

Provided such a statistical survey is correct, then it would mean that all stars including our Solar system may have dwarf companion, albeit its existence is undetected yet. Now, allow us to retell our own serendipity finding of its possible existence. It began around 15 years ago, while one of us (VC) investigating planetary orbits in Solar system. It came to him an idea that given the chemistry composition of Jovian planets are different from inner planets, therefore it is likely both series of planets have different origin. Initially, he read paper suggesting that our Solar system can be modelled as "*two fluid model*" of superfluidity. But he sought a simpler explanation to explain the orbits of Jovian planets.

By assuming inner planets orbits have different quantum number from Jovian planets, he tried to use "*least square difference*" method in order to seek the most optimal straight line for Jovian planets orbits in a different quantum number. Then it came out that such a straight line can only be modelled if we assume that the Jovian planets were originated from a twin star system: the Sun and its companion. Although his study was based on statistical optimization [6][7], it yields new prediction of 3 planetoids in the outer orbits beyond Pluto, which were discovered later by Mike Brown et al.

The Plausibility of Detecting a Dwarf Companion of the Sun

For observational purpose, we submit three possible ways to model the Solar system as binary star system:

a. Dipole magnetic view (Whitmire et al.): We found a compelling dipole magnetic view of our Solar System including Oort Cloud by D. Whitmire, whose view seems to support hypothesis of dwarf star as part of such a giant dipole system.

b. Statistical model of turbulence: A paper published in Turkish Physics Journal proves that Planck distribution of black-body radiation can be derived by assuming pairs of positive-negative mass. Considering that Planck blackbody law can be connected to statistical mechanics of turbulence (see Sohrab), then it seems plausible to come up with similar hypothesis with Sadavoy and Stahler, that all stars were once born in pairs [5].

c. Hydrogen-antihydrogen pairs model: Apart from two-fluid model of Solar System as we mention above, since we prefer a quantum-hydrodynamics model of Solar System, then it seems quite palatable to extend the Bohr-quantization of Solar System, by assuming it was not composed from large-scale hydrogen structure (check also Hydor model by ancient Greek philosopher), but to extend it by assuming a large scale "*hydrogen-antihydrogen pair*". Nonetheless, how we can model such hydrogen-antihydrogen system remain an open question.

Comparison with UVS model of solar system

Vincent Wee-Hoo is a developer of UVS model of solar system and galaxies (UVS stands for: unified vortice singularity). Based on UVS model, he is able to come up with barycenter drivers of solar cycle. He wrote [10]:

> Despite the gas giants when aligned directly with the Sun-SSB alignment were observed to have their pulsing effects for sunspot number, the ISN proposed solar minima or maxima in many cases were not dominantly modulated by any direct alignment of the gas giants. These cases assert that these solar cycles were dominantly driven by the more effective planetary barycenters that had aligned with the Sun-SSB alignment. This case study asserts solar cycle is dominantly driven by the oscillating alignments of the Sun, the barycenters, or their alignments with the gas giants that optimize the alignments of the barycenters. … There are 19 cases (2 Min, 10 Min, 10 Max, 11 Min, 11 Max, 12 Max, 13 Min, 13 Max, 14 Min, 14 Max, 16 Max, 17 Min, 17 Max, 18 Min, 18 Max, 19 Min, 20 Max, 21 Max, and 23 Max) on the ISN proposed solar minima or maxima that were apparently driven by the alignments of Sun-SSB and the effective planetary barycenters, and they were not modulated by any alignment of gas giants that aligns with the Sun at all. These peaks and troughs of the sunspots apparently were subliminally driven by the alignment of nonmaterial objects that periodically aligns with the Sun. In 31 of the 48 total cases, they demonstrated the ISN proposed solar minima or solar maxima, were dominantly driven by the alignments of Sun-SSB that have had periodically aligned with the more effective planetary barycenters. The rest of the cases were having other assortment of effective alignments. …As empirically observed, the magnetic field reversal of the Sun was not caused by the SSB is

> furthermost or being nearest to the center of the Sun in the solar cycles, nor were they consistently caused by any conjunction of the gas giants. The hypothesized spin mechanism that specifically drives the solar minima or solar maxima also could not be verified in this case study. It is still a work in progress on this case study and its extended case studies for some cases of solar minima and maxima.

Although his model is lacking quantitative method, yet he is able to predict solar cycle fluctuations. Moreover, his UVS model seems related to one of us (VC)'s model as proposed long time ago in *Apeiron Journal*, January 2004, called CSV model (Cantorian Superfluid vortex); see [9-10].

Conclusion

Nemesis is a theoretical dwarf star thought to be a companion to our sun. The theory was postulated to explain a perceived cycle of mass extinctions in Earth's history. Scientists speculated that such a star could affect the orbit of objects in the far outer solar system, sending them on a collision course with Earth. While recent astronomical surveys failed to find any evidence that such a star exists, we outline in this article some theoretical findings including our own serendipity finding, suggesting that such a dwarf star companion of the Sun remains a possibility. We also discuss possible theoretical framework to advance further this line of thought. Further observation is therefore recommended.

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Roles of Cymatics & Sound Therapy in Spirituality & Consciousness

Victor Christianto, Kasan Susilo, Florentin Smarandache (2020). Roles of Cymatics & Sound Therapy in Spirituality & Consciousness. *Journal of Consciousness Exploration & Research* 11(1), 100-113

Abstract

Sound is one of the types of waves that can be felt by the sense of hearing. In physics, the definition of sound is something that is produced from objects that vibrate. Objects that produce sound are called sound sources. The sound source will vibrate the molecules into the air around it. Sound is mechanical compression or longitudinal waves that propagate through the medium. This medium or intermediate agent can be liquid, solid, gas. So, sound waves can propagate for example in water, coal, or air. Most sounds are a combination of various vibratory signals composed of harmonic waves, but pure sound can theoretically be explained by oscillating vibrational speed or frequency measured in Hertz (Hz) vibration units and amplitude or loudness of sound with measurements in decibels (dB). The effect sounds can have on us humans is simply astounding. While everyday life becomes stressful, overcrowded and increasingly hectic, we seek ways for inner peace and balance. The power of music, felt through overtone-rich sound therapy and sound massage, can be a solution. No matter how much scientific explanation is behind it, healing through music remains fascinating, even magical. This paper introduces a new field called *cymatics* and sound healing with their plausible impact to spirituality and consciousness research.

Keywords: Cymatics, sound therapy, spirituality, consciousness.

Introduction

Humans hear sounds when sound waves, i.e. vibrations in the air or other medium, reach the human eardrum. The range of sound frequencies that can be heard by the human ear ranges from 20 Hz to 20 kHz at various amplitudes in the response curve. Sounds above 20 kHz are called ultrasonic and below 20 Hz are called infrasound.

Conditions occur and the sound is:

[1] A practicing doctor and well-being therapist, based in Indonesia.

There is a vibrating object (sound source);
There is a medium that propagates sound, as well;
There is a receiver within the range of the sound source; and
We know that there are physical studies about sound waves or resonance.

Sound as a wave has properties similar to those of a wave: Sound can be reflected when sound is applied to hard surfaces, such as rock, cement, iron, glass and zinc surfaces.

Example: Our voices are heard louder in the cave as a result of the sound of the cave walls; and our voices are in a building or music studio that uses no sound.

In addition to experiencing rejection, the sound is reflective. The sound reflection process is similar to the light reflection process. If the reflector wall is far enough away, then a bouncing sound will be heard after the original sound is transmitted (emitted). Bouncing sounds that are heard after the original are called echoes. The echo sounds as clear as the original sound. Echoes can occur on rugged slopes, cliffs and other places.

A mechanical wave is a wave that transmits energy and travels through a medium (air). Sound or sound is one example of a mechanical wave that travels through the air (closely spaced molecules - air molecules).

Waves in physics are known as slow-moving energy, or energy propagation will produce waves. The slowing / moving / traveling of the wave does not change the medium through which it passes, but only moves its energy. This means that according to the laws of physics, all objects have a natural frequency, which is the frequency with which an object can vibrate. When a fork is pressed into a glass of wine, the audible sound produced is the frequency.

Mechanical waves generated by sound or sound will cause *resonance*, so do we know what resonance is? Resonance is an event in which one object vibrates because another object vibrates. For easier understanding then let's take the resonance example of a fork.

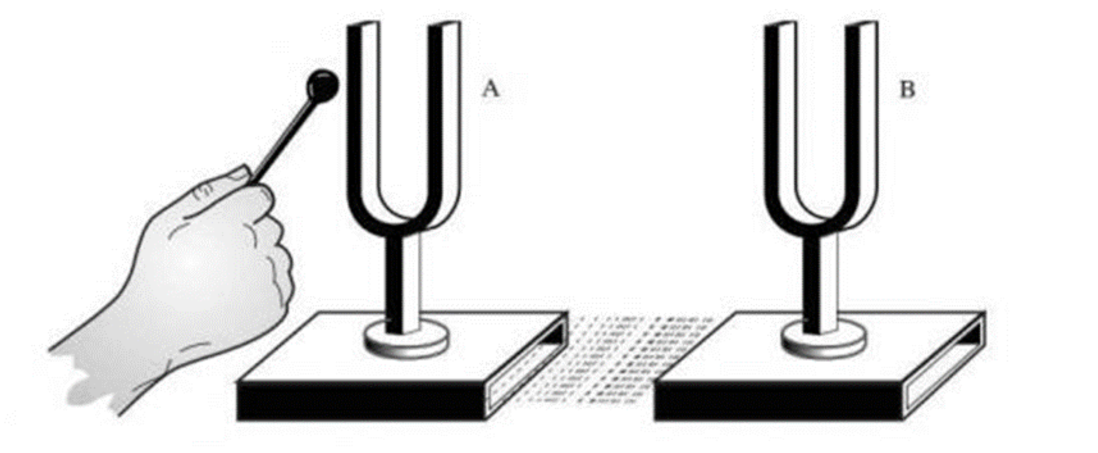

Figure 1. Tuning forks and resonance phenomenon

When we hit the tuning fork A, the tuning fork B will vibrate even when we hold or hold the tuning fork A so that it stops vibrating turns out that the tuning fork B is still vibrating. This is due to the resonance of object B. A simple example in everyday life when we play songs with a loud volume, the glass of our house will vibrate. That is caused by the resonance in the glass of the house because of the sound or sound that can cause other objects to vibrate. Many resonance events occur in everyday life. However, keep in mind that resonance can occur if the frequency of an object is equal to the frequency of the sound source or the sound that causes the object to vibrate.

The myth that develops that if there are voices of crickets, crickets, grasshoppers and other animals that exist in agricultural locations, it will speed up the harvest with good and quality harvests. It turns out that the growing myth has been proven true through research conducted by Dan Carlson. It turns out that after being examined by experts the frequency caused by garengpung is 3247 Hz which is known that this frequency is between 3000 - 5000 Hz. Where is Dan Carlson's sonic bloom[2] which states that sounds with frequencies between 3000 - 5000 Hz can increase plant growth and productivity. This is another example of the resonance method in daily life.

The influence of the singer's voice on the glass was proven in 2005. At that time rock singer Jaime Vendera was recruited by Discovery Channel in the Mythbusters program to test his strong voice.[3] "It took several tries, he was finally able to destroy the wine glass with his voice," wrote Live Science. Well, the human voice is also capable of producing similar natural frequencies that can vibrate glass. This phenomenon, known as resonance, occurs because the sound of a singer moves airborne particles nearby to hit the glass like invisible waves.

Another resonance event - Simple swing.Two pendulum given a rope with the same length of rope, will come to vibrate (swing) if one is swung. The condition for resonance in a simple swing is that the length of the rope must be the same. Swing in the image below, if pendulum A is swung, then pendulum C will participate even though it was not distorted before. Whereas pendulum B and D remain silent not to swing.

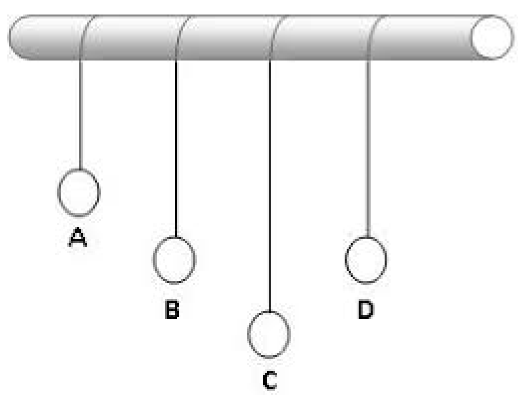

Figure 2. Pendulum and resonance

2 http://dancarlsonsonicbloom.com/
3 http://www.vocalist.org.uk/ultimate-vocal-workout.html

What is cymatics?

Cymatics, the study of wave phenomena and vibration, is a scientific methodology that demonstrates the vibratory nature of matter and the transformational nature of sound. It is sound science, and amazingly cool!

- See sound structure matter - as audible frequencies animate inert powders, pastes and liquids into life-like flowing forms and figures found in the sacred art and architecture of the world's Wisdom Traditions.
- Hear the mellifluous strains of MusiCure music specifically composed and clinically proven in therapeutic environments in Scandinavia, and now available in the USA.
- Feel revitalized as we bathe in the sonic atmosphere that we'll all compose.
- Understand the power we each have to create, sustain and destroy elements in our personal worlds.
- Implement practical approaches to creating greater harmony in life by becoming more aware of those unconscious vibrations that we constantly enliven throughout our day.

Cymatics shows how vibrations interact to create the world we experience *'out there'* and it brings to light hidden principles which underlie all natural processes. Understanding these principles can help us to 'cleanse the lens' through which we perceive our world, thereby clarifying our outlook on life. Once you've objectively observed the rhythmic interplay of chaos and re-integration in simple powder, you may view your own tumultuous circumstances a bit more objectively (and with far less apprehension) as a purposefully evolving process leading toward greater personal coherency and equanimity [1].

Therefore, in essence: Cymatics is the science of sound made visible.

It is based on the principle that when sound encounters a membrane such as your skin or the surface of water, it imprints an invisible pattern of energy. In other words, the periodic vibrations in the sound sample are converted and become periodic water ripples, creating beautiful geometric patterns that reveal the once hidden realm of sound. If we could see the sounds around us with our eyes we would see myriads of holographic bubbles, each with a kaleidoscopic-like pattern its surface. The CymaScope, in a sense, allows us to image a circular section through a holographic sound bubble. Developed by John Stuart Reid in the UK, the CymaScope reveals the once hidden realm of sound. Recently, an eminent physicist, Professor Brian Josephson, visited CymaScope lab.

Professor Brian Josephson, Nobel Laureate in physics, said of the CymaScope instrument to John Stuart Reid,“Having watched one of your lectures I think your (re) discovery is going to be of great importance to the future of physics”.

Subsequently, in his lecture at the Royal Society of Medicine, in July 2018, Professor Josephson presented a CymaScope video that may show water’s ability to remember a sonic input frequency.

He said, “Water exhibits remarkable structural and dynamic properties, including the ‘biological signal’ revealed by the investigations of Beneviste and Montagnier and the complex acoustically-induced structures in water revealed by the CymaScope. Organised dynamical behaviour is more the province of biology than of physics and will require different tools of investigation than are standard in physics. The CymaScope may be one such tool. It is not just a new scientific instrument but new science as well and I suspect a new field of maths.”

In 2019, Brian Josephson visited the CymaScope lab where John Stuart Reid showed several videos, including that of submerged air bubbles in water, excited by low frequency sound, which exhibit life-like behaviour in that they appear to chase each other around the CymaScope’s cuvette while a cymatic pattern forms on their spheroidal surfaces. The experiment was designed to begin to shed light on the origin of life, around hydrothermal vents in the primordial oceans. The research video captured Josephson’s interest and he commented, “This may help to clarify the way intelligence emerges in nature”. It was shown at the *Water Conference* in October 2019, at Bad Solen, Germany, where Reid presented on the subject of abiogenesis (the origins of life), among other topics [3].

Figure 3. Prof. Brian Josephson with John Stuart Reid in CymaScope Lab (source: [3])

A musical piece created by Professor Josephson entitled, “*Sweet and Sour Harmony*” is to be made visible in water, by CymaScope instrument, as a future project that marries art and science.[4]

Sound Healing & Prayer

Even in this modern era, sound is apparently still being used in positive efforts for human life. Many clinics and therapists have begun to use sound and music as a method of healing, even by exposing sounds to their patients. Not only in the field of health, education practitioners also began to realize that sound can affect brain activity, accelerate the learning process, even overcome disturbances in the learning process. Utilization of sound for human good is the modern term "sound healing". The basic principle of sound healing is vibration, said Jonathan Goldman, author of *Healing sounds*.[4] "Every object on earth shakes, thereby making a sound," Goldman said.

These objects include parts of our body, such as muscles, bones, internal organs and so on. " "When parts of our body vibrate / vibrate with the right frequency, we call this state healthy. But if there are parts of our body that vibrate at frequencies that don't match our bodies, we call that condition sick, "Goldman said. Good sound produced by human vocal cords or special musical instruments is used to maintain / restore the vibrations of the body parts in order to return to harmony, vibrate with the frequency of the body in a healthy condition.

The same thing was expressed by Kate Marks in her writing entitled "*Sound can Heal or Destroy*"[5], she said, "vibrations affect all cells and molecules in our body." She even added that vibrations in our bodies keep changing all the time. "We emit vibrations that affect the people around us, so do we receive vibrations from the environment around us. This ongoing exchange of vibrations affects electromagnetic energy in and around us, "That's why according to Marks, just like a musical instrument, our body also needs to be "tuned" regularly (tuning) so that all parts of our body emit harmonious vibrations and make a beautiful symphony.

Alfred Tomatis, a French doctor, made a fifty-year experiment on the human senses and came up with the result that the sense of hearing is the most important sense! He discovered that the ear controls the whole body, regulates its vital operations and the balance and coordination of its movements. He also found that the ear controls the nervous system. During his experiments, he discovered that the auditory nerve is connected to all the muscles of the body and this is the reason why the balance and flexibility of the body and the sense of sight are affected by sound. The inner ear is connected to all parts of the body such as the heart, lungs, liver, stomach and intestines; this explains why sound frequencies affect the whole body.

In 1960, Swiss scientist Hans Jenny discovered that sound affected various materials and renewed their particular parts, and that every cell of the body had its own voice and would be affected by the renewal of sound and the material within it. In 1974, researchers Fabien Maman and Sternheimer announced a very surprising discovery; They found that every part of the body has its own vibration system, according to the laws of physics. A few years later, Fabien and Grimal, other researchers, found that sounds affect cells, especially cancer cells, and that certain sounds have a strong influence; the miraculous thing found by the two researchers is that the voice that has the most powerful effect on the body's cells is the human voice itself!! Sound moves from the ear to the brain and affects brain cells; scientists have recently discovered that sound has miraculous healing powers and the amazing effects of brain cells that restore

[4] https://www.healingsounds.com/

[5] http://spiritweb.dk/artikler/sound-can-heal-or-destroy-it-must-be-used-wisely/

balance to the whole body! Reading the Bible has an extraordinary effect on brain cells and is able to restore its balance; the brain is the organ that controls the body and from here commands are sent to all organs of the body especially the immune system.

Fabien, a scientist and musician, places cells from a healthy body and applies them to various sounds; He found that each musical scale note affected the electromagnetic fields of the cell; when photographing this cell with a Kirlian camera, he found that the shape and value of the electromagnetic fields of the cell changed according to the sound frequency and type of the reader's voice. Then he conducted another experiment by taking a drop of blood from one patient; and then monitored the blood drop with a Kirlian camera and asked the patient to release various tones. He discovered, after processing the drawing, that certain tones in the blood droplets changed their electromagnetic fields and fully vibrated in response to their owners. He then concluded that there are certain tones that affect the body's cells and make them more vital and active, even renewing them. He came up with the important result that the human voice has a powerful and unique influence on the body's cells; this influence was not found in other instruments. This researcher says literally:

> The human voice has a special ring that makes it the most powerful treatment tool. Fabien found that some sounds easily destroy cancer cells, and at the same time activate healthy cells. Sound affects human blood cells which transfer this sound frequency throughout the body through blood circulation.

From the results of all this research there is a correlation with the Bible where as we know that praying is by saying the words. Means making sounds or sounds, and according to the research it will release energy. The energy can affect the surrounding objects as in the analysis of tuning forks and in simple research. Likewise in research on living things, energy is very influential, for more details in the following paragraph about water and blood will be explained.

In everyday life we sometimes do it, sometimes there are distant friends who talk about us so we don't feel it, instead we sometimes contact them and they just say we feel. This shows that the words we issue affect other people, especially if we pray with faith, are sure and persevere in praying it will surely be successful or if we help others as written in the Bible we pray not to be bored, meaning the energy of prayer it is done continuously but it all depends on the will of God to allow our time is not God's time, but certainly as in the study the *sound waves from our prayers will definitely affect*. There is research about water and blood molecules which may also be related to our prayers toward others and ourselves.

The results of Masuro Emoto's research, "*The True Power of Water*", is an amazing experience because it proves that water turns out to be "alive" and can respond to what humans say.[6] Emoto succeeded in proving that water is capable of carrying messages or information from what is given to him. Even water that is given a positive response, including prayer, will produce beautiful hexagonal crystal shapes.

[6] https://www.goodreads.com/book/show/38391.The_True_Power_of_Water

Emoto did research for 2 months with his best friend Kazuya Ishibashi (a scientist who is skilled in using microscopes). Masaru, who completed his education at Yokohama Municipal University Department of Humanities and Science, majoring in International Relations, managed to get a photograph of water crystals by freezing water at -25 degrees Celsius and using a high-speed photo tool. Then the water is examined using the response of words, images, and sounds. The results are amazing, as many people have read. Water, he said, can receive messages. Even in another book, "*The Hidden Message in Water*", Masaru said, water is like a magnetic tape or compact disk.[7]

Water recognizes the word not only as a simple design, but water can understand the meaning of the word. When water realizes that the word shown carries good information, water will form crystals. If a positive word is given, the crystals that are formed will blossom as extraordinary as a flower that is in full bloom, as if to describe the movements of a hand of water that is expressing its pleasure.

Conversely, if negative words are given, it will produce crystal fragments with an unbalanced size. It is also possible that water can feel the feelings of the person who wrote the word. So you can imagine what if the water was given a collection of words that are prayers? The human body is made up of 70% water, so it's not strange that our bodies react the fastest with our words that come out of our own mouths. If water molecules can record and react with good words, let alone our more complex organs, such as the heart and brain. So it is not wrong if we are encouraged to always say good and avoid vain words let alone bad words. Even good words can have a positive impact on our body.

Emoto was born July 22, 1943 and died on October 17, 2014, he was a researcher from the Hado Institute in Tokyo, Japan in 2003 who through his research revealed an oddity in the nature of water. Through his observation of more than two thousand examples of water crystal photographs collected from various parts of the world, Emoto found that water molecule particles can actually change depending on human feelings around them, which indirectly hints at the influence of feelings on the clustering of water molecules formed by the presence of bonds hydrogen.

Emoto also found that water crystal particles look "beautiful" and "amazing" when they get positive reactions around them, for example with excitement and happiness. But water crystal particles appear to be "bad" and "unsightly" if they have negative effects around them, such as sadness and disaster. More than two thousand photographs of water crystals are found in the book Hidden Message from Water, which he composed as proof of his conclusions so that this has the opportunity to become a breakthrough in believing in natural wonders. *Emoto concluded that water particles can be influenced by the sound of music, prayers and words written and dipped in the water.*

Until now Emoto and his work are still considered controversial. Ernst Braun of Burgistein in Thun, Switzerland, has tried in his laboratory the method of making crystal photographs as

[7] https://www.amazon.com/Hidden-Messages-Water-Masaru-Emoto/dp/0743289803

revealed by Emoto, unfortunately these results cannot be reproduced again, even under the same experimental conditions.

Figure 4. Emoto's results[8]

Blood Molecules

The blood composition consists of several types of korpuskula that form 45% of the blood, this figure is expressed in the value of Hermatocrit or the compacted red blood cell volume ranging from 40 to 47. Part 55% of the other is yellowish fluid that forms the medium of blood fluid called blood plasma. The corpuscus of blood consists of: Red blood cells or erythrocytes (about 99%).

This research was conducted by Rebecca Marina with Dr. Ferici who observed Rebecca's own blood cell behavior in each of the different emotional conditions that she felt (such as: feelings of love, fear, love / beauty, when praying), and strengthened by the potential of EFT (Emotional Freedom Technique) that is being studied for its effectiveness; blood samples are taken using the "Darkfield Microscope" which is connected to a Computer Monitor.

The following is a picture of Rebecca's blood in various emotional conditions that are felt:

1. Condition of blood cells when sick: Red blood cells clot and patches can occur on each blood cell that intersect.

2. Blood cells when healthy: Red blood cells become normal.

[8] https://www.goodreads.com/book/show/38391.The_True_Power_of_Water

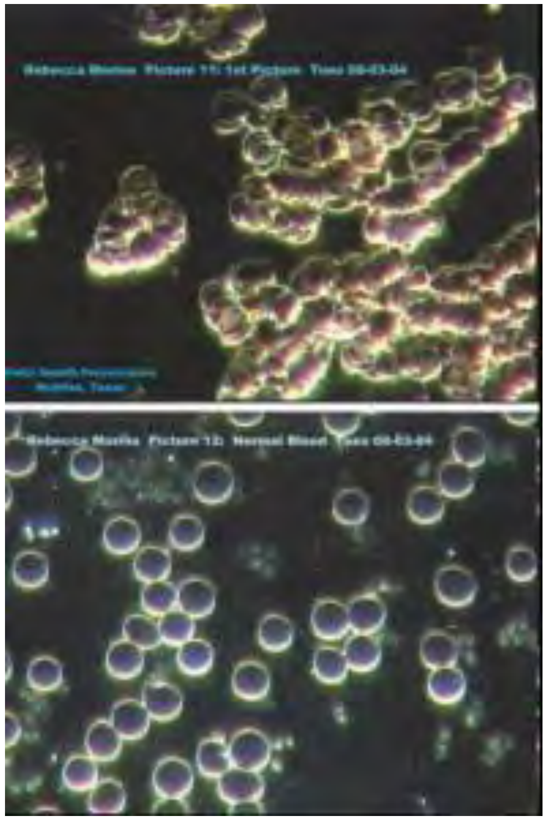

Figure 5. Blood molecules. (above: blood molecule of sick person, below: healthy person)

3. The condition of red blood cells when sad Red blood cells move quickly and form tears.

Figure 6. Blood molecules

4. The condition of red blood cells when emotions are full of love. The shape and formation of blood cells become normal, move beautifully and in an orderly fashion, sparkling substances arise in the blood fluid.

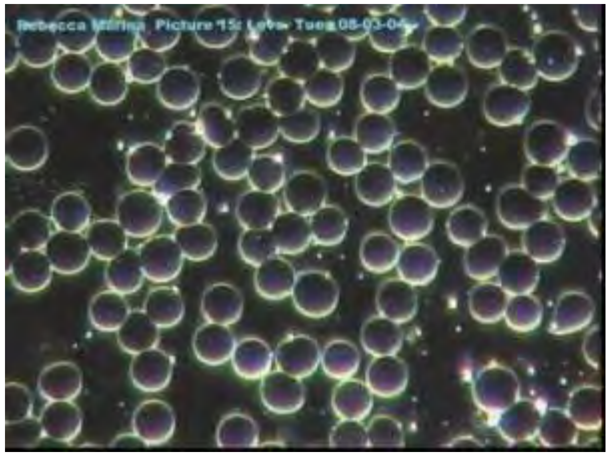

Figure 7. Blood molecules

5. The condition of red blood cells when feeling anxiety, and fear. Blood cells move irregularly very quickly.

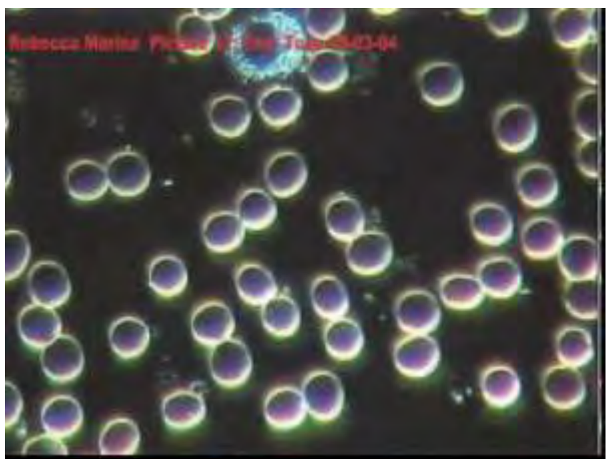

Figure 8. Blood molecules

6. The Condition of Red Blood Cells when a prayer is Prompted. What happened ...?, All those present in the laboratory were immediately speechless and stunned, because seeing the condition of the blood which was completely different from the others: Blood fluid is very bright.

   ♣ Regular red blood cell movements are very calm, as if moving peacefully.

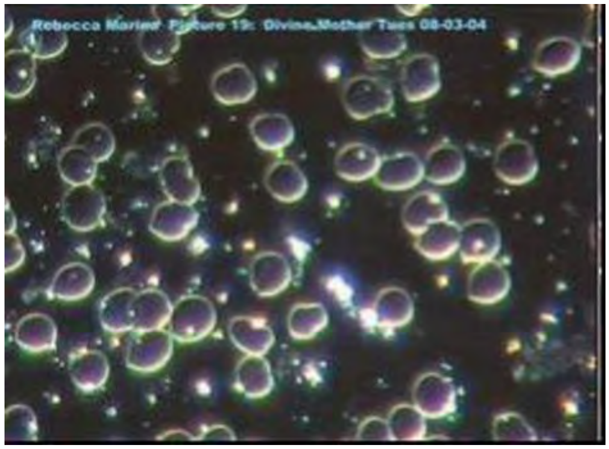

Figure 9. Blood molecules

♣ In the blood fluid appears sparkling substance.

Figure: blood molecule condition while a person is praying: In red blood cells there is a substance that glows and pulses like a heartbeat. After completing the research and when leaving the laboratory it was observed that there was a "miracle" that occurred namely: other blood samples had "stopped moving", except for blood samples that were given a prayer.

A folk-tale story: There are folklore about words - There is one habit that we can find in people who live around the Solomon Islands, which are located in the South Pacific, which is shouting at trees. This is what they can do if there are trees with very strong roots that are difficult to cut with an ax. They berate the tree with the aim that the tree died. The trick is, some residents who are stronger and braver will climb up to the top of the tree. Then, when they reach the top of the tree together with the people who are under the tree, they will shout at the top of their lungs to the tree. They shouted for hours, for about forty days. And, what happened was truly amazing. The tree that was shouted at by the negative words slowly began to dry out. After that the branches will also begin to fall out and slowly the tree will die and thus, it is easy to uproot.

If we look at what this primitive population did it was really strange. But we can learn one thing from them. They have proven that negative screams made against certain living things such as trees will cause the tree to lose its spirit. As a result, in a not too long time, a living creature or tree will die. Now, we have a very valuable lesson from the habits of the primitive population of the Solomon Islands. So remember well that *every time we shout with negative words at certain living creatures, it means we are killing the spirit*.

Remember, every time we scream at someone for being annoyed, angry, upset or other negative emotions, it will make others hurt and hurt and without us knowing it can kill the spirits of our loved ones. We also kill the spirit that links our relationship. The shouts, which we make out because our emotions slowly, will eventually kill the spirit (deadly feelings) that have embedded our relationship.

It would be nice if we talk carefully or discuss about what our expectations. Shouts or negative words that we say with uncontrolled emotions actually damage the atmosphere, making other people's hearts farther away from our hearts. Instead of being negative to others, positive words that we say with full sincerity will make the hearts of others closer to our hearts.

That's why angry and emotional people talk loudly or shouting when their distance is only a few dozen centimeters and it should be easy to explain. But even though they are physically close, their hearts are actually so far away that they have to shout to each other. So from now on every unpleasant thing happens, we quickly control our thoughts and emotions by shifting our main focus to the pleasant thing, shifting the discomfort to the feeling of comfort.

We must be able to control our thoughts and emotions in a balanced way, so that what we say is something positive (both for us and others). So that other people know our intentions and goals, there is no need to emotionally scream - screaming that ultimately makes us increasingly away from others. This does not mean we always have to yield and we cannot be firm. But we have to adjust, even though we are in the right position and we have the authority to reprimand or give allegations to others but not necessarily emotionally.

Conclusions

Based on physical science research and also on water and blood molecules it may have something to do with what we do in prayer especially the *intercessory prayer* for others and ourselves. Where we pray we say words whose words are the sounds that will produce a wave of energy that will affect ourselves and others. From all this to the point of undermining our faith, the author wants to show that prayer has power when it comes to sounding like a theory in physics and also affecting humanity, because the *essence of prayer is hope, request, praise to God*. While praying means to say (lift up) a prayer to God. It means prayer is an application directed to God in which there is hope, request and praise.

It is also important to remember that the Bible has been around since the early time of mankind history, but knowledge has developed after the time of AD. This shows that the Bible is a source of knowledge that is impossible but real. Then we as Christians must believe whatever is written in the Bible eventhough it is impossible. We humans are given the ability to think because of the grace of God by giving our brains to respond to the problem by acquiring the wisdom of God in uncovering the biblical mystery that is the word of God given to us and that nothing cannot be done if we really listen, reflect and also do everything as it is mentioned in the Bible.

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Remark on Lehnert's Revised Quantum Electrodynamics (RQED) as an Alternative to Francesco Celani's *et al.* Maxwell-Clifford equations: With an Outline of Chiral Cosmology model and its role to CMNS

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Abstract

In a recent paper published in *JCMNS* in 2017, Francesco Celani, Di Tommaso & Vassalo argued that Maxwell equations rewritten in Clifford algebra are sufficient to describe the electron and also ultra-dense deuterium reaction process proposed by Homlid *et al.* Apparently, Celani *et al.* believed that their Maxwell-Clifford equations are an excellent candidate to surpass both Classical Electromagnetic and *Zitterbewegung* QM.

Meanwhile, in a series of papers, Bo Lehnert proposed a novel and revised version of Quantum Electrodynamics (RQED) based on Proca equations. Therefore, in this paper, we gave an outline of Lehnert's RQED, as an alternative framework to Celani *et al.* Zitterbewegung-Classical EM. Moreover, in a rather old paper, Mario Liu described hydrodynamic Maxwell equations. While he also discussed potential implications of these new approaches to superconductors, such a discussion of electrodynamics of superconductors is made only after Tajmar's paper. Therefore, in this paper we present for the first time a derivation of fluidic Maxwell-Proca equations. The name of *fluidic Maxwell-Proca* is proposed because the equations were based on modifying Maxwell-Proca and Hirsch's theory of electrodynamics of superconductors. It is hoped that this paper may stimulate further investigations and experiments in superconductors. It may be expected to have some impact to cosmology modeling too, for instance we consider a hypothetical argument that the photon mass can be the origin of gravitation. Then, after combining with the so-called chiral modification of Maxwell equations (after Spröessig), we consider chiral Maxwell-Proca equations as a possible alternative of gravitation theory. Such a hypothesis has never considered in the literature to the best of our knowledge. In the last section, we discuss the plausible role of chiral Maxwell-Proca (RQED) in CMNS process. It is hoped that this paper may stimulate further investigations and experiments in particular for elucidating the physics of LENR and UDD reaction from classical electromagnetics.

Key Words: Maxwell equations, Proca equations, LENR, Revised QED, Hirsch theory, London equations, hydrodynamics Maxwell equations, Proca equations, electrodynamics of superconductor, chiral medium, chiral gravitation theory.

1. Introduction

In a recent paper published in *JCMNS* in 2017, Francesco Celani, Di Tommaso & Vassalo argued that Maxwell equations rewritten in Clifford algebra are sufficient to describe the electron and also ultra-dense deuterium reaction process proposed by Homlid *et al.* Apparently, Celani *et al.* believed that their Maxwell-Clifford equations are quite excellent candidate to surpass both Classical Electromagnetics theory and *Zitterbewegung* QM.[1]

In the meantime, it is known that conventional electromagnetic theory based on Maxwell's equations and quantum mechanics has been successful in its applications in numerous problems in physics, and has sometimes manifested itself in a good agreement with experiments. Nevertheless, as already stated by Feynman, there are unsolved problems leading to difficulties with Maxwell's equations that are not removed by and not directly associated with quantum mechanics [17-20]. Therefore QED, which is an extension of Maxwell's equations, also becomes subject to the typical shortcomings of electromagnetic in its conventional form. This reasoning makes a way for Revised Quantum Electrodynamics as proposed by Bo Lehnert.[17-19]

Meanwhile, according to J.E. Hirsch, from the outset of superconductivity research it was assumed that no electrostatic fields could exist inside superconductors and this assumption was incorporated into conventional London electrodynamics.[23] Hirsch suggests that there are difficulties with the two London equations. To summarize, London's equations together with Maxwell's equations lead to unphysical predictions.[22] Hirsch also proposes a new model for electrodynamics for superconductors. [22-23]

In this regard, in a rather old paper, Mario Liu described a hydrodynamic Maxwell equations.[25] While he also discussed potential implications of these new approaches to superconductors, such a discussion of electrodynamics of superconductors is made only after Tajmar's paper. Therefore, in this paper we present for the first time a derivation of fluidic Maxwell-Proca-Hirsch equations. The name of Maxwell-Proca-Hirsch is proposed because the equations were based on modifying Maxwell-Proca and Hirsch's theory of electrodynamics of superconductor. Therefore, the aim of the present paper is to propose a version of fluidic Maxwell-Proca model for electrodynamics of superconductor, along with an outline of a chiral cosmology model.

This may be expected to have some impact to cosmology modeling too, which will be discussed in the last section. It is hoped that this paper may stimulate further investigations and experiments in particular for fractal superconductor.

2. Lehnert's Revised Quantum Electrodynamics

In a series of papers Bo Lehnert proposed a novel and revised version of Quantum Electrodynamics, which he refers to as RQED. His theory is based on the hypothesis of a nonzero electric charge density in the vacuum, and it is based on *Proca-type* field equations [20, p. 23]:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)A_\mu=\mu_0 J_\mu, \mu=1,2,3,4 \tag{1}$$

Where

$$A_\mu=\left(A,\frac{i\phi}{c}\right), \tag{2}$$

With A and ϕ standing for the magnetic vector potential and the electrostatic potential in three-space. In three dimensions, we got [20, p.23]:

$$\frac{curlB}{\mu_0}=\varepsilon_0\left(divE\right)C+\frac{\varepsilon_0\partial E}{\partial t}, \tag{3}$$

$$curlE=-\frac{\partial B}{\partial t}, \tag{4}$$

$$B=curlA, divB=0, \tag{5}$$

$$E=-\nabla\phi-\frac{\partial A}{\partial t}, \tag{6}$$

$$divE=\frac{\bar{\rho}}{\varepsilon_0}. \tag{7}$$

These equations differ from the conventional form with a nonzero electric field divergence equation (7) and by the additional space-charge current density in addition to displacement current at equation (3). The extended field equations (3)-(7) are easily found also to become invariant to a gauge transformation.[20, p.23]

The main characteristic new features of the present theory can be summarized as follows [20, p. 24]:

a. The hypothesis of a nonzero electric field divergence in the vacuum introduces an additional degree of freedom, leading to new physical phenomena. The associated nonzero electric charge density thereby acts somewhat like a hidden variable.
b. This also abolishes the symmetry between the electric and magnetic fields, and then the field equations obtain the character of intrinsic linear symmetry breaking.
c. The theory is both Lorentz and gauge invariant.
d. The velocity of light is no longer a scalar quantity, but is represented by a velocity vector of the modulus c.
e. Additional results: Lehnert is also able to derive the mass of Z boson and Higgs-like boson.[21] These would pave an alternative way to new physics beyond the Standard Model.

Now it should be clear that Lehnert's RQED is a good alternative theory to QM/QED, and therefore it is also interesting to ask whether this theory can also explain some phenomena related to LENR and UDD reaction proposed by Homlid (as argued by Celani *et al.*).[1]

It should be noted too, that Proca equations can be considered as an extension of Maxwell equations, and they have been derived in various ways. It can be shown that Proca equations can be derived from first principles, and also that Proca equations may have a link with the Klein-Gordon equation.[6][7]

One persistent question concerning these Proca equations is how to measure the mass of the photon. This question has been discussed in lengthy by Tu, Luo & Gillies [12]. According to their report, there are various methods to estimate the upper bound limits of photon mass. In Table 1 below, some of upper bound limits of photon mass based on dispersion of speed of light are summarized.

Table 1. *Upper bound on the dispersion of the speed of light in different ranges of the electromagnetic spectrum, and the corresponding limits on the photon mass*. [12, p.94]

Author (year)	Type of measurement	Limits on m_γ (g)
Ross et al. (1937)	Radio waves transmission overland	5.9×10^{-42}
Mandelstam & Papalexi (1944)	Radio waves transmission over sea	5.0×10^{-43}
Al'pert et al. (1941)	Radio waves transmission over sea	2.5×10^{-43}
Florman (1955)	Radio-wave interferometer	5.7×10^{-42}
Lovell et al. (1964)	Pulsar observations on flour flare stars	1.6×10^{-42}
Frome (1958)	Radio-wave interferometer	4.3×10^{-40}
Warner et al. (1969)	Observations on Crab Nebula pulsar	5.2×10^{-41}
Brown et al. (1973)	Short pulses radiation	1.4×10^{-33}
Bay et al. (1972)	Pulsar emission	3.0×10^{-46}
Schaefer (1999)	Gamma ray bursts	4.2×10^{-44}
	Gamma ray bursts	6.1×10^{-39}

From this table and also from other results as reported in [12], it seems that we can expect that someday photon mass can be observed within experimental bounds.

3. Hirsch's proposed revision of London's equations

According to J.E. Hirsch, from the outset of superconductivity research it was assumed that no electrostatic fields could exist inside superconductors, and this assumption was incorporated into conventional London electrodynamics.[22] Hirsch suggests that there are difficulties with the two London equations. Therefore he concludes that London's equations together with Maxwell's equations lead to *unphysical* predictions.[1] However he still uses four- vectors J and A according to Maxwell's equations:

$$\square^2 A = -\frac{4\pi}{c} J, \tag{8}$$

and

$$J - J_0 = -\frac{c}{4\pi\lambda_L^2}(A - A_0). \tag{9}$$

Therefore, Hirsch proposes a new fundamental equation for electrodynamics for superconductors as follows: [22]

$$\square^2 (A - A_0) = \frac{1}{\lambda_L^2}(A - A_0), \tag{10a}$$

where

- London penetration depth λ_L is defined as follows:[23]

$$\frac{1}{\lambda_L^2} = \frac{4\pi n_s e^2}{m_e c^2}, \tag{10b}$$

- And d'Alembertian operator is defined as: [22]
-

$$\square^2 = \nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial_t^2} \ . \tag{10c}$$

Then he proposes the following equations: [22]

$$\square^2 (F - F_0) = \frac{1}{\lambda_L^2}(F - F_0), \tag{11}$$

and

$$\square^2 (J - J_0) = \frac{1}{\lambda_L^2}(J - J_0), \tag{12}$$

where F is the usual electromagnetic field tensor and F_0 is the field tensor with entries $\vec{E}_0$ and 0 from $\vec{E}$ and $\vec{B}$ respectively when expressed in the reference frame at rest with respect to the ions.

In the meantime, it is known that Proca equations can also be used to describe electrodynamics of superconductors, see [25-33]. The difference between Proca and Maxwell equations is that Maxwell equations and Lagrangian are based on the hypothesis that the photon has zero mass, but the Proca's Lagrangian is obtained by adding mass term to Maxwell's Lagrangian.[33] Therefore, the Proca equation can be written as follows:[33]

$$\partial_\mu F^{\mu\nu} + m_\gamma{}^2 A_\nu = \frac{4\pi}{c} J^\nu, \tag{13a}$$

where $m_\gamma = \frac{\omega}{c}$ is the inverse of the Compton wavelength associated with photon mass. [38] (Note: 'omega' is the angular frequency = 2 pi f, where f is the frequency, the definition of omega involves the "reduced" Compton wavelength, corresponding with the reduced Planck constant 'ĥ'.) It is also clear that in the particular case of m = 0, then that equation reduces to the Maxwell equation. In terms of the vector potentials, equation (13a) can be written as [33]:

$$(\square + m_\gamma) A_\mu = \frac{4\pi}{c} J_\mu. \tag{13b}$$

Similarly, according to Kruglov [31] the Proca equation for a free particle processing the mass m can be written as follows:

$$\partial_{\nu}\varphi_{\mu\nu}(x)+m^{2}\varphi_{\mu}(x)=0, \tag{14}$$

Now, the similarity between equations (8) and (13b) are remarkable, with the exception that equation (8) is in quadratic form. Therefore we propose to consider a modified form of Hirsch's model as follows:

$$(\square^{2}-m_{\gamma}^{2})(F-F_{0})=\frac{1}{\lambda_{L}^{2}}(F-F_{0}), \tag{15a}$$

and

$$(\square^{2}-m_{\gamma}^{2})(J-J_{0})=\frac{1}{\lambda_{L}^{2}}(J-J_{0}). \tag{15b}$$

The relevance of the proposed new equations in lieu of (11)-(14) should be verified by experiments with superconductors [37]. For convenience, the equations (15a)-(15b) can be given a name: *Maxwell-Proca-Hirsch equations.*

4. Fluidic Maxwell-Proca Equations

In this regard, in a rather old paper, Mario Liu described a hydrodynamic Maxwell equations.[24] While he also discussed the potential implications of these new approaches to superconductors, such a discussion of electrodynamics of superconductors is made only after Tajmar's paper. Therefore, in this section we present for the first time a derivation of *fluidic* Maxwell-Proca-Hirsch equations.

According to Blackledge, Proca equations can be written as follows [7]:

$$\nabla\cdot\vec{E}=\frac{\rho}{\varepsilon_{0}}-\kappa^{2}\phi, \tag{16}$$

$$\nabla\cdot\vec{B}=0, \tag{17}$$

$$\nabla\times\vec{E}=-\frac{\partial\vec{B}}{\partial t}, \tag{18}$$

$$\nabla\times\vec{B}=\mu_{0}j+\varepsilon_{0}\mu_{0}\frac{\partial\vec{E}}{\partial t}+\kappa^{2}\vec{A}, \tag{19}$$

Where (without assuming Planck constant, ĥ=1):

$$\nabla\phi = -\frac{\partial\vec{A}}{\partial t} - \vec{E}, \tag{20}$$

$$\vec{B} = \nabla\times\vec{A}, \tag{21}$$

$$\kappa = \frac{mc_0}{\hbar}. \tag{22}$$

Therefore, by using the definitions in equations (16)-(19), and by comparing with hydrodynamic Maxwell equations of Liu [24, eq. 2], now we can arrive at fluidic Maxwell-Proca equations, as follows:

$$\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0} - \kappa^2\phi, \tag{23}$$

$$\nabla\cdot\vec{B} = 0, \tag{24}$$

$$\dot{B} = -\nabla\times E - \nabla\times\left(\hat{\beta}\nabla\times H_0\right), \tag{25}$$

$$\varepsilon_0\mu_0\dot{E} = \nabla\times B - \mu_0 j - \kappa^2 A - \left(\hat{\sigma}E_0 + \rho_e v + \hat{\gamma}\nabla T\right) - \nabla\times\left(\hat{a}\nabla\times E_0\right), \tag{26}$$

where:

$$\nabla\phi = -\frac{\partial\vec{A}}{\partial t} - \vec{E}, \tag{27}$$

$$\vec{B} = \nabla\times\vec{A}, \tag{28}$$

$$\kappa = \frac{mc_0}{\hbar}. \tag{29}$$

Since according to Blackledge, the Proca equations can be viewed as a *unified wavefield* model of electromagnetic phenomena [7], therefore we can also regard the fluidic Maxwell-Proca equations as a *unified wavefield* model for electrodynamics of superconductor.

Now, having defined fluidic Maxwell-Proca equations, we are ready to write fluidic Maxwell—Proca equations using the same definition, as follows:

$$(\square_\alpha^2 - \kappa^2)(F - F_0) = \frac{1}{\lambda_L^2}(F - F_0), \tag{30}$$

And

$$(\square_\alpha^2 - \kappa^2)(J - J_0) = \frac{1}{\lambda_L^2}(J - J_0), \tag{31}$$

where

$$\square_{\alpha}^{2} = \nabla^{\alpha 2} - \frac{1}{c^{2}} \frac{\partial^{\alpha 2}}{\partial_{t}^{\alpha 2}}. \tag{32}$$

As far as we know, the above fluidic Maxwell-Proca equations have never been presented elsewhere before. Provided the above equations can be verified with experiments, they can be used to describe electrodynamics of superconductors.

As a last note, it seems interesting to remark here that Kruglov [31] has derived a square-root of Proca equations as a possible model for hadron mass spectrum, therefore perhaps equations (30)-(32) may be *factorized* too to find out a model for hadron masses. However, we leave this problem for future investigations.

5. Towards Chiral Cosmology model

The Maxwell-Proca electrodynamics corresponding to a finite photon mass causes a substantial change of the Maxwell stress tensor and, under certain circumstances, may cause the electromagnetic stresses to act effectively as "*negative pressure*." In a recent paper, Ryutov, Budker, Flambaum [34] suggest that such a negative pressure imitates gravitational pull, and may produce an effect similar to gravitation. In the meantime, there are other papers by Longo, Shamir etc. discussing observations indicating handedness of spiral galaxies, which seem to suggest chiral medium at large scale. However, so far there is no derivation of Maxwell-Proca equations in chiral medium.

In a recent paper, Ryutov, Budker, Flambaum suggest that Maxwell-Proca equations may induce a negative pressure imitates gravitational pull, and may produce effect similar to gravitation.[34]

In the meantime, there are other papers by Longo, Shamir etc. discussing observations indicating handedness of spiral galaxies, which seem to suggest chiral medium at large scale. As Shamir reported:

> "*A morphological feature of spiral galaxies that can be easily identified by the human eye is the handedness—some spiral galaxies spin clockwise, while other spiral galaxies rotate counterclockwise. Previous studies suggest large-scale asymmetry between the number of galaxies that rotate clockwise and the number of galaxies that rotate counterclockwise, and a large-scale correlation between the galaxy handedness and other characteristics can indicate an asymmetry at a cosmological scale*."[40]

However, so far there is no derivation of Maxwell-Proca equations in a chiral medium. Therefore, inspired by Ryutov *et al.* paper, in this paper, we present for the first time a possibility to extend Maxwell-Proca-type equations to chiral medium, which may be able to explain origin of handedness of spiral galaxies as reported by M. Longo *et al.*[39-40]

The present paper is intended to be a follow-up paper of our preceding paper, reviewing Shpenkov's interpretation of classical wave equation and its role to explain periodic table of elements and other phenomena [38].

6. Maxwell-Proca Equations in Chiral Medium

Proca equations can be considered as an extension of Maxwell equations, and they have been derived in various ways. It can be shown that Proca equations can be derived from first principles [6], and also that Proca equations may have link with Klein-Gordon equation [7].

It should be noted that the relations between flux densities and the electric and magnetic fields depend on the material. It is well known, for instance, that all organic materials contain carbon and realize in this way some kind of optical activity. Therefore, Lord Kelvin introduced the notion of the chirality measure of a medium. This coefficient expresses the optical activity of the underlying material. The correspondent constitutive laws are the following:[35]

$$D = \varepsilon E + \varepsilon\beta \text{ rot } E \text{ (Drude-Born-Feodorov laws),} \tag{33}$$

$$B = \mu H + \mu\beta \text{ rot } H \tag{34}$$

where e = E(t, x) is the electric permittivity, j = p(t, x) is the magnetic permeability and the coefficient β describes the *chirality measure* of the material.[35]

Now, since we want to obtain Maxwell-Proca equations in chiral medium, then eq. (28) should be replaced with eq. (34). But such a hypothetical assertion should be investigated in more detailed.

Since according to Blackledge, the Proca equations can be viewed as a unified wavefield model of electromagnetic phenomena [7], then we can also regard the Maxwell-Proca equations in chiral medium as a further generalization of his *unified wavefield picture.*

7. Plausible role of chiral superconductor model to LENR/CMNS

According to R.M. Kiehn, chirality already arises in electromagnetic equations, i.e. Maxwell equations:[41]

> "*From a topological viewpoint, Maxwell's electrodynamics indicates that the concept of Chirality is to be associated with a third rank tensor density of Topological Spin induced by the interaction of the 4 vector potentials {A, φ } and the field excitations (D,H). The distinct concept of Helicity is to be associated with the third rank tensor field of Topological Torsion induced by the interaction of the 4 vector potentials and field intensities (E,B). ...*
>
> *In the electromagnetic situation, the constitutive map is often considered to be (within a factor) a linear mapping between two six dimensional vector spaces. As such the constitutive map can have both a right- or a left-handed representation, implying that there are two topologically equivalent states that are not smoothly equivalent about the identity.*"

Therefore, here we will review some models of chirality in superconductors and other contexts, in hope that we may elucidate the chirality origin of spiraling wave as considered by Celani *et al.* for explaining UDD reaction (cf. Homlid).

Here, we summarize some reports on chirality as observed in experiments:

(a) F. Qin *et al*. reported "Superconductivity in a chiral nanotube."[42] Their abstract goes as follows: "Chirality of materials are known to affect optical, magnetic and electric properties, causing a variety of nontrivial phenomena such as circular dichiroism for chiral molecules, magnetic Skyrmions in chiral magnets and nonreciprocal carrier transport in chiral conductors. On the other hand, effect of chirality on superconducting transport has not been known. Here we report the nonreciprocity of superconductivity—unambiguous evidence of superconductivity reflecting chiral structure in which the forward and backward supercurrent flows are not equivalent because of inversion symmetry breaking. Such superconductivity is realized via ionic gating in individual chiral nanotubes of tungsten disulfide. The nonreciprocal signal is significantly enhanced in the superconducting state, being associated with unprecedented quantum Little-Parks oscillations originating from the interference of supercurrent along the circumference of the nanotube. The present results indicate that the nonreciprocity is a viable approach toward the superconductors with chiral or noncentrosymmetric structures." In other words, chirality may play a significant role in electromagnetic character of superconductors.

(b) In other paper, Kung *et al.* reported: "Using polarization-resolved resonant Raman spectroscopy, we explore collective spin excitations of the chiral surface states in a three dimensional topological insulator, Bi2Se3. We observe a sharp peak at 150 meV in the pseudovector A2 symmetry channel of the Raman spectra. By comparing the data with calculations, we identify this peak as the transverse *collective spin mode* of surface Dirac fermions. This mode, unlike a Dirac plasmon or a surface plasmon in the charge sector of excitations, is analogous to a spin wave in a partially polarized Fermi liquid, with spin-orbit coupling playing the role of an effective magnetic field." [43] What we would emphasize here is that the collective spin mode may alter the Dirac fermions, see also [44].

(c) Karimi *et al.* studied deviation from Larmor's theorem, their abstract begins as follows: "Larmor's theorem holds for magnetic systems that are invariant under spin rotation. In the presence of spin-orbit coupling this invariance is lost and Larmor's theorem is broken: for systems of interacting electrons, this gives rise to a subtle interplay between the spin-orbit coupling acting on individual single-particle states and *Coulomb many-body effects*."[45] What we would emphasize here is possible observation of Coulomb many-body effects, and this seems to attract considerable interests recently, see also [45a].

Concluding remarks

In a series of papers, Bo Lehnert proposed a novel and revised version of Quantum Electrodynamics (RQED) based on Proca equations. We submit a viewpoint that Lehnert's RQED is a good alternative theory to QM/QED, and therefore it is also interesting to ask: Can this theory also explain some phenomena related to LENR and UDD reaction of Homlid (as argued by Celani et al)? While we do not pretend to hold all the answers in this regard, we just gave an outline to Proca equations to electrodynamics of superconductors, then to chirality model.

Nonetheless, one of our aims with the present paper is to propose a combined version of London-Proca-Hirsch model for electrodynamics of superconductor. Considering that Proca equations may be used to explain electrodynamics in superconductor, the proposed fluidic

London-Proca equations may be able to describe electromagnetic of superconductors. It is hoped that this paper may stimulate further investigations and experiments in particular for superconductor. It may be expected to have some impact to cosmology modeling too. Another purpose is to submit a new model of gravitation based on a recent paper by Ryutov, Budker, Flambaum, who suggest that Maxwell-Proca equations may induce a negative pressure imitates gravitational pull, and may produce effect similar to gravitation. In the meantime, there are other papers by Longo, Shamir etc. discussing observations indicating handedness of spiral galaxies, which seem to suggest chiral medium at large scale.

However, so far there is no derivation of Maxwell-Proca equations in chiral medium. In this paper, we propose Maxwell-Proca-type equations in chiral medium, which may also explain (albeit hypothetically) origin of handedness of spiral galaxies as reported by M. Longo *et al.*

It may be expected that one can describe handedness of spiral galaxies by chiral Maxwell-Proca equations. This would need more investigations, both theoretically and empirically.

This paper is partly intended to stimulate further investigations and experiments of LENR inspired by classical electrodynamics, as a continuation with our previous report.

Postscript

It shall be noted that the present paper is not intended to be a complete description of physics of LENR and UDD reaction (Homlid et al). Nonetheless, we can remark on three things:

a. Although usually Proca equations are considered not gauge invariant, therefore some researchers tried to derive a gauge invariant massive version of Maxwell equations, a recent experiment suggest U(1) gauge invariance of Proca equations. See [46].
b. Since Proca equations can be related to electromagnetic Klein-Gordon equation, and from Klein-Gordon equation one can derive hydrino states of hydrogen atom, then one can also expect to derive ultradense hydrogen and hydrino states from Proca equation. This brings us to a consistent picture of hydrogen, see also Mills *et al.* [47-50]
c. Chirality effect of hydrino/UDD may be observed in experiments in the near future.

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Eureka Moment as Divine Spark

Victor Christianto, Florentin Smarandache

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Abstract

In the ancient world, the Greeks believed that all great insights came from one of nine muses, divine sisters who brought inspiration to mere mortals. In the modern world, few people still believe in the muses, but we all still love to hear stories of sudden inspiration, like Newton and the apple, or Archimedes and the bathtub. We are eager to hear and to share stories about flashes of insight. In this article, we point out some arguments suggesting that the *eureka* moment is divine spark.

Keywords: Eureka, divine spark, insight, creativity.

Introduction

Burkus, an educator of the executives management science, investigates innovativeness back to antiquated Greek fantasies. He contended that in Greek folklore, purported innovativeness was just controlled by a bunch of individuals who were honored by the divine beings' sprinkling of the "*divine fire*", so they in some cases experienced Eureka minutes [1].

As indicated by Burkus, there is nothing of the sort as an imaginative flash or aha minute. Genuine imagination is an iterative procedure, regularly comprising of moderate and steady changes and advancements for existing thoughts. Imaginative people seldom create in disengagement; actually, bunches are greater at advancement than people. Large thoughts are not constantly perceived from the start; many need a very long time to acknowledge, and others simply vanish.

Burkus likewise dismisses the organization's endeavors to empower innovativeness, contending that there is little proof of such endeavors bringing about more advancement. Inventive individuals are propelled by the work itself, which they feel is expressly fulfilling; Extrinsic sparks assume a moderately little job in their lives. The appropriate response, he proposed, was just giving individuals the work they needed to do, which they discovered fulfilling. He also believes that a happy workplace and a good team spirit, which is generally believed to be beneficial for creative thinking, can actually act as a barrier. "Excessive focus on cohesion…. actually can reduce team creativity," he wrote. "This can narrow down choices and cause those who have a unique perspective to censor themselves rather than take risks not to be considered part of the team."

What is *Eureka* moment?

Eureka's minute feels like a blaze of understanding since it frequently leaves periods when the brain isn't centered around the issue, which therapists call the hatching time frame. Brooding is where individuals step once again from their occupations. A significant number of the most beneficial innovative individuals purposely put aside activities and enjoy a physical reprieve from their work by accepting that this hatching stage is when thoughts start to blend beneath the limit of cognizant idea.

A few people shuffle different undertakings simultaneously under the conviction that while their cognizant psyche is centered around one anticipate, others are hatching their intuitive. The knowledge that emerges after hatching is the thing that feels like we are outfitting the intensity of delivering similar thoughts that help Newton and Archimedes [4].

An exploration group drove by Sophie Ellwood as of late discovered experimental proof for the intensity of hatching to upgrade imaginative understanding. The scientists isolated 90 undergrad brain research understudies into three gatherings. Each gathering is appointed to finish the Alternative Usage Test, which solicits members to make a rundown from the same number of uses of normal items as they can envision. Right now, were solicited to make a rundown from potential employments of paper. The quantity of unique thoughts delivered will fill in as an alternate proportion of thought, a significant component of imagination and a significant advance towards finding feasible bits of knowledge for Europeans.

The main gathering chipped away at the issue for 4 minutes persistently. The subsequent gathering was hindered following two minutes and approached to deliver equivalent words for each word from the rundown gave (considered another undertaking that did innovativeness), at that point given two additional minutes to finish the first test. The last gathering was hindered following two minutes, given the Myers-Briggs Type Indicator (thought about a random undertaking), and afterward requested to keep on taking a shot at the trial of utilizing the first option for an additional two minutes. Aside from the gathering, every member was given a similar measure of time (4 minutes) to deal with a rundown of potential uses for a bit of paper.

The research team can then compare the creativity that results from ongoing work, work with the incubation period in which the related tasks are completed, and work with the incubation period in which the unrelated tasks are completed. Interestingly, the researchers found that the group that was given a break to work on an unrelated task (the Myers-Briggs test) produced the majority of ideas, an average of 9.8 [4].

Burkus in his HBR article states [4]:

> One possible explanation for these findings is that when presented with complicated problems, the mind can often get stuck, finding itself tracing back through certain pathways of thinking again and again. When you work on a problem continuously, you can become fixated on previous solutions. You will just keep thinking of the same uses for that piece of paper instead of finding new possibilities. Taking a break from the problem and focusing on something else entirely gives the mind some time to release its fixation on the same solutions and let the old pathways fade from memory. Then, when you return to the original problem, your mind is more open to new possibilities – eureka moments.

Discussions

That creative spark or Eureka moment is indeed rare is true. But it is also not always true that working in groups produces more ideas. Although Burkus's analysis is quite interesting, it seems that he is too influenced by the management's perspective on creativity. More references are needed about methods of generating ideas and also the literature of creativity experts such as De Bono [2-3].

In addition to the task switching method as a way of incubation described above, there are actually various ways to generate fresh ideas and insights, see for example [3]. One quite interesting way is to provide regular intake to our minds, for example every morning, with two words combined at random (random).

Around 2002-2003, one of these authors (VC) made a small script that basically: (a) uses the Miriam-Webster or Oxford dictionary as a data source, (b) randomly selects two nouns from the dictionary, (c) displays both words as new phrase to users. Imagine, for example, one morning while you were having coffee and breakfast, knowing on your cellphone screen a strange phrase appeared: "ice cat" ... Your mind must have been searching for what was the meaning or application of the phrase “cat ice”? Maybe it can be a beautiful ice sculpture in the form of a cat (usually at a large party event there is “*ice carving*”). And so on, we tend to be more creative if our minds are routinely consumed with fresh things, which can be raised by such a method, which may be termed: RWPG method (*random word-pair generator*).

Another way, which might be closer to the original meaning of the Eureka moment as "*divine spark*," is to use time deliberately to experience and communicate with God and nature. This method is closer to *experiential learning* patterns. For example, if you take an hour each morning to take a walk in the woods or in the fields, observe the things you find along the way. And also take time to pray and communicate with God. See our previous papers [5-6].

Conclusion

Like Newton and the apple, or Archimedes and the bathtub (both another type of myth), we are eager to hear and to share stories about flashes of insight. But what does it take to be actually creative? How to have such a flash insight? Turns out, there is real science behind "*aha moments."* This article is our way to distinguish which is actual activity and which is myth in order to get such flash moments.

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A Short Remark on Vortex as Fluid Particle from Neutrosophic Logic perspective

(Towards "fluidicle" or "vorticle" model of QED.)

Victor Christianto, Florentin Smarandache

Victor Christianto, Florentin Smarandache (2020). A Short Remark on Vortex as Fluid Particle from Neutrosophic Logic perspective (Towards "fluidicle" or "vorticle" model of QED). *International Journal of Neutrosophic Science* 7(1), 38-46

Abstract

In a previous paper in this journal (IJNS), it is mentioned about a possible approach to re-describe QED without *renormalization* route. As it is known that in literature, there are some attempts to reconcile vortex-based fluid dynamics and particle dynamics. Some attempts are not quite as fruitful as others. As a follow up to previous paper, the present paper will discuss two theorems for developing unification theories, and then point out some new proposals including by Simula (2020) on how to derive Maxwell equations in superfluid dynamics setting; this could be a new alternative approach towards "*fluidicle*" or "*vorticle*" model of QED. Further research is recommended in this new direction.

Keywords: Neutrosophic logic, Vortex-based fluid dynamics, Fuidicle, Vorticle, QED, Renormalization, Maxwell-Proca equations.

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1.Introduction

In literature, there are some attempts to reconcile between vortex-based fluid dynamics and particle dynamics, see [15-21]. Some attempts are not quite fruitful as others, concerning describing classical electrodynamics.

This paper will continue our previous article, suggesting that it is possible to find a way out of the infinity problem in QED without renormalization route [14]. As in the previous paper [14], the role of neutrosophic Logic (developed by one of us, FS) here is to find a third way or intermediate solution between point particle and vortex, that is why it is suggested here a combined term: "*vorticle*" (from vortex and particle), or it may be called: "*fluidicle*" (from fluidic particle). These new words vorticle and fluidicle are intended to capture the essence of "middle way" representing the Neutrosophic Logic view.

Here three possible approaches by Tapio Simula, Lehnert's RQED, and also Carl Krafft, will also be discussed.

The present paper will point out some new papers including by Simula [7] on how to derive Maxwell equations in superfluid dynamics setting, this could be a new alternative approach towards "fluidicle" or "vorticle" model of QED.

2. A short review of progress QED theories in literature and two new theorems.

There are some progress in the literature of QED, beyond what is called "*renormalization*" route, for instance by Daywitt, using a 7-dimensional spacetime and spinor wave [22-24].

Other developments have been made by Prof. Bo Lehnert, which he calls: revised Quantum Electrodynamics. There are numerous possible ways to develop QED-like theories, and not only that some theoreticians have gone further to develop Unification Theories, SuperUnification, and even Theory of Everything (TOE).

But almost all of them boiled down to mounting complexities and ever-increasing difficult technicalities, so it appears to be more direct approach to start with writing down two theorems as follows:

2.a. Two new theorems and a corollary

Based on the above discussions, actually, it is suggested two theorems and a corollary over here:

Theorem 1:
The true unified theory between gravitation, particles, and electromagnetic (UTGPE) fields should be based on a consistent model of *vacuum*, preferably by a kind of ether fluid dynamics.

Theorem 2:
The true UTGPE, albeit it is quite difficult to find, shall be founded on no more than 3-dimensional space and 1-dimensional time (Newtonian space).

Corollary: It should be possible and indeed relatively easy to find theoretical ways to unify four fundamental forces by increasing spacetime dimensionality. Supra dimensional spacetime is one character of *anti-realism* theory of UTGPE.

2.b. **Implication.**

Therefore, a good candidate of true UTGPE, or at least a unification of gravitation and electromagnetic field in a quantum sense, should be better off based on such characteristics, as a consistent combination between a quantum feature of electrodynamics theory and/or quantum or *sub-quantum*[1] model of aether fluid.

3. Three possible alternatives on QED

Allow us to begin this section with a quote from Sonin's book [1], which can be paraphrased as follows:

> "The movement of vortices has been a region of study for over a century. During the old style time of vortex elements, from the late 1800s, many fascinating properties of vortices were found, starting with the outstanding Kelvin waves engendering along a disconnected vortex line (Thompson, 1880). The primary object of hypothetical investigations around then was a dissipationless immaculate fluid (Lamb, 1997). It was difficult for the hypothesis to find a shared opinion with try since any old style fluid shows gooey impacts. The circumstance changed after crafted by Onsager (1949) and Feynman (1955) who uncovered that turning superfluids are strung by a variety of vortex lines with quantized dissemination. With this revelation, the quantum time of vortex elements started."

Then it is possible find an expression that relates the topological and quantized vortices from the viewpoint of Bohr-Sommerfeld quantization rules, which seem to remind us to the Old Quantum Theory, albeit from a different perspective.

The quantization of circulation for nonrelativistic superfluid is given by [3]:

$$\oint v dr = N \frac{\hbar}{m_s} \tag{1}$$

Where $N, \hbar, m_s$ represents the winding number, reduced Planck constant, and superfluid particle's mass, respectively [3]. And the total number of vortices is given by [44]:

$$N = \frac{\omega . 2\pi r^2 m}{\hbar} \tag{2}$$

[1] Added note: Robert N. Boyd has suggested his sub-quantum kinetic model of aether and also electron, using some features of Kelvin-Helmholtz vortex theorem. See for instance: V. Christianto, F. Smarandache & R.N. Boyd, Electron Model Based Helmholtz's Electron Vortex & Kolmogorov's Theory of Turbulence. *Prespacetime J.* vol. 10 (1), 2019. url: https://prespacetime.com/index.php/pst/article/view/1516

Some implications:

a. Simula's approach

Provided it is acceptable that there is a neat correspondence between quantized vortices in superfluid helium and Bohr-Sommerfeld quantization rules, now let us quote from abstract of a recent paper where Tapio Simula wrote, which can be rephrased as follows [7]:

> "Right now, and electromagnetism have a similar starting point and are new properties of the superfluid universe, which itself rises up out of the hidden aggregate structure of progressively basic particles, for example, atoms. The Bose-Einstein condensate is identified as the tricky dull matter of the superfluid universe with vortices and phonons, separately, comparing to huge charged particles and massless photons."[7]

In Simula's model, Maxwell equations can be re-derived right from superfluid vortices.

b. Lehnert's RQED

And one more approach is worthy to mention here. Instead of Simula's model of electromagnetic and gravitation fields in terms of superfluid vortices, we can also come up with a model of electrodynamics by Lehnert's RQED from Proca equations. As Proca equations can be used to describe the electromagnetic field of superconductor, we find it as a possible approach too.

Conventional electromagnetic theory based on Maxwell's equations and quantum mechanics has been successful in its applications in numerous problems in physics and has sometimes manifested itself in a good agreement with experiments. Nevertheless, as already stated by Feynman, there are unsolved problems which lead to difficulties with Maxwell's equations that are not removed by and not directly associated with quantum mechanics [20]. Therefore QED, which is an extension of Maxwell's equations, also becomes subject to the *typical shortcomings* of electromagnetic in its conventional form. This reasoning makes a way for Revised Quantum Electrodynamics as proposed by Bo Lehnert. [11-13]
In a series of papers, Bo Lehnert proposed a novel and revised version of Quantum Electrodynamics, which he calls as RQED. His theory is based on the hypothesis of a nonzero electric charge density in the vacuum, and it is based on *Proca-type* field equations [10, p. 23]:

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2}-\nabla^2\right)A_\mu=\mu_0 J_\mu, \mu=1,2,3,4 \tag{3}$$

Where

$$A_\mu=\left(A,\frac{i\phi}{c}\right), \tag{4}$$

With A and ϕ standing for the magnetic vector potential and the electrostatic potential in three-space. In three dimensions, we got [20, p.23]:

$$\frac{curlB}{\mu_0} = \varepsilon_0 (divE)C + \frac{\varepsilon_0 \partial E}{\partial t}, \tag{5}$$

$$curlE = -\frac{\partial B}{\partial t}, \tag{6}$$

$$B = curlA, divB = 0, \tag{7}$$

$$E = -\nabla \phi - \frac{\partial A}{\partial t}, \tag{8}$$

$$divE = \frac{\overline{\rho}}{\varepsilon_0}. \tag{9}$$

These equations differ from the conventional form, by a nonzero electric field divergence equation (9) and by the additional space-charge current density in addition to displacement current at equation (5). The extended field equations (5)-(9) are easily found also to become invariant to a gauge transformation.[10, p.23]

The main characteristic new features of the present theory can be summarized as follows [10, p.24]:

a. The hypothesis of a nonzero electric field divergence in the vacuum introduces an additional degree of freedom, leading to new physical phenomena. The associated nonzero electric charge density thereby acts somewhat like a hidden variable.
b. This also abolishes the symmetry between the electric and magnetic fields, and then the field equations obtain the character of intrinsic linear symmetry breaking.
c. The theory is both Lorentz and gauge invariant.
d. The velocity of light is no longer a scalar quantity but is represented by a velocity vector of the modulus c.
e. Additional results: Lehnert is also able to derive the mass of Z boson and Higgs-like boson.[21] These would pave an alternative way to new physics beyond Standard Model.

Now it should be clear that Lehnert's RQED is a good alternative theory to QM/QED, and therefore it is also interesting to ask whether this theory can also explain some phenomena related to LENR and UDD reaction of Homlid (as argued by Celani *et al*).[8]

A recent paper [8] presented arguments in favor of extending RQED to become a fluidic Maxwell-Proca equations, as follows:

Now it appears possible to arrive at fluidic Maxwell-Proca equations, as follows [8]

$$\nabla\cdot\vec{E} = \frac{\rho}{\varepsilon_0} - \kappa^2\phi, \tag{10}$$

$$\nabla\cdot\vec{B} = 0, \tag{11}$$

$$\dot{B} = -\nabla\times E - \nabla\times\left(\hat{\beta}\nabla\times H_0\right), \tag{12}$$

$$\varepsilon_0\mu_0\dot{E} = \nabla\times B - \mu_0 j - \kappa^2 A - \left(\hat{\sigma}E_0 + \rho_e v + \hat{\gamma}\nabla T\right) - \nabla\times\left(\hat{a}\nabla\times E_0\right), \tag{13}$$

where:

$$\nabla\phi = -\frac{\partial\vec{A}}{\partial t} - \vec{E}, \tag{14}$$

$$\vec{B} = \nabla\times\vec{A}, \tag{15}$$

$$\kappa = \frac{mc_0}{\hbar}. \tag{16}$$

Since according to Blackledge, the Proca equations can be viewed as a *unified wavefield* model of electromagnetic phenomena [7], therefore the fluidic Maxwell-Proca equations can be considered as a *unified wavefield* model for electrodynamics of superconductor.

Now, having defined Maxwell-Proca equations, it is possible to write down *fluidic* Maxwell-Proca-Hirsch equations using the same definition, as follows:

$$(\Box_\alpha^2 - \kappa^2)(F - F_0) = \frac{1}{\lambda_L^2}(F - F_0), \tag{17}$$

And

$$(\Box_\alpha^2 - \kappa^2)(J - J_0) = \frac{1}{\lambda_L^2}(J - J_0), \tag{18}$$

where

$$\Box_\alpha^2 = \nabla^{\alpha 2} - \frac{1}{c^2}\frac{\partial^{\alpha 2}}{\partial_t^{\alpha 2}}. \tag{19}$$

In literature, the above fluidic Maxwell-Proca-Hirsch equations have never been presented elsewhere before. Provided the above equations can be verified with experiments, they can be used to describe electrodynamics of superconductors.

c. Krafft's approach

A third approach of describing elementary particles from aether vortices perspective is discussed by Carl F. Krafft [9]. See for example:

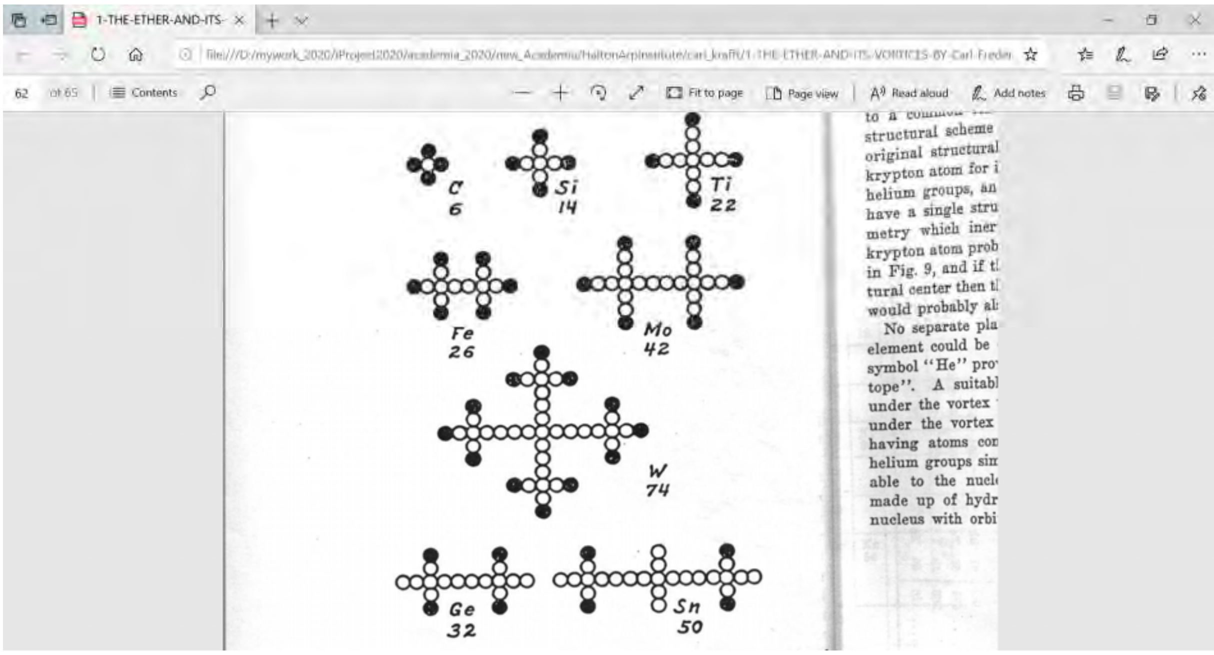

Figure 1. A few elementary particles, source: Carl Frederich Krafft [9]

4. Concluding remarks

In this paper, continuing our previous article, it is argued that it is possible to find a way out of the infinity problem in QED without renormalization route [14]. As a follow up to previous paper, in the present paper, first of all, two theorems for developing unification theories have been discussed, along with pointing out some new proposals including by Simula (2020) on how to derive Maxwell equations in superfluid dynamics setting. This could be a new alternative approach towards "*fluidicle*" or "*vorticle*" model of QED.

Three possible approaches: Tapio Simula, Lehnert's RQED and also Carl F. Krafft, have also been discussed. Nonetheless it should admitted that this article is not complete yet on possible ways to describe vorticle or fluidic as an alternative to QED.

Hopefully this article will inspire further investigations in this line of thoughts.

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Novel Concept of Energy in Bipolar Single-Valued Neutrosophic Graphs with Applications

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Abstract: The energy of a graph is defined as the sum of the absolute values of its eigenvalues. Recently, there has been a lot of interest in graph energy research. Previous literature has suggested integrating energy, Laplacian energy, and signless Laplacian energy with single-valued neutrosophic graphs (SVNGs). This integration is used to solve problems that are characterized by indeterminate and inconsistent information. However, when the information is endowed with both positive and negative uncertainty, then bipolar single-valued neutrosophic sets (BSVNs) constitute an appropriate knowledge representation of this framework. A BSVNs is a generalized bipolar fuzzy structure that deals with positive and negative uncertainty in real-life problems with a larger domain. In contrast to the previous study, which directly used truth and indeterminate and false membership, this paper proposes integrating energy, Laplacian energy, and signless Laplacian energy with BSVNs to graph structure considering the positive and negative membership degree to greatly improve decisions in certain problems. Moreover, this paper intends to elaborate on characteristics of eigenvalues, upper and lower bound of energy, Laplacian energy, and signless Laplacian energy. We introduced the concept of a bipolar single-valued neutrosophic graph (BSVNG) for an energy graph and discussed its relevant ideas with the help of examples. Furthermore, the significance of using bipolar concepts over non-bipolar concepts is compared numerically. Finally, the application of energy, Laplacian energy, and signless Laplacian energy in BSVNG are demonstrated in selecting renewable energy sources, while optimal selection is suggested to illustrate the proposed method. This indicates the usefulness and practicality of this proposed approach in real life.

Keywords: bipolar neutrosophic set; graph energy; laplacian energy; signless Laplacian energy; renewable energy source

1. Introduction

The graph spectrum is applicable in statistical physics and mathematical combinatorial optimization problems. Pattern recognition, modelling virus spread in computer networks, and safeguarding personal data in databases all benefit from the spectrum of a graph. The concept of graph energy is related to a graph's spectrum. This concept was originally introduced by Gutman [1] in 1978. It is defined as the sum of the absolute values of the eigen values of the graph's adjacency matrix. By linking the edge of a graph to the electron energy of a type of molecule, the energy of a graph is employed in quantum theory and many other applications in the context of energy. Later, Gutman and Zhou [2] defined the Laplacian energy of a graph as the sum of the absolute values of the differences of average vertex degree of G to the Laplacian eigenvalues of G. Details on the properties of graph energy and Laplacian energy can be found in [3–11].

Zadeh first introduced the fuzzy set theory in 1965 [12]. After the initiation of fuzzy sets, the concept of fuzzy graphs was developed by Kaufmann [13] and Rosenfeld [14] to deal with the fuzzy environment in graphs. Zhang [15] developed the concept of a bipolar fuzzy set where a positive membership function and a negative membership function are generalized from the traditional fuzzy set. Later, Smarandache [16] introduced the concept of the neutrosophic set, which is generalized from fuzzy set theory and intuitionistic fuzzy set.

Over the years, many researchers have studied graph energy in fuzzy and neutrosophic sets. Anjali and Mathew [17] defined the energy of a graph within the fuzzy set environment. In 2014, Sharbaf and Fayazi [18] introduced the concept of Laplacian energy of fuzzy graphs and some results on Laplacian energy bounds extended to fuzzy graphs. In the same year, Praba et al. [19] discussed the energy in the intuitionistic fuzzy graph. Laplacian energy in an intuitionistic fuzzy graph was defined by Basha and Kartheek [20]. Later, Akram and Naz [21] defined the energy and Laplacian energy of Pythagorean fuzzy graphs (PFGs) and Pythagorean fuzzy digraphs (PFDGs). Moreover, the study also derived the lower and upper bounds for the energy and Laplacian energy of PFGs. Later, Murugesan and Venkatesh [22] defined energy and Laplacian energy of a bipolar fuzzy graph. Furthermore, Naz et al. [23] introduced the concept of energy of bipolar fuzzy graph (BFG) and bipolar fuzzy digraph (BFDG). The study derived the maximal energy of BFGs and investigated their properties. In 2020, Ramesh and Basha [24] defined the signless Laplacian energy of an intuitionistic fuzzy graph by implementing the cosine similarity measure in solving decision-making problems. The same author [25] also computed signless Laplacian energy of an intuitionistic fuzzy graph with TOPSIS method using MATLAB software and applied it in group decision-making problems.

Recently, Broumi et al. [26] proposed an interval-valued neutrosophic graph using MATLAB to compute energy and spectrum analysis. Later, Mohsin et al. [27] extended the energy of a fuzzy graph, intuitionistic fuzzy graph, and single-valued neutrosophic graph concepts to a complex neutrosophic graph. Naz et al. [28] introduced the concept of energy, Laplacian energy, and signless Laplacian energy in single-valued neutrosophic graphs and constructed a relation between them. In 2020, Mullai et al. [29] introduced dominating energy of a neutrosophic graph, dominating neutrosophic adjacency matrix, eigen values for the dominating energy of neutrosophic graphs, and complement of neutrosophic graphs. Table 1 summarizes some significant influences towards energy graph, Laplacian energy graph and signless Laplacian energy graph. Previous literature suggested integrating of energy, Laplacian energy, and signless Laplacian energy with single-valued neutrosophic graphs (SVNGs). This integration is used to solve problems that are characterized by indeterminate and inconsistent information. However, when the information is endowed with both positive and negative uncertainty, then BSVNs constitute an appropriate knowledge representation of this framework. Moreover, there has been no discussion concerning energy graphs in BSVNs.

Bipolarity refers to the tendency of the human mind to analyze and take responsibility based on positive and negative outcomes. The positive analysis is all about reasonable, permitted, appropriate, or considered acceptable, while impossible, rejected or forbidden represents negative analyses. Furthermore, positive thoughts correspond to the preferences as they interpret which objects are preferable to others without rejecting those that do not meet the preferences. Still, negative thoughts correspond to the constraints as they interpret which values or objects must be declined. Based on these consequences, Deli et al. [30] proposed bipolar fuzzy sets and neutrosophic sets to bipolar neutrosophic sets in which positive membership degree, negative membership degree, and operations were studied. Bipolar fuzzy sets have a great value in dealing with uncertainty in real-life problems and useful in dealing with the positive and the negative membership values. Thus, in this paper, we combined BSVNs with an energy graph and applied them to selecting renewable energy sources. In particular, this paper aims to introduce the concepts of energy, Laplacian energy, and signless Laplacian energy in BSVNG, to investigate the properties on characteristics of eigenvalues, upper and lower bound of energy, Laplacian energy and signless Laplacian energy and to present the relationship among them.

The outline of this study is organized as follows: Section 2 gives the basic concepts related to neutrosophic and bipolar sets. Section 3 defines the concepts of energy in BSVNG while the concept of Laplacian energy in BSVNG is discussed in Section 4. On the other hand, Section 5 presents the concepts of signless Laplacian energy in BSVNG and the relation between energy, Laplacian energy, and signless Laplacian energy presented in Section 6. Moreover, to implement our proposed study, we discuss the application of the energy of BSVNG in the selection of renewable energy sources in Section 7, while Section 8 provides comparative results. Finally, this study is concluded by mentioning future potential research work in Section 9.

Table 1. Significance influences towards energy, Laplacian energy and signless Laplacian energy graph.

Author and References	Year	Fuzzy/Neutrosophic Sets	Significance Influences
Akram and Naz [21]	2018	Pythagorean fuzzy sets	Introduce the energy and Laplacian energy in Pythagorean fuzzy graphs and Pythagorean fuzzy digraphs.
Rajeshwari et al. [22]	2018	Bipolar fuzzy sets	Introduce Laplacian energy for a bipolar fuzzy graph.
Naz et al. [23]	2018	Bipolar fuzzy sets	Introduce the concept of energy in bipolar fuzzy graph (BFG) and bipolar fuzzy digraphs (BFDGs).
Naz et al. [28]	2018	Single-valued neutrosophic sets	Introduce the concept of energy, Laplacian energy and signless Laplacian energy in single-valued neutrosophic graphs (SVNGs).
Broumi et al. [26]	2019	Interval-valued neutrosophic sets	Compute the spectrum and energy of interval-valued neutrosophic graph (IVNG).
Mohsin et al. [27]	2019	Complex neutrosophic set	Compute Laplacian energy of a complex neutrosophic graph in terms of its adjacency matrix.
Ramesh and Basha [24]	2020	Intuitionistic fuzzy sets	Solve decision-making problem by signless Laplacian energy of intuitionistic fuzzy graph and cosine similarity measure.
Ramesh and Basha [25]	2020	Intuitionistic fuzzy sets	Solve group decision-making problem by signless Laplacian energy of intuitionistic fuzzy graph.
Mullai and Broumi [29]	2020	Single-valued neutrosophic sets	Introduce dominating sets and dominating numbers for energy graphs in single-valued neutrosophic graphs.

2. Preliminaries

In this section, some concepts related to neutrosophic set and bipolar neutrosophic set are presented. For further details, the readers are referred to [31–40].

Definition 1. [16] *Let X be a universal set. The neutrosophic set (NS) A in X categorized by membership functions $T_A(x)$, $I_A(x)$ and $F_A(x)$ denote the true, indeterminate and false contained in real standard and the non-standard subset of $]^-0, 1^+[$, respectively, such that:*

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle | x \in X\}.$$

There is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$:

$$^-0 \leq supT_A(x) + supI_A(x) + supF_A(x) \leq 3^+.$$

Definition 2. [30] *A bipolar neutrosophic set (BNS) A in X is defined as an object of the form:*

$$A = \left\{ \left\langle x, \left(T_A^+(x), I_A^+(x), F_A^+(x), T_A^-(x), I_A^-(x), F_A^-(x) \right) \right\rangle | x \in X \right\}$$

where $T_A^+(x), I_A^+(x), F_A^+(x) : A \to [0,1]$ *and* $T_A^-(x), I_A^-(x), F_A^-(x) : A \to [-1,0]$. *The positive membership degrees* T_A^+, I_A^+ *and* F_A^+ *denote the truth membership, indeterminate membership and false membership. On the other hand, negative membership degrees* T_A^-, I_A^- *and* F_A^- *denote the truth membership, indeterminate membership and false membership of an element* $x \in X$ *to some implicit counter-property corresponding to a bipolar neutrosophic set A.*

Definition 3. [30] *Let* $A_1 = \left\langle T_1^+, I_1^+, F_1^+, T_1^-, I_1^-, F_1^- \right\rangle$ *and* $A_2 = \left\langle T_2^+, I_2^+, F_2^+, T_2^-, I_2^-, F_2^- \right\rangle$ *be two bipolar neutrosophic numbers, then*

1. $\lambda A_1 = \left\langle 1-(1-T_1^+)^\lambda, (I_1^+)^\lambda, (F_1^+)^\lambda, -(-T_1^-)^\lambda, -(-I_1^-)^\lambda, -(1-(1-(F_1^-)))^\lambda \right\rangle$;
2. $A_1^\lambda = \left\langle (T_1^+)^\lambda, 1-(1-I_1^+)^\lambda, 1-(1-F_1^+)^\lambda, -(1-(1-(-T_1^-)))^\lambda, -(-I_1^-)^\lambda, -(-F_1^-)^\lambda \right\rangle$;
3. $A_1 + A_2 = \langle T_1^+ + T_2^+ - T_1^+T_2^+, I_1^+I_2^+, F_1^+F_2^+, -T_1^-T_2^-, -(-I_1^- - I_2^- - I_1^-I_2^-), -(-F_1^- - F_2^- - F_1^-F_2^-) \rangle$;
4. $A_1A_2 = \langle T_1^+T_2^+, I_1^+ + I_2^+ - I_1^+I_2^+, F_1^+ + F_2^+ - F_1^+F_2^+, -(-T_1^- - T_2^- - T_1^-T_2^-), -I_1^-I_2^-, -F_1^-F_2^- \rangle$;

   where $\lambda > 0$.

Definition 4. [30] *Let* $A_1 = \left\langle T_1^+, I_1^+, F_1^+, T_1^-, I_1^-, F_1^- \right\rangle$ *be a bipolar neutrosophic number. Then, the score function* $s(A_1)$, *accuracy function* $a(A_1)$ *and certainty function* $c(A_1)$ *are defined as follows:*

1. $s(A_1) = \left(T_1^+ + 1 - I_1^+ + 1 - F_1^+ + 1 + T_1^- - I_1^- - F_1^- \right)/6$;
2. $a(A_1) = T_1^+ - F_1^+ + T_1^- - I_1^-$;
3. $c(A_1) = T_1^+ - F_1^+$.

3. Energy of Bipolar Single-Valued Neutrosophic Graphs

In this section, we define and investigate the energy of a graph within the frameworks of BSVNG theory and discuss its properties.

Definition 5. *The adjacency matrix* $A(\overline{G})$ *of a BSVNG* $\overline{G} = (\alpha, \beta)$ *is defined as a square matrix* $A(\overline{G}) = [a_{pq}]$, $a_{pq} = \left\langle T_\beta^+(v_pv_q), I_\beta^+(v_pv_q), F_\beta^+(v_pv_q), T_\beta^-(v_pv_q), I_\beta^-(v_pv_q), F_\beta^-(v_pv_q) \right\rangle$ *where* $T_\beta^+(v_pv_q)$, $I_\beta^+(v_pv_q)$, $F_\beta^+(v_pv_q)$, $T_\beta^-(v_pv_q)$, $I_\beta^-(v_pv_q)$ *and* $F_\beta^-(v_pv_q)$ *represent the strength of a positive relationship, the strength of a positive undecided relationship, the strength of a positive non-relationship, the strength of a negative relationship, the strength of a negative undecided relationship and the strength of a negative non-relationship between* u_p *and* u_q, *respectively.*

The adjacency matrix of a BSVNG can be expressed as six matrices. The first matrix contains the elements as positive truth-membership values; the second matrix contains the elements as positive indeterminacy-membership values; the third matrix contains the elements as positive falsity-membership values; the fourth matrix contains the elements as negative truth-membership values; the fifth matrix contains the elements as negative indeterminacy-membership values; and the sixth matrix contains the elements as negative falsity-membership values, i.e.,

$$A(\overline{G}) = \left\langle A\left(T_\beta^+(v_pv_q)\right), A\left(I_\beta^+(v_pv_q)\right), A\left(F_\beta^+(v_pv_q)\right), A\left(T_\beta^-(v_pv_q)\right), A\left(I_\beta^-(v_pv_q)\right), A\left(F_\beta^-(v_pv_q)\right) \right\rangle.$$

Definition 6. *The spectrum of adjacency matrix of a BSVNG* $A(\overline{G})$ *is defined as* $\langle R, S, T, U, V, W \rangle$, *where* R, S, T, U, V *and* W *are the sets of eigenvalues of:*

$A\left(T_{\beta}^{+}\left(v_p v_q\right)\right)$, $A\left(I_{\beta}^{+}\left(v_p v_q\right)\right)$, $A\left(F_{\beta}^{+}\left(v_p v_q\right)\right)$, $A\left(T_{\beta}^{-}\left(v_p v_q\right)\right)$, $A\left(I_{\beta}^{-}\left(v_p v_q\right)\right)$, $A\left(F_{\beta}^{-}\left(v_p v_q\right)\right)$, *respectively.*

Example 1. *Consider a graph $G(V,E)$ where $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and $E = \{v_1v_2, v_1v_3, v_1v_4, v_1v_5, v_1v_6, v_1v_7, v_2v_3, v_3v_4, v_4v_5, v_5v_6, v_6v_7\}$. Let $\overline{G}(\alpha, \beta)$ be a BSVNG on V, as shown in Figure 1, defined by Tables 2 and 3 as follows:*

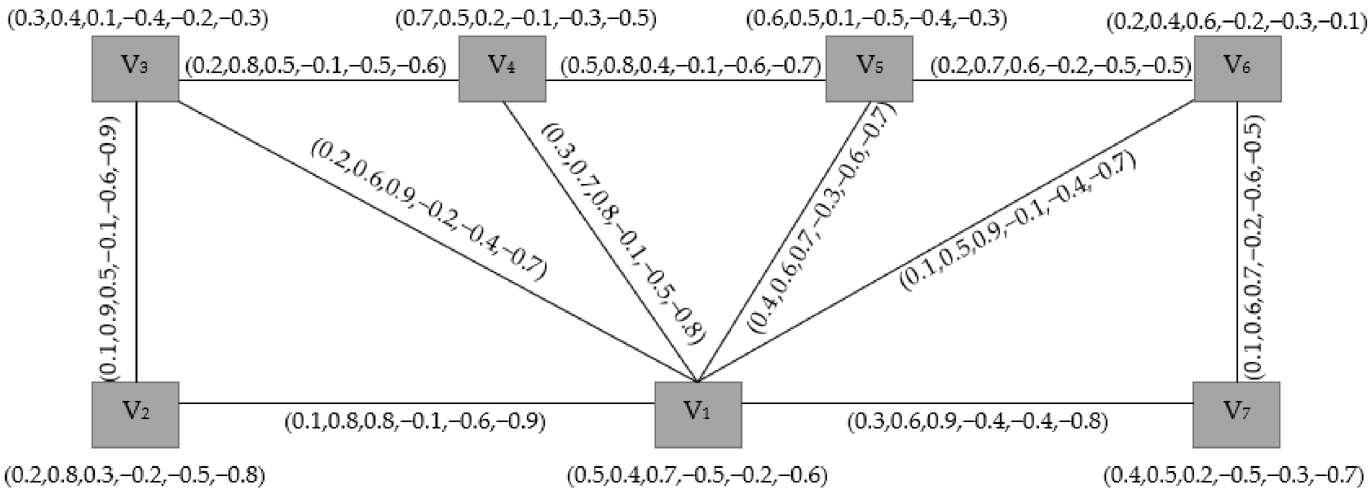


Figure 1. The energy of a bipolar single-valued neutrosophic graph.

Table 2. The energy of BSVNG set on V.

α	v_1	v_2	v_3	v_4	v_5	v_6	v_7
T_{α}^{+}	0.5	0.2	0.3	0.7	0.6	0.2	0.4
I_{α}^{+}	0.4	0.8	0.4	0.5	0.5	0.4	0.5
F_{α}^{+}	0.7	0.3	0.1	0.2	0.1	0.6	0.2
T_{α}^{-}	−0.5	−0.2	−0.4	−0.1	−0.5	−0.2	−0.5
I_{α}^{-}	−0.2	−0.5	−0.2	−0.3	−0.4	−0.3	−0.3
F_{α}^{-}	−0.6	−0.8	−0.3	−0.5	−0.3	−0.1	−0.7

Table 3. The energy of BSVNG relation on V.

β	v_1v_2	v_1v_3	v_1v_4	v_1v_5	v_1v_6	v_1v_7	v_2v_3	v_3v_4	v_4v_5	v_5v_6	v_6v_7
T_{β}^{+}	0.1	0.2	0.3	0.4	0.1	0.3	0.1	0.2	0.5	0.2	0.1
I_{β}^{+}	0.8	0.6	0.7	0.6	0.5	0.6	0.9	0.8	0.8	0.7	0.6
F_{β}^{+}	0.8	0.9	0.8	0.7	0.9	0.9	0.5	0.5	0.4	0.6	0.7
T_{β}^{-}	−0.1	−0.2	−0.1	−0.3	−0.1	−0.4	−0.1	−0.1	−0.1	−0.2	−0.2
I_{β}^{-}	−0.6	−0.4	−0.5	−0.6	−0.4	−0.4	−0.6	−0.5	−0.6	−0.5	−0.6
F_{β}^{-}	−0.9	−0.7	−0.8	−0.7	−0.7	−0.8	−0.9	−0.6	−0.7	−0.5	−0.5

The adjacency matrix of a BSVNG given in Figure 1 is

$$\begin{pmatrix}
\left\langle\begin{matrix}0,0,0,\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0.1,0.8,0.8,\\-0.1,-0.6,-0.9\end{matrix}\right\rangle & \left\langle\begin{matrix}0.2,0.6,0.9,\\-0.2,-0.4,-0.7\end{matrix}\right\rangle & \left\langle\begin{matrix}0.3,0.7,0.8,\\-0.1,-0.5,-0.8\end{matrix}\right\rangle & \left\langle\begin{matrix}0.4,0.6,0.7,\\-0.3,-0.6,-0.7\end{matrix}\right\rangle & \left\langle\begin{matrix}0.1,0.5,0.9,\\-0.1,-0.4,-0.7\end{matrix}\right\rangle & \left\langle\begin{matrix}0.3,0.6,0.9,\\-0.4,-0.4,-0.8\end{matrix}\right\rangle \\
\left\langle\begin{matrix}0.1,0.8,0.8,\\-0.1,-0.6,-0.9\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0,\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0.1,0.9,0.5,\\-0.1,-0.6,-0.9\end{matrix}\right\rangle & \langle 0,0,0,0,0,0\rangle & \left\langle\begin{matrix}0,0,0,\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0,\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0,\\0,0,0\end{matrix}\right\rangle \\
\left\langle\begin{matrix}0.2,0.6,0.9,\\-0.2,-0.4,-0.7\end{matrix}\right\rangle & \left\langle\begin{matrix}0.1,0.9,0.5,\\-0.1,-0.6,-0.9\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0,\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0.2,0.8,0.5,\\-0.1,-0.5,-0.6\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0,\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0,\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0,\\0,0,0\end{matrix}\right\rangle \\
\left\langle\begin{matrix}0.3,0.7,0.8,\\-0.1,-0.5,-0.8\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0,\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0.2,0.8,0.5,\\-0.1,-0.5,-0.6\end{matrix}\right\rangle & \langle 0,0,0,0,0,0\rangle & \left\langle\begin{matrix}0.5,0.8,0.4,\\-0.1,-0.6,-0.7\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0,\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0,\\0,0,0\end{matrix}\right\rangle \\
\left\langle\begin{matrix}0.4,0.6,0.7,\\-0.3,-0.6,-0.7\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0,\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0,\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0.5,0.8,0.4,\\-0.1,-0.6,-0.7\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0,\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0.2,0.7,0.6,\\-0.2,-0.5,-0.5\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0,\\0,0,0\end{matrix}\right\rangle \\
\left\langle\begin{matrix}0.1,0.5,0.9,\\-0.1,-0.4,-0.7\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0,\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0,\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0,\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0.2,0.7,0.6,\\-0.2,-0.5,-0.5\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0,\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0.1,0.6,0.7,\\-0.2,-0.6,-0.5\end{matrix}\right\rangle \\
\left\langle\begin{matrix}0.3,0.6,0.9,\\-0.4,-0.4,-0.8\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0,\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0,\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0,\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0,\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0.1,0.6,0.7,\\-0.2,-0.6,-0.5\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0,\\0,0,0\end{matrix}\right\rangle
\end{pmatrix}$$

The spectrum of a BSVNG $\overline{G}$, given in Figure 1, is as follows:

$$\begin{aligned}
Spec\Big(T_\beta^+(v_pv_q)\Big) &= \{-0.6163, -0.4338, -0.1208, -0.0406, 0.1252, 0.1495, 0.9369\}.\\
Spec\Big(I_\beta^+(v_pv_q)\Big) &= \{-1.4555, -1.1839, -0.7335, -0.3342, 0.4205, 0.9179, 2.3687\}.\\
Spec\Big(F_\beta^+(v_pv_q)\Big) &= \{-1.6620, -0.9839, -0.6946, -0.1998, 0.2355, 0.7439, 2.5609\}.\\
Spec\Big(T_\beta^-(v_pv_q)\Big) &= \{-0.7265, -0.1609, -0.0509, 0.0476, 0.1242, 0.1464, 0.6202\}.\\
Spec\Big(I_\beta^-(v_pv_q)\Big) &= \{-1.7702, -0.7044, -0.3299, 0.2795, 0.5615, 0.9685, 0.9950\}.\\
Spec\Big(F_\beta^-(v_pv_q)\Big) &= \{-2.5122, -0.8107, -0.4128, 0.3321, 0.7283, 1.1356, 1.5398\}.
\end{aligned}$$

Therefore,

$$Spec(\overline{G}) = \left\{\begin{matrix}
\langle -0.6163, -1.4555, -1.6620, -0.7265, -1.7702, -2.5122\rangle,\\
\langle -0.4338, -1.1839, -0.9839, -0.1609, -0.7044, -0.8107\rangle,\\
\langle -0.1208, -0.7335, -0.6946, -0.0509, -0.3299, -0.4128\rangle,\\
\langle -0.0406, -0.3342, -0.1998, 0.0476, 0.2795, 0.3321\rangle,\\
\langle 0.1252, 0.4205, 0.2355, 0.1242, 0.5615, 0.7283\rangle,\\
\langle 0.1495, 0.9179, 0.7439, 0.1464, 0.9685, 1.1356\rangle,\\
\langle 0.9369, 2.3687, 2.5609, 0.6202, 0.9950, 1.5398\rangle
\end{matrix}\right\}.$$

Definition 7. *The energy of a BSVNG $\overline{G}(\alpha, \beta)$ is defined as*

$$\begin{aligned}
E(\overline{G}) &= \Big\langle E\Big(T_\beta^+(v_pv_q)\Big), E\Big(I_\beta^+(v_pv_q)\Big), E\Big(F_\beta^+(v_pv_q)\Big), E\Big(T_\beta^-(v_pv_q)\Big), E\Big(I_\beta^-(v_pv_q)\Big), E\Big(F_\beta^-(v_pv_q)\Big)\Big\rangle\\
&= \left\langle \sum_{\substack{p=1\\ \lambda_p\in R}}^n |\lambda_p|, \sum_{\substack{p=1\\ \delta_p\in S}}^n |\delta_p|, \sum_{\substack{p=1\\ \gamma_p\in T}}^n |\gamma_p|, \sum_{\substack{p=1\\ \omega_p\in U}}^n |\omega_p|, \sum_{\substack{p=1\\ \xi_p\in V}}^n |\xi_p|, \sum_{\substack{p=1\\ \psi_p\in W}}^n |\psi_p| \right\rangle.
\end{aligned}$$

Definition 8. *Two BSVNG with the same number of vertices and the same energy are called equienergetic.*

Theorem 1. *Let $\overline{G}(\alpha, \beta)$ be a BSVNG and $A(\overline{G})$ be its adjacency matrix. If $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n, \delta_1 \geq \delta_2 \geq \ldots \geq \delta_n, \gamma_1 \geq \gamma_2 \geq \ldots \geq \gamma_n, \omega_1 \geq \omega_2 \geq \ldots \geq \omega_n, \xi_1 \geq \xi_2 \geq \ldots \geq \xi_n$ and $\psi_1 \geq \psi_2 \geq \ldots \geq \psi_n$ are the eigenvalues of*

$$A\Big(T_\beta^+(v_pv_q)\Big), A\Big(I_\beta^+(v_pv_q)\Big), A\Big(F_\beta^+(v_pv_q)\Big), A\Big(T_\beta^-(v_pv_q)\Big), A\Big(I_\beta^-(v_pv_q)\Big), A\Big(F_\beta^-(v_pv_q)\Big),$$

respectively.

Then,

1. $\sum\limits_{\substack{p=1\\ \lambda_p\in R}}^{n}\lambda_p=0, \sum\limits_{\substack{p=1\\ \delta_p\in S}}^{n}\delta_p=0, \sum\limits_{\substack{p=1\\ \gamma_p\in T}}^{n}\gamma_p=0, \sum\limits_{\substack{p=1\\ \omega_p\in U}}^{n}\omega_p=0, \sum\limits_{\substack{p=1\\ \xi_p\in V}}^{n}\xi_p=0$ and $\sum\limits_{\substack{p=1\\ \psi_p\in W}}^{n}\psi_p=0$.
2. $\sum\limits_{\substack{p=1\\ \lambda_p\in R}}^{n}{\lambda_p}^2=2\sum\limits_{1\le p<q\le n}\left(T_\beta^+(v_pv_q)\right)^2, \sum\limits_{\substack{p=1\\ \delta_p\in R}}^{n}{\delta_p}^2=2\sum\limits_{1\le p<q\le n}\left(I_\beta^+(v_pv_q)\right)^2, \sum\limits_{\substack{p=1\\ \gamma_p\in R}}^{n}{\gamma_p}^2=2\sum\limits_{1\le p<q\le n}\left(F_\beta^+(v_pv_q)\right)^2, \sum\limits_{\substack{p=1\\ \omega_p\in R}}^{n}{\omega_p}^2=2\sum\limits_{1\le p<q\le n}\left(T_\beta^-(v_pv_q)\right)^2, \sum\limits_{\substack{p=1\\ \xi_p\in R}}^{n}{\xi_p}^2=2\sum\limits_{1\le p<q\le n}\left(I_\beta^-(v_pv_q)\right)^2$ and $\sum\limits_{\substack{p=1\\ \psi_p\in R}}^{n}\psi_p=0$.

Proof.

1. Since $A(\overline{G})$ is a symmetric matrix with zero traces, its eigenvalues are real with a sum equal to zero.
2. By trace properties of a matrix, we have

$$tr\left(\left(A\left(T_\beta^+(v_pv_q)\right)\right)^2\right)=\sum_{\substack{p=1\\ \lambda_p\in R}}^{n}{\lambda_p}^2,$$

where

$$\begin{aligned}tr\left(\left(A\left(T_\beta^+(v_pv_q)\right)\right)^2\right)&=\left(0+\left(T_\beta^+(v_1v_2)\right)^2+\ldots+\left(T_\beta^+(v_1v_n)\right)^2\right)+\left(\left(T_\beta^+(v_2v_1)\right)^2+0+\ldots+\left(T_\beta^+(v_2v_n)\right)^2\right)+\ldots\\&\quad+\left(\left(T_\beta^+(v_nv_1)\right)^2+\left(T_\beta^+(v_nv_2)\right)^2+\ldots+0\right)\\&=2\sum_{1\le p<q\le n}^{n}\left(T_\beta^+(v_pv_q)\right)^2.\end{aligned}$$

Hence, $\sum\limits_{\substack{p=1\\ \lambda_p\in R}}^{n}{\lambda_p}^2=2\sum\limits_{1\le p<q\le n}\left(T_\beta^+(v_pv_q)\right)^2$. Similarly, we can show that

$\sum\limits_{\substack{p=1\\ \delta_p\in R}}^{n}{\delta_p}^2=2\sum\limits_{1\le p<q\le n}\left(I_\beta^+(v_pv_q)\right)^2, \sum\limits_{\substack{p=1\\ \gamma_p\in R}}^{n}{\gamma_p}^2=2\sum\limits_{1\le p<q\le n}\left(F_\beta^+(v_pv_q)\right)^2, \sum\limits_{\substack{p=1\\ \omega_p\in R}}^{n}{\omega_p}^2=2\sum\limits_{1\le p<q\le n}\left(T_\beta^-(v_pv_q)\right)^2,$
$\sum\limits_{\substack{p=1\\ \xi_p\in R}}^{n}{\xi_p}^2=2\sum\limits_{1\le p<q\le n}\left(I_\beta^-(v_pv_q)\right)^2$ *and* $\sum\limits_{\substack{p=1\\ \psi_p\in R}}^{n}{\psi_p}^2=2\sum\limits_{1\le p<q\le n}\left(F_\beta^-(v_pv_q)\right)^2.$

□

Example 2. *Consider a BSVNG $\overline{G}(\alpha,\beta)$ on $V=\{v_1,v_2,v_3,v_4,v_5,v_6,v_7\}$ as shown in Figure 1. Then*

$$E\left(T_\beta^+(v_pv_q)\right)=2.4231, E\left(I_\beta^+(v_pv_q)\right)=7.4142, E\left(F_\beta^+(v_pv_q)\right)=7.0806, E\left(T_\beta^-(v_pv_q)\right)=1.8767,$$
$$E\left(I_\beta^-(v_pv_q)\right)=5.609, E\left(F_\beta^-(v_pv_q)\right)=7.4715.$$

Therefore, $E(\overline{G})=\langle 2.4231, 7.4142, 7.0806, 1.8767, 5.6090, 7.4715\rangle$. Also, we have

$$\sum_{\substack{p=1\\ \lambda_p\in R}}^{7}\lambda_p=-0.6163-0.4338-0.1208-0.0406+0.1252+0.1495+0.9369=0,$$

$$\sum_{\substack{p=1\\ \delta_p\in S}}^{7}\delta_p=-1.4555-1.1839-0.7335-0.3342+0.4205+0.9179+2.3687=0,$$
$$\sum_{\substack{p=1\\ \gamma_p\in T}}^{7}\gamma_p=-1.6620-0.9839-0.6946-0.1998+0.2355+0.7439+2.5609=0,$$
$$\sum_{\substack{p=1\\ \omega_p\in U}}^{7}\omega_p=-0.7265-0.1609-0.0509+0.0476+0.1242+0.1464+0.6202=0,$$
$$\sum_{\substack{p=1\\ \xi_p\in V}}^{7}\xi_p=-1.7702-0.7044-0.3299+0.2795+0.5615+0.9685+0.9950=0,$$
$$\sum_{\substack{p=1\\ \psi_p\in W}}^{7}\psi_p=-2.5122-0.8107-0.4128+0.3321+0.7283+1.1356+1.5398=0.$$
$$\sum_{\substack{p=1\\ \lambda_p\in R}}^{7}\lambda_p^2=1.5000=2(0.7500)=2\sum_{1\le p<q\le 7}\left(T_\beta^+(v_pv_q)\right)^2,$$
$$\sum_{\substack{p=1\\ \delta_p\in S}}^{7}\delta_p^2=10.7999=2(5.3999)=2\sum_{1\le p<q\le 7}\left(I_\beta^+(v_pv_q)\right)^2,$$
$$\sum_{\substack{p=1\\ \gamma_p\in T}}^{7}\gamma_p^2=11.4197=2(5.7099)=2\sum_{1\le p<q\le 7}\left(F_\beta^+(v_pv_q)\right)^2,$$
$$\sum_{\substack{p=1\\ \omega_p\in U}}^{7}\omega_p^2=0.9800=2(0.4900)=2\sum_{1\le p<q\le 7}\left(T_\beta^-(v_pv_q)\right)^2,$$
$$\sum_{\substack{p=1\\ \xi_p\in V}}^{7}\xi_p^2=6.0600=2(3.0300)=2\sum_{1\le p<q\le 7}\left(I_\beta^-(v_pv_q)\right)^2,$$
$$\sum_{\substack{p=1\\ \psi_p\in W}}^{7}\psi_p^2=11.4401=2(5.7200)=2\sum_{1\le p<q\le 7}\left(F_\beta^-(v_pv_q)\right)^2.$$

We now give upper and lower bounds of energy of a BSVNG $\overline{G}$ in terms of the number of vertices and the sum of squares of positive truth-membership, positive indeterminacy-membership, positive falsity-membership, negative truth-membership values, negative indeterminacy-membership values, and negative falsity-membership values of the edges.

Theorem 2. *Let $\overline{G}(\alpha,\beta)$ be a BSVNG on n vertices with an adjacency matrix $A(\overline{G})=\left\langle A\left(T_\beta^+(v_pv_q)\right), A\left(I_\beta^+(v_pv_q)\right), A\left(F_\beta^+(v_pv_q)\right), A\left(T_\beta^-(v_pv_q)\right), A\left(I_\beta^-(v_pv_q)\right), A\left(F_\beta^-(v_pv_q)\right)\right\rangle$. Then*

i. $\sqrt{2\sum_{1\le p<q\le n}\left(T_\beta^+(v_pv_q)\right)^2+n(n-1)\left|T_\beta^+\right|^{\frac{2}{n}}}\le E\left(T_\beta^+(v_pv_q)\right)\le\sqrt{2n\sum_{1\le p<q\le n}\left(T_\beta^+(v_pv_q)\right)^2}$

ii. $\sqrt{2\sum_{1\le p<q\le n}\left(I_\beta^+(v_pv_q)\right)^2+n(n-1)\left|I_\beta^+\right|^{\frac{2}{n}}}\le E\left(I_\beta^+(v_pv_q)\right)\le\sqrt{2n\sum_{1\le p<q\le n}\left(I_\beta^+(v_pv_q)\right)^2}$

iii. $\sqrt{2\sum_{1\le p<q\le n}\left(F_\beta^+(v_pv_q)\right)^2+n(n-1)\left|F_\beta^+\right|^{\frac{2}{n}}}\le E\left(F_\beta^+(v_pv_q)\right)\le\sqrt{2n\sum_{1\le p<q\le n}\left(F_\beta^+(v_pv_q)\right)^2}$

iv. $\sqrt{2\sum_{1\le p<q\le n}\left(T_\beta^-(v_pv_q)\right)^2+n(n-1)\left|T_\beta^-\right|^{\frac{2}{n}}}\le E\left(T_\beta^-(v_pv_q)\right)\le\sqrt{2n\sum_{1\le p<q\le n}\left(T_\beta^-(v_pv_q)\right)^2}$

v. $\sqrt{2\sum_{1\le p<q\le n}\left(I_\beta^-(v_pv_q)\right)^2+n(n-1)\left|I_\beta^-\right|^{\frac{2}{n}}}\le E\left(I_\beta^-(v_pv_q)\right)\le\sqrt{2n\sum_{1\le p<q\le n}\left(I_\beta^-(v_pv_q)\right)^2}$

vi. $\sqrt{2\sum_{1\le p<q\le n}\left(F_\beta^-(v_pv_q)\right)^2+n(n-1)\left|F_\beta^-\right|^{\frac{2}{n}}}\le E\left(F_\beta^-(v_pv_q)\right)\le\sqrt{2n\sum_{1\le p<q\le n}\left(F_\beta^-(v_pv_q)\right)^2}$,

where $\left|T_\beta^+\right|, \left|I_\beta^+\right|, \left|F_\beta^+\right|, \left|T_\beta^-\right|, \left|I_\beta^-\right|$ and $\left|F_\beta^-\right|$ *are the determinant of* $A\left(T_\beta^+(v_p v_q)\right), A\left(I_\beta^+(v_p v_q)\right)$, $A\left(F_\beta^+(v_p v_q)\right), A\left(T_\beta^-(v_p v_q)\right), A\left(I_\beta^-(v_p v_q)\right)$ *and* $A\left(F_\beta^-(v_p v_q)\right)$, *respectively.*

Proof.

i. Upper bound:

Apply Cauchy-Schwarz inequality to the vectors $(1, 1, \ldots, 1)$ and $(|\lambda_1|, |\lambda_2|, \ldots, |\lambda_n|)$ with n entries, then

$$\sum_{p=1}^{n} |\lambda_p| \leq \sqrt{n} \sqrt{\sum_{p=}^{n} |\lambda_p|^2}, \tag{1}$$

$$\left(\sum_{p=1}^{n} \lambda_p\right)^2 = \sum_{p=1}^{n} |\lambda_p|^2 + 2 \sum_{1 \leq p < q \leq n} \lambda_p \lambda_q. \tag{2}$$

By comparing the coefficients of λ^{n-2} in the characteristic polynomial

$$\prod_{p=1}^{n} (\lambda - \lambda_p) = |A(\overline{G}) - \lambda I|,$$

we obtain

$$\sum_{1 \leq p < q \leq n} \lambda_p \lambda_q = - \sum_{1 \leq p < q \leq n} \left(T_\beta^+(v_p v_q)\right)^2. \tag{3}$$

Substituting (3) in (2), we obtain

$$\sum_{p=1}^{n} |\lambda_p|^2 = 2 \sum_{1 \leq p < q \leq n} \left(T_\beta^+(v_p v_q)\right)^2. \tag{4}$$

Meanwhile, substituting (4) in (1), we obtain

$$\sum_{p=1}^{n} |\lambda_p| \leq \sqrt{n} \sqrt{2 \sum_{1 \leq p < q \leq n} \left(T_\beta^+(v_p v_q)\right)^2} = \sqrt{2n \sum_{1 \leq p < q \leq n} \left(T_\beta^+(v_p v_q)\right)^2}.$$

Therefore,

$$E\left(T_\beta^+(v_p v_q)\right) \leq \sqrt{2n \sum_{1 \leq p < q \leq n} \left(T_\beta^+(v_p v_q)\right)^2}.$$

ii. Lower bound:

$$\begin{aligned}\left(E\left(T_\beta^+(v_p v_q)\right)\right)^2 &= \left(\sum_{p=1}^{n} |\lambda_p|\right)^2 \\ &= \sum_{p=1}^{n} |\lambda_p|^2 + 2 \sum_{1 \leq p < q \leq n} |\lambda_p \lambda_q| \\ &= 2 \sum_{1 \leq p < q \leq n} \left(T_\beta^+(v_p v_q)\right)^2 + \tfrac{2n(n-1)}{2} AM\{|\lambda_p \lambda_q|\}.\end{aligned}$$

Since $AM\{|\lambda_p \lambda_q|\} \geq GM\{|\lambda_p \lambda_q|\}, 1 \leq p < q \leq n, so$

$$E\left(T_\beta^+(v_p v_q)\right) \geq \sqrt{2 \sum_{1 \leq p < q \leq n} \left(T_\beta^+(v_p v_q)\right)^2 + n(n-1) GM\{|\lambda_p \lambda_q|\}}.$$

Also, since

$$GM\{|\lambda_p\lambda_q|\} = \left(\prod_{1\le p<q\le n}|\lambda_p\lambda_q|\right)^{\frac{2}{n(n-1)}} = \left(\prod_{p=1}^{n}|\lambda_p|^{n-1}\right)^{\frac{2}{n(n-1)}}$$
$$= \left(\prod_{p=1}^{n}|\lambda_p|\right)^{\frac{2}{n}} = \left|T_\beta^+\right|^{\frac{2}{n}},$$

then,

$$E\left(T_\beta^+(v_pv_q)\right) \ge \sqrt{2\sum_{1\le p<q\le n}\left(T_\beta^+(v_pv_q)\right)^2 + n(n-1)\left|T_\beta^+\right|^{\frac{2}{n}}}.$$

Thus,

$$\sqrt{2\sum_{1\le p<q\le n}\left(T_\beta^+(v_pv_q)\right)^2 + n(n-1)\left|T_\beta^+\right|^{\frac{2}{n}}} \le E\left(T_\beta^+(v_pv_q)\right) \le \sqrt{2n\sum_{1\le p<q\le n}\left(T_\beta^+(v_pv_q)\right)^2}.$$

Similarly, we can show that

$$\sqrt{2\sum_{1\le p<q\le n}\left(I_\beta^+(v_pv_q)\right)^2 + n(n-1)\left|I_\beta^+\right|^{\frac{2}{n}}} \le E\left(I_\beta^+(v_pv_q)\right) \le \sqrt{2n\sum_{1\le p<q\le n}\left(I_\beta^+(v_pv_q)\right)^2},$$
$$\sqrt{2\sum_{1\le p<q\le n}\left(F_\beta^+(v_pv_q)\right)^2 + n(n-1)\left|F_\beta^+\right|^{\frac{2}{n}}} \le E\left(F_\beta^+(v_pv_q)\right) \le \sqrt{2n\sum_{1\le p<q\le n}\left(F_\beta^+(v_pv_q)\right)^2},$$
$$\sqrt{2\sum_{1\le p<q\le n}\left(T_\beta^-(v_pv_q)\right)^2 + n(n-1)\left|T_\beta^-\right|^{\frac{2}{n}}} \le E\left(T_\beta^-(v_pv_q)\right) \le \sqrt{2n\sum_{1\le p<q\le n}\left(T_\beta^-(v_pv_q)\right)^2},$$
$$\sqrt{2\sum_{1\le p<q\le n}\left(I_\beta^-(v_pv_q)\right)^2 + n(n-1)\left|I_\beta^-\right|^{\frac{2}{n}}} \le E\left(I_\beta^-(v_pv_q)\right) \le \sqrt{2n\sum_{1\le p<q\le n}\left(I_\beta^-(v_pv_q)\right)^2},$$
$$\sqrt{2\sum_{1\le p<q\le n}\left(F_\beta^-(v_pv_q)\right)^2 + n(n-1)\left|F_\beta^-\right|^{\frac{2}{n}}} \le E\left(F_\beta^-(v_pv_q)\right) \le \sqrt{2n\sum_{1\le p<q\le n}\left(F_\beta^-(v_pv_q)\right)^2}.$$

□

Example 3. *(Illustration to Theorem 2) For the BSVNG $\overline{G}$, given in Figure 1,* $E\left(T_\beta^+(v_pv_q)\right) = 2.4231$*, lower bound = 1.8670 and upper bound = 3.2400, therefore,* $1.8670 \le 2.4231 \le 3.2400$. $E\left(I_\beta^+(v_pv_q)\right) = 7.4142$*, lower bound = 6.5423 and upper bound = 8.6940, therefore,* $6.5423 \le 7.4142 \le 8.6940$. $E\left(F_\beta^+(v_pv_q)\right) = 7.0806$*, lower bound = 5.7693 and upper bound = 8.9408, therefore,* $5.7693 \le 7.0806 \le 8.9408$. $E\left(T_\beta^-(v_pv_q)\right) = 1.8767$*, lower bound = 0.9899 and upper bound = 2.6192, therefore,* $0.9899 \le 1.8767 \le 2.6192$. $E\left(I_\beta^-(v_pv_q)\right) = 5.6090$*, lower bound = 2.4664 and upper bound = 6.5131, therefore,* $2.4664 \le 5.6090 \le 6.5131$. $E\left(F_\beta^-(v_pv_q)\right) = 7.4715$*, lower bound = 3.4926 and upper bound = 8.9487, therefore,* $3.4926 \le 7.4715 \le 8.9487$.

4. Laplacian Energy of Bipolar Single-Valued Neutrosophic Graphs (BSVNG)

Definition 9. *Let $\overline{G}(\alpha,\beta)$ be a BSVNG on n vertices. Then, the degree matrix*

$$D(\overline{G}) = \left\langle D\left(T_{\beta}^{+}(v_p v_q)\right), D\left(I_{\beta}^{+}(v_p v_q)\right), D\left(F_{\beta}^{+}(v_p v_q)\right), D\left(T_{\beta}^{-}(v_p v_q)\right), D\left(I_{\beta}^{-}(v_p v_q)\right), D\left(F_{\beta}^{-}(v_p v_q)\right)\right\rangle = \left[d_{pq}\right]$$

of $\overline{G}(\alpha, \beta)$ *is a* $n \times n$ *diagonal matrix defined as*:

$$d_{pq} = \begin{cases} d_{\overline{G}}(v_p) & if\ p = q \\ 0 & otherwise. \end{cases}$$

Definition 10. *The Laplacian matrix of a BSVNG* $\overline{G}(\alpha, \beta)$ *is defined as:*

$$L(\overline{G}) = \left\langle L\left(T_{\beta}^{+}(v_p v_q)\right), L\left(I_{\beta}^{+}(v_p v_q)\right), L\left(F_{\beta}^{+}(v_p v_q)\right), L\left(T_{\beta}^{-}(v_p v_q)\right), L\left(I_{\beta}^{-}(v_p v_q)\right), L\left(F_{\beta}^{-}(v_p v_q)\right)\right\rangle = D(\overline{G}) - A(\overline{G}),$$

where $D(\overline{G})$ *is a degree matrix of a BSVNG and* $A(\overline{G})$ *is an adjacency matrix.*

Definition 11. *The spectrum of the Laplacian matrix of a BSVNG* $L(\overline{G})$ *is defined as:* $\langle R_L, S_L, T_L, U_L, V_L, W_L\rangle$, *where* R_L, S_L, T_L, U_L, V_L *and* W_L *are the sets of Laplacian eigenvalues of* $L\left(T_{\beta}^{+}(v_p v_q)\right), L\left(I_{\beta}^{+}(v_p v_q)\right), L\left(F_{\beta}^{+}(v_p v_q)\right), L\left(T_{\beta}^{-}(v_p v_q)\right), L\left(I_{\beta}^{-}(v_p v_q)\right)$ *and* $L\left(F_{\beta}^{-}(v_p v_q)\right)$, *respectively.*

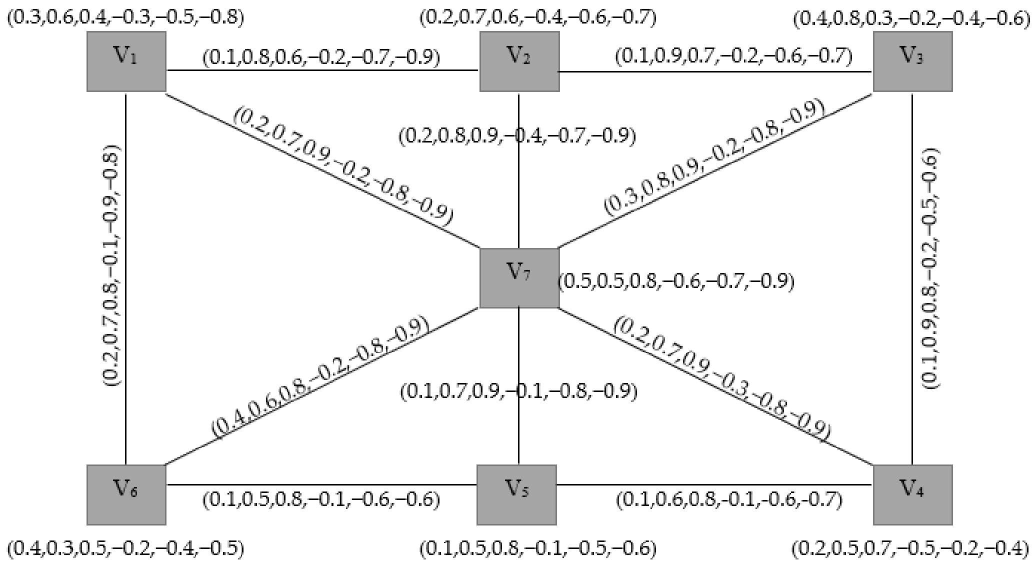
Example 4. *Consider a BSVNG* $\overline{G} = (\alpha, \beta)$ *of a graph* $G(V, E)$, *where* $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ *and* $E = \{v_1v_2, v_1v_6, v_1v_7, v_2v_3, v_2v_7, v_3v_4, v_3v_7, v_4v_5, v_4v_7, v_5v_6, v_5v_7, v_6v_7\}$, *as shown in* Figure 2, *defined by* Tables 4 and 5 *as follows:*

Table 4. Laplacian energy of BSVNG set on V.

α	v_1	v_2	v_3	v_4	v_5	v_6	v_7
T_{α}^{+}	0.3	0.2	0.4	0.2	0.1	0.4	0.5
I_{α}^{+}	0.6	0.7	0.8	0.5	0.5	0.3	0.5
F_{α}^{+}	0.4	0.6	0.3	0.7	0.8	0.5	0.8
T_{α}^{-}	−0.3	−0.4	−0.2	−0.5	−0.1	−0.2	−0.6
I_{α}^{-}	−0.5	−0.6	−0.4	−0.2	−0.5	−0.4	−0.7
F_{α}^{-}	−0.8	−0.7	−0.6	−0.4	−0.6	−0.5	−0.9

Table 5. Laplacian energy of BSVNG relation on V.

β	v_1v_2	v_1v_6	v_1v_7	v_2v_3	v_2v_7	v_3v_4	v_3v_7	v_4v_5	v_4v_7	v_5v_6	v_5v_7	v_6v_7
T_{β}^{+}	0.1	0.2	0.2	0.1	0.2	0.1	0.3	0.1	0.2	0.1	0.1	0.4
I_{β}^{+}	0.8	0.7	0.7	0.9	0.8	0.9	0.8	0.6	0.7	0.5	0.7	0.6
F_{β}^{+}	0.6	0.8	0.9	0.7	0.9	0.8	0.9	0.8	0.9	0.8	0.9	0.8
T_{β}^{-}	−0.2	−0.1	−0.2	−0.2	−0.4	−0.2	−0.2	−0.1	−0.3	−0.1	−0.1	−0.2
I_{β}^{-}	−0.7	−0.9	−0.8	−0.6	−0.7	−0.5	−0.8	−0.6	−0.8	−0.6	−0.8	−0.8
F_{β}^{-}	−0.9	−0.8	−0.9	−0.7	−0.9	−0.6	−0.9	−0.7	−0.9	−0.6	−0.9	−0.9



The adjacency, degree and the Laplacian matrices of the BSVNG shown in Figure 2

$$A(\overline{G})=\begin{pmatrix}
\left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0.1,0.8,0.6,\\-0.2,-0.7,-0.9\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0.2,0.7,0.8,\\-0.1,-0.9,-0.8\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0.2,0.7,0.9,\\-0.2,-0.8,-0.9\end{smallmatrix}\right\rangle\\
\left\langle\begin{smallmatrix}0.1,0.8,0.6\\-0.2,-0.7,-0.9\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0.1,0.9,0.7,\\-0.2,-0.6,-0.7\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0.2,0.8,0.9,\\-0.4,-0.7,-0.9\end{smallmatrix}\right\rangle\\
\left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0.1,0.9,0.7,\\-0.2,-0.6,-0.7\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0.1,0.9,0.8,\\-0.2,-0.5,-0.6\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0.3,0.8,0.9,\\-0.2,-0.8,-0.9\end{smallmatrix}\right\rangle\\
\left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0.1,0.9,0.8,\\-0.2,-0.5,-0.6\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0.1,0.6,0.8,\\-0.1,-0.6,-0.7\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0.2,0.7,0.9,\\-0.3,-0.8,-0.9\end{smallmatrix}\right\rangle\\
\left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0.1,0.6,0.8,\\-0.1,-0.6,-0.7\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0.1,0.5,0.8,\\-0.1,-0.6,-0.6\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0.1,0.7,0.9,\\-0.1,-0.8,-0.9\end{smallmatrix}\right\rangle\\
\left\langle\begin{smallmatrix}0.2,0.7,0.8,\\-0.1,-0.9,-0.8\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0.1,0.5,0.8,\\-0.1,-0.6,-0.6\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0.4,0.6,0.8,\\-0.2,-0.8,-0.9\end{smallmatrix}\right\rangle\\
\left\langle\begin{smallmatrix}0.2,0.7,0.9,\\-0.2,-0.8,-0.9\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0.2,0.8,0.9,\\-0.4,-0.7,-0.9\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0.3,0.8,0.9,\\-0.2,-0.8,-0.9\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0.2,0.7,0.9,\\-0.3,-0.8,-0.9\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0.1,0.7,0.9,\\-0.1,-0.8,-0.9\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0.4,0.6,0.8,\\-0.2,-0.8,-0.9\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle
\end{pmatrix}$$

$$D(\overline{G})=\begin{pmatrix}
\left\langle\begin{smallmatrix}0.5,2.2,2.3\\-0.5,-2.4,-2.6\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle\\
\left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0.4,2.5,2.2\\-0.8,-2.0,-2.5\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle\\
\left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0.5,2.6,2.4\\-0.6,-1.9,-2.2\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle\\
\left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0.4,2.2,2.5\\-0.6,-1.9,-2.2\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle\\
\left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0.3,1.8,2.5\\-0.3,-2.0,-2.2\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle\\
\left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0.7,1.8,2.4\\-0.4,-2.3,-2.3\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle\\
\left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}0,0,0\\0,0,0\end{smallmatrix}\right\rangle & \left\langle\begin{smallmatrix}1.4,4.3,5.3,\\-1.4,-4.7,-5.4\end{smallmatrix}\right\rangle
\end{pmatrix}$$

$$\mathrm{L}(\overline{\mathrm{G}}) = \begin{pmatrix}
\left\langle\begin{matrix}0.5,2.2,2.3\\-0.5,-2.4,-2.6\end{matrix}\right\rangle & \left\langle\begin{matrix}-0.1,-0.8,-0.6,\\0.2,0.7,0.9\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}-0.2,-0.7,-0.8,\\0.1,0.9,0.8\end{matrix}\right\rangle & \left\langle\begin{matrix}-0.2,-0.7,-0.9,\\0.2,0.8,0.9\end{matrix}\right\rangle \\
\left\langle\begin{matrix}-0.1,-0.8,-0.6,\\0.2,0.7,0.9\end{matrix}\right\rangle & \left\langle\begin{matrix}0.4,2.5,2.2\\-0.8,-2.0,-2.5\end{matrix}\right\rangle & \left\langle\begin{matrix}-0.1,-0.9,-0.7,\\0.2,0.6,0.7\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}-0.2,-0.8,-0.9,\\0.4,0.7,0.9\end{matrix}\right\rangle \\
\left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}-0.1,-0.9,-0.7,\\0.2,0.6,0.7\end{matrix}\right\rangle & \left\langle\begin{matrix}0.5,2.6,2.4\\-0.6,-1.9,-2.2\end{matrix}\right\rangle & \left\langle\begin{matrix}-0.1,-0.9,-0.8,\\0.2,0.5,0.6\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}-0.3,-0.8,-0.9,\\0.2,0.8,0.9\end{matrix}\right\rangle \\
\left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}-0.1,-0.9,-0.8,\\0.2,0.5,0.6\end{matrix}\right\rangle & \left\langle\begin{matrix}0.4,2.2,2.5\\-0.6,-1.9,-2.2\end{matrix}\right\rangle & \left\langle\begin{matrix}-0.1,-0.6,-0.8,\\0.1,0.6,0.7\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}-0.2,-0.7,-0.9,\\0.3,0.8,0.9\end{matrix}\right\rangle \\
\left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}-0.1,-0.6,-0.8,\\0.1,0.6,0.7\end{matrix}\right\rangle & \left\langle\begin{matrix}0.3,1.8,2.5\\-0.3,-2.0,-2.2\end{matrix}\right\rangle & \left\langle\begin{matrix}-0.1,-0.5,-0.8,\\0.1,0.6,0.6\end{matrix}\right\rangle & \left\langle\begin{matrix}-0.1,-0.7,-0.9,\\0.1,0.8,0.9\end{matrix}\right\rangle \\
\left\langle\begin{matrix}-0.2,-0.7,-0.8,\\0.1,0.9,0.8\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}-0.1,-0.5,-0.8,\\0.1,0.6,0.6\end{matrix}\right\rangle & \left\langle\begin{matrix}0.7,1.8,2.4\\-0.4,-2.3,-2.3\end{matrix}\right\rangle & \left\langle\begin{matrix}-0.4,-0.6,-0.8,\\0.2,0.8,0.9\end{matrix}\right\rangle \\
\left\langle\begin{matrix}-0.2,-0.7,-0.9,\\0.2,0.8,0.9\end{matrix}\right\rangle & \left\langle\begin{matrix}-0.2,-0.8,-0.9,\\0.4,0.7,0.9\end{matrix}\right\rangle & \left\langle\begin{matrix}-0.3,-0.8,-0.9,\\0.2,0.8,0.9\end{matrix}\right\rangle & \left\langle\begin{matrix}-0.2,-0.7,-0.9,\\0.3,0.8,0.9\end{matrix}\right\rangle & \left\langle\begin{matrix}-0.1,-0.7,-0.9,\\0.1,0.8,0.9\end{matrix}\right\rangle & \left\langle\begin{matrix}-0.4,-0.6,-0.8,\\0.2,0.8,0.9\end{matrix}\right\rangle & \left\langle\begin{matrix}1.4,4.3,5.3,\\-1.4,-4.7,-5.4\end{matrix}\right\rangle
\end{pmatrix}$$

The Laplacian spectrum of a BSVNG $\overline{G}$, given in Figure 2 is as follows:

$$\begin{aligned}
Laplacian\ Spec\left(T_\beta^+(v_p v_q)\right) &= \{-0.5438, -0.2286, -0.1241, -0.00737, 0.0889, 0.1449, 0.7364\}; \\
Laplacian\ Spec\left(I_\beta^+(v_p v_q)\right) &= \{-1.5403, -1.1802, -0.7856, -0.5981, 0.6368, 0.7856, 2.6779\}; \\
Laplacian\ Spec\left(F_\beta^+(v_p v_q)\right) &= \{-1.5746, -1.4898, -0.8029, -0.6783, 0.6928, 0.8096, 3.0432\}; \\
Laplacian\ Spec\left(T_\beta^-(v_p v_q)\right) &= \{-0.7907, -0.1534, -0.1218, 0.1157, 0.1516, 0.2558, 0.5428\}; \\
Laplacian\ Spec\left(I_\beta^-(v_p v_q)\right) &= \{-2.6887, -0.7102, -0.5976, 0.5962, 0.6587, 1.3399, 1.4016\}; \\
Laplacian\ Spec\left(F_\beta^-(v_p v_q)\right) &= \{-3.0418, -0.8027, -0.6307, 0.6283, 0.7733, 1.4673, 1.6062\}.
\end{aligned}$$

Therefore,

$$Laplacian\ Spec(\overline{\mathrm{G}}) = \left\{ \begin{matrix}
\langle -0.5438, -1.5403, -1.5746, -0.7907, -2.6887, -3.0418\rangle, \\
\langle -0.2286, -1.1802, -1.4898, -0.1534, -0.7102, -0.8027\rangle, \\
\langle -0.1241, -0.7856, -0.8029, -0.1218, -0.5976, -0.6307\rangle, \\
\langle -0.0737, -0.5981, -0.6783, 0.1157, 0.5962, 0.6283\rangle, \\
\langle 0.0889, 0.6368, 0.6928, 0.1516, 0.6587, 0.7733\rangle, \\
\langle 0.1449, 0.7856, 0.8096, 0.2558, 1.3399, 1.4673\rangle, \\
\langle 0.7364, 2.6779, 3.0432, 0.5428, 1.4016, 1.6062\rangle
\end{matrix} \right\}.$$

Theorem 3. *Let $\overline{\mathrm{G}} = (\alpha, \beta)$ be a BSVNG and let*

$$L(\overline{\mathrm{G}}) = \left\langle L\left(T_\beta^+(v_p v_q)\right), L\left(I_\beta^+(v_p v_q)\right), L\left(F_\beta^+(v_p v_q)\right), L\left(T_\beta^-(v_p v_q)\right), L\left(I_\beta^-(v_p v_q)\right), L\left(F_\beta^-(v_p v_q)\right)\right\rangle$$

be the Laplacian matrix of $\overline{\mathrm{G}}$. If $\phi_1 \geq \phi_2 \geq \ldots \geq \phi_n, \zeta_1 \geq \zeta_2 \geq \ldots \geq \zeta_n, \varphi_1 \geq \varphi_2 \geq \ldots \geq \varphi_n$, $\mu_1 \geq \mu_2 \geq \ldots \geq \mu_n, \varpi_1 \geq \varpi_2 \geq \ldots \geq \varpi_n$, and $\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_n$ are the eigenvalues of $L\left(T_\beta^+(v_p v_q)\right), L\left(I_\beta^+(v_p v_q)\right), L\left(F_\beta^+(v_p v_q)\right), L\left(T_\beta^-(v_p v_q)\right), L\left(I_\beta^-(v_p v_q)\right)$ and $L\left(F_\beta^-(v_p v_q)\right)$, respectively, then,

1. $\sum\limits_{\substack{p=1\\ \phi_p \in R_L}}^{n} \phi_p = 2\sum\limits_{1 \leq p < q \leq n} T_\beta^+(v_p v_q)$, $\sum\limits_{\substack{p=1\\ \zeta_p \in S_L}}^{n} \zeta_p = 2\sum\limits_{1 \leq p < q \leq n} I_\beta^+(v_p v_q)$, $\sum\limits_{\substack{p=1\\ \varphi_p \in T_L}}^{n} \varphi_p$ $= 2\sum\limits_{1 \leq p < q \leq n} F_\beta^+(v_p v_q)$, $\sum\limits_{\substack{p=1\\ \mu_p \in U_L}}^{n} \mu_p = 2\sum\limits_{1 \leq p < q \leq n} T_\beta^-(v_p v_q)$, $\sum\limits_{\substack{p=1\\ \varpi_p \in V_L}}^{n} \varpi_p = 2\sum\limits_{1 \leq p < q \leq n} I_\beta^-(v_p v_q)$ and $\sum\limits_{\substack{p=1\\ \sigma_p \in W_L}}^{n} \sigma_p = 2\sum\limits_{1 \leq p < q \leq n} F_\beta^-(v_p v_q)$.

2. $\sum\limits_{\substack{p=1\\ \phi_p\in R_L}}^{n} {\phi_p}^2 = 2\sum\limits_{1\le p<q\le n}\left(T_\beta^+(v_pv_q)\right)^2 + \sum\limits_{p=1}^{n} d^2_{T_\beta^+(v_pv_q)}(v_p)$, $\sum\limits_{\substack{p=1\\ \zeta_p\in S_L}}^{n} {\zeta_p}^2 = 2\sum\limits_{1\le p<q\le n}\left(I_\beta^+(v_pv_q)\right)^2 + \sum\limits_{p=1}^{n} d^2_{I_\beta^+(v_pv_q)}(v_p)$, $\sum\limits_{\substack{p=1\\ \varphi_p\in T_L}}^{n} {\varphi_p}^2 = 2\sum\limits_{1\le p<q\le n}\left(F_\beta^+(v_pv_q)\right)^2 + \sum\limits_{p=1}^{n} d^2_{F_\beta^+(v_pv_q)}(v_p)$, $\sum\limits_{\substack{p=1\\ \mu_p\in U_L}}^{n} {\mu_p}^2 = 2\sum\limits_{1\le p<q\le n}\left(T_\beta^-(v_pv_q)\right)^2 + \sum\limits_{p=1}^{n} d^2_{T_\beta^-(v_pv_q)}(v_p)$, $\sum\limits_{\substack{p=1\\ \varpi_p\in V_L}}^{n} {\varpi_p}^2 = 2\sum\limits_{1\le p<q\le n}\left(I_\beta^-(v_pv_q)\right)^2 + \sum\limits_{p=1}^{n} d^2_{I_\beta^-(v_pv_q)}(v_p)$ and $\sum\limits_{\substack{p=1\\ \sigma_p\in W_L}}^{n} {\sigma_p}^2 = 2\sum\limits_{1\le p<q\le n}\left(F_\beta^-(v_pv_q)\right)^2 + \sum\limits_{p=1}^{n} d^2_{F_\beta^-(v_pv_q)}(v_p)$.

Proof.

1. Since $L(\overline{G})$ is a symmetric matrix with non-negative Laplacian eigenvalues, we have

$$\sum_{\substack{p=1\\ \phi_p\in R_L}}^{n} \phi_p = tr\left(L(\overline{G})\right) = \sum_{p=1}^{n} d_{T_\beta^+(v_pv_q)}(v_p) = 2\sum_{1\le p<q\le n} T_\beta^+(v_pv_q).$$

Similarly, it is easy to show that

$$\sum_{\substack{p=1\\ \zeta_p\in S_L}}^{n} \zeta_p = 2\sum_{1\le p<q\le n} I_\beta^+(v_pv_q),\ \sum_{\substack{p=1\\ \varphi_p\in T_L}}^{n} \varphi_p = 2\sum_{1\le p<q\le n} F_\beta^+(v_pv_q),\ \sum_{\substack{p=1\\ \mu_p\in U_L}}^{n} \mu_p = 2\sum_{1\le p<q\le n} T_\beta^-(v_pv_q),$$
$$\sum_{\substack{p=1\\ \varpi_p\in V_L}}^{n} \varpi_p = 2\sum_{1\le p<q\le n} I_\beta^-(v_pv_q)\ and\ \sum_{\substack{p=1\\ \sigma_p\in W_L}}^{n} \sigma_p = 2\sum_{1\le p<q\le n} F_\beta^-(v_pv_q).$$

2. By definition of the Laplacian matrix, we have

$$L\left(T_\beta^+(v_pv_q)\right) = \begin{pmatrix} d_{T_\beta^+(v_pv_q)}(v_1) & -T_\beta^+(v_1v_2) & \cdots & -T_\beta^+(v_1v_n) \\ -T_\beta^+(v_2v_1) & d_{T_\beta^+(v_pv_q)}(v_1) & \cdots & -T_\beta^+(v_2v_n) \\ \vdots & \vdots & \ddots & \vdots \\ -T_\beta^+(v_nv_1) & -T_\beta^+(v_nv_2) & \cdots & d_{T_\beta^+(v_pv_q)}(v_n) \end{pmatrix}.$$

By trace properties of a matrix, we have

$$tr\left(\left(L\left(T_\beta^+(v_pv_q)\right)\right)^2\right) = \sum_{\substack{p=1\\ \phi_p\in R_L}}^{n} {\phi_p}^2,$$

where

$$\begin{aligned} tr\left(\left(L\left(T_\beta^+(v_pv_q)\right)\right)^2\right) &= \left(d^2_{T_\beta^+(v_pv_q)}(v_1) + T_\beta^+(v_1v_2)^2 + \ldots + T_\beta^+(v_1v_n)^2\right) \\ &+ \left(T_\beta^+(v_2v_1)^2 + d^2_{T_\beta^+(v_pv_q)}(v_2) + \ldots + T_\beta^+(v_2v_n)^2\right) \\ &+ \ldots + \left(T_\beta^+(v_nv_1)^2 + T_\beta^+(v_nv_2)^2 + \ldots + d^2_{T_\beta^+(v_pv_q)}(v_n)\right) \\ &= 2\sum_{1\le p<q\le n}\left(T_\beta^+(v_pv_q)\right)^2 + \sum_{p=1}^{n} d^2_{T_\beta^+(v_pv_q)}(v_p). \end{aligned}$$

Therefore,

$$\sum_{\substack{p=1\\ \phi_p \in R_L}}^{n} \phi_p{}^2 = 2\sum_{1\le p<q\le n}\left(T_\beta^+(v_pv_q)\right)^2 + \sum_{p=1}^{n} d^2_{T_\beta^+(v_pv_q)}(v_p).$$

Similarly, we can show that

$$\sum_{\substack{p=1\\ \zeta_p \in S_L}}^{n} \zeta_p{}^2 = 2\sum_{1\le p<q\le n}\left(I_\beta^+(v_pv_q)\right)^2 + \sum_{p=1}^{n} d^2_{I_\beta^+(v_pv_q)}(v_p),\ \sum_{\substack{p=1\\ \varphi_p \in T_L}}^{n} \varphi_p{}^2 = 2\sum_{1\le p<q\le n}\left(F_\beta^+(v_pv_q)\right)^2 + \sum_{p=1}^{n} d^2_{F_\beta^+(v_pv_q)}(v_p),$$
$$\sum_{\substack{p=1\\ \mu_p \in U_L}}^{n} \mu_p{}^2 = 2\sum_{1\le p<q\le n}\left(T_\beta^-(v_pv_q)\right)^2 + \sum_{p=1}^{n} d^2_{T_\beta^-(v_pv_q)}(v_p),\ \sum_{\substack{p=1\\ \varpi_p \in V_L}}^{n} \varpi_p{}^2 = 2\sum_{1\le p<q\le n}\left(I_\beta^-(v_pv_q)\right)^2 + \sum_{p=1}^{n} d^2_{I_\beta^-(v_pv_q)}(v_p),$$
$$\text{and } \sum_{\substack{p=1\\ \sigma_p \in W_L}}^{n} \sigma_p{}^2 = 2\sum_{1\le p<q\le n}\left(F_\beta^-(v_pv_q)\right)^2 + \sum_{p=1}^{n} d^2_{F_\beta^-(v_pv_q)}(v_p).$$

□

Definition 12. *The Laplacian energy of a BSVNG $\overline{G}(\alpha,\beta)$ is defined as*

$$\begin{aligned} LE(\overline{G}) &= \left\langle LE\left(T_\beta^+(v_pv_q)\right), LE\left(I_\beta^+(v_pv_q)\right), LE\left(F_\beta^+(v_pv_q)\right), LE\left(T_\beta^-(v_pv_q)\right), LE\left(I_\beta^-(v_pv_q)\right), LE\left(F_\beta^-(v_pv_q)\right)\right\rangle \\ &= \left\langle \sum_{p=1}^{n}|\varsigma_p|, \sum_{p=1}^{n}|\tau_p|, \sum_{p=1}^{n}|\upsilon_p|, \sum_{p=1}^{n}|\vartheta_p|, \sum_{p=1}^{n}|\xi_p|, \sum_{p=1}^{n}|\varepsilon_p| \right\rangle, \end{aligned}$$

where

$$\varsigma_p = \phi_p - \frac{2\sum\limits_{1\le p<q\le n} T_\beta^+(v_pv_q)}{n}, \tau_p = \zeta_p - \frac{2\sum\limits_{1\le p<q\le n} I_\beta^+(v_pv_q)}{n}, \upsilon_p = \varphi_p - \frac{2\sum\limits_{1\le p<q\le n} F_\beta^+(v_pv_q)}{n},$$
$$\vartheta_p = \mu_p - \frac{2\sum\limits_{1\le p<q\le n} T_\beta^-(v_pv_q)}{n}, \xi_p = \varpi_p - \frac{2\sum\limits_{1\le p<q\le n} I_\beta^-(v_pv_q)}{n}, \varepsilon_p = \sigma_p - \frac{2\sum\limits_{1\le p<q\le n} F_\beta^-(v_pv_q)}{n}.$$

Theorem 4. *Let $\overline{G} = (\alpha,\beta)$ be a BSVNG and let $L(\overline{G})$ be the Laplacian matrix of $\overline{G}$. If $\phi_1 \ge \phi_2 \ge \ldots \ge \phi_n$, $\zeta_1 \ge \zeta_2 \ge \ldots \ge \zeta_n$, $\varphi_1 \ge \varphi_2 \ge \ldots \ge \varphi_n$, $\mu_1 \ge \mu_2 \ge \ldots \ge \mu_n$, $\varpi_1 \ge \varpi_2 \ge \ldots \ge \varpi_n$, and $\sigma_1 \ge \sigma_2 \ge \ldots \ge \sigma_n$ are the eigenvalues of $L\left(T_\beta^+(v_pv_q)\right)$, $L\left(I_\beta^+(v_pv_q)\right)$, $L\left(F_\beta^+(v_pv_q)\right)$, $L\left(T_\beta^-(v_pv_q)\right)$, $L\left(I_\beta^-(v_pv_q)\right)$ and $L\left(F_\beta^-(v_pv_q)\right)$, respectively, and*

$$\varsigma_p = \phi_p - \frac{2\sum\limits_{1\le p<q\le n} T_\beta^+(v_pv_q)}{n}, \tau_p = \zeta_p - \frac{2\sum\limits_{1\le p<q\le n} I_\beta^+(v_pv_q)}{n}, \upsilon_p = \varphi_p - \frac{2\sum\limits_{1\le p<q\le n} F_\beta^+(v_pv_q)}{n},$$
$$\vartheta_p = \mu_p - \frac{2\sum\limits_{1\le p<q\le n} T_\beta^-(v_pv_q)}{n}, \xi_p = \varpi_p - \frac{2\sum\limits_{1\le p<q\le n} I_\beta^-(v_pv_q)}{n}, \varepsilon_p = \sigma_p - \frac{2\sum\limits_{1\le p<q\le n} F_\beta^-(v_pv_q)}{n}.$$

Then,

$$\sum_{p=1}^{n}\varsigma_p = 0, \sum_{p=1}^{n}\tau_p = 0, \sum_{p=1}^{n}\upsilon_p = 0, \sum_{p=1}^{n}\vartheta_p = 0, \sum_{p=1}^{n}\xi_p = 0, \sum_{p=1}^{n}\varepsilon_p = 0,$$
$$\sum_{p=1}^{n}\varsigma_p^2 = 2\Omega_T^+, \sum_{p=1}^{n}\tau_p^2 = 2\Omega_I^+, \sum_{p=1}^{n}\upsilon_p^2 = 2\Omega_F^+, \sum_{p=1}^{n}\vartheta_p^2 = 2\Omega_T^-, \sum_{p=1}^{n}\xi_p^2 = 2\Omega_I^-, \sum_{p=1}^{n}\varepsilon_p^2 = 2\Omega_F^-,$$

where

$$\Omega_T^+ = \sum_{1\le p<q\le n}\left(T_\beta^+(v_pv_q)\right)^2 + \frac{1}{2}\sum_{p=1}^{n}\left(d_{T_\beta^+(v_pv_q)}(v_p) - \frac{2\sum_{1\le p<q\le n} T_\beta^+(v_pv_q)}{n}\right)^2,$$
$$\Omega_I^+ = \sum_{1\le p<q\le n}\left(I_\beta^+(v_pv_q)\right)^2 + \frac{1}{2}\sum_{p=1}^{n}\left(d_{I_\beta^+(v_pv_q)}(v_p) - \frac{2\sum_{1\le p<q\le n} I_\beta^+(v_pv_q)}{n}\right)^2,$$
$$\Omega_F^+ = \sum_{1\le p<q\le n}\left(F_\beta^+(v_pv_q)\right)^2 + \frac{1}{2}\sum_{p=1}^{n}\left(d_{F_\beta^+(v_pv_q)}(v_p) - \frac{2\sum_{1\le p<q\le n} F_\beta^+(v_pv_q)}{n}\right)^2,$$
$$\Omega_T^- = \sum_{1\le p<q\le n}\left(T_\beta^-(v_pv_q)\right)^2 + \frac{1}{2}\sum_{p=1}^{n}\left(d_{T_\beta^-(v_pv_q)}(v_p) - \frac{2\sum_{1\le p<q\le n} T_\beta^-(v_pv_q)}{n}\right)^2,$$
$$\Omega_I^- = \sum_{1\le p<q\le n}\left(I_\beta^-(v_pv_q)\right)^2 + \frac{1}{2}\sum_{p=1}^{n}\left(d_{I_\beta^-(v_pv_q)}(v_p) - \frac{2\sum_{1\le p<q\le n} I_\beta^-(v_pv_q)}{n}\right)^2,$$
$$\Omega_F^- = \sum_{1\le p<q\le n}\left(F_\beta^-(v_pv_q)\right)^2 + \frac{1}{2}\sum_{p=1}^{n}\left(d_{F_\beta^-(v_pv_q)}(v_p) - \frac{2\sum_{1\le p<q\le n} F_\beta^-(v_pv_q)}{n}\right)^2.$$

Proof. The proof is similar to that of Theorem 1. □

Example 5. *Consider a BSVNG, $\overline{G} = (\alpha, \beta)$ on $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$, as shown in Figure 2. Then,*

$$LE\left(T_\beta^+(v_pv_q)\right) = 1.9404, LE\left(I_\beta^+(v_pv_q)\right) = 8.2045, LE\left(F_\beta^+(v_pv_q)\right) = 9.0912,$$
$$LE\left(T_\beta^-(v_pv_q)\right) = 2.1318, LE\left(I_\beta^-(v_pv_q)\right) = 7.9929, LE\left(F_\beta^-(v_pv_q)\right) = 8.9503.$$

Therefore, $LE(\overline{G}) = \langle 1.9404, 8.2045, 9.0912, 2.1318, 7.9929, 8.9503\rangle$. We also have

$$\sum_{p=1}^{7}\varsigma_p = -0.5438 - 0.2286 - 0.1241 - 0.0737 + 0.0889 + 0.1449 + 0.7364 = 0$$
$$\sum_{p=1}^{7}\tau_p = -1.5403 - 1.1802 - 0.7856 - 0.5981 + 0.6368 + 0.7856 + 2.6779 = 0$$
$$\sum_{p=1}^{7}\upsilon_p = -1.5746 - 1.4898 - 0.8029 - 0.6783 + 0.6928 + 0.8096 + 3.0432 = 0$$
$$\sum_{p=1}^{7}\vartheta_p = -0.7907 - 0.1534 - 0.1218 + 0.1157 + 0.1516 + 0.2558 + 0.5428 = 0$$
$$\sum_{p=1}^{7}\xi_p = -2.6887 - 0.7102 - 0.5976 + 0.5962 + 0.6587 + 1.3399 + 1.4016 = 0$$
$$\sum_{p=1}^{7}\varepsilon_p = -3.0418 - 0.8027 - 0.6307 + 0.6283 + 0.7733 + 1.4673 + 1.6062 = 0$$

$$\sum_{p=1}^{7} \varsigma_p = 0.93999 = 2(0.4699) = 2\Omega_T^+$$
$$\sum_{p=1}^{7} \tau_p = 12.9341 = 2(6.46705) = 2\Omega_I^+$$
$$\sum_{p=1}^{7} v_p = 16.2000 = 2(8.1000) = 2\Omega_F^+$$
$$\sum_{p=1}^{7} \vartheta_p = 1.0600 = 2(0.5300) = 2\Omega_T^-$$
$$\sum_{p=1}^{7} \xi_p = 12.6398 = 2(6.3199) = 2\Omega_I^-$$
$$\sum_{p=1}^{7} \varepsilon_p = 16.0203 = 2(8.0102) = 2\Omega_F^-$$

Theorem 5. *Let $\overline{G} = (\alpha, \beta)$ be a BSVNG on n vertices and let*

$$L(\overline{G}) = \left\langle L\left(T_\beta^+(v_pv_q)\right), L\left(I_\beta^+(v_pv_q)\right), L\left(F_\beta^+(v_pv_q)\right), L\left(T_\beta^-(v_pv_q)\right), L\left(I_\beta^-(v_pv_q)\right), L\left(F_\beta^-(v_pv_q)\right)\right\rangle$$

be the Laplacian matrix of $\overline{G}$. Then

i. $LE\left(T_\beta^+(v_pv_q)\right) \le \sqrt{2n\sum_{1\le p<q\le n}\left(T_\beta^+(v_pv_q)\right)^2 + n\sum_{p=1}^{n}\left(d_{T_\beta^+(v_pv_q)}(v_p) - \frac{2\sum_{1\le p<q\le n} T_\beta^+(v_pv_q)}{n}\right)^2}$;

ii. $LE\left(I_\beta^+(v_pv_q)\right) \le \sqrt{2n\sum_{1\le p<q\le n}\left(I_\beta^+(v_pv_q)\right)^2 + n\sum_{p=1}^{n}\left(d_{I_\beta^+(v_pv_q)}(v_p) - \frac{2\sum_{1\le p<q\le n} I_\beta^+(v_pv_q)}{n}\right)^2}$;

iii. $LE\left(F_\beta^+(v_pv_q)\right) \le \sqrt{2n\sum_{1\le p<q\le n}\left(F_\beta^+(v_pv_q)\right)^2 + n\sum_{p=1}^{n}\left(d_{F_\beta^+(v_pv_q)}(v_p) - \frac{2\sum_{1\le p<q\le n} F_\beta^+(v_pv_q)}{n}\right)^2}$;

iv. $LE\left(T_\beta^-(v_pv_q)\right) \le \sqrt{2n\sum_{1\le p<q\le n}\left(T_\beta^-(v_pv_q)\right)^2 + n\sum_{p=1}^{n}\left(d_{T_\beta^-(v_pv_q)}(v_p) - \frac{2\sum_{1\le p<q\le n} T_\beta^-(v_pv_q)}{n}\right)^2}$;

v. $LE\left(I_\beta^-(v_pv_q)\right) \le \sqrt{2n\sum_{1\le p<q\le n}\left(I_\beta^-(v_pv_q)\right)^2 + n\sum_{p=1}^{n}\left(d_{I_\beta^-(v_pv_q)}(v_p) - \frac{2\sum_{1\le p<q\le n} I_\beta^-(v_pv_q)}{n}\right)^2}$;

vi. $LE\left(F_\beta^-(v_pv_q)\right) \le \sqrt{2n\sum_{1\le p<q\le n}\left(F_\beta^-(v_pv_q)\right)^2 + n\sum_{p=1}^{n}\left(d_{F_\beta^-(v_pv_q)}(v_p) - \frac{2\sum_{1\le p<q\le n} F_\beta^-(v_pv_q)}{n}\right)^2}$.

Proof. We apply Cauchy-Schwarz inequality to the vectors $(1, 1, \ldots, 1)$ and $(|\varsigma_1|, |\varsigma_2|, \ldots, |\varsigma_n|)$ with n entries, which yields

$$\sum_{p=1}^{n} |\varsigma_p| \le \sqrt{n}\sqrt{\sum_{p=1}^{n} |\varsigma_p|^2}$$
$$LE\left(T_\beta^+(v_pv_q)\right) \le \sqrt{n}\sqrt{2\Omega_T^+} = \sqrt{2n\Omega_T^+}.$$

Since

$$\Omega_T^+ = \sum_{1\le p<q\le n}\left(T_\beta^+(v_pv_q)\right)^2 + \frac{1}{2}\sum_{p=1}^{n}\left(d_{T_\beta^+(v_pv_q)}(v_p) - \frac{2\sum_{1\le p<q\le n} T_\beta^+(v_pv_q)}{n}\right)^2,$$

therefore,

$$LE\Big(T_\beta^+(v_pv_q)\Big) \le \sqrt{2n\sum_{1\le p<q\le n}\Big(T_\beta^+(v_pv_q)\Big)^2 + n\sum_{p=1}^{n}\left(d_{T_\beta^+(v_pv_q)}(v_p) - \frac{2\sum_{1\le p<q\le n}T_\beta^+(v_pv_q)}{n}\right)^2}.$$

Similarly, we can show that

$$LE\Big(I_\beta^+(v_pv_q)\Big) \le \sqrt{2n\sum_{1\le p<q\le n}\Big(I_\beta^+(v_pv_q)\Big)^2 + n\sum_{p=1}^{n}\left(d_{I_\beta^+(v_pv_q)}(v_p) - \frac{2\sum_{1\le p<q\le n}I_\beta^+(v_pv_q)}{n}\right)^2},$$

$$LE\Big(F_\beta^+(v_pv_q)\Big) \le \sqrt{2n\sum_{1\le p<q\le n}\Big(F_\beta^+(v_pv_q)\Big)^2 + n\sum_{p=1}^{n}\left(d_{F_\beta^+(v_pv_q)}(v_p) - \frac{2\sum_{1\le p<q\le n}F_\beta^+(v_pv_q)}{n}\right)^2},$$

$$LE\Big(T_\beta^-(v_pv_q)\Big) \le \sqrt{2n\sum_{1\le p<q\le n}\Big(T_\beta^-(v_pv_q)\Big)^2 + n\sum_{p=1}^{n}\left(d_{T_\beta^-(v_pv_q)}(v_p) - \frac{2\sum_{1\le p<q\le n}T_\beta^-(v_pv_q)}{n}\right)^2},$$

$$LE\Big(I_\beta^-(v_pv_q)\Big) \le \sqrt{2n\sum_{1\le p<q\le n}\Big(I_\beta^-(v_pv_q)\Big)^2 + n\sum_{p=1}^{n}\left(d_{I_\beta^-(v_pv_q)}(v_p) - \frac{2\sum_{1\le p<q\le n}I_\beta^-(v_pv_q)}{n}\right)^2},$$

$$LE\Big(F_\beta^-(v_pv_q)\Big) \le \sqrt{2n\sum_{1\le p<q\le n}\Big(F_\beta^-(v_pv_q)\Big)^2 + n\sum_{p=1}^{n}\left(d_{F_\beta^-(v_pv_q)}(v_p) - \frac{2\sum_{1\le p<q\le n}F_\beta^-(v_pv_q)}{n}\right)^2}.$$

□

Theorem 6. *Let $\overline{G} = (\alpha, \beta)$ be a BSVNG on n vertices and let*

$$L(\overline{G}) = \Big\langle L\Big(T_\beta^+(v_pv_q)\Big), L\Big(I_\beta^+(v_pv_q)\Big), L\Big(F_\beta^+(v_pv_q)\Big), L\Big(T_\beta^-(v_pv_q)\Big), L\Big(I_\beta^-(v_pv_q)\Big), L\Big(F_\beta^-(v_pv_q)\Big)\Big\rangle$$

be the Laplacian matrix of $\overline{G}$. Then

i. $$LE\Big(T_\beta^+(v_pv_q)\Big) \ge 2\sqrt{\sum_{1\le p<q\le n}\Big(T_\beta^+(v_pv_q)\Big)^2 + \frac{1}{2}\sum_{p=1}^{n}\left(d_{T_\beta^+(v_pv_q)}(v_p) - \frac{2\sum_{1\le p<q\le n}T_\beta^+(v_pv_q)}{n}\right)^2};$$

ii. $$LE\Big(I_\beta^+(v_pv_q)\Big) \ge 2\sqrt{\sum_{1\le p<q\le n}\Big(I_\beta^+(v_pv_q)\Big)^2 + \frac{1}{2}\sum_{p=1}^{n}\left(d_{I_\beta^+(v_pv_q)}(v_p) - \frac{2\sum_{1\le p<q\le n}I_\beta^+(v_pv_q)}{n}\right)^2};$$

iii. $$LE\Big(F_\beta^+(v_pv_q)\Big) \ge 2\sqrt{\sum_{1\le p<q\le n}\Big(F_\beta^+(v_pv_q)\Big)^2 + \frac{1}{2}\sum_{p=1}^{n}\left(d_{F_\beta^+(v_pv_q)}(v_p) - \frac{2\sum_{1\le p<q\le n}F_\beta^+(v_pv_q)}{n}\right)^2};$$

iv. $$LE\Big(T_\beta^-(v_pv_q)\Big) \ge 2\sqrt{\sum_{1\le p<q\le n}\Big(T_\beta^-(v_pv_q)\Big)^2 + \frac{1}{2}\sum_{p=1}^{n}\left(d_{T_\beta^-(v_pv_q)}(v_p) - \frac{2\sum_{1\le p<q\le n}T_\beta^-(v_pv_q)}{n}\right)^2};$$

v. $$LE\Big(I_\beta^-(v_pv_q)\Big) \ge 2\sqrt{\sum_{1\le p<q\le n}\Big(I_\beta^-(v_pv_q)\Big)^2 + \frac{1}{2}\sum_{p=1}^{n}\left(d_{I_\beta^-(v_pv_q)}(v_p) - \frac{2\sum_{1\le p<q\le n}I_\beta^-(v_pv_q)}{n}\right)^2};$$

vi. $$LE\Big(F_\beta^-(v_pv_q)\Big) \ge 2\sqrt{\sum_{1\le p<q\le n}\Big(F_\beta^-(v_pv_q)\Big)^2 + \frac{1}{2}\sum_{p=1}^{n}\left(d_{F_\beta^-(v_pv_q)}(v_p) - \frac{2\sum_{1\le p<q\le n}F_\beta^-(v_pv_q)}{n}\right)^2}.$$

Proof. We have

$$\left(\sum_{p=1}^{n}|\varsigma_p|\right)^2 = \sum_{p=1}^{n}|\varsigma_p|^2 + 2\sum_{1\le p<q\le n}|\varsigma_p\varsigma_q| \ge 4\Omega_T^+$$
$$LE\left(T_\beta^+(v_pv_q)\right) \ge 2\sqrt{\Omega_T^+}.$$

Since

$$\Omega_T^+ = \sum_{1\le p<q\le n}\left(T_\beta^+(v_pv_q)\right)^2 + \frac{1}{2}\sum_{p=1}^{n}\left(d_{T_\beta^+(v_pv_q)}(v_p) - \frac{2\sum_{1\le p<q\le n}T_\beta^+(v_pv_q)}{n}\right)^2,$$

therefore,

$$LE\left(T_\beta^+(v_pv_q)\right) \ge 2\sqrt{\sum_{1\le p<q\le n}\left(T_\beta^+(v_pv_q)\right)^2 + \frac{1}{2}\sum_{p=1}^{n}\left(d_{T_\beta^+(v_pv_q)}(v_p) - \frac{2\sum_{1\le p<q\le n}T_\beta^+(v_pv_q)}{n}\right)^2}.$$

Similarly, it is easy to show that

$$LE\left(I_\beta^+(v_pv_q)\right) \ge 2\sqrt{\sum_{1\le p<q\le n}\left(I_\beta^+(v_pv_q)\right)^2 + \frac{1}{2}\sum_{p=1}^{n}\left(d_{I_\beta^+(v_pv_q)}(v_p) - \frac{2\sum_{1\le p<q\le n}I_\beta^+(v_pv_q)}{n}\right)^2},$$

$$LE\left(F_\beta^+(v_pv_q)\right) \ge 2\sqrt{\sum_{1\le p<q\le n}\left(F_\beta^+(v_pv_q)\right)^2 + \frac{1}{2}\sum_{p=1}^{n}\left(d_{F_\beta^+(v_pv_q)}(v_p) - \frac{2\sum_{1\le p<q\le n}F_\beta^+(v_pv_q)}{n}\right)^2},$$

$$LE\left(T_\beta^-(v_pv_q)\right) \ge 2\sqrt{\sum_{1\le p<q\le n}\left(T_\beta^-(v_pv_q)\right)^2 + \frac{1}{2}\sum_{p=1}^{n}\left(d_{T_\beta^-(v_pv_q)}(v_p) - \frac{2\sum_{1\le p<q\le n}T_\beta^-(v_pv_q)}{n}\right)^2},$$

$$LE\left(I_\beta^-(v_pv_q)\right) \ge 2\sqrt{\sum_{1\le p<q\le n}\left(I_\beta^-(v_pv_q)\right)^2 + \frac{1}{2}\sum_{p=1}^{n}\left(d_{I_\beta^-(v_pv_q)}(v_p) - \frac{2\sum_{1\le p<q\le n}I_\beta^-(v_pv_q)}{n}\right)^2},$$

$$LE\left(F_\beta^-(v_pv_q)\right) \ge 2\sqrt{\sum_{1\le p<q\le n}\left(F_\beta^-(v_pv_q)\right)^2 + \frac{1}{2}\sum_{p=1}^{n}\left(d_{F_\beta^-(v_pv_q)}(v_p) - \frac{2\sum_{1\le p<q\le n}F_\beta^-(v_pv_q)}{n}\right)^2}.$$

□

Theorem 7. *Let $\overline{G} = (\alpha, \beta)$ be a BSVNG on n vertices and let*

$$L(\overline{G}) = \left\langle L\left(T_\beta^+(v_pv_q)\right), L\left(I_\beta^+(v_pv_q)\right), L\left(F_\beta^+(v_pv_q)\right), L\left(T_\beta^-(v_pv_q)\right), L\left(I_\beta^-(v_pv_q)\right), L\left(F_\beta^-(v_pv_q)\right)\right\rangle$$

be the Laplacian matrix of $\overline{G}$. Then

i. $LE\left(T_\beta^+(v_pv_q)\right) \le |\varsigma_1| + \sqrt{(n-1)\left(2\sum_{1\le p<q\le n}\left(T_\beta^+(v_pv_q)\right)^2 + \sum_{p=1}^{n}\left(d_{T_\beta^+(v_pv_q)}(v_p) - \frac{2\sum_{1\le p<q\le n}T_\beta^+(v_pv_q)}{n}\right)^2 - (\varsigma_1)^2\right)}$;

ii. $LE\left(I_\beta^+(v_pv_q)\right) \le |\tau_1| + \sqrt{(n-1)\left(2\sum_{1\le p<q\le n}\left(I_\beta^+(v_pv_q)\right)^2 + \sum_{p=1}^{n}\left(d_{I_\beta^+(v_pv_q)}(v_p) - \frac{2\sum_{1\le p<q\le n}I_\beta^+(v_pv_q)}{n}\right)^2 - (\tau_1)^2\right)}$;

iii. $LE\left(F_\beta^+(v_pv_q)\right) \le |\upsilon_1| + \sqrt{(n-1)\left(2\sum_{1\le p<q\le n}\left(F_\beta^+(v_pv_q)\right)^2 + \sum_{p=1}^{n}\left(d_{F_\beta^+(v_pv_q)}(v_p) - \frac{2\sum_{1\le p<q\le n}F_\beta^+(v_pv_q)}{n}\right)^2 - (\upsilon_1)^2\right)}$;

iv. $LE\left(T_{\beta}^{-}(v_p v_q)\right) \leq |\vartheta_1| + \sqrt{(n-1)\left(2\sum_{1\leq p<q\leq n}\left(T_{\beta}^{-}(v_p v_q)\right)^2 + \sum_{p=1}^{n}\left(d_{T_{\beta}^{-}(v_p v_q)}(v_p) - \frac{2\sum_{1\leq p<q\leq n} T_{\beta}^{-}(v_p v_q)}{n}\right)^2 - (\vartheta_1)^2\right)}$;

v. $LE\left(I_{\beta}^{-}(v_p v_q)\right) \leq |\xi_1| + \sqrt{(n-1)\left(2\sum_{1\leq p<q\leq n}\left(I_{\beta}^{-}(v_p v_q)\right)^2 + \sum_{p=1}^{n}\left(d_{I_{\beta}^{-}(v_p v_q)}(v_p) - \frac{2\sum_{1\leq p<q\leq n} I_{\beta}^{-}(v_p v_q)}{n}\right)^2 - (\xi_1)^2\right)}$;

vi. $LE\left(F_{\beta}^{-}(v_p v_q)\right) \leq |\varepsilon_1| + \sqrt{(n-1)\left(2\sum_{1\leq p<q\leq n}\left(F_{\beta}^{-}(v_p v_q)\right)^2 + \sum_{p=1}^{n}\left(d_{F_{\beta}^{-}(v_p v_q)}(v_p) - \frac{2\sum_{1\leq p<q\leq n} F_{\beta}^{-}(v_p v_q)}{n}\right)^2 - (\varepsilon_1)^2\right)}$.

Proof. By using Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}
\sum_{p=1}^{n}|\varsigma_p| &\leq \sqrt{n\sum_{p=1}^{n}|\varsigma_p|^2}\\
\sum_{p=2}^{n}|\varsigma_p| &\leq \sqrt{(n-1)\sum_{p=2}^{n}|\varsigma_p|^2}\\
LE\left(T_{\beta}^{+}(v_p v_q)\right) - |\varsigma_1| &\leq \sqrt{(n-1)\left(2\Omega_T^{+} - (\varsigma_1)^2\right)}\\
LE\left(T_{\beta}^{+}(v_p v_q)\right) &\leq |\varsigma_1| + \sqrt{(n-1)\left(2\Omega_T^{+} - (\varsigma_1)^2\right)}.
\end{aligned}$$

Since

$$\Omega_T^{+} = \sum_{1\leq p<q\leq n}\left(T_{\beta}^{+}(v_p v_q)\right)^2 + \frac{1}{2}\sum_{p=1}^{n}\left(d_{T_{\beta}^{+}(v_p v_q)}(v_p) - \frac{2\sum_{1\leq p<q\leq n} T_{\beta}^{+}(v_p v_q)}{n}\right)^2,$$

therefore,

$$LE\left(T_{\beta}^{+}(v_p v_q)\right) \leq |\varsigma_1| + \sqrt{(n-1)\left(2\sum_{1\leq p<q\leq n}\left(T_{\beta}^{+}(v_p v_q)\right)^2 + \sum_{p=1}^{n}\left(d_{T_{\beta}^{+}(v_p v_q)}(v_p) - \frac{2\sum_{1\leq p<q\leq n} T_{\beta}^{+}(v_p v_q)}{n}\right)^2 - (\varsigma_1)^2\right)}. \quad (5)$$

Similarly, we can show that

$$LE\left(I_{\beta}^{+}(v_p v_q)\right) \leq |\tau_1| + \sqrt{(n-1)\left(2\sum_{1\leq p<q\leq n}\left(I_{\beta}^{+}(v_p v_q)\right)^2 + \sum_{p=1}^{n}\left(d_{I_{\beta}^{+}(v_p v_q)}(v_p) - \frac{2\sum_{1\leq p<q\leq n} I_{\beta}^{+}(v_p v_q)}{n}\right)^2 - (\tau_1)^2\right)};$$

$$LE\left(F_{\beta}^{+}(v_p v_q)\right) \leq |\upsilon_1| + \sqrt{(n-1)\left(2\sum_{1\leq p<q\leq n}\left(F_{\beta}^{+}(v_p v_q)\right)^2 + \sum_{p=1}^{n}\left(d_{F_{\beta}^{+}(v_p v_q)}(v_p) - \frac{2\sum_{1\leq p<q\leq n} F_{\beta}^{+}(v_p v_q)}{n}\right)^2 - (\upsilon_1)^2\right)};$$

$$LE\left(T_{\beta}^{-}(v_p v_q)\right) \leq |\vartheta_1| + \sqrt{(n-1)\left(2\sum_{1\leq p<q\leq n}\left(T_{\beta}^{-}(v_p v_q)\right)^2 + \sum_{p=1}^{n}\left(d_{T_{\beta}^{-}(v_p v_q)}(v_p) - \frac{2\sum_{1\leq p<q\leq n} T_{\beta}^{-}(v_p v_q)}{n}\right)^2 - (\vartheta_1)^2\right)};$$

$$LE\left(I_{\beta}^{-}(v_p v_q)\right) \leq |\xi_1| + \sqrt{(n-1)\left(2\sum_{1\leq p<q\leq n}\left(I_{\beta}^{-}(v_p v_q)\right)^2 + \sum_{p=1}^{n}\left(d_{I_{\beta}^{-}(v_p v_q)}(v_p) - \frac{2\sum_{1\leq p<q\leq n} I_{\beta}^{-}(v_p v_q)}{n}\right)^2 - (\xi_1)^2\right)};$$

$$LE\left(F_{\beta}^{-}(v_p v_q)\right) \leq |\varepsilon_1| + \sqrt{(n-1)\left(2\sum_{1\leq p<q\leq n}\left(F_{\beta}^{-}(v_p v_q)\right)^2 + \sum_{p=1}^{n}\left(d_{F_{\beta}^{-}(v_p v_q)}(v_p) - \frac{2\sum_{1\leq p<q\leq n} F_{\beta}^{-}(v_p v_q)}{n}\right)^2 - (\varepsilon_1)^2\right)}.$$

□

Theorem 8. *If the BSVNG $\overline{G} = (\alpha, \beta)$ is regular, then*

i. $LE\left(T^+_\beta(v_pv_q)\right) \le |\varsigma_1| + \sqrt{(n-1)\left(2\sum_{1\le p<q\le n}\left(T^+_\beta(v_pv_q)\right)^2 - (\varsigma_1)^2\right)}$;

ii. $LE\left(I^+_\beta(v_pv_q)\right) \le |\tau_1| + \sqrt{(n-1)\left(2\sum_{1\le p<q\le n}\left(I^+_\beta(v_pv_q)\right)^2 - (\tau_1)^2\right)}$;

iii. $LE\left(F^+_\beta(v_pv_q)\right) \le |\upsilon_1| + \sqrt{(n-1)\left(2\sum_{1\le p<q\le n}\left(F^+_\beta(v_pv_q)\right)^2 - (\upsilon_1)^2\right)}$;

iv. $LE\left(T^-_\beta(v_pv_q)\right) \le |\vartheta_1| + \sqrt{(n-1)\left(2\sum_{1\le p<q\le n}\left(T^-_\beta(v_pv_q)\right)^2 - (\vartheta_1)^2\right)}$;

v. $LE\left(I^-_\beta(v_pv_q)\right) \le |\xi_1| + \sqrt{(n-1)\left(2\sum_{1\le p<q\le n}\left(I^-_\beta(v_pv_q)\right)^2 - (\xi_1)^2\right)}$;

vi. $LE\left(F^-_\beta(v_pv_q)\right) \le |\varepsilon_1| + \sqrt{(n-1)\left(2\sum_{1\le p<q\le n}\left(F^-_\beta(v_pv_q)\right)^2 - (\varepsilon_1)^2\right)}$.

Proof. Let $\overline{G} = (\alpha, \beta)$ be a regular BSVNG, then

$$d_{T^+_\beta(v_pv_q)}(v_p) = \frac{2\sum_{1\le p<q\le n} T^+_\beta(v_pv_q)}{n} \tag{6}$$

Substituting (6) into (5), we obtain

$$LE\left(T^+_\beta(v_pv_q)\right) \le |\varsigma_1| + \sqrt{(n-1)\left(2\sum_{1\le p<q\le n}\left(T^+_\beta(v_pv_q)\right)^2 - (\varsigma_1)^2\right)}.$$

Similarly, it is easy to show that

$$LE\left(I^+_\beta(v_pv_q)\right) \le |\tau_1| + \sqrt{(n-1)\left(2\sum_{1\le p<q\le n}\left(I^+_\beta(v_pv_q)\right)^2 - (\tau_1)^2\right)};$$

$$LE\left(F^+_\beta(v_pv_q)\right) \le |\upsilon_1| + \sqrt{(n-1)\left(2\sum_{1\le p<q\le n}\left(F^+_\beta(v_pv_q)\right)^2 - (\upsilon_1)^2\right)};$$

$$LE\left(T^-_\beta(v_pv_q)\right) \le |\vartheta_1| + \sqrt{(n-1)\left(2\sum_{1\le p<q\le n}\left(T^-_\beta(v_pv_q)\right)^2 - (\vartheta_1)^2\right)};$$

$$LE\left(I^-_\beta(v_pv_q)\right) \le |\xi_1| + \sqrt{(n-1)\left(2\sum_{1\le p<q\le n}\left(I^-_\beta(v_pv_q)\right)^2 - (\xi_1)^2\right)};$$

$$LE\left(F^-_\beta(v_pv_q)\right) \le |\varepsilon_1| + \sqrt{(n-1)\left(2\sum_{1\le p<q\le n}\left(F^-_\beta(v_pv_q)\right)^2 - (\varepsilon_1)^2\right)}.$$

□

5. Signless Laplacian Energy of BSVNG

Definition 13. *The signless Laplacian matrix of a BSVNG $\overline{G} = (\alpha, \beta)$ is defined by*

$$\begin{aligned} L^*(\overline{G}) &= \left\langle L^*\left(T_\beta^+(v_pv_q)\right), L^*\left(I_\beta^+(v_pv_q)\right), L^*\left(F_\beta^+(v_pv_q)\right), L^*\left(T_\beta^-(v_pv_q)\right), L^*\left(I_\beta^-(v_pv_q)\right), L^*\left(F_\beta^-(v_pv_q)\right)\right\rangle \\ &= D(\overline{G}) + A(\overline{G}), \end{aligned}$$

where $D(\overline{G})$ is a degree matrix of a BSVNG and $A(\overline{G})$ is an adjacency matrix.

Definition 14. *The spectrum of signless Laplacian matrix of a BSVNG $L^*(\overline{G})$ is defined as $\langle R_{L^*}, S_{L^*}, T_{L^*}, U_{L^*}, V_{L^*}, W_{L^*}\rangle$, where $R_{L^*}, S_{L^*}, T_{L^*}, U_{L^*}, V_{L^*}$ and W_{L^*} are the sets of Laplacian eigenvalues of $L^*\left(T_\beta^+(v_pv_q)\right), L^*\left(I_\beta^+(v_pv_q)\right), L^*\left(F_\beta^+(v_pv_q)\right), L^*\left(T_\beta^-(v_pv_q)\right), L^*\left(I_\beta^-(v_pv_q)\right)$ and $L^*\left(F_\beta^-(v_pv_q)\right)$, respectively.*

Example 6. *Consider a BSVNG $\overline{G}(\alpha,\beta)$ of a graph $\overline{G}(V,E)$ where $V = \{v_1, v_2, v_3, v_4, v_5, v_6, v_7\}$ and $E = \{v_1v_2, v_1v_3, v_2v_3, v_2v_4, v_2v_5, v_3v_5, v_4v_5, v_4v_6, v_4v_7, v_5v_7, v_6v_7\}$ as shown in* Figure 3, *defined by* Tables 6 and 7 *as follows:*

Table 6. Signless Laplacian energy of BSVNG set on V.

α	v_1	v_2	v_3	v_4	v_5	v_6	v_7
T_α^+	0.2	0.7	0.1	0.5	0.3	0.4	0.8
I_α^+	0.6	0.1	0.3	0.2	0.4	0.6	0.7
F_α^+	0.4	0.5	0.2	0.1	0.7	0.7	0.6
T_α^-	−0.4	−0.1	−0.3	−0.4	−0.5	−0.2	−0.5
I_α^-	−0.6	−0.3	−0.2	−0.5	−0.3	−0.4	−0.4
F_α^-	−0.1	−0.6	−0.5	−0.2	−0.4	−0.1	−0.3

Table 7. Signless Laplacian energy of BSVNG relation on V.

β	v_1v_2	v_1v_3	v_2v_3	v_2v_4	v_2v_5	v_3v_5	v_4v_5	v_4v_6	v_4v_7	v_5v_7	v_6v_7
T_β^+	0.2	0.1	0.1	0.4	0.3	0.1	0.2	0.3	0.4	0.2	0.3
I_β^+	0.7	0.6	0.4	0.3	0.5	0.5	0.5	0.7	0.8	0.8	0.7
F_β^+	0.6	0.5	0.6	0.7	0.8	0.8	0.7	0.8	0.6	0.8	0.8
T_β^-	−0.1	−0.2	−0.1	−0.1	−0.1	−0.3	−0.3	−0.2	−0.3	−0.4	−0.1
I_β^-	−0.8	−0.7	−0.4	−0.6	−0.4	−0.5	−0.6	−0.7	−0.5	−0.5	−0.6
F_β^-	−0.7	−0.5	−0.6	−0.8	−0.7	−0.5	−0.4	−0.3	−0.4	−0.5	−0.4

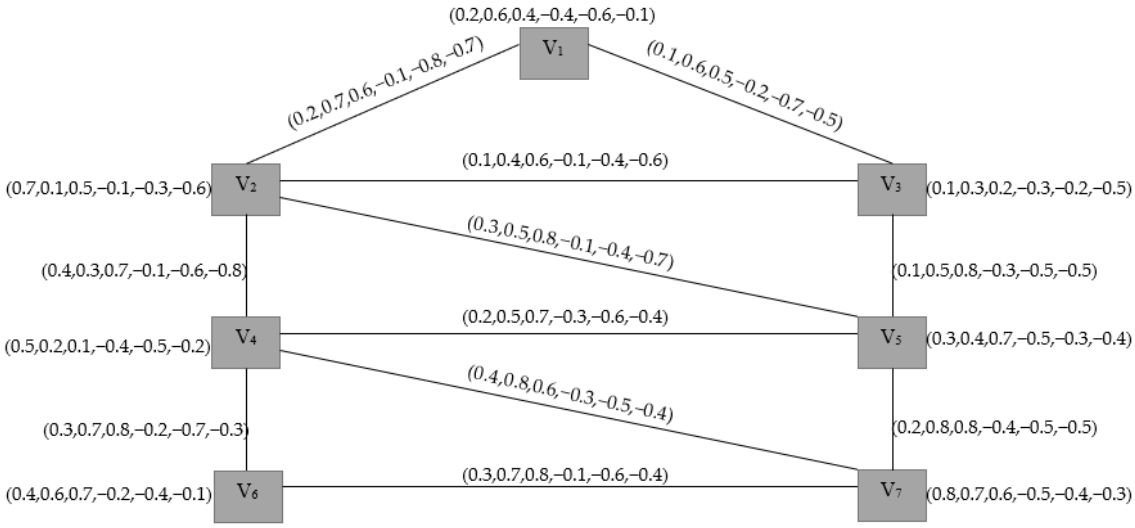


Figure 3. Signless Laplacian energy of a bipolar single-valued neutrosophic graph.

The adjacency, degree, and the signless Laplacian matrices of the BSVNG shown in Figure 3 are expressed as follows:

$$A(\overline{G}) = \begin{pmatrix}
\left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0.2,0.7,0.6,\\-0.1,-0.8,-0.7\end{matrix}\right\rangle & \left\langle\begin{matrix}0.1,0.6,0.5,\\-0.2,-0.7,-0.5\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle \\
\left\langle\begin{matrix}0.2,0.7,0.6,\\-0.1,-0.8,-0.7\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0.1,0.4,0.6,\\-0.1,-0.4,-0.6\end{matrix}\right\rangle & \left\langle\begin{matrix}0.4,0.3,0.7,\\-0.1,-0.6,-0.8\end{matrix}\right\rangle & \left\langle\begin{matrix}0.3,0.5,0.8,\\-0.1,-0.4,-0.7\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle \\
\left\langle\begin{matrix}0.1,0.6,0.5,\\-0.2,-0.7,-0.5\end{matrix}\right\rangle & \left\langle\begin{matrix}0.1,0.4,0.6,\\-0.1,-0.4,-0.6\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0.1,0.5,0.8,\\-0.3,-0.5,-0.5\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle \\
\left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0.4,0.3,0.7,\\-0.1,-0.6,-0.8\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0.2,0.5,0.7,\\-0.3,-0.6,-0.4\end{matrix}\right\rangle & \left\langle\begin{matrix}0.3,0.7,0.8,\\-0.2,-0.7,-0.3\end{matrix}\right\rangle & \left\langle\begin{matrix}0.4,0.8,0.6,\\-0.3,-0.5,-0.4\end{matrix}\right\rangle \\
\left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0.3,0.5,0.8,\\-0.1,-0.4,-0.7\end{matrix}\right\rangle & \left\langle\begin{matrix}0.1,0.5,0.8,\\-0.3,-0.5,-0.5\end{matrix}\right\rangle & \left\langle\begin{matrix}0.2,0.5,0.7,\\-0.3,-0.6,-0.4\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0.2,0.8,0.8,\\-0.4,-0.5,-0.5\end{matrix}\right\rangle \\
\left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0.3,0.7,0.8,\\-0.2,-0.7,-0.3\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0.3,0.7,0.8,\\-0.1,-0.6,-0.4\end{matrix}\right\rangle \\
\left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0.4,0.8,0.6,\\-0.3,-0.5,-0.4\end{matrix}\right\rangle & \left\langle\begin{matrix}0.2,0.8,0.8,\\-0.4,-0.5,-0.5\end{matrix}\right\rangle & \left\langle\begin{matrix}0.3,0.7,0.8,\\-0.1,-0.6,-0.4\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle
\end{pmatrix}$$

$$D(\overline{G}) = \begin{pmatrix}
\left\langle\begin{matrix}0.3,1.3,1.1,\\-0.3,-1.5,-1.2\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle \\
\left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}1.0,1.9,2.7,\\-0.4,-2.2,-2.8\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle \\
\left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0.3,1.5,1.9,\\-0.6,-1.6,-1.6\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle \\
\left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}1.3,2.3,2.8,\\-0.9,-2.4,-1.9\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle \\
\left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0.8,2.3,3.1,\\-1.1,-2.0,-2.1\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle \\
\left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0.6,1.4,1.6,\\-0.3,-1.3,-0.7\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle \\
\left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0,0,0\\0,0,0\end{matrix}\right\rangle & \left\langle\begin{matrix}0.9,2.3,2.2,\\-0.8,-1.6,-1.3\end{matrix}\right\rangle
\end{pmatrix}$$

$$\mathrm{L}^*(\overline{\mathrm{G}}) = \begin{pmatrix}
\left\langle\begin{array}{c}0.3,1.3,1.1,\\-0.3,-1.5,-1.2\end{array}\right\rangle & \left\langle\begin{array}{c}0.2,0.7,0.6,\\-0.1,-0.8,-0.7\end{array}\right\rangle & \left\langle\begin{array}{c}0.1,0.6,0.5,\\-0.2,-0.7,-0.5\end{array}\right\rangle & \left\langle\begin{array}{c}0,0,0\\0,0,0\end{array}\right\rangle & \left\langle\begin{array}{c}0,0,0\\0,0,0\end{array}\right\rangle & \left\langle\begin{array}{c}0,0,0\\0,0,0\end{array}\right\rangle & \left\langle\begin{array}{c}0,0,0\\0,0,0\end{array}\right\rangle \\
\left\langle\begin{array}{c}0.2,0.7,0.6,\\-0.1,-0.8,-0.7\end{array}\right\rangle & \left\langle\begin{array}{c}1.0,1.9,2.7,\\-0.4,-2.2,-2.8\end{array}\right\rangle & \left\langle\begin{array}{c}0.1,0.4,0.6,\\-0.1,-0.4,-0.6\end{array}\right\rangle & \left\langle\begin{array}{c}0.4,0.3,0.7,\\-0.1,-0.6,-0.8\end{array}\right\rangle & \left\langle\begin{array}{c}0.3,0.5,0.8,\\-0.1,-0.4,-0.7\end{array}\right\rangle & \left\langle\begin{array}{c}0,0,0\\0,0,0\end{array}\right\rangle & \left\langle\begin{array}{c}0,0,0\\0,0,0\end{array}\right\rangle \\
\left\langle\begin{array}{c}0.1,0.6,0.5,\\-0.2,-0.7,-0.5\end{array}\right\rangle & \left\langle\begin{array}{c}0.1,0.4,0.6,\\-0.1,-0.4,-0.6\end{array}\right\rangle & \left\langle\begin{array}{c}0.3,1.5,1.9,\\-0.6,-1.6,-1.6\end{array}\right\rangle & \left\langle\begin{array}{c}0,0,0\\0,0,0\end{array}\right\rangle & \left\langle\begin{array}{c}0.1,0.5,0.8,\\-0.3,-0.5,-0.5\end{array}\right\rangle & \left\langle\begin{array}{c}0,0,0\\0,0,0\end{array}\right\rangle & \left\langle\begin{array}{c}0,0,0\\0,0,0\end{array}\right\rangle \\
\left\langle\begin{array}{c}0,0,0\\0,0,0\end{array}\right\rangle & \left\langle\begin{array}{c}0.4,0.3,0.7,\\-0.1,-0.6,-0.8\end{array}\right\rangle & \left\langle\begin{array}{c}0,0,0\\0,0,0\end{array}\right\rangle & \left\langle\begin{array}{c}1.3,2.3,2.8,\\-0.9,-2.4,-1.9\end{array}\right\rangle & \left\langle\begin{array}{c}0.2,0.5,0.7,\\-0.3,-0.6,-0.4\end{array}\right\rangle & \left\langle\begin{array}{c}0.3,0.7,0.8,\\-0.2,-0.7,-0.3\end{array}\right\rangle & \left\langle\begin{array}{c}0.4,0.8,0.6,\\-0.3,-0.5,-0.4\end{array}\right\rangle \\
\left\langle\begin{array}{c}0,0,0\\0,0,0\end{array}\right\rangle & \left\langle\begin{array}{c}0.3,0.5,0.8,\\-0.1,-0.4,-0.7\end{array}\right\rangle & \left\langle\begin{array}{c}0.1,0.5,0.8,\\-0.3,-0.5,-0.5\end{array}\right\rangle & \left\langle\begin{array}{c}0.2,0.5,0.7,\\-0.3,-0.6,-0.4\end{array}\right\rangle & \left\langle\begin{array}{c}0.8,2.3,3.1,\\-1.1,-2.0,-2.1\end{array}\right\rangle & \left\langle\begin{array}{c}0,0,0\\0,0,0\end{array}\right\rangle & \left\langle\begin{array}{c}0.2,0.8,0.8,\\-0.4,-0.5,-0.5\end{array}\right\rangle \\
\left\langle\begin{array}{c}0,0,0\\0,0,0\end{array}\right\rangle & \left\langle\begin{array}{c}0,0,0\\0,0,0\end{array}\right\rangle & \left\langle\begin{array}{c}0,0,0\\0,0,0\end{array}\right\rangle & \left\langle\begin{array}{c}0.3,0.7,0.8,\\-0.2,-0.7,-0.3\end{array}\right\rangle & \left\langle\begin{array}{c}0,0,0\\0,0,0\end{array}\right\rangle & \left\langle\begin{array}{c}0.6,1.4,1.6,\\-0.3,-1.3,-0.7\end{array}\right\rangle & \left\langle\begin{array}{c}0.3,0.7,0.8,\\-0.1,-0.6,-0.4\end{array}\right\rangle \\
\left\langle\begin{array}{c}0,0,0\\0,0,0\end{array}\right\rangle & \left\langle\begin{array}{c}0,0,0\\0,0,0\end{array}\right\rangle & \left\langle\begin{array}{c}0,0,0\\0,0,0\end{array}\right\rangle & \left\langle\begin{array}{c}0.4,0.8,0.6,\\-0.3,-0.5,-0.4\end{array}\right\rangle & \left\langle\begin{array}{c}0.2,0.8,0.8,\\-0.4,-0.5,-0.5\end{array}\right\rangle & \left\langle\begin{array}{c}0.3,0.7,0.8,\\-0.1,-0.6,-0.4\end{array}\right\rangle & \left\langle\begin{array}{c}0.9,2.3,2.2,\\-0.8,-1.6,-1.3\end{array}\right\rangle
\end{pmatrix}$$

The signless Laplacian spectrum of a BSVNG $\overline{G}$, portrayed in Figure 3 is given by:

$$\begin{aligned}
&Signless\ Laplacian\ Spec\left(T_\beta^+(v_pv_q)\right) = \{0.1674, 0.3086, 0.3767, 0.5216, 0.7583, 1.1195, 1.9480\},\\
&Signless\ Laplacian\ Spec\left(I_\beta^+(v_pv_q)\right) = \{0.5469, 0.8376, 1.2085, 1.4716, 1.9603, 2.8532, 4.1218\},\\
&Signless\ Laplacian\ Spec\left(F_\beta^+(v_pv_q)\right) = \{0.6615, 0.8400, 1.4262, 1.7604, 2.4743, 3.2099, 5.0277\},\\
&Signless\ Laplacian\ Spec\left(T_\beta^-(v_pv_q)\right) = \{-1.6945, -0.8956, -0.6426, -0.4507, -0.3457, -0.2192, -0.1517\},\\
&Signless\ Laplacian\ Spec\left(I_\beta^-(v_pv_q)\right) = \{-3.8701, -2.8201, -1.9006, -1.5645, -1.1885, -0.6923, -0.5640\},\\
&Signless\ Laplacian\ Spec\left(F_\beta^-(v_pv_q)\right) = \{-4.1078, -2.2301, -1.8700, -1.2949, -0.9912, -0.6641, -0.4418\}.
\end{aligned}$$

Therefore,

$$Signless\ Laplacian\ \mathrm{Spec}(\overline{G}) = \left\{\begin{array}{c}
\langle 0.1674, 0.5469, 0.6615, -1.6945, -3.8701, -4.1078\rangle,\\
\langle 0.3086, 0.8376, 0.8400, -0.8956, -2.8201, -2.2301\rangle,\\
\langle 0.3767, 1.2085, 1.4262, -0.6426, -1.9006, -1.8700\rangle,\\
\langle 0.5216, 1.4716, 1.7604, -0.4507, -1.5645, -1.2949\rangle,\\
\langle 0.7583, 1.9603, 2.4743, -0.3457, -1.1885, -0.9912\rangle,\\
\langle 1.1195, 2.8532, 3.2099, -0.2192, -0.6923, -0.6641\rangle,\\
\langle 1.9480, 4.1218, 5.0277, -0.1517, -0.5640, -0.4418\rangle
\end{array}\right\}.$$

Theorem 9. *Let $\overline{G}(\alpha,\beta)$ be a BSVNG and let*

$$L^*(\overline{G}) = \left\langle L^*\left(T_\beta^+(v_pv_q)\right), L^*\left(I_\beta^+(v_pv_q)\right), L^*\left(F_\beta^+(v_pv_q)\right), L^*\left(T_\beta^-(v_pv_q)\right), L^*\left(I_\beta^-(v_pv_q)\right), L^*\left(F_\beta^-(v_pv_q)\right)\right\rangle$$

be the signless Laplacian matrix of $\overline{G}$. If, $\phi_1^* \geq \phi_2^* \geq \ldots \geq \phi_n^*, \zeta_1^* \geq \zeta_2^* \geq \ldots \geq \zeta_n^*, \varphi_1^* \geq \varphi_2^* \geq \ldots \geq \varphi_n^*, \mu_1^* \geq \mu_2^* \geq \ldots \geq \mu_n^*, \omega_1^* \geq \omega_2^* \geq \ldots \geq \omega_n^*$ and $\sigma_1^* \geq \sigma_2^* \geq \ldots \geq \sigma_n^*$ are the eigenvalues of $L^*\left(T_\beta^+(v_pv_q)\right), L^*\left(I_\beta^+(v_pv_q)\right), L^*\left(F_\beta^+(v_pv_q)\right), L^*\left(T_\beta^-(v_pv_q)\right), L^*\left(I_\beta^-(v_pv_q)\right)$ and $L^*\left(F_\beta^-(v_pv_q)\right)$, respectively, then,

1. $\sum\limits_{\substack{p=1\\ \phi_p^*\in R_{L^*}}}^{n} \phi_p^* = 2\sum\limits_{1\leq p<q\leq n} T_\beta^+(v_pv_q)$, $\sum\limits_{\substack{p=1\\ \zeta_p^*\in S_{L^*}}}^{n} \zeta_p^* = 2\sum\limits_{1\leq p<q\leq n} I_\beta^+(v_pv_q)$, $\sum\limits_{\substack{p=1\\ \varphi_p^*\in T_{L^*}}}^{n} \varphi_p^* = 2\sum\limits_{1\leq p<q\leq n} F_\beta^+(v_pv_q)$, $\sum\limits_{\substack{p=1\\ \mu_p^*\in U_{L^*}}}^{n} \mu_p^* = 2\sum\limits_{1\leq p<q\leq n} T_\beta^-(v_pv_q)$, $\sum\limits_{\substack{p=1\\ \omega_p^*\in W_{L^*}}}^{n} \omega_p^* = 2\sum\limits_{1\leq p<q\leq n} I_\beta^-(v_pv_q)$, $\sum\limits_{\substack{p=1\\ \sigma_p^*\in W_{L^*}}}^{n} \sigma_p^* = 2\sum\limits_{1\leq p<q\leq n} F_\beta^-(v_pv_q)$.

2. $\sum\limits_{\substack{p=1 \\ \phi_p^* \in R_{L^*}}}^{n} \left(\phi_p^*\right)^2 = 2\sum\limits_{1\leq p<q\leq n}\left(T_\beta^+(v_pv_q)\right)^2 + \sum\limits_{p=1}^{n} d^2_{T_\beta^+(v_pv_q)}(v_p)$, $\sum\limits_{\substack{p=1 \\ \zeta_p^* \in S_{L^*}}}^{n} \left(\zeta_p^*\right)^2 = 2\sum\limits_{1\leq p<q\leq n}\left(I_\beta^+(v_pv_q)\right)^2 + \sum\limits_{p=1}^{n} d^2_{I_\beta^+(v_pv_q)}(v_p)$, $\sum\limits_{\substack{p=1 \\ \varphi_p^* \in T_{L^*}}}^{n} \left(\varphi_p^*\right)^2 = 2\sum\limits_{1\leq p<q\leq n}\left(F_\beta^+(v_pv_q)\right)^2 + \sum\limits_{p=1}^{n} d^2_{F_\beta^+(v_pv_q)}(v_p)$, $\sum\limits_{\substack{p=1 \\ \mu_p^* \in U_{L^*}}}^{n} \left(\mu_p^*\right)^2 = 2\sum\limits_{1\leq p<q\leq n}\left(T_\beta^-(v_pv_q)\right)^2 + \sum\limits_{p=1}^{n} d^2_{T_\beta^-(v_pv_q)}(v_p)$, $\sum\limits_{\substack{p=1 \\ \varpi_p^* \in V_{L^*}}}^{n} \left(\varpi_p^*\right)^2 = 2\sum\limits_{1\leq p<q\leq n}\left(I_\beta^-(v_pv_q)\right)^2 + \sum\limits_{p=1}^{n} d^2_{I_\beta^-(v_pv_q)}(v_p)$, $\sum\limits_{\substack{p=1 \\ \sigma_p^* \in W_{L^*}}}^{n} \left(\sigma_p^*\right)^2 = 2\sum\limits_{1\leq p<q\leq n}\left(F_\beta^-(v_pv_q)\right)^2 + \sum\limits_{p=1}^{n} d^2_{F_\beta^-(v_pv_q)}(v_p)$.

Proof. The proof is similar to that of Theorem 3. □

Definition 15. *The signless Laplacian energy of BSVNG $\overline{G}(\alpha,\beta)$ is defined as*

$$\begin{aligned}LE^*(\overline{G}) &= \left\langle LE^*\left(T_\beta^+(v_pv_q)\right), LE^*\left(I_\beta^+(v_pv_q)\right), LE^*\left(F_\beta^+(v_pv_q)\right), LE^*\left(T_\beta^-(v_pv_q)\right), LE^*\left(I_\beta^-(v_pv_q)\right), LE^*\left(F_\beta^-(v_pv_q)\right)\right\rangle \\ &= \left\langle \sum_{p=1}^{n}\left|\varsigma_p^*\right|, \sum_{p=1}^{n}\left|\tau_p^*\right|, \sum_{p=1}^{n}\left|\upsilon_p^*\right|, \sum_{p=1}^{n}\left|\vartheta_p^*\right|, \sum_{p=1}^{n}\left|\xi_p^*\right|, \sum_{p=1}^{n}\left|\varepsilon_p^*\right| \right\rangle,\end{aligned}$$

where

$$\begin{aligned}&\varsigma_p^* = \phi_p^* - \frac{2\sum\limits_{1\leq p<q\leq n} T_\beta^+(v_pv_q)}{n}, \tau_p^* = \zeta_p^* - \frac{2\sum\limits_{1\leq p<q\leq n} I_\beta^+(v_pv_q)}{n}, \upsilon_p^* = \varphi_p^* - \frac{2\sum\limits_{1\leq p<q\leq n} F_\beta^+(v_pv_q)}{n}, \\ &\vartheta_p^* = \mu_p^* - \frac{2\sum\limits_{1\leq p<q\leq n} T_\beta^-(v_pv_q)}{n}, \xi_p^* = \varpi_p^* - \frac{2\sum\limits_{1\leq p<q\leq n} I_\beta^-(v_pv_q)}{n}, \varepsilon_p^* = \sigma_p^* - \frac{2\sum\limits_{1\leq p<q\leq n} F_\beta^-(v_pv_q)}{n}.\end{aligned}$$

Theorem 10. *Let $\overline{G}(\alpha,\beta)$ be a BSVNG and let $L^*(\overline{G})$ be the signless Laplacian matrix of $\overline{G}$. If*

$$\phi_1^* \geq \phi_2^* \geq \ldots \geq \phi_n^*, \zeta_1^* \geq \zeta_2^* \geq \ldots \geq \zeta_n^*, \varphi_1^* \geq \varphi_2^* \geq \ldots \geq \varphi_n^*, \mu_1^* \geq \mu_2^* \geq \ldots \geq \mu_n^*, \varpi_1^* \geq \varpi_2^* \geq \ldots \geq \varpi_n^*, \sigma_1^* \geq \sigma_2^* \geq \ldots \geq \sigma_n^*$$

are the eigenvalues of $L^*\left(T_\beta^+(v_pv_q)\right), L^*\left(I_\beta^+(v_pv_q)\right), L^*\left(F_\beta^+(v_pv_q)\right), L^*\left(T_\beta^-(v_pv_q)\right)$, $L^*\left(I_\beta^-(v_pv_q)\right)$ and$L^*\left(F_\beta^-(v_pv_q)\right)$, respectively, and

$$\begin{aligned}&\varsigma_p^* = \phi_p^* - \frac{2\sum\limits_{1\leq p<q\leq n} T_\beta^+(v_pv_q)}{n}, \tau_p^* = \zeta_p^* - \frac{2\sum\limits_{1\leq p<q\leq n} I_\beta^+(v_pv_q)}{n}, \upsilon_p^* = \varphi_p^* - \frac{2\sum\limits_{1\leq p<q\leq n} F_\beta^+(v_pv_q)}{n}, \\ &\vartheta_p^* = \mu_p^* - \frac{2\sum\limits_{1\leq p<q\leq n} T_\beta^-(v_pv_q)}{n}, \xi_p^* = \varpi_p^* - \frac{2\sum\limits_{1\leq p<q\leq n} I_\beta^-(v_pv_q)}{n}, \varepsilon_p^* = \sigma_p^* - \frac{2\sum\limits_{1\leq p<q\leq n} F_\beta^-(v_pv_q)}{n}.\end{aligned}$$

Then,

$$\sum_{p=1}^{n} \varsigma_p^* = 0, \sum_{p=1}^{n} \tau_p^* = 0, \sum_{p=1}^{n} \upsilon_p^* = 0, \sum_{p=1}^{n} \vartheta_p^* = 0, \sum_{p=1}^{n} \xi_p^* = 0, \sum_{p=1}^{n} \varepsilon_p^* = 0,$$
$$\sum_{p=1}^{n}\left(\varsigma_p^*\right)^2 = 2(\Omega_T^+)^*, \sum_{p=1}^{n}\left(\tau_p^*\right)^2 = 2(\Omega_I^+)^*, \sum_{p=1}^{n}\left(\upsilon_p^*\right)^2 = 2(\Omega_F^+)^*, \sum_{p=1}^{n}\left(\vartheta_p^*\right)^2 = 2(\Omega_T^-)^*, \sum_{p=1}^{n}\left(\xi_p^*\right)^2 = 2(\Omega_I^-)^*, \sum_{p=1}^{n}\left(\varepsilon_p^*\right)^2 = 2(\Omega_F^-)^*,$$

where

$$(\Omega_T^+)^* = \sum_{1\le p<q\le n}\left(T_\beta^+(v_pv_q)\right)^2 + \frac{1}{2}\sum_{p=1}^{n}\left(d_{T_\beta^+(v_pv_q)}(v_p) - \frac{2\sum_{1\le p<q\le n} T_\beta^+(v_pv_q)}{n}\right)^2,$$

$$(\Omega_I^+)^* = \sum_{1\le p<q\le n}\left(I_\beta^+(v_pv_q)\right)^2 + \frac{1}{2}\sum_{p=1}^{n}\left(d_{I_\beta^+(v_pv_q)}(v_p) - \frac{2\sum_{1\le p<q\le n} I_\beta^+(v_pv_q)}{n}\right)^2,$$

$$(\Omega_F^+)^* = \sum_{1\le p<q\le n}\left(F_\beta^+(v_pv_q)\right)^2 + \frac{1}{2}\sum_{p=1}^{n}\left(d_{F_\beta^+(v_pv_q)}(v_p) - \frac{2\sum_{1\le p<q\le n} F_\beta^+(v_pv_q)}{n}\right)^2,$$

$$(\Omega_T^-)^* = \sum_{1\le p<q\le n}\left(T_\beta^-(v_pv_q)\right)^2 + \frac{1}{2}\sum_{p=1}^{n}\left(d_{T_\beta^-(v_pv_q)}(v_p) - \frac{2\sum_{1\le p<q\le n} T_\beta^-(v_pv_q)}{n}\right)^2,$$

$$(\Omega_I^-)^* = \sum_{1\le p<q\le n}\left(I_\beta^-(v_pv_q)\right)^2 + \frac{1}{2}\sum_{p=1}^{n}\left(d_{I_\beta^-(v_pv_q)}(v_p) - \frac{2\sum_{1\le p<q\le n} I_\beta^-(v_pv_q)}{n}\right)^2,$$

$$(\Omega_F^-)^* = \sum_{1\le p<q\le n}\left(F_\beta^-(v_pv_q)\right)^2 + \frac{1}{2}\sum_{p=1}^{n}\left(d_{F_\beta^-(v_pv_q)}(v_p) - \frac{2\sum_{1\le p<q\le n} F_\beta^-(v_pv_q)}{n}\right)^2.$$

Proof. The proof is similar to that of Theorem 1. □

6. Relation between Energy, Laplacian Energy and Signless Laplacian Energy of BSVNG

Theorem 11. *Let $\overline{G}$ be a BSVNG on n vertices and let $A(\overline{G}), L(\overline{G})$ and $L^*(\overline{G})$ be the adjacency, the Laplacian and the signless Laplacian matrices of $\overline{G}$, respectively. Then, $\left|LE^*(\overline{G}) - LE(\overline{G})\right| \le 2E(\overline{G})$.*

Proof.

$$L^*\left(T_\beta^+(v_pv_q)\right) - \frac{2\sum_{1\le p<q\le n} T_\beta^+(v_pv_q)}{n} = D\left(T_\beta^+(v_pv_q)\right) + A\left(T_\beta^+(v_pv_q)\right) - \frac{2\sum_{1\le p<q\le n} T_\beta^+(v_pv_q)}{n}, \tag{7}$$

$$L\left(T_\beta^+(v_pv_q)\right) - \frac{2\sum_{1\le p<q\le n} T_\beta^+(v_pv_q)}{n} = D\left(T_\beta^+(v_pv_q)\right) - A\left(T_\beta^+(v_pv_q)\right) - \frac{2\sum_{1\le p<q\le n} T_\beta^+(v_pv_q)}{n}. \tag{8}$$

From (7) and (8), we obtain

$$\left(L^*\left(T_\beta^+(v_pv_q)\right) - \frac{2\sum_{1\le p<q\le n} T_\beta^+(v_pv_q)}{n}\right) - \left(L\left(T_\beta^+(v_pv_q)\right) - \frac{2\sum_{1\le p<q\le n} T_\beta^+(v_pv_q)}{n}\right) = 2A\left(T_\beta^+(v_pv_q)\right).$$

Then,

$$\left(L\left(T_\beta^+(v_pv_q)\right) - \frac{2\sum_{1\le p<q\le n} T_\beta^+(v_pv_q)}{n}\right) = \left(L^*\left(T_\beta^+(v_pv_q)\right) - \frac{2\sum_{1\le p<q\le n} T_\beta^+(v_pv_q)}{n}\right) - 2A\left(T_\beta^+(v_pv_q)\right).$$

Also,

$$\left(L^*\left(T_\beta^+(v_pv_q)\right) - \frac{2\sum_{1\le p<q\le n} T_\beta^+(v_pv_q)}{n}\right) = \left(L\left(T_\beta^+(v_pv_q)\right) - \frac{2\sum_{1\le p<q\le n} T_\beta^+(v_pv_q)}{n}\right) + 2A\left(T_\beta^+(v_pv_q)\right).$$

By the well-known property of energy of a graph, we obtain

$$\begin{aligned} LE\Big(T_\beta^+(v_pv_q)\Big) &= E\left(L\Big(T_\beta^+(v_pv_q)\Big) - \frac{2\sum\limits_{1\le p<q\le n} T_\beta^+(v_pv_q)}{n}\right) \le E\left(L^*\Big(T_\beta^+(v_pv_q)\Big) - \frac{2\sum\limits_{1\le p<q\le n} T_\beta^+(v_pv_q)}{n}\right) \\ &+E\Big(-2A\Big(T_\beta^+(v_pv_q)\Big)\Big) \\ &= LE^*\Big(T_\beta^+(v_pv_q)\Big) + 2E\Big(T_\beta^+(v_pv_q)\Big). \end{aligned} \tag{9}$$

$$\begin{aligned} LE^*\Big(T_\beta^+(v_pv_q)\Big) &= E\left(L^*\Big(T_\beta^+(v_pv_q)\Big) - \frac{2\sum\limits_{1\le p<q\le n} T_\beta^+(v_pv_q)}{n}\right) \le E\left(L\Big(T_\beta^+(v_pv_q)\Big) - \frac{2\sum\limits_{1\le p<q\le n} T_\beta^+(v_pv_q)}{n}\right) \\ &+E\Big(2A\Big(T_\beta^+(v_pv_q)\Big)\Big) \\ &= LE\Big(T_\beta^+(v_pv_q)\Big) + 2E\Big(T_\beta^+(v_pv_q)\Big). \end{aligned} \tag{10}$$

Combining (9) and (10), yields $\Big|LE^*\Big(T_\beta^+(v_pv_q)\Big) - LE\Big(T_\beta^+(v_pv_q)\Big)\Big| \le 2E\Big(T_\beta^+(v_pv_q)\Big)$. Similarly, we can show that

$$\begin{aligned} &\Big|LE^*\Big(I_\beta^+(v_pv_q)\Big) - LE\Big(I_\beta^+(v_pv_q)\Big)\Big| \le 2E\Big(I_\beta^+(v_pv_q)\Big), \Big|LE^*\Big(F_\beta^+(v_pv_q)\Big) - LE\Big(F_\beta^+(v_pv_q)\Big)\Big| \le 2E\Big(F_\beta^+(v_pv_q)\Big), \\ &\Big|LE^*\Big(T_\beta^-(v_pv_q)\Big) - LE\Big(T_\beta^-(v_pv_q)\Big)\Big| \le 2E\Big(T_\beta^-(v_pv_q)\Big), \Big|LE^*\Big(I_\beta^-(v_pv_q)\Big) - LE\Big(I_\beta^-(v_pv_q)\Big)\Big| \le 2E\Big(I_\beta^-(v_pv_q)\Big), \\ &\Big|LE^*\Big(F_\beta^-(v_pv_q)\Big) - LE\Big(F_\beta^-(v_pv_q)\Big)\Big| \le 2E\Big(F_\beta^-(v_pv_q)\Big). \end{aligned}$$

Hence, $\big|LE^*(\overline{G}) - LE(\overline{G})\big| \le 2E(\overline{G})$. □

7. Application of Energy of BSVNG

A group decision-making problem concerning the selecting of the most compatible renewable energy alternatives is solved to illustrate the applicability of the proposed concepts of energy of BSVNGs in practical scenarios. However, in order to reflect the relationship among the alternatives, we need to make pairwise comparisons for all the alternatives in the process of decision-making. If every element in the preference relations is a bipolar single-valued neutrosophic number (BSVNN), then the concept of the bipolar single-valued neutrosophic preference relation (BSVNPR) can be put forth as follows:

Definition 16. *A BSVNPR on the set $X = \{x_1, x_2, \ldots, x_n\}$ is represented by a matrix $M = (m_{pq})_{n\times n}$, where*

$$m_{pq} = \langle x_px_q, T^+(x_px_q), I^+(x_px_q), F^+(x_px_q), T^-(x_px_q), I^-(x_px_q), F^-(x_px_q)\rangle$$

for all $p, q = 1, 2, \ldots, n$. For convenience, let $m_{pq} = \Big\langle T_{pq}^+, I_{pq}^+, F_{pq}^+, T_{pq}^-, I_{pq}^-, F_{pq}^-\Big\rangle$ where T_{pq}^+ indicates the positive degree to which the object x_p is preferred by the object x_q, F_{pq}^+ denotes the positive degree to which the object x_p is not preferred by the object x_q, I_{pq}^+ is interpreted as an indeterminacy-membership positive degree, with the conditions:

$$T_{pq}^+, I_{pq}^+, F_{pq}^+ \in [0,1], T_{pq}^+ = F_{qp}^+, F_{pq}^+ = T_{qp}^+, I_{pq}^+ + I_{qp}^+ = 1, T_{pp}^+ = I_{pp}^+ = F_{pp}^+ = 0.5.$$

Here, T_{pq}^- indicates the negative degree to which the object x_p is preferred by the object x_q, F_{pq}^- denotes the negative degree to which the object x_p is not preferred by the object x_q, and I_{pq}^- is interpreted as an indeterminacy-membership negative degree, with the following conditions:

$$T_{pq}^-, I_{pq}^-, F_{pq}^- \in [-1,0], T_{pq}^- = F_{qp}^-, F_{pq}^- = T_{qp}^-, I_{pq}^- + I_{qp}^- = -1, T_{pp}^- = I_{pp}^- = F_{pp}^- = -0.5,$$

for all $p, q = 1, 2, \ldots, n$.

Selection of the Most Compatible Renewable Energy Source

Renewable energy sources involve biomass energy, geothermal energy, ocean energy, solar energy, wind energy, and hydropower energy. They have an enormous potential to meet the energy needs of the world. By doing that, the world's energy security can be powered by modern conversion technologies by reducing the long-term price of fuels from conventional sources and decreasing the use of fossil fuels. Using renewable energies does not only impact reducing air pollution, safety risks, and greenhouse gas emissions in the atmosphere but also are recycled in nature. Furthermore, it reduces dependence on imported fuels, creates new jobs, and provides regional employment.

We considered an issue, taken from [41], as an application for the proposed method in the present paper. The issue given is that the managers of a municipal close to sea cost want to invest in renewable energy technologies to self-meet their energy needs. After numerous consultations, six renewable energy sources were considered as an alternative. These are biomass energy plants (A_1), geothermal energy plants (A_2), hydro power plants (A_3), solar power plants (A_4), wave power plants (A_5), and wind power plants (A_6). To select the most effective renewable energy source, three experts $E_i(i = 1, 2, 3)$ are invited to participate in the decision analysis. These experts are from the operation management department, the engineering management department, and the human resource department. Based on their experience, the expert compares each pair of alternatives and gives individual judgements using the following BSVNPRs, where $M_i = \left(m_{pq}^{(i)}\right)_{6\times 6}(i = 1, 2, 3)$. The bipolar single-valued neutrosophic decision groups (BSVNDGs), D_i corresponding to BSVNPRs, M_i given in Tables 8–10.

Table 8. BSVNPR of the expert from the operation management department.

M_1	A_1	A_2	A_3	A_4	A_5	A_6
A_1	$\langle 0.5, 0.5, 0.5, -0.5, -0.5, -0.5 \rangle$	$\langle 0.4, 0.6, 0.3, -0.3, -0.1, -0.4 \rangle$	$\langle 0.3, 0.2, 0.6, -0.4, -0.6, -0.2 \rangle$	$\langle 0.7, 0.3, 0.4, -0.6, -0.5, -0.3 \rangle$	$\langle 0.4, 0.2, 0.8, -0.5, -0.3, -0.2 \rangle$	$\langle 0.9, 0.4, 0.6, -0.1, -0.6, -0.5 \rangle$
A_2	$\langle 0.3, 0.4, 0.4, -0.4, -0.9, -0.3 \rangle$	$\langle 0.5, 0.5, 0.5, -0.5, -0.5, -0.5 \rangle$	$\langle 0.8, 0.3, 0.2, -0.5, -0.2, -0.6 \rangle$	$\langle 0.3, 0.2, 0.7, -0.2, -0.3, -0.5 \rangle$	$\langle 0.4, 0.2, 0.3, -0.2, -0.5, -0.7 \rangle$	$\langle 0.2, 0.5, 0.6, -0.4, -0.6, -0.7 \rangle$
A_3	$\langle 0.6, 0.8, 0.3, -0.2, -0.4, -0.4 \rangle$	$\langle 0.2, 0.7, 0.8, -0.6, -0.8, -0.5 \rangle$	$\langle 0.5, 0.5, 0.5, -0.5, -0.5, -0.5 \rangle$	$\langle 0.8, 0.2, 0.4, -0.2, -0.3, -0.6 \rangle$	$\langle 0.3, 0.5, 0.2, -0.5, -0.5, -0.2 \rangle$	$\langle 0.1, 0.4, 0.7, -0.5, -0.2, -0.6 \rangle$
A_4	$\langle 0.4, 0.7, 0.7, -0.3, -0.5, -0.6 \rangle$	$\langle 0.7, 0.8, 0.3, -0.5, -0.7, -0.2 \rangle$	$\langle 0.4, 0.8, 0.8, -0.6, -0.7, -0.2 \rangle$	$\langle 0.5, 0.5, 0.5, -0.5, -0.5, -0.5 \rangle$	$\langle 0.3, 0.2, 0.1, -0.6, -0.3, -0.4 \rangle$	$\langle 0.7, 0.4, 0.5, -0.1, -0.2, -0.3 \rangle$
A_5	$\langle 0.8, 0.8, 0.4, -0.2, -0.7, -0.5 \rangle$	$\langle 0.3, 0.8, 0.4, -0.7, -0.5, -0.2 \rangle$	$\langle 0.2, 0.5, 0.3, -0.2, -0.5, -0.5 \rangle$	$\langle 0.1, 0.8, 0.3, -0.4, -0.7, -0.6 \rangle$	$\langle 0.5, 0.5, 0.5, -0.5, -0.5, -0.5 \rangle$	$\langle 0.4, 0.3, 0.8, -0.6, -0.3, -0.2 \rangle$
A_6	$\langle 0.6, 0.6, 0.9, -0.5, -0.4, -0.1 \rangle$	$\langle 0.6, 0.5, 0.2, -0.7, -0.4, -0.4 \rangle$	$\langle 0.7, 0.6, 0.1, -0.6, -0.8, -0.5 \rangle$	$\langle 0.5, 0.6, 0.7, -0.3, -0.8, -0.1 \rangle$	$\langle 0.8, 0.7, 0.4, -0.2, -0.7, -0.6 \rangle$	$\langle 0.5, 0.5, 0.5, -0.5, -0.5, -0.5 \rangle$

Table 9. BSVNPR of the expert from the engineering management department.

M_2	A_1	A_2	A_3	A_4	A_5	A_6
A_1	$\langle 0.5, 0.5, 0.5, -0.5, -0.5, -0.5 \rangle$	$\langle 0.5, 0.3, 0.8, -0.4, -0.2, -0.6 \rangle$	$\langle 0.9, 0.8, 0.7, -0.4, -0.7, -0.3 \rangle$	$\langle 0.8, 0.4, 0.2, -0.7, -0.4, -0.2 \rangle$	$\langle 0.4, 0.3, 0.7, -0.4, -0.4, -0.4 \rangle$	$\langle 0.7, 0.4, 0.8, -0.2, -0.3, -0.5 \rangle$
A_2	$\langle 0.8, 0.7, 0.5, -0.6, -0.8, -0.4 \rangle$	$\langle 0.5, 0.5, 0.5, -0.5, -0.5, -0.5 \rangle$	$\langle 0.2, 0.3, 0.4, -0.6, -0.4, -0.7 \rangle$	$\langle 0.4, 0.3, 0.6, -0.3, -0.4, -0.5 \rangle$	$\langle 0.3, 0.4, 0.6, -0.4, -0.6, -0.7 \rangle$	$\langle 0.4, 0.6, 0.7, -0.3, -0.4, -0.5 \rangle$
A_3	$\langle 0.7, 0.2, 0.9, -0.3, -0.3, -0.4 \rangle$	$\langle 0.4, 0.7, 0.2, -0.7, -0.6, -0.6 \rangle$	$\langle 0.5, 0.5, 0.5, -0.5, -0.5, -0.5 \rangle$	$\langle 0.7, 0.4, 0.6, -0.4, -0.3, -0.7 \rangle$	$\langle 0.7, 0.6, 0.3, -0.3, -0.4, -0.6 \rangle$	$\langle 0.2, 0.6, 0.8, -0.4, -0.3, -0.7 \rangle$
A_4	$\langle 0.2, 0.6, 0.8, -0.2, -0.6, -0.7 \rangle$	$\langle 0.6, 0.7, 0.4, -0.5, -0.6, -0.3 \rangle$	$\langle 0.6, 0.6, 0.7, -0.7, -0.7, -0.4 \rangle$	$\langle 0.5, 0.5, 0.5, -0.5, -0.5, -0.5 \rangle$	$\langle 0.4, 0.4, 0.2, -0.3, -0.4, -0.4 \rangle$	$\langle 0.6, 0.4, 0.4, -0.2, -0.4, -0.6 \rangle$
A_5	$\langle 0.7, 0.7, 0.4, -0.4, -0.6, -0.4 \rangle$	$\langle 0.6, 0.6, 0.3, -0.7, -0.4, -0.4 \rangle$	$\langle 0.3, 0.4, 0.7, -0.6, -0.6, -0.3 \rangle$	$\langle 0.2, 0.6, 0.4, -0.4, -0.6, -0.3 \rangle$	$\langle 0.5, 0.5, 0.5, -0.5, -0.5, -0.5 \rangle$	$\langle 0.3, 0.7, 0.2, -0.4, -0.4, -0.1 \rangle$
A_6	$\langle 0.8, 0.6, 0.7, -0.5, -0.7, -0.2 \rangle$	$\langle 0.7, 0.4, 0.4, -0.5, -0.6, -0.3 \rangle$	$\langle 0.8, 0.4, 0.2, -0.7, -0.7, -0.4 \rangle$	$\langle 0.4, 0.6, 0.6, -0.6, -0.6, -0.2 \rangle$	$\langle 0.2, 0.3, 0.3, -0.1, -0.6, -0.4 \rangle$	$\langle 0.5, 0.5, 0.5, -0.5, -0.5, -0.5 \rangle$

Table 10. BSVNPR of the expert from the human resource department.

M_3	A_1	A_2	A_3	A_4	A_5	A_6
A_1	$\langle 0.5, 0.5, 0.5, -0.5, -0.5, -0.5 \rangle$	$\langle 0.6, 0.6, 0.9, -0.5, -0.4, -0.1 \rangle$	$\langle 0.6, 0.5, 0.2, -0.7, -0.4, -0.4 \rangle$	$\langle 0.7, 0.6, 0.1, -0.6, -0.8, -0.5 \rangle$	$\langle 0.5, 0.6, 0.7, -0.3, -0.8, -0.1 \rangle$	$\langle 0.8, 0.7, 0.4, -0.2, -0.7, -0.6 \rangle$
A_2	$\langle 0.9, 0.4, 0.6, -0.1, -0.6, -0.5 \rangle$	$\langle 0.5, 0.5, 0.5, -0.5, -0.5, -0.5 \rangle$	$\langle 0.8, 0.8, 0.4, -0.2, -0.7, -0.5 \rangle$	$\langle 0.3, 0.8, 0.4, -0.7, -0.5, -0.2 \rangle$	$\langle 0.2, 0.5, 0.3, -0.2, -0.5, -0.5 \rangle$	$\langle 0.1, 0.8, 0.3, -0.4, -0.7, -0.6 \rangle$
A_3	$\langle 0.2, 0.5, 0.6, -0.4, -0.6, -0.7 \rangle$	$\langle 0.4, 0.2, 0.8, -0.5, -0.3, -0.2 \rangle$	$\langle 0.5, 0.5, 0.5, -0.5, -0.5, -0.5 \rangle$	$\langle 0.4, 0.7, 0.7, -0.3, -0.5, -0.6 \rangle$	$\langle 0.7, 0.8, 0.3, -0.5, -0.7, -0.2 \rangle$	$\langle 0.4, 0.8, 0.8, -0.6, -0.7, -0.2 \rangle$
A_4	$\langle 0.1, 0.4, 0.7, -0.5, -0.2, -0.6 \rangle$	$\langle 0.4, 0.2, 0.3, -0.2, -0.5, -0.7 \rangle$	$\langle 0.7, 0.3, 0.4, -0.6, -0.5, -0.3 \rangle$	$\langle 0.5, 0.5, 0.5, -0.5, -0.5, -0.5 \rangle$	$\langle 0.6, 0.8, 0.3, -0.2, -0.4, -0.4 \rangle$	$\langle 0.2, 0.7, 0.8, -0.6, -0.8, -0.5 \rangle$
A_5	$\langle 0.7, 0.4, 0.5, -0.1, -0.2, -0.3 \rangle$	$\langle 0.3, 0.5, 0.2, -0.5, -0.5, -0.2 \rangle$	$\langle 0.3, 0.2, 0.7, -0.2, -0.3, -0.5 \rangle$	$\langle 0.3, 0.2, 0.6, -0.4, -0.6, -0.2 \rangle$	$\langle 0.5, 0.5, 0.5, -0.5, -0.5, -0.5 \rangle$	$\langle 0.3, 0.4, 0.4, -0.4, -0.9, -0.3 \rangle$
A_6	$\langle 0.4, 0.3, 0.8, -0.6, -0.3, -0.2 \rangle$	$\langle 0.3, 0.2, 0.1, -0.6, -0.3, -0.4 \rangle$	$\langle 0.8, 0.2, 0.4, -0.2, -0.3, -0.6 \rangle$	$\langle 0.8, 0.3, 0.2, -0.5, -0.2, -0.6 \rangle$	$\langle 0.4, 0.6, 0.3, -0.3, -0.1, -0.4 \rangle$	$\langle 0.5, 0.5, 0.5, -0.5, -0.5, -0.5 \rangle$

The energy of a BSVNDG is the sum of absolute values of the real part of eigenvalues of D. The energy of each BSVNDG $D_i (i = 1, 2, 3)$ is calculated as follows:

$$\begin{aligned}
E(D_1) &= \langle 3.7606, 3.0000, 3.7606, 3.0000, 3.1361, 3.0000 \rangle, \\
E(D_2) &= \langle 4.0388, 3.0000, 4.0388, 3.4825, 3.0000, 3.4825 \rangle, \\
E(D_3) &= \langle 3.9621, 3.0000, 3.9621, 3.3062, 3.0000, 3.3062 \rangle.
\end{aligned}$$

Then, the weight of each expert can be determined as:

$$\begin{aligned}
w_i &= ((w_{T^+})_i, (w_{I^+})_i, (w_{F^+})_i, (w_{T^-})_i, (w_{I^-})_i, (w_{F^-})_i) \\
&= \left(\frac{E\big((D_{T^+})_i\big)}{\sum_{j=1}^{k} E\big((D_{T^+})_j\big)}, \frac{E\big((D_{I^+})_i\big)}{\sum_{j=1}^{k} E\big((D_{I^+})_j\big)}, \frac{E\big((D_{F^+})_i\big)}{\sum_{j=1}^{k} E\big((D_{F^+})_j\big)}, \frac{E\big((D_{T^-})_i\big)}{\sum_{j=1}^{k} E\big((D_{T^-})_j\big)}, \frac{E\big((D_{I^-})_i\big)}{\sum_{j=1}^{k} E\big((D_{I^-})_j\big)}, \frac{E\big((D_{F^-})_i\big)}{\sum_{j=1}^{k} E\big((D_{F^-})_j\big)} \right)
\end{aligned}$$

where $i = 1, 2, \ldots, k$,

$$\begin{aligned}
w_1 &= \langle 0.3197, 0.3333, 0.3197, 0.3065, 0.3433, 0.3065 \rangle, \\
w_2 &= \langle 0.3434, 0.3333, 0.3434, 0.3558, 0.3284, 0.3558 \rangle, \\
w_3 &= \langle 0.3368, 0.3333, 0.3368, 0.3378, 0.3284, 0.3378 \rangle.
\end{aligned}$$

Then, we utilize the aggregation operator to merge all the individual BSVNPRs, $M_i = \left(m_{pq}^{(i)}\right)_{6\times 6} (i = 1, 2, 3)$ into the collective BSVNPR $M = \left(m_{pq}\right)_{6\times 6}$. Here, we apply the bipolar single-valued neutrosophic weighted averaging (BSVNWA) operator [30] to merge all the individual BSVNPR.

$$\text{BSVNWA}\left(m_{pq}^{(1)}, m_{pq}^{(2)}, \ldots, m_{pq}^{(s)}\right) = \left\langle \begin{array}{c} 1 - \prod_{i=1}^{s}\left(1 - T_{pq}^{+(i)}\right)^{w_i}, \prod_{i=1}^{s}\left(I_{pq}^{+(i)}\right)^{w_i}, \prod_{i=1}^{s}\left(F_{pq}^{+(i)}\right)^{w_i}, -\prod_{i=1}^{s}\left(-T_{pq}^{-(i)}\right)^{w_i}, \\ -\left(1 - \prod_{i=1}^{s}\left(1 - \left(-I_{pq}^{-(i)}\right)\right)^{w_i}\right), -\left(1 - \prod_{i=1}^{s}\left(1 - \left(-F_{pq}^{-(i)}\right)\right)^{w_i}\right) \end{array} \right\rangle$$

We show the aggregated value for m_{12} as follows:

$$\begin{aligned}
T_{12}^{+} &= 1 - \prod_{i=1}^{3}\left(1 - T_{12}^{+(i)}\right)^{w_i} \\
&= 1 - \left[\left(1 - T_{12}^{+(1)}\right)^{w_1} \times \left(1 - T_{12}^{+(2)}\right)^{w_2} \times \left(1 - T_{12}^{+(3)}\right)^{w_3}\right] \\
&= 1 - \left[(1 - 0.4)^{0.3197} \times (1 - 0.5)^{0.3434} \times (1 - 0.6)^{0.3368}\right] \\
&= 1 - [(0.8493) \times (0.7882) \times (0.7345)] \\
&= 0.5083
\end{aligned}$$

$$\begin{aligned}
I_{12}^{+} &= \prod_{i=1}^{3}\left(I_{12}^{+(i)}\right)^{w_i} \\
&= \left(I_{12}^{+(1)}\right)^{w_1} \times \left(I_{12}^{+(2)}\right)^{w_2} \times \left(I_{12}^{+(3)}\right)^{w_3} \\
&= (0.6)^{0.3333} \times (0.3)^{0.3333} \times (0.6)^{0.3333} \\
&= (0.8435) \times (0.6695) \times (0.8435) \\
&= 0.4763
\end{aligned}$$

$$
\begin{aligned}
F_{12}^{+} &= \prod_{i=1}^{3}\left(F_{12}^{+(i)}\right)^{w_i} \\
&= \left(F_{12}^{+(1)}\right)^{w_1} \times \left(F_{12}^{+(2)}\right)^{w_2} \times \left(F_{12}^{+(3)}\right)^{w_3} \\
&= (0.3)^{0.3197} \times (0.8)^{0.3434} \times (0.9)^{0.3368} \\
&= (0.6805) \times (0.9262) \times (0.9651) \\
&= 0.6083
\end{aligned}
$$

$$
\begin{aligned}
T_{12}^{-} &= -\prod_{i=1}^{3}\left(-T_{12}^{-(i)}\right)^{w_i} \\
&= -\left[\left(-T_{12}^{-(1)}\right)^{w_1} \times \left(-T_{12}^{-(2)}\right)^{w_2} \times \left(-T_{12}^{-(3)}\right)^{w_3}\right] \\
&= -\left[(0.3)^{0.3065} \times (0.4)^{0.3558} \times (0.5)^{0.3378}\right] \\
&= -[(0.6914) \times (0.7218) \times (0.7912)] \\
&= -0.3949
\end{aligned}
$$

$$
\begin{aligned}
I_{12}^{-} &= -\left(1 - \prod_{i=1}^{3}\left(1 - \left(-I_{12}^{-(i)}\right)\right)^{w_i}\right) \\
&= -\left(1 - \left[\left(1 - \left(-I_{12}^{-(1)}\right)\right)^{w_1} \times \left(1 - \left(-I_{12}^{-(2)}\right)\right)^{w_2} \times \left(1 - \left(-I_{12}^{-(3)}\right)\right)^{w_3}\right]\right) \\
&= -\left(1 - \left[(1-0.1)^{0.3433} \times (1-0.2)^{0.3284} \times (1-0.4)^{0.3284}\right]\right) \\
&= -(1 - [(0.9645) \times (0.9293) \times (0.8456)]) \\
&= -0.2421
\end{aligned}
$$

$$
\begin{aligned}
F_{12}^{-} &= -\left(1 - \prod_{i=1}^{3}\left(1 - \left(-F_{12}^{-(i)}\right)\right)^{w_i}\right) \\
&= -\left(1 - \left[\left(1 - \left(-F_{12}^{-(1)}\right)\right)^{w_1} \times \left(1 - \left(-F_{12}^{-(2)}\right)\right)^{w_2} \times \left(1 - \left(-F_{12}^{-(3)}\right)\right)^{w_3}\right]\right) \\
&= -\left(1 - \left[(1-0.4)^{0.3065} \times (1-0.6)^{0.3558} \times (1-0.1)^{0.3378}\right]\right) \\
&= -(1 - [(0.8551) \times (0.7218) \times (0.9650)]) \\
&= -0.4044
\end{aligned}
$$

Then, we obtained

$$
m_{12} = \left\langle \begin{matrix} 0.5083, 0.4763, 0.6083, \\ -0.3949, -0.2421, -0.4044 \end{matrix} \right\rangle.
$$

Similarly, we can calculate other aggregated values using BSVNWA. Table 11 presents overall aggregated values.

Table 11. The collective BSVNPR of all the above individual BSVNPRs.

M	A_1	A_2	A_3
A_1	$\langle 0.5000, 0.5000, 0.5000, -0.5000, -0.5000, -0.5000 \rangle$	$\langle 0.5083, 0.4763, 0.6083, -0.3949, -0.2421, -0.4044 \rangle$	$\langle 0.7028, 0.4309, 0.4370 -0.4832, -0.5843, -0.3078 \rangle$
A_2	$\langle 0.7636, 0.4821, 0.4951, -0.2893, -0.8021, -0.4086 \rangle$	$\langle 0.5000, 0.5000, 0.5000, -0.5000, -0.5000, -0.5000 \rangle$	$\langle 0.6780, 0.4161, 0.3205, -0.3915, -0.4726, -0.6107 \rangle$
A_3	$\langle 0.5423, 0.4309, 0.5526, -0.2919, -0.4476, -0.5253 \rangle$	$\langle 0.3422, 0.4611, 0.4970, -0.5959, -0.6211, -0.4587 \rangle$	$\langle 0.5000, 0.5000, 0.5000, -0.5000, -0.5000, -0.5000 \rangle$
A_4	$\langle 0.2407, 0.5518, 0.7329, -0.3086, -0.4578, -0.6390 \rangle$	$\langle 0.5817, 0.4821, 0.3312, -0.3669, -0.6101, -0.4523 \rangle$	$\langle 0.5867, 0.5242, 0.6051, -0.6338, -0.6452, -0.3097 \rangle$
A_5	$\langle 0.7364, 0.6073, 0.4313, -0.2025, -0.5450, -0.4023 \rangle$	$\langle 0.4224, 0.6215, 0.2870, -0.6248, -0.4692, -0.2779 \rangle$	$\langle 0.2694, 0.3420, 0.5339, -0.2956, -0.4811, -0.4365 \rangle$
A_6	$\langle 0.6386, 0.4763, 0.7935, -0.5317, -0.4974, -0.1706 \rangle$	$\langle 0.5624, 0.3420, 0.2009, -0.5895, -0.4476, -0.3662 \rangle$	$\langle 0.7723, 0.3635, 0.2024, -0.4373, -0.6553, -0.5053 \rangle$
	A_4	A_5	A_6
A_1	$\langle 0.7390, 0.4161, 0.1977, -0.6338, -0.6071, -0.3448 \rangle$	$\langle 0.4357, 0.3302, 0.7306, -0.3886, -0.5590, -0.2485 \rangle$	$\langle 0.8158, 0.4821, 0.5778, -0.1617, -0.5627, -0.5363 \rangle$
A_2	$\langle 0.3361, 0.3635, 0.5499, -0.3527, -0.4042, -0.4140 \rangle$	$\langle 0.3030, 0.3420, 0.3807, -0.2559, -0.5354, -0.6435 \rangle$	$\langle 0.2459, 0.6215, 0.5009, -0.3610, -0.5843, -0.6035 \rangle$
A_3	$\langle 0.6671, 0.3826, 0.5552, -0.2935, -0.3733, -0.6390 \rangle$	$\langle 0.6066, 0.6215, 0.2636, -0.4169, -0.5512, -0.3749 \rangle$	$\langle 0.2460, 0.5769, 0.7666, -0.4911, -0.4452, -0.5437 \rangle$
A_4	$\langle 0.5000, 0.5000, 0.5000, -0.5000, -0.5000, -0.5000 \rangle$	$\langle 0.4501, 0.4000, 0.1837, -0.3235, -0.3674, -0.4000 \rangle$	$\langle 0.5392, 0.4821, 0.5426, -0.2344, -0.5383, -0.4880 \rangle$
A_5	$\langle 0.2058, 0.4579, 0.4183, -0.4000, -0.6376, -0.3831 \rangle$	$\langle 0.5000, 0.5000, 0.5000, -0.5000, -0.5000, -0.5000 \rangle$	$\langle 0.3336, 0.4380, 0.3935, -0.4529, -0.6488, -0.2026 \rangle$
A_6	$\langle 0.6090, 0.4763, 0.4354, -0.4562, -0.6041, -0.3437 \rangle$	$\langle 0.5338, 0.5014, 0.3289, -0.1792, -0.5271, -0.4701 \rangle$	$\langle 0.5000, 0.5000, 0.5000, -0.5000, -0.5000, -0.5000 \rangle$

Then, under the condition $T^{+}_{pq} \geq 0.5(p, q = 1, 2, \ldots, 6)$, a partial diagram is drawn, as shown in Figure 4. After that, the out-degree, $out - d(A_p)(p = 1, 2, \ldots, 6)$ of all criteria in a partial directed network are calculated as follows:

$$\begin{aligned}
out - d(A_1) &= \langle 2.7659, 2.1391, 1.8008, -1.6736, -1.9962, -1.5933\rangle \\
out - d(A_2) &= \langle 1.4416, 0.8982, 0.8156, -0.6808, -1.2747, -1.0193\rangle \\
out - d(A_3) &= \langle 1.8160, 1.4350, 1.3714, -1.0023, -1.3721, -1.5392\rangle \\
out - d(A_4) &= \langle 1.7076, 1.4884, 1.4789, -1.2351, -1.7936, -1.2500\rangle \\
out - d(A_5) &= \langle 0.7364, 0.6073, 0.4313, -0.2025, -0.5450, -0.4023\rangle \\
out - d(A_6) &= \langle 3.1161, 2.1595, 1.9611, -2.1939, -2.7315, -1.8559\rangle
\end{aligned}$$

According to positive membership degree of out-degree, $out - d(A_p)(p = 1, 2, \ldots, 6)$, the ranking of the alternatives $A_p(p = 1, 2, \ldots, 6)$ is written as:

$$A_6 \succ A_1 \succ A_3 \succ A_4 \succ A_2 \succ A_5.$$

Therefore, the best alternative is the wind power plant, A_6.

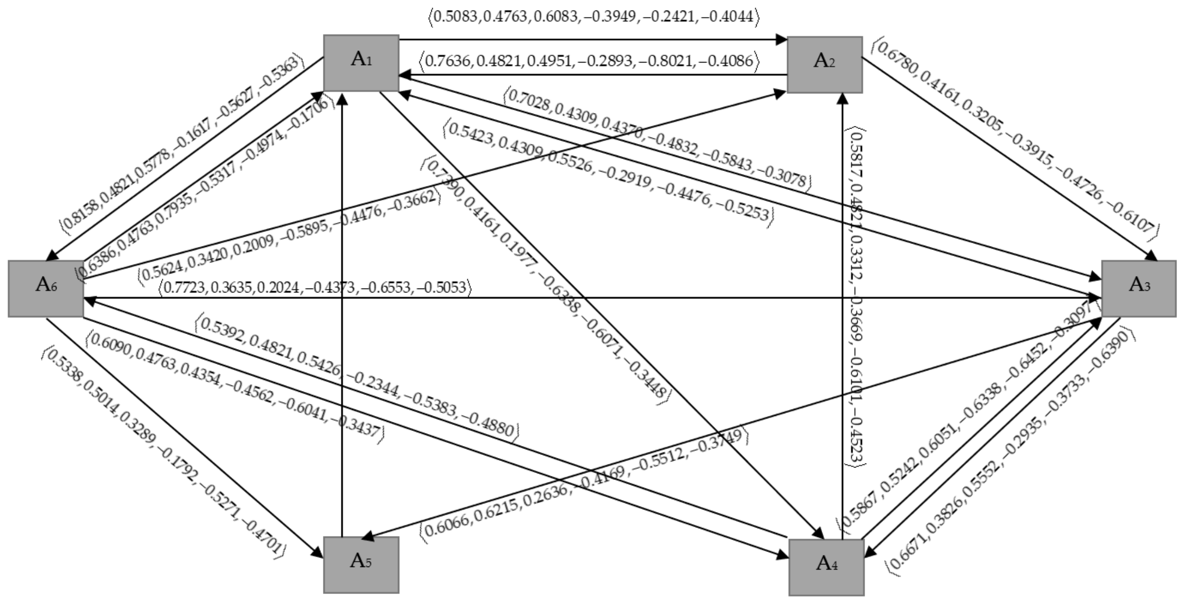


Figure 4. Partial directed network of the BSVN.

Now, the elements of the Laplacian matrices of the BSVNDGs, $L(D_i)$ corresponding to BSVNPRs, $M_i^L(i = 1, 2, 3)$ is given in Tables 12–14 while Table 15 presents overall aggregated values.

Table 12. Elements of the Laplacian matrix of the BSVNDG, D_1.

M_1	A_1	A_2	A_3	A_4	A_5	A_6
A_1	$\langle 2.7, 1.7, 2.7, -1.9, -2.1, -1.6\rangle$	$\langle -0.4, -0.6, -0.3, 0.3, 0.1, 0.4\rangle$	$\langle -0.3, -0.2, -0.6, 0.4, 0.6, 0.2\rangle$	$\langle -0.7, -0.3, -0.4, 0.6, 0.5, 0.3\rangle$	$\langle -0.4, -0.2, -0.8, 0.5, 0.3, 0.2\rangle$	$\langle -0.9, -0.4, -0.6, 0.1, 0.6, 0.5\rangle$
A_2	$\langle -0.3, -0.4, -0.4, 0.4, 0.9, 0.3\rangle$	$\langle 2.0, 1.6, 2.2, -1.7, -2.5, -2.8\rangle$	$\langle -0.8, -0.3, -0.2, 0.5, 0.2, 0.6\rangle$	$\langle -0.3, -0.2, -0.7, 0.2, 0.3, 0.5\rangle$	$\langle -0.4, -0.2, -0.3, 0.2, 0.5, 0.7\rangle$	$\langle -0.2, -0.5, -0.6, 0.4, 0.6, 0.7\rangle$
A_3	$\langle -0.6, -0.8, -0.3, 0.2, 0.4, 0.4\rangle$	$\langle -0.2, -0.7, -0.8, 0.6, 0.8, 0.5\rangle$	$\langle 2.0, 2.6, 2.4, -2.0, -2.2, -2.3\rangle$	$\langle -0.8, -0.2, -0.4, 0.2, 0.3, 0.6\rangle$	$\langle -0.3, -0.5, -0.2, 0.5, 0.5, 0.2\rangle$	$\langle -0.1, -0.4, -0.7, 0.5, 0.2, 0.6\rangle$
A_4	$\langle -0.4, -0.7, -0.7, 0.3, 0.5, 0.6\rangle$	$\langle -0.7, -0.8, -0.3, 0.5, 0.7, 0.2\rangle$	$\langle -0.4, -0.8, -0.8, 0.6, 0.7, 0.2\rangle$	$\langle 2.5, 2.9, 2.4, -2.1, -2.4, -1.7\rangle$	$\langle -0.3, -0.2, -0.1, 0.6, 0.3, 0.4\rangle$	$\langle -0.7, -0.4, -0.5, 0.1, 0.2, 0.3\rangle$
A_5	$\langle -0.8, -0.8, -0.4, 0.2, 0.7, 0.5\rangle$	$\langle -0.3, -0.8, -0.4, 0.7, 0.5, 0.2\rangle$	$\langle -0.2, -0.5, -0.3, 0.2, 0.5, 0.5\rangle$	$\langle -0.1, -0.8, -0.3, 0.4, 0.7, 0.6\rangle$	$\langle 1.8, 3.2, 2.2, -2.1, -2.7, -2.0\rangle$	$\langle -0.4, -0.3, -0.8, 0.6, 0.3, 0.2\rangle$
A_6	$\langle -0.6, -0.6, -0.9, 0.5, 0.4, 0.1\rangle$	$\langle -0.6, -0.5, -0.2, 0.7, 0.4, 0.4\rangle$	$\langle -0.7, -0.6, -0.1, 0.6, 0.8, 0.5\rangle$	$\langle -0.5, -0.6, -0.7, 0.3, 0.8, 0.1\rangle$	$\langle -0.8, -0.7, -0.4, 0.2, 0.7, 0.6\rangle$	$\langle 3.2, 3.0, 2.3, -2.3, -3.1, -1.7\rangle$

Table 13. Elements of the Laplacian matrix of the BSVNDG, D_2.

M_2	A_1	A_2	A_3	A_4	A_5	A_6
A_1	$\langle 3.3, 2.2, 3.2, -2.1, -2.0, -2.0 \rangle$	$\langle -0.5, -0.3, -0.8, 0.4, 0.2, 0.6 \rangle$	$\langle -0.9, -0.8, -0.7, 0.4, 0.7, 0.3 \rangle$	$\langle -0.8, -0.4, -0.2, 0.7, 0.4, 0.2 \rangle$	$\langle -0.4, -0.3, -0.7, 0.4, 0.4, 0.4 \rangle$	$\langle -0.7, -0.4, -0.8, 0.2, 0.3, 0.5 \rangle$
A_2	$\langle -0.8, -0.7, -0.5, 0.6, 0.8, 0.4 \rangle$	$\langle 2.1, 2.3, 2.8, -2.2, -2.6, -2.8 \rangle$	$\langle -0.2, -0.3, -0.4, 0.6, 0.4, 0.7 \rangle$	$\langle -0.4, -0.3, -0.6, 0.3, 0.4, 0.5 \rangle$	$\langle -0.3, -0.4, -0.6, 0.4, 0.6, 0.7 \rangle$	$\langle -0.4, -0.6, -0.7, 0.3, 0.4, 0.5 \rangle$
A_3	$\langle -0.7, -0.2, -0.9, 0.3, 0.3, 0.4 \rangle$	$\langle -0.4, -0.7, -0.2, 0.7, 0.6, 0.6 \rangle$	$\langle -0.4, -0.7, -0.2, 0.7, 0.6, 0.6 \rangle$	$\langle -0.7, -0.4, -0.6, 0.4, 0.3, 0.7 \rangle$	$\langle -0.7, -0.6, -0.3, 0.3, 0.4, 0.6 \rangle$	$\langle -0.2, -0.6, -0.8, 0.4, 0.3, 0.7 \rangle$
A_4	$\langle -0.2, -0.6, -0.8, 0.2, 0.6, 0.7 \rangle$	$\langle -0.6, -0.7, -0.4, 0.5, 0.6, 0.3 \rangle$	$\langle -0.6, -0.6, -0.7, 0.7, 0.7, 0.4 \rangle$	$\langle 2.4, 2.7, 2.5, -1.9, -2.7, -2.4 \rangle$	$\langle -0.4, -0.4, -0.2, 0.3, 0.4, 0.4 \rangle$	$\langle -0.6, -0.4, -0.4, 0.2, 0.4, 0.6 \rangle$
A_5	$\langle -0.7, -0.7, -0.4, 0.4, 0.6, 0.4 \rangle$	$\langle -0.6, -0.6, -0.3, 0.7, 0.4, 0.4 \rangle$	$\langle -0.3, -0.4, -0.7, 0.6, 0.6, 0.3 \rangle$	$\langle -0.2, -0.6, -0.4, 0.4, 0.6, 0.3 \rangle$	$\langle 2.1, 3.0, 2.0, -2.5, -2.6, -1.5 \rangle$	$\langle -0.3, -0.7, -0.2, 0.4, 0.4, 0.1 \rangle$
A_6	$\langle -0.8, -0.6, -0.7, 0.5, 0.7, 0.2 \rangle$	$\langle -0.7, -0.4, -0.4, 0.5, 0.6, 0.3 \rangle$	$\langle -0.8, -0.4, -0.2, 0.7, 0.7, 0.4 \rangle$	$\langle -0.4, -0.6, -0.6, 0.6, 0.6, 0.2 \rangle$	$\langle -0.2, -0.3, -0.3, 0.1, 0.6, 0.4 \rangle$	$\langle 2.9, 2.3, 2.2, -2.4, -3.2, -1.5 \rangle$

Table 14. Elements of the Laplacian matrix of the BSVNDG, D_3.

M_3	A_1	A_2	A_3	A_4	A_5	A_6
A_1	$\langle 3.2, 3.0, 2.3, -2.3, -3.1, -1.7 \rangle$	$\langle -0.6, -0.6, -0.9, 0.5, 0.4, 0.1 \rangle$	$\langle -0.6, -0.5, -0.2, 0.7, 0.4, 0.4 \rangle$	$\langle -0.7, -0.6, -0.1, 0.6, 0.8, 0.5 \rangle$	$\langle -0.5, -0.6, -0.7, 0.3, 0.8, 0.1 \rangle$	$\langle -0.8, -0.7, -0.4, 0.2, 0.7, 0.6 \rangle$
A_2	$\langle -0.9, -0.4, -0.6, 0.1, 0.6, 0.5 \rangle$	$\langle 2.3, 3.3, 2.0, -1.6, -3.0, -2.3 \rangle$	$\langle -0.8, -0.8, -0.4, 0.2, 0.7, 0.5 \rangle$	$\langle -0.3, -0.8, -0.4, 0.7, 0.5, 0.2 \rangle$	$\langle -0.2, -0.5, -0.3, 0.2, 0.5, 0.5 \rangle$	$\langle -0.1, -0.8, -0.3, 0.4, 0.7, 0.6 \rangle$
A_3	$\langle -0.2, -0.5, -0.6, 0.4, 0.6, 0.7 \rangle$	$\langle -0.4, -0.2, -0.8, 0.5, 0.3, 0.2 \rangle$	$\langle 2.1, 3.0, 3.2, -2.3, -2.8, -1.9 \rangle$	$\langle -0.4, -0.7, -0.7, 0.3, 0.5, 0.6 \rangle$	$\langle -0.7, -0.8, -0.3, 0.5, 0.7, 0.2 \rangle$	$\langle -0.4, -0.8, -0.8, 0.6, 0.7, 0.2 \rangle$
A_4	$\langle -0.1, -0.4, -0.7, 0.5, 0.2, 0.6 \rangle$	$\langle -0.4, -0.2, -0.3, 0.2, 0.5, 0.7 \rangle$	$\langle -0.7, -0.3, -0.4, 0.6, 0.5, 0.3 \rangle$	$\langle 2.0, 2.4, 2.5, -2.1, -2.4, -2.5 \rangle$	$\langle -0.6, -0.8, -0.3, 0.2, 0.4, 0.4 \rangle$	$\langle -0.2, -0.7, -0.8, 0.6, 0.8, 0.5 \rangle$
A_5	$\langle -0.7, -0.4, -0.5, 0.1, 0.2, 0.3 \rangle$	$\langle -0.3, -0.5, -0.2, 0.5, 0.5, 0.2 \rangle$	$\langle -0.3, -0.2, -0.7, 0.2, 0.3, 0.5 \rangle$	$\langle -0.3, -0.2, -0.6, 0.4, 0.6, 0.2 \rangle$	$\langle 1.9, 1.7, 2.4, -1.6, -2.5, -1.5 \rangle$	$\langle -0.3, -0.4, -0.4, 0.4, 0.9, 0.3 \rangle$
A_6	$\langle -0.4, -0.3, -0.8, 0.6, 0.3, 0.2 \rangle$	$\langle -0.3, -0.2, -0.1, 0.6, 0.3, 0.4 \rangle$	$\langle -0.8, -0.2, -0.4, 0.2, 0.3, 0.6 \rangle$	$\langle -0.8, -0.3, -0.2, 0.5, 0.2, 0.6 \rangle$	$\langle -0.4, -0.6, -0.3, 0.3, 0.1, 0.4 \rangle$	$\langle 2.7, 1.6, 1.8, -2.2, -1.2, -2.2 \rangle$

Table 15. The collective BSVNPR of all the above individual BSVNPRs.

M	A_1	A_2	A_3
A_1	$\langle 0.5000, 0.5000, 0.5000, -0.5000, -0.5000, -0.5000 \rangle$	$\langle 0.5066, 0.4763, 0.6051, -0.3918, -0.2440, -0.4071 \rangle$	$\langle 0.7062, 0.4309, 0.4440, -0.4794, -0.5839, -0.3046 \rangle$
A_2	$\langle 0.7603, 0.4821, 0.4934, -0.2948, -0.8000, -0.4054 \rangle$	$\langle 0.5000, 0.5000, 0.5000, -0.5000, -0.5000, -0.5000 \rangle$	$\langle 0.6738, 034161, 0.3196, -0.3965, -0.4758, -0.6115 \rangle$
A_3	$\langle 0.5478, 0.4309, 0.5533, -0.2888, -0.4482, -0.5205 \rangle$	$\langle 0.3415, 0.4611, 0.4903, -0.5973, -0.6174, -0.4619 \rangle$	$\langle 0.5000, 0.5000, 0.5000, -0.5000, -0.5000, -0.5000 \rangle$
A_4	$\langle 0.2428, 0.5518, 0.7338, -0.3068, -0.4571, -0.6386 \rangle$	$\langle 0.5845, 0.4821, 0.3320, -0.3718, -0.6085, -0.4443 \rangle$	$\langle 0.5845, 0.5242, 0.6098, -0.6336, -0.6443, -0.3078 \rangle$
A_5	$\langle 0.7369, 0.6073, 0.4299, -0.2042, -0.5421, -0.4054 \rangle$	$\langle 0.4255, 0.6215, 0.2887, -0.6278 - 0.4686, -0.2772 \rangle$	$\langle 0.2691, 0.3420, 0.5322, -0.2948, -0.4807, -0.4369 \rangle$
A_6	$\langle 0.6430, 0.4763, 0.7928, -0.5304, -0.4986, -0.1689 \rangle$	$\langle 0.5670, 0.3420, 0.2041, -0.5914, -0.4482, -0.3664 \rangle$	$\langle 0.7720, 0.3635, 0.2000, -0.4441, -0.6524, -0.5039 \rangle$

	A_4	A_5	A_6
A_1	$\langle 0.7400, 0.4161, 0.2000, -0.6336, -0.6085, -0.3419 \rangle$	$\langle 0.4344, 0.3302, 0.7309, -0.3918, -0.5620, -0.2491 \rangle$	$\langle 0.8156, 0.4821, 0.5825, -0.1599, -0.5620, -0.5348 \rangle$
A_2	$\langle 0.3371, 0.3635, 0.5531, -0.3461, -0.4056, -0.4179 \rangle$	$\langle 0.3047, 0.3420, 0.3831, -0.2555, -0.5358, -0.6461 \rangle$	$\langle 0.2492, 0.6215, 0.5063, -0.3614, -0.5839, -0.6057 \rangle$
A_3	$\langle 0.6708, 0.3826, 0.5531, -0.2913, -0.3742, -0.6386 \rangle$	$\langle 0.6054, 0.6215, 0.2631, -0.4176, -0.5518, -0.3736 \rangle$	$\langle 0.2428, 0.5769, 0.7662, -0.4902, -0.4482, -0.5477 \rangle$
A_4	$\langle 0.5000, 0.5000, 0.5000, -0.5000, -0.5000, -0.5000 \rangle$	$\langle 0.4469, 0.4000, 0.1822, -0.3293, -0.3683, -0.4000 \rangle$	$\langle 0.5440, 0.4821, 0.5380, -0.2281, -0.5421, -0.4847 \rangle$
A_5	$\langle 0.2041, 0.4579, 0.4155, -0.4000, -0.6365, -0.3901 \rangle$	$\langle 0.5000, 0.5000, 0.5000, -0.5000, -0.5000, -0.5000 \rangle$	$\langle 0.3341, 0.4380, 0.3918, -0.4561, -0.6524, -0.2013 \rangle$
A_6	$\langle 0.6036, 0.4763, 0.4420, -0.4520, -0.6000, -0.3359 \rangle$	$\langle 0.5345, 0.5014, 0.3292, -0.1786, -0.5237, -0.4737 \rangle$	$\langle 0.5000, 0.5000, 0.5000, -0.5000, -0.5000, -0.5000 \rangle$

The Laplacian energy of each BSVNDG $D_i(i = 1,2,3)$ is calculated as:

$$LE(D_1) = \langle 14.2000, 15.0000, 14.2000, 12.1000, 15.0000, 12.1000\rangle,$$
$$LE(D_2) = \langle 15.5000, 15.0000, 15.5000, 13.2000, 15.0000, 13.2000\rangle,$$
$$LE(D_3) = \langle 14.2000, 15.0000, 14.2000, 12.1000, 15.0000, 12.1000\rangle.$$

Then, the weight of each expert can be determined as:

$$w_i = ((w_{T^+})_i, (w_{I^+})_i, (w_{F^+})_i, (w_{T^-})_i, (w_{I^-})_i, (w_{F^-})_i)$$
$$= \left(\frac{LE\big((D_{T^+})_i\big)}{\sum\limits_{j=1}^{k} LE\big((D_{T^+})_j\big)}, \frac{LE\big((D_{I^+})_i\big)}{\sum\limits_{j=1}^{k} LE\big((D_{I^+})_j\big)}, \frac{LE\big((D_{F^+})_i\big)}{\sum\limits_{j=1}^{k} LE\big((D_{F^+})_j\big)}, \frac{LE\big((D_{T^-})_i\big)}{\sum\limits_{j=1}^{k} LE\big((D_{T^-})_j\big)}, \frac{LE\big((D_{I^-})_i\big)}{\sum\limits_{j=1}^{k} LE\big((D_{I^-})_j\big)}, \frac{LE\big((D_{F^-})_i\big)}{\sum\limits_{j=1}^{k} LE\big((D_{F^-})_j\big)} \right)$$

where $i = 1,2,\ldots,k$,

$$w_1 = \langle 0.3235, 0.3333, 0.3235, 0.3235, 0.3333, 0.3235\rangle,$$
$$w_2 = \langle 0.3531, 0.3333, 0.3531, 0.3529, 0.3333, 0.3529\rangle,$$
$$w_3 = \langle 0.3235, 0.3333, 0.3235, 0.3235, 0.3333, 0.3235\rangle.$$

After that, the out-degree, $out - d(A_p)(p = 1,2,\ldots,6)$ of all criteria in a partial directed network are calculated as follows:

$$out - d(A_1) = \langle 2.7684, 1.8053, 1.8316, -1.6646, -1.9985, -1.5884\rangle$$
$$out - d(A_2) = \langle 1.4341, 0.8981, 0.8130, -0.6912, -1.2758, -1.0169\rangle$$
$$out - d(A_3) = \langle 1.8241, 1.4350, 1.3695, -0.9976, -1.3742, -1.5327\rangle$$
$$out - d(A_4) = \langle 1.7130, 1.4883, 1.4798, -1.2334, -1.7948, -1.2368\rangle$$
$$out - d(A_5) = \langle 0.7369, 0.6073, 0.4299, -0.2042, -0.5421, -0.4054\rangle$$
$$out - d(A_6) = \langle 3.1200, 2.1594, 1.9681, -2.1965, -2.7229, -1.8488\rangle$$

According to the positive membership degree of out-degree, $out - d(A_p)(p = 1,2,\ldots,6)$, the ranking of the alternatives $A_p(p = 1,2,\ldots,6)$ is given by

$$A_6 \succ A_1 \succ A_3 \succ A_4 \succ A_2 \succ A_5.$$

Therefore, the best alternative is the wind power plant, A_6.

8. Comparative Study

In this section, the proposed energy BSVNG method is compared with BNSs developed by Deli et al. [30]. First, we construct the pair-wise comparison matrix provided by the decision-maker in Tables 8–10. Then, we compute weighted average operators and calculate the score function for each alternative. Lastly, we rank all the alternatives according to the score function. Table 16 shows the score function and rank for each alternative.

Table 16. The score function and rank for each alternative adopted from Deli et al. [30].

Alternatives	Weight/Score Function	Rank
Biomass energy plant (A_1)	0.5778	2
Geothermal energy plant (A_2)	0.5056	5
Hydro power plant (A_3)	0.5667	3
Solar power plant (A_4)	0.5556	4
Wave power plant (A_5)	0.4910	6
Wind power plant (A_6)	0.5860	1

Therefore, the ranking of alternatives is given as follows: $A_6 \succ A_1 \succ A_3 \succ A_4 \succ A_2 \succ A_5$. As we can see, after the ranking of alternatives according to score function in descending order, A_6 is still the best alternative as in our proposed energy BSVNG method. Hence, this comparative method shows the availability and effectiveness of our proposed method.

9. Conclusions

The energy of graphs has many mathematical properties that have been investigated. Certain bounds (upper and lower) on energy had been studied by many researchers. This paper proposed integrating bipolar single-valued neutrosophic set and the energy graph, Laplacian energy graph and signless Laplacian energy graph. Specifically, this paper developed the new concept of energy in BSVNG. It investigated its properties such as the characteristics of eigenvalues, lower and upper bound of energy graph, Laplacian energy graph and signless Laplacian energy graph. Moreover, the relation between them is also discussed, and the proposed method was applied to renewable energy sources selection in which the optimal solution is suggested. In this application, we suggest that a wind power plant (A_6) is the optimal alternative. This paper has proceeded with a comparative analysis whereing the rank of alternatives, using the proposed method, is similar to the method of comparison. Hence, it implies that the proposed method is valid and effective. In short, this study implies several significant contributions and modifications to the energy graph, Laplacian energy graph, and signless Laplacian energy graph. In future work, it is suggested to extend the graph energy to: (1) interval-bipolar neutrosophic graphs, (2) neutrosophic vague, (3) dominating energy in neutrosophic graph, etc. Furthermore, this paper does not evaluate the sensitivity of the experts' weights to the evaluation outcomes. As a result, any new sensitivity analysis, particularly on the experts' weights, could be pursued as a research topic in the future.

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An Explanation of Sedna Orbit from Condensed Matter or Superconductor Model of the Solar System: A New Perspective of TNOs

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Abstract. In a recent paper, we argued in favor of the Gross-Pitaevskii model as a complete depiction of both the close planetary system and winding worlds, particularly considering the idea of chirality and vortices in universes. In this paper, we apply the new model based on Bogoliubov-de Gennes equation correspondence with Bohr-Sommerfeld quantization rules. Then we put forth an argument that from Bohr-Sommerfeld quantization rules, we can come up with a model of quantized orbits of planets in our solar system, be it for inner planets and also for Jovian planets. In effect, we also tried to explain Sedna's orbit in the same scheme.

INTRODUCTION

A few abbreviations used in this paper: TNO: trans-Neptunian object; KBO: Kuiper-Belt Object. Every once in a while, cosmology and astronomy revelations have opened our eyes that the universe is substantially more entangled than what it appeared in 100-200 years prior. What's more, regardless of all invading fame of General Relativistic augmentation to Cosmology, considering antiquated Greek rationalists' theories, for example, hydor model (Thales) and streaming liquid model (Heracleitus) it appears to be as yet qualified to ask: does it imply that the Ultimate hypothesis that we attempt to discover ought to compare to hydrodynamics or a disturbance hypothesis [1-3].

Meanwhile, in a recent article, we presented some new contentions on the hypothetical small star thought to be an ally to our Sun, known as the Nemesis, which is proposed to clarify an apparent pattern of mass eradications in Earth's history. Some guessed that such a star could influence the circle of articles in the far external close planetary system, sending them on a crash course with Earth. While ongoing cosmic reviews neglected to discover any proof that such a star exists, we layout in this article some hypothetical discoveries including our own, suggesting that such a dwarf star companion of the Sun remains a possibility [4]. And one good indicator for such a dwarf companion of the Sun is Sedna, a planetoid which has been discovered around 2004 by Mike Brown and his Caltech team. Sedna location and eccentric orbit are such that it is not supposed to be there [5-10].

Therefore a physical explanation of why Sedna is located there can be a good start to begin to search the existence and location of the supposedly dwarf companion of the Sun.

METHOD

Methodology used in this paper: we develop a new conceptual/mathematical model then compare it with the supporting evidences.

Bohr-Sommerfeld Quantization Rules and Quantized Approach

Here we present Bohr-Sommerfeld quantization rules for planetary circle separations, which brings about a decent quantitative depiction of planetary circle separation in the Solar system [11-14].

First of all, let us point out some motivations for utilization of Bohr-Sommerfeld quantization rules: (a) the neat correspondence between Bohr-Sommerfeld quantization rules and topological quantization as found in superfluidity, and (b) there is neat correspondence between Bogoliubov de Gennes and generalized Bohr-Sommerfeld quantization can be applied to large scale systems like Solar system. (c) In the next section, we suggest another alternative approach, i.e. Eilenberger equation, which reduces to scalar model of Riccati equation [15]. As we have discussed how Riccati equation can be neatly linked to Newton equation, then it seems possible to use this approach too.

Eilenberger Equation Reduces to Scalar Riccati Equation

In this section, we suggest another alternative approach, i.e. Eilenberger equation, which reduces to scalar model of Riccati equation [15]. As we have discussed how Riccati equation can be neatly linked to Newton equation, then it seems possible to utilized this approach too [15]. Another parametrization of the Eilenberger conditions of superconductivity regarding the answers for a scalar differential condition of the Riccati type is presented. It is indicated that the quasiclassical propagator might be remade, without express information on any eigenfunctions and eigenvalues, by taking care of a straightforward beginning worth issue for the linearized Bogoliubov-de Gennes conditions. The Riccati parametrization of the quasiclassical propagator leads to a stable and fast numerical method to solve the Eilenberger equations [16].

Therefore it appears that we can utilize Eilenberger equation which is an alternative to Bogoliubov-De Gennes equation for description of superconductors. According to Schopol, the Eilenberger reduces to Riccati equation:

$$h_{vF}\frac{\partial}{\partial x}b_x+[2\bar{\varepsilon}_n+\Delta(x)b_x]+\Delta^{\dagger}(x)=0, \qquad (1)$$

which after some steps it will yield a system of coupled Riccati ODEs. Interestingly it can be shown that angular momentum conservation combined with power law potentials can be recast into a Riccati ODE:

$$\frac{1}{2}m\dot{r}^2+\left(\frac{1}{2}+\frac{1}{\epsilon}\right)\frac{1}{mr^2}-\frac{m\ddot{r}r}{\epsilon}-E = 0. \qquad (2)$$

Therefore, our hypothesis is that such a Riccati ODE (2) may be linked to scalar Riccati ODE as a reduction to Eilenberger equation. Numerical solution of equation (2) can be done with Mathematica or other computer algebra softwares.

In retrospect, we can recall the fact that there is a known Pioneer anomaly, which can be interpreted as an anomalous (scalar) acceleration after the Pioneer spacecraft enters the Jupiter's orbit and on. Therefore it can be interpreted as a possible indicator of the existence of scalar effect of Riccati ODE.

RESULT AND DISCUSSION

The quantization of circulation for nonrelativistic superfluid is given by [5]:

$$\oint v dr = N\frac{\hbar}{m_s} \qquad (3)$$

where $N, \hbar, m_s$ represent winding number, diminished Planck steady, and superfluid molecule's mass, individually. Also, the all out number of vortices is given by [5]:

$$N=\frac{\omega\, 2\pi\, r^2 m}{\hbar} \qquad (4)$$

Also, in light of the above condition (4), Sivaram & Arun [17] can give a gauge of the quantity of cosmic systems known to man, alongside a gauge of the number stars in a universe. In any case, they don't give clarification between the quantization of dissemination (5) and the quantization of rakish energy. As indicated by Fischer [8], the quantization of precise force is a relativistic augmentation of quantization of dissemination, and along these lines it yields Bohr-Sommerfeld quantization rules.

Besides, it was recommended that Bohr-Sommerfeld quantization rules can yield a clarification of planetary circle separations of the Solar framework and exoplanets [1,19]. Here, we start with Bohr-Sommerfeld's guess of quantization of rakish energy. As we probably am aware, for the wavefunction to be all around characterized and remarkable, the momenta must fulfill Bohr-Sommerfeld's quantization condition:

$$\oint_{\Gamma} p\, dx = 2\pi\, n\hbar, \quad (5)$$

for any closed classical orbit Γ. For the free particle of unit mass on the unit sphere the left-hand side is:

$$\int_0^T v^2\, d\tau = \omega^2 T = 2\pi\omega, \quad (6)$$

where $T = \frac{2\pi}{\omega}$ is the period of the orbit. Hence the quantization rule amounts to quantization of the rotation frequency (the angular momentum) $\omega = n\hbar$. Then we can write the force balance relation of Newton's equation of motion:

$$\frac{GMm}{r^2} = \frac{mv^2}{r}. \quad (7)$$

Using Bohr-Sommerfeld's hypothesis of quantization of angular momentum (6), a new constant g was introduced:

$$mvr = \frac{ng}{2\pi}. \quad (8)$$

Just like in the elementary Bohr theory (just before Schrodinger), this pair of equations yields known simple solution for the orbit radius for any quantum number of the form:

$$r = \frac{n^2 g^2}{4\pi^2 GMm^2}, \quad (9a)$$

or

$$r = \frac{n^2 GM}{v_0^2}, \quad (9b)$$

where r, n, G, M, v_0 represent orbit radii (semimajor axes), quantum number ($n = 1,2,3, \ldots$), Newton gravitation constant, mass of the nucleus of orbit, and specific velocity, respectively. In equation (10), we denote:

$$v_0 = \frac{2\pi}{g} GMm. \quad (10)$$

The value of m, g in equation (10) are adjustable parameters.

Strikingly, we can comment here that condition (9b) is actually the equivalent with what is gotten by Nottale utilizing his Schrodinger-Newton formula [17]. In this manner here we can check that the outcome is the equivalent, it is possible that one uses Bohr-Sommerfeld quantization rules of Schrodinger-Newton condition. The relevance of condition (9b) incorporates that one can anticipate new exoplanets (i.e., extrasolar planets) with noteworthy outcome.

Thusly, one can locate a flawless correspondence between Bohr-Sommerfeld quantization rules and movement of quantized vortices in consolidated issue frameworks, particularly in superfluid helium [1,20]. Here we propose a conjecture that superfluid vortices quantization rules also provide a good description for the planetary orbits in our Sollar System. An idea that given the chemistry composition of Jovian planets are different from inner olanets began around 15 years ago, therefore it is likely both series of planets have different origin. By assuming inner planets orbits have different quantum number from Jovian planets, here by using "least square difference" method in order to seek the most optimal straight line for Jovian planets orbits in a different quantum number. Then it came out that such a straight line can only be modelled if we assume that the Jovian planets were originated from a twin star system: the Sun and its companion, using the notion of $\mu = \frac{m_1 m_2}{m_c}$ is the reduced mass.

Although based on statistical optimization [20,21], it yields new prediction of 3 planetoids in the outer orbits beyond Pluto, from which prediction, Sedna. A table as shown below shows how our simple model based on large-scale quantization inspired by Bohr-Sommerfeld rule obtains a remarkably good prediction compared to observation:

TABLE 1. Comparison of prediction and observed orbit distance of planets in Solar system (in 0.1 AU unit) [22]

Object	No.	Titius	Nottale	CSV	Observ.	Δ, %
	1.		0.4	0.43		
	2.		1.7	1.71		
Mercury	3.	4	3.9	3.85	3.87	0.52
Venus	4.	7	6.8	6.84	7.32	6.50
Earth	5.	10	10.7	10.70	10.00	-6.95
Mars	6.	16	15.4	15.4	15.24	-1.05
Hungarias	7.		21.0	20.96	20.99	0.14
Asteroid	8.		27.4	27.38	27.0	1.40
Camilla	9.		34.7	34.6	31.5	-10.00
Jupiter	2.	52		45.52	52.03	12.51
Saturn	3.	100		102.4	95.39	-7.38
Uranus	4.	196		182.1	191.9	5.11
Neptune	5.			284.5	301	5.48
Pluto	6.	388		409.7	395	-3.72
2003EL61	7.			557.7	520	-7.24
Sedna	8.	722		728.4	760	4.16
2003UB31	9.			921.8	970	4.96
Unobserv.	10.			1138.1		
Unobserv.	11.			1377.1		

Source: *Apeiron*, vol. 23, July 2004 [23]

Further Evidences: Superfluidity of Solar Interior and Pairing of TNO Objects

In the aforementioned sections, we put forth an argument in favor of low temperature physics model of solar system, in particular using Bogoliubov-de Gennes equations which are normally utilized for superconductors. While this makes the model a bit simpler and comprehensible, one may ask: what are other evidences available to justify the BdG model for the Solar system. In this regards, allow us to submit three supporting evidences which seem to correspond to the conceptual model as we outlined above:

- Pairing of Pluto-Charon and other TNOs/KBOs seem to be attributed to the BCS/BdG pairing condition → pointing to low temperature physics model of Solar System.
- Solar interior has superfluid inner structure [24].
- Some literatures argue that G1.9 is remnant of supernovae, others argue that G1.9 cannot be supernovae, instead it is more plausible to argue that G1.9 is brown dwarf star.

First, the BdG model can be related to pairing of electrons, and as it has been discussed for instance in [25], when it is stated, which can be paraphrased as follows:

> "It is indicated that the Bogoliubov-de Gennes conditions pair the electrons in states which are direct blends of the typical states. For a homogeneous framework, we bring up that the part of the self-consistency condition got from the Bogoliubov-de Gennes conditions should be obliged by the BCS matching condition."

In this regard, we can point out that Pluto and Charon seem like evidences related to this pairing condition. Furthermore, Sedna also has a pair planetoid. We can expect that planetoids found around Kuiper Belt (or may be dubbed as TNOs) can take place in pairs.

Second, we can point out the Solar interior which has superfluid inner structure as another evidence [24-27].

Other hint for physical evidence of superconductor/superfluidity nature of solar system may be found in icy dwarf nature of some planetoids and other TNOs objects and other objects beyond Kuiper Belt.

As with potential location to find the dwarf companion of the Sun, we can mention briefly here that since 2017, there is an object dubbed as G1.9 which was observed around 60-66 AU (around Pluto/Kuiper Belt). We can also note here: while some literatures argue that G1.9 is remnant of supernovae [22,25,28], others argue that G1.9 cannot be a supernovae, instead it is more plausible to argue that G1.9 is brown dwarf star. Therefore it can be a good start to find out whether the G1.9 is indeed the dwarf companion that we're looking for all along. See Fig. 1 below.

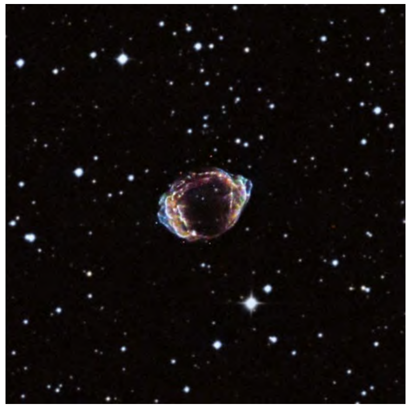

FIGURE 1. Gliese G1.9, a candidate of brown dwarf companion of the Sun [28-28a]

CONCLUSION

In this paper, we present an argument that Bohr-Sommerfeld quantization condition can be linked to Bogoliubov-de Gennes equations, and thus it tends to be indicated that such a Bohr-Sommerfeld quantization can be connected to huge scope structure quantization, for example, our nearby planetary group in Solar system.

As with potential location to find the dwarf companion of the Sun, we can mention briefly here that since 2017, there is an object dubbed as G1.9 which was observed around 60-66 AU (around Pluto/Kuiper Belt). Therefore it can be a good start to find out whether the G1.9 is indeed the dwarf companion that we're looking for all along.

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On Post-Empiricism doctrine and Neutrosophic way of doing science: From Principle of Parsimony to three examples in physics

Victor Christianto, Florentin Smarandache

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Abstract

Despite majority of theoretical physicists begin to accept the post-empiricism doctrine, still few physicists and mathematicians alike don't agree with such a doctrine, partly because it is against Popper's criterion of falsifiability for any theory in physics and other sciences. And partly because criteria like beauty or elegance seem rather subjective for a theory to be accepted as "physics'. Physicists like Peter Woit and Sabine Hossenfelder have wrote books on this topics. In this article, we don't repeat those arguments, we only argue in favor of principle of parsimony, or that Nature seems to prefer least action, or least energy either in modeling complexity, assumptions and free parameters involved, and minimizing calculational or computational entropy needed. Therefore, we arrive at conclusion that one shall find a balance among some criterion, of which we may call this point "*Ockham optimality*."

Keywords: Ockham razor, principle of parsimony, Neutrosophic Logic, Popperian epistemology, Ockham optimality.

Introduction

The present status of theoretical physics seems to face a dark cloud in the sky, because the highly acclaimed theories such as loop quantum gravity, superstring or M-theories cannot be verified by experiments, at least not within the present limit of measurement devices. Therefore, some theoreticians like Dawid began to argue in favor of releasing the verifiability criterion for any theory to be accepted as working physics theories. That kind of post-empiricism doctrine, as it is called, is supposed to supersede the conventional Popperian epistemology, which include falsifiability for any theory before it can be accepted.

However, some prominent cosmologists and theoreticians disagree with that doctrine. Attempts to exempt speculative theories of the Universe from experimental verification *undermine science*, argue George Ellis and Joe Silk. [1] They also wrote:

> "Faced with difficulties in applying fundamental theories to the observed Universe, some researchers called for a change in how theoretical physics is done. They began to argue — explicitly — that if a theory is sufficiently elegant and explanatory, it need not be tested experimentally, breaking with centuries of philosophical tradition of defining scientific knowledge as empirical. We disagree."

But, despite some physicists have emphasized on the virtue of empirical test and conceptual simplicity, such criteria appear not so clear to be applied on daily basis.

Therefore, there is a need to apply such criteria on simplicity or principle of parsimony in a more operational way.

Intuition and Neutrosophic way of doing science

In a recent article, we argue on the role of intuition in doing science, apart of the so-called Dirac's dictum that to *find new physics, we shall find new mathematics.* In our proposed "Neutrosophic way", it is intuition (or in German, *einfuhlung*) that should be given more emphasis.

Any effort to depict or map life or reality as an abstract substance needs to use real life or concrete experience to arrive at such an understanding. To choose the concrete experience and to connect it with the abstract domain, one needs intuition.

As this work emphasizes [3]:

> "More "right brain" activity, based on direct experiences, leads to direct experiences of the Divine. Your "inner vision" (the "mind's eye") can help readers in this, and in many other ways. The inner vision is also the seat of many of the intuitive faculties, which are experiencable facts, not imaginings. That means the information obtained by the intuitive faculty is verifiable and reproducibly observable.
>
> In order to do that, the Balanced Brain is the most efficacious way to function, as well as the most efficient, and the most comfortable. To obtain the Balanced Brain, the person usually needs to spend a great deal of their spare time being receptive, being the "receiver", being accepting and exploring, and not using the analytical intellect,

but instead, spending time in the Now and in the Senses and Sensitivities. This is best enjoyed in Natural settings."[3]

Therefore, to reply to the question concerning rectifying the problem of overemphasizing rationality in mathematics and beyond, McGilchrist's concept and Conceptual Linguistics theory can shed light.[2][4] From Neutrosophic Logic viewpoint, this article recommends that a combination

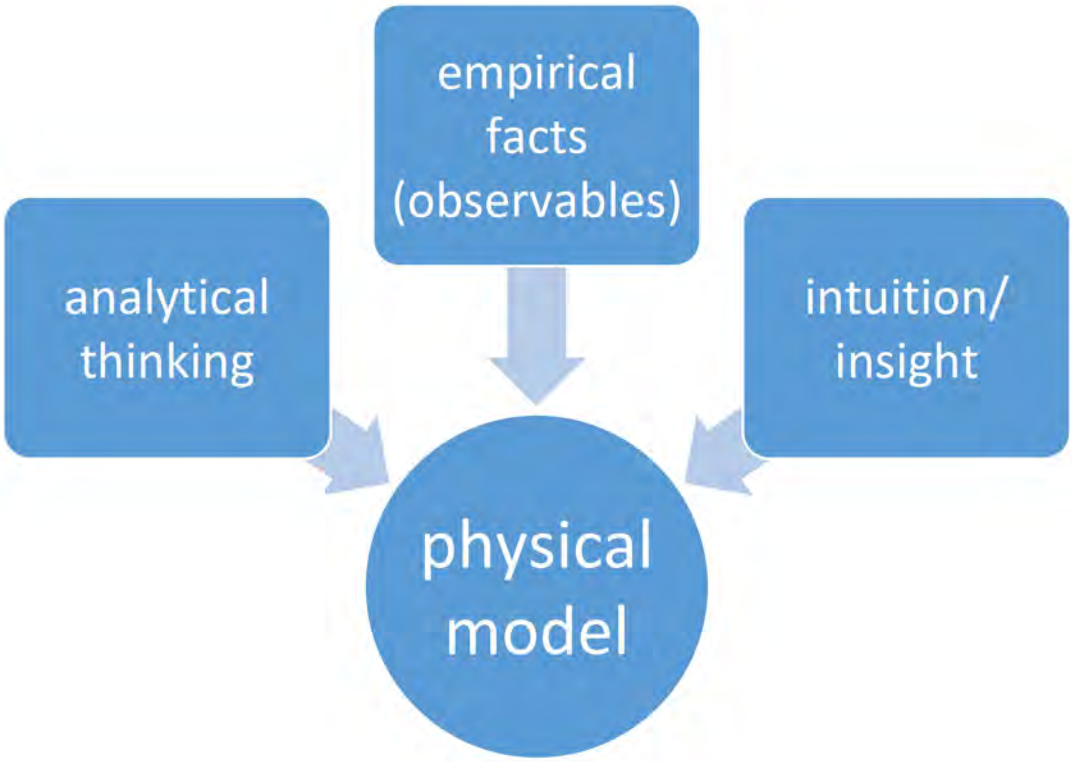

Diagram 1. The role of intuition, analytical thinking, and empirical facts

of both the intuitive aspect of the right hemisphere and the analytic or logical thinking processes of the human's left brain will be more adequate in creating a holistic approach. The article proposes a term: intuilytics to capture the essence of the **Balanced Brain** [3].

Therefore, in addition to the role of intuition and analytics/rational thinking, we need empirical facts as the basis of model building. To emphasize those triplet, let us put into a diagram:

Some criteria with respect to Principle of Parsimony and Ockham optimality

As we argued in a recent paper [5-6], this deep problem in philosophy of science can be viewed as another case that calls for implementation of Neutrosophic Logic: whenever there are two opposite sides, there is always a choice to find a neutral side, in order to reconcile those two opposite sides. We can also think of them starting from the *principle of contradiction*, proposed by Kolmogorov.[9] To summarize, he argues that there is fundamental problem in developing complex arguments, they always lead to contradiction. This was proven later by Gödel.

What can we conclude from Kolmogorov's principle of contradiction?

It is quite simple, i.e. developing a complicated theory from a number of postulates will very likely lead to messy contradictions, which are often called "paradoxes, " just like the twin paradox in general relativity, or cat paradox in quantum wave function.

To put this problem succinctly, we can paraphrase Arthur C. Clarke’s famous saying: “Any sufficiently advanced technology is indistinguishable from magic,” (Arthur C. Clarke, "Profiles of The Future", 1961[1]) to become "*Any sufficiently complicated theory will result in a number of contradictions and paradoxes.*"

Such a logical analysis derived from Kolmogorov’s principle of contradiction eventually remind us of the following:

(a) To keep humble mind before Nature (God's creation), and perhaps we should not rely too much on our logic system and mathematical prowess;

(b) In developing a theory one should keep complications and abstractions to a minimum;

(c) To build theory in the nearest correspondence to the facts; it is the best if each parameter can be mapped to a measurable quantity.

We hope the above three criteria can be a useful set of practical guidelines for building mathematical models in theoretical physics.

To emphasize the aforementioned argument, from Neutrosophic Logic perspective, the old tensions between mathematicians (opposite 1) and experimenters (opposite 2), can be reconciled if we can consider a third approach. Those the available approaches would be somewhere in the following spectrum:

Mathematics (opposite 1) – evidence-based mathematics -- experiments (opposite 2)

[1] Clarke's third law. url: http://www.quotationspage.com/quote/776.html

Therefore, the middle way that we submit as a plausible resolution to the present stagnation of modern physics, is to return to evidence-based mathematics.

At this point, some readers may ask: "But how can we apply such principle of parsimony into practice?"

To put the above three criteria into more practical guidelines, allow us to distinguish such a Principle of Parsimony (Ockham razor) into several possible approach:

1. To minimize assumptions involved (conceptual simplicity)
2. To minimize number of parameters (model simplicity)
3. To minimize calculation procedures (calculational simplicity)
4. To minimize computational/algorithm entropy (computational simplicity)
5. To maximize coverage of empirical facts to be explained (evidence based physics)

To make these criteria a bit more comprehensible, we can draw a diagram as follows:

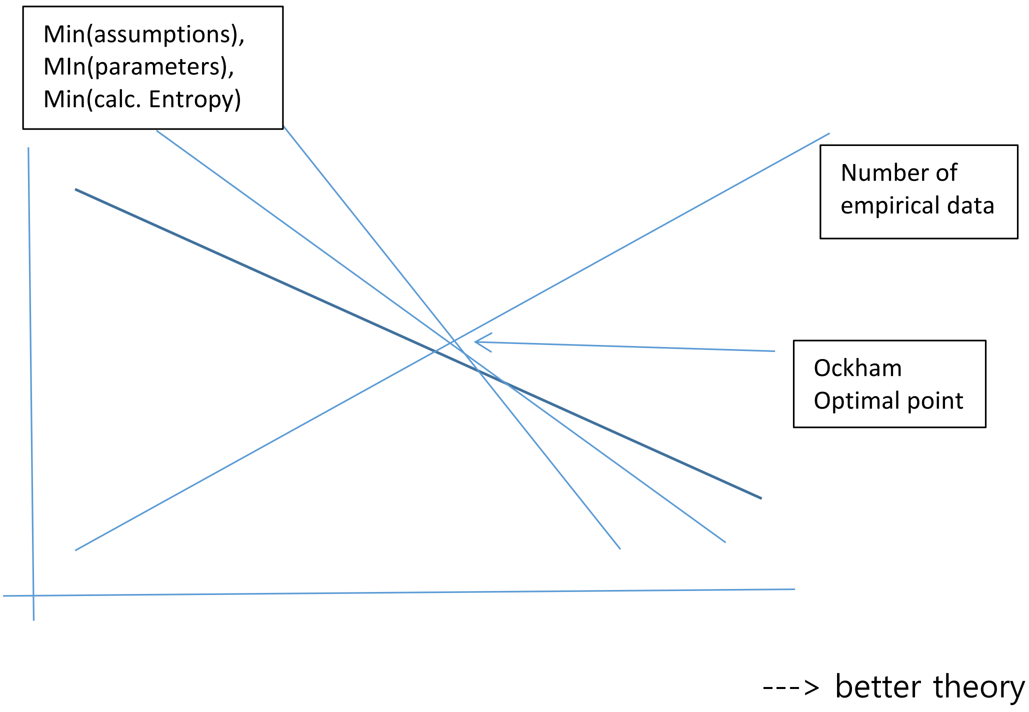

Diagram 2. To find Ockham optimality

Three examples

We have presented a more operational definition of Principle of Parsimony, allow us to give a few examples as illustrations, that sometimes: even the standard spacetime notion may be excluded to arrive at a good explanation of a set of observed phenomena.

A. Example 1 [14]

There are various models of electron which have been suggested, for instance see Chekh et al.

But we seek a more realistic electron model which is able to describe to experiments conducted by Winston Bostick et al. [17]. In our attempt to explain such experiments of electron creation in plasma, allow us to come up with a new model of electron, based on Helmholtz's electron vortex theory. In turn, we will discuss a plausible model of electron capture event inside Earth (matter creation), which can serve a basis to explain Le Sage/Laplace's push gravity. We will discuss its implications along with receding Moon effect in a forthcoming paper.

The Helmholtz vortex model of the electron as illustrated in the photo of a Helmholtz vortex (Fig. 1), is a toroid made of nested concentric toroidal flows of smaller particles, perhaps the inertons of Krasnoholovets, or aggregate particles made from Bhutatmas. (The "Bhutatma" infinitesimal particle of Vedic lore is the ultimate building block of everything, being the smallest unit of matter, and at the same time, the smallest unit of Consciousness.

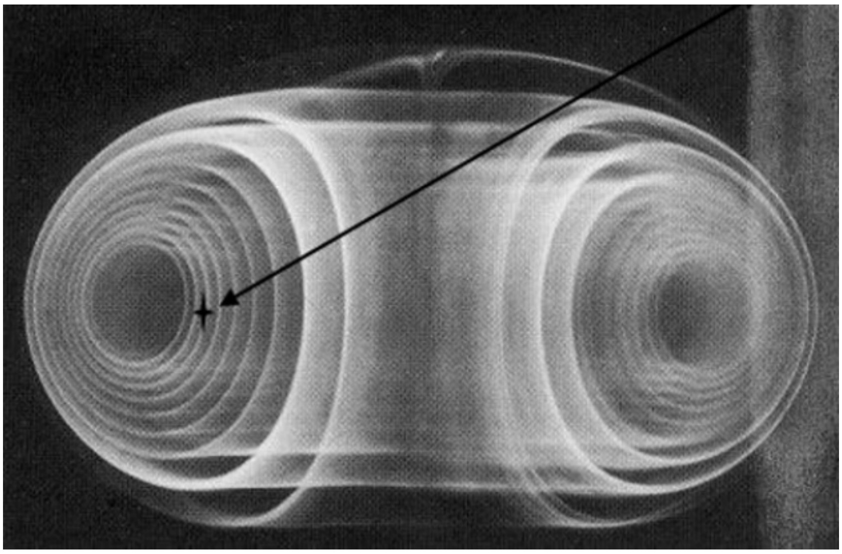

Figure 1. Helmholtz vortex model of electron (as verified by Bostick et al.)

B. Example 2 [15]

The golden ratio effectively enables multiple oscillators within a complex system to co-exist without blowing up the system. But it also leaves the oscillators within the system free to interact globally (by resonance), as observed in the coherence potentials that turn up frequently when the brain is processing information.

Obviously, this can be tied in to the creation of subatomic particles such as electrons and positrons. At a certain scale of smallness, the media in the local volume becomes isotropic, while larger volumes exhibit occupation by ever-larger turbulence formations and exhibit extremes of anisotropy in the media.

The Kolmogorov Limit is 10e -58 m, which is the smallest vortex that can exist in the aether media. Entities smaller than this, down to the SubQuantum infinitesimals (Bhutatmas) (vortex lines) are the primary cause of gravitation.

Shadow gravity is valid in the situation of gravitational interaction between two discrete masses that divert the ambient gravitational flux-density away from each other. This happens due to absorption (rare), scattering (more common), and refraction (most of the time) of gravitational infinitesimals. Gravitational flux density is a variable depending on stellar, interstellar, and intergalactic events.

A simplified model of vorticity fields in large scale structures of the Universe is depicted below:

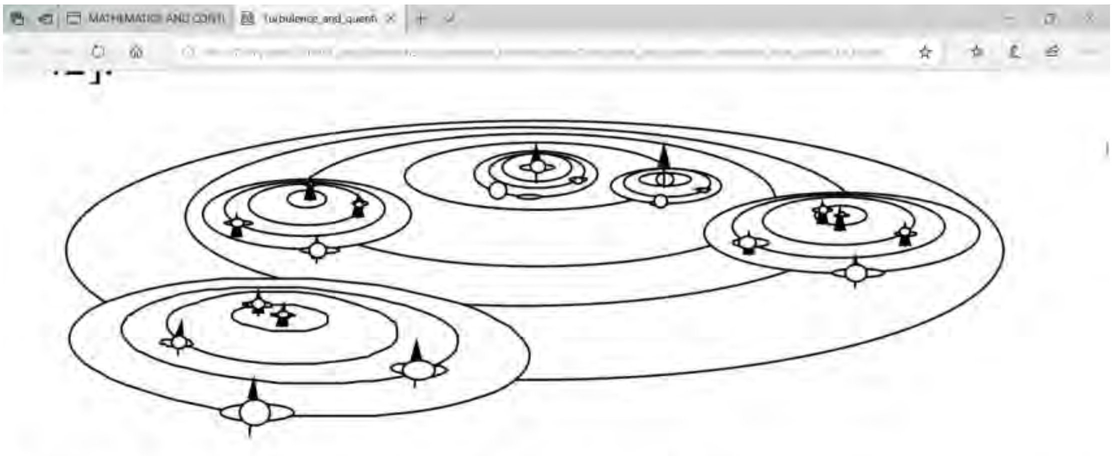

Fig.2 Description of internal (iso-spin) versus external vorticity fields in cosmology [41].

Figure 2. Vorticity fields in cosmology (after Siavash Sohrab [18])

The above diagram seems to be able to capture the turbulence phenomena from Planckian scale to cosmos.

What is more interesting here, is that it can be shown that there is correspondence between Golden section and in coupled oscillators and KAM Theorem, but also between Golden section and Burgers equation.

<u>How to write down Navier-Stokes equations on Cantor Sets</u>

Now we can extend further the Navier-Stokes equations to Cantor Sets, by keeping in mind their possible applications in cosmology.

By defining some operators as follows:

1. In Cantor coordinates :

$$\nabla^\alpha \cdot u = div^\alpha u = \frac{\partial^\alpha u_1}{\partial x_1^\alpha} + \frac{\partial^\alpha u_2}{\partial x_2^\alpha} + \frac{\partial^\alpha u_3}{\partial x_3^\alpha}, \tag{1}$$

$$\nabla^\alpha \times u = curl^\alpha u = \left(\frac{\partial^\alpha u_3}{\partial x_2^\alpha} - \frac{\partial^\alpha u_2}{\partial x_3^\alpha} \right) e_1^\alpha + \left(\frac{\partial^\alpha u_1}{\partial x_3^\alpha} - \frac{\partial^\alpha u_3}{\partial x_1^\alpha} \right) e_2^\alpha + \left(\frac{\partial^\alpha u_2}{\partial x_1^\alpha} - \frac{\partial^\alpha u_1}{\partial x_2^\alpha} \right) e_3^\alpha \quad .(2)$$

2. In Cantor-type cylindrical coordinates:

$$\nabla^\alpha \cdot r = \frac{\partial^\alpha r_R}{\partial R^\alpha} + \frac{1}{R^\alpha}\frac{\partial^\alpha r_\theta}{\partial \theta^\alpha} + \frac{r_R}{R^\alpha} + \frac{\partial^\alpha r_z}{\partial z^\alpha}, \tag{3}$$

$$\nabla^\alpha \times r = \left(\frac{1}{R^\alpha}\frac{\partial^\alpha r_\theta}{\partial \theta^\alpha} - \frac{\partial^\alpha r_\theta}{\partial z^\alpha} \right) e_R^\alpha + \left(\frac{\partial^\alpha r_R}{\partial z^\alpha} - \frac{\partial^\alpha r_z}{\partial R^\alpha} \right) e_\theta^\alpha + \left(\frac{\partial^\alpha r_\theta}{\partial R^\alpha} + \frac{r_R}{R^\alpha} - \frac{1}{R^\alpha}\frac{\partial^\alpha r_R}{\partial \theta^\alpha} \right) e_z^\alpha$$

Then Yang, Baleanu and Machado are able to obtain a general form of the Navier-Stokes equations on Cantor Sets as follows :

$$\rho \frac{D^\alpha \upsilon}{Dt^\alpha} = -\nabla^\alpha \cdot (pI) + \nabla^\alpha \left[2\mu\left(\nabla^\alpha \cdot \upsilon + \upsilon \cdot \nabla^\alpha\right) - \frac{2}{3}\mu\left(\nabla^\alpha \cdot \upsilon\right) I \right] + \rho b \tag{5}$$

The next task is how to find observational cosmology and astrophysical implications.

C. Example 3 [16]

The ideas of detecting CνB have been discussed since the 1960s. However the direct observations of the relic neutrinos is a great challenge to present experimental techniques due to the very low energy (~10−4 eV) of relic neutrinos at the present epoch.

It is therefore natural to ask: what are the prospects of a more direct, weak interaction based relic neutrino detection, sensitive in particular to the CνB in the present epoch. It is known, that all the existing measurements probe only the presence of the relic neutrinos at early stages in the cosmological evolution, and this often in a rather indirect way.

It is obvious that either WIMP or hot model of dark matter has not been observed yet. One of the most promising laboratory search, based on neutrino capture on beta decaying nuclei, may be done in future experiments designed to measure the neutrino mass through decay kinematics.

Another method is still underway, i.e. using PTolemy. According to Cocco:

"The PTolemy project aim at the direct detection of the Cosmological Relic Neutrino background by the use of a Tritium target. Cosmological Relic Neutrino produced in the early stage of the Big Bang are predicted to have thermally decoupled from other forms of matter at approximately 1 second after the Big Bang; they represent the oldest detectable Big Bang relics and as such they carry an invaluable content of information about the genesis and evolution of our Universe. ...In particular Tritium is among the nuclei having the most favorable detection conditions." [19]

Despite all of those progress in developing measures to directly detect relic neutrino background, there is one possibility why such a direct detection remains elusive: because there was no such thing as cosmic singularity. In other words, while we accept such an initial point of creation of the Universe, its beginning came through from a non-singular origin.

In two recent papers, we have outlined how a non-singular origin of the Universe is possible, if we consider a turbulence model of Early Universe, because the model includes nonlinear Ermakov equation instead of Friedman equation as usual.

Taking into considerations two other findings in recent years: (a) Earth Microwave Background by P-M. Robitaille, and (b) theories which suggest that cosmic singularity can be removed; then we submit the following hypothesis: Direct detection of Cosmic Neutrino Background is impossible because there is no such thing as Cosmic Singularity.

Concluding remarks

Despite majority of theoretical physicists begin to accept the *post-empiricism doctrine,* still few physicists and mathematicians alike don't agree with such a doctrine, partly because it is against Popper's criterion of *falsifiability* for any theory in physics and other sciences as well. And partly because criteria like beauty or elegance seem rather subjective for a theory to be accepted as "physics'.

In this article we have discussed several more operational criteria to apply the Principle of Parsimony into day to day model building processes. We also discuss *Ockham optimality* and also a number of examples.

Acknowledgment

Both authors wish to extend gratitude to Robert Neil Boyd, PhD, for discussions and insights, especially with regards to examples (1-3) provided here.

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Neutrosophic Battery: An Introduction to High-Performance Battery Configuration

Victor Christianto, Florentin Smarandache

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ABSTRACT Nowadays, great effort has been focused on various kinds of batteries commonly referred to as electric energy storage systems (EESS), such as lithium-related batteries, sodium-related batteries, zinc-related batteries, aluminum-related batteries and so on. Some cathodes can be used for these batteries, such as sulfur, oxygen, layered compounds. In the present article, let us consider the basic battery configuration, i.e. they are mostly composed of cathode and anode metals. Such a classic system can be considered as ”two elements” model. Based on Neutrosophic Logic as developed by one of us (FS), we consider in this paper that actually we can extend it further to become a Neutrosophic battery system, consists of three-elements (or may be more), i.e. cathode-anode-catalyst system. The catalytic electrolyte method is found to be significant to achieve high-performance battery.

KEYWORDS Neutrosophic Battery, Electric Energy Storage Systems (EESS)

INTRODUCTION

As it is known, the majority of energy usage in the world is predominated by fossil fuels. But it has resulted in many environmental problems, including ozone layers problem. Renewable energy sources have been promoted as an alternative to such fossil fuel sources. Among many kinds of renewable energy sources, solar energy has become the most favorite in many countries, along with water and wind. They are called WWS (wind, water, solar) energy. [4] Although it is still quite debatable, there is growing acceptance of potential contribution of renewable energy sources. Even there *are known projections* that renewable energy sources can be sufficient to meet future energy demands. See for instance Mark Jacobson’s book from Stanford University [5]. Among those plentitude alternatives of renewable energy types, it is known that “Sun, wind and tides have huge potential in providing us electricity in an environmental-friendly way." However, as Jianmin Ma et al., wrote: Its intermittency and non-dispatchability are major reasons preventing full-scale adoption of renewable energy generation. Energy storage systems will enable this adoption by enabling a constant and high-quality electricity supply from these systems, see [7]. But which storage technology should be considered? That is one of significant questions to be asked now-a-days. Nowadays, great effort has been focused on various kinds of batteries to store energy, lithium-related batteries, sodium- related batteries, zinc-related batteries, aluminum-related batteries and so on. Some cathodes can be used for these batteries, such as sulfur, oxygen, layered compounds. Now, let us consider the basic battery configuration, as composed of cathode and anode metals. Such a classic system can be considered as "two elements" model. Based on Neutrosophic Logic as developed by one of us (FS), we consider in this paper that actually we can extend it further to become a Neutrosophic battery system, consists of three-elements (or may be more), i.e. cathode-anode-catalyst system. The catalytic electrolyte method is found to be significant to achieve high-performance battery. Moreover, recently there is also research on "*interface engineering*." Therefore, the classic model of cathode- anode can still be improved further. This article is an introduction to this fast growing field in (physics-) chemistry engineering.

PROSPECT OF BATTERIES FOR ENERGY STORAGE

Let us cite from Nadeem *et al.*'s paper at IEEE Access: "It is an exciting time for power systems as there are many ground-breaking changes happening simultaneously. There is a global consensus in increasing the share of re-newable energy-based generation in the overall mix, transitioning to a more environmental-friendly transportation with electric vehicles as well as liberalizing the electricity markets, much to the distaste of traditional utility companies."

The generation affects distribution networks, renewables introduce intermittency, and liberalized markets need more competitive operation with the existing assets. All of these challenges require using some sort of storage device to develop viable power system operation solutions. There are different types of storage systems with different costs, operation characteristics, and potential applications. Understanding these is vital for the future design of power systems whether it be for short-term transient operation or long-term generation planning."[6] For instance, if we consider batteries for Large Scale EESS, then one obvious consideration is how to find quite abundance source in nature as well as cheap material. The effective use of electricity from renewable sources requires large-scale stationary electrical energy storage systems (EESS) with rechargeable high-energy-density, cheap batteries. While batteries using lithium, cadmium, lead-acid etc. have been widely used, notably there is an alternative source e.g. salt-water which is quite abundant in nature and known as electrolytes. In a recent paper, we reported a series of preliminary experiments on potential use of salt-water as cheap source of renewable battery with various kind of metals as anode and cathode. [4] Interestingly, a report by PreScouter (2018) also mentioned salt-water battery as one of potentially disruptive battery technologies. [2] They also wrote some key insights, such as follows: "Numerous chemistries are being developed to directly counter some of the disadvantages of Li-ion batteries, namely the high cost and sourcing for the raw materials, as well as degradation of the caused by dendrite formation in the solid-electrolyte interphase. Most of these technologies are still in either the prototype or research phase, and may not appear on the commercial market until at least 5-10 years from now (with notable exceptions, e.g. silicon-based chemistries). Most of these technologies are aiming to reduce the cost of energy storage and point to new opportunities in the energy sector."[2] Nonetheless, such a choice of salt-water battery (in German: "*salzwasser batterie*"), is not without hurdles. One of such a hurdle is low voltage produced in salt-water (EES) system. That is why nowadays we begin to consider several potential catalytic materials. Such a hurdle actually also leads us to the following new scheme of "Neutrosophic battery system," as we are going to discuss in subsequent section.

INTRODUCING NEUTROSOPHIC BATTERY SYSTEM

As we wrote in Introduction section, let us consider basic battery configuration, as composed of "cathode" and "anode" metals. Such a classic system can be considered as "two elements" model. Based on Neutrosophic Logic as developed by one of us, FS [3], we consider in this paper that actually we can extend it further to become a Neutrosophic battery system, consists of three-elements (or may be more), i.e. cathode-anode-catalyst system. Definition of catalyst (/kad()lst/): "a substance that increases the rate of a chemical reaction without itself undergoing any permanent chemical change." According to Merriam-Webster dic- tionary: "a substance that enables a chemical reaction to proceed at a usually faster rate or under different conditions (as at a lower temperature) than otherwise possible." (cf. https://www.merriam-webster.com/dictionary/catalyst). The catalytic electrolyte method is found to be significant to achieve high-performance battery configurations. Moreover, recently there is also research on "interface engineering." Therefore, the classic model of cathode-anode system can still be improved further. Although the use of catalyst was more known in fuel-cell research, see for instance [8], but recently it begins to be explored for applications in battery research, see for instance [9]. We admit that the proposed term "Neutrosophic battery" is still schematic as for now, but let us see the following diagram of configurations used in high-performance battery experiments, as follows (Figure 1).

Alternatively, one may consider the fourth element as extension of catalytic method to improve performance, i.e. interface. It is sometimes called as interface engineering; an example is as shown in the following diagram (Figure 2).

It is our hope that the above diagrams make clear what is the implication of the proposed new term "Neutrosophic battery" in actual experiments. It can be expected that high-performance battery research in the future shall include these additional elements.

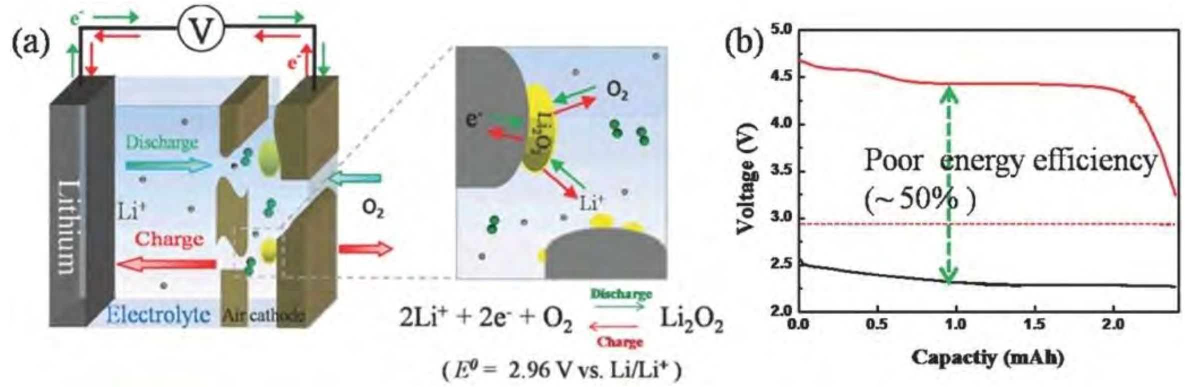

Figure 1: Sample configuration of electrocatalyst, from Chang's paper

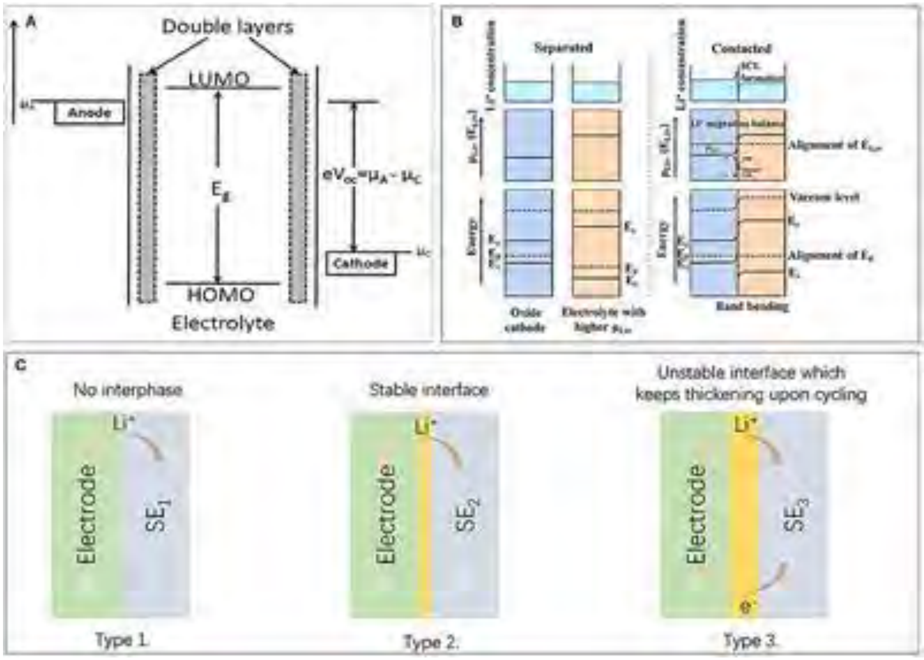

Figure 2: An example of interface in battery experiment.

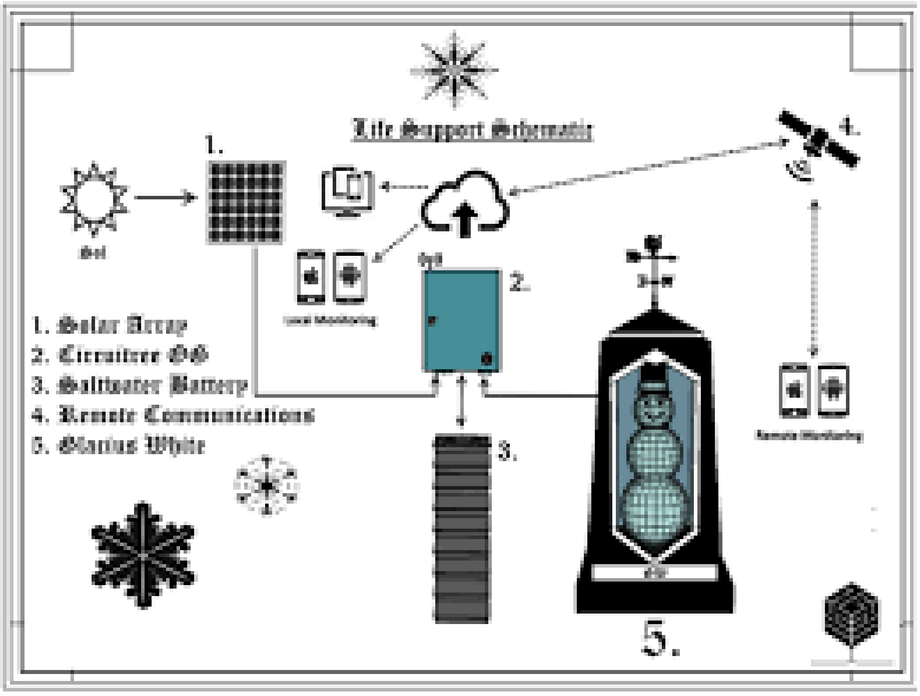

Figure 3: Example of Aquion Salt-Water battery for remote support.

COMMENTS ON INDUSTRIAL APPLICATIONS

As far as we can find in recent literatures, there are a number of companies which already put salt-water batteries into practical applications, including Aquion, SoNick, and also GreenRock. Nonetheless, there is news that it appears that Aquion battery went bankrupt then it restarts again recently. What is interesting here is that SWB (salt- water battery can also be used for powering off-grid house). See the following diagrams.

CONCLUDING REMARKS

There is growing acceptance of renewable energy contribution to meet world energy demands. However, intermittency problem of WWS requires solution to stabilize that intermittency i.e. battery as energy storage (EESS). Considering basic battery configuration, as composed of ″cathode″ and ″anode″ metals. Such a classic system can be considered as ″two elements″ model. Based on Neutrosophic Logic as developed by one of us (FS), we consider in this paper that actually we can extend it further to become a Neutrosophic battery system, consists of three-elements (or may be more), i.e. cathode-anode-catalyst system. The catalytic electrolyte method is found to be significant to achieve high-performance battery. Moreover, recently there is also research on "interface engineering." Therefore, the classic model of cathode-anode system can still be improved further. We admit that the proposed term ″Neutrosophic battery″ is still schematic as for now, but we cited several other configurations used in high-performance battery experiments. It is our hope that the above diagrams make clear what is the implication of the proposed new term ″Neutrosophic battery″ in actual experiments. To conclude, it can be expected that high-performance battery research in the future shall include these additional elements. We hope that this introductory article can be found useful for young physicists/chemistry scientists.

Figure 4: Salt-water battery can power off-grid house.

Figure 5: A Swedish school implements salt-water battery. Source: *PV Magazine*.

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Fantasia in Warp Drive - Part II: Plausible Steps to Make A Workable Warp Drive Machine, Someday in the Near Future: Discussion and Remark

Victor Christianto, Florentin Smarandache

Victor Christianto, Florentin Smarandache (2022). Fantasia in Warp Drive - Part II: Plausible Steps to Make A Workable Warp Drive Machine, Someday in the Near Future: Discussion and Remark. *Bulletin of Pure and Applied Science* 41D(1), 23-29;
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ABSTRACT There is a persistence interest among physicists, to investigate on possibility of FTL (faster than light) travels, even if those fields related to FTL are most likely categorized to fringe research. Especially after Alcubierre introduces a new notion called "warp drive" solution to FTL problem. For these writers, it is interesting because (one of us :) Smarandache's hypothesis says that there is no speed barrier of anything; by generalizing implications of EPR paradox, which is known in most QM experiments. But the question remains unsolved: how to make it possible? This short article reviews several progress in such a FTL drive, while it does not necessarily mean that we agree with Alcubierre nor GTR-based approaches. We will argue in terms of possible realization of trajectory in complex plane (i.e. Argand plane.)

KEYWORDS FTL travel, warp drive, Alcubierre hypothesis, warp bubble, Argand plane, Smarandache hypothesis, interstellar travel possibility

INTRODUCTION

This short article is intended -among other things - as anopen letter to Mr. Elon Musk, founder and chief of Tesla and SpaceX, as well as other scientists and experiment investigators in FTL drive. We admire him as one of great inventors in modern era. As we often hear, Mr.Musk wants to go to Mars as quickly as possible. Although we're rather skeptical if Mars will be our next destination for habitable planet, because it is already beyond Goldilocks zone, [1] nonetheless for experiment reasons…may be there are things we can learn if we get there. So we guess there

[1]Meaning of Goldilocks zone: "The habitable zone is the area around a star where it is not too hot and not too cold for liquid water to exist on the surface of surrounding planets." Ref. https://exoplanets.nasa.gov/faq/15/what-is-the-habitable-zone-or-goldilocks-zone/#:~:text=The%20habitable%20zone%20is%20the,the%20surface%20of%20surrounding%20planets.

are reasons to consider travelling to Mars is interesting.

Around several months ago, one of these writers wrote to him that if he plans to use “carbon capture method” to propel a rocket to go to Mars, that is more likely to fail, because CO2 only takes place around Earth atmosphere, not in outer space. And if he introduces certain nuclear engines to propel his starship, that may work, but it would take very long time to go to Mars (perhaps it would take months or may be a year or so). Therefore it leaves us to explore novel methods which so far belong to “fringe physics”, something like warp drive machines.

For us, it is also interesting because one of us’s hypothesis says that there is no speed barrier of anything; but the question remains unsolved: how to make it possible?

Other possible “real” warp drive machines in recent news

In 1994, physicist Miguel Alcubierre proposed an innovation that would permit quicker than light travel: warp drive, a theoretical method for avoiding around the universe's definitive speed limit by bowing the texture of the real world. It was an interesting thought - even NASA has been exploring it at the Eagleworks research facility - however Alcubierre's proposition contained issues that appeared to be unfavorable.[2]

He contended that the general relativity took into account "*warp bubbles*" - areas where matter and energy were organized so as to twist spacetime before the air pocket and grow it to the back in a manner that permitted a "level" region inside the air pocket to travel quicker than light. [2]

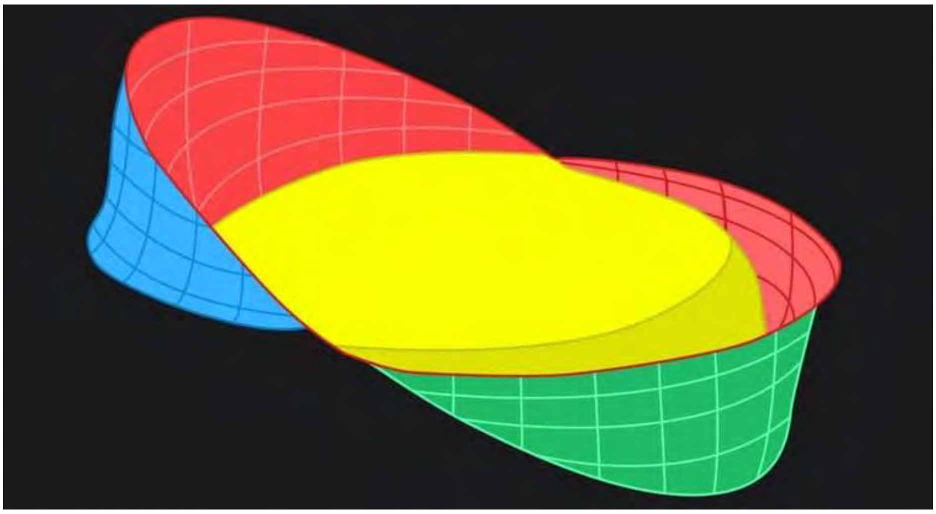

Illustration 1: Spacetime warp, as originally perceived by Alcubierre [2]

More awful, the negative energy necessities of Alcubierre's gadget are colossal. By certain appraisals, the whole energy in the realized universe would be required (however later work cuts the number down a little).Bobrick and Martire show a warp drive could be produced using positive energy (for example "ordinary" energy) or from a combination of negative and positive energy. All things considered, the energy necessities would in any case be gigantic.[4]

More recently, around December last year, Harold "Sonny" White, a NASA specialist at the Eagleworks Laboratory in Houston, Texas, distributed an examination paper with his group in July about the "conceivable design of the energy thickness present in a Casimir hole." As indicated in their report, the Eagleworks group went over

"*a miniature/nano-scale structure … that predicts negative energy thickness appropriation that intently matches necessities for the Alcubierre metric.*"[3]

White told a science and innovation magazine: "as far as anyone is concerned, this is the main paper in the companion surveyed writing that proposes a feasible nano-structure that is anticipated to show a genuine, yet modest, *warp bubble*." White and his group intend to direct more analyses into the chance of more modest models to more readily comprehend the chances of a forthcoming warp drive. Maybe the Eagleworks Laboratory can take us from sci-fi to the real warp starship model.[3]

Illustration 2: NASA illustration [3]

A comment by a reader to that article goes as follows: "They did not 'accidentally' create a warp bubble in the lab, that's a very misleading headline. What this, and the big-think article in another comment, is stating that the math works, in a very small scale albeit, but the numbers support the theory of a warp-bubble. But that's it, the numbers look good for a theoretical idea. But, no one has built anything in a lab, and 'accidentally' created a warp bubble." [3]

Beam me up, Scotty…

Other physicist, named Dr. Erik Lentz, tried to solve the problem, inspired partially by **Star Trek.** Lentz specifically examined the assumptions leading to the negative energy requirements in Alcubierre's work. Like his colleague, Lentz began by analyzing spacetime, modeling the multidimensional substance as a stack of very thin layers. He found that Alcubierre had only considered comparatively simple "linear" relationships between the equations for shifting one layer onto the next. At this point, choosing more complex "hyperbolic" relations, which typically express rapidly changing quantities, results in a different warp bubble than the one obtained by Alcubierre.[4]

Other people, including Agnew, argues that LIGO etc. provide interesting clue on such a realizable technology based on GTR: "The LIGO discovery a few years back was, in my opinion, a huge leap forward in science, since it proved, experimentally, that space-time can 'warp' and bend in the presence of enormous gravitational fields, and this is propagated out across the Universe in a way that we can measure. …."[5]

As we shall see later on, this is not the case. But before we discuss our new hypothesis, let us discuss Asaro's solution. She is also a researcher, although you may find her names more related to sci-fi books, but may be there are things that we can learn.

Catherine Asaro's hypothetical solution to warp Drive[1]

Physicist and science fiction author Catherine Asaro is tracing the mathematical background of this problem and came up with a way to modify Einstein's equations in order to make them compatible with an FTL drive. Warp speed, according to her, might be achievable by using a fairly simple solution involving imaginary numbers.

Overcoming the distance between planets and stars is something that science fiction needed to do, as it would otherwise significantly restrict the plots of any film or novel. We all know there's just no use trekking the stars when traveling from Earth to Mars takes half a year. To get around this issue at least some sort of warp drive is needed, allowing for faster-than-light travel.

Asaro explained at a panel at the Escape Velocity convention in Washington D.C. these days that most of the scientists base their FTL-research purely on Einstein's theory of relativity and **according to Einstein himself — there is no solution**. Asaro began by highlighting that:

"*The problem for light speed travel in sci-fi is you can never get to the speed of light […] As you approach the speed of light, your mass becomes infinite…* "

So okay — FTL travel might not be scientifically plausible right away, but Asaro was pretty determined to find some kind of mathematical justification for her own sci-fi novels. Therefore, she came up with a modified version of Einstein's equations as it's definitely a good idea to not ignore them. Instead she thought of speed as an imaginary number, a complex number defined by a real number multiplied by i, or the square root of -1.

And guess what — according to Asaro, this effectively solved the problem of FTL travel in some regard. Imaginary numbers don't have or need to have a physical equivalent in the real world. She opted to apply the same to the "complex speed," as she calls it in her paper published in the *American Journal of Physics.* What she did is treating the whole issue like a thought problem. The conclusion: If there were a physical equivalent to their imaginary number, it would theoretically allow FTL travel to work in real life.

Asaro wrote an essay for PBS explaining the thought process in even more detail:"I now had a framework based on relativistic physics that I could use to let my starships break the interstellar speed laws […] I called the process 'inversion' because if you are going faster than light, relativity predicts the constellations you see will be flipped around (ed) from their positions as seen at slower than light speeds."

…

For the time being, all of this remains the thought problem Asaro encountered, but who says we won't discover the physical analog for complex speed in real life now? Maybe we just can't fathom it yet.

On the other hand — imaginary numbers seems to suggest that that we can get back complex plane (Argand plane) without recourse to Einstein in even more ways than pure science.[2]

Another hint to a workable pathway: Solve it with 3D-symmetric plane

Let us recall one quote by one of great mathematicians of his time, J. Hadamard, something like this: "*a shortest path to a point (or to infinity) is through complex plane*."[2]

To put it in other words, we may consider that Asaro's idea may lead to something interesting…but the problem is: as we argue in a forthcoming paper, it is unlikely that the notion of *geometrodynamics* is true, and also the entire LIGO project.

To speak frankly, these authors wrote a number of papers in the past based on the notion of quaternion algebras and quaternion numbers. A senior math-physics professor once asked one of authors (VC): "Do you believe in such a universe of quaternion numbers?" At the time, he didn't think that much, because at the timeit seems like the simplest approach to find 6D-version of Maxwell equations." But it does not mean that we agree with the entire geometro-dynamics theories, includingAlcubierre hypothesis and also the remaining of GTR-based warp drive machine proposals so far.

As we wrote (cf. [8]):
"Therefore, it is understandable that the late John Wheeler himself, who coined term geometro-dynamics, later on abandoned many features of that approach. See J. Stachel's article: The rise and fall of Geometrodynamics, and also W. Misner. We can recall too, that there is senior professor in Germany who was brought to justice a few years ago, and then he admitted that there is no way to measure "spacetime curvature." This case file can be found in internet, related to LIGOproject. Actually, there are several criticisms on that observation project. … Of course, by

mentioning all of these, it does not mean that we are right all along, but let us face the truth as it is. Physics is more related to solid experimental evidences, not just made of tower of sand. ….Nonetheless, there is also possibility that the 6D geometry, or more exactly (3D, symmetric) can be found in nature, especially in quasicrystalline structure. Therefore all we can say is that may be this is still useful, although these 6D geometry may be seen more like a mathematical artefact. Zlabys *et al.* wrote their abstract as follows: "*Here we show that time and space crystalline structures can be combined together and even six-dimensional time-space lattices can be realized. As an example, we demonstrate that such time-space crystalline structures can reveal the six-dimensional quantum Hall effect quantified by the third Chern number*."[8]

CONCLUDING REMARK

To sum up our aforementioned arguments, what can be a simpler path toward making a working warp drive machine, are as follows:
(1) Find if it is possible at all to reconsider 6D (3D, symmetric) crystalline structure in such a way to allow "shortest path to infinity" - as J. Hadamard's famous phrase. (That is, our argument here is to do away from the geometro-dynamics approaches, such as Alcubierre's hypothesis.)
(2) Do experiments in small-scale version. We may call such experiments: *"lab scale warp drive" experiments (LSWDE)*. This is quite similar to what Harold White, a NASA specialist at the Eagleworks Laboratory, attempts to do with their warp bubble scenario.
(3) Computational simulation may be helpful, for instance, we shall check if the crystalline structure of the 3D space itself, can be related to Dr. Harold Aspden's space aether model.

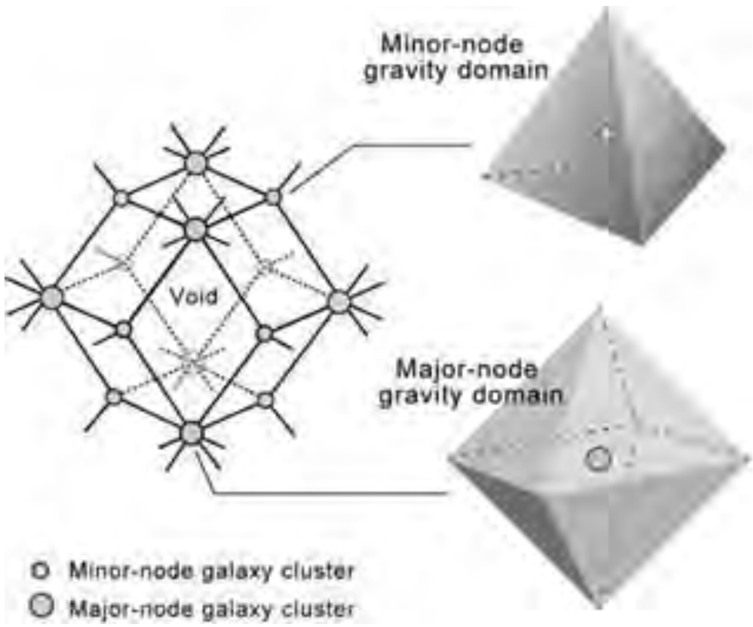

Illustration 3: Harold Aspden's space aether model

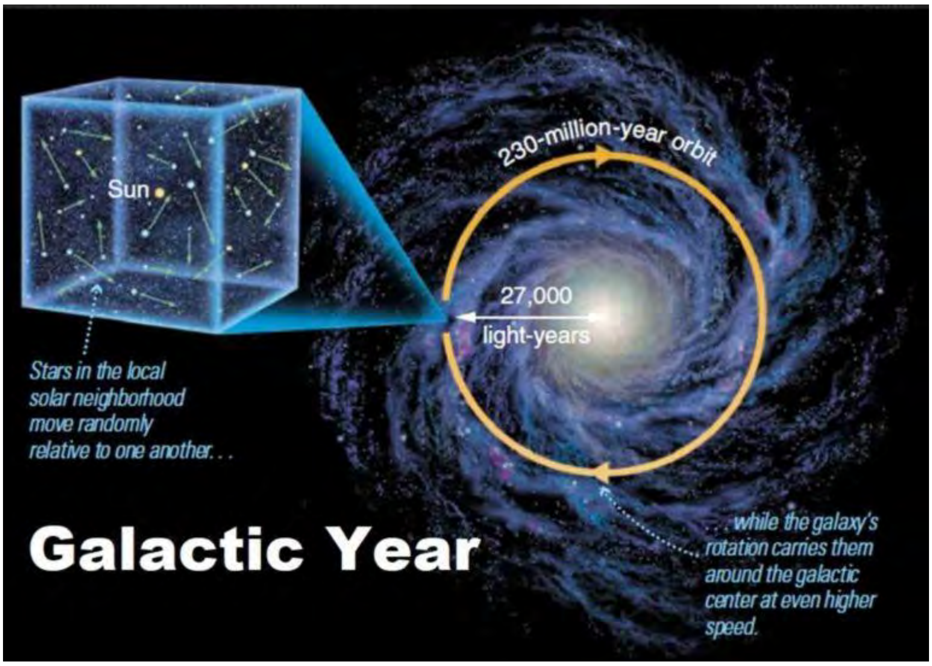

Illustration 4: Harold Aspden's space aether model.

(4) Once we find a "realistic" *small-scale version* / simulation of LSWDE, we can begin to be sure that what is required for small-scale version of warp drive machine, can be scaled up to larger prototype of WD.
(5) Then we can start to send *photon signal* or something like that to see if itis really workable warp drive as we sought for.

At this point, some readers may ask: why do we write this article? Just to make sense of all those nonsense?

Answer: No. You should know that mathematicians are more or less like a coffee making machine. As a wise word says: "A mathematician," the Hungarian mathematician Alfréd Rényi (1921-1970) used to say, "*is a machine for turning coffee into theorems*." Rényi's colleague Paul Erdős well embodied the statement. That is part of the reason we write this short article. Hopefully you will find something to learn from this short remark.

As with a question: how to generate required energy to travel over the complex plane? - we don't get an answer for that. May be, just may be, we can generate through extracting vacuum to become negative energy or negative masses. [7][9][10]

We write this remark quite short, and it is sketchy indeed. But, what is sketchy and seems like simple steps, can actually lead to human kind's real pathway to reach the stars, and go where we belong since the ancient past…along with the Galactic Federation.

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VC & FS

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ARTIFICIAL INTELLIGENCE

Unification of Fusion Theories (UFT)

Florentin Smarandache

Florentin Smarandache (2004). Unification of Fusion Theories (UFT). *International Journal of Applied Mathematics & Statistics,* 14

ABSTRACT

Since no fusion theory neither rule fully satisfy all needed applications, the author proposes a Unification of Fusion Theories and a combination of fusion rules in solving problems/applications. For each particular application, one selects the most appropriate model, rule(s), and algorithm of implementation.
We are working in the unification of the fusion theories and rules, which looks like a cooking recipe, better we'd say like a logical chart for a computer programmer, but we don't see another method to comprise/unify all things.
The unification scenario presented herein, which is now in an incipient form, should periodically be updated incorporating new discoveries from the fusion and engineering research.

Keywords: Distributive lattice, Boolean algebra, Conjunctive rule, Disjunctive rule, Partial and Total conflicts, Weighted Operator (WO), Proportional Conflict Redistribution (PCR) rules, Murphy's average rule, Dempster-Shafer Theory (DST), Yager's rule, Transferable Belief Model (TBM), Dubois-Prade's rule (DP), Dezert-Smarandache Theory (DSmT), static and dynamic fusion.

1. INTRODUCTION

Each theory works well for some applications, but not for all.

We extend the power and hyper-power sets from previous theories to a Boolean algebra obtained by the closure of the frame of discernment under union, intersection, and complement of sets (for non-exclusive elements one considers as complement the fuzzy or neutrosophic complement). All bbas and rules are redefined on this Boolean algebra.

A similar generalization has been previously used by Guan-Bell (1993) for the Dempster-Shafer rule using propositions in sequential logic, herein we reconsider the Boolean algebra for all fusion rules and theories but using sets instead of propositions, because generally it is harder to work in sequential logic with summations and inclusions than in the set theory.

2. FUSION SPACE

For $n \geq 2$ let Ė = {ė$_1$, ė$_2$, …, ė$_n$} be the frame of discernment of the fusion problem/application under consideration. Then (Ė, χ, 1, C), Ė closed under these three operations: union, intersection, and complementation of sets respectively, forms a Boolean algebra. With respect to the partial ordering relation, the inclusion φ, the minimum element is the empty set ϕ, and the maximal element is the total ignorance $I = \bigcup_{i=1}^{n} \theta i$.

Similarly one can define: (Ė, χ, 1, \) for sets, Ė closed with respect to each of these operations: union, intersection, and difference of sets respectively.

(Ė, χ, 1, C) and (Ė, χ, 1, \) generate the same super-power set S^{Ė} closed under χ, 1, C, and \ because for any A, B 0 S^{Ė} one has CA = I \ A and reciprocally A\B = A1CB.

If one considers propositions, then (Ė, ω, ϖ, 5) forms a Lindenbaum algebra in sequential logic, which is isomorphic with the above (Ė, χ, 1, C) Boolean algebra.

By choosing the frame of discernment Ė closed under χ only one gets DST, Yager's, TBM, DP theories. Then making Ė closed under both χ, 1 one gets DSm theory. While, extending Ė for closure under χ, 1, and C one also includes the complement of set (or negation of proposition if working in sequential logic); in the case of non-exclusive elements in the frame of discernment one considers a fuzzy or neutrosophic complement. Therefore the super-power set (Ė, χ, 1, C) includes all the previous fusion theories.

The power set 2^{Ė}, used in DST, Yager's, TBM, DP, which is the set of all subsets of Ė, is also a Boolean algebra, closed under χ, 1, and C, but does not contain intersections of elements from Ė.

The Dedekind distributive lattice D^{Ė}, used in DSmT, is closed under χ, 1, and if negations/complements arise they are directly introduced in the frame of discernment, say Ė', which is then closed under χ, 1. Unlike others, DSmT allows intersections, generalizing the previous theories.

The Unifying Theory contains intersections and complements as well.

Let's consider a frame of discernment Ė with exclusive or non-exclusive hypotheses, exhaustive or non-exhaustive, closed or open world (all possible cases).

We need to make the remark that in case when these $n \geq 2$ elementary hypotheses ė$_1$, ė$_2$, …, ė$_n$ are *exhaustive and exclusive* one gets the Dempster-Shafer Theory, Yager's, Dubois-Prade Theory, Dezert-Smarandache Theory, while for the case when the hypotheses are *non-exclusive* one gets Dezert-Smarandache Theory, but for *non-exhaustivity* one gets TBM.

An exhaustive frame of discernment is called *close world*, and a non-exhaustive frame of discernment is called *open world* (meaning that new hypotheses might exist in the frame of discernment that we are not aware of). Ė may be finite or infinite.

Let m$_j$: S^{Ė} → [0. 1], 1 j s, be s 2 basic belief assignments,

(when bbas are working with crisp numbers),

or with subunitary subsets, m$_j$: S^{Ė} → P([0. 1]), where P([0. 1]) is the set of all subsets of the interval [0,1] (when dealing with very imprecise information).

Normally the sum of crisp masses of a bba, m(.), is 1, i.e. $\sum_{X\in S^\wedge T} m(X) = 1$.

3. INCOMPLETE AND PARACONSISTENT INFORMATION

For incomplete information the sum of a bba components can be less than 1 (not enough information known), while in paraconsistent information the sum can exceed 1 (overlapping contradictory information).

The masses can be normalized (i.e. getting the sum of their components =1), or not (sum of components < 1 in incomplete information; or > 1 in paraconsistent information).

For a bba valued on subunitary subsets one can consider, as normalization of m(.),

either $\sum_{X\in S^\wedge T} \sup\{m(X)\} = 1$,

or that there exist crisp numbers x 0 X for each X 0 S^Ɵ such that $\sum_{\substack{X\in S^\wedge T\\ x\in X}} m(x) = 1$.

Similarly, for a bba m(.) valued on subunitary subsets dealing with paraconsistent and incomplete information respectively:

a) for incomplete information, one has $\sum_{X\in S^\wedge T} \sup\{m(X)\} < 1$,

b) while for paraconsistent information one has $\sum_{X\in S^\wedge T} \sup\{m(X)\} > 1$ and there do not exist crisp numbers x 0 X for each X 0 S^Ɵ such that $\sum_{\substack{X\in S^\wedge T\\ x\in X}} m(x) = 1$.

4. SPECIFICITY CHAINS

We use the min principle and the precocious/prudent way of computing and transferring the conflicting mass.

Normally by transferring the conflicting mass and by normalization we diminish the specificity. If A1B is empty, then the mass is moved to a less specific element A (also to B), but if we have a pessimistic view on A and B we move the mass m(A1B) to AχB (entropy increases, imprecision increases), and even more if we are very pessimistic about A and B: we move the conflicting mass to the total ignorance in a closed world, or to the empty set in an open world.

Specificity Chains:

a) From specific to less and less specific (in a closed world):

(A1B) δ A δ (AχB) δ I or (A1B) δ B δ (AχB) δ I.

Also from specific to unknown (in an open world):

A1B → ϕ.

b) And similarly for intersections of more elements: A1B1C, etc.

A1B1C δ A1B δ A δ (AχB) δ (AχBχC) δ I

or (A1B1C) δ (B1C) δ B δ (AχB) δ (AχBχC) δ I, etc. in a closed world.

Or A1B1C → ϕ in an open world.

c) Also in a closed world:

A1(BχC) δ BχC δ (BχC) δ (AχBχC) δ I or A1(BχC) δ A δ (AχB) δ (AχBχC) δ I.

Or A1(BχC) → ϕ in an open world.

5. STATIC AND DYNAMIC FUSION

According to Wu Li we have the following classification and definitions:

Static fusion means to combine all belief functions simultaneously.

Dynamic fusion means that the belief functions become available one after another sequentially, and the current belief function is updated by combining itself with a newly available belief function.

6. SCENARIO OF UNIFICATION OF FUSION THEORIES

Since everything depends on the application/problem to solve, this scenario looks like a logical chart designed by the programmer in order to write and implement a computer program, or like a cooking recipe.

Here it is the scenario attempting for a unification and reconciliation of the fusion theories and rules:

1) If all sources of information are reliable, then apply the conjunctive rule, which means consensus between them (or their common part):

2) If some sources are reliable and others are not, but we don't know which ones are unreliable, apply the disjunctive rule as a cautious method (and no transfer or normalization is needed).

3) If only one source of information is reliable, but we don't know which one, then use the exclusive disjunctive rule based on the fact that X_1 ξ X_2 ξ… ξX_n means either X_1 is reliable, or X_2, or and so on or X_n, but not two or more in the same time.

4) If a mixture of the previous three cases, in any possible way, use the mixed conjunctive-disjunctive rule.

As an example, suppose we have four sources of information and we know that: either the first two are telling the truth or the third, or the fourth is telling the truth.

The mixed formula becomes:

$m_{1\chi}(\phi) = 0$, and □ A0S$^{\dot{E}}$\ϕ, one has $m_{1\chi}(A) = \sum_{\substack{X_1, X_2, X_3, X_4 \in S^{\wedge}\Theta \\ ((X_1 \cap X_2) \cup X_3) \underline{\cup} X_4 = A}} m_1(X_1) m_2(X_2) m_3(X_3) m_4(X_4)$.

5) If we know the sources which are unreliable, we discount them. But if all sources are fully unreliable (100%), then the fusion result becomes vacuum bba (i.e. m(Ė) = 1, and the problem is indeterminate. We need to get new sources which are reliable or at least they are not fully unreliable.

6) If all sources are reliable, or the unreliable sources have been discounted (in the <u>default case</u>), then use the DSm classic rule (which is commutative, associative, Markovian) on Boolean algebra (Ė, χ, 1, C), no

matter what contradictions (or model) the problem has. I emphasize that the super-power set S^{Θ} generated by this Boolean algebra contains singletons, unions, intersections, and complements of sets.

7) If the sources are considered from a statistical point of view, use *Murphy's average rule* (and no transfer or normalization is needed).

8) In the case the model is not known (the <u>default case</u>), it is prudent/cautious to use the free model (i.e. all intersections between the elements of the frame of discernment are non-empty) and DSm classic rule on S^{Θ}, and later if the model is found out (i.e. the constraints of empty intersections become known), one can adjust the conflicting mass at any time/moment using the DSm hybrid rule.

9) Now suppose the model becomes known [i.e. we find out about the contradictions (= empty intersections) or consensus (= non-empty intersections) of the problem/application]. Then :

9.1) If an intersection A1B is not empty, we keep the mass m(A1B) on A1B, which means consensus (common part) between the two hypotheses A and B (i.e. both hypotheses A and B are right) [here one gets *DSmT*].

9.2) If the intersection A1B $=\phi$ is empty, meaning contradiction, we do the following :

9.2.1) if one knows that between these two hypotheses A and B one is right and the other is false, but we don't know which one, then one transfers the mass m(A1B) to m(AχB), since AχB means at least one is right [here one gets *Yager's* if n=2, or *Dubois-Prade*, or *DSmT*];

9.2.2) if one knows that between these two hypotheses A and B one is right and the other is false, and we know which one is right, say hypothesis A is right and B is false, then one transfers the whole mass m(A1B) to hypothesis A (nothing is transferred to B);

9.2.3) if we don't know much about them, but one has an optimistic view on hypotheses A and B, then one transfers the conflicting mass m(A1B) to A and B (the nearest specific sets in the Specificity Chains) [using *Dempster's*, *PCR2-5*]

9.2.4) if we don't know much about them, but one has a pessimistic view on hypotheses A and B, then one transfers the conflicting mass m(A1B) to AχB (the more pessimistic the further one gets in the Specificity Chains: (A1B) δ A δ (AχB) δ I); this is also the <u>default case</u> [using *DP's*, *DSm hybrid rule*, *Yager's*];

if one has a very pessimistic view on hypotheses A and B then one transfers the conflicting mass m(A1B) to the total ignorance in a closed world [*Yager's*, *DSmT*], or to the empty set in an open world [*TBM*];

9.2.5.1) if one considers that no hypothesis between A and B is right, then one transfers the mass m(A1B) to other non-empty sets (in the case more hypotheses do exist in the frame of discernment) - different from A, B, AχB - for the reason that: if A and B are not right then there is a bigger chance that other hypotheses in the frame of discernment have a higher subjective probability to occur; we do this transfer in a **closed world** [DSm hybrid rule]; but, if it is an **open world**, we can transfer the mass m(A1B) to the empty set leaving room for new possible hypotheses [here one gets *TBM*];

9.2.5.2) if one considers that none of the hypotheses A, B is right and no other hypothesis exists in the frame of discernment (i.e. n = 2 is the size of the frame of discernment), then one considers the **open world** and one transfers the mass to the empty set [here *DSmT* and *TBM* converge to each other].

Of course, this procedure is extended for any intersections of two or more sets: A1B1C, etc. and even for mixed sets: A1 (BχC), etc.

If it is a dynamic fusion in a real time and associativity and/or Markovian process are needed, use an algorithm which transforms a rule (which is based on the conjunctive rule and the transfer of the conflicting mass) into an associative and Markovian rule by storing the previous result of the conjunctive rule and, depending of the rule, other data. Such rules are called quasi-associative and quasi-Markovian.

Some applications require the necessity of **decaying the old sources** because their information is considered to be worn out.

If some bba is not normalized (i.e. the sum of its components is < 1 as in incomplete information, or > 1 as in paraconsistent information) we can easily divide each component by the sum of the components and normalize it. But also it is possible to fusion incomplete and paraconsistent masses, and then normalize them after fusion. Or leave them unnormalized since they are incomplete or paraconsistent.

PCR5 does the most mathematically exact (in the fusion literature) redistribution of the conflicting mass to the elements involved in the conflict, redistribution which exactly follows the tracks of the conjunctive rule.

7. EXAMPLES

7.1. Bayesian Example:

Let È = {A, B, C, D, E} be the frame of discernment.

	A	B	C	D	E	A1B	A1C	A1D	A1E	B1C	B1D
						φ	= φ	= φ	= φ	Not known if = or φ	= φ
						Consensus between A and B	Contradiction between A and C, but optimistic in both of them	One right, one wrong, but don't know which one	A is right, B is wrong	Don't know the exact model	Unknown any relation between B and D.
m_1	0.2	0	0.3	0.4	0.1						
m_2	0.5	0.2	0.1	0	0.2						
m_{12}	0.10	0	0.03	0	0.02	0.04	0.17	0.20	0.09	0.06	0.08
						A1B	A, C	AχB	A	B1C We keep the mass 0.06 on B1C till we find out more information on the model.	BχD
m_r						0.04	0.107, 0.063	0.20	0.09	0.06	0.08
$\mathbf{m_{UFT}}$	0.32	0.04	0.12	0	0.03	0.04	0	0	0	0.06	0
m_{lower} (closed world)	0.10	0	0.03	0	0.02						
m_{lower} (open world)	0.10	0	0.03	0	0.02						
m_{middle} (default)	0.10	0	0.03	0	0.02						
m_{upper}	0.40	0.08	0.18	0.23	0.11						

Table 1. Bayesian Example using the Unified Fusion Theories rule regarding a mixed redistribution of partial conflicting masses (Part 1).

	B1E	C1D	C1E	D1E	AχB	AχC	AχD	AχE	BχC
	ϕ	= ϕ	= ϕ	= ϕ					
	The intersection is not empty, but neither B1E nor BχE interest us	Pessimistic in both C and D	Very pessimistic in both C and E	Both D and E are wrong					
m_1									
m_2									
m_{12}	0.02	0.04	0.07	0.08					
	B, E	CχD	AχBχCχD χE	A,B,C					
m_r	0.013, 0.007	0.04	0.07	0.027, 0.027, 0.027					
m_{UFT}	**0**	**0**	**0**	**0**	**0**	**0**	**0.20**	**0**	**0**
m_{lower} (closed world)									
m_{lower} (open world)									
m_{middle} (default)					0.04	0.17	0.20	0.09	0.06
m_{upper}									

Table 1. Bayesian Example using the Unified Fusion Theories rule regarding a mixed redistribution of partial conflicting masses (Part 2).

	BχD	BχE	CχD	CχE	DχE	AχBχCχDχE	ϕ
m_1							
m_2							
m_{12}							
m_r							
m_{UFT}	**0.08**	**0**	**0.04**	**0**	**0**	**0.07**	**0**
m_{lower} (closed world)						0.85	
m_{lower} (open world)							0.85
m_{middle} (default)	0.08	0.02	0.04	0.07	0.08		
m_{upper}							

Table 1. Bayesian Example using the Unified Fusion Theories rule regarding a mixed redistribution of partial conflicting masses (Part 3).

We keep the mass m_{12}(B1C) = 0.06 on B1C (eleventh column in Table 1, part 1) although we don't know if the intersection B1C is empty or not (this is considered the default model), since in the case when it is empty one considers an open world because $m_{12}(\phi)$=0.06 meaning that there might be new possible hypotheses in the frame of discernment, but if B1C   ϕ one considers a consensus between B and C.

Later, when finding out more information about the relation between B and C, one can transfer the mass 0.06 to BχC, or to the total ignorance I, or split it between the elements B, C, or even keep it on B1C.

m_{12}(A1C)=0.17 is redistributed to A and C using the PCR5:

a1/0.2 = c1/0.1 = 0.02(0.2+0.1),

whence a1 = 0.2(0.02/0.3) = 0.013,

c1 = 0.1(0.02/0.3) = 0.007.

a2/0.5 = c2/0.3 = 0.15(0.5+0.3),

whence a2 = 0.5(0.15/0.8) = 0.094,

c2 = 0.3(0.15/0.8) = 0.056.

Thus A gains a1+a2 = 0.013+0.0.094 = 0.107 and C gains c1+c2 = 0.007+0.056 = 0.063.

m_{12}(B1E)=0.02 is redistributed to B and E using the PCR5:

b/0.2 = e/0.1 = 0.02/(0.2+0.1),

whence b = 0.2(0.02/0.3) = 0.013,

e = 0.1(0.02/0.3) = 0.007.

Thus B gains 0.013 and E gains 0.007.

Then one sums the masses of the conjunctive rule $\mathbf{m_{12}}$ and the redistribution of conflicting masses $\mathbf{m_r}$ (according to the information we have on each intersection, model, and relationship between conflicting hypotheses) in order to get the mass of the Unification of Fusion Theories rule m_{UFT}.

$\mathbf{m_{UFT}}$, the Unification of Fusion Theories rule, is a combination of many rules and gives the optimal redistribution of the conflicting mass for each particular problem, following the given model and relationships between hypotheses; this extra-information allows the choice of the combination rule to be used for each intersection. The algorithm is presented above.

$\mathbf{m_{lower}}$, the lower bound believe assignment, the most pessimistic/prudent belief, is obtained by transferring the whole conflicting mass to the total ignorance (Yager's rule) in a closed world, or to the empty set (Smets' TBM) in an open world herein meaning that other hypotheses might belong to the frame of discernment.

$\mathbf{m_{middle}}$, the middle believe assignment, half optimistic and half pessimistic, is obtained by transferring the partial conflicting masses m_{12}(X1Y) to the partial ignorance XχY (as in Dubois-Prade theory or more general as in Dezert-Smarandache theory).

Another way to compute a middle believe assignment would be to average the $\mathbf{m_{lower}}$ and $\mathbf{m_{upper}}$**.**

$\mathbf{m_{upper}}$, the lower bound believe assignment, the most optimistic (less prudent) belief, is obtained by transferring the masses of intersections (empty or non-empty) to the elements in the frame of discernment using the PCR5 rule of combination, i.e. m_{12}(X1Y) is split to the elements X, Y (see Table 2). We use PCR5 because it is more exact mathematically (following the backwards the tracks of the conjunctive rule) than Dempster's rule, minC, and PCR1-4.

X	$m_{12}(X)$	A	B	C	D	E
A1B	0.040	0.020	0.020			
A1C	0.170	0.107		0.063		
A1D	0.200	0.111			0.089	
A1E	0.090	0.020 0.042				0.020 0.008
B1C	0.060		0.024	0.036		
B1D	0.080		0.027		0.053	
B1E	0.020		0.013			0.007
C1D	0.040			0.008	0.032	
C1E	0.070			0.036 0.005		0.024 0.005
D1E	0.080				0.053	0.027
Total	0.850	0.300	0.084	0.148	0.227	0.091

Table 2. Redistribution of the intersection masses to the singletons A, B, C, D, E using the PCR5 rule only, needed to compute the upper bound belief assignment m_{upper}.

7.2. Negation/Complement Example:

Let È = {A, B, C, D} be the frame of discernment. Since (È, χ, 1, C) is Boolean algebra, the super-power set $S^{È}$ includes complements/negations, intersections and unions. Let's note by C(B) the complement of B.

	A	B	D	C(B)	A1C {Later in the dynamic fusion process we find out that A1C is empty }	BχC=B	A1B	A1 C(B)=A
					= ϕ		= ϕ	ϕ
					Unknown relationship between A and C		Optimistic in both A and B.	Consensus between A and C(B), but Aδ C(B)
m_1	0.2	0.3	0	0.1	0.1	0.3		
m_2	0.4	0.1	0	0.2	0.2	0.1		
m_{12}	0.08	0.09	0	0.02	0.17	0.03	0.14	0.08
					AχC	B	A,B	A
m_r					0.17	0.03	0.082, 0.058	0.08
m_{UFT}	**0.277**	**0.32**	**0.035**	**0.02**	**0**	**0**	**0**	**0**
m_{lower} (closed world)	0.16	0.26	0	0.02	0	0		
m_{lower} (open world)	0.16	0.26	0	0.02	0	0		
m_{middle} (default)	0.16	0.26	0	0.02	0	0		
m_{upper}	0.296	0.23 0	0	0.12 6	0.219	0.129		

Table 3. Negation/Complement Example using the Unified Fusion Theories rule regarding a mixed redistribution of partial conflicting masses (Part 1).

	A1 (BχC)	B1 C(B)	B1(A1C)	C(B)1(A1C)	C(B)1(BχC) = C(B)1C	Bχ(A1C)=B
	= ϕ	= ϕ	= ϕ	= ϕ	= ϕ	
	At least one is right between A and BχC	B is right, C(B) is wrong	No relationship known between B and A1C (default case)	Very pessimistic on C(B) and A1C	Neither C(B) nor BχC are right	
m_1						
m_2						
m_{12}	0.14	0.07	0.07	0.04	0.07	
	Aχ(BχC) .	B	Bχ(A1C)=B	AχBχCχD	A, D	B
m_r	0.14	0.07	0.07	0.04	0.035, 0.035	
m_{UFT}	**0**	**0**	**0**	**0**	**0**	**0**
m_{lower} (closed world)						
m_{lower} (open world)						
m_{middle} (default)						
m_{upper}						

Table 3. Negation/Complement Example using the Unified Fusion Theories rule regarding a mixed redistribution of partial conflicting masses (Part 2).

	AχB	AχC	AχD	BχC	BχD	CχD	AχBχC	AχBχCχD	ϕ
m_1									
m_2									
m_{12}									
m_r									
m_{UFT}	**0**	**0.170**	**0**	**0**	**0**	**0**	**0.140**	**0.040**	**0**
m_{lower} (closed world)								0.56	
m_{lower} (open world)									0.56
m_{middle} (default)	0.14	0.17		0.03			0.14	0.11	
m_{upper}									

Table 3. Negation/Complement Example using the Unified Fusion Theories rule regarding a mixed redistribution of partial conflicting masses (Part 3).

Model of Negation/Complement Example:

A1B = ϕ, CδB, Aδc(B).

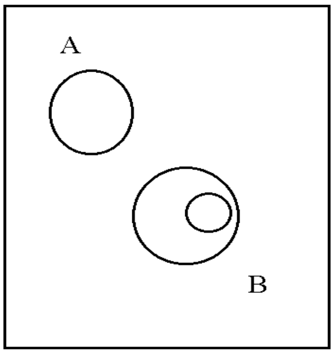

Fig. 1

m_{12}(A1B) = 0.14.

x1/0.2 = y1/0.1 = 0.02/0.3, whence x1 = 0.2(0.02/0.3) = 0.013, y1 = 0.1(0.02/0.3) = 0.007;

x2/0.4 = y2/0.3 = 0.12/0.7, whence x2 = 0.4(0.12/0.7) = 0.069, y2 = 0.3(0.12/0.7) = 0.051.

Thus, A gains 0.013+0.069 = 0.082 and B gains 0.007+0.051 = 0.058.

For the upper belief assignment m_{upper} one considered all resulted intersections from results of the conjunctive rule as empty and one transferred the partial conflicting masses to the elements involved in the conflict using PCR5.

All elements in the frame of discernment were considered non-empty.

7.3. Example with Intersection:

Look at this:

Suppose A={x<0.4}, B={0.3<x<0.6}, C={x>0.8}. The frame of discernment T={A, B, C} represents the possible cross section of a target, and there are two sensors giving the following bbas:

m1(A)=0.5, m1(B)=0.2, m1(C)=0.3.

m2(A)=0.4, m2(B)=0.4, m2(C)=0.2.

	A	B	C	A1B= {.3<x<.4}	AχC	BχC
m_1	.5	.2	.3			
m_2	.4	.4	.2			
m_1&m_2 DSmT	.20	.08	.06	.28	.22	.16

We have a DSm hybrid model (one intersection A&B=nonempty).

This example proves the necessity of allowing intersections of elements in the frame of discernment. [Shafer's model doesn't apply here.]

Dezert-Smarandache Theory of Uncertain and Paradoxist Reasoning (DSmT) is the only theory which accepts intersections of elements.

7.4. Another Multi-Example of UFT:

Cases:

1. Both sources reliable: use conjunctive rule [default case]:

1.1. A1B ϕ:

1.1.1. Consensus between A and B; mass 6 A1B;

1.1.2. Neither A1B nor AχB interest us; mass 6 A, B;

1.2. A1B=ϕ:

1.2.1. Contradiction between A and B, but optimistic in both of them; mass 6 A, B;

1.2.2. One right, one wrong, but don't know which one; mass 6 AχB;

1.2.3. Unknown any relation between A and B [default case]; mass 6 AχB;

1.2.4. Pessimistic in both A and B; mass 6 AχB;

1.2.5. Very pessimistic in both A and B;

1.2.5.1. Total ignorance ε AχB; mass 6 AχBχCχD (total ignorance);

1.2.5.2. Total ignorance = AχB; mass 6 ϕ (open world);

1.2.6. A is right, B is wrong; mass 6 A;

1.2.7. Both A and B are wrong; mass 6 C, D;

1.3. Don't know if A1B = or ϕ (don't know the exact model); mass 6 A1B (keep the mass on intersection till we find out more info) [default case];

2. One source reliable, other not, but not known which one: use disjunctive rule; no normalization needed.

3. S1 reliable, S2 not reliable 20%: discount S2 for 20% and use conjunctive rule.

	A	B	AχB	A1B	ϕ (open world)	AχBχCχD	C	D
S1	.2	.5	.3					
S2	.4	.4	.2					
S1&S2	.24	.42	.06	.28				
S1 or S2	.08	.20	.72	0				
UFT 1.1.1	.24	.42	.06	.28				
UFT 1.1.2 (PCR5)	.356	.584	.060	0				
UFT 1.2.1	.356	.584	.060	0				
UFT 1.2.2	.24	.42	.34	0				
UFT 1.2.3	.24	.42	.34	0				
UFT 1.2.4	.24	.42	.34	0				
UFT 1.2.5.1	.24	.42	.06	0	0	.28		
UFT 1.2.5.2	.24	.42	.06	0	.28			
80% S2	.32	.32	.16			.20		
UFT 1.2.6	.52	.42	.06					
UFT 1.2.7	.24	.42	.06	0			.14	.14
UFT 1.3	.24	.42	.06	.28				
UFT 2	.08	.20	.72	0				
UFT 3	.232	.436	.108	.224		0		

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A Position Indicator with Applications in The Field of Designing Forms with Artificial Intelligence

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The indicators used so far within the Theory of Extension, see papers [1], [2] and [4], can be synthetically expressed by the notion of "position indicators". More exactly, these indicators can be grouped in two main sub-categories: point-set position indicators and point-two sets position indicators. The secondary goal of this paper is to define these classifications, while the primary goal is that to extend the two notions to the most general notion of set-set position indicator.

Key words: Hausdorff measure, Extension theory, position indicators, computer vision, artificial intelligence.

1. Introduction

The first part of this paper aims at defining the notion of point-set position indicator and that of point-two sets position indicator, at discussing the main examples of such indicators and their relevance for the applicative field.

Point-set Position Indicators

For any point $x \in \mathbb{R}^n$ and any set $A \subset \mathbb{R}^n$, formula $\delta(x, A) = \inf\{d(x, a) | a \in A\}$, where d is the Euclidean distance on $\mathbb{R}^n$, defines (in classical mathematics) the distance from point x to set A. Based on the properties characteristic for the notion of distance, every time we have $\delta(x, A) > 0$ we can conclude that point x lies outside set $\overline{A}$ (closure of set A in relation to the usual topology of space $\mathbb{R}^n$) at distance $\delta(x, A)$ from the nearest point of set A. On this account expression $\delta(x, A)$ somehow takes the role of position indicator of x towards A, but not entirely, because in the case

$\delta(x,A)=0$ the sole information provided is that $x\in A$, with no further indication of how far or how close to frontier ∂A of set A this point lies. In many concrete situations the knowledge of these details can be more useful. For example, when A symbolizes the 2-dimensional representation of a risk zone for a human (a very deep lake, a region contaminated with toxic substances etc.) and x symbolizes the position vector of a human situated in the interior of that zone, it would be useful for that human to know how close or far the exit ways are. It thus, becomes necessary the extension of the classical notion of distance by taking into account some indicators that deal with more requirements. This aspect has been pointed out by other researchers in their papers, as well, see, for example [4]. The most general point-set position indicator aiming at fulfilling the requirements formulated above is given by [2] and takes the form

$$\mathfrak{s}(x,A)=\begin{cases}\delta(x,A), & x\in\complement A\\ -\delta(x,\complement A), & x\in A\end{cases}, \tag{1}$$

where $\complement A$ represents the absolute complement of A, i.e. $\complement A=\mathbb{R}^n\setminus A$. Given the way it has been defined, indicator $\mathfrak{s}(x,A)$ has the following properties: $x\in\complement\overline{A}$ $\Leftrightarrow$ $\mathfrak{s}(x,\ A)>0$; $x\in\overset{o}{A}$[4] $\Leftrightarrow$ $\mathfrak{s}(x,\ A)<0$; $x\in\partial A$[5] $\Leftrightarrow$ $\mathfrak{s}(x,\ A)=0$, respectively.

Observations: 1) The indicator defined in expression (1) does not represent a distance; it expresses the distance from x to A only when point x is exterior to set A.
2) Other examples of point-set position indicators can be found in [1] and [4].

Point-two Sets Position Indicators

Paper [2], mentioned above, provides us with an example of a point-two sets indicator, as well. This takes the expression

$$\mathfrak{S}(x,A,B)=\frac{\mathfrak{s}(x,A)}{\mathfrak{s}(x,B)-\mathfrak{s}(x,A)}, \tag{2}$$

---

[4] $\overset{o}{A}$ represents the interior of set A in the usual topology of space $\mathbb{R}^n$.

[5] ∂A represents the boundaries of set A, namely $\partial A=\overline{A}\setminus\overset{o}{A}$.

where $x \in \mathbb{R}^n$, A and B are two sets from $\mathbb{R}^n$ with the property $\overline{A} \subset \overset{o}{B}$, and $\mathfrak{s}$ is indicator (1). This indicator has the following properties: $\mathfrak{S}(x, A, B) < -1 \Leftrightarrow x \in \complement\overline{B}$; $-1 \leq \mathfrak{S}(x, A, B) < 0 \Leftrightarrow x \in \overline{B} \setminus \overline{A}$; $\mathfrak{S}(x, A, B) \geq 0 \Leftrightarrow x \in \overline{A}$.

Observations: 1) The properties presented earlier justify the designation of "point – two sets position indicator" given to indicator (2).
2) Other examples of point – two sets position indicators can be found in [1, 4].

2. Set-set Position Indicators

This paragraph focuses on the presentation of new results aiming at further developing and improving the existent theory of Extension. The frame we shall refer to in our discussion is that of any metric space (X, d). The mathematical apparatus we would like to advance further on requires to take into consideration the notion of "Hausdorff measure". To ensure a better understanding of the concepts presented, we have synthetized the minimum of knowledge required in this regard in the appendix.

Let A and B be two non-empty sets from X. About set A we additionally assume that it admits a Hausdorff measure of dimension $r \geq 0$, $\mathcal{H}^r(A)$ is finite and nonzero. Under these conditions, by using indicator $\mathfrak{s}$ defined by the generalized relation (1) from the Euclidean metric space $\mathbb{R}^n$ for the actual metric space (X, d), we are able to consider the expression

$$\mathsf{S}(A, B) = \frac{\mathcal{H}^r\left(\{a \in A \mid \mathfrak{s}(a, B) \leq 0\}\right)}{\mathcal{H}^r(A)}, \tag{3}$$

which accurately defines the indicator we wanted to introduce.

This indicator fulfills several mathematical properties important to the applicative field:

Proposition 1: $\mathsf{S}(A, B) = 0 \Leftrightarrow A \cap B = \varnothing$ $\mathcal{H}^r$- almost everywhere (or differently expressed, $\mathcal{H}^r(A \cap B) = 0$).

Demonstration: $\mathsf{S}(A, B) = 0 \Leftrightarrow \mathcal{H}^r\left(\{a \in A \mid \mathfrak{s}(a, B) \leq 0\}\right) = 0$. Since $\mathfrak{s}(a, B) \leq 0$ is equivalent to relation $a \in \overline{B}$, we deduce that $\mathcal{H}^r(A \cap B) = 0$.

Proposition 2: $\mathsf{S}(A, B) > 0 \Rightarrow A \cap \overline{B} \neq \varnothing$.

Demonstration: $\mathsf{S}(A, B) > 0 \Leftrightarrow \mathcal{H}^r\left(\left\{a \in A \mid \mathfrak{s}(a, B) \leq 0\right\}\right) > 0$. From relation $\mathcal{H}^r\left(\left\{a \in A \mid \mathfrak{s}(a, B) \leq 0\right\}\right) > 0$ we deduce that there is $a \in A$ so that $\mathfrak{s}(a, B) \leq 0$. But $\mathfrak{s}(a, B) \leq 0$ implies that $a \in \overline{B}$, so $A \cap \overline{B} \neq \varnothing$.

Proposition 3: If besides the initial hypothesis made over set *A* and *B,* we assume additionally that set $\overline{B}$ is measurable[6] with respect to Hausdorff measure $\mathcal{H}^r$ (regarded as an outer measure on $\mathcal{P}(X)$, the family of all subsets of *X*), then relation $\mathsf{S}(A, B) = 1$ is equivalently with $A \subseteq \overline{B}$ $\mathcal{H}^r$ - almost everywhere.

Demonstration: $\mathsf{S}(A, B) = 1 \Leftrightarrow \mathcal{H}^r\left(\left\{a \in A \mid \mathfrak{s}(a, B) \leq 0\right\}\right) = \mathcal{H}^r(A)$. The way how indicator $\mathfrak{s}$ has been defined implies that $\left\{a \in A \mid \mathfrak{s}(a, B) \leq 0\right\} = A \cap \overline{B}$. Because set $\overline{B}$ is measurable we have $\mathcal{H}^r(A) = \mathcal{H}^r\left(A \cap \overline{B}\right) + \mathcal{H}^r\left(A \cap \complement\overline{B}\right)$. But $\mathcal{H}^r\left(A \cap \overline{B}\right) = \mathcal{H}^r\left(\left\{a \in A \mid \mathfrak{s}(a, B) \leq 0\right\}\right) = \mathcal{H}^r(A) \Rightarrow \mathcal{H}^r\left(A \cap \complement\overline{B}\right) = 0$, namely $A \subseteq \overline{B}$ $\mathcal{H}^r$ - almost everywhere.

Corollary: Let A and B be two closed nonempty sets from X for which there exists a Hausdorff measure of dimension $r \geq 0$ so that $\mathcal{H}^r(A)$ and $\mathcal{H}^r(B)$ are finite and nonzero. If sets A and B are $\mathcal{H}^r$ - measurable and if $\mathsf{S}(A,B) = 1$ and $\mathsf{S}(B,A) = 1$, then $A = B$ $\mathcal{H}^r$ - almost everywhere, and reciprocally.

Demonstration: From proposition 3 it results that $\mathsf{S}(A, B) = 1 \Leftrightarrow A \subseteq B$ $\mathcal{H}^r$ - almost everywhere and $\mathsf{S}(B, A) = 1 \Leftrightarrow B \subseteq A$ $\mathcal{H}^r$ - almost everywhere. Then $A = B$ $\mathcal{H}^r$ - almost everywhere.

Observations: 1) From the definition of indicator S, (relation (3)) it can be easily deduced that $0 \leq \mathsf{S}(A,B) \leq 1$, for any pair of non-empty subsets A and B of

[6] By definition we say that set $\overline{B}$ is $\mathcal{H}^r$ measurable if for any $T \subseteq X$ relation $\mathcal{H}^r(T) = \mathcal{H}^r\left(T \cap \overline{B}\right) + \mathcal{H}^r\left(T \cap \complement\overline{B}\right)$ takes place.

space X for which set A admits a Hausdorff measure $\mathcal{H}^r$ of dimension $r \geq 0$, so that $\mathcal{H}^r(A) \neq 0$ and $\mathcal{H}^r(A) < \infty$.
2) The properties presented earlier within propositions 1 - 3 aim at justifying the designation of "set-set position indicator" that indicator (3) receives.
3) Another example of set-set position indicator can be found in [3].

3. Applications

Indicator S defined by us in this paper can be used as an example for computer vision while developing software applications regarding the automatic inclusion of a certain object $\mathcal{O}$ into a target region $\mathcal{R}$ of a given video image (*VIm*). To realize this, we propose an algorithm which in broad terms has the following content: by means of a set of isometries $\mathcal{I}_i$, $i \in I$ of the plan, we move object $\mathcal{O}$ to different regions and positions of image *VIm* by calculating the value of indicator $\mathsf{S}\left(\mathcal{I}_i(\mathcal{O}), \mathcal{R}\right)$, each time. Finding that index $i_0 \in I$ for which $\mathsf{S}\left(\mathcal{I}_{i_0}(\mathcal{O}), \mathcal{R}\right) = 1$, is equivalent to finding the solution to the problem.

Observations: 1) In some cases solution $\mathcal{I}_{i_0}$ found by using the method presented above can not be fully satisfactory because relation $\mathcal{I}_{i_0}(\mathcal{O}) \subseteq \mathcal{R}$, concerned in $\mathsf{S}\left(\mathcal{I}_{i_0}(\mathcal{O}), \mathcal{R}\right) = 1$, (see proposition 3, applied in the case when the sets by which objects $\mathcal{O}$ and $\mathcal{R}$ are being abstractized, are supposed to be compact[7]) is only guaranteed by $\mathcal{H}^r$ - almost everywhere.
2) Just like the algorithm presented in [3], this algorithm can be easily adapted to solving any similar problem in a space with three dimensions, becoming, thus, more useful to the field of designing forms with artificial intelligence.

4. Appendix

Hausdorff measure

Let (X, d) be a metric space, Y a subset from X, and δ a strictly positive real number. A finite or countable collection of sets $\{U_1, U_2, ...\}$ from X with diameter $D(U_1) \leq \delta$, $D(U_2) \leq \delta$,.., for which $Y \subset U_1 \cup U_2 \cup \cdots$, is called δ-cover of set Y. By virtue of this notion, for any subset Y from X and for any two real numbers $r \geq 0$, and $\delta > 0$, we can define indicator

[7] Given the application we analyze, these hypotheses are as natural as possible

$$\mathcal{H}_\delta^r(Y)=\inf_{\{U_1,\,U_2,\ldots\}\in\mathcal{C}_\delta(Y)}\left\{D^r(U_1)+D^r(U_2)+\cdots\right\},$$

where $\mathcal{C}_\delta(Y)=\left\{\{U_1,\,U_2,\ldots\}\mid\{U_1,\,U_2,\ldots\}\text{ is a }\delta-\text{cover of }Y\right\}$. This indicator defines a decreasing function $\delta\to\mathcal{H}_\delta^r(Y)$. This property guaranties the existence of the limit

$$\mathcal{H}^r(Y)=\lim_{\delta\to 0}\mathcal{H}_\delta^r(Y),$$

which, by definition, is called the *r*-dimensional Hausdorff measure of Y.

Among the properties of the Hausdorff measure, $\mathcal{H}^r(\cdot)$, we mention:

1) $\mathcal{H}^r(\varnothing)=0$;

2) $\mathcal{H}^r(Y_1)\le\mathcal{H}^r(Y_2)$, if $Y_1\subseteq Y_2$, $Y_1,\,Y_2\in\mathcal{P}(X)$;

3) $\mathcal{H}^r\left(\bigcup_{n=1}^{\infty}Y_n\right)\le\sum_{n=1}^{\infty}\mathcal{H}^r(Y_n)$, if $\{Y_n\mid n\in\mathbb{N}^*\}\subset\mathcal{P}(X)$, is any countable collection of sets.

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Inteligencia Artificial: retos, perspectivas y papel de la Neutrosofía

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RESUMEN: La Inteligencia Artificial ha avanzado hasta el punto en que tiene el poder de provocar una nueva revolución industrial. La neutrosofía por su parte es una nueva rama de la filosofía que estudia el origen, la naturaleza y el alcance de las neutralidades. Esto ha formado la base de una serie de teorías matemáticas que generalizan las teorías clásicas y difusas, como la lógica neutrosófica. En el presente trabajo se analizan los avances de la inteligencia artificial así como los retos que esta presenta. Además, se definirán los conceptos fundamentales de la inteligencia artificial y cómo la neutrosofía ha llegado a fortalecer esta disciplina.

PALABRAS CLAVES: Inteligenca Artificial, cognificación, neutrosofía, lógica neutrosófica, conjuntos neutrosóficos.

TITLE: Artificial intelligence: challenges, perspectives and neutrosophy role.

ABSTRACT: Artificial Intelligence has advanced to the point where it has the power to provoke a new industrial revolution. Neutrophics, on the other hand, is a new branch of philosophy that studies the origin, nature and scope of neutralities. This has formed the basis of a series of mathematical theories that generalize the classical and diffuse theories, such as the neutrosophical logic. In the present work, the advances of the artificial intelligence are analyzed as well as the challenges that this presents. In addition, the fundamental concepts of artificial intelligence and how neutrosophy has come to strengthen this discipline will be defined.

KEY WORDS: Artificial intelligence, cognification, neutrosophy, neutrosophic logic, neutrosophic sets.

INTRODUCCIÓN.

Hace aproximadamente 100 años, la electricidad transformó todas las industrias importantes. Hoy la Inteligencia Artificial (IA) ha avanzado hasta el punto en que tiene el poder de transformarlas de nuevo y provocar una nueva revolución industrial.

La Inteligencia Artificial ha llegado más allá de la ciencia ficción, actualmente es parte de nuestra vida cotidiana, desde el uso de un asistente personal virtual para organizar nuestra agenda, hasta que nuestros teléfonos sugieran canciones que nos pueden gustar. Más allá de facilitar nuestras vidas, los sistemas inteligentes nos están ayudando a resolver algunos de los mayores desafíos del mundo: tratar enfermedades crónicas, luchar contra el cambio climático y anticipar las amenazas meteorológicas. La IA es una de las tecnologías más estratégicas del siglo XXI, y con su llegada, se crearán numerosos puestos de trabajo, pero otros desaparecerán y la mayoría sufrirá transformaciones.

La inteligencia Artificial (IA) se puede catalogar, además, como la próxima carrera espacial. El desarrollo de la IA ha llegado a un punto crítico. Cualquier país que hace un avance en su aplicación puede tener una mayor oportunidad de liderar el mundo.

La neutrosofía, por su parte, es una nueva rama de la filosofía (Smarandache, 2002), la cual estudia el origen, naturaleza y alcance de las neutralidades, así como sus interacciones con diferentes espectros ideacionales: (A) es una idea, proposición, teoría, evento, concepto o entidad; anti (A) es el opuesto de (A); y (neut-A) significa ni (A) ni anti (A); es decir, la neutralidad entre los dos extremos (Bal, Shalla, & Olgun, 2018). Etimológicamente, neutron-sofía [Frances neutre < Latin neuter, neutral, y griego sophia, conocimiento] significa conocimiento de los pensamientos neutrales y comenzó en 1995. La neutrosofía está teniendo un impacto importante en la inteligencia artificial, y en especial, en el tratamiento de la incertidumbre.

En el presente trabajo se analizan los avances de la inteligencia artificial así como los retos que esta presenta. Adicionalmente, se discuten los conceptos fundamentales de la neutrosófica y su papel dentro de la inteligencia artificial.

La inteligencia artificial.

El término inteligencia artificial (IA) se le acredita a John McCarthy de la universidad de Stanford en una conferencia llevada a cabo en 1956 (Alandete, 2011). La IA surge a partir de algunos trabajos publicados en la década de los años 40, pero no es hasta el trabajo del científico Inglés Alan Turing en el año 1950 (A. Turing, 1950), que ocurre un despegue en la disciplina. El Test de Turing (A. M. Turing, 1950) es uno de los criterios de vida mental más debatidos y polémicos desde el punto de vista filosófico relacionado a la Inteligencia Artificial. Turing plantea, que si la máquina logra convencer a los jueces humanos, resulta justificado creer que es inteligente y pensante, debido a su capacidad para suplantar a humanos mediante comportamiento lingüístico (González, 2007) .

Inicialmente predominó el enfoque simbólico (Flasiński, 2016), que tuvo éxito comercial en los sistemas expertos (Heckerman, Horvitz, & Nathwani, 2016) de la década de los años 80, pero este sufrió debido a las limitantes que presenta en adquirir y actualizar el conocimiento. A mediados de los años 80, se inicia con éxito el enfoque basado en el aprendizaje automático (Witten, Frank, Hall,

& Pal, 2016), y en la actualidad, el éxito ha estado en los modelos de aprendizaje profundo (Vázquez, Jara, Riofrio, & Teruel, 2018) (Figura 1).

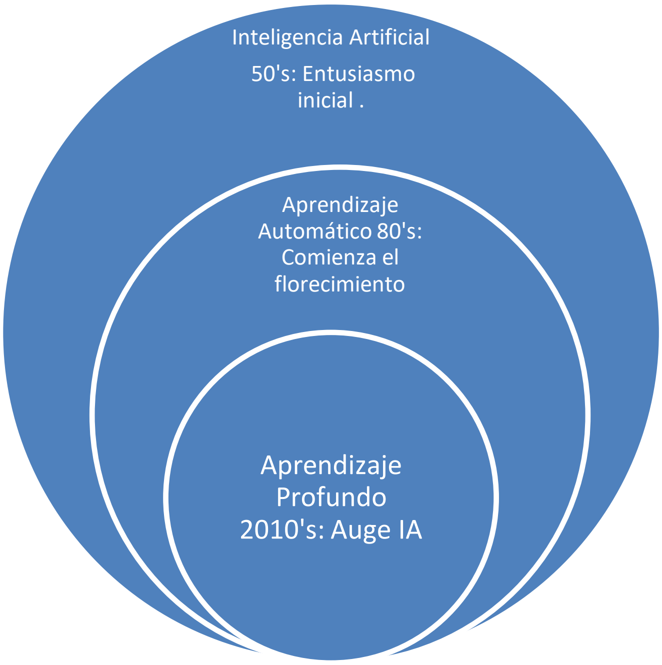

Figura 1. Principales hitos en la evolución de la IA.

El propósito fundamental de la inteligencia artificial puede ser definido como la creación de agentes racionales (S. Russell, Dewey, & Tegmark, 2015), que perciben el ambiente y toman decisiones para maximizar las oportunidades de alcanzar determinadas metas.

Existen dos enfoques hacia la inteligencia artificial. La denominada Inteligencia Artificial General (Goertzel & Pennachin, 2007) aspira a crear la capacidad de resolver tareas generales en las maquinas, tales como pensar y actuar semejante a la mente humana. Este enfoque ha sido muy bien representado en la ciencia ficción, como es el caso las Película Her y Ex-máquina. La otra, denominada Inteligencia Artificial Estrecha o IA débil (S. J. Russell & Norvig, 2016) pretende crear la capacidad en la máquinas de realizar tareas específicas, extremadamente bien, incluso superando a los humanos. Este enfoque está teniendo incontables éxitos y aplicaciones prácticas.

La inteligencia artificial no es un campo monolítico sino que está dividido en varias ramas (Figura 2), tales como: el aprendizaje automático, el procesamiento del lenguaje natural, los sistemas expertos, la visión por computadoras, el reconocimiento automático del habla, la planificación y la robótica (Vázquez et al., 2018).

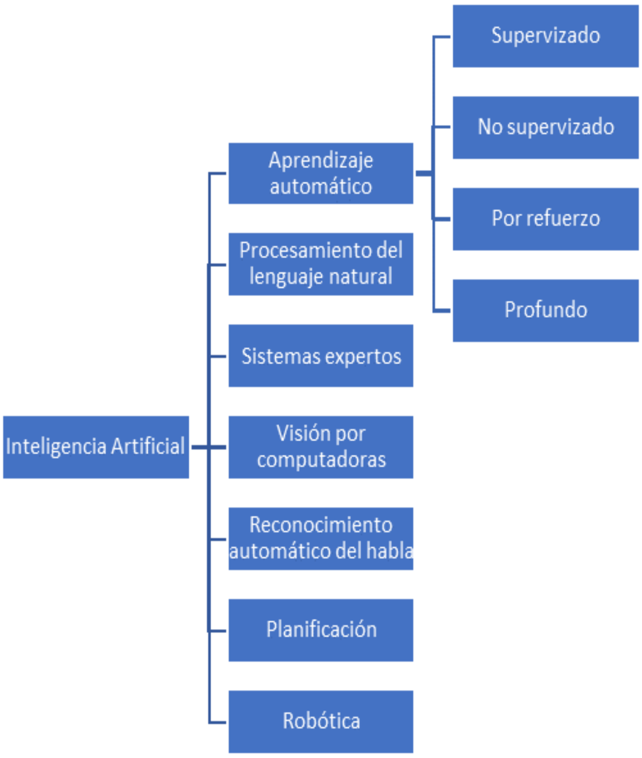

Figura 2. Ramas de la Inteligencia Artificial.

Cuando nos referimos a los agentes, puede tratarse de un Robot si este fuera un agente de hardware o a un Softbot (o simplemente bot) si fuera un agente de software. Entre los agentes que están teniendo un alto impacto encontramos a los chatbots y los asistentes personales. Un chatbot es un bot que interactúa con usuarios a través de un canal de mensajería, mientras que un asistente personal inteligente es un bot que puede realizar determinadas tareas u ofrecer servicios. Entre estos se encuentra: Cortana, Siri, Google Now y Alexa (Rabelo, Romero, & Zambiasi, 2018).

Los agentes conversacionales responden a guiones predeterminados de diálogo, y los agentes virtuales responden a preguntas más complejas; adicionalmente, los primeros son distribuidos fundamentalmente por aplicaciones de mensajería. Los chatbots, por su parte, pueden ser definidos como robots que interactúan con usuarios a través de un chat, simulando ser un operador o una persona en tiempo real; excelentes para optimizar la experiencia del usuario, gestionar pedidos y resolver sus necesidades (McTear, Callejas, & Griol, 2016). Un agente virtual, por su parte, es un asistente personal inteligente que puede realizar tareas u ofrecer servicios a un individuo generalmente controlados mediante la voz (Pant, 2016).

Un elemento importante en la Inteligencia Artificial es el aprendizaje automático. El aprendizaje automático es una rama de la Inteligencia Artificial que tiene como objetivo lograr que las computadoras aprendan. Existen 5 paradigmas fundamentales de la aprendizaje automático (Domingos, 2015):

1. Algoritmos evolutivos.
2. Conexionismo y redes neuronales.
3. Simbolismo.
4. Redes bayesianas.
5. Razonamiento por analogía.

El aprendizaje automático (Witten et al., 2016) tiene como propósito dotar a las computadoras de la capacidad de aprender sin haber sido explícitamente programadas para ello; sin embargo, presenta imitaciones en cuanto a la necesidad de realizar la selección de variables a procesar y en cómo escalan en presencia de grandes volúmenes de datos tales como imágenes y videos.

Un nuevo enfoque toma fuerza actualmente denominado: aprendizaje profundo, dónde "profundo" (LeCun, Bengio, & Hinton, 2015) se refiere al número de capas ocultas en una red neuronal artificial. Este nuevo enfoque revitalizó las redes neuronales y ha sido empujado por 3 grandes catalizadores:

- Hardware, con la disponibilidad de GPU (Unidades de Procesamiento Gráfico, por sus siglas en inglés).
- Datos, debido a la alta disponibilidad de datos que existe actualmente que permite alimentar los modelos.
- Algoritmos, debido a los avances en el aprendizaje en redes de múltiples capas.

El enfoque del aprendizaje profundo ha permitido importantes éxitos en ramas donde antes no se obtenían buenos resultados aplicables en la práctica, como por ejemplo: la visión por computadoras (Ioannidou, Chatzilari, Nikolopoulos, & Kompatsiaris, 2017) y la traducción automática de texto (Young, Hazarika, Poria, & Cambria, 2018).

En este tipo de algoritmos no se requiere la intervención humana para la selección de las variables; sin embargo, una de sus principales limitaciones está en su interpretabilidad, ya que funciona como cajas negras (Shwartz-Ziv & Tishby, 2017). A pesar de ello, ya se han obtenido algunos resultados promisorios para lograr la interpretación de estos modelos (Samek, Wiegand, & Müller, 2017).

El aprendizaje profundo tiene y tendrá, sin lugar a dudas, un gran impacto en la sociedad en áreas tales como la creación de vehículos autónomos, el diagnóstico médico, el marketing, etc. El auge de la Inteligencia Artificial ha llevado a que aumente el número de actividades en que las computadoras han superado a los humanos, tales como el diagnóstico médico, juegos por computadora (Ajedrez, Trivia, Go, etc.), entre otras. Esto ha llevado a incrementar los temores en relación a la pérdida de empleos (Figura 3). En algunos sitios se puede calcular el riesgo de ser sustituido por robots en el trabajo. Este mensaje del miedo se impone cuando se cree erróneamente que las máquinas arrebatarán el trabajo; sin embargo, los estudios y estadísticas señalan todo lo contrario.

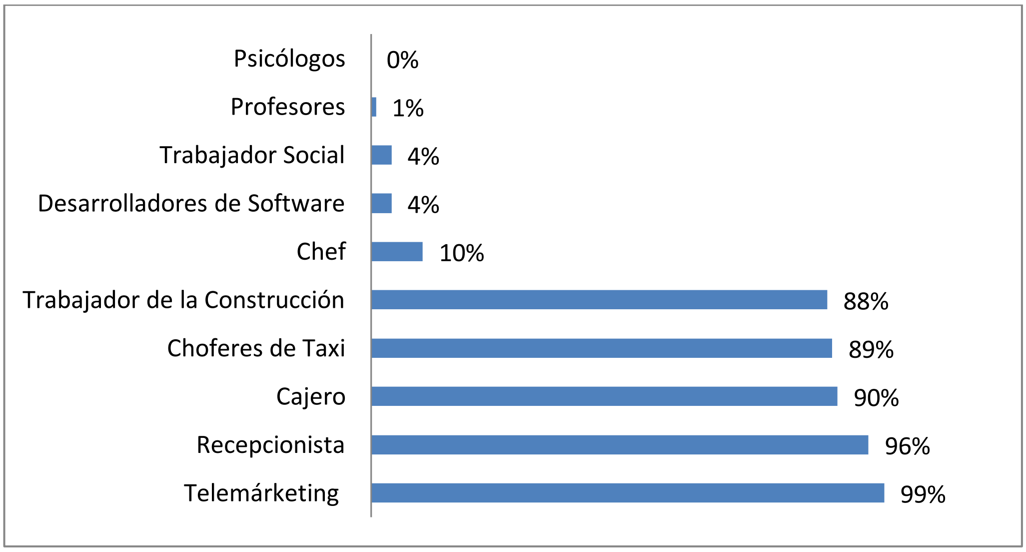

Figura 3. Probabilidad que tienen distintas profesiones de ser sustituidas por robot/IA en los próximos 20 años. Fuente: www.willrobotstakemyjob.com

El mensaje del miedo se impone cuando llega el mensaje de que las máquinas nos arrebatarán el trabajo; sin embargo las estadísticas también señalan que la robótica y la IA son responsables de la creación de muchos puestos de trabajo.

La consultora Metra Martech (Seitz, 2017) señala, que los robots actualmente en operaciones han sido responsables de la creación de al menos 8 millones de empleos, a los que se sumarán un millón más en los próximos años. Se crean más puestos de trabajo de los que destruyen. Se necesitarán nuevas profesiones; por ejemplo: las bases de datos usadas para entrenarlos deben carecer de sesgos. El chatbot Tay, en menos de un día y a través de su interacción con trolls adquirió tendencias de extrema derecha, antifeministas y transfóbicas, por lo que Microsoft lo desconectó. Harán falta especialistas en 'educación' de IA para eliminar este tipo de problemas.

Existe otro fenómeno del cual el llamado Ajedrez Centauro constituye su paradigma. Este surgió en el año 1998 a partir de la derrota del campeón mundial de ajedrez Kasparov por Deep Blue. Es la combinación entre la inteligencia humana y artificial a través del uso y combinación de hardware, software y bases de datos. Bajo este enfoque, una persona normal + una máquina es superior a un único ordenador potente. La inteligencia artificial es mejor en la búsqueda de respuestas y los humanos en la realización de preguntas; en ese sentido, resultan complementarios.

Adicionalmente, la inteligencia humana no es unidimensional y se encuentra exteriorizada, en gran medida, no en nuestro cerebro sino en nuestra civilización. Desde ese punto de vista, la IA es una herramienta cognitiva (Figura 4), que puede ayudar a las personas a ser mejores médicos, jueces, maestros, etc.

Figura 4. Ciclo de las herramientas cognitivas.

Otro fenómeno es reflejado por el término Cognificación (agregar conocimiento con inteligencia artificial a nuestros diseños) (Figura 5) acuñado por Kevin Kelly, el que consiste en que aún una pequeña cantidad de inteligencia artificial -barata (o gratis), poderosa y ubicua- embebida dentro de un proceso ya existente, lleva su efectividad hacia todo un nuevo nivel (Juncu, 2017) .

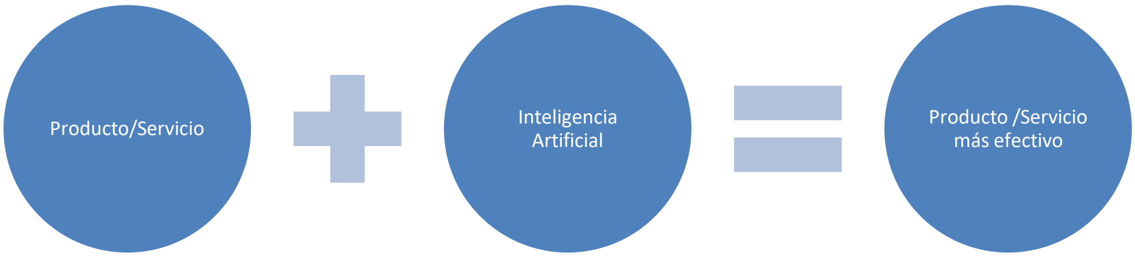

Figura 5. Proceso de Cognificación.

Actualmente, existe una carrera por el dominio de la inteligencia artificial entre las compañías tecnológicas, entre ellas Google y Facebook, en las cuales han entrado con fuerza las compañías chinas como Baidu. A nivel de países, se destaca Estado Unidos y China, y la iniciativa de la Unión Europea liderada por Francia con la iniciativa del presidente francés Emmanuel Macro: “IA para la humanidad” (Olhede & Wolfe, 2018).

Neutrosofía.

La neutrosofía es una nueva rama de la filosofía (Smarandache, 2002), la cual estudia el origen, naturaleza y alcance de las neutralidades, así como sus interacciones con diferentes espectros ideacionales: (A) es una idea, proposición, teoría, evento, concepto o entidad; anti (A) es el opuesto de (A); y (neut-A) significa ni (A) ni anti (A); es decir, la neutralidad entre los dos extremos (Bal et al., 2018). Etimológicamente, neutron-sofía [Frances neutre < Latin neuter, neutral, y griego sophia, conocimiento] significa conocimiento de los pensamientos neutrales y su inicio se reporta en el año 1995 (Vázquez & Smarandache, 2018).

Su teoría fundamental afirma que toda idea < A > tiende a ser neutralizada, disminuida, balaceada por <noA>, las ideas (no solo <antiA> como Hegel (2017) planteó)- como un estado de equilibrio.

<noA> = lo que no es <A>,

<antiA> = lo opuesto a <A>, y

<neutA> = lo que no es <A> ni <antiA>.

En su forma clásica <A>, <neutA>, <antiA> son disjuntos de dos en dos.

Como en varios casos, los límites entre conceptos son vagos e imprecisos, es posible que <A>, <neutA>, <antiA> (y <nonA> por supuesto) tengan partes comunes de dos en dos.

Esta teoría ha constituido la base para la lógica neutrosófica (Smarandache, 1999), los conjuntos neutrosóficos (Haibin, Smarandache, Zhang, & Sunderraman, 2010), la probabilidad neutrosófica, y la estadística neutrosófica, y multiples aplicaciones prácticas (Smarandache, 2003).

Se propuso el término "neutrosófico", porque "neutrosófico" proviene etimológicamente de la "neutrosofía", que significa conocimiento del pensamiento neutro, y este tercer / neutral representa la distinción principal; es decir, la parte neutra/indeterminada/desconocida (además de la "verdad"/ "pertenencia" y "falsedad", componentes de "no pertenencia" que aparecen en la lógica borrosa/ conjunto). NL es una generalización de la lógica difusa de Zadeh (LD), y especialmente de la lógica difusa intuitiva (LDI) de Atanassov, y de otras lógicas multivaluadas (Figura 6).

.

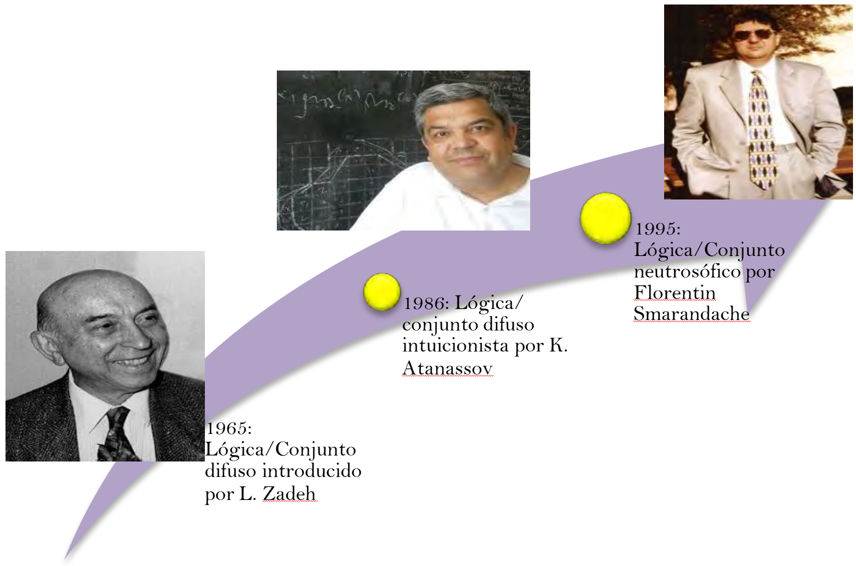

Figura 6. Neutrosófica y sus antecedentes fundamentales.

Sea U un universo de discurso, y M un conjunto incluido en U. Un elemento x de U se anota con respecto al conjunto M como x (T, I, F) y pertenece a M de la siguiente manera: es t % verdadero en el conjunto, i% indeterminado (desconocido) en el conjunto, y f% falso, donde t varía en T, i varía en I y f varía en F. Estáticamente, T, I, F son subconjuntos, pero dinámicamente T, I, F son funciones/operadores que dependen de muchos parámetros conocidos o desconocidos.

Los conjuntos neutrosóficos generalizan el conjunto difuso (especialmente el conjunto difuso e intuicionista), el conjunto paraconsistente, el conjunto intuitivo, etc, pero permite manejar un mayor número de situaciones que se dan en la realidad (Figura 7).

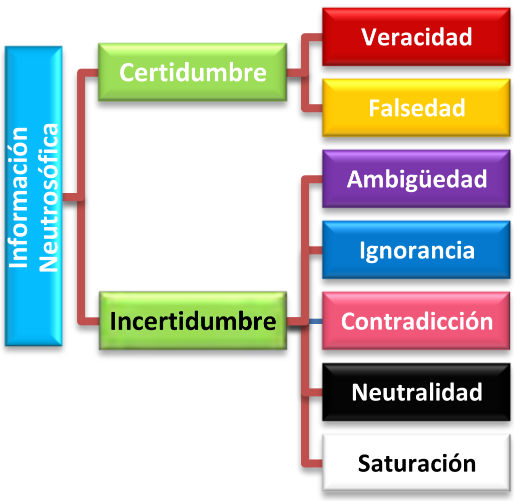

Figura 7. Estructura de la información neutrosófica.

Es en este campo de la representación de la incertidumbre en que la neutrosofía ha realizado aportes fundamentales a la IA. Adicionalmente, y de forma general, la neutrosofía construye un campo unificado de la lógica para un estudio transdisciplinario que traspase las fronteras entre las ciencias naturales y sociales. La neutrosofía trata de resolver los problemas de indeterminación que aparecen universalmente, con vistas a reformar las ciencias actuales, naturales o sociales, con una metodología abierta para promover la innovación (Smarandache & Liu, 2004).

CONCLUSIONES.

Hoy, la Inteligencia Artificial (IA) ha avanzado hasta el punto en que tiene el poder de provocar una nueva revolución industrial. La neutrosofía, por su parte, es una nueva rama de la filosofía, la cual estudia el origen, naturaleza y alcance de las neutralidades. Esta ha formado las bases para una serie de teorías matemáticas que generalizan las teorías clásicas y difusas tales como los conjuntos neutrosóficos y la lógica neutrosófica. En el trabajo se presentaron los conceptos fundamentales relacionados con la neutrosofía y sus antecedentes. Adicionalmente, se definieron conceptos fundamentales de la inteligencia artificial y cómo la neutrosofía ha venido a fortalecer esta disciplina y el tratamiento de la incertidumbre en la inteligencia artificial.

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Smart mobile application to recognize tomato leaf diseases using Convolutional Neural Networks

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Azeddine Elhassouny, Florentin Smarandache (2019). Smart mobile application to recognize tomato leaf diseases using Convolutional Neural Networks. IEEE/ICCSRE2019, 22-24 July, Agadir, Morocco, 4

***Abstract*—The automatic identification and diagnosis of tomato leaves diseases are highly desired in field of agriculture information. Recently Deep Convolutional Neural networks (CNN) has made tremendous advances in many fields, close to computer vision such as classification, object detection, segmentation, achieving better accuracy than human-level perception. In spite of its tremendous advances in computer vision tasks, CNN face many challenges, such as computational burden and energy, to be used in mobile phone and embedded systems. In this study, we propose an efficient smart mobile application model based on deep CNN to recognize tomato leaf diseases. To build such application, our model has been inspired from MobileNet CNN model and can recognize the 10 most common types of Tomato leaf disease. Trained on tomato leafs dataset, to build our application 7176 images of tomato leaves are used in the smart mobile system, to perform a Tomato disease diagnostics.**

***Index Terms*—Convolutional Neural networks, Tomato disease detection, Smart Mobile application, MobileNet.**

I. Introduction

Even if the tremendous effort deployed by the world to decreases the plant loss and food security, several references [1], [2], confirm that more than 20% of crop losses in global scenario is due to plant diseases. This issue has become serious in recent decade due to the impact of pollution and climate change. With recent development in various farming technologies, farmers opt for plant diseases databases or consult local pathologists through phones, instead of the classical procedure to send the plants to diagnostic laboratory in order to propose the appropriate treatment. Furthermore, there are many attempts to use ICT tools to improve efficiency of agricultural development, taking advantage of the wide use of mobile phones.

Regarding plant disease detection, there are many papers introduced this application using one of standard design architectures of CNN [3] such as SqueezeNet [4], ResNeXt (Aggregated Residual Transformations for Deep Neural Networks) [5], ResNet (Deep Residual Learning for Image Recognition) [6], NiN (Network In Network) [7], GoogLeNet [8], VGGNet [9], ZFNet [10], AlexNet [11], etc. Numerous techniques and applications have been made up to reduce crop loss because of diseases. But a few methods and applications have been proposed to identify plants diseases in general and Tomato diseases using Convolutional Neural Networks in particular. All related methods proposed below are developed and works on computer with powerful computation resources.

[12] proposed a CNN method to identify Tomato disease based on wavelengths and RGB channels. The authors demonstrated that there is a relationship between specific wavelet color sensitivity of each disease. [13] introduced a Plant Species Recognition method that is different from the existing feature extraction based Deep Convolutional Neural Networks recognition approaches. [14] built an intelligent alerting system based on CNN for recognition for disease-pest of fruit-melon. [15] used CNNs to build a robust diagnostic system for all possible viral diseases for cucumber. [16] trained tow CNNs Models AlexNet and GoogleNet to identify 14 crop species and 26 diseases. A dataset of 7176 images of diseased and healthy plant leaves, have been used. It was the first attempt has paved the way for smartphone-assisted disease diagnosis, even if the both models used AlexNet and Google Net didn't work on Mobile phone.

All existing leaf plant diseases detection methods rely on the use of computer with power of calculation. One drawback of these methods is unable to be used in Mobile and embedded system with minimum resources of calculation. Although the number of researches made in this field, all smart systems have been built to detect plant diseases, based on the recent deep CNN, are mostly limited to the use on computer with a large storage capacity and resources computation. But with the development of Quantized CNN Models embed of smart application on mobile phone with limit resources become possible. Thus, in this work, we propose an embedded smart application to recognize tomato leaves diseases. Due to the impact of smartphones and mobile devices on human life, it has become necessary to made up an accurate, easy and inexpensive automated diagnostic system for plant diseases. Until now, there is no smart mobile application detecting Tomato diseases in particularly.

The remainder of this paper is organized as follows. In Section 2 , we present some related works. Section 3 introduces the adapted MobileNet architecture and learning algorithm of CNNs. The Dataset, experimentation and discussion of the developed CNN model to recognize the Tomato leaves diseases are given in Section 4. Conclusions and perspectives are presented in Section 5.

A. Advances in CNN Architectures: Embedded and mobile systems ConvNets

The application of deep convolutional networks in embedded and mobile systems is limited by algorithm's power consumption, bandwidth requirements, tight energy budgets, requirement of very low latency and limited processing resources [17], [18]. To solve these challenges, many algorithms are proposed: [17] introduced a fast CNN's based on Winograd's minimal filtering algorithms (Winograd's algorithm), which a generalization of minimal filtering algorithms of Toom and Cook. However [18] proposed a combination of ASP sensor with CNN backend Angle Sensitive Pixels (ASP) Vision. ASPs is a bio-inspired CMOS(complementarymetaloxide semiconductor) image sensors, that have Gabor wavelet impulse responses, and perform optical convolution for the CNN first layer. The idea is based on the hardcoding of first layer of CNNs to lead to significant energy savings.

[19] introduced Quantized-CNN for Mobile Devices is a framework of convolutional neural network (CNN), that achieves $3.03\times$ speed-up against the Caffe implementation.

II. Adaptation of MobileNet

The based CNN's Model can work on mobile phone with acceptable speed and obtain a high classification accuracy. The principle contribution of this work is : (1) The CNN model of MobileNet is first adapted and applied to the question of diagnostic of Tomato leaves diseases. The built application can identify and recognize the 10 most common types of Tomato leaves diseases. (2) The test and the use of the application show that MobileNet CNN model, in addition to the high recognition ratio, works speedily on popular mobile phone using its own camera. Below we described the CNN MobileNets Model.

1) MobileNets: A team of Google researchers [20] introduced an efficient models named MobileNets for mobile and embedded vision applications. The MobileNet model consists on depthwise separable convolutions, which reduces between 8 to 9 times less computation than standard convolutions, and then smaller and faster MobileNets using width multiplier and resolution.

A. Brief description of adapted model

MobileNets is based on the both concepts depthwise separable filters and factorization thing reduces the amount of computation in the first layers. To reduce the computation, the convolution is factorized into a depthwise that applies each filter to each channel and then pointwise which uses a 1×1 convolution to combine the depthwise.In addition to the both operations, MobileNets use ReLU activation function for both layers. Here we compare the cost of computation between the use of standard convolution and depthwise technique. The fact that MobileNet uses 3×3 depthwise separable convolutions, reduces the computation 9 times less. Let consider that W is the size of input image with M the input channels, F ($D_k \times D_k$) is size of the kernel convolution (generally are 3×3 or 1×1), N is the number expected output channels, s is the stride of convolution (that is set to 1 or 2), P is usually the used padding, then we get the tensor (image after convolution), $D_K \times D_K \times M \times D_F \times D_F$ instead of the Standard convolution $DK \times DK \times M \times N \times DF \times DF$ convolution), $D_K \times D_K \times N \times M \times D_F \times D_F$.

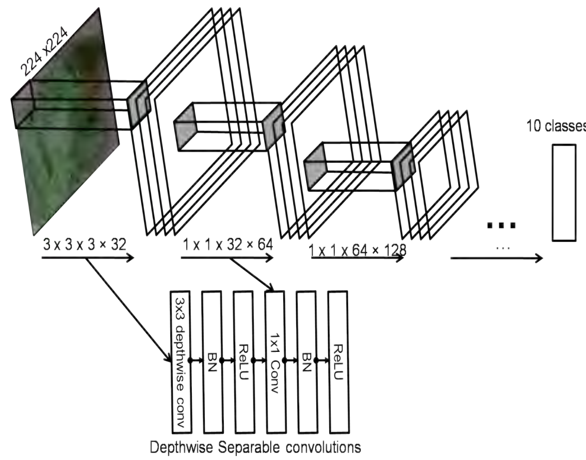

Fig. 1. The structure of deep convolutional neural networks

Architecture of deep convolutional neural networks model Inspired by successful model in mobile and embedded systems that is MobileNet CNN architecture.

The figure 1 illustrates the details of CNNs-based architecture and its parameters. The model accepts an input color image of size 224×224. it consists on a series of Depthwise Separable convolutions with Depthwise and Pointwise layers followed by batchnorm(BN) and ReLU. One layer correspond to 3×3 Depthwise Conv, BN, 1×1 Convolution, BN and ReLU activation function. To get classification of diseases, the model is ended by average Pooling of pooling layers, full connected and the Softmax function with 10 classes.

III. Dataset and experimentation

To build our application we have used a dataset of images on plant health, a 7176 images of tomato leaves are used to train the model, then made up a smart mobile system able to perform a Tomato disease diagnostics.

The figure 2 below shows Tomato leaves with the most common kinds of diseases and Table 1 presents images number of each class.

We have tested our application on some test images example, and we have got the results on below table (II). Our results of recognition of the excepted diseases are got with significant gaps compared to others diseases. for example, we get Fungi leaf mold passalora Fulvia-fulva disease with 0.99% compared to others diseases as shown in the table(II). The reason behind the larger gap, maybe due to the big difference of appearance between diseases.

Figures 3 and 4 show the average loss and accuracy curves of that CNN model.

We demonstrate, as shown in Table IV, that our adapted model based on MobileNets model might achieve a encouraged results on tomato diseases, which achieves using

(a) Bacterial Spot Xanthomonas campestris

(b) Early Blight Alternaria solani

(c) Late Blight Phytophthora infestans

(d)Leaf Mold Passalora fulva

(e) Septoria Leaf Spot Septoria lycopersici

(f) Two Spotted Spider Mite Tetranychus urticae

(g)Target Spot Corynespora cassiicola

(h)Mosaic Virus

(k) Yellow Leaf Curl Virus

(l) Tomato leaf healthy

Fig. 2. Tomato leaves with diseases

Class(disease type)	Number of images
Bacterial Spot Xanthomonas campestris	793
Early Blight Alternaria solani	406
Late Blight Phytophthora infestans	727
Leaf Mold Passalora fulva	361
Septoria Leaf Spot Septoria lycopersici	735
Two Spotted Spider Mite Tetranychus urticae	721
Target Spot Corynespora cassiicola	548
Mosaic Virus	140
Yellow Leaf Curl Virus	2101
Tomato leaves healthy	644
Total	7176

TABLE I
IMAGES NUMBER OF EACH CLASS

Stochastic gradient descent optimizer a 88.4% accuracy on classification.

The application works on real time using a standard phone with its own camera, even if until now, automatic plant diseases recognition on mobile devices has not used.

There are many ways to increase the overall accuracy of a CNN Model. Our adapted CNN Model have tested with many

Image example	Obtained result	Expected results
	- leaf mold passalora fulva 0.99999 - septoria leaf spot septoria lycopersici 5.87832e-06 - mosaic virus 2.34571e-06 - late blight phytophthora infestans 1.02782e-06 - early blight alternaria solani 2.06558e-07	- leaf mold passalora fulva

TABLE II
TOMATO SAMPLE TEST IMAGE WITH PREDICTED AND EXPECTED RESULTS

Fig. 3. Average accuracy curve

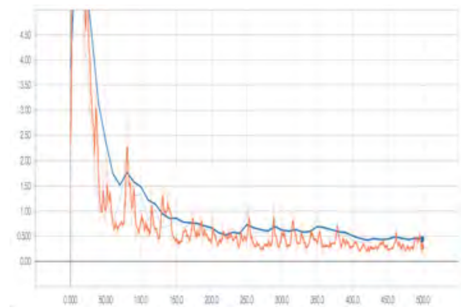

Fig. 4. Average loss curve

optimization algorithms such as Stocastic gradient descent, adadelta optimizer, adagrad, adagradDA, Momentum, Adam, Ftrl, proximaladagrad and RMSprop optimizers. Some of those results figure out in table III.

Optimization method	Final test accuracy
Stocastic gredient descent	88.4%
Adagrad	88.3%
SGD with Momentum	87.6%
Adam	88.5%
Proximal Gradient Descent	89.2%
Proximal Adagrad	88.8%
RMSProp	85.9%

TABLE III
TOMATO DISEASES RECOGNITION PERFORMANCE WITH RESPECT OF OPTIMIZATION ALGORITHM

In addition our adapted model have tested too with many training steps and with changing many configuration parameters. For example, we have adjusted learning rate, that while its value is decreased, the training will take longer, but improve the overall accuracy as shown in table 4.

learning rate	0.01	0.005	0.001
Accuracy	86.7%	88.9%	90.3%

TABLE IV
TOMATO DISEASES RECOGNITION PERFORMANCE IN FUNCTION OF VALUES OF LEARNING RATE

IV. CONCLUSION

Our contribution and results while being a new using of mobile device on farms to recognize the plants diseases based on deep convolutional neural networks, compared to the previously published, is still preliminary, and deriving from a rather straightforward adaptation of MobileNets goolge Model. Deep convolution neural networks have achieved great performance breakthroughs in machine learning fields, but there still exist some research challenges. The proposed CNN based model can effectively classify 10 common tomato leaves diseases through image recognition. We can extend the model for fault diagnosis. To improve tomato diseases identification accuracy, we still need to provide thousands of high-quality tomato diseases images samples. We believe that the simple use of deep convolutional neural networks in computer vision and its applications, specially smart mobile plant diseases recognition is going to decrease the lack of food because of diseases, so increase production and save life of so many hungry people in the world.

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HEALTH ISSUES

Wireless Technologies (4G, 5G) Are Very Harmful to Human Health and Environment: A Preliminary Review

Victor Christianto, Robert Neil Boyd, Florentin Smarandache

Victor Christianto, Robert Neil Boyd, Florentin Smarandache (2019). Wireless Technologies (4G, 5G) Are Very Harmful to Human Health and Environment: A Preliminary Review. *BAOJ Cancer Research & Therapy* 5(2), 066, 3

Abstract

The intent of this article is to show that wireless technology is, without remedy other than termination, one of the most devastating environmental and health threats and threats to personal liberty ever created. It is becoming widely known that 4G and 5G technologies cause many harms to human health. Cancer is only one problem, and one that is easily solved. 4G and 5G cause 720! (factorial) different maladies in human beings, and can kill everything that lives but some forms of micro organisms. Some pathogens and certain parasites are made more virulent by selected frequencies of RF. Insects and birds are already being killed by the RF broadcasts. The broadcasts can be controlled to give selected individuals selected maladies. All this needs to be stopped. There are other ways to communicate that do not require radio waves, nor wires, which cause no damage to any form of life. We need to make those methods available to the public, while all the RF systems are being phased out.

Introduction

So many people are more and more accustomed to a wide variety of wireless technologies.

However, allow us to argue on 4 reasons why wireless technologies should be stopped:

- Wireless technologies disrupt family relations

- Wireless technologies induce 720! Different maladies, cancer included

- Wireless technologies especially 4G and 5G potentially damage environments

- Wireless technologies steal privacy from everyone using those technologies

It is becoming widely known that 4G and 5G technologies cause many harms to human health. Cancer is only one problem, and one that is easily solved. 4G and 5G cause 720! (factorial) different maladies in human beings, and can kill everything that lives but some forms of micro organisms. Some pathogens and certain parasites are made more virulent by selected frequencies of RF. Insects and birds are already being killed by the RF broadcasts. The broadcasts can be controlled to give selected individuals selected maladies. All this needs to be stopped. There are other ways to communicate that do not require radio waves, nor wires, which cause no damage to any form of life. We need to make those methods available to the public, while all the RF systems are being phased out.

Figure 01: Illustration, after Peter Tocci [1]

Potential Harmful Effects of Wireless Technology to Human Bodies, Carcinogenic Etc

According to Peter Tocci:[1] "By all appearance, world governments, world organizations such as the WHO and UN, and international agencies—even the supposedly independent International Commission on Non-Ionizing Radiation Protection (ICNIRP), which issued draft guidelines on 7/11/18 for exposure to electromagnetic fields (100 kHz to 300 GHz)—knowingly participate in a dangerous deception based on scientific fraud: The arbitrary presumption and single-minded assertion as an operating principle that the only potential danger from ICMR is tissue heating. Included is the extreme effect, 'electro-stimulation,' comprising shocks and burns. As of this writing (December 2018), worldwide telecom exposure limits are based on the stultified parameter of tissue heating/electro-stimulation."Furthermore, Tocci also wrote:[1]"Also, it's not unusual to see argument to the effect that, "Some studies show harm, some don't," with the implication or assertion that wireless should continue, because the latter 'cancels out' the former, or makes the situation 'inconclusive.' This conflates scientific principles and 'legal-speak.' 'Weight–of–evidence' is foreign to science, and such rationalization is used for deception or out of ignorance."However, there was a testimony in Toronto, several years ago. From a presentation given at the Toronto Whole Life Expo 2009 by Andrew Michrowski, PhD: [1] see also [2] "It is not generally appreciated that the advanced nature of wireless gadgets being currently marketed is founded on devices that have been around since the 1940s. … Precise, quality, straightforward medical and scientific research since 1950s details radiofrequency and microwave effects – without influence of stocks, PR and lawyers. By 1970s, electromagnetic, electrochemical, cascade effect equations were well defined for tissues, cells, intracellular & extracellular fluids and macromolecular effects on living systems… Analysis of 1950-1974 mortality of 40,000 Korean War veterans shows that microwave exposure effect is cumulative [emphasis added] it affects all deaths … doubling to tripling cancers of eye, brain and central nervous system, lymphatic and hematopoietic [blood-cell/platelet-forming] and digestive systems. This means that even 'weak' and short exposures from wireless systems accumulate over the years and decades to engender serious diseases [emphasis added].…[a] flow chart prepared [by] the National Research Council of Canada Control Systems Laboratory in 1973 [indicated] 22 non-thermal effects documented and generally understood by the scientific community more than 30 [40] years ago. Now, scientists daring to describe a part of such phenomena risk their career and income."Corroborating Michrowski, Trower asserts that the dangers were fully known by mid-1970. A big reason, he says, is that telecom microwave technology was not originally developed for telecom, but, among other things, as a military stealth weapon for inducing illness.[1] Trower presents proof that Government knew of the follicle-DNA threat before promoting WiFi in schools. In 20 to 25 years (2038-2043), we could easily have a generation with a high percentage of genetically damaged kids [1]. Moreover, in a 3/17/15 phone conversation, Dr. Carlo shared with me his understanding about no-safe-dose, which arose from his WTR experience: Information (data) 'riding' on the microwave 'carrier' frequencies (called modulation) manifests as pulses. These must exist at all power levels to transmit any data. They are sensed by cell membranes. Carlo said that because cells don't recognize the stimulus, pulses provoke, for one thing, a defensive and pathogenic membrane response: Transport channel shutdown, preventing exchange between cell and extracellular medium. It also interrupts intercellular communication, a very serious consequence.[4]"…pulsed EMFs are, in most cases, much more biologically active than are non-pulsed (often called continuous wave) EMFs." – Professor Martin Pall, PhD (Page 45, Chapter 6, first par.). See [3] According to Peter Tocci, known ICMR effects include endocrine disruption (host of illnesses), breakdown of blood-brain barrier, DNA strand breaks, inhibition of

DNA repair, reproductive problems, autism, Alzheimer's – and many more. Though not to be dismissed, cancer, the 'popular' concern, is actually a lesser one in the panoply of effects – as in, ecocide and eventual termination of reproduction [5].

Possible Solutions

Other than RF wireless technologies, which were actually a forbidden weapon grade method by international treaties, we can come up with alternative methods based on known electromagnetic theories. We suppose we can give information regarding one of 3 ways to accomplish new communications technologies that do not require wires, nor RF. The first one, one of us (RNB) already gave to the US government. That involves modulation of curl-free (CF) magnetic field lines which go in a line to infinity and penetrate all intervening matter. Detection of CF information is accomplished by Josephson-Atto-Weber switches (JAWS) which require cryogenic temperatures to operate properly. CF communications are exceedingly directional. Any lack of accuracy between sending and receiving the CF line results in no information transfer. There is the advantage that CF communications exhibit faster than light propagation. However, these devices are not suited for use by the general public. The other two methods, we are going to contemplate giving out. Maybe writing a paper would be a better way, because we can at least get credit for the idea and establish prior claim for legal purposes. In a separate article, we describe basic principle of superluminal wave, that is quantum communication, as an alternative to RF based wireless communication technology.[7] This communications method can provide an infinite number of infinite bandwidth communications channels for each user. Communication using this method travels much faster than light. It does not use radio waves and does not need wires. It cannot be monitored nor tracked nor interfered with. It cannot be regulated due to the infinities involved, and due to the fact that it is unmonitorable. Each user benefits personally from the perfect information security provided by quantum communications. Quantum communications does not harm any form of life, nor the environment, in any way, as quantum events are, and always have been, constantly a part of the Natural Environment.

Concluding Remarks

The intent of this article is to show that wireless technology is, without remedy other than termination, one of the most devastating environmental and health threats—and threats to personal liberty—ever created. It is becoming widely known that 4G and 5G technologies cause many harms to human health. Cancer is only one problem, and one that is easily solved. 4G and 5G cause 720! (Factorial) different maladies in human beings and can kill everything that lives but some forms of micro organisms. According to Peter Tocci, known ICMR effects include endocrine disruption (host of illnesses), breakdown of blood-brain barrier, DNA strand breaks, inhibition of DNA repair, sperm damage, reproductive problems, autism, Alzheimer's – and many more. Though not to be dismissed, cancer, the 'popular' concern, is actually a lesser one in the panoply of effects – as in, ecocide and eventual termination of reproduction.[6] All this needs to be stopped. There are other ways to communicate that do not require radio waves, nor wires, which cause no damage to any form of life. We need to make those methods available to the public, while all the RF systems are being phased out.

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A Review on how an Ancient Forgiveness Way Called *Ho'oponopono* Can Boost Human Health and Immune System

Victor Christianto, Florentin Smarandache

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Abstract

In life, everyone goes through hurtful events caused by significant others: a deceiving friend, a betraying partner, or an unjustly blaming parent. In response to painful emotions, individuals may react with anger, hostility, and the desire for revenge. Experimental evidence suggests that when people are transgressed against interpersonally, they often react by experiencing unforgiveness. Unforgiveness is conceptualized as a stress reaction. As an alternative, they may decide to forgive the wrongdoer and relinquish resentment. Forgiveness is one (of many) ways people reduce unforgiveness. Forgiveness is conceptualized as an emotional juxtaposition of positive emotions (i.e., empathy, sympathy, compassion, or love) against the negative emotions of unforgiveness. Forgiveness can thus be used as an emotion-focused coping strategy to reduce a stressful reaction to a transgression. More evidences have shown that forgiveness can result in better health and boost human immune system. In this article, we discuss an ancient Hawaiian way of forgiveness, called *Ho'oponopono*. We hope this article may be found useful for healthcare practitioners and therapists as well.

Keywords: *Ho'oponopono; Human Health; Immune System*

Introduction

In life, everyone goes through hurtful events caused by significant others: a deceiving friend, a betraying partner, or an unjustly blaming parent. In response to painful emotions, individuals may react with anger, hostility, and the desire for revenge. Forgiveness can thus be used as an emotion-focused coping strategy to reduce a stressful reaction to a transgression [3,4]. More evidences have shown that forgiveness can result in better health and boost human immune system. Therefore, many doctors and counselors advise people to practice forgiveness.

As David Hawkins wrote in preface of his book [1], which can be paraphrased as follows: "The present work describes a simple and effective means by which to let go of negative feelings and become free. The letting go technique is a pragmatic system of eliminating obstacles and attachments. It can also be called a mechanism of surrender. There is scientific proof of its efficacy, an explanation of which is included in one of the chapters".

Those people who forgive actually they choose to forgive, no matter how hard it is. There is no easy way, forgiveness should be done out of a committed decision. They choose to end hatred and anger and they learn to forget their motive to punish others [2]. They realized that failing to do so, only makes their hearts and bodies to suffer more pain. But where can we find the strength to forgive others? It seems that forgiveness can only be done by recognizing the fact that God is always waiting us up there to forgive and let us learn to live in His ways.

Despite the fact that forgiveness is not easy, an ancient way of forgiveness called *Ho'oponopono* teach simple ways to forgive others. This ancient way can be considered as an alternative to methods of D. Hawkins etc. We discuss it in the next section.

Forgiveness and human immune system

In their article, Worthington JR and Scherer some possible mechanisms which may explain how forgiveness can boost human health, which can be rephrased as follows [4].

Direct mechanism 2: Forgiveness could affect the immune system at the cellular level

Another conceivable instrument for influencing wellbeing results is that the density of unforgiveness may influence the resistant framework. A sizable writing has created about how negative feelings (of which unforgiveness is one) is identified with and can cause dysregulation in the insusceptible framework. Kiecolt-Glaser., *et al.* (2002) have completely inspected the writing on how stress and negative feeling can influence cytokines. Cytokines are protein-like materials that are discharged when a disease or injury is supported or when stress is experienced. They basically help battle disease and give an early reaction to injury. They draw in insusceptible cells to the site of harm and initiate them. They likewise influence digestion and help direct internal heat level. At the point when an individual is under consistent pressure, master fiery cytokines are raised. This can dysregulate the intercellular insusceptible framework. No analyst who is considering absolution has yet analyzed cytokine creation as a proportion of responsiveness to unforgiveness or to pardoning, however such research is in progress by Temoshok and Wald (2002).

Direct mechanism 3: Forgiveness could affect the immune system at the neuro-endocrine level

Absolution could influence wellbeing by boosting the resistant framework. In particular, absolution could lessen HPA reactivity and diminish discharged cortisol. Over-creation of cortisol has been appeared to have malicious wellbeing consequences for the cardiovascular and resistant frameworks and on subjective and cerebrum working (for audits, see Sapolsky, 1994; McEwen, 2002). Just one examination has tended to cortisol and unforgiveness. Berry and Worthington (2001) discovered base-rate creation of salivary cortisol to be somewhat raised in individuals who were high in characteristic unforgiveness....

Direct mechanism 4: Forgiveness could affect the immune system through release of antibodies

The investigation of stress recommends that antibodies are stifled during incessant pressure. Given that pardoning is conjectured to compare positive feelings against unforgiveness, we propose that discoveries from the investigation of positive feelings can give speculations about absolution. Salovey., *et al.* (2000) explored the writing on the connection among negative and positive feelings and physical wellbeing. They concentrated basically on the connection between negative feelings and concealment of secretory Immunoglobulin-A (sIg-A) hindrance. Levels of sIg-A have been found to influence invulnerable framework working. At the present, sIg-A concealment has not been researched comparable to the unforgiveness. It is theorized that when individuals use absolution as an adapting system to diminish unforgiveness, at that point sIg-A levels would come back to benchmark. This has not been researched".

Therefore, it shall be clear that there is sufficient ground to believe that forgiveness can lead to better health and immune system.

Introducing *Ho'oponopono*

According to Thakurdas, *Ho'oponopono* is an ancient Hawaiian method of stress reduction, conflict resolution and energy clearing from people and places. In her article, she refers to the transforming wisdom and its modern psychology and spiritual based applications. *Ho'oponopono* provides a profound method of healing that promotes the Universal Law of the Interconnection of Life. It is a powerful method that can meet the stressful challenges of our time [5].

Moreover, according to Thakurdas, which can be paraphrased as follows [5]: "*Ho'oponopono* signifies "to make right," or "to redress a mistake". Viably, it intends to make it directly with the predecessors, or your family, companions or anybody or anything with whom or which you have a relationship. This can incorporate the creature, vegetable or mineral realms and to be sure our planet. Initially *Ho'oponopono* was utilized to address the wrongs that had happened.

Na kala: The Fortune of Forgiveness. Hawaiian qualities incorporate a significant code of pardoning. They accept that when we excuse others, we our additionally sympathetic ourselves. Kala signifies "to loosen, unbind and set free." The individual, gathering or country to whom the transgressor is obliged free themselves and the 'others' of the past obligations or bad behavior. It doesn't exist any longer. This must be finished by unbinding connections to the past wrongs; by making right what's to come".

Thakurdas also emphasizes: "In Ihaleakala's version of H'oponopono "the therapist must be willing to be 100% responsible for having created the problem situation, that is, he must be willing to see that the source of the problem is erroneous thoughts within him, not within the client. Therapists never seem to notice that every time there is a problem, they are always present!" [5].

In a different study, Matthew James reports, which can be rephrased as follows [6]: "The outcomes exhibited that the individuals who occupied with the *Ho'oponopono* procedure along these lines encountered a measurably critical decrease in unforgiveness, while those in the benchmark group indicated no factually huge change in negative effect throughout the examination. In view of these discoveries and by approving *Ho'oponopono* as a powerful helpful absolution strategy, this examination lays the foundation for future research of this particular pardoning process. Solid ramifications for constructive social change through the use of *Ho'oponopono* incorporate improved wellbeing, and improved relational and intrapersonal connections".

Is there a theory behind *Ho'oponopono*?

According to Thakrudas, which can be rephrased as follows [5]: "We convey inside us as fragments of the Unconscious Mind, interfacing us with all the critical individuals in our lives. These sections look like Carl Jung's models. The procedure of *Ho'oponopono* is to line up with and clear up remaining energies resounding through our family history just as to clear up our associations with others in our present lives. Making this one stride further, we can clear up similar issues for customers and even gatherings of people. Right now, is no compelling reason to work out, understand, oversee or adapt to issues. Since the Divine made everything, you can simply go legitimately to Divine and ask that it be rectified and purged" (see also Carl G Jung [7] and also B Lipton [8]).

Linguistic origin: "As a matter of first importance, *Ho'oponopono* might be thought as compromise, reestablishing profound arrangement, to make right. In the event that you look in the Hawaiian word reference, the meaning of pono, it takes practically a large portion of the page to decipher. Pono intends to be right, in concordance, otherworldly arrangement. To be well.

Pono is being sound. In trustworthiness. Remaining in balance. It includes everything we need to have and require in the entirety of our bodies: profound, mental, passionate and physical.

In the event that you put Ho'o before pono, *Ho'o gives activity.* On the off chance that we don't deal with things when soul gives a little thump on of entryway of our otherworldly mindfulness, messages will go on. On the off chance that we overlook the message in our psychological mindfulness, it will advance right to the physical body. When it gets to the physical body at that point here and there it will take more work. It will take more Ho'o to get that pono" [9].

A bit of history: "One of the most prestigious educators of *Ho'oponopono* nowadays is Dr. Ihaleakala Hew Len. Dr. Cut Len first concentrated with a Hawaiian Kahuna and healer named Morrnah Simeona. Albeit a relative of Hawaiian Royalty and showed the old ways, Simeona was affected by Christian lessons joined with an enthusiasm for all encompassing practices from India, China and the profound educator Edgar Cayce. Crossing over her Hawaiian and Christian comprehension of pardoning and compromise, Simeona started investigating how past obligations and circumstances and logical results attached every individual to past injury, negative contemplations and encounters, and even activated resurrection. She instructed that the correct petitions, fundamentally the same as those educated by Science of Mind and other New Thought religions, were the most ideal approach to clear and purge those negative energies or "stones" that are obstructing the Light inside us and inside the world.

Upon Simeona's passing, Dr. Ihaleakala S. Cut Len proceeded with the educating and made adjustments en route. From Dr. Slash Len's point of view, *Ho'oponopono* is best comprehended as an arrival to our clear province of Light. He calls that state, "Zero" and accepts that once there, we exist with zero cutoff points. Be that as it may, a major key is the mindfulness that in the event that all Life is interconnected and we are One with it, at that point we should acknowledge 100% duty regarding everything that happens to us and in our reality. Slash Len veered away from Simeona's attention on utilizing petition as the best device to reach *Ho'oponopono*, and rather started to utilize a mantra that anybody could utilize, whenever, anyplace. "I love you. I'm heartbroken. Kindly pardon me. Much obliged to you" (Or any variety of that request)" [10].

4 simple steps to practice *Ho'oponopono* [11]

Stage 1: Repentance - I'M SORRY

As we notice above, you are answerable for everything in your psyche, regardless of whether it is by all accounts "out there". Once you understand that, it's extremely normal to feel sorry. I know I sure do. On the off chance that I know about a tornado, I am so loaded with regret that something in my cognizance has made that thought. I'm so extremely heartbroken that somebody I know has a messed up bone that I understand I have caused.

This acknowledgment can be agonizing, and you will probably oppose tolerating duty regarding the "out there" sort of issues until you begin to rehearse this strategy on your progressively self-evident "in here" issues and get results.

So pick something that you definitely realize you've caused for yourself? Dependent on nicotine, liquor or some other substance? Do you have outrage issues? Medical issues? Start there and state you're grieved. That is the entire advance: I'M SORRY. In spite of the fact that I think it is all the more impressive on the off chance that you state it all the more plainly: "I understand that I am liable for the (issue) in my life and I feel awful regret that something in my awareness has caused this".

Stage 2: Ask Forgiveness - Please FORGIVE ME

Try not to stress over who you're inquiring. Simply inquire! It would be ideal if you FORGIVE ME. Let's assume it again and again. Would not joke about this. Recall your regret from stage 1 as you request to be excused.

Stage 3: Gratitude - THANK YOU

State "THANK YOU" - again it doesn't generally make a difference who or what you're expressing gratitude toward. Thank your body for everything it accomplishes for you. Express gratitude toward yourself for being as well as can be expected be. Express gratitude toward God. Thank the Universe. Thank whatever it was that just excused you. Simply continue saying THANK YOU.

Stage 4: Love - I LOVE YOU

This can likewise be stage 1. Let's assume I LOVE YOU. Let's assume it to your body, say it to God. Let's assume I LOVE YOU to the air you inhale, to the house that covers you. Let's assume I LOVE YOU to your difficulties. Let's assume it again and again. Would not joke about this. Feel it. There is nothing as amazing as Love.

That is it. The entire practice more or less.

Concluding Remarks

In this article we discuss an ancient and very simple way of forgiving, called *Ho'oponopono*. It is advised here that forgiveness can lead to better health and improved immune system, either to cytokines or other mechanisms.

To these authors, the message of this ancient practice brings us back to the Lord's prayer: forgive us our debts; see Appendix.

Version 1.0: 3.04.2020 pk. 11:52

Version 1.1: 3.04.2020 pk. 15:51

VC & FS

Appendix: Lord's Prayer in English and in Arabic

In English

After this manner therefore pray ye: Our Father which art in heaven, Hallowed be thy name.

Thy kingdom come, Thy will be done in earth, as it is in heaven.

Give us this day our daily bread.

And forgive us our debts, as we forgive our debtors.

And lead us not into temptation, but deliver us from evil: For thine is the kingdom, and the power, and the glory, for ever. Amen.

In Arabic: Lord's Prayer read in Arabic:

Aba na alathie fi asamawat,

Li yatakadas ismoka, Li ya'atie malakotoka,

Litakon mashia toka,

Kama fisama' kathaleka 'ahla al a'ard.

A'atinia khubzana kafafa yawmina,

Wa igfer lana khatayana,

Kama naahnu naghfer la man akhta'a elayna,

Wa la tudkhilna fit a jareeb;

Laken najjina min ashireer.

Lia'anna laka al kowata wal majd, al aan wa ila abad al aabideen.

Amin.

Arabic - transliteration.

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A Review of Major Role of Vitamin D3 in Human Immune System and its Possible Use for Novel Corona Virus Treatment

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Abstract

Considering growing concerns of the world's authorities on the spreading of novel corona virus (Covid-19), in this paper we review the evidences showing major role of Vitamin D3 in human immune system and its potential use for novel corona virus treatment. Our argument is based on research finding that corona virus has viral envelope glycoproteins. In this regard, Vitamin D3 proves to offer various beneficial effects, including immunomodulatory effect, in order to break the glycoproteins envelope of the virus. One of the greatest benefit of vitamin D3 is the fact that it is easy to get 10,000 - 20,000 IU of daily intake requirement, by sunbathing for more or less twenty minutes. Such a method is likely applicable in many tropical countries. Therefore, we submit a view which can be rephrased as follows: "Twenty minutes sunbathing a day may likely keep Covid-19 away".

"*It is better to light a candle than curse the darkness*" - a Chinese proverb[1].

Keywords: *Vitamin D3; Human Immune System; Corona Virus*

Introduction

The present paper reflects a growing concern on the spreading Covid-19 virus over more than 60 countries to date. The SARS-like coronavirus that appears to have originated in Wuhan, China has now infected thousands of people.

According to Worldometers, as of March 4, 2020, there have been 93,191 confirmed cases around the world, 3,203 death cases, and 50,984 recovered. The COVID-19 is affecting 80 countries and territories around the world and 1 international conveyance (the Diamond Princess cruise ship harbored in Yokohama, Japan)[2].

The virus was initially named novel coronavirus of 2019 (nCoV-2019 or 2019-nCoV) as of now. This has now been renamed as COVID-19. Sequencing of the virus has determined it to be 75 to 80 percent match to SARS-CoV and more than 85 percent similar to multiple coronaviruses found in bats. SARS stands for severe acute respiratory syndrome. It is also a coronavirus or CoV [2].

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[1]Note: some authors attributed this quote to William Watkinson, 1907.

[2]Source: https://www.worldometers.info/coronavirus/

Despite this gloomy picture, there is also news reporting that all corona patients in 3 countries, including Vietnam and Nepal have recovered[3]. Therefore, there is ground to be hopeful that cure does exist.

Inspired by a recent video lecture by virus expert, Prof. Luc Montagnier [1], in this paper we will review some evidences suggesting immunomodulatory effect and other benefits of Vitamin D/D3 as potential way to Covid-19 treatment.

A short review of the basics of respiratory medicine

Covid-19 belongs to respiratory diseases related to viral pneumonia. Studies show that there are glycoprotein shells covering the corona viruses, which make it difficult to break the virus.

First of all, let us review some basic facts from textbooks of respiratory medicine:

- Oxford Handbook of Respiratory Medicine wrote regarding viral pneumonia, which can be rephrased as follows: Viral URTIs are normal, yet commonly self-restricting, and are typically overseen in the network. Viral pneumonia is less normal yet is progressively genuine and typically requires hospitalization. Viral pneumonia in the immunocompetent is uncommon and regularly influences kids or the old; flu strains are the commonest cause in grown-ups. Studies recommend that infections are perceivable in 15 - 30% of patients hospitalized with pneumonia. Infections may cause genuine respiratory disease in the immunocompromised (especially patients with discouraged T-cell work, for example following organ transplantation). CMV is the commonest genuine viral pathogen that influences immunocompromised patients. Flu, parainfluenza, rSV, measles, and adenovirus may likewise cause pneumonia in the immunocompromised, despite the fact that analysis of these infections is troublesome and contamination is generally undetected [3].

- Shen Wei Lim in ERS Handbook of Respiratory Medicine wrote on influenza and pandemic which can rephrased as follows: Flu is profoundly transmissible. Human-to human transmission happens through huge bead spread and direct contact with emissions (or fomites).

Treatment: There are two principle classes of medication that are dynamic against flu. The M2 particle channel inhibitors, amantadine and rimantadine, are viable against flu A. Be that as it may, their utilization is thwarted by the quick rise of protection from these medications together with a high rate of symptoms.

Antibiotics are generally prompted for patients with flu related pneumonia or patients with serious flu disease who are at high danger of creating auxiliary bacterial contaminations. The utilization of corticosteroids in serious flu can't be routinely supported dependent on current information; observational accomplice contemplates led during the 2009 H1N1 pandemic have detailed blended outcomes including expanded mischief [4].

Therefore, it seems we can conclude for now that even though there are recommended for such a viral pneumonia, there is no clear suggestion for treatment in a pandemic situation.

In this regards, inspired by a recent video lecture by Prof. Luc Montagnier, a renowned virus expert, we will discuss immunomodulatory effect and other advantages of Vitamin D, especially D3.

[3] Source: https://japantoday.com/category/world/all-16-of-vietnam%27s-coronavirus-sufferers-cured

Short review of immunomodulatory effect and other advantages of vitamin D/D3

In this section, we will review some advantages of Vitamin D and D3 for possible treatment of Covid-19.

Jeremy Beard., *et al*. wrote on major role of Vitamin D in regulating immune system, which can be rephrased as follows: Nutrient D has for some time been perceived as fundamental to the skeletal framework. More current proof recommends that it additionally assumes a significant job managing the invulnerable framework, maybe including safe reactions to viral contamination. Interventional and observational epidemiological investigations give proof that nutrient D insufficiency may give expanded danger of flu and respiratory tract contamination. Cell culture tests bolster the postulation that nutrient D has direct anti-viral impacts especially against enveloped viruses. In spite of the fact that nutrient D's enemy of viral instrument has not been completely settled, it might be connected to nutrient D's capacity to up-manage the counter microbial peptides LL-37 and human beta defensin 2. Extra examinations are important to completely clarify the efficacy and mechanism of vitamin D as an anti-viral agent [6].

We continue with a review of inflammatory research. Mag Mangin., *et al*. wrote on relation between Vitamin D and inflammatory disease, which can be rephrased as follows: Inflammation is accepted to be a contributing element to numerous constant maladies. The influence of nutrient D deficiency on inflammation is being investigated however examines have not exhibited a causative impact. Inflammation is associated with numerous incessant illnesses and concern has been raised about the influence of nutrient D deficiency on inflammatory forms. At the point when studies found a relationship between inflammatory infections and low serum 25-hydroxyvitamin D (25(OH)D), further research discovered proof of low nutrient D in a huge fragment of everybody. This drove a few specialists to announce an overall pandemic of nutrient D deficiency and to prescribe nutrient D supplementation. Specialists are discussing the definition of nutrient D deficiency and the fitting nutrient D portions, while further research is being done to decide whether nutrient D supplementation has the expected impact [5].

A possible mechanism of vitamin D metabolism is shown in figure 1 below.

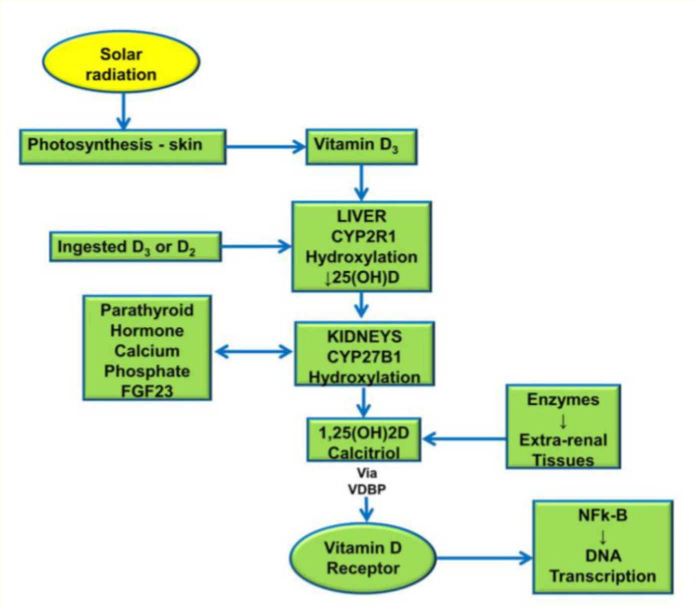

Figure 1: *Synthesis and metabolism of Vitamin D. Sequential metabolic processes convert biologically inactive, parental vitamin D into active metabolites [5].*

Another paper studies the interplay between vitamin D and viral infections and its metabolism mechanism is as shown in figure 2 below.

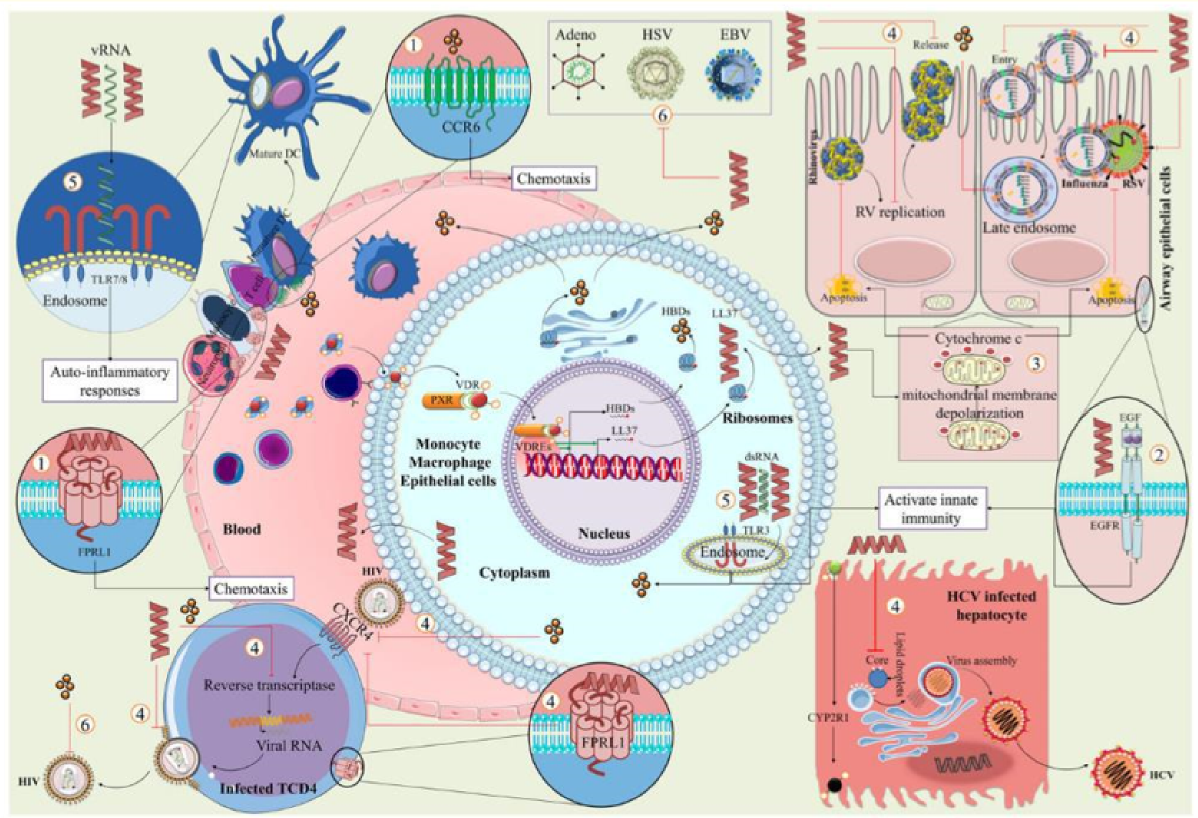

Figure 2: V*itamin D and induction of antimicrobial peptides against viral infection. Vitamin D-induced antimicrobial peptides (LL37 and HBDs) via VDR and RXR dimerization act against viral infections through (1) binding to FPRL1 and CCR6 for recruiting immune cells to site of infection; (2) activation of innate immunity by transactivation of the EGFR; (3) clearance of viral infection through mitochondria! membrane depolarization and release of cytochrome c; (4) down-regulation of cell entry, replication, and viral release; (5) protection of viral RNAs from degradation and induced immunity response through TLRs activation; (6) and direct effects on virions. Abbreviation: Epidermal growth factor receptor (EGFR), formyl peptide receptor-like 1 (FPRL1), human p-defensins (HBDs), retinoic X receptor (RXR), toll-like receptor (TLR), and vitamin D receptor (VDR). After Teymoori-Rad., et al [9].*

And for the last quote is from a not-peer-reviewed manuscript by William B Grant., *et al*. which can be rephrased as follows: Low nutrient D status in winter licenses viral pandemics. During winter, individuals who don't take nutrient D supplements are probably going to have low serum 25-hydroxyvitamin D [25(OH)D] fixations. Nutrient D can diminish the danger of viral plagues and pandemics in a few different ways. To start with, higher 25(OH)D focuses decrease the danger of numerous constant ailments, including tumors, cardiovascular ailment, interminable respiratory tract diseases (RTIs), diabetes mellitus, and hypertension. Patients with interminable illnesses have fundamentally higher danger of death from RTIs than in any case solid individuals. Second, nutrient D decreases danger of RTIs through three components: keeping up tight intersections, killing encompassed infections through acceptance of cathelicidin and defensins, and diminishing creation of proinflammatory cytokines by the inborn safe framework, along these lines lessening the danger of a cytokine storm prompting pneumonia [7].

In their abstract, they conclude: "From the accessible proof, we estimate that raising serum 25(OH)D fixations through nutrient D supplementation could lessen the rate, seriousness, and danger of death from flu, pneumonia, and the current COVID-19 scourge" [7].

Despite all of the above advantages of Vitamin D/D3 for human immune system, one of the problems is that there is seemingly deficiency of Vitamin D, as reported in Poland's recent guideline, which can be rephrased as follows: Nutrient D insufficiency is a significant general medical issue around the world. Nutrient D lack presents a noteworthy hazard for both skeletal and non-skeletal disarranges and various long lasting negative wellbeing results. The targets of this proof based rules archive are to give social insurance experts in Poland, a refreshed suggestion for the counteraction, finding and treatment of nutrient D lack [8].

How to get a daily intake of Vitamin D and D3 as per required [10]

We have reviewed possible efficacy of Vitamin D/D3 in improving human immune system, including from Covid-19. Now the question is how we can obtain rapid medicine for daily intake of the entire population. The answer could be quite simple: sunbathing (exposing your body to sunlight in the morning) and nutrition and/or with vegetables.

Sunbathing

At the point when our progenitors were living in the tropics, daylight on-skin was their fundamental wellspring of nutrient D. At the point when daylight sparkles on skin it produces nutrient D3. All things considered, just a segment of ultra-violet in daylight carries out the responsibility, known as UV-B. The frequencies for UV-B go from 280 to 315 nanometres and nutrient D is best-delivered by light in the range 295 to 297 nanometres. Uncovering the vast majority of your skin to solid daylight for around 20 minutes can deliver somewhere in the range of 10,000 and 20,000 IU of nutrient D3, in the event that your skin is reasonable. By and large (In the event that you are darker looking, you may take an hour or more to create a similar sum).

Vegetables

Nutrient D nourishment sources can be either vegetable or creature. Vegetable sources are chiefly green growth (which a great many people don't eat, yet it gets into the natural way of life through fish) mushrooms. Wild mushrooms can create very considerable measures of nutrient D2 (ergocalciferol) in the event that they have been presented to solid daylight.

Business mushrooms are developed inside and not exposed to UVB light. Therefore they contain no nutrient D. Nonetheless, it merits realizing that on the off chance that you illuminate developed mushrooms that you purchase at a store, by setting them in solid daylight for 30 minutes or something like that, you can create around 1500 IU of nutrient D2 per medium-sized punnet. So, this sort of nutrient D can happen normally in our eating routine. We can utilize it, yet it's not the perfect structure for us. Our bodies are upgraded for nutrient D3, the thoughtful our skin produces. In any case, getting all your nutrient D as vegetable-sourced nutrient D2 would be a whole lot better than getting no nutrient D by any means.

Animal source of vitamin D

Cod liver oil (just a couple of teaspoons daily for a grown-up) can be utilized to give some nutrient D3. Common cod liver oil contains around 5000 IU of nutrient A, however just 500 IU of nutrient D per teaspoon. If you somehow managed to drink a few teaspoons every day (not suggested), the nutrient A would arrive at dangerous levels before you got enough nutrient D. Some cod liver oils are handled to expel the regular nutrients. At that point in the wake of preparing, manufactured nutrient An and nutrient D2 are included go into the oil. The regular extents of nutrient A to nutrient D may likewise be changed in such oils. Likewise, nutrient A may vie for ingestion with nutrient D, so on the off chance that you are attempting to raise your nutrient D levels, taking additional nutrient A (retinol) could be counter-profitable.

Vitamin D supplement

The most straightforward and maybe most secure approach to build your nutrient D level is to utilize nutrient D3 (cholecalciferol) supplements. Enhancements of nutrient D2 (ergocalciferol) are likewise accessible, yet are presumably less powerful. Nutrient D3 supplements are broadly accessible in many nations. In most first-world nations you would now be able to get nutrient D3 in case estimates extending from 400 IU to 10,000 IU, or even 50,000 IU. Unfortunately, a few nations limit the most extreme nutrient D case size to 400 IU or 1000 IU. This makes it harder, and progressively costly to take a successful portion. Yet, even in those nations, you can for the most part request nutrient D on the web.

Sunlamps

One final other option, for individuals with profound pockets – you can utilize a sheltered tanning bed to give UV-B to your skin lasting through the year, in absolutely controlled portions. UV lights are likewise accessible to do something very similar. The gear needs to give precisely the correct frequencies of UV light, in the correct extents, so as to be sheltered. Purchase from a trustworthy maker and use carefully as per their directions. Additionally, know that tanning beds and sunlamps may not create a similar nature of light for an incredible duration. So, follow producer's proposal for recharging tubes.

We hope that we made it quite clear that certain treatments do exist for Covid-19 based on existing knowledge on the major role of Vitamin D in human immune system, in particular for anti-viral treatment.

So, what is the best way to keep daily intake of Vitamin D/D3?

Presently you recognize what your choices are for getting the nutrient D your body needs. The vast majority should focus on daylight and supplementation, which isn't to dishearten you from eating fish! Fish - from unpolluted waters - is a solid nourishment, yet it can just go so far in meeting your nutrient D needs, except if it frames an enormous piece of your eating regimen. Both daylight and supplementation should be utilized with care and comprehension, for various reasons. Daylight can harm your skin in the event that you take a lot at once. Learn and read more about safe Vitamin D from sunlight. Nutrient D Supplements must be taken in suitable dosages, which are not the equivalent for everybody. At last, while expanding your degree of nutrient D to approach ideal, it is significant that different supplements are likewise taken in the correct amounts, especially calcium, magnesium, nutrient K and nutrient A [10].

For other sources of information regarding nutrition and phytomedicine, see [11-13].

Concluding Remarks

We reviewed a major role of Vitamin D and D3 in human immune system. This is just an early schematic paper, it would need more study to establish which the suggested medicinal plants or nutrition related to Vitamin D/D3 are the most beneficial for improving human immune system.

We hope that this short review article can be found useful for policy makers of health in reducing the effect of Covid-19 in many countries.

Acknowledgement

The authors are grateful to Sibylle Gabriel who informed us a recent video lecture by Prof Luc Montagnier [1].

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A Survey on Deep Transfer Learning and Edge Computing for Mitigating the COVID-19 Pandemic

Abu Sufian, Anirudha Ghosh, Ali Safaa Sadiq, Florentin Smarandache

Abu Sufian, Anirudha Ghosh, Ali Safaa Sadiq, Florentin Smarandache (2020). A Survey on Deep Transfer Learning and Edge Computing for Mitigating the COVID-19 Pandemic. Journal of Systems Architecture 108, 101830, 30

Abstract

Global Health sometimes faces pandemics as are currently facing COVID-19 disease. The spreading and infection factors of this disease are very high. A huge number of people from most of the countries are infected within six months from its first report of appearance and it keeps spreading. The required systems are not ready up to some stages for any pandemic; therefore, mitigation with existing capacity becomes necessary. On the other hand, modern-era largely depends on Artificial Intelligence(AI) including Data Science; Deep Learning(DL) is one of the current flag-bearer of these techniques. It could use to mitigate COVID-19 like pandemics in terms of stop spread, diagnosis of the disease, drug & vaccine discovery, treatment, and many more. But this DL requires large datasets as well as powerful computing resources. A shortage of reliable datasets of a running pandemic is a common phenomenon. So, Deep Transfer Learning(DTL) would be effective as it learns from one task and could work on another task. In addition, Edge Devices(ED) such as IoT, Webcam, Drone, Intelligent Medical Equipment, Robot, etc. are very useful in a pandemic situation. These types of equipment make the infrastructures sophisticated and automated which helps to cope with an outbreak. But these are equipped with low computing resources, so, applying DL is also a bit challenging; therefore, DTL also would be effective there. This article scholarly studies the potentiality and challenges of these issues. It has described relevant technical backgrounds and reviews of the related recent state-of-the-art. This article also draws a pipeline of DTL over Edge Computing as a future scope to assist the mitigation of any pandemic.

Keywords: AI for Good, COVID-19, Deep Learning, Edge Computing, Pandemic, Review, Transfer Learning.

1. Introduction

The COVID-19 is a disease caused by a novel coronavirus called 'SARS-CoV-2'. This virus is transferable from human to human and its spreading, and infection factors are very high [1, 2]. Over nine million people are infected and approx half million are died within just six months from it's fist report, and it is increasing steadily [1]. The World Health Organization (WHO) has declared it a pandemic [3, 4]. But this is not the only pandemic human civilization is facing, there are many outbreaks had come in the past or it may come in the future [5, 6]. The appropriate drugs, vaccines, infrastructure, etc. are not available up to some stages of any outbreaks. Therefore, mitigate these types of diseases with existing capacity becomes most important in those stages [7, 8]. Many researchers from all over the world trying hard to develop such kind of techniques to cope with such challenges [9, 10, 11].

Modern-era largely depends on Artificial Intelligence(AI) including Data Science; Deep Learning (DL) is one of the current flag-bearer of these techniques [12]. Therefore, these techniques could also assist to mitigate COVID-19 like pandemics in terms of stop spread, diagnosis of the disease, drug & vaccine discovery, treatment, and many more [13, 14]. But to trained this DL, large datasets as well as powerful computing resources are required. For a new pandemic, data insufficiency and it's variation over different geographic regions is a huge problem, so here Deep Transfer Learning (DTL) would be effective as it learns from one task and could apply in another task after required fine-tuning [15]. On the other hand edge devices such as IoT, Webcam, Drone, Intelligent Medical Equipment, Robot, etc. are very useful in any pandemic situation. These types of equipment make the infrastructures sophisticated and automated which helps to cope with an outbreaks [16]. Though, such devices are equipped with low computing resources which represent the main challenges of Edge Computing(EC) [17]. As a way to overcome this challenge, transfer learning could be a possible way to consolidate the needed computational power and facilitate more efficient EC. Therefore, DTL in edge devices as an EC could be smart techniques to mitigate a new pandemic [18]. This survey article has tried to report all these issues scholarly as potentialities and challenges with relevant technical backgrounds. Here, we also proposed a possible pipeline architecture for future scopes to brings DTL over EC to assists mitigation in any outbreaks.

[1] https://www.worldometers.info/coronavirus/

1.1. Contributions of this Article

Some highlights of the contributions of this article are as follows:

- Presented a systematic study of Deep Learning (DL), Deep Transfer Learning(DTL) and Edge Computing(EC) to mitigate COVID-19.
- Surveyed on existing DL, DTL, EC, and Dataset to mitigate pandemics with potentialities and challenges.
- Drawn a precedent pipeline model of DTL over EC for a future scope to mitigate any outbreaks.
- Given brief analyses and challenges wherever relevant in perspective of COVID-19.

1.2. Organization of the Article

Starting from the introduction in section 1, the remainder of the article organized follows. Section 2 for technical background whereas review of generic state-of-the-art of DTL in EC in section 3. Existing computing(DL, DTL, EC & Dataset) to mitigate pandemic in section 4. A proposed pipeline of DTL in EC to mitigate pandemics in section 5. Finally, conclusion in section 6.

2. Technical Backgrounds

The main focus of this article is how DL, DTL, EC, and it's associate could assist to mitigate any pandemics. The possible roles and challenges of these techniques in a pandemic, especially for COVID-19, are mentioned in section 4. This section has tried to bring an overview and general progress of DL, DTL, and EC in the following three subsections.

2.1. Deep Learning(DL)

Deep learning (DL) (also known as hierarchical learning or deep structured learning) is one of the effective inventions for modern-era of Artificial Intelligence (AI) [12]. Until the decade '90s, classical machine learning techniques were used for making inferences on data and prediction. Nevertheless it had several drawbacks such as depend on handcrafted features, bounded by human-level accuracy, etc [19]. But in DL, handcrafted feature engineering is not required rather features are extracted from data during training. In addition, DL can make more accurate classifications and predictions with the help of innovative algorithms, computing power of modern machines, and the availability of Big Datasets [20]. Nowadays, DL methods have been successfully applied for several AI-based medical applications such as Magnetic Resonance Imaging (MRI) images analysis for cancer and diabetes diagnoses, conjunction with biometric characteristics, etc [21].

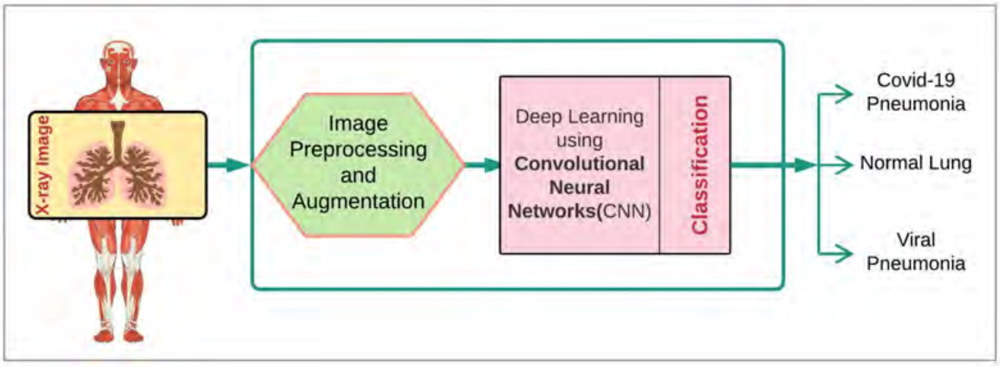

Figure 1: A overall block diagram of a Deep learning-based screening system.

DL is a kind of learning algorithm or model under the umbrella of AI which is based on Artificial Neural Networks(ANN) [22]. These models are trained using dataset through backpropagation algorithm [23] and a suitable optimizer method [24]. The inherent capacities of such DL model such massive parallelism, non-linearity, and capabilities of feature extraction have made them powerful and widely used [20]. There are several variety of DL algorithm such as Convolutional Neural Networks(CNN) [25, 26], Recurrent Neural Networks (RNN) [27], Long Short Term Memory(LSTM) [28], GAN [29], etc. After success of a CNN-based model, called AlexNet [30], many deep learning model has proposed such ZFNet [31], VGGNet [32], GooglNet [33], ResNet [34], DenseNet [35], etc specially for computer vision tasks [36]. In figure 1 we try to illustrate a typical methodology of a DL based screening system, where the system uses a DL algorithm (CNN) to predict whether the X-ray images of suspected patient's lung is normal or having viral pneumonia or COVID-19 pneumonia.

In the time of the COVID-19 crisis, when the numbers of infected patients are at a time very high and the disease is still spreading, many research groups are using the DL techniques for screening COVID-19 patients by detection fever temperature, viral and COVID-19 pneumonia, etc. In addition, DL is using or could be used for other purposes such as patient care, detection systematic social distancing violation, etc [37]. As for reference, S. Wang et.al used a CNN-based DL for screening COVID-19 patients with an accuracy, sensitivity, and specificity of 89.5%, 87%, and 88% respectively by using their computed tomography (CT) images [38]. Similarly, in another study [39] L.

Wang et.al used chest X-ray images for a screening of COVID-19 cases with 83.5% accuracy. The description of such works is in section 4.1.1.

2.2. Deep Transfer Learning(DTL)

Transfer Learning is a technique that effectively uses knowledge of an already learned model to solve another new task (possibly related or little related) with require of minimal re-training or fine-tuning [40, 41]. Since DL requires a massive training data compared to traditional machine learning methods. So, the requirement of a large amount of labeled data is a big problem in solving some critical domain-specific tasks, specifically the applications for the medical domain, where the making of large-scale, high-quality annotated medical datasets is very complex, and expensive [42]. In addition, the usual DL model requires large computing power such as GPU enabled sever, although researchers are trying hard to optimizing it [43, 44]. Therefore, Deep Transfer Learning (DTL), a DL based Transfer Learning try to overcome this problem [45]. DTL significantly reduces the demand for training data and training time for a target domain-specific task by choosing a pre-trained model (trained on another large dataset of same target domain) for a fixed feature extractor [46] or for further fine-tuning [47]. Figure 2 demon-

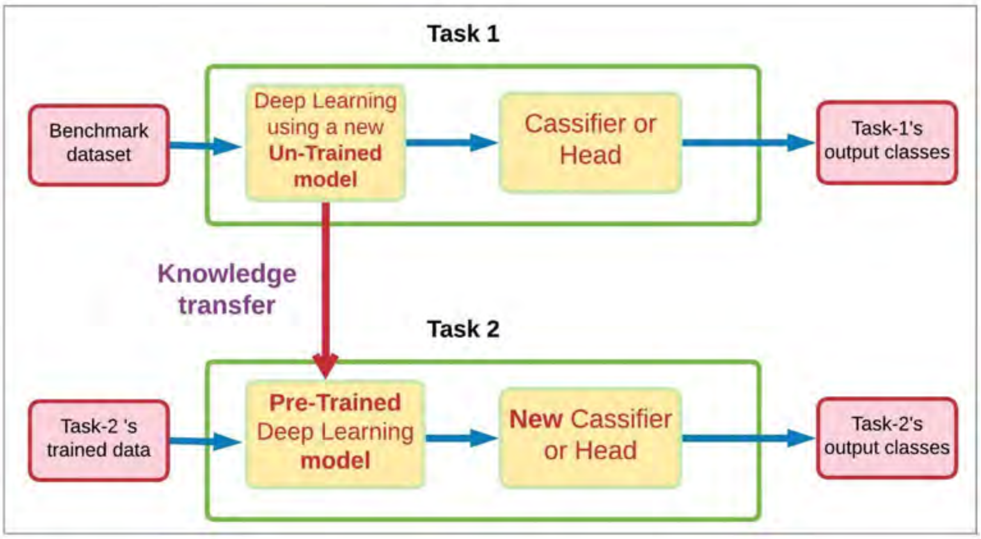

Figure 2: Block diagram of an example of Deep Transfer Learning.

strating the main steps methodology of a Deep Transfer Learning approach, where an untrained model is trained using benchmark dataset (task-1) for feature extraction. Then that pre-trained model is further used to tackle a new challenge such as the task (task-2) of COVID-19 by just replacing only few last layers in the head of the architecture and required fine-tuning.

So far, many DTL models have been proposed [15]. A few recent are reported and discussed in the article. In a research study [45], Mingsheng Long et.al proposed a joint adaptation network. It learns a transfer network by aligning the joint distributions of multiple domain-specific layers across domains based on a joint maximum mean discrepancy. In another study [48], Yuqing Gao and Khalid M. Mosalam proposed a state-of-the-art transfer learning model based on VGG model [32]. They have used ImagNet [49] dataset for features extractor and their hand label construction images for fine-tuning. Abnormality classification in MR images through DTL proposed in a study [50]. The authors of that study also have used pre-trained ResNet34 model with fine-tuning. In a research practice [51], a DTL for diagnosing faults in target applications without labeling was proposed. Their framework used condition distribution adaptation. Q-TRANSFER [52], another DTL framework proposed by Trung V. Phan et.al. To mitigate the dataset insufficiency problem in the context of communication networking, a DTL-based reinforcement learning approach is used.

As the COVID-19 disease spread is terrifying all over the world, screening, quarantine, and providing appropriate treatment to COVID-19 patients has become the first priority in the current scenario. But the global standard diagnostic pathogenic laboratory testing is massive time consuming and more costly with significant false-negative results [53]. At the same time, tests are are hardly to be taken place in the common healthcare centres or hospital due to limited resources and places compared with the high volume of cases at one time. To combat this kind of situation, the researcher from this domain are trying hard to develop some possible DTL models to mitigate this challenges [54, 55]. As for example M. Loey et.al in [54] use DTL along with the GAN model on their very limited, only 307 chest X-ray images to test COVID-19 disease based patient chest X-ray. Here, they have three pre-trained state-of-the-art model namely AlexNet [30], GoogleNet [33], and ResNet18 [34]. Among these three pre-trained GoogleNet give the highest accuracy in their experimental studies.

2.3. Edge Computing(EC)

In the era of cloud computing, maximum technology depend organization in the world rely on very few selected cloud providers for hosting and computation. The user's data from millions of devices around the world is being delivered to some centralized cloud providers for processing or computation. This data transformation always resulted in extra latency and extra bandwidth consumption [56]. The explosive proliferation of IoT devices along with the requirement of real-time computing power have forced to move the scenario of computing paradigm towards Edge computing. Therefore, instead of relying on doing all the work at a cloud, it focuses to start the computational process close to the IoT devices (near to the source of data) in order to reduce the utilized bandwidth and latency [57, 58]. Sometime in Edge computing, an additional nearby server called Fog is associated between the cloud and the Edge or IoT devices. It locally stores the copy of densely used data from the cloud and it provides additional functionality to IoT devices to analyze and process their data locally with real-time working capability. Hence only the relevant data from IoT devices is need to transferred to the cloud through the Fog server [59].

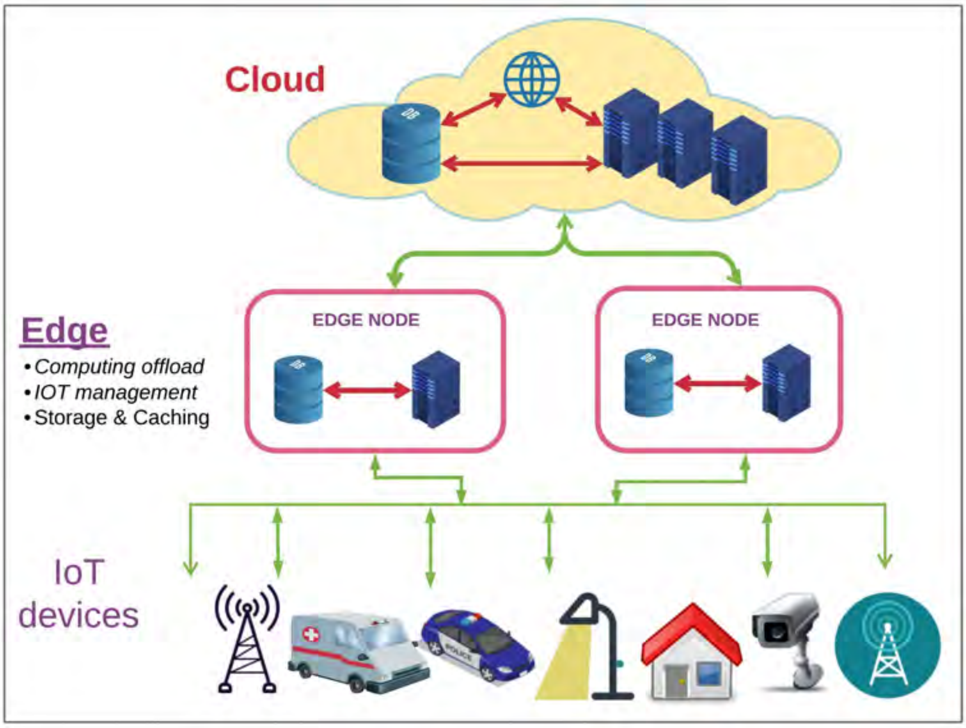

Figure 3: Hierarchy of Edge, Fog, and Cloud Computing.

In figure 3 the hierarchy of a possible framework for Edge Computing association with Fog and Cloud computing is illustrated. The data are collected from various IoT devices are being pre-processed before sending by Edge to Fog server for the analysis and computation with the real-time speed (because of the minimal distance between Edge layer and IoT devices and the local database of Fog). While the cloud holds the central control system and it manages the whole database of the system. The database on the cloud is continuously uploaded by the Fog only when it has important data or information.

Although EC is not a new concept but it becomes popular in the last five years in the era of IoT [60, 61]. Few recent different type of state-of-the-art of EC are mentioned in this section as a way to familiarize the reader with the recent development with the era of EC and its potential benefits in mitigating COVID-19 as pandemic. EdgeIoT [62], a study of mobile EC proposed by X. Sun et.al. It is a SDN-based EC work with Fog Computing(FC) [63] to provide computational load locally. In a study [64], F. Wang et.al have proposed a joint offloading strategy of mobile EC and wireless power transfer. This scheme tried to address energy consumption, latency, and access point issues in IoT. In another study [65], Wei Ding et.al propose a field-programmable gate array-based depth-wise separable CNN accelerator to improve the system throughput and performance. They have used double-buffering-based memory channels to handle the data-flow between adjacent layers for mobile EC. On the other hand, G. Premsankar et.al in their case study [66] have discussed how efficient mobile gaming can run through EC. In a study [67], S. Wang et.al have proposed a mobile edge computing with an edge server placement strategy. In their multi-objective constraint optimization-based EC have tried reduced delay between a mobile user and an edge server. In-Edge AI [67], an integrate the deep reinforcement learning techniques and Federated Learning framework with mobile edge systems are proposed by X. Wang et.al. This framework intelligently utilizes the collaboration among devices and edge nodes to exchange the learning parameters for betterment. In another recent study [68], an integrated two key technologies, ETSI and 3GPP are introduced to enhanced slicing capabilities to the edge of the 5G network. In the case of COVID-19 like pandemics, discussion of the possible role of EC is done in section 4.3.

3. Review of Related State-of-the-art

Although the whole article is referred and cited current relevant state-of-the-art wherever relevant, this section is dedicated to provide a review on some of the very generic recent state-of-the-art works related to transfer learning approaches over edge computing. As mentioned in section 2.1, the progress of DL is very fast but when it comes to application in Edge or IoT devices then a huge gap is noticeable [69]. However, researchers are working hard to cope with the challenges, as results in many computing ideas, optimized model, as well as some computing accelerator devices, comes in picture [70, 71]. Deep Transfer Learning as mentioned in section 2.2 is one such area that is useful where the size of datasets is not sufficient [45]. This transfer learning is also useful where computing resources are not sufficient such as Edge or IoT devices [72]. Since edge computing becomes popular in the last few years, so, we restricted this review to the last five years with chronological order.

Lorenzo Valerio et.al have studied the trade-off between accuracy and traffic load of computing in edge-based on transfer learning [73]. They have suggested that sometimes the partial model needs to move across edge devices and data will stay at those edge devices and vice-versa. In a study [74], Tingting Hou et.al proposed a transfer learning approach in edge computing for proactive content caching. In their learningbased cooperative caching technique they have used a greedy algorithm for solving the problem of cache content placement. On the other hand, Junjue Wang et.al proposed a model [75] for live video analytic through drone using edge computing. They have used a transfer learning approach to formulate a pre-trained model to apply a few aerial view image classification. In another study [76], Ragini Sharma et.al proposed a teacher(large networks) student(small network at edge) model using transfer learning. The applied different transfer learning techniques of teacher-student with considering accuracy and convergence speed.

Qiong Chen et.al used a multitask transfer learning in their work [77]. In their data-driven cooperative task allocation scheme, they have used the concepts of the Knapsack problem to prioritized the tasks before transferring them for use in another task. In a study [78], Wen Sun et.al suggested an edge-cloud framework. Here, pre-trained networks used in their framework that are trained in the cloud. In other work, Rih-Teng Wu et.al proposed an edge computing strategy for autonomous robots [79]. They have used CNN with pruning through the transfer learning technique. In their presented work, pr-trained VGG16 [32] and ResNet18 [34] are used for classification after fine-tuning. Cartel [80], a model of collaborative transfer learning approach for edge computing was proposed by Harshit Daga et.al. Here, they have created a model-sharing environment where a pre-trained model was adapted by each edge according to the needs. In a study [81], Yiqiang Chen et.al proposed a framework using Federated Transfer Learning for Wearable Healthcare (FedHealth). They have first performed data aggregation using federated learning and then created personalized models for each edge using transfer learning. OpenEI [82], an edge intelligence framework that was proposed by Xingzhou Zhang et.al. This framework with lightweight software equips with the edges as well as intelligent computing and data sharing capability.

In a research study [83], Changyang She et.al proposed a reliable low latency communication and edge computing system. They have adopted deep transfer learning in the architecture to fine-tune the pre-trained networks in non-stationary networks. This proposed work was designed for future 6G networks systems. On the other hand, Guangshun Li et.al proposed a task allocation load balancing strategy for edge computing [84]. They have used the concept of transfer learning from cloud to intermediate node to edge. In another study [85], Gary White and Siobhan Clarke have proposed a deep transfer learning-based edge computing for urban intelligent systems. They have also used VGG16 pretrained network at edge devices and experimented to classify Dog vs. Cat images. MobileDA [86], a domain adaption framework in edge computing was proposed by Jianfei Yang et.al. Here, a teacher network was trained in a server and transfer knowledge or feature to student networks was implemented at the edge side. Their model was evaluated and obtained promising results on an IoT-based WiFi gesture recognition scenario. Davy Preuveneers et.al proposed a resource and performance trade-off strategy for a smart environment [87]. They have used a transfer learning model for less training efforts in smart edge devices. In their study, multi-objective optimization also was utilized to optimize the trade-off between computing resources uses and performances.

4. Existing Computing (DL, DTL, EC & Dataset) to Mitigate Pandemic

As mentioned in section 1, the appropriate drugs, vaccines, infrastructure are not ready up to some stages of any pandemic. Therefore, to cope with challenges existing knowledge, infrastructures, AI-based models could be exploited to mitigate such pandemic. This section tried to bring four insights of the discussion topics and their roles in mitigating pandemics. Each of them is systematically discussed with potentiality with recent state-of-the-art and challenges.

4.1. Deep Learning Approaches to Mitigate Pandemic

4.1.1. Potentiality

As described in section 2.1, Deep Learning(DL) can extract features directly from labeled data. In COVID-19 like pandemic data are new, so, handcrafted feature engineering might be difficult. But for DL, no feature engineering required, so that problem could be solved. The DL can assist in many ways to mitigate COVID-19 like pandemics along with other healthcare issues [88]. Some of them are Testing Sample Classification, Medical Image Understanding, Forecasting, etc [89]. Some recent DL based models have already proposed to cope with pandemics are listed and their main features are highlighted in table 1. This table brings some proposed peer-reviewed as well as few promising pre-print works. Table 1 has placed some recent works in upper rows.

Table 1: Recent DL based works to mitigate pandemics

Reference of Proposed Works	**Dedicated task of a Pandemic**	**Main Contributions**
F.Ucar and D.Korkmaz [90]	Deep Bayes-SqueezeNet based diagnosis of COVID-19 from chest X-ray images.	• Develop an intelligent diagnosis system for COVID-19 using practical DL model for medical image processing. • A new decision-making system for detecting COVID-19 with the integration of conventional and state-of-the-art methods for chest X-ray.
D. Das et.al [91]	Screening chest X-ray images to classify COVID-19 positive or negative using a CNN model.	• Proposed a CNN based model called Truncated Inception Net for the task. • Their works pointed out a good accuracy on different datasets to classify COVID-19 pneumonia, Normal Pneumonia, Tuberculosis, and healthy cases.
C.Butt et.al [92]	Screen COVID-2019 pneumonia with multiple CNN-based model.	• A study that compared results of multiple CNN models to classify CT samples with COVID-19, influenza viral pneumonia, and no-infection. •Mentioned an accuracy of 0.996 (95%CI: 0.9891.00) for COVID-19 vs Non-COVID-19 cases in CT studies, and calculated a sensitivity: 98.2% and specificity: 92.2%.
H. Mukherjee et.al [93]	A shallow CNN-based automatic COVID-19 cases detected using chest X-rays.	•Proposed a light-weight shallow CNN-tailored model that can detect COVID-19 positive cases in chest X-rays. • They considered different dataset including MERS, SARS, and pointed out a good accuracy on automatic detection of these cases.
S.Hu et.al [94]	COVID-19 Infection Detection and Classification from CT Images.	• Weakly supervised DL for detecting and classifying COVID-19 infection from CT images. • Minimize the requirements of labeling of CT images.
T.Ozturka et.al [95]	An automated detection of COVID-19 cases using raw chest X-ray.	• A DL-based COVID-19 vs No-finding as well as COVID-19 vs No-finding vs. Pneumonia as binary and multi-classification model. • A combined of YOLO and DarkNet model [96] used as the backbone of the model to achieved a good accuracy.
E.Luz et.al [97]	COVID-19 Patterns Detection in X-ray Images.	• Identification of COVID-19 disease. • A resource efficient model with overall accuracy of 91.4%, COVID-19, sensitivity of 90% and positive prediction of 100% in the dataset from [98].
M. Zhou et.al [99]	Differentiating novel coronavirus and influenza pneumonia using CT images.	• An early diagnosis tool on chest CT images for differentiate COVID-19 pneumonia and normal Influenza with transferability.
O.Gozes et.al [100]	Automated Detection & Patient Monitoring using Deep Learning on CT Image Analysis.	• A model utilizing 2D-3D DL for clinical understanding. •Proposed a systematic continuous monitoring methodology for COVID-19 patients and their clinical data to make a statistical Corona score for monitoring their progress.
S.M. Ayyoubzadeh et.al [101]	Predict the incidence of COVID-19 in Iran.	• Data were mined that taken from Google Trends. • Linear regression and LSTM models were used to estimate COVID-19 positive cases
L. Li et.al [102]	Fully automatic DL framework to detect COVID-19 using CT images of chest.	• Developed a DL-based model, COVNet to detect COVID-19 by extracted features from chest CT images. • Collected dataset consisted of 4356 chest CT images from 3,322 patients from six different hospital.
L. Wang et.al [39]	An publicly available chest X-Ray image dataset and a CNN for detection of COVID-19.	• Proposed a publicly available COVID-Net, a deep CNN for the detection of COVID-19 cases from CXR images. • COVIDx, an open access chest X-ray(CSR) dataset consisting 13,800 CSR images of 13,725 patients.
S.J. Fong [103]	A forecasting model of COVID-19.	• A Composite Monte-Carlo simulation forecasting model. • A case study of using the above simulation through DL.
A. Lopez-Rincon et.al [104]	Identification of SARS-CoV-2 from Viral Genome Sequences.	• Interaction between viromics and DL model. • A DL-based model for assisted detection tests for SARS-CoV-2.
S. Chae [105]	Predicting infectious disease using DL and Big data.	• A study on DL and LSTM model over ARIMA model to predict future infectious diseases. • Proposed model tried to improve existing surveillance systems to detect future infectious diseases.

4.1.2. Analysis and Challenges

From table 1 it could be drawn one conclusion that the majority of the works are for assisting radiologists to diagnose diseases. Some of are mentioned forecasting, fake news alert, etc, but more critical parts of this pandemic maybe addressed by this DL approach. Successfully apply DL in COVID-19 or any running pandemic has three main challenges. The first one is a shortage of reliable datasets. As data collection and validation are a time-consuming process as well as privacy issues also there whereas a pandemic or epidemic comes suddenly. The second one is the variety of data of a pandemic virus. This COVID-19 virus 'SARS-CoV-2' has mutating itself over different geographic regions, environments, and time [106, 107]. Therefore, the pandemic dataset collected from one region may not be work to drawn inference on the pandemic of other regions. The third one high computational resources required for a DL model whereas to cope with an outbreak IoT or Edge Device (ED) are useful for many purposes [16]. Though these types of equipments have low computing resources.

In order to overcome such challenges, cleaver implementation of relevant AI strategies is required. For the first two challenges, DTL or few shot learning and GAN [29] could be a possible approach towards possible solutions. DTL has described in section 4.2 whereas details about GAN are out of the scope of this article. The third challenge could be mitigated using Cloud Computing, Fog Computing, and Edge Computing [108]. However, for Cloud or even Fog Computing latency and data security & privacy could be a problem. Therefore, Edge Computing could be effective for the third challenge, which has described in section 4.3.

4.2. Deep Transfer Learning to Mitigate Pandemic

4.2.1. Potentiality

Section 2.2 has described about Deep Transfer Learning (DTL) in general. In this sections, how DTL could help to mitigate COVID-19 like pandemics is described. As mentioned, sufficient datasets of COVID-19 or any running pandemic are difficult to develop in a short period of time. Therefore, to exploit the benefit of DL to cope with COVID-19 or other pandemics are a bit challenging. Therefore, DTL could be effective in this case. As through DTL a DL model could be trained using a large scale benchmark dataset and learned features could be used in the domain of COVID-19 [55]. Many researchers are trying hard to use this DTL in the domain COVID-19 for many purposes. We have tried to summarize in table 2 some of the recent state-of-the-art along with their main contribution towards mitigation of pandemics. As the number of peer-reviewed work is limited as this pandemic is new, so this table also has listed some pre-print works, which have tried to introduce some of the contributions in mitigating this current pandemic.

Table 2: Recent DTL based works to mitigate pandemics.

Reference of Proposed Works	**Dedicated tasks of a pandemic**	**Main contributions**
J. P. Cohen et.al [109]	A severity score prediction model for COVID-19 pneumonia for frontal chest X-ray images using beside tool.	• A DTL model that was pre-trained on large size non-COVID-19 chest X-ray datasets for predicting COVID-19 pneumonia. This study uses a pre-trained model predicts a geographic extent score of range 0-8 with 1.14 MAE and lung opacity score of range 0-6 with 0.78 MAE. • A COVID-19 chest image dataset from a public COVID-19 database were scored retrospectively by three experts.
S.Minaee et.al [110]	Predicting COVID-19 using transfer learning from Chest X-Ray Images.	• DTL on a subset of 2000 of 5000 radiograms was used to train four popular CNN, including ResNet18, ResNet50, SqueezeNet, and DenseNet-121, to identify COVID-19 disease. • Evaluated these trained models on the remaining 3000 radiograms and achieved a sensitivity rate of 97%(5%), while a specificity rate of 90% approx.
S.Basu et.al [111]	Screening COVID-19 using chest X-Ray images.	• A domain extension transfer learning with pre-trained deep CNN, that is tuned for classifying among four classes: normal, other diseases, pneumonia, and Covid19. • A 5-fold cross-validation has experimented and overall accuracy measured 95.3%0.02.
N.E.M Khalifa et.al [112]	An Experimental Case on a limited COVID-19 chest X-Ray dataset.	• A study on neutrosophic and DTL models on limited COVID-19 chest X-Ray dataset. • They first converted grayscale X-ray images into neutrosophic images then applied pre-trained AlexNet, GooglNet, and ResNet18 to classify four classes: COVID-19, Normal, bacterial, and virus pneumonia.
B.R. Beck et.al [113]	Predicting commercially available antiviral drugs that may act on SARS-CoV-2 through the DL model and a drug-target interaction.	• Drug-target interaction model called MT-DTI to recognize commercially available drugs that could use for SARS-CoV-2. • Proposed a list of antiviral drugs identified by this MT-DTI model.
A. Narin et.al [114]	Automatic Detection of COVID-19.	• Three different CNN (ResNet50, InceptionV3, and Inception-ResNetV2)-based models for the detection of COVID-19 infection using X-ray radiography. • Proposed pre-trained ResNet50 has given the best result among these three.
I.D. Apostolopoulos et.al [55]	Performance evaluation of state-of-the-art CNN architectures through TL over medical image classification.	• Suggested DL with X-ray imaging may extract significant bio-markers related to the COVID-19 disease. • A dataset of 1427 X-ray images consisting of 224 images of Covid-19 disease, 700 images of common bacterial pneumonia, and 504 images of normal conditions.
B. Subirana et.al [115]	New crowd-source AI approach to support health care dealing with COVID-19.	• A proposed transfer learning works on recognition of cough sound recording by phone as a diagnostic test for COVID-19.
N.E.M. Khalifa et.al [116]	Detection COVID-19 using GAN and TL method.	• A combination of GAN and DTL models for enhancing accuracy of testing. • Their ResNet18-based combined model achieved state-of-the-art accuracy in a chest x-ray dataset.

4.2.2. Analysis and Challenges

DTL does task adaption that is very necessary for analyzing, diagnosing as well as mitigating COVID-19 like pandemics. The number of studies is not large; in addition most of the existing studies and experiments on COVID-19 were applied for chest image analysis as cited in table 2. Only a few among them are proposed for target drug interaction, cough sound classification, etc. Lots of work could be done to mitigate this pandemic such as Intensive Care Unit(ICU) Monitoring, Patient Care, Hygienic Practice Monitoring, Wearing Personal Protective Equipment(PPE) Monitoring, Monitoring Systematic Social Distancing, Automatic fever detection, rumor detection, economical and social impact, etc. Most of these works could be easier when AI is cooperating and forming such a model along with IoT or ED [37]. Some issues could be solved by EC as described in section 4.3. Though a better system could be delivered when the most suitable algorithm applied on EC. One possibility of archiving this when DTL implemented alongside with EC which as conceptually describes in section 5.

4.3. Edge Computing to Mitigate Pandemic

4.3.1. Potentiality

Edge or IoT devices-based sophisticated equipments such as smart medical equipment, webcam, drone, wearable sensors, etc are very useful in a pandemic like situations [117]. As mentioned in section 2.3, edge computing brings the computation to near edge devices. It reduces latency, security & privacy issue, etc. Therefore, this computing paradigm will be very effective to mitigate a pandemic situation [118]. The researchers from all over the world are trying hard to bring this along with other AI techniques to mitigate current COVID-19 pandemic [16, 119]. So far only a limited number of studies have investigated the use of EC in obtain an efficient and effective mitigation system of COVID-19. This subsection tried to bring some potentiality and scopes which shall help to mitigate COVID-19 like pandemics. Table 3 has mentioned some EC based studies on COVID-19 and related healthcare.

4.3.2. Analysis and Challenges

EC works on site, so, many benefits could draw from EC with IoT or ED. Nevertheless as mentioned IoT or ED has limited computing resources. Therefore, to get the benefit of modern AI algorithm such as DL it is still challenging. To cope with these challenges researchers from all over the world are working hard to propose many ideas [70, 126, 18]. But so far only a few studies on EC in pandemic are proposed in limited areas of application as mentioned in table 3. Assisting many critical COVID-19 related tasks such

Table 3: Recent EC based works to mitigate pandemics.

Reference of Proposed Works	**Dedicated task related to a pandemic or healthcare**	**Main contributions**
A.Sufian et.al [37]	EC based model to stop spread COVID-19	• An EC based ICU, Critical Areas monitoring model. • Proposed DL and Computer Vision-based surveillance model.
C.Hegde et.al [120]	An open-source EC for clinical screening system.	• Fever and Cyanosis detection using visible and far-infrared cameras emergency department. • This image segmentation-based EC uses open source hardware.
A.A.Abdellatif et.al [121]	Data and application-specific energy-efficient smart health systems	• An optimizes medical data transmission from edge nodes to a healthcare provider with energy efficiency and quality-of-service. • Managing a heterogeneous wireless network through EC to provide fast emergency response.
A.H. Sudhro et.al [122]	QoS optimization in medical healthcare applications.	• A window-based Rate Control Algorithm to QoS in mobile EC. • A framework for Mobile EC based Medical Applications.
M.Chen et.al [123]	Smart Healthcare System.	• Edge cognitive computing-based smart healthcare mechanism to dynamic resource allocation in healthcare.
P.Pace et.al [124]	Efficient Applications for Healthcare Industry 4.0	• Proposed BodyEdge, an architecture suited for human-centric applications in context of the emerging healthcare industry. • A tiny mobile client module with EC for better health service.
H.Zhang et.al [125]	Smart Hospitals Using Narrowband-IoT.	•An architecture to connect intelligent things in smart hospitals based on Narrowband IoT. • Smart hospital by connecting intelligent with low latency.

as remote sensing-based COVID-19 patient monitoring, Hygienic practice monitoring, systematic social distancing monitoring in a crowded area, etc could be done through EC [37]. This article brings a conceptual model of EC with DTL in section 5 as a future scope to cope with such challenges.

4.4. Dataset to Mitigate Pandemic

4.4.1. Potentiality

Data is the fuel of a modern computing. Whether it is medical field or retailer market, in every field data are the most precious things. Recent AI techniques are mostly follow data driven approaches [127, 128]. DL or DTL based algorithms almost fully depend on the dataset. Therefore, to cope with a pandemic, data is one of the driving forces. For a pandemic as COVID-19, the dataset could be chest X-ray, CT images, pathological images, geographical region based spreading patterns, seasonal behavior, regional mortality rates, impact on the economy, etc. [129]. In table 4 some available datasets related to COVID-19 like pandemics are mentioned with brief description.

Table 4: Some Dataset for COVID-19 like pandemics.

Name of dataset and Reference	**Brief description**
COVID-CT-Dataset [130].	• A publicly CT scan dataset consisting of 275 positives for COVID-19 cases.
COVID-19 X-ray image dataset with two different combinations for applying with DTL models.[55]	• One dataset of 1427 X-ray images consisting of 224 images of Covid-19 disease, 700 images of common bacterial pneumonia, and 504 images of normal conditions. • Another dataset of 1442 X-ray images consisting of 224 images of Covid-19 disease, 714 images of common bacterial pneumonia, and 504 images of normal conditions
COVID-19 Image Data Collection [98].	• It was a crowd-sourcing hosting currently contains 123 frontal X-rays images.
Chest CT Images [102]	• A dataset consisted of 4356 chest CT exams from 3,322 patients. • Data are collected from six hospitals of average age is 49 years, among them 1838 male patients.
Coronavirus Twitter Dataset [131].	• A multilingual COVID-19 Twitter dataset that has been continuously collecting since Jan 22, 2020. • It consists of an online conversation about COVID-19 to track scientific misinformation, rumors, etc.
COVIDx CXR Dataset [39].	• This dataset consisting of 13,800 images of chest radiography across 13,725 patients.
Epidemiological COVID-19 data [132].	• Individual-level data from municipal, provincial, and national health reports, as well as additional information from online reports. • All data are geo-coded including where available, including symptoms, key dates, and travel history.
H1N1 Fever Dataset [133].	• Two datasets collected at Narita International Airport during the H1N1 pandemic 2009. • The first dataset only 16 candidates and the second one is 1049 collected using infrared thermal scanners.
Registry data from the 191820 pandemic. [134].	• A high-quality vital registration data with mortality for the 191820 pandemic from all countries.

4.4.2. Analysis and Challenges

As mentioned data is driving force to which bring the knowledge but it not easily available. Specially COVID-19 or a sudden pandemic, gathering data and arrange it in a knowledgeable form are not expected as an easy task. Although for COVID-19, many sectors are very active as a result many data sources are quickly oriented towards COVID-19 pandemic. Some data sources are listed in table 5 where COVID-19, as well as other pandemic data, are available, so, researchers may use them for many purposes. The main challenges are sufficient datasets especially machine-readable datasets in every affected sector are yet to be available. Therefore, that are the challenges for data-driven AL algorithms or models, hence existing studies on real data and analysis are few. Although some related datasets mentioned in table 4 are available but most of them are for clinical purposes. As said this novel coronavirus is behaving differently across geographic regions, different environments, etc. Therefore, data of one region may not be effective to enhance knowledge in other regions. Data privacy and security also are considered ones of the big issues. To this reason this article suggesting transfer learning

Table 5: Some Data Sources for COVID-19 like pandemics.

Sources and/or Reference	**Brief description**
World Health Organization(WHO) [3]	• WHO leading this battle by providing each and every possible data and information. • Most of the data are unstructured so it bit challenging to feed into an AI model.
Johns Hopkins University is in the forefront to provide COVID-19 dataset [135] through their portal: https://coronavirus.jhu.edu	• A machine-readable dataset that aggregates relevant data from country-level governmental, academic sources, journalistic, etc. • Some notable COVID-19 dataset are 'county-level time-series', 'healthcare system-related metrics', 'climate', 'transit scores', 'hospital', etc.
University of Oxford dataset regarding COVID-19 at their portal: https://www.bsg.ox.ac.uk/news/coronavirus-research-blavatnik-school.	• Oxford Covid-19 Government Response Tracker (OxCGRT), an index-based data indication which govt. taking what kind of policy. • Several policy data of different govt. taken during pandemic including education policy and their impact.
European Union provides an open data portal: https://www.europeandataportal.eu/en/highlights/covid-19.	• Open data and COVID-19: provide many dataset medical data, spreading data, etc. • An Interactive map are provided and by clicking region specific dataset can be downloaded.
European Center for Disease Prevention and Control(ECDC): https://qap.ecdc.europa.eu/public/extensions/COVID-19/COVID-19.html.	• Many datasets about infectious diseases including COVID-19. • Enhanced surveillance dataset including daily update dataset, medical dataset, public health in communicable diseases.
Google: https://google.com/ covid19-map/.	• Different statistical, numeral data including number active cases, number of death, number of recovered. • Provide COVID-19 interactive map in addition dedicated dataset search engine that is also available.
GitHub: https://github.com/open-covid-19/data.	• An open repository where many datasets is stored. • Many research projects stores their data and mentioned links to their article, but they provide a link to see and access the COVID-19 dataset.
Kaggle: https://www.kaggle.com/c/covid19-global-forecasting-week-#.	• An online community of data scientists and machine learning practitioners • Forecasting dataset and other COVID-19, or pandemic dataset available.

approaches to be used in developing models for mitigating COVID-19 like pandemics or epidemics.

5. A Precedent Pipeline of DTL over EC to Mitigate COVID-19.

As mentioned 4.1.2, DL has some limitations to cope with the challenges of a pandemic whereas section 2.2 has described the task adaptability through DTL where data shortages are there. Section 2.3 mentioned the potentialities of EC where computing power is low. Therefore, the merging of these three computing models could be more effective in assisting the mitigation of pandemic situations. This combined model, that is, Deep Transfer Learning over Edge Computing (DTLEC) will take the power of DL through DTL as well as would be applicable in critical sectors by EC to cope with a sudden pandemic. There are some studies that exist in DTLEC as in [70] and some related work mentioned in section 3. However, these works are still in general concept or their proposed methods were targeted to some other application areas. As per literature studies, this idea has not been studied or experimented to mitigate COVID-19 pandemic. This section tried to present a precedent working pipeline of DTLEC to assist mitigation of pandemic or epidemic.

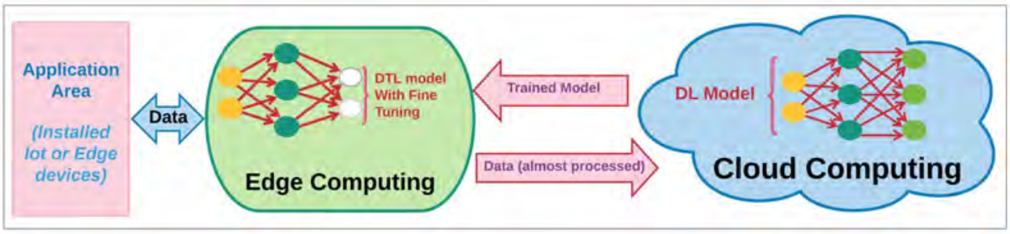

Figure 4: Proposed pipeline for DTL in EC.

5.1. Model Description

DTLEC model could be helpful in the healthcare sector, quarantine center, or other critical areas where an outbreak may arise. This methodology shall also use in remote health monitoring including elder people care at home. As in figure 4 edge or IoT devices that are set up in those areas may be embedded with EC, and then it could be connected with a cloud server. A state-of-the-art DL model shall train in GPU enabled cloud server by using a publicly available dataset for feature extraction. Then a pre-trained model (with extracted weight or features except for classification layer) shall push down to the edge devices. In edge devices required fine-tuning mechanism to be implemented into that model with some real data. In this way, the model may ready to work in some critical areas where outbreaks are affected such as hospitals, crowded places, and many more.

In figure 5 a typical current COVID-19 outbreaks situation and possible working model are shown. This figure illustrating the proposed framework to tackle the COVID-19 situation by using DTL in EC in both COVID-19 patient care and management systematic social distancing. In the first scenario, we may use several healthcare sensors like blood pressure sensors, body temperature sensors, webcam, etc to sense the data about the running health condition of each patient. Then all of the collected data would be sent to the EC layer where a pre-trained DL based model will be used to process the captured data and making an inference out of it. If the generated report is a critical health condition then an automatic system alert message will be sent with all the details to the hospital control room and also to

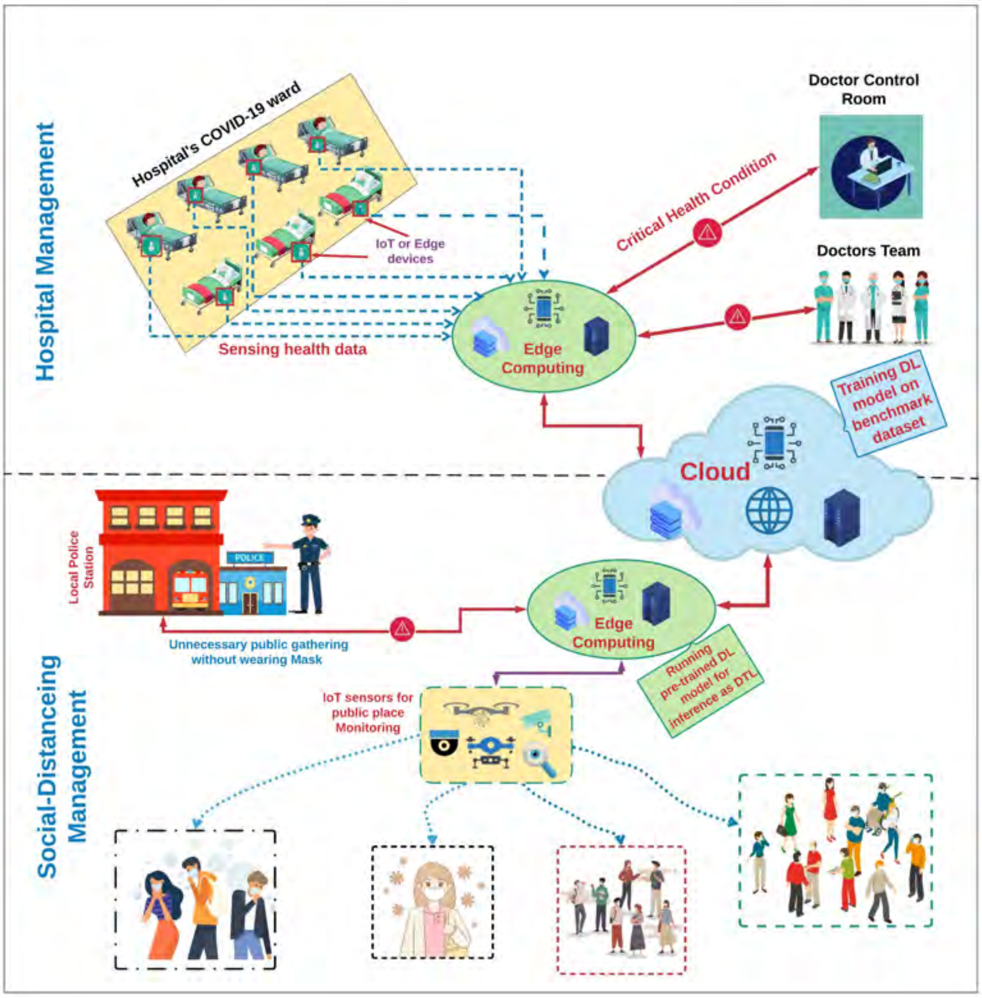

Figure 5: Framework for Edge Computing.

all the doctors of associated team. In the second scenario, several public place monitoring sensors (like a drone, CCTV, traffic cameras, etc) could be used to detect unnecessary illegal crowd with or without wearing personal protective equipment (PPE) with help from the DTLEC-based model. If the model finds any such activities then an automatic alert message will be sent with details to the nearby police station or any respective authority.

5.2. Future Challenges

This model may be successfully deployed in some critical sectors such as hospitals, airports, markets, emergency service areas, and those areas which are the primary hotshots for spreading infectious deceases. This methodology may also use for remote health monitoring with further study. The model has to be work on real data to draw the inference. In order to make it successfully deployed, lots of collaborative work need to be done, which may face many challenges. Some challenges need to be addressed such as (i.) At first, IoT or Edge devices need to be connected with each other and a cloud server, hence am optimized sensor type of networking protocol will be required. (ii.) EC through DTL need be implemented, for that appropriate pre-trained deep learning need be carefully selected after some studies. (iii.) For the transfer learning approach, using only EC is not sufficient, while the adoption of EC-Fog-Cloud combined model would be more useful. A deep learning model shall be trained at a cloud server using a publicly available dataset for feature extraction. After that pre-trained model will push down to the edge where limited re-training (or fine-tuning) shall be carried out to orient a few last layers for required inference. So, at least a small size pandemic dataset with ground truth label needs to be created. Here, the Fog server could work as a cluster. (iv.) Security and privacy issues of data need to be addressed. This inquires much more attention in analyzing numerous vulnerabilities associated with such an outbreak due to rumors and fake news. Besides, the privacy of captured data from multiple sources (things in IoT or individuals) will open a new research direction for the near coming future. (v.) A new simulation model is required for experimental studies. These are a few of the many challenges we can work for.

6. Conclusion

This article has tried to bring potentialities and challenges of Deep Transfer Learning, Edge Computing and their related issues to mitigate COVID-19 pandemic. It has also proposed a conceptual combined model with its scope and future challenges of working at critical sites and real data. As the running pandemic is new, so, there is a limited number of peer-reviewed studies and experimental results. Therefore, this systematic study article also considered some pre-print studies which are tried to make some contributions in mitigating running pandemic. The running pandemic definitely will be mitigated but there will be a leftover impact on global health, economics, education, etc. Therefore, mitigation of this pandemic as early as possible becomes necessary to restrict further worsen. In addition, every scientific community of the world needs to think wisely to get prepared to cope with such kind of crisis in a case similar outbreaks appear in the future. This article will definitely assist the research community; especially the practitioners of deep transfer learning and edge computing, to work further in developing many methodologies, tools, and applications, towards the mitigation of running pandemic or any future pandemic if that arises.

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A Study of the Neutrosophic Set Significance on Deep Transfer Learning Models: An Experimental Case on a Limited COVID-19 Chest X-Ray Dataset

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Nour Eldeen M. Khalifa, Florentin Smarandache, Mohamed Loey (2021). A Study of the Neutrosophic Set Significance on Deep Transfer Learning Models: An Experimental Case on a Limited COVID-19 Chest X-Ray Dataset. *Cognitive Computation*, 17; DOI: 10.1007/s12559-020-09802-9

Abstract: Coronavirus, also known as COVID-19, has spread to several countries around the world. It was announced as a pandemic disease by The World Health Organization (WHO) in 2020 for its devastating impact on humans. With the advancements in computer science algorithms, the detection of this type of virus in the early stages is urgently needed for the fast recovery of patients. In this paper, a study of neutrosophic set significance on deep transfer learning models over a limited COVID-19 chest x-ray dataset will be presented. The study relies on neutrosophic set theory, as it shows a huge potential for solving many computers problems related to the detection, and the classification domains. The neutrosophic set in this study is used for converting the medical images from the grayscale spatial domain to the neutrosophic domain. The neutrosophic domain consists of three types of images and they are, the True (T) images, the Indeterminacy (I) images, and the Falsity (F) images. The dataset used in this research has been collected from different sources as there is no benchmark dataset for COVID-19 chest X-ray until the writing of this research. The dataset consists of four classes and they are COVID-19, Normal, Pneumonia bacterial, and Pneumonia virus. The selected deep learning models for investigation are Alexnet, Googlenet, and Restnet18. Those models are selected as they have a small number of layers on their architectures. To test the performance of the conversion to the neutrosophic domain, 36 experimental trails have been conducted and presented. A combination of training and testing strategies have been applied into dataset by splitting it to (90%-10%, 80%-20%, and 70%-30%) accordingly. According to the experimental results, the Indeterminacy (I) neutrosophic domain achieves the highest accuracy possible in the testing accuracy and performance metrics such as Precision, Recall, and F1 Score. The study concludes that using the neutrosophic set with deep learning models may be an encouraging transition to achieve higher testing accuracy, especially with limited datasets such as COVID-19 chest x-ray dataset which is investigated throughout this study.

Keywords: Coronavirus; Neutrosophic; deep transfer learning; COVID-19; SARS-CoV-2; CNN;

1. Introduction

Severe acute respiratory syndrome-related coronavirus (SARSr-CoV) is a kind of B-coronavirus that infects bats and some other mammals. In 2002–2004 SARSr-CoV flare-up was an epidemic covering severe acute respiratory syndrome (SARS). The Canton in South China was infected with B-coronavirus that lead to SARSr-CoV. SARSr-CoV was a kind of coronavirus as a family of the B-coronavirus (B- CoV) subgroup and was title as SARSr-CoV [1]. The subclasses of the coronaviruses family are A-coronavirus (A-CoV), B-coronavirus (B-CoV), C-coronavirus (C-CoV), and D-coronavirus (D-CoV) coronavirus [2]. Historically, SARSr-CoV, across 29 countries in the world, infected over 8000 humans and at least 750 died. The 2019–2020 coronavirus epidemic is an ongoing scourge of coronavirus malady 2019 (COVID-19) created by severe acute respiratory syndrome coronavirus-2 (SARS-CoV-2) [3]. The International Committee of Viruses and World health organization (WHO) official names the B-coronavirus of 2019 as SARS-CoV-2 and the malady as COVID-19 in the International Classification of Diseases (ICD) [4–6]. However, SARS-CoV-2 infected more than two million of humans with more than 200000 deaths, across 230 states, until the date of this lettering. It elucidates that the propagate rate of SARS-CoV-2 is greater than SRAS-CoV [7,8]. The SARS-CoV-2 transmission has been assured by the World Health Organization (WHO) with evidence of human-to-human conveyance from different cases outside China, namely in Vietnam [9], Italy [10], Germany [11], US [12]. On 26 April 2020, SARS-CoV-2 confirmed more than 2,920,660 cases, 829,075 recovered cases, and 203,622 death cases [13].

The theory of neutrosophic logic was proposed by Smarandache in 1995. Afterward, it has been unified and generalized by its founder in 1999 [14]. Since that date, neutrosophic logic has been used in many computer science fields including pattern recognition [15], image segmentation, and processing [16] and more. It contributes to solving many research and practical real-life problems in a lot of domains such as medicine [17], economics [18], space satellite[19], and agriculture…etc. Neutrosophy leads to a whole family of novel mathematical theories with an overview of not only classical but also fuzzy counterparts [20]. The term neutro-sophy means knowledge of neutral thought and this neutral represents the main difference between fuzzy and intuitionistic fuzzy logic and set [21,22]. Neutrosophic set has the required potentials of being a general framework for uncertainty analysis in data sets[21] and especially with images in the field of Artificial Intelligence and deep learning.

Deep Learning (DL) is a type of Artificial Intelligence (AI) concerned with methods inspired by the functions of people's brain [23]. For the time being, DL is quickly becoming an important method in image/video detection and diagnosis. DL used in medical x-ray/computed tomography diagnoses. DL upgrade medical diagnosis system (MDS) to realize great results, and implementing applicable real-time medical diagnosis system [24,25]. Convolutional Neural Network (ConvNet or CNN) is a mathematical type of DL architectures used originally to recognize and diagnose images. CNN's have masterful unusual success for medicinal image/video diagnoses and detection. In 2012, [26,27] introduced how ConvNets can boost many image/vision databases such as Modified National Institute of Standards and Technology database (MNIST) [28], Arabic digits recognition (ADR) [29], Arabic handwritten characters recognition (AHCR) [30], and big-scale ImageNet [31]. Deep Transfer Learning (DTL) is a CNN architecture that storing learning parameters gained while solving the DL problem and execute DTL to various DL problem. Many DTL models were introduced like VGG [32], Google CNN [33], residual neural network [34], Xception [35], Inception-V3 [36], and densely connected CNN [37].

This part is dedicated to works on the recent x-ray academic researches for applying DL in the field of MDS in chest x-ray diagnosis. In [38], proposed an early medical diagnosis system for Pneumonia chest X-ray images based on DTL models. In this academic research, an x-ray data [39] containing about 1600 healthy case, 4200 un-healthy pneumonia case. The trial score introduced that VGG DTL networks better than X-ception DTL network with an error rate of 19%. In [40], it introduced a new method of diagnosing the existence of pneumonia from chest X-ray database samples based on a CNN architecture with augmentation algorithms trained based on an x-ray database [39]. The results the model improves medical x-ray diagnosis with a miss-classification rate of 12.88% in training miss-classification rate is 18.35% in the validation. In [41], introduced DTL architectures as feature extractors followed by various classifiers (k-nearest neighbors, naïve Bayes, support vector machine, and random forest algorithm) for the diagnose of healthy/unhealthy chest X-rays data. They used an x-ray database called ChestX-ray14 proposed by Wang et al. [42]. In [43], introduced a Compressed Sensing (CS) with DTL architectures for automatic identification of pneumonia on the X-ray database to assist the medical physicians. The chest x-ray database used for this research contained about 5800 X-ray images of (healthy /unhealthy). The suggested simulation results have shown that the proposed DTL architectures diagnose pneumonia from a chest x-ray with an error rate of 2.66%. Finally, in [44] proposed an ensemble DTL architecture that combines results from all DTL architectures for the identification of chest pneumonia x-ray based on the concept of DL. The suggested model based on [39] database reached a miss-classification error of 3.6%.

The modernity of this research is the outcome as follows: a) the proposed DTL models have end-to-end structure without hand-craft feature extraction and selection techniques. b) We show that neutrosophic is an effective method to generate x-ray data. c) Chest X-ray data are one of the best diagnostic methods for COVID19. d) The DTL architectures outcome high-performance measurement in a limited COVID19 chest x-ray. The rest of the paper is organized as follows. Segment 2 discusses the dataset used in our paper. Segment 3 presents the proposed DTL architectures, while Segment 4 identifies the carried-out results and our discussion. Finally, Segment 5 provides conclusions and directions for additional study.

2. Dataset

The COVID19 x-ray database applied in this paper [45] was introduced by Dr. Joseph Cohen. The Pneumonia [39] Chest X-Ray Images was used to create the introduced database with Cohen Covid-19 dataset. The Cohen dataset was collected from websites like Italian Society of Medical, Radiopaedia web and online publications. The created dataset [46] is organized into four categories normal, pneumonia bacterial, pneumonia virus, and COVID19. The dataset contains 306 x-ray images divided to 69 images for the COVID-19 class, 79 images for the normal class, 79 images for the pneumonia bacterial class, and 79 images for pneumonia virus class. Figure 1 illustrates samples of images used for this research. Figure 1 also illustrates that there is a lot of variation of x-ray image sizes and features that may reflect on the accuracy of the proposed model which will be presented in the next section.

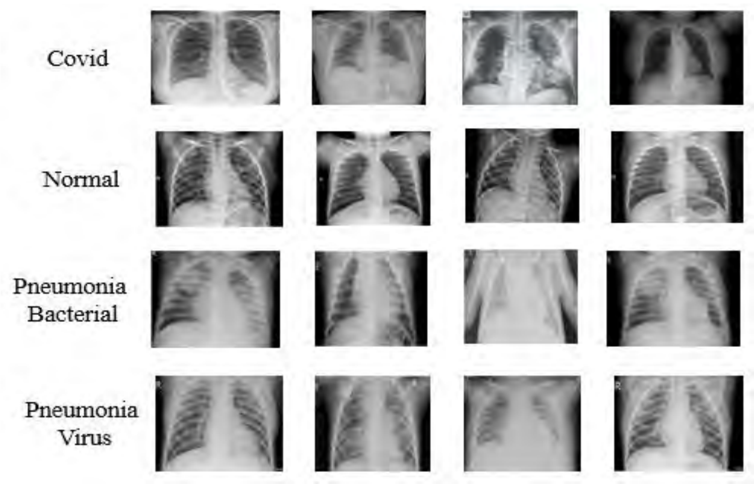

Figure 1. Samples of the used images in this research.

3. The Proposed Model

The proposed model includes two main components, the first component is neutrosophic domain conversion while the second component is the transfer learning architectures. Figure 2 illustrates the proposed Neutrosophic/DTL model for the study. The neutrosophic image domain conversion used as a preprocessing step while the DTL architectures used in the training, and the testing steps.

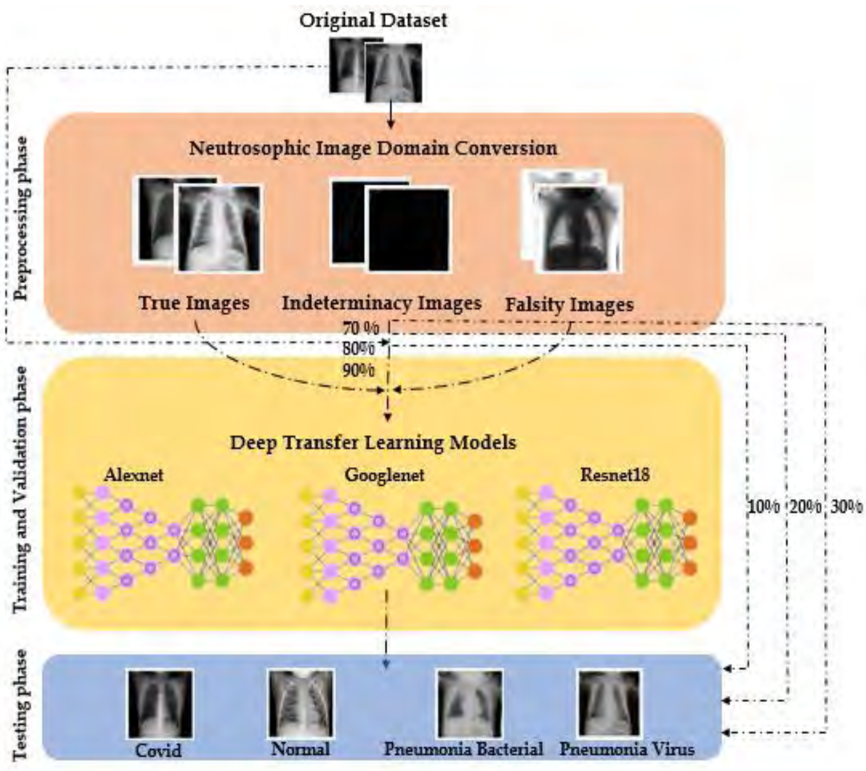

Figure 2. The introduced Neutrosophic /DTL model for the study.

3.1. Neutrosophic Image Domain Conversion

Neutrosophy (NS) is a theory sophisticated and created by Florentin Smarandache [47–49], NS is a useful and successful theory in analyzing uncertain situations. In NS theory, events are analyzed by subset them into three sets as true (T) significance the status is percentage of true, indeterminacy (I) significance the status is percentage of indefinite and falsity (F) significance the status is percentage of false, where t varies in T subsets. In image processing such as object and edge detection, all pixels of the image are subdivided into T, I and F subsets. Then, the edge detection/object process of the image is performed through necessary operations on these subsets. The input image converts to the neutrosophic domain as shown in equations 1-5. $P(n,m)$ pixel in the image domain is converted to neutrosophic domain $P2NS(n,m)$ [50,51]:

$$P2NS_{NS}(n,m) = \{T_{n,m}, I_{n,m}, F_{n,m}\} \tag{1}$$

$$T_{n,m} = \frac{f(\bar{n},m) - \overline{f_{min}}}{\overline{f_{max}} - \overline{f_{min}}} \tag{2}$$

Where $I(\bar{n},m)$ is the local average value of related pixels. $\overline{f_{min}}$ and $\overline{f_{max}}$ variables correspond to the last and first peaks measured from those pixels with a value higher than the maximum local average of the histogram.

$$I_{n,m} = 1 - \frac{H(n,m) - \overline{H_{min}}}{\overline{H_{max}} - \overline{H_{min}}} \tag{3}$$

$$H(n,m) = abs(\, I(n,m) \; - \; I(\bar{n},m)\,) \tag{4}$$

Where $H(n,m)$ is the homogeneity value of T at (n,m), which is measured by the absolute value of the difference between intensity $f(n,m)$ and its local mean value $f(\bar{n},m)$. While $\overline{H_{max}}$ and $\overline{H_{min}}$ are the last and first peaks respectively, measured from the homogeneity image.

$$F_{n,m} = 1 - T_{n,m} \tag{5}$$

After the image is converted to the NS domain, the COVID19 chest x-ray (object) is kept in the $T_{n,m}$ domain, the edges are in the $I_{n,m}$ domain and the background is kept in the $F_{n,m}$ domain. Figure 3 presents samples of images after the conversion neutrosophic image domain in the different domains for every class in the dataset.

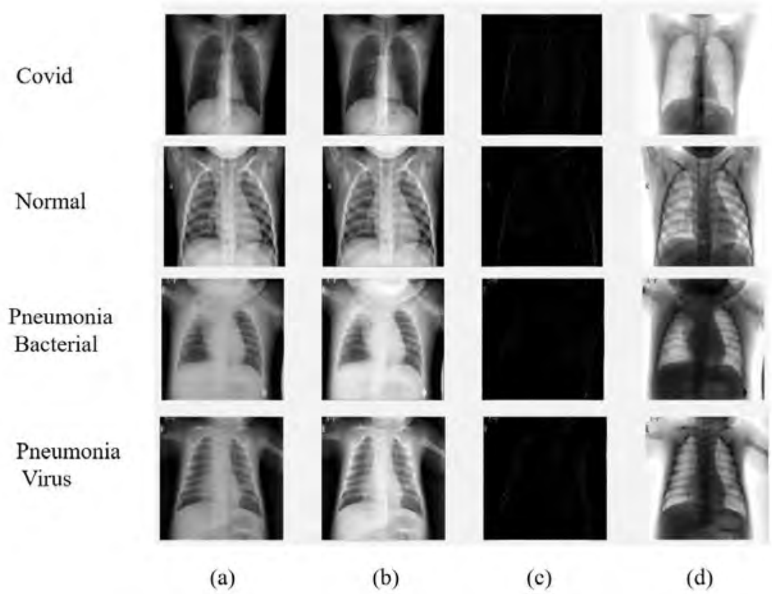

Figure 3. Different neutrosophic images domain for 4 classes in the dataset were (a) original images, (b) True significance, (c) Indeterminacy significance, and (d) Falsity significance images.

3.2. Deep Transfer Learning Model

Algorithm 1 introduces the proposed DTL model in detail. Each DTL model is trained with the COVID-19 x-ray Images database (I, Z); where I the set of N input data, each of size, 512 lengths × 512 widths, and Z have the corresponding label, $Z = \{z/z \in$ {normal; pneumonia bacterial; pneumonia virus; COVID19}}. Let V is a DTL architecture where $V =$ {Alexnet, Googlenet, Resnet18} be the set of CNN models. The x-ray database divided to train and test, training set $(\mathrm{I}_{\mathrm{train}}; \mathrm{Z}_{\mathrm{train}})$ for 80% for the training and then validation while 10% for the validating proved it is efficient in many types of academic papers [52–56]. The training data then divided into mini-batches, each of size $s = 32$, such that $(I_t;\ Z_t) \in (\mathrm{I}_{\mathrm{train}}; \mathrm{Z}_{\mathrm{train}});\ t = 1,2,\ldots,\frac{N}{n}$ and iteratively optimizes the CNN model $v \in V$ to reduce the functional loss as illustrated in Equation (6).

$$f_{loss}(w, I_u) = \frac{1}{s} \sum_{i \in I_u, z \in Z_u} f(v(I, w), Z) \tag{6}$$

where $v(I, w)$ is the ConvNet model that true label Z for input I given w is a weight and $f(.)$ is the multi-class entropy loss function.

Algorithm 1 Introduced algorithm.

Input images: COVID-19 Chest x-ray Images (I, Z); where $Z = \{z/z \in$ {normal; pneumonia bacterial; pneumonia virus; COVID-19}}
Output model: The DTL model that diagnosed COVID19 x-ray images $i \in I$
Pre-processing steps:
resize input images to dimension 512 height × 512 width
Generate neutrosophic x-ray images
normalize each image
install and reuse DTLmodels D= {Alexnet, Googlenet, Resnet18}

Update the last layer by 4 × 1 layer dimension.
foreach $\forall v \in v$ **do**
$\sigma = 0.01$
for epochs = 1 to 50 **do**
foreach mini-batch $(I_j; Z_j) \in (\mathrm{I_{train}}; \mathrm{Z_{train}})$ do
    Update the coefficients of the DTL model $v(\cdot)$
        if the miss-classification rate is increased for six epochs **then**
            $\sigma = \sigma \times 0.01$
        end
    end
end
end
foreach $\forall i \in I_{test}$ **do**
Result of all DTL architectures, $v \in V$
end

4. Experimental Results

The introduced model for the evaluation of the neutrosophic sets with deep transfer models was implemented using a software package (MATLAB). The development was CPU specific. All outcomes were directed on a computer server equipped by an Intel Xeon processor (2 GHz), 96 GB of RAM. About 36 recorded experiments were conducted in this study. The experiments included the following trails specifications:

- Different Image domains
    - The Original dataset domain (grayscale).
    - The True (T) neutrosophic domain.
    - The Indeterminacy (I) neutrosophic domain
    - The Falsity (F) neutrosophic domain.
- Different training and testing strategies
    - 70% for the training – 30% for the testing.
    - 80% for the training – 20% for the testing.
    - 90% for the training – 10% for the testing.
- Different deep transfer models
    - Alexnet.
    - Googlenet.
    - Restnet18.

The authors of this research tried first to build their deep neural networks based on the works presented [52,53,55], but the testing accuracy wasn't acceptable. So, the alternative way is to use deep transfer learning models. Using deep transfer models proved its efficiency in many types of research such as work presented in [56,57]. The Alexnet, Googlenet, and Restnet18 models are selected in this study as they have a small number of layers on their architectures which will reflect on decreasing the training time, consumed memory, and processing time. All the experimental results have been tested according to the following hyperparameters for the training, and the testing phases:

- Batch size [58]: 32
- Momentum: 0.9
- Epochs: 60
- Learning Rate [58]: 0.001
- Optimizer [59]: adaboost
- Early stopping [60]: 10 epochs

A large number of trials were performed to draw a full picture of the effectiveness and significance of using neutrosophic sets in different experimental environments with different deep learning models. To evaluate the performance of the neutrosophic set in deep transfer learning models, performance matrices are needed to be investigated through this study. The most common performance measures in the field of deep learning are Accuracy, Precision, Recall, and F1 Score [61] and they are presented from Equation (7) to Equation (10).

$$\text{Accuracy} = \frac{\text{TP+TN}}{(\text{TP+FP})+(TN+FN)} \tag{7}$$

$$\text{Precision} = \frac{\text{TP}}{(\text{TP+FP})} \tag{8}$$

$$\text{Recall} = \frac{\text{TP}}{(\text{TP+FN})} \tag{9}$$

$$\text{F1 Score} = 2 * \frac{\text{Precision}*\text{Recall}}{(\text{Precision}+\text{Recall})} \tag{10}$$

Where TP is the count of True Positive samples, TN is the count of True Negative samples, FP is the count of False Positive samples, and FN is the count of False Negative samples from a confusion matrix. The experimental results will be presented in three subsections, the first subsection will discuss the experimental results for the original dataset. The second subsection will introduce the experimental results for the different neutrosophic domains. Finally, the third subsection will illustrate a comparative results analysis for the original, the neutrosophic domain according to the confusion matrix for the testing accuracy.

4.1. Original Dataset Experimental Results

As mentioned above different experimental environments were selected in this research for the research experiment. Table 1 presents the testing accuracy and performance metrics for the original dataset. The table clearly shows that in the 90%-10% strategy, the Resnet18 model achieves the highest testing accuracy with 74.19% with tight scores in performance metrics. In the 80%-20% strategy, Googlenet achieves the highest testing accuracy with 64.52%, while in a 70%-30% strategy, the Googlenet model achieves the highest testing accuracy with 62.47% and close to Resnet18 which achieved 61.29%.

Table 1. Testing accuracy and performance metrics for the original dataset

Training / Testing	Deep Transfer Model	Recall	Precision	F Score	Testing Accuracy
90%-10%	**Alexnet**	0.6875	0.7582	0.7211	0.6774
	Googlenet	0.7188	0.8167	0.7646	0.7097
	Resnet18	0.7500	0.7639	0.7569	**0.7419**
80%-20%	**Alexnet**	0.5781	0.6222	0.5994	0.5645
	Googlenet	0.6563	0.7477	0.6990	**0.6452**
	Resnet18	0.6250	0.681	0.6518	0.6129
70%-30%	**Alexnet**	0.5506	0.5706	0.5604	0.5376
	Googlenet	0.6354	0.6814	0.6576	**0.6237**
	Resnet18	0.6250	0.6929	0.6572	0.6129

Table 1 illustrates interesting facts and they are **1)** The more data the deep learning models have, the higher testing accuracy they will achieve. **2)** The 80%-20%, and 70%-30% strategy achieved very close results for the testing accuracy which means that those strategies are enough to reflect an accurate testing accuracy for the proposed evaluation model.

4.2. Neutrosophic Domains Experimental Results

The Neutrosophic domains included three types, and they are the True (T) neutrosophic domain, the Indeterminacy (I) neutrosophic domain, the Falsity (F) neutrosophic domain. Those neutrosophic domains will be experimented on in this section to measure their performance under different experimental conditions. Table 2 presents the testing accuracy and performance metrics for the True (T) neutrosophic domain. As illustrated in section 3.1, The True (T) image is the averaging of the original image as every pixel is averaged by it is neighbors with a window of choice. The choice of the window in the study is 5 pixels.

Table 2 illustrates that in the 90%-10% strategy, both Alexnet and Googlenet model achieves similar highest testing accuracy with 64.52% with an advantage for the Googlenet model in the achieved performance metrics. In the 80%-20% strategy, also both Alexnet and Googlenet model achieves similar highest testing accuracy with 54.84% with an advantage for the Googlenet model in the achieved performance metrics while in 70%-30% strategy, Resnet18 model achieves the highest testing accuracy with 67.74%.

Table 2. Testing accuracy and performance metrics for the True (T) neutrosophic domain

Training / Testing	Deep Transfer Model	Recall	Precision	F Score	Testing Accuracy
90%-10%	**Alexnet**	0.6563	0.6979	0.6764	**0.6452**
	Googlenet	0.6563	0.7500	0.7000	**0.6452**
	Resnet18	0.5938	0.6556	0.6231	0.5806
80%-20%	**Alexnet**	0.5625	0.5868	0.5744	**0.5484**
	Googlenet	0.5625	0.6139	0.5871	**0.5484**
	Resnet18	0.5156	0.5893	0.5500	0.5000
70%-30%	**Alexnet**	0.6310	0.7433	0.6825	0.6237
	Googlenet	0.6860	0.7462	0.7149	0.6774
	Resnet18	0.6979	0.7565	0.7260	**0.6882**

Table 2 illustrates interesting facts and they are **1)** In the True (T) neutrosophic domain, more data doesn't mean higher accuracy in those deep learning architectures' as in the 70%-30% strategy, the highest testing accuracy is achieved by 68.82% all over the other strategies. **2)** The images on the True (T) neutrosophic domain are averaged images, which means that some of the important features of images are concealed which negatively affect the achieved testing accuracy if it is compared to the original experimental results presented in Table 1 for the 90%-10%, and the 80%-20% strategy.

The second neutrosophic domain to be experimented on is the Falsity (F) neutrosophic domain. This domain is the opposite of the True (T) neutrosophic domain. In the Falsity (F) domain, all pixel's values are inverted, it is expected that some features will be concealed, and other features will be revealed in images. Table 3 presents the testing accuracy and performance metrics for the (F) Falsity domain.

Table 3. Testing accuracy and performance metrics for the Falsity (F) domain

Training / Testing	Deep Transfer Model	Recall	Precision	F Score	Testing Accuracy
90%-10%	**Alexnet**	0.6563	0.6714	0.6638	**0.6452**
	Googlenet	0.6563	0.7404	0.6958	**0.6452**
	Resnet18	0.5938	0.6408	0.6164	0.5806
80%-20%	**Alexnet**	0.5781	0.7153	0.6394	**0.5645**
	Googlenet	0.5781	0.6249	0.6006	**0.5645**
	Resnet18	0.5469	0.5759	0.5610	0.5323
70%-30%	**Alexnet**	0.6131	0.7250	0.6644	0.6022
	Googlenet	0.6667	0.7083	0.6869	**0.6559**
	Resnet18	0.6548	0.7036	0.6783	0.6452

Table 3 illustrates that in the 90%-10% strategy, both Alexnet and Googlenet model achieves similar highest testing accuracy with 64.52% with an advantage for the Googlenet model in the achieved performance metrics. In the 80%-20% strategy, also both Alexnet and Googlenet model achieves similar highest testing accuracy with 56.45% with an advantage for the Googlenet model in the achieved performance metrics while in 70%-30% strategy, Googlenet model achieves the highest testing accuracy with 65.59%.

Table 3 also shows interesting facts and they are **1)** In the Falsity (F) neutrosophic domain, more data doesn’t mean higher accuracy in those deep learning architectures’ as in the 70%-30% strategy, the highest testing accuracy is achieved by 65.59% all over the other strategies. **2)** The images on the Falsity (F) neutrosophic domain are the inversion of True (T) domain, which means that some of the important features of images are concealed which negatively affect the achieved testing accuracy if it is compared to the original experimental results are presented in Table 1 for the 90%-10%, and the 80%-20% strategy. **3)** The results presented in Table 3 are very close to results presented in Table 2, which means the Falsity (F) neutrosophic domains don’t add extra value for the grayscale images and can be discarded in some applications depending on their nature.

The Third neutrosophic domain to be experimented on is the Indeterminacy (I) neutrosophic domain. This domain contains the absolute edges in the image. In the Indeterminacy (I) domain, all pixel values are resulted from subtracting the original pixel value from the average pixel value in the True (T) neutrosophic domain. Table 5 presents the testing accuracy and performance metrics for the Indeterminacy (I) domain.

Table 5 illustrates that in the 90%-10% strategy, Alexnet achieves the highest testing accuracy with 87.10% with the highest achieved performance metrics scores. In the 80%-20% strategy, Googlenet achieved the highest testing accuracy with 66.13%, while in 70%-30% strategy, both Googlenet and Resnet18 models achieve similar highest testing accuracy with 73.12% with advantage for Googlenet model in the achieved performance metrics.

Table 4. Testing accuracy and performance metrics for the Indeterminacy (I) domain

Training / Testing	Deep Transfer Model	Recall	Precision	F Score	Testing Accuracy
90%-10%	Alexnet	0.8750	0.9167	0.8953	**0.8710**
	Googlenet	0.8125	0.8458	0.8288	0.8065
	Resnet18	0.7813	0.8534	0.8157	0.7742
80%-20%	Alexnet	0.6406	0.8386	0.7264	0.6290
	Googlenet	0.6719	0.8116	0.7352	**0.6613**
	Resnet18	0.6406	0.7688	0.6989	0.6290
70%-30%	Alexnet	0.7158	0.7440	0.7296	0.7097
	Googlenet	0.7336	0.8464	0.7860	**0.7312**
	Resnet18	0.7396	0.8294	0.7819	**0.7312**

Table 4 illustrates interesting facts and they are **1)** In the Indeterminacy (I) neutrosophic domain, all the achieved testing accuracies are better than all the achieved testing accuracies in the Falsity (F), the True (T), and the original domain. **2)** The images on Indeterminacy (I) neutrosophic domain are the absolute difference between the original and the averaged image in the True (T) domain. Those are very important features that have been revealed and helped the deep transfer models to achieve higher testing accuracy. **3)** More data doesn't mean achieving higher testing accuracy in Indeterminacy (I) neutrosophic domain, as in the 70%-30% strategy, the achieved testing accuracy was better than the achieved accuracy in 80%-20% strategy with 6.99% enhancement using Googlenet.

4.3 Comparative Result of Indeterminacy (I) neutrosophic domain with the original domain

Section 4.2 concluded that the Indeterminacy (I) neutrosophic domain achieved the highest possible testing accuracy in all experiment's trails. This section is dedicated for presenting a comparison result between the Indeterminacy (I) neutrosophic domain with the original domain with deeper performance metrics to evaluate the performance of the Indeterminacy (I) domain. Table 5 presents a comparative result of the achieved testing accuracy between the Indeterminacy (I) and the original domain. Table 5 is a summary for the highest achieved testing accuracy selected from Table 1 and Table 4.

Table 5. Testing accuracy for the Indeterminacy (I) and original domain

Training / Testing	Domain	Deep Transfer Learning Model	Highest Testing Accuracy
90%-10%	**Original**	Resnet18	0.7419
	Indeterminacy (I)	Alexnet	**0.8710**
80%-20%	**Original**	Googlenet	0.6452
	Indeterminacy (I)	Googlenet	**0.6613**
70%-30%	**Original**	Googlenet	0.6237
	Indeterminacy (I)	Googlenet	**0.7312**

Table 5 shows that the Indeterminacy (I) neutrosophic domain achieved the highest testing accuracy in all training and testing strategies with 87.10%(Alexnet), 66.13%(Googlenet), and 73.12%(Googlenet) in the 90%-10%, 80%-20%, and 70%-30% accordingly.

Table 5 also shows interesting facts and they are **1)** in the Indeterminacy (I) neutrosophic or the original domain, Googlenet model is the most dominant model in achieving the highest accuracy possible as it contains 20 layers in its architecture if it is compared with Alexnet and Resnet18 which contains 8, and 18 layers. **2)** The Indeterminacy (I) neutrosophic greatly affects the testing accuracy, in the 90%-10%, the Indeterminacy (I) neutrosophic domain achieves better accuracy with 12.91% more than the achieved testing accuracy in original domain. In the 70%-30% the Indeterminacy (I) neutrosophic domain achieves better accuracy with 10.75% than the original domain. **3)** In the Indeterminacy (I) neutrosophic domain, deep transfer models can learn with fewer data as illustrated in the 70%-30% strategy, the Googlenet achieves better accuracy than the 80%-20% strategy. That means the model can generalize whatever the amount of the data existed. While in the original domain, more data means higher testing accuracy.

All the experimental outcomes show that converting to the Indeterminacy (I) neutrosophic domain from the original domain grantees achieving higher testing accuracy. Therefore, the Indeterminacy (I) neutrosophic need further investigations to prove it is efficient for the detection of COVID-19 among the other classes. The confusion matrices for the Indeterminacy (I) neutrosophic domain for the different deep transfer models are presented in Figures 4, 5, and 6. The figures show that the testing accuracy for the COVID-19 class in the different training and testing strategies are acceptable. For the 90%-10% strategy, The Alexnet model was able to detect COVID-19 with testing accuracy 100% and for the normal class with 100%. In the 80%-20% strategy, The Googlenet model was able to detect COVID-19 with testing accuracy 77.8% and for the normal class with 100%. While in the 70%-30% strategy, The Googlenet model was able to detect COVID-19 with testing accuracy 100% and for the normal class with 87.5%.

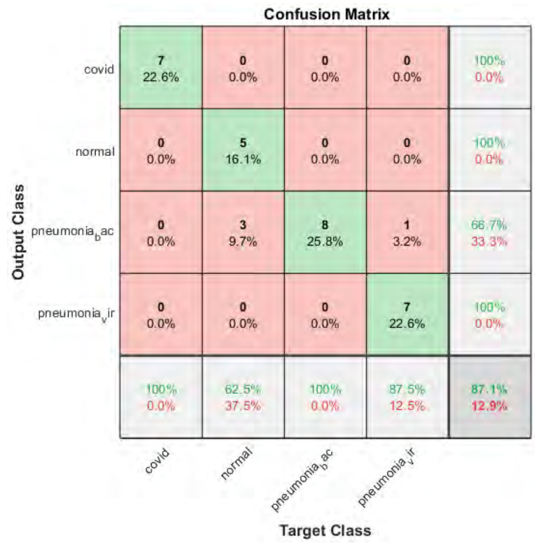

Figure 4. Confusion matrix for Alexnet in 90%-10% strategy for Indeterminacy (I) neutrosophic domain

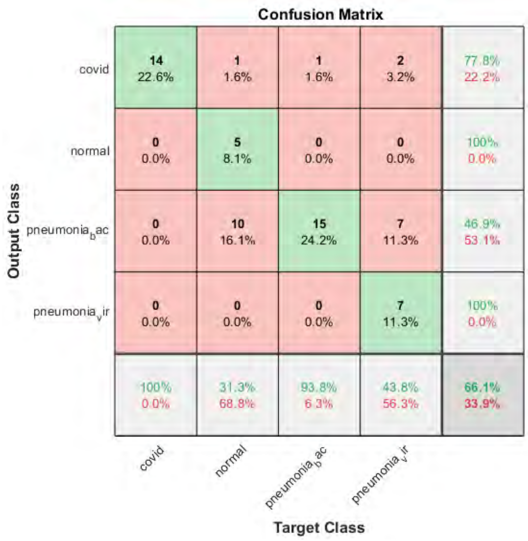

Figure 5. Confusion matrix for Googlenet in 80%-20% strategy for Indeterminacy (I) neutrosophic domain

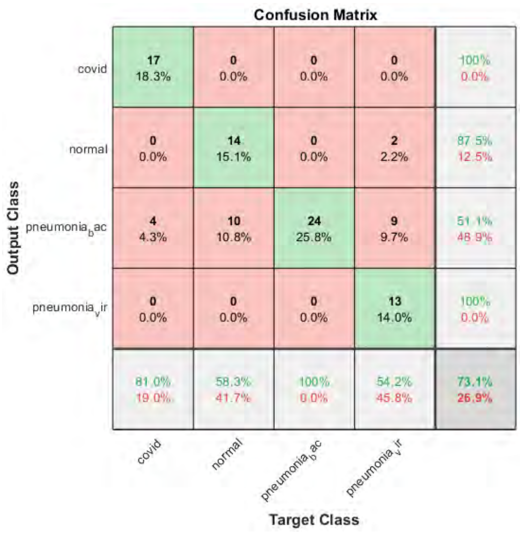

Figure 6. Confusion matrix for Googlenet in 70%-30% strategy for Indeterminacy (I) neutrosophic domain

This study concludes that using the neutrosophic set with deep learning models might be an encouraging transition to achieve higher testing accuracy, especially with limited datasets such as COVID-19 chest x-ray dataset which is investigated throughout this research.

5. Conclusion and future works

According to the World Health Organization (WHO), coronaviruses are a family of viruses that lead to sicknesses ranging from the common cold to more severe diseases. With the advancements in computer science, detection of this type of virus is urgently needed. In this paper, a study of neutrosophic significance on the deep transfer learning model is presented. The neutrosophic domain consisted of three types of images and they are, the True (T) images, the Indeterminacy (I) images, and the Falsity (F) images. The dataset used in this research had been collected from different sources as there is no benchmark dataset for COVID-19 chest X-ray until the writing of this research. The dataset consisted of four classes and they are COVID-19, Normal, Pneumonia bacterial, and Pneumonia virus. This study aimed to review the effect of neutrosophic sets on deep transfer learning models. The selected deep learning models in this study were Alexnet, Googlenet, and Restnet18. Those models were selected as they had a small number of layers on their architectures that will reflect on reducing the consumed memory and training time. To test the performance of the conversion to the neutrosophic domain, about 36 trails had been conducted and recorded. A combination of training and testing strategies by splitting the dataset into (90%-10%, 80%-20%, and70%-30%) were included in the experiments. Four domains of images are tested, and they were, the original images, the True (T) neutrosophic images, the Indeterminacy (I) neutrosophic images, and the Falsity (F) neutrosophic images. The four domains with the different training and testing strategies were tested using Alexnet, Googlenet, and Restnet18 deep transfer models. According to the experimental results, the Indeterminacy (I) neutrosophic domain achieved the highest accuracy possible in the testing accuracy and performance metrics such as Precision, Recall, and F1 Score. The study concluded that using the neutrosophic set with deep learning models might be an encouraging transition to achieve better testing accuracy, especially with limited datasets such as COVID-19 dataset.

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DECISION MAKING

Three Non-linear α-Discounting MCDM-Method Examples

Florentin Smarandache

Florentin Smarandache (2013). Three Non-linear α-Discounting MCDM-Method Examples. Proceedings of the 2013 International Conference on Advanced Mechatronic Systems, Luoyang, China, September 25-27, 3

Abstract— **In this paper we present three new examples of using the α-Discounting Multi-Criteria Decision Making Method in solving non-linear problems involving algebraic equations and inequalities in the decision making process.**

Keywords—α-Discounting MCDM-Method, non-linear decision making problems

I. INTRODUCTION

This We have defined [see 1] a new procedure called *α-Discounting Method for Multi-Criteria Decision Making (α-D MCDM)*, which is as an alternative and extension of Saaty's Analytical Hierarchy Process (AHP). We have also defined the *degree of consistency* (and implicitly a *degree of inconsistency*) of a decision-making problem [1].

The α-D MCDM can deal with any set of preferences that can be transformed into an algebraic system of linear and/or non-linear homogeneous and/or non-homogeneous equations and/or inequalities.

We discuss below three new examples of non-linear decision making problems.

II. EXAMPLE 1

The Set of References is $\{C_1, C_2, C_3\}$.

1. C_1 is as important as the product of C_2 and C_3.
2. The square of C_2 is as important as C_3.
3. C_3 is less important than C_2.

We denote $C_1 = x,\ C_2 = y,\ C_3 = z$, and we'll obtain the following non-linear algebraic system of two equations and one inequality:

$$\begin{cases} x = yz \\ y^2 = z \\ z < y \end{cases} \tag{1}$$

and of course the conditions

$$\begin{cases} x + y + z = 1 \\ x, y, z \in [0,1] \end{cases} \tag{2}$$

From the first two equations we have:

$$x = yz = y \cdot y^2 = y^3 \tag{3}$$

and

$$z = y^2 \tag{4}$$

whence the general priority vector is

$$\langle y^3\ y\ y^2 \rangle. \tag{5}$$

We consider $y \neq 0$, because if $y = 0$ the priority vector becomes $\langle 0\ 0\ 0 \rangle$ which does not make sense.

Dividing by y we have $\langle y^2\ 1\ y \rangle$, and normalized:

$$\left\langle \frac{y^2}{y^2+y+1}\ \ \frac{1}{y^2+y+1}\ \ \frac{y}{y^2+y+1} \right\rangle. \tag{6}$$

From $y^2 = z$ and $z < y$

we have

$$y^2 < y \text{ or } y^2 - y < 0 \text{ or } y(y-1) < 0,$$

hence $y \in (0,1)$. (7)

For $y \in (0,1)$ we have the order:

$$\frac{1}{y^2+y+1} > \frac{y}{y^2+y+1} > \frac{y^2}{y^2+y+1} \tag{8}$$

so

$$C_2 > C_3 > C_1. \tag{9}$$

III. EXAMPLE 2

The Set of References is also $\{C_1, C_2, C_3\}$.

1. C_1 is as important as the square of C_2.
2. C_2 is as important as double C_3.

3. The square of C_3 is as important as triple C_1.

We again denote $C_1 = x,\ C_2 = y,\ C_3 = z$, and we'll obtain the following non-linear algebraic system of three equations:

$$\begin{cases} x = y^2 \\ y = 2z \\ z^2 = 3x \end{cases}. \tag{10}$$

If we solve it, we get:

$$\left.\begin{matrix} x = 4z^2 \\ x = \dfrac{z^2}{3} \end{matrix}\right\} \Rightarrow 4z^2 = \frac{z^2}{3} \Rightarrow z = 0 \Rightarrow y = 0 \Rightarrow x = 0. \tag{11}$$

Algebraically the only solution is $\langle 0\ 0\ 0 \rangle$, but the null solution is not convenient for MCDM.

Let's parameterize, i.e. "discount" each equality:

$$\left.\begin{matrix} \left.\begin{matrix} x = \alpha_1 y^2 \\ y = 2\alpha_2 z \end{matrix}\right\} \Rightarrow x = \alpha_1 (2\alpha_2 z)^2 = 4\alpha_1\alpha_2^2 z^2 \\ z^2 = 3\alpha_3 x \Rightarrow x = \dfrac{1}{3\alpha_3} z^2 \end{matrix}\right\} \Rightarrow 4\alpha_1\alpha_2^2 = \frac{1}{3\alpha_3}$$

or

$$12\alpha_1\alpha_2^2\alpha_3 = 1, \tag{12}$$

which is the characteristic parametric equation needed for the consistency of this algebraic system.
The general algebraic solution in this parameterized case is:

$<4\alpha_1\alpha_2^2 z^2 \quad 2\alpha_2 z \quad z>$.

Using the Fairness Principle as in the α-Discounting MCDM Method, we set all parameters equal:

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha \tag{13}$$

whence, from the characteristic parametric equation, we obtain that

$$12\alpha^4 = 1, \tag{14}$$

therefore

$$\alpha = \sqrt[4]{\frac{1}{12}} = \frac{1}{\sqrt[4]{12}} \approx 0.537. \tag{15}$$

Thus the general solution for the Fairness Principle is:

$$<4\alpha^3 z^2 \quad 2\alpha z \quad z> \tag{16}$$

and, after substituting $\alpha \approx 0.537$, it results:

$$\begin{cases} x = 4\alpha^3 z^2 \simeq 0.619 z^2 \\ y = 2\alpha z = 1.074 z. \end{cases} \tag{17}$$

Whence the general solution for the Fairness Principle becomes:

$$\langle 4\alpha^3 z^2 \quad 2\alpha z \quad z \rangle \simeq \langle 0.619 z^2 \quad 1.074 z \quad z \rangle$$

and dividing by $z \neq 0$ one has:

$$\langle 0.619z,\ 1.074,\ 1 \rangle. \tag{18}$$

But y = 1.074 > 1 = z, hence y > z. (19)

Discussion:

a) If $z < \dfrac{1}{0.619} \approx 1.616$, then y > z > x. (20)

b) If $z = \dfrac{1}{0.619} \approx 1.616$, then y > z = x. (21)

c) If $\dfrac{1}{0.619} < z < \dfrac{1.074}{0.619}$ or 1.616 < z < 1.735, then y > x > z. (22)

d) If $z = \dfrac{1.074}{0.619} \approx 1.735$, then x = y > z. (23)

e) If $z > \dfrac{1.074}{0.619} \approx 1.735$, then x > y > z. (24)

From the orders of x, y, and z it results the corresponding orders between the preferences C_1, C_2, and C_3.

IV. EXAMPLE 3.

Let's suppose that the sources are not equally reliable. First source is five times less reliable than the second, while the third source is twice more reliable that the second one. Then the parameterized system:

$$\begin{cases} x = \alpha_1 y^2 \\ y = 2\alpha_2 z \\ z^2 = 3\alpha_3 x \end{cases} \text{ becomes } \begin{cases} x = 3\alpha_2 y^2 \\ y = 2\alpha_2 z \\ z^2 = 3\dfrac{\alpha_2}{4} x \end{cases} \tag{25}$$

since $\alpha_1 = 3\alpha_2$ which means that we need to discount the first equation three times more than the second, and $\alpha_3 = \dfrac{\alpha_2}{4}$ which means that we need to discount the third equation a quarter of the second equation's discount.

Denote $\alpha_2 = \alpha$, then:

$$\begin{cases} x = 3\alpha y^2 \\ y = 2\alpha z \\ z^2 = \dfrac{3\alpha}{4} x \end{cases} \tag{26}$$

whence

$$x = 3\alpha(2\alpha z)^2 = 12\alpha^3 z^2 \tag{27}$$

and

$$x = \frac{4}{3\alpha} z^2 \tag{28}$$

therefore

$$12\alpha^3 z^2 = \frac{4}{3\alpha} z^2 \text{, or } 36\alpha^4 = 4 \tag{29}$$

Thus

$$\alpha = \frac{1}{\sqrt[4]{9}} \simeq 0.485. \tag{30}$$

The algebraic general solution is:

$$\left\langle 12\alpha^3 z^2 \;\; 2\alpha z \;\; z \right\rangle = \left\langle 1.557 z^2 \;\; 0.970 z \;\; z \right\rangle = \left\langle \frac{1.557 z}{1.557 z + 1.854} \;\; 0.854 \;\; 1 \right\rangle. \tag{31}$$

And in a similar way, as we did for Example 2, we may discuss upon parameter z > 0 the order of preferences C_1, C_2, and C_3.

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An Extended TOPSIS Method for Multiple Attribute Decision Making based on Interval Neutrosophic Uncertain Linguistic Variables

Said Broumi, Jun Ye, Florentin Smarandache

Said Broumi, Flornetin Smarandache (2015). An Extended TOPSIS Method for Multiple Attribute Decision Making based on Interval Neutrosophic Uncertain Linguistic Variables. *Neutrosophic Sets and Systems* 8, 22-31

Abstract: The interval neutrosophic uncertain linguistic variables can easily express the indeterminate and inconsistent information in real world, and TOPSIS is a very effective decision making method more and more extensive applications. In this paper, we will extend the TOPSIS method to deal with the interval neutrosophic uncertain linguistic information, and propose an extended TOPSIS method to solve the multiple attribute decision making problems in which the attribute value takes the form of the interval neutrosophic uncertain linguistic variables and attribute weight is unknown. Firstly, the operational rules and properties for the interval neutrosophic variables are introduced. Then the distance between two interval neutrosophic uncertain linguistic variables is proposed and the attribute weight is calculated by the maximizing deviation method, and the closeness coefficients to the ideal solution for each alternatives. Finally, an illustrative example is given to illustrate the decision making steps and the effectiveness of the proposed method.

Keywords: The interval neutrosophic linguistic, multiple attribute decision making, TOPSIS, maximizing deviation method

I-Introduction

F. Smarandache [7] proposed the neutrosophic set (NS) by adding an independent indeterminacy-membership function. The concept of neutrosophic set is generalization of classic set, fuzzy set [25], intuitionistic fuzzy set [22], interval intuitionistic fuzzy set [23,24] and so on. In NS, the indeterminacy is quantified explicitly and truth-membership, indeterminacy membership, and false-membership are completely independent. From scientific or engineering point of view, the neutrosophic set and set-theoretic view, operators need to be specified .Otherwise, it will be difficult to apply in the real applications. Therefore, H. Wang et al [8] defined a single valued neutrosophic set (SVNS) and then provided the set theoretic operations and various properties of single valued neutrosophic sets. Furthermore, H. Wang et al.[9] proposed the set theoretic operations on an instance of neutrosophic set called interval valued neutrosophic set (IVNS) which is more flexible and practical than NS. The works on neutrosophic set (NS) and interval valued neutrosophic set (IVNS), in theories and application have been progressing rapidly (e.g, [1,2,4,6,7,8,9,10,11,12,13,14,15,16,17, ,18,19,20,21,27,28,29,30,31,32,33,35,36,37,38,39,40,41,42 ,43,44,45,46,47,48,53].

Multiple attribute decision making (MADM) problem are of importance in most kinds of fields such as engineering, economics, and management. In many situations decision makers have incomplete , indeterminate and inconsistent information about alternatives with respect to attributes. It is well known that the conventional and fuzzy or intuitionistic fuzzy decision making analysis [26, 50, 51,] using different techniques tools have been found to be inadequate to handle indeterminate an inconsistent data. So, Recently, neutrosophic multicriteria decision making problems have been proposed to deal with such situation.

TOPSIS (Technique for Order Performance by Similarity to Ideal Solution) method, initially introduced by C. L. Hwang and Yoon [3], is a widely used method for dealing with MADM problems, which focuses on choosing the alternative with the shortest distance from the positive ideal solution (PIS) and the farthest distance from the negative ideal solution (NIS). The traditional TOPSIS is only used to solve the decision making problems with crisp numbers, and many extended TOPSIS were proposed to deal with fuzzy information. Z. Yue [55] extended TOPSIS to deal with interval numbers, G. Lee et al.[5] extend TOPSIS to deal wit fuzzy numbers, P. D. Liu and Su [34], Y. Q. Wei and Liu [49] extended TOPSIS to linguistic information environments, Recently, Z. Zhang and C. Wu [53] proposed the single valued neutrosophic or interval neutrosophic TOPSIS method to calculate the relative closeness coefficient of each alternative to the single valued neutrosophic or interval neutrosophic positive ideal solution, based on which the considered alternatives are ranked and then the most desirable one is selected. P. Biswas et al. [32] introduced single –valued neutrosophic multiple attribute decision making problem with incompletely known or completely unknown attribute weight information based on modified GRA.

Based on the linguistic variable and the concept of interval neutrosophic sets, J. Ye [19] defined interval neutrosophic linguistic variable, as well as its operation principles, and developed some new aggregation operators for the interval neutrosophic linguistic information, including interval neutrosophic linguistic arithmetic weighted average (INLAWA) operator, linguistic geometric weighted average(INLGWA) operator and discuss some properties. Furthermore, he proposed the decision making method for multiple attribute decision making (MADM) problems with an illustrated example to show the process of decision making and the effectiveness of the proposed method. In order to process incomplete, indeterminate and inconsistent information more efficiency and precisely J. Ye [20] further proposed the interval neutrosophic uncertain linguistic variables by combining uncertain linguistic variables and interval neutrosophic sets, and proposed the operational rules, score function , accuracy functions ,and certainty function of interval neutrosophic uncertain linguistic variables. Then the interval neutrosophic uncertain linguistic weighted arithmetic averaging (INULWAA) and the interval neutrosophic uncertain linguistic weighted arithmetic averaging (INULWGA) operator are developed, and a multiple attribute decision method with interval neutrosphic uncertain linguistic information was developed.

To do so, the remainder of this paper is set out as follows. Section 2 briefly recall some basic concepts of neutrosphic sets, single valued neutrosophic sets (SVNSs), interval neutrosophic sets(INSs), interval neutrosophic linguistic variables and interval neutrosophic uncertain linguistic variables. In section 3, we develop an extended TOPSIS method for the interval neutrosophic uncertain linguistic variables, In section 4, we give an application example to show the decision making steps, In section 5, a comparison with existing methods are presented. Finally, section 6 concludes the paper.

II-Preliminaries

In the following, we shall introduce some basic concepts related to uncertain linguistic variables, single valued neutrosophic set, interval neutrosophic sets, interval neutrosophic uncertain linguistic sets, and interval neutrosophic uncertain linguistic set.

2.1 Neutrosophic sets

Definition 2.1 [7]

Let U be a universe of discourse then the neutrosophic set A is an object having the form

$A = \{< x: T_A(x), I_A(x), F_A(x) >, x \in X \}$,

Where the functions $T_A(x), I_A(x), F_A(x)$: $U \rightarrow]^-0,1+[$define respectively the degree of membership, the degree of indeterminacy, and the degree of non-membership of the element $x \in X$ to the set A with the condition.

$$^-0 \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+. \quad \textbf{(1)}$$

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^-0,1^+[$. So instead of $]^-0,1^+[$ we need to take the interval [0,1] for technical applications, because $]^-0,1^+[$will be difficult to apply in the real applications such as in scientific and engineering problems.

2.2 Single valued Neutrosophic Sets

Definition 2.2 [8]

Let X be an universe of discourse, then the neutrosophic set A is an object having the form

$A = \{< x: T_A(x), I_A(x), F_A(x) >, x \in X \}$,

where the functions $T_A(x), I_A(x), F_A(x) : U \rightarrow [0,1]$define respectively the degree of membership , the degree of indeterminacy, and the degree of non-membership of the element x ∈ X to the set A with the condition.

$0 \le T_A(x) + I_A(x) + F_A(x) \le 3$ **(2)**

Definition 2.3 [8]

A single valued neutrosophic set A is contained in another single valued neutrosophic set B i.e. A ⊆ B if ∀x ∈ U, $T_A(x) \le T_B(x)$, $I_A(x) \ge I_B(x)$, $F_A(x) \ge F_B(x)$.
(3)

2.3 Interval Neutrosophic Sets

Definition 2.4[9]

Let X be a space of points (objects) with generic elements in X denoted by x. An interval valued neutrosophic set (for short IVNS) A in X is characterized by truth-membership function $T_A(x)$, indeteminacy-membership function $I_A(x)$ and falsity-membership function $F_A(x)$. For each point x in X, we have that $T_A(x), I_A(x), F_A(x) \subseteq [0,1]$.

For two IVNS, A_{IVNS}= {<x, $[T_A^L(x), T_A^U(x)]$, $[I_A^L(x), I_A^U(x)]$, $[F_A^L(x), F_A^U(x)]$ > | x ∈ X } **(4)**

And B_{IVNS}= {<x, $[T_B^L(x), T_B^U(x)]$, $[I_B^L(x), I_B^U(x)]$, $[F_B^L(x), F_B^U(x)]$> | x ∈ X } the two relations are defined as follows:

(1) $A_{IVNS} \subseteq B_{IVNS}$ If and only if $T_A^L(x) \le T_B^L(x)$, $T_A^U(x) \le T_B^U(x)$, $I_A^L(x) \ge I_B^L(x)$, $I_A^U(x) \ge I_B^U(x)$, $F_A^L(x) \ge F_B^L(x)$, $F_A^U(x) \ge F_B^U(x)$

(2) $A_{IVNS} = B_{IVNS}$ if and only if , $T_A(x) = T_B(x)$, $I_A(x) = I_B(x)$, $F_A(x) = F_B(x)$ for any x ∈ X

The complement of A_{IVNS} is denoted by A_{IVNS}^o and is defined by

A_{IVNS}^o= {<x, $[F_A^L(x), F_A^U(x)]$>, $[1 - I_A^U(x), 1 - I_A^L(x)]$, $[T_A^L(x), T_A^U(x)]$ | x ∈ X }

A∩B ={ <x , [min($T_A^L(x)$,$T_B^L(x)$), min($T_A^U(x)$,$T_B^U(x)$)], [max($I_A^L(x)$,$I_B^L(x)$), max($I_A^U(x)$,$I_B^U(x)$], [max($F_A^L(x)$,$F_B^L(x)$), max(F(x),$F_B^U(x)$)] >: x ∈ X }

A∪B ={ <x , [max($T_A^L(x)$,$T_B^L(x)$), max($T_A^U(x)$,$T_B^U(x)$)], [min($I_A^L(x)$,$I_B^L(x)$), min($I_A^U(x)$,$I_B^U(x)$], [min($F_A^L(x)$,$F_B^L(x)$), min($F_A^U(x)$,$F_B^U(x)$)] >: x ∈ X }

2.4 Uncertain linguistic variable.

A linguistic set is defined as a finite and completely ordered discreet term set,

S=(s_0, s_1,…, s_{l-1}), where l is the odd value. For example, when l=7, the linguistic term set S can be defined as follows: S={s_0(extremely low); s_1(very low); s_2(low); s_3(medium); s_4(high); s_5(very high); s_6(extermley high)}

Definition 2.5. Suppose $\tilde{s} = [s_a, s_b]$, where $s_a, s_b \in \tilde{S}$ with a ≤ b are the lower limit and the upper limit of S, respectively. Then $\tilde{s}$ is called an uncertain linguitic varaible.

Definition 2.6. Suppose $\tilde{s}_1 = [s_{a_1}, s_{b_1}]$ and $\tilde{s}_2 = [s_{a_2}, s_{b_2}]$ are two uncertain linguistic variable ,then the distance between $\tilde{s}_1$ and $\tilde{s}_2$ is defined as follows.

$d(\tilde{s}_1, \tilde{s}_2) = \frac{1}{2(l-1)} (|a_2 - a_1| + |b_2 - b_1|)$ **(5)**

2.5 Interval neutrosophic linguistic set

Based on interval neutrosophic set and linguistic variables, J. Ye [18] presented the extension form of the linguistic set, i.e, interval neutroosphic linguistic set, which is shown as follows:

Definition 2.7 :[19] An interval neutrosophic linguistic set A in X can be defined as

A ={<x, $s_{\theta(x)}$, (T_A(x), I_A(x), F_A(x))>| x ∈ X} **(6)**

Where $s_{\theta(x)} \in \hat{s}$, $T_A(x) = [T_A^L(x), T_A^U(x)] \subseteq [0.1]$, $I_A(x) = [I_A^L(x), I_A^U(x)] \subseteq [0.1]$, and $F_A(x) = [F_A^L(x), F_A^U(x)] \subseteq [0.1]$ with the condition $0 \le T_A^U(x) + I_A^U(x) + F_A^U(x) \le 3$ for any x ∈ X. The function $T_A(x), I_A(x)$ and $F_A(x)$ express, respectively, the truth-membership degree, the indeterminacy –membership degree, and the falsity-membership degree with interval values of the element x in X to the linguistic variable $s_{\theta(x)}$.

2.6 Interval neutrosophic uncertain linguistic set.

Based on interval neutrosophic set and uncertain linguistic variables, J.Ye [20] presented the extension form of the uncertain linguistic set, i.e, interval neutrsophic uncertain linguistic set, which is shown as follows:

Definition 2.8 :[20] An interval neutrosophic uncertain linguistic set A in X can be defined as

A ={<x,[$s_{\theta(x)}$, $s_{\rho(x)}$], (T_A(x), I_A(x), F_A(x))>| x ∈ X} **(7)**

Where $s_{\theta(x)} \in \hat{s}$, $T_A(x) = [T_A^L(x), T_A^U(x)] \subseteq [0.1]$, $I_A(x) = [I_A^L(x), I_A^U(x)] \subseteq [0.1]$, and $F_A(x) = [F_A^L(x), F_A^U(x)] \subseteq [0.1]$ with the condition $0 \le T_A^U(x) + I_A^U(x) + F_A^U(x) \le 3$ for any x ∈ X. The function $T_A(x), I_A(x)$ and $F_A(x)$ express, respectively, the truth-membership degree, the indeterminacy–membership degree, and the falsity-membership degree with interval values of the element x in X to the uncertain linguistic variable [$s_{\theta(x)}$, $s_{\rho(x)}$].

Definition 2.9 Let $\tilde{a}_1$=< [$s_{\theta(\tilde{a}_1)}$, $s_{\rho(\tilde{a}_1)}$], ([$T^L(\tilde{a}_1)$,$T^U(\tilde{a}_1)$], [$I^L(\tilde{a}_1)$,$I^U(\tilde{a}_1)$], [$F^L(\tilde{a}_1)$,$F^U(\tilde{a}_1)$])> and $\tilde{a}_2$={<x, [$s_{\theta(\tilde{a}_2)}$, $s_{\rho(\tilde{a}_2)}$], ([$T^L(\tilde{a}_2)$,$T^U(\tilde{a}_2)$], [$I^L(\tilde{a}_2)$,$I^U(\tilde{a}_2)$], [$F^L(\tilde{a}_2)$,$F^U(\tilde{a}_2)$])>

be two INULVs and λ ≥ 0, then the operational laws of INULVs are defined as follows:

$\tilde{a}_1 \oplus \tilde{a}_2$ =< $[s_{\theta(\tilde{a}_1)+\theta(\tilde{a}_2)}, s_{\rho(\tilde{a}_1)+\rho(\tilde{a}_2)}]$, ([$T^L(\tilde{a}_1)$+ $T^L(\tilde{a}_2)$- $T^L(\tilde{a}_1)$ $T^L(\tilde{a}_2)$,$T^U(\tilde{a}_1)$+ $T^U(\tilde{a}_2)$- $T^U(\tilde{a}_1)$ $T^U(\tilde{a}_2)$], [$I^L(\tilde{a}_1)$ $I^L(\tilde{a}_2)$,$I^U(\tilde{a}_1)$ $I^U(\tilde{a}_2)$], [$F^L(\tilde{a}_1)$ $F(\tilde{a}_2)$,$F^U(\tilde{a}_1)$ $F^L(\tilde{a}_2)$])> **(8)**

$\tilde{a}_1 \otimes \tilde{a}_2$ =< $[s_{\theta(\tilde{a}_1)\times\theta(\tilde{a}_2)}]$, ([$T^L(\tilde{a}_1)$ $T^L(\tilde{a}_2)$, $T^U(\tilde{a}_1)$ $T^U(\tilde{a}_2)$], [$I^L(\tilde{a}_1)$+ $I^L(\tilde{a}_2)$ - $I^L(\tilde{a}_1)$ $I^L(\tilde{a}_2)$, $I^U(\tilde{a}_1)$+ $I^U(\tilde{a}_2)$- $I^U(\tilde{a}_1)$ $I^U(\tilde{a}_2)$], [$F^L(\tilde{a}_1)$+ $F^L(\tilde{a}_2)$ - $F^L(\tilde{a}_1)$ $F(\tilde{a}_2)$, $F^U(\tilde{a}_1)$+ $F^U(\tilde{a}_2)$ - $F^U(\tilde{a}_1)$ $F^U(\tilde{a}_2)$])> **(9)**

$\lambda\tilde{a}_1$=<$[s_{\lambda\theta(\tilde{a}_1)}, s_{\lambda\rho(\tilde{a}_1)}]$,([1-$(1-T^L(\tilde{a}_1))^\lambda$,1-$(1-T^U(\tilde{a}_1))^\lambda$], [$(I^L(\tilde{a}_1))^\lambda$,$(I^U(\tilde{a}_1))^\lambda$], [$(F^L(\tilde{a}_1))^\lambda$,$(F^U(\tilde{a}_1))^\lambda$]>

(10)

$\tilde{a}_1^\lambda$=< $[s_{\theta^\lambda(\tilde{a}_1)}, s_{\rho^\lambda(\tilde{a}_1)}]$, ([$(T^L(\tilde{a}_1))^\lambda$,$(T^U(\tilde{a}_1))^\lambda$], [1-$(1-I^L(\tilde{a}_1))^\lambda$, 1-$(1-I^U(\tilde{a}_1))^\lambda$], [1-$(1-F^L(\tilde{a}_1))^\lambda$, 1-$(1-F^U(\tilde{a}_1))^\lambda$]> **(11)**

Obviously, the above operational results are still INULVs.

III. The Extended TOPSIS for the Interval Neutrosophic Uncertain Linguistic Variables

A. The description of decision making problems with interval neutrosphic uncertain linguistic information.

For the MADM problems with interval neutrosophic uncertain variables, there are m alternatives A= $(A_1, A_2,\ldots, A_m)$ which can be evaluated by n attributes C=$(C_1, C_2,\ldots, C_n)$ and the weight of attributes A_i is w_i, and meets the conditions $0 \le w_i \le 1$, $\sum_{j=1}^{n} w_j$=1.Suppose z_{ij} (i=1, 2,…, n; j=1, 2,…, m) is the evaluation values of alternative A_i with respect to attribute C_j

And it can be represented by interval neutrosophic uncertain linguistic variable z_{ij}= <$[x_{ij}^L, x_{ij}^U]$,([T_{ij}^L, T_{ij}^U], [I_{ij}^L, I_{ij}^U], [F_{ij}^L, F_{ij}^U])>, where $[x_{ij}^L, x_{ij}^U]$ is the uncertain linguistic variable, and $x_{ij}^L, x_{ij}^U \in$ S, S =$(s_0, s_1,\ldots, s_{l-1})$, T_{ij}^L, T_{ij}^U, I_{ij}^L, I_{ij}^U and F_{ij}^L, $F_{ij}^U \in$ [0, 1] and $0 \le T_{ij}^U + I_{ij}^U + F_{ij}^U \le 3$. Suppose attribute weight vector W=$(w_1, w_2,\ldots w_n)$ is completely unknown, according to these condition, we can rank the alternatives $(A_1, A_2,\ldots, A_m)$

B. Obtain the attribute weight vector by the maximizing deviation.

In order to obtain the attribute weight vector, we firstly define the distance between two interval neutrosophic uncertain variables.

Definition 3.1

Let $\tilde{s}_1$ = <$[s_{a_1}, s_{b_1}]$,([T_A^L, T_A^U], [I_A^L, I_A^U], [F_A^L, F_A^U])>, $\tilde{s}_2$ = <$[s_{a_2}, s_{b_2}]$,([T_B^L, T_B^U], [I_B^L, I_B^U], [F_B^L, F_B^U])> and $\tilde{s}_3$ = <$[s_{a_3}, s_{b_3}]$,([T_C^L, T_C^U], [I_C^L, I_C^U], [F_C^L, F_C^U])>, be any three interval neutrosophic uncertain linguistic variables, and $\tilde{S}$ be the set of linguistic variables, f is a map, and $f: \tilde{S} \times \tilde{S} \longrightarrow$ R. If d$(\tilde{s}_1, \tilde{s}_2)$ meets the following conditions

(1) $0 \le d_{INULV}(\tilde{s}_1, \tilde{s}_2) \le 1$, $d_{INULV}(\tilde{s}_1, \tilde{s}_1)$= 0

(2) $d_{INULV}(\tilde{s}_1, \tilde{s}_2) = d_{INULV}(\tilde{s}_2, \tilde{s}_1)$

(3) $d_{IVNS}(\tilde{s}_1, \tilde{s}_2) + d_{INULV}(\tilde{s}_2, \tilde{s}_3) \ge d_{INULV}(\tilde{s}_1, \tilde{s}_3)$

then $d_{INULV}(\tilde{s}_1, \tilde{s}_2)$ is called the distance between two interval neutrosophic uncertain linguistic variables $\tilde{s}_1$

Definition 3.2:

Let $\tilde{s}_1$ = <$[s_{a_1}, s_{b_1}]$,([T_A^L, T_A^U], [I_A^L, I_A^U], [F_A^L, F_A^U])>, and $\tilde{s}_2$ = <$[s_{a_2}, s_{b_2}]$,([T_B^L, T_B^U], [I_B^L, I_B^U], [F_B^L, F_B^U])>, be any two interval neutrosophic uncertain linguistic variables, then the Hamming distance between $\tilde{s}_1$ and $\tilde{s}_2$ can be defined as follows.

$$d_{INULV}(\tilde{s}_1, \tilde{s}_2) = \frac{1}{12(l-1)} (|a_1 \times T_A^L - a_2 \times T_B^L|+|a_1 \times T_A^U - a_2 \times T_B^U|+|a_1 \times I_A^L - a_2 \times I_B^L|+ |a_1 \times I_A^U - a_2 \times I_B^U|+|a_1 \times F_A^L - a_2 \times F_B^L|+|a_1 \times F_A^U - a_2 \times F_B^U|+ +|b_1 \times T_A^L - b_2 \times T_B^L|+|b_1 \times T_A^U - b_2 \times T_B^U|+|b_1 \times I_A^L - b_2 \times I_B^L|+ |b_1 \times I_A^U - b_2 \times I_B^U|+|b_1 \times F_A^L - b_2 \times F_B^L|+|b_1 \times F_A^U - b_2 \times F_B^U|) \quad \textbf{(12)}$$

In order to illustrate the effectiveness of definition 3.2, the distance defined above must meet the three conditions in definition 3.1

Proof

Obviously, the distance defined in (12) can meets the conditions (1) and (2) in definition 3.1

In the following, we will prove that the distance defined in (12) can also meet the condition (3) in definition 3.1

For any one interval neutrosophic uncertain linguistic variable $\tilde{s}_3$ = <$[s_{a_3}, s_{b_3}]$,([T_C^L, T_C^U], [I_C^L, I_C^U], [F_C^L, F_C^U])>,

$$d_{IVNS}(\tilde{s}_1, \tilde{s}_3) = \frac{1}{12(l-1)} (|a_1 \times T_A^L - a_3 \times T_C^L|+|a_1 \times T_A^U - a_3 \times T_C^U|+|a_1 \times I_A^L - a_3 \times I_C^L|+|a_1 \times I_A^U - a_3 \times I_C^U|+|a_1 \times F_A^L - a_3 \times F_C^L|+|a_1 \times F_A^U - a_3 \times F_C^U|+|b_1 \times T_A^L - b_3 \times T_C^L|+|b_1 \times T_A^U - b_3 \times T_C^U|+|b_1 \times I_A^L - b_3 \times I_C^L|+|b_1 \times I_A^U - b_3 \times I_C^U|+|b_1 \times F_A^L - b_3 \times F_C^L|+|b_1 \times F_A^U - b_3 \times F_C^U|)$$

$=\frac{1}{12(l-1)}$ $(|a_1 \times T_A^L - a_2 \times T_B^L + a_2 \times T_B^L - a_3 \times T_C^L|+|a_1 \times T_A^U - a_2 \times T_B^U + a_2 \times T_B^U - a_3 \times T_C^U|+|a_1 \times I_A^L - a_2 \times I_B^L + a_2 \times I_B^L - a_3 \times I_C^L|+|a_1 \times I_A^U - a_2 \times I_B^U + a_2 \times I_B^U - a_3 \times I_C^U|$
$+|a_1 \times F_A^L - a_2 \times F_B^L + a_2 \times F_B^L - a_3 \times F_C^L|+|a_1 \times F_A^U - a_2 \times F_B^U + a_2 \times F_B^U - a_3 \times F_C^U|$

$+|b_1 \times T_A^L - b_2 \times T_B^L + b_2 \times T_B^L - b_3 \times T_C^L|+|b_1 \times T_A^U - b_2 \times T_B^U + b_2 \times T_B^U - b_3 \times T_C^U|+|b_1 \times I_A^L - b_2 \times I_B^L + b_2 \times I_B^L - b_3 \times I_C^L|+|b_1 \times I_A^U - b_2 \times I_B^U + b_2 \times I_B^U - b_3 \times I_C^U|$
$+|b_1 \times F_A^L - b_2 \times F_B^L + b_2 \times F_B^L - a_3 \times F_C^L|+|b_1 \times F_A^U - b_2 \times F_B^U + b_2 \times F_B^U - b_3 \times F_C^U|$

And
$\frac{1}{12(l-1)}(|a_1 \times T_A^L - a_2 \times T_B^L|+|a_2 \times T_B^L - a_3 \times T_C^L|+|a_1 \times T_A^U - a_2 \times T_B^U|+|a_2 \times T_B^U - a_3 \times T_C^U|+|a_1 \times I_A^L - a_2 \times I_B^L|+|a_2 \times I_B^L - a_3 \times I_C^L|+|a_1 \times I_A^U - a_2 \times I_B^U|+|a_2 \times I_B^U - a_3 \times I_C^U|+$

$|a_1 \times F_A^L - a_2 \times F_B^L|+|a_2 \times F_B^L - a_3 \times F_C^L|+|a_1 \times F_A^U - a_2 \times F_B^U|+|a_2 \times F_B^U - a_3 \times F_C^U|+$

$+|b_2 \times T_B^L - b_3 \times T_C^L|+|b_1 \times T_A^U - b_2 \times T_B^U|+|b_2 \times T_B^U - b_3 \times T_C^U|+|b_1 \times I_A^L - b_2 \times I_B^L|+|b_2 \times I_B^L - b_3 \times I_C^L|+|b_1 \times I_A^U - b_2 \times I_B^U|+|b_2 \times I_B^U - b_3 \times I_C^U|+|b_1 \times F_A^L - b_2 \times F_B^L|+|b_2 \times F_B^L - b_3 \times F_C^L|+|b_1 \times F_A^U - b_2 \times F_B^U|+|b_2 \times F_B^U - b_3 \times F_C^U|)$

$=\frac{1}{12(l-1)}(|a_1 \times T_A^L - a_2 \times T_B^L|+|a_1 \times T_A^U - a_2 \times T_B^U|+|a_1 \times I_A^L - a_2 \times I_B^L|+|a_1 \times I_A^U - a_2 \times I_B^U|+|a_1 \times F_A^L - a_2 \times F_B^L|$
$+|a_1 \times F_A^U - a_2 \times F_B^U|+|b_1 \times T_A^L - b_2 \times T_B^L|+|b_1 \times T_A^U - b_2 \times T_B^U|+|b_1 \times I_A^L - b_2 \times I_B^L|+|b_1 \times I_A^U - b_2 \times I_B^U| +|b_1 \times F_A^L - b_2 \times F_B^L|+|b_1 \times F_A^U - b_2 \times F_B^U|+$
$|a_2 \times T_B^L - a_3 \times T_C^L|+|a_2 \times T_B^U - a_3 \times T_C^U|+|a_2 \times I_B^L - a_3 \times I_C^L|+|a_2 \times I_B^U - a_3 \times I_C^U|+|a_2 \times F_B^L - a_3 \times F_C^L|+|a_2 \times F_B^U - a_3 \times F_C^U|+|b_2 \times T_B^L - b_3 \times T_C^L|+|b_2 \times T_B^U - b_3 \times T_C^U|+|b_2 \times I_B^L - b_3 \times I_C^L|+|b_2 \times I_B^U - b_3 \times I_C^U|+|b_2 \times F_B^L - b_3 \times F_C^L|+|b_2 \times F_B^U - b_3 \times F_C^U|)$

$=\frac{1}{12(l-1)}(|a_1 \times T_A^L - a_2 \times T_B^L|+|a_1 \times T_A^U - a_2 \times T_B^U|+|a_1 \times I_A^L - a_2 \times I_B^L|+|a_1 \times I_A^U - a_2 \times I_B^U|+|a_1 \times F_A^L - a_2 \times F_B^L|$
$+|a_1 \times F_A^U - a_2 \times F_B^U|+|b_1 \times T_A^L - b_2 \times T_B^L|+|b_1 \times T_A^U - b_2 \times T_B^U|+|b_1 \times I_A^L - b_2 \times I_B^L|+|b_1 \times I_A^U - b_2 \times I_B^U| +|b_1 \times F_A^L - b_2 \times F_B^L|+|b_1 \times F_A^U - b_2 \times F_B^U|)+$
$\frac{1}{12(l-1)}(|a_2 \times T_B^L - a_3 \times T_C^L|+|a_2 \times T_B^U - a_3 \times T_C^U|+|a_2 \times I_B^L - a_3 \times I_C^L|+|a_2 \times I_B^U - a_3 \times I_C^U|+|a_2 \times F_B^L - a_3 \times F_C^L|+|a_2 \times F_B^U - a_3 \times F_C^U|+|b_2 \times T_B^L - b_3 \times T_C^L|+|b_2 \times T_B^U - b_3 \times T_C^U|+|b_2 \times I_B^L - b_3 \times I_C^L|+|b_2 \times I_B^U - b_3 \times I_C^U|+|b_2 \times F_B^L - b_3 \times F_C^L|+|b_2 \times F_B^U - b_3 \times F_C^U|)$

$=d_{INULV}(\tilde{s}_1, \tilde{s}_2) + d_{INULV}(\tilde{s}_2, \tilde{s}_3)$
So , $d_{INULV}(\tilde{s}_1, \tilde{s}_2) + d_{INULV}(\tilde{s}_2, \tilde{s}_3) \geq d_{INULV}(\tilde{s}_1, \tilde{s}_3)$

Especially, when T_A^L=T_A^U, I_A^L=I_A^U, F_A^L=F_A^U,and T_B^L=T_B^U, I_B^L=I_B^U, and F_B^L=F_B^U the interval neutrosophic uncertain linguistic variables $\tilde{s}_1, \tilde{s}_2$ can be reduced to single valued uncertain linguistic variables. So the single valued neutrosophic uncertain linguistic variables are the special case of the interval neutrosophic uncertain linguistic variables.

Because the attribute weight is fully unknown, we can obtain the attribute weight vector by the maximizing deviation method. Its main idea can be described as follows. If all attribute values z_{ij} (j=1, 2,…, n) in the attribute C_j have a small difference for all alternatives, it shows that the attribute C_j has a small importance in ranking all alternatives, and it can be assigned a small attribute weight, especially, if all attribute values z_{ij} (j=1, 2,…,n) in the attribute C_j are equal, then the attribute C_j has no effect on sorting, and we can set zero to the weight of attribute C_j. On the contrary, if all attribute values z_{ij} (j=1, 2,…, n) in the attribute C_j have a big difference, the attribute C_j will have a big importance in ranking all alternatives, and its weight can be assigned a big value.
Here, based on the maximizing deviation method, we construct an optimization model to determine the optimal relative weights of criteria under interval neutrosophic uncertain linguistic environment. For the criterion $C_i \in$ C, we can use the distance $d(z_{ij}, z_{kj})$ to represent the deviation between attribute values z_{ij} and z_{kj}, and D_{ij} $=\sum_{k=1}^{m} d(z_{ij}, z_{kj})\, w_j$ can present the weighted deviation sum for the alternative A_i to all alternatives, then

D_j (w_j)=$\sum_{i=1}^{m} D_{ij}(w_j)$= $\sum_{i=1}^{m}\sum_{k=1}^{m} d(z_{ij}, z_{kj})\, w_j$ presents the weighted deviation sum for all alternatives, D (w_j)= $\sum_{j=1}^{n} D_j(w_j)$= $\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m} d(z_{ij}, z_{kj})\, w_j$, presents total weighted deviations for all alternatives with respect to all attributes.

Based on the above analysis, we can construct a non linear programming model to select the weight vector w by maximizing D (w),as follow:

$$\begin{cases} \text{Max } D(w_j) = \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m} d(z_{ij}, z_{kj})\, w_j \\ s.t\ \sum_{j=1}^{n} w_j^2,\ w_j \in [0,1], j = 1,2,\dots,n \end{cases} \quad \textbf{(13)}$$

Then we can build Lagrange multiplier function, and get

$$L(w_j,\lambda) = \sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m} d(z_{ij}, z_{kj})\, w_j + \lambda\left(\sum_{j=1}^{n} w_j^2 - 1\right)$$

$$\text{Let} \begin{cases} \dfrac{\partial L(w_j,\lambda)}{\partial w_j} = \sum_{i=1}^{m}\sum_{k=1}^{m} d(z_{ij}, z_{kj})\, w_j + 2\lambda w_j = 0 \\ \dfrac{\partial L(w_j,\lambda)}{\partial w_j} = \sum_{j=1}^{n} w_j^2 - 1 = 0 \end{cases}$$

We can get

$$\begin{cases} 2\lambda = \sqrt{\sum_{j}^{n}\left(\sum_{i=1}^{m}\sum_{k=1}^{m} d(z_{ij}, z_{kj})\right)^2} \\ w_j = \dfrac{\sum_{i=1}^{m}\sum_{k=1}^{m} d(z_{ij}, z_{kj})}{\sqrt{\sum_{j}^{n}\left(\sum_{i=1}^{m}\sum_{k=1}^{m} d(z_{ij}, z_{kj})\right)^2}} \end{cases} \quad \textbf{(14)}$$

Then we can get the normalized attribute weight, and have

$$w_j = \frac{\sum_{i=1}^{m}\sum_{k=1}^{m} d(z_{ij}, z_{kj})}{\sum_{j=1}^{n}\sum_{i=1}^{m}\sum_{k=1}^{m} d(z_{ij}, z_{kj})} \quad \textbf{(15)}$$

C. The Extended TOPSIS Method for the Interval Neutrosophic Uncertain linguistic Information.

The standard TOPSIS method can only process the real numbers, and cannot deal with the interval neutrosophic uncertain linguistic information. In the following, we will extend TOPSIS to process the interval neutrosophic uncertain linguistic variables. The steps are shown as follows

(1) Normalize the decision matrix

Considering the benefit or cost type of the attribute values, we can give the normalized matrix R=(r_{ij}), where r_{ij}=<$[r_{ij}^L, r_{ij}^U]$,],($[\dot{T}_{ij}^L, \dot{T}_{ij}^U]$, $[\dot{I}_{ij}^L, \dot{I}_{ij}^U]$, $[\dot{F}_{ij}^L, \dot{F}_{ij}^U]$)>,The normalization can be made shown as follows.

(i) For benefit type,

$$\begin{cases} r_{ij}^L = x_{ij}^L, r_{ij}^U = x_{ij}^U \quad \text{for } (1 \le i \le m,\ 1 \le j \le n) \\ \dot{T}_{ij}^L = T_{ij}^L,\ \dot{T}_{ij}^U = T_{ij}^U,\ \dot{I}_{ij}^L = I_{ij}^L,\ \dot{I}_{ij}^U = I_{ij}^U,\ \dot{F}_{ij}^L = F_{ij}^L, \dot{F}_{ij}^U = F_{ij}^U \end{cases} \quad \textbf{(16)}$$

(ii) For cost type,

$$\begin{cases} r_{ij}^L = \text{neg}(x_{ij}^U), r_{ij}^U = \text{neg}(x_{ij}^L) \quad \text{for } (1 \le i \le m,\ 1 \le j \le n) \\ \dot{T}_{ij}^L = T_{ij}^L,\ \dot{T}_{ij}^U = T_{ij}^U,\ \dot{I}_{ij}^L = I_{ij}^L,\ \dot{I}_{ij}^U = I_{ij}^U,\ \dot{F}_{ij}^L = F_{ij}^L, \dot{F}_{ij}^U = F_{ij}^U \end{cases} \quad \textbf{(17)}$$

(2) Construct the weighted normalize matrix

Y=$[y_{ij}]_{m\times n}$

$$\begin{bmatrix} <[y_{11}^L, y_{11}^U],],([\ddot{T}_{11}^L, \ddot{T}_{11}^U],[\ddot{I}_{11}^L, \ddot{I}_{11}^U],[\ddot{F}_{11}^L, \ddot{F}_{11}^U])> & \cdots & <[y_{11}^L, y_{11}^U],],([\ddot{T}_{1n}^L, \ddot{T}_{1n}^U],[\ddot{I}_{1n}^L, \ddot{I}_{1n}^U],[\ddot{F}_{1n}^L, \ddot{F}_{1n}^U])> \\ <[y_{21}^L, y_{21}^U],],([\ddot{T}_{21}^L, \ddot{T}_{21}^U],[\ddot{I}_{21}^L, \ddot{I}_{21}^U],[\ddot{F}_{21}^L, \ddot{F}_{21}^U])> & \cdots. & <[y_{2n}^L, y_{2n}^U],],([\ddot{T}_{2n}^L, \ddot{T}_{2n}^U],[\ddot{I}_{2n}^L, \ddot{I}_{2n}^U],[\ddot{F}_{2n}^L, \ddot{F}_{2n}^U])> \\ \cdots & & \cdots\cdots \\ <[y_{mn}^L, y_{mn}^U],],([\ddot{T}_{mn}^L, \ddot{T}_{mn}^U],[\ddot{I}_{mn}^L, \ddot{I}_{mn}^U],[\ddot{F}_{mn}^L, \ddot{F}_{mn}^U])> & \cdots & <[y_{mn}^L, y_{mn}^U],],([\ddot{T}_{mn}^L, \ddot{T}_{mn}^U],[\ddot{I}_{mn}^L, \ddot{I}_{mn}^U],[\ddot{F}_{mn}^L, \ddot{F}_{mn}^U])> \end{bmatrix}$$

Where

$$\begin{cases} y_{ij}^L = w_j r_{ij}^L, y_{ij}^U = w_j r_{ij}^U \\ \ddot{T}_{ij}^L = 1-(1-\dot{T}_{ij}^L)^{w_j}, \ddot{T}_{ij}^U = 1-(1-\dot{T}_{ij}^U)^{w_j}, \ddot{I}_{ij}^L = (\dot{I}_{ij}^L)^{w_j}, \ddot{I}_{ij}^U = (\dot{I}_{ij}^U)^{w_j},\ \ddot{F}_{ij}^L = (\dot{F}_{ij}^L)^{w_j}, \ddot{F}_{ij}^U = (\dot{F}_{ij}^U)^{w_j} \end{cases} \quad \textbf{(18)}$$

(3) Identify, the sets of the positive ideal solution Y^+= $(y_1^+, y_2^+,\dots, y_m^+)$ and the negative ideal solution Y^-= $(y_1^-, y_2^-,\dots, y_m^-)$, then we can get

$$Y^+ = (y_1^+, y_2^+,\dots, y_m^+) = (<[y_1^{L+}, y_1^{U+}],([\ddot{T}_1^{L+}, \ddot{T}_1^{U+}],[\ddot{I}_1^{L+}, \ddot{I}_1^{U+}],[\ddot{F}_1^{L+}, \ddot{F}_1^{U+}])>, <[y_2^{L+}, y_2^{U+}],([\ddot{T}_2^{L+}, \ddot{T}_2^{U+}],[\ddot{I}_2^{L+}, \ddot{I}_2^{U+}],[\ddot{F}_2^{L+}, \ddot{F}_2^{U+}])>,\dots, <[y_n^{L+}, y_n^{U+}],([\ddot{T}_n^{L+}, \ddot{T}_n^{U+}],[\ddot{I}_n^{L+}, \ddot{I}_n^{U+}],[\ddot{F}_n^{L+}, \ddot{F}_n^{U+}])> \quad (19)$$

$Y^{-}=(y_1^{-}, y_2^{-},\ldots, y_m^{-})=$
$$)=(<[\boldsymbol{y_1^{L-}}, \boldsymbol{y_1^{U-}}],([\boldsymbol{\ddot{T}_1^{L-}}, \boldsymbol{\ddot{T}_1^{U-}}],[\boldsymbol{\ddot{I}_1^{L-}}, \boldsymbol{\ddot{I}_1^{U-}}],[\boldsymbol{\ddot{F}_1^{L-}}, \boldsymbol{\ddot{F}_1^{U-}}])>,<[\boldsymbol{y_2^{L-}}, \boldsymbol{y_2^{U-}}],([\boldsymbol{\ddot{T}_2^{L-}}, \boldsymbol{\ddot{T}_2^{U-}}],[\boldsymbol{\ddot{I}_2^{L-}}, \boldsymbol{\ddot{I}_2^{U-}}],[\boldsymbol{\ddot{F}_2^{L-}}, \boldsymbol{\ddot{F}_2^{U-}}])>,\ldots,<[\boldsymbol{y_n^{L-}}, \boldsymbol{y_n^{U-}}],([\boldsymbol{\ddot{T}_n^{L-}}, \boldsymbol{\ddot{T}_n^{U-}}],[\boldsymbol{\ddot{I}_n^{L-}}, \boldsymbol{\ddot{I}_n^{U-}}],[\boldsymbol{\ddot{F}_n^{L-}}, \boldsymbol{\ddot{F}_n^{U-}}])> \quad \textbf{(20)}$$

Where

$$\begin{cases} \boldsymbol{y_j^{L+}} = \boldsymbol{\max}_i(\boldsymbol{y_{ij}^{L}}), \boldsymbol{y_j^{U+}} = \boldsymbol{\max}_i(\boldsymbol{y_{ij}^{U}}), \\ \boldsymbol{\ddot{T}_j^{L+}} = \boldsymbol{\max}_i(\boldsymbol{\ddot{T}_{ij}^{L}}), \boldsymbol{\ddot{T}_j^{U+}} = \boldsymbol{\max}_i(\boldsymbol{\ddot{T}_{ij}^{U}}), \boldsymbol{\ddot{I}_j^{L+}} = \boldsymbol{\min}_i(\boldsymbol{\ddot{I}_{ij}^{L}}), \boldsymbol{\ddot{I}_j^{U+}} = \boldsymbol{\min}_i(\boldsymbol{\ddot{I}_{ij}^{U}}), \boldsymbol{\ddot{F}_j^{L+}} = \boldsymbol{\min}_i(\boldsymbol{\ddot{F}_{ij}^{L}}), \boldsymbol{\ddot{F}_j^{U+}} = \boldsymbol{\min}_i(\boldsymbol{\ddot{F}_{ij}^{U}}), \\ \boldsymbol{y_j^{L-}} = \boldsymbol{\min}_i(\boldsymbol{y_{ij}^{L}}), \boldsymbol{y_j^{U-}} = \boldsymbol{\min}_i(\boldsymbol{y_{ij}^{U}}), \\ \boldsymbol{\ddot{T}_j^{L-}} = \boldsymbol{\min}_i(\boldsymbol{\ddot{T}_{ij}^{L}}), \boldsymbol{\ddot{T}_j^{U-}} = \boldsymbol{\min}_i(\boldsymbol{\ddot{T}_{ij}^{U}}), \boldsymbol{\ddot{I}_j^{L-}} = \boldsymbol{\max}_i(\boldsymbol{\ddot{I}_{ij}^{L}}), \boldsymbol{\ddot{I}_j^{U-}} = \boldsymbol{\max}_i(\boldsymbol{\ddot{I}_{ij}^{U}}), \boldsymbol{\ddot{F}_j^{L-}} = \boldsymbol{\max}_i(\boldsymbol{\ddot{F}_{ij}^{L}}), \boldsymbol{\ddot{F}_j^{U-}} = \boldsymbol{\max}_i(\boldsymbol{\ddot{F}_{ij}^{U}}), \end{cases}$$
(21)

(4) Obtain the distance between each alternative and the positive ideal solution, and between each alternative and the negative ideal solution, then we can get

$$D^{+}=(d_1^{+}, d_2^{+},\ldots, d_m^{+})$$
$$D^{-}=(d_1^{-}, d_2^{-},\ldots, d_m^{-}) \quad \textbf{(22)}$$

Where,

$$\begin{cases} d_i^{+} = \left[\sum_{j=1}^{n}\left(d(y_{ij}, y_j^{+})\right)^2\right]^{\frac{1}{2}} \\ d_i^{-} = \left[\sum_{j=1}^{n}\left(d(y_{ij}, y_j^{-})\right)^2\right]^{\frac{1}{2}} \end{cases} \quad \textbf{(23)}$$

Where , $d(y_{ij}, y_j^{+})$is the distance between the interval valued neutrosophic uncertain linguistic variables y_{ij} and y_j^{+} and $d(y_{ij}, y_j^{-})$ is the distance between the interval valued neutrosophic uncertain linguistic variables y_{ij} and y_j^{-} which can be calculated by (12)

(5) Obtain the closeness coefficients of each alternative to the ideal solution, and then we can get

$$cc_i=\frac{d_i^{+}}{d_i^{+}+d_i^{-}} \quad (i=1,2,\ldots,m) \quad \textbf{(24)}$$

(6) Rank the alternatives

According to the closeness coefficient above, we can choose an alternative with minimum cc_i or rank alternatives according to cc_i in ascending order

IV. An illustrative example

In this part, we give an illustrative example adapted from J. Ye [20] for the extended TOPSIS method to multiple attribute decision making problems in which the attribute values are the interval neutrosophic uncertain linguistic variables.

Suppose that an investment company, wants to invest a sum of money in the best option. To invest the money, there is a panel with four possible alternatives: (1) A_1 is car company; (2) A_2 is food company; (3) A_3 is a computer company; (4) A_4 is an arms company. The investement company must take a decision according to the three attributes: (1) C_1 is the risk; (2) C_2 is the growth; (3) C_3 is a the environmental impact. The weight vector of the attributes is ω= $(0.35, 0.25, 0.4)^{\mathrm{T}}$.The expert evaluates the four possible alternatives of $\mathrm{A_i}$ (i=1,2,3,4) with respect to the three attributes of $\mathrm{C_j}$ (i=1,2,3), where the evaluation information is expressed by the form of INULV values under the linguistic term set S={s_0=extremely poor, s_1=very poor, s_2= poor, s_3= medium, s_4= good, s_5= very good, s_6= extermely good}.

The evaluation information of an alternative $\mathrm{A_i}$ (i=1, 2, 3) with respect to an attribute $\mathrm{C_j}$ (j=1, 2, 3) can be given by the expert. For example, the INUL value of an alternative $\mathrm{A_1}$ with respect to an attribute $\mathrm{C_1}$ is given as <$[s_4, s_5]$, ([0.4, 0.5],[0.2, 0.3], [0.3, 0.4])> by the expert, which indicates that the mark of the alternative $\mathrm{A_1}$ with respect to the attribute $\mathrm{C_1}$ is about the uncertain linguistic value $[s_4, s_5,]$ with the satisfaction degree interval [0.4 ,0.5], indeterminacy degree interval [0.2, 0.3], and dissatisfaction degree interval [0.3, 0.4]. similarly, the four possible alternatives with respect to the three attributes can be evaluated by the expert, thus we can obtain the following interval neutrosophic uncertain linguistic decision matrix:

$(R)_{\mathrm{m\times n}}=$

$$\begin{bmatrix} <([s_4,s_5],([0.4,0.5],[0.2,0.3],[0.3,0.4])> & <([s_5,s_6],([0.4,0.6],[0.1,0.2],[0.2,0.4])> & <([s_4,s_5],([0.2,0.3],[0.1,0.2],[0.5,0.6])> \\ <([s_5,s_6],([0.5,0.7],[0.1,0.2],[0.2,0.3])> & <([s_4,s_5],([0.6,0.7],[0.1,0.2],[0.2,0.3])> & <([s_4,s_5],([0.5,0.7],[0.2,0.2],[0.1,0.2])> \\ <([s_5,s_6],([0.3,0.5],[0.1,0.2],[0.3,0.4])> & <([s_5,s_6],([0.5,0.6],[0.1,0.3],[0.3,0.4])> & <([s_4,s_4],([0.5,0.6],[0.1,0.3],[0.1,0.3])> \\ <([s_3,s_4],([0.7,0.8],[0.0,0.1],[0.1,0.2])> & <([s_3,s_4],([0.5,0.7],[0.1,0.2],[0.2,0.3])> & <([s_5,s_6],([0.3,0.4],[0.1,0.2],[0.1,0.2])> \end{bmatrix}$$

A. Decision steps

To get the best an alternatives, the following steps are involved:

Step 1: Normalization
Because the attributes are all the benefit types, we don't need the normalization of the decision matrix X
Step 2: Determine the attribute weight vector W, by formula (24), we can get

w_1= 0.337 , w_2= 0.244 , w_3=0.379
Step 3: Construct the weighted normalized matrix, by formula (18), we can get

$$Y=\begin{bmatrix} <([s_{1.508}, s_{1.885}],([0.175,0.229],[0.545,0.635],[0.635,0.708])> & <([s_{1.225}, s_{1.467}],([0.117,0.201],[0.570,0.675],[0.675,0.800])> & <([s_{1.508}, s_{1.885}],([0.081,0.126],[0.420,0.545],[0.77,0.825])> \\ <([s_{1.885}, s_{2.262}],([0.229,0.365],[0.42,0.545],[0.545,0.635])> & <([s_{0.98},, s_{1.225}],([0.201,0.255],[0.570,0.675],[0.675,0.745])> & <([s_{1.508}, s_{1.885}],([0.231,0.365],[0.545,0.545],[0.420,0.545])> \\ <([s_{1.885}, s_{2.262}],([0.125,0.23],[0.42,0.545],[0.635,0.708])> & <([s_{0.98}, s_{1.225}],([0.156,0.201],[0.570,0.745],[0.745,0.800])> & <([s_{1.508}, s_{1.508}],([0.231,0.292],[0.420,0.635],[0.420,0.635])> \\ <([s_{1.131}, s_{1.508}],([0.364,0.455],[0.0,0.42],[0.42,0.545])> & <([s_{0.735}, s_{0.98}],([0.156,0.255],[0.570,0.674],[0.675,0.745])> & <([s_{1.885}, s_{2.262}],([0.126,0.175],[0.420,0.545],[0.420,0.545])> \end{bmatrix}$$

Step 4: Identify the sets of the positive ideal solution $Y^{+}=(y_1^{+}, y_2^{+}, y_3^{+})$ and the negative ideal solution $Y^{-}=(y_1^{-}, y_2^{-}, y_3^{-})$, by formulas (19)- (21), we can get then we can get

$Y^{+}=(<([s_{1.885}, s_{2.262}],([0.365,0.455],[0,0.42],[0.42,0.545])>$
$,<([s_{1.225}, s_{1.47}],([0.201,0.255],[0.569,0.674],[0.674,0.745])>,$
$<([s_{1.885}, s_{2.262}],([0.230,0.365],[0.420,0.545],[0.420,0.545])>)$

$Y^{-}=(<([s_{1.131}, s_{1.508}],([0.126,0.230],[0.545,0.635],[0.635,0.708])>$
$,<([s_{0.735}, s_{0.98}],([0.117,0.201],[0.569,0.745],[0.745,0.799])>,<$
$([s_{1.508}, s_{1.508}],([0.081,0.126],[0.545,0.635],[0.770,0.825])>)$

Step 5: Obtain the distance between each alternative and the positive ideal solution, and between each alternative and the negative ideal solution, by formulas (22)-(23), we can get
D^{+}= (0.402, 0.065, 0.089, 0.066)

D^{-}= (0.052, 0.073, 0.080, 0.065)

Step 6: Calculate the closeness coefficients of each alternative to the ideal solution, by formula (24) and then we can get

cc_i = (0.885, 0.472, 0.527, 0.503)
Step 7: Rank the alternatives
According to the closeness coefficient above, we can choose an alternative with minimum to cc_i in ascending order. We can get

$$A_2 \geq A_4 \geq A_3 \geq A_1$$

So, the most desirable alternative is A_2

V-Comparison analysis with the existing interval neutrosophic uncertain linguistic multicriteria decision making method.

Recently, J. Ye [20] developed a new method for solving the MCDM problems with interval neutrosophic uncertain linguistic information. In this section, we will perform a comparison analysis between our new method and the existing method, and then highlight the advantages of the new method over the existing method.

(1) Compared with method proposed proposed by J. Ye [20], the method in this paper can solve the MADM problems with unknown weight, and rank the alternatives by the closeness coefficients. However, the method proposed by J. Ye [20] cannot deal with the unknown weight It can be seen that the result of the proposed method is same to the method proposed in [20].

(2) Compared with other extended TOPSIS method
Because the interval neutrosophic uncertain linguistic variables are the generalization of interval neutrosophic linguistic variables (INLV), interval neutrosophic variables (INV),and intuitionistic uncertain linguistic variable. Obviously, the extended TOPSIS method proposed by J. Ye [19], Z. Wei [54], Z. Zhang and C. Wu [3], are the special cases of the proposed method in this paper.
In a word, the method proposed in this paper is more generalized. At the same time, it is also simple and easy to use.

VI-**Conclusion**

In real decision making, there is great deal of qualitative information which can be expressed by uncertain linguistic variables. The interval neutrosophic uncertain linguistic variables were produced by combining the uncertain linguistic variables and interval neutrosophic set, and could easily express the indeterminate and inconsistent information in real world. TOPSIS had been proved to be a very effective decision making method and has been achieved more and more extensive applications. However, the standard TOPSIS method can only process the real numbers. In this paper, we extended TOPSIS method to deal with the interval neutrosophic uncertain linguistic variables information, and proposed an extended TOPSIS method with respect to the MADM problems in which the attribute values take the form of the interval neutrosophic and attribute weight unknown. Firstly, the operational rules and properties for the interval neutrosophic uncertain linguistic variables were presented. Then the distance between two interval neutrosophic uncertain linguistic variables was proposed and the attribute weight was calculated by the maximizing deviation method, and the closeness coefficient to the ideal solution for each alternative used to rank the alternatives. Finally, an illustrative example was given to illustrate the decision making steps, and compared with the existing method and proved the effectiveness of the proposed method. However, we hope that the concept presented here will create new avenue of research in current neutrosophic decision making area.

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Multi-attribute Decision Making based on Rough Neutrosophic Variational Coefficient Similarity Measure

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Abstract: The purpose of this study is to propose new similarity measures namely rough variational coefficient similarity measure under the rough neutrosophic environment. The weighted rough variational coefficient similarity measure has been also defined. The weighted rough variational coefficient similarity measures between the rough ideal alternative and each alternative are calculated to find the best alternative. The ranking order of all the alternatives can be determined by using the numerical values of similarity measures. Finally, an illustrative example has been provided to show the effectiveness and validity of the proposed approach. Comparisons of decision results of existing rough similarity measures have been provided.

Keywords: Neutrosophic set, Rough neutrosophic set; Rough variation coefficient similarity measure; Decision making.

1 Introduction

In 1965, L. A. Zadeh grounded the concept of degree of membership and defined fuzzy set [1] to represent/manipulate data with non-statistical uncertainty. In 1986, K. T. Atanassov [2] introduced the degree of non-membership as independent component and proposed intuitionistic fuzzy set (IFS). F. Smarandache introduced the degree of indeterminacy as independent component and defined the neutrosophic set [3, 4, 5]. For purpose of solving practical problems, Wang et al. [6] restricted the concept of neutrosophic set to single valued neutrosophic set (SVNS), since single value is an instance of set value. SVNS is a subclass of the neutrosophic set. SVNS consists of the three independent components namely, truth-membership, indeterminacy-membership and falsity-membership functions.

The concept of rough set theory proposed by Z. Pawlak [7] is an extension of the crisp set theory for the study of intelligent systems characterized by inexact, uncertain or insufficient information. The hybridization of rough set theory and neutrosophic set theory produces the rough neutrosophic set theory [8, 9], which was proposed by Broumi, Dhar and Smarandache [8, 9]. Rough neutrosophic set theory is also a powerful mathematical tool to deal with incompleteness.

Literature review reflects that similarity measures play an important role in the analysis and research of clustering analysis, decision making, medical diagnosis, pattern recognition, etc. Various similarity measures [10, 11, 12, 13, 14, 15, 16, 17, 18] of SVNSs and hybrid SVNSs are available in the literature. The concept of similarity measures in rough neutrosophic environment [19, 20, 21] has been recently proposed.

Pramanik and Mondal [19] proposed cotangent similarity measure of rough neutrosophic sets. In the same study [19],Pramanik and Mondal established its basic properties and provided its application to medical diagnosis. Pramanik and Mondal [20] also proposed cosine similarity measure of rough neutrosophic sets and its application in medical diagnosis. The same authors [21] also studied Jaccard similarity measure and Dice similarity measures in rough neutrosophic environment and provided their applications to multi attribute decision making. Mondal and Pramanik [22] presented tri-complex rough neutrosophic similarity measure and its application in multi-attribute decision making. Together with F. Smarandache and S. Pramnik, K. Mondal [23] presented hypercomplex rough neutrosophic similarity measure and its application in multi-attribute decision making. Mondal, Pramanik, and Smarandache [24] presented several trigonometric Hamming similarity measures of rough neutrosophic sets and their applications in multi attribute decision making problems.

Different methods for multiattribute decision making (MADM) and multicriteria decision making (MCDM) problems are available in the literature in different environment such as crisp environment [25, 26, 27, 28, 29], fuzzy environment [30, 31], intuitionistic fuzzy environment [32, 33, 34, 35, 36, 37, 38, 39, 40], neutrosophic environment [41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62], interval neutrosophic environment [63, 65, 66, 67, 68], neutrosophic soft expert environment [69], neutrosophic bipolar environment [70, 71], neutrosophic soft environment [72, 73, 74, 75, 76], neutrosophic hesitant fuzzy environment [77, 78, 79], rough neutrolsophic environment [80, 81], etc. In neutrosophic environment Biswas, Pramanik and Giri [82] studied hybrid vector similarity measure and its application in multi-attribute decision making. Getting motivation from the work of Biswas, Pramanik and Giri [82], for hybrid vector similarity measure in neutrosophic envionment, we extend the concept in rough neutrosophic environment.

In this paper, a new similarity measurement is proposed, namely rough variational coefficient similarity measure under rough neutrosophic environment. A numerical example is also provided.

Rest of the paper is structured as follows. Section 2 presents neutrosophic and rough neutrosophic preliminaries. Section 3 discusses various similarity measures and varional coefficient similarity measure in crisp environment. Section 4 presents various similarity measures and variational similarity measure for single valued neutrosophic sets. Section 5 presents variational coefficient similarity measure and weighted variational coefficient similarity measure for rough neutrosophic sets and establishes their basic properties. Section 6 is devoted to present multi attribute decision making based on rough neutrosophic variational coefficient similarity measure. Section 7 demonstrates the application of rough variational coefficient similarity measures to investment problem Finally, section 8 concludes the paper with stating the future scope of research.

2 Neutrosophic preliminaries

Definition 2.1 [3, 4, 5] Neutrosophic set

Let X be a space of points (objects) with generic element in X denoted by x. Then a neutrosophic set A in X is denoted by $A=\{x(T_A(x), I_A(x), F_A(x)): x \in X\}$ where, $T_A(x)$ is the truth membership function, $I_A(x)$ is the indeterminacy membership function and $F_A(x)$ is the falsity membership function. The functions $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or non-standard subsets of $]^-0, 1^+[$. There is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$ i.e. $^-0 \le T_A(x)+I_A(x)+F_A(x) \le 3^+$.

Definition 2.2 [6] (**Single-valued neutrosophic set**).

Let X be a universal space of points (objects), with a generic element $x \in X$. A single-valued neutrosophic set (SVNS) $N \subset X$ is denoted by

$N=\int_x \langle T_N(x), I_N(x), F_N(x)\rangle / x, \forall x \in X$, when X is continuous;

$N=\sum_{i=1}^{m} \langle T_N(x), I_N(x), F_N(x)\rangle / x, \forall x \in X$, when X is discrete.

SVNS is characterized by a true membership function $T_N(x)$, a falsity membership function $F_N(x)$ and an indeterminacy function $I_N(x)$ ith $T_N(x)$, $F_N(x)$, $I_N(x)$ $\in$ [0, 1] for all $x \in X$. For each $x \in X$, of a SVNS N $0 \le T_N(x)+I_N(x)+F_N(x) \le 3$.

2.1 Some operational rules and properties of SVNSs

Let $N_A=\langle T_A, I_A, F_A\rangle$ and $N_B=\langle T_B, I_B, F_B\rangle$ be two SVNSs in X. Then the following operations are defined as follows:

I. Complement: $N_{A^c}=\langle F_A, 1\text{-}I_A, T_A\rangle \ \forall x \in X$.

II. Addition: $N_A \oplus N_B=\langle T_A+T_B-T_AT_B, I_AI_B, F_AF_B\rangle$

III. Multiplication:
$N_A \otimes N_B=\langle T_AT_B, I_A+I_B\text{-}I_AI_B, F_A+F_B\text{-}F_AF_B\rangle$

IV. Scalar Multiplication:
$\lambda N_A=\langle 1-(1-T_A)^{\lambda}, I_A^{\lambda}, F_A^{\lambda}\rangle$ for $\lambda>0$.

V. $\langle N_A\rangle^{\lambda}=\langle (T_A)^{\lambda}, 1-(1-I_A)^{\lambda}, 1-(1-F_A)^{\lambda}\rangle$ for $\lambda>0$.

Definition 2.3 [6]

Complement of a SVNS N is denoted by N^c and is defined by

$T_{N^c}(x)=F_N(x)$; $I_{N^c}(x)=1-I_N(x)$; $F_{N^c}(x)=T_N(x)$

Definition 2.4 [6]

A SVNS N_A is contained in the other SVNS N_B, denoted as $N_A \subseteq N_B$, if and only if

$T_{N_A}(x) \le T_{N_B}(x)$; $I_{N_A}(x) \ge I_{N_B}(x)$; $F_{N_A}(x) \ge F_{N_B}(x)$ $\forall x \in X$

Definition 2.5 [6]

Two SVNSs N_A and N_B are equal, i.e. $N_A = N_B$, if and only if $N_A \supseteq N_B$ and $N_A \subseteq N_B$

Definition 2.6 [6]

Union of two SVNSs N_A and N_B is a SVNS N_C, written as $N_C = N_A \cup N_B$. Its truth membership, indeterminacy-membership and falsity membership functions are related to those of N_A and N_B by

$T_{N_C}(x)=\max(T_{N_A}(x), T_{N_B}(x))$; $I_{N_C}(x)=\min(I_{N_A}(x), I_{N_B}(x))$; $F_{N_C}(x)=\min(F_{N_A}(x), F_{N_B}(x))$ for all x in X.

Definition 2.7 [6] Intersection of two SVNSs N_A and N_B is a SVNS N_D, written as $N_D = N_A \cap N_B$, whose truth membership, indeterminacy-membership and falsity membership functions are related to those of N_A and N_B by

$T_{N_C}(x)=\min(T_{N_A}(x), T_{N_B}(x))$; $I_{N_C}(x)=\max(I_{N_A}(x), I_{N_B}(x))$; $F_{N_C}(x)=\max(F_{N_A}(x), F_{N_B}(x))$ for all x in X.

Definition 2.8 Rough Neutrosophic Sets [8, 9]

Let Z be a non-null set and R be an equivalence relation on Z. Let P be neutrosophic set in Z with the membership function T_P indeterminacy function I_P and non-membership function F_P. The lower and the upper approximations of P in the approximation (Z, R) denoted by $\underline{N}(P)$ and $\overline{N}(P)$ are respectively defined as follows:

$\underline{N}(P)=\left\langle <x, T_{\underline{N}(P)}(x), I_{\underline{N}(P)}(x), F_{\underline{N}(P)}(x)> / z\in[x]_R, x\in Z\right\rangle$,

$\overline{N}(P)=\left\langle <x, T_{\overline{N}(P)}(x), I_{\overline{N}(P)}(x), F_{\overline{N}(P)}(x)> / z\in[x]_R, x\in Z\right\rangle$,

Here, $T_{\underline{N}(P)}(x)=\wedge_z\in[x]_R\, T_P(z)$, $I_{\underline{N}(P)}(x)=\wedge_z\in[x]_R\, I_P(z)$,

$F_{\underline{N}(P)}(x)=\wedge_z\in[x]_R\, F_P(z)$, $T_{\overline{N}(P)}(x)=\vee_z\in[x]_R\, T_P(z)$,

$I_{\overline{N}(P)}(x)=\vee_z\in[x]_R\, I_P(z)$, $F_{\overline{N}(P)}(x)=\vee_z\in[x]_R\, F_P(z)$

So, $0\le \sup T_{\underline{N}(P)}(x)+\ \sup I_{\underline{N}(P)}(x)+\ \sup F_{\underline{N}(P)}(x)\ \le 3$

$0\le \sup T_{\overline{N}(P)}(x)+\ \sup I_{\overline{N}(P)}(x)+\ \sup F_{\overline{N}(P)}(x)\ \le 3$

Here $\vee$ and $\wedge$ denote "max" and "min'' operators respectively. $T_P(z)$, $I_P(z)$ and $F_P(z)$ denote respectively the membership, indeterminacy and non-membership function of z with respect to P. It is easy to see that $\underline{N}(P)$ and $\overline{N}(P)$ are two neutrosophic sets in Z.

Thus NS mappings $\underline{N}$, $\overline{N}$: $N(Z)\rightarrow N(Z)$ are, respectively, referred to as the lower and the upper rough NS approximation operators, and the pair $(\underline{N}(P), \overline{N}(P))$ is called the rough neutrosophic set [8, 9] in (Z, R).

From the above definition, it is seen that $\underline{N}(P)$ and $\overline{N}(P)$ have constant membership on the equivalence classes of R. if $\underline{N}(P)$ = $\overline{N}(P)$ i.e. $T_{\underline{N}(P)}(x)=$ $T_{\overline{N}(P)}(x)$, $I_{\underline{N}(P)}(x)=$ $I_{\overline{N}(P)}(x)$ and $F_{\underline{N}(P)}(x)=$ $F_{\overline{N}(P)}(x), \forall x\in Z$.

P is said to be a definable neutrosophic set in the approximation (Z, R). It can be easily proved that zero neutrosophic set (0_N = (0, 1, 1)) and unit neutrosophic sets (1_N = (1, 0, 0)) are definable neutrosophic sets.

Definition 2.9 [8, 9]

If $N(P)=(\underline{N}(P), \overline{N}(P))$ is a rough neutrosophic set in (Z, R) , the rough complement [8, 9] of $N(P)$ is the rough neutrosophic set denoted by $\sim N(P)=(\underline{N}(P)^c, \overline{N}(P)^c)$ where $\underline{N}(P)^c$, $\overline{N}(P)^c$ are the complements of neutrosophic sets of $\underline{N}(P)$, $\overline{N}(P)$ respectively.

$\underline{N}(P)^c=\left\langle <x, F_{\underline{N}(P)}(x), 1-I_{\underline{N}(P)}(x), T_{\underline{N}(P)}(x)> / x\in Z\right\rangle$ and

$\overline{N}(P)^c=\left\langle <x, F_{\overline{N}(P)}(x), 1-I_{\overline{N}(P)}(x), T_{\overline{N}(P)}(x)> / x\in Z\right\rangle$

Definition 2.10 [8, 9]

If $N(P_1)$ and $N(P_2)$ are the two rough neutrosophic sets of the neutrosophic set P respectively in Z, then the following definitions [8, 9] hold:

$N(P_1)=N(P_2)\Leftrightarrow \underline{N}(P_1)=\underline{N}(P_2)\wedge \overline{N}(P_1)=\overline{N}(P_2)$

$N(P_1)\subseteq N(P_2)\Leftrightarrow \underline{N}(P_1)\subseteq \underline{N}(P_2)\wedge \overline{N}(P_1)\subseteq \overline{N}(P_2)$

$N(P_1)\cup N(P_2)=<\underline{N}(P_1)\cup \underline{N}(P_2), \overline{N}(P_1)\cup \overline{N}(P_2)>$

$N(P_1)\cap N(P_2)=<\underline{N}(P_1)\cap \underline{N}(P_2), \overline{N}(P_1)\cap \overline{N}(P_2)>$

$N(P_1)+N(P_2)=<\underline{N}(P_1)+\underline{N}(P_2), \overline{N}(P_1)+\overline{N}(P_2)>$

$N(P_1).N(P_2)=<\underline{N}(P_1).\underline{N}(P_2), \overline{N}(P_1).\overline{N}(P_2)>$

If N, M, L are the rough neutrosophic sets in (Z, R), then the following proposition are stated from definitions [8, 9].

Proposition 1 [8, 9]

1. $\sim(\sim N)=N$
2. $N\cup M=M\cup N, N\cap M=M\cap N$
3. $(L\cup M)\cup N=L\cup(M\cup N)$,
   $(L\cap M)\cap N=L\cap(M\cap N)$
4. $(L\cup M)\cap N=(L\cup M)\cap(L\cup N)$,
   $(L\cap M)\cup N=(L\cap M)\cup(L\cap N)$

Proposition 2 [8, 9]

De Morgan's Laws are satisfied for rough neutrosophic sets .

1. $\sim(N(P_1)\cup N(P_2))=(\sim(N(P_1))\cap(\sim N(P_2))$
2. $\sim(N(P_1)\cap N(P_2))=(\sim(N(P_1)\cup(\sim N(P_2))$

Proposition 3 [8, 9]

If P_1 and P_2 are two neutrosophic sets in U such that $P_1\subseteq P_2$ *then* $N(P_1)\subseteq N(P_2)$

1. $N(P_1\cap P_2)\subseteq N(P_1)\cap N(P_2)$
2. $N(P_1\cup P_2)\supseteq N(P_1)\cup N(P_2)$

Proposition 4 [8, 9]

1. $\underline{N}(P)=\sim\overline{N}(\sim P)$
2. $\overline{N}(P)=\sim\underline{N}(\sim P)$
3. $\underline{N}(P)\subseteq\overline{N}(P)$

3 Similarity measures and variational coefficient similarity measure in crisp environment

The vector similarity measure is one of the important tools for the degree of similarity between objects. However, the Jaccard, Dice, and cosine similarity measures are often used for this purpose. Jaccard [83] , Dice [84] , and cosine [85] similarity measures between two vectors are stated below.

Let X = (x_1, x_2, …, x_n) and Y = (y_1, y_2, …, y_n) be two n-dimensional vectors with positive co-ordinates.

Definition 3.1 [83]

Jeccard index of two vectors (measuring the "similarity" of these vectors) can be defined as follows:

$$J(X, Y) = \frac{X.Y}{\|X\|^2+\|Y\|^2-X.Y} = \frac{\sum_{i=1}^n x_i y_i}{\sum_{i=1}^n x_i^2+\sum_{i=1}^n y_i^2-\sum_{i=1}^n x_i y_i} \quad (1)$$

where $\|X\|^2 = \sum_{i=1}^n x_i^2$ and $\|Y\|^2 = \sum_{i=1}^n y_i^2$ are the Euclidean norm of X and Y, $X.Y=\sum_{i=1}^n x_i y_i$ is the inner product of the vector X and Y.

Proposition 5 [83]

Jaccard index satisfies the following properties:

1. $0\le$ J(X, Y) ≤ 1

2. J(X, Y) = J(Y, X)
3. J(X, Y) = 1, for $X = Y$ i.e, $x_i = y_i$ (i = 1, 2, …, n) for every $x_i \in X$ and $y_i \in Y$

Definition 3.2 [84]

The Dice similarity measure can be defined as follows:

$$\mathrm{E}(X, Y) = \frac{2\,X.Y}{\|X\|^2+\|Y\|^2} = \frac{2\sum_{i=1}^{n} x_i\, y_i}{\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2} \tag{2}$$

Proposition 6 [84]

The Dice similarity measure satisfies the following properties:

1. 0 ≤ E(X, Y) ≤ 1
2. E(X, Y) = E(Y, X)
3. J(X, Y) = 1, for $X = Y$ i.e, $x_i = y_i$ (i = 1, 2, …, n) for every $x_i \in X$ and $y_i \in Y$.

Definition 3.3 [85]

The cosine similarity measure between two vectors X and Y is the inner product of these two vectors divided by the product of their lengths and can be defined as follows:

$$\mathrm{C}(X, Y) = \frac{X.Y}{\|X\|.\|Y\|} = \frac{\sum_{i=1}^{n} x_i\, y_i}{\sqrt{\sum_{i=1}^{n} x_i^2}\sqrt{\sum_{i=1}^{n} y_i^2}} \tag{3}$$

Proposition 7 [85]

The cosine similarity measure satisfies the following properties

1. 0 ≤ C(X, Y) ≤ 1
2. C(X, Y) = C(Y, X)
3. C(X, Y) = 1, for $X = Y$ i.e, $x_i = y_i$ (i = 1, 2, …, n) for every $x_i \in X$ and $y_i \in Y$.

These three formulas are similar in the sense that they take values in the interval [0, 1]. Jaccard and Dice similarity measures are undefined when $x_i = 0$, and $y_i = 0$ for i = 1, 2, …, n and cosine similarity measure is undefined when $x_i = 0$ or $y_i = 0$ for i = 1, 2, …, n.

Definition 3.4 [86]

Variational co-efficient similarity measure can be defined as follows:

$$\mathrm{V}(X, Y) = \lambda\frac{2\,X.Y}{\|X\|^2+\|Y\|^2} + (1-\lambda)\frac{X.Y}{\|X\|.\|Y\|}$$

$$= \lambda\frac{2\sum_{i=1}^{n} x_i\, y_i}{\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2} + (1-\lambda)\frac{\sum_{i=1}^{n} x_i\, y_i}{\sqrt{\sum_{i=1}^{n} x_i^2}\sqrt{\sum_{i=1}^{n} y_i^2}} \tag{4}$$

Proposition 8 [86]

Variational co-efficient similarity measure satisfies the following properties:

1. 0 ≤ V(X, Y) ≤ 1
2. V(X, Y) = V(Y, X)
3. V(X, Y) = 1, for $X = Y$ i.e, $x_i = y_i$ (i = 1, 2, …, n) for every $x_i \in X$ and $y_i \in Y$.

4. Various similarity measures for single valued neutrosophic sets.

Assume $N_A = \langle T_A, I_A, F_A \rangle$ and $N_B = \langle T_B, I_B, F_B \rangle$ be two SVNSs in a universe of discourse $X = (x_1, x_2, \ldots, x_n)$. $T_A, I_A, F_A \in [0,1]$ for any $x_i \in X$ in N_A or $T_B, I_B, F_B \in [0,1]$ for any $x_i \in X$ in N_B can be considered as a vector representation with three elements. Let $w_i \in [0,1]$ be the weight of each element x_i for i = 1, 2, …, n such that $\sum_{i=1}^{n} w_i = 1$, then Jaccard, Dice and cosine similarity measures can be presented as follows:

Definition 4.1[10] Jaccard similarity measure between $N_A = \langle T_A, I_A, F_A \rangle$ and $N_B = \langle T_B, I_B, F_B \rangle$ can be defined as follows:

$Jac(N_A, N_B)$ =

$$\frac{1}{n}\sum_{i=1}^{n}\frac{(T_A(x_i)\,T_B(x_i) + I_A(x_i)\,I_B(x_i) + F_A(x_i)\,F_B(x_i))}{\left\langle \begin{array}{l} \left[(T_A(x_i))^2+(I_A(x_i))^2+(F_A(x_i))^2\right] \\ +\left[(T_B(x_i))^2+(I_B(x_i))^2+(F_B(x_i))^2\right] \\ -\{T_A(x_i)\,T_B(x_i)+I_A(x_i)\,I_B(x_i) \\ +F_A(x_i)\,F_B(x_i)\} \end{array} \right\rangle} \tag{5}$$

Proposition 9 [10]

Jaccard similarity measure satiefies the following properites:

1. $0 \le Jac(N_A, N_B) \le 1$;
2. $Jac(N_A, N_B) = Jac(N_B, N_A)$;
3. $Jac(N_A, N_B) = 1$; if $N_A = N_B$ i.e., $T_A(x_i) = T_B(x_i)$, $I_A(x_i) = I_B(x_i)$, and $F_A(x_i) = F_B(x_i)$, for every x_i (i = 1, 2, …, n) in X.

Definition 4.1.1 [10] Weighted Jaccard similarity measure between $N_A = \langle T_A, I_A, F_A \rangle$ and $N_B = \langle T_B, I_B, F_B \rangle$ can be defined as follows:

$Jac_w(N_A, N_B)$=

$$\sum_{i=1}^{n} w_i \frac{(T_A(x_i)\,T_B(x_i) + I_A(x_i)\,I_B(x_i) + F_A(x_i)\,F_B(x_i))}{\left\langle \begin{array}{l} \left[(T_A(x_i))^2+(I_A(x_i))^2+(F_A(x_i))^2\right] \\ +\left[(T_B(x_i))^2+(I_B(x_i))^2+(F_B(x_i))^2\right] \\ -\{T_A(x_i)\,T_B(x_i)+I_A(x_i)\,I_B(x_i) \\ +F_A(x_i)\,F_B(x_i)\} \end{array} \right\rangle} \tag{6}$$

Proposition 10 [10]

Weighted Jaccard similarity measure satisfies the following properties:

1. $0 \le Jac_w(N_A, N_B) \le 1$;
2. $Jac_w(N_A, N_B) = Jac_w(N_B, N_A)$;
3. $Jac_w(N_A, N_B) = 1$; if $N_A = N_B$ i.e., $T_A(x_i) = T_B(x_i)$, $I_A(x_i) = I_B(x_i)$, and $F_A(x_i) = F_B(x_i)$, for every x_i (i = 1, 2, …, n) in X.

Definition 4.2 [11]

Dice similarity measure between $N_A=\langle T_A, I_A, F_A\rangle$ and $N_B=\langle T_B, I_B, F_B\rangle$ is defined as:

$$Dic(N_A, N_B) = \frac{1}{n}\sum_{i=1}^{n}\frac{2\begin{bmatrix}T_A(x_i)T_B(x_i)+I_A(x_i)I_B(x_i)\\ +F_A(x_i)F_B(x_i)\end{bmatrix}}{\left\langle\begin{matrix}\left[(T_A(x_i))^2+(I_A(x_i))^2+(F_A(x_i))^2\right]\\ +\left[(T_B(x_i))^2+(I_B(x_i))^2+(F_B(x_i))^2\right]\end{matrix}\right\rangle} \quad (7)$$

Proposition 11 [11]

Dice similarity measure satisfies the following properties:

1. $0 \le Dic(N_A, N_B) \le 1$;
2. $Dic(N_A, N_B) = Dic(N_B, N_A)$;
3. $Dic(N_A, N_B)=1$; if $N_A=N_B$ i.e., $T_A(x_i)=T_B(x_i)$, $I_A(x_i)=I_B(x_i)$, and $F_A(x_i)=F_B(x_i)$, for every x_i (i = 1, 2, …, n) in X.

Definition 4.2.1 [11]

Weighted Dice similarity measure between $N_A=\langle T_A, I_A, F_A\rangle$ and $N_B=\langle T_B, I_B, F_B\rangle$ can be defined as follows:

$$Dic_w(N_A, N_B) = \sum_{i=1}^{n} w_i\frac{2\begin{bmatrix}T_A(x_i)T_B(x_i)+I_A(x_i)I_B(x_i)\\ +F_A(x_i)F_B(x_i)\end{bmatrix}}{\left\langle\begin{matrix}\left[(T_A(x_i))^2+(I_A(x_i))^2+(F_A(x_i))^2\right]\\ +\left[(T_B(x_i))^2+(I_B(x_i))^2+(F_B(x_i))^2\right]\end{matrix}\right\rangle} \quad (8)$$

Proposition 12 [11]

Weighted Dice similarity measure

1. $0 \le Dic_w(N_A, N_B) \le 1$;
2. $Dic_w(N_A, N_B) = Dic_w(N_B, N_A)$;
3. $Dic_w(N_A, N_B)=1$; if $N_A = N_B$ i.e., $T_A(x_i)=T_B(x_i)$, $I_A(x_i)=I_B(x_i)$, and $F_A(x_i)=F_B(x_i)$, for every x_i (i = 1, 2, …, n) in X.

Definition 4.3 [12]

Cosine similarity measure between $N_A=\langle T_A, I_A, F_A\rangle$ and $N_B=\langle T_B, I_B, F_B\rangle$ can be defined as follows:

$$Cos(N_A, N_B) = \frac{1}{n}\sum_{i=1}^{n}\frac{\begin{matrix}(T_A(x_i)T_B(x_i)+I_A(x_i)I_B(x_i)\\ +F_A(x_i)F_B(x_i))\end{matrix}}{\sqrt{(T_A(x_i))^2+(I_A(x_i))^2+(F_A(x_i))^2}\sqrt{(T_B(x_i))^2+(I_B(x_i))^2+(F_B(x_i))^2}} \quad (9)$$

Proposition 13 [12]

Cosine similarity measure satisfies the following properties:

1. $0 \le Cos_w(N_A, N_B) \le 1$;
2. $Cos_w(N_A, N_B) = Cos_w(N_B, N_A)$
3. $Cos_w(N_A, N_B)=1$; if $N_A = N_B$ i.e., $T_A(x_i)=T_B(x_i)$, $I_A(x_i)=I_B(x_i)$, and $F_A(x_i)=F_B(x_i)$, for every x_i (i = 1, 2, …, n) in X.

Definition 4.3.1 [12]

Weighted cosine similarity measure between $N_A=\langle T_A, I_A, F_A\rangle$ and $N_B=\langle T_B, I_B, F_B\rangle$ can be defined as follows:

$$Cos_w(N_A, N_B)= \sum_{i=1}^{n} w_i\frac{\begin{matrix}(T_A(x_i)T_B(x_i)+I_A(x_i)I_B(x_i)\\ +F_A(x_i)F_B(x_i))\end{matrix}}{\sqrt{(T_A(x_i))^2+(I_A(x_i))^2+(F_A(x_i))^2}\sqrt{(T_B(x_i))^2+(I_B(x_i))^2+(F_B(x_i))^2}} \quad (10)$$

Proposition 14 [12]

Weighted cosine similarity measure satisfies the following properties:

1. $0 \le Cos_w(N_A, N_B) \le 1$;
2. $Cos_w(N_A, N_B) = Cos_w(N_B, N_A)$
3. $Cos_w(N_A, N_B)=1$; if $N_A = N_B$ i.e., $T_A(x_i)=T_B(x_i)$, $I_A(x_i)=I_B(x_i)$, and $F_A(x_i)=F_B(x_i)$, for every x_i (i = 1, 2, …, n) in X.

Jaccard and Dice similarity measures between two neutrosophic sets $N_A=\langle T_A, I_A, F_A\rangle$ and $N_B=\langle T_B, I_B, F_B\rangle$ are undefined when $T_A(x_i)=I_A(x_i)=F_A(x_i)=0$ and $T_B(x_i)=I_B(x_i)=F_B(x_i)=0$ for all i = 1, 2, …, n. Similarly the cosine formula for two neutrosophic sets $N_A=\langle T_A, I_A, F_A\rangle$ and $N_B=\langle T_B, I_B, F_B\rangle$ is undefined when $T_A(x_i)=I_A(x_i)=F_A(x_i)=0$ or $T_B(x_i)=I_B(x_i)=F_B(x_i)=0$ for all i = 1, 2, …, n.

5 Variational similarity measures for rough neutrosophic sets

The notion of rough neutrosophic set (RNS) is used as vector representations in 3D-vector space. Assume that $X = (x_1, x_2, \ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$ be two n-dimensional vectors with positive co-ordinates. Jaccard, Dice, cosine and cotangent similarity measures between two vectors are stated as follows.

Definition 5.1 [21] Jaccard similarity measure under rough neutrosophic environment

Assume that $A=\left\langle(\underline{T}_A(x_i),\underline{I}_A(x_i),\underline{F}_A(x_i)),(\overline{T}_A(x_i),\overline{I}_A(x_i),\overline{F}_A(x_i))\right\rangle$ and $B=\left\langle(\underline{T}_B(x_i),\underline{I}_B(x_i),\underline{F}_B(x_i)),(\overline{T}_B(x_i),\overline{I}_B(x_i),\overline{F}_B(x_i))\right\rangle$ in $X = (x_1, x_2, \ldots, x_n)$ be any two rough neutrosophic sets. Jacard similarity measure [21] between rough neutrosophic sets A and B can be defined as follows:

$Jac_{RNS}(A, B) =$

$$\frac{1}{n}\sum_{i=1}^{n}\frac{\begin{array}{l}(\delta T_A(x_i)\,\delta T_B(x_i)+\delta I_A(x_i)\,\delta I_B(x_i)\\ +\delta F_A(x_i)\,\delta F_B(x_i))\end{array}}{\left\langle\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\ +\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\\ -[\delta T_A(x_i)\,\delta T_B(x_i)+\delta I_A(x_i)\,\delta I_B(x_i)\\ +\delta F_A(x_i)\,\delta F_B(x_i)]\end{array}\right\rangle} \quad (11)$$

Proposition 15 [21]

Jaccard similarity measure [21] between A and B satisfies the following properties:

1. $0\le Jac_{RNS}(A,B)\le 1$;
2. $Jac_{RNS}(A,B)=Jac_{RNS}(B,A)$;
3. $Jac_{RNS}(A,B)=1$; iff $A = B$
4. If C is a RNS in Y and $A\subset B\subset C$ then,

$Jac_{RNS}(A, C)\le Jac_{RNS}(A, B)$, and $Jac_{RNS}(A, C) \le Jac_{RNS}(B, C)$

Definition 5.1.1 [21]

If we consider the weights of each element x_i, weighted rough Jaccard similarity measure [21] between rough neutrosophic sets A and B can be defined as follows:

$Jac_{WRNS}(A, B)$ =

$$\sum_{i=1}^{n} w_i\frac{\begin{array}{l}(\delta T_A(x_i)\,\delta T_B(x_i)+\delta I_A(x_i)\,\delta I_B(x_i)\\ +\delta F_A(x_i)\,\delta F_B(x_i))\end{array}}{\left\langle\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\ +\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\\ -[\delta T_A(x_i)\,\delta T_B(x_i)+\delta I_A(x_i)\,\delta I_B(x_i)\\ +\delta F_A(x_i)\,\delta F_B(x_i)]\end{array}\right\rangle} \quad (12)$$

$w_i\in[0,1]$, $i = 1, 2,\ldots, n$ and $\sum_{i=1}^{n} w_i=1$. If we take $w_i=\frac{1}{n}$, $i = 1, 2,\ldots, n$, then $Jac_{WRNS}(A, B) = Jac_{RNS}(A,B)$

Proposition 16 [21]

The weighted rough Jaccard similarity [21] measure between two rough neutrosophic sets A and B also satisfies the following properties:

1. $0\le Jac_{WRNS}(A,B)\le 1$;
2. $Jac_{WRNS}(A,B)=Jac_{WRNS}(B,A)$;
3. $Jac_{WRNS}(A,B)=1$; iff $A = B$
4. If C is a WRNS in Y and $A\subset B\subset C$ then, $Jac_{WRNS}(A, C) \le Jac_{WRNS}(A, B)$, and $Jac_{WRNS}(A, C) \le Jac_{WRNS}(B, C)$

Definition 5.2 [21] **Dice similarity measure under rough neutrosophic environment**

In this section, Dice similarity measure and the weighted Dice similarity measure for rough neutrosophic sets have been stated due to Pramanik and Mondal [21].

Suppose that

$A=\left\langle(\underline{T}_A(x_i),\underline{I}_A(x_i),\underline{F}_A(x_i)),(\overline{T}_A(x_i),\overline{I}_A(x_i),\overline{F}_A(x_i))\right\rangle$ and

$B=\left\langle(\underline{T}_B(x_i),\underline{I}_B(x_i),\underline{F}_B(x_i)),(\overline{T}_B(x_i),\overline{I}_B(x_i),\overline{F}_B(x_i))\right\rangle$ be any two rough neutrosophic sets in $X = (x_1, x_2,\ldots, x_n)$. Dice similarity measure between rough neutrosophic sets A and B can be defined as follows:

$DIC_{RNS}(A, B)$=

$$\frac{1}{n}\sum_{i=1}^{n}\frac{2\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]}{\left\langle\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\ +\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\end{array}\right\rangle} \quad (13)$$

Proposition 17 [21]

Dice similarity measure [21] satisfies the following properties.

1. $0\le DIC_{RNS}(A,B)\le 1$;
2. $DIC_{RNS}(A,B)=DIC_{RNS}(B,A)$;
3. $DIC_{RNS}(A,B)=1$; iff $A = B$
4. If C is a RNS in Y and $A\subset B\subset C$ then,

$DIC_{RNS}(A, C) \le DIC_{RNS}(A, B)$, and $DIC_{RNS}(A, C) \le DIC_{RNS}(B, C)$,

For proofs of the above mentioned four properties, see [21].

Definition 5.2.1

If we consider the weights of each element x_i, a weighted rough Dice similarity measure between rough neutrosophic sets A and B can be defined as follows:

$DIC_{WRNS}(A, B)$ =

$$\sum_{i=1}^{n} w_i\frac{2\begin{bmatrix}\delta T_A(x_i)\,\delta T_B(x_i)+\delta I_A(x_i)\,\delta I_B(x_i)\\ +\delta F_A(x_i)\,\delta F_B(x_i)\end{bmatrix}}{\left\langle\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\ +\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\end{array}\right\rangle} \quad (14)$$

$w_i\in[0,1]$, $i = 1, 2,\ldots, n$ and $\sum_{i=1}^{n} w_i=1$. If we take $w_i=\frac{1}{n}$, $i = 1, 2,\ldots, n$, then $DIC_{WRNS}(A, B) = DIC_{RNS}(A,B)$

Proposition 18 [21]

The weighted rough Dice similarity [21] measure between two rough neutrosophic sets A and B also satisfies the following properties:

1. $0\le DIC_{WRNS}(A,B)\le 1$;
2. $DIC_{WRNS}(A,B)=DIC_{WRNS}(B,A)$;
3. $DIC_{WRNS}(A,B)=1$; iff $A = B$
4. If C is a RNS in Y and $A\subset B\subset C$ then,

$DIC_{WRNS}(A, C) \le DIC_{WRNS}(A, B)$, and
$DIC_{WRNS}(A, C) \le DIC_{WRNS}(B, C)$.

For proofs of the above mentioned four properties, see [21].

Definition 5.3 [20]

Cosine similarity measure can be defined as the inner product of two vectors divided by the product of their lengths. It is the cosine of the angle between the vector representations of two rough neutrosophic sets. The cosine similarity measure is a fundamental measure used in information technology. Pramanik and Mondal [20] defined cosine similarity measure between rough neutrosophic sets in 3-D vector space.

Assume that $A=\left\langle\left(\underline{T}_A(x_i),\underline{I}_A(x_i),\underline{F}_A(x_i)\right),\left(\overline{T}_A(x_i),\overline{I}_A(x_i),\overline{F}_A(x_i)\right)\right\rangle$ and $B=\left\langle\left(\underline{T}_B(x_i),\underline{I}_B(x_i),\underline{F}_B(x_i)\right),\left(\overline{T}_B(x_i),\overline{I}_B(x_i),\overline{F}_B(x_i)\right)\right\rangle$ in $X = (x_1, x_2,\ldots, x_n)$ be any rough neutrosophic sets. Pramanik and Mondal [20] defined cosine similarity measure between rough neutrosophic sets A and B as follows:

C_{RNS}(A, B) =

$$\frac{1}{n}\sum_{i=1}^{n}\frac{\delta T_A(x_i)\delta T_B(x_i)+\delta I_A(x_i)\delta I_B(x_i)+\delta F_A(x_i)\delta F_B(x_i)}{\sqrt{\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]}\sqrt{\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]}} \tag{15}$$

Here, $\delta T_A(x_i)=\frac{\underline{T}_A(x_i)+\overline{T}_A(x_i)}{2}$, $\delta T_B(x_i)=\frac{\underline{T}_B(x_i)+\overline{T}_B(x_i)}{2}$,

$\delta I_A(x_i)=\frac{\underline{I}_A(x_i)+\overline{I}_A(x_i)}{2}$, $\delta I_B(x_i)=\frac{\underline{I}_B(x_i)+\overline{I}_B(x_i)}{2}$,

$\delta F_A(x_i)=\frac{\underline{F}_A(x_i)+\overline{F}_A(x_i)}{2}$, $\delta F_B(x_i)=\frac{\underline{F}_B(x_i)+\overline{F}_B(x_i)}{2}$

Proposition 19 [20]

Let A and B be rough neutrosophic sets. Cosine similarity measure [20] between A and B satisfies the following properties.

1. $0\le C_{RNS}(A,B)\le 1$;
2. $C_{RNS}(A,B)=C_{RNS}(B,A)$;
3. $C_{RNS}(A,B)=1$; iff $A = B$
4. If C is a RNS in Y and $A\subset B\subset C$ then, $C_{RNS}(A, C) \le C_{RNS}(A, B)$, and $C_{RNS}(A, C) \le C_{RNS}(B, C)$.

Definition 5.3.1 [20]

If we consider the weights of each element x_i, a weighted rough cosine similarity measure between rough neutrosophic sets A and B can be defined as follows:

$$C_{WRNS}(A,B)=\sum_{i=1}^{n}w_i\frac{\delta T_A(x_i)\delta T_B(x_i)+\delta I_A(x_i)\delta I_B(x_i)+\delta F_A(x_i)\delta F_B(x_i)}{\sqrt{\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]}\sqrt{\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]}} \tag{16}$$

$w_i\in[0,1]$, i = 1, 2,…, n and $\sum_{i=1}^{n}w_i=1$. If we take $w_i=\frac{1}{n}$, i = 1, 2,…, n, then $C_{WRNS}(A, B) = C_{RNS}(A, B)$

Proposition 20 [20]

The weighted rough cosine similarity measure [20] between two rough neutrosophic sets A and B also satisfies the following properties:

1. $0\le C_{WRNS}(A,B)\le 1$;
2. $C_{WRNS}(A,B)=C_{WRNS}(B,A)$;
3. $C_{WRNS}(A,B)=1$; iff $A = B$
4. If C is a WRNS in Y and $A\subset B\subset C$ then, $C_{WRNS}(A, C) \le C_{WRNS}(A, B)$, and $C_{WRNS}(A, C) \le C_{WRNS}(B, C)$.

For proofs of the above mentioned four properties, see [20].

Definition 5.4 [19] **Cotangent similarity measures of rough neutrosophic sets**

Assume that $A=\left\langle\left(\underline{T}_A(x_i),\underline{I}_A(x_i),\underline{F}_A(x_i)\right),\left(\overline{T}_A(x_i),\overline{I}_A(x_i),\overline{F}_A(x_i)\right)\right\rangle$ and $B=\left\langle\left(\underline{T}_B(x_i),\underline{I}_B(x_i),\underline{F}_B(x_i)\right),\left(\overline{T}_B(x_i),\overline{I}_B(x_i),\overline{F}_B(x_i)\right)\right\rangle$ in $X = (x_1, x_2,\ldots, x_n)$ be any two rough neutrosophic sets. Pramanik and Mondal [19] defined cotangent similarity measure between rough neutrosophic sets A and B as follows:

$COT_{RNS}(A, B)$ =

$$\frac{1}{n}\sum_{i=1}^{n}\left\langle\cot\left(\frac{\pi}{12}\left(\begin{array}{l}3+\left|\delta T_A(x_i)-\delta T_B(x_i)\right|\\+\left|\delta I_A(x_i)-\delta I_B(x_i)\right|\\+\left|\delta I_A(x_i)-\delta I_B(x_i)\right|\end{array}\right)\right)\right\rangle \tag{17}$$

Here, $\delta T_A(x_i)=\frac{\underline{T}_A(x_i)+\overline{T}_A(x_i)}{2}$, $\delta T_B(x_i)=\frac{\underline{T}_B(x_i)+\overline{T}_B(x_i)}{2}$,

$\delta I_A(x_i)=\frac{\underline{I}_A(x_i)+\overline{I}_A(x_i)}{2}$, $\delta I_B(x_i)=\frac{\underline{I}_B(x_i)+\overline{I}_B(x_i)}{2}$,

$\delta F_A(x_i)=\frac{\underline{F}_A(x_i)+\overline{F}_A(x_i)}{2}$, $\delta F_B(x_i)=\frac{\underline{F}_B(x_i)+\overline{F}_B(x_i)}{2}$

Proposition 21 [19]

Cotangent similarity measure satisfies the following properties:

1. $0\le COT_{RNS}(A,B)\le 1$;
2. $COT_{RNS}(A,B)=COT_{RNS}(B,A)$;
3. $COT_{RNS}(A,B)=1$; iff $A = B$
4. If C is a RNS in Y and $A\subset B\subset C$ then, $COT_{RNS}(A, C) \le COT_{RNS}(A, B)$, and $COT_{RNS}(A, C) \le COT_{RNS}(B, C)$.

Definition 5.4.1

If we consider the weights of each element x_i, a weighted rough cotangent similarity measure [19] between rough neutrosophic sets A and B can be defined as follows:

$COT_{WRNS}(A, B)$ =

$$\sum_{i=1}^{n}w_i\left\langle\cot\left(\frac{\pi}{12}\left(\begin{array}{l}3+\left|\delta T_A(x_i)-\delta T_B(x_i)\right|\\+\left|\delta I_A(x_i)-\delta I_B(x_i)\right|\\+\left|\delta I_A(x_i)-\delta I_B(x_i)\right|\end{array}\right)\right)\right\rangle \tag{18}$$

$w_i\in[0,1]$, i = 1, 2,…, n and $\sum_{i=1}^{n}w_i=1$. If we take $w_i=\frac{1}{n}$, i = 1, 2,…, n, then $COT_{WRNS}(A, B) = COT_{RNS}(A, B)$

Proposition 22 [19]

The weighted rough cosine similarity measure between two rough neutrosophic sets A and B also satisfies the following properties:

1. $0\le COT_{WRNS}(A,B)\le 1$;
2. $COT_{WRNS}(A,B)=COT_{WRNS}(B,A)$;
3. $COT_{WRNS}(A,B)=1$; iff $A = B$
4. If C is a WRNS in Y and $A\subset B\subset C$ then, $COT_{WRNS}(A, C) \le COT_{WRNS}(A, B)$, and $COT_{WRNS}(A, C) \le COT_{WRNS}(B, C)$

Definition 5.5 (Variational co-efficient similarity measure between rough neutrosophic sets)

Let $A=\left\langle\left(\underline{T}_A(x_i),\underline{I}_A(x_i),\underline{F}_A(x_i)\right),\left(\overline{T}_A(x_i),\overline{I}_A(x_i),\overline{F}_A(x_i)\right)\right\rangle$ and $B=\left\langle\left(\underline{T}_B(x_i),\underline{I}_B(x_i),\underline{F}_B(x_i)\right),\left(\overline{T}_B(x_i),\overline{I}_B(x_i),\overline{F}_B(x_i)\right)\right\rangle$ be two rough neutrosophic sets. Variational co-efficient similarity measure between rough neutrosophic sets can be presented as follows:

$Var_{RNS}(A, B) =$

$$\frac{1}{n}\left[\begin{array}{l}\lambda\sum_{i=1}^{n}\dfrac{2\left\{\begin{array}{l}\delta T_A(x_i)\delta T_B(x_i)+\delta I_A(x_i)\delta I_B(x_i)\\+\delta F_A(x_i)\delta F_B(x_i)\end{array}\right\}}{\left\{\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\+\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\end{array}\right\}}\\+(1-\lambda)\sum_{i=1}^{n}\dfrac{\delta T_A(x_i)\delta T_B(x_i)+\delta I_A(x_i)\delta I_B(x_i)+\delta F_A(x_i)\delta F_B(x_i)}{\sqrt{\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\end{array}}}\end{array}\right] \tag{19}$$

Here, $\delta T_A(x_i)=\dfrac{\underline{T}_A(x_i)+\overline{T}_A(x_i)}{2}$, $\delta T_B(x_i)=\dfrac{\underline{T}_B(x_i)+\overline{T}_B(x_i)}{2}$,

$\delta I_A(x_i)=\dfrac{\underline{I}_A(x_i)+\overline{I}_A(x_i)}{2}$, $\delta I_B(x_i)=\dfrac{\underline{I}_B(x_i)+\overline{I}_B(x_i)}{2}$,

$\delta F_A(x_i)=\dfrac{\underline{F}_A(x_i)+\overline{F}_A(x_i)}{2}$, $\delta F_B(x_i)=\dfrac{\underline{F}_B(x_i)+\overline{F}_B(x_i)}{2}$

Proposition 23

The variational co-efficient similarity measure $Var_{RNS}(A, B)$ between two rough neutrosophic sets A and B, satisfies the following properties:

1. $0\le Var_{RNS}(A,B)\le 1$;
2. $Var_{RNS}(A,B)=Var_{RNS}(B,A)$;
3. $Var_{RNS}(A,B)=1$; if $A = B$ i.e., $\delta T_A(x_i)=\delta T_B(x_i)$, $\delta I_A(x_i)=\delta I_B(x_i)$, and $\delta F_A(x_i)=\delta F_B(x_i)$, for every x_i ($i = 1, 2, \ldots, n$) in X.

Proof.

(1.) It is obvious that $Var_{RNS}(A,B)\ge 0$. Thus it is required to prove that $Var_{RNS}(A,B)\le 1$.

From rough neutrosophic dice similarity measure it can be witten that

$$0\le\frac{1}{n}\sum_{i=1}^{n}\frac{2\left[\begin{array}{l}\delta T_A(x_i)\delta T_B(x_i)+\delta I_A(x_i)\delta I_B(x_i)\\+\delta F_A(x_i)\delta F_B(x_i)\end{array}\right]}{\left\langle\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\+\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\end{array}\right\rangle}\le 1 \tag{20}$$

and from rough neutrosophic cosine similarity measure it can be written that

$$0\le\frac{1}{n}\sum_{i=1}^{n}\frac{\delta T_A(x_i)\delta T_B(x_i)+\delta I_A(x_i)\delta I_B(x_i)+\delta F_A(x_i)\delta F_B(x_i)}{\sqrt{\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\end{array}}}\le 1 \tag{21}$$

Combining Eq.(20) and Eq.(21) , we obtain

$Var_{RNS}(A, B) =$

$$\frac{1}{n}\left[\begin{array}{l}\lambda\sum_{i=1}^{n}\dfrac{2\left\{\begin{array}{l}\delta T_A(x_i)\delta T_B(x_i)+\delta I_A(x_i)\delta I_B(x_i)\\+\delta F_A(x_i)\delta F_B(x_i)\end{array}\right\}}{\left\{\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\+\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\end{array}\right\}}\\+(1-\lambda)\sum_{i=1}^{n}\dfrac{\delta T_A(x_i)\delta T_B(x_i)+\delta I_A(x_i)\delta I_B(x_i)+\delta F_A(x_i)\delta F_B(x_i)}{\sqrt{\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\.\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\end{array}}}\end{array}\right] \tag{22}$$

$\le\lambda+(1-\lambda)=1$

Thus, $0\le Var_{RNS}(A,B)\le 1$;

(2.) $Var_{RNS}(A, B) =$

$$\frac{1}{n}\left[\begin{array}{l}\lambda\sum_{i=1}^{n}\dfrac{2\left\{\begin{array}{l}\delta T_A(x_i)\delta T_B(x_i)+\delta I_A(x_i)\delta I_B(x_i)\\+\delta F_A(x_i)\delta F_B(x_i)\end{array}\right\}}{\left\{\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\+\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\end{array}\right\}}\\+(1-\lambda)\sum_{i=1}^{n}\dfrac{\delta T_A(x_i)\delta T_B(x_i)+\delta I_A(x_i)\delta I_B(x_i)+\delta F_A(x_i)\delta F_B(x_i)}{\sqrt{\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\end{array}}}\end{array}\right]$$

$$=\frac{1}{n}\left[\begin{array}{l}\lambda\sum_{i=1}^{n}\dfrac{2\left\{\begin{array}{l}\delta T_B(x_i)\delta T_A(x_i)+\delta I_B(x_i)\delta I_A(x_i)\\+\delta F_B(x_i)\delta F_A(x_i)\end{array}\right\}}{\left\{\begin{array}{l}\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\\+\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\end{array}\right\}}\\+(1-\lambda)\sum_{i=1}^{n}\dfrac{\delta T_B(x_i)\delta T_A(x_i)+\delta I_B(x_i)\delta I_A(x_i)+\delta F_B(x_i)\delta F_A(x_i)}{\sqrt{\begin{array}{l}\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\\\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\end{array}}}\end{array}\right]$$

$=Var_{RNS}(B,A)$

(3.) If $A = B$ i.e., $\delta T_A(x_i)=\delta T_B(x_i)$, $\delta I_A(x_i)=\delta I_B(x_i)$, and $\delta F_A(x_i)=\delta F_B(x_i)$, for every x_i ($i = 1, 2, \ldots, n$) in X,

$Var_{RNS}(A, A) =$

$$\frac{1}{n}\left[\begin{array}{l}\lambda\sum_{i=1}^{n}\dfrac{2\left\{\begin{array}{l}\delta T_A(x_i)\delta T_A(x_i)+\delta I_A(x_i)\delta I_A(x_i)\\+\delta F_A(x_i)\delta F_A(x_i)\end{array}\right\}}{\left\{\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\+\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\end{array}\right\}}\\+(1-\lambda)\sum_{i=1}^{n}\dfrac{\begin{array}{c}\delta T_A(x_i)\delta T_A(x_i)+\delta I_A(x_i)\delta I_A(x_i)\\+\delta F_A(x_i)\delta F_A(x_i)\end{array}}{\sqrt{\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\end{array}}}\end{array}\right]$$

$$=\frac{1}{n}\left[n\lambda+n(1-\lambda)\right]=1$$

These results show the completion of the proofs of the three properties.

Definition 5.6 (Weighted variational co-efficient similarity measure between rough neutrosophic sets)

Let $A=\left\langle\left(\underline{T}_A(x_i),\underline{I}_A(x_i),\underline{F}_A(x_i)\right),\left(\overline{T}_A(x_i),\overline{I}_A(x_i),\overline{F}_A(x_i)\right)\right\rangle$ and $B=\left\langle\left(\underline{T}_B(x_i),\underline{I}_B(x_i),\underline{F}_B(x_i)\right),\left(\overline{T}_B(x_i),\overline{I}_B(x_i),\overline{F}_B(x_i)\right)\right\rangle$ be any two rough neutrosophic sets. Rough variational co-efficient similarity measure between rough neutrosophic setsA and B in 3-D vector space can be presented as follows:

$Var_{WRNS}(A, B)$ =

$$\left[\begin{array}{l}\lambda\sum_{i=1}^{n}w_i\dfrac{2\left\{\begin{array}{l}\delta T_A(x_i)\delta T_B(x_i)+\delta I_A(x_i)\delta I_B(x_i)\\+\delta F_A(x_i)\delta F_B(x_i)\end{array}\right\}}{\left\{\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\+\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\end{array}\right\}}\\+(1-\lambda)\sum_{i=1}^{n}w_i\dfrac{\begin{array}{c}\delta T_A(x_i)\delta T_B(x_i)+\delta I_A(x_i)\delta I_B(x_i)\\+\delta F_A(x_i)\delta F_B(x_i)\end{array}}{\sqrt{\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\\cdot\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\end{array}}}\end{array}\right] \quad (23)$$

If $w=\left[\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n}\right]^T$, then Eq.(23) is reduced to Eq.(19).

Proposition 24

The weighted variational co-efficient similarity measure also satisfies the following properties:

1. $0\le Var_{WRNS}(A,B)\le 1$;
2. $Var_{WRNS}(A,B)=Var_{WRNS}(B,A)$;
3. $Var_{WRNS}(A, B) = 1$; if $A = B$ i.e., $\delta T_A(x_i)=\delta T_B(x_i)$, $\delta I_A(x_i)=\delta I_B(x_i)$, and $\delta F_A(x_i)=\delta F_B(x_i)$, for every x_i (i = 1, 2, …, n) in X.

Proof:

(1.) It is obvious that $Var_{WRNS}(A,B)\ge 0$. Thus it is required to prove that $Var_{WRNS}(A,B)\le 1$.

From rough neutrosophic weighted dice similarity measure, it can be written that

$$0\le\frac{1}{n}\sum_{i=1}^{n}w_i\frac{2\left[\begin{array}{l}\delta T_A(x_i)\delta T_B(x_i)+\delta I_A(x_i)\delta I_B(x_i)\\+\delta F_A(x_i)\delta F_B(x_i)\end{array}\right]}{\left\langle\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\+\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\end{array}\right\rangle}\le 1 \quad (24)$$

and from rough neutrosophic weighted cosine similarity measure it can be written that

$$0\le\frac{1}{n}\sum_{i=1}^{n}w_i\frac{\begin{array}{c}\delta T_A(x_i)\delta T_B(x_i)+\delta I_A(x_i)\delta I_B(x_i)\\+\delta F_A(x_i)\delta F_B(x_i)\end{array}}{\sqrt{\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\end{array}}}\le 1 \quad (25)$$

Combining Eq.(24) and Eq.(25), we obtain

Var_{WRNS}(A, B) =

$$\left[\begin{array}{l}\lambda\sum_{i=1}^{n}w_i\dfrac{2\left\{\begin{array}{l}\delta T_A(x_i)\delta T_B(x_i)+\delta I_A(x_i)\delta I_B(x_i)\\+\delta F_A(x_i)\delta F_B(x_i)\end{array}\right\}}{\left\{\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\+\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\end{array}\right\}}\\+(1-\lambda)\sum_{i=1}^{n}w_i\dfrac{\begin{array}{c}\delta T_A(x_i)\delta T_B(x_i)+\delta I_A(x_i)\delta I_B(x_i)\\+\delta F_A(x_i)\delta F_B(x_i)\end{array}}{\sqrt{\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\end{array}}}\end{array}\right] \quad (26)$$

$$\le\lambda+(1-\lambda)=1$$

Thus, $0\le Var_{WRNS}(A,B)\le 1$;

(2.) $Var_{WRNS}(A, B)$ =

$$\left[\begin{array}{l}\lambda\sum_{i=1}^{n}w_i\dfrac{2\left\{\begin{array}{l}\delta T_A(x_i)\delta T_B(x_i)+\delta I_A(x_i)\delta I_B(x_i)\\+\delta F_A(x_i)\delta F_B(x_i)\end{array}\right\}}{\left\{\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\+\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\end{array}\right\}}\\+(1-\lambda)\sum_{i=1}^{n}w_i\dfrac{\begin{array}{c}\delta T_A(x_i)\delta T_B(x_i)+\delta I_A(x_i)\delta I_B(x_i)\\+\delta F_A(x_i)\delta F_B(x_i)\end{array}}{\sqrt{\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\end{array}}}\end{array}\right]$$

$$=\left[\begin{array}{l}\lambda\sum_{i=1}^{n}w_i\dfrac{2\left\{\begin{array}{l}\delta T_B(x_i)\delta T_A(x_i)+\delta I_B(x_i)\delta I_A(x_i)\\+\delta F_B(x_i)\delta F_A(x_i)\end{array}\right\}}{\left\{\begin{array}{l}\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\\+\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\end{array}\right\}}\\+(1-\lambda)\sum_{i=1}^{n}w_i\dfrac{\begin{array}{c}\delta T_B(x_i)\delta T_A(x_i)+\delta I_B(x_i)\delta I_A(x_i)\\+\delta F_B(x_i)\delta F_A(x_i)\end{array}}{\sqrt{\begin{array}{l}\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\\\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\end{array}}}\end{array}\right]$$

$$=Var_{WRNS}(B,A)$$

(3.) If $A = B$ i.e.,

$\delta T_A(x_i) = \delta T_B(x_i)$, $\delta I_A(x_i) = \delta I_B(x_i)$, and $\delta F_A(x_i) = \delta F_B(x_i)$, for every x_i (i = 1, 2, …, n) in X,

$Var_{WRNS}(A, A) =$

$$\left[\begin{array}{l}\lambda \sum_{i=1}^{n} w_i \dfrac{2\left\{\begin{array}{l}\delta T_A(x_i)\delta T_A(x_i)+\delta I_A(x_i)\delta I_A(x_i)\\+\delta F_A(x_i)\delta F_A(x_i)\end{array}\right\}}{\left\{\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\+\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\end{array}\right\}}\\ +(1-\lambda)\sum_{i=1}^{n} w_i \dfrac{\begin{array}{l}\delta T_A(x_i)\delta T_A(x_i)+\delta I_A(x_i)\delta I_A(x_i)\\+\delta F_A(x_i)\delta F_A(x_i)\end{array}}{\sqrt{\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\ \sqrt{(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2}\end{array}}}\end{array}\right]$$

$$= \left[\lambda \sum_{i=1}^{n} w_i + (1-\lambda)\sum_{i=1}^{n} w_i\right] = 1$$

These results show the completion of the proofs of the three proiperties.

6. Multi attribute decision making based on rough neutrosophic variational coefficient similarity measure

In this section, a rough variational co-efficient similarity measure is employed to multi-attribute decision making in rough neutrosophic environment. Assume that $A = \{A_1, A_2,..., A_m\}$ be the set of alternatives and $C = \{C_1, C_2,..., C_n\}$ be the set of attributes in a multi-attribute decision making problem. Assune that w_j be the weight of the attribute C_j provided by the decision maker such that each $w_i \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$ However, in real situation decision maker may often face difficulty to evaluate alternatives over the attributes due to vague or incomplete information about alternatives in a decision making situation. Rough neutrosophic set can be used in MADM to deal with incomplete information of the alternatives. In this paper, the assessment values of all the alternatives with respect to attributes are considered as the rough neutrosophic values (see Table 1).

Table1: Rough neutrosophic decision matrix

$$D_{RNS} = \left\langle \underline{d}_{ij}, \overline{d}_{ij} \right\rangle_{m\times n} =$$

$$\begin{array}{c|cccc} & C_1 & C_2 & . & C_n \\ \hline A_1 & \langle \underline{d}_{11}, \overline{d}_{11}\rangle & \langle \underline{d}_{12}, \overline{d}_{12}\rangle & \cdots & \langle \underline{d}_{1n}, \overline{d}_{1n}\rangle \\ A_2 & \langle \underline{d}_{21}, \overline{d}_{21}\rangle & \langle \underline{d}_{22}, \overline{d}_{22}\rangle & \cdots & \langle \underline{d}_{2n}, \overline{d}_{2n}\rangle \\ . & \cdots & \cdots & \cdots & \cdots \\ . & \cdots & \cdots & \cdots & \cdots \\ A_m & \langle \underline{d}_{m1}, \overline{d}_{m1}\rangle & \langle \underline{d}_{m2}, \overline{d}_{m2}\rangle & \cdots & \langle \underline{d}_{mn}, \overline{d}_{mn}\rangle \end{array} \quad (27)$$

Here $\langle \underline{d}_{ij}, \overline{d}_{ij} \rangle$ is the rough neutrosophic number for the i-th alternative and the j-th attribute.

Definition 6.1: Transforming operator for SVNSs [80]

The rough neutrosophic decision matrix (27) can be transformed to single valued neutrosophic decision matrix whose ij-th element α_{ij} can be presented as follows:

$$\alpha_{ij} = \left\langle \frac{\underline{d}_{ij} + \overline{d}_{ij}}{2} \right\rangle_{m\times n}, \text{ for } i = 1, 2, 3,..., m;$$

j = 1, 2, 3, ..., n. (28)

Step1. Determine the neutrosophic relative positive ideal solution

In multi-criteria decision-making environment, the concept of ideal point has been used to help identify the best alternative in the decision set.

Definition 6.2 [51].

Let H be the collection of two types of attributes, namely, benefit type attribute (P) and cost type attribute (L) in the MADM problems. The relative positive ideal neutrosophic solution (RPINS) $Q_S^+ = [\delta_{q_S}^+, \delta_{q_S}^+,..., \delta_{q_S}^+]$ is the solution of the decision matrix $D_S = \langle \delta T_{ij}, \delta I_{ij}, \delta F_{ij} \rangle_{m\times n}$ where, every component of Q_S^+ has the following form:

for benefit type attribute, every component of Q_S^+ has the following form:

$$\begin{aligned} q_S^+ &= \langle \delta T_j^+, \delta I_j^+, \delta F_j^+ \rangle \\ &= \left\langle \max_i \{\delta T_{ij}\}, \min_i \{\delta I_{ij}\}, \min_i \{\delta F_{ij}\} \right\rangle \text{ for } j \in P \end{aligned} \quad (29)$$

and for cost type attribute, every component of Q_S^+ has the following form

$$\begin{aligned} q_S^+ &= \langle \delta T_j^+, \delta I_j^+, \delta F_j^+ \rangle \\ &= \left\langle \min_i \{\delta T_{ij}\}, \max_i \{\delta I_{ij}\}, \max_i \{\delta F_{ij}\} \right\rangle \text{ for } j \in L \end{aligned} \quad (30)$$

Step 2. Determine the weighted variational co-efficient similarity measure between ideal alternative and each alternative.

The variational co-efficient similarity measure between ideal alternative Q_S^+ and each alternative A_i for i = 1, 2, …, m can be determined by the following equation as follows:

$Var_{WRNS}(Q_S^+, D_S) =$

$$\left[\begin{array}{l}\lambda \sum_{i=1}^{n} w_i \dfrac{2\left\{\delta T_j^+.\delta T_{ij} + \delta I_j^+.\delta I_{ij} + \delta F_j^+.\delta F_{ij}\right\}}{\left\{\begin{array}{l}\left[(\delta T_j^+)^2+(\delta I_j^+)^2+(\delta F_j^+)^2\right]+\\ \left[(\delta T_{ij})^2+(\delta I_{ij})^2+(\delta F_{ij})^2\right]\end{array}\right\}}\\ +(1-\lambda)\sum_{i=1}^{n} w_i \dfrac{\left\{\delta T_j^+.\delta T_{ij} + \delta I_j^+.\delta I_{ij} + \delta F_j^+.\delta F_{ij}\right\}}{\sqrt{\begin{array}{l}\left[(\delta T_j^+)^2+(\delta I_j^+)^2+(\delta F_j^+)^2\right]\\ \sqrt{(\delta T_{ij})^2+(\delta I_{ij})^2+(\delta F_{ij})^2}\end{array}}}\end{array}\right] \quad (31)$$

Step3. Rank the alternatives.

According to the values obtained from Eq.(31), the ranking order of all the alternatives can be easily determined. Highest value indicates the best alternative.

Step 4. End.

7 Numerical example

In this section, rough neutrosophic MADM regarding investment problem is considered to demonstrate the applicability and the effectiveness of the proposed approach. However, investment problem is not easy to solve. It not only requires oodles of patience and discipline, but also a great deal of research and a sound understanding of the market, mathematical tools, among others. Suppose an investment company wants to invest a sum of money in the best option. Assume that there are four possible alternatives to invest the money: (1) A_1 is a computer company; (2) A_2 is a garment company; (3) A_3 is a telecommunication company; and (4) A_4 is a food company. The investment company must take a decision based on the following three criteria: (1) C_1 is the growth factor; (2) C_2 is the environmental impact; and (3) C_3 is the risk factor. The four possible alternatives are to be evaluated under the attribute by the rough neutrosophic assessments provided by the decision maker. These assessment values are given in the rough neutrosophic decision matrix (see the table 2).

Table2. Rough neutrosophic decision matrix

$$D=\left\langle \underline{N}_{ij}(P),\overline{N}_{ij}(P)\right\rangle_{4\times 3}=$$

$$\begin{array}{c|ccc}
 & C_1 & C_2 & C_3 \\
\hline
A_1 & \left\langle \begin{matrix}(0.1,0.2,0.2),\\(0.3,0.2,0.2)\end{matrix}\right\rangle & \left\langle \begin{matrix}(0.6,0.4,0.3),\\(0.8,0.2,0.3)\end{matrix}\right\rangle & \left\langle \begin{matrix}(0.3,0.2,0.3),\\(0.5,0.2,0.1)\end{matrix}\right\rangle \\
A_2 & \left\langle \begin{matrix}(0.2,0.4,0.3),\\(0.4,0.2,0.3)\end{matrix}\right\rangle & \left\langle \begin{matrix}(0.6,0.3,0.3),\\(0.8,0.1,0.1)\end{matrix}\right\rangle & \left\langle \begin{matrix}(0.1,0.4,0.3),\\(0.3,0.2,0.3)\end{matrix}\right\rangle \\
A_3 & \left\langle \begin{matrix}(0.3,0.2,0.3),\\(0.5,0.2,0.1)\end{matrix}\right\rangle & \left\langle \begin{matrix}(0.5,0.2,0.3),\\(0.7,0.2,0.1)\end{matrix}\right\rangle & \left\langle \begin{matrix}(0.0,0.2,0.4),\\(0.2,0.2,0.2)\end{matrix}\right\rangle \\
A_4 & \left\langle \begin{matrix}(0.0,0.4,0.4),\\(0.2,0.2,0.2)\end{matrix}\right\rangle & \left\langle \begin{matrix}(0.5,0.4,0.4),\\(0.7,0.2,0.2)\end{matrix}\right\rangle & \left\langle \begin{matrix}(0.2,0.3,0.3),\\(0.4,0.1,0.1)\end{matrix}\right\rangle
\end{array} \quad (32)$$

The known weight information is given as follows: $W = [w_1, w_2, w_3]^T = [0.3, 0.3, 0.4]$ and $\sum_{i=1}^{3} w_i = 1$.

Step1. Determine the types of criteria.

First two types i.e. C_1 and C_2 of the given criteria are benefit type criteria and the last one criterion i.e. C_3 is the cost type criteria.

Step2. Determine the relative neutrosophic positive ideal solution

Using Eq. (29), Eq.(30), the relative positive ideal neutrosophic solution for the given matrix defined in Eq.(32) can be obtained as:

$$Q_S^{+}=[(0.4,0.2,0.2),(0.7,0.2,0.2),(0.1,0.3,0.3)]$$

Step3. Determine the weighted variational similarity measure

The weighted variational co-efficient similarity measure is determined by using Eq.(28), Eq.(31) and Eq.(32). The results obtained for different values of have been shown in the Table-3.

Table-3. Results of rough variational similarity measure for different values of , $0 \le \lambda \le 1$

Similarity measure method	Values of s	Measure values	Ranking order
$Var_{WRNS}(Q_S^{+}, D_S)$	0.10	0.8769; 0.9741; 0.9917; 0.8107	$A_3 > A_2 > A_1 > A_4$
	0.25	0.8740, 0.9739 0.9905 0.8078	$A_3 > A_2 > A_1 > A_4$
	0.50	0.8692; 0.9735; 0.9887; 0.8028	$A_3 > A_2 > A_1 > A_4$
	0.75	0.8643; 0.9730; 0.9868; 0.7979	$A_3 > A_2 > A_1 > A_4$
	0.90	0.8614; 0.9728; 0.9857; 0.7949	$A_3 > A_2 > A_1 > A_4$

Step 4. Rank the alternatives.

According to the different values of , the results obtained in Table-3 reflects that A_3 is the best alternative.

8. Comparisons of different rough similarity measure with rough variation similarity measure

In this section, four existing rough similarity measures - namely: rough cosine similarity measure, rough dice similarity measure, rough cotangent similarity measure and rough Jaccard similarity measure - have been compared with proposed rough variational co-efficient similarity measure for different values of λ. The comparison results are listed in the Table 3 and Table 4.

Table-4. Results of existing rough neutrosophic similarity measure methods.

Rough similarity measure methods	Values of s	Measure values	Ranking order
$JAC_{WRNS}(Q_S^+, D_S)$ [21]	...	0.7870, 0.9471; 0.9739; 0.6832	$A_3 > A_2 > A_1 > A_4$
$DIC_{WRNS}(Q_S^+, D_S)$ [21]	...	0.8595; 0.9726; 0.9873; 0.7929	$A_3 > A_2 > A_1 > A_4$
$C_{WRNS}(Q_S^+, D_S)$ [20]	...	0.8788; 0.9738; 0.9920; 0.9132	$A_3 > A_2 > A_4 > A_1$
$COT_{WRNS}(Q_S^+, D_S)$ [19]	...	0.8472; 0.9358; 0.9643; 0.8103	$A_3 > A_2 > A_1 > A_4$

Conclusion

In this paper, we have proposed rough variational coefficient similarity measures. We also proved some of their basic properties. We have presented an application of rough neutrosophic variational coefficient similarity measure for a decision making problem on investment. The concept presented in the paper can be applied to deal with other multi attribute decision making problems in rough neutrosophic environment.

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MCDM Method for n-Wise Criteria Comparisons and Inconsistent Problems

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The purpose of this paper is to present an e[xtension and alternative of the hybrid method based on Saaty's Analytical Hierarchy Process and Technique for Order Preference by Similarity to Ideal Solution method (AHP-TOPSIS), that based on the AHP and its use of pairwise comparisons, to a new method called α -D MCDM-TOPSIS(α -Discounting Method for multicriteria decision making-TOPSIS). The new method overcomes limits of AHP which work only for pairwise comparisons of criteria to any-wise (n-wise) comparisons, with crisp coefficients or with interval-valued coefficients.

α-D MCDM-TOPSIS is verified by some examples to demonstrate how it allows for consistency, inconsistent, weak, inconsistent, and strong inconsistent problems.

Keywords: α-D MCDM-TOPSIS, N-wise criteria comparisons, AHP, TOPSIS, Consistency, Inconsistency.

1. INTRODUCTION

The economic, social and technological problems have been widely resolvedin recent years andmulticriteria decision making methods have played a keyrole [8]. However, the quantity of data, the complexity of the modern world and the recent technological advances have made obviously MCDM methods more challenging than ever, hence the necessity of methods able giving quality solution.

Among the complete, simple and the most often MCDM methods used to improve the reliability of the decision making [10, 11, 15] process is the combined method AHP-TOPSIS [2-4, 8,12-14, 16].

In literature, AHP-TOPSIS is a useful and most applied MCDM method to resolve difficult decision making problems and to select the best one of the alternatives. Its applications are several, [8] developed a support for management and planning of flight mission at NASAbased on AHP-TOPSIS, using AHP-TOPSIS, [14] developed a study how the traffic congestion of urban road are evaluated, [3] established a TOPSIS-AHP solution, tried in the mobile phone industry domains, to choose logistics service provider, [12] summarizing a e-SCM performance with AHP-TOPSIS, for management of supply chain [13] proposed a Topsis-AHP simulation model, [2] developed an AHP and TOPSIS Method to evaluate faculty performance in engineering education, the sharing capacity assessment knowledge of supply chain is evaluated using AHP-TOPSIS method in the [4].

The paper is organized as follows. In the next section, the literature survey for consistency is given. Section 3 and 4 will focus on AHP-TOPSIS and the proposed α -D MCDM-TOPSIS model respectively, in a step by- step fashion. Afterwards, the proposed method is tested on the consistent, weak inconsistent and strong inconsistent examples (section 5). AHP method used one to rank the preferences is considered in section 6. In this section, we discuss developments via the use of an example to compare all methods. Finally, conclusions and perspectives are shown.

2. a-D MCDM METHOD

The general idea of the α-D MCDM is to transforman MCDM inconsistent problem (in that AHP does not work) to an MCDM consistent problem, by discounting each coefficient by the same percentages

Let us assume that $C = \{C_1, C_2, \cdots, C_n\}$, with $n \geq 2$,. are a set of Criteria.

Construct a linear homogeneous system of equations,

Each criterion $C_i$can be expressed as linear homogeneous equation, or as non-linear equation, with crisp coefficients or with interval-valued coefficients of other criteria$C_1, \cdots C_j \cdots C_n$.

$$C_i = f(C_1, \cdots C_j \cdots C_n)$$

Consequently a comparaisons matrix associated tothis linear homogeneous system is constracted.

To determine the weights w_i of the criteria, we solve the previous system.

The α-D MCDM method procedure cited above that introduced by Smarandache is not designed to rank preferences P_i based on C_i criteria, as AHP method do, but to detrmine only weights of criteria in any types of problems (consistent, inconsistent)

AHP as cited above is a complete method designed to calculates the weights of criteria C_i and to rank the preferences P_i .In addition, when the AHP is used with TOPSIS, or other MCDM method, we just benefit from the part of weight calculation criteria and we used TOPSIS to rank preferences or other MCDM methods.

The same, for α-D MCDM, in the first time, is just used to calculate the weight of criteria, that will be used later by TOPSIS to rank preferences and, in the second time, we extended α-D MCDM to a complete method to rank the preferences.

In the first time, we will used α-D MCDM for just calculate weight of criteria C_i and not to rank P_i preferences. In this case, when we will calculate the weights of criteria C_i .

We should have $C_i = f(\{C\} \setminus C_i)$

Then criteria C_i is a linear equation of C_j such $C_i = \sum_{j=1 j \neq i}^{n} x_{ij} C_j$

So the comparisons criteria matrix has the number of criteria by rows and columns (rows number n = number of criteria and columns number also m = number of equations). In the result, we havea square matrix ($n = m$), consequently we can calculate the determinant of this matrix. At this point, we have an $n \times n$ linear homogeneous system and its associated matrix.

$$\begin{cases} x_{1,1}w_1 + x_{1,2}w_2 + \cdots + x_{1,n}w_n = 0 \\ \vdots \\ x_{n,1}w_1 + x_{m,2}w_2 + \cdots + x_{n,n}w_n = 0 \end{cases}$$

$$X = \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix}$$

The difference between AHP and α-D MCDM is the ability of the latter to work with consistent and inconsistent problems, and if the problem is inconsistent α-D MCDM method transform it to a consistent problem while AHP is unable and works only for consistent problem.

In the follwing, relationship beteween determinant of matrix and consistency and parameterization of system by α_i in order to get a consistent problem.

Properties 1:

* If $det(X) = 0$, the system has a solutions (i.e MCDM problem is consistent,).

* If$det(X) \neq 0$, the system has a only the null solution solutions (i.e MCDM problem is inconsistent).

• If problem is inconsistence, then construct parameterized matrix denoted $X(\alpha)$ by parameterize the right-hand in order to get $\det(X(\alpha)) = 0$ and use Fairness principe (set equal parameters to all criteria $\alpha = \alpha_1 = \alpha_2 = \cdots = \alpha_k > 0$). To get priority vector, resove the new system obtained and set 1 to secondary variable and normalize the vector by deviding on sum of all components.

3. AHP-TOPSIS METHOD

In the real word decisions problems (case 1., section 3.) we have a multiples preferences and diverse criteria. The MCDM problem can summarized as follow :

- calculate weights w_i of criteria C_i.
- Rank preferences (alternatives) A_i.

Let us assume there are n criteria and theirs pairwise relative importance x_{ij}.

TOPSIS assumes that we have n alternatives (preferences) $A_i (i = 1, 2, \cdots, m)$ and n attributes/criteria $C_j (j = 1, 2, \cdots, n)$ and comparison matrix a_{ij} of preference i with respect to criterion j.

The AHP-TOPSIS method is described in the following steps :

Step 2.1. Construct decision matrix denoted by $A = (a_{ij})_{m\times n}$

Table 1: Decision matrix

	C_1	C_2	$\cdots$	C_n
	w_1	w_2	$\cdots$	w_n
A_1	a_{11}	a_{12}	$\cdots$	a_{1n}
A_2	a_{21}	a_{22}	$\cdots$	a_{2n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A_m	a_{m1}	a_{m2}	$\cdots$	a_{mn}

Step 2.2. Determine weights (w_i) of each criterion using AHP Method

Where $\sum_{j=1}^{n} w_j = 1,\ j = 1, 2, \cdots, n$

Step 2.2.1. Build a pairwise comparison matrix of criteria

The pairwise comparison of criterion i with respect criterion j gives a square matrix $(X)_{n\times n} = (x_{ij})$ where x_{ij} represents the relative importance of criterion i over the criterion j. In the matrix, $x_{ij} = 1$ when $i = j$ and $x_{ij} = 1/x_{ji}$. So we get an $n \times n$ pair-wise comparison matrix$(X)_{n\times n}$.

Step 2.2.2. Find the relative normalized weight (w_j) of each criterion defined by follow formula.

$$w_j = \frac{\prod_{j=1}^{n} (x_{ij})^{1/n}}{\sum \prod_{j=1}^{n} (x_{ij})^{1/n}}$$

Then get w_i weight of thei^{th} criterion.

Step 2.2.3. Calculate matrix X_3 and X_4 such that $X_3 = X_1 \times X_2$ and where $X_4 = X_3/X_2$ $X2 = [w_1, w_2, \cdots, w_j]^T$

Step 2.2.4. Findthe largest eigen value of pairwise comparison matrix.

For simplfied the calcul, the largest eigen value of pairwise comparison matrix is the average of X_4.

Furthermore, according to the Perron-Frobenius theorem, principal eigenvalue λ_{max} always exists for the Saaty's matrix and it holds $\lambda_{max} \geq n$; for fully consistent matrix $\lambda_{max} = n$.

Consistency check is then performed to ensure that the evaluation of the pair-wise comparison matrix is reasonable and acceptable.

Step 2.2.5. Determine the consistency ratio CR

After calculation consistency ratio (RC) using equation (eq.), and in order to verify the consistency of the matrix that is considered to be consistent if CR is less than threshold and not otherwise,according to Saaty and search.

At this point, we have the weights of criteria and if the consistency is cheked , we will using TOPSIS to rank preferences.

Step 2.3. Normalize decision matrix

The normalize decision matrix is obtained, which is given here with r_{ij}

$$r_{ij} = a_{ij} / \left(\sum_{i=1}^{m} a_{ij}^2 \right)^{0.5} ; j = 1, 2, \cdots, n; i = 1, 2 \cdots, m$$

Step 2.4. Calculate the weighted decision matrix

Weighting each column of obtained matrix by its associated weight.

$$v_{ij} = w_j r_{ij}; j = 1, 2, \cdots, n; i = 1, 2 \cdots, m$$

Step 2.5. Determine the positive ideal solution (PIS) and negative ideal solution (NIS)

$$A^+ = (v_1^+, v_2^+, \cdots, v_n^+) = \left\{ \left(max_i \left\{ v_{ij} \mid j \in B \right\} \right), \left(min_i \left\{ v_{ij} \mid j \in C \right\} \right) \right\}$$

$$A^- = (v_1^-, v_2^-, \cdots, v_n^-) = \left\{ \left(min_i \left\{ v_{ij} \mid j \in B \right\} \right), \left(max_i \left\{ v_{ij} \mid j \in C \right\} \right) \right\}$$

The benefit and cost solutions are represent B and C respectively

Step 2.6. Calculate thedistance measure for each alternative from the PIS and NIS

Thedistance measure for each alternative from the PIS is

$$S_i^+ = \left\{ \sum_{j=1}^{n} (v_{ij} - v_j^+)^2 \right\}^{0.5} ; i = 1, 2 \cdots, m$$

Also, the distance measure for each alternative from the NIS is

$$S_i^- = \left\{ \sum_{j=1}^{n} (v_{ij} - v_j^-)^2 \right\}^{0.5} ; i = 1, 2 \cdots, m$$

Step 2.7. Determine the values of relative closeness mesure

For each alternative we calculate the relative closeness mesure as follow :

$$T_i = \frac{S_i^-}{(S_i^+ + S_i^-)}; i = 1, 2 \cdots, m$$

Ranking alternatives set according to the ordre of relative closeness mesure values T_i.

4. α-D MCDM-TOPSIS METHOD

The MCDM problem description is same of problem used in AHP-TOPSIS method (section 4.), but in this case we have n-wise comparisons matrix of criteria.

Let us assume that $C = \{C_1, C_2, \cdots, C_n\}$, with $n \geq 2$, and $\{A_1, A_2, \cdots, A_m\}$, with $m \geq 1$, are a set of criteria and the set of preferences, respectively.

Let us assume each criteria C_i is a linear homogeneous equation of the other criteria $C_1, C_2, \cdots, C_n$: $C_i = f(\{C\} \setminus C_i)$

The -D MCDM-TOPSIS method is described in the following steps :

Step 3.1. Construct decision matrix denoted by $A=(a_{ij})_{m\times n}$

Table 2: Decision matrix

α

	C_1	C_2	$\cdots$	C_n
	w_1	w_2	$\cdots$	w_n
A_1	a_{11}	a_{12}	$\cdots$	a_{1n}
A_2	a_{21}	a_{22}	$\cdots$	a_{2n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A_m	a_{m1}	a_{m2}	$\cdots$	a_{mn}

Step 3.2. Determine weights (w_i) of each criterion using α-D MCDM Method

Step 3.2.1. Using α**-D MCDM** to determine the importance weight (w_i) of the criteria

Where $\sum_{j=1}^{n} w_j = 1,\ j=1,2,\cdots,n$

Step 3.2.2. Build asystem of equations and its associated matrix

To construct linear system of equations, each criteria C_i be expressed as a linear equation of C_j such as. $C_i = \sum_{j=1\, j\neq i}^{n} x_{ij} C_j$

Consequently, we have a system of n linear equations (one equation of each criterion) with n variables (variablew_i is weight of criterion).

$$\begin{cases} x_{1,1}w_1 + x_{1,2}w_2 + \cdots + x_{1,n}w_n = 0 \\ \vdots \\ x_{n,1}w_1 + x_{m,2}w_2 + \cdots + x_{n,n}w_n = 0 \end{cases}$$

In mathematic, each linear system can associeted to a matrix, in this case,denoted by $X=(x_{ij})$, $1\le i\le n$ and $1\le j\le n$ where

$$X = \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix}$$

Step 3.2.3. Solve system of equation using whose associeted matrix

Solve the system of equation, the differentes cases are discussed in proporites 1 in that we compute the determinant of X (find strictly positive solution $w_i > 0$).

Solving this homogeneous linear system, in differents cases above the general solution that we set as a solution vector $S=[s_1,s_2,\cdots,s_n]$, and set 1 to secondary variable we get $W=[w_1,w_2,\cdots,w_n]$

Deviding each vector element on sum of all components of vector to get normalized vector where

$$w_j = \frac{s_j}{\sum_{k=1}^{n} s_k};\ i=1,2\cdots,n$$

Step 3.2.4. Build a pairwise comparison matrix of criteria

The pairwise comparison of criterion i with respect criterion j gives a square matrix $(X)_{n\times n}=(x_{ij})$ where x_{ij} represents the relative importance of criterion i over the criterion j. In the matrix, $x_{ij}=1$ when

$i = j$ and $x_{ij} = 1/x_{ji}$. So we get an $n \times n$ pair-wise comparison matrix$(X)_{n\times n}$.

Step 3.2.5. Find the relative normalized weight (w_j) of each criterion defined by follow formula.

$$w_j = \frac{\prod_{j=1}^{n}(x_{ij})^{1/n}}{\sum \prod_{j=1}^{n}(x_{ij})^{1/n}}$$

Then get w_i weight of thei^{th} criterion.

Step 2.2.6. Calculate matrix X_3 and X_4 such that $X_3 = X_1 \times X_2$ and where $X_4 = X_3/X_2$

$$X2 = [w_1, w_2, \cdots, w_j]^T$$

Step 3.2.7. Findthe largest eigen value of pairwise comparison matrix.

For simplfied the calcul, the largest eigen value of pairwise comparison matrix is the average of X_4. Furthermore, according to the Perron-Frobenius theorem, principal eigenvalue λ_{max} always exists for the Saaty's matrix and it holds $\lambda_{max} \geq n$; for fully consistent matrix $\lambda_{max} = n$.

Consistency check is then performed to ensure that the evaluation of the pair-wise comparison matrix is reasonable and acceptable.

Step 3.2.8. Determine the consistency ratio (CR).

After calculation consistency ratio (RC) using equation (eq.), and in order to verify the consistency of the matrix that is considered to be consistent if CR is less than threshold and not otherwise,according to Saaty and search.

At this point, we have the weights of criteria and if the consistency is cheked , we will using TOPSIS to rank preferences.

Step 3.3. Normalize decision matrix

The normalize decision matrix is obtained, which is given here with r_{ij}

$$r_{ij} = a_{ij} / \left(\sum_{i=1}^{m} a_{ij}^2\right)^{0.5} ; j = 1, 2, \cdots, n; i = 1, 2 \cdots, m$$

Step 3.4. Calculate the weighted decision matrix

Weighting each column of obtained matrix by its associated weight.

$$v_{ij} = w_j r_{ij}; j = 1, 2, \cdots, n; i = 1, 2 \cdots, m$$

Step 3.5. Determine the positive ideal solution (PIS) and negative ideal solution (NIS)

$$A^+ = (v_1^+, v_2^+, \cdots, v_n^+) = \left\{\left(max_i \left\{v_{ij} \mid j \in B\right\}\right), \left(min_i \left\{v_{ij} \mid j \in C\right\}\right)\right\}$$

$A^- = (v_1^-, v_2^-, \cdots, v_n^-) = \left\{\left(min_i \left\{v_{ij} \mid j \in B\right\}\right), \left(max_i \left\{v_{ij} \mid j \in C\right\}\right)\right\}$ The benefit and cost solutions are represent B and C respectively

Step 3.6. Calculate thedistance measure for each alternative from the PIS and NIS

The distance measure for each alternative from the PIS is $S_i^+ = \left\{\sum_{j=1}^{n}(v_{ij} - v_j^+)^2\right\}^{0.5} ; i = 1, 2 \cdots, m$

Also, the distance measure for each alternative from the NIS is $S_i^- = \left\{\sum_{j=1}^{n}(v_{ij} - v_j^-)^2\right\}^{0.5} ; i = 1, 2 \cdots, m$

Step 3.7. Determine the values of relative closeness mesure

For each alternative we calculate the relative closeness mesure as follow :

$$T_i = \frac{S_i^-}{(S_i^+ + S_i^-)}; i = 1, 2 \cdots, m$$

Ranking alternatives set according to the ordre of relative closeness mesure values T_i.

5. NUMERICAL EXAMPLES

We examined a numerical example in which a synthetic evaluation desire to rank four alternatives A_1, A_2, A_3 and A_4 with respect to three benefit attribute C_1, C_2 and C_3.

Table 3: Decision matrix

	C_1	C_2	C_3
	w_1	w_2	w_3
A_1	7	9	9
A_2	8	7	8
A_3	9	6	8
A_4	6	7	8

In the examples below we used -D MCDM and AHP (if it works) to calculate the weights of the criteria w_1, w_2 and w_3. After we used TOPSIS to rank the four alternatives, the decision matrix (Table 6) below is used for the three following examples.

a. Consistent example 1

We use the -D MCDM

Let the Set of Criteria be $\{C_1, C_2, C_3\}$ with $w_1 = w(C_1) = x$ and $w_3 = w(C_3) = z$.

Let consider the system of equations associated to MCDM problem and its associated matrix.

$$\alpha \quad \begin{cases} \alpha \\ x = 4y \\ y = 3z \\ z = \dfrac{x}{12} \end{cases} \qquad X1 = \begin{pmatrix} 1 & 4 & 0 \\ 0 & 1 & 3 \\ \dfrac{1}{12} & 0 & 1 \end{pmatrix}$$

We calcutalte $\det(X)$ (in this cas equal= 0) then MCDM problem is consisten we solve the system we get follow solution $S = \begin{bmatrix} 12z & 3z & z \end{bmatrix}$

Seting 1 to secondary varaible the general solution becomes $S = \begin{bmatrix} 12 & 3 & 1 \end{bmatrix}$

and normalizing vector is (deviding by sum=12+3+1) the weights vector is $W = \begin{bmatrix} \dfrac{12}{16} & \dfrac{3}{16} & \dfrac{1}{16} \end{bmatrix}$

Using AHP, we get the same result

The pairwise comparison matrix of criteria is : $X1 = \begin{pmatrix} 1 & 4 & 12 \\ \dfrac{1}{4} & 1 & 3 \\ \dfrac{1}{12} & \dfrac{1}{3} & 1 \end{pmatrix}$

whose maximum eigenvalue is $\lambda_{max} = 3$ and its corresponding normalized eigenvector (Perron-Frobenius vector) is $W = \begin{bmatrix} \dfrac{12}{16} & \dfrac{3}{16} & \dfrac{1}{16} \end{bmatrix}$

We use TOPSIS to rank the four alternatives

Table 4: Calculate (a_{ij}^2) for each column and divide each column by $(\sum_{i=1}^{n} a_{ij}^2)^{1/2}$ to get r_{ij}

a_{ij}^2	C_1	C_2	C_3
w_i	*12/16*	*3/16*	*1/16*
A_1	*49*	*81*	*81*
A_2	*64*	*49*	*64*
A_3	*81*	*36*	*64*
A_4	*36*	*49*	*64*
$\sum_{i=1}^{n} a_{ij}^2$	*230*	*215*	*273*

Table 5: Multiply each column by w_j to get v_{ij}

v_{ij}	C_1	C_2	C_3
w_i	*12/16*	*3/16*	*1/16*
A_1	*0.3462*	*0.1151*	*0.0340*
A_2	*0.3956*	*0.0895*	*0.0303*
A_3	*0.4451*	*0.0767*	*0.0303*
A_4	*0.2967*	*0.0895*	*0.0303*
v_{max}	*0.4451*	*0.1151*	*0.0340*
v_{min}	*0.2967*	*0.0767*	*0.0303*

Table 6: The separation mesure values and the final rankings for decision matrix (Table 4) using AHP-TOPSIS and -D MCDM-TOPSIS

	S_i^+	S_i^-	T_i	*Rank*
A_1	*0.0989*	*0.0627*	*0.3880*	*3*
A_2	*0.0558*	*0.0997*	*0.6412*	*2*
A_3	*0.0385*	*0.1484*	*0.7938*	*1*
A_4	*0.1506*	*0.0128*	*0.0783*	*4*

α

Table 10 presents the rank of alternatives (A_1, A_2, A_3, A_4) and separation measure values of each alternative from the PIS and from NIS in wish the weighted values are calculated by AHP or -D MCDM. The both methods AHP and -D MCDM with fairnessp rinciple give the same weights as proven above methods together give same result in consistent problem.

b. Weak inconsistent example 2 where AHP does not work

Consider another example investigated by [7] for which AHP does not work (i.e AHP-TOPSIS doses not work too), **then we use the -D MCDM to calculate the weights values and ranking the four alternatives by TOPSIS (see Table 14)**.

Let the Set of Criteria be $\{C_1, C_2, C_3\}$ with $w_1 = w(C_1) = x$ and $w_3 = w(C_3) = z$.

Let consider the system of equations associated to MCDM problem and its associated matrix.

α α

α

$$\begin{cases} x = & 2y + 3z \\ y = & \frac{x}{2} \\ z = & \frac{x}{3} \end{cases} \qquad X1 = \begin{pmatrix} 1 & -2 & -3 \\ \frac{-1}{2} & 1 & 0 \\ \frac{-1}{3} & 0 & 1 \end{pmatrix}$$

The solution of this system is $x = y = z = 0$ sine the sum of weights should be = 1, then this solution is not acceptable.

Parameterizing the right-hand side coefficient of each equations by α_i we get

$$x = 2\alpha_1 y + 3\alpha_2 z$$

$$y = \frac{\alpha_3 x}{2}$$

$$z = \frac{\alpha_4 x}{3}$$

we solve the system we get follow solution $S = \begin{cases} y = \frac{\alpha_3 x}{2} \\ z = \frac{\alpha_4 x}{3} \end{cases}$ or $S = \begin{bmatrix} x & \frac{\alpha_3 x}{2} & \frac{\alpha_4 x}{3} \end{bmatrix}$

Seting 1 to secondary varaible the general solution becomes $S = \begin{bmatrix} 1 & \frac{\alpha_3}{2} & \frac{\alpha_4}{3} \end{bmatrix}$

Applying Fairness Principle: then replace $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha$ in whence $\alpha = \frac{\sqrt{2}}{2}$

$$S = \begin{bmatrix} 1 & \frac{\sqrt{2}}{4} & \frac{\sqrt{2}}{6} \end{bmatrix}$$

and normalizing vector is (deviding by sum) the weights vector is $W = \begin{bmatrix} 0.62923 & 0.22246 & 0.14831 \end{bmatrix}$

TOPSIS is used to rank the four alternative: application of TOPSIS method is in the same manner as in the previous example (the four alternatives (Ai)are ranked in following table 10)

Table 7: Calculate (a_{ij}^2) for each column

a_{ij}^2	C_1	C_2	C_3
	0.62923	*0.22246*	*0.14831*
A_1	*49*	*81*	*81*
A_2	*64*	*49*	*64*
A_3	*81*	*36*	*64*
A_4	*36*	*49*	*64*
$\sum_{i=1}^{n} a_{ij}^2$	*230*	*215*	*273*

Table 8: Divide each column by $(\sum_{i=1}^{n} a_{ij}^2)^{1/2}$ to get r_{ij}

r_{ij}	C_1	C_2	C_3
	0.62923	0.22246	0.14831
A_1	0.4616	0.6138	0.5447
A_2	0.5275	0.4774	0.4842
A_3	0.5934	0.4092	0.4842
A_4	0.3956	0.4774	0.4842
$\sum_{i=1}^{n} a_{ij}^2$	230	215	273

Table 9: Multiply each column by w_j to get v_{ij}

v_{ij}	C_1	C_2	C_3
	0.62923	0.22246	0.14831
A_1	0.2904	0.1365	0.0808
A_2	0.3319	0.1062	0.0718
A_3	0.3734	0.0910	0.0718
A_4	0.2489	0.1062	0.0718
v_{max}	0.3734	0.1365	0.0808
v_{min}	0.2489	0.0910	0.0718

Table 10: The separation mesure values and the final rankings for decision matrix (Table 4) using -D MCDM-TOPSIS

	S_i^+	S_i^-	T_i	Rank
A_1	0.0830	0.0622	0.4286	3
A_2	0.0522	0.0844	0.6178	2
A_3	0.0464	0.1245	0.7285	1
A_4	0.1284	0.0152	0.1057	4

α

c. Jean Dezert's strong inconsistent example 3

Smarandache [7] introduced a Jean Dezert's Strong Inconsistent example, Let consider the system of equations associated to MCDM problem and its associated matrix.

$$X = \begin{pmatrix} 1 & 9 & \frac{1}{9} \\ \frac{1}{9} & 1 & 9 \\ 9 & \frac{1}{9} & 1 \end{pmatrix} \qquad \begin{cases} x = 9y, x > y \\ x = \frac{1}{9}z, x < z \\ y = 9z, y > z \end{cases}$$

We followthe same process asthe example above we get the general solution

$$W = \begin{bmatrix} \frac{1}{6643} & \frac{81}{6643} & \frac{6561}{6643} \end{bmatrix}$$

We use TOPSIS to rank the four alternatives

Table 11: Calculate (a_{ij}^2) for each column

a_{ij}^2	C_1	C_2	C_3
	0.0002	0.0122	0.9877
A_1	49	81	81
A_2	64	49	64
A_3	81	36	64
A_4	36	49	64
$\sum_{i=1}^{n} a_{ij}^2$	230	215	273

Table 12: Divide each column by $(\sum_{i=1}^{n} a_{ij}^2)^{1/2}$ to get r_{ij}

r_{ij}	C_1	C_2	C_3
	0.0002	0.0122	0.9877
A_1	0.503	0.699	0.623
A_2	0.574	0.543	0.553
A_3	0.646	0.466	0.553
A_4	0.431	0.543	0.553
$\sum_{i=1}^{n} a_{ij}^2$	230	215	273

Table 13: Multiply each column by w_j to get v_{ij}

v_{ij}	C_1	C_2	C_3
	0.0002	0.0122	0.9877
A_1	0.0001	0.0075	0.5380
A_2	0.0001	0.0058	0.4782
A_3	0.0001	0.0050	0.4782
A_4	0.0001	0.0058	0.4782
v_{max}	0.0001	0.0075	0.5380
v_{min}	0.0001	0.0050	0.4782

Table 14: The separation mesure values and the final rankings for decision matrix (Table 4) using -D MCDM-TOPSIS

	S_i^+	S_i^-	T_i	Rank
A_1	0.0000	0.0598	0.99966	1
A_1	0.0598	0.0008	.01372	2
A_1	0.0598	0.0000	0.00049	4
A_1	0.0598	0.0008	.013715	3

α

Table 15: Summary of the results of three examples of all methods

Example	Alternative	AHP-TOPSIS		α-DMCDM-TOPSIS	
Consistent example 1	A_1	*0.3880*	*3*	*0.3880*	*3*
	A_2	*0.6412*	*2*	*0.6412*	*2*
	A_3	*0.7938*	*1*	*0.7938*	*1*
	A_4	*0.0783*	*4*	*0.0783*	*4*
Weak Inconsistent Example 2	A_1	*Does not works*		*0.3880*	*3*
	A_2			*0.6412*	*2*
	A_3			*0.7938*	*1*
	A_4			*0.0783*	*4*
Strong Inconsistent Example 3	A_1	*Does not works*		*0.999668*	*1*
	A_2			*0.013719*	*2*
	A_3			*0.000497*	*4*
	A_4			*0.013715*	*3*

For the three examples presented in this paper, the table 25 (that summarized all results of all methods) illustrate that the AHP and AHP-TOPSIS methods works just for the first example in wish the criteria and alternatives are consistent in their pairwise comparisons. Ours proposed methods α-D MCDM-TOPSIS works not only for consistent example 1, in that gives the same results as AHP and AHP-TOPSIS methods, but for weak inconsistent and strong inconsistent examples.

The results have claimed that AHP-TOPSIS and our α-D MCDM-TOPSIS methods preserves the ranking order of the alternatives and overcome the nearness scored values problem. By using AHP-TOPSIS and our alpha-D MCDM-TOPSIS methods, the scored value of the A3 was changed from 0.2789 to 0.7938, the scored value of the A2 was changed from 0.2604 to 0.6412, and the scored value of the A4 was changed from 0.2104 to 0.0783. The bigger differences between the score values of alternatives 0.7155 ((A3) 0.7938-(A4) 0.0783) is also subject to gain additional insight into this phenomenon.

In the two last examples (weak inconsistent and strong inconsistent), one sees that the importance discounting ours approaches (what we call the α-D MCDM-TOPSIS approachs) will suggest, that can uses to solve real-life problems in wish criteria are not only pairwise but n-wise comparisons and the problems are not perfectly consistent. It is however worth to note that the ranking order of the four alternatives obtained by both methods is similar but scored values of are slightly different, in wish the same remarks mentioned above are available between extended α-D MCDM and α-D MCDM-TOPSIS. α-D MCDM-TOPSIS method allows to take into account also any nembers of alternatives and any weights of criteria.

6. CONCLUSION

We have proposed tow multicritere decision making methods, Extended- α-D MCDM and α-D MCDM-TOPSIS models that allows to works for consistent and inconsistent MCDM problem In addition, three examples have demonstrated the α-D MCDM-TOPSIS model is efficient and robust.

Ours approch α-D MCDM-TOPSIS give the same result as AHP-TOPSIS and AHP in consistent MCDM problems and elements of decision matrix are pairwise comparisons, but for weak inconsistent and strong inconsistent MCDM problems in which AHP and AHP-TOPSIS are limited and unable, ours proposed methods Extended- α-D MCDM and α-D MCDM -TOPSIS give a justifiable results.

Furthermore ours proposed approaches α-D MCDM-TOPSIS can uses to solve real-life problems in wish criteria are not only pairwise but n-wise comparisons and the problems are not perfectly consistent.

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Similarity Measure of Refined Single-Valued Neutrosophic Sets and Its Multicriteria Decision Making Method

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Abstract. This paper introduces a refined single-valued neutrosophic set (RSVNS) and presents a similarity measure of RSVNSs. Then a multicriteria decision-making method with RSVNS information is developed based on the similarity measure of RSVNSs. By the similarity measure between each alternative and the ideal solution (ideal alternative), all the alternatives can be ranked and the best one can be selected as well. Finally, an actual example on the selecting problems of construction projects demonstrates the application and effectiveness of the proposed method.

Keywords: Refined single-valued neutrosophic set, similarity measure, decision making.

1 Introduction

To deal with indeterminate and inconsistent information, Smarandache [1] proposed a neutrosophic set, which is composed of the neutrosophic components of truth, indeterminacy, and falsity denoted by T, I, F. Then, Wang et al. [2] constrained the neutrosophic set to a single-valued neutrosophic set (SVNS) as a subclass of the neutrosophic set for convenient actual applications. Further, Smarandache [3] extended the classical neutrosophic logic to n-valued refined neutrosophic logic, in which neutrosophic components T, I, F are refined (splitted) into $T_1, T_2, ..., T_p$ and $I_1, I_2, ..., I_r$, and $F_1, F_2, ..., F_t$, respectively, and constructed as a n-valued refined neutrosophic set. In existing literature [4-7], neutrosophic refined sets were studied and applied to medical diagnosis and decision making. However, the existing neutrosophic refined set is also a single-valued neutrosophic multiset [6] in the concept. In this paper, we present a refined single-valued neutrosophic set (RSVNS), then its concept is different from the concept of single-valued neutrosophic multisets (neutrosophic refined sets) [4-7]. In fact, RSVNSs are scarcely studied and applied in science and engineering fields. Therefore, it is necessary to propose a similarity measure between RSVNSs and its decision making method in this paper.

The rest of the paper is constructed as follows. Section 2 reviews basic concepts of a SVNS and a neutrosophic refined set (single-valued neutrosophic multiset). Section 3 introduces a RSVNS and a similarity measure of RSVNSs. Section 4 presents a multicriteria decision-making method based on the similarity method under a RSVNS environment. In section 5, an actual example is provided for the decision-making problem of selecting construction projects to illustrate the application of the proposed method. Section 6 contains conclusions and future research.

2 Preliminaries

Definition 1 [2]. Let U be a universe of discourse, then a SVNS A in U is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$, such that $T_A(x), I_A(x), F_A(x) \in [0, 1]$ and $0 \le T_A(x) + T_A(x) + T_A(x) \le 3$. Thus, a SVNS A can be expressed as $A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle \mid x \in U\}$.

Let $U = \{x_1, x_2, \ldots, x_n\}$ be the universe of discourse, and A and B be two (non-refined) single-valued neutrosophic sets, $A = \{\langle x_i, T_A(x_i), I_A(x_i), F_A(x_i)\rangle \mid x_i \in U\}$ and $B = \{\langle x_i, T_B(x_i), I_B(x_i), F_B(x_i)\rangle \mid x_i \in U\}$. Majumdar and Samanta's similarity method of two (non-refined) single-valued neutrosophic sets A and B is:

$$M_{MS}(A,B) = \frac{\sum_{i=1}^{n}\left\{\begin{array}{l}\min(T_{iA}(x_i), T_{iB}(x_i)) \\ +\min(I_{iA}(x_i), I_{iB}(x_i)) \\ +\min(F_{iA}(x_i), F_{iB}(x_i))\end{array}\right\}}{\sum_{i=1}^{n}\left\{\begin{array}{l}\max(T_{iA}(x_i), T_{iB}(x_i)) \\ +\max(I_{iA}(x_i), I_{iB}(x_i)) \\ +\max(F_{iA}(x_i), F_{iB}(x_i))\end{array}\right\}}. \quad (1)$$

Based on n-valued refined neutrosophic sets [3], Ye and Ye [6] introduced a single-valued neutrosophic multisets (also called a single-valued neutrosophic refined set (SVNRS) [4, 5, 7]) and defined it below.

Definition 2. Let U be a universe of discourse, then a SVNRS R in U can be defined as follows:

$$R = \left\{ \left\langle \begin{matrix} x, (T_{1R}(x), T_{2R}(x), ..., T_{pR}(x)), (I_{1R}(x), I_{2R}(x), \\ ..., I_{pR}(x)), (F_{1R}(x), F_{2R}(x), ..., F_{pR}(x)) \end{matrix} \right\rangle | x \in U \right\},$$

where p is a positive integer, $T_{1R}(x), T_{2R}(x), ..., T_{pR}(x) : U \to [0,1]$, $I_{1R}(x), I_{2R}(x), ..., I_{pR}(x) : U \to [0,1]$, and $F_{1R}(x), F_{2R}(x), ..., F_{pR}(x) : U \to [0,1]$, and there are $0 \le T_{jR}(x) + I_{jR}(x) + F_{jR}(x) \le 3$ for $j = 1, 2, …, p$.

Definition 3. Let two SVNRS R and S in U be:

$$R = \left\{ \left\langle \begin{matrix} x, (T_{1R}(x), T_{2R}(x), ..., T_{pR}(x)), (I_{1R}(x), I_{2R}(x), \\ ..., I_{pR}(x)), (F_{1R}(x), F_{2R}(x), ..., F_{pR}(x)) \end{matrix} \right\rangle | x \in U \right\},$$

$$S = \left\{ \left\langle \begin{matrix} x, (T_{1S}(x), T_{2S}(x), ..., T_{pS}(x)), (I_{1S}(x), I_{2S}(x), \\ ..., I_{pS}(x)), (F_{1S}(x), F_{2S}(x), ..., F_{pS}(x)) \end{matrix} \right\rangle | x \in U \right\}.$$

Then there are the following relations of R and S:

(1) Containment:

$R \subseteq S$, if and only if $T_{jR}(x) \le T_{jS}(x)$, $I_{jR}(x) \ge I_{jS}(x)$, $F_{jR}(x) \ge F_{jS}(x)$ for $j = 1, 2, …, p$;

(2) Equality:

$R = S$, if and only if $T_{jR}(x) = T_{jS}(x)$, $I_{jR}(x) = I_{jS}(x)$, $F_{jR}(x) = F_{jS}(x)$ for $j = 1, 2, …, p$;

(3) Union:

$$R \cup S = \left\{ \left\langle \begin{matrix} x, (T_{1R}(x) \vee T_{1S}(x), T_{2R}(x) \vee T_{2S}(x), ..., T_{pR}(x) \vee T_{pS}(x)), \\ (I_{1R}(x) \wedge I_{1S}(x), I_{2R}(x) \wedge I_{2S}(x), ..., I_{pR}(x) \wedge I_{pS}(x)), \\ (F_{1R}(x) \wedge F_{1S}(x), F_{2R}(x) \wedge F_{2S}(x), ..., F_{pR}(x) \wedge F_{pS}(x)) \end{matrix} \right\rangle | x \in U \right\};$$

(4) Intersection:

$$R \cap S = \left\{ \left\langle \begin{matrix} x, (T_{1R}(x) \wedge T_{1S}(x), T_{2R}(x) \wedge T_{2S}(x), \\ ..., T_{pR}(x) \wedge T_{pS}(x)), (I_{1R}(x) \vee I_{1S}(x), \\ I_{2R}(x) \vee I_{2S}(x), ..., I_{pR}(x) \vee I_{pS}(x)), \\ (F_{1R}(x) \vee F_{1S}(x), F_{2R}(x) \vee F_{2S}(x), \\ ..., F_{pR}(x) \vee F_{pS}(x)) \end{matrix} \right\rangle | x \in U \right\}.$$

3 Similarity Methods of RSVNSs

In this section, we introduce a RSVNS and propose a similarity method between RSVNSs based on the extension of Majumdar and Samanta's similarity method of two (non-refined) single-valued neutrosophic sets [8].

Definition 4. Let R and S in the universe of discourse $U = \{x_1, x_2, …, x_n\}$ be two refined single-valued neutrosophic sets, which are defined as

$$R = \left\{ \left\langle \begin{matrix} x_i, (T_{1R}(x_i), T_{2R}(x_i), ..., T_{p_iR}(x_i)), \\ (I_{1R}(x_i), I_{2R}(x_i), ..., I_{p_iR}(x_i)), \\ (F_{1R}(x_i), F_{2R}(x_i), ..., F_{p_iR}(x_i)) \end{matrix} \right\rangle | x_i \in U \right\},$$

$$S = \left\{ \left\langle \begin{matrix} x_i, (T_{1S}(x_i), T_{2S}(x_i), ..., T_{p_iS}(x_i)), \\ (I_{1S}(x_i), I_{2S}(x_i), ..., I_{p_iS}(x_i)), \\ (F_{1S}(x_i), F_{2S}(x_i), ..., F_{p_iS}(x_i)) \end{matrix} \right\rangle | x_i \in U \right\},$$

where p_i is a positive integer, and all $T_{jR}(x_i)$, $I_{jR}(x_i)$, $F_{jR}(x_i)$ and $T_{jS}(x_i)$, $I_{jS}(x_i)$, $F_{jS}(x_i)$ ($i = 1, 2, …, n$; $j = 1, 2, …, p_i$) belong to [0, 1].

As an extension of Majumdar and Samanta's similarity method of SVNSs [8], we present a similarity method between two RSVNSs R and S as follows:

$$M(R,S) = \frac{\sum_{i=1}^{n} \sum_{j=1}^{p_i} \left\{ \begin{matrix} \min(T_{jR}(x_i), T_{jS}(x_i)) \\ + \min(I_{jR}(x_i), I_{jS}(x_i)) \\ + \min(F_{jR}(x_i), F_{jS}(x_i)) \end{matrix} \right\}}{\sum_{i=1}^{n} \sum_{j=1}^{p_i} \left\{ \begin{matrix} \max(T_{jR}(x_i), T_{jS}(x_i)) \\ + \max(I_{jR}(x_i), I_{jS}(x_i)) \\ + \max(F_{jR}(x_i), F_{jS}(x_i)) \end{matrix} \right\}}. \quad (2)$$

Obviously, the above similarity measure $M(R, S)$ satisfies the following properties:

(1) $0 \le M(R, S) \le 1$;

(2) $M(R, S) = M(S, R)$

(3) $M(R, S) = 1$ if and only if $R = S$.

In general, we usually consider the weights of criteria. Assume that the weight of each criterion x_i is w_i ($i = 1, 2, …, n$), with $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. Then, we can introduce the weighted similarity formula:

$$M_w(R,S)=\frac{\sum_{i=1}^{n} w_i \sum_{j=1}^{p_i}\left\{\begin{array}{l}\min(T_{jR}(x_i),T_{jS}(x_i))\\+\min(I_{jR}(x_i),I_{jS}(x_i))\\+\min(F_{jR}(x_i),F_{jS}(x_i))\end{array}\right\}}{\sum_{i=1}^{n} w_i \sum_{j=1}^{p_i}\left\{\begin{array}{l}\max(T_{jR}(x_i),T_{jS}(x_i))\\+\max(I_{jR}(x_i),I_{jS}(x_i))\\+\max(F_{jR}(x_i),F_{jS}(x_i))\end{array}\right\}}. \quad (3)$$

4 Decision-making method using the similarity measure

In a decision making problem, there is a set of alternatives $R = \{R_1, R_2, \ldots, R_m\}$, which needs to satisfies a set of criteria $C = \{C_1, C_2, \ldots, C_n\}$, where C_i ($i = 1, 2, \ldots, n$) may be splitted into some sub-criteria C_{ij} ($i = 1, 2, \ldots, n$; $j = 1, 2, \ldots, p_i$). If the decision maker provides the suitability evaluation values of the criteria for C_i (i =1,2,..., n) on the alternative R_k (k = 1, 2,…, m) by using a RSVNS:

$$R_k=\left\{\left\langle\begin{array}{l}C_i,(T_{1R_k}(C_i),T_{2R_k}(C_i),\ldots,T_{p_iR_k}(C_i)),\\(I_{1R_k}(C_i),I_{2R_k}(C_i),\ldots,I_{p_iR_k}(C_i)),\\(F_{1R_k}(C_i),F_{2R_k}(C_i),\ldots,F_{p_iR_k}(C_i))\end{array}\right\rangle \mid C_i\in C\right\}.$$

Then for convenience, each basic element in the RSVNS R_k is represented by the refined single-valued neutrosophic number (RSVNN):

$$\left\langle(T_{1R_k},T_{2R_k},\ldots,T_{p_iR_k}),(I_{1R_k},I_{2R_k},\ldots,I_{p_iR_k}),(F_{1R_k},F_{2R_k},\ldots,F_{p_iR_k})\right\rangle$$

for $i = 1, 2, \ldots, n$; $k = 1, 2, \ldots, m$. Hence, we can construct the refined single-valued neutrosophic decision matrix D, as shown in Table 1.

When the weights of criteria are considered as the different importance of each criterion C_i ($i = 1, 2, \ldots, n$), the weight vector of the three criteria is given by $W = (w_1, w_2, \ldots, w_n)$ with $w_i \geq 0$ and $\sum_{i=1}^{n} w_i = 1$. Thus, the decision-making steps are described as follows:

Step 1: Based on the refined single-valued neutrosophic decision matrix D, we can determine the ideal solution (ideal RSVNN) by

$$s_i^*=\left\langle\begin{array}{l}(\max_k(T_{1R_k}),\max_k(T_{2R_k}),\ldots,\max_k(T_{p_iR_k})),\\(\min_k(I_{1R_k}),\min_k(I_{2R_k}),\ldots,\min_k(I_{p_iR_k})),\\(\min_k(F_{1R_k}),\min_k(F_{2R_k}),\ldots,\min_k(F_{p_iR_k}))\end{array}\right\rangle, \quad (4)$$

which is constructed as the ideal alternative $S^* = \{s_1^*, s_2^*, \ldots, s_n^*\}$.

Step 2: The similarity measure between each alternative R_k ($k = 1, 2, \ldots, m$) and the ideal alternative S^* can be calculated according to Eq. (3) and the values of $M_w(R_k, S^*)$ for $k = 1, 2, \ldots, m$ can be obtained.

Step 3: The alternatives are ranked in a descending order according to the values of $M_w(R_k, S^*)$ for $k = 1, 2, \ldots, m$. The greater value of $M_w(R_k, S^*)$ means the better alternative R_k.

Step 4: End.

5 Actual example on the selection of construction projects

In this section, we give the application of the decision making method for the selection of construction projects.

A construction company needs to determine the selecting problem of construction projects. Decision makers provide four construction projects as a set of four alternatives $R = \{R_1, R_2, R_3, R_4\}$. Then, the selection of these construction projects is dependent on three main criteria and seven sub-criteria: (1) Financial state (C_1): budget control (C_{11}) and risk/return ratio (C_{12}); (2) Environmental protection (C_2): public relation (C_{21}), geographical location (C_{22}), and health and safety (C_{23}); (3) Technology (C_3): technical knowhow (C_{31}), technological capability (C_{32}).

Experts or decision makers are required to evaluate the four possible alternatives under the above three criteria (seven sub-criteria) by suitability judgments, which are represented by RSVNSs. Thus we can construct the following refined single-valued neutrosophic decision matrix D, as shown in Table 2.

Table 1. The refined single-valued neutrosophic decision matrix D

	$C_1(C_{11},C_{12},\ldots,C_{1p_1})$	…	$C_n(C_{n1},C_{n2},\ldots,C_{np_n})$
R_1	$\langle(T_{1R_1},T_{2R_1},\ldots,T_{p_1R_1}),(I_{1R_1},I_{2R_1},\ldots,I_{p_1R_1}),(F_{1R_1},F_{2R_1},\ldots,F_{p_1R_1})\rangle$	…	$\langle(T_{1R_1},T_{2R_1},\ldots,T_{p_nR_1}),(I_{1R_1},I_{2R_1},\ldots,I_{p_nR_1}),(F_{1R_1},F_{2R_1},\ldots,F_{p_nR_1})\rangle$
R_2	$\langle(T_{1R_2},T_{2R_2},\ldots,T_{p_1R_2}),(I_{1R_2},I_{2R_2},\ldots,I_{p_1R_2}),(F_{1R_2},F_{2R_2},\ldots,F_{p_1R_2})\rangle$	...	$\langle(T_{1R_2},T_{2R_2},\ldots,T_{p_nR_2}),(I_{1R_2},I_{2R_2},\ldots,I_{p_nR_2}),(F_{1R_2},F_{2R_2},\ldots,F_{p_nR_2})\rangle$
…	…	...	…
R_m	$\langle(T_{1R_m},T_{2R_m},\ldots,T_{p_1R_m}),(I_{1R_m},I_{2R_m},\ldots,I_{p_1R_m}),(F_{1R_m},F_{2R_m},\ldots,F_{p_1R_m})\rangle$	...	$\langle(T_{1R_m},T_{2R_m},\ldots,T_{p_nR_m}),(I_{1R_m},I_{2R_m},\ldots,I_{p_nR_m}),(F_{1R_m},F_{2R_m},\ldots,F_{p_nR_m})\rangle$

Table 2. Defined single-valued neutrosophic decision matrix D for the four alternatives on three criteria (seven sub-criteria)

	C_1 (C_{11}, C_{12})	C_2 (C_{21}, C_{22}, C_{23})	C_3 (C_{31}, C_{32})
R_1	<(0.6, 0.7), (0.2, 0.1), (0.2, 0.3)>	<(0.9, 0.7, 0.8), (0.1, 0.3, 0.2), (0.2, 0.2, 0.1)>	<(0.6, 0.8), (0.3, 0.2), (0.3, 0.4)>
R_2	<(0.8, 0.7), (0.1, 0.2), (0.3, 0.2)>	<(0.7, 0.8, 0.7), (0.2, 0.4, 0.3), (0.1, 0.2, 0.1)>	<(0.8, 0.8), (0.1, 0.2), (0.1, 0.2)>
R_3	<(0.6, 0.8), (0.1, 0.3), (0.3, 0.4)>	<(0.8, 0.6, 0.7), (0.3, 0.1, 0.1), (0.2, 0.1, 0.2)>	<(0.8, 0.7), (0.4, 0.3), (0.2, 0.1)>
R_4	<(0.7, 0.6), (0.1, 0.2), (0.2, 0.3)>	<(0.7, 0.8, 0.7), (0.2, 0.2, 0.1), (0.1, 0.2, 0.2)>	<(0.7, 0.7), (0.2, 0.3), (0.2, 0.3)>

Then, the weight vector of the three criteria is given by W = (0.4, 0.3, 0.3). Thus, the proposed decision making method is applied to the selecting problem of the construction projects. Consequently, the decision-making steps are described as follows:

Step 1: By Eq. (4), the ideal solution (ideal RSVNS) can be determined as the following ideal alternative:

S^* ={<(0.8, 0.8), (0.1, 0.1), (0.2, 0.2)>, <(0.9, 0.8, 0.8), (0.1, 0.1, 0.1), (0.1, 0.1, 0.1)>, <(0.8, 0.8), (0.1, 0.2), (0.1, 0.1)>}.

Step 2: According to Eq. (3), the weighted similarity measure values between each alternative R_k (k = 1, 2, 3, 4) and the ideal alternative S^* can be obtained as follows:

$M_w(R_1, S^*) = 0.7743$, $M_w(R_2, S^*) = 0.8370$, $M_w(R_3, S^*) = 0.7595$, and $M_w(R_4, S^*) = 0.7778$.

Step 3: Since the measure values are $M_w(R_2, S^*) > M_w(R_4, S^*) > M_w(R_1, S^*) > M_w(R_3, S^*)$, the ranking order of the four alternatives is $R_2 > R_4 > R_1 > R_3$. Hence, the alternative R_2 is the best choice among all the construction projects.

6 Conclusion

This paper introduced RSVNSs and presented the similarity measure of RSVNSs. Then, we proposed a similarity measure-based multicriteria decision-making method under a RSVNS environment. In the decision-making process, through the similarity measure between each alternative and the ideal alternative, the ranking order of all alternatives can be determined and the best alternative can be selected as well. Finally, an actual example on the selecting problem of construction projects demonstrated the application of the proposed method. The main advantage of the proposed approach is easy evaluation and more suitable for actual applications in decision-making problems with RSVNS information. In the future, we shall extend the proposed decision-making method to medical diagnosis and fault diagnosis.

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Multi-Criteria Decision Making Method for *n*-wise Criteria Comparisons and Inconsistent Problems

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Abstract

The purpose of this paper is to present an alternative of a hybrid method based on Saaty's Analytical Hierarchy Process and on the Technique for Order Preference by using the Similarity to Ideal Solution method (AHP-TOPSIS) and, based on the AHP and its use of pairwise comparisons, to extend it to a new method called α-D MCDM-TOPSIS (α-Discounting Method for multi-criteria decision making-TOPSIS). The new method overcomes the limits of AHP, which works only for pairwise comparisons of criteria, to any-wise (*n*-wise) comparisons, with crisp coefficients or with interval-valued coefficients. An extended MCMD method (called Extended α-D MCDM) of α-D MCDM, introduced by Smarandache to solve decision making problems, is developed. α-D MCDM-TOPSIS and Extended α-D MCDM are verified on several examples, to demonstrate how they work with consistent, weak inconsistent or strong inconsistent problems. Finally, we discuss and compare all methods.

Keyword

Decision making, Extended α-D MCDM, Consistency, Inconsistency, n-wise criteria comparisons, AHP TOPSIS.

1 Introduction

Many economic, social or technological problems have been widely discussed and resolved in recent years by multi-criteria decision making methods [8]. However, the quantity of data, the complexity of the modern world and the recent technological advances have made obviously that MCDM methods are more challenging than ever, hence the necessity of developing other methods, able to give quality solutions.

Among MCDM methods, the most often used to improve the reliability of the decision making process is the combined method AHP-TOPSIS [12], [3], [2], [8], [10], [11] and [4].

AHP-TOPSIS is indeed a useful MCDM method to resolve difficult decision making problems and to select the best of the alternatives. Its applications are significant [8]: a support for management and planning of flight mission at NASA [12]; a method to study how the traffic congestion of urban roads is evaluated [3]; choosing logistics service provider in the mobile phone industry domains [10]; summarizing an e-SCM performance for management of supply chain [11]; evaluating faculty performance in engineering education, or sharing capacity assessment knowledge of supply chain [4].

Our paper is organized as follows. In the next section (*Section 2*), a literature survey for consistency problems is given. *Section 3* and *Section 4* focus on AHP-TOPSIS, and on the proposed α-D MCDM-TOPSIS model, respectively. The proposed method is tested on consistent, weak inconsistent and strong inconsistent examples (in *Section 5*). AHP method employed to rank the preferences is considered in *Section 6*. An extended α-D MCDM is introduced in *Section 7*, and it is shown how it can be applied for ranking preferences. We discuss developments via the use of an example to compare all methods. Finally, we draw conclusions and envisage some perspectives.

2 Comparison of characteristics between AHP and α-D MCDM: Consistency

2.1 A brief overview of Analytic Hierarchy Process (AHP)

AHP, introduced by the Saaty [6], is one of the most complete methods of multi-criteria decision making technique, determining the weights of criteria and ranking alternatives. The use of AHP only, or its hybrid use with other methods, proved its capacity to solve MCDM problems and to be a popular technique for determining weights — see more than a thousand references in [9]. Besides the performance of AHP and its added value at both levels, theoretical and practical, this method functions only if the problem is perfectly consistent, which is rarely checked in real MCDM problems.

2.2 Description of α-D MCDM

α-D MCDM (α-Discounting Method for Multi-Criteria Decision Making) was introduced by Smarandache — see [7]. The new method overcomes the limits of AHP, which work only for pairwise comparisons of criteria, expanding to any-wise (n-wise) comparisons.

Smarandache used the homogeneous linear mathematical equations to express the relationship between criteria with crisp coefficients or with interval-valued coefficients also for non-linear equations, with crisp coefficients or with interval-valued coefficients.

The two aims of α-D MCDM method were: firstly, to transform the equations of each criterion with respect to other criteria that has only a null solution into a linear homogeneous system having a non-null solution by multiplying each criteria of the right hand by non-null positive parameters $\alpha_1, \alpha_2, \cdots, \alpha_k$; secondly, to apply the "Fairness Principle" on the general solution of the above system by discounting each parameter by the same value $(\alpha = \alpha_1 = \alpha_2 = \cdots = \alpha_k)$.

2.2.1 α-D MCDM method

The general idea of the α-D MCDM is to transform any MCDM inconsistent problem (in which AHP does not work) to a MCDM consistent problem, by discounting each coefficient by the same percentage.

Let us assume that $C = \{C_1, C_2, \cdots, C_n\}$, with $n \geq 2$, is a set of criteria, and let's construct a linear homogeneous system of equations.

Each criterion C_i can be expressed as linear homogeneous equation, or as non-linear equation, with crisp coefficients or with interval-valued coefficients of other criteria $C_1, \cdots C_j \cdots C_n$ —

$$C_i = f(C_1, \cdots C_j \cdots C_n).$$

Consequently, a comparisons matrix associated to this linear homogeneous system is constructed.

To determine the weights w_i of the criteria, we solve the previous system.

The α-D MCDM method is not designed to rank preferences P_i based on C_i criteria, as AHP method does, but to determine only the weights of criteria in any type of problems (consistent, inconsistent).

AHP as cited above is a complete method designed to calculate the weights of criteria C_i and to rank the preferences P_i. In addition, when the AHP is used with TOPSIS, or other MCDM method, we just benefit from the part of weight calculation criteria and we use TOPSIS to rank preferences — or other MCDM methods.

The same for α-D MCDM: firstly, it is just used to calculate the weight of criteria, that will be used later by TOPSIS to rank preferences, and, secondly, the α-D MCDM is extended to a complete method, in order to rank the preferences.

Therefore, we use α-D MCDM for calculating the weight of criteria C_i and not to rank P_i preferences.

We have —

$$C_i = f(\{C\} \setminus C_i).$$

Then, criteria C_i is a linear equation of C_j such as —

$$C_i = \sum_{j=1 j \neq i}^{n} x_{ij} C_j .$$

So, the comparisons criteria matrix has the number of criteria by rows and columns (rows number n = number of criteria, and columns number m = number of equations). In the result, we have a square matrix ($n = m$), consequently we can calculate the determinant of this matrix. At this point, we have an $n \times n$ linear homogeneous system and its associated matrix —

$$\begin{cases} x_{1,1}w_1 + x_{1,2}w_2 + \cdots + x_{1,n}w_n = 0 \\ \vdots \\ x_{n,1}w_1 + x_{m,2}w_2 + \cdots + x_{n,n}w_n = 0 \end{cases}$$

$$X = \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix}.$$

The difference between AHP and α-D MCDM is the ability of the latter to work with consistent and inconsistent problems, and if the problem is inconsistent, α-D MCDM method transforms it in a consistent problem, while AHP is unable to do that, managing strictly consistent problems.

In the following, we deal with the relationship between determinant of matrix and consistency, and the parameterization of system by α_i, in order to get a consistent problem.

Property 1

— If $det(X) = 0$, the system has a solution (i.e. MCDM problem is consistent).
— If $det(X) \neq 0$, the system has only the null solution (i.e. MCDM problem is inconsistent).

If the problem is inconsistent, then one constructs the parameterized matrix, denoted $X(\alpha)$, by parameterizing the right-hand in order to get $\det(X(\alpha)) = 0$ and use Fairness principe (set equal parameters to all criteria $\alpha = \alpha_1 = \alpha_2 = \cdots = \alpha_k > 0$). To get priority vector, one resolves the new system obtained and set 1 to secondary variable, then normalize the vector by dividing the sum of all components.

2.3 Consistency of decision making problems

In this section, we discuss the consistency of the MCDM problems for both methods (α-D MCDM and AHP).

For resolving a linear system of equations, in mathematics we use raw operations, such as substitution, interchange, … .

Definition 1 [7]

Applying any substitution raw operations on two equations, if it does not influence the system consistency and there is an agreement of all equations, we say that the linear system of equations (of the linear MCDM problem) is consistent.

Definition 2 [7]

Applying any substitution raw operations on two equations, if equation result is in disagreement with another, we say that the linear system of equations (of the linear MCDM problem) is weakly consistent.

Definition 3 [7]

Applying any substitution raw operations on two equations, if equation result is in opposition with another, we say that the linear system of equations (of the linear MCDM problem) is strongly inconsistent.

2.4 Consistency

AHP provides the decision maker with a way of examining the consistency of entries in a pairwise comparison matrix; the problem of accepting/rejecting matrices has been largely discussed [5], [1], [13], especially regarding the relation between the consistency and the scale used to represent the decision maker’s judgments. AHP is too restrictive when the size of the matrix increases, and when order n of judgment matrix is large; the satisfying consistency is more difficult to be met [5], [1].

This problem may become a very difficult one when the decision maker is not perfectly consistent, moreover, it seems impossible (AHP does not work) when there are not pairwise comparisons, but all kind of comparisons between criteria, such as $n-$wise, because there is set a strict consistency condition in the AHP, in order to keep the rationality of preference intensities between compared elements.

In addition, the inconsistency exists in all judgments [5]; comparing three alternatives — or more, it is possible that inconsistency exists when there are more than 25 percent of the $3-by-3$ reciprocal matrices with a consistency ratio less than or equal to ten percent. Consequently, as the matrix size increases, the percentage of inconsistency decreases dramatically [1], [5].

Furthermore, the AHP method sets a consistency ratio (CR) threshold ($CR(X) > 0.1$), which should not be exceeded, by examining the inconsistency of the pairwise comparison matrix, but this requirement for the Saaty's matrix is not achievable in the real situations.

In order to overcome this deficiency, instead of the AHP we suggest employing an α-D MCDM, which is very natural and more suitable for the linguistic descriptions of the Saaty's scale and, as a result of it, it is easier to reach this requirement in the real situations.

Moreover, the attractiveness of α-D MCDM is due to its potential to overcome limits of AHP, which works only for pairwise comparisons of criteria, expanding to n-wise (with $n \geq 2$) comparisons, with crisp coefficients or with interval-valued coefficients. Therefore, α-D MCDM method works for inconsistent, weak inconsistent and strong inconsistent problems.

As previously shown, in α-D MCDM method, in order to transform a inconsistent MCDM problem to a consistent problem — we multiply each criteria of the right hand by non-null positive parameters $\alpha_1, \alpha_2, \cdots, \alpha_k$ and we use "Fairness Principle" assigning to each parameter the same value ($\alpha = \alpha_1 = \alpha_2 = \cdots = \alpha_k$).

Property 2

In α-D MCDM (and Fairness-Principle for coefficients α_i), the parameter α (or $\frac{1}{\alpha}$) signifies the degree of consistency and β ($\beta = f(\alpha)$) represents the degree of inconsistency.

— If $0 < \alpha < 1$, then α and $\beta = 1 - \alpha$ represent the degree of consistency and the degree of inconsistency, respectively, of the decision-making problem.

— If $\alpha > 1$, then $\frac{1}{\alpha}$ and $\beta = 1 - \frac{1}{\alpha}$ represent the degree of consistency and the degree of inconsistency, respectively, of the decision-making problem.

Property 3

In AHP method, RI — consistency index, CR — consistency ratio and λ (λ_{max} largest) — the eigenvalue of the $(X)_{n\times n}$ pairwise comparison matrix.

— We say that MCDM problem is consistent (pairwise comparison matrix X is consistent), if $Rank(X) = 1$ and $\lambda = n$ (ideal case).
— We say that MCDM problem is consistent too (pairwise comparison matrix X is consistent), if consistency ratio $CR(X) \le 0.1$, $CR(X) = \frac{CI(X)}{RI(X)}$, where $CI(X) = \frac{\lambda_{max} - n}{n-1}$, and RI values are given (simulation parameter).
— If $CR(X) \le 0.1$, the MCDM problem is inconsistent and the pairwise comparison matrix should be improved.

Characteristics	**AHP**	**α -D MCDM**
Weight elicitation	Pairwise comparison	n -wise comparison ($n \ge 2$)
Number of attributes accommodated	7 ± 2	Large inputs
Consistent problems	Provided	Not provided and α-D MCDM gives same result as AHP
Weakly inconsistent problems	Does not work	Justifiable results
Strongly inconsistent problems	Does not work	Justifiable results

Table 1: Comparison of characteristics of both methods (AHP, α-D MCDM)

3 Description of data structure decision problems under consideration

Taking into account that pertinent data is frequently very high-priced to collect, we can't change real life problems to obtain a specific form of data. In addition, information from real world certainly includes imperfection — such as uncertainty, conflict, etc.

Hence, the choice of the MCDM method is based, firstly, on the structure of decision problem considered, secondly, on the types of data that can be obtained, and, finally, on the capability to get accurate results. For this reason, we detail the different types of all data structure decision problem, for example:

— If decision matrix illustrates the importance of alternatives with respect of criteria, the pairwise (or n-wise) comparison can't be used directly in the hybrid AHP-TOPSIS approach. Firstly, priority weights for criteria are calculated using AHP technique, and then the alternatives are prioritized using TOPSIS approach.

The derivation of weights is a central step in eliciting the decision-maker's preferences, but the hybrid AHP-TOPSIS method is more difficult to be met: on one hand, AHP does not work in inconsistent problems, on the other hand it cannot be employed for the n-wise comparisons criteria cases.

The problem can be abstracted as how to derive weights for a set of activities according to their impact on the situation and the objective of decisions to be made.

Hence, this study will extend AHP-TOPSIS to a MCDM to fit real world. A complete and efficient procedure for decision making will then be provided. The developed model has been analyzed to select the best alternative using α-D MCDM and the technique for order preference by similarity to ideal solution (TOPSIS) as a hybrid approach.

Let us assume that $C = \{C_1, C_2, \cdots, C_n\}$ is a set of Criteria, with $n \geq 2$, and $A = \{A_1, A_2, \cdots, A_m\}$ is the set of Preferences (Alternatives), with $m \geq 1$.

$$
\begin{array}{ccc}
\begin{array}{ccc} C_1 & & C_n \\ \downarrow & \cdots & \downarrow \end{array} & \\
\begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & & \vdots \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix} & \begin{array}{c} \leftarrow C_1 \\ \vdots \\ \leftarrow C_n \end{array}
\end{array}
$$

	C_1	C_2	$\cdots$	C_n
	w_1	w_2	$\cdots$	w_n
A_1	a_{11}	a_{12}	$\cdots$	a_{1n}
A_2	a_{21}	a_{22}	$\cdots$	a_{2n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A_m	a_{m1}	a_{m2}	$\cdots$	a_{mn}

Table 2: Decision matrix

If the data cannot be obtained directly to construct the decision matrix $A=(a_{ij})$ above, we should have, for each criteria C_i, a pairwise (or n-wise) comparison matrix of the preferences (not just for the criterion).

The comparison matrix of the preferences gives the relative importance (b_{ij}) of each alternative A_i compared with another A_j with respect to criterion C_k.

As mentioned, the comparison matrices of the preferences should be given, but for comparing the results, we will demonstrate how we can obtain it from decision matrix.

For each criterion C_k, the comparison matrix of the preferences is defined by $B_k=(b_{ij})$ such as $b_{ij}=\frac{a_{ik}}{a_{jk}}$, with $i=1,2\cdots,n$ (for each criterion a comparison matrix of preferences, consequently n comparisons preferences matrices will be constructed).

C_i	A_1	A_2	$\cdots$	A_n
A_1	b_{11}	b_{12}	$\cdots$	b_{1m}
A_2	b_{21}	b_{22}	$\cdots$	b_{2m}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A_m	b_{m1}	b_{m2}	$\cdots$	b_{mm}

Table 3: Relative importance of alternative comparison matrix

Hence, we need to construct n (number of criteria) matrices with pairwise or n-wise comparisons of size $m*m$ each, with m the number of preferences, in these cases we can use AHP or α-D MCDM.

In this case, AHP method is used both to calculate the weights of criteria and to ranking preferences by calculate the priority.

AHP being more difficult to be met, we will extend α-D MCDM to work for the calculation of the weights criteria and ranking preferences.

4 AHP-TOPSIS method

In the real word decisions problems (*case 1, Section 3*) we have multiple preferences and diverse criteria. The MCDM problem can be summarized as it follows:

— Calculate weights w_i of criteria C_i;

— Rank preferences (alternatives) A_i.

Let us assume there are n criteria and their pairwise relative importance is x_{ij}.

TOPSIS assumes that we have n alternatives (preferences) $A_i (i = 1, 2, \cdots, m)$ and n attributes/criteria $C_j (j = 1, 2, \cdots, n)$ and comparison matrix a_{ij} of preference i with respect to criterion j.

The AHP-TOPSIS method is described in the following steps:

Step 4.1. *Construct decision matrix denoted by* $A = (a_{ij})_{m \times n}$

	C_1	C_2	$\cdots$	C_n
	w_1	w_2	$\cdots$	w_n
A_1	a_{11}	a_{12}	$\cdots$	a_{1n}
A_2	a_{21}	a_{22}	$\cdots$	a_{2n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A_m	a_{m1}	a_{m2}	$\cdots$	a_{mn}

Table 4: Decision matrix

Step 4.2. *Determine weights* (w_j) *of each criterion using AHP Method*, where

$$\sum_{j=1}^{n} w_j = 1, \; j = 1, 2, \cdots, n.$$

Step 4.2.1. *Build a pairwise comparison matrix of criteria*

The pairwise comparison of criterion i with respect to criterion j gives a square matrix $(X)_{n \times n} = (x_{ij})$ where x_{ij} represents the relative importance of criterion i over the criterion j. In the matrix, $x_{ij} = 1$ when $i = j$ and $x_{ij} = 1 / x_{ji}$. So, we get a $n \times n$ pairwise comparison matrix $(X)_{n \times n}$.

Step 4.2.2. *Find the relative normalized weight* (w_j) *of each criterion* defined by following formula —

$$w_j = \frac{\prod_{j=1}^{n} (x_{ij})^{1/n}}{\sum \prod_{j=1}^{n} (x_{ij})^{1/n}}$$

Then, get w_i weight of the i^{th} criterion.

Step 4.2.3. *Calculate matrix* X_3 *and* X_4 — such that $X_3 = X_1 \times X_2$ and where $X_4 = X_3/X_2$ —

$$X2 = [w_1, w_2, \cdots, w_j]^T .$$

Step 4.2.4. *Find the largest eigenvalue of pairwise comparison matrix*

For simplified calculus, the largest eigenvalue of pairwise comparison matrix is the average of X_4. Furthermore, according to the Perron-Frobenius theorem, principal eigenvalue λ_{max} always exists for the Saaty's matrix and it holds $\lambda_{max} \geq n$; for fully consistent matrix $\lambda_{max} = n$.

Consistency check is then performed to ensure that the evaluation of the pair-wise comparison matrix is reasonable and acceptable.

Step 4.2.5. *Determine the consistency ratio* (CR)

After calculation consistency ratio (RC) using equation (eq.), and in order to verify the consistency of the matrix that is considered to be consistent if CR is less than threshold and not otherwise, according to Saaty and search. At this point, we have the weights of criteria and if the consistency is checked, we will be using TOPSIS to rank preferences.

Step 4.3. *Normalize decision matrix*

The normalize decision matrix is obtained, which is given here with r_{ij}

$$r_{ij} = a_{ij} / \left(\sum_{i=1}^{m} a_{ij}^2 \right)^{0.5} ; j = 1, 2, \cdots, n; i = 1, 2 \cdots, m .$$

Step 4.4. *Calculate the weighted decision matrix*

Weighting each column of obtained matrix by its associated weight.

$$v_{ij} = w_j r_{ij}; j = 1, 2, \cdots, n; i = 1, 2 \cdots, m.$$

Step 4.5. *Determine the positive ideal solution (PIS) and negative ideal solution (NIS)*

$$A^+ = (v_1^+, v_2^+, \cdots, v_n^+) = \left\{ \left(max_i \left\{ v_{ij} \mid j \in B \right\} \right), \left(min_i \left\{ v_{ij} \mid j \in C \right\} \right) \right\};$$

$$A^- = (v_1^-, v_2^-, \cdots, v_n^-) = \left\{ \left(min_i \left\{ v_{ij} \mid j \in B \right\} \right), \left(max_i \left\{ v_{ij} \mid j \in C \right\} \right) \right\}.$$

The benefit and cost solutions are represented by B and C respectively.

Step 4.6. *Calculate the distance measure for each alternative from the PIS and NIS*

The distance measure for each alternative from the PIS is —

$$S_i^+ = \left\{ \sum_{j=1}^{n} (v_{ij} - v_j^+)^2 \right\}^{0.5} ; i = 1, 2 \cdots, m$$

Also, the distance measure for each alternative from the NIS is —

$$S_i^- = \left\{ \sum_{j=1}^{n} (v_{ij} - v_j^-)^2 \right\}^{0.5} ; i = 1, 2 \cdots, m$$

Step 4.7. *Determine the values of relative closeness measure*

For each alternative we calculate the relative closeness measure as it follows:

$$T_i = \frac{S_i^-}{(S_i^+ + S_i^-)} ; i = 1, 2 \cdots, m .$$

Rank alternatives set according to the order of relative closeness measure values T_i.

5 α-D MCDM-TOPSIS method

The MCDM problem description is the same as the one used in AHP-TOPSIS method (*Section 4*), but in this case we have n-wise comparisons matrix of criteria. Let us assume that $C = \{C_1, C_2, \cdots, C_n\}$, with $n \geq 2$, and $\{A_1, A_2, \cdots, A_m\}$, with $m \geq 1$, are a set of criteria and a set of preferences, respectively. Let us assume that each criterion C_i is a linear homogeneous equation of the other criteria $C_1, C_2, \cdots, C_n$:

$$C_i = f(\{C\} \setminus C_i).$$

The α-D MCDM-TOPSIS method is described in the following steps:

Step 5.1. *Construct decision matrix denoted by* $A = (a_{ij})_{m \times n}$

	C_1	C_2	$\cdots$	C_n
	w_1	w_2	$\cdots$	w_n
A_1	a_{11}	a_{12}	$\cdots$	a_{1n}
A_2	a_{21}	a_{22}	$\cdots$	a_{2n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A_m	a_{m1}	a_{m2}	$\cdots$	a_{mn}

Table 5: Decision matrix

Step 5.2. *Determine weights (w_i) of each criterion using α-D MCDM Method*

Step 5.2.1. *Using α-D MCDM to determine the importance weight* (w_i) *of the criteria*, where —

$$\sum_{j=1}^{n} w_j = 1,\ j = 1, 2, \cdots, n\,.$$

Step 5.2.2. *Build a system of equations and its associated matrix*

To construct linear system of equations, each criterion C_i is expressed as a linear equation of C_j such as —

$$C_i = \sum_{j=1\, j\neq i}^{n} x_{ij} C_j$$

Consequently, we have a system of *n* linear equations (one equation of each criterion) with *n* variables (variable w_i is weight of criterion).

$$\begin{cases} x_{1,1}w_1 + x_{1,2}w_2 + \cdots + x_{1,n}w_n = 0 \\ \vdots \\ x_{n,1}w_1 + x_{m,2}w_2 + \cdots + x_{n,n}w_n = 0 \end{cases}$$

In mathematics, each linear system can be associated to a matrix, in this case, denoted by $X = (x_{ij})$, $1 \le i \le n$ and $1 \le j \le n$ where —

$$X = \begin{pmatrix} x_{1,1} & \cdots & x_{1,n} \\ \vdots & \ddots & \vdots \\ x_{n,1} & \cdots & x_{n,n} \end{pmatrix}.$$

Step 5.2.3. *Solve system of equation using whose associated matrix*

Solve the system of equation; the different cases are discussed in *Property 1* in that we compute the determinant of X (find strictly positive solution $w_i > 0$).

Solving this homogeneous linear system, in different cases above the general solution that we set as a solution vector —

$$S = [s_1, s_2, \cdots, s_n]$$

and set 1 to secondary variable, we get —

$$W = [w_1, w_2, \cdots, w_n].$$

Dividing each vector element on sum of all components of vector to get normalized vector, where —

$$w_j = \frac{s_j}{\sum_{k=1}^{n} s_k}; i = 1, 2 \cdots, n.$$

Step 5.2.4. *Build a pairwise comparison matrix of criteria*

The pairwise comparison of criterion i with respect to criterion j gives a square matrix $(X)_{n \times n} = (x_{ij})$ where x_{ij} represents the relative importance of criterion i over the criterion j. In the matrix, $x_{ij} = 1$ when $i = j$ and $x_{ij} = 1 / x_{ji}$. So we get a $n \times n$ pairwise comparison matrix $(X)_{n \times n}$.

Step 5.2.5. *Find the relative normalized weight (w_j) of each criterion* defined by the following formula —

$$w_j = \frac{\prod_{j=1}^{n} (x_{ij})^{1/n}}{\sum \prod_{j=1}^{n} (x_{ij})^{1/n}}.$$

Then, get w_i weight of the i^{th} criterion.

Step 5.2.6. *Calculate matrix* X_3 *and* X_4 — such that $X_3 = X_1 \times X_2$ and where $X_4 = X_3 / X_2$ —

$$X2 = [w_1, w_2, \cdots, w_j]^T.$$

Step 5.2.7. *Find the largest eigenvalue of pairwise comparison matrix*

For simplifying the calculus, the largest eigenvalue of pairwise comparison matrix is the average of X_4. Furthermore, according to the Perron-Frobenius theorem, principal eigenvalue λ_{max} always exists for the Saaty's matrix and it holds $\lambda_{max} \geq n$; for fully consistent matrix $\lambda_{max} = n$.

Consistency check is then performed to ensure that the evaluation of the pair-wise comparison matrix is reasonable and acceptable.

Step 5.2.8. *Determine the consistency ratio* (CR)

After calculating consistency ratio (RC) using equation (eq.), and in order to verify the consistency of the matrix that is considered to be consistent if CR is less than threshold and not otherwise, according to Saaty and search.

At this point, we have the weights of criteria and if the consistency is checked, we will be using TOPSIS to rank preferences.

Step 5.3. *Normalize decision matrix*

The normalized decision matrix is obtained, which is given here with r_{ij}

$$r_{ij} = a_{ij} / \left(\sum_{i=1}^{m} a_{ij}^2 \right)^{0.5}; j = 1, 2, \cdots, n; i = 1, 2 \cdots, m.$$

Step 5.4. *Calculate the weighted decision matrix*

Weighting each column of obtained matrix by its associated weight —

$$v_{ij} = w_j r_{ij}; j = 1, 2, \cdots, n; i = 1, 2 \cdots, m.$$

Step 5.5. *Determine the positive ideal solution (PIS) and negative ideal solution (NIS)*

$$A^+ = (v_1^+, v_2^+, \cdots, v_n^+) = \left\{ \left(max_i \left\{ v_{ij} \mid j \in B \right\} \right), \left(min_i \left\{ v_{ij} \mid j \in C \right\} \right) \right\};$$

$$A^- = (v_1^-, v_2^-, \cdots, v_n^-) = \left\{ \left(min_i \left\{ v_{ij} \mid j \in B \right\} \right), \left(max_i \left\{ v_{ij} \mid j \in C \right\} \right) \right\}.$$

The benefit and cost solutions are represented by B and C, respectively.

Step 5.6. *Calculate the distance measure for each alternative from the PIS and NIS*

The distance measure for each alternative from the PIS is —

$$S_i^+ = \left\{ \sum_{j=1}^{n} (v_{ij} - v_j^+)^2 \right\}^{0.5}; i = 1, 2 \cdots, m.$$

Also, the distance measure for each alternative from the NIS is —

$$S_i^- = \left\{ \sum_{j=1}^{n} (v_{ij} - v_j^-)^2 \right\}^{0.5}; i = 1, 2 \cdots, m.$$

Step 5.7. *Determine the values of relative closeness measure*

For each alternative, we calculate the relative closeness measure as it follows:

$$T_i = \frac{S_i^-}{(S_i^+ + S_i^-)}; i = 1, 2 \cdots, m.$$

Rank alternatives set according to the order of relative closeness measure values T_i.

6 Numerical examples

We examine a numerical example in which a synthetic evaluation desire to rank four alternatives A_1, A_2, A_3 and A_4 with respect to three benefit attribute C_1, C_2 and C_3.

	C_1	C_2	C_3
	w_1	w_2	w_3
A_1	7	9	9
A_2	8	7	8
A_3	9	6	8
A_4	6	7	8

Table 6: Decision matrix

In the examples below, we use α-D MCDM and AHP (if it works) to calculate the weights of the criteria w_1, w_2 and w_3. After we used TOPSIS to rank the four alternatives, the decision matrix (*Table 6*) is used for the three following examples.

6.1 Consistent Example 1

We use the α-D MCDM. Let the Set of Criteria be $\{C_1, C_2, C_3\}$ with $w_1 = w(C_1) = x$ and $w_3 = w(C_3) = z$.

Let us consider the system of equations associated to MCDM problem and its associated matrix.

$$\begin{cases} x = & 4y \\ y = & 3z \\ z = & \dfrac{x}{12} \end{cases} \quad X1 = \begin{pmatrix} 1 & 4 & 0 \\ 0 & 1 & 3 \\ \dfrac{1}{12} & 0 & 1 \end{pmatrix}.$$

We calculate $\det(X)$ (in this case, equal = 0); then MCDM problem is consistent; we solve the system; we get the following solution —

$$S = [12z \quad 3z \quad z].$$

Setting 1 to secondary variable, the general solution becomes —

$$S = [12 \quad 3 \quad 1],$$

and normalizing the vector (dividing by sum=12+3+1), the weights vector is:

$$W = \begin{bmatrix} \frac{12}{16} & \frac{3}{16} & \frac{1}{16} \end{bmatrix}.$$

Using AHP, we get the same result.

The pairwise comparison matrix of criteria is:

$$X1 = \begin{pmatrix} 1 & 4 & 12 \\ \frac{1}{4} & 1 & 3 \\ \frac{1}{12} & \frac{1}{3} & 1 \end{pmatrix},$$

whose maximum eigenvalue is $\lambda_{max} = 3$ and its corresponding normalized eigenvector (Perron-Frobenius vector) is —

$$W = \begin{bmatrix} \frac{12}{16} & \frac{3}{16} & \frac{1}{16} \end{bmatrix}.$$

We use TOPSIS to rank the four alternatives.

a_{ij}^2	C_1	C_2	C_3
w_i	12/16	3/16	1/16
A_1	49	81	81
A_2	64	49	64
A_3	81	36	64
A_4	36	49	64
$\sum_{i=1}^{n} a_{ij}^2$	230	215	273

Table 7: Calculate (a_{ij}^2) for each column

r_{ij}	C_1	C_2	C_3
w_i	12/16	3/16	1/16
A_1	0.4616	0.6138	0.5447
A_2	0.5275	0.4774	0.4842
A_3	0.5934	0.4092	0.4842
A_4	0.3956	0.4774	0.4842
$\sum_{i=1}^{n} a_{ij}^2$	230	215	273

Table 8: Divide each column by $(\sum_{i=1}^{n} a_{ij}^2)^{1/2}$ to get r_{ij}

v_{ij}	C_1	C_2	C_3
w_i	12/16	3/16	1/16
A_1	0.3462	0.1151	0.0340
A_2	0.3956	0.0895	0.0303
A_3	0.4451	0.0767	0.0303
A_4	0.2967	0.0895	0.0303
v_{max}	0.4451	0.1151	0.0340
v_{min}	0.2967	0.0767	0.0303

Table 9: Multiply each column by w_j to get v_{ij}

Alternative	S_i^+	S_i^-	T_i	Rank
A_1	0.0989	0.0627	0.3880	3
A_2	0.0558	0.0997	0.6412	2
A_3	0.0385	0.1484	0.7938	1
A_4	0.1506	0.0128	0.0783	4

Table 10: The separation measure values and the final rankings for decision matrix (*Table 4*) using AHP-TOPSIS and α-D MCDM-TOPSIS

Table 10 presents the rank of alternatives (A_1, A_2, A_3, A_4) and separation measure values of each alternative from the PIS and from NIS in which the weighted values are calculated by AHP or α-D MCDM. Both methods, AHP and α-D MCDM with Fairness Principle, give the same weights as proven above methods together give same result in consistent problem.

6.2 Weak inconsistent Example 2 where AHP does not work

Let us consider another example investigated by [7] for which AHP does not work (i.e. AHP-TOPSIS does not work too); we use the α-D MCDM to calculate the weights values and ranking the four alternatives by TOPSIS (see *Table 14*).

Let the Set of Criteria be $\{C_1, C_2, C_3\}$ with $w_1 = w(C_1) = x$ and $w_3 = w(C_3) = z$. Let us consider the system of equations associated to MCDM problem and its associated matrix.

$$\begin{cases} x = & 2y + 3z \\ y = & \dfrac{x}{2} \\ z = & \dfrac{x}{3} \end{cases}$$

$$X1 = \begin{pmatrix} 1 & -2 & -3 \\ \frac{-1}{2} & 1 & 0 \\ \frac{-1}{3} & 0 & 1 \end{pmatrix}.$$

The solution of this system is $x = y = z = 0$; be the sum of weights = 1, then this solution is not acceptable.

Parameterizing the right-hand side coefficient of each equation by α_i we get:

$$\begin{cases} x = 2\alpha_1 y + 3\alpha_2 z \\ y = \dfrac{\alpha_3 x}{2} \\ z = \dfrac{\alpha_4 x}{3} \end{cases}$$

We solve the system and we get the following solution —

$$S = \begin{cases} y = \frac{\alpha_3 x}{2} \\ z = \frac{\alpha_4 x}{3} \end{cases} \text{ or } S = \begin{bmatrix} x & \dfrac{\alpha_3 x}{2} & \dfrac{\alpha_4 x}{3} \end{bmatrix}.$$

Setting 1 to secondary variable, the general solution becomes —

$$S = \begin{bmatrix} 1 & \dfrac{\alpha_3}{2} & \dfrac{\alpha_4}{3} \end{bmatrix}.$$

Applying Fairness Principle, then replacing $\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \alpha$, whence $\alpha = \dfrac{\sqrt{2}}{2}$.

$$S = \begin{bmatrix} 1 & \dfrac{\sqrt{2}}{4} & \dfrac{\sqrt{2}}{6} \end{bmatrix}.$$

Normalizing vector (dividing by sum), the weights vector is:

$$W = \begin{bmatrix} 0.62923 & 0.22246 & 0.14831 \end{bmatrix}.$$

TOPSIS is used to rank the four alternative: application of TOPSIS method is in the same manner as in the previous example (the four alternatives (Ai) are ranked in the following *Table 14*).

a_{ij}^2	C_1	C_2	C_3
	0.62923	0.22246	0.14831
A_1	49	81	81
A_2	64	49	64
A_3	81	36	64
A_4	36	49	64
$\sum_{i=1}^{n} a_{ij}^2$	230	215	273

Table 11: Calculate (a_{ij}^2) for each column

r_{ij}	C_1	C_2	C_3
	0.62923	0.22246	0.14831
A_1	0.4616	0.6138	0.5447
A_2	0.5275	0.4774	0.4842
A_3	0.5934	0.4092	0.4842
A_4	0.3956	0.4774	0.4842
$\sum_{i=1}^{n} a_{ij}^2$	230	215	273

Table 12: Divide each column by $(\sum_{i=1}^{n} a_{ij}^2)^{1/2}$ to get r_{ij}

v_{ij}	C_1	C_2	C_3
	0.62923	0.22246	0.14831
A_1	0.2904	0.1365	0.0808
A_2	0.3319	0.1062	0.0718
A_3	0.3734	0.0910	0.0718
A_4	0.2489	0.1062	0.0718
v_{max}	0.3734	0.1365	0.0808
v_{min}	0.2489	0.0910	0.0718

Table 13: Multiply each column by w_j to get v_{ij}

Alternative	S_i^+	S_i^-	T_i	Rank
A_1	0.0830	0.0622	0.4286	3
A_2	0.0522	0.0844	0.6178	2
A_3	0.0464	0.1245	0.7285	1
A_4	0.1284	0.0152	0.1057	4

Table 14: The separation measure values and the final rankings for decision matrix (*Table 4*) using α -D MCDM-TOPSIS

6.3 Jean Dezert's strong inconsistent Example 3

Smarandache [7] introduced a Jean Dezert's Strong Inconsistent example. Let us consider the system of equations associated to MCDM problem and its associated matrix.

$$X = \begin{pmatrix} 1 & 9 & \frac{1}{9} \\ \frac{1}{9} & 1 & 9 \\ 9 & \frac{1}{9} & 1 \end{pmatrix} \qquad \begin{cases} x = & 9y, x > y \\ x = & \frac{1}{9}z, x < z \\ y = & 9z, y > z \end{cases}$$

We follow the same process as in the example above to get the general solution:

$$W = \begin{bmatrix} \frac{1}{6643} & \frac{81}{6643} & \frac{6561}{6643} \end{bmatrix}.$$

We use TOPSIS to rank the four alternatives.

a_{ij}^2	C_1	C_2	C_3
	0.0002	0.0122	0.9877
A_1	49	81	81
A_2	64	49	64
A_3	81	36	64
A_4	36	49	64
$\sum_{i=1}^{n} a_{ij}^2$	230	215	273

Table 15: Calculate (a_{ij}^2) for each column

r_{ij}	C_1	C_2	C_3
	0.0002	0.0122	0.9877
A_1	0.503	0.699	0.623
A_2	0.574	0.543	0.553
A_3	0.646	0.466	0.553
A_4	0.431	0.543	0.553
$\sum_{i=1}^{n} a_{ij}^2$	230	215	273

Table 16: Divide each column by $(\sum_{i=1}^{n} a_{ij}^2)^{1/2}$ to get r_{ij}

v_{ij}	C_1	C_2	C_3
	0.0002	0.0122	0.9877
A_1	0.0001	0.0075	0.5380
A_2	0.0001	0.0058	0.4782
A_3	0.0001	0.0050	0.4782
A_4	0.0001	0.0058	0.4782
v_{max}	0.0001	0.0075	0.5380
v_{min}	0.0001	0.0050	0.4782

Table 17: Multiply each column by w_j to get v_{ij}

Alternative	S_i^+	S_i^-	T_i	Rank
A_1	0.0000	0.0598	0.999668	1
A_1	0.0598	0.0008	0.013719	2
A_1	0.0598	0.0000	0.000497	4
A_1	0.0598	0.0008	0.013715	3

Table 18: The separation measure values and the final rankings for decision matrix (*Table 4*) using α-D MCDM-TOPSIS

7 AHP method

As we proved in *Section 3. Description of data structure decision problems under consideration*, AHP method can be used in the second case, in which data is structured as pairwise comparisons of matrices of preferences.

Let us assume that X is the comparison matrix of criteria; for each criterion C_k $(k = 1, 2, \cdots, n)$ we have a comparison matrix of preferences B_k and the consistency condition is perfect.

We use AHP to determine the importance weight (w_i) of the criteria.

We apply again AHP method for each comparison matrices of preferences B_k to determine the maximum eigenvalue and its associate eigenvector (same that is used to determine the weights of criteria).

For each matrix B_k (associated to C_k), we calculate the λ_{max} and priority vector (eigenvector).

C_i	A_1	A_2	$\cdots$	A_n	Priority vector
A_1	a_{11}	a_{12}	$\cdots$	a_{1m}	p_{1i}
A_2	a_{21}	a_{22}	$\cdots$	a_{2m}	p_{2i}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A_m	a_{m1}	a_{m2}	$\cdots$	a_{mm}	p_{mi}

Table 19: Comparison matrix of the preferences with priority vector in latest column

We get a decision matrix (different from the decision matrix above), formed using priority vectors, in which the entries of the decision matrix are p_{ij}, and not a_{ij}.

	C_1	C_2	$\cdots$	C_n
w_i	w_1	w_2	$\cdots$	w_n
A_1	p_{11}	p_{12}	$\cdots$	p_{1n}
A_2	p_{21}	p_{22}	$\cdots$	p_{2n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A_m	p_{m1}	p_{m2}	$\cdots$	p_{mn}

Table 20: Decision matrix of priority of preferences

The last step of AHP method is to rank the preferences using the following formula —

$$\sum_{j=1}^{n} w_j p_{ij}.$$

	C_1	C_2	$\cdots$	C_n	R_i
w_i	w_1	w_2	$\cdots$	w_n	
A_1	p_{11}	p_{12}	$\cdots$	p_{1n}	$\sum_{j=1}^{n} w_j p_{1j}$
A_2	p_{21}	p_{22}	$\cdots$	p_{2n}	$\sum_{j=1}^{n} w_j p_{2j}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A_m	p_{m1}	p_{m2}	$\cdots$	p_{mn}	$\sum_{j=1}^{n} w_j p_{mj}$

Table 21: Ranking decision matrix

7.1 Numerical examples

Let us consider the three numerical examples (*Section 6*) and the decision matrix mentioned above.

For weak inconsistent and strong inconsistent examples, AHP does not work, as proved above, consequently the AHP method will be applied on consistent example 1.

The weights of criteria are calculated by using AHP (consistent example 1, *Section 6*).

As mentioned above (case 2, *Section 3*), for applying AHP we need to construct three pairwise comparisons matrices of size $4*4$ each.

C_1	A_1	A_2	A_3	A_4
A_1	1	7/8	7/9	7/6
A_2	8/7	1	8/9	8/6
A_3	9/7	7/8	1	9/6
A_4	6/7	7/8	6/9	1

C_2	A_1	A_2	A_3	A_4
A_1	1	9/7	9/6	9/7
A_2	7/9	1	7/6	1
A_3	6/9	6/7	1	6/7
A_4	7/9	1	7/6	1

C_3	A_1	A_2	A_3	A_4
A_1	1	9/8	9/8	9/8
A_2	8/9	1	1	1
A_3	8/9	1	1	1
A_4	8/9	1	1	1

Table 22: Relative importance comparison matrix of alternatives for each criteria

We apply AHP method on three pairwise comparison matrices to rank the preferences, and calculate priority (i.e., normalized eigenvector), consistency index (CI) and consistency ratio (CR) for each matrix.

The eigenvalue, consistency index (CI) and consistency ratio (CR) for each matrix are: C1 — (CR =0, CI=0, $\lambda = 4$), C2 — (CR =0, CI=0, $\lambda = 4$) and C3 — (CR =0, CI=0, $\lambda = 4$).

The priority vectors for three matrices are listed respectively in *Table 23*.

C_1	A_1	A_2	A_3	A_4	*Priority vector*
A_1	1	7/8	7/9	7/6	0.2333
A_2	8/7	1	8/9	8/6	0.2667
A_3	9/7	7/8	1	9/6	0.3000
A_4	6/7	7/8	6/9	1	0.2000
C_2	A_1	A_2	A_3	A_4	*Priority vector*
A_1	1	9/7	9/6	9/7	0.3103

A_2	*7/9*	*1*	*7/6*	*1*	*0.2414*
A_3	*6/9*	*6/7*	*1*	*6/7*	*0.2069*
A_4	*7/9*	*1*	*7/6*	*1*	*0.2414*

C_3	A_1	A_2	A_3	A_4	*Priority vector*
A_1	*1*	*9/8*	*9/8*	*9/8*	*0.2727*
A_2	*8/9*	*1*	*1*	*1*	*0.2424*
A_3	*8/9*	*1*	*1*	*1*	*0.2424*
A_4	*8/9*	*1*	*1*	*1*	*0.2424*

Table 23: Relative importance comparison matrix of alternatives for each criteria

The last step of AHP method is applying it to rank the preferences using the following formula $\sum_{j=1}^{n} w_j p_{ij}$ (resulted as listed in the last column of the matrix above).

	C_1	C_2	C_3	$\sum_{j=1}^{n} w_j p_{ij}$	*Rank*
w_i	*12/16*	*3/16*	*1/16*		
A_1	*0.2333*	*0.3103*	*0.2727*	*0.2502*	*3*
A_2	*0.2667*	*0.2414*	*0.2424*	*0.2604*	*2*
A_3	*0.3000*	*0.2069*	*0.2424*	*0.2789*	*1*
A_4	*0.2000*	*0.2414*	*0.2424*	*0.2104*	*4*

Table 24: Decision matrix of priority of preferences and its ranking

8 Extended α-D MCDM

α-D MCDM introduced by Smarandache is not designed to rank preferences, but only for generating the weights for preferences or criteria, based on their n-wise matrix comparison. Hence, we proposed above a new method, called α-D MCDM-TOPSIS, employed to calculate the criteria weights for pairwise comparison matrices and for n-wise comparison matrices of criteria, in which α-D MCDM is used for calculate criteria weights, and TOPSIS — to rank the preferences.

In this section, we do not focus on criteria weights problem of α-D MCDM, as discussed above and calculated for the three examples, but we propose an extension of α-D MCDM benefiting the skills of α-D MCDM to calculate maximum eigenvalue and its associate eigenvector of n-wise comparison matrix, in order to apply it again on α-wise matrices of preferences.

An extended α-D MCDM can be described as it follows:

Let us consider the second case of data structure decision problem (*Section 3*) — for each criterion C_i corresponds a pairwise (or n-wise) comparison matrix of preferences and criteria.

Step 8.1 *We use α-D MCDM to calculate the weight (w_i) of the criteria*

Step 8.2 *We apply again α-D MCDM method* for each comparison matrices of preferences B_k to determine the maximum eigenvalue and its associate eigenvector (the same that is used to determine the weights of criteria).

For each matrix B_k (associated to C_k), we calculate the λ_{max} and priority vector (eigenvector).

We get —

C_i	A_1	A_2	$\cdots$	A_n	Priority vector
A_1	a_{11}	a_{12}	$\cdots$	a_{1m}	p_{1i}
A_2	a_{21}	a_{22}	$\cdots$	a_{2m}	p_{2i}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A_m	a_{m1}	a_{m2}	$\cdots$	a_{mm}	p_{mi}

Table 25: Comparison matrix of the preferences with priority vector in latest column

Step 8.3 *We get a decision matrix* (different from the decision matrix above), formed using priority vectors, in which the entries of the decision matrix are p_{ij}, and not a_{ij}.

	C_1	C_2	$\cdots$	C_n
w_i	w_1	w_2	$\cdots$	w_n
A_1	p_{11}	p_{12}	$\cdots$	p_{1n}
A_2	p_{21}	p_{22}	$\cdots$	p_{2n}
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A_m	p_{m1}	p_{m2}	$\cdots$	p_{mn}

Table 26: Decision matrix of priority of preferences

Step 8.4 *We employ the following formula (simple additive weighting) to rank the preferences —*

$$\sum_{j=1}^{n} w_j p_{ij}.$$

	C_1	C_2	$\cdots$	C_n	R_i
w_i	w_1	w_2	$\cdots$	w_n	
A_1	p_{11}	p_{12}	$\cdots$	p_{1n}	$\sum_{j=1}^{n} w_j p_{1j}$
A_2	p_{21}	p_{22}	$\cdots$	p_{2n}	$\sum_{j=1}^{n} w_j p_{2j}$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
A_m	p_{m1}	p_{m2}	$\cdots$	p_{mn}	$\sum_{j=1}^{n} w_j p_{mj}$

Table 27: Ranking decision matrix

8.1 Numerical examples

Let us consider the three examples mentioned above (numerical consistent, weak inconsistent and strong inconsistent examples (*Section 6*).

We do not repeat the calculation of weights criteria by using α-D MCDM, because it was already done in *Section 6*, and we got the following priority vectors of criteria:

Consistent example 1 —

$$W = \begin{bmatrix} \frac{12}{16} & \frac{3}{16} & \frac{1}{16} \end{bmatrix}.$$

Weak inconsistent example 2 —

$$W = \begin{bmatrix} 0.62923 & 0.22246 & 0.14831 \end{bmatrix}.$$

Strong inconsistent example 3 —

$$W = \begin{bmatrix} \frac{1}{6643} & \frac{81}{6643} & \frac{6561}{6643} \end{bmatrix}.$$

We construct three comparisons matrices of size $4 * 4$ each (or three linear homogeneous systems), based on decision matrix of *Section 6*, and apply extended α-D MCDM to the three examples.

Let

$$m(A_1) = x \text{ , } m(A_2) = y \text{ , } m(A_3) = z \text{ and } m(A_4) = t.$$

For criteria C_1 —

- A_2 is 8 seventh as important as A_1 ,

- A_3 is 9 seventh as important as A_1 ,
- A_4 is 6 seventh as important as A_1 .

The linear homogeneous system associated is —

$$\begin{cases} y \quad \frac{8}{7}x \\ z \quad \frac{9}{7}x \\ t \quad \frac{6}{7}x \\ \end{cases}$$

and its general solution is —

$$W = \begin{bmatrix} \frac{7}{30} & \frac{8}{30} & \frac{9}{30} & \frac{6}{30} \end{bmatrix}.$$

For criteria C_2 —

- A_1 is 9 sixth as important as A_3 ,
- A_3 is 7 sixth as important as A_3 ,
- A_4 is 7 sixth as important as A_3 .

The linear homogeneous system associated is —

$$\begin{cases} x \quad \frac{9}{6}z \\ y \quad \frac{7}{6}z \\ t \quad \frac{7}{6}z \\ \end{cases}$$

and its general solution is —

$$W = \begin{bmatrix} \frac{9}{29} & \frac{7}{29} & \frac{6}{29} & \frac{7}{29} \end{bmatrix}.$$

For criteria C_3 —

- A_1 is 9 eighth as important as A_2 ,
- A_2 , A_3 and A_4 have the same importance.

The associated linear homogeneous system is —

$$\begin{cases} x \quad \frac{9}{8}y \\ y \quad z \\ t \quad z \end{cases}$$

and its general solution is —

$$W = \begin{bmatrix} \frac{9}{33} & \frac{8}{33} & \frac{8}{33} & \frac{8}{33} \end{bmatrix}.$$

The results of extended α-D MCDM are summarized in the *Table 24*:

Consistent example 1 —

	C_1	C_2	C_3	R_i	*Rank*
w_i	*12/16*	*3/16*	*1/16*		
A_1	*7/30*	*9/29*	*9/33*	*0.2502*	*3*
A_2	*8/30*	*7/29*	*8/33*	*0.2604*	*2*
A_3	*9/30*	*6/29*	*8/33*	*0.2789*	*1*
A_4	*6/30*	*7/29*	*8/33*	*0.2104*	*4*

Weak inconsistent example 2 —

	C_1	C_2	C_3	R_i	*Rank*
w_i	*0.62923*	*0.22246*	*0.14831*		
A_1	*7/30*	*9/29*	*9/33*	*0.2563*	*3*
A_2	*8/30*	*7/29*	*8/33*	*0.2574*	*2*
A_3	*9/30*	*6/29*	*8/33*	*0.2707*	*1*
A_4	*6/30*	*7/29*	*8/33*	*0.2155*	*4*

Strong inconsistent example 3 —

	C_1	C_2	C_3	R_i	*Rank*
w_i	*1/6643*	*81/6643*	*6561/6643*		
A_1	*7/30*	*9/29*	*9/33*	*0.27318*	*1*
A_2	*8/30*	*7/29*	*8/33*	*0.24242*	*2*
A_3	*9/30*	*6/29*	*8/33*	*0.24200*	*4*
A_4	*6/30*	*7/29*	*8/33*	*0.24241*	*3*

Table 24: Decision matrix of priority of preferences and its ranking using Extended -DMCDM

Example	Alternative	AHP-TOPSIS		α-DMCDM-TOPSIS		AHP		Extended α-DMCDM	
Consistent example 1	A_1	*0.3880*	*3*	*0.3880*	*3*	*0.2502*	*3*	*0.2502*	*3*
	A_2	*0.6412*	*2*	*0.6412*	*2*	*0.2604*	*2*	*0.2604*	*2*
	A_3	*0.7938*	*1*	*0.7938*	*1*	*0.2789*	*1*	*0.2789*	*1*
	A_4	*0.0783*	*4*	*0.0783*	*4*	*0.2104*	*4*	*0.2104*	*4*
Weak Inconsistent Example 2	A_1	*Does not works*		*0.3880*	*3*	*Does not works*		*0.2563*	*3*
	A_2			*0.6412*	*2*			*0.2574*	*2*
	A_3			*0.7938*	*1*			*0.2707*	*1*
	A_4			*0.0783*	*4*			*0.2155*	*4*
Strong Inconsistent Example 3	A_1	*Does not works*		*0.999668*	*1*	*Does not works*		*0.27318*	*1*
	A_2			*0.013719*	*2*			*0.24242*	*2*
	A_3			*0.000497*	*4*			*0.24200*	*4*
	A_4			*0.013715*	*3*			*0.24241*	*3*

Table 25: Summary of the results of three examples of all methods

For the three examples presented in this paper, the *Table 25*, summarizing all results of all methods, illustrates that the AHP and AHP-TOPSIS methods work just for the first example, in which criteria and alternatives are consistent in their pairwise comparisons. Our proposed methods — Extended α-D MCDM and α-D MCDM-TOPSIS — work not only for consistent example 1, giving the same results as AHP and AHP-TOPSIS methods, but also for weak inconsistent and strong inconsistent examples.

In the example 1, it is recorded that the alternative A3 has the first rank with the value (0.2789), the alternative A2 gets second rank with the coefficient value (0.2604), the alternative A1 realizes following rank with value (0.2502) and the alternative A4 lowers rank with the coefficient (0.2104), by using the AHP and our proposed method Extended α-DMCDM. Results indicate that all considered alternatives have near score values, for example 0.0685 ((A3)0.2789 - (A4) 0.2104). As a difference between the first and the latest ranking alternative, it is not sufficient to make a founded decision making, hence that can have a strong impact in practice to choose the best alternatives.

The results claimed that AHP-TOPSIS and our α-D MCDM-TOPSIS methods preserves the ranking order of the alternatives and overcome the near score values problem. By using AHP-TOPSIS and our α-D MCDM-TOPSIS methods, the score value of A3 was changed from 0.2789 to 0.7938, the score value of A2 was changed from 0.2604 to 0.6412, and the score value of A4 was changed from 0.2104 to 0.0783.

The bigger differences between the score values of alternatives 0.7155 ((A3) 0.7938 - (A4) 0.0783) is also subject to gain additional insights.

In the two last examples (weak inconsistent and strong inconsistent), one sees that the importance of discounting in our approaches suggest that they can be used to solve real-life problems in which criteria are not only pairwise, but *n*-wise comparisons, and the problems are not perfectly consistent. It is however worth to note that the ranking order of the four alternatives obtained by both methods is similar, but score values are slightly different. Both Extended α-D MCDM and α-D MCDM-TOPSIS methods allow taking into consideration any numbers of alternatives and any weights of criteria.

9 Conclusions

We have proposed two multi-criteria decision making methods, Extended α-D MCDM and α-D MCDM-TOPSIS models that allow to work for consistent and inconsistent MCDM problems. In addition, three examples have demonstrated that the α-D MCDM-TOPSIS model is efficient and robust.

Our approaches, Extended α-D MCDM and α-D MCDM-TOPSIS, give the same result as AHP-TOPSIS and AHP in consistent MCDM problems and elements of decision matrix are pairwise comparisons, but for weak inconsistent and strong inconsistent MCDM problems in which AHP and AHP-TOPSIS are limited and unable, our proposed methods — Extended α-D MCDM and α-D MCDM-TOPSIS — give justifiable results.

Furthermore, our proposed approaches — α-D MCDM-TOPSIS and Extended α-D MCDM — can be used to solve real-life problems in which criteria are not only pairwise, but *n*-wise comparisons, and the problems are not perfectly consistent.

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The Use of The Pivot Pairwise Relative Criteria Importance Assessment Method for Determining the Weights of Criteria

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Abstract

The weights of evaluation criteria could have a significant impact on the results obtained by applying multiple criteria decision-making methods. Therefore, the two extensions of the SWARA method that can be used in cases when it is not easy, or even is impossible to reach a consensus on the expected importance of the evaluation criteria are proposed in this paper. The primary objective of the proposed extensions is to provide an understandable and easy-to-use approach to the collecting of respondents' real attitudes towards the significance of evaluation criteria and to also provide an approach to the checking of the reliability of the data collected.

Keyword: PIvot Pairwise RElative Criteria Importance Assessment, PIPRECIA, PIPRECIA-E, SWARA, criteria weights, group decision-making,

1. Introduction

Multiple Criteria Decision Making (MCDM) is becoming one of the most important and fastest-growing areas of operational research. Therefore, the result of rapid development has led to the creation of many MCDM methods (Mardani et al. 2015, Zavadskas et al. 2014, Stanujkic et al. 2013), such as: SAW, AHP, TOPSIS, PROMETHEE, ELECTRE, LINMAP, COPRAS, VIKOR, ARAS, MOORA, MULTIMOORA, WASPAS, SWARA, EDAS, FARE and so on. The overview of the previously mentioned methods, as well as of their application, is considered by Zavadskas et al. (2015), Gul et al. (2016), Stefano et al. (2015), Govindan and Jepsen (2016), Behzadian et al. (2010), Baležentis T. and Baležentis A. (2013).

The above-mentioned methods have successfully been applied over time for solving numerous multicriteria problems (Mardani et al. 2017, Zare et al. 2016, Kou et al. 2016, Zhou et al. 2017, Stanujkic 2016, Turskis and Juodagalviene 2016, Kaklauskas 2016, Keshavarz Ghorabaee 2016). The increasing use of multiple criteria decision-making methods indicates a good approach to problems requiring optimal decision-making and the adoption of sustainable solutions (Ignatius et al. 2016, Ou 2016).

The significance of evaluation criteria, more often called the weights of criteria or shorter – weights, can have a significant influence on the ranking results obtained by using MCDM methods. Therefore, numerous authors have proposed a variety of procedures for determining the weights of criteria (Ma, 1999), such as the LINMAP technique (Ma et al., 1999.) pairwise comparisons taken from the AHP method (Saaty 1977, 1980), the Entropy approach (Hwang and Yoon 1981), the Delphi method (Hwang and Lin 1987), CILOS, IDOCRIW (Zavadskas and Podvezko 2016), and so on.

The new Step-Wise Weight Assessment Ratio Analysis (SWARA), proposed by Kersuliene et al. (2010), can also be used for determining the weights of criteria, as well as for solving various MCDM problems.

Despite the fact that the SWARA method can be mentioned as a newly-proposed MCDM method, the same has successfully been used to solve many decision-making problems, such as those shown in Table 1.

In these decision-making problems, the SWARA method has mainly been used in combination with other MCDM methods, whereby it was used to determine the weights of criteria.
In comparison with the other methods intended to determine the weights of criteria, the SWARA method is less complex to use from the standpoint of the questioned persons [44], which has certainly had an effect on its applicability. For example, in comparison to commonly used pairwise comparisons taken from the AHP method, the SWARA method requires significantly fewer comparisons. In addition, the computational procedures of the SWARA method are also less complex compared to the computational procedures used in similar methods.

For the reason of the above said, a rise in the use of SWARA methods for solving MCDM problems can be expected to continue in the future.

Table 1

The Usage of the SWARA Method

Decision-making problems	**Authors**
Rational dispute resolution	Kersuliene et al. (2010)
An architect selection	Kersuliene and Turskis (2011)
A shopping mall location selection	Hashemkhani Zolfani et al. (2013)
A machine tool selection	Aghdaie et al. (2013a)
Market segmentation and selection	Aghdaie et al. (2013b)
A supplier selection	Alimardani et al. (2013) Yazdani et al. (2016a)
Sustainability assessment	Hashemkhani Zolfani and Saparauskas (2013)

Decision-making problems	Authors
Investment prioritizing	Hashemkhani Zolfani and Bahrami (2014)
The evaluation of an external wall insulation	Ruzgys et al. (2014)
House locating	Haghnazar Kouchaksaraei et al. (2015) Hashemkhani Zolfani and Banihashemi (2014)
Personnel selection	Karabasevic et al. (2015b, 2015b, 2016) Urosevic et al. (2017) Karabašević et al. (2016)
The packaging design selection	Stanujkic et al. (2015)
Assessment of landslide hazard areas	Dehnavi et al. (2015)
The ranking of companies	Karabasevic et al. (2015a)
Evaluation of light supply in the public underground safe spaces	Nakhaei et al. (2016a)
Materials selection	Yazdani et al. (2016b)
Vulnerability of office buildings	Nakhaei et al. (2016b)
ERP system selection	Shukla et al. (2016)
Evaluation of third-party reverse logistic provider	Mavi et al. (2017)

Additionally, in some cases, the precise determination of a weight of criteria could require the participation of a large number of respondents. Therefore, in this manuscript, an approach that should facilitate the usage of the SWARA method in a group environment, especially when used in order to obtain attitudes from the largest number of respondents and/or when a list of the criteria sorted on the basis of their expected importance is difficult to form, is considered.

For that reason, the remaining part of the paper is organized as follows: In Section 2, the computational procedure of the SWARA method is presented and in Section 3, an innovative approach to the determination of the weights of criteria based on the SWARA method is proposed. In Section 4, an approach to checking the reliability of data collected is considered. These are followed by Section 5, in which the innovative approach proposed in Section 3 is further extended in order to enable the checking of the reliability of the data collected. In Section 6, two numerical illustrations are the subject of consideration with the aim of presenting the usability of the proposed extensions. Finally, the conclusions are given.

2. The Computational Procedure of the SWARA Method

Based on Stanujkic et al. (2015) and Kersuliene et al. (2010), the computational procedure of the ordinary SWARA method can accurately be shown through the following steps:

Step 1. Determine the set of the relevant evaluation criteria and sort them in descending order, based on their expected significances.

Step 2. Starting from the second criterion, determine the relative importance sj of the criterion j in relation to the previous j-1 criterion, and do that for each particular criterion.

Step 3. Determine the coefficient kj as follows:

$$k_j = \begin{cases} 1 & j = 1 \\ s_j + 1 & j > 1 \end{cases} . \quad (1)$$

Step 4. Determine the recalculated weight qj as follows:

$$q_j = \begin{cases} 1 & j = 1 \\ \frac{q_{j-1}}{k_j} & j > 1 \end{cases} . \quad (2)$$

Step 5. Determine the relative weights of the evaluation criteria as follows:

$$w_j = \frac{q_j}{\sum_{k=1}^{n} q_k} , \quad (3)$$

where: wj denotes the relative weight of the criterion j.

In cases when the SWARA method is used for determining the weights of criteria, it is very important, or precisely said – necessary, that a consensus on the expected significances of the evaluation criteria should be ensured.

3. An Extension of the SWARA Method Adapted for Group Decision-making

A real difficulty that can occur when the SWARA method is used for solving real-world decision-making problems in a group environment is the forming of the list of the evaluation criteria sorted according to their expected importance.

In such cases, the respondents involved in an evaluation may have different attitudes towards the importance of the evaluation criteria. Conducting an interview with accidental passers-by with the aim of determining the factors that are relevant to the selection of something, which usually includes various categories of respondents, can be specified as one of such typical cases.

Therefore, an extended approach based on the use of the SWARA method, which is able to cope with the previously unsorted list of evaluation criteria, is considered in this section. For the purpose of application of the same, it is necessary to implement the following modifications in the ordinary SWARA computational procedure:

As the first and crucial modification of the ordinary SWARA approach, the way of allocating the values of sj should be mentioned, and they should be assigned as follows:

$$s_j = \begin{cases} >1 & when\, C_j \succ C_{j-1} \\ 1 & when\, C_j = C_{j-1} \\ <1 & when\, C_j \prec C_{j-1} \end{cases} . \quad (4)$$

The previously proposed modification also has an impact on the determining of the value of kj, which should be determined in the following manner:

$$k_j = \begin{cases} 1 & j = 1 \\ 2 - s_j & j > 1 \end{cases} . \quad (5)$$

In the proposed approach, the remaining part of the computational procedure remains the same as in the ordinary SWARA method, i.e. Eqs (2) and (3) remain unchanged.

3.1 An Innovative Approach to Determining the Weights of Evaluation Criteria

Taking into account the previously proposed modifications, the computational procedure of the innovative PIvot Pairwise RElative Criteria Importance Assessment (PIPRECIA) approach can be accurately presented as follows:

Step 1. Determine the set of the relevant evaluation criteria and sort them in descending order, based on their expected significances.

Step 2. Starting from the second criterion, determine the relative importance sj as follows:

$$s_j = \begin{cases} >1 & when\, C_j \succ C_{j-1} \\ 1 & when\, C_j = C_{j-1} \\ <1 & when\, C_j \prec C_{j-1} \end{cases}. \tag{6}$$

Step 3. Determine the coefficient kj as follows:

$$k_j = \begin{cases} 1 & j = 1 \\ 2 - s_j & j > 1 \end{cases}. \tag{7}$$

Step 4. Determine the recalculated weight qj as follows:

$$q_j = \begin{cases} 1 & j = 1 \\ \dfrac{q_{j-1}}{k_j} & j > 1 \end{cases}. \tag{8}$$

Step 5. Determine the relative weights of the evaluation criteria as follows:

$$w_j = \frac{q_j}{\sum_{k=1}^{n} q_k}. \tag{9}$$

3.2 The Use of the PIPRECIA Method in a Group Decision-making Environment

The use of the PIPRECIA method in a group decision-making environment can accurately be demonstrated through the following steps:

Step 1. Form or define a group of the respondents who will participate in the evaluation. Depending on the objectives, the number of the participants that should be included in the research study could vary from a small number of experts and/or decision-makers to a larger number of ordinary respondents.

Step 2. Determine the set of the relevant evaluation criteria and sort them in descending order, based on their expected significances.

This step is similar to that of the ordinary SWARA method when research includes a small number of experts and/or decision-makers. However, a consensus on the expected significances of the evaluation criteria is not so easy to reach when a research study involves a larger number of respondents. In such cases, the proposed PIPRECIA method could have some advantages because it does not require the use of the sorted list of the evaluation criteria.

Step 3. Determine the relative importance of the evaluation criteria, and do so for each respondent. In this steep, each respondent expresses his/her attitudes towards the relative importance of the criteria by using Eq. (6), as follows:

$$s_j^r = \begin{cases} >1 & when\ C_j \succ C_{j-1} \\ 1 & when\ C_j = C_{j-1} \\ <1 & when\ C_j \prec C_{j-1} \end{cases} \quad (10)$$

where: s_j^r denotes the relative importance of the criterion j obtained from the respondent r.

Step 4. Determine the relative weights of the evaluation criteria, and do so for each respondent. The relative weights should be calculated by using Eqs (7), (8) and (9), for each respondent, in the following manner:

$$k_j^r = \begin{cases} 1 & j=1 \\ 2-s_j^r & j>1 \end{cases}, \quad (11)$$

$$q_j^r = \begin{cases} 1 & j=1 \\ \dfrac{q_{j-1}^r}{k_j^r} & j>1 \end{cases}, \quad (12)$$

$$w_j^r = \frac{q_j^r}{\sum_{j=1}^{n} q_j^r}, \quad (13)$$

where: k_j^r, q_j^r and w_j^r.denote the coefficient, the recalculated weight and the weight of the criterion j, respectively, determined on the basis of the respondent r.

Step 5. Determine the group relative weights of the evaluation criteria.

As a result of the evaluation that includes the R respondents, the R different weights could be obtained for each criterion, and the weights obtained in that manner could be used in accordance with the computational procedure of the MCDM method chosen for the further evaluation and selection of the most suitable alternative.

The weights so obtained can also be used for forming different types of group weights, such as fuzzy, grey or crisp group weights of criteria. Finally, as one of the simplest transformations of an individual weight of criteria to a group one could be done as follows:

$$w_j^* = \left(\prod_{r=1}^{R} w_j^r \right)^{1/R}, \quad (14)$$

$$w_j = \frac{w_j^*}{\sum_{j=1}^{n} w_j^*}, \quad (15)$$

where: w_j^* denotes the geometric mean of the weights of the criterion j obtained by surveying the respondents and R denotes the number of the respondents.

4. Checking the Reliability of Data Collected

A lack of a built-in mechanism for the calculation of the consistency of the performed pairwise comparisons can be identified as a weakness of the SWARA method, as well as of the PIPRECIA method, compared to the commonly used AHP method.

Checking the reliability of the data obtained from the respondents can be performed in two diverse ways, namely:

by using correlation techniques, such as: Pearson's correlation, Kendall's rank correlation or Spearman's correlation; and

by using the extended PIPRECIA method, i.e. by applying the bidirectional procedure for the determination of the weights of criteria.

4.1. Checking Reliability by Using Spearman's Rank Correlation Coefficient

In this approach, Spearman's rank correlation, proposed by Spearman (1904, 1906) and Kendall (1948), is chosen for the purpose of determining the reality of the data collected.

Spearman's rank correlation coefficient between two independent data series is as follows:

$$\rho = 1 - \frac{6\sum_{i=1}^{n} d_i^2}{n\,(n^2 - 1)} \tag{16}$$

where: ρ denotes the correlation the coefficient, di denotes the distance between the ranks for each xi, yi stands for the data pairs, x and y denote the data series, whereas n is the number of the elements in each data series, and $\rho \in [-1, 1]$.

The correlation is calculated between the weights obtained on the basis of the attitudes of the respondents involved in the evaluation and the weights are determined by using Eq. (14).

The greater value of ρr indicates higher correlation between the weights obtained from the respondent r and those obtained on the basis of the attitudes of all the respondents. The least allowable value for ρ can be defined depending on the objectives of the research study and so the surveys of the respondents whose attitudes differ significantly from those of the group can be identified.

5. The Extended PIPRECIA Method

The extended PIPRECIA (PIPRECIA-E) method is based on the use of a bidirectional approach to the determination of the weights of criteria, and it comprises the following stages:

Phase 1. Determining the weights of the criteria by using the PIPRECIA method.

Phase 2. Determining the weights of the criteria by using the inverse PIPRECIA method, wherein the assigning of the relative importance of the criteria is performed in the inverse direction, i.e. starting from the second least significant to the most important criterion.

Phase 3. Checking the reliability of the data obtained from the respondents.

Phase 4. Determining the resulting weight of the criteria.

The first phase of the PIPRECIA-E is already explained in subsection 3.1. The remaining phases of the PIPRECIA-E are explained in details in the subsections 5.1 to the 5.3, while its usage in a group environment is considered in the subsection section 5.4.

5.1. The Inverse PIPRECIA Method

In order to provide inverse pairwise comparisons, starting from the second least significant to the most significant criterion, the following modifications should be made in computational procedure of the PIPRECIA method: instead of the relative importance sj used in the inverse PIPRECIA method, the inverse relative importance of the criteria s'_j should be used, and the same should be determined as follows:

$$s'_j = \begin{cases} >1 & when\ C_j \succ C_{j+1} \\ 1 & when\ C_j = C_{j+1} \\ <1 & when\ C_j \prec C_{j+1} \end{cases} . \quad (17)$$

This modification also has an impact on the process of calculating the values of the coefficient kj and the recalculated weight qj. Therefore, the inverse coefficient k'_j and the inverse recalculated weight q'_j should be calculated as follows:

$$k'_j = \begin{cases} 1 & j = n \\ 2 - s'_j & j < n \end{cases} \text{, and} \quad (18)$$

$$q'_j = \begin{cases} 1 & j = n \\ \dfrac{q'_{j+1}}{k'_j} & j < n \end{cases} . \quad (19)$$

Taking into consideration the aforementioned adaptations, the computational procedure of the inverse PIPRECIA approach can accurately be presented as follows:

Step 1. Starting from the second least significant criterion, determine the inverse relative importance s'_j as follows:

$$s'_j = \begin{cases} >1 & when\ C_j \succ C_{j+1} \\ 1 & when\ C_j = C_{j+1} \\ <1 & when\ C_j \prec C_{j+1} \end{cases} . \quad (20)$$

Step 2. Determine the inverse coefficient k'_j as follows:

$$k'_j = \begin{cases} 1 & j = n \\ 2 - s'_j & j < n \end{cases} . \quad (21)$$

Step 3. Determine the inverse recalculated weight q'_j as follows:

$$q'_j = \begin{cases} 1 & j = n \\ \dfrac{q'_{j+1}}{k'_j} & j < n \end{cases} . \quad (22)$$

Step 4. Determine the inverse relative weights of the evaluation criteria as follows:

$$w'_j = \frac{q'_j}{\sum_{k=1}^{n} q'_k}, \tag{23}$$

where: w'_j denotes the inverse weight of the criterion j.

5.2. Checking the Reliability of the Data Obtained from the Respondents

In the PIPRECIA-E method, the checking of the reliability of the data collected is not mandatory and this phase could be omitted.

Unlike the checking procedure proposed for being used with the PIPRECIA method, in the PIPRECIA-E method the reliability of the data collected is determined on the basis of the data obtained from the same respondent by applying the PIPRECIA and the inverse PIPRECIA methods.

Checking the reliability of the data collected can be performed by using the Spearman, or any other, correlation coefficient.

5.3. Determining the Resulting Weight of Criteria.

The weights w″ of criteria in PIPRECIA-E should be calculated in the following manner:

$$w''_j = \frac{1}{2}(w_j + w'_j). \tag{23}$$

where: w_j, w'_j and w''_j denote the weights of the criterion j, obtained by applying the PIPRECIA, inverse PIPRECIA and PIPRECIA-E methods, respectively.

5.4. The Use of PIPRECIA-E for Obtaining Respondents' Attitudes

The PIPRECIA-E method is proposed with the aim of providing a simple and effective approach to obtaining the real attitudes of the respondents that are not prepared for being interviewed. By using the PIPRECIA-E method, as well as the procedure proposed for checking the reliability of the data obtained, all inadequate surveys can be discarded.

After that, group weights could be calculated as follows:

$$w^*_j = \left(\prod_{r=1}^{R} w''^r_j\right)^{1/R}, \tag{24}$$

$$w_j = \frac{w^*_j}{\sum_{j=1}^{n} w^*_j}. \tag{25}$$

where: wj denotes the group weight of the criterion j, w''^r_j denotes the weight of the criterion j obtained from the respondent r, and R denotes the number of the respondents.

6. Numerical Illustrations

In this section, the two numerical illustrations borrowed from two case studies are considered in order to explain the proposed approach and present its usability.

6.1. The First Numerical Illustration

In the first numerical illustration, some partial results adopted from the case study conducted for the purpose of selecting the most appropriate promoter are presented. For that purpose, the results obtained from the three Human Resource Managers (HRM) involved in the research study are discussed. In the referenced case study, the team consisted of the three Human Resource Managers (HRMs) made the evaluation and selection of the most appropriate promoter on the basis of the six criteria.

The weights of the evaluation criteria obtained from the first of the three HRMs, by using the PIPRECIA method, are accounted for in Table 2.

Table 2
The Responses Obtained from the First of the Three HRMs and the Weights of the Criteria

	Criteria	sj	kj	qj	wj
C1	Personality		1	1	0.19
C2	Self-confidence	1	1.00	1.00	0.19
C3	Communication and presentation skills	1	1.00	1.00	0.19
C4	Interview preparedness	0.75	1.25	0.80	0.15
C5	Education	1	1.00	0.80	0.15
C6	Past experience	0.8	1.20	0.67	0.13

In Table 3, the weights of the criteria obtained by using the ordinary SWARA method, also obtained on the basis of the responses obtained from the first of the three HRMs, are presented.

Table 3
The Responses and the Weights of the Criteria Obtained from the First HRM by Using the Ordinary SWARA Method

	Criteria	sj	kj	qj	wj
C1	Personality		1	1	0.19
C2	Self-confidence	0	1.00	1.00	0.19
C3	Communication and presentation skills	0	1.00	1.00	0.19
C4	Interview preparedness	0.25	1.25	0.80	0.15
C5	Education	0	1.00	0.80	0.15
C6	Past experience	0.2	1.20	0.67	0.13

As one may see from Tables 2 and 3, the obtained weights of the criteria are similar to each other, which confirms the validity of the proposed PIPRECIA method.

The advantages of the proposed approach can be more precisely identified on the basis of the response obtained from the responses given by the second and the third HRMs, as is shown in Tables 4 and 5.

Table 4

The Responses Obtained from the Second of the Three HRMs and the Weights of the Criteria

	Criteria	sj	kj	qj	wj
C1	Personality		1	1	0.19
C2	Self-confidence	1	1.00	1.00	0.19
C3	Communication and presentation skills	0.95	1.05	0.95	0.18
C4	Interview preparedness	0.85	1.15	0.83	0.16
C5	Education	0.85	1.15	0.72	0.14
C6	Past experience	1.1	0.90	0.80	0.15

Table 5

The Responses Obtained from the Third of the Three HRMs and the Weights of the Criteria

	Criteria	sj	kj	qj	wj
C1	Personality		1	1	0.18
C2	Self-confidence	1	1.00	1.00	0.18
C3	Communication and presentation skills	1.05	0.95	1.05	0.19
C4	Interview preparedness	0.9	1.10	0.96	0.17
C5	Education	0.7	1.30	0.74	0.13
C6	Past experience	1	1.00	0.74	0.13

As one may see from Table 4, the HRM2 used the advantages of the proposed approach when evaluating Criterion C6 by assigning greater significance to it in relation to Criterion C5. The mentioned advantage was also used by the HRM3 when Criterion C3 was evaluated.

Finally, the weights of the evaluation criteria obtained from the three HRMs are summarily shown in Table 6.

Table 6

The Weights of the Criteria Obtained from the Three HRMs

	Criteria	w_j^1	w_j^2	w_j^3	w_j^*	w_j
C1	Personality	0.190	0.189	0.182	0.187	0.187
C2	Self-confidence	0.190	0.189	0.182	0.187	0.187
C3	Communication and presentation skills	0.190	0.180	0.192	0.187	0.187
C4	Interview preparedness	0.152	0.156	0.175	0.161	0.161
C5	Education	0.152	0.136	0.134	0.140	0.141
C6	Past experience	0.127	0.151	0.134	0.137	0.137
	ρ	0.914	0.771	0.971		

The usability of the procedure proposed for checking the reliability of the data collected proposed in Subsection 3.1. is also presented in Table 6. The high values of Spearman's ρ confirm the reliability of the data collected.

6.2. The Second Numerical Illustration

In the second numerical illustration, some partial results adopted from the case study conducted for the purpose of the evaluation of traditional Serbian restaurants, known as kafana6, in the city of Zaječar, Serbia, are presented. On the basis of the consultations conducted with the local experts from within the field of tourism and hospitality, the following set of criteria was chosen for the purpose of making the evaluation:
C1, the interior of the building and friendly atmosphere,
C2, the courtesy and friendliness of the staff,
C3, the variety of traditional food and drinks,
C4, the quality and taste of food and drinks, including the manner of serving,
C5, the reasonable price for the quality of the services provided, and
C6, Other.

Out of about 50 persons interviewed through an e-mail questionnaire, positive feedback and the correctly completed questionnaires were received from 35 interviewed people.

The responses, the computational details and the weight of the criteria obtained from the first of the three randomly selected respondents are accounted for in Tables 7, 8 and 9.

Table 7

The Responses, the Computational Details and the Weight of the Criteria Obtained from the First of the Three Respondents, Computed by Applying the PIPRECIA Approach

Criterion	sj	kj	qj	wj
C1		1	1	0.13
C2	1.10	0.90	1.11	0.15
C3	1.20	0.80	1.39	0.19
C4	1.05	0.95	1.46	0.20
C5	0.95	1.05	1.39	0.19
C6	0.70	1.30	1.07	0.14

Table 8

The Responses, the Computational Details and the Weight of the Criteria Obtained from the First of the Three Respondents, Computed by Applying the Inverse PIPRECIA Approach

Criterion	sj	kj	qj	wj
C1	0.8	1.20	1.00	0.14
C2	0.9	1.10	1.20	0.17
C3	0.95	1.05	1.32	0.19
C4	1.2	0.80	1.39	0.20
C5	1.1	0.90	1.11	0.16
C6	0.8	1	1	0.14

6 The word kafana originates from the Turkish word kahvehane, which means a place for drinking coffee. Over time, under the influence of different cultures, kafana has obtained its own specificity in the Balkan Peninsula, also having been recognized as a place for food consumption and later for serving alcohol drinks. Nowadays, “kafanas” are still places in which people meet their friends, have celebrations, make conversations and discuss things, and so on.

The weights of the criteria obtained by using the PIPRECIA, the inverse PIPRECIA and finally the PIPRECIA-E approaches are presented in Table 9. The relatively high value of Spearman's ρ confirms the reliability of the data collected, i.e. it confirms that the first respondent gave a relatively consistent response.

The responses, the computational details and the weights obtained from the second and the third respondents are shown in Tables 10 and 11.

Table 9

The Responses, the Computational Details and the Weight of the Criteria Obtained from the First of the Three Respondents, Computed by Applying the PIPRECIA-E Approach

Criterion	w_j	Rank	w'_j	Rank	w''_j
C1	0.135	6	0.143	5	0.139
C2	0.150	4	0.171	3	0.160
C3	0.187	3	0.188	2	0.188
C4	0.197	1	0.198	1	0.197
C5	0.188	2	0.158	4	0.173
C6	0.144	5	0.142	6	0.143
				ρ	0.771

Table 10

The Responses and the Weight of the Criteria Obtained from the Second of the Three Respondents, Computed by Applying the PIPRECIA-E Approach

Criterion	s_j	w_j	Rank	s'_j	w'_j	Rank	w''_j
C1		0.11	6	0.65	0.13	6	0.12
C2	1.2	0.13	4	1.05	0.18	2	0.15
C3	0.9	0.12	5	0.80	0.17	4	0.14
C4	1.5	0.24	1	1.15	0.20	1	0.22
C5	0.9	0.22	2	1.10	0.17	3	0.19
C6	0.8	0.18	3		0.15	5	0.17
						ρ	0.714

Table 11

The Responses and the Weight of the Criteria Obtained from the Third of the Three Respondents, Computed by Applying the PIPRECIA-E Approach

Criterion	s_j	w_j	Rank	s'_j	w'_j	Rank	w''_j
C1		0.09	6	0.50	0.12	5	0.11
C2	1.4	0.15	4	1.10	0.19	3	0.17
C3	0.7	0.12	5	0.80	0.17	4	0.14
C4	1.5	0.24	1	1.00	0.20	1	0.22
C5	1	0.24	1	1.40	0.20	1	0.22
C6	0.6	0.17	3		0.12	6	0.14
						ρ	0.657

The relatively high values of Spearman's ρ also confirm the reliability of the data collected from the second and the third respondents.

Finally, the group weights calculated by using Eqs (24) and (25) are shown in Table 12.

Table 12

The Weights of the Criteria Obtained from the Three HRMs

	Criterion	w''^{1}_{j}	w''^{2}_{j}	w''^{3}_{j}	w^{*}_{j}	w_j
C1	The interior of the building and friendly atmosphere	0.14	0.12	0.11	0.12	0.12
C2	A courtesy and friendliness of staff	0.16	0.15	0.17	0.16	0.16
C3	The variety of traditional food and drinks	0.19	0.14	0.14	0.16	0.16
C4	The quality and taste of food in drinks, including the manner of serving	0.20	0.22	0.22	0.21	0.21
C5	Reasonable price for the quality of services provided	0.17	0.19	0.22	0.19	0.20
C6	Other	0.14	0.17	0.14	0.15	0.15
	ρ	0.771	0.714	0.657		

Conclusion

In this paper, the two extensions of the SWARA method formed with the aim of enabling an easier use of this particular method in solving group multiple criteria decision-making problems are proposed.

Compared with the ordinary SWARA method, the proposed extensions of the SWARA method do not require a consensus during the formation of the lists of the evaluation criteria sorted according to their expected importance. The proposed approaches provide greater flexibility compared to the SWARA method and also enable its much wider usage in solving different MCGDM problems.

The usability of the proposed extensions has been tested and proved based on the two case studies. In the first case study, a classical paper questionnaire was used, whereas in the second, an interactive questionnaire prepared in Excel was used. The second approach to examination also enabled the participants to see the calculated weights of the criteria, in both the numerical and the graphic forms, and allowed them to make certain changes if they were not satisfied with the results obtained.

In addition to that, the non-mandatory procedures for checking the reliability of data collected are also given in the proposed extensions, which can assign certain advantages to them in comparison with the SWARA method.

In the PIPRECIA method, the correlation coefficient is calculated relative to the geometric mean calculated on the basis of the responses obtained from the surveyed respondents. Such an approach can be specified as an efficient and easy-to-use approach. However, this manner of checking the reliability of the data collected could unreasonably exclude the responses of those respondents whose attitudes substantially differ from the attitudes expressed by the majority, especially when a high value of the correlation coefficient is required.

The computational procedure of the PIPRECIA-E method is more complex compared to the computational procedure of the PIPRECIA method and is based on the bidirectional approach to the determination of the weights of criteria.

Unlike the checking procedure proposed for being used with the PIPRECIA method, in PIPRECIA-E the reliability of the data collected is determined on the basis of the data collected from the same respondent by applying the PIPRECIA and the inverse PIPRECIA approaches. Therefore, the use of the PIPRECIA-E method could have some advantages when a non-homogeneous group of respondents is involved in a research study.

While doing so, we should always keep in mind the fact that the use of the inverse PIPRECIA method is not so simple for and understandable to all respondents. Therefore, higher values for the correlation coefficient should not be required when the PIRRECIA-E method is being applied.

Finally, the initial investigations conducted in order to verify the proposed extensions of the SWARA method, the PIPRECIA and PIPRECIA-E methods, confirm their usability and point to their advantages.

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Multi criteria decision making using correlation coefficient under rough neutrosophic environment

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Surapati Pramanik, Rumi Roy, Tapan Kumar Roy, Florentin Smarandache (2017). Multi criteria decision making using correlation coefficient under rough neutrosophic environment. *Neutrosophic Sets and Systems* 17, 29-36

Abstract In this paper, we define correlation coefficient measure between any two rough neutrosophic sets. We also prove some of its basic properties.. We develop a new multiple attribute group decision making method based on the proposed correlation coefficient measure. An illustrative example of medical diagnosis is solved to demonstrate the applicability and effecriveness of the proposed method.

Keywords: Multi-attribute group decision making; Neutrosophic set; Rough set; Rough neutrosophic set; Correlation coefficient.

1 Introduction

Smarandache established the concept of neutrosophic set and neutrosophic logic [1] to deal uncertainty, inconsistency, incompleteness and indeterminacy in 1998. Smarandache [1] and Wang et. al. [2] studied single valued neutrosophic set (SVNS), a subclass of neutrosophic set to deal realistic problems in 2010. SVNSs have been widely studied and applied in different fields such as medical diagnosis [3], multi criteria decision making [4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17], image processing [18, 19, 20], etc.

Pawlak [21] defined rough set to study intelligence systems characterized by inexact, uncertain or insufficient information. Broumi et al. [22, 23] defined rough neutrosophic set by combining the rough set and single valued neutrosophic set to deal with problems involving uncertain, imprecise, incomplete and inconsistent information existing in real world problems.

Decision making in rough neutrosophic environment is a new subfield of operational resesarch. In rough neutrosophic environment, Mondal and Pramanik [24] defined accumulated geometric operator to transform rough neutrosophic number (neutrosophic pair) to single valued neutrosophic number and developed a new multi-attribute decision-making (MADM) method based on grey relational analysis. Mondal and Pramanik [25] defined accuracy score function and proved its basic properties. In the same study, Mondal and Pramanik [25] presented a new MADM method in rough neutrosophic environment. Pramanik and Mondal [26] defined cotangent similarity measure of rough neutrosophic sets and proved its basic properties. In the same study, Pramanik and Mondal [26] presented its application to medical diagnosis. Pramanik and Mondal [27] proposed cosine similarity measure of rough neutrosophic sets and its application in medical diagnosis. Pramanik and Mondal [28] also proposed Dice and Jaccard similarity measures in rough neutrosophic environment and applied them for MADM. Mondal and Pramanik [29] studied cosine, Dice and Jaccard similarity measures for interval rough neutrosophic sets and presented MADM methods based on proposed rough cosine, Dice and Jaccard similarity measures in interval rough neutrosophic environment Mondal et al. [30] presented rough trigonometric Hamming similarity measures such as cosine, sine and cotangent rough similarity measures and proved their basic properties. In the same study, Mondal et al. [30] presented new MADM methods based on cosine, sine and cotangent rough similarity measures with illustrative example. Mondal et al. [31] proposed variational coefficient similarity measures under rough neutrosophic environment and proved some of their basic properties. In the same study, Mondal et al. [31] developed a new MADM method based on the proposed variational coefficient similarity measures and presented a comparison with four existing rough similarity measures namely, rough cosine similarity measure, rough dice similarity measure, rough cotangent similarity measure and rough Jaccard similarity measure for different values of the parameter λ . Mondal et al. [32] proposed rough neutrosophic aggregate operator and weighted rough neutrosophic aggregate operator to develop TOPSIS based MADM method in rough neutrosophic environment. Pramanik et al. [33] defined projection and bidirectional projection measures between rough neutrosophic sets. In the same study, Pramanik et al. [33] proposed two new multi criteria decision making (MCDM) methods based on neutrosophic projection and bidirectional projection measures respectively.

Mondal and Pramanik [34] proposed rough tri-complex similarity measure based MADM method in rough neutrosophic environment and proved some of its basic properties. In the same study, Mondal and Pramanik [34] presented comparison of obtained results for an illustrative MADM problem with other existing rough neutrosophic similarity measures.

Mondal et al. [35] defined rough neutrosophic hyper-complex set and rough neutrosophic hyper-complex cosine function and proved some of their basic properties. In the same study, Mondal et al. [35] also proposed rough neutrosophic hyper-complex similarity measure based MADM method.

Pramanik and Mondal [36] defined bipolar rough neutrosophic sets and proved it basic properties.

The correlation coefficient is an important tool to judge the relation between two objects. The correlation coefficients [37, 38, 39, 40, 41, 42] have been widely employed to data analysis and classification, decision making, pattern recognition, and so on. Many researchers pay attention to correlation coefficients under fuzzy environments. Chiang and Lin [43] introduced the correlation of fuzzy sets. Hong [44] proposed fuzzy measures for a correlation coefficient of fuzzy numbers under Tw (the weakest t-norm)-based fuzzy arithmetic operations. As an extension of fuzzy correlations, Wang and Li [45] introduced the correlation and information energy of interval-valued fuzzy numbers. Gerstenkorn and Manko [46] developed the correlation coefficients of intuitionistic fuzzy sets IFSs). Hung and Wu [47] also proposed a method to calculate the correlation coefficients of IFSs by centroid method. Xu [48] developed another correlation measure of interval-valued intuitionistic fuzzy environment, and applied it to medical diagnosis. Ye [49] studied the fuzzy decision-making method based on the weighted correlation coefficient under intuitionistic fuzzy environment. Bustince and Burillo [50] and Hong [51] further developed the correlation coefficients for interval-valued intuitionistic fuzzy sets (IVIFSs). Hanafy et al. [52] introduced the correlation of neutrosophic data. Ye [53] presented the correlation coefficient of SVNSs based on the extension of the correlation coefficient of IFSs and proved that the cosine similarity measure of SVNSs is a special case of the correlation coefficient of SVNSs. Hanafy et al. [54] presented the centroid-based correlation coefficient of neutrosophic sets and investigated its properties. Broumi and Smarandache [55] defined correlation coefficient of interval neutrosophic set and investigated its properties.

In the literature no studies have been reported on MADM using correlation coefficient under rough neutrosophic environment. To fill the research gap, we propose correlation coefficient under rough neutrosophic environment and proved some of its basic properties. We also present a new MADM method based on proposed measure. We also present an illustrative numerical example to show the effectiveness and applicability of the proposed method.

Rest of the paper is organized as follows: Section 2 describes preliminaries of neutrosophic sets, SVNSs and rough neutrosophic set (RNS). Section 3 describes the correlation coefficient between SVNSs. Section 4 presents definition and properties of proposed correlation coefficient between RNSs. Section 5 presents a rough neutrosophic decision making method based on correlation coefficient. Section 6 presents an illustrative hypothetical medical diagnostic problem based on the proposed MADM method. Finally, section 7 presents concluding remarks and future scope of research.

2 Preliminaries

2.1 Neutrosophic sets In 1998, Smarandache offered the following definition of neutrosophic set(NS)[1].

Definition 2.1.1 [1]

Let X be a space of points(objects) with generic element in X denoted by x. A NS A in X is characterized by a truth-membership function T_A, an indeterminacy membership function I_A and a falsity membership function F_A. The functions T_A, I_A and F_A are real standard or non-standard subsets of $]0^-,1^+[$ that is $T_A : X \to]0^-,1^+[, I_A : X \to]0^-,1^+[$ and $F_A : X \to]0^-,1^+[$. It should be noted that there is no restriction on the sum of T_A, I_A and F_A i.e $0^- \le T_A + I_A + F_A \le 3^+$.

Definition 2.1.2 [1]

(Complement) The complement of a neutrosophic set A is denoted by C(A) and is defined by $T_{c(A)}(x)=\{1^+\}-T_A(x)$, $I_{c(A)}(x)=\{1^+\}-I_A(x)$, $F_{c(A)}(x)=\{1^+\}-F_A(x)$.

Definition 2.1.3 [1]

A neutrosophic set A is contained in another neutrosophic set B, denoted by $A \subseteq B$ iff inf $T_A(x) \le$ inf $T_B(x)$, sup $T_A(x) \le$ sup $T_B(x)$, inf $I_A(x) \ge$ inf $I_B(x)$, sup $I_A(x) \ge$ inf $I_B(x)$, inf $F_A(x) \ge$ inf $F_B(x)$ and sup $F_A(x) \ge$ sup $F_B(x)$ for all x in X.

Definition 2.1.4 [2]

Let X be a universal space of points (objects) with a generic element of X denoted by x. A single valued neutrosophic set A is characterized by a truth membership function $T_A(x)$, a falsity membership function $F_A(x)$ and indeterminacy function $I_A(x)$ with $T_A(x)$, $I_A(x)$ and $F_A(x) \in [0,1]$ for all x in X.

When X is continuous, a SNVS A can be written as follows: A= $\int_X < T_A(x), I_A(x), F_A(x) > / x$ for all $x \in X$ and when X is discrete, a SVNS A can be written as follows : A= $\Sigma < T_A(x), I_A(x), F_A(x) > / x$ for all $x \in X$.

For a SVNS S, $0 \le \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \le 3$.

Definition 2.1.5 [2]

The complement of a single valued neutrosophic set A is denoted by c(A) and is defined by $T_{c(A)}(x) = F_A(x)$, $I_{c(A)}(x) = 1 - I_A(x)$, $F_{c(A)}(x) = T_A(x)$.

Definition 2.1.6 [2]

A SVNS A is contained in the other SVNS B, denoted as A $\subseteq$ B iff, $T_A(x) \le T_B(x)$, $I_A(x) \ge I_B(x)$, $F_A(x) \ge F_B(x)$ for all x in X.

2.2 Rough Neutrosophic sets

Rough neutrosophic sets [22, 23] are the generalization of rough fuzzy sets [56, 57, 58] and rough intuitionistic fuzzy sets [59].

Definition 2.2.1 [22]

Let Y be a non-null set and R be an equivalence relation on Y. Let P be a neutrosophic set in Y with the membership function T_P, indeterminacy function I_P and non-membership function F_P. The lower and the upper approximations of P in the approximation space (Y, R) are respectively defined as:

$$\underline{N(P)} = << x, T_{\underline{N(P)}}(x), I_{\underline{N(P)}}(x), F_{\underline{N(P)}}(x) > / y \in [x]_R, x \in \overline{Y} >$$

and

$$\overline{N(P)} = << x, T_{\overline{N(P)}}(x), I_{\overline{N(P)}}(x), F_{\overline{N(P)}}(x) > / y \in [x]_R, x \in Y >$$

where,

$$T_{\underline{N(P)}}(x) = \wedge z \in [x]_R T_P(Y), I_{\underline{N(P)}}(x) = \wedge z \in [x]_R I_P(Y), F_{\underline{N(P)}}(x) = \wedge z \in [x]_R F_P(Y)$$

and

$$T_{\overline{N(P)}}(x) = \vee z \in [x]_R T_P(Y), I_{\overline{N(P)}}(x) = \vee z \in [x]_R I_P(Y), F_{\overline{N(P)}}(x) = \vee z \in [x]_R F_P(Y).$$

So,

$0 \le T_{\underline{N(P)}}(x) + I_{\underline{N(P)}}(x) + F_{\underline{N(P)}}(x) \le 3$ and $0 \le T_{\overline{N(P)}}(x) + I_{\overline{N(P)}}(x) + F_{\overline{N(P)}}(x) \le 3$.

Here $\vee$ and $\wedge$ denote "max" and "min" operators respectively, $T_P(y)$, $I_P(y)$, and $F_P(y)$ are the degrees of membership, indeterminacy and non-membership of Y with respect to P.

Thus NS mapping, $\underline{N}, \overline{N} : N(Y) \to N(Y)$ are, respectively, referred to as the lower and upper rough NS approximation operators, and the pair $(\underline{N(P)}, \overline{N(P)})$ is called the rough neutrosophic set in (Y, R).

Definition 2.2.2 [22]

If $N(P) = (\underline{N}(P), \overline{N}(P))$ is a rough neutrosophic set in (Y, R), the rough complement of N(P) is the rough neutrosophic set denoted by $\sim(N(P)) = ((\underline{N}(P))^c, (\overline{N}(P))^c)$,where $(\underline{N}(P))^c$ and $(\overline{N}(P))^c$ are the complements of neutrosophic sets $\underline{N}(P)$ and $\overline{N}(P)$ respectively.

3 Correlation coefficient of SVNSs

Based on the correlation of intuitionistic fuzzy sets, Ye [53] defined the informational energy of a SVNS A, the correlation of two SVNSs A and B, and the correlation coefficient of two SVNSs A and B.

Definition 3.1 [53]

For a SVNS A in the universe of discourse X = {$x_1, x_2, \ldots, x_n$}, the informational energy of the SVNS A is defined by

$$I(A) = \sum_{i=1}^{n} [T_A^2(x_i) + I_A^2(x_i) + F_A^2(x_i)]$$

Definition 3.2 [53]

For two SVNSs A and B in the universe of discourse X = {$x_1, x_2, \ldots, x_n$}, correlation of the SVNSs A and B is defined as

$$C(A,B) = \sum_{i=1}^{n} [T_A(x_i) T_B(x_i) + I_A(x_i) I_B(x_i) + F_A(x_i) F_B(x_i)]$$

Definition 3. 3 [53]

The correlation coefficient of the SVNSs A and B is defined by the following formula: (1)

$$K(A, B) = \frac{C(A,B)}{[C(A,A).C(B,B)]^{1/2}} = \frac{\sum_{i=1}^{n} [T_A(x_i) T_B(x_i) + I_A(x_i) I_B(x_i) + F_A(x_i) F_B(x_i)]}{[\sum_{i=1}^{n} [(T_A(x_i))^2 + (I_A(x_i))^2 + (F_A(x_i))^2]]^{\frac{1}{2}} [\sum_{i=1}^{n} [(T_B(x_i))^2 + (I_B(x_i))^2 + (F_B(x_i))^2]]^{\frac{1}{2}}}$$

The correlation coefficient K(A, B) satisfies the following properties :

(1) K(A, B) = K(B, A);

(2) $0 \le K(A, B) \le 1$;

(3) K(A, B) = 1, if A = B.

4 Correlation coefficient of rough neutrosophic sets

Correlation coefficient between rough neutrosophic sets (RNSs) is yet to define in the literature. Therefore in this paper, we define correlation coefficient between RNSs.

Definition4.1. Assume that there are any two RNSs

$A = <(\underline{T_A(x_i)}, \underline{I_A(x_i)}, \underline{F_A(x_i)}), (\overline{T_A(x_i)}, \overline{I_A(x_i)}, \overline{F_A(x_i)}),>$ and

$B = <(\underline{T_B(x_i)}, \underline{I_B(x_i)}, \underline{F_B(x_i)}), (\overline{T_B(x_i)}, \overline{I_B(x_i)}, \overline{F_B(x_i)})>$. Then the correlation between the RNSs A and B is defined as

$$C(A,B) = \sum_{i=1}^{n} [\delta T_A(x_i).\delta T_B(x_i) + \delta I_A(x_i).\delta I_B(x_i) + \delta F_A(x_i).\delta F_B(x_i)]$$

where

$$\delta T_A(x_i) = \frac{\underline{T}_A(x_i) + \overline{T}_A(x_i)}{2},$$

$$\delta I_A(x_i) = \frac{\underline{I}_A(x_i) + \overline{I}_A(x_i)}{2},$$

$$\delta F_A(x_i) = \frac{\underline{F}_A(x_i) + \overline{F}_A(x_i)}{2},$$

$$\delta T_B(x_i) = \frac{\underline{T}_B(x_i) + \overline{T}_B(x_i)}{2},$$

$$\delta I_B(x_i) = \frac{\underline{I_B}(x_i) + \overline{I_B}(x_i)}{2} \quad \text{and}$$

$$\delta F_B(x_i) = \frac{\underline{F_B}(x_i) + \overline{F_B}(x_i)}{2}.$$

Definition 4.2.The correlation coefficient of the RNSs A and B is defined as

$$K(A,B) = \frac{C(A,B)}{[C(A,A).C(B,B)]^{1/2}}$$

$$= \frac{\sum_{i=1}^{n}[\delta T_A(x_i).\delta T_B(x_i) + \delta I_A(x_i).\delta I_B(x_i) + \delta F_A(x_i).\delta F_B(x_i)]}{(\sum_{i=1}^{n}[(\delta T_A(x_i))^2 + (\delta I_A(x_i))^2 + (\delta F_A(x_i))^2]^{\frac{1}{2}})(\sum_{i=1}^{n}[(\delta T_B(x_i))^2 + (\delta I_B(x_i))^2 + (\delta F_B(x_i))^2]^{\frac{1}{2}})} \quad ...(2)$$

The correlation coefficient K(A, B) satisfies the following properties :

(1) K(A, B) = K(B, A);

(2) $0 \le K(A,B) \le 1$;

(3) K(A, B) = 1, if A = B.

Proof

(i)

$$K(A,B) = \frac{C(A,B)}{[C(A,A).C(B,B)]^{1/2}}$$

$$= \frac{C(B,A)}{[C(B,B).C(A,A)]^{1/2}} = K(B,A)$$

(ii) As C(A, B) ≥ 0, C(A, A) ≥ 0, C(B, B) ≥ 0 so K(A, B) ≥ 0.

According to the Cauchy–Schwarz inequality:

$$(a_1b_1 + ...+ a_nb_n)^2 \le (a_1^2 + ...+ a_n^2)(b_1^2 + ...+ b_n^2)$$

where a_i, $b_i \in R$ for i=1,......,n,

So $\frac{(a_1b_1 + ...+ a_nb_n)}{(a_1^2 + ...+ a_n^2)^{\frac{1}{2}}(b_1^2 + ...+ b_n^2)^{\frac{1}{2}}} \le 1$

Replacing a_i by $\delta T_A(x_i)$ and b_i by $\delta T_B(x_i)$ we obtain K(A, B) ≤1.

Therefore, 0≤ K(A, B) ≤ 1.

(iii) If A = B

then K(A,B) = K(A,A) = $\frac{C(A,A)}{[C(A,A).C(A,A)]^{1/2}}$

$$= \frac{C(A,A)}{C(A,A)} = 1$$

Hence proved.

Considering n = 1, we get the following: (3)

K(A,B)=

$$\frac{\delta T_A(x_i).\delta T_B(x_i) + \delta I_A(x_i).\delta I_B(x_i) + \delta F_A(x_i).\delta F_B(x_i)}{((\delta T_A(x_i))^2 + (\delta I_A(x_i))^2 + (\delta F_A(x_i))^2)^{\frac{1}{2}}((\delta T_B(x_i))^2 + (\delta I_B(x_i))^2 + (\delta F_B(x_i))^2)^{\frac{1}{2}}}$$

Which is the cosine similarity measure between two RNSs A and B [27].

Weighted correlation coefficient:

Let w = {w_1, w_2, …, w_n} be the weight vector of the elements x_i (i = 1, 2, …, n).

Then the weighted correlation coefficient between A and B is defined by the following formula:

$$K_W(A,B) = \frac{\sum_{i=1}^{n} w_i[\delta T_A(x_i).\delta T_B(x_i) + \delta I_A(x_i).\delta I_B(x_i) + \delta F_A(x_i).\delta F_B(x_i)]}{[(\sum_{i=1}^{n}\{w_i[(\delta T_A(x_i))^2 + (\delta I_A(x_i))^2 + (\delta F_A(x_i))^2]\}^{\frac{1}{2}})(\sum_{i=1}^{n}\{w_i[(\delta T_B(x_i))^2 + (\delta I_B(x_i))^2 + (\delta F_B(x_i))^2]\}^{\frac{1}{2}})]} \quad (4)$$

If w = {1/n, 1/n, …, 1/n}, then equation (4) reduces to equation (2).

Weighted correlation coefficient K_w(A, B) also satisfies the following properties:

(1) K_w(A, B) = K_w(B, A);

(2) $0 \le K_w(A,B) \le 1$;

(3) K_w(A, B) = 1, if A = B.

Proof

(i)

$$K_W(A,B) = \frac{\sum_{i=1}^{n} w_i[\delta T_A(x_i).\delta T_B(x_i) + \delta I_A(x_i).\delta I_B(x_i) + \delta F_A(x_i).\delta F_B(x_i)]}{[(\sum_{i=1}^{n}\{w_i[(\delta T_A(x_i))^2 + (\delta I_A(x_i))^2 + (\delta F_A(x_i))^2]\}^{\frac{1}{2}})(\sum_{i=1}^{n}\{w_i[(\delta T_B(x_i))^2 + (\delta I_B(x_i))^2 + (\delta F_B(x_i))^2]\}^{\frac{1}{2}})]}$$

$$= \frac{\sum_{i=1}^{n} w_i[\delta T_B(x_i).\delta T_A(x_i) + \delta I_B(x_i).\delta I_A(x_i) + \delta F_B(x_i).\delta F_A(x_i)]}{[(\sum_{i=1}^{n}\{w_i[(\delta T_B(x_i))^2 + (\delta I_B(x_i))^2 + (\delta F_B(x_i))^2]\}^{\frac{1}{2}})(\sum_{i=1}^{n}\{w_i[(\delta T_A(x_i))^2 + (\delta I_A(x_i))^2 + (\delta F_A(x_i))^2]\}^{\frac{1}{2}})]}$$

$$= K_w(B,A)$$

(ii) As $\sum_{i=1}^{n} w_i[\delta T_A(x_i).\delta T_B(x_i) + \delta I_A(x_i).\delta I_B(x_i) + \delta F_A(x_i).\delta F_B(x_i)] \ge 0$,

$$\sum_{i=1}^{n}\{w_i[(\delta T_A(x_i))^2 + (\delta I_A(x_i))^2 + (\delta F_A(x_i))^2]\}^{\frac{1}{2}} \ge 0$$

and $\sum_{i=1}^{n}\{w_i[(\delta T_B(x_i))^2 + (\delta I_B(x_i))^2 + (\delta F_B(x_i))^2]\}^{\frac{1}{2}} \ge 0$

so K_W(A,B) ≥ 0.

Using the weighted Cauchy–Schwarz inequality [60], we have

$$(w_1a_1b_1 + ...+ w_na_nb_n)^2 \le (w_1a_1^2 + ...+ w_1a_n^2)(w_1b_1^2 + ...+ w_nb_n^2)$$

where w_i, a_i, $b_i \in R$ for i =1, ... , n.

So $\frac{(w_1a_1b_1 + ...+ w_na_nb_n)}{(w_1a_1^2 + ...+ w_na_n^2)^{\frac{1}{2}}(w_1b_1^2 + ...+ w_nb_n^2)^{\frac{1}{2}}} \le 1$

Replacing a_i by $w_i\delta T_A(x_i)$ and b_i by $w_i\delta T_B(x_i)$ we obtain

K_W(A, B) ≤ 1.

Therefore, 0 ≤ K(A, B) ≤ 1.

(iii) If A = B, then

K(A,B) = K(A,A)

$$= \frac{\sum_{i=1}^{n} w_i[\delta T_A(x_i).\delta T_A(x_i) + \delta I_A(x_i).\delta I_A(x_i) + \delta F_A(x_i).\delta F_A(x_i)]}{[(\sum_{i=1}^{n}\{w_i[(\delta T_A(x_i))^2 + (\delta I_A(x_i))^2 + (\delta F_A(x_i))^2]\}^{\frac{1}{2}})(\sum_{i=1}^{n}\{w_i[(\delta T_A(x_i))^2 + (\delta I_A(x_i))^2 + (\delta F_A(x_i))^2]\}^{\frac{1}{2}})]}$$

$$= \frac{\sum_{i=1}^{n}\{w_i[(\delta T_A(x_i))^2 + (\delta I_A(x_i))^2 + (\delta F_A(x_i))^2]\}}{\sum_{i=1}^{n}\{w_i[(\delta T_A(x_i))^2 + (\delta I_A(x_i))^2 + (\delta F_A(x_i))^2]\}} = 1$$

Hence proved.

5 Rough neutrosophic decision making based on correlation coefficient

Let $A_1, A_2, \dots, A_m$ be a set of elements (/objects / persons), $C_1, C_2, \dots, C_n$ be a set of criteria for each element and $E_1, E_2, \dots, E_k$ are the alternatives for each element.

Step 1. The relation between elements A_i ($i = 1, 2, \dots, m$) and the criteria C_j ($j = 1, 2, \dots, n$) is presented in Table 1 in terms of RNSs.

Table1 : Relation between elements and criteria

$$\begin{array}{c} \\ A_1 \\ A_2 \\ \dots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \dots & C_n \end{array} \\ \begin{bmatrix} X_{11} & X_{12} & \dots & X_{1n} \\ X_{21} & X_{22} & \dots & X_{2n} \\ \dots & \dots & \dots. & \dots \\ X_{m1} & X_{m2} & \dots & X_{mn} \end{bmatrix} \end{array}$$

where

$$X_{ij} = \langle (\underline{T}_{ij}, \underline{I}_{ij}, \underline{F}_{ij}), (\overline{T}_{ij}, \overline{I}_{ij}, \overline{F}_{ij}) \rangle$$

with $0 \le \underline{T}_{ij} + \underline{I}_{ij} + \underline{F}_{ij} \le 3$ and $0 \le \overline{T}_{ij} + \overline{I}_{ij} + \overline{F}_{ij} \le 3.$

The relation between criterion C_i ($i = 1, 2, \dots, n$) and the alternative E_j ($j = 1, 2, \dots, k$) is presented in Table 2 in terms of RNSs.

Table 2 : Relation between criteria and alternatives

$$\begin{array}{c} \\ C_1 \\ C_2 \\ \dots \\ C_m \end{array} \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & \dots & E_k \end{array} \\ \begin{bmatrix} Y_{11} & Y_{12} & \dots & Y_{1k} \\ Y_{21} & Y_{22} & \dots & Y_{2k} \\ \dots & \dots & \dots. & \dots \\ Y_{n1} & Y_{n2} & \dots & Y_{nk} \end{bmatrix} \end{array}$$

where

$$Y_{ij} = \langle (\underline{T}_{ij}, \underline{I}_{ij}, \underline{F}_{ij}), (\overline{T}_{ij}, \overline{I}_{ij}, \overline{F}_{ij}) \rangle$$

with

$0 \le \underline{T}_{ij} + \underline{I}_{ij} + \underline{F}_{ij} \le 3$ and $0 \le \overline{T}_{ij} + \overline{I}_{ij} + \overline{F}_{ij} \le 3.$

Step 2. Determine the correlation measure between Table 1 and Table 2 using equation 2. The obtained values are presented in Table 3.

Table 3 : Correlation coefficient between table1 and table2

$$\begin{array}{c} \\ A_1 \\ A_2 \\ \dots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} E_1 & E_2 & \dots & E_k \end{array} \\ \begin{bmatrix} p_{11} & p_{12} & \dots & p_{1k} \\ p_{21} & p_{22} & \dots & p_{2k} \\ \dots & \dots & \dots. & \dots \\ p_{m1} & p_{m2} & \dots & p_{mk} \end{bmatrix} \end{array}$$

Step 3. From Table 3, for each element A_i ($i = 1, 2, \dots, m$), find the maximum correlation value of the i-th row ($i = 1, 2, \dots, m$). If the maximum value occurs at j-th column ($j = 1, 2, \dots, k$) (see Table 3), then E_j will be the best alternative for the element A_i ($i = 1, 2, \dots, m$).

Step 4. End.

6 Medical Diagnosis Problem

We consider a medical diagnosis problem for illustration of the proposed method. Medical diagnosis comprises of inconsistent, indeterminate and incomplete information though increased volume of information available to doctors from new medical technologies. The proposed correlation coefficients among the patients versus symptoms and symptoms versus diseases will provide medical diagnosis. Let $P = \{P_1, P_2, P_3\}$ be a set of patients, D = {Viral fever, Malaria, Stomach problem, Chest problem} be a set of diseases and S = {Temperature, Headache, Stomach pain, Cough, Chest pain} be a set of symptoms. Using proposed method the doctor is to examine the patient and to determine the disease of the patient in rough neutrosophic environment.

Based on the proposed approach the considered problem is solved using the following steps:

Step 1. Construction of the rough neutrosophic decision matrix

Table 4: (Relation-1) The relation between Patients and Symptoms

	Temperature	Headache	Stomach pain	cough	Chest pain
P1	<(.6,.4,.3), (.8,.2,.1)>	<(.4,.4,.4), (.6,.2,.2)>	<(.5,.3,.2), (.7,.1,.2)>	<(.6,.2,.4), (.8,.0,.2)>	<(.4,.4,.4), (.6,.2,.2)>
P2	<(.5,.3,.4), (.7,.3,.2)>	<(.5,.3,.3), (.7,.3,.3)>	<(.5,.3,.4), (.7,.1,.4)>	<(.5,.3,.3), (.9,.1,.3)>	<(.5,.3,.3), (.7,.1,.3)>
P3	<(.6,.4,.4), (.8,.2,.2)>	<(.5,.2,.3), (.7,.0,.1)>	<(.4,.3,.4), (.8,.1,.2)>	<(.6,.1,.4), (.8,.1,.2)>	<(.5,.3,.3), (.7,.1,.1)>

Table 5: (Relation-2) The relation among Symptoms and Diseases

	Viral Fever	Malaria	Stomach problem	Chest problem
Temperature	<(.6,.5,.4), (.8,.3,.2)>	<(.1,.4,.4), (.5,.2,.2)>	<(.3,.4,.4), (.5,.2,.2)>	<(.2,.4,.6), (.4,.4,.4)>
Headache	<(.5,.3,.4), (.7,.3,.2)>	<(.2,.3,.4), (.6,.3,.2)>	<(.2,.3,.3), (.4,.1,.1)>	<(.1,.5,.5), (.5,.3,.3)>
Stomach pain	<(.2,.3,.4), (.4,.3,.2)>	<(.1,.4,.4), (.3,.2,.2)>	<(.4,.3,.4), (.6,.1,.2)>	<(.1,.4,.6), (.3,.2,.4)>
cough	<(.4,.3,.3), (.6,.1,.1)>	<(.3,.3,.3), (.5,.1,.3)>	<(.1,.6,.6), (.3,.4,.4)>	<(.5,.3,.4), (.7,.1,.2)>

Chest pain	<(.2,.4,.4), (.4,.2,.2)>	<(.1,.3,.3), (.3,.1,.1)>	<(.1,.4,.4), (.3,.2,.2)>	<(.4,.4,.4), (.6,.2,.3)>

Step 2. Determination of correlation coefficient between table 1 and table 2

Table 6: The correlation measure between Relation-1 and Relation-2

	Viral Fever	Malaria	Stomach problem	Chest problem
P_1	**0.95135**	0.91141	0.84518	0.87465
P_2	**0.95033**	0.94374	0.86228	0.91731
P_3	**0.93473**	0.89549	0.82559	0.85937

Step 3. Ranking the alternatives

According to the values of correlation coefficient of each alternative shown in Table 3, the highest correlation measure occurs in column1(i.e. for the diseases viral fever. Therefore, all three patients P_1, P_2, P_3 suffer from viral fever.

7 Conclusion

In this paper, we have proposed correlation coefficient and weighted correlation coefficient between rough neutrosophic sets and proved some of their basic properties. We have developed a new multi criteria decision making method based on the correlation coefficient measure. We presented an illustrative example in medical diagnosis. We hope that the proposed method can be applied in solving realistic multi criteria group decision making problems in rough neutrosophic environment.

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Neutrosophic Cubic MCGDM Method Based on Similarity Measure

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Abstract. The notion of neutrosophic cubic set is originated from the hybridization of the concept of neutrosophic set and interval valued neutrosophic set. We define similarity measure for neutrosophic cubic sets and prove some of its basic properties. We present a new multi criteria group decision making method with linguistic variables in neutrosophic cubic set environment. Finally, we present a numerical example to demonstrate the usefulness and applicability of the proposed method.

Keywords: Cubic set, Neutrosophic cubic set, similarity measure, multi criteria group decision making.

1. Introduction

In practical life we frequently face decision making problems with uncertainty that cannot be dealt with the classical methods. Therefore sophisticated techniques are required for modification of classical methods to deal decision making problems with uncertainty. L. A. Zadeh [1] first proposed the concept of fuzzy set to deal non-statistical uncertainty called fuzziness. K. T. Atanassov [2, 3] introduced the concept of intuitionistic fuzzy set (IFS) to deal with uncertainty by introducing the non-membership function as an independent component. F. Smarandache [4, 5, 6, 7, 8] introduced the notion of neutrosophic set by introducing indeterminacy as independent component. The theory of neutrosophic sets is a powerful tool to deal with incomplete, indeterminate and inconsistent information involed in real world decision making problem. Wang et al. [9] defined single valued neutrosophic set (SVNS) which is an instance of neutrosophic set. SVNS can independently express a truth-membership degree, an indeterminacy-membership degree and non-membership (falsity-membership) degree. SVNS is capable of representing human thinking due to the imperfection of knowledge received from real world problems. SVNS is obviously suitable for representing incomplete, inconsistent and indeterminate information.

Neutrosophic sets and SVNSs have become hot research topics in different areas of research such as conflict resolution [10], clustering analysis [11, 12], decision making [13-41], educational problem [42, 43], image processing [44, 45, 46], medical diagnosis [47], optimization [48-53], social problem [54, 55].

By combining neutrosophic sets and SVNS with other sets, several neutrosophic hybrid sets have been proposed in the literature such as neutrosophic soft sets [56, 57, 58, 59, 60, 61], neutrosophic soft expert set [62, 63], single valued neutrosophic hesitant fuzzy sets [64, 65, 66, 67, 68], interval neutrosophic hesitant sets [69], interval neutrosophic linguistic sets [70], single valued neutrosophic linguistic sets [71], rough neutrosophic set [72, 73, 74, 75, 76, 77, 78, 79], interval rough neutrosophic set [80, 81, 82], bipolar neutrosophic set [83, 84], bipolar rough neutrosophic set [85] Tri-complex rough neutrosophic set[86], hyper complex rough neutrosophic set [87]. Neutrosophic refined set [88, 89, 90, 91, 92, 93], Bipolar neutrosophic refined sets [94], rough complex set neutrosophic cubic set [95].

Jun et al. [96] put forward the concept of cubic set in fuzzy environment and defined external and internal cubic set. Ali et al. [95] proposed neutrosophic cubic set and defined external and internal neutrosophic cubic sets and their basic properties.

Similarity measure is a vital topic in fuzzy set theory, Chen and Hsiao [97] presented comparisons of similarity measures of fuzzy sets. Pramanik and Mondal [98] studied weighted fuzzy similarity measure based on tangent function and presented its application to medical diagnosis. Hwang and Yang [99] constructed a new similarity measure between intuitionistic fuzzy sets based on lower, upper and middle fuzzy sets. Pramanik and Mondal [100] developed tangent similarity measures in intuitionistic fuzzy environment and applied to medical diagnosis. Ren and Wang [101] proposed similarity measures in interval-valued intuitionistic fuzzy environment and applied it to multi attribute decision making problems. Baccour et al. [102] presented survey of similarity measures for intuitionistic fuzzy sets. Baroumi and Smarandache [103] disicussed several similarity measures of neutrosophic sets. Mondal and Pramanik [104] extended the concept of intuitionistic tangent similarity measure to neutrosophic environment. Biswas et al. [105] studied cosine similarity measure with trapezoidal fuzzy neutrosophic number and its applied to multi attribute decision making problems. Pramanik and Mondal [106] proposed cosine similarity measure of rough neutrosophic set and applied it to medical diagnosis problems. Pramanik and Mondal [107]developed ccotangent similarity measure of rough neutrosophic sets and its application to medical diagnosis J. Ye [108] proposed a similarity measures under interval neutrosophic domain using hamming distance and Euclidean distance. P. Majumdar and S. K. Samanta [109] introduced some measures of similarity and entropy of single valued neutrosophic sets. Ali aydogdu [110] proposed similarity and entropy measure of single valued neutrosophic sets. Ali aydogdu [111] also defined entropy and similarity measures of interval neutrosophic sets. Mukherjee and Sarkar [112] proposed similarity measures, weighted similarity measure and developed an algorithm in interval valued neutrosophic soft set setting for supervised pattern recognition problem. In neutrosophic cubic set environment, similarity measure is yet to appear.

In this paper we define similarity measures in neutrosophic cubic set environment and develop a multi criteria group decision making (MCGDM) method in neutrosophic cubic set setting. The decision makers' weights and criteria (attributes) weights are described by neutrosophic cubic numbers using linguistic variables. The ranking of alternatives is presented in descending order. Finally, illustrate numerical example MCGDM problem in neutrosophic cubic set environment is dolved to show the effectiveness of the proposed method.

Rest of the paper is presented as follows. Section 2 presents some basic definition of fuzzy sets, interval-valued fuzzy sets, neutrosophic sets, interval valued neutrosophic sets, cubic set, neutrosophic cubic sets and their basic operations. Section 3 is devoted to prove the basic properties of similarity measure for neutrosophic cubic sets. Section 4 presents a MCGDM method based on similarity measure in neutrosophic cubic set environment. Section 5 presents a numerical example for a MCGDM problem. Finally, section 6 presents conclusion and future scope of research.

2 Preliminaries

In this section, we recall some basic definitions which are relevant to develop the paper.

Definition 2.1 [1] Fuzzy set

Let U be a universal set. Then a fuzzy set Z over U is defined by $Z = \{(u, \mu_Z(u)): u \in U\}$

Where $\mu_Z : U \rightarrow [0, 1]$ is called membership function of Z and $\mu_Z(u)$ specifies the grade or degree to which any element u in Z, $\mu_Z(u) \in [0, 1]$. Larger values of $\mu_Z(u)$ indicate higher degrees of membership.

Definition 2.2 [113] Interval valued fuzzy set

Let U be a universal set, then an interval valued fuzzy set $\tilde{Z}$ over U is defined by $\tilde{Z} = \{[Z^-(u), Z^+(u)] /u: u \in U\}$, where $Z^-(u)$, $Z^+(u)$ represent respectively the lower and upper degrees of membership values for $u \in U$ and $0 \leq Z^-(u) + Z^+(u) \leq 1$.

Definition 2.3 [96] Cubic set

Let G be a non-empty set. A cubic set C (G) in G is defined by

$C(G) = \{g, \tilde{Z}(g), Z(g)/g \in G\}$

Where $\tilde{Z}(g)$ and $Z(g)$ be the interval valued fuzzy set and fuzzy set in G.

Definition 2.4 [4] Neutrosophic set (NS)

Let U be a space of points (objects) with a generic element in U denoted by u i.e. $u \in U$. A neutrosophic set R in U is characterized by truth-membership function t_R, a indeterminacy membership function i_R and falsity-membership function f_R. Where t_R, i_R, f_R are the functions from U to $]^{-}0, 1^{+}[$ i.e. $t_R, i_R, f_R: U \rightarrow\ ]^{-}0, 1^{+}[$ that means $t_R(u), i_R(u), f_R(u)$ are the real standard or non-standard subset of $]^{-}0, 1^{+}[$. Neutrosophic set can be expressed as $R = \{<u, (t_R(u), i_R(u), f_R(u))>: u \in U\}$. Since $t_R(u), i_R(u), f_R(u)$ are the subset of $]^{-}0, 1^{+}[$ then the sum $(t_R(u) + i_R(u) + f_R(u))$ lies between $^{-}0$ and 3^{+}, where $^{-}0 = 0 - \varepsilon$ and $3^{+} = 3 + \varepsilon$, $\varepsilon > 0$ and $\varepsilon \rightarrow 0$.

Definition 2.5 [9] Single valued neutrosophic set
Let U be a space of points (objects) with a generic element in U denoted by u. A single valued neutrosophic set H in U is expressed by $H = \{<u, (t_H(u), i_H(u), f_H(u))>, u \in U\}$, where $t_H(u), i_H(u), f_H(u): U \rightarrow [0, 1]$
Therefore for each $u \in U$, $t_H(u), i_H(u), f_H(u) \in [0, 1]$ and $0 \leq t_H(u) + i_H(u) + f_H(u) \leq 3$.

Definition 2.6 [4] Complement of neutrosophic set
The complement of neutrosophic set R denoted by R' and defined as $R' = \{< u, t_{R'}(u), i_{R'}(u), f_{R'}(u)>: u \in U\}$, where $t_{R'}(u) = f_R(u)$, $i_{R'}(u) = \{1^{+}\} - i_R(u)$, $f_{R'}(u) = t_R(u)$.

Definition 2.7 [8]Containment
A neutrosophic set R_1 is contained in another neutrosophic set R_2 i.e. $R_1 \subseteq R_2$ iff $t_{R_1}(u) \leq t_{R_2}(u)$, $i_{R_1}(u) \leq i_{R_2}(u)$ and $f_{R_1}(u) \geq f_{R_2}(u)$, $\forall u \in U$.

Definition 2.8 [4] Equality
Two single valued neutrosophic set R_1 and R_2 are equal iff $R_1 \subseteq R_2$ and $R_2 \subseteq R_1$.

Definition 2.9 [4] Union
The union of two single valued neutrosophic set R_1 and R_2 is a neutrosophic set R_3 (say) written as $R_3 = R_1 \cup R_2$.
$t_{R_3}(u) = \max\{t_{R_1}(u), t_{R_2}(u)\}$, $i_{R_3}(u) = \max\{i_{R_1}(u), i_{R_2}(u)\}$, $f_{R_1}(u) = \min\{f_{R_1}(u), f_{R_2}(u)\}$, $\forall u \in U$.

Definition 2.10 [4] Intersection
The intersection of two single valued neutrosophic set R_1 and R_2 denoted by R_4 and written as $R_4 = R_1 \cap R_2$defined by $t_{R_4}(u) = \min\{t_{R_1}(u), t_{R_2}(u)\}$, $i_{R_4}(u) = \min\{i_{R_1}(u), i_{R_2}(u)\}$
$f_{R_4}(u) = \max\{f_{R_1}(u), f_{R_2}(u)\}$, $\forall u \in U$.

Definition 2.11 [114] Interval neutrosophic set (INS)
Let G be a non-empty set. An interval neutrosophic set $\tilde{G}$ in G is characterized by truth-membership function $t_{\tilde{G}}$, the indeterminacy function $i_{\tilde{G}}$ and falsity membership function $f_{\tilde{G}}$. For each $g \in G$, $t_{\tilde{G}}(g)$, $i_{\tilde{G}}(g)$, $f_{\tilde{G}}(g) \subseteq [0, 1]$ and $\tilde{G}$ defined as
$\tilde{G} = \{<g; [t^{-}_{\tilde{G}}(g), t^{+}_{\tilde{G}}(g)], [i^{-}_{\tilde{G}}(g), i^{+}_{\tilde{G}}(g)], [f^{-}_{\tilde{G}}(g), f^{+}_{\tilde{G}}(g)]: g \in G\}$.

Definition 2.12 [114] Containment
Let G_1 and G_2 be two interval neutrosophic set defined by
$\tilde{G}_1 = \{<g, [t^{-}_{\tilde{G}_1}(g), t^{+}_{\tilde{G}_1}(g)], [i^{-}_{\tilde{G}_1}(g), i^{+}_{\tilde{G}_1}(g)], [f^{-}_{\tilde{G}_1}(g), f^{+}_{\tilde{G}_1}(g)]>: g \in G\}$
and $\tilde{G}_2 = \{<g, [t^{-}_{\tilde{G}_2}(g), t^{+}_{\tilde{G}_2}(g)], [i^{-}_{\tilde{G}_2}(g), i^{+}_{\tilde{G}_2}(g)], [f^{-}_{\tilde{G}_2}(g), f^{+}_{\tilde{G}_2}(g)]>: g \in G\}$
then, (i) $\tilde{G}_1 \subseteq \tilde{G}_2$ defined as
$t^{-}_{\tilde{G}_1}(g) \leq t^{-}_{\tilde{G}_2}(g)$, $t^{+}_{\tilde{G}1}(g) \leq t^{+}_{\tilde{G}_2}(g)$
$i^{-}_{\tilde{G}_1}(g) \leq i^{-}_{\tilde{G}_2}(g)$, $i^{+}_{\tilde{G}_1}(g) \leq i^{+}_{\tilde{G}_2}(g)$
$f^{-}_{\tilde{G}_1}(g) \geq f^{-}_{\tilde{G}_2}(g)$, $f^{+}_{\tilde{G}_1}(g) \geq f^{+}_{\tilde{G}_2}(g)$ for all $g \in G$.

Definition 2.13 [114] Equality
$\tilde{G}_1 = \tilde{G}_2$ iff $\tilde{G}_1 \subseteq \tilde{G}_2$ and $\tilde{G}_2 \subseteq \tilde{G}_1$ that means $t^{-}_{\tilde{G}_1}(g) = t^{-}_{\tilde{G}_2}(g)$, $t^{+}_{\tilde{G}1}(g) = t^{+}_{\tilde{G}_2}(g)$, $i^{-}_{\tilde{G}_1}(g) = i^{-}_{\tilde{G}_2}(g)$, $i^{+}_{\tilde{G}_1}(g) = i^{+}_{\tilde{G}_2}(g)$, $f^{-}_{\tilde{G}_1}(g) = f^{-}_{\tilde{G}_2}(g)$, $f^{+}_{\tilde{G}_1}(g) = f^{+}_{\tilde{G}_2}(g)$ for all $g \in G$.

Definition 2.14 [114] Compliment
Compliment of an interval neutrosophic set $\tilde{G}_1$ denoted by $\tilde{G}_1'$ and defined by
$\tilde{G}_1' = \{<g, [t^{-}_{\tilde{G}_1'}(g), t^{+}_{\tilde{G}_1'}(g)], [i^{-}_{\tilde{G}_1'}(g), i^{+}_{\tilde{G}_1'}(g)], [f^{-}_{\tilde{G}_1'}(g), f^{+}_{\tilde{G}_1'}(g)]>: g \in G\}$, Where, $t^{-}_{\tilde{G}_1'}(g) = f^{-}_{\tilde{G}_1}(g)$, $t^{+}_{\tilde{G}_1'}(g) = f^{+}_{\tilde{G}_1}(g)$, $i^{-}_{\tilde{G}_1'}(g) = \{1\} - i^{-}_{\tilde{G}_1}(g)$, $i^{+}_{\tilde{G}_1'}(g) = \{1\} - i^{+}_{\tilde{G}_1}(g)$,
$f^{-}_{\tilde{G}_1'}(g) = t^{-}_{\tilde{G}_1}(g)$, $f^{+}_{\tilde{G}_1'}(g) = f^{+}_{\tilde{G}_1}(g)$.

Definition 2.15 [114] Union
The union of two interval neutrosophic sets $\tilde{G}_1$, and $\tilde{G}_2$ is denoted by $\tilde{G}_3 = \tilde{G}_1 \cup \tilde{G}_2$ and defined as

$\tilde{G}_3$ = {<g, [max { $t^-_{\tilde{G}_1}$ (g), $t^-_{\tilde{G}_2}$ (g)},max { $t^+_{\tilde{G}1}$ (g), $t^+_{\tilde{G}_2}$ (g)}], [max { $i^-_{\tilde{G}_1}$ (g), $i^-_{\tilde{G}_2}$ (g)}, max { $i^+_{\tilde{G}_1}$ (g), $i^+_{\tilde{G}_2}$ (g)}], [min { $f^-_{\tilde{G}_1}$ (g), $f^-_{\tilde{G}_2}$ (g)}, min { $f^+_{\tilde{G}_1}$ (g), $f^+_{\tilde{G}_2}$ (g)}]>: g $\in$ G}.

Definition 2.16 [114] Intersection

The intersection of two interval neutrosophic set $\tilde{G}_1$, $\tilde{G}_2$ is denoted by $\tilde{G}_4 = \tilde{G}_1 \cap \tilde{G}_2$ and defined as

$\tilde{G}_4$ = {<g, [min { $t^-_{\tilde{G}_1}$ (g), $t^-_{\tilde{G}_2}$ (g)},min { $t^+_{\tilde{G}1}$ (g), $t^+_{\tilde{G}_2}$ (g)}], [min { $i^-_{\tilde{G}_1}$ (g), $i^-_{\tilde{G}_2}$ (g)}, min { $i^+_{\tilde{G}_1}$ (g), $i^+_{\tilde{G}_2}$ (g)}], [max { $f^-_{\tilde{G}_1}$ (g), $f^-_{\tilde{G}_2}$ (g)}, max { $f^+_{\tilde{G}_1}$ (g), $f^+_{\tilde{G}_2}$ (g)}]>: g $\in$ G}.

Definition 2.17 [95] Neutrosophic cubic set (NCS)

A neutrosophic cubic set Q (N) in a universal set G is defined as

Q (N) = {<g, $\tilde{G}$ (g), R (g)>: g $\in$ G}, where $\tilde{G}$ is an interval neutrosophic set and R is a neutrosophic set in G. In this paper,we represent neutrosophic cubic set in the following form:

Q (N) = < $\tilde{G}$, R > as order pair, set of all neutrosophic cubic sets in G, we denote it by NCS (G).

Definition 2.18 Another definition of neutrosophic cubic set

Let G be a universal set, then the neutrosophic cubic set Q (N) in G is expressed as the pair

< $\tilde{G}$, R > , where $\tilde{G}$ and R be the mappings represented by

$\tilde{G}$: G $\to$ INS (G), R: $\to$ NS (G)

Combining the two mappings, NCS can be expressed as Q (N) = $\tilde{G}^R$: G $\to$ [INS (G), NS (G)] and defined as Q (N) = $\tilde{G}^R$ = {< g/ < $\tilde{G}$ (g), R (g)>>: g $\in$ G}.

Definition 2.19 [95] Containment

Let Q_1 (N) = ($\tilde{G}_1^{R_1}$) and Q_2 (N) = ($\tilde{G}_2^{R_2}$) be any two NCSs in G, then Q_1 (N) contained in Q_2 (N) i.e. Q_1 (N) $\subseteq Q_2$ (N) iff $\tilde{G}_1 \subseteq \tilde{G}_2$ and $R_1 \subseteq R_2$.

Definition 2.20 [95] Equality

Assume that Q_1 (N) = ($\tilde{G}_1^{R1}$) and Q_2 (N) = ($\tilde{G}_2^{R2}$) be the two NCSs in G. They are said to be equal iff Q_1 (N) $\subseteq$ Q_2 (N) and Q_2 (N) $\subseteq$ Q_1 (N) that means $\tilde{G}_1 = \tilde{G}_2$ and $R_1 = R_2$.

Definition 2.21 [95] Union

The union of two NCSs Q_1 (N) = ($\tilde{G}_1^{R_1}$) and Q_2 (N) = ($\tilde{G}_2^{R_2}$) in G is denoted by

Q_1 (N) $\cup$ Q_2 (N) = Q_3 (N) (say) and defined as

Q_3 (N) = {<g, ($\tilde{G}_1 \cup \tilde{G}_2$) (g), ($R_1 \cup R_2$) (g)>: g $\in$ G}.

Definition 2.22 [95] Intersection

The intersection of two NCS Q_1 (N) = ($\tilde{G}_1^{R_1}$) and Q_2 (N) = ($\tilde{G}_2^{R_2}$) in G is denoted by Q_1 (N) $\cup$ Q_2 (N) = Q_4 (N) (say) and defined as Q_4 (N) = {<g, ($\tilde{G}_1 \cap \tilde{G}_2$) (g), ($R_1 \cap R_2$) (g)>: g $\in$ G}.

Definition 2.23 [95]Complement

Let Q_1 (N) be a NCS. Then complement of Q_1 (N) is denoted by Q'_1 (N) = {<g, $\tilde{G}_1'$ (g), $\tilde{R}_1'$ (g)>: g $\in$ G}.

3 Similarity measure of NCS

We define similarity measure for neutrosophic cubic set.

Definition3.1

Let Q_1 and Q_2 be two NCSs in G. Similarity measure for $Q_1$and Q_2 is defined as a mapping

SM: NCS (G) × NCS (G) $\to$ [0, 1] that satisfies the following conditions:

(1) $0 \le$ SM (Q_1, Q_2) ≤ 1
(2) SM (Q_1, Q_2) = 1 iff $Q_1 = Q_2$
(3) SM (Q_1, Q_2) = SM (Q_2, Q_1)
(4) If $Q_1 \subseteq Q_2 \subseteq Q_3$ then SM (Q_1, Q_3) $\le$ SM (Q_1, Q_2) and SM (Q_1, Q_3) $\le$ SM (Q_2, Q_3) for all Q_1, Q_2, Q_3 $\in$ NCS (G).

Similarity measure for two NCSs Q_1 and Q_2 expressed as

$$SM (Q_1, Q_2) = \frac{1}{n}\sum_{i=1}^{n}(1-\frac{D_i}{9}),$$

where $D_i = (\left| t^-_{\tilde{G}_1}(g_i) - t^-_{\tilde{G}_2}(g_i)\right| + \left| t^+_{\tilde{G}1}(g_i) - t^+_{\tilde{G}_2}(g_i)\right| + \left| i^-_{\tilde{G}_1}(g_i) - i^-_{\tilde{G}_2}(g_i)\right| + \left| i^+_{\tilde{G}_1}(g_i) - i^+_{\tilde{G}_2}(g_i)\right| + \left| f^-_{\tilde{G}_1}(g_i) - f^-_{\tilde{G}_2}(g_i)\right| + \left| f^+_{\tilde{G}_1}(g_i) - f^+_{\tilde{G}_2}(g_i)\right| + \left| t_{R_1}(g_i) - t_{R_2}(g_i)\right| + \left| i_{R_1}(g_i) - i_{R_2}(g_i)\right| + \left| f_{R_1}(g_i) - f_{R_2}(g_i)\right|)$.

We now prove that the similarity measure satisfies the four stated conditions:

(1) $0 \le$ SM (Q_1, Q_2) ≤ 1

Proof: If D_i has extreme value i.e. $D_i = 0$ or 9, then SM $(Q_1, Q_2) = 1$ or 0 (1)

If D_i lies between 0 and 9 i.e. $0<D_i<9$, then $0<\frac{D_i}{9}<1$

$$\Rightarrow 0> -\frac{D_i}{9} > -1$$

Adding 1 each part of the above inequality, we obtain

$$0< 1-\frac{D_i}{9}<1$$

$$\frac{1}{n}\sum_{i=1}^{n} 0 < \frac{1}{n}\sum_{i=1}^{n}\left(1-\frac{D_i}{9}\right) < \frac{1}{n}\sum_{i=1}^{n} 1 = 1$$

$$\Rightarrow 0<\frac{1}{n}\sum_{i=1}^{n}\left(1-\frac{D_i}{9}\right)<1$$

$\Rightarrow 0<SM(Q_1, Q_2)<1$ (2)

Combining (1) and (2), we get $0\le SM(Q_1, Q_2)\le 1$

(2) $SM(Q_1, Q_2) = 1$ iff $Q_1 = Q_2$

Proof:

If $Q_1 = Q_2$, then $D_i = 0$ by the definition of equality.

$$SM(Q_1, Q_2) = \frac{1}{n}\sum_{i=1}^{n}\left(1-\frac{D_i}{9}\right) = 1.$$

(3) $SM(Q_1, Q_2) = SM(Q_2, Q_1)$

Proof: $SM(Q_1, Q_2) = \frac{1}{n}\sum_{i=1}^{n}\left(1-\frac{D_i}{9}\right)$,

where $D_i(Q_1, Q_2) = (|t^-_{\tilde{G}_1}(g_i) - t^-_{\tilde{G}_2}(g_i)| + |t^+_{\tilde{G}_1}(g_i) - t^+_{\tilde{G}_2}(g_i)| + |i^-_{\tilde{G}_1}(g_i) - i^-_{\tilde{G}_2}(g_i)| + |i^+_{\tilde{G}_1}(g_i) - i^+_{\tilde{G}_2}(g_i)| + |f^-_{\tilde{G}_1}(g_i) - f^-_{\tilde{G}_2}(g_i)| + |f^+_{\tilde{G}_1}(g_i) - f^+_{\tilde{G}_2}(g_i)| + |t_{R_1}(g_i) - t_{R_2}(g_i)| + |i_{R_1}(g_i) - i_{R_2}(g_i)| + |f_{R_1}(g_i) - f_{R_2}(g_i)|)$

since, $|t^-_{\tilde{G}_1}(g_i) - t^-_{\tilde{G}_2}(g_i)| = |t^-_{\tilde{G}_2}(g_i) - t^-_{\tilde{G}_1}(g_i)|$, $|t^+_{\tilde{G}_1}(g_i) - t^+_{\tilde{G}_2}(g_i)| = |t^+_{\tilde{G}_2}(g_i) - t^+_{\tilde{G}_1}(g_i)|$, $|i^-_{\tilde{G}_1}(g_i) - i^-_{\tilde{G}_2}(g_i)| = |i^-_{\tilde{G}_2}(g_i) - i^-_{\tilde{G}_1}(g_i)|$, $|i^+_{\tilde{G}_1}(g_i) - i^+_{\tilde{G}_2}(g_i)| = |i^+_{\tilde{G}_2}(g_i) - i^+_{\tilde{G}_1}(g_i)|$, $|f^-_{\tilde{G}_1}(g_i) - f^-_{\tilde{G}_2}(g_i)| = |f^-_{\tilde{G}_2}(g_i) - f^-_{\tilde{G}_1}(g_i)|$, $|f^+_{\tilde{G}_1}(g_i) - f^+_{\tilde{G}_2}(g_i)| = |f^+_{\tilde{G}_2}(g_i) - f^+_{\tilde{G}_1}(g_i)|$, $|t_{R_1}(g_i) - t_{R_2}(g_i)| = |t_{R_2}(g_i) - t_{R_1}(g_i)|$, $|i_{R_1}(g_i) - i_{R_2}(g_i)| = |i_{R_2}(g_i) - i_{R_1}(g_i)|$, $|f_{R_1}(g_i) - f_{R_2}(g_i)| = |f_{R_2}(g_i) - f_{R_1}(g_i)|$.

$$\Rightarrow D_i(Q_1, Q_2) = D_i(Q_2, Q_1)$$

Therefore, $SM(Q_1, Q_2) = SM(Q_2, Q_1)$.

(4) If $Q_1 \subseteq Q_2 \subseteq Q_3$, then $SM(Q_1, Q_3) \le SM(Q_1, Q_2)$ and $SM(Q_1, Q_3) \le SM(Q_2, Q_3)$ for all $Q_1, Q_2, Q_3 \in$ NCS (G).

Proof:

Let $Q_1 \subseteq Q_2 \subseteq Q_3$ then,

$t^-_{\tilde{G}_1}(g_i) \le t^-_{\tilde{G}_2}(g_i) \le t^-_{\tilde{G}_3}(g_i)$, $t^+_{\tilde{G}_1}(g_i) \le t^+_{\tilde{G}_2}(g_i) \le t^+_{\tilde{G}_3}(g_i)$,

$i^-_{\tilde{G}_1}(g_i) \le i^-_{\tilde{G}_2}(g_i) \le i^-_{\tilde{G}_3}(g_i)$

$i^+_{\tilde{G}_1}(g_i) \le i^+_{\tilde{G}_2}(g_i), \le i^+_{\tilde{G}_3}(g_i)$,

$f^-_{\tilde{G}_1}(g_i) \ge f^-_{\tilde{G}_2}(g_i) \ge f^-_{\tilde{G}_3}(g_i)$, $f^+_{\tilde{G}_1}(g_i) \ge f^+_{\tilde{G}_2}(g_i) \ge f^+_{\tilde{G}_3}(g_i)$

$t_{R_1}(g_i) \le t_{R_2}(g_i) \le t_{R_3}(g_i)$, $i_{R_1}(g_i) \le i_{R_2}(g_i) \le i_{R_3}(g_i)$, $f_{R_1}(g_i) \ge f_{R_2}(g_i) \ge f_{R_3}(g_i)$ (3)

Now $D_i(Q_1, Q_2) = (|t^-_{\tilde{G}_1}(g_i) - t^-_{\tilde{G}_2}(g_i)| + |t^+_{\tilde{G}_1}(g_i) - t^+_{\tilde{G}_2}(g_i)| + |i^-_{\tilde{G}_1}(g_i) - i^-_{\tilde{G}_2}(g_i)| + |i^+_{\tilde{G}_1}(g_i) - i^+_{\tilde{G}_2}(g_i)| + |f^-_{\tilde{G}_1}(g_i) - f^-_{\tilde{G}_2}(g_i)| + |f^+_{\tilde{G}_1}(g_i) - f^+_{\tilde{G}_2}(g_i)| + |t_{R_1}(g_i) - t_{R_2}(g_i)| + |i_{R_1}(g_i) - i_{R_2}(g_i)| + |f_{R_1}(g_i) - f_{R_2}(g_i)|)$

And $D_i(Q_1, Q_3) = (|t^-_{\tilde{G}_1}(g_i) - t^-_{\tilde{G}_3}(g_i)| + |t^+_{\tilde{G}_1}(g_i) - t^+_{\tilde{G}_3}(g_i)| + |i^-_{\tilde{G}_1}(g_i) - i^-_{\tilde{G}_3}(g_i)| + |i^+_{\tilde{G}_1}(g_i) - i^+_{\tilde{G}_3}(g_i)| + |f^-_{\tilde{G}_1}(g_i) - f^-_{\tilde{G}_3}(g_i)| + |f^+_{\tilde{G}_1}(g_i) - f^+_{\tilde{G}_3}(g_i)| + |t_{R_1}(g_i) - t_{R_2}(g_i)| + |i_{R_1}(g_i) - i_{R_3}(g_i)| + |f_{R_1}(g_i) - f_{R_3}(g_i)|)$

From (3), we conclude that

$D_i(Q_1, Q_3) \ge D_i(Q_1, Q_2)$

$$\Rightarrow \frac{D_i(Q_1, Q_3)}{9} \ge \frac{D_i(Q_1, Q_2)}{9}$$

$$\Rightarrow -\frac{D_i(Q_1, Q_3)}{9} \le -\frac{D_i(Q_1, Q_2)}{9}$$

$$\Rightarrow \left[1-\frac{D_i(Q_1, Q_3)}{9}\right] \le \left[1-\frac{D_i(Q_1, Q_2)}{9}\right]$$

$$\Rightarrow \frac{1}{n}\sum_{i=1}^{n}\left[1-\frac{D_i(Q_1, Q_3)}{9}\right] \le \frac{1}{n}\sum_{i=1}^{n}\left[1-\frac{D_i(Q_1, Q_3)}{9}\right]$$

$\Rightarrow SM(Q_1, Q_3) \le SM(Q_1, Q_2)$

Similarly we can shows that $SM(Q_1, Q_3) \le SM(Q_2, Q_3)$, hence the proof.

4 MCGDM methods based on similarity measure in NCS environment

In this section we propose a new MCGDM method based on similarity measure in NCS environment. Assume that

$\alpha=\{\alpha_1,\alpha_2,\alpha_3,...,\alpha_n\}$ be a set of n alternatives with criteria $\beta=\{\beta_1,\beta_2,\beta_3,...,\beta_m\}$ and $\gamma=\{\gamma_1,\gamma_2,\gamma_3,...,\gamma_r\}$ be the r decision makers. Let $\Psi=\{\Psi_1,\Psi_2,\Psi_3,...,\Psi_r\}$ be the weight vestor of decision makers, where $\Psi_k>0$ and $\sum_{k=1}^{r}\Psi_k=1$. Proposed MCGDM method is presented using the following steps.

Step1. Formation of ideal NCS decision matrix

Ideal NCS decision matrix is an important matrix for similarity measure of MCGDM. Here we construct an ideal NCS matrix in the form

$$M=\begin{pmatrix} & \beta_1 & \beta_2 & \dots\dots & \beta_m \\ \alpha_1 & Q_{11} & Q_{12} & \dots & Q_{1m} \\ \alpha_2 & Q_{21} & Q_{22} & & Q_{2m} \\ . & . & & \dots & . \\ \alpha_n & Q_{n1} & Q_{n2} & \dots & .Q_{nm} \end{pmatrix} \quad (4)$$

Where $Q_{ij}=\langle G_{ij}, R_{ij}\rangle$, i = 1, 2, 3,…, n. j = 1, 2, 3, …, m.

Step 2. Construction of NCS decision matrix

Since r decision makers are involved in the decision making process, the k-th (k = 1, 2, 3,…, r) decision maker provides the evaluation information of the alternative α_i (i= 1, 2, 3,…, n) with respect to criteria β_j (j= 1, 2, 3,…, m) in terms of the NCS. The k-th decision matrix denoted by M^k (See eq. (5)) is construected as follows:

$$M^k=\langle Q_{ij}^{k}\rangle=\begin{pmatrix} & \beta_1 & \beta_2 & \dots\dots & \beta_m \\ \alpha_1 & Q_{11}^k & Q_{12}^k & \dots & Q_{1m}^k \\ \alpha_2 & Q_{21}^k & Q_{22}^k & & Q_{2m}^k \\ . & . & & \dots & . \\ \alpha_n & Q_{n1}^k & Q_{n2}^k & \dots & .Q_{nm}^k \end{pmatrix} \quad (5)$$

Where k = 1, 2, 3,…, r. i = 1, 2, 3,…, n. j = 1, 2, 3,…, m.

Step 3. Determination of attribute weight

All attribute are not equally important in decision making situation. Every decision maker provides their own opinion regarding to the attribute weight in terms of linguistic variables that can be converted into NCS. Let $w_k(\beta_j)$ be the attribute weight for the attribute β_j given by the k-th decision maker in term of NCS. We convert $w_k(\beta_j)$ into fuzzy number as follows:

$$w_k^F(\beta_j)=\begin{cases}(1-\sqrt{\dfrac{V_{kj}}{9}}),\text{if } \beta_j\in\beta \\ 0 \quad \text{otherwise}\end{cases} \quad (6)$$

where V_{kj} =

$$\sqrt{\left\{\begin{matrix}(1-t_k^-(\beta_j))^2+(1-t_k^+(\beta_j))^2+(i_k^-(\beta_j))^2+(i_k^+(\beta_j))^2 \\ +(f_k^-(\beta_j))^2+(f_k^+(\beta_j))^2+(1-t_k(\beta_j))^2 \\ +(i_k(\beta_j))^2+(f_k(\beta_j))^2\end{matrix}\right\}}.$$

Then aggregate weight for the criteria β_j can be determined as:

$$W_j=\frac{(1-\prod_{k=1}^{r}(1-w_k^F(\beta_j))}{\sum_{k=1}^{r}(1-\prod_{k=1}^{r}(1-w_k^F(\beta_j))} \quad (7)$$

Here $\sum_{k=1}^{r}W_j=1$.

Step 4. Calculation of weighted similarity measure

We now calculate weighted similarity measure between idel matrix M and M^k as follows:

$$S^w(M,M^k)=\langle\lambda_i^k\rangle$$

$$=(\lambda_1^k,\lambda_2^k,...,\lambda_n^k)^T=\left(\frac{1}{m}\sum_{j=1}^{m}(1-\frac{D_{ij}^k}{9})W_j\right)_{i=1}^{n} \quad (8)$$

Here, k =1, 2, 3,…, r.

Step 5. Ranking of alternatives

In order to rank alternatives, we propose the formula (see eq.9):

$$\rho_i=\sum_{k=1}^{r}\psi_k\lambda_i^k \quad (9)$$

We arrange alternatives according to the descending order values of ρ_i. The highest value of ρ_i (i= 1, 2, 3,…, n) reflects the best alternative.

5 Numerical example

We solve a MCGDM problem adapted from [108] to demonstrate the applicability and effectiveness of the proposed method. Assume that an investment company wants to invest a sum of money in the best option. The investment company forms a decision making committee comprising of three members (k_1, k_2, k_3) to make a panel of four alternatives to invest money. The alternatives are Car company (α_1), Food company (α_2), Computer company

(α_3) and Arm company (α_4). Decision makers take decision based on the criteria namely, risk analysis (β_1), growth analysis (β_2), environment impact (β_3) and criterion weights are provided by the decision makers in terms of linguistic variables that can be converted into NCS.(See Table 1).

Table 1: Linguistic term for rating of attribute/ criterion

Linguistic terms	NCS
Very important (VI)	<[.7, .9], [.1, .2], [.1, .2], (.9, .2,.2)>
Important (I)	<[.6, .8], [.2, .3], [.2, .4], (.8, .3, .4)>
Medium (M)	<[.4, .5], [.4, .5], [.4, .5], (.5, .5, .5)>
Unimportant (UI)	<[.3, .4], [.5, .6], [.5, .7], (.4, .6, .7)>
Very unimportant (VUI)	<[.1, .2], [.6, .8], [.7, .9], (.2, .8, .9)>

Step1. Formation of ideal NCS decision matrix

We construct ideal NCS decision matrix (see eq.(10).

$$M=\begin{array}{c} \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{array}\begin{pmatrix} \beta_1 & \beta_2 & \beta_3 \\ \langle [1,1],[0,0],[0,0],(1,0,0)\rangle & \langle [1,1],[0,0],[0,0],(1,0,0)\rangle & \langle [1,1],[0,0],[0,0],(1,0,0)\rangle \\ \langle [1,1],[0,0],[0,0],(1,0,0)\rangle & \langle [1,1],[0,0],[0,0],(1,0,0)\rangle & \langle [1,1],[0,0],[0,0],(1,0,0)\rangle \\ \langle [1,1],[0,0],[0,0],(1,0,0)\rangle & \langle [1,1],[0,0],[0,0],(1,0,0)\rangle & \langle [1,1],[0,0],[0,0],(1,0,0)\rangle \\ \langle [1,1],[0,0],[0,0],(1,0,0)\rangle & \langle [1,1],[0,0],[0,0],(1,0,0)\rangle & \langle [1,1],[0,0],[0,0],(1,0,0)\rangle \end{pmatrix} \quad (10)$$

Step 2. Construction of NCS decision matrix

The NCS decision matrices are constructed for four alternatives with respect to the three criteria.

Decision matrix for k_1 in NCS form

M^1 =

$$M^1=\begin{array}{c} \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4 \end{array}\begin{pmatrix} \beta_1 & \beta_2 & \beta_3 \\ \langle [.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)\rangle & \langle [.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)\rangle & \langle [.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)\rangle \\ \langle [.6,.8],[.2,.3],[.2,.4],(.8,.3,.4)\rangle & \langle [.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)\rangle & \langle [.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)\rangle \\ \langle [.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)\rangle & \langle [.6,.8],[.2,.3],[.2,.4],(.8,.3,.4)\rangle & \langle [.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)\rangle \\ \langle [.3,.4],[.5,.6],[.5,.7],(.4,.6,.7)\rangle & \langle [.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)\rangle & \langle [.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)\rangle \end{pmatrix}$$

Decision matrix for k_2 in NCS form

M^2 =

$$
\begin{array}{c}
 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4
\end{array}
\begin{pmatrix}
\beta_1 & \beta_2 & \beta_3 \\
\langle[.3,.4],[.5,.6],[.5,.7],(.4,.6,.7)\rangle & \langle[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)\rangle & \langle[.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)\rangle \\
\langle[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)\rangle & \langle[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)\rangle & \langle[.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)\rangle \\
\langle[.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)\rangle & \langle[.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)\rangle & \langle[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)\rangle \\
\langle[.6,.8],[.2,.3],[.2,.4],(.8,.3,.4)\rangle & \langle[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)\rangle & \langle[.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)\rangle
\end{pmatrix}
$$

Decision matrix for k_3 in NCS form

$M^3 =$

$$
\begin{array}{c}
 \\ \alpha_1 \\ \alpha_2 \\ \alpha_3 \\ \alpha_4
\end{array}
\begin{pmatrix}
\beta_1 & \beta_2 & \beta_3 \\
\langle[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)\rangle & \langle[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)\rangle & \langle[.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)\rangle \\
\langle[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)\rangle & \langle[.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)\rangle & \langle[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)\rangle \\
\langle[.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)\rangle & \langle[.6,.8],[.2,.3],[.2,.4],(.8,.3,.4)\rangle & \langle[.6,.8],[.2,.3],[.2,.4],(.8,.3,.4)\rangle \\
\langle[.7,.9],[.1,.2],[.1,.2],(.9,.2,.2)\rangle & \langle[.4,.5],[.4,.5],[.4,.5],(.5,.5,.5)\rangle & \langle[.3,.4],[.5,.6],[.5,.7],(.4,.6,.7)\rangle
\end{pmatrix}
$$

Step 3. Determination of attribute weight

The linguistic terms shown in Table 1 are used to evaluate each attribute. The importance of each attribute for every decision maker is rated with linguistic terms shown in Table 2. Linguistic terms are converted into NCS (See Table 3.) .

Table 2. Attribute rating in linguistic variables

	β_1	β_2	β_3
K_1	VI	M	I
K_2	VI	VI	M
K_3	M	VI	M

Table 3. Attribute rating in NCS

	β_1	β_2	β_3
K_1	<[.7, .9], [.1, .2], [.1, .2], (.9, .2,.2)>	<[.4, .5], [.4, .5], [.4, .5], (.5, .5, .5)>	<[.6, .8], [.2, .3], [.2, .4], (.8, .3, .4)>
K_2	<[.7, .9], [.1, .2], [.1, .2], (.9, .2,.2)>	<[.7, .9], [.1, .2], [.1, .2], (.9, .2,.2)>	<[.4, .5], [.4, .5], [.4, .5], (.5, .5, .5)>
K_3	<[.4, .5], [.4, .5], [.4, .5], (.5, .5, .5)>	<[.7, .9], [.1, .2], [.1, .2], (.9, .2,.2)>	<[.4, .5], [.4, .5], [.4, .5], (.5, .5, .5)>

Using eq. (6) and eq. (7), we obtain the attribute weights as follows: $w_1 = .36, w_2 = .37, w_3 = .27$. (11)

Step 4. Calculation of weighted similarity measures

We now calculate weighted similarity measures using the formula (8).

$$
S^w(M,M^1) = \begin{pmatrix} .25 \\ .22 \\ .19 \\ .24 \end{pmatrix}, S^w(M,M^2) = \begin{pmatrix} .18 \\ .20 \\ .25 \\ .22 \end{pmatrix}, S^w(M,M^3) = \begin{pmatrix} .20 \\ .21 \\ .25 \\ .20 \end{pmatrix} \qquad (12)
$$

Step 5. Ranking of alternatives

We rank the alternatives according to the descending value of ρ_i (i = 1, 2, 3, 4) using eq.(10), eq.(11), and eq. (12).

We obtain $\rho_1 = .202, \rho_2 = .206, \rho_3 = .232, \rho_4 = .216$, Therefore the ranking order is

$\rho_3 > \rho_4 > \rho_2 > \rho_1 \Rightarrow \alpha_3 > \alpha_4 > \alpha_2 > \alpha_1$.

Hence Computer company (α_3) is the best alternative for money investment.

6 Conclusion

In this paper we have defined similarity measure between neutrosophic cubic sets and proved its basic properties . We have developed a new multi criteria group decision making method basd on the proposed similarity measure. We also provide an illustrative example for multi criteria group decision making method to show its applicability and effectiveness. We have employed linguistic variables to present criteria weights and presented conversion of linguistic variables into neutrosophic cubic numbers. We have also proposed a conversion formula for neutrosophic cubic number into fuzzy number. The poposed method can be applied to other MCGDM making problems in neutrosophic cubic set environment such as banking system, engineering problems, school choice problems, teacher selection problem, etc. We also hope that the proposed method will open up a new direction of research work in neutrosophic cubic set environment.

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Multi Attribute Decision Making Strategy on Projection and Bidirectional Projection Measures of Interval Rough Neutrosophic Sets

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Abstract. In this paper, we define projection and bidirectional projection measures between interval rough neutrosophic sets and prove their basic properties. Then two new multi attribute decision making strategies are proposed based on interval rough neutrosophic projection and bidirectional projection measures respectively. Then the proposed methods are applied for solving multi attribute decision making problems. Finally, a numerical example is solved to show the feasibility, applicability and effectiveness of the proposed strategies.

Keywords: Projection measure, Bidirectional projection measure, Interval rough neutrosophic set, MADM problem.

1 Introduction

The concept of neutrosophic set[1, 2, 3, 4, 5] introduced by Smarandache is a generalization of crisp set[6], fuzzy set[7] and intuitionistic fuzzy set[8]. To use neutrosophic set in real fields, Wang et al. extended it to single valued neutrosophic set[9].

Broumi et al. introduced rough neutrosophic set[10, 11] by combining the concept of rough set[12] and neutrosophic set.

Broumi and Smarandache defined interval rough neutrosophic set[13] by combining the concept of rough set and interval neutrosophic set theory[14].

Projection measure is a very useful for solving decision making problems because it takes into account the distance as well as the included angle between points. Yue [15] studied projection based MADM problem in crisp environment.Yue also[16] presented a projection method to obtain weights of the experts in a group decision making problem. Xu and Da [17] and Xu [18] studied projection method for decision making in uncertain environment with preference information. Yang et al. [19] develop projection method for material selection in fuzzy environment. Xu and Hu [20] developed two projection based models for MADM in intuitionistic fuzzy and interval valued intuitionistic fuzzy environment. Zeng et al. [21] provided weighted projection algorithm for intuitionistic fuzzy MADM problems and interval-valued intuitionistic fuzzy MADM problems. Chen and Ye [22] developed the projection based model for solving MADM problem and applied it to select clay-bricks in construction field.

To overcome the shortcomings of the general projection measure Ye [23] introduced a bidirectional projection measure between single valued neutrosophic numbers and developed MADM method for selecting problems of mechanical design schemes under a single valued neutrosophic environment. Ye [24] also presented the bidirectional projection method for multiple attribute group decision making with neutrosophic numbers. Dey et al. [25] defined weighted projection measure with interval neutrosophic environment and applied it to solve MADM problems with interval valued neutrosophic information. Yue [26] proposed a projection based approach for partner selection in a group decision making problem with linguistic value and intuitionistic fuzzy information.Dey et al. [27] defined projection, bidirectional projection and hybrid projection measures between bipolar neutrosophic sets and presented bipolar neutrosophic projection based models for MADM problems. Pramanik et al. [28] defined projection and bidirectional projection measure between rough neutrosophic sets and proposed the decision making methods based on them.

Research gap MADM strategy using projection and bidirectional projection measures under interval rough neutrosophic environment.

Research questions

(i) Is it possible to define two new projection and bidirectional projection measure between interval rough neutrosophic sets?

(ii) Is it possible to develop two new MADM strategies based on the proposed measures in interval rough neutrosophic environment?

The objectives of the paper are

(i) To define two new projection and bidirectional projection measure between interval rough neutrosophic sets.

(ii) To develop two new MADM strategies based on the proposed measures in interval rough neutrosophic environment.

Contributions

(i) In this paper, we propose projection and bidirectional projection measures under interval rough neutrosophic environment.

(ii) In this paper, we develop two new MADM strategies based on the proposed measures in interval rough neutrosophic environment.

(iii) We also present numerical example to show the effectiveness and applicability of the proposed measures.

Rest of the paper is organized as follows: Section 2 describes preliminaries of neutrosophic number, SVNS, RNS and IRNS. Section 3 presents definitions and properties of proposed projection and bidirectional projection measure between IRNSs. Section 4 describes the MADM methods based on projection and bidirectional projection measures of IRNSs. In section 5 we describe a numerical example. Finally, section 6 presents the conclusion.

2 Preliminaries

In this Section, we provide some basic definitions regarding SVNSs, IRNSs which are useful in the paper.

2.1 Neutrosophic set:

In 1999, Smarandache gave the following definition of neutrosophic set(NS) [1].

Definition 2.1.1. Let X be a space of points (objects) with generic element in X denoted by x. A NS A in X is characterized by a truth-membership function T_A, an indeterminacy membership function I_A and a falsity membership function F_A. The functions T_A , I_A and F_A are real standard or non-standard subsets of $(^-0,1^+)$ that is $T_A:X\rightarrow (^-0, 1^+)$, $I_A:X\rightarrow (^-0, 1^+)$ and $F_A:X\rightarrow (^-0, 1^+)$. It should be noted that there is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$ i.e. $^-0 \le T_A(X)+I_A(X)+F_A(X)\le 3^+$

Definition 2.1.2: (complement)

The complement of a neutrosophic set A is denoted by C(A) and is defined by $T_{c(A)}(x) = \{1^+\}-T_A(x), I_{c(A)}(x)=\{1^+\}-I_A(x), F_{c(A)}(x)=\{1^+\}-F_A(x)$.

Definition 2.1.3: (Containment)

A neutrosophic set A is contained in the other neutrosophic set B, denoted by $A\subseteq B$ iff

$$\inf T_A(x)\le \inf T_B(x), \sup T_A(x)\le \sup T_B(x),$$
$$\inf I_A(x)\ge \inf I_B(x), \sup I_A(x)\ge \sup I_B(x),$$
$$\inf F_A(x)\ge \inf F_B(x), \sup F_A(x)\ge \sup F_B(x)\ \forall x\in X$$

Definition 2.1.4: (Single-valued neutrosophic set).

Let X be a universal space of points (objects) with a generic element of X denoted by x. A single valued neutrosophic set A is characterized by a truth membership function $T_A(x)$, a falsity membership function $F_A(x)$ and indeterminacy function $I_A(x)$ with

$$T_A(x), I_A(x) \text{ and } F_A(x) \in [0,1]\ \forall x \text{ in } X$$

When X is continuous, a SNVS S can be written as follows

$$A=\int_x < T_A(x), F_A(x), I_A(x) > /x\forall x\in X$$

and when X is discrete, a SVNS S can be written as follows

$$A=\sum < T_A(x), F_A(x), I_A(x) > /x\forall x\in X$$

For a SVNS S, $0\le \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \le 3$.

Definition2.1.5:

The complement of a single valued neutrosophic set A is denoted by c(A) and is defined by $T_{c(A)}(x) = F_A(x)$, $I_{c(A)}(x) = 1-I_A(x)$, $F_{c(A)}(x) = T_A(x)$.

Definition 2.1.6: A SVNS A is contained in the other SVNS B, denoted as $A\subseteq B$ iff,

$$T_A(x)\le T_B(x), I_A(x)\ge I_B(x)$$

and $F_A(x)\ge F_B(x)\forall x\in X$.

2.2Rough neutrosophic set

Rough neutrosophic sets [10, 11] are the generalization of rough fuzzy sets [29, 30] and rough intuitionistic fuzzy sets [31].

Definition 2.2.1:

Let Y be a non-null set and R be an equivalence relation on Y. Let P be neutrosophic set in Y with the membership function T_P , indeterminacy function I_P and non-membership function F_P . The lower and the upper approximations of P in the approximation (Y, R) denoted by are respectively defined as:

$\underline{N(P)} = \ll x, T_{\underline{N(P)}}(x), I_{\underline{N(P)}}(x), F_{\underline{N(P)}}(x) > / y \in [x]_R, x \in Y >$

and

$\overline{N(P)} = \ll x, T_{\overline{N(P)}}(x), I_{\overline{N(P)}}(x), F_{\overline{N(P)}}(x) > / y \in [x]_R, x \in Y >$

where,

$T_{\underline{N(P)}}(x) = \wedge z \in [x]_R\, T_P(Y),$
$I_{\underline{N(P)}}(x) = \wedge z \in [x]_R\, I_P(Y),$
$F_{\underline{N(P)}}(x) = \wedge z \in [x]_R\, F_P(Y)$

and

$T_{\overline{N(P)}}(x) = \vee z \in [x]_R\, T_P(Y),$
$I_{\overline{N(P)}}(x) = \vee z \in [x]_R\, I_P(Y),$.
$F_{\overline{N(P)}}(x) = \vee z \in [x]_R\, F_P(Y)$

So,

$0 \le T_{\underline{N(P)}}(x) + I_{\underline{N(P)}}(x) + F_{\underline{N(P)}}(x) \le 3$

and

$0 \le T_{\overline{N(P)}}(x) + I_{\overline{N(P)}}(x) + F_{\overline{N(P)}}(x) \le 3$

Here $\vee$ and $\wedge$ denote "max" and "min" operators respectively, $T_P(y)$, $I_P(y)$ and $F_P(y)$ are the membership, indeterminacy and non-membership of Y with respect to P.

Thus NS mapping,

$\underline{N}, \overline{N} : N(Y) \to N(Y)$ are, respectively, referred to as the lower and upper rough NS approximation operators, and the pair $(\underline{N(P)}, \overline{N(P)})$ is called the rough neutrosophic set in (Y, R).

Definition 2.2.2 If $N(P) = (\underline{N(P)}, \overline{N(P)})$ is a rough neutrosophic set in (Y, R), the rough complement of N(P) is the rough neutrosophic set denoted by

$\sim N(P) = ((\overline{N(P)})^C, (\underline{N(P)})^C)$

,where

$(\underline{N(P)})^C$ and $(\overline{N(P)})^C$

are the complements of neutrosophic sets $\underline{N}(P)$ and $\overline{N(P)}$ respectively.

2.3 Interval rough neutrosophic set

Interval neutrosophic rough set is the hybrid structure of rough sets and interval neutrosophic sets. According to Broumi and Smarandache interval neutrosophic roughset is the generalizations of interval valued intuitionistic fuzzy rough set.

Definition 2.3.1

Let R be an equivalence relation on the universal set U.Then the pair (U, R) is called a Pawlak approximationspace. An equivalence class of R containing x will bedenoted by $[x]_R$ for X $\in$ U, the lower and upper approximationof X with respect to (U, R) are denoted by respectively,

$\underline{R}X$ and $\overline{R}X$ and are defined by

$\underline{R}X = \{x \in U : [x]_R \subseteq X\},$

$\overline{R}X = \{x \in U : [x]_R \cap X \neq \emptyset\}.$

Now if $\underline{R}X = \overline{R}X$, then X is called definable; otherwise Xis called a rough set.

Definition 2.3.2

Let U be a universe and X, a rough set in U. An intuitionistic fuzzy rough set A in U is characterized by a membership function $\mu A: U \to [0, 1]$ and non-membership functionν_A: $U \to [0, 1]$ such that $\mu_A(\underline{R}X)=1$and $\nu_A(\underline{R}X) = 0$ ie, $[\mu_A(x), \nu_A(x)]=[1,0]$ if $x \in (\underline{R}X)$ and $\mu_A(U - \overline{R}X) = 0$, $\nu_A(U - \overline{R}X)=1$

ie,

$0 \le \mu_A(\overline{R}X - \underline{R}X) + \nu_A(\overline{R}X - \underline{R}X) \le 1$

Definition 2.3.3

Assume that, (U, R) be a Pawlak approximation space, for an interval neutrosophic set

$A = \{<x, [T_A^L(x), T_A^U(x)], [I_A^L(x), I_A^U(x)], [F_A^L(x), F_A^U(x)]> : x \in U\}$

The lower approximation $\underline{A}_R$ and the upper approximation $\overline{A}_R$ of A in the Pawlak approximation space (U, R) are expressed as follows:

$\underline{A}_R = \{<x, [\wedge_{y\in[x]_R}\{T_A^L(y)\}, \ _{y\ [x]}\ \{T_A(y)\}],$
$[\vee_{y\in[x]_R}\{I_A^L(y)\}, \ _{y\ [x]}\ \{I_A(y)\}],$
$[\vee_{y\in[x]_R}\{F_A^L(y)\}, \vee_{y\ [x]}\{F_A(y)\}]> : x \in U\}$

$\overline{A}_R = \{<x, [\vee_{y\in[x]_R}\{T_A^L(y)\}, \ _{y\ [x]}\ \{T_A(y)\}],$
$[\wedge_{y\in[x]_R}\{I_A^L(y)\}, \ _{y\ [x]}\ \{I_A(y)\}],$
$[\wedge_{y\in[x]_R}\{F_A^L(y)\}, \wedge_{y\ [x]}\{F_A(y)\}]> : x \in U\}$

The symbols $\wedge$ and $\vee$ indicate "min" and "max" operators respectively. R denotes an equivalence relation for interval neutrosophic set A. Here $[x]_R$ is the equivalence class of the element x. It is obvious that

$[\wedge_{y\in[x]_R}\{T_A(y)\}, \wedge_{y\ [x]}\{T_A^L(y)\}] \subset [0,1],$
$[\vee_{y\in[x]_R}\{I_A^U(y)\}, \vee_{y\ [x]}\{I_A(y)\}] \subset [0,1],$
$[\vee_{y\in[x]_R}\{F_A^U(y)\}, \vee_{y\ [x]}\{F_A(y)\}] \subset [0,1].$

and $0 \le \wedge_{y\in[x]_R}\{T_A^U(y)\} + \vee_{y\in[x]_R}\{I_A^U(y)\} + \vee_{y\in[x]_R}\{F_A^U(y)\} \le 3$

Then $\underline{A}_R$ is an interval neutrosophic set (INS)

Similarly, we have

$$[\vee_{y\in[x]_R}\{T_A^L(y)\},\vee_{y\in[x]_R}\{T_A^U(y)\}]\subset[0,1],$$
$$[\wedge_{y\in[x]_R}\{I_A^L(y)\},\wedge_{y\in[x]_R}\{I_A^U(y)\}]\subset[0,1],$$
$$[\wedge_{y\in[x]_R}\{F_A^L(y)\},\wedge_{y\in[x]_R}\{F_A^U(y)\}]\subset[0,1]$$

and

$$0\le \vee_{y\in[x]_R}\{T_A^U(y)\}+\wedge_{y\in[x]_R}\{I_A^U(y)\}+\wedge_{y\in[x]_R}\{F_A^U(y)\}]\le 3$$

Then $\underline{A}_R$ is an interval neutrosophic set.

If $\underline{A}_R = \overline{A}_R$ then A is a definable set, otherwise A is an interval valued neutrosophic rough set. Here, $\underline{A}_R$ and $\overline{A}_R$ are called the lower and upper approximations of interval neutrosophic set with respect to approximation space (U,R) respectively. $\underline{A}_R$ and $\overline{A}_R$ are simply denoted by $\underline{A}$ and $\overline{A}$ respectively.

3 Projection and Bidirectional projection measure of interval rough neutrosophic sets :

Existing projection and bidirectional projection measure does not deal with interval rough neutrosophic set(IRNS)s. Therefore, a new projection and bidirectional projection measure between IRNSs is proposed.

Assume that there are two IRNSs

$$M=\{<x_i,([\underline{T}_{iM}^-,\underline{T}_{iM}^+],[\underline{I}_{iM}^-,\underline{I}_{iM}^+],[\underline{F}_{iM}^-,\underline{F}_{iM}^+],[\overline{T}_{iM}^-,\overline{T}_{iM}^+],[\overline{I}_{iM}^-,\overline{I}_{iM}^+],[\overline{F}_{iM}^-,\overline{F}_{iM}^+]>: i=1,2,...,n\}$$

and

$$N=\{<x_i,[\underline{T}_{iN}^-,\underline{T}_{iN}^+],[\underline{I}_{iN}^-,\underline{I}_{iN}^+],[\underline{F}_{iN}^-,\underline{F}_{iN}^+],[\overline{T}_{iN}^-,\overline{T}_{iN}^+],[\overline{I}_{iN}^-,\overline{I}_{iN}^+],[\overline{F}_{iN}^-,\overline{F}_{iN}^+]>: i=1,2,...,n\}$$

Then the inner product of M and N denoted by M.N can be defined as

$$M.N=\sum_{i=1}^{n}[\underline{T}_{iM}^-.\underline{T}_{iN}^- + \underline{T}_{iM}^+\underline{T}_{iN}^+ + \underline{I}_{iM}^-\underline{I}_{iN}^- + \underline{I}_{iM}^+\underline{I}_{iN}^+ + \underline{F}_{iM}^-\underline{F}_{iN}^- + \underline{F}_{iM}^+\underline{F}_{iN}^+ + \overline{T}_{iM}^-\overline{T}_{iN}^- + \overline{T}_{iM}^+\overline{T}_{iN}^+ + \overline{I}_{iM}^-\overline{I}_{iN}^- + \overline{I}_{iM}^+\overline{I}_{iN}^+ + \overline{F}_{iM}^-\overline{F}_{iN}^- + \overline{F}_{iM}^+\overline{F}_{iN}^+]$$

The modulus of M can be defined as

$$\|M\|=\sqrt{\sum_{i=1}^{n}\begin{bmatrix}(\underline{T}_{iM}^-)^2+(\underline{T}_{iM}^+)^2+(\underline{I}_{iM}^-)^2+(\underline{I}_{iM}^+)^2\\ +(\underline{F}_{iM}^-)^2+(\underline{F}_{iM}^+)^2+(\overline{T}_{iM}^-)^2+(\overline{T}_{iM}^+)^2\\ +(\overline{I}_{iM}^-)^2+(\overline{I}_{iM}^+)^2+(\overline{F}_{iM}^-)^2+(\overline{F}_{iM}^+)\end{bmatrix}}$$

and the modulus of N can be defined as

$$\|N\|=\sqrt{\sum_{i=1}^{n}\begin{bmatrix}(\underline{T}_{iN}^-)^2+(\underline{T}_{iN}^+)^2+(\underline{I}_{iN}^-)^2+(\underline{I}_{iN}^+)^2\\ +(\underline{F}_{iN}^-)^2+(\underline{F}_{iN}^+)^2+(\overline{T}_{iN}^-)^2+(\overline{T}_{iN}^+)^2\\ +(\overline{I}_{iN}^-)^2+(\overline{I}_{iN}^+)^2+(\overline{F}_{iN}^-)^2+(\overline{F}_{iN}^+)\end{bmatrix}}$$

Definition4.1.The projection of M on N can be defined as

$$\Pr oj(M)_N=\frac{1}{\|N\|}M.N.$$

Definition4.2.The bidirectional projection measure between the RNSs M and N is defined as

$$B\Pr oj(M,N)=\frac{1}{1+\big|\|M\|-\|N\|\big|M.N}=\frac{\|M\|\|N\|}{\|M\|\|N\|+\big|\|M\|-\|N\|\big|M.N}$$

Here also the bidirectional projection measure satisfies the following properties :

(1) BProj(M,N) = BProj(N,M);

(2) $0\le BProj(M,N)\le 1$;

(3) BProj(M,N) = 1, iff M = N.

Proof:

(i)

$$B\Pr oj(M,N)=\frac{1}{1+\big|\|M\|-\|N\|\big|M.N}=\frac{1}{1+\big|\|N\|-\|M\|\big|N.M}=B\Pr oj(N,M)$$

(ii) As

$$\frac{1}{1+\big|\|M\|-\|N\|\big|M.N}\ge 0$$

and

$$\frac{1}{1+\big|\|M\|-\|N\|\big|M.N}\le 1$$

so, $0\le BProj(M,N)\le 1$;

(iii) If M=N then

$$B\Pr oj(M,N)=B\Pr oj(M,M)=\frac{1}{1+\big|\|M\|-\|M\|\big|M.M}=1$$

4. Projection And Bidirectional Projection Based Decision Making Methods For MADM Problems With Interval Rough Neutrosophic Information

In this section, we develop projection and bidirectional projection based decision making models to solve MADM problems with interval rough neutrosophic information. Consider C={C_1, ...,C_m} be the set of attributes and A={A_1,……, A_n} be a set of alternatives. Now we provide two algorithms for MADM problems involving interval rough neutrosophic information.

4.1. Algorithm 1.(see Fig 1)

Step 1. The value of alternative A_i(i=1,…..,n) for the attribute C_j(j=1,……,m) is evaluated by the decision maker in terms of IRNSs and the interval rough neutrosophic decision matrix is constructed as:

$$D = \langle z_{ij} \rangle_{n\times m} = \begin{pmatrix} z_{11}\ z_{12}\ \ldots\ldots\ldots z_{1m} \\ z_{21}\ z_{22}\ \ldots\ldots\ldots z_{2m} \\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots \\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots \\ z_{n1}\ z_{n2}\ \ldots\ldots\ldots z_{nm} \end{pmatrix}$$

where

$$\underline{z_{ij}} = \langle([\underline{T}^-_{iM}, \underline{T}^+_{iM}], [\underline{I}^-_{iM}, \underline{I}^+_{iM}], [\underline{F}^-_{iM}, \underline{F}^+_{iM}],$$
$$[T^-_{iM}, T^+_{iM}], [I^-_{iM}, I^+_{iM}], [F^-_{iM}, F^+_{iM}])\rangle$$

with

$$0 \le \vee_{y\in[x]_R} \{T^U_A(y)\} + \wedge_{y\in[x]_R} \{I^U_A(y)\} + \wedge_{y\in[x]_R} \{F^U_A(y)\}] \le 3$$

Step 2. Calculate the weighted alternative decision matrix
For the attribute C_j (j=1,……,m) the weight vector of attribute is considered as : W = (w_1, w_2, … , w_m) with

$$w_j \ge 0 \quad \text{and} \quad \sum_{i=1}^{n} w_j = 1$$

On calculating

$$s_{ij} = \langle([w_j \underline{T}^-_{iM}, w_j \underline{T}^+_{iM}], [w_j \underline{I}^-_{iM}, w_j \underline{I}^+_{iM}],$$
$$[w_j \underline{F}^-_{iM}, w_j \underline{F}^+_{iM}], [w_j T^-_{iM}, w_j T^+_{iM}],$$
$$[w_j I^-_{iM}, w_j I^+_{iM}], [w_j F^-_{iM}, w_j F^+_{iM}])\rangle$$

for i=1, 2, … , n and j=1, 2, … , m , we obtain the weighted alternative decision matrix

$$S = \langle s_{ij} \rangle_{n\times m} = \begin{pmatrix} s_{11}\ s_{12}\ \ldots\ldots\ldots s_{1m} \\ s_{21}\ s_{22}\ \ldots\ldots\ldots s_{2m} \\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots \\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots \\ s_{n1}\ s_{n2}\ \ldots\ldots\ldots s_{nm} \end{pmatrix}$$

Step 3. Determine the ideal solution S^*.
For benefit type attribute,

$$S^* = \{(\min_i \underline{T}_{ij}, \max_i \underline{I}_{ij}, \max_i \underline{F}_{ij}), (\max_i \overline{T}_{ij}, \min_i \overline{I}_{ij}, \min_i \overline{F}_{ij})\}$$

For cost type attribute,

$$S^* = \{(\max_i \underline{T}_{ij}, \min_i \underline{I}_{ij}, \min_i \underline{F}_{ij}), (\min_i \overline{T}_{ij}, \max_i \overline{I}_{ij}, \max_i \overline{F}_{ij})\}$$

Step 4. Compute the projection measure between S^* and Z_i = $\langle Z_{ij} \rangle_{nxm}$ for all i = 1, ….., n and j = 1, ….., m.
Step 5. Ranking of alternatives is prepared based on the values of projection measure. The highest value reflects the best alternatives.
Step 6. End.

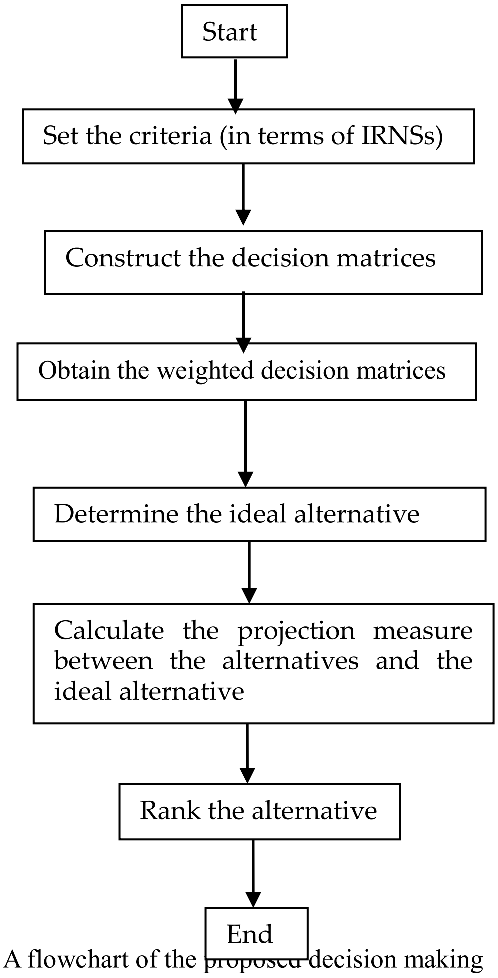

Fig 1. A flowchart of the proposed decision making method

4.2. Algorithm 2.(see Fig 2)

Step 1. The value of alternative A_i(i=1,…..,n) for the attribute C_j(j=1,……,m) is evaluated by the decision maker in terms of IRNSs and the interval rough neutrosophic decision matrix is constructed as:

$$D = \langle z_{ij} \rangle_{n\times m} = \begin{pmatrix} z_{11}\ z_{12}\ \ldots\ldots\ldots z_{1m} \\ z_{21}\ z_{22}\ \ldots\ldots\ldots z_{2m} \\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots \\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots\ \ldots \\ z_{n1}\ z_{n2}\ \ldots\ldots\ldots z_{nm} \end{pmatrix}$$

where

$$\underline{z_{ij}} = \langle([\underline{T}^-_{iM}, \underline{T}^+_{iM}], [\underline{I}^-_{iM}, \underline{I}^+_{iM}], [\underline{F}^-_{iM}, \underline{F}^+_{iM}],$$
$$[T^-_{iM}, T^+_{iM}], [I^-_{iM}, I^+_{iM}], [F^-_{iM}, F^+_{iM}])\rangle$$

with

$$0 \le \vee_{y\in[x]_R}\{T^U_A(y)\} + \wedge_{y\in[x]_R}\{I^U_A(y)\} +$$
$$\wedge_{y\in[x]_R}\{F^U_A(y)\}] \le 3$$

Step 2. Calculate the weighted alternative decision matrix

For the attribute C_j (j=1,……,m) the weight vector of attribute is considered as : $W = (w_1, w_2, \dots, w_m)$ with

$$w_j \ge 0 \quad \text{and} \quad \sum_{i=1}^{n} w_j = 1$$

On calculating

$$s_{ij} = <([w_j \underline{T}^-_{iM}, w_j \underline{T}^+_{iM}], [w_j \underline{I}^-_{iM}, w_j \underline{I}^+_{iM}],$$
$$[w_j \underline{F}^-_{iM}, w_j \underline{F}^+_{iM}], [w_j \overline{T}^-_{iM}, w_j \overline{T}^+_{iM}],$$
$$[w_j \overline{I}^-_{iM}, w_j \overline{I}^+_{iM}], [w_j \overline{F}^-_{iM}, w_j \overline{F}^+_{iM}])>$$

for i=1, 2, … , n and j=1, 2, … , m , we obtain the weighted alternative decision matrix

$$S = < s_{ij} >_{n\times m} = \begin{pmatrix} s_{11}\ s_{12} \,\dots\dots\dots\, s_{1m} \\ s_{21}\ s_{22} \,\dots\dots\dots\, s_{2m} \\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots \\ \dots\ \dots\ \dots\ \dots\ \dots\ \dots \\ s_{n1}\ s_{n2} \,\dots\dots\dots\, s_{nm} \end{pmatrix}$$

Step 3. Determine the ideal solution S^*.

For benefit type attribute,

$$S^* = \{(\min_i \underline{T}_{ij}, \max_i \underline{I}_{ij}, \max_i \underline{F}_{ij}), (\max_i \overline{T}_{ij}, \min_i \overline{I}_{ij}, \min_i \overline{F}_{ij})\}$$

For cost type attribute,

$$S^* = \{(\max_i \underline{T}_{ij}, \min_i \underline{I}_{ij}, \min_i \underline{F}_{ij}), (\min_i \overline{T}_{ij}, \max_i \overline{I}_{ij}, \max_i \overline{F}_{ij})\}$$

.

Step 4. Compute the bidirectional projection measure between S^* and $Z_i = <Z_{ij}>_{nxm}$ for all i = 1, ….., n and j = 1, ….., m.

Step 5. Ranking of alternatives is prepared based on the values of bidirectional projection measure. The highest value reflects the best alternatives.

Step 6. End.

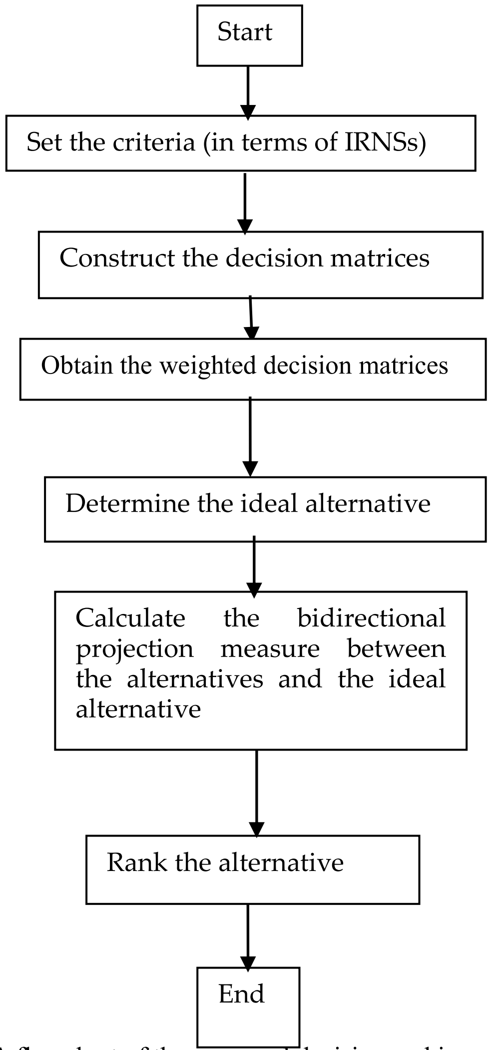

Fig 2. A flowchart of the proposed decision making method

5. A Numerical Example:

Assume that a decision maker intends to select the most suitable laptop for random use from the three initially chosen laptops (A_1, A_2, A_3) by considering four attributes namely: features C_1, reasonable price C_2, customer care C_3, risk factor C_4. Based on the proposed approach discussed in section 5, the considered problem is solved by the following steps:

Step1: Construct the decision matrix with interval rough neutrosophic number

The decision maker construct the decision matrix with respect to the three alternatives and four attributes in terms of interval rough neutrosophic number.

	C_1	C_2	C_3	C_4
A_1	<([.6, .7], [.3, .5], [.3, .4]), ([.8, .9], [.1, .3], [.1, .2])>	<([.5, .7], [.3, .4], [.1, .2]), ([.7, .9], [.3, .5], [.3, .4])>	<([.5, .6], [.4, .5], [.4, .6]), ([.7, .8], [.2, .4], [.3, .4])>	<([.8, .9], [.3, .4], [.5, .6]), ([.7, .8], [.3, .5], [.3, .5])>
A_2	<([.7, .8], [.2, .3], [.0, .2]), ([.7, .9], [.1, .2], [.1, .2])>	<([.6, .7], [.1, .2], [.0, .2]), ([.6, .7], [.1, .3], [.1, .3])>	<([.5, .7], [.2, .3], [.1, .2]), ([.6, .9], [.3, .5], [.2 .4])>	<([.7, .8], [.3, .5], [.1, .3]), ([.5, .7], [.5, .6], [.2, .3])>
A_3	<([.6, .7], [.3, .4], [.0, .3]), ([.6, .9], [.1, .2], [.1, .2])>	<([.5, .7], [.2, .4], [.2, .4]), ([.6, .8], [.1, .3], [.1, .2])>	<([.6, .8], [.2, .4], [.3, .4]), ([.6, .8], [.2, .5], [.3, .5])>	<([.4, .7], [.2, .4], [.4, .5]), ([.5, .8], [.2, .5], [.0, .2])>

Step 2: The weight vectors considered by the decision maker are 0.35, 0.25, 0.25 and 0.15 respectively. The weighted decision matrix is:

	C_1	C_2	C_3	C_4
S_1	<([0.21,0.245], [0.105,0.175], [0.105,0.14]), ([0.28,0.315], [0.035,0.105], [0.035,0.07])>	<([0.125,0.175], [0.075,0.1], [0.025,0.05]), ([0.175,0.225], [0.075,0.125], [0.075,0.1])>	<([0.125,0.15], [0.1,0.125], [0.1,0.15]), ([0.175,0.2], [0.05,0.1], [0.075,0.1])>	<([0.12,0.135], [0.045,0.06], [0.075,0.09]), ([0.105,0.12], [0.045,0.075], [0.045,0.75])>
S_2	<([0.245,0.28], [0.07,0.105], [0.0,0.07]), ([0.245,0.315], [0.035,0.07], [0.035,0.07])>	<([0.15,0.175], [0.025,0.05], [0.0,0.05]), ([0.15,0.175], [0.025,0.075], [0.025,0.075])>	<([0.125,0.175], [0.05,0.075], [0.025,0.05]), ([0.15,0.225], [0.075,0.125], [0.05,0.1])>	<([0.105,0.12], [0.045,0.75], [0.015,0.045]), ([0.075,0.105], [0.075,0.09], [0.03,0.045])>
S_3	<([0.21,0.245], [0.105,0.14], [0.0,0.105]), ([0.21,0.315], [0.035,0.7], [0.035,0.7])>	<([0.125,0.175], [0.05,0.1], [0.05,0.1]), ([0.15,0.2], [0.025,0.075], [0.025,0.05])>	<([0.15,0.2], [0.05,0.1], [0.075,0.1]), ([0.15,0.2], [0.05,0.125], [0.075,0.125])>	<([0.06,0.105], [0.03,0.06], [0.06,0.075]), ([0.075,0.12], [0.03,0.075], [0.0,0.03])>

Step3: Determine the benefit type attribute and cost type attribute

Here three benefit type attributes C_1, C_2, C_3 and one cost type attribute C_4. We calculate the ideal alternative as follows:

$$S^* = \{<([.21,.245],[.07,.175],[.105,.14]), ([.28,.315],[.035,.07],[.035,.07])>,$$
$$<([.15,.175],[.075,.1],[.05,.1]), ([.175,.225],[.025,.075],[.025,.05])>,$$
$$<([.15,.15],[.1,.1],[.1,.1]), ([.175,.225],[.075,.125],[.075,.125])>,$$
$$<([.12,.135],[.03,.06],[.015,.045]), ([.075,.105],[.075,.09],[.045,.075])>)>\}$$

Step4:Calculate the projection and bidirectional projection measure of the alternatives

$\|S_1\| = 0.918273$,
$\|S_2\| = 0.829533$,
$\|S_3\| = 0.832331$.
$\|S^*\| = 0.818175$.
$S_1.S^* = 0.815425$,
$S_2.S^* = 0.563137$,
$S_3.S^* = 0.7337$.

$\operatorname{Proj}(S_1)_{S^*} = 0.99663886$,
$\operatorname{Proj}(S_2)_{S^*} = 0.68828490$,
$\operatorname{Proj}(S_3)_{S^*} = 0.89675192$.
$\Rightarrow \operatorname{Proj}(S_1)_{S^*} > \operatorname{Proj}(S_3)_{S^*} > \operatorname{Proj}(S_2)_{S^*}$.
$\operatorname{BProj}(S_1, S^*) = 0.92453705$,
$\operatorname{BProj}(S_2, S^*) = 0.99364454$,
$\operatorname{BProj}(S_3, S^*) = 0.98972051$.
$\Rightarrow \operatorname{BProj}(S_2, S^*) > \operatorname{BProj}(S_3, S^*) > \operatorname{BProj}(S_1, S^*)$.

Step5: Rank the alternatives

Ranking of alternatives is prepared based on the descending order of projection and bidirectional measures. The highest value reflects the best alternatives.

Hence, according to the projection measure, the laptop A_1 is the best alternative and according to the bidirectional projection measure, the laptop A_2 is the best alternative. As bidirectional projection measure gives better result than projection measure, so A_2 is the best laptop for random use.

6. Comparative study and discussions:

Mondal and Pramanik study the MADM method in interval rough neutrosophic environment using cosine, dice and Jaccard similarity measure [32]. We take the same problem and solve the problem using projection and bidirectional projection measure based decision making method. In the existing methods, S_2 is the best alternatives. But in new method S_1 is the best alternative.

7. Conclusion:

In this paper, we have defined projection measure, weighted projection measure, bidirectional projection measure, weighted bidirectional projection measure between interval rough neutrosophic sets. We have also proved their basic properties. We have developed two new MADM strategies based on the proposed projection and bidirectional projection measures respectively. Finally, we have solved a numerical example to demonstrate the feasiblity, applicability and effectiveness of the proposed strategies. The proposed strategies can be applied to solve different MADM problems such as teacher selection [33, 34, 35], school selection [36], weaver selection [37, 38, 39], brick field selection [40, 41], logistics center location selection [42, 43], data mining [44] etc. The proposed strategies can also be extended for MAGDM in interval rough neutrosophic environment.

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Multi-Attribute Decision Making Based on Several Trigonometric Hamming Similarity Measures under Interval Rough Neutrosophic Environment

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Abstract. In this paper, the sine, cosine and cotangent similarity measures of interval rough neutrosophic sets is proposed. Some properties of the proposed measures are discussed. We have proposed multi attribute decision making approaches based on proposed similarity measures. To demonstrate the applicability, a numerical example is solved.

Keywords: sine hamming similarity measure, cosine hamming similarity measure, cotangent hamming similarity measure, interval rough neutrosophic set.

1 Introduction

The basic concept of neutrosophic set grounded by Smarandache [1, 2, 3, 4, 5] is a generalization of classical set or crisp set [6], fuzzy set [7], intuitionistic fuzzy set [8]. Wang et al.[9] extended the concept of neutrosophic set to single valued neutrosophic sets (SVNSs). Broumi et al. [10, 11] proposed new hybrid intelligent structure namely, rough neutrosophic set combing the concept of rough set theory [12] and the concept of neutrosophic set theory to deal with uncertainty and incomplete information. Rough neutrosophic set is the generalization of rough fuzzy sets [13, 14] and rough intuitionistic fuzzy sets [15]. Several studies of rough neutrosophic sets have been reported in the literature. Mondal and Pramanik [16] applied the concept of rough neutrosophic set in multi-attribute decision making based on grey relational analysis. Pramanik and Mondal [17] presented cosine similarity measure of rough neutrosophic sets and its application in medical diagnosis. Pramanik and Mondal [18] also proposed some rough neutrosophic similarity measures namely Dice and Jaccard similarity measures of rough neutrosophic environment. Mondal and Pramanik [19] proposed rough neutrosophic multi attribute decision making based on rough score accuracy function. Pramanik and Mondal [20] presented cotangent similarity measure of rough neutrosophic sets and its application to medical diagnosis. Pramanik and Mondal [21] presented trigonometric Hamming similarity measure of rough neutrosophic sets. Pramanik et al. [22] proposed rough neutrosophic multi attribute decision making based on correlation coefficient. Pramanik et al. [23] also proposed rough neutrosophic projection and bidirectional projection measures. Mondal et al. [24] presented multi attribute decision making based on rough neutrosophic variational coefficient similarity measures. Mondal at al. [25] also presented rough neutrosophic TOPSIS for multi attribute group decision making. Mondal and Pramanik [26] presented tri-complex rough neutrosophic similarity measure and its application in multi-attribute decision making. In 2015, Broumi and Smarandache [27] combined the concept of rough set theory [12] and interval neutrosophic set theory [28] and defined interval rough neutrosophic set. Pramanik et al. [29] presented multi attribute decision making based on projection and bidirectional projection measures under interval rough neutrosophic environment.

Multi-attribute decision making using trigonometric Hamming similarity measures under interval rough neutrosophic environment is not addressed in the literature.

Research gap MADM strategy using sine, cosine and cotangent similarity measures under interval rough neutrosophic environment.

Research questions

(i) Is it possible to define sine, cosine and cotangent similarity measures between interval rough neutrosophic sets?

(ii) Is it possible to develop new MADM strategies based on the proposed measures in interval rough neutrosophic environment?

The objectives of the paper are

i. to define sine, cosine and cotangent similarity measures between interval rough neutrosophic sets.

ii. to prove the basic properties of sine, cosine and cotangent similarity measures of interval rough neutrosophic sets.

iii. to develop new MADM strategies based on the proposed measures in interval rough neutrosophic environment.

Contributions

(i) In this paper, we propose sine, cosine and cotangent similarity measures under interval rough neutrosophic environment.

(ii) We develop new MADM strategy based on the proposed measures in interval rough neutrosophic environment.

(iii) We also present numerical example to show the feasibility and applicability of the proposed measures.

Rest of the paper is organized in the following way. Section 2 describes preliminaries of neutrosophic sets and rough neutrosophic sets and interval rough neutrosophic sets. Section 3, Section 4 and Section 5 presents definitions and propositions of the proposed measures. Section 6 presents multi attribute decision-making strategies based on the similarity measures. Section 7 provides a numerical example. Section 8 presents the conclusion and future scopes of research.

2 Preliminaries

In this Section, we provide some basic definitions regarding SVNSs, IRNSs which are useful in the paper.

In 1999, Smarandache presented the following definition of neutrosophic set (NS) [1].

Definition 2.1.1. Let X be a space of points (objects) with generic element in X denoted by x. A NS A in X is characterized by a truth-membership function T_A, an indeterminacy membership function I_A and a falsity membership function F_A. The functions T_A , I_A and F_A are real standard or non-standard subsets of $(^-0,1^+)$ that is $T_A:X \to (^-0, 1^+)$, $I_A:X \to (^-0, 1^+)$ and $F_A:X \to (^-0, 1^+)$. It should be noted that there is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$ i.e. $^-0 \le T_A(X) + I_A(X) + F_A(X) \le 1^+$

Definition 2.1.2: (Single-valued neutrosophic set) [9]. Let X be a universal space of points (objects) with a generic element of X denoted by x. A single valued neutrosophic set A is characterized by a truth membership function $T_A(x)$, a falsity membership function $F_A(x)$ and indeterminacy function $I_A(x)$ with

$$T_A(x), I_A(x) \text{ and } F_A(x) \in [0,1] \ \forall \ x \text{ in } X$$

When X is continuous, a SNVS S can be written as follows

$$A = \int_x < T_A(x), F_A(x), I_A(x) > / \forall x \in X$$

and when X is discrete, a SVNS S can be written as follows

$$A = \sum < T_A(x), F_A(x), I_A(x) > / \forall x \in X$$

For a SVNS S, $0 \le \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \le 3$.

2.2 Rough neutrosophic set

Rough neutrosophic sets [10, 11] are the generalization of rough fuzzy sets [13, 14] and rough intuitionistic fuzzy sets [15].

Definition 2.2.1: Let Y be a non-null set and R be an equivalence relation on Y. Let P be neutrosophic set in Y with the membership function T_P , indeterminacy function I_P and non-membership function F_P . The lower and the upper approximations of P in the approximation (Y, R) denoted by are respectively defined as:

$$\underline{N(P)} = << x, T_{\underline{N(P)}}(x), I_{\underline{N(P)}}(x), F_{\underline{N(P)}}(x) > / y \in [x]_R, \ x \in Y >$$

and

$$\overline{N(P)} = << x, T_{\overline{N(P)}}(x), I_{\overline{N(P)}}(x), F_{\overline{N(P)}}(x) > / y \in [x]_R, \ x \in Y >$$

where,

$T_{\underline{N(P)}}(x) = \wedge z \in [x]_R T_P(Y), I_{\underline{N(P)}}(x) = \wedge z \in [x]_R I_P(Y),$
$F_{\underline{N(P)}}(x) = \wedge z \in [x]_R F_P(Y)$

and

$T_{\overline{N(P)}}(x) = \vee z \in [x]_R T_P(Y), I_{\overline{N(P)}}(x) = \vee z \in [x]_R I_P(Y),$
$F_{\overline{N(P)}}(x) = \vee z \in [x]_R F_P(Y)$.

So,

$0 \le T_{\underline{N(P)}}(x) + I_{\underline{N(P)}}(x) + F_{\underline{N(P)}}(x) \le 3$

and

$0 \le T_{\overline{N(P)}}(x) + I_{\overline{N(P)}}(x) + F_{\overline{N(P)}}(x) \le 3$

Here $\vee$ and $\wedge$ denote "max" and "min" operators respectively, $T_P(y)$, $I_P(y)$ and $F_P(y)$ are the membership, indeterminacy and non-membership of Y with respect to P.

Thus NS mapping,

$\underline{N}, \overline{N} : N(Y) \to N(Y)$ are, respectively, referred to as the lower and upper rough NS approximation operators, and the pair $(\underline{N(P)}, \overline{N(P)})$ is called the rough neutrosophic set in (Y, R).

2.3 Interval rough neutrosophic set

Interval rough neutrosophic set (IRNS) [22] is the hybrid structure of rough sets and interval neutrosophic sets. According to Broumi and Smarandache, IRNS is the generalizations of interval valued intuitionistic fuzzy rough set.

Definition 2.3.1

Let R be an equivalence relation on the universal set U.Then the pair (U, R) is called a Pawlak approximationspace. An equivalence class of R containing x will bedenoted by $[x]_R$ for X $\in$ U, the lower and upper approximationof X with respect to (U, R) are denoted by respectively

$\underline{R}X$ and $\overline{R}X$ and are defined by

$\underline{R}X = \{x \in U : [x]_R \subseteq X\}$,

$\overline{R}X = \{x \in U : [x]_R \cap X \neq \emptyset\}$.

Now if $\underline{R}X = \overline{R}X$, then X is called definable; otherwise Xis called a rough set.

Definition 2.3.2

Let U be a universe and X, a rough set in U. An intuitionistic fuzzy rough set A in U is characterized by a membership function $\mu A: U \to [0, 1]$ and non-membership functionν_A: $U \to [0, 1]$ such that $\mu_A(\underline{R}X)=1$and $\nu_A(\underline{R}X) = 0$

ie, $[\mu_A(x), \nu_A(x)]=[1,0]$ if $x \in (\underline{R}X)$ and $\mu_A(U-\overline{R}X)= 0$, $\nu_A(U-\overline{R}X)=1$

ie,

$0 \le \mu_A(\overline{R}X - \underline{R}X) + \nu_A(\overline{R}X - \underline{R}X) \le 1$

Definition 2.3.3

Assume that, (U, R) be a Pawlak approximation space, for an interval neutrosophic set

A = {<x, $[T_A{}^L(x), T_A{}^U(x)]$, $[I_A{}^L(x), I_A{}^U(x)]$, $[F_A{}^L(x), F_A{}^U(x)]$> : $x \in U$}

The lower approximation $\underline{A}_R$ and the upper approximation $\overline{A}_R$ of A in the Pawlak approximation space (U, R) are expressed as follows:

$\underline{A}_R = \{<x, [\wedge_{y\in[x]_R} \{T^L_A(y)\}, \ _{y\ [x]} \{T_A(y)\}],$
$[\vee_{y\in[x]_R} \{I^L_A(y)\}, \ _{y\ [x]} \{I_A(y)\}],$
$[\vee_{y\in[x]_R} \{F^L_A(y)\}, \vee_{y\ [x]} \{F_A(y)\}]> : x \in U\}$

$\overline{A}_R = \{<x, [\vee_{y\in[x]_R} \{T^L_A(y)\}, \ _{y\ [x]} \{T_A(y)\}],$
$[\wedge_{y\in[x]_R} \{I^L_A(y)\}, \ _{y\ [x]} \{I_A(y)\}],$
$[\wedge_{y\in[x]_R} \{F^L_A(y)\}, \wedge_{y\ [x]} \{F_A(y)\}]> : x \in U\}$

The symbols $\wedge$ and $\vee$ indicate "min" and "max" operators respectively. R denotes an equivalence relation for interval neutrosophic set A. Here $[x]_R$ is the equivalence class of the element x. It is obvious that

$[\wedge_{y\in[x]_R} \{T^U_A(y)\}, \wedge_{y\ [x]} \{T_A(y)\}] \subset [0,1],$
$[\vee_{y\in[x]_R} \{I_A(y)\}, \vee_{y\ [x]} \{I^L_A(y)\}] \subset [0,1],$
$[\vee_{y\in[x]_R} \{F_A(y)\}, \vee_{y\ [x]} \{F^L_A(y)\}] \subset [0,1].$
and $0 \le \wedge_{y\in[x]_R} \{T_A(y)\} + \vee_{y\ [x]} \{I^U_A(y)\} +$
$\vee_{y\in[x]_R} \{F^U_A(y)\} \quad 3$

Then $\underline{A}_R$ is an interval neutrosophic set (INS)
Similarly, we have

$[\vee_{y\in[x]_R} \{T^L_A(y)\}, \vee_{y\ [x]} \{T_A(y)\}] \subset [0,1],$
$[\wedge_{y\in[x]_R} \{I^L_A(y)\}, \wedge_{y\ [x]} \{I_A(y)\}] \subset [0,1],$
$[\wedge_{y\in[x]_R} \{F^L_A(y)\}, \wedge_{y\ [x]} \{F_A(y)\}] \subset [0,1]$ and
$0 \le \vee_{y\in[x]_R} \{T^U_A(y)\} + \wedge_{y\ [x]} \{I_A(y)\} +$
$\wedge_{y\in[x]_R} \{F^U_A(y)\}] \quad 3$

Then $\underline{A}_R$ is an interval neutrosophic set.

If $\underline{A}_R = \overline{A}_R$ then A is a definable set, otherwise A is an interval valued neutrosophic rough set. Here, $\underline{A}_R$ and $\overline{A}_R$ are called the lower and upper approximations of interval neutrosophic set with respect to approximation space (U,R) respectively. $\underline{A}_R$ and $\overline{A}_R$ are simply denoted by $\underline{A}$ and $\overline{A}$ respectively.

2.4 Hamming distance

Hamming distance between two neutrosophic sets $M=(T_M(x), I_M(x), F_M(x))$ and $N=(T_N(x), I_N(x), F_N(x))$ is defined as

$$H(M,N)=$$
$$\frac{1}{3}\sum_{i=1}^{n}(|T_M(x_i)-T_N(x_i)|+|I_M(x_i)-I_N(x_i)|$$
$$+|F_M(x_i)-F_N(x_i)|).$$

3. Cosine Hamming Similarity Measure of IRNS

Assume that

$$M=\{<x_i, ([\underline{T}^-_{iM}, \underline{T}^+_{iM}], [\underline{I}^-_{iM}, \underline{I}^+_{iM}], [\underline{F}^-_{iM}, \underline{F}^+_{iM}],$$
$$[\overline{T}^-_{iM}, \overline{T}^+_{iM}], [\overline{I}^-_{iM}, \overline{I}^+_{iM}], [\overline{F}^-_{iM}, \overline{F}^+_{iM}]>: i=1,2,...,n\}$$

and

$$N=\{<x_i, [\underline{T}^-_{iN}, \underline{T}^+_{iN}], [\underline{I}^-_{iN}, \underline{I}^+_{iN}], [\underline{F}^-_{iN}, \underline{F}^+_{iN}],$$
$$[\overline{T}^-_{iN}, \overline{T}^+_{iN}], [\overline{I}^-_{iN}, \overline{I}^+_{iN}], [\overline{F}^-_{iN}, \overline{F}^+_{iN}]>: i=1,2,...,n\}$$

in X = {x_1, x_2, …, x_n}be any two IRNSs. A cosine Hamming similarity operator between IRNS M and N is defined as follows:

$$\cos(M,N)=\frac{1}{n}\sum_{i=1}^{n}\cos(\frac{\Pi}{6}(|\Delta T_M(x_i)-\Delta T_N(x_i)|+$$
$$|\Delta I_M(x_i)-\Delta I_N(x_i)|+|\Delta F_M(x_i)-\Delta F_N(x_i)|)).$$

$$\Delta T_M(x_i)=\frac{(\underline{T}^-_{iM}+\underline{T}^+_{iM}+\overline{T}^-_{iM}+\overline{T}^+_{iM})}{4},$$
$$\Delta I_M(x_i)=\frac{(\underline{I}^-_{iM}+\underline{I}^+_{iM}+\overline{I}^-_{iM}+\overline{I}^+_{iM})}{4},$$
$$\Delta F_M(x_i)=\frac{(\underline{F}^-_{iM}+\underline{F}^+_{iM}+\overline{F}^-_{iM}+\overline{F}^+_{iM})}{4},$$
$$\Delta T_N(x_i)=\frac{(\underline{T}_{iN}+\underline{T}_{iN}+\overline{T}_{iN}+\overline{T}_{iN})}{4},$$
$$\Delta I_N(x_i)=\frac{(\underline{I}^-_{iN}+\underline{I}^+_{iN}+\overline{I}^-_{iN}+\overline{I}^+_{iN})}{4},$$
$$\Delta F_N(x_i)=\frac{(\underline{F}^-_{iN}+\underline{F}^+_{iN}+\overline{F}^-_{iN}+\overline{F}^+_{iN})}{4}.$$

Properties 3.1

The defined rough neutrosophic cosine hamming similarity operator cos(M, *N*) between IRNSs M and N satisfies the following properties:

1. $0\le\cos(M,N)\le 1$.
2. cos(M,N) = 1 if and only if M = N.
3. cos(M,N) = cos(N,M).

Proof:

1. Since the functions $\Delta T_M(x), \Delta I_M(x), \Delta F_M(x), \Delta T_N(x), \Delta I_N(x)$ and $\Delta F_N(x)$ the value of the cosine function are within [0, 1], the similarity measure based on interval rough neutrosophic cosine Hamming similarity function also lies within [0, 1]. Hence $0\le\cos(M,N)\le 1$.

This completes the proof.

2. For any two RNSs M and N, if M = N, then the following relations hold

$$\Delta T_M(x_i)=\Delta T_N(x_i), \Delta I_M(x_i)=\Delta I_N(x_i),$$
$$\Delta F_M(x_i)=\Delta F_N(x_i).$$

Hence,

$$|\Delta T_M(x_i)-\Delta T_N(x_i)|=0, |\Delta I_M(x_i)-\Delta I_N(x_i)|=0,$$
$$|\Delta F_M(x_i)-\Delta F_N(x_i)|=0.$$

Thus cos(M,N) = 1

Conversely,

If cos(M,N) = 1, then

$$|\Delta T_M(x_i)-\Delta T_N(x_i)|=0, |\Delta I_M(x_i)-\Delta I_N(x_i)|=0,$$
$$|\Delta F_M(x_i)-\Delta F_N(x_i)|=0.$$

Since cos(0) = 1. So we can write

$$\Delta T_M(x_i)=\Delta T_N(x_i), \Delta I_M(x_i)=\Delta I_N(x_i),$$
$$\Delta F_M(x_i)=\Delta F_N(x_i).$$

Hence M = N**.**

3. As

$$\cos(M,N)=\frac{1}{n}\sum_{i=1}^{n}\cos(\frac{\Pi}{6}(|\Delta T_M(x_i)-\Delta T_N(x_i)|+$$
$$|\Delta I_M(x_i)-\Delta I_N(x_i)|+|\Delta F_M(x_i)-\Delta F_N(x_i)|))$$
$$=\frac{1}{n}\sum_{i=1}^{n}\cos(\frac{\Pi}{6}(|\Delta T_N(x_i)-\Delta T_M(x_i)|+$$
$$|\Delta I_N(x_i)-\Delta I_M(x_i)|+|\Delta F_N(x_i)-\Delta F_M(x_i)|))$$
$$=\cos(N,M)$$

This completes the proof.

4. Sine Hamming Similarity Measure of IRNS

Assume that

$$M=\{<x_i, ([\underline{T}^-_{iM}, \underline{T}^+_{iM}], [\underline{I}^-_{iM}, \underline{I}^+_{iM}], [\underline{F}^-_{iM}, \underline{F}^+_{iM}],$$
$$[\overline{T}^-_{iM}, \overline{T}^+_{iM}], [\overline{I}^-_{iM}, \overline{I}^+_{iM}], [\overline{F}^-_{iM}, \overline{F}^+_{iM}]>: i=1,2,...,n\}$$

and

$$N=\{<x_i, [\underline{T}^-_{iN}, \underline{T}^+_{iN}], [\underline{I}^-_{iN}, \underline{I}^+_{iN}], [\underline{F}^-_{iN}, \underline{F}^+_{iN}],$$
$$[\overline{T}^-_{iN}, \overline{T}^+_{iN}], [\overline{I}^-_{iN}, \overline{I}^+_{iN}], [\overline{F}^-_{iN}, \overline{F}^+_{iN}]>: i=1,2,...,n\}$$

in $X = \{x_1, x_2, \ldots, x_n\}$ be any two IRNSs. A sine Hamming similarity operator between IRNS M and N is defined as follows:

$$\sin(M,N) = 1 - [\frac{1}{n}\sum_{i=1}^{n} \sin(\frac{\Pi}{}(\left|\Delta T_M(x_i) - \Delta T_N(x_i)\right| + \left|\Delta I_M(x_i) - \Delta I_N(x_i)\right| + \left|\Delta F_M(x_i) - \Delta F_N(x_i)\right|))].$$

Here,

$$\Delta T_M(x_i) = \frac{(\underline{T_{iM}^-} + \underline{T_{iM}^+} + \overline{T_{iM}^-} + \overline{T_{iM}^+})}{4},$$

$$\Delta I_M(x_i) = \frac{(\underline{I_{iM}^-} + \underline{I_{iM}^+} + \overline{I_{iM}^-} + \overline{I_{iM}^+})}{4},$$

$$\Delta F_M(x_i) = \frac{(\underline{F_{iM}^-} + \underline{F_{iM}^+} + \overline{F_{iM}^-} + \overline{F_{iM}^+})}{4},$$

$$\Delta T_N(x_i) = \frac{(\underline{T_{iN}^-} + \underline{T_{iN}^+} + \overline{T_{iN}^-} + \overline{T_{iN}^+})}{4},$$

$$\Delta I_N(x_i) = \frac{(\underline{I_{iN}^-} + \underline{I_{iN}^+} + \overline{I_{iN}^-} + \overline{I_{iN}^+})}{4},$$

$$\Delta F_N(x_i) = \frac{(\underline{F_{iN}^-} + \underline{F_{iN}^+} + \overline{F_{iN}^-} + \overline{F_{iN}^+})}{4}.$$

Properties 4.1

The defined rough neutrosophic sine hamming similarity operator sin (M, *N*) between IRNSs M and N satisfies the following properties:

1. $0 \le \sin(M,N) \le 1$.
2. sin (M,N) = 1 if and only if M = N.
3. sin (M,N) = sin (N,M).

Proof:

1.Since the functions $\Delta T_M(x), \Delta I_M(x), \Delta F_M(x), \Delta T_N(x), \Delta I_N(x)$ and $\Delta F_N(x)$ the value of the sine function are within [0,1], the similarity measure based on interval rough neutrosophic cosine Hamming similarity function also lies within [0,1]. Hence $0 \le \sin(M,N) \le 1$.

This completes the proved.

2.For any two RNSs M and N, if M = N, then the following relations hold

$$\Delta T_M(x_i) = \Delta T_N(x_i), \Delta I_M(x_i) = \Delta I_N(x_i),$$
$$\Delta F_M(x_i) = \Delta F_N(x_i).$$

Hence,

$$\left|\Delta T_M(x_i) - \Delta T_N(x_i)\right| = 0, \left|\Delta I_M(x_i) - \Delta I_N(x_i)\right| = 0,$$
$$\left|\Delta F_M(x_i) - \Delta F_N(x_i)\right| = 0.$$

Thus sin(M,N) = 1

Conversely,

If sin(M,N) = 1, then

$$\left|\Delta T_M(x_i) - \Delta T_N(x_i)\right| = 0, \left|\Delta I_M(x_i) - \Delta I_N(x_i)\right| = 0,$$
$$\left|\Delta F_M(x_i) - \Delta F_N(x_i)\right| = 0.$$

Since sin(0) = 1. So we can write

$$\Delta T_M(x_i) = \Delta T_N(x_i), \Delta I_M(x_i) = \Delta I_N(x_i),$$
$$\Delta F_M(x_i) = \Delta F_N(x_i).$$

Hence M = N**.**

3. As

$$\begin{aligned}\sin(M,N) &= 1 - [\frac{1}{n}\sum_{i=1}^{n} \sin(\frac{\Pi}{}(\left|\Delta T_M(x_i) - \Delta T_N(x_i)\right| + \left|\Delta I_M(x_i) - \Delta I_N(x_i)\right| + \left|\Delta F_M(x_i) - \Delta F_N(x_i)\right|))]\\ &= 1 - [\frac{1}{n}\sum_{i=1}^{n} \sin(\frac{\Pi}{}(\left|\Delta T_N(x_i) - \Delta T_M(x_i)\right| + \left|\Delta I_N(x_i) - \Delta I_M(x_i)\right| + \left|\Delta F_N(x_i) - \Delta F_M(x_i)\right|))]\\ &= \sin(N,M).\end{aligned}$$

This completes the proof.

5. Cotangent Hamming Similarity Measure of IRNS

Assume that

$$M = \{< x_i, ([\underline{T_{iM}^-}, \underline{T_{iM}^+}], [\underline{I_{iM}^-}, \underline{I_{iM}^+}], [\underline{F_{iM}^-}, \underline{F_{iM}^+}], [\overline{T_{iM}^-}, \overline{T_{iM}^+}], [\overline{I_{iM}^-}, \overline{I_{iM}^+}], [\overline{F_{iM}^-}, \overline{F_{iM}^+}] >: i = 1, 2, \ldots, n\}$$

and

$$N = \{< x_i, [\underline{T_{iN}^-}, \underline{T_{iN}^+}], [\underline{I_{iN}^-}, \underline{I_{iN}^+}], [\underline{F_{iN}^-}, \underline{F_{iN}^+}], [\overline{T_{iN}^-}, \overline{T_{iN}^+}], [\overline{I_{iN}^-}, \overline{I_{iN}^+}], [\overline{F_{iN}^-}, \overline{F_{iN}^+}] >: i = 1, 2, \ldots, n\}$$

in $X = \{x_1, x_2, \ldots, x_n\}$ be any two IRNSs. A cosine Hamming similarity operator between IRNS M and N is defined as follows:

$$\cot(M,N) = \frac{1}{n}\sum_{i=1}^{n} \cot(\frac{\Pi}{4} + \frac{\Pi}{12}(\left|\Delta T_M(x_i) - \Delta T_N(x_i)\right| + \left|\Delta I_M(x_i) - \Delta I_N(x_i)\right| + \left|\Delta F_M(x_i) - \Delta F_N(x_i)\right|)).$$

Here,

$$\Delta T_M(x_i) \quad \frac{(\underline{T_{iM}^-} + \underline{T_{iM}} \quad \overline{T_{iM}} \quad \overline{T_{iM}})}{4},$$

$$\Delta I_M(x_i) \quad \frac{(\underline{I_{iM}^-} + \underline{I_{iM}} \quad \overline{I_{iM}} \quad \overline{I_{iM}})}{4},$$

$$\Delta F_M(x_i) = \frac{(\underline{F_{iM}^-} + \underline{F_{iM}^+} + \overline{F_{iM}^-} + \overline{F_{iM}^+})}{4},$$

$$\Delta T_N(x_i) = \frac{(\underline{T_{iN}^-} + \underline{T_{iN}^+} + \overline{T_{iN}^-} + \overline{T_{iN}^+})}{4},$$

$$\Delta I_N(x_i) = \frac{(\underline{I_{iN}^-} + \underline{I_{iN}^+} + \overline{I_{iN}^-} + \overline{I_{iN}^+})}{4},$$

$$\Delta F_N(x_i) = \frac{(\underline{F_{iN}^-} + \underline{F_{iN}^+} + \overline{F_{iN}^-} + \overline{F_{iN}^+})}{4}.$$

Properties 5.1

The defined rough neutrosophic cosine hamming similarity operator cot(M, *N*) between IRNSs M and N satisfies the following properties:

1. cot(M, N) = 1 if and only if M = N.
2. cot(M, N) = cot(N, M).

Proof:

1.For any two RNSs M and N, if M = N, then the following relations hold

$\Delta T_M(x_i) = \Delta T_N(x_i), \Delta I_M(x_i) = \Delta I_N(x_i), \Delta F_M(x_i) = \Delta F_N(x_i).$

Hence,

$|\Delta T_M(x_i) - \Delta T_N(x_i)| = 0, |\Delta I_M(x_i) - \Delta I_N(x_i)| = 0,$

$|\Delta F_M(x_i) - \Delta F_N(x_i)| = 0.$

Thus cot(M,N) = 1

Conversely,

If cot(M,N) = 1, then

$|\Delta T_M(x_i) - \Delta T_N(x_i)| = 0, |\Delta I_M(x_i) - \Delta I_N(x_i)| = 0,$

$|\Delta F_M(x_i) - \Delta F_N(x_i)| = 0.$

Since cot($\frac{\Pi}{4}$) = 1. So we can write

$\Delta T_M(x_i) = \Delta T_N(x_i), \Delta I_M(x_i) = \Delta I_N(x_i),$

$\Delta F_M(x_i) = \Delta F_N(x_i).$

Hence M = N**.**

2. As,

$$\cot(M,N) = \frac{1}{n}\sum_{i=1}^{n}\cot\left(\frac{\Pi}{4} + \frac{\Pi}{12}\left(|\Delta T_M(x_i) - \Delta T_N(x_i)| + |\Delta I_M(x_i) - \Delta I_N(x_i)| + |\Delta F_M(x_i) - \Delta F_N(x_i)|\right)\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\cot\left(\frac{\Pi}{4} + \frac{\Pi}{12}\left(|\Delta T_N(x_i) - \Delta T_M(x_i)| + |\Delta I_N(x_i) - \Delta I_M(x_i)| + |\Delta F_N(x_i) - \Delta F_M(x_i)|\right)\right)$$

$= \cot(N, M).$

This completes the proof.

6. Decision making under trigonometric interval rough neutrosophic Hamming similarity measures

In this section, we apply interval rough cosine, sine and cotangent Hamming similarity measures between IRNSs to the multi-attribute decision making problem. Consider C={$C_1, C_2, \ldots, C_m$} be the set of attributes and A={$A_1, A_2, \ldots, A_n$} be a set of alternatives. Now we provide an algorithm for MADM problems involving interval rough neutrosophic information.

Algorithm 1. (see Fig 1)

Step 1: Construction of the decision matrix with interval rough neutrosophic number

Decision maker considers the decision matrix with respect to m alternatives and n attributes in terms of interval rough neutrosophic numbers as follows:

Table1: Interval Rough neutrosophic decision matrix

$$D = \langle Z_{ij} \rangle_{n\times m} = \begin{pmatrix} Z_{11} & Z_{12} & \ldots\ldots\ldots & Z_{1m} \\ Z_{21} & Z_{22} & \ldots\ldots\ldots & Z_{2m} \\ \ldots & \ldots & \ldots\ldots\ldots & \ldots \\ \ldots & \ldots & \ldots\ldots\ldots & \ldots \\ \ldots & \ldots & \ldots\ldots\ldots & \ldots \\ Z_{n1} & Z_{n2} & \ldots\ldots\ldots & Z_{nm} \end{pmatrix}$$

Where

$Zij = \langle ([\underline{T_{iM}^-}, \underline{T_{iM}^+}], [\underline{I_{iM}^-}, \underline{I_{iM}^+}], [\underline{F_{iM}^-}, \underline{F_{iM}^+}],$ with $[\overline{T_{iM}^-}, \overline{T_{iM}^+}], [\overline{I_{iM}^-}, \overline{I_{iM}^+}], [\overline{F_{iM}^-}, \overline{F_{iM}^+}]) \rangle$

$0 \le \vee_{y\in[x]_R}\{T_A^U(y)\} + \wedge_{y\in[x]_R}\{I_A^U(y)\} + \wedge_{y\in[x]_R}\{F_A^U(y)\}] \le 3$

Step 2: Determination of the ideal alternative

Generally, the evaluation attribute can be categorized into two types: benefit type attribute and cost type attribute. We define an ideal alternative S^* .

For benefit type attribute,

$S^*=$

$\{(\min_i \underline{T_{ij}}, \max_i \underline{I_{ij}}, \max_i \underline{F_{ij}}), (\max_i \overline{T_{ij}}, \min_i \overline{I_{ij}}, \min_i \overline{F_{ij}})\}$.

For cost type attribute,

$S^*=$

$\{(\max_i \underline{T_{ij}}, \min_i \underline{I_{ij}}, \min_i \underline{F_{ij}}), (\min_i \overline{T_{ij}}, \max_i \overline{I_{ij}}, \max_i \overline{F_{ij}})\}$.

Step 3: Determination of the interval rough trigonometric neutrosophic Hamming similarity function of the alternatives

We compute interval rough trigonometric neutrosophic similarity measure between the ideal alternative S^* and each alternative $Z_i = \langle Z_{ij} \rangle_{n\times m}$ for all i = 1, ….., n and j = 1, ….., m.

Step 4: Ranking the alternatives

Using the interval rough trigonometric neutrosophic similarity measure between each alternative and the ideal alternative, the ranking order of all alternatives can be determined and the best alternative is selected with the highest similarity value.

Step 5: End.

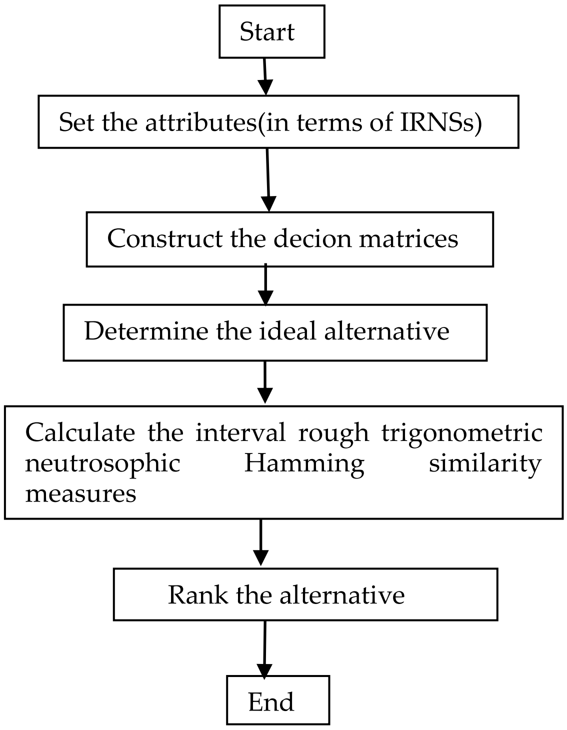

Fig 1. A flowchart of the proposed decision making method

7. Numerical example

Assume that a decision maker intends to select the most suitable laptop for random use from the three initially chosen laptops (S_1, S_2, S_3) by considering four attributes namely: features C_1, reasonable price C_2, customer care C_3, risk factor C_4. Based on the proposed approach discussed in section 5, the considered problem is solved by the following steps:

Step1: Construct the decision matrix with interval rough neutrosophic number

The decision maker construct the decision matrix with respect to the three alternatives and four attributes in terms of interval rough neutrosophic number.

	C_1	**C_2**	**C_3**	**C_4**
S_1	<([.6, .7], [.3, .5], [.3, .4]), ([.8, .9], [.1, .3], [.1, .2])>	<([.5, .7], [.3, .4], [.1, .2]), ([.7, .9], [.3, .5], [.3, .4])>	<([.5, .6], [.4, .5], [.4, .6]), ([.7, .8], [.2, .4], [.3, .4])>	<([.8, .9], [.3, .4], [.5, .6]), ([.7, .8], [.3, .5], [.3, .5])>
S_2	<([.7, .8], [.2, .3], [.0, .2]), ([.7, .9], [.1, .2], [.1, .2])>	<([.6, .7], [.1, .2], [.0, .2]), ([.6, .7], [.1, .3], [.1, .3])>	<([.5, .7], [.2, .3], [.1, .2]), ([.6, .9], [.3, .5], [.2 .4])>	<([.7, .8], [.3, .5], [.1,.3]), ([.5, .7], [.5, .6], [.2, .3])>
S_3	<([.6, .7], [.3, .4], [.0, .3]), ([.6, .9], [.1, .2], [.1, .2])>	<([.5, .7], [.2, .4], [.2, .4]), ([.6, .8], [.1, .3], [.1, .2])>	<([.6, .8], [.2, .4], [.3, .4]), ([.6, .8], [.2, .5], [.3, .5])>	<([.4, .7], [.2, .4], [.4, .5]), ([.5, .8], [.2, .5], [.0, .2])>

Step 2: Determine the benefit type attribute and cost type attribute

Here three benefit type attributes C_1, C_2, C_3 and one cost type attribute C_4. We calculate the ideal alternative as follows:

$$S^* =$$
$$\{<([.6,.7],[.3,.5],[.3,.4]),([.8,.9],[.1,.2],[.1,.2])>,$$
$$<([.5,.7],[.3,.4],[.2,.4]),([.7,.9],[.1,.3],[.1,.2])>,$$
$$<([.5,.6],[.4,.5],[.4,.6]),([.7,.9],[.2,.4],[.2,.4])>,$$
$$<([.8,.9],[.2,.4],[.1,.3]),([.5,.7],[.5,.6],[.3,.5])>)>\}$$

Step3: Calculate the interval rough trigonometric neutrosophic Hamming similarity measure of the alternatives

$$\cos(S_1, S^*) = 0.999998923,$$
$$\cos(S_1, S^*) = 0.999997135,$$
$$\cos(S_1, S^*) = 0.999998505,$$

$$\sin(S_1, S^*) = 0.999531651,$$
$$\sin(S_1, S^*) = 0.997658256,$$

$\sin(S_1, S^*) = 0.998343644$,
$\cot(S_1, S^*) = 70.25049621$,
$\cot(S_1, S^*) = 67.22363275$,
$\cot(S_1, S^*) = 68.81008448$.

Step 4: Rank the alternatives

Ranking of alternatives is prepared based on the descending order of similarity measures. The highest value reflects the best alternatives.
Here,
$\cos(S_1, S^*) > \cos(S_3, S^*) > \cos(S_2, S^*)$.
$\sin(S_1, S^*) > \sin(S_3, S^*) > \sin(S_2, S^*)$.
$\cot(S_1, S^*) > \cot(S_3, S^*) > \cot(S_2, S^*)$.

Hence, the laptop S_1 is the best alternative for random use.

8. Conclusions

In this paper, we have proposed interval rough trigonometric Hamming similarity measures and proved their properties. We have developed three MADM strategies base on sine, cosine and cotangent similarity measures under interval rough neutrosophic environment. Then we solved an illustrative numerical example to demonstrate the feasibility, applicability of the developed strategies. The concept presented in this paper can be applied other multiple attribute decision making problems such as teacher selection [30, 31, 32], school selection [33], weaver selection [34, 35, 36], brick field selection [37, 38], logistics center location selection [39, 40], data mining [41] etc. under interval rough neutrosophic environment.

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On Multi-Criteria Decision Making problem via Bipolar Single-Valued Neutrosophic Settings

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Abstract. In this paper, the idea bipolar single-valued neutrosophic (BSVN) set was introduced. We also introduce bipolar single-valued neutrosophic topological space and some of its properties were characterized. Comparing Bipolar single-valued neutrosophic sets with score function, certainty function and accuracy function .Bipolar single-valued neutrosophic weighted average operator (A_ω) and bipolar single-valued neutrosophic weighted geometric operator (G_ω) were developed and based on Bipolar single-valued neutrosophic set, a multiple decision making problem were evaluated through an example to select the desirable one.

Keywords: Bipolar single-valued neutrosophic set, bipolar single-valued neutrosophic topological space, bipolar single-valued neutrosophic average operator, bipolar single-valued neutrosophic geometric operator, score, certainty and accuracy functions.

1. Introduction

Fuzzy Logic resembles the human decision making methodology.Zadeh [39] who was considered as the Father of Fuzzy Logic introduced the fuzzy sets in 1965 and it is a tool in learning logical subject. He put forth the concept of fuzzy sets to deal with contrasting types of uncertainties. Using single value $\mu_A(x)\in[0, 1]$, the degree of membership of the fuzzy set is in classic fuzzy, which is defined on a universal scale, they cannot grasp convinced cases where it is hard to define μ_A by one specific value.

Intuitionistic fuzzy sets which was proposed by Atanassov [2] is the extension of Zadeh's Fuzzy Sets to overthrown the lack of observation of non-membership degrees. Intuitionistic fuzzy sets generally tested in solving multi-criteria decision making problems. Intuitionistic fuzzy sets detailed into the membership degree, non-membership degree and simultaneously with degree of indeterminancy.

Neutrosophic is the base for the new mathematical theories derives both their classical and fuzzy depiction. Smarandache [4,5] introduced the neutrosophic set . Neutrosophic set has the capability to induce classical sets, fuzzy set, Intuitionistic fuzzy sets. Introduceing the components of the neutrosophic set are True(T), Indeterminacy(I), False(F) which represent the membership, indeterminacy, and non-membership values respectively.The notion of classical set, fuzzy set [17], interval-valued fuzzy set [39], Intuitionistic fuzzy [2], etc were generalized by the neutrosophic set. Majumdar & Samant [19] recommended the Single-valued neutrosophic sets (SVNSs), which is a variation of Neutrosophic Sets. Wang, et.al [38] describe an example of neutrosophic set and sgnify single valued Neutrosophic set (SVNs).They give many properties of Single-Valued Neutrosophic Set, which are associated to the operations and relations by Single-Valued Neutrosophic Sets.The correlation coefficient of SVNSs placed on the development of the correlation coefficient of Intuitionistic fuzzy sets and tested that the cosine similarity measure of SVNS is a special case of the correlation coefficient and correlated it to single valued neutrosophic multicriteria decision-making problems which was presented by Jun Ye [7]. For solving multi-criteria decision-making problems, he overworked similarity measure for interval valued neutrosophic set. Single valued neutrosophic sets (SVNSs) can handle the undetermined and uncertain information and also symbolize, which fuzzy sets and Intuitionistic fuzzy sets cannot define and finalize.

Turksen [37] proposed the Interval-valued fuzzy set is similar as Intuitionistic fuzzy set. The concept is to hook the anxiety of class of membership . Interval- valued fuzzy set need an interval value $[\mu_A{}^L(a), \mu_A{}^U(a)]$ with $0\leq\mu_A{}^L(a)\leq\mu_A{}^U(a)\leq1$ to represent the class of membership of a fuzzy set A. But it is not suffient to take only the membership function, but also to have the non-membership function .

Bipolar fuzzy relations was given by Bosc and Pivert [3] where a pair of satisfaction degrees is made with each tuple. In 1994, an development of fuzzy set termed bipolar fuzzy was given by Zhang [40].By the notion of fuzzy sets, Lee [16] illustrate bipolar fuzzy sets. Manemaran and Chellappa [20] provide some applications in groups are called the bipolar fuzzy groups, fuzzy d-ideals of groups under (T-S) norm. They also explore few properties of the groups and the relations. Bipolar fuzzy subalgebras and bipolar fuzzy ideals of BCK/BCI-algebras were researched by K. J. Lee[17]. Multiple attribute decision-making method situated on single-valued neutrosophic was granted by P. Liu and Y. Wang[18].

In bipolar neutrosophic environment, bipolar neutrosophic sets(BNS) was developed by Irfan Deli [6] and et.al. The application based on multi-criteria decision making problems were also given by them in bipolar neutrosophic set. To collect bipolar neutrosophic information, they defined score, accuracy, and certainty functions to compare BNS and developed bipolar neutrosophic weighted average (BNWA) and bipolar neutrosophic weighted geometric (BNWG) operators. In the study, a Multi Criteria Decision Making approach were discussed on the basis of score, accuracy, and certainty functions, bipolar Neutrosophic Weighted Average and bipolar Neutrosophic Weighted Geometric operators were calculated. Fuzzy neutrosophic sets and its Topological spaces was introduced by I.Arockiarani and J.Martina Jency [1].

Positive and Negative effects count on Decision making . Multiple decision-making problems have gained very much attention in the area of systemic optimization, urban planning, operation research, management science and many other fields. Correlation Coefficient between Single Valued Neutrosophic Sets and its Multiple Attribute Decision Making Method given by Jun Ye [7]. A Neutrosophic Multi-Attribute Decision making with Unknown Weight data was investigated by Pranab Biswas, Surapati Pramanik, Bibhas C. Giri[30]. Neutrosophic Tangent Similarity Measure and its Application was given by Mondal, Surapati Pramanik [11]. Many of the authors[8-14,21,22,24-29,31,32,33,35,36] studied and examine different and variation of neutrosophic set theory in Decision making problems.

Here, we introduce bipolar single-valued neutrosophic set which is an expansion of the fuzzy sets, Intuitionistic fuzzy sets, neutrosophic sets and bipolar fuzzy sets. Bipolar single-valued neutrosophic topological spaces were also proposed. Bipolar single-valued neutrosophic topological spaces characterized a few of its properties and a numerical example were illustrated. Bipolar single-valued neutrosophic sets were compared with score function, certainty function and accuracy function. Then,the bipolar single-valued Neutrosophic weighted average operator (A_ω) and bipolar single-valued neutrosophic weighted geometric operator (G_ω) are developed to aggregate the data.To determine the application and the performance of this method to choose the best one, atlast a numerical example of the method was given.

2 Preliminaries

2.1 Definition [34]: Let *X* be a non-empty fixed set. *A* neutrosophic set B is an object having the form $B=\{<x,\mu_B(x),\sigma_B(x),\gamma_B(x)>x\in X\}$ Where $\mu_B(x),\sigma_B(x)$ and $\gamma_B(x)$ which represent the degree of membership function , the degree of indeterminacy and the degree of non-membership respectively of each element $x\in X$ to the set B .

2.2 Definition [38]: Let a universe X of discourse. Then $A_{NS}=\{<x,F_A(x),T_A(x)I_A(x)>x\in X\}$ defined as a single-valued neutrosophic set where truth-membership function $T_A:X\to[0,1]$,an indeterminacy-membership function $I_A: X\to[0,1]$ and a falsity-membership function $F_A: X \to [0,1]$.No restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$, so $0\le\sup T_A(x) \le\sup I_A(x) \le\sup F_A(x) \le 3$. $\widetilde{A}$ =<T, I, F> is denoted as a single-valued neutrosophic number.

2.3 Definition [23]: Let two single-valued neutrosophic number be $\widetilde{A}_1=<T_1, I_1, F_1>$ and $\widetilde{A}_2=<T_2, I_2, F_2>$. Then,the operations for NNs are defined as follows:

i. $\lambda \widetilde{A} =<1-(1-T_1)^\lambda,I_1^\lambda,F_1^\lambda>$
ii. $\widetilde{A}_1^\lambda =<T_1^\lambda,1-(1-I_1)^\lambda,1-(1-F_1)^\lambda>$
iii. $\widetilde{A}_1+\widetilde{A}_2=<T_1+T_2-T_1T_2,I_1I_2,F_1F_2>$
iv. $\widetilde{A}_1.\widetilde{A}_2=<T_1T_2,I_1+I_2-I_1I_2,F_1+F_2-F_1F_2>$

2.4 Definition [15]: Let a single-valued neutrosophic number be $\widetilde{B}_1=<T_1, I_1, F_1>$. Then, SNN are defined as

i. score function $s(\widetilde{B}_1) = (T_1+1- I_1+ 1-F_1)/3$;
ii. accuracy function $a(\widetilde{B}_1) = T_1-F_1$;
iii. certainty function $c(\widetilde{B}_1) = T_1$.

2.5 Definition [23]: Let two single-valued neutrosophic number be $\widetilde{B}_1=<T_1, I_1, F_1>$ and $\widetilde{B}_2=<T_2, I_2, F_2>$. The comparison method defined as:

i. if $s(\widetilde{B}_1) > s(\widetilde{B}_2)$, then $\widetilde{B}_1$ is greater than $\widetilde{B}_2$, that is, $\widetilde{B}_1$ is superior to $\widetilde{B}_2$, denoted by $\widetilde{B}_1>\widetilde{B}_2$.
ii. if $s(\widetilde{B}_1) = s(\widetilde{B}_2)$ and $a(\widetilde{B}_1)>a(\widetilde{B}_2)$, then $\widetilde{B}_1$ is greater than $\widetilde{B}_2$, that is, $\widetilde{B}_1$ is superior to $\widetilde{B}_2$, denoted by

$\widetilde{B}_1 < \widetilde{B}_2$.

iii.if s($\widetilde{B}_1$) = s($\widetilde{B}_2$) and a($\widetilde{B}_1$)=a($\widetilde{B}_2$) and c($\widetilde{B}_1$) >c($\widetilde{B}_2$), then $\widetilde{B}_1$ is greater than $\widetilde{B}_2$, that is, $\widetilde{B}_1$ is superior to $\widetilde{B}_2$, denoted by $\widetilde{B}_1 > \widetilde{B}_2$.

iv.if s($\widetilde{B}_1$) = s($\widetilde{B}_2$) and a($\widetilde{B}_1$)=a($\widetilde{B}_2$) and c($\widetilde{B}_1$)=c($\widetilde{B}_2$), then $\widetilde{B}_1$ is equal to $\widetilde{B}_2$, that is, $\widetilde{B}_1$ is indifferent to $\widetilde{B}_2$, denoted by $\widetilde{B}_1 = \widetilde{B}_2$.

2.6 Definition [6]: In *X,* a bipolar neutrosophic set B is defined in the form

$$B=\langle x, (T^+(x), I^+(x), F^+(x), T^-(x), I^-(x), F^-(x)) : x\in X\rangle$$

Where T^+, I^+, F^+: X→ [1, 0] and T^- , I^- , F^- : X [-1, 0].The positive membership degree denotes the truth membership $T^+(x)$, indeterminate membership I^+ (x) and false membership F^+ (x) of an element x∈X corresponding to the set *A* and the negative membership degree denotes the truth membership $T^-(x)$, indeterminate membership $I^-(x)$ and false membership $F^-(x)$ of an element x∈ X to some implicit counter-property corresponding to a bipolar neutrosophic set .

2.7 Definition [39, 2]: Each element had a degree of membership (T) in the fuzzy set . The Intuitionistic fuzzy set on a universe, where the degree of membership $\mu_B(x)$ ∈[0,1] of each element x∈X to a set B, there was a degree of non-membership $\nu_B(x)$∈[0,1],such that ∀ x∈X, $\mu_B(x) + \nu_B(x) \leq 1$.

2.8 Definition [15, 20]: Let a non-empty set be X. Then, B_{BF}= {<x, $\mu^+_B(x)$, $\mu^-_B(x)$>: x∈X}is a bipolar-valued fuzzy set denoted by B_{BF} ,where μ^+_B: X → [0, 1] and μ^-_B: X → [0, 1]. The positive Membership degree $\mu^+_B(x)$ denotes the satisfaction degree of an element x to the property corresponding to B_{BF} and the negative membership degree $\mu_B^-(x)$ denotes the satisfaction degree of x to some implicit counter property of B_{BF} .

In this section, we give the concept bipolar single-valued neutrosophic set and its operations. We also developed the bipolar single-valued neutrosophic weighted (A_ω) average operator and geometric operator (G_ω). Some of it is quoted from [2, 5, 7, 10, and 14].

3. Bipolar single-valued Neutrosophic set(BSVN):

3.1 Definition : A Bipolar Single-Valued Neutrosophic set (BSVN) S in X is defined in the form of

$$\textbf{BSVN (S)} = \langle v,(T_{BSVN}^+,T_{BSVN}^-),(I_{BSVN}^+,I_{BSVN}^-),(F_{BSVN}^+,F_{BSVN}^-):v\in X\rangle$$

where (T_{BSVN}^+,I_{BSVN}^+,F_{BSVN}^+):X→[0,1] and (T_{BSVN}^-,I_{BSVN}^-,F_{BSVN}^-):X→ [-1,0] .In this definition, there T_{BSVN}^+ and T_{BSVN}^- are acceptable and unacceptable in past. Similarly I_{BSVN}^+ and I_{BSVN}^- are acceptable and unacceptable in future. F_{BSVN}^+ and F_{BSVN}^- are acceptable and unacceptable in present respectively.

3.2 Example : Let X={s_1,s_2,s_3}.Then a bipolar single-valued neutrosophic subset of *X* is

$$S=\begin{Bmatrix} < s_1,(0.1,-0.1),(0.2,-0.3),(0.3,-0.5) > \\ < s_2,(0.2,-0.3),(0.4,-0.4),(0.6,-0.5) > \\ < s_3,(0.2,-0.8),(0.6,-0.4),(0.7,-0.7) > \end{Bmatrix}$$

3.3 Definition : Let two bipolar single-valued neutrosophic sets $BSVN_1$(S) and $BSVN_2$(S) in X defined as
$BSVN_1$ (S)=<v,($T_{BSVN}^+(1)$,$T_{BSVN}^-(1)$),($I_{BSVN}^+(1)$,$I_{BSVN}^-(1)$),($F_{BSVN}^+(1)$,$F_{BSVN}^-(1)$):v∈X> and
$BSVN_2$(S)=<v,($T_{BSVN}^+(2)$,$T_{BSVN}^-(2)$),($I_{BSVN}^+(2)$,$I_{BSVN}^-(2)$),($F_{BSVN}^+(2)$,$F_{BSVN}^-(2)$):v∈X>. Then the operators are defined as follows:

(i) Complement

$BSVN^c(S)$ = {< v,(1-T_{BSVN}^+), (-1-T_{BSVN}^-), (1-I_{BSVN}^+), (-1-I_{BSVN}^-), (1-F_{BSVN}^+), (-1-F_{BSVN}^-): v∈X >}

(ii) Union of two BSVN

$BSVN_1$(S) U $BSVN_2$(S) =

$$\left\langle \begin{matrix} \max(T^+_{BSVN}(1),T^+_{BSVN}(2)),\min(I^+_{BSVN}(1),I^+_{BSVN}(2)),\min(F^+_{BSVN}(1),F^+_{BSVN}(2)) \\ \max(T^-_{BSVN}(1),T^-_{BSVN}(2)),\min(I^-_{BSVN}(1),I^-_{BSVN}(2)),\min(F^-_{BSVN}(1),F^-_{BSVN}(2)) \end{matrix} \right\rangle$$

(iii) Intersection of two BSVN

BSVN₁(S) ∩BSVN₂(S) =

$$\left\langle \begin{array}{l} \min(T^{+}_{BSVN}(1),T^{+}_{BSVN}(2)),\max(I^{+}_{BSVN}(1),I^{+}_{BSVN}(2)),\max(F^{+}_{BSVN}(1),F^{+}_{BSVN}(2)) \\ \min(T^{-}_{BSVN}(1),T^{-}_{BSVN}(2)),\max(I^{-}_{BSVN}(1),I^{-}_{BSVN}(2)),\max(F^{-}_{BSVN}(1),F^{-}_{BSVN}(2)) \end{array} \right\rangle$$

3.4 Example : Let X={s_1,s_2,s_3}.Then the bipolar single-valued neutrosophic subsets S_1 and S_2 of X,

$$S_1=\left\{\begin{array}{l} <s_1,(0.1,-0.1),(0.2,-0.3),(0.3,-0.5)> \\ <s_2,(0.2,-0.3),(0.4,-0.4),(0.6,-0.5)> \\ <s_3,(0.2,-0.8),(0.6,-0.4),(0.7,-0.7)> \end{array}\right\} \text{ and } S_2=\left\{\begin{array}{l} <s_1,(0.2,-0.1),(0.3,-0.5),(0.4,-0.5)> \\ <s_2,(0.3,-0.3),(0.3,-0.5),(0.4,-0.6)> \\ <s_3,(0.5,-0.3),(0.6,-0.3),(0.8,-0.7)> \end{array}\right\}$$

(i) Complement of S_1 is $S_1^c=\left\{\begin{array}{l} <s_1,(0.9,-0.9),(0.8,-0.7),(0.7,-0.5> \\ <s_2,(0.8,-0.7),(0.6,-0.6),(0.4,-0.5> \\ <s_3,(0.8,-0.2),(0.4,-0.6),(0.3,-0.3> \end{array}\right\}$

(ii) Union of S_1 and S_2 is S_1US_2= $\left\{\begin{array}{l} <s_1,(0.2,-0.1),(0.2,-0.5),(0.3,-0.5> \\ <s_2,(0.3,-0.3),(0.3,-0.5),(0.4,-0.6)> \\ <s_3,(0.5,-0.3),(0.6,-0.4),(0.7,-0.7)> \end{array}\right\}$

(iii) Intersection of S_1 and S_2 is $S_1 \cap S_2$= $\left\{\begin{array}{l} <s_1,(0.1,-0.1),(0.3,-0.3),(0.4,-0.5)> \\ <s_2,(0.2,-0.3),(0.4,-0.4),(0.6,-0.5)> \\ <s_3,(0.2,-0.8),(0.6,-0.3),(0.8,-0.7)> \end{array}\right\}$

3.5 Definition : Let two bipolar single-valued neutrosophic sets be BSVN₁(S) and BSVN₂(S) in X defined as
$BSVN_1(S)=<v,(T_{BSVN}^{+}(1),T_{BSVN}^{-}(1)),(I_{BSVN}^{+}(1),I_{BSVN}^{-}(1)),(F_{BSVN}^{+}(1),F_{BSVN}^{-}(1)):v\in X>$ and
$BSVN_2(S)=<v, (T_{BSVN}^{+}(2),T_{BSVN}^{-}(2)),(I_{BSVN}^{+}(2),I_{BSVN}^{-}(2)), (F_{BSVN}^{+}(2),F_{BSVN}^{-}(2)):v\in X>$.

Then $S_1=S_2$ if and only if

$T_{BSVN}^{+}(1) = T_{BSVN}^{+}(2)$, $I_{BSVN}^{+}(1) = I_{BSVN}^{+}(2)$, $F_{BSVN}^{+}(1) = F_{BSVN}^{+}(2)$,
$T_{BSVN}^{-}(1) = T_{BSVN}^{-}(2)$, $I_{BSVN}^{-}(1) = I_{BSVN}^{-}(2)$, $F_{BSVN}^{-}(1) = F_{BSVN}^{-}(2)$ for all v∈X.

3.6 Definition : Let two bipolar single-valued neutrosophic sets be $BSVN_1$ and $BSVN_2$ in X defined as
$BSVN_1(S)=<v,(T_{BSVN}^{+}(1),T_{BSVN}^{-}(1)),(I_{BSVN}^{+}(1),I_{BSVN}^{-}(1)),(F_{BSVN}^{+}(1),F_{BSVN}^{-}(1)):v\in X>$ and
$BSVN_2(S)=<v, (T_{BSVN}^{+}(2),T_{BSVN}^{-}(2)),(I_{BSVN}^{+}(2),I_{BSVN}^{-}(2)), (F_{BSVN}^{+}(2),F_{BSVN}^{-}(2)):v\in X>$.

Then $S_1 \subseteq S_2$ if and only if

$T_{BSVN}^{+}(1) \le T_{BSVN}^{+}(2)$, $I_{BSVN}^{+}(1) \ge I_{BSVN}^{+}(2)$, $F_{BSVN}^{+}(1) \ge F_{BSVN}^{+}(2)$,
$T_{BSVN}^{-}(1) \le T_{BSVN}^{-}(2)$, $I_{BSVN}^{-}(1) \ge I_{BSVN}^{-}(2)$, $F_{BSVN}^{-}(1) \ge F_{BSVN}^{-}(2)$ for all v∈X.

4. Bipolar single-valued Neutrosophic Topological space:

4.1 Definition : A bipolar single-valued neutrosophic topology on a non-empty set X is a τ of BSVN sets satisfying the axioms

(i) 0_{BSVN} , 1_{BSVN} ∈ τ
(ii) $S_1 \cap S_2 \in \tau$ for any $S_1,S_2 \in \tau$
(iii) $US_i \in \tau$ for any arbitrary family $\{S_i : i\in j\} \in \tau$

The pair (X,τ) is called BSVN topological space. Any BSVN set in τ is called as BSVN open set in X. The complement S^c of BSVN set in BSVN topological space (X,τ) is called a BSVN closed set.

4.2 Definition : Null or Empty bipolar single-valued neutrosophic set of a Bipolar single-valued Neutrosophic set S over X is said to be if <v, (0, 0), (0, 0), (0, 0)> for all v∈X and it is denoted by 0_{BSVN}.

4.3 Definition : Absolute Bipolar single-valued neutrosophic set denoted by 1_{BSVN} of a Bipolar single-valued Neutrosophic set S over X is said to be if <v, (1,-1), (1,-1), (1,-1)> for all v∈X.

4.4 Example : Let X={s_1,s_2,s_3} and τ= {0_{BSVN},1_{BSVN},P,Q,R,S}Then a bipolar single-valued neutrosophic subset of *X* is

$$P=\left\{\begin{matrix} < s_1,(0.3,-0.5),(0.4,-0.2),(0.5,-0.3)> \\ < s_2,(0.3,-0.6),(0.7,-0.1),(0.4,-0.4)> \\ < s_3,()0.2,-0.7,(0.4,-0.3),(0.4,-0.1)> \end{matrix}\right\} \quad Q=\left\{\begin{matrix} < s_1,(0.5,-0.2),(0.5,-0.2),(0.3,-0.2)> \\ < s_2,(0.3,-0.4),(0.4,-0.2),(0.4,-0.2)> \\ < s_3,(0.3,-0.2),(0.4,-0.3),(0.4,-0.4)> \end{matrix}\right\}$$

$$R=\left\{\begin{matrix} < s_1,(0.5,-0.2),(0.4,-0.2),(0.3,-0.3)> \\ < s_2,(0.3,-0.4),(0.4,-0.2),(0.4,-0.4)> \\ < s_3,(0.3,-0.2),(0.4,-0.3),(0.4,-0.4)> \end{matrix}\right\} \quad S=\left\{\begin{matrix} < s_1,(0.3,-0.5),(0.5,-0.2),(0.5,-0.2)> \\ < s_2,(0.3,-0.6),(0.7,-0.1),(0.4,-0.2)> \\ < s_3,(0.2,-0.7),(0.4,-0.3),(0.4,-0.1)> \end{matrix}\right\}$$

Then (X,τ) is called BSVN topological space on X.

4.5 Definition : Let (X,τ) be a BSVN topological space and
BSVN (S) =<v, (T_{BSVN}^+,T_{BSVN}^-), (I_{BSVN}^+,I_{BSVN}^-),(F_{BSVN}^+,F_{BSVN}^-):v∈X> be a BSVN set in X. Then the closure and interior of A is defined as

Int (S) = U {F: F is a BSVN open set (BSVNOs) in X and F⊆S}

Cl (S) = ∩ {F: F is a BSVN closed set (BSVNCs) in X and S⊆F}.

Here cl(S) is a BSVNCs and int (S) is a BSVNOs in X.

(a) S is a BSVNCs in X iff cl (S) =S.
(b) S is a BSVNOs in X iff int (S) =S.

4.6 Example : Let X={s_1,s_2,s_3} and τ={0_{BSVN},1_{BSVN},P,Q,R,S}. Then a bipolar single-valued neutrosophic subset of *X* is

$$P=\left\{\begin{matrix} < s_1,(0.3,-0.5),(0.4,-0.2),(0.5,-0.3)> \\ < s_2,(0.3,-0.6),(0.7,-0.1),(0.4,-0.4)> \\ < s_3,(0.2,-0.7),(0.4,-0.3),(0.4,-0.1)> \end{matrix}\right\} Q=\left\{\begin{matrix} < s_1,(0.5,-0.2),(0.5,-0.2),(0.3,-0.2)> \\ < s_2,(0.3,-0.4),(0.4,-0.2),(0.4,-0.2)> \\ < s_3,(0.3,-0.2),(0.4,-0.3),(0.4,-0.4)> \end{matrix}\right\}$$

$$R=\left\{\begin{matrix} < s_1,(0.5,-0.2),(0.4,-0.2),(0.3,-0.3)> \\ < s_2,(0.3,-0.4),(0.4,-0.2),(0.4,-0.4)> \\ < s_3,(0.3,-0.2),(0.4,-0.3),(0.4,-0.4)> \end{matrix}\right\} S=\left\{\begin{matrix} < s_1,(0.3,-0.5),(0.5,-0.2),(0.5,-0.2)> \\ < s_2,(0.3,-0.6),(0.7,-0.1),(0.4,-0.2)> \\ < s_3,(0.2,-0.7),(0.4,-0.3),(0.4,-0.1)> \end{matrix}\right\}$$

$$T=\left\{\begin{matrix} < s_1,0.7,0.3,0.3,-0.5,-0.2,-0.4> \\ < s_2,0.6,0.6,0.3,-0.3,-0.5,-0.5> \\ < s_3,0.5,0.2,0.3,-0.5,-0.5,-0.6> \end{matrix}\right\}$$

Then int(T)=P and cl(T)=1_{BSVN}.

4.7 Proposition : Let BSVNTS of (X,τ) and S,T be BSVN's in X. Then the properties hold:

i. int (S) ⊆ S and S ⊆ cl(S)
ii. S⊆ T ⇒int(S) ⊆ int(T)
S⊆ T ⇒cl(S) ⊆ cl(T)

iii. int(int(S))=int(S)
 cl(cl(S))=cl(S)
iv. int(S∩T)=int(S)∩int(T)
 cl(SUT)=cl(S)Ucl(T)
v. $int(1_{BSVN})=1_{BSVN}$
 $cl(0_{BSVN})=0_{BSVN}$

Proof: The proof is obvious.

4.8 Proposition : Let BSVN sets of S_i's and T in X, then $S_i⊆T$ for each $i∈J⇒$ (a). $US_i⊆T$ and (b). $T ⊆ ∩S_i$.
Proof: (a).Let $S_i⊆B$ (i.e) $S_1⊆B$, $S_2⊆B$,….., $S_n⊆B$.

$⇒ \{T_{BSVN}^+(S_1) \le T_{BSVN}^+(T)$, $T_{BSVN}^-(S_1) \le T_{BSVN}^-(T)$, $I_{BSVN}^+(S_1) \ge I_{BSVN}^+(T)$, $I_{BSVN}^-(S_1) \ge I_{BSVN}^-(T)$, $F_{BSVN}^+(S_1) \ge F_{BSVN}^+(T)$, $F_{BSVN}^-(S_1) \ge F_{BSVN}^-(T)$,$T_{BSVN}^+(S_2) \le T_{BSVN}^+(T)$, $T_{BSVN}^-(S_2) \le T_{BSVN}^-(T)$, $I_{BSVN}^+(S_2) \ge I_{BSVN}^+(T)$,$I_{BSVN}^-(S_2) \ge I_{BSVN}^-(T)$,$F_{BSVN}^+(S_2) \ge F_{BSVN}^+(T)$,$F_{BSVN}^-(S_2) \ge F_{BSVN}^-(T)$…..................., $T_{BSVN}^+(S_n) \le T_{BSVN}^+(T)$, $T_{BSVN}^-(S_n) \le T_{BSVN}^-(T)$, $I_{BSVN}^+(S_n) \ge I_{BSVN}^+(T)$, $I_{BSVN}^-(S_n) \ge I_{BSVN}^-(T)$, $F_{BSVN}^+(S_n) \ge F_{BSVN}^+(T)$, $F_{BSVN}^-(S_n) \ge F_{BSVN}^-(T)$ }

$⇒\max\{(T_{BSVN}^+(S_1), T_{BSVN}^+(S_2) ,...,T_{BSVN}^+(S_n)),(T_{BSVN}^-(S_1) ,T_{BSVN}^-(S_2) ,..,T_{BSVN}^-(S_n))\}\le(T_{BSVN}^+(T),T_{BSVN}^-(T))$
$\min\{(I_{BSVN}^+(S_1) ,I_{BSVN}^+(S_2) ,…,I_{BSVN}^+(S_n)), (I_{BSVN}^-(S_1) ,I_{BSVN}^-(S_2) ,…,I_{BSVN}^-(S_n))\} \ge (I_{BSVN}^+(T) ,I_{BSVN}^-(T))$
$\min\{(F_{BSVN}^+(S_1) ,F_{BSVN}^+(S_2) ,…,F_{BSVN}^+(S_n)),(F_{BSVN}^-(S_1),F_{BSVN}^-(S_2) ,…,F_{BSVN}^-(S_n))\} \ge (F_{BSVN}^+(T),F_{BSVN}^-(T))$

where UA_i = <x, $\max\{(T_{BSVN}^+(S_1), T_{BSVN}^+(S_2) ,...,T_{BSVN}^+(S_n)),(T_{BSVN}^-(S_1) ,T_{BSVN}^-(S_2) ,..,T_{BSVN}^-(S_n))\}$
$\min\{(I_{BSVN}^+(S_1) ,I_{BSVN}^+(S_2) ,…,I_{BSVN}^+(S_n)), (I_{BSVN}^-(S_1) ,I_{BSVN}^-(S_2) ,…,I_{BSVN}^-(S_n))\}$
$\min\{(F_{BSVN}^+(S_1) ,F_{BSVN}^+(S_2) ,…,F_{BSVN}^+(S_n)), (F_{BSVN}^-(S_1) ,F_{BSVN}^-(S_2) ,…,F_{BSVN}^-(S_n))\}$ >
∴ $U S_i ⊆ T$ .Hence proved.

(b)Let $T⊆S_i$ (i.e) $T⊆S_1$, $T⊆S_2$, … $T⊆S_i$.

$⇒<$ $T_{BSVN}^+(T) \le T_{BSVN}^+(S_1)$, $T_{BSVN}^-(T) \le T_{BSVN}^-(S_1)$, $I_{BSVN}^+(T) \ge I_{BSVN}^+(S_1)$, $I_{BSVN}^-(T) \ge I_{BSVN}^-(S_1)$, $F_{BSVN}^+(T) \ge F_{BSVN}^+(S_1)$, $F_{BSVN}^-(T) \ge F_{BSVN}^-(S_1)$, $T_{BSVN}^+(T) \le T_{BSVN}^+(S_2)$, $T_{BSVN}^-(T) \le T_{BSVN}^-(S_2)$, $I_{BSVN}^+(T) \ge I_{BSVN}^+(S_2)$, $I_{BSVN}^-(T) \ge I_{BSVN}^-(S_2)$,$F_{BSVN}^+(T) \ge F_{BSVN}^+(S_2)$,$F_{BSVN}^-(T) \ge F_{BSVN}^-(S_2)$,………………., $T_{BSVN}^+(T) \le T_{BSVN}^+(S_n)$, $T_{BSVN}^-(T) \le T_{BSVN}^-(S_n)$,$I_{BSVN}^+(T) \ge I_{BSVN}^+(S_n)$, $I_{BSVN}^-(T) \ge I_{BSVN}^-(S_n)$, $F_{BSVN}^+(T) \ge F_{BSVN}^+(S_n)$, $F_{BSVN}^-(T) \ge F_{BSVN}^-(S_n)$ }>

$⇒(T_{BSVN}^+(T),T_{BSVN}^-(T)) \le \min\{(T_{BSVN}^+(S_1),T_{BSVN}^+(S_2) ,...,T_{BSVN}^+(S_n)),(T_{BSVN}^-(S_1) ,T_{BSVN}^-(S_2) ,..,T_{BSVN}^-(S_n))\}$
$(I_{BSVN}^+(T) ,I_{BSVN}^-(T)) \ge \max \{(I_{BSVN}^+(S_1) ,I_{BSVN}^+(S_2) ,…,I_{BSVN}^+(S_n)), (I_{BSVN}^-(S_1) ,I_{BSVN}^-(S_2) ,…,I_{BSVN}^-(S_n))\}$
$(F_{BSVN}^+(T),F_{BSVN}^-(T))\ge \max\{(F_{BSVN}^+(S_1) ,F_{BSVN}^+(S_2) ,…,F_{BSVN}^+(S_n)),(F_{BSVN}^-(S_1) ,F_{BSVN}^-(S_2) ,…, F_{BSVN}(S_n))\}$

Where $∩A_i$= <x, $\min\{(T_{BSVN}^+(S_1), T_{BSVN}^+(S_2) ,...,T_{BSVN}^+(S_n)),(T_{BSVN}^-(S_1) ,T_{BSVN}^-(S_2) ,..,T_{BSVN}^-(S_n))\}$
$\max\{(I_{BSVN}^+(S_1) ,I_{BSVN}^+(S_2) ,…,I_{BSVN}^+(S_n)), (I_{BSVN}^-(S_1) ,I_{BSVN}^-(S_2) ,…,I_{BSVN}^-(S_n))\}$
$\max\{(F_{BSVN}^+(S_1) ,F_{BSVN}^+(S_2) ,…,F_{BSVN}^+(S_n)), (F_{BSVN}^-(S_1) ,F_{BSVN}^-(S_2) ,…,F_{BSVN}^-(S_n))\}$ >
∴ $T ⊆ ∩S_i$ .Hence proved.

4.9 Proposition : Let S_i's and T are BSVN sets in X then (i). $(US_i)^c = ∩S_i^c$, (ii). $(∩S_i)^c = US_i^c$ and (iii). $(S^c)^c = S$.
Proof: (i) Let US_i = <x, $\max\{(T_{BSVN}^+(S_1), T_{BSVN}^+(S_2) ,...,T_{BSVN}^+(S_n)),(T_{BSVN}^-(S_1) ,T_{BSVN}^-(S_2) ,..,T_{BSVN}^-(S_n))\}$
$\min\{(I_{BSVN}^+(S_1) ,I_{BSVN}^+(S_2) ,…,I_{BSVN}^+(S_n)), (I_{BSVN}^-(S_1) ,I_{BSVN}^-(S_2) ,…,I_{BSVN}^-(S_n))\}$
$\min\{(F_{BSVN}^+(S_1) ,F_{BSVN}^+(S_2) ,…,F_{BSVN}^+(S_n)),(F_{BSVN}^-(S_1) ,F_{BSVN}^-(S_2) ,..,F_{BSVN}^-(S_n))\}$ >

$(US_i)^c$ = <x,$\min\{(1-T_{BSVN}^+(S_1),1-T_{BSVN}^+(S_2) ,..,1-T_{BSVN}^+(S_n)),(-1- T_{BSVN}^-(S_1) ,-1-T_{BSVN}^-(S_2) ,..,-1- T_{BSVN}^-(S_n))\}$
$\max\{(1-I_{BSVN}^+(S_1) ,1-I_{BSVN}^+(S_2) ,…,1-I_{BSVN}^+(S_n)), (-1-I_{BSVN}^-(S_1) ,-1-I_{BSVN}^-(S_2) ,…,-1-I_{BSVN}^-(S_n))\}$
$\max\{(1-F_{BSVN}^+(S_1) ,1-F_{BSVN}^+(S_2) ,..,1-F_{BSVN}^+(S_n)), (-1-F_{BSVN}^-(S_1) ,-1-F_{BSVN}^-(S_2) ,..,-1-F_{BSVN}^-(S_n))$ >
------------------->(1)

S_i^c = <x,$(1-T_{BSVN}^+(S_1),1- T_{BSVN}^+(S_2) ,...,1-T_{BSVN}^+(S_n)),(-1- T_{BSVN}^-(S_1) ,-1-T_{BSVN}^-(S_2) ,..,-1-T_{BSVN}^-(S_n))$
$(1-I_{BSVN}^+(S_1) ,1-I_{BSVN}^+(S_2) ,…,1-I_{BSVN}^+(S_n)), (-1-I_{BSVN}^-(S_1) ,-1-I_{BSVN}^-(S_2) ,…,-1-I_{BSVN}^-(S_n))$
$(1-F_{BSVN}^+(S_1) ,1-F_{BSVN}^+(S_2) ,…,1-F_{BSVN}^+(S_n)), (-1-F_{BSVN}^-(S_1) ,-1-F_{BSVN}^-(S_2) ,…,-1-F_{BSVN}^-(S_n)$>

$\cap S_i^c = \langle x, \min\{(1-T_{BSVN}^+(S_1), 1-T_{BSVN}^+(S_2), ..., 1-T_{BSVN}^+(S_n)), (-1-T_{BSVN}^-(S_1), -1-T_{BSVN}^-(S_2), .., -1-T_{BSVN}^-(S_n))\}$

$\max\{(1-I_{BSVN}^+(S_1), 1-I_{BSVN}^+(S_2), \ldots, 1-I_{BSVN}^+(S_n)), (-1-I_{BSVN}^-(S_1), -1-I_{BSVN}^-(S_2), \ldots, -1-I_{BSVN}^-(S_n))\}$

$\max\{(1-F_{BSVN}^+(S_1), 1-F_{BSVN}^+(S_2), .., 1-F_{BSVN}^+(S_n)), (-1-F_{BSVN}^-(S_1), -1-F_{BSVN}^-(S_2), .., -1-F_{BSVN}^-(S_n)\}\rangle$

------------------->(2)

From (1) and (2), $(US_i)^c = \cap S_i^c$ .Hence proved.

(ii). Similar as proof of (i).

(iii). Let $S=\langle (T_{BSVN}^+(S), T_{BSVN}^-(S)), (I_{BSVN}^+(S), I_{BSVN}^-(S)), (F_{BSVN}^+(S), F_{BSVN}^-(S))\rangle$ be a BSVN set in X, then
$S^c = \langle (1-T_{BSVN}^+(S), -1-T_{BSVN}^-(S)), (1-I_{BSVN}^+(S), -1-I_{BSVN}^-(S)), (1-F_{BSVN}^+(S), -1-F_{BSVN}^-(S)\rangle$
$(S^c)^c = \langle (T_{BSVN}^+(S), T_{BSVN}^-(S)), (I_{BSVN}^+(S), I_{BSVN}^-(S)), (F_{BSVN}^+(S), F_{BSVN}^-(S))\rangle$
$(S^c)^c = S$. Hence proved.

5. Bipolar single-valued Neutrosophic Number (BSVNN)

5.1 Definition : Let two bipolar single-valued neutrosophic number(BSVNN) be
$\widetilde{S}_1=\langle T_{BSVN}^+(1), T_{BSVN}^-(1)), (I_{BSVN}^+(1), I_{BSVN}^-(1)), (F_{BSVN}^+(1), F_{BSVN}^-(1)\rangle$ and
$\widetilde{S}_2=\langle T_{BSVN}^+(2), T_{BSVN}^-(2)), (I_{BSVN}^+(2), I_{BSVN}^-(2)), (F_{BSVN}^+(2), F_{BSVN}^-(2)\rangle$. Then the operations are

i. $\lambda\widetilde{S}_1=\langle 1-(1-T_{BSVN}^+(1))^\lambda, -(-T_{BSVN}^-(1))^\lambda, (I_{BSVN}^+(1))^\lambda, -(-I_{BSVN}^-(1))^\lambda, (F_{BSVN}^+(1))^\lambda, -(1-(1-(-F_{BSVN}^-(1)))^\lambda)\rangle$

ii. $\widetilde{S}_1^\lambda = \langle (T_{BSVN}^+(1))^\lambda, -(1-(1-(-T_{BSVN}^-(1)))^\lambda), 1-(1-I_{BSVN}^+(1))^\lambda, -(-I_{BSVN}^-(1))^\lambda, 1-(1-F_{BSVN}^+(1))^\lambda, -(-F_{BSVN}^-(1))^\lambda\rangle$

iii. $\widetilde{S}_1+\widetilde{S}_2= \langle T_{BSVN}^+(1)+ T_{BSVN}^+(2)- T_{BSVN}^+(1)\, T_{BSVN}^+(2), - T_{BSVN}^-(1)\, T_{BSVN}^-(2),$
$I_{BSVN}^+(1)\, I_{BSVN}^+(2), - (- I_{BSVN}^-(1) - I_{BSVN}^-(2) - I_{BSVN}^-(1)\, I_{BSVN}^-(2)),$
$F_{BSVN}^+(1)\, F_{BSVN}^+(2), -(- F_{BSVN}^-(1) - F_{BSVN}^-(2) - F_{BSVN}^-(1) F_{BSVN}^-(2))\rangle$

iv. $\widetilde{S}_1 . \widetilde{S}_2 = \langle T_{BSVN}^+(1)\, T_{BSVN}^+(2), -(- T_{BSVN}^-(1)- T_{BSVN}^-(2)- T_{BSVN}^-(1) T_{BSVN}^-(2)),$
$I_{BSVN}^+(1)+ I_{BSVN}^+(2)- I_{BSVN}^+(1)\, I_{BSVN}^+(2), - I_{BSVN}^-(1)\, I_{BSVN}^-(2),$
$F_{BSVN}^+(1)+ F_{BSVN}^+(2)- F_{BSVN}^+(1)\, F_{BSVN}^+(2), - F_{BSVN}^-(1)\, F_{BSVN}^-(2)\rangle$

5.2 Definition : Let a bipolar single-valued neutrosophic number(BSVNN) be
$\widetilde{S}_1=\langle T_{BSVN}^+(1), T_{BSVN}^-(1)), (I_{BSVN}^+(1), I_{BSVN}^-(1)), (F_{BSVN}^+(1), F_{BSVN}^-(1)\rangle$.Then

i. score function: $s(\widetilde{S}_1)=(T_{BSVN}^+(1)+1- I_{BSVN}^+(1)+1- F_{BSVN}^+(1)+1+T_{BSVN}^-(1)- I_{BSVN}^-(1)-F_{BSVN}^-(1))/6$

ii. accuracy function: $a(\widetilde{S}_1)= T_{BSVN}^+(1)-F_{BSVN}^+(1)+T_{BSVN}^-(1)-F_{BSVN}^-(1)$

iii. certainty function : $c(\widetilde{S}_1)= T_{BSVN}^+(1)-F_{BSVN}^+(1)$

5.3 Definition : The two bipolar single-valued neutrosophic numbers (BSVNN) are compared
$\widetilde{S}_1=\langle T_{BSVN}^+(1), T_{BSVN}^-(1)), (I_{BSVN}^+(1), I_{BSVN}^-(1)), (F_{BSVN}^+(1), F_{BSVN}^-(1)\rangle$
$\widetilde{S}_2=\langle T_{BSVN}^+(2), T_{BSVN}^-(2)), (I_{BSVN}^+(2), I_{BSVN}^-(2)), (F_{BSVN}^+(2), F_{BSVN}^-(2)\rangle$ can be defined as

i. If $s(\widetilde{S}_1)>s(\widetilde{S}_2)$, $\widetilde{S}_1$ is superior to $\widetilde{S}_2$,(i.e.) $\widetilde{S}_1$ is greater than $\widetilde{S}_2$ denoted as $\widetilde{S}_1>\widetilde{S}_2$.

ii. If $s(\widetilde{S}_1)=s(\widetilde{S}_2)$ and $\widetilde{S}\ (\widetilde{S}_1)> \widetilde{S}\ (\widetilde{S}_2)$, $\widetilde{S}_1$is superior to $\widetilde{S}_2$,(i.e.) $\widetilde{S}_1$ is greater than $\widetilde{S}_2$ denoted as $\widetilde{S}_1 < \widetilde{S}_2$.

iii. If $s(\widetilde{S}_1)=s(\widetilde{S}_2)$ and $\widetilde{S}\ (\widetilde{S}_1)= \widetilde{S}\ (\widetilde{S}_2)$ and $c(\widetilde{S}_1)>c(\widetilde{S}_2)$, $\widetilde{S}_1$ is greater than $\widetilde{S}_2$,that is $\widetilde{S}_1$ is superior to $\widetilde{S}_2$,denoted as $\widetilde{S}_1>\widetilde{S}_1$.

iv. If $s(\widetilde{S}_1)=s(\widetilde{S}_2)$ and $\widetilde{S}\ (\widetilde{S}_1)= \widetilde{S}\ (\widetilde{S}_2)$ and $c(\widetilde{S}_1)=c(\widetilde{S}_2)$, $\widetilde{S}_1$ is equal to $\widetilde{S}_2$,that is $\widetilde{S}_1$ is indifferent to $\widetilde{S}_2$,denoted as $\widetilde{S}_1=\widetilde{S}_1$.

5.4 Definition : Let a family of bipolar single-valued neutrosophic numbers(BSVNN) be $\widetilde{S}_j=\langle T_{BSVN}^+(j), T_{BSVN}^-(j)), (I_{BSVN}^+(j), I_{BSVN}^-(j)), (F_{BSVN}^+(j), F_{BSVN}^-(j)\rangle (j=1,2,3,\ldots,n)$. A mapping $A_\omega : F_n \rightarrow F$ is called bipolar single-valued Neutrosophic weighted average ($BSVNWA_\omega$) operator if satisfies

$$A_\omega(\widetilde{s}_1, \widetilde{s}_2, .., \widetilde{s}_n) = \sum_{j=1}^{n} \omega_j \widetilde{s}_j = < 1-\prod_{j=1}^{n}(1-T_{BSVN}^{+}(j))^{\omega_j}, -\prod_{j=1}^{n}(-T_{BSVN}^{-}(j))^{\omega_j}, \prod_{j=1}^{n} I_{BSVN}^{+}(j)^{\omega_j},$$

$$-(1-\prod_{j=1}^{n}(1-(-I_{BSVN}^{-}))^{\omega_j}), \prod_{j=1}^{n} F_{BSVN}^{+}(j)^{\omega_j}, -(1-\prod_{j=1}^{n}(1-(-F_{BSVN}^{-}))^{\omega_j})>$$

Here ω_j is the weight of $\widetilde{s}_j$ (j=1,2,…n), $\sum_{j=1}^{n} \omega_j=1$ and $\omega_j \in [0,1]$.

5.5 Definition : Let a family of bipolar single-valued neutrosophic numbers(BSVNN) be $\widetilde{s}_j =<T_{BSVN}^{+}(j), T_{BSVN}^{-}(j)),(I_{BSVN}^{+}(j),I_{BSVN}^{-}(j)),(F_{BSVN}^{+}(j),F_{BSVN}^{-}(j)>$(j=1,2,3,…,n) . A mapping $G_\omega:F_n \to F$ is called bipolar single-valued neutrosophic weighted geometric(BSVNWG$_\omega$) operator if it satisfies

$$G_\omega(\widetilde{s}_1, \widetilde{s}_2, .., \widetilde{s}_n) = \prod_{j=1}^{n} \widetilde{s}_j^{\ \omega_j} = < \prod_{j=1}^{n} T_{BSVN}^{+}(j))^{\omega_j}, -(1-\prod_{j=1}^{n}(1-(-T_{BSVN}^{-}(j)))^{\omega_j}),$$

$$1-\prod_{j=1}^{n}(1-I_{BSVN}^{+}(j))^{\omega_j}, -\prod_{j=1}^{n}(-I_{BSVN}^{-}))^{\omega_j}, 1-\prod_{j=1}^{n}(1-F_{BSVN}^{+}(j)^{\omega_j}, -\prod_{j=1}^{n}(-F_{BSVN}^{-})^{\omega_j}>$$ where ω_j is the weight of $\widetilde{s}_j$ (j=1,2,…n), $\sum_{j=1}^{n} \omega_j=1$ and $\omega_j \in [0,1]$.

5.6. Decision making problem:

Here, with bipolar single-valued neutrosophic data, we develop decision making problem based on A_ω operator Suppose the set of alternatives is S = {$S_1,S_2,...S_m$} and the set of all criterions (or attributes) are G= {$G_1,G_2,….,G_n$}.Let $\omega=(\omega_1,\omega_2,….\omega_n)^T$ be the weight vector of attributes such that $\sum_{j=1}^{n} \omega_j=1$ and $\omega_j \geq 0$ (j=1,2,…n) and ω_j assign to the weight of attribute G_j.An alternative on criterions is calculated by the decision maker and the assess values are represented by the design of bipolar single-valued neutrosophic numbers.

Assume the decision matrix $(\widetilde{s}_{ij})_{m\times n} =(<T_{BSVN}^{+}(ij),T_{BSVN}^{-}(ij)),(I_{BSVN}^{+}(ij),I_{BSVN}^{-}(ij)),(F_{BSVN}^{+}(ij),F_{BSVN}^{-}(ij)>)_{mxn}$ contributed by the decision maker, for Alternative S_i with criterion G_j ,the bipolar single-valued neutrosophic number is $\widetilde{s}_{ij}$. The conditions are $T_{BSVN}^{+}(ij),T_{BSVN}^{-}(ij),(I_{BSVN}^{+}(ij),I_{BSVN}^{-}(ij),F_{BSVN}^{+}(ij),F_{BSVN}^{-}(ij) \in [0,1]$ such that $0 \leq T_{BSVN}^{+}(ij) -T_{BSVN}^{-}(ij) +I_{BSVN}^{+}(ij) -I_{BSVN}^{-}(ij) +F_{BSVN}^{+}(ij) -F_{BSVN}^{-}(ij) \leq 6$ for i=1,2,3,…m and j=1,2,…n.

Algorithm:

STEP 1: Construct the decision matrix by the decision maker.

$$(\widetilde{s}_{ij})_{m\times n} =(<T_{BSVN}^{+}(ij),T_{BSVN}^{-}(ij)),(I_{BSVN}^{+}(ij),I_{BSVN}^{-}(ij)),(F_{BSVN}^{+}(ij),F_{BSVN}^{-}(ij)>)_{mxn}$$

STEP 2: Compute $\widetilde{s}_i=A_\omega(\widetilde{s}_{i1}, \widetilde{s}_{i2},… \widetilde{s}_{in})$ for each i=1,2,….m.

STEP 3: Using the set of overall bipolar single-valued neutrosophic number of $\widetilde{s}_i$ (i=1,2,…m),calculate the score values $\widetilde{s}(\widetilde{s}_i)$.

STEP 4: Rank all the structures of $\widetilde{s}_i$ (i=1,2,…m) according to the score values.

Example (5.7): A patient is intending to analyze which disease is caused to him. Four types of diseases S_i(i=1,2,3,4) are Cancer, Asthuma, Hyperactive, Typhoid. The set of symptoms are G_1=cough, G_2=Headache,G_3=stomach pain,G_4=blood cloting. To evaluate the 4 diseases (alternatives) S_i(i=1,2,3,4) under the above four symptoms(attributes) using the bipolar single-valued neutrosophic values.The weight vector of the attributes G_j (j=1, 2, 3, 4) is ω= $(0.25,0.35,0.20,0.20)^T$.

STEP 1: The decision matrix provided by the patient is constructed as below:

S_i /G_i	G_1	G_2	G_3	G_4
S_1	(0.3,-0.5)(0.4,-0.4) (0.4,-0.2)	(0.3,-0.3)(0.5,-0.2) (0.3,-0.4)	(0.6,-0.4)(0.4,-0.3) (0.3,-0.5)	(0.1,-0.3)(0.6,-0.4) (0.5,-0.3)
S_2	(0.3,-0.4)(0.7,-0.5) (0.4,-0.5)	(0.1,-0.3)(0.2,-0.4) (0.3,-0.5)	(0.3,-0.5)(0.2,-0.4) (0.1,-0.3)	(0.4,-0.2)(0.2,-0.3) (0.1,-0.2)
S_3	(0.3,-0.4)(0.4,-0.5) (0.5,-0.6)	(0.1,-0.2)(0.2,-0.3) (0.3,-0.4)	(0.5,-0.4)(0.4,-0.5) (0.5,-0.6)	(0.1,-0.3)(0.2,-0.4) (0.3,-0.6)
S_4	(0.3,-0.2)(0.2,-0.1) (0.1,-0.2)	(0.3,-0.1)(0.4,-0.2) (0.5,-0.3)	(0.2,-0.3)(0.4,-0.7) (0.7,-0.8)	(0.1,-0.3)(0.2,-0.5) (0.3,-0.7)

STEP 2: Compute $\widetilde{S}_i=A_\omega(\widetilde{S}_{i1}, \widetilde{S}_{i2}, \widetilde{S}_{i3}, \widetilde{S}_{i4})$ for each i=1,2,3,4;

$\widetilde{S}_1$=< (0.3,-0.4) (0.5,-0.3) (0.4,-0.4)>
$\widetilde{S}_2$=< (0.2,-0.3) (0.3,-0.4) (0.2,-0.4)>
$\widetilde{S}_3$=< (0.2,-0.3) (0.3,-0.4) (0.4,-0.5)>
$\widetilde{S}_4$=< (0.2,-0.2) (0.3,-0.4) (0.3,-0.5)>

STEP 3: The score value of $\widetilde{S}(\widetilde{S}_i)$ (i=1, 2, 3, 4) are computed for the set of overall bipolar single-valued neutrosophic number .

$\widetilde{S}(\widetilde{S}_1)=0.45$
$\widetilde{S}(\widetilde{S}_2)=0.53$
$\widetilde{S}(\widetilde{S}_3)=0.51$
$\widetilde{S}(\widetilde{S}_4)=0.55$

STEP 4: According to the score values rank all the software systems of S_i (i=1, 2, 3, and 4)

$$\mathbf{S_4 > S_2 > S_3 > S_1}$$

Thus S_4 is the most affected disease (alternative) . Typhoid(S_4) is affected to him.

Conclusion:

In this paper, bipolar single-valued neutrosophic sets were developed. Bipolar single-valued neutrosophic topological spaces were also introduced and characterized some of its properties. Further score function, certainty function and accuracy functions of the Bipolar single-valued neutrosophic were given. We proposed the average and geometric operators (A_ω and G_ω) for bipolar single-valued neutrosophic information. To calculate the integrity of alternatives on the attributes taken, a bipolar single-valued neutrosophic decision making approach using the score function, certainty function and accuracy function were refined.

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Triangular Cubic Hesitant Fuzzy Einstein Hybrid Weighted Averaging Operator and Its Application to Decision Making

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Abstract: In this paper, triangular cubic hesitant fuzzy Einstein weighted averaging (TCHFEWA) operator, triangular cubic hesitant fuzzy Einstein ordered weighted averaging (TCHFEOWA) operator and triangular cubic hesitant fuzzy Einstein hybrid weighted averaging (TCHFEHWA) operator are proposed. An approach to multiple attribute group decision making with linguistic information is developed based on the TCHFEWA and the TCHFEHWA operators. Furthermore, we establish various properties of these operators and derive the relationship between the proposed operators and the existing aggregation operators. Finally, a numerical example is provided to demonstrate the application of the established approach.

Keywords: triangular cubic hesitant fuzzy set (TCFS); Einstein t-norm; arithmetic averaging operator; multi-attribute decision making (MADM)

1. Introduction

Multicriteria decision-making (MCDM) problems seek great attention to practical fields, whose target is to find the best alternative(s) among the feasible options. In primitive times, decisions were framed without handling the uncertainties in the data, which may lead to inadequate results toward the real-life operating situations. Since all these facilitate the uncertainties to a great extent, they cannot withstand situations where the decision the maker has to consider the falsity corresponding to the truth a value ranging over an interval. IFSs have the advantage that permits the user to model some uncertainty on the membership function of the elements. That is, fuzzy sets require a membership degree for each element in the reference set, whereas an IFS permits us to include some hesitation on this value. Atanassov [1] originated the notion of intuitionistic fuzzy set (IFS), which is a generalization of Zadeh's fuzzy sets [2]. The intuitionistic fuzzy set has three main parts: membership function, non-membership function, and hesitancy function. According to Zadeh [3], a conditional statement "If $x = A$ then $y =$" describes a relation between the two fuzzy variables x and y. He therefore suggests that the conditional statement should be represented by a fuzzy relation from U and V. Bustince et al. [4] analysed the existent relations between the structures of a relation and the structures of its complementary one. Deschrijver et al. [5] introduced the many theories to establish the relationships between intuitionistic fuzzy sets, interval-valued fuzzy sets, and interval-valued intuitionistic fuzzy sets. Through the last period, the AIFS theory has been suitably associated with aggregate intuitionistic fuzzy information [6,7]. Xu [8] developed some basic arithmetic aggregation operators, such as the intuitionistic fuzzy weighted averaging (IFWA) operator, intuitionistic fuzzy ordered weighted averaging (IFOWA) operator, and intuitionistic fuzzy hybrid averaging (IFHA) operator, for aggregating intuitionistic fuzzy values (IFVs). Chen and Xu [9] presented a structure which generalizes the idea of HFS [10] to IVHFS that makes possible the membership evaluation of a component into many realizable interval numbers. Torra [10] proposed the hesitant fuzzy set that permitted the membership having a set of possible values and discussed the relationship between the hesitant fuzzy set and intuitionistic fuzzy set. The hesitant fuzzy set can be applied to many decision-making problems. He also proved that the operations he proposed are consistent with the ones of the intuitionistic fuzzy set when applied to the envelope of the hesitant fuzzy set. To get the optimal alternative in a decision making problem with multiple attributes and multiple persons, there are usually two ways: (1) aggregate the decision makers' opinions under each attribute for alternatives, then aggregate the collective values of attributes for each alternative; (2) aggregate the attribute values given by the decision makers for each alternative, and then aggregate the decision makers' opinions for each alternative. Xia et al. [11] developed some operations and aggregation operators for hesitant fuzzy elements. Xia et al. [12] defined the reflect the correlation of the aggregation arguments, two methods are proposed to determine the aggregation weight vectors. Zeng et al. [13] developed an intuitionistic fuzzy ordered weighted distance (IFOWD) operator.

Decision-making is an integral part of modern management. Essentially, rational or sound decision making is taken as a primary function of management. Every manager takes hundreds and hundreds of decisions subconsciously or consciously making it as the key component in the role of a manager. Decisions play important roles as they determine both organizational and managerial activities. A decision can be defined as a course of action purposely chosen from a set of alternatives to achieve organizational or managerial objectives or goals. A decision-making process is a continuous and indispensable component of managing any organization or business activities. Decisions are made to sustain the activities of all business activities and organizational functioning. Decision theory has its root in economic theory, with the assumption that people make decisions to maximize utility on the basis of self-interest and rationality. This, however, does not consider the possibilities or effects of moderating or intervening factors that make decisions reference-dependent. Nonetheless, expected utility theory has been applied in the construction industry with some success and has been the predominant model for normative decision making. The theory is considered idealistic, however, because it focuses on how managers should make decisions rather than how they actually make decisions.

Technical people in the construction industry have been observed to exhibit a tendency for a normative approach to decision making, thereby weakening their ability to deal with uncertainty. Program management is dominated by technical staff and probably more than a few are struggling with tendencies toward this normative thinking phenomenon. An alternative approach is the descriptive decision theory.

Comparative analysis needs to be distinguished from the juxtaposition of descriptions of a series of cases. While sequential presentations of descriptive data are undoubtedly informative about the cases concerned they are only comparative in the weak sense of making the reader aware of differences and similarities. They whet the appetite to know more. Based on the algebraic sum, algebraic product, and operational laws on AIFSs, some aggregation operators have been proposed to overcome the lack of precision in the final results using the binary operations to carry the combination process. Xu and Yager [14] developed some basic geometric aggregation operators such as intuitionistic fuzzy weighted geometric (IFWG) operator, intuitionistic fuzzy ordered weighted geometric (IFOWG) operator, and intuitionistic fuzzy hybrid geometric (IFHG) operator, and applied them to multiple attribute decision making (MADM) based on AIFSs. Xu and Yager [15] defined dynamic IFWA operator and developed a procedure to solve the dynamic intuitionistic fuzzy MADM problems. Zhao et al. [16]

developed some hesitant triangular fuzzy aggregation operators based on the Einstein operation: the hesitant triangular fuzzy Einstein weighted averaging (HTFEWA) operator, hesitant triangular fuzzy Einstein weighted geometric (HTFEWG) operator, hesitant triangular fuzzy Einstein ordered weighted averaging (HTFEOWA) operator, hesitant triangular fuzzy Einstein ordered weighted geometric (HTFEOWG) operator, hesitant triangular fuzzy Einstein hybrid average (HTFEHA) operator and hesitant triangular fuzzy Einstein hybrid geometric (HTFEHG) operator. Li et al. [17] developed to introduce some concepts of fuzzy measure and interval-valued intuitionistic uncertain linguistic variables based on Archimedean *t*-norm. Then, interval-valued intuitionistic uncertain linguistic weighted average(geometric) and interval-valued intuitionistic uncertain linguistic ordered weighted average operator based on Archimedean *t*-norm are developed. Furthermore, some desirable properties of these operators, such as commutativity, idempotency, and monotonicity have been studied, and interval-valued intuitionistic uncertain linguistic hybrid average operator based on Archimedean *t*-norm are developed.

Cubic set was introduced by Jun et al. [18]. Cubic sets are the extensions of fuzzy sets and intuitionistic fuzzy sets, in which there are two representations, one is utilized for the degree of membership and other is utilized for the degree of non-membership. The membership function is grip as interval while non-membership is thoroughly considered the ordinary fuzzy set.

Medina et al. [19] showed that the information common to both concept lattices can be seen as a sublattice of the Cartesian product of both concept lattices. Pozna et al. [20] proposed the attractive applications of signatures related to the modeling of fuzzy inference systems suggested and discussed. Jankowski et al. [21] proposed model revealed that a growing level of persuasion can increase results only to a certain extent. Kumar et al. [22] proposed the Artificial Bee Colony (ABC) algorithm is a swarm-based algorithm inspired by the intelligent foraging behavior of honey bees. In order to make use of the merits of both algorithms, a hybrid algorithm (IABCFCM) based on improved ABC and FCM algorithms.

Chen et al. [23] developed some hesitant triangular intuitionistic fuzzy aggregation operators and standardized hesitant triangular intuitionistic fuzzy aggregation operators. Xu et al. [24] provided a survey of the aggregation techniques of intuitionistic fuzzy information, and their applications in various fields, such as decision making, cluster analysis, medical diagnosis, forecasting, and manufacturing grid. Zhang [25] developed a series of aggregation operators for interval-valued intuitionistic hesitant fuzzy information.

Fahmi et al. [26] developed the hamming distance for the triangular cubic fuzzy number and weighted averaging operator. Fahmi et al. [27] proposed the cubic TOPSIS method and grey relational analysis set. Fahmi et al. [28] defined the triangular cubic fuzzy number and operational laws. The authors developed the triangular cubic fuzzy hybrid aggregation (TCFHA) administrator to total all individual fuzzy choice structure provided by the decision makers into the aggregate cubic fuzzy decision matrix. Amin et al. [29] defined the generalized triangular cubic linguistic hesitant fuzzy weighted geometric (GTCHFWG) operator, generalized triangular cubic linguistic hesitant fuzzy ordered weighted average (GTCLHFOWA) operator, generalized triangular cubic linguistic hesitant fuzzy ordered weighted geometric (GTCLHFOWG) operator, generalized triangular cubic linguistic hesitant fuzzy hybrid averaging (GTCLHFHA) operator and generalized triangular cubic linguistic hesitant fuzzy hybrid geometric (GTCLHFHG) operator. Fahmi et al. [30] developed Trapezoidal linguistic cubic hesitant fuzzy TOPSIS method to solve the MCDM method based on trapezoidal linguistic cubic hesitant fuzzy TOPSIS method. Fahmi et al. [31] define aggregation operators for triangular cubic linguistic hesitant fuzzy sets which include cubic linguistic fuzzy (geometric) operator, triangular cubic linguistic hesitant fuzzy weighted geometric (TCLHFWG) operator, triangular cubic linguistic hesitant fuzzy ordered weighted geometric (TCHFOWG) operator and triangular cubic linguistic hesitant fuzzy hybrid geometric (TCLHFHG) operator. Fahmi et al. [32] defined the trapezoidal cubic fuzzy weighted arithmetic averaging operator and weighted geometric averaging operator. Expected values, score function, and accuracy function of trapezoidal cubic fuzzy numbers are defined. Fahmi et al. [33] developed three arithmetic averaging operators, that is trapezoidal cubic fuzzy Einstein weighted averaging (TrCFEWA) operator, trapezoidal cubic fuzzy Einstein ordered weighted averaging (TrCFEOWA) operator and trapezoidal cubic fuzzy Einstein hybrid weighted averaging (TrCFEHWA) operator, for aggregating trapezoidal cubic fuzzy information. Fahmi et al. [34] defined some Einstein operations on cubic fuzzy set (CFS) and develop three arithmetic averaging operators, which are cubic fuzzy Einstein weighted averaging (CFEWA) operator, cubic fuzzy Einstein ordered weighted averaging (CFEOWA) operator and cubic fuzzy Einstein hybrid weighted averaging (CFEHWA) operator, for aggregating cubic fuzzy data. Amin et al. [35] introduced the new concept of the trapezoidal cubic hesitant fuzzy TOPSIS method.

Despite having a bulk of related literature on the problem under consideration, the following aspects related to triangular cubic hesitant fuzzy numbers (TCHFNs) and their aggregation operators motivated the researchers to carry it an in depth inquiry into the current study.

(1) The main advantages of the proposed operators are these aggregation operators provided more accurate and precious result as compare to the above mention operators.

(2) We generalized the concept of triangular cubic hesitant fuzzy numbers (TCHFNs), triangular intuitionistic fuzzy sets and introduce the concept of triangular cubic hesitant fuzzy numbers. If we take only one element in the membership degree of the triangular cubic hesitant fuzzy number, i.e., instead of the interval we take a fuzzy number, then we get triangular intuitionistic fuzzy numbers, similarly, if we take membership degree as the fuzzy number and non-membership degree equal to zero, then we get triangular fuzzy numbers.

(3) The objective of the study include:

Propose triangular cubic hesitant fuzzy number, operational laws, score value and accuracy value of TCHFNs.

Propose three aggregation operators, namely Triangular cubic hesitant fuzzy Einstein weighted averaging operator, Triangular cubic hesitant fuzzy Einstein ordered weighted averaging operator and Triangular cubic hesitant fuzzy Einstein hybrid weighted averaging operator.

Establish MADM program approach based triangular cubic hesitant fuzzy numbers.

Provide illustrative examples of MADM program.

(4) In order to testify the application of the developed method, we apply the triangular cubic hesitant fuzzy numbers in the decision making.

(5)The initial decision matrix is composed of LVs. In order to fully consider the randomness and ambiguity of the linguistic term, we convert LVs into the triangular cubic hesitant fuzzy numbers, and the decision matrix is transformed into the triangular cubic hesitant fuzzy matrix.

(6)The operator can fully express the uncertainty of the qualitative concept and triangular cubic hesitant fuzzy operators can capture the interdependencies among any multiple inputs or attributes by a variable parameter. The aggregation operators can take into account the importance of the attribute weights. Nevertheless, sometimes, for some MAGDM problems, the weights of the attributes are important factors for the decision process.

(7) As we have discussed earlier that Cubic sets are the generalization of intuitionistic hesitant fuzzy sets and a powerful tool to deal with fuzziness. Also, triangular intuitionistic hesitant fuzzy numbers are suitable to deal with fuzziness. However, there may be a situation where the decision maker may provide the degree of membership and nonmembership of a particular attribute in such a way that membership degree is a triangular interval hesitant fuzzy number and non-membership degree is a triangular hesitant fuzzy number. Therefore, to overcome this shortcoming we generalize the concept of triangular intuitionistic hesitant fuzzy numbers and introduce the concept of triangular cubic hesitant fuzzy sets which are very suitable to be used for depicting uncertain or hesitant fuzzy information. If we take only one element in the membership degree of the triangular cubic hesitant fuzzy number, i.e., instead of interval we take a hesitant fuzzy number, then we get triangular intuitionistic hesitant fuzzy numbers, similarly if we take membership degree as hesitant fuzzy number and nonmemberíship degree equal to zero, than we get triangular hesitant fuzzy numbers.

Thus motivating by the idea proposed by Zhao et al. [16], in this paper we first proposed triangular cubic hesitant fuzzy number and including the Triangular cubic hesitant fuzzy Einstein weighted averaging (TCHFEWA) operator, triangular cubic hesitant fuzzy einstein ordered weighted averaging (TCHFEOWA) operator and triangular cubic hesitant fuzzy einstein hybrid weighted averaging (TCHFEHWA) operator.

This paper is organized as follows. In Section 2, we discuss some basic ideas to the fuzzy set and cubic set. In Section 3, we define the triangular cubic hesitant fuzzy numbers (TCHFNs) and operational laws. In Section 4, we present some Einstein operations on triangular cubic hesitant fuzzy numbers (TCHFNs) and analysis some desirable properties of the suggested operations. In Section 5, we first develop some novel arithmetic averaging operators, such as the triangular cubic hesitant fuzzy Einstein weighted averaging (TCHFEWA) operator, triangular cubic hesitant fuzzy Einstein ordered weighted averaging (TCHFEOWA) operator and triangular cubic hesitant fuzzy Einstein hybrid weighted averaging (TCHFEHWA) operator, for aggregating a collection of triangular cubic hesitant fuzzy numbers (TCHFNs). In Section 6, we relate the TCHFEHWA operator to MADM with triangular cubic hesitant fuzzy material. In Section 7, gives an example to illustrate the application of the developed method. In Section 8, we propose the comparison method. In Section 9, we consume a conclusion.

2. Preliminaries

Definition 1. *[2] Let H be a universe of discourse. Then the fuzzy set can be defined as:$J = \{\ddot{h}, \Gamma_J(\ddot{h}) | \ddot{h} \in H\}$. A fuzzy set in a set H is denoted by $\Gamma_J : H \to I$. The function $\Gamma_J(\ddot{h})$ denoted the degree of membership of the element $\ddot{h}$ to the set H, where $I = [0,1]$. The gathering of all fuzzy subsets of H is denoted by I^H. Define a relation on I^H as follows: $(\forall \Gamma, \eta \in I^H)(\Gamma \leq \eta \Leftrightarrow (\forall \ddot{h} \in \ddot{h})(\Gamma(\ddot{h}) \leq \eta(\ddot{h})))$.*

Definition 2. *[1] Let a set H be fixed, an AIFS A in H is defined as:*

$A = \{\langle \ddot{h}, \Gamma_A(\ddot{h}), \eta_A(\ddot{h})\rangle | \ddot{h} \in H\}$ where Γ_A and η_A are mapping from H to the closed interval $[0,1]$ such that $0 \leq \Gamma_A \leq 1, 0 \leq \eta_A \leq 1$ and $0 \leq \Gamma_A(\ddot{h}) + \eta_A(\ddot{h}) \leq 1$, for all $\ddot{h} \in H$, and they denote the degrees of membership and non-membership of element $\ddot{h} \in H$ to set A, respectively. Let $\pi_A(\ddot{h}) = 1 - \Gamma_A(\ddot{h}) - \eta_A(\ddot{h})$,then it is usually called the intuitionistic fuzzy index of element $\ddot{h} \in H$ to set A, representing the degree of indeterminacy or hesitation of $\ddot{h}$ to A. It is obvious that $0 \leq \pi_A(\ddot{h}) \leq 1$ for every $\ddot{h} \in H$.

Definition 3. *[18] Let H is a nonempty set. By a cubic set in H we mean a structure $F = \{h, \alpha(h), \ddot{\beta}(h) : h \in H\}$ in which α is an IVF set in H and $\ddot{\beta}$ is a fuzzy set in H. A cubic set $F = \{h, \alpha(h), \ddot{\beta}(h) : h \in H\}$ is simply denoted by $F = \langle \alpha, \ddot{\beta} \rangle$. Denote by $\ddot{c}^H$ the collection of all cubic sets in H. A cubic set $F = \langle \alpha, \ddot{\beta} \rangle$ in which $\alpha(h) = 0$ and $\ddot{\beta}(h) = 1$(resp.$\alpha(h) = 1$ And $\ddot{\beta}(h) = 0$ for all $h \in H$ is denoted by 0 (resp. 1). A cubic set $\ddot{D} = \langle \lambda, \xi \rangle$ in which $\lambda(h) = 0$ and $\xi(h) = 0$ (resp.$\lambda(h) = 1$ and $\xi(h) = 1$) for all $h \in H$ is denoted by 0 (resp. 1).*

Definition 4. *[18] Let H is a non-empty set. A cubic set $F = (\ddot{c}, \lambda)$ in H is said to be an internal cubic set if $\ddot{c}^-(h) \leq \lambda(h) \leq \ddot{c}^+(h)\ \forall h \in H$.*

Definition 5. *[18] Let H is a non-empty set. A cubic set $F = (\ddot{c}, \lambda)$ in H is said to be an external cubic set if $\lambda(h) \notin (\ddot{c}^-(h), \ddot{c}^+(h))\ \forall h \in H$.*

Definition 6. *[36] Triangular fuzzy number is defined as $A = \{a, b, c\}$, where a, b, c are real number and its membership function is given below*

$$\mu_A(x) = \begin{cases} 0 \quad \text{for } x < a \\ \{\frac{(x-a)}{(b-a)}\} \ a \le x \le b \\ 1 \quad \text{for} \quad x = b \\ \{\frac{(c-x)}{(c-b)}\} \ b < x \le c \\ 0 \quad \text{for } x > c \end{cases}$$

Definition 7. *[36] Let X be a fixed set, a triangular hesitant fuzzy set defined by $A = \{< x, Th_A(x) > \backslash x \in X\}$,where $Th_A(x)$ is a set of some triangular values in $\left(a^L, a^M, a^U\right)$.*

3. Triangular Cubic Hesitant Fuzzy Number and Operational Laws

Definition 8. *Let $\tilde{b}$ be the triangular cubic hesitant fuzzy number on the set of real numbers, its interval-valued triangular hesitant fuzzy set is defined as follows:*

$$\lambda_{\tilde{b}}(h) = \begin{cases} \{\frac{(h-p^-)}{(q^- - p^-)}, \frac{(h-p^+)}{(q^+ - p^+)}\} \ (p^-, q^-) \le h < (p^+, q^+) \\ \{\frac{(r^- - h)}{(r^- - q^-)}, \frac{(r^+ - h)}{(r^+ - q^+)}\} \ (q^-, r^-) \le h < (q^+, r^+) \\ 0 \qquad \text{otherwise} \end{cases}$$

and its triangular hesitant fuzzy set is
$\Gamma_{\tilde{b}}(h) = \begin{cases} \{\frac{(q-h)}{(q-p)}\} \ p \le h < q \\ \{\frac{(r-h)}{(r-q)}\} \ q < h \le r \\ 0 \quad \text{otherwise} \end{cases}$ *where $0 \le \lambda_{\tilde{b}}(h) \le 1, 0 \le \Gamma_{\tilde{b}}(h) \le 1$. Then the TCHFN $\tilde{b}$ is given by $\tilde{b} = \left\langle \begin{array}{c} [p^-, q^-, r^-], \\ [p^+, q^+, r^+], \\ [p, q, r] \end{array} \right\rangle$ and $\tilde{b}$ is called triangular cubic hesitant fuzzy number.*

Definition 9. *Let $\ddot{A}_1 = \left\{ \begin{array}{c} \langle [p_1^-(\ddot{z}), \\ q_1^-(\ddot{z}), r_1^-(\ddot{z})], \\ [p_1^+(\ddot{z}), \\ q_1^+(\ddot{z}), r_1^+(\ddot{z})], \\ [p_1(\ddot{z}), \\ q_1(\ddot{z}), r_1(\ddot{z})] \rangle \\ |\ddot{z} \in Z \end{array} \right\}$ and $\ddot{A}_2 = \left\{ \begin{array}{c} \langle [p_2^-(\ddot{z}), \\ q_2^-(\ddot{z}), r_2^-(\ddot{z})], \\ [p_2^+(\ddot{z}), \\ q_2^+(\ddot{z}), r_2^+(\ddot{z})], \\ [p_2(\ddot{z}), \\ q_2(\ddot{z}), r_2(\ddot{z})] \rangle \\ |\ddot{z} \in Z \end{array} \right\}$ are two triangular cubic hesitant fuzzy numbers (TCHFNs), some operations on triangular cubic hesitant fuzzy numbers (TCHFSs) are defined as follows:*

$$\left\{ \begin{array}{c} \ddot{A}_1 \subseteq \ddot{A}_2 iff \forall \ddot{z} \in Z, p_1^-(\ddot{z}) \ge p_2^-(\ddot{z}), q_1^-(\ddot{z}) \ge q_2^-(\ddot{z}), \\ r_1^-(\ddot{z}) \ge r_1^-(\ddot{z}), p_1^+(\ddot{z}) \ge p_2^+(\ddot{z}), q_1^+(\ddot{z}) \ge q_2^+(\ddot{z}), \\ r_1^+(\ddot{z}) \ge r_1^+(\ddot{z}) \text{ and } p_1(\ddot{z}) \le p_2(\ddot{z}), q_1(\ddot{z}) \le q_2(\ddot{z}), \\ r_1(\ddot{z}) \le r_2(\ddot{z}) \end{array} \right\} \tag{1}$$

$$\ddot{A}_1 \cap_{T,S} \ddot{A}_2 = \left\langle \begin{array}{c} T[p_1^-(\ddot{z}), p_2^-(\ddot{z})], T[q_1^-(\ddot{z}), q_2^-(\ddot{z})], \\ T[r_1^-(\ddot{z}), r_2^-(\ddot{z})], T[p_1^+(\ddot{z}), p_2^+(\ddot{z})], \\ T[q_1^+(\ddot{z}), q_2^+(\ddot{z})], T[r_1^+(\ddot{z}), r_2^+(\ddot{z})], \\ S[p_1(\ddot{z}), p_2(\ddot{z})], S[q_1(\ddot{z}), q_2(\ddot{z})], \\ S[r_1(\ddot{z}), r_2(\ddot{z})] \end{array} \right\rangle, \tag{2}$$

$$\ddot{A}_1 \cup_{T,S} \ddot{A}_2 = \left\langle \begin{array}{c} S[p_1^-(\ddot{z}), p_2^-(\ddot{z})], S[q_1^-(\ddot{z}), q_2^-(\ddot{z})], \\ S[r_1^-(\ddot{z}), r_2^-(\ddot{z})], S[p_1^+(\ddot{z}), p_2^+(\ddot{z})], \\ S[q_1^+(\ddot{z}), q_2^+(\ddot{z})], S[r_1^+(\ddot{z}), r_2^+(\ddot{z})], \\ T[p_1(\ddot{z}), p_2(\ddot{z})], T[q_1(\ddot{z}), q_2(\ddot{z})], \\ T[r_1(\ddot{z}), r_2(\ddot{z})] \end{array} \right\rangle. \tag{3}$$

Example 1. *Let* $\ddot{A}_1 = \left\langle \begin{array}{c} [0.4, 0.6, 0.8], \\ [0.6, 0.8, 0.10], \\ [0.5, 0.7, 0.9] \end{array} \right\rangle$ *and* $\ddot{A}_2 = \left\langle \begin{array}{c} [0.1, 0.3, 0.5], \\ [0.3, 0.5, 0.7], \\ [0.2, 0.4, 0.6] \end{array} \right\rangle$ *be two TCHFNs*

(a) $\ddot{A}_1 \subseteq \ddot{A}_2$ *iff* $\forall \ddot{z} \in Z, 0.4 \geq 0.1, 0.6 \geq 0.3, 0.8 \geq 0.5, 0.6 \geq 0.3, 0.8 \geq 0.5,$ $0.10 \geq 0.7$ *and* $0.5 \leq 0.2, 0.7 \leq 0.4, 0.9 \leq 0.6,$

(b) $\ddot{A}_1 \cap_{T,S} \ddot{A}_2 = \left\langle \begin{array}{c} T[0.4, 0.1], T[0.6, 0.3], T[0.8, 0.5], T[0.6, 0.3], T[0.8, 0.5], \\ T[0.10, 0.7] \text{ and } S[0.5, 0.2], S[0.7, 0.4], S[0.9, 0.6] \end{array} \right\rangle,$

(c) $\ddot{A}_1 \cap_{T,S} \ddot{A}_2 = \left\langle \begin{array}{c} S[0.4, 0.1], S[0.6, 0.3], S[0.8, 0.5], S[0.6, 0.3], S[0.8, 0.5], \\ S[0.10, 0.7] \text{ and } T[0.5, 0.2], T[0.7, 0.4], T[0.9, 0.6] \end{array} \right\rangle.$

where T denotes a t-norm and S a so-called t-conorm dual to the t-norm T, defined by $S(\ddot{z}, \tau) = 1 - T(1 - \ddot{z}, 1 - \tau)$.

Definition 10. *Let* $\ddot{A} = \left\langle \begin{array}{c} [a^-_{\ddot{A}}(\ddot{z}), b^-_{\ddot{A}}(\ddot{z}), c^-_{\ddot{A}}(\ddot{z})], \\ [a^+_{\ddot{A}}(\ddot{z}), b^+_{\ddot{A}}(\ddot{z}), c^+_{\ddot{A}}(\ddot{z})], \\ [a_{\ddot{A}}(\ddot{z}), b_{\ddot{A}}(\ddot{z}), c_{\ddot{A}}(\ddot{z})], \\ |\ddot{z} \in Z \end{array} \right\rangle$ *be an TCHFN and then the score function* $S(\ddot{A})$, *accuracy function* $H(\ddot{A})$, *membership uncertainty index* $T(\ddot{A})$ *and hesitation uncertainty index* $G(\ddot{A})$ *of an TCHFN* $\ddot{A}$ *are defined by*

$$S(\ddot{A}) = \bigcup_{\substack{p^- \in a^-, q^- \in b^-, r^- \in c^-, \\ p^+ \in a^+, q^+ \in b^+, r^+ \in c^+, \\ p \in a, q \in b, r \in c}} \left\langle \frac{\begin{array}{c} [[p^-_{\ddot{A}}(\ddot{z}) + q^-_{\ddot{A}}(\ddot{z}) + r^-_{\ddot{A}}(\ddot{z})] \\ +[p^+_{\ddot{A}}(\ddot{z}) + q^+_{\ddot{A}}(\ddot{z}) + r^+_{\ddot{A}}(\ddot{z})]] \\ -[p_{\ddot{A}}(\ddot{z}) + q_{\ddot{A}}(\ddot{z}) + r_{\ddot{A}}(\ddot{z})] \end{array}}{9} \right\rangle, \tag{4}$$

$$H(\ddot{A}) = \bigcup_{\substack{p^- \in a^-, q^- \in b^-, r^- \in c^-, \\ p^+ \in a^+, q^+ \in b^+, r^+ \in c^+, \\ p \in a, q \in b, r \in c}} \left\langle \frac{\begin{array}{c} [[p^-_{\ddot{A}}(\ddot{z}) + q^-_{\ddot{A}}(\ddot{z}) + r^-_{\ddot{A}}(\ddot{z})] \\ +[p^+_{\ddot{A}}(\ddot{z}) + q^+_{\ddot{A}}(\ddot{z}) + r^+_{\ddot{A}}(\ddot{z})]] \\ +[p_{\ddot{A}}(\ddot{z}) + q_{\ddot{A}}(\ddot{z}) + r_{\ddot{A}}(\ddot{z})] \end{array}}{9} \right\rangle, \tag{5}$$

$$T(\ddot{A}) = \bigcup_{\substack{p^- \in a^-, q^- \in b^-, r^- \in c^-, \\ p^+ \in a^+, q^+ \in b^+, r^+ \in c^+, \\ p \in a, q \in b, r \in c}} \left\langle \begin{array}{c} [p^+_{\ddot{A}}(\ddot{z}) + q^+_{\ddot{A}}(\ddot{z}) + r^+_{\ddot{A}}(\ddot{z})] + \\ [p_{\ddot{A}}(\ddot{z}) + q_{\ddot{A}}(\ddot{z}) + r_{\ddot{A}}(\ddot{z})] - \\ [p^-_{\ddot{A}}(\ddot{z}) + q^-_{\ddot{A}}(\ddot{z}) + r^-_{\ddot{A}}(\ddot{z})] \end{array} \right\rangle, \tag{6}$$

$$G(\ddot{A}) = \bigcup_{\substack{p^- \in a^-, q^- \in b^-, r^- \in c^-, \\ p^+ \in a^+, q^+ \in b^+, r^+ \in c^+, \\ p \in a, q \in b, r \in c}} \left\langle \begin{array}{c} [p^+_{\ddot{A}}(\ddot{z}) + q^+_{\ddot{A}}(\ddot{z}) + r^+_{\ddot{A}}(\ddot{z})] + \\ [p^-_{\ddot{A}}(\ddot{z}) + q^-_{\ddot{A}}(\ddot{z}) + r^-_{\ddot{A}}(\ddot{z})] - \\ [p_{\ddot{A}}(\ddot{z}) + q_{\ddot{A}}(\ddot{z}) + r_{\ddot{A}}(\ddot{z})] \end{array} \right\rangle. \tag{7}$$

Example 2. *Let* $\ddot{A} = \left\langle \begin{array}{c} [0.9, 0.11, 0.13], \\ [0.11, 0.13, 0.15], \\ [0.10, 0.12, 0.14] \end{array} \right\rangle$ *be a TCHFN. Then the score function* $S(\ddot{A})$*, accuracy function* $H(\ddot{A})$*, membership uncertainty index* $T(\ddot{A})$ *and hesitation uncertainty index* $G(\ddot{A})$ *of the TCHFN* $\ddot{A}$ *are defined by*

$$S(\ddot{A}) = \left\langle \frac{\begin{array}{c} [0.9 + 0.11 + 0.13]+ \\ [0.11 + 0.13 + 0.15] \\ -[0.10 - 0.12 - 0.14] \end{array}}{9} \right\rangle = 0.1877,$$

$$H(\ddot{A}) = \left\langle \frac{\begin{array}{c} [0.9 + 0.11 + 0.13] \\ +[0.11 + 0.13 + 0.15] \\ +[0.10 + 0.12 + 0.14] \end{array}}{9} \right\rangle = 0.21,$$

$$T(\ddot{A}) = \left\langle \begin{array}{c} [0.11 + 0.13 + 0.15]+ \\ [0.10 + 0.12 + 0.14]- \\ [0.9 - 0.11 - 0.13] \end{array} \right\rangle = 0.09,$$

$$G(\ddot{A}) = \left\langle \begin{array}{c} [0.11 + 0.13 + 0.15]+ \\ [0.9 + 0.11 + 0.13]- \\ [0.10 - 0.12 - 0.14] \end{array} \right\rangle = 1.69.$$

Figure 1 represents the values of the the score function, accuracy function , membership uncertainty index and hesitation uncertainty index of the TCHFN $\ddot{A}$.

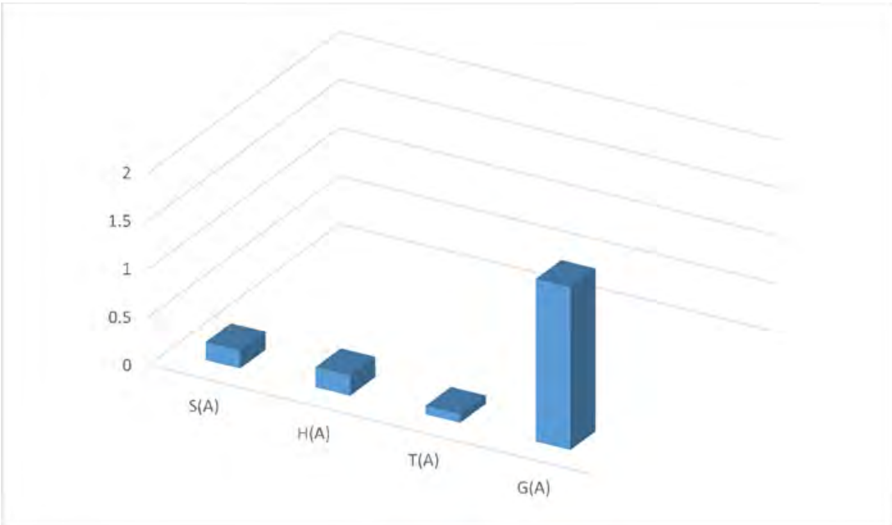

Figure 1. Graph of functions of example 2.

4. Some Einstein Operations on Triangular Cubic Hesitant Fuzzy Numbers

In this section, we introduce the Einstein t-norm T $(T(\ddot{z}, \tau) = \frac{\ddot{z}\tau}{1+(1-\ddot{z})(1-\tau)})$ and its dual t-conorm S

$$(S(\ddot{z}, \tau) = \frac{\ddot{z} + \tau}{1 + \ddot{z}\tau})$$

then the generalized union $\ddot{D}$) on TCHFNs $\ddot{A}, \ddot{A}_1$ and $\ddot{A}_2$ become to the Einstein sum (denoted by $\ddot{A}_1 + \ddot{A}_2$) on $\ddot{A}_1$ and $\ddot{A}_2$ as follows:

$$\ddot{A}_1+\ddot{A}_2=\left\langle \begin{array}{c} \bigcup\limits_{\substack{p_1^-\in a_1^-,p_2^-\in a_2^-,q_1^-\in b_1^-,\\ q_2^-\in b_2^-,r_1^-\in c_1^-,r_2^-\in c_2^-,\\ p_1^+\in a_1^+,p_2^+\in a_2^+,q_1^+\in b_1^+,\\ q_2^+\in b_2^+,r_1^+\in c_1^+,r_2^+\in c_2^+,}} \begin{bmatrix} \frac{p_1^-(\ddot{z})+p_2^-(\ddot{z})}{1+p_1^-(\ddot{z})p_2^-(\ddot{z}))}, \frac{p_1^+(\ddot{z})+p_2^+(\ddot{z})}{1+p_1^+(\ddot{z}).p_2^+(\ddot{z}))}, \\ \frac{q_1^-(\ddot{z})+q_2^-(\ddot{z})}{1+q_1^-(\ddot{z}).q_2^-(\ddot{z}))}, \frac{q_1^+(\ddot{z})+q_2^+(\ddot{z})}{1+q_1^+(\ddot{z}).q_2^+(\ddot{z}))}, \\ \frac{r_1^-(\ddot{z})+r_2^-(\ddot{z})}{1+r_1^-(\ddot{z}).r_2^-(\ddot{z}))}, \frac{r_1^+(\ddot{z})+r_2^+(\ddot{z})}{1+r_1^+(\ddot{z}).r_2^+(\ddot{z}))} \end{bmatrix}, \\ \bigcup\limits_{\substack{p_1\in a_1,p_2\in a_2,q_1\in b_1,\\ q_2\in b_2,r_1\in c_1,r_2\in c_2}} \begin{bmatrix} \frac{p_1 \cdot p_2}{(1+(1-p_1(\ddot{z}))(1-p_2(\ddot{z})))}, \\ \frac{q_1 \cdot q_2}{(1+(1-q_1(\ddot{z}))(1-q_2(\ddot{z})))}, \\ \frac{r_1 \cdot r_2}{(1+(1-r_1(\ddot{z}))(1-r_2(\ddot{z})))} \end{bmatrix} \end{array} \right\rangle$$

By mathematical induction we can derive the multiplication operation on TCHFNs as follows:

$$\lambda\ddot{A}=\left\langle \begin{array}{c} \bigcup\limits_{\substack{p^-\in a^-,q^-\in b^-,r^-\in c^-,\\ p^+\in a^+,q^+\in b^+,r^+\in c^+}} \begin{bmatrix} \frac{[1+p_{\ddot{A}}^-(\ddot{z})]^\lambda-[1-p_{\ddot{A}}^-(\ddot{z})]^\lambda}{[1+p_{\ddot{A}}^-(\ddot{z})]^\lambda+[1-p_{\ddot{A}}^-(\ddot{z})]^\lambda}, \frac{[1+p_{\ddot{A}}^+(\ddot{z})]^\lambda-[1-p_{\ddot{A}}^+(\ddot{z})]^\lambda}{[1+p_{\ddot{A}}^+(\ddot{z})]^\lambda+[1-p_{\ddot{A}}^+(\ddot{z})]^\lambda}, \\ \frac{[1+q_{\ddot{A}}^-(\ddot{z})]^\lambda-[1-q_{\ddot{A}}^-(\ddot{z})]^\lambda}{[1+q_{\ddot{A}}^-(\ddot{z})]^\lambda+[1-q_{\ddot{A}}^-(\ddot{z})]^\lambda}, \frac{[1+q_{\ddot{A}}^+(\ddot{z})]^\lambda-[1-q_{\ddot{A}}^+(\ddot{z})]^\lambda}{[1+q_{\ddot{A}}^+(\ddot{z})]^\lambda+[1-q_{\ddot{A}}^+(\ddot{z})]^\lambda}, \\ \frac{[1+r_{\ddot{A}}^-(\ddot{z})]^\lambda-[1-r_{\ddot{A}}^-(\ddot{z})]^\lambda}{[1+r_{\ddot{A}}^-(\ddot{z})]^\lambda+[1-r_{\ddot{A}}^-(\ddot{z})]^\lambda}, \frac{[1+r_{\ddot{A}}^+(\ddot{z})]^\lambda-[1-r_{\ddot{A}}^+(\ddot{z})]^\lambda}{[1+r_{\ddot{A}}^+(\ddot{z})]^\lambda+[1-r_{\ddot{A}}^+(\ddot{z})]^\lambda} \end{bmatrix}, \\ \bigcup\limits_{p\in a,q\in b,r\in c} \begin{bmatrix} \frac{2[p_{\ddot{A}}(\ddot{z})]^\lambda}{[(2-p_{\ddot{A}}(\ddot{z})]^\lambda+[p_{\ddot{A}}(\ddot{z})]^\lambda}, \frac{2[q_{\ddot{A}}(\ddot{z})]^\lambda}{[(2-q_{\ddot{A}}(\ddot{z})]^\lambda+[q_{\ddot{A}}(\ddot{z})]^\lambda}, \\ \frac{2[r_{\ddot{A}}(\ddot{z})]^\lambda}{[(2-r_{\ddot{A}}(\ddot{z})]^\lambda+[r_{\ddot{A}}(\ddot{z})]^\lambda} \end{bmatrix} \end{array} \right\rangle.$$

Definition 11. *Let* $\ddot{A}=\left\langle \begin{array}{c} [a^-(\ddot{z}),b^-(\ddot{z}),c^-(\ddot{z})], \\ [a^+(\ddot{z}),b^+(\ddot{z}),c^+(\ddot{z})], \\ [a(\ddot{z}),b(\ddot{z}),c(\ddot{z})] \\ |\ddot{z}\in Z \end{array} \right\rangle$, $\ddot{A}_1=\left\langle \begin{array}{c} [a_1^-(\ddot{z}),b_1^-(\ddot{z}),c_1^-(\ddot{z})], \\ [a_1^+(\ddot{z}),b_1^+(\ddot{z}),c_1^+(\ddot{z})], \\ [a_1(\ddot{z}),b_1(\ddot{z}),c_1(\ddot{z})] \\ |\ddot{z}\in Z \end{array} \right\rangle$ *and* $\ddot{A}_2=\left\langle \begin{array}{c} [a_2^-(\ddot{z}),b_2^-(\ddot{z}),c_2^-(\ddot{z})], \\ [a_2^+(\ddot{z}),b_2^+(\ddot{z}),c_2^+(\ddot{z})], \\ [a_2(\ddot{z}),b_2^+(\ddot{z}),c_2^+(\ddot{z})], \\ |\ddot{z}\in Z \end{array} \right\rangle$ *be any three TCHFNs. Then some Einstein operations of* $\ddot{A}_1$ *and* $\ddot{A}_2$ *can be defined as:*

$$\ddot{A}_1+\ddot{A}_2=\left\langle \begin{array}{c} \bigcup\limits_{\substack{p_1^-\in a_1^-,p_2^-\in a_2^-,q_1^-\in b_1^-,q_2^-\in b_2^-,\\ r_1^-\in c_1^-,r_2^-\in c_2^-,p_1^+\in a_1^+,p_2^+\in a_2^+,\\ q_1^+\in b_1^+,q_2^+\in b_2^+,r_1^+\in c_1^+,r_2^+\in c_2^+,}} \begin{bmatrix} \frac{p_1^-(\ddot{z})+p_2^-(\ddot{z})}{1+p_1^-(\ddot{z}).p_2^-(\ddot{z}))}, \frac{p_1^+(\ddot{z})+p_2^+(\ddot{z})}{1+p_1^+(\ddot{z}).p_2^+(\ddot{z}))}, \\ \frac{q_1^-(\ddot{z})+q_2^-(\ddot{z})}{1+q_1^-(\ddot{z}).q_2^-(\ddot{z}))}, \frac{q_1^+(\ddot{z})+q_2^+(\ddot{z})}{1+q_1^+(\ddot{z}).q_2^+(\ddot{z}))}, \\ \frac{r_1^-(\ddot{z})+r_2^-(\ddot{z})}{1+r_1^-(\ddot{z}).r_2^-(\ddot{z}))}, \frac{r_1^+(\ddot{z})+r_2^+(\ddot{z})}{1+r_1^+(\ddot{z}).r_2^+(\ddot{z}))} \end{bmatrix}, \\ \bigcup\limits_{\substack{p_1\in a_1,p_2\in a_2,q_1\in b_1,\\ q_2\in b_2,r_1\in c_1,r_2\in c_2}} \begin{bmatrix} \frac{p_1 \cdot p_2}{(1+(1-p_1(\ddot{z}))(1-p_2(\ddot{z})))}, \\ \frac{q_1 \cdot q_2}{(1+(1-q_1(\ddot{z}))(1-q_2(\ddot{z})))}, \\ \frac{r_1 \cdot r_2}{(1+((1-r_1(\ddot{z}))(1-r_2(\ddot{z})))} \end{bmatrix} \end{array} \right\rangle, \tag{8}$$

$$\lambda\ddot{A} = \left\langle \begin{array}{c} \bigcup\limits_{\substack{p^- \in a^-, q^- \in b^-, r^- \in c^-, \\ p^+ \in a^+, q^+ \in b^+, r^+ \in c^+}} \left[\begin{array}{c} \frac{[1+p^-_{\ddot{A}}(\dddot{z})]^\lambda - [1-p^-_{\ddot{A}}(\dddot{z})]^\lambda}{[1+p^-_{\ddot{A}}(\dddot{z})]^\lambda + [1-p^-_{\ddot{A}}(\dddot{z})]^\lambda}, \frac{[1+p^+_{\ddot{A}}(\dddot{z})]^\lambda - [1-p^+_{\ddot{A}}(\dddot{z})]^\lambda}{[1+p^+_{\ddot{A}}(\dddot{z})]^\lambda + [1-p^+_{\ddot{A}}(\dddot{z})]^\lambda}, \\ \frac{[1+q^-_{\ddot{A}}(\dddot{z})]^\lambda - [1-q^-_{\ddot{A}}(\dddot{z})]^\lambda}{[1+q^-_{\ddot{A}}(\dddot{z})]^\lambda + [1-q^-_{\ddot{A}}(\dddot{z})]^\lambda}, \frac{[1+q^+_{\ddot{A}}(\dddot{z})]^\lambda - [1-q^+_{\ddot{A}}(\dddot{z})]^\lambda}{[1+q^+_{\ddot{A}}(\dddot{z})]^\lambda + [1-q^+_{\ddot{A}}(\dddot{z})]^\lambda}, \\ \frac{[1+r^-_{\ddot{A}}(\dddot{z})]^\lambda - [1-r^-_{\ddot{A}}(\dddot{z})]^\lambda}{[1+r^-_{\ddot{A}}(\dddot{z})]^\lambda + [1-r^-_{\ddot{A}}(\dddot{z})]^\lambda}, \frac{[1+r^+_{\ddot{A}}(\dddot{z})]^\lambda - [1-r^+_{\ddot{A}}(\dddot{z})]^\lambda}{[1+r^+_{\ddot{A}}(\dddot{z})]^\lambda + [1-r^+_{\ddot{A}}(\dddot{z})]^\lambda} \end{array} \right], \\ \bigcup\limits_{p \in a, q \in b, r \in c} \left[\begin{array}{c} \frac{2[p_{\ddot{A}}(\dddot{z})]^\lambda}{[(2-p_{\ddot{A}}(\dddot{z})]^\lambda + [p_{\ddot{A}}(\dddot{z})]^\lambda}, \frac{2[q_{\ddot{A}}(\dddot{z})]^\lambda}{[(2-q_{\ddot{A}}(\dddot{z})]^\lambda + [q_{\ddot{A}}(\dddot{z})]^\lambda}, \\ \frac{2[r_{\ddot{A}}(\dddot{z})]^\lambda}{[(2-r_{\ddot{A}}(\dddot{z})]^\lambda + [r_{\ddot{A}}(\dddot{z})]^\lambda} \end{array} \right] \end{array} \right\rangle. \tag{9}$$

Example 3. *Let* $\ddot{A} = \left\langle \begin{array}{c} [0.16, 0.18, 0.20], \\ [0.18, 0.20, 0.22], \\ [0.17, 0.19, 0.21] \end{array} \right\rangle$, $\ddot{A}_1 = \left\langle \begin{array}{c} [0.2, 0.4, 0.6], \\ [0.4, 0.6, 0.8], \\ [0.3, 0.5, 0.7] \end{array} \right\rangle$ *and* $\ddot{A}_2 = \left\langle \begin{array}{c} [0.13, 0.15, 0.17], \\ [0.15, 0.17, 0.19], \\ [0.14, 0.16, 0.18] \end{array} \right\rangle$ *be any three TCHFNs. Then some Einstein operations of* $\ddot{A}_1$ *and* $\ddot{A}_2$ *can be defined as follows:*

$$\ddot{A}_1 + \ddot{A}_2 = \left\langle \begin{array}{c} \left[\begin{array}{c} \frac{0.2+0.13}{(1+(0.2).(0.13))}, \\ \frac{0.4+0.15}{(1+(0.4).(0.15))} \end{array}, \begin{array}{c} \frac{0.4+0.15}{(1+(0.4).(0.15))}, \\ \frac{0.6+0.17}{(1+(0.6).(0.17))} \end{array}, \begin{array}{c} \frac{0.6+0.17}{(1+(0.6).(0.17))}, \\ \frac{0.8+0.19}{(1+(0.8).(0.19))} \end{array} \right], \\ \left[\frac{0.3.0.14}{(1+((1-0.3)(1-0.14))}, \frac{0.5.0.16}{1+((1-0.5)(1-0.16))}, \frac{0.7.0.18}{(1+((1-0.7)(1-0.18))} \right] \end{array} \right\rangle$$

$$= \left\langle \begin{array}{c} [0.3216, 0.5188, 0.6987], [0.5188, 0.6987, 0.8593], \\ [0.0262, 0.0563, 0.1011] \end{array} \right\rangle,$$

$$\lambda\ddot{A} = \left\langle \begin{array}{c} \left[\begin{array}{c} \frac{[1+0.16]^{0.25}-[1-0.16]^{0.25}}{[1+0.16]^{0.25}+[1-0.16]^{0.25}}, \frac{[1+0.18]^{0.50}-[1-0.18]^{0.50}}{[1+0.18]^{0.50}+[1-0.18]^{0.50}}, \frac{[1+0.20]^{0.25}-[1-0.20]^{0.25}}{[1+0.20]^{0.25}+[1-0.20]^{0.25}}, \\ \frac{[1+0.18]^{0.25}-[1-0.18]^{0.25}}{[1+0.18]^{0.25}+[1-0.18]^{0.25}}, \frac{[1+0.20]^{0.50}-[1-0.20]^{0.50}}{[1+0.20]^{0.50}+[1-0.20]^{0.50}}, \frac{[1+0.22]^{0.25}-[1-0.22]^{0.25}}{[1+0.22]^{0.25}+[1-0.22]^{0.25}} \end{array} \right] \\ , \frac{2[0.17]^{0.25}}{[(2-0.17]^{0.25}+[0.17]^{0.25}}, \frac{2[0.19]^{0.50}}{[(2-0.19]^{0.50}+[0.19]^{0.50}}, \frac{2[0.21]^{0.25}}{[(2-0.21]^{0.25}+[0.21]^{0.25}} \end{array} \right\rangle$$

$$= \langle [0.0402, 0.0907, 0.0505], [0.0907, 0.0505, 0.0558], [0.6776, 0.4893, 0.7383] \rangle.$$

Proposition 1. *Lat* $\ddot{A}$, $\ddot{A}_1$ *and* $\ddot{A}_2$ *be three TCHFNs,* $\lambda, \lambda_1, \lambda_2 > 0$, *then we have:*

(1) $\ddot{A}_1 + \ddot{A}_2 = \ddot{A}_2 + \ddot{A}_1$,
(2) $\lambda(\ddot{A}_1 + \ddot{A}_2) = \lambda\ddot{A}_2 + \lambda\ddot{A}_1$,
(3) $\lambda_1\ddot{A} + \lambda_2\ddot{A} = (\lambda_1 + \lambda_2)\ddot{A}$.

Proof. (1) $A_1 + A_2 = A_2 + A_1$;

$$A_1 + A_2 = \left\{ \begin{array}{c} \langle \bigcup\limits_{\substack{p_1^- \in a_1^-, p_2^- \in a_2^-, q_1^- \in b_1^- \\ , q_2^- \in b_2^-, r_1^- \in c_1^-, r_2^- \in c_2^-}} [\frac{p_1^-(h)+p_2^-(h)}{(1+p_1^-(h))(1-p_1^-(h))}, \frac{q_1^-(h)+q_2^-(h)}{(1+q_1^-(h))(1-q_1^-(h))}, \\ \frac{r_1^-(h)+r_2^-(h)}{(1+r_1^-(h))(1-r_1^-(h))}], \bigcup\limits_{\substack{p_1^+ \in a_1^+, p_2^+ \in a_2^+, q_1^+ \in b_1^+, \\ q_2^+ \in b_2^+, r_1^+ \in c_1^+, r_2^+ \in c_2^+,}} [\frac{p_1^+(h)+p_2^+(h)}{(1+p_1^+(h))(1-p_2^+(h))}, \\ \frac{q_1^+(h)+q_2^+(h)}{(1+q_1^+(h))(1-q_2^+(h))}, \frac{r_1^+(h)+r_1^+(h)}{(1+r_1^+(h))(1-r_1^+(h))}], \bigcup\limits_{\substack{p_1 \in a_1, p_2 \in a_2, q_1 \in b_1, \\ q_2 \in b_2, r_1 \in c_1, r_2 \in c_2}} \\ [\frac{p_1(h).p_2(h)}{1+((1-p_1(h))(1-p_2(h)))}, \frac{q_1(h).q_2(h)}{1+((1-q_1(h))(1-q_2(h)))}, \\ \frac{r_1(h).r_2(h)}{1+((1-r_1(h))(1-r_2(h)))}] \rangle \end{array} \right\} =$$

$$\left\{\begin{array}{l} \langle \bigcup\limits_{\substack{p_2^-\in a_2^-,p_1^-\in a_1^-,q_2^-\in b_2^-,\\ q_1^-\in b_1^-,r_2^-\in c_2^-,r_1^-\in c_1^-,}} [\frac{p_2^-(h)+p_1^-(h)}{(1+p_2^-(h))(1-p_1^-(h))}, \frac{q_2^-(h)+q_1^-(h)}{(1+q_2^-(h))(1-q_1^-(h))}, \\ \frac{r_2^-(h)+r_1^-(h)}{(1+r_2^-(h))(1-r_1^-(h))}], \bigcup\limits_{\substack{p_2^+\in a_2^+,p_1^+\in a_1^+,q_2^+\in b_2^+,\\ q_1^+\in b_1^+,r_2^+\in c_2^+,r_1^+\in c_1^+,}} [\frac{p_2^+(h)+p_1^+(h)}{(1+p_2^+(h))(1-p_1^+(h))}, \\ \frac{q_2^+(h)+q_1^+(h)}{(1+q_2^+(h))(1-q_1^+(h))}, \frac{r_2^+(h)+r_1^+(h)}{(1+r_2^+(h))(1-r_1^+(h))}], \\ \bigcup\limits_{\substack{p_2\in a_2,p_1\in a_1,q_2\in b_2,\\ q_1\in b_1,r_2\in c_2,r_1\in c_1}} [\frac{p_2(h).p_1(h)}{1+((1-p_2(h))(1-p_1(h)))}, \frac{q_2(h).q_1(h)}{1+((1-q_2(h))(1-q_1(h)))}, \\ \frac{r_2(h).r_1(h)}{1+((1-r_2(h))(1-r_1(h)))}]\rangle \end{array}\right\}$$

$= A_2 + A_1$

Hence $A_1 + A_2 = A_2 + A_1$.

(2) $\lambda(A_1 + A_2) = \lambda A_2 + \lambda A_1$

$$\lambda(A_1+A_2) = \left\{\begin{array}{l} \langle \bigcup\limits_{\substack{p_2^-\in a_2^-,p_1^-\in a_1^-,q_2^-\in b_2^-,\\ q_1^-\in b_1^-,r_2^-\in c_2^-,r_1^-\in c_1^-}} [\frac{[(1+p_1^-(h))(1-p_1^-(h))]^\lambda[(1+p_2^-(h))(1-p_2^-(h))]^\lambda}{[(1+p_1^-(h))(1-p_1^-(h))]^\lambda[(1+p_2^-(h))(1-p_2^-(h))]^\lambda}, \\ \frac{[(1+q_1^-(h))(1-q_1^-(h))]^\lambda[(1+q_2^-(h))(1-q_2^-(h))]^\lambda}{[(1+q_1^-(h))(1-q_1^-(h))]^\lambda[(1+q_2^-(h))(1-q_2^-(h))]^\lambda}, \\ \frac{[(1+r_1^-(h))(1-r_1^-(h))]^\lambda[(1+r_2^-(h))(1-r_2^-(h))]^\lambda}{[(1+r_1^-(h))(1-r_1^-(h))]^\lambda[(1+r_2^-(h))(1-r_2^-(h))]^\lambda}], \\ \bigcup\limits_{\substack{p_2^+\in a_2^+,p_1^+\in a_1^+,q_2^+\in b_2^+,\\ q_1^+\in b_1^+,r_2^+\in c_2^+,r_1^+\in c_1^+}} [\frac{[(1+p_1^+(h))(1-p_1^+(h))]^\lambda[(1+u_{p2}^+(h))(1-p_2^+(h))]^\lambda}{[(1+p_1^+(h))(1-p_1^+(h))]^\lambda[(1+p_2^+(h))(1-p_2^+(h))]^\lambda}, \\ \frac{[(1+q_1^+(h))(1-q_1^+(h))]^\lambda[(1+q_2^+(h))(1-q_2^+(h))]^\lambda}{[(1+q_1^+(h))(1-q_1^+(h))]^\lambda[(1+q_2^+(h))(1-q_2^+(h))]^\lambda}, \\ \frac{[(1+r_1^+(h))(1-r_1^+(h))]^\lambda[(1+r_2^+(h))(1-r_2^+(h))]^\lambda}{[(1+r_1^+(h))(1-r_1^+(h))]^\lambda[(1+r_2^+(h))(1-r_2^+(h))]^\lambda}], \\ \bigcup\limits_{\substack{p_2\in a_2,p_1\in a_1,q_2\in b_2,\\ q_1\in b_1,r_2\in c_2,r_1\in c_1}} [\frac{2[p_1(h)p_2(h)]^\lambda}{[(4-2p_1(h)-2p_2(h)-p_1(h)p_2(h)]^\lambda+[p_1(h)p_2(h)]^\lambda}, \\ \frac{2[q_1(h)q_2(h)]^\lambda}{[(4-2q_1(h)-2q_2(h)-q_1(h)q_2(h)]^\lambda+[q_1(h)q_2(h)]^\lambda}, \\ \frac{2[r_1(h)r_2(h)]^\lambda}{[(4-2r_1(h)-2r_2(h)-r_1(h)r_2(h)]^\lambda+[r_1(h)r_2(h)]^\lambda}]\rangle \end{array}\right\}$$

$$\text{and we have } \lambda A_1 = \left\{\begin{array}{l} \langle \bigcup\limits_{p_1^-\in a_1^-,q_1^-\in b_1^-,r_1^-\in c_1^-} [\frac{[(1+p_1^-(h))^\lambda-(1-p_1^-(h))^\lambda]}{[(1+p_1^-(h))^\lambda+(1-p_1^-(h))^\lambda]}, \\ \frac{[(1+q_1^-(h))^\lambda-(1-q_1^-(h))^\lambda]}{[(1+q_1^-(h))^\lambda+(1-q_1^-(h))^\lambda]}, \frac{[(1+r_1^-(h))^\lambda-(1-r_1^-(h))^\lambda]}{[(1+r_1^-(h))^\lambda+(1-r_1^-(h))^\lambda]}], \\ \bigcup\limits_{p_1^+\in a_1^+,q_1^+\in b_1^+,r_1^+\in c_1^+} [\frac{[(1+p_1^+(h))^\lambda-(1-p_1^+(h))^\lambda]}{[(1+p_1^+(h))^\lambda+(1-p_1^+(h))^\lambda]}, \\ \frac{[(1+q_1^+(h))^\lambda-(1-q_1^+(h))^\lambda]}{[(1+q_1^+(h))^\lambda+(1-q_1^+(h))^\lambda]}, \frac{[(1+r_1^+(h))^\lambda-(1-r_1^+(h))^\lambda]}{[(1+r_1^+(h))^\lambda+(1-r_1^+(h))^\lambda]}], \\ \bigcup\limits_{p_1\in a_1,q_1\in b_1,r_1\in c_1} [\frac{2p_1^\lambda(h)}{[(2-p_1(h)]^\lambda+[p_1(h)]^\lambda}, \frac{2q_1^\lambda(h)}{[(2-q_1(h)]^\lambda+[q_1(h)]^\lambda}, \\ \frac{2r_1^\lambda(h)}{[(2-r_1(h)]^\lambda+[r_1(h)]^\lambda}]\rangle \end{array}\right\}$$

$$\lambda A_2=\left\{\begin{array}{l}\langle\bigcup\limits_{p_2^-\in a_2^-,q_2^-\in b_2^-,r_2^-\in c_2^-,}[\frac{[(1+p_2^-(h))^\lambda-(1-p_2^-(h))^\lambda]}{[(1+p_2^-(h))^\lambda+(1-p_2^-(h))^\lambda]},\frac{[(1+q_2^-(h))^\lambda-(1-q_2^-(h))^\lambda]}{[(1+q_2^-(h))^\lambda+(1-q_2^-(h))^\lambda]},\\ \frac{[(1+r_2^-(h))^\lambda-(1-r_2^-(h))^\lambda]}{[(1+r_2^-(h))^\lambda+(1-r_2^-(h))^\lambda]}],\bigcup\limits_{p_2^+\in a_2^+,q_2^+\in b_2^+,r_2^+\in c_2^+,}[\frac{[(1+p_2^+(h))^\lambda-(1-p_2^+(h))^\lambda]}{[(1+p_2^+(h))^\lambda+(1-p_2^+(h))^\lambda]},\\ \frac{[(1+q_2^+(h))^\lambda-(1-q_2^+(h))^\lambda]}{[(1+q_2^+(h))^\lambda+(1-q_2^+(h))^\lambda]},\frac{[(1+r_2^+(h))^\lambda-(1-r_2^+(h))^\lambda]}{[(1+r_2^+(h))^\lambda+(1-r_2^+(h))^\lambda]}],\\ \bigcup\limits_{p_2\in a_2,q_2\in b_2,r_2\in c_2}[\frac{2p_2^\lambda(h)}{[(2-p_2(h)]^\lambda+[p_2(h)]^\lambda},\frac{2q_2^\lambda(h)}{[(2-q_2(h)]^\lambda+[q_2(h)]^\lambda},\\ \frac{2r_2^\lambda(h)}{[(2-r_2(h)]^\lambda+[r_2(h)]^\lambda}]\rangle\end{array}\right\}$$

$$\lambda A_2+\lambda A_1=\left\{\begin{array}{l}\langle\bigcup\limits_{\substack{p_2^-\in a_2^-,p_1^-\in a_1^-,q_2^-\in b_2^-,\\ q_1^-\in b_1^-,r_2^-\in c_2^-,r_1^-\in c_1^-}}[\frac{[(1+p_2^-(h))(1-p_2^-(h))]^\lambda[(1+p_1^-(h))(1-p_1^-(h))]^\lambda}{[(1+p_2^-(h))(1-p_2^-(h))]^\lambda[(1+p_1^-(h))(1-p_1^-(h))]^\lambda},\\ \frac{[(1+q_2^-(h))(1-q_2^-(h))]^\lambda[(1+q_1^-(h))(1-q_1^-(h))]^\lambda}{[(1+q_2^-(h))(1-q_2^-(h))]^\lambda[(1+q_1^-(h))(1-q_1^-(h))]^\lambda},\\ \frac{[(1+r_2^-(h))(1-r_2^-(h))]^\lambda[(1+r_1^-(h))(1-r_1^-(h))]^\lambda}{[(1+r_2^-(h))(1-r_2^-(h))]^\lambda[(1+r_1^-(h))(1-r_1^-(h))]^\lambda}],\\ \bigcup\limits_{\substack{p_2^+\in a_2^+,p_1^+\in a_1^+,q_2^+\in b_2^+,\\ q_1^+\in b_1^+,r_2^+\in c_2^+,r_1^+\in c_1^+}}[\frac{[(1+p_2^+(h))(1-p_2^+(h))]^\lambda[(1+p_1^+(h))(1-p_1^+(h))]^\lambda}{[(1+p_2^+(h))(1-p_2^+(h))]^\lambda[(1+p_1^+(h))(1-p_1^+(h))]^\lambda},\\ \frac{[(1+q_2^+(h))(1-q_2^+(h))]^\lambda[(1+q_1^+(h))(1-q_1^+(h))]^\lambda}{[(1+q_2^+(h))(1-q_2^+(h))]^\lambda[(1+q_1^+(h))(1-q_1^+(h))]^\lambda},\\ \frac{[(1+r_2^+(h))(1-r_2^+(h))]^\lambda[(1+r_1^+(h))(1-r_1^+(h))]^\lambda}{[(1+r_2^+(h))(1-r_2^+(h))]^\lambda[(1+r_1^+(h))(1-r_1^+(h))]^\lambda}],\\ \bigcup\limits_{\substack{p_2\in a_2,p_1\in a_1,q_2\in b_2,\\ q_1\in b_1,r_2\in c_2,r_1\in c_1}}[\frac{2[p_2(h)p_1(h)]^\lambda}{[(4-2p_2(h)-2p_1(h)-p_2(h)p_1(h)]^\lambda+[p_2(h)p_1(h)]^\lambda},\\ \frac{2[q_2(h)q_1(h)]^\lambda}{[(4-2q_2(h)-2q_1(h)-q_2(h)q_1(h)]^\lambda+[q_2(h)q_1(h)]^\lambda},\\ \frac{2[r_2(h)r_1(h)]^\lambda}{[(4-2r_2(h)-2r_1(h)-r_2(h)r_1(h)]^\lambda+[r_2(h)r_1(h)]^\lambda}]\rangle\end{array}\right\}$$

so, we have $\lambda(A_1+A_2)=\lambda A_2+\lambda A_1$.

(3) $\lambda_1 A+\lambda_2 A=(\lambda_1+\lambda_2)A$

$$\lambda_1 A=\left\{\begin{array}{l}\langle\bigcup\limits_{p^-\in a^-,q^-\in b^-,r^-\in c^-}[\frac{[1+p_A^-(h)]^{\lambda_1}-[1-p_A^-(h)]^{\lambda_1}}{[1+p_A^-(h)]^{\lambda_1}+[1-p_A^-(h)]^{\lambda_1}},\frac{[1+q_A^-(h)]^{\lambda_1}-[1-q_A^-(h)]^{\lambda_1}}{[1+q_A^-(h)]^{\lambda_1}+[1-q_A^-(h)]^{\lambda_1}},\\ \frac{[1+r_A^-(h)]^{\lambda_1}-[1-r_A^-(h)]^{\lambda_1}}{[1+r_A^-(h)]^{\lambda_1}+[1-r_A^-(h)]^{\lambda_1}}],\bigcup\limits_{p^+\in a^+,q^+\in b^+,r^+\in c^+}[\frac{[1+p_A^+(h)]^{\lambda_1}-[1-p_A^+(h)]^{\lambda_1}}{[1+p_A^+(h)]^{\lambda_1}+[1-p_A^+(h)]^{\lambda_1}},\\ \frac{[1+q_A^+(h)]^{\lambda_1}-[1-q_A^+(h)]^{\lambda_1}}{[1+q_A^+(h)]^{\lambda_1}+[1-q_A^+(h)]^{\lambda_1}},\frac{[1+r_A^+(h)]^{\lambda_1}-[1-r_A^+(h)]^{\lambda_1}}{[1+r_A^+(h)]^{\lambda_1}+[1-r_A^+(h)]^{\lambda_1}}],\bigcup\limits_{p\in a,q\in b,r\in c}\\ [\frac{2[p_A(h)]^{\lambda_1}}{[(2-p_A(h)]^{\lambda_1}+[p_A(h)]^{\lambda_1}},\frac{2[q_A(h)]^{\lambda_1}}{[(2-q_A(h)]^{\lambda_1}+[q_A(h)]^{\lambda_1}},\frac{2[r_A(h)]^{\lambda_1}}{[(2-r_A(h)]^{\lambda_1}+[r_A(h)]^{\lambda_1}}]\rangle\end{array}\right\}$$

$$\text{and } \lambda_2 A=\left\{\begin{array}{l}\langle\bigcup\limits_{p^-\in a^-,q^-\in b^-,r^-\in c^-}[\frac{[1+p_A^-(h)]^{\lambda_2}-[1-p_A^-(h)]^{\lambda_2}}{[1+p_A^-(h)]^{\lambda_2}+[1-p_A^-(h)]^{\lambda_2}},\frac{[1+q_A^-(h)]^{\lambda_2}-[1-q_A^-(h)]^{\lambda_2}}{[1+q_A^-(h)]^{\lambda_2}+[1-q_A^-(h)]^{\lambda_2}},\\ \frac{[1+r_A^-(h)]^{\lambda_2}-[1-r_A^-(h)]^{\lambda_2}}{[1+r_A^-(h)]^{\lambda_2}+[1-r_A^-(h)]^{\lambda_2}}],\bigcup\limits_{p^+\in a^+,q^+\in b^+,r^+\in c^+}[\frac{[1+p_A^+(h)]^{\lambda_2}-[1-p_A^+(h)]^{\lambda_2}}{[1+p_A^+(h)]^{\lambda_2}+[1-p_A^+(h)]^{\lambda_2}},\\ \frac{[1+q_A^+(h)]^{\lambda_2}-[1-q_A^+(h)]^{\lambda_2}}{[1+q_A^+(h)]^{\lambda_2}+[1-q_A^+(h)]^{\lambda_2}},\frac{[1+r_A^+(h)]^{\lambda_2}-[1-r_A^+(h)]^{\lambda_2}}{[1+r_A^+(h)]^{\lambda_2}+[1-r_A^+(h)]^{\lambda_2}}],\\ \bigcup\limits_{p\in a,q\in b,r\in c}[\frac{2[p_A(h)]^{\lambda_2}}{[(2-p_A(h)]^{\lambda_2}+[p_A(h)]^{\lambda_2}},\frac{2[q_A(h)]^{\lambda_2}}{[(2-q_A(h)]^{\lambda_2}+[q_A(h)]^{\lambda_2}},\\ \frac{2[r_A(h)]^{\lambda_2}}{[(2-r_A(h)]^{\lambda_2}+[r_A(h)]^{\lambda_2}}]\rangle\end{array}\right\}$$

$$= \left\{ \begin{array}{c} \langle \bigcup_{p^- \in a^-, q^- \in b^-, r^- \in c^-} [\frac{[1+p_A^-(h)]^{\lambda_1+\lambda_2} - [1-p_A^-(h)]^{\lambda_1+\lambda_2}}{[1+p_A^-(h)]^{\lambda_1+\lambda_2} + [1-p_A^-(h)]^{\lambda_1+\lambda_2}}, \frac{[1+q_A^-(h)]^{\lambda_1+\lambda_2} - [1-q_A^-(h)]^{\lambda_1+\lambda_2}}{[1+q_A^-(h)]^{\lambda_1+\lambda_2} + [1-q_A^-(h)]^{\lambda_1+\lambda_2}}, \\ \frac{[1+r_A^-(h)]^{\lambda_1+\lambda_2} - [1-r_A^-(h)]^{\lambda_1+\lambda_2}}{[1+r_A^-(h)]^{\lambda_1+\lambda_2} + [1-r_A^-(h)]^{\lambda_1+\lambda_2}}], \bigcup_{p^+ \in a^+, q^+ \in b^+, r^+ \in c^+} [\frac{[1+p_A^+(h)]^{\lambda_1+\lambda_2} - [1-p_A^+(h)]^{\lambda_1+\lambda_2}}{[1+p_A^+(h)]^{\lambda_1+\lambda_2} + [1-p_A^+(h)]^{\lambda_1+\lambda_2}}, \\ \frac{[1+q_A^+(h)]^{\lambda_1+\lambda_2} - [1-q_A^+(h)]^{\lambda_1+\lambda_2}}{[1+q_A^+(h)]^{\lambda_1+\lambda_2} + [1-q_A^+(h)]^{\lambda_1+\lambda_2}}, \frac{[1+r_A^+(h)]^{\lambda_1+\lambda_2} - [1-r_A^+(h)]^{\lambda_1+\lambda_2}}{[1+r_A^+(h)]^{\lambda_1+\lambda_2} + [1-r_A^+(h)]^{\lambda_1+\lambda_2}}], \bigcup_{p \in a, q \in b, r \in c} \\ [\frac{2[p_A(h)]^{\lambda_1+\lambda_2}}{[(2-p_A(h)]^{\lambda_1+\lambda_2} + [p_A(h)]^{\lambda_1+\lambda_2}}, \frac{2[q_A(h)]^{\lambda_1+\lambda_2}}{[(2-q_A(h)]^{\lambda_1+\lambda_2} + [q_A(h)]^{\lambda_1+\lambda_2}}, \\ \frac{2[r_A(h)]^{\lambda_1+\lambda_2}}{[(2-r_A(h)]^{\lambda_1+\lambda_2} + [r_A(h)]^{\lambda_1+\lambda_2}}] \rangle \end{array} \right\}$$

$= (\lambda_1 + \lambda_2)A.$ $\square$

Remark 1. *If $\alpha_1 \leq_{L_{TCHFN}} \alpha_2$, then $\alpha_1 \leq \alpha_2$, that is the total order contains the usual partial order on L_{TCHFN}.*

5. Triangular Cubic Hesitant Fuzzy Arithmetic Averaging Operators Based on Einstein Operations

Definition 12. *An aggregation function $f : [0,1]^n \to [0,1]$ is a function non-decreasing in each argument, that is $\ddot{A}_j \leq \ddot{B}_j$ for all $j \in \{1,2,...,n\}$ implies $f(\ddot{A}_1, \ddot{A}_2, ..., \ddot{A}_n) \leq f(\ddot{B}_1, \ddot{B}_2, ..., \ddot{B}_n)$ and satisfying $f(0,0,...,0) = 0$ and $f(1,1,...,1) = 1$.*

Definition 13. *$f_{L_{TCHFN}} : L^n_{TCHFN} \to L_{TCHFN}$ is an aggregation function if it is monotone with respect to $\leq L_{TCHFN}$ and satisfies $f_{L_{TCHFN}}(0_{L_{TCHFN}}, ..., 0_{L_{TCHFN}}) = 0_{L_{TCHFN}}$ and $f_{L_{TCHFN}}(1_{L_{TCHFN}}, ..., 1_{L_{TCHFN}}) = 1_{L_{TCHFN}}$.*

5.1. Triangular Cubic Hesitant Fuzzy Einstein Weighted Averaging Operator

Definition 14. *Let $\ddot{A} = \left\langle \begin{array}{c} [a_1^-(\ddot{z}), b_1^-(\ddot{z}), c_1^-(\ddot{z})], \\ [a_1^+(\ddot{z}), b_1^+(\ddot{z}), c_1^+(\ddot{z})], \\ [a_1(\ddot{z}), b_1(\ddot{z}), c_1(\ddot{z})] \\ |\ddot{z} \in Z \end{array} \right\rangle$ be a collection of TCHFNs in L_{TCHFN} and $\ddot{\omega} = (\ddot{\omega}_1, \ddot{\omega}_2, ..., \ddot{\omega}_n)^T$ is the weight vector of $\ddot{A}_j (j = 1,2,...,n)$ such that $\ddot{\omega}_j \in [0,1]$ and $\sum_{j=1}^{n} \ddot{\omega}_j = 1$. Then triangular cubic hesitant fuzzy Einstein weighted averaging operator of dimension n is a mapping TCHFEWA: $L^n_{TCHFN} \to L_{TCHFN}$ and defined by*

$$TCHFEWA(\ddot{A}_1, \ddot{A}_2, ..., \ddot{A}_n) = \ddot{\omega}_1 \ddot{A}_1 + \ddot{\omega}_2 \ddot{A}_2, ..., \ddot{\omega}_n \ddot{A}_n. \tag{10}$$

If $\ddot{\omega} = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})^T$. Then the TCHFEWA operator is reduced to triangular cubic fuzzy Einstein averaging operator of dimension n. Which is defined as follows:

$$TCHFEA(\ddot{A}_1, \ddot{A}_2, ..., \ddot{A}_n) = \frac{1}{n}(\ddot{A}_1, \ddot{A}_2, ..., \ddot{A}_n). \tag{11}$$

Theorem 1. *Let $\ddot{A} = \left\langle \begin{array}{c} [a_1^-(\ddot{z}), b_1^-(\ddot{z}), c_1^-(\ddot{z})], \\ [a_1^+(\ddot{z}), b_1^+(\ddot{z}), c_1^+(\ddot{z})], \\ [a_1(\ddot{z}), b_1(\ddot{z}), c_1(\ddot{z})] \\ |\ddot{z} \in Z \end{array} \right\rangle$ be a collection of TCHFNs in L_{TCHFN}. Then their aggregated value by using the TCHFEWA operator is also a TCHFN and $TCHFEWA(\ddot{A}_1, \ddot{A}_2, ..., \ddot{A}_n) =$*

$$\left\langle \begin{array}{c} \bigcup\limits_{p^- \in a^-, q^- \in b^-, r^- \in c^-} \left[\begin{array}{c} \frac{\prod_{j=1}^{n}[1+p_1^-(\ddot{z})]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-p_1^-(\ddot{z})]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+p_1^-(\ddot{z})]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-p_1^-(\ddot{z})]^{\ddot{\omega}}}, \frac{\prod_{j=1}^{n}[1+q_1^-(\ddot{z})]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-q_1^-(\ddot{z})]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+q_1^-(\ddot{z})]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-q_1^-(\ddot{z})]^{\ddot{\omega}}}, \\ \frac{\prod_{j=1}^{n}[1+r_1^-(\ddot{z})]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-r_1^-(\ddot{z})]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+r_1^-(\ddot{z})]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-r_1^-(\ddot{z})]^{\ddot{\omega}}} \end{array} \right], \\ \bigcup\limits_{p^+ \in a^+, q^+ \in b^+, r^+ \in c^+} \left[\begin{array}{c} \frac{\prod_{j=1}^{n}[1+p_1^+(\ddot{z})]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-p_1^+(\ddot{z})]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+p_1^+(\ddot{z})]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-p_1^+(\ddot{z})]^{\ddot{\omega}}}, \frac{\prod_{j=1}^{n}[1+q_1^+(\ddot{z})]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-q_1^+(\ddot{z})]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+q_1^+(\ddot{z})]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-q_1^+(\ddot{z})]^{\ddot{\omega}}}, \\ \frac{\prod_{j=1}^{n}[1+r_1^+(\ddot{z})]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-r_1^+(\ddot{z})]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+r_1^+(\ddot{z})]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-r_1^+(\ddot{z})]^{\ddot{\omega}}} \end{array} \right], \\ \bigcup\limits_{p \in a, q \in b, r \in c} \left[\begin{array}{c} \frac{2\prod_{j=1}^{n}[p_1(\ddot{z})]^{\ddot{\omega}}}{\prod_{j=1}^{n}[(2-p_1(\ddot{z})]^{\ddot{\omega}} + \prod_{j=1}^{n}[p_1(\ddot{z})]^{\ddot{\omega}}}, \frac{2\prod_{j=1}^{n}[q_q(\ddot{z})]^{\ddot{\omega}}}{\prod_{j=1}^{n}[(2-q_q(\ddot{z})]^{\ddot{\omega}} + \prod_{j=1}^{n}[q_1(\ddot{z})]^{\ddot{\omega}}}, \\ \frac{2\prod_{j=1}^{n}[r_1(\ddot{z})]^{\ddot{\omega}}}{\prod_{j=1}^{n}[(2-r_1(\ddot{z})]^{\ddot{\omega}} + \prod_{j=1}^{n}[r_1(\ddot{z})]^{\ddot{\omega}}} \end{array} \right] \end{array} \right\rangle.$$

where $\ddot{\omega} = (\ddot{\omega}_1, \ddot{\omega}_2, ..., \ddot{\omega}_n)^T$ *is the weight vector of* $\ddot{A}_j (j = 1, 2, ..., n)$ *such that* $\ddot{\omega}_j \in [0, 1]$ *and* $\sum_{j=1}^{n} \ddot{\omega}_j = 1$.

Proof. Assume that $n = 1$,TCHFEWA$(A_1, A_2, ..., A_n) = \bigoplus_{j=1}^{k} w_1 A_1$

$\langle(\lambda(A_1 + A_2) = \lambda A_2 + \lambda A_1$

$$\lambda(A_1 + A_2) = \left\{ \begin{array}{c} \langle \bigcup\limits_{p^- \in a^-, q^- \in b^-, r^- \in c^-} [\frac{[(1+p_1^-(h))(1-p_1^-(h))]^\lambda [(1+p_2^-(h))(1-p_2^-(h))]^\lambda}{[(1+p_1^-(h))(1-p_1^-(h))]^\lambda [(1+p_2^-(h))(1-p_2^-(h))]^\lambda}, \\ \frac{[(1+q_1^-(h))(1-q_1^-(h))]^\lambda [(1+q_2^-(h))(1-q_2^-(h))]^\lambda}{[(1+q_1^-(h))(1-q_1^-(h))]^\lambda [(1+q_2^-(h))(1-q_2^-(h))]^\lambda} \\ , \frac{[(1+r_1^-(h))(1-r_1^-(h))]^\lambda [(1+r_2^-(h))(1-r_2^-(h))]^\lambda}{[(1+r_1^-(h))(1-r_1^-(h))]^\lambda [(1+r_2^-(h))(1-r_2^-(h))]^\lambda}], \\ \bigcup\limits_{p^+ \in a^+, q^+ \in b^+, r^+ \in c^+} [\frac{[(1+p_1^+(h))(1-p_1^+(h))]^\lambda [(1+p_2^+(h))(1-p_2^+(h))]^\lambda}{[(1+p_1^+(h))(1-p_1^+(h))]^\lambda [(1+p_2^+(h))(1-p_2^+(h))]^\lambda} \\ , \frac{[(1+q_1^+(h))(1-q_1^+(h))]^\lambda [(1+q_2^+(h))(1-q_2^+(h))]^\lambda}{[(1+q_1^+(h))(1-q_1^+(h))]^\lambda [(1+q_2^+(h))(1-q_2^+(h))]^\lambda}, \\ \frac{[(1+r_1^+(h))(1-r_1^+(h))]^\lambda [(1+r_2^+(h))(1-r_2^+(h))]^\lambda}{[(1+r_1^+(h))(1-r_1^+(h))]^\lambda [(1+r_2^+(h))(1-r_2^+(h))]^\lambda}] \\ , \bigcup\limits_{p \in a, q \in b, r \in c} [\frac{2[p_1(h)p_2(h)]^\lambda}{[(4-2p_1(h)-2p_2(h)-p_1(h)p_2(h)]^\lambda + [p_1(h)p_2(h)]^\lambda}, \\ \frac{2[q_1(h)q_2(h)]^\lambda}{[(4-2q_1(h)-2q_2(h)-q_1(h)q_2(h)]^\lambda + [q_1(h)q_2(h)]^\lambda}, \\ \frac{2[r_1(h)r_2(h)]^\lambda}{[(4-2r_1(h)-2r_2(h)-r_1(h)r_2(h)]^\lambda + [r_1(h)r_2(h)]^\lambda}]\rangle \end{array} \right\}$$

$$\text{and we have } \lambda A_1 = \left\{ \begin{array}{c} \langle \bigcup_{p^-\in a^-, q^-\in b^-, r^-\in c^-} [\frac{[(1+p_1^-(h))^\lambda-(1-p_1^-(h))^\lambda]}{[(1+p_1^-(h))^\lambda+(1-p_1^-(h))^\lambda]}, \frac{[(1+q_1^-(h))^\lambda-(1-q_1^-(h))^\lambda]}{[(1+q_1^-(h))^\lambda+(1-q_1^-(h))^\lambda]}, \\ \frac{[(1+r_1^-(h))^\lambda-(1-r_1^-(h))^\lambda]}{[(1+r_1^-(h))^\lambda+(1-r_1^-(h))^\lambda]}] \bigcup_{p^+\in a^+, q^+\in b^+, r^+\in c^+} [\frac{[(1+p_1^+(h))^\lambda-(1-p_1^+(h))^\lambda]}{[(1+p_1^+(h))^\lambda+(1-p_1^+(h))^\lambda]}, \\ \frac{[(1+q_1^+(h))^\lambda-(1-q_1^+(h))^\lambda]}{[(1+q_1^+(h))^\lambda+(1-q_1^+(h))^\lambda]}, \frac{[(1+r_1^+(h))^\lambda-(1-r_1^+(h))^\lambda]}{[(1+r_1^+(h))^\lambda+(1-r_1^+(h))^\lambda]}] \\ , \bigcup_{p\in a, q\in b, r\in c} [\frac{2p_1^\lambda(h)}{[(2-p_1(h)]^\lambda+[p_1(h)]^\lambda}, \frac{2q_1^\lambda(h)}{[(2-q_1(h)]^\lambda+[q_1(h)]^\lambda}, \\ \frac{2r_1^\lambda(h)}{[(2-r_1(h)]^\lambda+[r_1(h)]^\lambda}]\rangle \end{array} \right\}$$

$$\lambda A_2 = \left\{ \begin{array}{c} \langle \bigcup_{p^-\in a^-, q^-\in b^-, r^-\in c^-} [\frac{[(1+p_2^-(h))^\lambda-(1-p_2^-(h))^\lambda]}{[(1+p_2^-(h))^\lambda+(1-p_2^-(h))^\lambda]}, \frac{[(1+q_2^-(h))^\lambda-(1-q_2^-(h))^\lambda]}{[(1+q_2^-(h))^\lambda+(1-q_2^-(h))^\lambda]}, \\ \frac{[(1+r_2^-(h))^\lambda-(1-r_2^-(h))^\lambda]}{[(1+r_2^-(h))^\lambda+(1-r_2^-(h))^\lambda]}] \bigcup_{p^+\in a^+, q^+\in b^+, r^+\in c^+} [\frac{[(1+p_2^+(h))^\lambda-(1-p_2^+(h))^\lambda]}{[(1+p_2^+(h))^\lambda+(1-p_2^+(h))^\lambda]}, \\ \frac{[(1+q_2^+(h))^\lambda-(1-q_2^+(h))^\lambda]}{[(1+q_2^+(h))^\lambda+(1-q_2^+(h))^\lambda]}, \frac{[(1+r_2^+(h))^\lambda-(1-r_2^+(h))^\lambda]}{[(1+r_2^+(h))^\lambda+(1-r_2^+(h))^\lambda]}] \\ , \bigcup_{p\in a, q\in b, r\in c} [\frac{2p_2^\lambda(h)}{[(2-p_2(h)]^\lambda+[p_2(h)]^\lambda}, \frac{2q_2^\lambda(h)}{[(2-q_2(h)]^\lambda+[q_2(h)]^\lambda}, \\ \frac{2r_2^\lambda(h)}{[(2-r_2(h)]^\lambda+[r_2(h)]^\lambda}]\rangle \end{array} \right\}$$

$$\lambda A_2 + \lambda A_1 = \left\{ \begin{array}{c} \langle \bigcup_{p^-\in a^-, q^-\in b^-, r^-\in c^-} [\frac{[(1+p_2^-(h))(1-p_2^-(h))]^\lambda[(1+p_1^-(h))(1-p_1^-(h))]^\lambda}{[(1+p_2^-(h))(1-p_2^-(h))]^\lambda[(1+p_1^-(h))(1-p_1^-(h))]^\lambda}, \\ \frac{[(1+q_2^-(h))(1-q_2^-(h))]^\lambda[(1+q_1^-(h))(1-q_1^-(h))]^\lambda}{[(1+q_2^-(h))(1-q_2^-(h))]^\lambda[(1+q_1^-(h))(1-q_1^-(h))]^\lambda}, \\ \frac{[(1+r_2^-(h))(1-r_2^-(h))]^\lambda[(1+r_1^-(h))(1-r_1^-(h))]^\lambda}{[(1+r_2^-(h))(1-r_2^-(h))]^\lambda[(1+r_1^-(h))(1-r_1^-(h))]^\lambda}], \\ \bigcup_{p^+\in a^+, q^+\in b^+, r^+\in c^+} [\frac{[(1+p_2^+(h))(1-p_2^+(h))]^\lambda[(1+p_1^+(h))(1-p_1^+(h))]^\lambda}{[(1+p_2^+(h))(1-p_2^+(h))]^\lambda[(1+p_1^+(h))(1-p_1^+(h))]^\lambda}, \\ \frac{[(1+q_2^+(h))(1-q_2^+(h))]^\lambda[(1+q_1^+(h))(1-q_1^+(h))]^\lambda}{[(1+q_2^+(h))(1-q_2^+(h))]^\lambda[(1+q_1^+(h))(1-q_1^+(h))]^\lambda}, \\ \frac{[(1+r_2^+(h))(1-r_2^+(h))]^\lambda[(1+r_1^+(h))(1-r_1^+(h))]^\lambda}{[(1+r_2^+(h))(1-r_2^+(h))]^\lambda[(1+r_1^+(h))(1-r_1^+(h))]^\lambda}] \\ , \bigcup_{p\in a, q\in b, r\in c} [\frac{2[p_2(h)p_1(h)]^\lambda}{[(4-2p_2(h)-2p_1(h)-p_2(h)p_1(h)]^\lambda+[p_2(h)p_1(h)]^\lambda}, \\ \frac{2[q_2(h)q_1(h)]^\lambda}{[(4-2q_2(h)-2q_1(h)-q_2(h)q_1(h)]^\lambda+[q_2(h)q_1(h)]^\lambda}, \\ \frac{2[r_2(h)r_1(h)]^\lambda}{[(4-2r_2(h)-2r_1(h)-r_2(h)r_1(h)]^\lambda+[s_2(h)s_1(h)]^\lambda}]\rangle \end{array} \right\}$$

so, we have $\lambda(A_1 + A_2) = \lambda A_2 + \lambda A_1$.

$\lambda_1 A + \lambda_2 A = (\lambda_1 + \lambda_2) A$

$$\lambda_1 A = \left\{ \begin{array}{c} \langle \bigcup_{p^-\in a^-, q^-\in b^-, r^-\in c^-} [\frac{[1+p_A^-(h)]^{\lambda_1}-[1-p_A^-(h)]^{\lambda_1}}{[1+p_A^-(h)]^{\lambda_1}+[1-p_A^-(h)]^{\lambda_1}}, \frac{[1+q_A^-(h)]^{\lambda_1}-[1-q_A^-(h)]^{\lambda_1}}{[1+q_A^-(h)]^{\lambda_1}+[1-q_A^-(h)]^{\lambda_1}}, \\ \frac{[1+r_A^-(h)]^{\lambda_1}-[1-r_A^-(h)]^{\lambda_1}}{[1+r_A^-(h)]^{\lambda_1}+[1-r_A^-(h)]^{\lambda_1}}] \bigcup_{p^+\in a^+, q^+\in b^+, r^+\in c} [\frac{[1+p_A^+(h)]^{\lambda_1}-[1-p_A^+(h)]^{\lambda_1}}{[1+p_A^+(h)]^{\lambda_1}+[1-p_A^+(h)]^{\lambda_1}}, \\ \frac{[1+q_A^+(h)]^{\lambda_1}-[1-q_A^+(h)]^{\lambda_1}}{[1+q_A^+(h)]^{\lambda_1}+[1-q_A^+(h)]^{\lambda_1}}, \frac{[1+r_A^+(h)]^{\lambda_1}-[1-r_A^+(h)]^{\lambda_1}}{[1+r_A^+(h)]^{\lambda_1}+[1-r_A^+(h)]^{\lambda_1}}], \\ \bigcup_{p\in a, q\in b, r\in c} [\frac{2[p_A(h)]^{\lambda_1}}{[(2-p_A(h)]^{\lambda_1}+[p_A(h)]^{\lambda_1}}, \frac{2[q_A(h)]^{\lambda_1}}{[(2-q_A(h)]^{\lambda_1}+[q_A(h)]^{\lambda_1}}, \\ \frac{2[r_A(h)]^{\lambda_1}}{[(2-r_A(h)]^{\lambda_1}+[r_A(h)]^{\lambda_1}}]\rangle \end{array} \right\}$$

$$\text{and } \lambda_2 A=\left\{\begin{array}{c}\langle\bigcup\limits_{p^-\in a^-,q^-\in b^-,r^-\in c^-}[\frac{[1+p_A^-(h)]^{\lambda_2}-[1-p_A^-(h)]^{\lambda_2}}{[1+p_A^-(h)]^{\lambda_2}+[1-p_A^-(h)]^{\lambda_2}},\frac{[1+q_A^-(h)]^{\lambda_2}-[1-q_A^-(h)]^{\lambda_2}}{[1+q_A^-(h)]^{\lambda_2}+[1-q_A^-(h)]^{\lambda_2}},\\ \frac{[1+r_A^-(h)]^{\lambda_2}-[1-r_A^-(h)]^{\lambda_2}}{[1+r_A^-(h)]^{\lambda_2}+[1-r_A^-(h)]^{\lambda_2}}],\bigcup\limits_{p^+\in a^+,q^+\in b^+,r^+\in c^+}[\frac{[1+p_A^+(h)]^{\lambda_2}-[1-p_A^+(h)]^{\lambda_2}}{[1+p_A^+(h)]^{\lambda_2}+[1-p_A^+(h)]^{\lambda_2}},\\ \frac{[1+q_A^+(h)]^{\lambda_2}-[1-q_A^+(h)]^{\lambda_2}}{[1+q_A^+(h)]^{\lambda_2}+[1-q_A^+(h)]^{\lambda_2}},\frac{[1+r_A^+(h)]^{\lambda_2}-[1-r_A^+(h)]^{\lambda_2}}{[1+r_A^+(h)]^{\lambda_2}+[1-r_A^+(h)]^{\lambda_2}}],\\ \bigcup\limits_{p\in a,q\in b,r\in c}[\frac{2[p_A(h)]^{\lambda_2}}{[(2-p_A(h)]^{\lambda_2}+[p_A(h)]^{\lambda_2}},\frac{2[q_A(h)]^{\lambda_2}}{[(2-q_A(h)]^{\lambda_2}+[q_A(h)]^{\lambda_2}},\\ \frac{2[r_A(h)]^{\lambda_2}}{[(2-r_A(h)]^{\lambda_2}+[r_A(h)]^{\lambda_2}}]\rangle\end{array}\right\}$$

$$=\left\{\begin{array}{c}\langle\bigcup\limits_{p^-\in a^-,q^-\in b^-,r^-\in c^-}[\frac{[1+p_A^-(h)]^{\lambda_1+\lambda_2}-[1-p_A^-(h)]^{\lambda_1+\lambda_2}}{[1+p_A^-(h)]^{\lambda_1+\lambda_2}+[1-p_A^-(h)]^{\lambda_1+\lambda_2}},\frac{[1+q_A^-(h)]^{\lambda_1+\lambda_2}-[1-q_A^-(h)]^{\lambda_1+\lambda_2}}{[1+q_A^-(h)]^{\lambda_1+\lambda_2}+[1-q_A^-(h)]^{\lambda_1+\lambda_2}},\\ \frac{[1+r_A^-(h)]^{\lambda_1+\lambda_2}-[1-r_A^-(h)]^{\lambda_1+\lambda_2}}{[1+r_A^-(h)]^{\lambda_1+\lambda_2}+[1-r_A^-(h)]^{\lambda_1+\lambda_2}}],\bigcup\limits_{p^+\in a^+,q^+\in b^+,r^+\in c^+}[\frac{[1+p_A^+(h)]^{\lambda_1+\lambda_2}-[1-p_A^+(h)]^{\lambda_1+\lambda_2}}{[1+p_A^+(h)]^{\lambda_1+\lambda_2}+[1-p_A^+(h)]^{\lambda_1+\lambda_2}},\\ \frac{[1+q_A^+(h)]^{\lambda_1+\lambda_2}-[1-q_A^+(h)]^{\lambda_1+\lambda_2}}{[1+q_A^+(h)]^{\lambda_1+\lambda_2}+[1-q_A^+(h)]^{\lambda_1+\lambda_2}},\frac{[1+r_A^+(h)]^{\lambda_1+\lambda_2}-[1-r_A^+(h)]^{\lambda_1+\lambda_2}}{[1+r_A^+(h)]^{\lambda_1+\lambda_2}+[1-r_A^+(h)]^{\lambda_1+\lambda_2}}],\\ \bigcup\limits_{p\in a,q\in b,r\in c}[\frac{2[p_A(h)]^{\lambda_1+\lambda_2}}{[(2-p_A(h)]^{\lambda_1+\lambda_2}+[p_A(h)]^{\lambda_1+\lambda_2}},\frac{2[q_A(h)]^{\lambda_1+\lambda_2}}{[(2-q_A(h)]^{\lambda_1+\lambda_2}+[q_A(h)]^{\lambda_1+\lambda_2}},\\ \frac{2[r_A(h)]^{\lambda_1+\lambda_2}}{[(2-r_A(h)]^{\lambda_1+\lambda_2}+[r_A(h)]^{\lambda_1+\lambda_2}}]\rangle\end{array}\right\}$$

$$=\bigcup_{p^-\in a^-,q^-\in b^-,r^-\in c^-}\left[\begin{array}{c}\frac{[[1+p_1^-(h)]^{\varpi_1}-[1-p_1^-(h)]^{\varpi_1}}{[1+p_1^-(h)]^{\varpi_1}+[1-p_1^-(h)]^{\varpi_1}},\frac{[1+q_1^-(h)]^{\varpi_1}-[1-q_1^-(h)]^{\varpi_1}}{[1+q_1^-(h)]^{\varpi_1}+[1-q_1^-(h)]^{\varpi_1}},\\ \frac{[1+r_1^-(h)]^{\varpi_1}-[1-r_1^-(h)]^{\varpi_1}}{[1+r_1^-(h)]^{\varpi_1}+[1-r_1^-(h)]^{\varpi_1}}\end{array}\right];$$

$$\bigcup_{p^+\in a^+,q^+\in b^+,r^+\in c^+}\left[\begin{array}{c}\frac{[1+p_1^+(h)]^{\varpi_1}-[1-p_1^+(h)]^{\varpi_1}}{[1+p_1^+(h)]^{\varpi_1}+[1-p_1^+(h)]^{\varpi_1}},\frac{[1+q_1^+(h)]^{\varpi_1}-[1-q_1^+(h)]^{\varpi_1}}{[1+q_1^+(h)]^{\varpi_1}+[1-q_1^+(h)]^{\varpi_1}},\\ \frac{[1+r_1^+(h)]^{\varpi_1}-[1-r_1^+(h)]^{\varpi_1}}{[1+r_1^+(h)]^{\varpi_1}+[1-r_1^+(h)]^{\varpi_1}}\end{array}\right];$$

$$\bigcup_{p\in a,q\in b,r\in c}\left[\begin{array}{c}\frac{2[p_1(h)]^{\varpi_1}}{[(2-p_1(h)]^{\varpi_1}+[p_1(h)]^{\varpi_1}},\frac{2[q_1(h)]^{\varpi_1}}{[(2-q_1(h)]^{\varpi_1}+[q_1(h)]^{\varpi_1}},\\ \frac{2[r_1(h)]^{\varpi_1}}{[(2-r_1(h)]^{\varpi_1}+[r_1(h)]^{\varpi_1}}\end{array}\right].$$

Assume that $n=k,\text{TCHFEWA}(A_1,A_2,\ldots,A_n)=\bigoplus\limits_{j=1}^{k}w_jA_j$

$$\langle\bigcup_{p^-\in a^-,q^-\in b^-,r^-\in c^-}\left[\begin{array}{c}\frac{[\prod_{j=1}^{k}[1+p_1^-(h)]^{\varpi}-\prod_{j=1}^{k}[1-p_1^-(h)]^{\varpi}}{\prod_{j=1}^{k}[1+p_1^-(h)]^{\varpi}+\prod_{j=1}^{k}[1-p_1^-(h)]^{\varpi}},\frac{\prod_{j=1}^{k}[1+q_1^-(h)]^{\varpi}-\prod_{j=1}^{k}[1-q_1^-(h)]^{\varpi}}{\prod_{j=1}^{k}[1+q_1^-(h)]^{\varpi}+\prod_{j=1}^{k}[1-q_1^-(h)]^{\varpi}},\\ \frac{\prod_{j=1}^{k}[1+r_1^-(h)]^{\varpi}-\prod_{j=1}^{k}[1-r_1^-(h)]^{\varpi}}{\prod_{j=1}^{k}[1+r_1^-(h)]^{\varpi}+\prod_{j=1}^{k}[1-r_1^-(h)]^{\varpi}}\end{array}\right],$$

$$\bigcup_{p^+\in a^+,q^+\in b^+,r^+\in c^+}\left[\begin{array}{c}\frac{\prod_{j=1}^{k}[1+p_1^+(h)]^{\varpi}-\prod_{j=1}^{k}[1-p_1^+(h)]^{\varpi}}{\prod_{j=1}^{k}[1+p_1^+(h)]^{\varpi}+\prod_{j=1}^{k}[1-p_1^+(h)]^{\varpi}},\frac{\prod_{j=1}^{k}[1+q_1^+(h)]^{\varpi}-\prod_{j=1}^{k}[1-q_1^+(h)]^{\varpi}}{\prod_{j=1}^{k}[1+q_1^+(h)]^{\varpi}+\prod_{j=1}^{k}[1-q_1^+(h)]^{\varpi}},\\ \frac{\prod_{j=1}^{k}[1+r_1^+(h)]^{\varpi}-\prod_{j=1}^{k}[1-r_1^+(h)]^{\varpi}}{\prod_{j=1}^{k}[1+r_1^+(h)]^{\varpi}+\prod_{j=1}^{k}[1-r_1^+(h)]^{\varpi}}\end{array}\right];$$

$$\bigcup_{p\in a,q\in b,r\in c}\left[\begin{array}{c}\dfrac{2\prod\limits_{j=1}^{k}[p_1(h)]^{\varpi}}{\prod\limits_{j=1}^{k}[(2-p_1(h)]^{\varpi}+\prod\limits_{j=1}^{k}[p_1(h)]^{\varpi}},\dfrac{2\prod\limits_{j=1}^{k}[q_1(h)]^{\varpi}}{\prod\limits_{j=1}^{k}[(2-q_1(h)]^{\varpi}+\prod\limits_{j=1}^{k}[q_1(h)]^{\varpi}},\\ \dfrac{2\prod\limits_{j=1}^{k}[r_1(h)]^{\varpi}}{\prod\limits_{j=1}^{k}[(2-r_1(h)]^{\varpi}+\prod\limits_{j=1}^{k}[r_1(h)]^{\varpi}}\end{array}\right].$$

Then when $n=k+1$,we have TCHFEWA$(A_1,A_2,...,A_{k+1})$ =TCHFEWA$(A_1,A_2,...,A_k)\oplus A_{k+1})$

$$\left\langle\left[\begin{array}{c}\dfrac{[\prod\limits_{j=1}^{k}[1+p_1^-(h)]^{\varpi}-\prod\limits_{j=1}^{k}[1-p_1^-(h)]^{\varpi}}{\prod\limits_{j=1}^{k}[1+p_1^-(h)]^{\varpi}+\prod\limits_{j=1}^{k}[1-p_1^-(h)]^{\varpi}},\dfrac{\prod\limits_{j=1}^{k}[1+q_1^-(h)]^{\varpi}-\prod\limits_{j=1}^{k}[1-q_1^-(h)]^{\varpi}}{\prod\limits_{j=1}^{k}[1+q_1^-(h)]^{\varpi}+\prod\limits_{j=1}^{k}[1-q_1^-(h)]^{\varpi}},\\ \dfrac{\prod\limits_{j=1}^{k}[1+r_1^-(h)]^{\varpi}-\prod\limits_{j=1}^{k}[1-r_1^-(h)]^{\varpi}}{\prod\limits_{j=1}^{k}[1+r_1^-(h)]^{\varpi}+\prod\limits_{j=1}^{k}[1-r_1^-(h)]^{\varpi}}\end{array}\right];\right.$$

$$\left[\begin{array}{c}\dfrac{\prod\limits_{j=1}^{k}[1+p_1^+(h)]^{\varpi}-\prod\limits_{j=1}^{k}[1-p_1^+(h)]^{\varpi}}{\prod\limits_{j=1}^{k}[1+p_1^+(h)]^{\varpi}+\prod\limits_{j=1}^{k}[1-p_1^+(h)]^{\varpi}},\dfrac{\prod\limits_{j=1}^{k}[1+q_1^+(h)]^{\varpi}-\prod\limits_{j=1}^{k}[1-q_1^+(h)]^{\varpi}}{\prod\limits_{j=1}^{k}[1+q_1^+(h)]^{\varpi}+\prod\limits_{j=1}^{k}[1-q_1^+(h)]^{\varpi}},\\ \dfrac{\prod\limits_{j=1}^{k}[1+r_1^+(h)]^{\varpi}-\prod\limits_{j=1}^{k}[1-r_1^+(h)]^{\varpi}}{\prod\limits_{j=1}^{k}[1+r_1^+(h)]^{\varpi}+\prod\limits_{j=1}^{k}[1-r_1^+(h)]^{\varpi}}\end{array}\right];$$

$$\left[\begin{array}{c}\dfrac{2\prod\limits_{j=1}^{k}[p_1(h)]^{\varpi}}{\prod\limits_{j=1}^{k}[(2-p_1(h)]^{\varpi}+\prod\limits_{j=1}^{k}[p_1(h)]^{\varpi}},\dfrac{2\prod\limits_{j=1}^{k}[q_1(h)]^{\varpi}}{\prod\limits_{j=1}^{k}[(2-q_1(h)]^{\varpi}+\prod\limits_{j=1}^{k}[q_1(h)]^{\varpi}},\\ \dfrac{2\prod\limits_{j=1}^{k}[r_1(h)]^{\varpi}}{\prod\limits_{j=1}^{k}[(2-r_1(h)]^{\varpi}+\prod\limits_{j=1}^{k}[r_1(h)]^{\varpi}}\end{array}\right]\oplus_{k+1}$$

$$\left\langle\left[\begin{array}{c}\dfrac{[\prod\limits_{j=1}^{k+1}[1+p_1^-(h)]^{\varpi}-\prod\limits_{j=1}^{k+1}[1-p_1^-(h)]^{\varpi}}{\prod\limits_{j=1}^{k+1}[1+p_1^-(h)]^{\varpi}+\prod\limits_{j=1}^{k+1}[1-p_1^-(h)]^{\varpi}},\dfrac{\prod\limits_{j=1}^{k+1}[1+q_1^-(h)]^{\varpi}-\prod\limits_{j=1}^{k+1}[1-q_1^-(h)]^{\varpi}}{\prod\limits_{j=1}^{k+1}[1+q_1^-(h)]^{\varpi}+\prod\limits_{j=1}^{k+1}[1-q_1^-(h)]^{\varpi}},\\ \dfrac{\prod\limits_{j=1}^{k+1}[1+r_1^-(h)]^{\varpi}-\prod\limits_{j=1}^{k+1}[1-r_1^-(h)]^{\varpi}}{\prod\limits_{j=1}^{k+1}[1+r_1^-(h)]^{\varpi}+\prod\limits_{j=1}^{k+1}[1-r_1^-(h)]^{\varpi}}\end{array}\right];\right.$$

$$\left[\begin{array}{c}\dfrac{\prod\limits_{j=1}^{k+1}[1+p_1^+(h)]^{\varpi}-\prod\limits_{j=1}^{k+1}[1-p_1^+(h)]^{\varpi}}{\prod\limits_{j=1}^{k+1}[1+p_1^+(h)]^{\varpi}+\prod\limits_{j=1}^{k+1}[1-p_1^+(h)]^{\varpi}},\dfrac{\prod\limits_{j=1}^{k+1}[1+q_1^+(h)]^{\varpi}-\prod\limits_{j=1}^{k+1}[1-q_1^+(h)]^{\varpi}}{\prod\limits_{j=1}^{k+1}[1+q_1^+(h)]^{\varpi}+\prod\limits_{j=1}^{k+1}[1-q_1^+(h)]^{\varpi}},\\ \dfrac{\prod\limits_{j=1}^{k+1}[1+r_1^+(h)]^{\varpi}-\prod\limits_{j=1}^{k+1}[1-r_1^+(h)]^{\varpi}}{\prod\limits_{j=1}^{k+1}[1+r_1^+(h)]^{\varpi}+\prod\limits_{j=1}^{k+1}[1-r_1^+(h)]^{\varpi}}\end{array}\right];$$

$$\left[\begin{array}{c} \frac{2\prod_{j=1}^{k+1}[p_1(h)]^{\varpi}}{\prod_{j=1}^{k+1}[(2-p_1(h)]^{\varpi}+\prod_{j=1}^{k+1}[p_1(h)]^{\varpi}}, \frac{2\prod_{j=1}^{k+1}[q_1(h)]^{\varpi}}{\prod_{j=1}^{k+1}[(2-q_1(h)]^{\varpi}+\prod_{j=1}^{k+1}[q_1(h)]^{\varpi}}, \\ \frac{2\prod_{j=1}^{k+1}[r_1(h)]^{\varpi}}{\prod_{j=1}^{k+1}[(2-r_1(h)]^{\varpi}+\prod_{j=1}^{k+1}[r_1(h)]^{\varpi}} \end{array}\right]$$

$$= \left[\begin{array}{c} \frac{[\prod_{j=1}^{k+1}[1+p_1^-(h)]^{\varpi}-\prod_{j=1}^{k}[1-p_1^-(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+p_1^-(h)]^{\varpi}+\prod_{j=1}^{k}[1-p_1^-(h)]^{\varpi}}, \frac{\prod_{j=1}^{k+1}[1+q_1^-(h)]^{\varpi}-\prod_{j=1}^{k}[1-q_1^-(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+q_1^-(h)]^{\varpi}+\prod_{j=1}^{k}[1-q_1^-(h)]^{\varpi}}, \\ \frac{\prod_{j=1}^{k+1}[1+r_1^-(h)]^{\varpi}-\prod_{j=1}^{k}[1-r_1^-(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+r_1^-(h)]^{\varpi}+\prod_{j=1}^{k}[1-r_1^-(h)]^{\varpi}} \end{array}\right],$$

$$\left[\begin{array}{c} \frac{\prod_{j=1}^{k+1}[1+p_1^+(h)]^{\varpi}-\prod_{j=1}^{k}[1-p_1^+(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+p_1^+(h)]^{\varpi}+\prod_{j=1}^{k}[1-p_1^+(h)]^{\varpi}}, \frac{\prod_{j=1}^{k+1}[1+q_1^+(h)]^{\varpi}-\prod_{j=1}^{k}[1-q_1^+(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+q_1^+(h)]^{\varpi}+\prod_{j=1}^{k}[1-q_1^+(h)]^{\varpi}}, \\ \frac{\prod_{j=1}^{k+1}[1+r_1^+(h)]^{\varpi}-\prod_{j=1}^{k}[1-r_1^+(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+r_1^+(h)]^{\varpi}+\prod_{j=1}^{k}[1-r_1^+(h)]^{\varpi}} \end{array}\right],$$

$$\left[\begin{array}{c} \frac{2\prod_{j=1}^{k+1}[p_1(h)]^{\varpi}}{\prod_{j=1}^{k+1}[(2-p_1(h)]^{\varpi}+\prod_{j=1}^{k+1}[p_1(h)]^{\varpi}}, \frac{2\prod_{j=1}^{k+1}[q_1(h)]^{\varpi}}{\prod_{j=1}^{k+1}[(2-q_1(h)]^{\varpi}+\prod_{j=1}^{k+1}[q_1(h)]^{\varpi}}, \\ \frac{2\prod_{j=1}^{k+1}[r_1(h)]^{\varpi}}{\prod_{j=1}^{k+1}[(2-r_1(h)]^{\varpi}+\prod_{j=1}^{k+1}[r_1(h)]^{\varpi}} \end{array}\right].$$

Especially, if $w = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})^T$, then the TCHFEWA operator is reduced to the triangular cubic hesitant fuzzy Einstein weighing averaging operator, which is shown as follows:

$$\langle \left[\begin{array}{c} \frac{[\prod_{j=1}^{n}[1+p_1^-(h)]^{\frac{1}{n}}-\prod_{j=1}^{n}[1-p_1^-(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[1+p_1^-(h)]^{\frac{1}{n}}+\prod_{j=1}^{n}[1-p_1^-(h)]^{\frac{1}{n}}}, \frac{\prod_{j=1}^{n}[1+q_1^-(h)]^{\frac{1}{n}}-\prod_{j=1}^{n}[1-q_1^-(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[1+q_1^-(h)]^{\frac{1}{n}}+\prod_{j=1}^{n}[1-q_1^-(h)]^{\frac{1}{n}}}, \\ \frac{\prod_{j=1}^{n}[1+r_1^-(h)]^{\frac{1}{n}}-\prod_{j=1}^{n}[1-r_1^-(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[1+r_1^-(h)]^{\frac{1}{n}}+\prod_{j=1}^{n}[1-r_1^-(h)]^{\frac{1}{n}}} \end{array}\right];$$

$$\left[\begin{array}{c} \frac{\prod_{j=1}^{n}[1+p_1^+(h)]^{\frac{1}{n}}-\prod_{j=1}^{n}[1-p_1^+(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[1+p_1^+(h)]^{\frac{1}{n}}+\prod_{j=1}^{n}[1-p_1^+(h)]^{\frac{1}{n}}}, \frac{\prod_{j=1}^{n}[1+q_1^+(h)]^{\frac{1}{n}}-\prod_{j=1}^{n}[1-q_1^+(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[1+q_1^+(h)]^{\frac{1}{n}}+\prod_{j=1}^{n}[1-q_1^+(h)]^{\frac{1}{n}}}, \\ \frac{\prod_{j=1}^{n}[1+r_1^+(h)]^{\frac{1}{n}}-\prod_{j=1}^{n}[1-r_1^+(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[1+r_1^+(h)]^{\frac{1}{n}}+\prod_{j=1}^{n}[1-r_1^+(h)]^{\frac{1}{n}}} \end{array}\right];$$

$$\left[\begin{array}{c} \dfrac{2\prod_{j=1}^{n}[p_1(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[(2-p_1(h)]^{\frac{1}{n}}+\prod_{j=1}^{n}[p_1(h)]^{\frac{1}{n}}}, \dfrac{2\prod_{j=1}^{n}[q_1(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[(2-q_1(h)]^{\frac{1}{n}}+\prod_{j=1}^{n}[q_1(h)]^{\frac{1}{n}}}, \\ \dfrac{2\prod_{j=1}^{n}[r_1(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[(2-r_1(h)]^{\frac{1}{n}}+\prod_{j=1}^{n}[r_1(h)]^{\frac{1}{n}}} \end{array}\right]. \quad \square$$

Example 4. *Let* $\ddot{A}_1 = \left\langle \begin{array}{c} [0.2, 0.4, 0.6], \\ [0.4, 0.6, 0.8], \\ [0.3, 0.5, 0.7] \end{array} \right\rangle, \ddot{A}_2 = \left\langle \begin{array}{c} [0.1, 0.3, 0.5], \\ [0.3, 0.5, 0.7], \\ [0.2, 0.4, 0.6] \end{array} \right\rangle$ *and* $\ddot{A}_3 = \left\langle \begin{array}{c} [0.5, 0.7, 0.9], \\ [0.7, 0.9, 0.11], \\ [0.6, 0.8, 0.10] \end{array} \right\rangle$ *be a collection of TCHFNs in* L_{TCHFN} *and* $\ddot{\omega} = (0.2, 0.2, 0.6)^T$ *is the weight vector of* $\ddot{A}_j (j = 1, 2, ..., n)$ *such that* $\ddot{\omega}_j \in [0, 1]$ *and* $\sum_{j=1}^{n} \ddot{\omega}_j = 1$. *Then their aggregated value by using the TCHFEWA operator is also a TCHFN and* $TCHFEWA(\ddot{A}_1, \ddot{A}_2, ..., \ddot{A}_n) =$

$$\left\langle \begin{array}{c} \left[\begin{array}{c} \frac{[1+0.2]^{0.2}[1+0.1]^{0.2}[1+0.5]^{0.2}-[1-0.2]^{0.2}[1-0.1]^{0.2}[1-0.5]^{0.2}}{[1+0.2]^{0.2}[1+0.1]^{0.2}[1+0.5]^{0.2}+[1-0.2]^{0.2}[1-0.1]^{0.2}[1-0.5]^{0.2}}, \\ \frac{[1+0.4]^{0.2}[1+0.3]^{0.2}[1+0.5]^{0.2}-[1-0.4]^{0.2}[1-0.3]^{0.2}[1-0.5]^{0.2}}{[1+0.4]^{0.2}[1+0.3]^{0.2}[1+0.5]^{0.2}+[1-0.4]^{0.2}[1-0.3]^{0.2}[1-0.5]^{0.2}}, \\ \frac{[1+0.6]^{0.2}[1+0.5]^{0.2}[1+0.9]^{0.2}-[1-0.6]^{0.2}[1-0.5]^{0.2}[1-0.9]^{0.2}}{[1+0.6]^{0.2}[1+0.5]^{0.2}[1+0.9]^{0.2}+[1-0.6]^{0.2}[1-0.5]^{0.2}[1-0.9]^{0.2}} \end{array}\right], \\ \left[\begin{array}{c} \frac{[1+0.4]^{0.2}[1+0.3]^{0.2}[1+0.5]^{0.2}-[1-0.4]^{0.2}[1-0.3]^{0.2}[1-0.5]^{0.2}}{[1+0.4]^{0.2}[1+0.3]^{0.2}[1+0.5]^{0.2}+[1-0.4]^{0.2}[1-0.3]^{0.2}[1-0.5]^{0.2}}, \\ \frac{[1+0.6]^{0.2}[1+0.5]^{0.2}[1+0.9]^{0.2}-[1-0.6]^{0.2}[1-0.5]^{0.2}[1-0.9]^{0.2}}{[1+0.6]^{0.2}[1+0.5]^{0.2}[1+0.9]^{0.2}+[1-0.6]^{0.2}[1-0.5]^{0.2}[1-0.9]^{0.2}}, \\ \frac{[1+0.8]^{0.2}[1+0.7]^{0.2}[1+0.11]^{0.2}-[1-0.8]^{0.2}[1-0.7]^{0.2}[1-0.11]^{0.2}}{[1+0.8]^{0.2}[1+0.7]^{0.2}[1+0.11]^{0.2}+[1-0.8]^{0.2}[1-0.7]^{0.2}[1-0.11]^{0.2}} \end{array}\right], \\ \left[\begin{array}{c} \frac{2[(0.3)^{0.6}(0.2)^{0.6}(0.6)^{0.6}]}{[(2-0.3)^{0.6}(2-0.2)^{0.6}(2-0.6)^{0.6}]+[(0.3)^{0.6}(0.2)^{0.6}(0.6)^{0.6}]}, \\ \frac{2[(0.5)^{0.6}(0.4)^{0.6}(0.8)^{0.6}]}{[(2-0.5)^{0.6}(2-0.4)^{0.6}(2-0.8)^{0.6}]+[(0.5)^{0.6}(0.4)^{0.6}(0.8)^{0.6}]}, \\ \frac{2[(0.7)^{0.6}(0.6)^{0.6}(0.10)^{0.6}]}{[(2-0.7)^{0.6}(2-0.6)^{0.6}(2-0.10)^{0.6}]+[(0.7)^{0.6}(0.6)^{0.6}(0.10)^{0.6}]} \end{array}\right] \end{array} \right\rangle =$$

$$\left\langle \begin{array}{c} [0.1687, 0.2509, 0.4951], \\ [0.2509, 0.4951, 0.3928], \\ [0.1075, 0.3001, 0.1324] \end{array} \right\rangle.$$

Proposition 2. *Let* $\ddot{A} = \left\langle \begin{array}{c} [a_1^-(\dddot{z}), b_1^-(\dddot{z}), c_1^-(\dddot{z})], \\ [a_1^+(\dddot{z}), b_1^+(\dddot{z}), c_1^+(\dddot{z})], \\ [a_1(\dddot{z}), b_1(\dddot{z}), c_1(\dddot{z})] \\ |\dddot{z} \in Z \end{array} \right\rangle$ *be a collection of TCHFNs in* L_{TCHFN} *and where* $\ddot{\omega} = (\ddot{\omega}_1, \ddot{\omega}_2, ..., \ddot{\omega}_n)^T$ *is the weight vector of* $\ddot{A}_j (j = 1, 2, ..., n)$ *such that* $\ddot{\omega}_j \in [0, 1]$ *and* $\sum_{j=1}^{n} \ddot{\omega}_j = 1$.

Then (1) (Idempotency): If all $A_j, j = 1, 2, ..., n$ *are equal, i.e.,* $A_j = A$, *for all* $j = 1, 2, ..., n$, *then* $TCHFEWA(A_1, A_2, ..., A_n) = A$.

(2) (Boundary):If $a_{\min}^- = \min_{1 \le j \le n} a_j^-, b_{\min}^- = \min_{1 \le j \le n} b_j^-, c_{\min}^- = \min_{1 \le j \le n} c_j^-,$
$a_{\min}^+ = \min_{1 \le j \le n} a_j^+, b_{\min}^+ = \min_{1 \le j \le n} b_j^+, c_{\min}^+ = \min_{1 \le j \le n} c_j^+,$
$a_{\max} = \max_{1 \le j \le n} a_j, b_{\max} = \max_{1 \le j \le n} b_j, c_{\max} = \max_{1 \le j \le n} c_j,$
$a_{\max}^- = \max_{1 \le j \le n} a_j^-, b_{\max}^- = \max_{1 \le j \le n} b_j^-, c_{\max}^- = \max_{1 \le j \le n} c_j^-,$
$a_{\max}^+ = \max_{1 \le j \le n} a_j^+, b_{\max}^+ = \max_{1 \le j \le n} b_j^+, c_{\max}^+ = \max_{1 \le j \le n} c_j^+,$
$, a_{\min} = \min_{1 \le j \le n} a_j, b_{\min} = \min_{1 \le j \le n} b_j, c_{\min} = \min_{1 \le j \le n} c_j,$ *for all* $j = 1, 2, .., n$, *we can obtain that*

$$\left\langle \begin{array}{c} [a_{\min}^-(h), b_{\min}^-(h), c_{\min}^-(h)], \\ [a_{\min}^+(h), b_{\min}^+(h), c_{\min}^+(h)], \\ [a_{\max}(h), b_{\max}(h), c_{\max}(h) \\ |h \in H \end{array} \right\rangle$$

$$\leq TCHFEWA(A_1, A_2, ..., A_n) \leq \left\langle \begin{array}{c} [a^-_{\max}(h), b^-_{\max}(h), c^-_{\max}(h)], \\ [a^+_{\max}(h), b^+_{\max}(h), c^+_{\max}(h)], \\ [a_{\min}(h), b_{\min}(h), c_{\min}(h)] \\ h \in H \end{array} \right\rangle.$$

(3) (Monotonicity): $A = \left\langle \begin{array}{c} [\{a^-_A(h), b^-_A(h), c^-_A(h)] \\ , [a^+_A(h), b^+_A(h), c^+_A(h)], \\ [a_A(h), b_A(h), c_A(h)]\rangle \\ |h \in H \end{array} \right\rangle$ *and* $B = \left\langle \begin{array}{c} [a^-_B(h), b^-_B(h), c^-_B(h)], \\ [a^+_B(h), b^+_B(h), c^+_B(h)], \\ [a_B(h), b_B(h), c_B(h)] \\ |h \in H \end{array} \right\rangle$ *be two collection of TCHFNs in* L_{TCHFN} *and* $A_j \leq L_{TCHFN} B_j$ *i.e.,* $a^-_A(h) \leq a^-_B(h), b^-_A(h) \leq b^-_B(h), c^-_A(h) \leq c^-_B(h)$, $a^+_A(h) \leq a^+_B(h), b^+_A(h) \leq b^+_B(h), c^+_A(h) \leq c^+_B(h)$ *and* $a_A(h) \leq a_B(h), b_A(h) \leq b_B(h), c_A(h) \leq c_B(h)$ *then* $TCHFEWA(A_1, A_2, ..., A_n) \leq TCHFEWA(B_1, B_2, ..., B_n)$.

5.2. Triangular Cubic Hesitant Fuzzy Einstein Ordered Weighted Averaging Operator

We also develop a type of triangular cubic fuzzy Einstein ordered weighted averaging (TCFEOWA) operator.

Definition 15. *Let* $\ddot{A} = \left\langle \begin{array}{c} [a^-_{\ddot{A}}(\ddot{z}), b^-_{\ddot{A}}(\ddot{z}), c^-_{\ddot{A}}(\ddot{z})], \\ [a^+_{\ddot{A}}(\ddot{z}), b^-_{\ddot{A}}(\ddot{z}), c^+_{\ddot{A}}(\ddot{z})], \\ [a_{\ddot{A}}(\ddot{z}), b_{\ddot{A}}(\ddot{z}), c_{\ddot{A}}(\ddot{z})] \\ |\ddot{z} \in Z \end{array} \right\rangle$ *be a collection of TCHFNs in* L_{TCHFN}, *an TCHFEOWA operator of dimension n is a mapping* $TCHFEOWA : L^n_{TCHFN} \to L_{TCHFN}$ *, that has an associated vector* $\ddot{\omega} = (\ddot{\omega}_1, \ddot{\omega}_2, ..., \ddot{\omega}_n)^T$ *such that* $\ddot{\omega}_j \in [0, 1]$ *and* $\sum_{j=1}^{n} \ddot{\omega}_j = 1$. $TCHFEOWA(\ddot{A}_1, \ddot{A}_2, ..., \ddot{A}_n) = \ddot{\omega}_1 \ddot{A}_{(\sigma)1} + \ddot{\omega}_2 \ddot{A}_{(\sigma)2}, ..., \ddot{\omega}_n \ddot{A}_{(\sigma)n}$ *, where* $(\sigma(1), \sigma(2), ..., \sigma(n))$ *is a permutation of* $(1, 2, ..., n)$ *such that* $\ddot{A}_{\sigma(1)} \leq \ddot{A}_{\sigma(j-1)}$ *for all* $j = 2, 3, ..., n$ *(i.e.,* $\ddot{A}_{\sigma(j)}$ *is the jth largest value in the collection* $(\ddot{A}_1, \ddot{A}_2, ..., \ddot{A}_n)$*. If* $\ddot{\omega} = (\ddot{\omega}_1, \ddot{\omega}_2, ..., \ddot{\omega}_n)^T = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})^T$*. Then the TCHFEOWA operator is reduced to the TCHFA operator (2) of dimension n.*

Theorem 2. *Let* $\ddot{A} = \left\langle \begin{array}{c} [a^{-}_{\ddot{A}}(\ddot{z}), b^{-}_{\ddot{A}}(\ddot{z}), c^{-}_{\ddot{A}}(\ddot{z})], \\ [a^{+}_{\ddot{A}}(\ddot{z}), b^{-}_{\ddot{A}}(\ddot{z}), c^{+}_{\ddot{A}}(\ddot{z})], \\ [a_{\ddot{A}}(\ddot{z}), b_{\ddot{A}}(\ddot{z}), c_{\ddot{A}}(\ddot{z})] \\ |\ddot{z} \in Z \end{array} \right\rangle$ *be a collection of TCHFNs in* L_{TCHFN}. *Then their aggregated value by using the TCHFEOWA operator is also a TCHFN and* $TCHFEOWA(\ddot{A}_1, \ddot{A}_2, ..., \ddot{A}_n) =$

$$\left\langle \begin{array}{c} \bigcup_{p^{-}\in a^{-}, q^{-}\in b^{-}, r^{-}\in c^{-}} \left[\begin{array}{c} \frac{\prod_{j=1}^{n}[1+p^{-}_{\sigma(j)}(\ddot{z})]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-p^{-}_{\sigma(j)}(\ddot{z})]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+p^{-}_{\sigma(j)}(\ddot{z})]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-p^{-}_{\sigma(j)}(\ddot{z})]^{\ddot{\omega}}}, \\ \frac{\prod_{j=1}^{n}[1+q^{-}_{\sigma(j)}(\ddot{z})]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-q^{-}_{\sigma(j)}(\ddot{z})]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+q^{-}_{\sigma(j)}(\ddot{z})]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-q^{-}_{\sigma(j)}(\ddot{z})]^{\ddot{\omega}}}, \\ \frac{\prod_{j=1}^{n}[1+r^{-}_{\sigma(j)}(\ddot{z})]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-r^{-}_{\sigma(j)}(\ddot{z})]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+r^{-}_{\sigma(j)}(\ddot{z})]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-r^{-}_{\sigma(j)}(\ddot{z})]^{\ddot{\omega}}} \end{array} \right], \\ \bigcup_{p^{+}\in a^{+}, q^{+}\in b^{+}, r^{+}\in c^{+}} \left[\begin{array}{c} \frac{\prod_{j=1}^{n}[1+p^{+}_{\sigma(j)}(\ddot{z})]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-p^{+}_{\sigma(j)}(\ddot{z})]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+p^{+}_{\sigma(j)}(\ddot{z})]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-p^{+}_{\sigma(j)}(\ddot{z})]^{\ddot{\omega}}}, \\ \frac{\prod_{j=1}^{n}[1+q^{+}_{\sigma(j)}(\ddot{z})]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-q^{+}_{\sigma(j)}(\ddot{z})]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+q^{+}_{\sigma(j)}(\ddot{z})]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-q^{+}_{\sigma(j)}(\ddot{z})]^{\ddot{\omega}}}, \\ \frac{\prod_{j=1}^{n}[1+r^{+}_{\sigma(j)}(\ddot{z})]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-r^{+}_{\sigma(j)}(\ddot{z})]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+r^{+}_{\sigma(j)}(\ddot{z})]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-r^{+}_{\sigma(j)}(\ddot{z})]^{\ddot{\omega}}} \end{array} \right] \\ \bigcup_{p\in a, q\in b, r\in c} \left[\begin{array}{c} \frac{2\prod_{j=1}^{n}[p_{\sigma(j)}(\ddot{z})]^{\ddot{\omega}}}{\prod_{j=1}^{n}[(2-p_{\sigma(j)}(\ddot{z})]^{\ddot{\omega}} + \prod_{j=1}^{n}[p_{\sigma(j)}(\ddot{z})]^{\ddot{\omega}}}, \\ \frac{2\prod_{j=1}^{n}[q_{\sigma(j)}(\ddot{z})]^{\ddot{\omega}}}{\prod_{j=1}^{n}[(2-q_{\sigma(j)}(\ddot{z})]^{\ddot{\omega}} + \prod_{j=1}^{n}[q_{\sigma(j)}(\ddot{z})]^{\ddot{\omega}}}, \\ \frac{2\prod_{j=1}^{n}[r_{\sigma(j)}(\ddot{z})]^{\ddot{\omega}}}{\prod_{j=1}^{n}[(2-r_{\sigma(j)}(\ddot{z})]^{\ddot{\omega}} + \prod_{j=1}^{n}[r_{\sigma(j)}(\ddot{z})]^{\ddot{\omega}}} \end{array} \right] \end{array} \right\rangle, \tag{12}$$

where $(\sigma(1), \sigma(2), ..., \sigma(n))$ *is a permutation of* $(1, 2, ..., n)$ *such that* $\ddot{A}_{\sigma(1)} \le \ddot{A}_{\sigma(j-1)}$ *for all* $j = 2, 3, ..., n$, $\ddot{\omega} = (\ddot{\omega}_1, \ddot{\omega}_2, ..., \ddot{\omega}_n)^T$ *is the weight vector of* $\ddot{A}_j (j = 1, 2, ..., n)$ *such that* $\ddot{\omega}_j \in [0, 1]$, *and* $\sum_{j=1}^{n} \ddot{\omega}_j = 1$. *If* $\ddot{\omega} = (\ddot{\omega}_1, \ddot{\omega}_2, ..., \ddot{\omega}_n)^T = (\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n})^T$. *Then the TCHFEOWA operator is reduced to the TCHFA operator (2) of dimension n. Where* $\ddot{\omega} = (\ddot{\omega}_1, \ddot{\omega}_2, ..., \ddot{\omega}_n)^T$ *is the weight vector of* $\ddot{A}_j (j = 1, 2, ..., n)$ *such that* $\ddot{\omega}_j \in [0, 1]$ *and* $\sum_{j=1}^{n} \ddot{\omega}_j = 1$.

Example 5. *Let* $\ddot{A}_1 = \left\langle \begin{array}{c} [0.4, 0.6, 0.8], \\ [0.6, 0.8, 0.10], \\ [0.5, 0.7, 0.9] \end{array} \right\rangle, \ddot{A}_2 = \left\langle \begin{array}{c} [0.5, 0.7, 0.9], \\ [0.7, 0.9, 0.11], \\ [0.6, 0.8, 0.10] \end{array} \right\rangle$ *and* $\ddot{A}_3 = \left\langle \begin{array}{c} [0.2, 0.4, 0.6], \\ [0.4, 0.6, 0.8], \\ [0.3, 0.5, 0.7] \end{array} \right\rangle$ *be a collection of TCHFNs in* L_{TCHFN} *and* $\ddot{\omega} = (0.4, 0.4, 0.2)^T$ *is the weight vector of* $\ddot{A}_j (j = 1, 2, ..., n)$ *such that* $\ddot{\omega}_j \in [0, 1]$ *and* $\sum_{j=1}^{n} \ddot{\omega}_j = 1$*. Then their aggregated value by using the TCHFEOWA operator is also a TCHFN and* $TCHFEOWA(\ddot{A}_1, \ddot{A}_2, ..., \ddot{A}_n) =$

$$\left\langle \begin{array}{c} \left[\begin{array}{c} \frac{[1+0.4]^{0.4}[1+0.5]^{0.4}[1+0.2]^{0.4} - [1-0.4]^{0.4}[1-0.5]^{0.4}[1-0.2]^{0.4}}{[1+0.4]^{0.4}[1+0.5]^{0.4}[1+0.2]^{0.4} + [1-0.4]^{0.4}[1-0.5]^{0.4}[1-0.2]^{0.4}}, \\ \frac{[1+0.6]^{0.4}[1+0.7]^{0.4}[1+0.4]^{0.4} - [1-0.6]^{0.4}[1-0.7]^{0.4}[1-0.4]^{0.4}}{[1+0.6]^{0.4}[1+0.7]^{0.4}[1+0.4]^{0.4} + [1-0.6]^{0.4}[1-0.7]^{0.4}[1-0.4]^{0.4}}, \\ \frac{[1+0.8]^{0.4}[1+0.9]^{0.4}[1+0.6]^{0.4} - [1-0.8]^{0.4}[1-0.9]^{0.4}[1-0.6]^{0.4}}{[1+0.8]^{0.4}[1+0.9]^{0.4}[1+0.6]^{0.4} + [1-0.8]^{0.4}[1-0.9]^{0.4}[1-0.6]^{0.4}} \end{array} \right], \\ \left[\begin{array}{c} \frac{[1+0.6]^{0.4}[1+0.7]^{0.4}[1+0.4]^{0.4} - [1-0.6]^{0.4}[1-0.7]^{0.4}[1-0.4]^{0.4}}{[1+0.6]^{0.4}[1+0.7]^{0.4}[1+0.4]^{0.4} + [1-0.6]^{0.4}[1-0.7]^{0.4}[1-0.4]^{0.4}}, \\ \frac{[1+0.8]^{0.4}[1+0.9]^{0.4}[1+0.6]^{0.4} - [1-0.8]^{0.4}[1-0.9]^{0.4}[1-0.6]^{0.4}}{[1+0.8]^{0.4}[1+0.9]^{0.4}[1+0.6]^{0.4} + [1-0.8]^{0.4}[1-0.9]^{0.4}[1-0.6]^{0.4}}, \\ \frac{[1+0.10]^{0.4}[1+0.11]^{0.4}[1+0.8]^{0.4} - [1-0.10]^{0.4}[1-0.11]^{0.4}[1-0.8]^{0.4}}{[1+0.10]^{0.4}[1+0.11]^{0.4}[1+0.8]^{0.4} + [1-0.10]^{0.4}[1-0.11]^{0.4}[1-0.8]^{0.4}} \end{array} \right], \\ \left[\begin{array}{c} \frac{2[(0.5)^{0.2}(0.6)^{0.2}(0.3)^{0.2}]}{[(2-0.5)^{0.2}(2-0.6)^{0.2}(2-0.3)^{0.2}] + [(0.5)^{0.2}(0.6)^{0.2}(0.3)^{0.2}]}, \\ \frac{2[(0.7)^{0.2}(0.8)^{0.2}(0.5)^{0.2}]}{[(2-0.7)^{0.2}(2-0.4)^{0.2}(2-0.8)^{0.2}] + [(0.7)^{0.2}(0.4)^{0.2}(0.8)^{0.2}]}, \\ \frac{2[(0.9)^{0.2}(0.10)^{0.2}(0.7)^{0.2}]}{[(2-0.9)^{0.2}(2-0.10)^{0.2}(2-0.7)^{0.2}] + [(0.9)^{0.2}(0.10)^{0.2}(0.7)^{0.2}]} \end{array} \right] \end{array} \right\rangle$$

$$= \left\langle \begin{array}{c} [0.4384, 0.6604, 0.8631], \\ [0.6604, 0.8631, 0.4788], \\ [0.6477, 0.7983, 0.6404] \end{array} \right\rangle.$$

Proposition 3. *Let* $\ddot{A} = \left\langle \begin{array}{c} [a^-_{\ddot{A}}(\dddot{z}), b^-_{\ddot{A}}(\dddot{z}), c^-_{\ddot{A}}(\dddot{z})], \\ [a^+_{\ddot{A}}(\dddot{z}), b^-_{\ddot{A}}(\dddot{z}), c^+_{\ddot{A}}(\dddot{z})], \\ [a_{\ddot{A}}(\dddot{z}), b_{\ddot{A}}(\dddot{z}), c_{\ddot{A}}(\dddot{z})] \\ |\dddot{z} \in Z \end{array} \right\rangle$ *be a collection of TCHFNs in* L_{TCHFN} *and where* $\ddot{\omega} = (\ddot{\omega}_1, \ddot{\omega}_2, ..., \ddot{\omega}_n)^T$ *be the weighting vector of the TCHFEOWA operator, such that* $\ddot{\omega}_j \in [0, 1]$ *and* $\sum_{j=1}^{n} \ddot{\omega}_j = 1$*. Then*

$$TCHFEOWA(\ddot{A}_1, \ddot{A}_2, ..., \ddot{A}_n) \leq TCHFEOWA(\ddot{B}_1, \ddot{B}_2, ..., \ddot{B}_n) \quad (13)$$

where $(\ddot{A}_1, \ddot{A}_2, ..., \ddot{A}_n)$ *is any permutation of* $(\ddot{B}_1, \ddot{B}_2, ..., \ddot{B}_n)$*.*

Corollary 1. *Let* $\ddot{A} = \left\langle \begin{array}{c} [a^-_{\ddot{A}}(\dddot{z}), b^-_{\ddot{A}}(\dddot{z}), c^-_{\ddot{A}}(\dddot{z})], \\ [a^+_{\ddot{A}}(\dddot{z}), b^-_{\ddot{A}}(\dddot{z}), c^+_{\ddot{A}}(\dddot{z})], \\ [a_{\ddot{A}}(\dddot{z}), b_{\ddot{A}}(\dddot{z}), c_{\ddot{A}}(\dddot{z})] \\ |\dddot{z} \in Z \end{array} \right\rangle$ *be a collection of TCHFNs in* L_{TCHFN} *and where* $\ddot{\omega} = (\ddot{\omega}_1, \ddot{\omega}_2, ..., \ddot{\omega}_n)^T$ *is the weight vector of TCHFEOWA such that* $\ddot{\omega}_j \in [0, 1]$ *and* $\sum_{j=1}^{n} \ddot{\omega}_j = 1$*. Then*

$$TCHFEOWA(\ddot{A}_1, \ddot{A}_2, ..., \ddot{A}_n) \leq TCHFEOWA(\ddot{A}_1, \ddot{A}_2, ..., \ddot{A}_n). \quad (14)$$

5.3. Triangular Cubic Hesitant Fuzzy Einstein Hybrid Weighted Averaging Operator

Deliberate that the TCHFEWA operator weights individual the TCHFNs and the TCHFEOWA operator weights individual the ordered positions of the TCHFNs. In what follows, we elaborate triangular cubic hesitant fuzzy hybrid averaging (TCHFEHWA) operator, which weights together the given TCHFN and its well-ordered position.

Definition 16. *Let* $\ddot{A} = \left\langle \begin{array}{c} [a^-_{\ddot{A}}(\ddot{z}), b^-_{\ddot{A}}(\ddot{z}), c^-_{\ddot{A}}(\ddot{z})], \\ [a^+_{\ddot{A}}(\ddot{z}), b^-_{\ddot{A}}(\ddot{z}), c^+_{\ddot{A}}(\ddot{z})], \\ [a_{\ddot{A}}(\ddot{z}), b_{\ddot{A}}(\ddot{z}), c_{\ddot{A}}(\ddot{z})] \\ |\ddot{z} \in Z \end{array} \right\rangle$ *be a collection of TCHFNs in* L_{TCHFN} *and* $\ddot{\omega} = (\ddot{\omega}_1, \ddot{\omega}_2, ..., \ddot{\omega}_n)^T$ *is the weight vector of* $\ddot{A}_j (j = 1, 2, ..., n)$ *such that* $\ddot{\omega}_j \in [0, 1]$ *and* $\sum_{j=1}^{n} \ddot{\omega}_j = 1$. *Then triangular cubic hesitant fuzzy Einstein hybrid weighted averaging operator of dimension n is a mapping TCHFEHWA:* $L^n_{TCHFN} \to L_{TCHFN}$, *that has an associated vector* $w = (w_1, w_2, ..., w_n)^T$ *such that* $w_j \in [0, 1]$ *and* $\sum_{j=1}^{n} w_j = 1$. $TCHFEHWA(\ddot{A}_1, \ddot{A}_2, ..., \ddot{A}_n) = p_1 \ddot{A}_{\sigma(1)} + p_2 \ddot{A}_{\sigma(1)}, ..., p_n \ddot{A}_{\sigma(1)}$. *If* $p = \theta \ddot{\omega}_{\sigma(j)} + (1 - \theta) w_{\sigma(j)}$ *with a balancing coefficient* $\theta \in [0, 1]$, $(\sigma(1), \sigma(2), ..., \sigma(n))$ *is a permutation of* $(1, 2, ..., n)$ *such that* $\ddot{A}_{\sigma(j)} \leq \ddot{A}_{\sigma(j-1)}$ *for all* $j = 2, 3, ..., n$ *(i.e.,*$\ddot{A}_{\sigma(j)}$ *is the jth largest value in the collection* $(\ddot{A}_1, \ddot{A}_2, ..., \ddot{A}_n)$.

Theorem 3. *Let* $\ddot{A} = \left\langle \begin{array}{c} [a^-_{\ddot{A}}(\ddot{z}), b^-_{\ddot{A}}(\ddot{z}), c^-_{\ddot{A}}(\ddot{z})], \\ [a^+_{\ddot{A}}(\ddot{z}), b^-_{\ddot{A}}(\ddot{z}), c^+_{\ddot{A}}(\ddot{z})], \\ [a_{\ddot{A}}(\ddot{z}), b_{\ddot{A}}(\ddot{z}), c_{\ddot{A}}(\ddot{z})] \\ |\ddot{z} \in Z \end{array} \right\rangle$ *be a collection of TCHFNs in* L_{TCHFN} *and* $\ddot{\omega} = (\ddot{\omega}_1, \ddot{\omega}_2, ..., \ddot{\omega}_n)^T$ *is the weight vector of* $\ddot{A}_j (j = 1, 2, ..., n)$ *such that* $\ddot{\omega}_j \in [0, 1]$ *and* $\sum_{j=1}^{n} \ddot{\omega}_j = 1$. *Then their aggregated value by using the TCHFEHWA operator, which has an associated vector* $\varpi = (\varpi_1, \varpi_2, ..., \varpi_n)^T$ *is the weight vector of* $\ddot{A}_j (j = 1, 2, ..., n)$ *such that* $\varpi_j \in [0, 1]$ *and* $\sum_{j=1}^{n} \varpi_j = 1$, *is also an TCHFN and*

$TCHFEHWA(\ddot{A}_1, \ddot{A}_2, ..., \ddot{A}_n) =$

$$\left\langle \begin{array}{c} \bigcup\limits_{p^- \in a^-, q^- \in b^-, r^- \in c^-} \left[\frac{\prod_{j=1}^{n}[1+p^-_{\sigma(j)}]^{\varpi} - \prod_{j=1}^{n}[1-p^-_{\sigma(j)}]^{\varpi}}{\prod_{j=1}^{n}[1+p^-_{\sigma(j)}]^{\varpi} + \prod_{j=1}^{n}[1-p^-_{\sigma(j)}]^{\varpi}}, \frac{\prod_{j=1}^{n}[1+q^-_{\sigma(j)}]^{\varpi} - \prod_{j=1}^{n}[1-q^-_{\sigma(j)}]^{\varpi}}{\prod_{j=1}^{n}[1+q^-_{\sigma(j)}]^{\varpi} + \prod_{j=1}^{n}[1-q^-_{\sigma(j)}]^{\varpi}}, \right. \\ \left. \frac{\prod_{j=1}^{n}[1+r^-_{\sigma(j)}]^{\varpi} - \prod_{j=1}^{n}[1-r^-_{\sigma(j)}]^{\varpi}}{\prod_{j=1}^{n}[1+r^-_{\sigma(j)}]^{\varpi} + \prod_{j=1}^{n}[1-r^-_{\sigma(j)}]^{\varpi}} \right], \bigcup\limits_{p^+ \in a^+, q^+ \in b^+, r^+ \in c^+} \left[\frac{\prod_{j=1}^{n}[1+p^+_{\sigma(j)}]^{\varpi} - \prod_{j=1}^{n}[1-p^+_{\sigma(j)}]^{\varpi}}{\prod_{j=1}^{n}[1+p^+_{\sigma(j)}]^{\varpi} + \prod_{j=1}^{n}[1-p^+_{\sigma(j)}]^{\varpi}}, \right. \\ \left. \frac{\prod_{j=1}^{n}[1+q^+_{\sigma(j)}]^{\varpi} - \prod_{j=1}^{n}[1-q^+_{\sigma(j)}]^{\varpi}}{\prod_{j=1}^{n}[1+q^+_{\sigma(j)}]^{\varpi} + \prod_{j=1}^{n}[1-q^+_{\sigma(j)}]^{\varpi}}, \frac{\prod_{j=1}^{n}[1+r^+_{\sigma(j)}]^{\varpi} - \prod_{j=1}^{n}[1-r^+_{\sigma(j)}]^{\varpi}}{\prod_{j=1}^{n}[1+r^+_{\sigma(j)}]^{\varpi} + \prod_{j=1}^{n}[1-r^+_{\sigma(j)}]^{\varpi}} \right. \\ \bigcup\limits_{p \in a, q \in b, r \in c} \left[\begin{array}{c} \frac{2\prod_{j=1}^{n}[p_{\sigma(j)}]^{\varpi}}{\prod_{j=1}^{n}[(2-p_{\sigma(j)}]^{\varpi} + \prod_{j=1}^{n}[p_{\sigma(j)}]^{\varpi}}, \frac{2\prod_{j=1}^{n}[q_{\sigma(j)}]^{\varpi}}{\prod_{j=1}^{n}[(2-q_{\sigma(j)}]^{\varpi} + \prod_{j=1}^{n}[q_{\sigma(j)}]^{\varpi}}, \\ \frac{2\prod_{j=1}^{n}[r_{\sigma(j)}]^{\varpi}}{\prod_{j=1}^{n}[(2-r_{\sigma(j)}]^{\varpi} + \prod_{j=1}^{n}[r_{\sigma(j)}]^{\varpi}} \end{array} \right] \end{array} \right\rangle. \quad (15)$$

If $p = \theta \ddot{\omega}_{\sigma(j)} + (1 - \theta) \varpi_{\sigma(j)}$ *with a balancing coefficient* $\theta \in [0, 1]$, $(\sigma(1), \sigma(2), ..., \sigma(n))$ *is a permutation of* $(1, 2, ..., n)$ *such that* $\ddot{A}_{\sigma(j)} \leq \ddot{A}_{\sigma(j-1)}$ *for all* $j = 2, 3, ..., n$ *(i.e.,* $\ddot{A}_{\sigma(j)}$ *is the jth largest value in the collection* $(\ddot{A}_1, \ddot{A}_2, ..., \ddot{A}_n)$.

Example 6. *Let* $\ddot{A}_1 = \left\langle \begin{array}{c} [0.8, 0.10, 0.12], \\ [0.10, 0.12, 0.14], \\ [0.9, 0.11, 0.13] \end{array} \right\rangle, \ddot{A}_2 = \left\langle \begin{array}{c} [0.7, 0.9, 0.11], \\ [0.9, 0.11, 0.13], \\ [0.8, 0.10, 0.12] \end{array} \right\rangle$ *and* $\ddot{A}_3 = \left\langle \begin{array}{c} [0.202, 0.204, 0.206], \\ [0.204, 0.206, 0.208], \\ [0.203, 0.205, 0.207] \end{array} \right\rangle$ *be a collection of TCHFNs in* L_{TCHFN} *and* $\ddot{\omega} = (0.4, 0.4, 0.2)^T$ *is the weight vector of* $\ddot{A}_j (j = 1, 2, ..., n)$ *such that* $\ddot{\omega}_j \in [0, 1]$ *and* $\sum_{j=1}^{n} \ddot{\omega}_j = 1$. *Then their aggregated value by using the TCHFEHWA operator is also a TCHFN and* $TCHFEHWA(\ddot{A}_1, \ddot{A}_2, ..., \ddot{A}_n) =$

$$\left\langle \begin{array}{c} \left[\begin{array}{c} \frac{[1+0.8]^{0.4}[1+0.7]^{0.4}[1+0.202]^{0.4}-[1-0.8]^{0.4}[1-0.7]^{0.4}[1-0.202]^{0.4}}{[1+0.8]^{0.4}[1+0.7]^{0.4}[1+0.202]^{0.4}+[1-0.8]^{0.4}[1-0.7]^{0.4}[1-0.202]^{0.4}}, \\ \frac{[1+0.10]^{0.4}[1+0.9]^{0.4}[1+0.204]^{0.4}-[1-0.10]^{0.4}[1-0.9]^{0.4}[1-0.204]^{0.4}}{[1+0.10]^{0.4}[1+0.9]^{0.4}[1+0.204]^{0.4}+[1-0.10]^{0.4}[1-0.9]^{0.4}[1-0.204]^{0.4}}, \\ \frac{[1+0.12]^{0.4}[1+0.11]^{0.4}[1+0.206]^{0.4}-[1-0.12]^{0.4}[1-0.11]^{0.4}[1-0.206]^{0.4}}{[1+0.12]^{0.4}[1+0.11]^{0.4}[1+0.206]^{0.4}+[1-0.12]^{0.4}[1-0.11]^{0.4}[1-0.206]^{0.4}}, \end{array} \right], \\ \left[\begin{array}{c} \frac{[1+0.10]^{0.4}[1+0.9]^{0.4}[1+0.204]^{0.4}-[1-0.10]^{0.4}[1-0.9]^{0.4}[1-0.204]^{0.4}}{[1+0.10]^{0.4}[1+0.9]^{0.4}[1+0.204]^{0.4}+[1-0.10]^{0.4}[1-0.9]^{0.4}[1-0.204]^{0.4}}, \\ \frac{[1+0.12]^{0.4}[1+0.11]^{0.4}[1+0.206]^{0.4}-[1-0.12]^{0.4}[1-0.11]^{0.4}[1-0.206]^{0.4}}{[1+0.12]^{0.4}[1+0.11]^{0.4}[1+0.206]^{0.4}+[1-0.12]^{0.4}[1-0.11]^{0.4}[1-0.206]^{0.4}}, \\ \frac{[1+0.14]^{0.4}[1+0.13]^{0.4}[1+0.208]^{0.4}-[1-0.14]^{0.4}[1-0.13]^{0.4}[1-0.208]^{0.4}}{[1+0.14]^{0.4}[1+0.13]^{0.4}[1+0.208]^{0.4}+[1-0.14]^{0.4}[1-0.13]^{0.4}[1-0.208]^{0.4}} \end{array} \right], \\ \left[\begin{array}{c} \frac{2[(0.9)^{0.2}(0.8)^{0.2}(0.203)^{0.2}]}{[(2-0.9)^{0.2}(2-0.8)^{0.2}(2-0.203)^{0.2}]+[(0.9)^{0.2}(0.8)^{0.2}(0.203)^{0.2}]}, \\ \frac{2[(0.11)^{0.2}(0.10)^{0.2}(0.205)^{0.2}]}{[(2-0.11)^{0.2}(2-0.10)^{0.2}(2-0.205)^{0.2}]+[(0.11)^{0.2}(0.10)^{0.2}(0.205)^{0.2}]}, \\ \frac{2[(0.13)^{0.2}(0.12)^{0.2}(0.207)^{0.2}]}{[(2-0.13)^{0.2}(2-0.12)^{0.2}(2-0.207)^{0.2}]+[(0.13)^{0.2}(0.12)^{0.2}(0.207)^{0.2}]} \end{array} \right] \end{array} \right\rangle$$

$$= \left\langle \begin{array}{c} [0.7005, 0.6117, 0.1742] \\ [0.6117, 0.1742, 0.1907], \\ [0.7283, 0.3383, 0.3602] \end{array} \right\rangle.$$

Proposition 4. *The TCHFEHWA operator (4) satisfy the following properties:*

(1) The weighting vector $p = (p_1, p_2, ..., p_n)^T$ satisfies $\sum_{j=1}^{n} p_j = 1$.

(2) The TCHFEHWA operator (4) is a TCHFEOWA operator with weights p.

(3) If $\theta = 1$ then the TCHFEHWA operator (4) reduces to a TCHFEWA operator (1) with a weighting vector $\ddot{\omega}$.

(4) If $\theta = 0$ then the TCHFEHWA operator (4) reduces to a TCHFEOWA operator (3) with a weighting vector w.

Theorem 4. *The TCHFEHWA operator (4) generalizes both the TCHFEWA operator (1) and the TCHFEOWA operator (3).*

6. An Approach to Multiple Attribute Decision Making with Triangular Cubic Hesitant Fuzzy Information

In this section, we shall utilize the triangular cubic hesitant fuzzy operator to multiple attribute decision making with linguistic information.

There are different factors affecting the chain of command in a program, such as the geographical location of projects, the capability of the engineers, staff, and workers, and the similarity of projects comprising the program. Other factors of much importance in constructing a program are the complexity of the projects, the level of the design, the availability of resources, and the technical know-how. In terms of program organization, delegation is very important to keep a tight control over a large number of projects comprising the program.

Alcantud et al. [37] developed two real implementations: (i) new metarankings of world academic institutions that build on real data from three reputed agencies, and (ii) a new procedure for improving teaching performance assessments which we illustrate with real data collected by ourselves.

Author(s)	Tool(s)/method(s)
Based on HFEs scores	
Xia and Xu [11]	Generalized hesitant fuzzy weighted averaging operator (GHFWA), Generalized hesitant fuzzy weighted geometric operator (GHFWG)
Farhadinia [38]	Series of score functions for hesitant fuzzy sets
Xia, Xu and Zhu [39]	Weighted hesitant fuzzy geometric Bonferroni mean (WHFGBM) Weighted hesitant fuzzy Choquet geometric Bonferroni mean (WHFCGBM)
Wei [40]	Hesitant fuzzy prioritized operators

Based on distance measures	
Xu and Xia [41]	Distance measures of hesitant fuzzy elements
Xu and Xia [42]	Generalized hesitant weighted distance
Li, Zeng and Li [43]	Distance and similarity measures considering hesitancy degree

A multiple attribute decision making (MADM) problem is to find the best compromise solution from all feasible alternatives assessed on multiple attributes. Let $\dddot{z} = \{\dddot{z}_1, \dddot{z}_2, ..., \dddot{z}_n\}$ be a set of alternatives and $G = \{g_1, g_2, ..., g_n\}$ be the set of attributes. Suppose the rating of alternatives $\dddot{z}_i$ $(i = 1, 2, ..., n)$ on attributes g_j $(j = 1, 2, ..., m)$ given by decision maker are TCHFNs in L_{TCHFN}: $\ddot{A} = \left\langle \begin{matrix} [a^-_{\ddot{A}}(\dddot{z}), b^-_{\ddot{A}}(\dddot{z}), c^-_{\ddot{A}}(\dddot{z})], \\ [a^+_{\ddot{A}}(\dddot{z}), b^-_{\ddot{A}}(\dddot{z}), c^+_{\ddot{A}}(\dddot{z})], \\ [a_{\ddot{A}}(\dddot{z}), b_{\ddot{A}}(\dddot{z}), c_{\ddot{A}}(\dddot{z})] \\ |\dddot{z} \in Z \end{matrix} \right\rangle$,where $\left\langle \begin{matrix} [a^-_{\ddot{A}}(\dddot{z}), b^-_{\ddot{A}}(\dddot{z}), c^-_{\ddot{A}}(\dddot{z})], \\ [a^+_{\ddot{A}}(\dddot{z}), b^+_{\ddot{A}}(\dddot{z}), c^+_{\ddot{A}}(\dddot{z})], \end{matrix} \right\rangle$ indicates the interval value triangular hesitant fuzzy number that the alternative $\dddot{z}_i$ satisfies the attribute g_j and $\langle [p_{\ddot{A}}(\dddot{z}), q_{\ddot{A}}(\dddot{z}), r_{\ddot{A}}(\dddot{z})] \rangle$ indicates the triangular hesitant fuzzy number that the alternative $\dddot{z}_i$ does not satisfy the attribute g_j, such that, an MADM problem can be concisely expressed in the triangular cubic hesitant fuzzy decision matrix $\ddot{D} = (\ddot{A}_{ij})_{m \times n} = \left\langle \begin{matrix} [a^-_{\ddot{A}}(\dddot{z}), b^-_{\ddot{A}}(\dddot{z}), c^-_{\ddot{A}}(\dddot{z})], \\ [a^+_{\ddot{A}}(\dddot{z}), b^-_{\ddot{A}}(\dddot{z}), c^+_{\ddot{A}}(\dddot{z})], \\ [a_{\ddot{A}}(\dddot{z}), b_{\ddot{A}}(\dddot{z}), c_{\ddot{A}}(\dddot{z})] \\ |\dddot{z} \in Z \end{matrix} \right\rangle$.

Then, we utilize the triangular cubic hesitant fuzzy number to develop an approach to multiple attribute decision-making problems with linguistic information, which can be described as following:

Step 1: Define the triangular cubic hesitant fuzzy decision matrix

Step 2: Calculate the normalized triangular cubic hesitant fuzzy decision matrix. The following normalization formula $E^k = (\ddot{\beta}^k_{ij})_{n \times 1}$,

Accordingly, we attain the normalized triangular cubic hesitant fuzzy decision matrix $\dddot{E} = (\ddot{\beta}_{ij})_{n \times m}$.

Step 3: Utilize the TCHFEHWA operator to aggregate all the rating values $\ddot{\beta}_{ij}$ $(j = 1, 2, ..., m)$ of the ith line and get the overall rating value $\ddot{\beta}_{ij}$ corresponding to the alternative $\dddot{z}_i$ $(i = 1, 2, ..., n)$, i.e., $\ddot{\beta}_{ij}$ =TCHFHWA$(\ddot{\beta}_{i1}, \ddot{\beta}_{i2}, ..., \ddot{\beta}_{im})$ =

$$
\begin{aligned}
\Bigg\langle & \bigcup_{p^- \in a^-, q^- \in b^-, r^- \in c^-} \left[\begin{array}{c} \dfrac{\prod_{i=1}^{n}[1+p_i^-(\ddot{z})]^{\ddot{\omega}} - \prod_{i=1}^{n}[1-p_i^-(\ddot{z})]^{\ddot{\omega}}}{\prod_{i=1}^{n}[1+p_i^-(\ddot{z})]^{\ddot{\omega}} + \prod_{i=1}^{n}[1-p_i^-(\ddot{z})]^{\ddot{\omega}}}, \dfrac{\prod_{i=1}^{n}[1+q_i^-(\ddot{z})]^{\ddot{\omega}} - \prod_{i=1}^{n}[1-q_i^-(\ddot{z})]^{\ddot{\omega}}}{\prod_{i=1}^{n}[1+q_i^-(\ddot{z})]^{\ddot{\omega}} + \prod_{i=1}^{n}[1-q_i^-(\ddot{z})]^{\ddot{\omega}}}, \\ \dfrac{\prod_{i=1}^{n}[1+r_i^-(\ddot{z})]^{\ddot{\omega}} - \prod_{i=1}^{n}[1-r_i^-(\ddot{z})]^{\ddot{\omega}}}{\prod_{i=1}^{n}[1+r_i^-(\ddot{z})]^{\ddot{\omega}} + \prod_{i=1}^{n}[1-r_i^-(\ddot{z})]^{\ddot{\omega}}} \end{array} \right], \\
& \bigcup_{p^+ \in a^+, q^+ \in b^+, r^+ \in c^+} \left[\begin{array}{c} \dfrac{\prod_{i=1}^{n}[1+p_i^+(\ddot{z})]^{\ddot{\omega}} - \prod_{i=1}^{n}[1-p_i^+(\ddot{z})]^{\ddot{\omega}}}{\prod_{i=1}^{n}[1+p_i^+(\ddot{z})]^{\ddot{\omega}} + \prod_{i=1}^{n}[1-p_i^+(\ddot{z})]^{\ddot{\omega}}}, \dfrac{\prod_{i=1}^{n}[1+q_i^+(\ddot{z})]^{\ddot{\omega}} - \prod_{i=1}^{n}[1-q_i^+(\ddot{z})]^{\ddot{\omega}}}{\prod_{i=1}^{n}[1+q_i^+(\ddot{z})]^{\ddot{\omega}} + \prod_{i=1}^{n}[1-q_i^+(\ddot{z})]^{\ddot{\omega}}}, \\ \dfrac{\prod_{i=1}^{n}[1+r_i^+(\ddot{z})]^{\ddot{\omega}} - \prod_{i=1}^{n}[1-r_i^+(\ddot{z})]^{\ddot{\omega}}}{\prod_{i=1}^{n}[1+r_i^+(\ddot{z})]^{\ddot{\omega}} + \prod_{i=1}^{n}[1-r_i^+(\ddot{z})]^{\ddot{\omega}}} \end{array} \right], \Bigg\rangle \\
& \bigcup_{p \in a, q \in b, r \in c} \left[\begin{array}{c} \dfrac{2\prod_{i=1}^{n}[p_i(\ddot{z})]^{\ddot{\omega}}}{\prod_{i=1}^{n}[(2-p_i(\ddot{z})]^{\ddot{\omega}} + \prod_{i=1}^{n}[p_i(\ddot{z})]^{\ddot{\omega}}}, \dfrac{2\prod_{i=1}^{n}[q_i(\ddot{z})]^{\ddot{\omega}}}{\prod_{i=1}^{n}[(2-q_i(\ddot{z})]^{\ddot{\omega}} + \prod_{i=1}^{n}[q_i(\ddot{z})]^{\ddot{\omega}}}, \\ \dfrac{2\prod_{i=1}^{n}[r_i(\ddot{z})]^{\ddot{\omega}}}{\prod_{i=1}^{n}[(2-r_i(\ddot{z})]^{\ddot{\omega}} + \prod_{i=1}^{n}[r_i(\ddot{z})]^{\ddot{\omega}}} \end{array} \right]
\end{aligned} \tag{16}
$$

where $\ddot{\omega} = (\ddot{\omega}_1, \ddot{\omega}_2, ..., \ddot{\omega}_n)^T$ is the attribute weight vector of g_j $(j = 1, 2, ..., m)$ such that $\ddot{\omega}_j \in [0, 1]$, $(j = 1, 2, ..., m)$ and $\sum_{j=1}^{n} \ddot{\omega}_j = 1$. $w = (w_1, w_2, ..., w_n)^T$ is the associated vector of the TCHFEHWA operator, such that $w_j \in [0, 1], j = 1, 2, ..., n$ and $\sum_{j=1}^{n} w_j = 1$.

$\ddot{\beta}_{ij}$ =TCHFHWA$(\ddot{\beta}_{j1}, \ddot{\beta}_{j2}, ..., \ddot{\beta}_{jm})$ =

$$\left\langle \begin{array}{c} \bigcup_{p^-\in a^-,q^-\in b^-,r^-\in c^-} \left[\frac{\prod_{j=1}^{n}[1+p^-_{\sigma(j)}]^{\varpi}-\prod_{j=1}^{n}[1-p^-_{\sigma(j)}]^{\varpi}}{\prod_{j=1}^{n}[1+p^-_{\sigma(j)}]^{\varpi}+\prod_{j=1}^{n}[1-p^-_{\sigma(j)}]^{\varpi}}, \frac{\prod_{j=1}^{n}[1+q^-_{\sigma(j)}]^{\varpi}-\prod_{j=1}^{n}[1-q^-_{\sigma(j)}]^{\varpi}}{\prod_{j=1}^{n}[1+q^-_{\sigma(j)}]^{\varpi}+\prod_{j=1}^{n}[1-q^-_{\sigma(j)}]^{\varpi}}, \right. \\ \left. \frac{\prod_{j=1}^{n}[1+r^-_{\sigma(j)}]^{\varpi}-\prod_{j=1}^{n}[1-r^-_{\sigma(j)}]^{\varpi}}{\prod_{j=1}^{n}[1+r^-_{\sigma(j)}]^{\varpi}+\prod_{j=1}^{n}[1-r^-_{\sigma(j)}]^{\varpi}}\right], \bigcup_{p^+\in a^+,q^+\in b^+,r^+\in c^+} \left[\frac{\prod_{j=1}^{n}[1+p^+_{\sigma(j)}]^{\varpi}-\prod_{j=1}^{n}[1-p^+_{\sigma(j)}]^{\varpi}}{\prod_{j=1}^{n}[1+p^+_{\sigma(j)}]^{\varpi}+\prod_{j=1}^{n}[1-p^+_{\sigma(j)}]^{\varpi}}, \right. \\ \frac{\prod_{j=1}^{n}[1+q^+_{\sigma(j)}]^{\varpi}-\prod_{j=1}^{n}[1-q^+_{\sigma(j)}]^{\varpi}}{\prod_{j=1}^{n}[1+q^+_{\sigma(j)}]^{\varpi}+\prod_{j=1}^{n}[1-q^+_{\sigma(j)}]^{\varpi}}, \frac{\prod_{j=1}^{n}[1+r^+_{\sigma(j)}]^{\varpi}-\prod_{j=1}^{n}[1-r^+_{\sigma(j)}]^{\varpi}}{\prod_{j=1}^{n}[1+r^+_{\sigma(j)}]^{\varpi}+\prod_{j=1}^{n}[1-r^+_{\sigma(j)}]^{\varpi}} \\ \bigcup_{p\in a,q\in b,r\in c} \left[\begin{array}{c} \frac{2\prod_{j=1}^{n}[p_{\sigma(j)}]^{\varpi}}{\prod_{j=1}^{n}[(2-p_{\sigma(j)}]^{\varpi}+\prod_{j=1}^{n}[p_{\sigma(j)}]^{\varpi}}, \frac{2\prod_{j=1}^{n}[q_{\sigma(j)}]^{\varpi}}{\prod_{j=1}^{n}[(2-q_{\sigma(j)}]^{\varpi}+\prod_{j=1}^{n}[q_{\sigma(j)}]^{\varpi}}, \\ \frac{2\prod_{j=1}^{n}[r_{\sigma(j)}]^{\varpi}}{\prod_{j=1}^{n}[(2-r_{\sigma(j)}]^{\varpi}+\prod_{j=1}^{n}[r_{\sigma(j)}]^{\varpi}} \end{array} \right] \end{array} \right\rangle$$

Step 4: Calculate the scores function to aggregate the value of each alternative $\ddot{\beta}_i$ $(i = 1, 2, ..., n)$, then we have must find out the score value of the aggregate value of each alternative.

Step 5: Calculate all the score values of the alternatives in the form of descending order and select the best alternative $\ddot{\beta}_i$ which is the highest degree of the score value.

Next, we shall apply the TCHFEHWA operator to deal with the MADM problem, which involves the following algorithm steps.

Algorithm. A MAGDM approach based on the triangular cubic hesitant fuzzy number

Output: The Ranking of all the alternatives

Step 1: Define the triangular cubic hesitant fuzzy matrix

Step 2: Calculate the normalized triangular cubic hesitant fuzzy decision matrix to obtain the integrated information by aggregating individual information for all DMs.

Step 3: Utilize the TCHFEHWA operator to aggregate all the rating values $\ddot{\beta}_{ij}$ $(j = 1, 2, ..., m)$ of the ith line and get the overall rating value $\ddot{\beta}_{ij}$ corresponding to the alternative $\ddot{z}_i$ $(i = 1, 2, ..., n)$.

Step 4: Calculate the scores function based on the (Eq. 4) of each alternative $\ddot{\beta}_i$ $(i = 1, 2, ..., n)$, then we have must find out the score value of the aggregate value of each alternative.

Step 5: Rank all the alternatives in the form of descending order and select the best alternative $\ddot{\beta}_i$ which is the highest degree of the score value.

Figure 2 is a flow chart describing the steps of the algorithm.

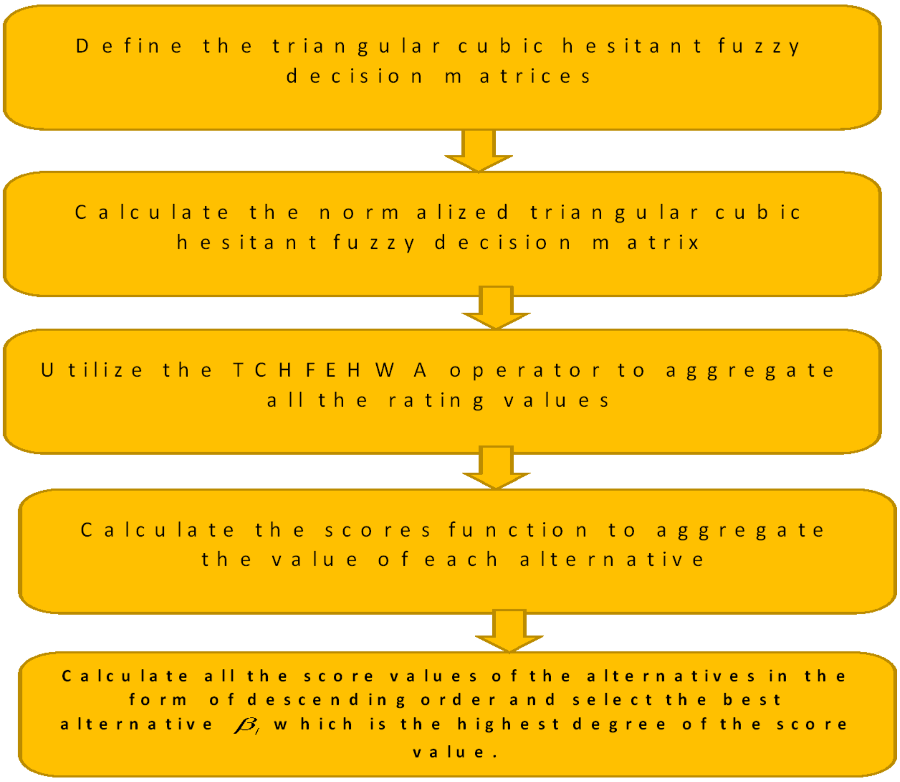

Figure 2. Flowchart of the triangular cubic hesitant fuzzy information.

7. Illustrative Example

Regular exposures and disasters are yonder gentleman's controller. Whatever can take the defensive events before its incidence or can ensure less injury by expressing an efficient disaster administration strategy so as to evade loss of humanoid life and additional essential resources is unique. Consider Pakistan where deluges hit numerous conditions and principal to vast loss of life and stuff. Directing on the disaster management, related particularly to four Indian States, taken as alternatives $A_i, i = 1,2,3$, viz., (Lahore, Mansehra, and Islamabad) which remained largely devastated during first half of the year 2018, suppose the Administration of Pakistan is trying to make the best decision on allocating reserves to these three conditions. When the whole situation is examined for major areas of fund allocation, it was found that money should be allocated in such a way that three major factors represented by $C_j, j = 1,2,3$ namely: "Food Shortage", "Amount of persons saved" and "Absence of substructure Rebuilding amenities" needed to be coped up. Suppose, different factors are prioritized state-wise in accordance with the weight vector $\ddot{\omega} = (0.25, 0.50, 0.25)$. Thus, the aim of the problem is to determine the order in which states should be allocated the relief fund.

It ends with a discussion on how to aggregate information that is expressed in terms of linguistic variables. This is a crucial step of the process of decision making, as we need to be able to aggregate the various bits of "linguistic" information (that is expressed in linguistic format) on the issue at hand under the condition of fulfilling certain criteria. The following section then explicates how to model the decision-making process in various succeeding stages. One way to avoid the above problems associated with using membership functions is to use linguistic variables instead of fuzzy numbers, that is, to use variables whose values are not numbers but words or sentences in a natural or artificial language. Since words are in general less precise than numbers, the concept of a linguistic variable serves the purpose of providing a means to approximately characterize phenomena that are complex or ill-defined and hence not amenable to a crisp description in quantitative terms. Of course, in order for linguistic variables to be useful tools of analysis, one ought to be able to manipulate them through various operations. Generally, manipulating linguistic variables can be done in one of two ways. First, one can use linguistic variables by associating membership functions to them and then manipulating the membership functions, since linguistic variables are but a special category of fuzzy sets. Second, one can directly symbolically manipulate the linguistic variables themselves.

In the former case, the manipulation of linguistic variables is done by using the extension principle, which in essence allows the extension of usual algebraic and arithmetic manipulations to fuzzy sets. However, using extended arithmetic operations to handle fuzzy numbers increases the vagueness of results at every step of the calculation, and the shape of the membership functions is not preserved. The final results of this kind of method are fuzzy sets with membership functions that are often hard to interpret, thereby making the corresponding final linguistic variables outside the original set of linguistic terms that one starts from. In the second approach, based on using algebraic operations on the linguistic variables themselves, there is no need to manipulate any sort of membership functions at all. In the following, a summary of the contours of the symbolic approach is presented. The approach is introduced and illustrated by discussing a hypothetical situation in which state leaders have to make choices on how to build a security alignment. This is a fictitious example, hence there is no attempt to justify the posited underlying substantive assumptions for why certain variables are chosen.

The linguistic fuzzy analysis is mathematically rigorous, that is, it is built on a coherent mathematical framework of definitions, theorems, lemmas, and the like. A key idea behind this approach is to directly manipulate the available linguistic information and knowledge. In this pursuit, the first step is to choose the basic ingredients that are used in the symbolic manipulation. This means that the analyst has to choose a context-dependent linguistic term set to describe vague or imprecise information. For example, linguistic terms set for the linguistic variable Importance denoted by H(Importance), can be defined as H(Importance) = { important, not important, very important, not very important, fairly important}. Normally, in any one situation of a decision under multiple criteria, one is faced with a number of linguistic variables, not just one. It is easier to opt for the same linguistic terms set to describe a variation of the various linguistic concepts in the problem at hand, although this is not a requirement. The elements in the set will determine the level of distinction among different parts of the available information, called the granularity of uncertainty. Each value of the linguistic variables is characterized by a syntactic label and a semantic value or meaning. The label is a word or a sentence belonging to the chosen linguistic terms set. The meaning of the label is a fuzzy subset in a universe of discourse (a finite set of words and phrases). The choice of the linguistic terms set with its semantic is thus the first step of any linguistic approach to solving a problem.

Assume that the decision-maker procedures the linguistic terms to the evaluating estimates of the alternatives with reverence to different attributes individually as shown in Table 1. The relationship between the linguistic terms and the comparing TCHFNs in L_{TCHFN} as appeared in Tables 2 and 3.

Table 1. Linguistic terms of triangular cubic hesitant fuzzy values.

Linguistic Terms	TCHFVs
Very High (VH)	$\langle [0.6, 0.8, 0.10], [0.8, 0.10, 0.12], [0.7, 0.9, 0.11] \rangle$
Very Low (VL)	$\langle [0.3, 0.5, 0.7], [0.5, 0.7, 0.9], [0.4, 0.6, 0.8] \rangle$
Low (L)	$\langle [0.10, 0.12, 0.14], [0.12, 0.14, 0.16], [0.11, 0.13, 0.15] \rangle$
Medium Low (ML)	$\langle [0.2, 0.4, 0.6], [0.4, 0.6, 0.8], [0.3, 0.5, 0.7] \rangle$
Medium (M)	$\langle [0.4, 0.6, 0.8], [0.6, 0.8, 0.10], [0.5, 0.7, 0.9] \rangle$
Medium High (MH)	$\langle [0.7, 0.9, 0.11], [0.9, 0.11, 0.13], [0.8, 0.10, 0.12] \rangle$

Step 1: Define triangular cubic hesitant fuzzy Decision matrix Tables 2 and 3.

Table 2. Triangular cubic hesitant fuzzy Decision matrix.

$\ddot{D}_1 =$	$\ddot{c}_1$	$\ddot{c}_2$	$\ddot{c}_3$
$\ddot{A}_1$	$\{\langle[0.6, 0.8, 0.10], [0.8, 0.10, 0.12], [0.7, 0.9, 0.11]\rangle\}$	$\{\langle[0.3, 0.5, 0.7], [0.5, 0.7, 0.9], [0.4, 0.6, 0.8]\rangle\}$	$\{\langle[0.10, 0.12, 0.14], [0.12, 0.14, 0.16], [0.11, 0.13, 0.15]\rangle\}$
$\ddot{A}_2$	$\{\langle[0.3, 0.5, 0.7], [0.5, 0.7, 0.9], [0.4, 0.6, 0.8]\rangle\}$	$\{\langle[0.10, 0.12, 0.14], [0.12, 0.14, 0.16], [0.11, 0.13, 0.15]\rangle\}$	$\{\langle[0.6, 0.8, 0.10], [0.8, 0.10, 0.12], [0.7, 0.9, 0.11]\rangle\}$
$\ddot{A}_3$	$\{\langle[0.10, 0.12, 0.14], [0.12, 0.14, 0.16], [0.11, 0.13, 0.15]\rangle\}$	$\{\langle[0.6, 0.8, 0.10], [0.8, 0.10, 0.12], [0.7, 0.9, 0.11]\rangle\}$	$\{\langle[0.3, 0.5, 0.7], [0.5, 0.7, 0.9], [0.4, 0.6, 0.8]\rangle\}$

Table 3. Triangular cubic hesitant fuzzy Decision matrix.

$\ddot{D}_2 =$	$\ddot{c}_1$	$\ddot{c}_2$	$\ddot{c}_3$
$\ddot{A}_1$	$\{\langle[0.2, 0.4, 0.6], [0.4, 0.6, 0.8], [0.3, 0.5, 0.7]\rangle\}$	$\{\langle[0.4, 0.6, 0.8], [0.6, 0.8, 0.10], [0.5, 0.7, 0.9]\rangle\}$	$\{\langle[0.7, 0.9, 0.11], [0.9, 0.11, 0.13], [0.8, 0.10, 0.12]\rangle\}$
$\ddot{A}_2$	$\{\langle[0.4, 0.6, 0.8], [0.6, 0.8, 0.10], [0.5, 0.7, 0.9]\rangle\}$	$\{\langle[0.7, 0.9, 0.11], [0.9, 0.11, 0.13], [0.8, 0.10, 0.12]\rangle\}$	$\{\langle[0.2, 0.4, 0.6], [0.4, 0.6, 0.8], [0.3, 0.5, 0.7]\rangle\}$
$\ddot{A}_3$	$\{\langle[0.7, 0.9, 0.11], [0.9, 0.11, 0.13], [0.8, 0.10, 0.12]\rangle\}$	$\{\langle[0.2, 0.4, 0.6], [0.4, 0.6, 0.8], [0.3, 0.5, 0.7]\rangle\}$	$\{\langle[0.4, 0.6, 0.8], [0.6, 0.8, 0.10], [0.5, 0.7, 0.9]\rangle\}$

Step 2: Get the normalized triangular cubic hesitant fuzzy decision matrix. The following normalization formula $E^k = (\ddot{\beta}^k_{ij})_{n\times 1}$. The normalized decision matrix is presented in Tables 4 and 5.

Table 4. Normalized decision matrix.

$\ddot{D}_1 =$	$\ddot{c}_1$	$\ddot{c}_2$	$\ddot{c}_3$
$\ddot{A}_1$	$\{\langle[0.4, 0.2, 0.9], [0.2, 0.9, 0.88], [0.3, 0.1, 0.89]\rangle\}$	$\{\langle[0.7, 0.5, 0.3], [0.5, 0.3, 0.1], [0.6, 0.4, 0.2]\rangle\}$	$\{\langle[0.9, 0.88, 0.86], [0.88, 0.86, 0.84], [0.89, 0.87, 0.85]\rangle\}$
$\ddot{A}_2$	$\{\langle[0.7, 0.5, 0.3], [0.5, 0.3, 0.1], [0.6, 0.4, 0.2]\rangle\}$	$\{\langle[0.9, 0.88, 0.86], [0.88, 0.86, 0.84], [0.89, 0.87, 0.85]\rangle\}$	$\{\langle[0.4, 0.2, 0.9], [0.2, 0.9, 0.88], [0.3, 0.1, 0.89]\rangle\}$
$\ddot{A}_3$	$\{\langle[0.9, 0.88, 0.86], [0.88, 0.86, 0.84], [0.89, 0.87, 0.85]\rangle\}$	$\{\langle[0.4, 0.2, 0.9], [0.2, 0.9, 0.88], [0.3, 0.1, 0.89]\rangle\}$	$\{\langle[0.7, 0.5, 0.3], [0.5, 0.3, 0.1], [0.6, 0.4, 0.2]\rangle\}$

Table 5. Normalized decision matrix table.

$\ddot{D}_2 =$	$\ddot{c}_1$	$\ddot{c}_2$	$\ddot{c}_3$
$\ddot{A}_1$	$\{\langle[0.8, 0.6, 0.4], [0.6, 0.4, 0.2], [0.7, 0.5, 0.3]\rangle\}$	$\{\langle[0.6, 0.4, 0.2], [0.4, 0.2, 0.9], [0.5, 0.3, 0.1]\rangle\}$	$\{\langle[0.3, 0.1, 0.89], [0.1, 0.89, 0.87], [0.2, 0.9, 0.88]\rangle\}$
$\ddot{A}_2$	$\{\langle[0.6, 0.4, 0.2], [0.4, 0.2, 0.9], [0.5, 0.3, 0.1]\rangle\}$	$\{\langle[0.3, 0.1, 0.89], [0.1, 0.89, 0.87], [0.2, 0.9, 0.88]\rangle\}$	$\{\langle[0.8, 0.6, 0.4], [0.6, 0.4, 0.2], [0.7, 0.5, 0.3]\rangle\}$
$\ddot{A}_3$	$\{\langle[0.3, 0.1, 0.89], [0.1, 0.89, 0.87], [0.2, 0.9, 0.88]\rangle\}$	$\{\langle[0.8, 0.6, 0.4], [0.6, 0.4, 0.2], [0.7, 0.5, 0.3]\rangle\}$	$\{\langle[0.6, 0.4, 0.2], [0.4, 0.2, 0.9], [0.5, 0.3, 0.1]\rangle\}$

Hence, we acquire the standardized triangular cubic hesitant fuzzy decision matrix $E = (\ddot{\beta}_{ij})_{n\times m}$.

Step 3: Utilize the TCHFEHWA operator rating values in Table 6 and get the general rating value $\ddot{\beta}_{ij}$ as displayed in Table 7 to be compared to the alternative $\ddot{z}_i, w = (0.25, 0.50, 0.25)$.

Table 6. Rating value of the TCHFEHWA operator.

	$\ddot{c}_1$	$\ddot{c}_2$	$\ddot{c}_3$
$\ddot{A}_1$	$\{\langle[0.3631, 0.2203, 0.4413], [0.2203, 0.4413, 0.3753], [0.7139, 0.5336, 0.7603]\rangle\}$	$\{\langle[0.3716, 0.2385, 0.1273], [0.2385, 0.1273, 0.3741], [0.7614, 0.6285, 0.4332]\rangle\}$	$\{\langle[0.4181, 0.3531, 0.5907], [0.3531, 0.5907, 0.5638], [0.7065, 0.9422, 0.9321]\rangle\}$
$\ddot{A}_2$	$\{\langle[0.6528, 0.4514, 0.2506], [0.4514, 0.2506, 0.6562], [0.5485, 0.3471, 0.1421]\rangle\}$	$\{\langle[0.7118, 0.6279, 0.4669], [0.6279, 0.4669, 0.8557], [0.4597, 0.8849, 0.8649]\rangle\}$	$\{\langle[0.6264, 0.4201, 0.7388], [0.4201, 0.7388, 0.6579], [0.4712, 0.2339, 0.5466]\rangle\}$
$\ddot{A}_3$	$\{\langle[0.4180, 0.3531, 0.7387], [0.3531, 0.7387, 0.5638], [0.7065, 0.9422, 0.9321]\rangle\}$	$\{\langle[0.3631, 0.2202, 0.4413], [0.2202, 0.4413, 0.3753], [0.7139, 0.53336, 0.7603]\rangle\}$	$\{\langle[0.3714, 0.2385, 0.1273], [0.2385, 0.1273, 0.3740], [0.7614, 0.6285, 0.4332\rangle\}$

$w = (0.25, 0.25, 0.50)$

Table 7. General rating value.

$\ddot{A}_1$	$\{\langle[0.2949, 0.2060, 0.3095], [0.2060, 0.3095, 0.2279], [0.7930, 0.7660, 0.7656]\rangle\}$
$\ddot{A}_2$	$\{\langle[0.5382, 0.3952, 0.4032], [0.3952, 0.4032, 0.6125], [0.6031, 0.5563, 0.5557]\rangle\}$
$\ddot{A}_3$	$\{\langle[0.4816, 0.3952, 0.6496], [0.3952, 0.6496, 0.6125], [0.6029, 0.5562, 0.5556]\rangle\}$

Step 4: Calculate the scores function to aggregate the value of each alternative $\ddot{\beta}_i$ $(i = 1, 2, ..., n)$, then we have must to find out the score value of the aggregate value of each alternative. $\ddot{s}_1 = \frac{2.2924}{9} = 0.2547, \ddot{s}_2 = \frac{3.2564}{9} = 0.3618, \ddot{s}_3 = \frac{3.6926}{9} = 0.4102$.

Step 5: Arrange the scores of all alternatives in the form of descending order and select that alternative which is the highest score function. Since $\ddot{s}_3 > \ddot{s}_2 > \ddot{s}_1$, thus the most wanted alternative is $\ddot{s}_3$.

Figure 3 is the graph illustrating the three ranking.

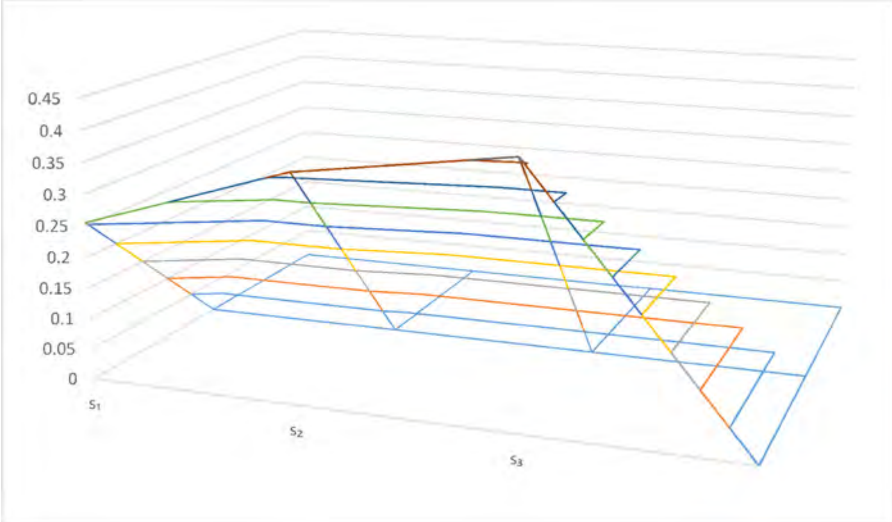

Figure 3. Illustration of the ranking.

8. Comparison Analysis

In order to verify the validity and effectiveness of the proposed approach, a comparative study is conducted using the methods of interval-valued intuitionistic hesitant fuzzy number [25] and hesitant triangular intuitionistic fuzzy number [23] , which are special cases of TCHFNs, to the same illustrative example.

8.1. A Comparison Analysis of the Existing MCDM Interval-Valued Intuitionistic Hesitant Fuzzy Number with Our Proposed Methods

An interval-valued intuitionistic hesitant fuzzy number can be considered as a special case of triangular cubic hesitant fuzzy numbers when there is the only element in membership and non-membership degree [25]. For comparison, the interval-valued intuitionistic hesitant fuzzy number can be transformed to the triangular cubic hesitant fuzzy number (TCHFN) by calculating the average value of the membership and nonmembership degrees. After transformation, the interval-valued intuitionistic hesitant fuzzy number are given in Tables 8 and 9.

Table 8. Interval-valued intuitionistic hesitant fuzzy decision matrix.

	$\ddot{c}_1$	$\ddot{c}_2$	$\ddot{c}_3$
$\ddot{A}_1$	$\left\{\begin{matrix}[0.6,0.8], \\ [0.10,0.12]\end{matrix}\right\}$	$\left\{\begin{matrix}[0.3,0.5], \\ [0.7,0.9]\end{matrix}\right\}$	$\left\{\begin{matrix}\langle[0.10,0.12], \\ [0.14,0.16]\end{matrix}\right\}$
$\ddot{A}_2$	$\left\{\begin{matrix}[0.3,0.5], \\ [0.7,0.9]\end{matrix}\right\}$	$\left\{\begin{matrix}[0.10,0.12], \\ [0.14,0.16]\end{matrix}\right\}$	$\left\{\begin{matrix}[0.6,0.8], \\ [0.10,0.12]\end{matrix}\right\}$
$\ddot{A}_3$	$\left\{\begin{matrix}\langle[0.10,0.12], \\ [0.14,0.16]\end{matrix}\right\}$	$\left\{\begin{matrix}[0.6,0.8], \\ [0.10,0.12]\end{matrix}\right\}$	$\left\{\begin{matrix}[0.3,0.5], \\ [0.7,0.9]\end{matrix}\right\}$

Step 1: Calculate the interval-valued intuitionistic hesitant fuzzy weighted averaging (IVIHFWA) operator $\omega = (0.3, 0.2, 0.5)^T$.

Table 9. IVIHFWA operator.

$\ddot{A}_1$	$\{[0.3386, 0.5176], [0.2496, 0.2959]\}$
$\ddot{A}_2$	$\{[0.3146, 0.3849], [0.3965, 0.4441]\}$
$\ddot{A}_3$	$\{[0.4981, 0.7033], [0.0989, 0.1314]\}$

Step 2: Calculate the score value $s_1 = 0.1553, s_2 = -0.0705, s_3 = 0.4855$.

Step 3: Rank all the alternatives. According to the ranking of score function $S(z_i)$, the ranking is $s_3 > s_1 > s_2$.

Figure 4 is the graph illustrating the position of the three scores.

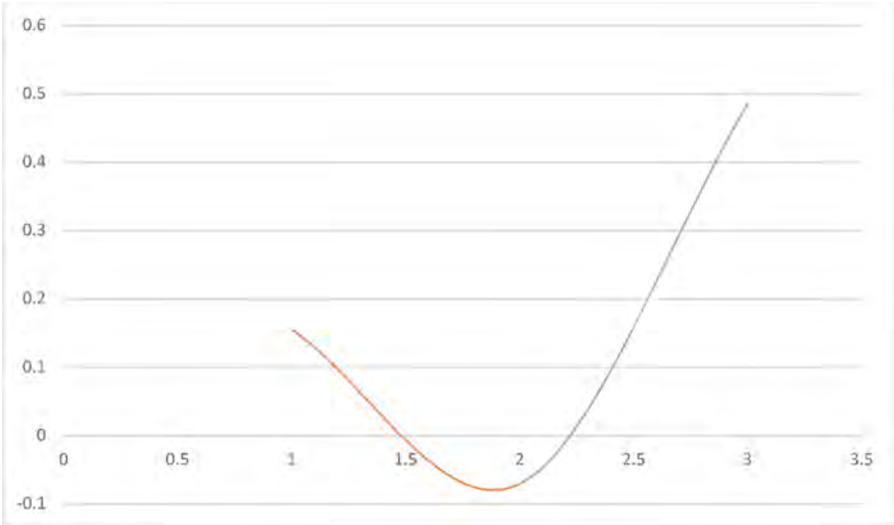

Figure 4. Ranking values in the interval-valued intuitionistic hesitant fuzzy number.

The ranking of all alternatives $s_3 > s_1 > s_2$ and s_3 is the best selection. Obviously, the ranking is derived from the method proposed by Zhang [25], is different from the result of the proposed method. The main reasons are that an interval-valued intuitionistic hesitant fuzzy number only consider the triangular number, membership degrees of an element and nonmembership degrees, which may result in information interval-valued intuitionistic hesitant fuzzy number are not equal.

8.2. A Comparison Analysis of the Existing MCDM Hesitant Triangular Intuitionistic Fuzzy Number with Our Proposed Methods

The hesitant triangular intuitionistic fuzzy number can be considered as a special case of triangular cubic hesitant fuzzy numbers when there is the only element in membership and non-membership degree [23]. For comparison, the hesitant triangular intuitionistic fuzzy number can be transformed into the triangular cubic hesitant fuzzy number (TCHFN) by calculating the average value of the membership and nonmembership degrees. After transformation, the hesitant triangular intuitionistic fuzzy number are given in Tables 10 and 11.

Table 10. Hesitant triangular intuitionistic fuzzy decision matrix.

	$\ddot{c}_1$	$\ddot{c}_2$	$\ddot{c}_3$
$\ddot{A}_1$	$\left\{\begin{array}{c}[0.6, 0.8, 0.10], \\ [0.8, 0.10],\end{array}\right\}$	$\left\{\begin{array}{c}[0.3, 0.5, 0.7], \\ [0.4, 0.6]\end{array}\right\}$	$\left\{\begin{array}{c}[0.10, 0.12, 0.14], \\ [0.12, 0.14]\end{array}\right\}$
$\ddot{A}_2$	$\left\{\begin{array}{c}[0.3, 0.5, 0.7], \\ [0.4, 0.6]\end{array}\right\}$	$\left\{\begin{array}{c}[0.10, 0.12, 0.14], \\ [0.12, 0.14]\end{array}\right\}$	$\left\{\begin{array}{c}[0.6, 0.8, 0.10], \\ [0.8, 0.10],\end{array}\right\}$
$\ddot{A}_3$	$\left\{\begin{array}{c}[0.10, 0.12, 0.14], \\ [0.12, 0.14]\end{array}\right\}$	$\left\{\begin{array}{c}[0.6, 0.8, 0.10], \\ [0.8, 0.10],\end{array}\right\}$	$\left\{\begin{array}{c}[0.3, 0.5, 0.7], \\ [0.4, 0.6]\end{array}\right\}$

Step 1: Calculate the Hesitant triangular intutionistic fuzzy weighted geometric (HTIFWG) opeator $\omega = (0.3, 0.2, 0.5)^T$.

Table 11. HTIFWG operator.

$\ddot{A}_1$	$\{[0.2996, 0.4021, 0.12496], [0.3761, 0.2965]\}$
$\ddot{A}_2$	$\{[0.4477, 0.5448, 0.3965], [0.5211, 0.2091]\}$
$\ddot{A}_3$	$\{[0.1341, 0.2190, 0.0989], [0.1959, 0.4435]\}$

Step 2: Calculate the score value $s_1 = 0.0165, s_2 = 0.1083, s_3 = -0.0279$.

Step 3: Rank all the alternatives. According to the ranking of score function $S(z_i)$, the ranking is $s_2 > s_1 > s_3$.

Figure 5 is the graph illustrating the position of the three scores. The ranking of all alternatives $s_2 > s_1 > s_3$ and s_2 is the best selection. Obviously, the ranking is derived from the method proposed by Chen et al. [23] in Table 12, is different from the result of the proposed method. The main reasons are that hesitant triangular intuitionistic fuzzy number only consider the triangular number, membership degrees of an element and nonmembership degrees, which may result in information hesitant triangular intuitionistic fuzzy number are not equal.

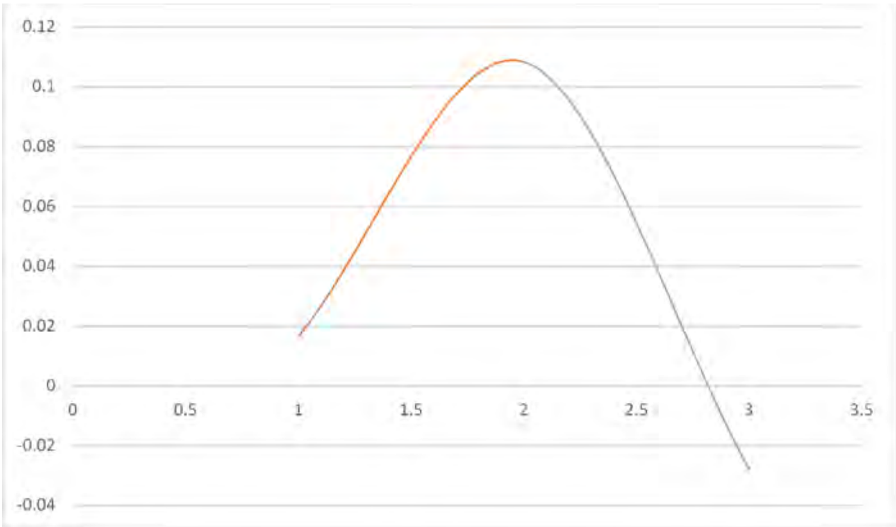

Figure 5. Ranking values in the hesitant triangular intuitionistic fuzzy number.

Table 12. Comparison analysis with existing methods.

Method	Ranking
triangular cubic hesitant fuzzy number	$ddots_3 > \ddot{s}_2 > \ddot{s}_1$
interval-valued intuitionistic hesitant fuzzy number [25]	$s_3 > s_1 > s_2$
hesitant triangular intuitionistic fuzzy number [23]	$s_2 > s_1 > s_3$

The following advantages of our proposal can be summarized on the basis of the above comparison analyses. Triangular cubic hesitant fuzzy number (TCHFN) are very suitable for illustrating uncertain or fuzzy information in MCDM problems because the membership and non-membership degrees can be two sets of several possible values, which cannot be achieved by interval-valued intuitionistic hesitant fuzzy number and intuitionistic triangular hesitant fuzzy number. On the basis of basis operations, aggregation operators and comparison method of triangular cubic hesitant fuzzy number (TCHFN) can be also used to process interval-valued intuitionistic hesitant fuzzy number and intuitionistic triangular hesitant fuzzy number after slight adjustments, because triangular cubic hesitant fuzzy number (TCHFN) can be considered as the generalized form of interval-valued intuitionistic hesitant fuzzy number and intuitionistic triangular hesitant fuzzy number. The defined operations of Triangular cubic hesitant fuzzy number (TCHFN) give us more accurate than the existing operators. This further enhanced the idea of decision making besides the established Q-fuzzy sets [44–46] and vague sets [47–49].

9. Conclusions

In this paper, we develop the new idea of triangular cubic hesitant fuzzy number and operational laws. We develop the score function S(A), accuracy function H(A), membership uncertainty index T(A) and hesitation uncertainty index G(A). We develop the Einstein operations on Triangular cubic hesitant fuzzy sets. We develop three arithmetic averaging operators, that are triangular cubic hesitant fuzzy Einstein weighted averaging (TCHFEWA) operator, triangular cubic hesitant fuzzy Einstein ordered weighted averaging (TCHFEOWA) operator and triangular cubic hesitant fuzzy Einstein hybrid weighted averaging (TCHFEHWA) operator, for gathering cubic hesitant fuzzy data. The TCHFEHWA operator simplifies both the TCHFEWA and TCHFEOWA operators. Moreover, we apply the developed aggregation operators to multiple attribute group decision-making with triangular cubic hesitant fuzzy information. We apply on the TCHFEHWA operator to multiple attribute decision making with the triangular cubic hesitant fuzzy material. Finally, a numerical example is used to illustrate the validity of the proposed approach in group decision-making problems. The advantages of this new method are that: (1) it is more reliable and reasonable to aggregate the Einstein information under the triangular cubic hesitant fuzzy numbers environment; (2) it offers an effective and powerful mathematics tool for the MADM under uncertainty and can provide more reliable and flexible aggregation results in decision making; (3) it not only considers relationships between the input arguments or the attributes, but also takes into account the correlation between input argument and itself or the interrelations between the attribute and itself, furthermore, interrelationships between input arguments or the attributes are tackled once. The new methods provide some reasonable and reliable MADM aggregation operators, which broaden the selection scope of the decision makers and offer theory Einstein for the MADM methods. In group decision-making problems, because the experts usually come from different speciality fields and have different backgrounds and levels of knowledge, they usually have diverging opinions. Thus, in future work, we will present a consensus model for group decision-making with trapezoidal cubic linguistic neutrosophic hesitant fuzzy information.

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Some Interval Neutrosophic Dombi Power Bonferroni Mean Operators and Their Application in Multi–Attribute Decision–Making

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Abstract: The power Bonferroni mean (PBM) operator is a hybrid structure and can take the advantage of a power average (PA) operator, which can reduce the impact of inappropriate data given by the prejudiced decision makers (DMs) and Bonferroni mean (BM) operator, which can take into account the correlation between two attributes. In recent years, many researchers have extended the PBM operator to handle fuzzy information. The Dombi operations of T-conorm (TCN) and T-norm (TN), proposed by Dombi, have the supremacy of outstanding flexibility with general parameters. However, in the existing literature, PBM and the Dombi operations have not been combined for the above advantages for interval-neutrosophic sets (INSs). In this article, we first define some operational laws for interval neutrosophic numbers (INNs) based on Dombi TN and TCN and discuss several desirable properties of these operational rules. Secondly, we extend the PBM operator based on Dombi operations to develop an interval-neutrosophic Dombi PBM (INDPBM) operator, an interval-neutrosophic weighted Dombi PBM (INWDPBM) operator, an interval-neutrosophic Dombi power geometric Bonferroni mean (INDPGBM) operator and an interval-neutrosophic weighted Dombi power geometric Bonferroni mean (INWDPGBM) operator, and discuss several properties of these aggregation operators. Then we develop a multi-attribute decision-making (MADM) method, based on these proposed aggregation operators, to deal with interval neutrosophic (IN) information. Lastly, an illustrative example is provided to show the usefulness and realism of the proposed MADM method. The developed aggregation operators are very practical for solving MADM problems, as it considers the interaction among two input arguments and removes the influence of awkward data in the decision-making process at the same time. The other advantage of the proposed aggregation operators is that they are flexible due to general parameter.

Keywords: interval neutrosophic sets; Bonferroni mean; power operator; multi-attribute decision making (MADM)

1. Introduction

While dealing with any real world problems, a decision maker (DM) often feels discomfort when expressing his\her evaluation information by utilizing a single real number in multi-attribute decision making (MADM) or multi-attribute group decision making (MAGDM) problems due to the intellectual fuzziness of DMs. For this cause, Zadeh [1] developed fuzzy sets (FSs), which are assigned by a truth-membership degree (TMD) in $[0,1]$ and are a better tool to present fuzzy information for handling MADM or MAGDM problems. After the introduction of FSs, different fuzzy modelling approaches were developed to deal with uncertainty in various fields [2–4]. However, in some situations, it is difficult to express truth-membership degree with an exact number. In order to overcome this defect and to express TMD in a more appropriate way, Turksen [5] developed interval valued FSs (IVFSs), in which TMD is represented by interval numbers instead of exact numbers. Since only TMD was considered in FSs or IVFSs and the falsity-membership degree (FMD) came automatically by subtracting TMD from one, it is hard to explain some complicated fuzzy information, for example, for the selection of Dean of a faculty, if the results received from five professors are in favor, two are against and three are neither in favor nor against. Then, this type of information cannot be expressed by FSs. So, in order to handle such types of information, Atanassov [6] developed intuitionistic fuzzy sets (IFSs), which were assigned by TMD and FMD. Atanassov et al. [7] further enlarged IFSs and developed the interval valued IFS (IVIFSs). However, the shortcoming of FSs, IVFSs, IFSs and IVIFSs are that they cannot deal with unreliable or indefinite information. To solve such problems, Smarandache [8,9] developed neutrosophic sets (NSs). In neutrosophic set, every member of the domain set has TMD, an indeterminacy-membership degree (IMD) and FMD, which capture values in $]0^-, 1^+[$. Due to the containment of subsets of $]0^-, 1^+[$ in NS, it is hard to utilize NS in real world and engineering problems. To make NSs helpful in these cases, some authors developed subclasses of NSs, such as single valued neutrosophic sets (SVNSs) [10], interval neutrosophic sets (INSs) [11,12], simplified neutrosophic sets (SNSs) [13,14] and so forth. In recent years, INSs have gained much attention from the researchers and a great number of achievement have been made, such as distance measures [15–17], entropies of INS [18–20], correlation coefficient [21–23]. The theory of NSs has been extensively utilized to handle MADM and MAGDM problems.

For the last many years, information aggregation operators [24–27] have stimulated much awareness of authors and have become very dominant research topic of MADM and MAGDM problems. The conventional aggregation operators (AGOs) proposed by Xu, Xu and Yager [28,29] can only aggregate a group of real numbers into a single real number. Now these conventional AGOs were further extended by many authors, for example, Sun et al. [30] proposed the interval neutrosophic number Choquet integral operator for MADM and Liu et al. [31] developed prioritized ordered weighted AGOs for INSs and applied them to MADM. In addition, some decision-making methods were also developed for MADM problems, for example, Mukhametzyanov et al. [32] developed a statistically based model for sensitivity analysis in MADM problems. Petrovic et al. [33] developed a model for the selection of aircrafts based on decision making trial and evaluation laboratory and analytic hierarchy process (DEMATEL-AHP). Roy et al. [34] proposed a rough relational DEMATEL model to analyze the key success factor of hospital quality. Sarkar et al. [35] developed an optimization technique for national income determination model with stability analysis of differential equation in discrete and continuous process under uncertain environment. These methods can only give a ranking result, however, AGOs can not only give the ranking result, but also give the comprehensive value of each alternative by aggregating its attribute values.

It is obvious that, different aggregation operators have distinct functions, a few of them can reduce the impact of some awkward data produced by predispose DMs, such as power average (PA) operator proposed by Yager [36]. The PA operator can aggregate the input data by designating the weight vector based on the support degree among the input arguments, and can attain this function. Now the PA operator was further extended by many researchers into different environments. Liu et al. [37] proposed some generalized PA operator for INNs, and applied them to MADM. Consequently, some aggregation operators can include the interrelationship between the aggregating parameters, such as the Bonferroni mean (BM) operators developed by Bonferroni [38], the Heronian mean (HM) operator introduced by Sykora [39], Muirhead Mean (MM) operator [40], Maclaurin symmetric mean [41] operators. In addition, these aggregation operators have also been extended by many authors to deal with fuzzy information [42–46].

For aggregating INNs, some AGOs are developed by utilizing different T-norms (TN) and T-conorms (TCN), such as algebraic, Einstein and Hamacher. Usually, the Archimedean TN and TCN are the generalizations of various TNs and TCNs such as algebraic, Einstein, Hamacher, Frank, and Dombi [47] TNs and TCNs. Dombi TN and TCN have the characteristics of general TN and TCN by a general parameter, and this can make the aggregation process more flexible. Recently, several authors defined some operational laws for IFSs [48], SVNSs [49], hesitant fuzzy sets (HFSs) [50,51] based on Dombi TN and TCN. In practical decision making, we generally need to consider interrelationship among attributes and eliminated the influence of awkward data. For this purpose, some researchers combined BM and PA operators to propose some PBM operators and extended them to various fields [52–55]. The PBM operators have two characteristics. Firstly, it can consider the interaction among two input arguments by BM operator, and secondly, it can remove the effect of awkward data by PA operator. The Dombi TN and TCN have a general parameter, which makes the decision-making process more flexible. From the existing literatures, we know that PBM operators are combined with algebraic operations to aggregate IFNs, or IVIFNs, and there is no research on combining PBM operator with Dombi operations to aggregate INNs.

In a word, by considering the following advantages. (1) Since INSs are the more précised class by which one can handle the vague information in a more accurate way when compared with FSs and all other extensions like IVFSs, IFSs, IVIFSs and so forth, they are more suitable to describe the attributes of MADM problems, so in this study, we will select the INSs as information expression; (2) Dombi TN and TCN are more flexible in the decision making process due to general parameter which is regarded as decision makers' risk attitude; (3) The PBM operators have the properties of considering interaction between two input arguments and vanishes the effect of awkward data at the same time. Hence, the purpose and motivation are that we try to combine these three concepts to take the above defined advantages and proposed some new powerful tools to aggregate INNs. (1) we define some Dombi operational laws for INNs; (2) we propose some new PBM aggregation operators based on these new operational laws; (3) we develop a novel MADM based on these developed aggregation operators.

The following sections of this article are shown as follows. In Section 2, we review some basic concepts of INSs, PA operators, BM operators, and GBM operators. In Section 3, we review basic concept of Dombi TN and TCN. After that, we propose some Dombi operations for INNs, and discuss some properties. In Section 4, we define INDPBM operator, INWDPBM operator, INDPGBM operator and INWDPGBM operator and discuss their properties. In Section 5, we propose a MADM method based on the proposed aggregation operators with INNs. In Section 6, we use an illustrative example to show the effectiveness of the proposed MADM method. The conclusion is discussed in Section 7.

2. Preliminaries

In this part, some basic definitions, properties about INSs, BM operators and PA operators are discussed.

2.1. The INSs and Their Operational Laws

Definition 1. *Let* Ω *be the domain set* [8,9], *with a non-specific member in* Ω *expressed by* $\overline{\overline{v}}$. *A NS* $\overline{\overline{NS}}$ *in* Ω *is expressed* by

$$\overline{\overline{NS}} = \left\{ \left\langle \overline{\overline{v}}, t_{\overline{\overline{NS}}}\left(\overline{\overline{v}}\right), i_{\overline{\overline{NS}}}\left(\overline{\overline{v}}\right), f_{\overline{\overline{NS}}}\left(\overline{\overline{v}}\right) \right\rangle \mid \overline{\overline{v}} \in \Omega \right\}, \quad (1)$$

where, $t_{\overline{\overline{NS}}}\left(\overline{\overline{v}}\right), i_{\overline{\overline{NS}}}\left(\overline{\overline{v}}\right)$ *and* $f_{\overline{\overline{NS}}}\left(\overline{\overline{v}}\right)$ *respectively express the TMD, IMD and FMD of the element* $\overline{\overline{v}} \in U$ *to the set* $\overline{\overline{NS}}$. *For each point* $\overline{\overline{v}} \in U$ *, we have,* $t_{\overline{\overline{NS}}}\left(\overline{\overline{v}}\right), i_{\overline{\overline{NS}}}\left(\overline{\overline{v}}\right)\ f_{\overline{\overline{NS}}}\left(\overline{\overline{v}}\right) \in \left]0^-, 1^+\right[$ *and* $0^- \leq t_{\overline{\overline{NS}}}\left(\overline{\overline{v}}\right) + i_{\overline{\overline{NS}}}\left(\overline{\overline{v}}\right) + f_{\overline{\overline{NS}}}\left(\overline{\overline{v}}\right) \leq 3^+$.

The NS was predominantly developed from philosophical perspective, and it is hard to be applied to engineering problems due to the containment of subsets of $\left]0^-, 1^+\right[$. So, in order to use it more easily in real life or engineering problem, Wang et al. [8] presented a subclass of NS by changing $\left]0^-, 1^+\right[$ to $[0,1]$ and was named SVNS, and is defined as follow:

Definition 2. *Let* Ω *be the domain set [10], with a non-specific member in* Ω *expressed by* $\overline{\overline{v}}$. *A SVNS* $\overline{\overline{SV}}$ *in* Ω *is expressed by*

$$\overline{\overline{SV}} = \left\{\left\langle \overline{\overline{u}}, t_{\overline{\overline{SV}}}\left(\overline{\overline{v}}\right), i_{\overline{\overline{SV}}}\left(\overline{\overline{v}}\right), f_{\overline{\overline{SV}}}\left(\overline{\overline{v}}\right)\right\rangle \mid \overline{\overline{v}} \in \Omega\right\}, \tag{2}$$

where $t_{\overline{\overline{SV}}}\left(\overline{\overline{v}}\right), i_{\overline{\overline{SV}}}\left(\overline{\overline{v}}\right)$ *and* $f_{\overline{\overline{SV}}}\left(\overline{\overline{v}}\right)$ *express the TMD, IMD and FMD of the element* $\overline{\overline{v}} \in \Omega$ *to the set* $\overline{\overline{SV}}$ *respectively. For each point* $\overline{\overline{v}} \in \Omega$ *, we have,* $t_{\overline{\overline{SV}}}\left(\overline{\overline{v}}\right), i_{\overline{\overline{SV}}}\left(\overline{\overline{v}}\right), f_{\overline{\overline{SV}}}\left(\overline{\overline{v}}\right)$ *,* $\in [0,1]$ *and* $0 \leq t_{\overline{\overline{SV}}}\left(\overline{\overline{v}}\right) + i_{\overline{\overline{SV}}}\left(\overline{\overline{v}}\right) + f_{\overline{\overline{SV}}}\left(\overline{\overline{v}}\right) \leq 3$.

In order to define more complex information, Wang et al. [9] further developed INS which is define as follows:

Definition 3. *Let* Ω *be the domain set and* $\overline{\overline{v}} \in \Omega$ [11]. *Then an INS* $\overline{\overline{IN}}$ *in* Ω *is expressed by*

$$\overline{\overline{IN}} = \left\{\left\langle \overline{\overline{v}}, \overline{\overline{TR}}_{\hat{I}N}\left(\overline{\overline{v}}\right), \overline{\overline{ID}}_{\hat{I}N}\left(\overline{\overline{v}}\right), \overline{\overline{FL}}_{\hat{I}N}\left(\overline{\overline{v}}\right)\right\rangle \mid \overline{\overline{v}} \in \Omega\right\}, \tag{3}$$

where, $\overline{\overline{TR}}_{\hat{I}N}\left(\overline{\overline{v}}\right), \overline{\overline{ID}}_{\hat{I}N}\left(\overline{\overline{v}}\right)$ *and* $\overline{\overline{FL}}_{\hat{I}N}\left(\overline{\overline{v}}\right)$ *respectively, express the TMD, IMD and FMD of the element* $\overline{\overline{v}} \in \Omega$ *to the set* $\overline{\overline{IN}}$. *For each point* $\overline{\overline{v}} \in U$ *, we have,* $\overline{\overline{TR}}_{IN}\left(\overline{\overline{v}}\right), \overline{\overline{ID}}_{IN}\left(\overline{\overline{v}}\right)$ *,* $\overline{\overline{FL}}_{IN}\left(\overline{\overline{v}}\right) \subseteq [0,1]$ *and* $0 \leq \max \overline{\overline{ID}}_{IN}\left(\overline{\overline{v}}\right) + \max \overline{\overline{ID}}_{IN}\left(\overline{\overline{v}}\right) + \max \overline{\overline{FL}}_{IN}\left(\overline{\overline{v}}\right) \leq 3.$

For computational simplicity, we can use $\overline{\overline{in}} = \left\langle \left[\overline{\overline{TR}}^L, \overline{\overline{TR}}^U\right], \left[\overline{\overline{ID}}^L, \overline{\overline{ID}}^U\right], \left[\overline{\overline{FL}}^L, \overline{\overline{FL}}^U\right]\right\rangle$ to express an element $\overline{\overline{in}}$ in an INS, and the element $\overline{\overline{in}}$ is called an interval neutrosophic number (INN). Where $\left[\overline{\overline{TR}}^L, \overline{\overline{TR}}^U\right] \subseteq [0,1]$, $\left[\overline{\overline{ID}}^L, \overline{\overline{ID}}^U\right] \subseteq [0,1]$, $\left[\overline{\overline{FL}}^L, \overline{\overline{FL}}^U\right] \subseteq [0,1]$ and $0 \leq \overline{\overline{TR}}^U + \overline{\overline{ID}}^U + \overline{\overline{FL}}^U \leq 3$.

Definition 4. *Let* $\overline{\overline{in}}_1 = \left\langle \left[\overline{\overline{TR}}_1^L, \overline{\overline{TR}}_1^U\right], \left[\overline{\overline{ID}}_1^L, \overline{\overline{ID}}_1^U\right], \left[\overline{\overline{FL}}_1^L, \overline{\overline{FL}}_1^U\right]\right\rangle$ *and* $\overline{\overline{in}}_2 = \left\langle \left[\overline{\overline{TR}}_2^L, \overline{\overline{TR}}_2^U\right], \left[\overline{\overline{ID}}_2^L, \overline{\overline{ID}}_2^U\right], \left[\overline{\overline{FL}}_2^L, \overline{\overline{FL}}_2^U\right]\right\rangle$ *be any two INNs* [12], *and* $\zeta > 0$. *Then the operational laws of INNs can be defined as follows:*

(1) $$\overline{\overline{in}}_1 \oplus \overline{\overline{in}}_2 = \left\langle \left[\overline{\overline{TR}}_1^L + \overline{\overline{TR}}_2^L - \overline{\overline{TR}}_1^L \overline{\overline{TR}}_2^L, \overline{\overline{TR}}_1^U + \overline{\overline{TR}}_2^U - \overline{\overline{TR}}_1^U \overline{\overline{TR}}_2^U\right], \left[\overline{\overline{ID}}_1^L \overline{\overline{ID}}_2^L, \overline{\overline{ID}}_1^U \overline{\overline{ID}}_2^U\right], \left[\overline{\overline{FL}}_1^L \overline{\overline{FL}}_2^L, \overline{\overline{FL}}_1^U \overline{\overline{FL}}_2^U\right]\right\rangle; \tag{4}$$

(2) $$\overline{\overline{in}}_1 \otimes \overline{\overline{in}}_2 = \left\langle \left[\overline{\overline{TR}}_1^L \overline{\overline{TR}}_2^L, \overline{\overline{TR}}_1^U \overline{\overline{TR}}_2^U\right], \left[\overline{\overline{ID}}_1^L + \overline{\overline{ID}}_2^L - \overline{\overline{ID}}_1^L \overline{\overline{ID}}_2^L, \overline{\overline{ID}}_1^U + \overline{\overline{ID}}_2^U - \overline{\overline{ID}}_1^U \overline{\overline{ID}}_2^U\right], \left[\overline{\overline{FL}}_1^L + \overline{\overline{FL}}_2^L - \overline{\overline{FL}}_1^L \overline{\overline{FL}}_2^L, \overline{\overline{FL}}_1^U + \overline{\overline{FL}}_2^U - \overline{\overline{FL}}_1^U \overline{\overline{FL}}_2^U\right]\right\rangle; \tag{5}$$

(3) $\overline{\overline{in}}_1^{\zeta} = \left\langle \left[\left(\overline{\overline{TR}}_1^L\right)^{\zeta}, \left(\overline{\overline{TR}}_1^U\right)^{\zeta} \right], \left[1-\left(1-\overline{\overline{ID}}_1^L\right)^{\zeta}, 1-\left(1-\overline{\overline{ID}}_1^U\right)^{\zeta} \right], \left[1-\left(1-\overline{\overline{FL}}_1^L\right)^{\zeta}, 1-\left(1-\overline{\overline{FL}}_1^U\right)^{\zeta} \right] \right\rangle;$ (6)

(4) $\zeta \overline{\overline{in}}_1 = \left\langle \left[1-\left(1-\overline{\overline{TR}}_1^L\right)^{\zeta}, 1-\left(1-\overline{\overline{TR}}_1^U\right)^{\zeta} \right], \left[\left(\overline{\overline{ID}}_1^L\right)^{\zeta}, \left(\overline{\overline{ID}}_1^U\right)^{\zeta} \right], \left[\left(\overline{\overline{FL}}_1^L\right)^{\zeta}, \left(\overline{\overline{FL}}_1^U\right)^{\zeta} \right] \right\rangle.$ (7)

Definition 5. *Let* $\overline{\overline{in}} = \left\langle \left[\overline{\overline{TR}}^L, \overline{\overline{TR}}^U\right], \left[\overline{\overline{ID}}^L, \overline{\overline{ID}}^U\right], \left[\overline{\overline{FL}}^L, \overline{\overline{FL}}^U\right] \right\rangle$ [42], *be an INN. Then the score function* $S\left(\overline{\overline{in}}\right)$ *and accuracy function* $A\left(\overline{\overline{in}}\right)$ *can be defined as follows:*

(i) $S\left(\overline{\overline{in}}\right) = \frac{\overline{\overline{TR}}^L + \overline{\overline{TR}}^U}{2} + 1 - \frac{\overline{\overline{ID}}^L + \overline{\overline{ID}}^U}{2} + 1 - \frac{\overline{\overline{FL}}^L + \overline{\overline{FL}}^U}{2};$ (8)

(ii) $A\left(\overline{\overline{in}}\right) = \frac{\overline{\overline{TR}}^L + \overline{\overline{TR}}^U}{2} + 1 - \frac{\overline{\overline{ID}}^L + \overline{\overline{ID}}^U}{2} + \frac{\overline{\overline{FL}}^L + \overline{\overline{FL}}^U}{2}.$ (9)

In order to compare two INNs, the comparison rules were defined by Liu et al. [36], which can be stated as follows.

Definition 6. *Let* $\overline{\overline{in}}_1 = \left\langle \left[\overline{\overline{TR}}_1^L, \overline{\overline{TR}}_1^U\right], \left[\overline{\overline{ID}}_1^L, \overline{\overline{ID}}_1^U\right], \left[\overline{\overline{FL}}_1^L, \overline{\overline{FL}}_1^U\right] \right\rangle$ *and* $\overline{\overline{in}}_2 = \left\langle \left[\overline{\overline{TR}}_2^L, \overline{\overline{TR}}_2^U\right], \left[\overline{\overline{ID}}_2^L, \overline{\overline{ID}}_2^U\right], \left[\overline{\overline{FL}}_2^L, \overline{\overline{FL}}_2^U\right] \right\rangle$ *be any two INNs [42]. Then we have:*

(1) If $S\left(\overline{\overline{in}}_1\right) > S\left(\overline{\overline{in}}_2\right)$, *then* $\overline{\overline{in}}_1$ *is better than* $\overline{\overline{in}}_2$, *and denoted by* $\overline{\overline{in}}_1 > \overline{\overline{in}}_2$;

(2) If $S\left(\overline{\overline{in}}_1\right) = S\left(\overline{\overline{in}}_2\right)$, *and* $A\left(\overline{\overline{in}}_1\right) > A\left(\overline{\overline{in}}_2\right)$, *then* $\overline{\overline{in}}_1$ *is better than* $\overline{\overline{in}}_2$, *and denoted by* $\overline{\overline{in}}_1 > \overline{\overline{in}}_2$;

(3) If $S\left(\overline{\overline{in}}_1\right) = S\left(\overline{\overline{in}}_2\right)$, *and* $A\left(\overline{\overline{in}}_1\right) = A\left(\overline{\overline{in}}_2\right)$, *then* $\overline{\overline{in}}_1$ *is equal to* $\overline{\overline{in}}_2$, *and denoted by* $\overline{\overline{in}}_1 = \overline{\overline{in}}_2$.

Definition 7. *Let* $\overline{\overline{in}}_1 = \left\langle \left[\overline{\overline{TR}}_1^L, \overline{\overline{TR}}_1^U\right], \left[\overline{\overline{ID}}_1^L, \overline{\overline{ID}}_1^U\right], \left[\overline{\overline{FL}}_1^L, \overline{\overline{FL}}_1^U\right] \right\rangle$ *and* $\overline{\overline{in}}_2 = \left\langle \left[\overline{\overline{TR}}_2^L, \overline{\overline{TR}}_2^U\right], \left[\overline{\overline{ID}}_2^L, \overline{\overline{ID}}_2^U\right], \left[\overline{\overline{FL}}_2^L, \overline{\overline{FL}}_2^U\right] \right\rangle$ *be any two INNs [15]. Then the normalized Hamming distance between* n_1 *and* n_2 *is described as follows.*

$$D\left(\overline{\overline{in}}_1, \overline{\overline{in}}_2\right) = \frac{1}{6}\left(\left|\overline{\overline{TR}}_1^L - \overline{\overline{TR}}_2^L\right| + \left|\overline{\overline{TR}}_1^U - \overline{\overline{TR}}_2^U\right| + \left|\overline{\overline{ID}}_1^L - \overline{\overline{ID}}_2^L\right| + \left|\overline{\overline{ID}}_1^U - \overline{\overline{ID}}_2^U\right| + \left|\overline{\overline{FL}}_1^L - \overline{\overline{FL}}_2^L\right| + \left|\overline{\overline{FL}}_1^U - \overline{\overline{FL}}_2^U\right| \right) \tag{10}$$

2.2. The PA Operator

The PA operator was first presented by Yager [36] and it is described as follows.

Definition 8. *For positive real numbers* $\wp_h (h = 1,2,...,l)$ [36], *the PA operator is described as*

$$PA\left(\wp_1, \wp_2, ..., \wp_l\right) = \frac{\sum_{h=1}^{l}\left(1+T(\wp_h)\right)\wp_h}{\sum_{h=1}^{l}\left(1+T(\wp_h)\right)}, \tag{11}$$

where, $T\left(\wp_h\right) = \sum_{y=1, h\neq y}^{l} sup\left(\wp_h, \wp_y\right)$, *and* $sup\left(\wp_h, \wp_y\right)$ *is the degree to which* $\wp_h$ *supports* $\wp_y$. *The support degree (SPD) satisfies the following properties.*

(1) $sup\left(\wp_h, \wp_y\right) = sup\left(\wp_y, \wp_h\right)$;

(2) $sup\left(\wp_h, \wp_y\right) \in [0,1]$;

(3) $sup\left(\wp_h, \wp_y\right) \geq sup\left(\wp_c, \wp_d\right)$, *if* $\left|\wp_h - \wp_y\right| \leq \left|\wp_c - \wp_d\right|$.

2.3. The BM Operator

The BM operator was initially presented by Bonferroni [38], and it was explained as follows:

Definition 9. *For non-negative real numbers* $\wp_h (h = 1,2,...,l)$ *, and* $x, y \geq 0$ *[38], the BM operator is described as*

$$BM^{x,y}\left(\wp_1, \wp_2, ..., \wp_l\right) = \left(\frac{1}{l^2 - l}\sum_{h=1}^{l}\sum_{s=1, h\neq s}^{l} \wp_h^x \wp_s^y\right)^{\frac{1}{x+y}}. \tag{12}$$

The BM operator ignores the importance degree of each input argument, which can be given by decision makers according to their interest. To overcome this shortcoming of BM operator, He et al. [52] defined the weighted Bonferroni mean (WBM) operators which can be explained as follows:

Definition 10. *For positive real numbers* $\wp_h (h = 1,2,...,l)$ *and* $x, y \geq 0$ *[52], then the weighted BM operator (WBM) is described as*

$$WBM^{x,y}\left(\wp_1, \wp_2, ..., \wp_l\right) = \left(\frac{1}{l^2 - l}\sum_{h=1}^{l}\sum_{s=1, h\neq s}^{l} \frac{\kappa_h \kappa_s}{1 - \kappa_h} \wp_h^x \wp_s^y\right)^{\frac{1}{x+y}}, \tag{13}$$

where $\kappa = \left(\kappa_1, \kappa_2, ..., \kappa_l\right)^T$ *is the importance degree of every* $\wp_h (h = 1,2,....,l)$.

The WBM operator has the following characteristics:

Theorem 1. *(Reducibility) If the weight vector is* $\kappa = \left(\frac{1}{l}, \frac{1}{l}, ..., \frac{1}{l}\right)^T$*, then*

$$\begin{aligned} WBM^{x,y}\left(\wp_1, \wp_2, ..., \wp_l\right) &= \left(\frac{1}{l^2 - l}\sum_{h=1}^{l}\sum_{s=1, z\neq s}^{l} \wp_h^x \wp_s^y\right)^{\frac{1}{x+y}} \\ &= BM^{x,y}\left(\wp_1, \wp_2, ..., \wp_m\right). \end{aligned} \tag{14}$$

Theorem 2. *(Idempotency) Let* $\wp_h = \wp, (h = 1,2,...,l)$ *. Then* $BM^{x,y}\left(\wp_1, \wp_2, ..., \wp_l\right) = \wp$.

Theorem 3. *(Permutation) Let* $\left(\wp_1, \wp_2, ..., \wp_l\right)$ *be any permutation of* $\left(Z_1', Z_2', ..., Z_l'\right)$*. Then*

$$WMB^{x,y}\left(Z_1', Z_2', ..., Z_l'\right) = WBM\left(\wp_1, \wp_2, ..., \wp_l\right). \tag{15}$$

Theorem 4. *(Monotonicity) Let* $\wp_h \geq K_h' (h = 1,2,...,l)$ *. Then*

$$WBM^{x,y}\left(\wp_1, \wp_2, ..., \wp_l\right) \geq WBM^{x,y}\left(K_1', K_2', ..., K_l'\right). \tag{16}$$

Theorem 5. *(Boundedness) The* $WBM^{x,y}$ *lies in the min and max operators, that is,*

$$\min\left(\wp_1, \wp_2, ..., \wp_l\right) \leq WBM^{x,y}\left(\wp_1, \wp_2, ..., \wp_l\right) \leq \max\left(\wp_1, \wp_2, ..., \wp_l\right). \tag{17}$$

Similar to BM operator, the geometric BM operator also considers the correlation among the input arguments. It can be explained as follows:

Definition 11. *For positive real numbers* $\wp_h (h = 1, 2, ..., l)$ *and* $x, y \geq 0$ [53], *the geometric BM operator (GBM) is described as*

$$GBM^{x,y}\left(\wp_1, \wp_2, ..., \wp_l\right) = \frac{1}{x+y} \prod_{h=1}^{l} \prod_{s=1, h \neq s}^{l} \left(x\wp_h + y\wp_s\right)^{\frac{1}{l^2 - l}}. \tag{18}$$

The GBM operator ignores the importance degree of each input argument, which can be given by decision makers according to their interest. In a similar way to WBM, the weighted geometric BM (WGBM) operator was also presented. The extension process is same as that of WBM, so it is omitted here.

The definition of power Bonferroni mean (PBM) and power geometric Bonferroni mean (PGBM) operators are given in Appendix A.

3. Some Operations of INSs Based on Dombi TN and TCN

Dombi TN and TCN

Dombi operations consist of the Dombi sum and Dombi product.

Definition 12. *Let* $\Im$ *and* $\aleph$ *be any two real numbers* [47]. *Then the Dombi TN and TCN among* $\Im$ *and* $\aleph$ *are explained as follows:*

$$T_D(\Im, \aleph) = \frac{1}{1 + \left\{\left(\frac{1-\Im}{\Im}\right)^{\lambda} + \left(\frac{1-\aleph}{\aleph}\right)^{\lambda}\right\}^{\frac{1}{\lambda}}}; \tag{19}$$

$$T_D^{*}(\Im, \aleph) = 1 - \frac{1}{1 + \left\{\left(\frac{\Im}{1-\Im}\right)^{\lambda} + \left(\frac{\aleph}{1-\aleph}\right)^{\lambda}\right\}^{\frac{1}{\lambda}}}, \tag{20}$$

where, $\lambda \geq 1,$ *and* $(\Im, \aleph) \in [0,1] \times [0,1].$

According to the Dombi TN and TCN, we develop a few operational rules for INNs.

Definition 13. *Let* $\overline{\overline{in}} = \left\langle \left[\overline{\overline{TR}}^L, \overline{\overline{TR}}^U\right], \left[\overline{\overline{ID}}^L, \overline{\overline{ID}}^U\right], \left[\overline{\overline{FL}}^L, \overline{\overline{FL}}^U\right] \right\rangle,$ $\overline{\overline{in}}_1 = \left\langle \left[\overline{\overline{TR}}_1^L, \overline{\overline{TR}}_1^U\right], \left[\overline{\overline{ID}}_1^L, \overline{\overline{ID}}_1^U\right], \left[\overline{\overline{FL}}_1^L, \overline{\overline{FL}}_1^U\right] \right\rangle$ *and* $\overline{\overline{in}}_2 = \left\langle \left[\overline{\overline{TR}}_2^L, \overline{\overline{TR}}_2^U\right], \left[\overline{\overline{ID}}_2^L, \overline{\overline{ID}}_2^U\right], \left[\overline{\overline{FL}}_2^L, \overline{\overline{FL}}_2^U\right] \right\rangle$ *be any three INNs and* $\Phi > 0$. *Then, based on Dombi TN and TCN, the following operational laws are developed for INNs.*

(1)

$$\begin{aligned} \overline{\overline{in}}_1 \oplus \overline{\overline{in}}_2 = \Bigg\langle & \left[1 - \frac{1}{1 + \left(\left(\frac{\overline{\overline{TR}}_1^L}{1-\overline{\overline{TR}}_1^L}\right)^{\gamma} + \left(\frac{\overline{\overline{TR}}_2^L}{1-\overline{\overline{TR}}_2^L}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, 1 - \frac{1}{1 + \left(\left(\frac{\overline{\overline{TR}}_1^U}{1-\overline{\overline{TR}}_1^U}\right)^{\gamma} + \left(\frac{\overline{\overline{TR}}_2^U}{1-\overline{\overline{TR}}_2^U}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}\right], \left[\frac{1}{1 + \left(\left(\frac{1-\overline{\overline{ID}}_1^L}{\overline{\overline{ID}}_1^L}\right)^{\gamma} + \left(\frac{1-\overline{\overline{ID}}_2^L}{\overline{\overline{ID1}}_2^L}\right)^{\gamma}\right)^{\frac{1}{\gamma}}},\right. \\ & \left.\frac{1}{1 + \left(\left(\frac{1-\overline{\overline{ID}}_1^U}{\overline{\overline{ID}}_1^U}\right)^{\gamma} + \left(\frac{1-\overline{\overline{ID}}_2^U}{\overline{\overline{ID}}_2^U}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}\right], \left[\frac{1}{1 + \left(\left(\frac{1-\overline{\overline{FL}}_1^L}{\overline{\overline{FL}}_1^L}\right)^{\gamma} + \left(\frac{1-\overline{\overline{FL}}_2^L}{\overline{\overline{FL}}_2^L}\right)^{\gamma}\right)^{\frac{1}{v}}}, \frac{1}{1 + \left(\left(\frac{1-\overline{\overline{FL}}_1^U}{\overline{\overline{FL}}_1^U}\right)^{\gamma} + \left(\frac{1-\overline{\overline{FL}}_2^U}{\overline{\overline{FL}}_2^U}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}\right] \Bigg\rangle; \end{aligned} \tag{21}$$

(2)
$$\overline{\overline{in}}_1 \otimes \overline{\overline{in}}_2 = \left\langle \begin{array}{l} \left[\frac{1}{1+\left(\left(\frac{1-\overline{\overline{TR}}_1^L}{\overline{\overline{TR}}_1^L}\right)^{\gamma}+\left(\frac{1-\overline{\overline{TR}}_2^L}{\overline{\overline{TR}}_2^L}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, \frac{1}{1+\left(\left(\frac{1-\overline{\overline{TR}}_1^U}{\overline{\overline{TR}}_1^U}\right)^{\gamma}+\left(\frac{1-\overline{\overline{TR}}_2^U}{\overline{\overline{TR}}_2^U}\right)^{\gamma}\right)^{\frac{1}{\gamma}}} \right], \left[1-\frac{1}{1+\left(\left(\frac{\overline{\overline{ID}}_1^L}{1-\overline{\overline{ID}}_1^L}\right)^{\gamma}+\left(\frac{\overline{\overline{ID}}_2^L}{1-\overline{\overline{ID}}_2^L}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, \right. \\ \left. 1-\frac{1}{1+\left(\left(\frac{\overline{\overline{ID}}_1^U}{1-\overline{\overline{ID}}_1^U}\right)^{\gamma}+\left(\frac{\overline{\overline{ID}}_2^U}{1-\overline{\overline{ID}}_2^U}\right)^{\gamma}\right)^{\frac{1}{\gamma}}} \right], \left[1-\frac{1}{1+\left(\left(\frac{\overline{\overline{FL}}_1^L}{1-\overline{\overline{FL}}_1^L}\right)^{\gamma}+\left(\frac{\overline{\overline{FL}}_2^L}{1-\overline{\overline{FL}}_2^L}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, 1-\frac{1}{1+\left(\left(\frac{\overline{\overline{FL}}_1^U}{1-\overline{\overline{FL}}_1^U}\right)^{\gamma}+\left(\frac{\overline{\overline{FL}}_2^U}{1-\overline{\overline{FL}}_2^U}\right)^{\gamma}\right)^{\frac{1}{\gamma}}} \right] \end{array} \right\rangle; \tag{22}$$

(3)
$$\Phi\overline{\overline{in}} = \left\langle \begin{array}{l} \left[1-\frac{1}{1+\left(\Phi\left(\frac{\overline{\overline{TR}}^L}{1-\overline{\overline{TR}}^L}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, 1-\frac{1}{1+\left(\Phi\left(\frac{\overline{\overline{TR}}^U}{1-\overline{\overline{TR}}^U}\right)^{\gamma}\right)^{\frac{1}{\gamma}}} \right], \left[\frac{1}{1+\left(\Phi\left(\frac{1-\overline{\overline{ID}}^L}{\overline{\overline{ID}}^L}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, \frac{1}{1+\left(\Phi\left(\frac{1-\overline{\overline{ID}}^U}{\overline{\overline{ID}}^U}\right)^{\gamma}\right)^{\frac{1}{\gamma}}} \right] \\ \left[\frac{1}{1+\left(\Phi\left(\frac{1-\overline{\overline{FL}}^L}{\overline{\overline{FL}}^L}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, \frac{1}{1+\left(\Phi\left(\frac{1-\overline{\overline{FL}}^U}{\overline{\overline{FL}}^U}\right)^{\gamma}\right)^{\frac{1}{\gamma}}} \right] \end{array} \right\rangle; \tag{23}$$

(4)
$$\overline{\overline{in}}^{\Phi} = \left\langle \begin{array}{l} \left[\frac{1}{1+\left(\Phi\left(\frac{1-\overline{\overline{TR}}^L}{\overline{\overline{TR}}^L}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, \frac{1}{1+\left(\Phi\left(\frac{1-\overline{\overline{TR}}^U}{\overline{\overline{TR}}^U}\right)^{\gamma}\right)^{\frac{1}{\gamma}}} \right], \left[1-\frac{1}{1+\left(\Phi\left(\frac{\overline{\overline{ID}}^L}{1-\overline{\overline{ID}}^L}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, 1-\frac{1}{1+\left(\Phi\left(\frac{\overline{\overline{ID}}^U}{1-\overline{\overline{ID}}^U}\right)^{\gamma}\right)^{\frac{1}{\gamma}}} \right] \\ \left[1-\frac{1}{1+\left(\Phi\left(\frac{\overline{\overline{FL}}^L}{1-\overline{\overline{FL}}^L}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}, 1-\frac{1}{1+\left(\Phi\left(\frac{\overline{\overline{FL}}^U}{1-\overline{\overline{FL}}^U}\right)^{\gamma}\right)^{\frac{1}{\gamma}}} \right] \end{array} \right\rangle \tag{24}$$

Now, based on these new operational laws for INNs, we develop some aggregation operators to aggregate IN information in the preceding sections.

4. The INPBM Operator Based on Dombi TN and Dombi TCN

In this part, based on the Dombi operational laws for INNs, we combine PA operator and BM to introduce interval neutrosophic Dombi power Bonferroni mean (INDPBM), interval neutrosophic weighted Dombi power Bonferroni mean, interval neutrosophic Dombi power geometric Bonferroni mean (INDPGBM) and interval neutrosophic weighted Dombi power Bonferroni mean (INWDPGBM) operators and discuss some related properties.

4.1. The INDPBM Operator and INWDPBM Operator

Definition 14. *Let* $\overline{\overline{in}}_i = \left\langle \left[\overline{\overline{TR}}_i^L, \overline{\overline{TR}}_i^U\right], \left[\overline{\overline{ID}}_i^L, \overline{\overline{ID}}_i^U\right], \left[\overline{\overline{FL}}_i^L, \overline{\overline{FL}}_i^L\right] \right\rangle, (i=1,2,\ldots,l)$ *, be a group of INNs, and* $x, y \ge 0.$ *If*

$$INDPBM^{x,y}\left(\overline{\overline{in}}_1, \overline{\overline{in}}_2, \ldots, \overline{\overline{in}}_l\right) = \left(\frac{1}{l^2-l}\left(\underset{\substack{i,j=1 \\ i\neq j}}{\overset{l}{\oplus}}\left(\left(\frac{l\left(1+T(\overline{\overline{in}}_i)\right)}{\overset{l}{\underset{u=1}{\oplus}}\left(1+T(\overline{\overline{in}}_u)\right)}\overline{\overline{in}}_i\right)^{x} \otimes_D \left(\frac{l\left(1+T(\overline{\overline{in}}_j)\right)}{\overset{l}{\underset{u=1}{\oplus}}\left(1+T(\overline{\overline{in}}_u)\right)}\overline{\overline{in}}_j\right)^{y}\right)\right)\right)^{\frac{1}{x+y}}. \tag{25}$$

then $INDPBM^{x,y}$ *is said to be IN Dombi power Bonferroni mean (INDPBM) operator, where* $T\left(\overline{\overline{in}}_z\right)=\underset{s=1,s\neq z}{\overset{l}{\oplus}} Sup\left(\overline{\overline{in}}_z,\overline{\overline{in}}_s\right)$. $Sup\left(\overline{\overline{in}}_z,\overline{\overline{in}}_s\right)$ *is the support degree for* $\overline{\overline{in}}_z$ *from* $\overline{\overline{in}}_s$, *which satisfies the following axioms: (1)* $Sup\left(\overline{\overline{in}}_z,\overline{\overline{in}}_s\right)\in[0,1]$; *(2)* $Sup\left(\overline{\overline{in}}_z,\overline{\overline{in}}_s\right)=Sup\left(\overline{\overline{in}}_s,\overline{\overline{in}}_z\right)$; *(3)* $Sup\left(\overline{\overline{in}}_z,\overline{\overline{in}}_s\right)\geq Sup\left(\overline{\overline{in}}_a,\overline{\overline{in}}_b\right)$, *if* $D\left(\overline{\overline{in}}_z,\overline{\overline{in}}_s\right)<D\left(\overline{\overline{in}}_a,\overline{\overline{in}}_b\right)$, *in which* $D\left(\overline{\overline{in}}_a,\overline{\overline{in}}_b\right)$ *is the distance measure between INNs* $\overline{\overline{in}}_a$ *and* $\overline{\overline{in}}_b$ *defined in Definition 7.*

In order to simplify Equation (25), we can give

$$\Lambda_z=\frac{\left(1+T\left(\overline{\overline{in}}_z\right)\right)}{\overset{l}{\underset{z=1}{\oplus}}\left(1+T\left(\overline{\overline{in}}_z\right)\right)} \tag{26}$$

and call $\Lambda=(\Lambda_1,\Lambda_2,\ldots,\Lambda_l)^T$ is the power weight vector (PWV), such that $\Lambda_z\geq 0,\overset{l}{\underset{z=1}{\oplus}}\Lambda_z=1$. It turns Equation (25) into the following form

$$INDPBM^{x,y}\left(\overline{\overline{in}}_1,\overline{\overline{in}}_2,\ldots,\overline{\overline{in}}_l\right)=\left(\frac{1}{l^2-l}\underset{\substack{i,j=1\\ i\neq j}}{\overset{l}{\oplus}}\left(l\Lambda_i\overline{\overline{in}}_i\right)^x\otimes_D\left(l\Lambda_j\overline{\overline{in}}_j\right)^y\right)^{\frac{1}{x+y}}. \tag{27}$$

Theorem 6. *Let* $\overline{\overline{in}}_i=\left\langle\left[\overline{\overline{TR}}_i^L,\overline{\overline{TR}}_i^U\right],\left[\overline{\overline{ID}}_i^L,\overline{\overline{ID}}_i^U\right],\left[\overline{\overline{FL}}_i^L,\overline{\overline{FL}}_i^U\right]\right\rangle,(i=1,2,\ldots,l)$ *be a group of INNs. Then the value obtained by utilizing Equation (25) is expressed as*

$$\begin{aligned}
&INDPBM^{x,y}\left(\overline{\overline{in}}_1,\overline{\overline{in}}_2,\ldots,\overline{\overline{in}}_e\right)\\
&=\left\langle\left[1\Bigg/\left(1+\frac{l^2-l}{x+y}\times 1\Bigg/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{e}1\Bigg/\left(\frac{x}{l\Lambda_i\left(\frac{\overline{\overline{TR}}_i^L}{1-\overline{\overline{TR}}_i^L}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{\overline{\overline{TR}}_j^L}{1-\overline{\overline{TR}}_j^L}\right)^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right),\ 1\Bigg/\left(1+\frac{l^2-l}{x+y}\times 1\Bigg/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Bigg/\left(\frac{x}{l\Lambda_i\left(\frac{\overline{\overline{TR}}_i^U}{1-\overline{\overline{TR}}_i^U}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{\overline{\overline{TR}}_j^U}{1-\overline{\overline{TR}}_j^U}\right)^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right],\right.\\
&\left[1-1\Bigg/\left(1+\frac{l^2-l}{x+y}\times 1\Bigg/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Bigg/\left(\frac{x}{l\Lambda_i\left(\frac{1-\overline{\overline{ID}}_i^L}{\overline{\overline{ID}}_i^L}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{1-\overline{\overline{ID}}_j^L}{\overline{\overline{ID}}_j^L}\right)^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right),1-1\Bigg/\left(1+\frac{l^2-l}{x+y}\times 1\Bigg/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Bigg/\left(\frac{x}{l\Lambda_i\left(\frac{1-\overline{\overline{ID}}_i^U}{\overline{\overline{ID}}_i^U}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{1-\overline{\overline{ID}}_j^U}{\overline{\overline{ID}}_j^U}\right)^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right],\\
&\left.\left[1-1\Bigg/\left(1+\frac{l^2-l}{x+y}\times 1\Bigg/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Bigg/\left(\frac{x}{l\Lambda_i\left(\frac{1-\overline{\overline{FL}}_i^L}{\overline{\overline{FL}}_i^L}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{1-\overline{\overline{FL}}_j^L}{\overline{\overline{FL}}_j^L}\right)^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right),1-1\Bigg/\left(1+\frac{l^2-l}{x+y}\times 1\Bigg/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Bigg/\left(\frac{x}{l\Lambda_i\left(\frac{1-\overline{\overline{FL}}_i^U}{\overline{\overline{FL}}_i^U}\right)^{\gamma}}+\frac{y}{l\Lambda_i\left(\frac{1-\overline{\overline{FL}}_j^U}{\overline{\overline{FL}}_j^U}\right)^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right]\right\rangle.
\end{aligned} \tag{28}$$

Proof. Proof of Theorem 6 is given in Appendix B. □

In order to determine the PWV Λ, we firstly need to determine the support degree among INNs. In general, the similarity measure among INNs can replace the support degree among INNs. That is,

$$Sup\left(\overline{\overline{in}}_i,\overline{\overline{in}}_m\right)=1-D\left(\overline{\overline{in}}_i,\overline{\overline{in}}_m\right)(i,m=1,2,....,l). \tag{29}$$

Example 1. *Let* $\overline{\overline{in}}_1=\langle[0.3,0.7],[0.2,0.4],[0.3,0.5]\rangle,\overline{\overline{in}}_2=\langle[0.4,0.6],[0.1,0.3],[0.2,0.4]\rangle$ *and* $\overline{\overline{in}}_3=\langle[0.1,0.3],[0.4,0.6],[0.2,0.4]\rangle$ *be any three INNs,* $x=1,y=1,\gamma=3$. *Then by Theorem 6 in Equation (28), we can aggregate these three INNs and generate the comprehensive value* $\overline{\overline{in}}=\left\langle\left[\overline{\overline{TR}}^L,\overline{\overline{TR}}^U\right],\left[\overline{\overline{ID}}^L,\overline{\overline{ID}}^U\right],\left[\overline{\overline{FL}}^L,\overline{\overline{FL}}^U\right]\right\rangle$ *which is calculated as follows:*

Step 1. *Determine the supports* $Sup\left(\overline{\overline{in}}_i,\overline{\overline{in}}_j\right),i,j=1,2,3$ *by using Equation (29), and then we get*

$$Sup\left(\overline{\overline{in}}_1,\overline{\overline{in}}_2\right)=Sup\left(\overline{\overline{in}}_2,\overline{\overline{in}}_1\right)=0.9,Sup\left(\overline{\overline{in}}_1,\overline{\overline{in}}_3\right)=Sup\left(\overline{\overline{in}}_3,\overline{\overline{in}}_1\right)=0.933,Sup\left(\overline{\overline{in}}_2,\overline{\overline{in}}_3\right)=Sup\left(\overline{\overline{in}}_3,\overline{\overline{in}}_2\right)=1.$$

Step 2. *Determine the PWV*

Because $T\left(\overline{\overline{in}}_z\right)=\sum_{s=1,s\neq z}^{3}Sup\left(\overline{\overline{in}}_z,\overline{\overline{in}}_s\right)$, *and*

$$T\left(\overline{\overline{in}}_1\right)=Sup\left(\overline{\overline{in}}_1,\overline{\overline{in}}_2\right)+Sup\left(\overline{\overline{in}}_1,\overline{\overline{in}}_3\right)=1.833,T\left(\overline{\overline{in}}_2\right)=Sup\left(\overline{\overline{in}}_2,\overline{\overline{in}}_1\right)+Sup\left(\overline{\overline{in}}_2,\overline{\overline{in}}_3\right)=1.9,T\left(\overline{\overline{in}}_3\right)=Sup\left(\overline{\overline{in}}_3,\overline{\overline{in}}_1\right)+Sup\left(\overline{\overline{in}}_3,\overline{\overline{in}}_2\right)=1.933,$$

then

$$\Lambda_1=\frac{\left(T\left(\overline{\overline{in}}_1\right)+1\right)}{\left(T\left(\overline{\overline{in}}_1\right)+1\right)+\left(T\left(\overline{\overline{in}}_2\right)+1\right)+\left(T\left(\overline{\overline{in}}_3\right)+1\right)}=0.3269,$$

$$\Lambda_2=\frac{\left(T\left(\overline{\overline{in}}_2\right)+1\right)}{\left(T\left(\overline{\overline{in}}_1\right)+1\right)+\left(T\left(\overline{\overline{in}}_2\right)+1\right)+\left(T\left(\overline{\overline{in}}_3\right)+1\right)}=0.3346,$$

$$\Lambda_3=\frac{\left(T\left(\overline{\overline{in}}_3\right)+1\right)}{\left(T\left(\overline{\overline{in}}_1\right)+1\right)+\left(T\left(\overline{\overline{in}}_2\right)+1\right)+\left(T\left(\overline{\overline{in}}_3\right)+1\right)}=0.3385.$$

Step 3. *Determine the comprehensive value* $\overline{\overline{in}}=\left\langle\left[\overline{\overline{TR}}^L,\overline{\overline{TR}}^U\right],\left[\overline{\overline{ID}}^L,\overline{\overline{ID}}^U\right],\left[\overline{\overline{FL}}^L,\overline{\overline{FL}}^U\right]\right\rangle$ *by using Equation (28), we have*

$$\sum_{\substack{i,j=1\\i\neq j}}^{3}1\Bigg/\left(\frac{x}{3\Lambda_i\left(\frac{\overline{\overline{TR}}_i^L}{1-\overline{\overline{TR}}_i^L}\right)^3}+\frac{y}{3\Lambda_j\left(\frac{\overline{\overline{TR}}_j^L}{1-\overline{\overline{TR}}_j^L}\right)^3}\right)=1\Bigg/\left(\frac{1}{3\Lambda_1\left(\frac{\overline{\overline{TR}}_1^L}{1-\overline{\overline{TR}}_1^L}\right)^3}+\frac{2}{3\Lambda_2\left(\frac{\overline{\overline{TR}}_2^L}{1-\overline{\overline{TR}}_2^L}\right)^3}\right)+1\Bigg/\left(\frac{1}{3\Lambda_1\left(\frac{\overline{\overline{TR}}_1^L}{1-\overline{\overline{TR}}_1^L}\right)^3}+\frac{2}{3\Lambda_3\left(\frac{\overline{\overline{TR}}_3^L}{1-\overline{\overline{TR}}_3^L}\right)^3}\right)+1\Bigg/\left(\frac{1}{3\Lambda_2\left(\frac{\overline{\overline{TR}}_2^L}{1-\overline{\overline{TR}}_2^L}\right)^3}+\frac{2}{3\Lambda_1\left(\frac{\overline{\overline{TR}}_1^L}{1-\overline{\overline{TR}}_1^L}\right)^3}\right)$$

$$+1\Bigg/\left(\frac{1}{3\Lambda_2\left(\frac{\overline{\overline{TR}}_2^L}{1-\overline{\overline{TR}}_2^L}\right)^3}+\frac{2}{3\Lambda_3\left(\frac{\overline{\overline{TR}}_3^L}{1-\overline{\overline{TR}}_3^L}\right)^3}\right)+1\Bigg/\left(\frac{1}{3\Lambda_3\left(\frac{\overline{\overline{TR}}_3^L}{1-\overline{\overline{TR}}_3^L}\right)^3}+\frac{2}{3\Lambda_1\left(\frac{\overline{\overline{TR}}_1^L}{1-\overline{\overline{TR}}_1^L}\right)^3}\right)+1\Bigg/\left(\frac{1}{3\Lambda_3\left(\frac{\overline{\overline{TR}}_3^L}{1-\overline{\overline{TR}}_3^L}\right)^3}+\frac{2}{3\Lambda_2\left(\frac{\overline{\overline{TR}}_2^L}{1-\overline{\overline{TR}}_2^L}\right)^3}\right)=0.1281$$

$$1\Bigg/\left(1+\left(\frac{3^2-3}{1+1}\times 1\Bigg/\left(\sum_{\substack{i,j=1\\i\neq j}}^{3}1\Bigg/\left(\frac{1}{3\Lambda_i\left(\frac{\overline{\overline{TR}}_i^L}{1-\overline{\overline{TR}}_i^L}\right)^\gamma}+\frac{1}{3\Lambda_j\left(\frac{\overline{\overline{TR}}_j^L}{1-\overline{\overline{TR}}_j^L}\right)^\gamma}\right)\right)\right)^{\frac{1}{3}}\right)=0.2590$$

Similarly, we can get $n=\langle[0.2590,0.5525],[0.2221,0.4373],[0.2334,0.4365]\rangle$.

Theorem 7. *(Idempotency) Let* $\overline{\overline{in}}_i=\left\langle\left[\overline{\overline{TR}}_i^L,\overline{\overline{TR}}_i^U\right],\left[\overline{\overline{ID}}_i^L,\overline{\overline{ID}}_i^U\right]\left[\overline{\overline{FL}}_i^L,\overline{\overline{FL}}_i^L\right]\right\rangle,(i=1,2,...,l)$, *be a group of INNs, if all* $\overline{\overline{in}}_i\,(i=1,2,...,l)$ *are equal, that is* $\overline{\overline{in}}_i=\overline{\overline{in}}=\left\langle\left[\overline{\overline{TR}}^L,\overline{\overline{TR}}^U\right],\left[\overline{\overline{ID}}^L,\overline{\overline{ID}}^U\right],\left[\overline{\overline{FL}}^L,\overline{\overline{FL}}^U\right]\right\rangle,(i=1,2,...,l),$. *Then*

$$INDPBM^{x,y}\left(\overline{\overline{in}}_1,\overline{\overline{in}}_2,...,\overline{\overline{in}}_l\right)=\overline{\overline{in}}. \tag{30}$$

Proof. Proof of Theorem 7 is given in Appendix C. □

Theorem 8. *(Commutativity) Assume that* $\overline{\overline{in'}}_u$ *is any permutation of* $\overline{\overline{in}}_u\ (u=1,2,...,l)$, *then*

$$INDPBM^{x,y}\left(\overline{\overline{in'}}_1,\overline{\overline{in'}}_2,...,\overline{\overline{in'}}_l\right)=INDPBM^{x,y}\left(\overline{\overline{in}}_1,\overline{\overline{in}}_2,...,\overline{\overline{in}}_l\right). \tag{31}$$

Proof. From Definition 14, we have

$$INDPBM^{x,y}\left(\overline{\overline{in'}}_1,\overline{\overline{in'}}_2,...,\overline{\overline{in'}}_e\right)=\left(\frac{1}{l^2-l}\sum_{\substack{i,j=1\\i\neq j}}^{l}\left(l\Lambda'_i\overline{\overline{in'}}_i\right)^x\otimes_D\left(l\Lambda'_j\overline{\overline{in'}}_j\right)^y\right)^{\frac{1}{x+y}},$$

and

$$INDPBM^{x,y}\left(\overline{\overline{in}}_1,\overline{\overline{in}}_2,...,\overline{\overline{in}}_l\right)=\left(\frac{1}{l^2-l}\sum_{\substack{i,j=1\\i\neq j}}^{l}\left(l\Lambda_i\overline{\overline{in}}_i\right)^x\otimes_D\left(l\Lambda_j\overline{\overline{in}}_j\right)^y\right)^{\frac{1}{x+y}}.$$

Because,

$$\sum_{\substack{i,j=1\\i\neq j}}^{l}\left(l\Lambda'_i\overline{\overline{in'}}_i\right)^x\otimes_D\left(l\Lambda'_j\overline{\overline{in'}}_j\right)^y=\sum_{\substack{i,j=1\\i\neq j}}^{l}\left(l\Lambda_i\overline{\overline{in}}_i\right)^x\otimes_D\left(l\Lambda_j\overline{\overline{in}}_j\right)^y,$$

hence, $INDPBM^{x,y}\left(\overline{\overline{in'}}_1,\overline{\overline{in'}}_2,...,\overline{\overline{in'}}_l\right)=INDPBM^{x,y}\left(\overline{\overline{in}}_1,\overline{\overline{in}}_2,...,\overline{\overline{in}}_l\right)$. □

Theorem 9. *(Boundedness)* Let $\overline{\overline{in}}_i=\left\langle\left[\overline{\overline{TR}}_i^L,\overline{\overline{TR}}_i^U\right],\left[\overline{\overline{ID}}_i^L,\overline{\overline{ID}}_i^U\right],\left[\overline{\overline{FL}}_i^L,\overline{\overline{FL}}_i^L\right]\right\rangle,(i=1,2,...,l)$ be a group of INNs, and $\overline{\overline{in}}^+=\left\langle\max_{i=1}^{l}\left[\overline{\overline{TR}}_i^L,\overline{\overline{TR}}_i^U\right],\min_{i=1}^{l}\left[\overline{\overline{ID}}_i^L,\overline{\overline{ID}}_i^U\right],\min_{i=1}^{l}\left[\overline{\overline{FL}}_i^L,\overline{\overline{FL}}_i^U\right]\right\rangle,\overline{\overline{in}}^-=\left\langle\min_{i=1}^{l}\left[\overline{\overline{TR}}_i^L,\overline{\overline{TR}}_i^U\right],\max_{i=1}^{l}\left[\overline{\overline{ID}}_i^L,\overline{\overline{ID}}_i^U\right],\max_{i=1}^{l}\left[\overline{\overline{FL}}_i^L,\overline{\overline{FL}}_i^U\right]\right\rangle,$. Then

$$\overline{\overline{in}}^-\leq INDPBM\left(\overline{\overline{in}}_1,\overline{\overline{in}}_2,...,\overline{\overline{in}}_l\right)\leq\overline{\overline{in}}^+. \tag{32}$$

Proof. Proof of Theorem 9 is given in Appendix D. □

Now, we shall study a few special cases of the $INDPBM^{x,y}$ with respect to x and y.

(1) When $y\rightarrow 0,\gamma>0,$ then we can get

$$INDPBM^{x,0}\left(\overline{\overline{in}}_1,\overline{\overline{in}}_2,....,\overline{\overline{in}}_l\right)$$

$$= \lim_{y\to 0}\left\langle \begin{array}{l} \left[\left(1\Big/\left(1+\left(\frac{l^2-l}{x}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Big/\left(\frac{x}{l\Lambda_i\left(\frac{\overline{\overline{TR}}_i^L}{1-\overline{\overline{TR}}_i^L}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right), \left(1\Big/\left(1+\left(\frac{l^2-l}{x}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Big/\left(\frac{x}{l\Lambda_i\left(\frac{\overline{\overline{TR}}_i^U}{1-\overline{\overline{TR}}_i^U}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\right], \\ \left[\left(1-1\Big/\left(1+\left(\frac{l^2-l}{x}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Big/\left(\frac{x}{l\Lambda_i\left(\frac{1-\overline{\overline{ID}}_i^L}{\overline{\overline{ID}}_i^L}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right), \left(1\Big/\left(1+\left(\frac{l^2-l}{x}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Big/\left(\frac{x}{l\Lambda_i\left(\frac{1-\overline{\overline{ID}}_i^U}{\overline{\overline{ID}}_i^U}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\right], \\ \left[\left(1-1\Big/\left(1+\left(\frac{l^2-l}{x}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Big/\left(\frac{x}{l\Lambda_i\left(\frac{1-\overline{\overline{ID}}_i^L}{\overline{\overline{ID}}_i^L}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right), \left(1\Big/\left(1+\left(\frac{l^2-l}{x}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Big/\left(\frac{x}{l\Lambda_i\left(\frac{1-\overline{\overline{ID}}_i^U}{\overline{\overline{ID}}_i^U}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\right]. \end{array}\right. \tag{33}$$

(2) When $x=1, y\to 0, \gamma>0,$ then we can get

$$INDPBM^{1,0}\left(\overline{\overline{in}}_1, \overline{\overline{in}}_2, \ldots, \overline{\overline{in}}_l\right)$$

$$= \lim_{y\to 0}\left\langle \begin{array}{l} \left[\left(1\Big/\left(1+\left(l^2-l\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Big/\left(\frac{1}{l\Lambda_i\left(\frac{\overline{\overline{TR}}_i^L}{1-\overline{\overline{TR}}_i^L}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right), \left(1\Big/\left(1+\left(l^2-l\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Big/\left(\frac{1}{l\Lambda_i\left(\frac{\overline{\overline{TR}}_i^U}{1-\overline{\overline{TR}}_i^U}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\right], \\ \left[\left(1-1\Big/\left(1+\left(l^2-l\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Big/\left(\frac{1}{l\Lambda_i\left(\frac{1-\overline{\overline{ID}}_i^L}{\overline{\overline{ID}}_i^L}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right), \left(1\Big/\left(1+\left(l^2-l\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Big/\left(\frac{1}{l\Lambda_i\left(\frac{1-\overline{\overline{ID}}_i^U}{\overline{\overline{ID}}_i^U}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\right], \\ \left[\left(1-1\Big/\left(1+\left(l^2-l\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Big/\left(\frac{1}{l\Lambda_i\left(\frac{1-\overline{\overline{FL}}_i^L}{\overline{\overline{FL}}_i^L}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right), \left(1\Big/\left(1+\left(l^2-l\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Big/\left(\frac{1}{l\Lambda_i\left(\frac{1-\overline{\overline{FL}}_i^U}{\overline{\overline{FL}}_i^U}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\right]. \end{array}\right. \tag{34}$$

(3) When $x=y=1, \gamma>0,$ then we can get

$$INDPBM^{1,1}\left(\overline{\overline{in}}_1, \overline{\overline{in}}_2, \ldots, \overline{\overline{in}}_e\right)$$

$$
= \left\langle \left[1 \Big/ \left(1 + \left(\frac{l^2 - l}{2} \times 1 \Big/ \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{l} 1 \Big/ \left(\frac{x}{l\Lambda_i \left(\frac{\overline{\overline{TR}}_i^L}{1 - \overline{\overline{TR}}_i^L} \right)^{\gamma}} + \frac{y}{l\Lambda_j \left(\frac{\overline{\overline{TR}}_j^L}{1 - \overline{\overline{TR}}_j^L} \right)^{\gamma}} \right) \right) \right)^{\frac{1}{\gamma}} \right), 1 \Big/ \left(1 + \left(\frac{l^2 - l}{2} \times 1 \Big/ \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{l} 1 \Big/ \left(\frac{x}{l\Lambda_i \left(\frac{\overline{\overline{TR}}_i^U}{1 - \overline{\overline{TR}}_i^U} \right)^{\gamma}} + \frac{y}{l\Lambda_j \left(\frac{\overline{\overline{TR}}_j^U}{1 - \overline{\overline{TR}}_j^U} \right)^{\gamma}} \right) \right) \right)^{\frac{1}{\gamma}} \right) \right], \right.
$$

$$
\left[1 - 1 \Big/ \left(1 + \left(\frac{l^2 - l}{2} \times 1 \Big/ \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{l} 1 \Big/ \left(\frac{x}{l\Lambda_i \left(\frac{1 - \overline{\overline{ID}}_i^L}{\overline{\overline{ID}}_i^L} \right)^{\gamma}} + \frac{y}{l\Lambda_j \left(\frac{1 - \overline{\overline{ID}}_j^L}{\overline{\overline{ID}}_j^L} \right)^{\gamma}} \right) \right) \right)^{\frac{1}{\gamma}} \right), 1 - 1 \Big/ \left(1 + \left(\frac{l^2 - l}{2} \times 1 \Big/ \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{l} 1 \Big/ \left(\frac{x}{l\Lambda_i \left(\frac{1 - \overline{\overline{ID}}_i^U}{\overline{\overline{ID}}_i^U} \right)^{\gamma}} + \frac{y}{l\Lambda_j \left(\frac{1 - \overline{\overline{ID}}_j^U}{\overline{\overline{ID}}_j^U} \right)^{\gamma}} \right) \right) \right)^{\frac{1}{\gamma}} \right) \right], \quad (35)
$$

$$
\left. \left[1 - 1 \Big/ \left(1 + \left(\frac{l^2 - l}{2} \times 1 \Big/ \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{l} 1 \Big/ \left(\frac{x}{l\Lambda_i \left(\frac{1 - \overline{\overline{FL}}_i^L}{\overline{\overline{FL}}_i^L} \right)^{\gamma}} + \frac{y}{l\Lambda_j \left(\frac{1 - \overline{\overline{FL}}_j^L}{\overline{\overline{FL}}_j^L} \right)^{\gamma}} \right) \right) \right)^{\frac{1}{\gamma}} \right), 1 - 1 \Big/ \left(1 + \left(\frac{l^2 - l}{2} \times 1 \Big/ \left(\sum_{\substack{i,j=1 \\ i \neq j}}^{l} 1 \Big/ \left(\frac{x}{l\Lambda_i \left(\frac{1 - \overline{\overline{FL}}_i^U}{\Upsilon_i^U} \right)^{\gamma}} + \frac{y}{l\Lambda_i \left(\frac{1 - \overline{\overline{FL}}_j^U}{\overline{\overline{FL}}_j^U} \right)^{\gamma}} \right) \right) \right)^{\frac{1}{\gamma}} \right) \right] \right\rangle.
$$

In the INDPBM operator, we can only take the correlation among the input arguments and cannot consider the importance degree of input arguments. In what follows, the INWPDBM operator shall be proposed to overcome the shortcoming of the INDPBM operator.

Definition 15. *Let* $\overline{\overline{in}}_i = \left\langle \left[\overline{\overline{TR}}_i^L, \overline{\overline{TR}}_i^U \right], \left[\overline{\overline{ID}}_i^L, \overline{\overline{ID}}_i^U \right], \left[\overline{\overline{FL}}_i^L, \overline{\overline{FL}}_i^L \right] \right\rangle, (i = 1, 2, \ldots, l)$ *, be a group of INNs, then the INWDPBM operator is defined as*

$$
INWDPBM^{x,y}\left(\overline{\overline{in}}_1, \overline{\overline{in}}_2, \ldots, \overline{\overline{in}}_l \right) = \left(\frac{1}{l^2 - l} \sum_{\substack{i,j=1 \\ i \neq j}}^{l} \left(\left(\frac{lw_i \left(T(\overline{\overline{in}}_i) + 1 \right)}{\sum_{z=1}^{l} w_z \left(T(\overline{\overline{in}}_z + 1) \right)} \overline{\overline{in}}_i \right)^{x} \otimes_D \left(\frac{lw_i \left(T(\overline{\overline{in}}_i) + 1 \right)}{\sum_{z=1}^{l} w_z \left(T(\overline{\overline{in}}_z + 1) \right)} \overline{\overline{in}}_j \right)^{y} \right) \right)^{\frac{1}{x+y}}, \quad (36)
$$

where, $T\left(\overline{\overline{in}}_i \right) = \sum_{j=1, i \neq j}^{l} Sup\left(\overline{\overline{in}}_i, \overline{\overline{in}}_j \right), x, y > 0, w = \left(w_1, w_2, \ldots., w_l \right)^T$ *is the importance degree of the INNs, such that* $0 \leq w_z \leq 1 \left(z = 1, 2, \ldots, l \right)$ *and* $\sum_{k=1}^{l} w_k = 1.$

Theorem 10. *Let* $\overline{\overline{in}}_i = \left\langle \left[\overline{\overline{TR}}_i^L, \overline{\overline{TR}}_i^U \right], \left[\overline{\overline{ID}}_i^L, \overline{\overline{ID}}_i^U \right], \left[\overline{\overline{FL}}_i^L, \overline{\overline{FL}}_i^L \right] \right\rangle, (i = 1, 2, \ldots, l)$ *be a group of INNs. Then the value obtained using Definition 15, is represented by*

$$
INDPBM^{x,y}\left(\overline{\overline{in}}_1, \overline{\overline{in}}_2, \ldots., \overline{\overline{in}}_l \right)
$$

$$
\begin{aligned}
=\Bigg\langle &\left[1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Big/\left(\frac{x}{\frac{lw_i\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l}w_z\left(T(\overline{\overline{in}}_z+1)\right)}\left(\frac{\overline{\overline{TR}}_i^{L}}{1-\overline{\overline{TR}}_i^{L}}\right)^{\gamma}}+\frac{y}{\frac{lw_j\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l}w_z\left(T(\overline{\overline{in}}_z+1)\right)}\left(\frac{\overline{\overline{TR}}_j^{L}}{1-\overline{\overline{TR}}_j^{L}}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right),\right.\\
&\left.1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Big/\left(\frac{x}{\frac{lw_i\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l}w_z\left(T(\overline{\overline{in}}_z+1)\right)}\left(\frac{\overline{\overline{TR}}_i^{U}}{1-\overline{\overline{TR}}_i^{U}}\right)^{\gamma}}+\frac{y}{\frac{lw_j\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l}w_z\left(T(\overline{\overline{in}}_z+1)\right)}\left(\frac{\overline{\overline{TR}}_j^{U}}{1-\overline{\overline{TR}}_j^{U}}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right],\\
&\left[1-1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Big/\left(\frac{x}{\frac{lw_i\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l}w_z\left(T(\overline{\overline{in}}_z+1)\right)}\left(\frac{1-\overline{\overline{ID}}_i^{L}}{\overline{\overline{ID}}_i^{L}}\right)^{\gamma}}+\frac{y}{\frac{lw_j\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l}w_z\left(T(\overline{\overline{in}}_z+1)\right)}\left(\frac{1-\overline{\overline{ID}}_j^{L}}{\overline{\overline{ID}}_j^{L}}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right),\right.\\
&\left.1-1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Big/\left(\frac{x}{\frac{lw_i\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l}w_z\left(T(\overline{\overline{in}}_z+1)\right)}\left(\frac{1-\overline{\overline{ID}}_i^{U}}{\overline{\overline{ID}}_i^{U}}\right)^{\gamma}}+\frac{y}{\frac{lw_j\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l}w_z\left(T(\overline{\overline{in}}_z+1)\right)}\left(\frac{1-\overline{\overline{ID}}_j^{U}}{\overline{\overline{ID}}_j^{U}}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right],\\
&\left[1-1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Big/\left(\frac{x}{\frac{lw_i\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l}w_z\left(T(\overline{\overline{in}}_z+1)\right)}\left(\frac{1-\overline{\overline{FL}}_i^{L}}{\overline{\overline{FL}}_i^{L}}\right)^{\gamma}}+\frac{y}{\frac{lw_j\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l}w_z\left(T(\overline{\overline{in}}_z+1)\right)}\left(\frac{1-\overline{\overline{FL}}_j^{L}}{\overline{\overline{FL}}_j^{L}}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right),\right.\\
&\left.1-1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l} 1\Big/\left(\frac{x}{\frac{lw_i\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l}w_z\left(T(\overline{\overline{in}}_z+1)\right)}\left(\frac{1-\overline{\overline{FL}}_i^{U}}{\overline{\overline{FL}}_i^{U}}\right)^{\gamma}}+\frac{y}{\frac{lw_j\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l}w_z\left(T(\overline{\overline{in}}_z+1)\right)}\left(\frac{1-\overline{\overline{FL}}_j^{U}}{\overline{\overline{FL}}_j^{U}}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right]\Bigg\rangle.
\end{aligned}
\tag{37}
$$

Proof. Proof of Theorem 10 is similar to Theorem 6. □

Similar to the INDPBM operator, the INWDPBM operator has the properties of boundedness, idempotency and commutativity.

4.2. The INDPGBM Operator and INWDPGBM Operator

In this subpart, we develop INDPGBM and INWDPGBM operators.

Definition 16. *Let* $\overline{\overline{in}}_i=\left\langle\left[\overline{\overline{TR}}_i^{L},\overline{\overline{TR}}_i^{U}\right],\left[\overline{\overline{ID}}_i^{L},\overline{\overline{ID}}_i^{U}\right],\left[\overline{\overline{FL}}_i^{L},\overline{\overline{FL}}_i^{L}\right]\right\rangle,(i=1,2,\ldots,l)$ *be a group of INNs. Then the INDPGBM operator is defined as*

$$
INDPGBM^{x,y}\left(\overline{\overline{in}}_1,\overline{\overline{in}}_2,\ldots,\overline{\overline{in}}_l\right)=\frac{1}{x+y}\left(\prod_{\substack{i,j=1\\ i\neq j}}^{l}\left(x\overline{\overline{in}}_i^{\frac{l\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l}\left(T(\overline{\overline{in}}_z)+1\right)}}+y\overline{\overline{in}}_j^{\frac{l\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l}\left(T(\overline{\overline{in}}_z)+1\right)}}\right)\right)^{\frac{1}{l^2-l}}. \tag{38}
$$

Then, $INDPGBM^{x,y}$ *is said to be an interval neutrosophic Dombi power geometric Bonferroni mean (INDPGBM) operator. Where* $T\left(\overline{\overline{in}}_z\right)=\sum_{s=1,s\neq z}^{l}Sup\left(\overline{\overline{in}}_z,\overline{\overline{in}}_s\right)$, $Sup\left(\overline{\overline{in}}_z,\overline{\overline{in}}_s\right)$ *is the support degree for* n_z *from* n_s, *which satisfies the following axioms: (1)* $Sup\left(\overline{\overline{in}}_z,\overline{\overline{in}}_s\right)\in[0,1]$ *; (2)* $Sup\left(\overline{\overline{in}}_z,\overline{\overline{in}}_s\right)=Sup\left(\overline{\overline{in}}_s,\overline{\overline{in}}_z\right)$; *(3)* $Sup\left(\overline{\overline{in}}_z,\overline{\overline{in}}_s\right)\geq Sup\left(\overline{\overline{in}}_a,\overline{\overline{in}}_b\right)$, *if* $D\left(\overline{\overline{in}}_z,\overline{\overline{in}}_s\right)<D\left(\overline{\overline{in}}_a,\overline{\overline{in}}_b\right)$, *in which* $D(n_a,n_b)$ *is the distance measure between INNs* $\overline{\overline{in}}_a$ *and* $\overline{\overline{in}}_b$ *defined in Definition 7.*

In order to simplify Equation (38), we can describe

$$
\Lambda_z=\frac{\left(1+T\left(\overline{\overline{in}}_z\right)\right)}{\sum_{z=1}^{l}\left(1+T\left(\overline{\overline{in}}_z\right)\right)}, \tag{39}
$$

and call $\Lambda=\left(\Lambda_1,\Lambda_2,\ldots,\Lambda_l\right)^T$ is the power weight vector (PWV), such that $\Lambda_z\geq 0,\sum_{z=1}^{l}\Lambda_z=1$. Then Equation (38) can be written as follows:

$$INDPGBM^{x,y}\left(\overline{\overline{in}}_1,\overline{\overline{in}}_2,...,\overline{\overline{in}}_l\right)=\frac{1}{x+y}\left(\prod_{\substack{i,j=1\\ i\neq j}}^{l}\left(x\overline{\overline{in}}_i^{\,l\Lambda_i}+y\overline{\overline{in}}_j^{\,l\Lambda_j}\right)\right)^{\frac{1}{l^2-l}}. \tag{40}$$

Theorem 11. *Let* $\overline{\overline{in}}_i=\left\langle\left[\overline{\overline{TR}}_i^L,\overline{\overline{TR}}_i^U\right],\left[\overline{\overline{ID}}_i^L,\overline{\overline{ID}}_i^U\right],\left[\overline{\overline{FL}}_i^L,\overline{\overline{FL}}_i^L\right]\right\rangle,(i=1,2,...,l)$ *be a group of INNs. Then the result obtained from Equation (38) is expressed as*

$$INDPGBM^{x,y}\left(\overline{\overline{in}}_1,\overline{\overline{in}}_2,....,\overline{\overline{in}}_l\right)$$

$$\begin{aligned}=\Bigg\langle&\left[\left(1-1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Big/\left(\frac{x}{l\Lambda_i\left(\frac{1-\overline{\overline{TR}}_i^L}{\overline{\overline{TR}}_i^L}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{1-\overline{\overline{TR}}_j^L}{\overline{\overline{TR}}_j^L}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right),\left(1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Big/\left(\frac{x}{l\Lambda_i\left(\frac{1-\overline{\overline{TR}}_i^U}{\overline{\overline{TR}}_i^U}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{1-\overline{\overline{TR}}_j^U}{\overline{\overline{TR}}_j^U}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\right],\\
&\left[\left(1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Big/\left(\frac{x}{l\Lambda_i\left(\frac{\overline{\overline{ID}}_i^L}{1-\overline{\overline{ID}}_i^L}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{\overline{\overline{ID}}_j^L}{1-\overline{\overline{ID}}_j^L}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right),\left(1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Big/\left(\frac{x}{l\Lambda_i\left(\frac{\overline{\overline{ID}}_i^U}{1-\overline{\overline{ID}}_i^U}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{\overline{\overline{ID}}_j^U}{1-\overline{\overline{ID}}_j^U}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\right],\\
&\left[\left(1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Big/\left(\frac{x}{l\Lambda_i\left(\frac{\overline{\overline{FL}}_i^L}{1-\overline{\overline{FL}}_i^L}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{\overline{\overline{FL}}_j^L}{1-\overline{\overline{FL}}_j^L}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right),\left(1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Big/\left(\frac{x}{l\Lambda_i\left(\frac{\overline{\overline{FL}}_i^U}{1-\overline{\overline{FL}}_i^U}\right)^{\gamma}}+\frac{y}{l\Lambda_i\left(\frac{\overline{\overline{FL}}_j^U}{1-\overline{\overline{FL}}_j^U}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\right]\Bigg\rangle.\end{aligned} \tag{41}$$

Theorem 12. *(Idempotency) Let* $\overline{\overline{in}}_i=\left\langle\left[\overline{\overline{TR}}_i^L,\overline{\overline{TR}}_i^U\right],\left[\overline{\overline{ID}}_i^L,\overline{\overline{ID}}_i^U\right],\left[\overline{\overline{FL}}_i^L,\overline{\overline{FL}}_i^L\right]\right\rangle,(i=1,2,...,l)$ *be a group of INNs, if all* $\overline{\overline{in}}_i\ (i=1,2,...,l)$ *are equal, that is* $\overline{\overline{in}}_i=\overline{\overline{in}}=\left\langle\left[\overline{\overline{TR}}^L,\overline{\overline{TR}}^U\right],\left[\overline{\overline{ID}}^L,\overline{\overline{ID}}^U\right],\left[\overline{\overline{FL}}^L,\overline{\overline{FL}}^U\right]\right\rangle,(i=1,2,...,l)$, *then*

$$INDPGBM^{x,y}\left(\overline{\overline{in}}_1,\overline{\overline{in}}_2,...,\overline{\overline{in}}_l\right)=\overline{\overline{in}}. \tag{42}$$

Theorem 13. *(Commutativity) Assume that* $\overline{\overline{in'}}_u$ *is any permutation of* $\overline{\overline{in}}_u\ (u=1,2,...,l)$, *then*

$$INDPGBM^{x,y}\left(\overline{\overline{in'}}_1,\overline{\overline{in'}}_2,...,\overline{\overline{in'}}_l\right)=INDPGBM^{x,y}\left(\overline{\overline{in}}_1,\overline{\overline{in}}_2,...,\overline{\overline{in}}_l\right). \tag{43}$$

Theorem 14. *(Boundedness) Let* $\overline{\overline{in}}_i=\left\langle\left[\overline{\overline{TR}}_i^L,\overline{\overline{TR}}_i^U\right],\left[\overline{\overline{ID}}_i^L,\overline{\overline{ID}}_i^U\right],\left[\overline{\overline{FL}}_i^L,\overline{\overline{FL}}_i^L\right]\right\rangle,(i=1,2,...,l)$ *be a group of INNs, and* $\overline{\overline{in}}^+=\left\langle\max_{i=1}^{l}\left[\overline{\overline{TR}}_i^L,\overline{\overline{TR}}_i^U\right],\min_{i=1}^{l}\left[\overline{\overline{ID}}_i^L,\overline{\overline{ID}}_i^U\right],\min_{i=1}^{l}\left[\overline{\overline{FL}}_i^L,\overline{\overline{FL}}_i^U\right]\right\rangle,\overline{\overline{in}}^-=\left\langle\min_{i=1}^{l}\left[\overline{\overline{TR}}_i^L,\overline{\overline{TR}}_i^U\right],\max_{i=1}^{l}\left[\overline{\overline{ID}}_i^L,\overline{\overline{ID}}_i^U\right],\max_{i=1}^{l}\left[\overline{\overline{FL}}_i^L,\overline{\overline{FL}}_i^U\right]\right\rangle$, *then*

$$\overline{\overline{in}}^-\leq INDPGBM\left(\overline{\overline{in}}_1,\overline{\overline{in}}_2,...,\overline{\overline{in}}_l\right)\leq\overline{\overline{in}}^+. \tag{44}$$

Definition 17. *Let* $\overline{\overline{in}}_i=\left\langle\left[\overline{\overline{TR}}_i^L,\overline{\overline{TR}}_i^U\right],\left[\overline{\overline{ID}}_i^L,\overline{\overline{ID}}_i^U\right],\left[\overline{\overline{FL}}_i^L,\overline{\overline{FL}}_i^L\right]\right\rangle,(i=1,2,...,l)$ *be a group of INNs, then the INWDPGBM operator is defined as*

$$INWDPGBM^{x,y}\left(\overline{\overline{in}}_1,\overline{\overline{in}}_2,...,\overline{\overline{in}}_l\right)=\frac{1}{x+y}\left(\prod_{\substack{i,j=1\\ i\neq j}}^{l}\left(x\overline{\overline{in}}_i^{\frac{lw_i\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l}w_z\left(T(\overline{\overline{in}}_z)+1\right)}}+y\overline{\overline{in}}_j^{\frac{lw_j\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l}w_z\left(T(\overline{\overline{in}}_z)+1\right)}}\right)\right)^{\frac{1}{l^2-l}}. \quad (45)$$

Theorem 15. *Let* $\overline{\overline{in}}_i=\left\langle\left[\overline{\overline{TR}}_i^L,\overline{\overline{TR}}_i^U\right],\left[\overline{\overline{ID}}_i^L,\overline{\overline{ID}}_i^U\right],\left[\overline{\overline{FL}}_i^L,\overline{\overline{FL}}_i^L\right]\right\rangle,(i=1,2,...,l)$ *be a group of INNs. Then the aggregated result from Equation (45) is expressed as*

$$INWDPGBM^{x,y}\left(\overline{\overline{in}}_1,\overline{\overline{in}}_2,....,\overline{\overline{in}}_l\right)$$

$$=\left\langle\left[1-\frac{1}{1+\left(\frac{l^2-l}{x+y}\times\frac{1}{\sum_{\substack{i,j=1\\ i\neq j}}^{l}\frac{1}{\frac{x}{\frac{lw_i\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l}w_z\left(T(\overline{\overline{in}}_z)+1\right)}\left(\frac{1-\overline{\overline{TR}}_i^L}{\overline{\overline{TR}}_i^L}\right)^{\gamma}}+\frac{y}{\frac{lw_j\left(T(\overline{\overline{in}}_j)+1\right)}{\sum_{z=1}^{l}w_z\left(T(\overline{\overline{in}}_z)+1\right)}\left(\frac{1-\overline{\overline{TR}}_j^L}{\overline{\overline{TR}}_j^L}\right)^{\gamma}}}}\right)^{\frac{1}{\gamma}}},\;1-\frac{1}{1+\left(\frac{l^2-l}{x+y}\times\frac{1}{\sum_{\substack{i,j=1\\ i\neq j}}^{l}\frac{1}{\frac{x}{\frac{lw_i\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l}w_z\left(T(\overline{\overline{in}}_z)+1\right)}\left(\frac{1-\overline{\overline{TR}}_i^U}{\overline{\overline{TR}}_i^U}\right)^{\gamma}}+\frac{y}{\frac{lwj\left(T(\overline{\overline{in}}_j)+1\right)}{\sum_{z=1}^{l}w_z\left(T(\overline{\overline{in}}_z)+1\right)}\left(\frac{1-\overline{\overline{TR}}_j^U}{\overline{\overline{TR}}_j^U}\right)^{\gamma}}}}\right)^{\frac{1}{\gamma}}}\right],\right.$$

$$\left[\frac{1}{1+\left(\frac{l^2-l}{x+y}\times\frac{1}{\sum_{\substack{i,j=1\\ i\neq j}}^{l}\frac{1}{\frac{x}{\frac{lw_i\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l}w_z\left(T(\overline{\overline{in}}_z)+1\right)}\left(\frac{\overline{\overline{ID}}_i^L}{1-\overline{\overline{ID}}_i^L}\right)^{\gamma}}+\frac{y}{\frac{lw_j\left(T(\overline{\overline{in}}_j)+1\right)}{\sum_{z=1}^{l}w_z\left(T(\overline{\overline{in}}_z)+1\right)}\left(\frac{\overline{\overline{ID}}_j^L}{1-\overline{\overline{ID}}_j^L}\right)^{\gamma}}}}\right)^{\frac{1}{\gamma}}},\;\frac{1}{1+\left(\frac{l^2-l}{x+y}\times\frac{1}{\sum_{\substack{i,j=1\\ i\neq j}}^{l}\frac{1}{\frac{x}{\frac{lw_i\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l}w_z\left(T(\overline{\overline{in}}_z)+1\right)}\left(\frac{\overline{\overline{ID}}_i^U}{1-\overline{\overline{ID}}_i^U}\right)^{\gamma}}+\frac{y}{\frac{lw_j\left(T(\overline{\overline{in}}_j)+1\right)}{\sum_{z=1}^{l}w_z\left(T(\overline{\overline{in}}_z)+1\right)}\left(\frac{\overline{\overline{ID}}_j^U}{1-\overline{\overline{ID}}_j^U}\right)^{\gamma}}}}\right)^{\frac{1}{\gamma}}}\right], \quad (46)$$

$$\left.\left[\frac{1}{1+\left(\frac{l^2-l}{x+y}\times\frac{1}{\sum_{\substack{i,j=1\\ i\neq j}}^{l}\frac{1}{\frac{x}{\frac{lw_i\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l}w_z\left(T(\overline{\overline{in}}_z)+1\right)}\left(\frac{\overline{\overline{FL}}_i^L}{1-\overline{\overline{FL}}_i^L}\right)^{\gamma}}+\frac{y}{\frac{lw_j\left(T(\overline{\overline{in}}_j)+1\right)}{\sum_{z=1}^{n}w_z\left(T(\overline{\overline{in}}_z)+1\right)}\left(\frac{\overline{\overline{FL}}_j^L}{1-\overline{\overline{FL}}_j^L}\right)^{\gamma}}}}\right)^{\frac{1}{\gamma}}},\;\frac{1}{1+\left(\frac{l^2-l}{x+y}\times\frac{1}{\sum_{\substack{i,j=1\\ i\neq j}}^{l}\frac{1}{\frac{x}{\frac{lw_i\left(T(\overline{\overline{in}}_i)+1\right)}{\sum_{z=1}^{l}w_z\left(T(\overline{\overline{in}}_z)+1\right)}\left(\frac{\overline{\overline{FL}}_i^U}{1-\overline{\overline{FL}}_i^U}\right)^{\gamma}}+\frac{y}{\frac{lw_j\left(T(\overline{\overline{in}}_j)+1\right)}{\sum_{z=1}^{l}w_z\left(T(\overline{\overline{in}}_z)+1\right)}\left(\frac{\overline{\overline{FL}}_j^U}{1-\overline{\overline{FL}}_j^U}\right)^{\gamma}}}}\right)^{\frac{1}{\gamma}}}\right]\right\rangle.$$

5. MADM Approach Based on the Developed Aggregation Operator

In this section, based upon the developed INWDPBM and INWDPGBM operators, we will propose a novel MADM method, which is defined as follows.

Assume that in a MADM problem, we need to evaluate u alternatives $M=\left\{M_1,M_2,...,M_u\right\}$ with respect to v attributes $C=\left\{C_1,C_2,...,C_v\right\}$, and the importance degree of the attributes is represented by $\varpi=\left(\varpi_1,\varpi_2,...,\varpi_v\right)^T$, satisfying the condition $\varpi_h\in[0,1],\sum_{h=1}^{v}\varpi_h=1$. The decision matrix for this decision problem is denoted by $D=\left[d_{gh}\right]_{m\times n}$, where $d_{gh}=\left\langle\left[\overline{\overline{TR}}_{gh}^L,\overline{\overline{TR}}_{gh}^U\right],\left[\overline{\overline{ID}}_{gh}^L,\overline{\overline{ID}}_{gh}^U\right],\left[\overline{\overline{FL}}_{gh}^L\overline{\overline{FL}}_{gh}^U\right]\right\rangle$ is an INN for the alternative M_g with respect to the attribute $C_h,\left(g=1,2,...,u;h=1,2,...,v\right)$. Then the main purpose is to rank the alternative and select the best alternative.

In the following, we will use the proposed INWDPBM and INWDPGBM operators to solve this MADM problem, and the detailed decision steps are shown as follows:

Step 1. Standardize the attribute values. Normally, in real problems, the attributes are of two types, (1) cost type, (2) benefit type. To get right result, it is necessary to change cost type of attribute values to benefit type using the following formula:

$$d_{gh} = \left\langle \left[\overline{\overline{FL}}_{gh}^{L}, \overline{\overline{FL}}_{gh}^{U} \right], \left[1 - \overline{\overline{ID}}_{gh}^{U}, 1 - \overline{\overline{ID}}_{gh}^{L} \right], \left[\overline{\overline{TR}}_{gh}^{L}, \overline{\overline{TR}}_{gh}^{U} \right] \right\rangle. \tag{47}$$

Step 2. Calculate the supports

$$Supp\left(d_{gh}, d_{gl}\right) = 1 - D\left(d_{gh}, d_{gl}\right),\ (g = 1,2,\ldots,u; h,l = 1,2,\ldots,v), \tag{48}$$

where, $D\left(d_{gh}, d_{gl}\right)$ is the distance measure defined in Equation (10).

Step 3. Calculate $T\left(d_{gh}\right)$

$$T\left(d_{gh}\right) = \sum_{\substack{l=1 \\ l \neq h}}^{u} Supp\left(d_{gh}, d_{gl}\right), (g = 1,2,\ldots,u; h,l = 1,2,\ldots,v). \tag{49}$$

Step 4. Aggregate all the attribute values $d_{gh}\ (h = 1,2,\ldots,v)$ to the comprehensive value R_g by using INWDPBM or INWDPGBM operators shown as follows.

$$R_g = INWDPBM\left(d_{g1}, d_{g2}, \ldots., d_{gv}\right); \tag{50}$$

or

$$R_g = INWDPGBM\left(d_{g1}, d_{g2}, \ldots., d_{gv}\right). \tag{51}$$

Step 5. Determine the score values, accuracy values of $R_g\ (g = 1,2,\ldots,u)$, using Definition 5.

Step 6. Rank all the alternatives according to their score and accuracy values, and select the best alternative using Definition 6.

Step 7. End.

This decision steps are also described in Figure 1.

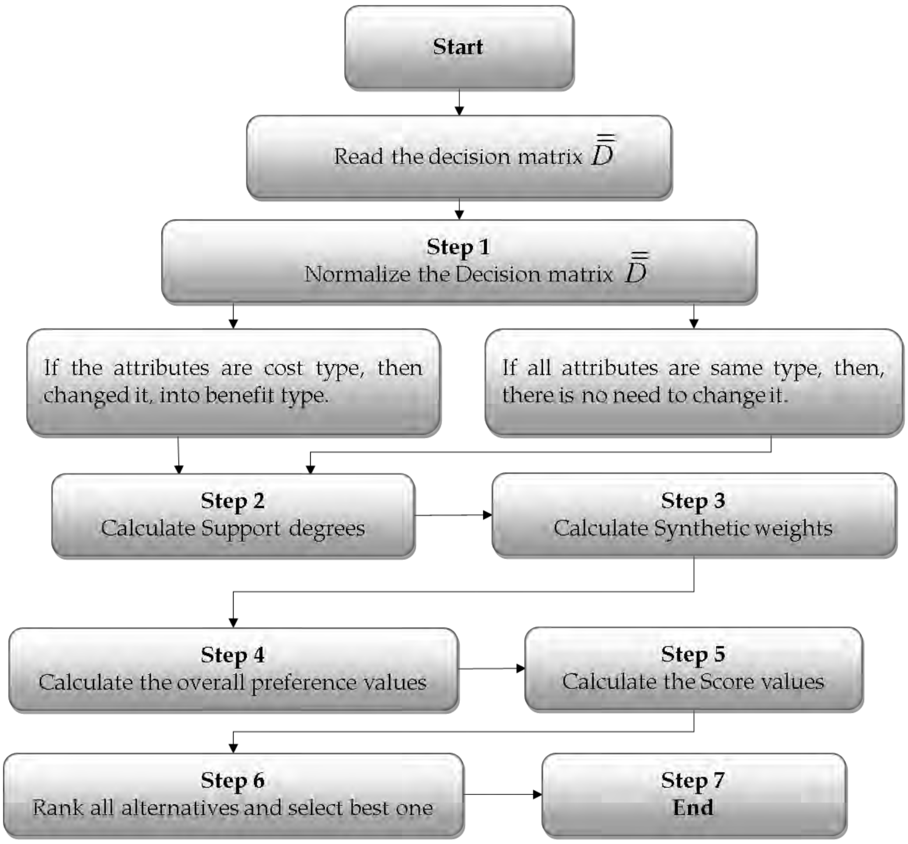

Figure 1. Flow chart for developed approach.

6. Illustrative Example

In this part, an example adapted from [42] is used to illustrate the application and effectiveness of the developed method in MADM problem.

An investment company wants to invest a sum of money in the best option. The company must invest a sum of money in the following four possible companies (alternatives): (1) car company M_1; (2) food company M_2; (3) Computer company M_3; (4) An arm company M_4, and the attributes under consideration are (1) risk analysis C_1; (2) growth analysis C_2; (3) environmental impact analysis C_3. The importance degree of the attributes is $\varpi = (0.35, 0.4, 0.25)^T$. The four possible alternatives $M_g (g = 1,2,3,4)$ are evaluated with respect to the above attributes $C_h (h = 1,2,3)$ by the form of INN, and the IN decision matrix D is listed in Table 1. The purpose of this decision-making problem is to rank the alternatives.

Table 1. The IN decision matrix $\overline{\overline{D}}$.

Alternatives/Attributes	C_1	C_2	C_3
M_1	$\langle[0.4,0.5],[0.2,0.3],[0.3,0.4]\rangle$	$\langle[0.4,0.6],[0.1,0.3],[0.2,0.4]\rangle$	$\langle[0.7,0.9],[0.7,0.8],[0.4,0.5]\rangle$
M_2	$\langle[0.6,0.8],[0.1,0.2],[0.1,0.2]\rangle$	$\langle[0.6,0.7],[0.15,0.25],[0.2,0.3]\rangle$	$\langle[0.3,0.6],[0.2,0.3],[0.8,0.9]\rangle$
M_3	$\langle[0.3,0.6],[0.2,0.3],[0.3,0.4]\rangle$	$\langle[0.5,0.6],[0.2,0.3],[0.3,0.4]\rangle$	$\langle[0.4,0.5],[0.2,0.4],[0.7,0.9]\rangle$
M_4	$\langle[0.7,0.8],[0.01,0.1],[0.2,0.3]\rangle$	$\langle[0.6,0.7],[0.1,0.2],[0.3,0.4]\rangle$	$\langle[0.4,0.6],[0.5,0.6],[0.8,0.9]\rangle$

6.1. The Decision-Making Steps

Step 1. Since C_1, C_2 are of benefit type, and C_3 is of cost type. So, C_3 will be changed into benefit type using Equation (47). So, the normalize decision matrix $\overline{\overline{D}}$ is given in Table 2.

Table 2. The Normalize IN decision matrix $\overline{\overline{D}}$.

Alternatives/Attributes	C_1	C_2	C_3
M_1	$\langle[0.4,0.5],[0.2,0.3],[0.3,0.4]\rangle$	$\langle[0.4,0.6],[0.1,0.3],[0.2,0.4]\rangle$	$\langle[0.4,0.5],[0.2,0.3],[0.7,0.9]\rangle$
M_2	$\langle[0.6,0.8],[0.1,0.2],[0.1,0.2]\rangle$	$\langle[0.6,0.7],[0.15,0.25],[0.2,0.3]\rangle$	$\langle[0.8,0.9],[0.6,0.7],[0.3,0.6]\rangle$
M_3	$\langle[0.3,0.6],[0.2,0.3],[0.3,0.4]\rangle$	$\langle[0.5,0.6],[0.2,0.3],[0.3,0.4]\rangle$	$\langle[0.7,0.9],[0.6,0.8],[0.4,0.5]\rangle$
M_4	$\langle[0.7,0.8],[0.01,0.1],[0.2,0.3]\rangle$	$\langle[0.6,0.7],[0.1,0.2],[0.3,0.4]\rangle$	$\langle[0.8,0.9],[0.4,0.5],[0.4,0.6]\rangle$

Step 2. Determine the supports $Supp\left(d_{gh}, d_{gl}\right), (g = 1,2,3,4; h,l = 1,2,3)$ by Equation (48) (for simplicity we denote $Supp\left(d_{gh}, d_{gl}\right)$ with $S^g_{gh,gl}$), we have

$$S^1_{11,12} = S^1_{12,11} = 0.950; S^1_{12,13} = S^1_{13,12} = 0.800; S^1_{11,13} = S^1_{13,11} = 0.85; S^2_{11,12} = S^2_{12,11} = 0.933; S^2_{12,13} = S^2_{13,12} = 0.717; S^2_{11,13} = S^2_{13,11} = 0.683;$$
$$S^3_{11,12} = S^3_{12,11} = 0.967; S^3_{12,13} = S^3_{13,12} = 0.733; S^3_{11,13} = S^3_{13,11} = 0.700; S^4_{11,12} = S^4_{12,11} = 0.902; S^4_{12,13} = S^4_{13,12} = 0.783; S^4_{11,13} = S^4_{13,11} = 0.752;$$

Step 3. Determine $T\left(d_{gh}\right); (g = 1,2,3,4; h = 1,2,3)$ by Equation (49), and we get

$$T^1_{11} = 1.800, T^1_{12} = 1.750, T^1_{13} = 1.650, T^2_{11} = 1.617, T^2_{12} = 1.650, T^2_{13} = 1.400,$$
$$T^3_{11} = 1.667, T^3_{12} = 1.700, T^3_{13} = 1.433, T^4_{11} = 1.653, T^4_{12} = 1.685, T^4_{13} = 1.535.$$

Step 4. (a) Determine the comprehensive value of every alternative using the INWDPBM operator, that is, Equation (50) $\left(\textit{Assume that } x = y = 1; \gamma = 3\right)$, we have

$$R_1 = \langle[0.3974, 0.5195], [0.1823, 0.3023], [0.3353, 0.4796]\rangle;$$
$$R_2 = \langle[0.6457, 0.7954], [0.1700, 0.2885], [0.2044, 0.3265]\rangle;$$

$$R_3 = \langle [0.4846, 0.6503], [0.2556, 0.3711], [0.3376, 0.4394] \rangle;$$
$$R_4 = \langle [0.6938, 0.7953], [0.1062, 0.2154], [0.3069, 0.4278] \rangle.$$

(b) Determine the comprehensive value of every alternative using the INWDPGBM operator, that is Equation (51), $(Assume\,that\, x = y = 1; \gamma = 3)$, we have

$$R_1 = \langle [0.4026, 0.5381], [0.1570, 0.2977], [0.2998, 0.4520] \rangle;$$
$$R_2 = \langle [0.6654, 0.8193], [0.1558, 0.2686], [0.1836, 0.3035] \rangle;$$
$$R_3 = \langle [0.5159, 0.6732], [0.2366, 0.3473], [0.3265, 0.4279] \rangle;$$
$$R_4 = \langle [0.5159, 0.8193], [0.0938, 0.1952], [0.2862, 0.4037] \rangle.$$

Step 5. **(a)** Determine the score values of $R_g (g = 1, 2, 3, 4)$ by Definition 5, we have

$$S(R_1) = 1.8087, S(R_2) = 2.2259, S(R_3) = 1.8656, S(R_4) = 2.2164;$$

(b) Determine the score values of $R_g (g = 1, 2, 3, 4)$ by Definition 5, we have

$$S(R_1) = 1.8671, S(R_2) = 2.2866, S(R_3) = 1.9254, S(R_4) = 2.1781;$$

Step 6. **(a)** According to their score and accuracy values, by using Definition 6, the ranking order is $M_2 > M_4 > M_3 > M_1$. So the best alternative is M_2, while the worst alternative is M_1.
(b) According to their score and accuracy values, by using Definition 6, the ranking order is $M_2 > M_4 > M_3 > M_1$. So the best alternative is M_2, while the worst alternative is M_1.

So, by using INWDPBM or INWDPGBM operators, the best alternative is M_2, while the worst alternative is M_1.

6.2. Effect of Parameters γ, x and y on Ranking Result of this Example

In order to show the effect of the parameters x and y on the ranking result of this example, we set different parameter values for x and y, and $\gamma = 3$ is fixed, to show the ranking results of this example. The ranking results are given in Table 3.

As we know from Tables 3 and 4, the score values and ranking order are different for different values of the parameters x and y, when we use INWDPBM operator and INWDPGBM operator. We can see from Tables 3 and 4, when the parameter values x = 1 or 0 and y = 0 or 1, the best choice is M_4 and the worst one is M_1. In simple words, when the interrelationship among attributes are not considered, the best choice is M_4 and the worst one is M_1. On the other hand, when different values for the parameters x and y are utilized, for INWPBM and INWDPGBM operators, the ranking result is changed. That is, from Table 4, we can see that when the parameter values $x = 1, y = 1$, the ranking results are changed as the one obtained for x = 1 or 0 and y = 0 or 1. In this case the best alternative is M_2 while the worst alternative remains the same.

Table 3. Ranking orders of decision result using different values for x and y for INWDPBM.

Parameter Values	INWDPBM Operator	Ranking Orders
$x=1, y=0, \gamma=3$	$S(R_1)=1.9319, S(R_2)=2.4172,$ $S(R_3)=2.0936, S(R_4)=2.4222;$	$M_4 > M_2 > M_3 > M_1.$
$x=1, y=5, \gamma=3$	$S(R_1)=1.8338, S(R_2)=2.2684,$ $S(R_3)=1.9049, S(R_4)=2.2666;$	$M_2 > M_4 > M_3 > M_1.$
$x=3, y=7, \gamma=3$	$S(R_1)=1.8169, S(R_2)=2.2398,$ $S(R_3)=1.8777, S(R_4)=2.2327;$	$M_2 > M_4 > M_3 > M_1.$
$x=5, y=10, \gamma=3$	$S(R_1)=1.8143, S(R_2)=2.2354,$ $S(R_3)=1.8738, S(R_4)=2.2275;$	$M_4 > M_2 > M_3 > M_1.$
$x=1, y=10, \gamma=3$	$S(R_1)=1.8501, S(R_2)=2.2966,$ $S(R_3)=1.9355, S(R_4)=2.3012;$	$M_4 > M_2 > M_3 > M_1.$
$x=10, y=4, \gamma=3$	$S(R_1)=1.8182, S(R_2)=2.2419,$ $S(R_3)=1.8796, S(R_4)=2.2352;$	$M_4 > M_2 > M_3 > M_1.$
$x=3, y=12, \gamma=3$	$S(R_1)=1.8285, S(R_2)=2.2592,$ $S(R_3)=1.8958, S(R_4)=2.2557;$	$M_2 > M_4 > M_3 > M_1.$

From Tables 3 and 4, we can observe that when the values of the parameter increase, the score values obtained using INWDPBM decrease. While using the INWDPGBM operator, the score values increase but the best choice is M_2 for $x=y\geq 1$.

Table 4. Ranking orders of decision result using different values for x and y for INWDPGBM.

Parameter Values	INWDPGBM Operator	Ranking Orders
$x=1, y=0, \gamma=3$	$S(R_1)=1.5032, S(R_2)=1.7934,$ $S(R_3)=1.5136, S(R_4)=1.8037;$	$M_4 > M_2 > M_3 > M_1.$
$x=1, y=5, \gamma=3$	$S(R_1)=1.8220, S(R_2)=2.2256,$ $S(R_3)=1.8717, S(R_4)=2.1140;$	$M_2 > M_4 > M_3 > M_1.$
$x=3, y=7, \gamma=3$	$S(R_1)=1.8539, S(R_2)=2.2686,$ $S(R_3)=1.9094, S(R_4)=2.1584;$	$M_2 > M_4 > M_3 > M_1.$
$x=5, y=10, \gamma=3$	$S(R_1)=1.8583, S(R_2)=2.2745,$ $S(R_3)=1.9146, S(R_4)=2.1647;$	$M_2 > M_4 > M_3 > M_1.$
$x=1, y=10, \gamma=3$	$S(R_1)=1.7814, S(R_2)=2.1710,$ $S(R_3)=1.8248, S(R_4)=2.0632;$	$M_2 > M_4 > M_3 > M_1.$
$x=10, y=4, \gamma=3$	$S(R_1)=1.8087, S(R_2)=2.2259,$ $S(R_3)=1.8656, S(R_4)=2.2164;$	$M_2 > M_4 > M_3 > M_1.$
$x=3, y=12, \gamma=3$	$S(R_1)=1.8671, S(R_2)=2.2866,$ $S(R_3)=1.9254, S(R_4)=2.1781;$	$M_4 > M_2 > M_3 > M_1.$

From Table 5, we can see that different ranking orders are obtained for different values of γ. When $\gamma=0.5$ and $\gamma=2$, the best choice is M_4 by the INWPBM operator; when we use the INWPGBM operator, it is M_2. Similarly, for other values of $\gamma>2$, the best choice is M_2 while the worst is M_1.

Table 5. Ranking orders of decision result using different values for γ.

Parameter Values	INWDPBM Operator	INWDPGBM Operator	Ranking Orders
$x=1, y=1, \gamma=0.5$	$S(R_1)=1.6662, S(R_2)=2.1025,$ $S(R_3)=1.7606, S(R_4)=2.1972;$	$S(R_1)=1.7870, S(R_2)=2.2347,$ $S(R_3)=1.9103, S(R_4)=2.1812;$	$M_4>M_2>M_3>M_1$. $M_2>M_4>M_3>M_1$.
$x=1, y=1, \gamma=2$	$S(R_1)=1.7783, S(R_2)=2.2015,$ $S(R_3)=1.8408, S(R_4)=2.2091;$	$S(R_1)=1.8491, S(R_2)=2.2786,$ $S(R_3)=1.9213, S(R_4)=2.1799;$	$M_4>M_2>M_3>M_1$. $M_2>M_4>M_3>M_1$.
$x=1, y=1, \gamma=4$	$S(R_1)=1.8229, S(R_2)=2.2363,$ $S(R_3)=1.8803, S(R_4)=2.2219;$	$S(R_1)=1.8740, S(R_2)=2.2856,$ $S(R_3)=1.9275, S(R_4)=2.1751;$	$M_2>M_4>M_3>M_1$. $M_2>M_4>M_3>M_1$.
$x=1, y=1, \gamma=7$	$S(R_1)=1.8375, S(R_2)=2.2455,$ $S(R_3)=1.9037, S(R_4)=2.2315;$	$S(R_1)=1.8747, S(R_2)=2.2763,$ $S(R_3)=1.9331, S(R_4)=2.1669;$	$M_2>M_4>M_3>M_1$. $M_2>M_4>M_3>M_1$.
$x=1, y=1, \gamma=10$	$S(R_1)=1.8418, S(R_2)=2.2477,$ $S(R_3)=1.9160, S(R_4)=2.2365;$	$S(R_1)=1.8701, S(R_2)=2.2698,$ $S(R_3)=1.9373, S(R_4)=2.1622;$	$M_2>M_4>M_3>M_1$. $M_2>M_4>M_3>M_1$.
$x=1, y=1, \gamma=15$	$S(R_1)=1.8447, S(R_2)=2.2488,$ $S(R_3)=1.9270, S(R_4)=2.2409;$	$S(R_1)=1.8642, S(R_2)=2.2637,$ $S(R_3)=1.9414, S(R_4)=2.1582;$	$M_2>M_4>M_3>M_1$. $M_2>M_4>M_3>M_1$.
$x=1, y=1, \gamma=20$	$S(R_1)=1.8460, S(R_2)=2.2492,$ $S(R_3)=1.9328, S(R_4)=2.2432;$	$S(R_1)=1.8608, S(R_2)=2.2604,$ $S(R_3)=1.9435, S(R_4)=2.1562;$	$M_2>M_4>M_3>M_1$. $M_2>M_4>M_3>M_1$.

6.3. Comparing with the Other Methods

To illustrate the advantages and effectiveness of the developed method in this article, we solve the above example by four existing MADM methods, including IN weighted averaging operator, IN weighted geometric operator [12], the similarity measure defined by Ye [15], Muirhead mean operators developed by Liu et al. [42], IN power aggregation operator developed by Liu et al. [37].

From Table 6, we can see that the ranking orders are the same as the ones produced by the existing aggregation operators when the parameter values $x=1, y=0, \gamma=3$, but the ranking orders are different when the interrelationship among attributes are considered. That is why the developed method based on the proposed aggregation operators is more flexible due the parameter and practical as it can consider the interrelationship among input arguments.

Table 6. Ranking order of the alternatives using different aggregation operators.

Aggregation Operator	Parameter	Score Values	Ranking Order
INWA operator [12]	No	$S(R_1)=1.8430, S(R_2)=2.2497,$ $S(R_3)=1.9151, S(R_4)=2.2788;$	$M_4 > M_2 > M_3 > M_1.$
INWGA operator [12]	No	$S(R_1)=1.7286, S(R_2)=2.0991,$ $S(R_3)=1.7751, S(R_4)=2.1608;$	$M_4 > M_2 > M_3 > M_1.$
Similarity measure Hamming distance [15]	No	$D_1(R^*,R_1)=0.7948, D_1(R^*,R_2)=0.9581,$ $D_1(R^*,R_3)=0.8805, D_1(R^*,R_4)=0.9725;$	$M_4 > M_2 > M_3 > M_1.$
Generalized power Aggregation operator [37]	Yes $\lambda=1$	$S(R_1)=1.8460, S(R_2)=2.2543,$ $S(R_3)=1.9163, S(R_4)=2.2799;$	$M_4 > M_2 > M_3 > M_1.$
INWMM operator [42]	Yes $P(1,1,1)$	$S(R_1)=1.8054, S(R_2)=2.2321,$ $S(R_3)=1.9172, S(R_4)=2.2773;$	$M_4 > M_2 > M_3 > M_1.$
INWDMM operator [42]	Yes $P(1,1,1)$	$S(R_1)=1.6260, S(R_2)=1.9202,$ $S(R_3)=1.7061, S(R_4)=2.0798;$	$M_4 > M_2 > M_3 > M_1.$
Proposed INWDPBM $x=1, y=0, \gamma=3$	Yes	$S(R_1)=1.9319, S(R_2)=2.4172,$ $S(R_3)=2.0936, S(R_4)=2.4222;$	$M_4 > M_2 > M_3 > M_1.$
Proposed INWDPGBM $x=1, y=0, \gamma=3$	Yes	$S(R_1)=1.5032, S(R_2)=1.7934,$ $S(R_3)=1.5136, S(R_4)=1.8037;$	$M_4 > M_2 > M_3 > M_1.$
INWDPBM operator in this article	Yes $x=y=1, \gamma=3$	$S(R_1)=1.8087, S(R_2)=2.2259,$ $S(R_3)=1.8656, S(R_4)=2.2164;$	$M_2 > M_4 > M_3 > M_1.$
INWDPBM operator in this article	Yes $x=y=1, \gamma=3$	$S(R_1)=1.8671, S(R_2)=2.2866,$ $S(R_3)=1.9254, S(R_4)=2.1781;$	$M_2 > M_4 > M_3 > M_1.$

From the above comparative analysis, we can know the proposed method has the following advantages, that is, it can consider the interrelationship among the input arguments and can relieve the effect of the awkward data by PWV at the same time, and it can permit more precise ranking order than the existing methods. The proposed method can take the advantages of PA operator and BM operator concurrently, these factors makes it a little complex in calculations.

The score values and ranking orders by these methods are shown in Table 6.

7. Conclusions

The PBM operator can take the advantage of PA operator, which can eliminate the impact of awkward data given by the predisposed DMs, and BM operator, which can consider the correlation between two attributes. The Dombi operations of TN and TCN proposed by Dombi have the edge of good flexibility with general parameter. In this article, we combined PBM with Dombi operation and proposed some aggregation operators to aggregate INNs. Firstly, we defined some operational laws for INSs based on Dombi TN and TCN and discussed some properties of these operations. Secondly, we extended PBM operator based on Dombi operations to introduce INDPBM operator, INWDPBM operator, INDPGBM operator, INWDPGBM operator and discussed some properties of these aggregation operators. The developed aggregation operators have the edge that they can take the correlation among the attributes by BM operator, and can also remove the effect of awkward data by PA operator at the same and due to general parameter, so they are more flexible in the aggregation process. Further, we developed a novel MADM method based on developed aggregation operators to deal with interval neutrosophic information. Finally, an illustrative example is used to show the effectiveness and practicality of the proposed MADM method and comparison were made with the existing methods. The proposed aggregation operators are very useful to solve MADM problems.

In future research, we shall define some distinct aggregation operators for SVHFSs, INHFSs, double valued neutrosophic sets and so on based on Dombi operations and apply them to MAGDM and MADM.

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Appendix A. Basic Concept of PBM Operator

Definition A1. *For positive real numbers* $\wp_h (h = 1, 2, ..., l)$ *and* $x, y > 0$ *the aggregation mapping [54]*

$$PBM^{x,y}(\wp_1, \wp_2, ..., \wp_l) = \left(\frac{1}{l^2 - l} \sum_{\substack{i=1, j=1 \\ i \neq j}}^{l} \left(\left(\frac{l(T(\wp_i)+1)}{\sum_{o=1}^{l}(T(\wp_o)+1)} \wp_i \right)^x \times \left(\frac{l(T(\wp_i)+1)}{\sum_{o=1}^{l}(T(\wp_o)+1)} \wp_j \right)^y \right) \right)^{\frac{1}{x+y}} \quad \text{(A1)}$$

is said to be power Bonferroni mean (PBM) mean operator.

Definition A2. *For positive real numbers* $\wp_h (h = 1, 2, ..., l)$ *and* $x, y > 0$ *the aggregation mapping [54]*

$$PBM^{x,y}(\wp_1, \wp_2, ..., \wp_l) = \frac{1}{x+y} \left(\prod_{\substack{i=1, j=1 \\ i \neq j}}^{l} \left(x \wp_i^{\frac{l(T(\wp_i)+1)}{\sum_{o=1}^{l}(T(\wp_o)+1)}} + y \wp_j^{\frac{l(T(\wp_j)+1)}{\sum_{o=1}^{l}(T(\wp_o)+1)}} \right) \right)^{\frac{1}{l^2-l}} \quad \text{(A2)}$$

is said to be power geometric Bonferroni mean (PGBM) mean operator.

In Definitions A1 and A2, $T(\wp_i) = \sum_{j=1, j \neq i}^{l} \text{supp}(\wp_i, \wp_j)$, and $\text{supp}(\wp_i, \wp_j)$ is the SPD for $\wp_i$ from $\wp_j$ satisfying the axioms as;

(1) $sup(\wp_i, \wp_j) = 1 - D(\wp_i, \wp_j)$, so $sup(\wp_i, \wp_j) \in [0,1]$;

(2) $sup(\wp_i, \wp_j) = sup(\wp_j, \wp_i)$;

(3) $sup(\wp_i, \wp_j) \geq sup(\wp_c, \wp_d)$, if $|\wp_i - \wp_y| \leq |\wp_c - \wp_d|$.

where $D(\wp_i, \wp_j)$ is the distance measure among $\wp_i$ and $\wp_j$.

Appendix B. Proof of Theorem 6

Proof. Since

$$l\Lambda_i \overline{\overline{in}}_i = \left\langle \left[1 - \frac{1}{1 + \left(l\Lambda_i \left(\frac{\overline{\overline{TR}}_i^L}{1 - \overline{\overline{TR}}_i^L} \right)^\gamma \right)^{\frac{1}{\gamma}}}, 1 - \frac{1}{1 + \left(l\Lambda_i \left(\frac{\overline{\overline{TR}}_i^U}{1 - \overline{\overline{TR}}_i^U} \right)^\gamma \right)^{\frac{1}{\gamma}}} \right], \left[\frac{1}{1 + \left(l\Lambda_i \left(\frac{1 - \overline{\overline{ID}}_i^L}{\overline{\overline{ID}}_i^L} \right)^\gamma \right)^{\frac{1}{\gamma}}}, 1 - \frac{1}{1 + \left(l\Lambda_i \left(\frac{1 - \overline{\overline{ID}}_i^U}{\overline{\overline{ID}}_i^U} \right)^\gamma \right)^{\frac{1}{\gamma}}} \right], \left[\frac{1}{1 + \left(l\Lambda_i \left(\frac{1 - \overline{\overline{FL}}_i^L}{\overline{\overline{FL}}_i^L} \right)^\gamma \right)^{\frac{1}{\gamma}}}, 1 - \frac{1}{1 + \left(l\Lambda_i \left(\frac{1 - \overline{\overline{FL}}_i^U}{\overline{\overline{FL}}_i^U} \right)^\gamma \right)^{\frac{1}{\gamma}}} \right] \right\rangle;$$

and

$$l\Lambda_j\overline{\overline{in}}_j=\left\langle\left[1-\frac{1}{1+\left(l\Lambda_j\left(\frac{\overline{\overline{TR}}_j^L}{1-\overline{\overline{TR}}_j^L}\right)^{\gamma}\right)^{\frac{1}{\gamma}}},1-\frac{1}{1+\left(l\Lambda_j\left(\frac{\overline{\overline{TR}}_j^U}{1-\overline{\overline{TR}}_j^U}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}\right],\left[\frac{1}{1+\left(l\Lambda_j\left(\frac{1-\overline{\overline{ID}}_j^L}{\overline{\overline{ID}}_j^L}\right)^{\gamma}\right)^{\frac{1}{\gamma}}},1-\frac{1}{1+\left(l\Lambda_j\left(\frac{1-\overline{\overline{ID}}_j^U}{\overline{\overline{ID}}_j^U}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}\right],\left[\frac{1}{1+\left(l\Lambda_j\left(\frac{1-\overline{\overline{FL}}_j^L}{\overline{\overline{FL}}_j^L}\right)^{\gamma}\right)^{\frac{1}{\gamma}}},1-\frac{1}{1+\left(l\Lambda_j\left(\frac{1-\overline{\overline{FL}}_j^U}{\overline{\overline{FL}}_j^U}\right)^{\gamma}\right)^{\frac{1}{\gamma}}}\right]\right\rangle.$$

Let

$$a_i=\frac{\overline{\overline{TR}}_i^L}{1-\overline{\overline{TR}}_i^L},b_i=\frac{\overline{\overline{TR}}_i^U}{1-\overline{\overline{TR}}_i^U},c_i=\frac{1-\overline{\overline{ID}}_i^L}{\overline{\overline{ID}}_i^L},d_i=\frac{1-\overline{\overline{ID}}_i^U}{\overline{\overline{ID}}_i^U},g_i=\frac{1-\overline{\overline{FL}}_i^L}{\overline{\overline{FL}}_i^L},h_i=\frac{1-\overline{\overline{FL}}_i^U}{\overline{\overline{FL}}_i^U},a_j=\frac{\overline{\overline{TR}}_j^L}{1-\overline{\overline{TR}}_j^L},b_j=\frac{\overline{\overline{TR}}_j^U}{1-\overline{\overline{TR}}_j^U},$$

$$c_j=\frac{1-\overline{\overline{ID}}_j^L}{\overline{\overline{ID}}_j^L},d_j=\frac{1-\overline{\overline{ID}}_j^U}{\overline{\overline{ID}}_j^U},g_j=\frac{1-\overline{\overline{FL}}_j^L}{\overline{\overline{FL}}_j^L},h_j=\frac{1-\overline{\overline{FL}}_j^U}{\overline{\overline{FL}}_j^U}.$$

then, we have

$$l\Lambda_i\overline{\overline{in}}_i=\left\langle\left[1-\frac{1}{1+\left(l\Lambda_i\right)^{\frac{1}{\gamma}}a_i},1-\frac{1}{1+\left(l\Lambda_i\right)^{\frac{1}{\gamma}}b_i}\right],\left[\frac{1}{1+\left(l\Lambda_i\right)^{\frac{1}{\gamma}}c_i},\frac{1}{1+\left(l\Lambda_i\right)^{\frac{1}{\gamma}}d_i}\right],\left[\frac{1}{1+\left(l\Lambda_i\right)^{\frac{1}{\gamma}}g_i},\frac{1}{1+\left(l\Lambda_i\right)^{\frac{1}{\gamma}}h_i}\right]\right\rangle;$$

$$l\Lambda_j\overline{\overline{in}}_j=\left\langle\left[1-\frac{1}{1+\left(l\Lambda_j\right)^{\frac{1}{\gamma}}a_j},1-\frac{1}{1+\left(l\Lambda_j\right)^{\frac{1}{\gamma}}b_j}\right],\left[\frac{1}{1+\left(l\Lambda_j\right)^{\frac{1}{\gamma}}c_j},\frac{1}{1+\left(l\Lambda_j\right)^{\frac{1}{\gamma}}d_j}\right],\left[\frac{1}{1+\left(l\Lambda_j\right)^{\frac{1}{\gamma}}g_j},\frac{1}{1+\left(l\Lambda_j\right)^{\frac{1}{\gamma}}h_j}\right]\right\rangle.$$

and

$$\left(l\Lambda_i\overline{\overline{in}}_i\right)^x=\left\langle\left[\frac{1}{1+x^{\frac{1}{\gamma}}\Big/\left(l\Lambda_i\right)^{\frac{1}{\gamma}}a_i},\frac{1}{1+x^{\frac{1}{\gamma}}\Big/\left(l\Lambda_i\right)^{\frac{1}{\gamma}}b_i}\right],\left[1-\frac{1}{1+x^{\frac{1}{\gamma}}\Big/\left(l\Lambda_i\right)^{\frac{1}{\gamma}}c_i},1-\frac{1}{1+x^{\frac{1}{\gamma}}\Big/\left(l\Lambda_i\right)^{\frac{1}{\gamma}}d_i}\right],\left[1-\frac{1}{1+x^{\frac{1}{\gamma}}\Big/\left(l\Lambda_i\right)^{\frac{1}{\gamma}}g_i},1-\frac{1}{1+x^{\frac{1}{\gamma}}\Big/\left(l\Lambda_i\right)^{\frac{1}{\gamma}}h_i}\right]\right\rangle;$$

$$\left(l\omega_jn_j\right)^y=\left\langle\left[\frac{1}{1+y^{\frac{1}{\gamma}}\Big/\left(l\omega_j\right)^{\frac{1}{\gamma}}a_j},\frac{1}{1+y^{\frac{1}{\gamma}}\Big/\left(l\omega_j\right)^{\frac{1}{\gamma}}b_j}\right],\left[1-\frac{1}{1+y^{\frac{1}{\gamma}}\Big/\left(l\omega_j\right)^{\frac{1}{\gamma}}c_j},1-\frac{1}{1+y^{\frac{1}{\gamma}}\Big/\left(l\omega_j\right)^{\frac{1}{\gamma}}d_j}\right],\left[1-\frac{1}{1+y^{\frac{1}{\gamma}}\Big/\left(l\omega_j\right)^{\frac{1}{\gamma}}g_j},1-\frac{1}{1+y^{\frac{1}{\gamma}}\Big/\left(l\omega_j\right)^{\frac{1}{\gamma}}h_j}\right]\right\rangle.$$

Moreover, we have

$$\left(l\Lambda_i\overline{\overline{in}}_i\right)^x\otimes_D\left(l\Lambda_j\overline{\overline{in}}_j\right)^y=\left\langle\left[\frac{1}{1+\left(x/l\Lambda_ia_i^{\gamma}+y/l\Lambda_ja_j^{\gamma}\right)^{\frac{1}{\gamma}}},\frac{1}{1+\left(x/l\Lambda_ib_i^{\gamma}+y/l\Lambda_jb_j^{\gamma}\right)^{\frac{1}{\gamma}}}\right],\left[1-\frac{1}{1+\left(x/l\Lambda_ic_i^{\gamma}+y/l\Lambda_jc_j^{\gamma}\right)^{\frac{1}{\gamma}}},1-\frac{1}{1+\left(x/l\Lambda_id_i^{\gamma}+y/l\Lambda_jd_j^{\gamma}\right)^{\frac{1}{\gamma}}}\right],\right.$$

$$\left.\left[1-\frac{1}{1+\left(x/l\Lambda_ig_i^{\gamma}+y/l\Lambda_jg_j^{\gamma}\right)^{\frac{1}{\gamma}}},1-\frac{1}{1+\left(x/l\Lambda_ih_i^{\gamma}+y/l\Lambda_jh_j^{\gamma}\right)^{\frac{1}{\gamma}}}\right]\right\rangle,$$

and

$$\sum_{\substack{i,j=1\\i\neq j}}^{l}\left(l\Lambda_i\overline{\overline{in}}_i\right)^x\otimes_D\left(l\Lambda_j\overline{\overline{in}}_j\right)^y$$

$$=\left\langle\left[1-1\Bigg/\left(1+\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}\left(\frac{1}{1+\left(x/l\Lambda_ia_i^{\gamma}+y/l\omega_ja_j^{\gamma}\right)^{\frac{1}{\gamma}}}\Bigg/\left(1-\frac{1}{1+\left(x/l\Lambda_ia_i^{\gamma}+y/l\Lambda_ja_j^{\gamma}\right)^{\frac{1}{\gamma}}}\right)\right)\right)^{\gamma}\right)^{\frac{1}{\gamma}},1-1\Bigg/\left(1+\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}\left(\frac{1}{1+\left(x/l\Lambda_ib_i^{\gamma}+y/l\Lambda_jb_j^{\gamma}\right)^{\frac{1}{\gamma}}}\Bigg/\left(1-\frac{1}{1+\left(x/l\Lambda_ib_i^{\gamma}+y/l\Lambda_jb_j^{\gamma}\right)^{\frac{1}{\gamma}}}\right)\right)\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right]\right.$$

$$\left[1\Bigg/\left(1+\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}\left(\left(1-1+\frac{1}{1+\left(x/l\Lambda_ic_i^{\gamma}+y/l\Lambda_jc_j^{\gamma}\right)^{\frac{1}{\gamma}}}\right)\Bigg/\left(1-\frac{1}{1+\left(x/l\Lambda_ic_i^{\gamma}+y/l\Lambda_jc_j^{\gamma}\right)^{\frac{1}{\gamma}}}\right)\right)\right)^{\gamma}\right)^{\frac{1}{\gamma}},\ 1\Bigg/\left(1+\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}\left(\left(1-1+\frac{1}{1+\left(x/l\Lambda_id_i^{\gamma}+y/l\omega_jd_j^{\gamma}\right)^{\frac{1}{\gamma}}}\right)\Bigg/\left(1-\frac{1}{1+\left(x/l\Lambda_id_i^{\gamma}+y/l\Lambda_jd_j^{\gamma}\right)^{\frac{1}{\gamma}}}\right)\right)\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right]$$

$$\left.\left[1\Bigg/\left(1+\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}\left(\left(1-1+\frac{1}{1+\left(x/l\Lambda_ig_i^{\gamma}+y/e\Lambda_jg_j^{\gamma}\right)^{\frac{1}{\gamma}}}\right)\Bigg/\left(1-\frac{1}{1+\left(x/l\Lambda_ig_i^{\gamma}+y/l\Lambda_jg_j^{\gamma}\right)^{\frac{1}{\gamma}}}\right)\right)\right)^{\gamma}\right)^{\frac{1}{\gamma}},\ 1\Bigg/\left(1+\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}\left(\left(1-1+\frac{1}{1+\left(x/l\Lambda_ih_i^{\gamma}+y/l\Lambda_jh_j^{\gamma}\right)^{\frac{1}{\gamma}}}\right)\Bigg/\left(1-\frac{1}{1+\left(x/l\Lambda_ih_i^{\gamma}+y/lv_jh_j^{\gamma}\right)^{\frac{1}{\gamma}}}\right)\right)\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right]\right\rangle$$

$$
\begin{aligned}
&=\left\langle\left[1-1\Big/\left(1+\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Big/\left(\frac{x}{l\Lambda_i a_i^{\gamma}}+\frac{y}{l\Lambda_j a_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right),1-1\Big/\left(1+\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Big/\left(\frac{x}{l\Lambda_i b_i^{\gamma}}+\frac{y}{l\Lambda_j b_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right],\left[1\Big/\left(1+\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Big/\left(\frac{x}{l\Lambda_i c_i^{\gamma}}+\frac{y}{l\Lambda_j c_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right),\right.\right.\\
&\left.\left.1\Big/\left(1+\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Big/\left(\frac{x}{l\Lambda_i d_i^{\gamma}}+\frac{y}{l\Lambda_j d_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right],\left[1\Big/\left(1+\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Big/\left(\frac{x}{l\Lambda_i g_i^{\gamma}}+\frac{y}{l\Lambda_j g_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right),1\Big/\left(1+\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Big/\left(\frac{x}{l\Lambda_i h_i^{\gamma}}+\frac{y}{l\Lambda_j h_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right]\right\rangle.
\end{aligned}
$$

So, we can have

$$
\frac{1}{l^2-l}\sum_{\substack{i,j=1\\ i\neq j}}^{l}\left(l\Lambda_i\overline{\overline{in}}_i\right)^{x}\otimes_D\left(l\Lambda_j\overline{\overline{in}}_j\right)^{y}
$$

$$
\begin{aligned}
&=\left\langle\left[\left(1-1\Big/\left(1+\left(\frac{1}{l^2-l}\left(\left(1-1\Big/1+\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Big/\left(\frac{x}{l\Lambda_i a_i^{\gamma}}+\frac{y}{l\Lambda_j a_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\Big/\left(1-1+1\Big/1+\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Big/\left(\frac{x}{e\Lambda_i a_i^{\gamma}}+\frac{y}{e\Lambda_j a_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)\right),\right.\right.\\
&\left.\left(1-1\Big/\left(1+\left(\frac{1}{l^2-l}\left(\left(1-1\Big/1+\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Big/\left(\frac{x}{l\Lambda_i b_i^{\gamma}}+\frac{y}{l\Lambda_j b_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\Big/\left(1-1+1\Big/1+\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Big/\left(\frac{x}{l\Lambda_i b_i^{\gamma}}+\frac{y}{l\Lambda_j b_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)\right)\right],\\
&\left[\left(1\Big/\left(1+\left(\frac{1}{l^2-l}\left(\left(1-1\Big/1+\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Big/\left(\frac{x}{l\Lambda_i c_i^{\gamma}}+\frac{y}{l\Lambda_j c_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\Big/\left(1\Big/1+\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Big/\left(\frac{x}{l\Lambda_i c_i^{\gamma}}+\frac{y}{l\Lambda_j c_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)\right),\right.\\
&\left.\left(1\Big/\left(1+\left(\frac{1}{l^2-l}\left(\left(1-1\Big/1+\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Big/\left(\frac{x}{l\Lambda_i d_i^{\gamma}}+\frac{y}{l\Lambda_j d_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\Big/\left(1\Big/1+\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Big/\left(\frac{x}{l\Lambda_i d_i^{\gamma}}+\frac{y}{l\Lambda_j d_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)\right)\right]\\
&\left[\left(1\Big/\left(1+\left(\frac{1}{l^2-l}\left(\left(1-1\Big/1+\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Big/\left(\frac{x}{l\Lambda_i e_i^{\gamma}}+\frac{y}{l\Lambda_j e_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\Big/\left(1\Big/1+\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Big/\left(\frac{x}{l\Lambda_i e_i^{\gamma}}+\frac{y}{l\Lambda_j e_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)\right),\right.\\
&\left.\left.\left(1\Big/\left(1+\left(\frac{1}{l^2-l}\left(\left(1-1\Big/1+\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Big/\left(\frac{x}{l\Lambda_i h_i^{\gamma}}+\frac{y}{l\Lambda_j h_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\Big/\left(1\Big/1+\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Big/\left(\frac{x}{l\Lambda_i h_i^{\gamma}}+\frac{y}{l\Lambda_j h_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right)^{\gamma}\right)^{\frac{1}{\gamma}}\right)\right)\right].
\end{aligned}
$$

$$
=\left\langle \left[\left(1-1\middle/\left(1+\left(\frac{1}{l^2-l}\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\middle/\left(\frac{x}{l\Lambda_i a_i^{\gamma}}+\frac{y}{l\Lambda_j a_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right),\left(1-1\middle/\left(1+\left(\frac{1}{l^2-l}\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\middle/\left(\frac{x}{l\Lambda_i b_i^{\gamma}}+\frac{y}{l\Lambda_j b_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right)\right],\left[\left(1\middle/\left(1+\left(\frac{1}{l^2-l}\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\middle/\left(\frac{x}{l\Lambda_i c_i^{\gamma}}+\frac{y}{l\Lambda_j c_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right),\right.\right.
$$

$$
\left.\left.\left(1-1\middle/\left(1+\left(\frac{1}{l^2-l}\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\middle/\left(\frac{x}{l\Lambda_i d_i^{\gamma}}+\frac{y}{l\Lambda_j d_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right)\right],\left[\left(1\middle/\left(1+\left(\frac{1}{l^2-l}\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\middle/\left(\frac{x}{l\Lambda_i g_i^{\gamma}}+\frac{y}{l\Lambda_j g_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right),\left(1\middle/\left(1+\left(\frac{1}{l^2-l}\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\middle/\left(\frac{x}{l\Lambda_i h_i^{\gamma}}+\frac{y}{l\Lambda_j h_j^{\gamma}}\right)\right)^{\frac{1}{\gamma}}\right)\right)\right]\right\rangle
$$

Then,

$$
\left(\frac{1}{l^2-l}\sum_{\substack{i,j=1\\ i\neq j}}^{l}\left(l\Lambda_i\overline{\overline{in}}_i\right)^{x}\otimes_D\left(l\Lambda_j\overline{\overline{in}}_j\right)^{y}\right)^{\frac{1}{x+y}}
$$

$$
=\left\langle\left[\left(1\middle/\left(1+\left(\frac{l^2-l}{x+y}\times 1\middle/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\middle/\left(\frac{x}{l\Lambda_i a_i^{\gamma}}+\frac{y}{l\Lambda_j a_j^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right),\left(1\middle/\left(1+\left(\frac{l^2-l}{x+y}\times 1\middle/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\middle/\left(\frac{x}{l\Lambda_i b_i^{\gamma}}+\frac{y}{l\Lambda_j b_j^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\right],\left[\left(1-1\middle/\left(1+\left(\frac{l^2-l}{x+y}\times 1\middle/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\middle/\left(\frac{x}{l\Lambda_i c_i^{\gamma}}+\frac{y}{l\Lambda_j c_j^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right),\right.\right. \tag{A3}
$$

$$
\left.\left.\left(1-1\middle/\left(1+\left(\frac{l^2-l}{x+y}\times 1\middle/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\middle/\left(\frac{x}{l\Lambda_i d_i^{\gamma}}+\frac{y}{l\Lambda_j d_j^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\right],\left[\left(1-1\middle/\left(1+\left(\frac{l^2-l}{x+y}\times 1\middle/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\middle/\left(\frac{x}{l\Lambda_i g_i^{\gamma}}+\frac{y}{l\Lambda_j g_j^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right),\left(1-1\middle/\left(1+\left(\frac{l^2-l}{x+y}\times 1\middle/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\middle/\left(\frac{x}{l\Lambda_i h_i^{\gamma}}+\frac{y}{l\Lambda_j h_j^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\right]\right\rangle.
$$

Now, put

$$
a_i=\frac{\overline{\overline{TR}}_i^L}{1-\overline{\overline{TR}}_i^L},b_i=\frac{\overline{\overline{TR}}_i^U}{1-\overline{\overline{TR}}_i^U},c_i=\frac{1-\overline{\overline{ID}}_i^L}{\overline{\overline{ID}}_i^L},d_i=\frac{1-\overline{\overline{ID}}_i^U}{\overline{\overline{ID}}_i^U},g_i=\frac{1-\overline{\overline{FL}}_i^L}{\overline{\overline{FL}}_i^L},h_i=\frac{1-\overline{\overline{FL}}_i^U}{\overline{\overline{FL}}_i^U},a_j=\frac{\overline{\overline{TR}}_j^L}{1-\overline{\overline{TR}}_j^L},b_j=\frac{\overline{\overline{TR}}_j^U}{1-\overline{\overline{TR}}_j^U},
$$

$$
c_j=\frac{1-\overline{\overline{ID}}_j^L}{\overline{\overline{ID}}_j^L},d_j=\frac{1-\overline{\overline{ID}}_j^U}{\overline{\overline{ID}}_j^U},g_j=\frac{1-\overline{\overline{FL}}_j^L}{\overline{\overline{FL}}_j^L}\ ,h_j=\frac{1-\overline{\overline{FL}}_j^U}{\overline{\overline{FL}}_j^U}.
$$

in Equation (A3), we can get

$$
\left(\frac{1}{l^2-l}\sum_{\substack{i,j=1\\ i\neq j}}^{l}\left(l\Lambda_i\overline{\overline{in}}_i\right)^{x}\otimes_D\left(l\Lambda_j\overline{\overline{in}}_j\right)^{y}\right)^{\frac{1}{x+y}}
$$

$$
=\left\langle\left[1\middle/\left(1+\left(\frac{l^2-l}{x+y}\times 1\middle/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\middle/\left(\frac{x}{l\Lambda_i\left(\frac{\overline{\overline{TR}}_i^L}{1-\overline{\overline{TR}}_i^L}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{\overline{\overline{TR}}_j^L}{1-\overline{\overline{TR}}_j^L}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right),1\middle/\left(1+\left(\frac{l^2-l}{x+y}\times 1\middle/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\middle/\left(\frac{x}{l\Lambda_i\left(\frac{\overline{\overline{TR}}_i^U}{1-\overline{\overline{TR}}_i^U}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{\overline{\overline{TR}}_j^U}{1-\overline{\overline{TR}}_j^U}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right],\right.
$$

$$
\left[1-1\middle/\left(1+\left(\frac{l^2-l}{x+y}\times 1\middle/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\middle/\left(\frac{x}{l\Lambda_i\left(\frac{1-\overline{\overline{ID}}_i^L}{\overline{\overline{ID}}_i^L}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{1-\overline{\overline{ID}}_j^L}{\overline{\overline{ID}}_j^L}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right),1-1\middle/\left(1+\left(\frac{l^2-l}{x+y}\times 1\middle/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\middle/\left(\frac{x}{l\Lambda_i\left(\frac{1-\overline{\overline{ID}}_i^U}{\overline{\overline{ID}}_i^U}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{1-\overline{\overline{ID}}_j^U}{\overline{\overline{ID}}_j^U}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right],
$$

$$
\left.\left[1-1\middle/\left(1+\left(\frac{l^2-l}{x+y}\times 1\middle/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\middle/\left(\frac{x}{l\Lambda_i\left(\frac{1-\overline{\overline{FL}}_i^L}{\overline{\overline{FL}}_i^L}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{1-\overline{\overline{FL}}_j^L}{\overline{\overline{FL}}_j^L}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right),1-1\middle/\left(1+\left(\frac{l^2-l}{x+y}\times 1\middle/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\middle/\left(\frac{x}{l\Lambda_i\left(\frac{1-\overline{\overline{FL}}_i^U}{\overline{\overline{FL}}_i^U}\right)^{\gamma}}+\frac{y}{l\Lambda_i\left(\frac{1-\overline{\overline{FL}}_j^U}{\overline{\overline{FL}}_j^U}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right].\right.
$$

This is the required proof of the Theorem 6. □

Appendix C. Proof of Theorem 7

Proof. Since all $\overline{\overline{in}}_i=\overline{\overline{in}}=\left\langle\left[\overline{\overline{TR}}^{L},\overline{\overline{TR}}^{U}\right],\left[\overline{\overline{ID}}^{L},\overline{\overline{ID}}^{U}\right],\left[\overline{\overline{FL}}^{L},\overline{\overline{FL}}^{U}\right]\right\rangle,(i=1,2,...,l)$, so we have $Supp\left(\overline{\overline{in}}_p,\overline{\overline{in}}_q\right)=1$, for all $p,q=1,2,...,l$, so $\Lambda_p=\frac{1}{l}$, for all $p=1,2,...,l$. Then

$$INDPBM^{x,y}\left(\overline{\overline{in}}_1,\overline{\overline{in}}_2,...,\overline{\overline{in}}_l\right)=INDPBM^{x,y}\left(\overline{\overline{in}},\overline{\overline{in}},...,\overline{\overline{in}}\right)$$

$$=\left\langle\left[\left(1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Big/\left(\frac{x}{l\frac{1}{l}\left(\frac{\overline{\overline{TR}}^{L}}{1-\overline{\overline{TR}}^{L}}\right)^{\gamma}}+\frac{x}{l\frac{1}{l}\left(\frac{\overline{\overline{TR}}^{L}}{1-\overline{\overline{TR}}^{L}}\right)^{\gamma}}\right)\right)\right)\right)^{\frac{1}{\gamma}}\right),\left(1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Big/\left(\frac{x}{l\frac{1}{l}\left(\frac{\overline{\overline{TR}}^{U}}{1-\overline{\overline{TR}}^{U}}\right)^{\gamma}}+\frac{x}{l\frac{1}{l}\left(\frac{\overline{\overline{TR}}^{U}}{1-\overline{\overline{TR}}^{U}}\right)^{\gamma}}\right)\right)\right)\right)^{\frac{1}{\gamma}}\right)\right],$$

$$\left[\left(1-1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Big/\left(\frac{x}{l\frac{1}{l}\left(\frac{1-\overline{\overline{ID}}^{L}}{\overline{\overline{ID}}^{L}}\right)^{\gamma}}+\frac{x}{l\frac{1}{l}\left(\frac{1-\overline{\overline{ID}}^{L}}{\overline{\overline{ID}}^{L}}\right)^{\gamma}}\right)\right)\right)\right)^{\frac{1}{\gamma}}\right),\left(1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Big/\left(\frac{x}{l\frac{1}{l}\left(\frac{1-\overline{\overline{ID}}^{U}}{\overline{\overline{ID}}^{U}}\right)^{\gamma}}+\frac{x}{l\frac{1}{l}\left(\frac{1-\overline{\overline{ID}}^{U}}{\overline{\overline{ID}}^{U}}\right)^{\gamma}}\right)\right)\right)\right)^{\frac{1}{\gamma}}\right)\right],$$

$$\left[\left(1-1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Big/\left(\frac{x}{l\frac{1}{l}\left(\frac{1-\overline{\overline{FL}}^{L}}{\overline{\overline{FL}}^{L}}\right)^{\gamma}}+\frac{x}{l\frac{1}{l}\left(\frac{1-\overline{\overline{FL}}^{L}}{\overline{\overline{FL}}^{L}}\right)^{\gamma}}\right)\right)\right)\right)^{\frac{1}{\gamma}}\right),\left(1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Big/\left(\frac{x}{l\frac{1}{l}\left(\frac{1-\overline{\overline{FL}}^{U}}{\overline{\overline{FL}}^{U}}\right)^{\gamma}}+\frac{x}{l\frac{1}{l}\left(\frac{1-\overline{\overline{FL}}^{U}}{\overline{\overline{FL}}^{U}}\right)^{\gamma}}\right)\right)\right)\right)^{\frac{1}{\gamma}}\right)\right].,$$

$$=\left\langle\left[\left(1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Big/\left(\frac{x+y}{\left(\frac{\overline{\overline{TR}}^{L}}{1-\overline{\overline{TR}}^{L}}\right)^{\gamma}}\right)\right)\right)\right)^{\frac{1}{\gamma}}\right),\left(1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Big/\left(\frac{x+y}{\left(\frac{\overline{\overline{TR}}^{U}}{1-\overline{\overline{TR}}^{U}}\right)^{\gamma}}\right)\right)\right)\right)^{\frac{1}{\gamma}}\right)\right],$$

$$\left[\left(1-1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Big/\left(\frac{x+y}{\left(\frac{1-\overline{\overline{ID}}^{L}}{\overline{\overline{ID}}^{L}}\right)^{\gamma}}\right)\right)\right)\right)^{\frac{1}{\gamma}}\right),\left(1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Big/\left(\frac{x+y}{\left(\frac{1-\overline{\overline{ID}}^{U}}{\overline{\overline{ID}}^{U}}\right)^{\gamma}}\right)\right)\right)\right)^{\frac{1}{\gamma}}\right)\right],$$

$$\left[\left(1-1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Big/\left(\frac{x+y}{\left(\frac{1-\overline{\overline{FL}}^{L}}{\overline{\overline{FL}}^{L}}\right)^{\gamma}}\right)\right)\right)\right)^{\frac{1}{\gamma}}\right),\left(1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\i\neq j}}^{l}1\Big/\left(\frac{x+y}{\left(\frac{1-\overline{\overline{FL}}^{U}}{\overline{\overline{FL}}^{U}}\right)^{\gamma}}\right)\right)\right)\right)^{\frac{1}{\gamma}}\right)\right],$$

$$=\left\langle\left[\left(\frac{1}{1+\left(\frac{l^2-l}{x+y}\times\frac{1}{\frac{1}{\frac{x+y}{\left(\frac{\overline{\overline{TR}}^{L}}{1-\overline{\overline{TR}}^{L}}\right)^{\gamma}}}}\right)}\right)^{\frac{1}{\gamma}},\left(\frac{1}{1+\left(\frac{l^2-l}{x+y}\times\frac{1}{\frac{1}{\frac{x+y}{\left(\frac{\overline{\overline{TR}}^{U}}{1-\overline{\overline{TR}}^{U}}\right)^{\gamma}}}}\right)}\right)^{\frac{1}{\gamma}}\right],\right.$$

$$\left[1-\left(\frac{1}{1+\left(\frac{l^2-l}{x+y}\times\frac{1}{\frac{1}{\frac{x+y}{\left(\frac{1-\overline{\overline{ID}}^{L}}{\overline{\overline{ID}}^{L}}\right)^{\gamma}}}}\right)}\right)^{\frac{1}{\gamma}},1-\left(\frac{1}{1+\left(\frac{l^2-l}{x+y}\times\frac{1}{\frac{1}{\frac{x+y}{\left(\frac{1-\overline{\overline{ID}}^{U}}{\overline{\overline{ID}}^{U}}\right)^{\gamma}}}}\right)}\right)^{\frac{1}{\gamma}}\right],$$

$$\left.\left[1-\left(\frac{1}{1+\left(\frac{l^2-l}{x+y}\times\frac{1}{\frac{1}{\frac{x+y}{\left(\frac{1-\overline{\overline{FL}}^{L}}{\overline{\overline{FL}}^{L}}\right)^{\gamma}}}}\right)}\right)^{\frac{1}{\gamma}},1-\left(\frac{1}{1+\left(\frac{l^2-l}{x+y}\times\frac{1}{\frac{1}{\frac{x+y}{\left(\frac{1-\overline{\overline{FL}}^{U}}{\overline{\overline{FL}}^{U}}\right)^{\gamma}}}}\right)}\right)^{\frac{1}{\gamma}}\right]\right\rangle,$$

$$\left\langle\left[\overline{\overline{TR}}^{L},\overline{\overline{TR}}^{U}\right],\left[\overline{\overline{ID}}^{L},\overline{\overline{ID}}^{U}\right],\left[\overline{\overline{FL}}^{L},\overline{\overline{FL}}^{U}\right]\right\rangle=\overline{\overline{in}}.$$

□

Appendix D. Proof of Theorem 9

Proof. Since

$$\overline{\overline{in}}^{+}=\left\langle\max_{i=1}^{l}\left[\overline{\overline{TR}}_i^{L},\overline{\overline{TR}}_i^{U}\right],\min_{i=1}^{l}\left[\overline{\overline{ID}}_i^{L},\overline{\overline{ID}}_i^{U}\right],\min_{i=1}^{l}\left[\overline{\overline{FL}}_i^{L},\overline{\overline{FL}}_i^{U}\right]\right\rangle,\overline{\overline{in}}^{-}=\left\langle\min_{i=1}^{l}\left[\overline{\overline{TR}}_i^{L},\overline{\overline{TR}}_i^{U}\right],\max_{i=1}^{l}\left[\overline{\overline{ID}}_i^{L},\overline{\overline{ID}}_i^{U}\right],\max_{i=1}^{l}\left[\overline{\overline{FL}}_i^{L},\overline{\overline{FL}}_i^{U}\right]\right\rangle,$$

then, there are

$$\overline{\overline{TR}}^{L-}\le\overline{\overline{TR}}^{L}(n_i)\le\overline{\overline{TR}}^{L+},\overline{\overline{TR}}^{U-}\le\overline{\overline{TR}}^{U}(n_i)\le\overline{\overline{TR}}^{U+},\overline{\overline{ID}}^{L-}\le\overline{\overline{ID}}^{L}(n_i)\le\overline{\overline{ID}}^{L+},\overline{\overline{ID}}^{U-}\le\overline{\overline{ID}}^{U}(n_i)\le\overline{\overline{ID}}^{U+},\overline{\overline{FL}}^{L-}\le\overline{\overline{FL}}^{L}(n_i)\le\overline{\overline{FL}}^{L+},$$

$$\overline{\overline{FL}}^{U-}\le\overline{\overline{FL}}^{U}(n_i)\le\overline{\overline{FL}}^{U+},$$

for all $i=1,2,\ldots,l$. We have

$$\overline{\overline{TR}}^{L}(\overline{\overline{in}}_i)=\left(\frac{1}{1+\left(\frac{l^2-l}{x+y}\times\frac{1}{\sum_{\substack{i,j=1\\i\ne j}}^{l}\frac{1}{\frac{x}{l\Lambda_i\left(\frac{\overline{\overline{TR}}_i^{L}}{1-\overline{\overline{TR}}_i^{L}}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{\overline{\overline{TR}}_j^{L}}{1-\overline{\overline{TR}}_j^{L}}\right)^{\gamma}}}}\right)}\right)^{\frac{1}{\gamma}}\ge\left(\frac{1}{1+\left(\frac{l^2-l}{x+y}\times\frac{1}{\sum_{\substack{i,j=1\\i\ne j}}^{l}\frac{1}{\frac{x}{l\Lambda_i\left(\frac{\overline{\overline{TR}}_i^{L-}}{1-\overline{\overline{TR}}_i^{L-}}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{\overline{\overline{TR}}_j^{L-}}{1-\overline{\overline{TR}}_j^{L-}}\right)^{\gamma}}}}\right)}\right)^{\frac{1}{\gamma}}=\overline{\overline{TR}}^{L-},$$

$$\overline{\overline{TR}}^{U}(\overline{\overline{in}}_i)=\left(\frac{1}{1+\left(\frac{l^2-l}{x+y}\times\frac{1}{\sum_{\substack{i,j=1\\i\ne j}}^{l}\frac{1}{\frac{x}{l\Lambda_i\left(\frac{\overline{\overline{TR}}_i^{U}}{1-\overline{\overline{TR}}_i^{U}}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{\overline{\overline{TR}}_j^{U}}{1-\overline{\overline{TR}}_j^{U}}\right)^{\gamma}}}}\right)}\right)^{\frac{1}{\gamma}}\ge\left(\frac{1}{1+\left(\frac{l^2-l}{x+y}\times\frac{1}{\sum_{\substack{i,j=1\\i\ne j}}^{l}\frac{1}{\frac{x}{l\Lambda_i\left(\frac{\overline{\overline{TR}}_i^{U-}}{1-\overline{\overline{TR}}_i^{U-}}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{\overline{\overline{TR}}_j^{U-}}{1-\overline{\overline{TR}}_j^{U-}}\right)^{\gamma}}}}\right)}\right)^{\frac{1}{\gamma}}=\overline{\overline{TR}}^{U-},$$

$$\overline{\overline{ID}}^{L}\left(\overline{\overline{in}}_i\right)=1-\left(\frac{1}{1+\left(\frac{l^2-l}{x+y}\times\frac{1}{\sum_{\substack{i,j=1\\i\ne j}}^{l}\frac{1}{\frac{x}{l\Lambda_i\left(\frac{1-\overline{\overline{ID}}_i^{L}}{\overline{\overline{ID}}_i^{L}}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{1-\overline{\overline{ID}}_j^{L}}{\overline{\overline{ID}}_j^{L}}\right)^{\gamma}}}}\right)}\right)^{\frac{1}{\gamma}}\le1-\left(\frac{1}{1+\left(\frac{l^2-l}{x+y}\times\frac{1}{\sum_{\substack{i,j=1\\i\ne j}}^{l}\frac{1}{\frac{x}{l\Lambda_i\left(\frac{1-\overline{\overline{ID}}_i^{L+}}{\overline{\overline{ID}}_i^{L+}}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{1-\overline{\overline{ID}}_j^{L+}}{\overline{\overline{ID}}_j^{L+}}\right)^{\gamma}}}}\right)}\right)^{\frac{1}{\gamma}}=\overline{\overline{ID}}^{L+}$$

$$\overline{\overline{ID}}^{U}\left(\overline{\overline{in}}_i\right)=1-\left(\frac{1}{1+\left(\frac{l^2-l}{x+y}\times\frac{1}{\sum_{\substack{i,j=1\\i\ne j}}^{l}\frac{1}{\frac{x}{l\Lambda_i\left(\frac{1-\overline{\overline{ID}}_i^{U}}{\overline{\overline{ID}}_i^{U}}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{1-\overline{\overline{ID}}_j^{U}}{\overline{\overline{ID}}_j^{U}}\right)^{\gamma}}}}\right)}\right)^{\frac{1}{\gamma}}\le1-\left(\frac{1}{1+\left(\frac{l^2-l}{x+y}\times\frac{1}{\sum_{\substack{i,j=1\\i\ne j}}^{l}\frac{1}{\frac{x}{l\Lambda_i\left(\frac{1-\overline{\overline{ID}}_i^{U+}}{\Psi_i^{U+}}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{1-\overline{\overline{ID}}_j^{U+}}{\overline{\overline{ID}}_j^{U+}}\right)^{\gamma}}}}\right)}\right)^{\frac{1}{\gamma}}=\overline{\overline{ID}}^{U+}$$

$$\overline{\overline{FL}}^{L}\left(\overline{\overline{in}}_i\right)=\left(1-1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Big/\left(\frac{x}{l\Lambda_i\left(\frac{1-\overline{\overline{FL}}_i^{L}}{\overline{\overline{FL}}_i^{L}}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{1-\overline{\overline{FL}}_j^{L}}{\overline{\overline{FL}}_j^{L}}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\leq\left(1-1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Big/\left(\frac{x}{l\Lambda_i\left(\frac{1-\overline{\overline{FL}}_i^{L+}}{\overline{\overline{FL}}_i^{L+}}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{1-\overline{\overline{FL}}_j^{L+}}{\overline{\overline{FL}}_j^{L+}}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)=\overline{\overline{FL}}^{L+}$$

$$\Upsilon^{U}\left(\overline{\overline{in}}_i\right)=\left(1-1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Big/\left(\frac{x}{l\Lambda_i\left(\frac{1-\Upsilon_i^{U}}{\Upsilon_i^{U}}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{1-\Upsilon_j^{U}}{\Upsilon_j^{U}}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\leq\left(1-1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Big/\left(\frac{x}{l\Lambda_i\left(\frac{1-\Upsilon_i^{U+}}{\Upsilon_i^{U+}}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{1-\Upsilon_j^{U+}}{\Upsilon_j^{U+}}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)=\Upsilon^{U+}$$

Then there are the following scores

$$\frac{\overline{\overline{TR}}^{L}+\overline{\overline{TR}}^{U}}{2}+1-\frac{\overline{\overline{ID}}^{L}+\overline{\overline{ID}}^{U}}{2}+1-\frac{\overline{\overline{FL}}^{L}+\overline{\overline{FL}}^{U}}{2}=$$

$$\left(\left(1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Big/\left(\frac{x}{l\Lambda_i\left(\frac{\overline{\overline{TR}}_i^{L}}{1-\overline{\overline{TR}}_i^{L}}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{\overline{\overline{TR}}_j^{L}}{1-\overline{\overline{TR}}_j^{L}}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)+1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Big/\left(\frac{x}{l\Lambda_i\left(\frac{\overline{\overline{TR}}_i^{U}}{1-\overline{\overline{TR}}_i^{U}}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{\overline{\overline{TR}}_j^{U}}{1-\overline{\overline{TR}}_j^{U}}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\Big/2\right.$$

$$+1-\left(\left(1-1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Big/\left(\frac{x}{l\Lambda_i\left(\frac{1-\overline{\overline{ID}}_i^{L}}{\overline{\overline{ID}}_i^{L}}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{1-\overline{\overline{ID}}_j^{L}}{\overline{\overline{ID}}_j^{L}}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)+\left(1-1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Big/\left(\frac{x}{l\Lambda_i\left(\frac{1-\overline{\overline{ID}}_i^{U}}{\overline{\overline{ID}}_i^{U}}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{1-\overline{\overline{ID}}_j^{U}}{\overline{\overline{ID}}_j^{U}}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\Big/2\right)$$

$$\left.+1-\left(\left(1-1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Big/\left(\frac{x}{l\Lambda_i\left(\frac{1-\overline{\overline{FL}}_i^{L}}{\overline{\overline{FL}}_i^{L}}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{1-\overline{\overline{FL}}_j^{L}}{\overline{\overline{FL}}_j^{L}}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)+\left(1-1\Big/\left(1+\left(\frac{l^2-l}{x+y}\times 1\Big/\left(\sum_{\substack{i,j=1\\ i\neq j}}^{l}1\Big/\left(\frac{x}{l\Lambda_i\left(\frac{1-\overline{\overline{FL}}_i^{U}}{\overline{\overline{FL}}_i^{U}}\right)^{\gamma}}+\frac{y}{l\Lambda_j\left(\frac{1-\overline{\overline{FL}}_j^{U}}{\overline{\overline{FL}}_j^{U}}\right)^{\gamma}}\right)\right)\right)^{\frac{1}{\gamma}}\right)\right)\Big/2\right)$$

$$\geq\frac{\overline{\overline{TR}}^{L-}+\overline{\overline{TR}}^{U-}}{2}+1-\frac{\overline{\overline{ID}}^{L+}+\overline{\overline{ID}}^{U+}}{2}+1-\frac{\overline{\overline{FL}}^{L+}+\overline{\overline{FL}}^{U+}}{2}.$$

Therefore according to the Definition 6Equation, we have

$$\overline{\overline{in}}^{-}\leq INDPBM\left(\overline{\overline{in}}_1,\overline{\overline{in}}_2,\ldots,\overline{\overline{in}}_l\right).$$

In a similar way, the other part can be proved. That is $INDPBM\left(\overline{\overline{in}}_1,\overline{\overline{in}}_2,\ldots,\overline{\overline{in}}_m\right)\leq\overline{\overline{in}}^{+}$. Hence

$$\overline{\overline{in}}^{-}\leq INDPBM\left(\overline{\overline{in}}_1,\overline{\overline{in}}_2,\ldots,\overline{\overline{in}}_m\right)\leq\overline{\overline{in}}^{+}.$$

□

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Decision Making Methods for Evaluation of Efficiency of General Insurance Companies in Malaysia: A Comparative Study

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ABSTRACT This paper proposes an integration of two neutrosophic based multi-criteria decision making methods, namely the neutrosophic data analytical hierarchy process (NDAHP) and the Technique of Order Preference by Similarity to Ideal Solution (TOPSIS) with maximizing deviation method, both based on the single-valued neutrosophic set (SVNS) to evaluate the efficiency of general insurance companies in Malaysia. The level of efficiency of insurance companies is a subjective and vague matter, as the efficiency can be further branched into operational efficiency, investment efficiency, underwriting efficiency, and risk management efficiency. Hence relying on entirely objective decision making methods based on crisp data might not address the problem effectively, and therefore fuzzy based decision making methods are highly appropriate to be used in this situation. Our proposed decision making algorithm uses an integrated weighting mechanism that takes into consideration both the objective and subjective weights of the data attributes. The objective weighting mechanism handles the actual datasets that were used which consists of crisp values, whereas the subjective weighing mechanism handles the opinions of the experts in the general insurance industry who were surveyed in this study. This makes the proposed method a more holistic approach to evaluate the efficiency of general insurance companies in Malaysia as previous researches in this area are generally based on the actual datasets without consideration of the opinions and evaluations of the industry experts, or vice-versa. The proposed decision making algorithm is applied on actual datasets of management expenses, net commission, net earned premium and the net investment income for 19 selected general insurance companies in Malaysia over a two-year period from 2016 to 2017. The results obtained are then discussed and the possible reasons for the results are analyzed. A comprehensive comparative study of the results obtained via our proposed method and two other commonly used methods are then presented, analyzed and discussed.

INDEX TERMS Single-valued neutrosophic set, analytic hierarchy process (AHP), multi-criteria decision making, neutrosophic data AHP, neutrosophic decision making, efficiency of general insurance companies, TOPSIS, maximizing deviation method.

I. INTRODUCTION

General insurance is basically a type of insurance product that protects the insured against losses and damages other than those covered in the category of life insurance, which pays the benefit or sum assured upon death or expiry of the policy. The risk covered in general insurance includes property loss, liability loss or damage caused by a third party as well as acci-dental death or injury. Generally, products in the category of general insurance are insurances which are related to property and casualty. The general insurance industry in Malaysia is still in the development process compared to other developing countries. According to the official website of the General Insurances Association of Malaysia (PIAM), the number of general insurance companies which have been registered as a member of the Association currently stands at 21. The intense competition has meant that there is a need for the insurance companies to strengthen their position in the market and to operate in the most efficient manner. This is also true in other developing and developed markets, and this has led to an increasing number of researches on the evaluation of the efficiency of financial institutions in the past few years.

Fuzzy set is modified from the classical set theory which was firstly introduced by Zadeh in 1965 [1]. In classical set theory, the way to classify each element in a set was dichotomous, which means the belongingness of an element is either positive or negative in the set. On the contrary, in fuzzy set theory, the set is able to deal with a certain magnitude of membership. The concept of fuzziness generally deals with a degree of truthfulness of values, uncertainty, vagueness and imprecision. The ability of fuzzy sets to model the uncertainty of objects and its ability in capturing the imprecision in defining a sharp criteria of class membership functions has led to the rapid progress in research pertaining to the fuzzy set theory. This phenomenon led to the development of many similar models.

The most commonly used ones which are relevant to this research project are the intuitionistic fuzzy set [2] and neutrosophic set [3]. In 1986, Atanassov [2] proposed the concept of intuitionistic fuzzy set (IFS) which was modified from the general fuzzy set model. In Zadeh's fuzzy set theory, the membership function is single-valued and assumes values between zero and one, while the IFS model has the characteristic of having a dual membership function system which comprises of a membership function and a non-membership function. In the real world, it is not always true that the magnitude to which an object does not belong to a set fully complements the magnitude to which an object belongs to the set. Since the IFS model takes a degree of non-membership into account, it has been powerful enough to attract researchers and authors to apply the IFS model in various fields of study.

In 1998, Smarandache introduced a more advanced form of fuzzy sets called neutrosophic sets that is able to handle the fuzziness that exists in real-life situations in a more practical manner compared to Zadeh's fuzzy set model [1] and the IFS model [2]. The neutrosophic set is essentially an extension of the IFS model, albeit a more accurate one. The basis of neutrosophic set is neutrosophic logic that can describe a degree of membership of an indeterminable element that neither belongs nor does not belong, cases which are not taken into consideration in the ordinary fuzzy set model and the IFS model. The distinctive characteristics of neutrosophic set is that it has a unique triple membership structure that describes the degree of truth (T), falsity (F) and indeterminacy (I) for each model. In the neutrosophic model, the values from the three membership functions are not influenced by one another, that is, the sum of the three membership functions do not necessarily need to sum up to 1, and need not necessarily complement to 1. Unlike other fuzzy sets, the interval of the membership function for the original neutrosophic sets is defined to be the non-standard interval of $]_0^-0, 1^+[$. Using this non-standard interval, it enables users to distinguish whether the grade of membership of the set is relative or absolute. In the event that the truth is relative, it is represented as 1 and the truth is absolute, it is represented as 1^+. Likewise, in the event that the falsehood is relative, it is represented as 0, and in the event that the falsehood is absolute, it is represented as $_0^-0$.

The non-standard interval of $]_0^-0, 1^+[$ in which the membership functions of neutrosophic sets are generated makes the application to solve the problems existing in real world exceedingly impractical especially in the areas of engineering and science. This led to the conceptualization of a type of neutrosophic sets called the single-valued neutrosophic sets (SVNSs) [4]. The SVNS model's membership structure is similar to the original model, but all of these membership functions take on values between 0 and 1, similar to the membership function in fuzzy sets and IFSs. This also makes it more compatible with the other fuzzy decision making methods. In this sense, the SVNS model would be more suitable to be used to solve problems that arise in real-life with indeterminate and incomplete information.

The analytic hierarchy process (AHP) is a technique to handle complex criteria which could be quantifiable or intangible in decision-making. The AHP was first introduced by Thomas L. Saaty in 1990 [5]. The AHP analyses a set of criteria to be assessed and chooses the optimal alternative. The most powerful feature of AHP is that AHP can create a weight of every single criterion in a set according to the decision maker's pairwise comparisons of the criteria. In addition, when making decisions, the AHP considers human experiences and knowledge [6].

In the real-world, all of the information is not necessarily quantifiable nor complete. To resolve the problem, the fuzzy analytic hierarchy process (FAHP) was developed by Van Laarhoven and Pedrycz in 1983 [7]. The FAHP is an integration of fuzzy set theory and the AHP method, which is an effective tool to deal with such vagueness that may exist in the decision-making process. As the popularity of AHP grew, it was adapted to other forms of fuzzy sets and this led to the introduction of intuitionistic fuzzy AHP (IFAHP) by Xu and Liao [8], interval-valued fuzzy AHP (IVFAHP) by Mirzaei et al. [9], interval-valued intuitionistic fuzzy AHP (IVIFAHP) by Abdullah and Najib [10], and neutrosophic AHP (NAHP) by Radwan *et al.* [11].

In this paper, a comprehensive method that utilizes an objective and subjective weighting mechanism is proposed. Our proposed method involves the use of NDAHP, the integration of the weightage of each data component both objectively and subjectively using the maximizing deviation method, followed by the evaluation of the performance of the general insurance companies in Malaysia with respect to their efficiency level via the neutrosophic TOPSIS method.

The remainder of this paper is organized as follows. In Section 2, we recapitulate some of the fundamental concepts related to SVNSs and some of the recent developments related to SVNS based decision making. In Section 3, we introduce a decision making algorithm that integrates the concepts of converting raw numerical data into fuzzy number using the NDAHP method, followed by the derivation of weightage for each criteria based on the maximizing deviation method, and subsequently the ranking of the alternatives based on the SVNS TOPSIS approach to evaluate the efficiency of 19 general insurance companies in Malaysia over the period of 2016- 2017 based on four selected financial parameters. In Section 4, the results obtained from the proposed decision making algorithm are presented, along with a brief discussion on certain observations. Furthermore, a comparative study that compares and analyzes the results obtained through our proposed method and two other conventionally used decision making methods to evaluate the efficiency of general insurance companies are provided. Concluding remarks are given in Section 5, followed by acknowledgements and the list of references.

II. RELATED WORKS

Some of the recent research development on FAHP, IFAHP and NAHP are expounded below. Putra et al. [12] proposed a FAHP model to determine the quality of gemstones. This research has created a system which can evaluate the quality of gemstones accurately and effectively, based on the data criteria which consist of the specific gravity, color, hardness, cutting and clarity of the gemstones. Recognizing that medical practitioners may have different opinions during medical procedures, particularly in the diagnosis of patients, and that this may lead to different actions and decisions, Moghaddam et al. [13] proposed a diagnosis system based on FAHP and Fuzzy Inference System (FIS). This system was used to evaluate the conditions of patients who were being examined for heart disease. Biswas et al. [14] on the other hand, developed a FAHP based method to determine the apparel item that should be manufactured among a wide range of apparels in order to obtain maximum profit, with the aim of helping investors looking to invest in a garment factory in Bangladesh. Basar [15] utilized FAHP to design a strategic decision making method to assess the supplier selection problem in the Turkish construction industry. The proposed model considered three main supplier selection criteria, namely product features, suppliers' features and delivery conditions in the study. The proposed method has shortened and simplified the complicated and generally tangled process of choosing the right construction supplier. Tan et al. [16] has proposed a FAHP approach, as a systematic and simple methodology in the multi- criteria evaluation of alternatives for the harvesting and drying process of microalgae production. Several criteria related to the harvesting and drying methods as well as technology capability, cost and the environmental impact of these methods were considered in this study. Kaur [17] proposed a triangular intuitionistic fuzzy number-based approach for vendor selection problem using the AHP method and the five criteria that were considered were the cost, quality, cycle time, service and reputation. Abdullah et al. [18] has ranked the human capital indicators using a hybridization of the AHP and two- sided evaluation using the IFAHP method. In this study, linguistic data were obtained from experts in the area of human capital management via questionnaires. The proposed IFAHP method was used to evaluate the four main indicators of human capital, and subsequently determine the most important criteria in human capital management. Abdullah and Liana [19] also proposed a new IFAHP method that is characterized by a new preference scale of pair- wise comparison matrix measurement. The new preference scale has considered the degree of hesitation of IFS in expressing the conversion of consistency to triangular intuitionistic fuzzy numbers, and the proposed method were applied on three MCDM problems. Burak and Faruk [20] has proposed an integrated multi- criteria decision making method for personnel selection with perfect multiplicative consistent intuitionistic preference relation under an intuitionistic fuzzy environment. Ouyang and Guo [21] have developed an IFAHP method to select the mangrove paradigm which is optimal for municipal wastewater treatment. The entropy weights in this research were entrained by the valid evaluation of 64 experts and representatives via an online survey. An extension of the NAHP called the Neutrosophic Data Analytic Hierarchy Process (NDAHP) was introduced by Tey et al. in [22]. The NDAHP method is based on an objective weighting mechanism that is designed to handle actual data sets that consists of crisp values. This framework was used to assess the financial performance of five petrochemical companies listed on the main board of the Kuala Lumpur Stock Exchange (KLSE).

The maximizing deviation approach was first introduced by Wang [23] with the goal of utilizing this method to quantify the importance of attributes which are completely unknown or only partially known input arguments. This approach involves the magnitude of the membership function of each alternative for each data attribute in the derivation of the weight coefficients. It has been used to determine the objective weightage of data attributes combined with the subjective weightage in an integrated manner to solve a multitude of MADM problems [24].

The technique for order preference by similarity to ideal solution (TOPSIS) was introduced by Hwang and Yoon in 1981 [25]. In this method, there are two artificial hypothesized alternatives which are the positive ideal alternative and the negative ideal alternative. The positive ideal alternative is the alternative that can generate the maximum benefit criteria and the minimum cost criteria, whereas the negative ideal alternative is the alternative that performs on the contrary to the former. In TOPSIS, the optimal alternative should have the shortest distance from the positive ideal alternative and the farthest distance from the negative ideal solution. Bulgurcu [26] proposed a TOPSIS based group decision making method to identify the company with the best green supply chain management among six different tyre companies. Five criteria were considered namely green design, green transformation, green purchasing, green logistics and reverse logistics. Sahin and Yigider [27] has extended the TOPSIS approach to multi- criteria group decision making and applied this method in a supplier selection problem, whereby the importance of the criteria and alternatives were identified by aggregating individual opinions of the decision makers via single-valued neutrosophic weighted averaging (SVNWA) operator. Balioti et al. [28] introduced an integrated model of fuzzy TOPSIS and AHP to select the optimal spillway for a dam in the district of Kilkis in Northern Greece. A modified TOPSIS method called the D-TOPSIS method was introduced by Fei et al. [29] and applied on the selection of candidates in human resource management. This method is based on the concept of D-numbers which has been generalized from the Dempster-Shafer evidence theory to represent uncertain information which can denote fuzzy conditions more effectively. Forgahani et al. [30] applied principal component analysis, TOPSIS and mixed integer linear programming to solve a pharmaceutical supply chain problem, particularly on the selection of the suppliers. Siddique et al. [31] applied fuzzy TOPSIS in ranking the decision criteria of flexible manufacturing system (FMS) with the aim of assisting the management of a company in their decision making process during the implementation of FMS. Huang and Jiang [32] extended the fuzzy TOPSIS method by introducing a component known as optimism coefficient in solving problems related to investment, with the optimism coefficient used to describe the attitudes of investors towards risk and profit.

III. PRELIMINARIES

In this section, we recapitulate some important concepts pertaining to the theory of SVNSs, and some of the recent developments related to SVNS based decision making.

The formal definition of the classical neutrosophic set introduced by Smarandache [3], [4] in 1998 is outlined as below:

Definition 1: Set B is a neutrosophic set drawn from a universal set Y. Then B is as follows

$$B = \{\langle y, T_{\mathrm{B}}(y), I_{\mathrm{B}}(y), F_{\mathrm{B}}(y)\rangle \mid y \in Y\},$$

where $T_{\mathrm{B}}(y)$ is a membership function which describes the degree of truth of element y to set B, $I_{\mathrm{B}}(y)$ is a membership function which describes the level of indeterminacy between truth and false of element y to set B and $F_{\mathrm{B}}(y)$ is a membership function which describes the degree of falsity of element y to set B and T_{B}, I_{B} and F_{B}: Y→P( ] (_0^-)0,1^+ [ ), which is the power set of](_0^-)0,1^+ [Y → P ($]_0^-0, 1^+[$), which is the power set of $]_0^-0, 1^+[$. The sum of the values assigned by $T_{\mathrm{B}}(y)$, $I_{\mathrm{B}}(y)$, $F_{\mathrm{B}}(y)$ is not restricted, which means the summation of the values is between 0 and 3.

Let B and C be neutrosophic sets defined as given below:

$$B = \{\langle y, T_{\mathrm{B}}(y), I_{\mathrm{B}}(y), F_{\mathrm{B}}(y)\rangle \mid y \in Y\}$$

$$C = \{\langle y, T_{\mathrm{C}}(y), I_{\mathrm{C}}(y), F_{\mathrm{C}}(y)\rangle \mid y \in \mathrm{Y}$$

Definition 2: [3] For neutrosophic sets B and C, the basic concepts are as below:

(i) B is contained in C, represented by the notation, $B \subseteq C$, if and only if $\inf T_{\mathrm{B}}(y) \leq \inf T_{\mathrm{C}}(y)$, $\sup T_{\mathrm{B}}(y) \leq \sup T_{\mathrm{C}}(y)$, $\inf F_{\mathrm{B}}(y) \leq \inf F_{\mathrm{C}}(y)$ and $\sup F_{\mathrm{B}}(y) \leq \sup F_{\mathrm{C}}(y)$

(ii) B is equivalent to C if $B \subseteq C$ and $B \supseteq C$

Single-valued neutrosophic set (SVNS), a particular case of neutrosophic sets was proposed by Wang et al. [4] as a means of overcoming the difficulties in applying the classical neutrosophic sets which have membership functions defined in the non-standardized interval of 0 and 1. The SVNS also has a triple membership structure that consists of a truth membership, indeterminacy membership and falsity membership functions. However, the membership functions in the SVNS model assigns values within the standard interval of [0,1]. The SVNS model is one of the most commonly used versions of the neutrosophic set model, and a lot of research related to SVNS based decision making have been done. SVNS based decision making provides a way of adequately handling the ambiguities that often exists in the data of most of the problems encountered in everyday life.

Definition 3: [4] Let B be a SVNS defined over a universal set Y, where B is expressed as

$$B = \{\langle y, T_{\mathrm{B}}(y), I_{\mathrm{B}}(y), F_{\mathrm{B}}(y)\rangle \mid y \in Y\}.$$

Here, $T_{\mathrm{B}}(y)$, $I_{\mathrm{B}}(y)$ and $F_{\mathrm{B}}(y)$ are functions that assign the magnitude of truth membership, indeterminacy membership and falsity membership respectively, $T_{\mathrm{B}}(y), I_{\mathrm{B}}(y), F_{\mathrm{B}}(y) \in [0, 1]$, and the membership functions must conform to this condition $0 \leq T_{\mathrm{B}}(y) + I_{\mathrm{B}}(y) + F_{\mathrm{B}}(y) \leq 3$. A triplet of the form $(T_{\mathrm{B}}(y), I_{\mathrm{B}}(y), F_{\mathrm{B}}(y))$ is called a single-valued neutrosophic number (SVNS).

Let B and C be SvNSs that are defined as follows:

$$B = \{\langle y, T_B(y), I_B(y), F_B(y)\rangle \mid y \in Y\}$$
$$C = \{\langle y, T_C(y), I_C(y), F_C(y)\rangle \mid y \in \mathrm{Y}\}$$

Definition 4: [4] Let B and C be two SVNSs over a universe Y. Some of the basic operations for B and C are as given below:

(i) B is contained in C, if $T_B(y) \leq T_C(y), I_B(y) \geq I_C(y)$, and $F_B(y) \geq F_C(y)$, for all $y \in Y$. This relationship is denoted as $B \subseteq C$.

(ii) B and C are said to be equal if $B \subseteq C$ and $C \subseteq B$.

(iii) $B^c = (y, (F_A(y), 1 - I_A(y), T_A(y)))$, for all $y \in Y$.

(iv) $B \cup C = (y, (\max(T_B(y), T_C(y)), \min(I_B(y), I_C(y)), \min(F_B(y), F_C(y))))$, $\forall y \in Y$.

(v) $B \cap C = (y, (\min(T_B(y), T_C(y)), \max(I_B(y), I_C(y)), \max(F_B(y), F_C(y))))$, $\forall y \in Y$.

Majumdar and Samanta [33] introduced the information measures of distance, similarity and entropy for SVNSs. Here we only present the definition of the distance measures between SVNSs as it is the only component that is relevant to this paper.

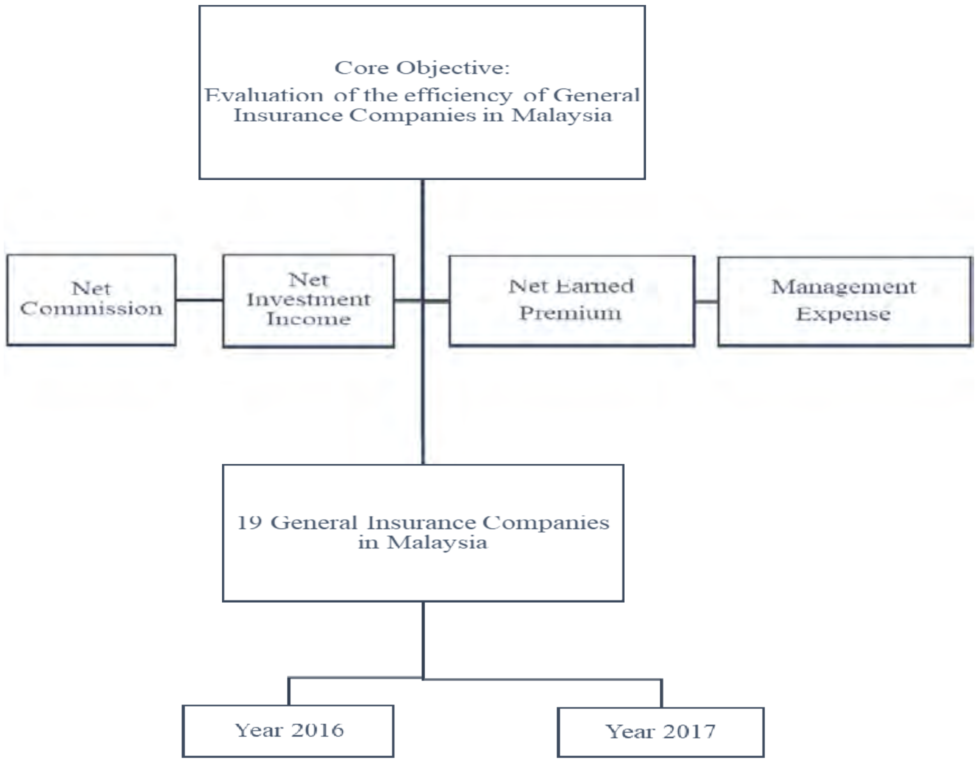

FIGURE 1. **The NDAHP framework for this study.**

Definition 5: [33] Let B and C be two SVNSs over a finite universe $Y = \{y_1, y_2, \ldots, y_n\}$. The various distance measures between B and C are as given below:

(i) The Hamming distance between B and C are defined as:

$$d_H(B, C) = \sum_{i=1}^{n} \{|T_B(y_i) - T_C(y_i)| + |I_B(y_i) - I_C(y_i)| + |F_B(y_i) - F_C(y_i)|\} \quad (1)$$

(ii) The normalized Hamming distance between B and C are defined as:

$$d_H^N(B, C) = \frac{1}{3n} \sum_{i=1}^{n} \{|T_B(y_i) - T_C(y_i)| + |I_B(y_i) - I_C(y_i)| + |F_B(y_i) - F_C(y_i)|\} \quad (2)$$

(iii) The Euclidean distance between B and C are defined as:

(iv) The normalized Euclidean distance between B and C are defined as:

IV. PROPOSED WORK

This section presents the application of the neutrosophic data analytic hierarchy process (NDAHP), the integration of the weightage of each data component both objectively and subjectively using the maximizing deviation method, followed by the evaluation of the performance of the general insurance companies in Malaysia with respect to their efficiency level via the neutrosophic TOPSIS method.

A. DESCRIPTION OF PROBLEM AND PARAMETERS

In this section, let $G = \{G_1, G_2, G_3, \ldots G_m\}$ be a set of alternatives that represents Malaysia's general insurance companies that are considered in this study, $C = \{C_1, C_2, .., C_n\}$ be the set of criteria or data attributes that are considered in this study, $O = \{O_1, O_2, .., O_n\}$ and $S = \{S_1, S_2, .., S_n\}$ be the set of objective weightage and subjective weights, respectively for each data attribute in the evaluation of each criteria, and let $\alpha = \{\alpha_1, \alpha_2, .., \alpha_n\}$ be the set of integrated weights for each attribute.

B. SETTING UP THE HIERARCHICAL MODEL OF THE STUDY

The concept of neutrosophic data analytic hierarchy process (NDAHP) that was introduced by Tey et al. [22] is used to determine the weightage of each data attribute. The overall framework and visualization of model is important to present a proper understanding of the concepts to the decision maker on the problem that is being studied. In this stage, the core focus of the whole model has to be identified. This is followed by the selection of criteria that are relevant to the objective of the model. Each definition of criteria is studied in depth before it is selected in order to reduce the possibility of involving irrelevant criteria in the computation process which might affect the accuracy and completeness of the results obtained and/or increase the computational complexity and computation time of the decision making process.

Out of all the criterion which are made available, the four data attributes which are selected as the criteria for further evaluation are the net earned premium of the general insurance business, the net investment income, the management expenses and the net commission of the company. Evaluation of efficiency of the general insurance industry involves fuzziness as it involves a lot of subjective judgement. From all the criteria that are available from the sources, these data are selected as these figures are able to justify both the underwriting and operational efficiency of a general insurance business. These four criteria are as explained below:

(i) The net earned premium represents the total amount of premium an insurance business collects from the policyholders or the insured at which that portion is considered "earned" when there no claim request has been made since the policy inception date until the date where the figure is extracted.

(ii) The net investment income justifies the return of investment for the portfolio that the general insurance business has invested in after the deduction of the cost of investment which may include costs such as transaction costs and fund management charges.

(iii) The management expenses represents the operational expenses of running a general insurance company. These include expenses such as office rental, staff wages and directors' fee.

(iv) The net commission represents the amount of expenses incurred during the distribution of the insurance products such as the commission paid to the agencies and general insurance agents, and remuneration.

The NDAHP framework for this study is illustrated in Figure 1.

C. SOURCE OF DATA

The data for the criteria that has been selected in Section 2 is extracted from the annual statistical report published by the Insurance Services Malaysia Berhad (ISM) which is a registered company with the local insurance regulatory body, the Central Bank of Malaysia (BNM). The statistical year books are published by ISM on a yearly basis based on the audited financial performance reports provided by all the insurance and takaful (Islamic insurance) operators in Malaysia. The extraction of financial figures from the annual report from each company is not preferable due to the difference in the financial reporting periods as different companies have different start and end dates for their financial years. Therefore, using data extracted from the annual statistical reports published by ISM ensures that all of the data that we are using are consistent and in compliance with the regulations of the BNM.

D. CONVERSION OF CRISP DATA INTO SVNSs

Before the derivation of weightage for each data attribute, the crisp data obtained from the original source needs to be converted into single-valued neutrosophic numbers (SVNS). In this study, we have used the Vector-Normalization Method (VNM) introduced by Nirmal and Bhatt [34] that is outlined below in Eq. (7) and (8).

The four criteria which have been presented in the previous section can be differentiated into two categories of criteria, namely the beneficial criteria and non-beneficial criteria, respectively. The criteria which can be included in the category of beneficial criteria are those which are relatively better if the value is higher, and the criteria which are better if the value is lower will be grouped into the category of non-beneficial criteria.

In Eq. (5) and (6) below, u_{ij} denotes the crisp numbers for each criterion for each alternative, whereas a_{ij} denotes the fuzzy numbers for each corresponding criteria for each alternative, and $= 1, 2, \ldots m, j = 1, 2, \ldots n$.

For beneficial criteria:

$$a_{ij} = \frac{u_{ij}}{\sum_{i=0}^{m} u_{ij}^2} \tag{5}$$

For non-beneficial criteria:

$$a_{ij} = 1 - \frac{u_{ij}}{\sum_{i=0}^{m} u_{ij}^2} \tag{6}$$

The fuzzy numbers a_{ij} are then converted to SVNSs by applying Eq. (7) and (8), both of which were also introduced by Nirmal and Bhatt [34]:

$$\text{Beneficial criteria: } \left(t_{ij}, i_{ij}, f_{ij}\right) = \left(a_{ij}, 1 - a_{ij}, 1 - a_{ij}\right) \tag{7}$$

$$\text{Non-beneficial criteria: } \left(t_{ij}, i_{ij}, f_{ij}\right) = \left(1 - a_{ij}, 1 - a_{ij}, a_{ij}\right) \tag{8}$$

E. ATTRIBUTE WEIGHT DETERMINATION

In the decision-making process, especially in the subject matter dealing with fuzziness, two main types of weight

$$d_E(B, C) = \sqrt{\sum_{i=1}^{n} \left\{ (T_B(y_i) - T_C(y_i))^2 + (I_B(y_i) - I_C(y_i))^2 + (F_B(y_i) - F_C(y_i))^2 \right\}} \tag{3}$$

$$d_E^N(B, C) = \sqrt{\frac{1}{3n} \sum_{i=1}^{n} \left\{ (T_B(y_i) - T_C(y_i))^2 + (I_B(y_i) - I_C(y_i))^2 + (F_B(y_i) - F_C(y_i))^2 \right\}} \tag{4}$$

coefficients need to be taken into consideration, namely the objective weight and subjective weight. Objective weight refers to the weightage of the data attribute that is derived through mathematical concepts, and is calculated based on the actual datasets that are used in the study or decision making process. Subjective weights refer to the weightage for each attribute that is assigned by the decision makers based on their expert judgement, preference and individual points of view. In our study, it would not be adequate to use only one type of weight as the evaluation of the efficiency of a company deals with both objective and subjective judgement. Therefore, both the objective weights and subjective weights for each criteria for each attribute will be calculated/obtained, and subsequently used to determine an integrated weight. This is more appropriate in reflecting the level of importance of each data attribute.

The objective weight o_j for each data attribute, j is calculated using Eq. (9) which is based on the modified maximizing deviation method for the SvNS model introduced in [24]:

$$o_j = \frac{\sum_{i=1}^{m} \sum_{k=1}^{m} d(a_{ij}, a_{ij})}{\sum_{j=1}^{n} \sum_{i=1}^{m} \sum_{k=1}^{m} d(a_{ij}, a_{ij})} \tag{9}$$

In Eq. (9), d refers to the normalized Euclidean distance measure given in Eq. (4), as shown at the bottom of the previous page.

To determine the subjective weight of each data attribute, a short survey was conducted on a group of people with extensive industrial experience in the general insurance industry. The respondents are given a scale of 1-4 to rank the importance of each criterion in the evaluation of the efficiency of general insurance companies'. Here, a score of 1 indicates that a criteria is a least important criteria, whereas a score of 4 indicates that a criteria is a most important criteria. The subjective weight, S for each criterion for each attribute is then computed by using Eq. (10) which is due to [24].

$$s_j = \frac{\sum_{i=0}^{v} K_j}{v} \tag{10}$$

where K_j represents the score given by each respondent and v represents the number of respondents.

The integrated weight, α for each criterion is then calculated by using Eq. (11) which was introduced in [24]:

$$\alpha_j = \frac{o_j s_j}{\sum_{j=1}^{n} o_j s_j} \tag{11}$$

F. IDENTIFYING THE RELATIVE IDEAL SOLUTION

In this stage, the relative neutrosophic positive ideal solution (RNPIS) and relative neutrosophic negative ideal solution (RNNIS) are to be determined.

Let the RNPIS and RNNIS be denoted by p^+ and p^- respectively where the derivations are as defined below:

$$p^+ = \left\{ \left(\max T_{ij}, \min I_{ij}, \min F_{ij} \right) \mid j = 1, 2 \ldots, n \right\} \tag{12}$$

$$p^- = \left\{ \left(\min T_{ij}, \max I_{ij}, \max F_{ij} \right) \mid j = 1, 2 \ldots, n \right\} \tag{13}$$

G. CALCULATING THE DISTANCE OF EACH ALTERNATIVE AND THEIR RELATIVE IDEAL SOLUTION

The distance between each element and the RNPIS denoted by d_i^+, and the distance between each element and the RNNIS denoted by d_i^- are then computed using the normalized Euclidean distance formula presented in Eq. (4) of Definition 5. In the formula, α_j represents the integrated weight for each criterion.

$$d_i^+ = \sum_{j=1}^{n} \alpha_j d_E^N \left(p_{ij}, p^+ \right) \tag{14}$$

$$d_i^- = \sum_{j=1}^{n} \alpha_j d_E^N \left(p_{ij}, p^- \right) \tag{15}$$

H. COMPUTING THE RELATIVE CLOSENESS COEFFICIENT FOR EACH ALTERNATIVE

The optimal alternative can be found by computing the relative closeness coefficient of each alternative denoted by C_i using the formula outlined below which is due to [24].

$$C_i = \frac{d_i^-}{max d_j^-} - \frac{d_i^+}{min d_j^+} \tag{16}$$

where $i, j = 1, 2, .., m$.

I. RANKING THE ALTERNATIVES BASED ON THEIR CLOSENESS COEFFICIENT

The general insurance companies are then ranked based on their closeness coefficient. The rule of thumb for this process is the alternative with the maximum or largest closeness coefficient would be the optimal alternative. In the context of our study the optimal alternative is the general insurance company with the highest efficiency among the 19 general insurance companies in Malaysia that were considered in this study.

V. RESULTS

In this section, the level of importance for each data attribute which have been quantified through the integrated weighing mechanism proposed in this study are presented, and followed by the ranking of the efficiency of 19 general insurance companies based on their closeness coefficient. A discussion on the results that are obtained is also included in this section.

A. WEIGHTAGE OF THE CRITERIA STUDIED IN THE PROJECT

The justification on the level of efficiency of the general insurance businesses is not as easy as extracting the figure directly from the financial reports of the companies. It involves a multitude of complex financial criteria which has to be studied in depth, and the weightage of each criterion will then have to be evaluated based on the objective and subjective weighting methods. For the objective weighting mechanism, the maximizing deviation method was used to objectively evaluate the weightage of the criteria. For the subjective weighting mechanism, a short survey was conducted

TABLE 1. Objective, subjective and integrated weight of each criteria.

Criteria	2016			2017		
	Objective weight	Subjective weight	Integrated weight	Objective weight	Subjective weight	Integrated weight
Net Investment Income	0.2656	0.22	0.2374	0.2665	0.22	0.2389
Net Earned Premium	0.2751	0.30	0.3353	0.2740	0.30	0.3350
Management Expense	0.2044	0.34	0.2824	0.2014	0.34	0.2790
Net Commissions	0.2549	0.14	0.1450	0.2580	0.14	0.1472

among professionals in the general insurance industry. The respondents involved in this short survey are all professionals with extensive experience in different areas of the general insurance industry, with 3 of the respondents from general insurance companies and the remaining 2 respondents from actuarial consulting firms. Table 1 provides a summary of the objective, subjective and the integrated weights for each criterion from year 2016 to 2017.

B. RANKING OF THE GENERAL INSURANCE COMPANIES BASED ON THEIR LEVEL OF EFFICIENCY

The results of the ranking of the 19 general insurance companies are given in Table 2. A rank of 1 indicates that based on the proposed model, the company has the highest level of efficiency compared to the other 18 companies; a rank of 19 indicates that based on the proposed model the company has the lowest level of efficiency compared to the other 18 companies that were studied.

C. DISCUSSION OF RESULTS OBTAINED VIA OUR PROPOSED METHOD

From Table 1, the significant observation is that both the net earned premium and management expenses of a general insurance business have consistently remained as the most important attributes in the evaluation of the level of efficiency of general insurance companies. This information infers that the data figures for these two criteria have material impact on the ranking of the general insurance companies with regards to the level of efficiency.

From Table 2, for two consecutive years i.e. from 2016 to 2017, Lonpaq, P&O and Allianz General have emerged as the top 3 most efficient general insurance companies compared to their peers in the general insurance industry. One of the key reasons for this is because the combined net earned premiums of these general insurance companies' portfolio achieves 23% to 24% out of the whole industry performance. As the net earned premium has been identified as the most important evaluation criteria from our objective and subjective weighting mechanisms, the high net earned premium figures of these companies have contributed to the high overall ranking of these 3 companies.

Based on our proposed model, AIA had consistently shown the lowest efficiency for the two observed years. This could possibly be due to their consistently low net earned premium generated and a low net investment income achieved. In the general insurance industry, a low net earned premium can be justified by two main factors, which are due to high claims observed from their underwriting portfolio, or because the business underwritten is relatively smaller compared to the other competitors. Low net investment income could be due the company having invested in less risky investment portfolios which typically yields lower returns, or because the company might have invested a significant amount of funds in the long-term investment tools such as bonds for which the return of investment is not likely to be observed in the short-term period, and therefore the returns are not recognized or captured in their annual financial reports.

Another important observation on the movement in the rankings of the 19 companies is that Tokio Marine had risen by 5 places from being ranked at 16 in 2016 to being ranked at 11 in 2017 due to their higher growth of net earned premium which was 2.26% compared to the overall industry growth of 1.6%.

From Table 2, we can infer that our proposed fuzzy model yields a consistent output, at which the fluctuation of the ranking of the companies is not considered extremely volatile. There are only a few companies which showed some significant changes in their year to year rankings due to their better performance in the beneficial criteria compared to their peers and the performance of the industry as a whole.

D. COMPARATIVE STUDIES USING THE DEA AND SFA METHODS

In existing literature, the evaluation of efficiency of insurance companies is mostly focused on two types of approaches that are methods based on parametric model and non-parametric

TABLE 2. Ranking results of the 19 general insurance companies using our proposed method.

Ranking	2016	2017
1	Lonpac	Lonpac
2	P&O	P&O
3	Allianz General	Allianz General
4	Progressive	Tune
5	Tune	Progressive
6	MSIG	MPI Generali
7	Pacific	Pacific
8	Berjaya Sompo	Berjaya Sompo
9	MPI Generali	RHB
10	RHB	MSIG
11	Etiqa	Tokio Marine
12	GE General	GE General
13	QBE	ETIQA
14	AMGeneral	QBE
15	AXA Affin General	AXA Affin General
16	Tokio Marine	AIG
17	Zurich	AMGeneral
18	AIG	Zurich
19	AIA	AIA

TABLE 3. Results of the ranking of the 19 general insurance companies using the DEA approach.

Ranking	2016	2017
1	Etiqa	Pacific
2	Pacific	Tune
3	RHB	Tokio Marine
4	AXA Affin General	QBE
5	GE General	MPI Generali
6	Lonpac	RHB
7	Progressive	AIG
8	MSIG	Allianz General
9	MPI Generali	Lonpac
10	Zurich	Progressive
11	Allianz General	Berjaya Sompo
12	AIG	MSIG
13	QBE	GE General
14	Berjaya Sompo	P&O
15	AMGeneral	AIA
16	P&O	AXA Affin General
17	Tokio Marine	Zurich
18	Tune	AMGeneral
19	AIA	Etiqa

model. In the parametric model, the most commonly used methods are the Stochastic Frontier Approach (SFA), Distribution Free Approach (DFA) and Thick Frontier Approach (TFA). In our study, we have the model of Data Envelopment Analysis (DEA) and Stochastic Frontier Analysis (SFA) as a benchmark of comparison.

TABLE 4. Results of the ranking of the 19 general insurance companies using the SFA approach.

Ranking	2016	2017
1	Lonpac	Lonpaq
2	MSIG	MSIG
3	Allianz General	Allianz General
4	Berjaya Sompo	Berjaya Sompo
5	P&O	P&O
6	MPI Generali	MPI Generali
7	Tune	Tune
8	AXA Affin General	AXA Affin General
9	QBE	QBE
10	Tokio Marine	Tokio Marine
11	RHB	RHB
12	Pacific	Pacific
13	AMGeneral	AMGeneral
14	Progressive	Progressive
15	GE General	GE General
16	AIA	AIA
17	Etiqa	Etiqa
18	Zurich	Zurich
19	AIG	AIG

DEA is a non-parametric method which evaluates the efficiency of objects based on the assumptions of constant returns to scale and variable returns to scale. The DEA method was first introduced by Charnes, Cooper and Rhodes in 1994 [35]. DEA is a non-parametric technique to evaluate productive efficiency with numerous inputs and outputs. DEA generates an efficient frontier which is developed by decision-making units (DMUs) that shows the best performance. This approach has a basic assumption that there is a linear assumption between the observed input combinations on an isoquant [36]. It determines the DMU's relative efficiency that has multiple inputs and outputs [37]. The DEA works by categorizing the actual financial data into two categories, which are output and input, similar to the concept included in our suggested fuzzy model which have differentiated the data into beneficial and non- beneficial components. However, the DEA methodology is completely objective, and the efficiency score can only be derived if the data from the previous year is available. The most important component in the DEA, which is the Malmquist productivity index is evaluated based on a company's performance from the previous year and the changes in the productivity frontier which is different from our suggested fuzzy model that actually evaluates the efficiency of a company based on individual performance and the industry's overall performance, irrespective and independent of the performance of the company in the previous year [38].

The SFA is a parametric method where a functional form is assumed and random factors are introduced to fit in the production function and enable the frontier to shift around the fitted function for companies [39]. One of the key process of this method is the composition of error term. For error term, it is subdivided into one-sided error term which is used to measure the firm-specific inefficiency and two-sided error term which shows the random fluctuations. Here, the two-sided error term is identically and independently distributed among the firms.

As the DEA is a non-parametric method, the risk of imposing the wrong functional form is avoided. SFA requires assumptions about the form of the relationship between the inputs and outputs, and the distribution of the random error and inefficiency terms. Due to the nature of the SFA to incorporate the error term or noise component in the evaluation, the DEA appears to be sensitive to outliers as it is a deterministic approach and assumes no random noise in the data. A more detailed and comprehensive analysis of the comparison of the DEA and SFA methods, and our proposed models will be presented in Section 5.

In our study, the computer software which we have used to study the results based on the DEA and SFA model are the DEAP Version 2.1 and FRONTIER Version 4.1. The results of the ranking of the efficiency of the 19 general insurance companies based on the DEA and SFA approaches are presented in the Table 3 and Table 4, respectively.

As observed from Table 3 and Table 4, both the DEA and SFA approaches yield inconsistent results which is due to differences in the methodology that is used in both of these approaches. The DEA approach being a deterministic and non-parametric model does not include an error component in the evaluation. The SFA has an error component which is commonly known as noise included in the assumption of functional form in the evaluation process.

The rankings obtained via the SFA method for the top three most efficient general insurance companies are almost the same with the rankings obtained via our proposed method, whereas the results obtained via the DEA method is very different from the other two methods. The first and third ranked companies according to the SFA method and our method are Lonpaq and Allianz General, respectively.

Both the DEA and SFA methods do not have any mechanism to incorporate the opinions and inputs of industry experts in the decision making process. Thus, these methods are not able to take the subjective weights into consideration in the decision making process, and hence the results obtained is less reliable compared to the results obtained via our proposed method that takes the integrated weights into consideration. Moreover, although the DEA and SFA can assume multiple inputs and outputs in the analysis and decision making process, both of these approaches do not take into consideration the level of importance of each factor or criteria that contributes to the efficiency of the organizations. The failure of the DEA and SFA methods in determining the importance of each criteria in the evaluation of the efficiency of a company means that both of these approaches are entirely based on relative comparison of the criteria, making the results obtained via these methods less reliable compared to the results obtained via our proposed method. However, in the industry, it is pertinent to determine the level of importance of each factor or criteria to enable the company to implement the necessary strategies to improve the overall efficiency of the companies. These prove the reliability and accuracy of the results from our proposed method, and by extension the superiority of our proposed method compared to the DEA and SFA methods in evaluating the efficiency of general insurance companies.

In the future, the insurance companies can possibly extend our proposed method into their yearly exercise to benchmark their efficiency level against their competitors. The companies can involve more decision makers, for example: experts from the various departments in their organizations such as the finance, economic planning and business development departments to quantify the main factors which lead to their overall level of efficiency. By incorporating quantitative and qualitative mechanisms for efficiency measurement, it can help the companies to strategize their directions and adopt best corporate practices to improve their operating efficiency, strengthen their market standing and give them competitive advantage over their competitors.

VI. CONCLUSION AND REMARKS

The concluding remarks and the significant contributions of this paper are summarized below:

(i) A novel integrated model for the single- valued neutrosophic set (SVNS) model was introduced. The key operations from the NDAHP method was used to convert the raw data which comprised of the net earned premium, net investment income, management expenses and net commissions for 19 selected general insurance companies over the year 2016 to 2017 into single-valued neutrosophic numbers (SVNN). The maximizing deviation method was then applied to quantify the objective weight of the criteria before integrating it with the subjective weight which was derived from the evaluation of five selected experienced professionals from the general insurance industry. The companies are then ranked to determine their level of efficiency via the Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) [40]–[49].

(ii) The key components extracted from the NDAHP method to convert the crisp values in the raw numerical data to SVNNs are able to differentiate between the beneficial criteria and non-beneficial criteria. This key feature has justified that our analytical procedure is indeed objective, and will not be impacted by the differing opinions of different decision makers in which subjectivity and biasness would be a major concern. As the concept of efficiency of a business does involves a certain level of subjectivity, the subjective weighting mechanism was also introduced in our study, and this was integrated with the objective weights obtained using the maximizing deviation method. The integration of the objective and subjective weighting mechanisms, and the integration of methods proposed in this project makes the analysis more comprehensive compared to most of the existing research on the study of efficiency of general insurance companies as our proposed method is able to handle actual datasets that are objective with expert opinions which are subjective and biased in an effective manner. Furthermore, as the concept of efficiency of general insurance companies is in itself a very subjective and vague concept, it is most appropriate to use fuzzy tools and decision making methods to study this concept in a fuzzy environment.

(iii) The concept of efficiency of general insurance companies is in itself a very subjective and vague concept. It is most appropriate to use fuzzy tools and decision making methods to study this concept in a fuzzy environment. For the time being, the evaluation of efficiency has been only performed mainly for academic research purpose due to the limitation that over-reliance on quantitative data might not be comprehensive enough to conduct a benchmark analysis on the efficiency of all the general insurance companies. The proposed concept of integrating the neutrosophic concept does provide an alternative and it can be possibly extended for use in the general industry use by actual practitioners.

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On Soft Rough Topology with Multi-Attribute Group Decision Making

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Abstract: Rough set approaches encounter uncertainty by means of boundary regions instead of membership values. In this paper, we develop the topological structure on soft rough set ($\mathcal{SR}$-set) by using pairwise $\mathcal{SR}$-approximations. We define $\mathcal{SR}$-open set, $\mathcal{SR}$-closed sets, $\mathcal{SR}$-closure, $\mathcal{SR}$-interior, $\mathcal{SR}$-neighborhood, $\mathcal{SR}$-bases, product topology on $\mathcal{SR}$-sets, continuous mapping, and compactness in soft rough topological space ($\mathcal{SRTS}$). The developments of the theory on $\mathcal{SR}$-set and $\mathcal{SR}$-topology exhibit not only an important theoretical value but also represent significant applications of $\mathcal{SR}$-sets. We applied an algorithm based on $\mathcal{SR}$-set to multi-attribute group decision making (MAGDM) to deal with uncertainty.

Keywords: rough set; $\mathcal{SR}$-set; $\mathcal{SR}$-topology; $\mathcal{SR}$-continuity; $\mathcal{SR}$-compactness; decision making

1. Introduction

The problem of imperfect knowledge has been the center of attention for many years. In the field of mathematics, computer science, and artificial intelligence, researchers have used different methods to tackle the problem of uncertain and incomplete data, including probability theory, fuzzy set [1], and rough set [2,3] and soft set techniques [4–6]. Molodstov [6] introduced soft set as an effective tool to manage imprecision; it includes a set of parameters to describe the set properly. Maji et al. (2002–2003) [4,5] extended some operations of soft set and effectively used this technique in a decision-making problem. Soft set with decision making have studied by many researchers [7–11]. In 2011, Shabir and Naz [12] and Cagman et al. [13] independently worked on the topological structure of soft set. Chen [14] presented a new definition related to the reduction of soft parameterization. The study of hybrid structures, having emerged from the fusion of soft sets with other mathematical approaches, is becoming an active topic for research nowadays. Aktas and Cagman (2007) [15] efficiently related the three concepts of soft set, rough set, and fuzzy set. Riaz et al. [16–18] established some results of soft algebra, soft metric spaces, and measurable soft mappings. Riaz and Masooma [19–23] introduced fuzzy parameterized fuzzy soft set ($\mathfrak{fpfs}$-set), $\mathfrak{fpfs}$-topology, and $\mathfrak{fpfs}$-compact spaces, with some important applications of $\mathfrak{fpfs}$-set to decision-making problems. They presented $\mathfrak{fns}$-mappings and fixed points of $\mathfrak{fns}$-mapping. Different researchers have tackled the problem of incomplete or uncertain information in the system in different ways. Shang worked on robust statistics, and he investigated the robustness of a system under different circumstances and analyzed the robustness properties of subgraphs under attack in complex networks [24,25]. The rough set concept presented by Pawlak presents a systematic approach for the classification of objects. It characterizes a set of objects by two exact concepts, known as its approximations. Here, vagueness is expressed in the form of a boundary region, where empty boundary region implies that the set is crisp, and a non-empty boundary region implies that our knowledge is insufficient to explain the set precisely. By using equivalence relations, Thivagar et al. [26] generated the topology on rough set which includes approximations and the boundary region. Equivalence relation plays an important role in Pawlak's rough set model, and by replacing it with a soft set, soft rough set $\mathcal{SR}$-sets were introduced by Feng [27]. Feng et al. [28] presented some properties related to $\mathcal{SR}$-approximations. Xue et al. [29] presented some decision-making algorithms regarding hybrid soft models. Zou and Xiao [30] analyzed data in soft sets under incomplete information systems. There are mainly two streams of study connecting soft rough set theory and topology theory. At the same time, according to the topological properties on the topological $\mathcal{SR}$-space, some applications for image processing and some topological diagrams are introduced. The remainder of the paper is composed as follows. In Section 2, we briefly define the notions of rough set $\mathcal{R}$-set and soft rough set $\mathcal{SR}$-set. In Section 3, we present a novel topological structure of $\mathcal{SR}$-set. We present some new results of $\mathcal{SR}$-set theory and $\mathcal{SR}$-topology. A topological structure on soft rough set was defined by Bakier et al. [31]. Malik and Riaz [32,33] studied action of modular group on real quadratic fields. Soft sets, neutrosophic set and rough sets with decision making problems have studied by many researchers [34–40]. We define $\mathcal{SR}$-topology on soft rough set in the form of the pair $\tau_{\mathcal{SR}} = (\tau_{\mathcal{SR}\star}, \tau_{\mathcal{SR}}{}^{\star})$, where $\tau_{\mathcal{SR}\star}$ is the lower $\mathcal{SR}$-topology and $\tau_{\mathcal{SR}}{}^{\star}$ is the upper $\mathcal{SR}$-topology on set $\mathcal{Y}$. This $\mathcal{SR}$-topology is more appropriate, as it looks like a natural soft rough topology on a soft rough set. In Section 4, continuity, homeomorphism, and projection mappings in $\mathcal{SR}$-set are discussed. Section 5 describes the compactness in $\mathcal{SR}$-set. In Section 6, $\mathcal{SR}$ approximations are employed to solve multi-attribute group decision-making problem.

2. Preliminaries

In this section, we illustrate some basic notions related to $\mathcal{SR}$-theory. First we define rough set $\mathcal{R}$-set and soft rough set $\mathcal{SR}$-set and then explain a few related operations on $\mathcal{SR}$-set.

Definition 1 ([2]). *Suppose we have an object set $\mathcal{V}$ known as universe, and an indiscernibility relation $\Re \subseteq \mathcal{V} \times \mathcal{V}$ which represents knowledge about elements of $\mathcal{V}$. We take $\Re$ as an equivalence relation and denote it by $\Re(y)$. The pair $(\mathcal{V}, \Re)$ is called the approximation space. Let $\mathcal{Y}$ be any subset of $\mathcal{V}$. We characterize the set $\mathcal{Y}$ with respect to $\Re$.*

(1) The union of all granules which are entirely included in the set $\mathcal{Y}$ is called the lower approximation of the set $\mathcal{Y}$ w.r.t $\Re$, mathematically defined as

$$\Re_\star(\mathcal{Y}) = \bigcup_{y \in \mathcal{V}} \{\Re(y) : \Re(y) \subseteq \mathcal{Y}\}$$

(2) The union of all the granules having a non-empty intersection with the set $\mathcal{Y}$ is called the upper approximation of the set $\mathcal{Y}$ w.r.t $\Re$, mathematically defined as

$$\Re^\star(\mathcal{Y}) = \bigcup_{y \in \mathcal{V}} \{\Re(y) : \Re(y) \cap \mathcal{Y} \neq \emptyset\}$$

(3) The difference between the upper and lower approximations is called the boundary region of the set $\mathcal{Y}$ w.r.t $\Re$, mathematically defined as

$$B_{\Re}(\mathcal{Y}) = \Re^\star(\mathcal{Y}) - \Re_\star(\mathcal{Y})$$

If $\Re^\star(\mathcal{Y}) = \Re_\star(\mathcal{Y})$, the set $\mathcal{Y}$ is said to be defined. If $\Re^\star(\mathcal{Y}) \neq \Re_\star(\mathcal{Y})$, i.e., $B_R(\mathcal{Y}) \neq \emptyset$, the set $\mathcal{Y}$ is said to be a (imprecise) rough set w.r.t $\Re$.

We denote a rough set $\mathcal{Y}$ by a pair comprising a lower approximation and upper approximation $\mathcal{Y} = (\Re_\star(\mathcal{Y}), \Re^\star(\mathcal{Y}))$

Definition 2 ([28]). *Consider a soft set $\mathcal{S} = (\mathcal{T}, \mathcal{A})$ over the universe $\mathcal{V}$, where $\mathcal{A} \subseteq \mathcal{E}$ and $\mathcal{T}$ is a mapping defined as $\mathcal{T} : \mathcal{A} \to \mathcal{P}(\mathcal{V})$. Here, soft approximation space is the pair $\mathcal{P} = (\mathcal{V}, \mathcal{S})$. Following the soft approximation space $\mathcal{P}$, we define two operations as follows:*

$$\Re_{P_\star}(\mathcal{Y}) = \{v \in \mathcal{V} : \exists a \in \mathcal{A}, [v \in \mathcal{T}(a) \subseteq \mathcal{Y}]\},$$

$$\Re_{P}{}^{\star}(\mathcal{Y}) = \{v \in \mathcal{V} : \exists a \in \mathcal{A}, [v \in \mathcal{T}(a) \cap \mathcal{Y} \neq \emptyset]\}$$

regarding every subset $\mathcal{Y} \subseteq \mathcal{V}$, two sets $\Re_{P_\star}(\mathcal{Y})$ and $\Re_{P}{}^{\star}(\mathcal{Y})$, which are called the soft P-lower approximation and soft P-upper approximation of $\mathcal{Y}$, respectively. In general, we refer to $\Re_{P_\star}(\mathcal{Y})$ and $\Re_{P}{}^{\star}(\mathcal{Y})$ as $\mathcal{SR}$-approximations of $\mathcal{Y}$ w.r.t P. If $\Re_{P_\star}(\mathcal{Y}) = \Re_{P}{}^{\star}(\mathcal{Y})$, then $\mathcal{Y}$ is said to be soft P-definable; otherwise, $\mathcal{Y}$ is a soft P-rough set. Then, $Bnd_P = \Re_{P}{}^{\star}(\mathcal{Y}) - \Re_{P_\star}(\mathcal{Y})$ is the $\mathcal{SR}$-boundary region.

We denote $\mathcal{SR}$-set ($\mathcal{SR}$-set) $\mathcal{Y}$ by a pair comprising $\mathcal{SR}$-lower approximation and $\mathcal{SR}$-upper approximation $\mathcal{Y} = (\Re_{P_\star}(\mathcal{Y}), \Re_{P}{}^{\star}(\mathcal{Y}))$.

Example 1. *Let $\mathcal{V} = \{s_1, s_2, s_3, s_4, s_5\}$ be the set of perfumes, and let $\mathcal{A} = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\} = \mathcal{E}$ be the qualities which Miss Amal wants in her perfume. Let $\mathcal{S} = (\mathcal{T}, \mathcal{A})$ be a soft set over $\mathcal{V}$. $\mathcal{T}(\zeta_1) = \{s_3, s_5\}, \mathcal{T}(\zeta_2) = \{s_2, s_4, s_5\}, \mathcal{T}(\zeta_3) = \{s_1, s_2, s_5\}, \mathcal{T}(\zeta_4) = \{s_2, s_3\}$ and the soft approximation space $\mathcal{P} = (\mathcal{V}, \mathcal{S})$.The tabular form of soft set $(\mathcal{T}, \mathcal{A})$ is given in Table 1.*

Table 1. Soft set $(\mathcal{T}, \mathcal{A})$.

$(\mathcal{T}, \mathcal{A})$	s_1	s_2	s_3	s_4	s_5
ζ_1	0	1	1	0	0
ζ_2	0	1	0	1	1
ζ_3	1	1	0	0	1
ζ_4	0	0	1	0	1

For $\mathcal{Y} = \{s_3, s_4, s_5\} \subseteq \mathcal{V}$, we have $\Re_{P_\star}(\mathcal{Y}) = \{s_3, s_5\}$ and $\Re_{P}{}^{\star}(\mathcal{Y}) = \{s_1, s_2, s_3, s_4, s_5\}$. Since $\Re_{P_\star}(\mathcal{Y}) \neq \Re_{P}{}^{\star}(\mathcal{Y})$; therefore, $\mathcal{Y}$ is a soft P-rough set and is denoted by $\mathcal{Y} = (\{s_3, s_5\}, \{s_1, s_2, s_3, s_4, s_5\})$

Definition 3. *Let $\mathcal{A} = (\Re_{P_\star}(\mathcal{A}), \Re_{P}{}^{\star}(\mathcal{A}))$ and $\mathcal{B} = (\Re_{P_\star}(\mathcal{B}), \Re_{P}{}^{\star}(\mathcal{B}))$ be two arbitrary $\mathcal{SR}$-sets and $\mathcal{P} = (\mathcal{V}, \mathcal{S})$ be soft approximation space. Then, $\mathcal{A}$ is a $\mathcal{SR}$-subset of $\mathcal{B}$ if $\Re_{P_\star}(\mathcal{A}) \subseteq \Re_{P_\star}(\mathcal{B})$ and $\Re_{P}{}^{\star}(\mathcal{A}) \subseteq \Re_{P}{}^{\star}(\mathcal{B})$.*

Example 2. *Suppose $\mathcal{V} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$ and $\mathcal{E} = \{\xi_1, \xi_2, \xi_3, \xi_4\}$. Let $\mathcal{S} = (\mathcal{T}, \mathcal{E})$ be a soft set over $\mathcal{V}$,*

$\mathcal{T}(\xi_1) = \{\alpha_2, \alpha_8\}$
$\mathcal{T}(\xi_2) = \{\alpha_2, \alpha_3, \alpha_6, \alpha_8\}$
$\mathcal{T}(\xi_3) = \{\alpha_2, \alpha_5, \alpha_7\}$
$\mathcal{T}(\xi_4) = \{\alpha_3, \alpha_4, \alpha_6\}$

and $\mathcal{P} = (\mathcal{V}, \mathcal{S})$ be soft approximation space. Consider $\mathcal{A} = \{\alpha_2, \alpha_4, \alpha_5, \alpha_7\} \subseteq \mathcal{V}$ and $\mathcal{B} = \{\alpha_3, \alpha_4\}$ then $\Re_{P_\star}(\mathcal{A}) = \{\alpha_2, \alpha_5, \alpha_7\}$ and $\Re_{P}{}^{\star}(\mathcal{A}) = \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\}$, while $\Re_{P_\star}(\mathcal{B}) = \emptyset$ and $\Re_{P}{}^{\star}(\mathcal{B}) = \{\alpha_2, \alpha_3, \alpha_4, \alpha_6, \alpha_8\}$.

So we have two $\mathcal{SR}$-sets $\mathcal{A} = (\Re_{P_\star}(\mathcal{A}), \Re_{P}{}^{\star}(\mathcal{A})) = (\{\alpha_2, \alpha_5, \alpha_7\}, \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\})$ and $\mathcal{B} = (\Re_{P_\star}(\mathcal{B}), \Re_{P}{}^{\star}(\mathcal{B})) = (\emptyset, \{\alpha_2, \alpha_3, \alpha_4, \alpha_6, \alpha_8\})$. Since $\Re_{P_\star}(\mathcal{B}) \subseteq \Re_{P_\star}(\mathcal{A})$ and $\Re_{P}{}^{\star}(\mathcal{B}) \subseteq \Re_{P_\star}(\mathcal{B})$. Thus, $\mathcal{B}$ is $\mathcal{SR}$-subset of $\mathcal{A}$.

Definition 4. *Let $\mathcal{A} = (\Re_{P_\star}(\mathcal{A}), \Re_{P}{}^{\star}(\mathcal{A}))$, $\mathcal{B} = (\Re_{P_\star}(\mathcal{B}), \Re_{P}{}^{\star}(\mathcal{B}))$ be taken as two arbitrary $\mathcal{SR}$ sets and let $(\mathcal{V}, \mathcal{S})$ be soft approximation space. Then, the union of $\mathcal{A}$ and $\mathcal{B}$ is defined as $\mathcal{A} \cup \mathcal{B} = (\Re_{P_\star}(\mathcal{A}) \cup \Re_{P_\star}(\mathcal{B}), \Re_{P}{}^{\star}(\mathcal{A}) \cup \Re_{P}{}^{\star}(\mathcal{B}))$.*

Definition 5. *Let $\mathcal{A} = (\Re_{P_\star}(\mathcal{A}), \Re_P{}^\star(\mathcal{A}))$, $\mathcal{B} = (\Re_{P_\star}(\mathcal{B}), \Re_P{}^\star(\mathcal{B}))$ be taken as two arbitrary $\mathcal{SR}$ sets and $(\mathcal{V}, \mathcal{S})$ be soft approximation space. Then, the intersection of $\mathcal{A}$ and $\mathcal{B}$ is defined as $\mathcal{A} \cap \mathcal{B} = (\Re_{P_\star}(\mathcal{A}) \cap \Re_{P_\star}(\mathcal{B}), \Re_P{}^\star(\mathcal{A}) \cap \Re_P{}^\star(\mathcal{B}))$.*

Example 3. *By using Example 2, we obtain*
$\mathcal{A} \cup \mathcal{B} = (\Re_{P_\star}(\mathcal{A}) \cup \Re_{P_\star}(\mathcal{B}), \Re_P{}^\star(\mathcal{A}) \cup \Re_{P_\star}(\mathcal{B})) = (\{\alpha_2, \alpha_5, \alpha_7\}, \{\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7, \alpha_8\})$
and
$\mathcal{A} \cap \mathcal{B} = (\Re_{P_\star}(\mathcal{A}) \cup \Re_{P_\star}(\mathcal{B}), \Re_P{}^\star(\mathcal{A}) \cup \Re_{P_\star}(\mathcal{B})) = (\emptyset, \{\alpha_2, \alpha_3, \alpha_4, \alpha_6, \alpha_8\})$.

Definition 6 ([31]). *Let $\mathcal{V}$ be the universe of discourse and $\mathcal{P} = (\mathcal{V}, \mathcal{S})$ is soft approximation space; then, $\mathcal{SR}$-topology is defined as*

$$\tau_{\mathcal{SR}}(\mathcal{Y}) = \{\mathcal{V}, \emptyset, \Re_{P_\star}(\mathcal{Y}), \Re_P{}^\star(\mathcal{Y}), Bd(\mathcal{Y})\}$$

where $\mathcal{Y} \subseteq \mathcal{V}$. $\tau_{\mathcal{SR}}(\mathcal{Y})$ satisfies the following axioms:

(i) $\mathcal{V}$ and $\emptyset$ belong to $\tau_{\mathcal{SR}}(X)$.

(ii) Union of elements of any subcollection of $\tau_{\mathcal{SR}}(\mathcal{Y})$ belongs to $\tau_{\mathcal{SR}}(\mathcal{Y})$.

(iii) Intersection of elements of finite subcollection of $\tau_{\mathcal{SR}}(\mathcal{Y})$ belongs to $\tau_{\mathcal{SR}}(\mathcal{Y})$.

The topology defined by $\tau_{\mathcal{SR}}(\mathcal{Y})$ on $\mathcal{V}$ is called $\mathcal{SR}$-topology on $\mathcal{V}$ w.r.t $\mathcal{Y}$ and $(\mathcal{V}, \tau_{\mathcal{SR}}(\mathcal{Y}), \mathcal{E})$ is said to be $\mathcal{SR}$-topological space. Soft rough set with the topology $\tau_{\mathcal{SR}}$ is called a topological $\mathcal{SR}$-set.

Example 4. *Let $\mathcal{V} = \{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_5, \vartheta_6\}$ be the set of cars under consideration, and let $\mathcal{E} = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4, \zeta_5\}$ be the set of all parameters and $\mathcal{A} = \{\zeta_1, \zeta_2, \zeta_3\} \subseteq \mathcal{E}$. Consider the soft approximation $P = (\mathcal{V}, \mathcal{S})$, where $\mathcal{S} = (\mathcal{T}, \mathcal{A})$ is a soft set over $\mathcal{U}$ given by:*
$\mathcal{T}(\zeta_1) = \{\vartheta_1, \vartheta_3\}, \mathcal{T}(\zeta_2) = \{\vartheta_1, \vartheta_3, \vartheta_6\}$ *and* $\mathcal{T}(\zeta_3) = \{\vartheta_2, \vartheta_4\}$.

For $\mathcal{Y} = \{\vartheta_2, \vartheta_3, \vartheta_4, \vartheta_6\}$, we obtain $\Re_{P_\star}(\mathcal{Y}) = \{\vartheta_2, \vartheta_4\}$, $\Re_P{}^\star(\mathcal{Y}) = \{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_6\}$ and $Bd(\mathcal{Y}) = \{\vartheta_1, \vartheta_3, \vartheta_6\}$. Then,

$$\tau_{\mathcal{SR}}(\mathcal{Y}) = \{\mathcal{V}, \emptyset, \{\vartheta_2, \vartheta_4\}, \{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_6\}, \{\vartheta_1, \vartheta_3, \vartheta_6\}\}$$

is a $\mathcal{SR}$-topology.

Definition 7. *Let $(\mathcal{V}, \tau_{\mathcal{SR}}(\mathcal{Y}), \mathcal{E})$ be a $\mathcal{SR}$-topological space. Any subset $\mathcal{A}$ such that $\mathcal{A} \in \tau_{\mathcal{SR}}$ $\mathcal{A}$ is said to be $\mathcal{SR}$-open, and any subset $\mathcal{A}$ is $\mathcal{SR}$-closed if and only if $\mathcal{A}^c \in \tau_{\mathcal{SR}}$.*

Example 5. *In Example 2, we can see that $\{\vartheta_2, \vartheta_4\}, \{\vartheta_1, \vartheta_2, \vartheta_3, \vartheta_4, \vartheta_6\}, \{\vartheta_1, \vartheta_3, \vartheta_6\}$ are $\mathcal{SR}$-open sets, and their relative complements $\{\vartheta_1, \vartheta_3, \vartheta_5, \vartheta_6\}, \{\vartheta_5\}, \{\vartheta_2, \vartheta_4, \vartheta_5\}$ are $\mathcal{SR}$-closed sets, while $\mathcal{V}$ and $\emptyset$ are both $\mathcal{SR}$-open and $\mathcal{SR}$-closed.*

3. Topological Structure of $\mathcal{SR}$-Sets

In this section, we define a new topological structure on $\mathcal{SR}$-sets. We define $\mathcal{SR}$-open set, $\mathcal{SR}$-closed sets, $\mathcal{SR}$-closure, $\mathcal{SR}$-interior, $\mathcal{SR}$-neighborhood, and $\mathcal{SR}$-bases.

Definition 8. *Let $\mathcal{Y} = (\Re_{P_\star}(\mathcal{Y}), \Re_P{}^\star(\mathcal{Y}))$ be a $\mathcal{SR}$-subset, where $\mathcal{P} = (\mathcal{V}, \mathcal{S})$. Let $\tau_{\mathcal{SR}_\star}$ and $\tau_{\mathcal{SR}}{}^\star$ be two topologies which contain only exact subsets of $\Re_{P_\star}(\mathcal{Y})$ and $\Re_P{}^\star(\mathcal{Y})$, respectively. Then, the pair $\tau_{\mathcal{SR}} = (\tau_{\mathcal{SR}_\star}, \tau_{\mathcal{SR}}{}^\star)$ is called a $\mathcal{SR}$-topology on the $\mathcal{SR}$-set $\mathcal{Y}$ and the pair $(\mathcal{Y}, \tau_{\mathcal{SR}})$ is known as a soft rough topological space (SRTS). Soft rough set $\mathcal{Y}$ with the topology $\tau_{\mathcal{SR}} = (\tau_{\mathcal{SR}_\star}, \tau_{\mathcal{SR}}{}^\star)$ is known as topological $\mathcal{SR}$-set. Also, in a $\mathcal{SR}$-topology, $\tau_{\mathcal{SR}} = (\tau_{\mathcal{SR}_\star}, \tau_{\mathcal{SR}}{}^\star)$, $\tau_{\mathcal{SR}_\star}$ is the lower $\mathcal{SR}$-topology and $\tau_{\mathcal{SR}}{}^\star$ is the upper $\mathcal{SR}$-topology on X.*

Remark 1. *Since $\Re_{P_\star}(\mathcal{Y})$ and $\Re_{P^\star}(\mathcal{Y})$ are only exactly defined sets in $\mathcal{SR}$-approximation space, we restrict the elements of $\tau_\star$ and $\tau^\star$ to the set of all exact or definable subsets of $\Re_{P_\star}(\mathcal{Y})$ and $\Re_{P^\star}(\mathcal{Y})$, respectively. However, when they are grouped to form the $\mathcal{SR}$-topology $\tau_{\mathcal{SR}} = (\tau_{\mathcal{SR}\star}, \tau_{\mathcal{SR}}{}^\star)$, indefinable sets can also be $\mathcal{SR}$-open. The point to be noted is that a subset of $\mathcal{Y}$, either exact or inexact, is $\mathcal{SR}$-open iff its lower approximation is in the lower $\mathcal{SR}$-topology and its upper approximation is in the upper $\mathcal{SR}$-topology.*

Definition 9. *Let $(\mathcal{Y}, \tau_{\mathcal{SR}})$ be an SRTS, where $\tau_{\mathcal{SR}} = (\tau_{\mathcal{SR}\star}, \tau_{\mathcal{SR}}{}^\star)$. Let $\mathcal{A} = (\Re_{P_\star}(\mathcal{A}), \Re_{P^\star}(\mathcal{A}))$ be any $\mathcal{SR}$-subset of $\mathcal{Y} = (\Re_{P_\star}(\mathcal{Y}), \Re_{P^\star}(\mathcal{Y}))$. Then, $\mathcal{A}$ is said to be lower $\mathcal{SR}$-open if the lower approximation of $\mathcal{A}$ belongs to the lower $\mathcal{SR}$-topology. That is, $\Re_{P_\star}(\mathcal{A}) \in \tau_{\mathcal{SR}\star}$. Also, $\mathcal{A}$ is said to be upper $\mathcal{SR}$-open if the upper approximation of $\mathcal{A}$ belongs to the upper $\mathcal{SR}$-topology. That is, $\Re_{P^\star}(\mathcal{A}) \in \tau_{\mathcal{SR}}{}^\star$. $\mathcal{A}$ is said to be $\mathcal{SR}$-open iff $\mathcal{A}$ is both lower $\mathcal{SR}$-open and upper $\mathcal{SR}$-open, i.e., $\Re_{P_\star}(\mathcal{A}) \in \tau_{\mathcal{SR}\star}$ and $\Re_{P^\star}(\mathcal{A}) \in \tau_{\mathcal{SR}}{}^\star$.*

Theorem 1. *Consider an SRTS $(\mathcal{Y}, \tau_{\mathcal{SR}})$, where $\mathcal{Y} = (\Re_{P_\star}(\mathcal{Y}), \Re_{P^\star}(\mathcal{Y}))$ and $\tau_{\mathcal{SR}} = (\tau_{\mathcal{SR}\star}, \tau_{\mathcal{SR}}{}^\star)$. Let T be a collection of $\mathcal{SR}$-open subsets of $(\mathcal{Y}, \tau_{\mathcal{SR}})$. Then, T is a topology on $\mathcal{Y}$.*

Proof. Consider $\mathcal{Y} = (\Re_{P_\star}(\mathcal{Y}), \Re_{P^\star}(\mathcal{Y}))$ and $\tau_{\mathcal{SR}} = (\tau_{\mathcal{SR}\star}, \tau_{\mathcal{SR}}{}^\star)$.
(i) We have $\emptyset \in \tau_{\mathcal{SR}\star}$ and $\emptyset \in \tau_{\mathcal{SR}}{}^\star$. Therefore, $\emptyset\ (\emptyset, \emptyset) \in T$. Also, $\Re_{P_\star}(\mathcal{Y}) \in \tau_{\mathcal{SR}\star}$ and $\Re_{P^\star}(\mathcal{Y}) \in \tau_{\mathcal{SR}}{}^\star$ and, hence, $\mathcal{Y} = (\Re_{P_\star}(\mathcal{Y}), \Re_{P^\star}(\mathcal{Y})) \in T$.
(ii) Let $\mathcal{A} = (\Re_{P_\star}(\mathcal{A}), \Re_{P^\star}(\mathcal{A}))$ and $\mathcal{B} = (\Re_{P_\star}(\mathcal{B}), \Re_{P^\star}(\mathcal{B}))$ be any two elements of T, implying that both $\mathcal{A}$ and $\mathcal{B}$ are $\mathcal{SR}$-open subsets of $\mathcal{Y}$. Therefore, $\Re_{P_\star}(\mathcal{A}) \in \tau_{\mathcal{SR}\star}$, $\Re_{P^\star}(\mathcal{A}) \in \tau_{\mathcal{SR}}{}^\star$ and $\Re_{P_\star}(\mathcal{B}) \in \tau_{\mathcal{SR}\star}$, $\Re_{P^\star}(\mathcal{B}) \in \tau_{\mathcal{SR}}{}^\star$. Being topologies, $\tau_{\mathcal{SR}\star}$ and $\tau_{\mathcal{SR}}{}^\star$ are closed under finite intersection; therefore, $\Re_{P_\star}(\mathcal{A}) \cap \Re_{P_\star}(\mathcal{B}) \in \tau_{\mathcal{SR}\star}$ and $\Re_{P^\star}(\mathcal{A}) \cap \Re_{P^\star}(\mathcal{B}) \in \tau_{\mathcal{SR}}{}^\star$. Hence, $\mathcal{A} \cap \mathcal{B} = (\Re_{P_\star}(\mathcal{A}) \cap \Re_{P_\star}(\mathcal{B}), \Re_{P^\star}(\mathcal{A}) \cap \Re_{P^\star}(\mathcal{B}))$ is an $\mathcal{SR}$-open subset of $\mathcal{Y}$, which shows that $\mathcal{A} \cap \mathcal{B} \in T$. Since $\mathcal{A}$ and $\mathcal{B}$ are arbitrary, T is closed under finite intersections.
(iii) Let $\{\mathcal{A}_\mu = (\Re_{P_\star}(\mathcal{A}_\mu), \Re_{P^\star}(\mathcal{A}_\mu)) \mid \mu \in \Omega\}$ be an arbitrary family of $\mathcal{SR}$-open subsets of $\mathcal{Y}$, and belongs to the subcollection T. $\mathcal{A}_\mu = (\Re_{P_\star}(\mathcal{A}_\mu), \Re_{P^\star}(\mathcal{A}_\mu)) \in T$ implies $\Re_{P_\star}(\mathcal{A}_\mu) \in \tau_{\mathcal{SR}\star}$ and $\Re_{P^\star}(\mathcal{A}_\mu) \in \tau_{\mathcal{SR}}{}^\star$ for all $\mu \in \Omega$. Since $\tau_{\mathcal{SR}\star}$ and $\tau_{\mathcal{SR}}{}^\star$ are closed under arbitrary union, we have $\bigcup_{\mu\in\Omega} \Re_{P_\star}(\mathcal{A}_\mu) \in \tau_{\mathcal{SR}\star}$ and $\bigcup_{\mu\in\Omega} \Re_{P^\star}(\mathcal{A}_\mu) \in \tau_{\mathcal{SR}}{}^\star$, which shows that $\bigcup_{\mu\in\Omega} \mathcal{A}_\mu = \left(\bigcup_{\mu\in\Omega} \Re_{P_\star}(\mathcal{A}_\mu), \bigcup_{\mu\in\Omega} \Re_{P^\star}(\mathcal{A}_\mu)\right)$ is an $\mathcal{SR}$-open subset of $\mathcal{Y}$. Thus, T is closed under arbitrary union.

From (i), (ii), and (iii), the family T of $\mathcal{Y}$ forms a topology on $\mathcal{Y}$. □

Definition 10. *In any $\mathcal{SR}$-set $\mathcal{Y} = (\Re_{P_\star}(\mathcal{Y}), \Re_{P^\star}(\mathcal{Y}))$, define $\tau_{\mathcal{SR}\star} = \{\mathcal{A} \subseteq \Re_{P_\star}(\mathcal{Y}) \mid \mathcal{A} \text{ is } P - definable\}$ and $\tau_{\mathcal{SR}}{}^\star = \{\mathcal{B} \subseteq \Re_{P^\star}(\mathcal{Y}) / \mathcal{B} \text{ is } P - definable\}$. Then, $\tau_{\mathcal{SR}\star}$ and $\tau_{\mathcal{SR}}{}^\star$ are topologies on $\Re_{P_\star}(\mathcal{Y})$ and $\Re_{P^\star}(\mathcal{Y})$, respectively, and the $\mathcal{SR}$-topology $\tau_{\mathcal{SR}} = (\tau_{\mathcal{SR}\star}, \tau_{\mathcal{SR}}{}^\star)$ is known as the Discrete $\mathcal{SR}$-topology on $\mathcal{Y}$, and the topological space $(\mathcal{Y}, \tau_{\mathcal{SR}})$ is known as the Discrete $\mathcal{SR}$-Topological Space on $\mathcal{Y}$.*

Definition 11. *In an $\mathcal{SR}$-set $\mathcal{Y} = (\Re_{P_\star}(\mathcal{Y}), \Re_{P^\star}(\mathcal{Y}))$, take $\tau_{\mathcal{SR}\star} = (\emptyset, \Re_{P_\star}(\mathcal{Y}))$ and $\tau_{\mathcal{SR}}{}^\star = (\emptyset, \Re_{P^\star}(\mathcal{Y}))$, then $\tau_{\mathcal{SR}\star}$ and $\tau_{\mathcal{SR}}{}^\star$ are topologies on $\Re_{P_\star}(\mathcal{Y})$ and $\Re_{P^\star}(\mathcal{Y})$, respectively, and the $\mathcal{SR}$-topology $\tau_{\mathcal{SR}} = (\tau_{\mathcal{SR}\star}, \tau_{\mathcal{SR}}{}^\star)$ on $\mathcal{Y}$ is known as the indiscrete $\mathcal{SR}$-topology on $\mathcal{Y}$, and $(\mathcal{Y}, \tau_{\mathcal{SR}})$ is known as the indiscrete $\mathcal{SR}$-topological space on $\mathcal{Y}$.*

Definition 12. *In an SRTS $(\mathcal{Y}, \tau_{\mathcal{SR}})$, where $\mathcal{Y} = (\Re_{P_\star}(\mathcal{Y}), \Re_{P^\star}(\mathcal{Y}))$ and $\tau_{\mathcal{SR}} = (\tau_{\mathcal{SR}\star}, \tau_{\mathcal{SR}}{}^\star)$. Consider a subcollection $\beta_\star$ of subsets of $\Re_{P_\star}(\mathcal{Y})$; if every element of $\tau_{\mathcal{SR}\star}$ can be expressed as the union of some elements of $\beta_\star$, then $\beta_\star$ is said to be a base for $\tau_{\mathcal{SR}\star}$. If every member of $\tau_{\mathcal{SR}}{}^\star$ can be expressed as the union of some members of $\beta^\star$ for another subcollection $\beta^\star$ of subsets of $\Re_{P^\star}(\mathcal{Y})$, then $\beta^\star$ is said to be a base for $\tau_{\mathcal{SR}}{}^\star$. If the above conditions are satisfied, then the pair $\beta_{\mathcal{SR}} = (\beta_\star, \beta^\star)$ is known as a $\mathcal{SR}$-base for the $\mathcal{SR}$-topology $\tau_{\mathcal{SR}}$ on $\mathcal{Y}$.*

Theorem 2. *Consider the SRTS $(\mathcal{Y}, \tau_{\mathcal{SR}})$, where $\mathcal{Y} = (\Re_{P_\star}(\mathcal{Y}), \Re_{P^\star}(\mathcal{Y}))$ and $\tau_{\mathcal{SR}} = (\tau_{\mathcal{SR}\star}, \tau_{\mathcal{SR}}{}^\star)$. $\beta_{\mathcal{SR}} = (\beta_\star, \beta^\star)$ is an $\mathcal{SR}$-base for $\tau_{\mathcal{SR}}$ iff for any $\mathcal{SR}$-open set $\mathcal{A} = (\Re_{P_\star}(\mathcal{A}), \Re_{P^\star}(\mathcal{A}))$ of $(\mathcal{Y}, \tau_{\mathcal{SR}})$ and $(x, y) \in \mathcal{A}$ such that $x \in \Re_{P_\star}(\mathcal{A})$ and $y \in \Re_{P^\star}(\mathcal{A})$, then there exist $\mathfrak{B}_\star \in \beta_\star$ and $\mathfrak{B}^\star \in \beta^\star$ such that $x \in \mathfrak{B}_\star \subseteq \Re_{P_\star}(\mathcal{A})$ and $y \in \mathfrak{B}^\star \subseteq \Re_{P^\star}(\mathcal{A})$.*

Proof. Consider the SRTS $(\mathcal{Y}, \tau_{\mathcal{SR}})$, where $\mathcal{Y} = (\Re_{P_\star}(\mathcal{Y}), \Re_{P^\star}(\mathcal{Y}))$ and $\tau_{\mathcal{SR}} = (\tau_{\mathcal{SR}\star}, \tau_{\mathcal{SR}}{}^\star)$. Let $\beta_\star$ and $\beta^\star$ be families of subsets of $\Re_{P_\star}(\mathcal{Y})$ and $\Re_{P_\star}(\mathcal{Y})$, respectively, such that $\beta_{\mathcal{SR}} = (\beta_\star, \beta^\star)$ is a $\mathcal{SR}$-base for $\tau_{\mathcal{SR}}$. Also, consider any $\mathcal{SR}$-subset $\mathcal{A} = (\Re_{P_\star}(\mathcal{A}), \Re_{P^\star}(\mathcal{A}))$ and let $(x, y) \in \mathcal{A}$ be an arbitrary point such that $x \in \Re_{P_\star}(\mathcal{A})$ and $y \in \Re_{P^\star}(\mathcal{A})$. Now, $x \in \Re_{P_\star}(\mathcal{A}), \Re_{P_\star}(\mathcal{A}) \in \tau_{\mathcal{SR}\star}$ and $\beta_\star$ is a base for $\tau_{\mathcal{SR}\star}$, which implies that $\Re_{P_\star}(\mathcal{A})$ can be written as the union of elements of $\beta_\star$. Hence, $\exists \mathfrak{B}_\mu \in \beta_\star$ such that $x \in \mathfrak{B}_\mu$ and $\mathfrak{B}_\mu \subseteq \Re_{P^\star}(\mathcal{A})$. Choose such a $\mathfrak{B}_\mu$ as $\mathfrak{B}_\star$. Therefore, $x \in \mathfrak{B}_\star \subseteq \Re_{P_\star}(\mathcal{A})$.

Similarly, by the same argument, there exists $\mathfrak{B}^\star \in \beta^\star$ such that $y \in \mathfrak{B}^\star \subseteq \Re_{P^\star}(\mathcal{A})$.

Conversely, suppose that $\beta_\star$ and $\beta^\star$ are families of subsets of $\Re_{P_\star}(\mathcal{Y})$ and $\Re_{P^\star}(\mathcal{Y})$, respectively, such that for any $\mathcal{SR}$-open set $\mathcal{A} = (\Re_{P_\star}(\mathcal{A}), \Re_{P^\star}(\mathcal{A}))$ of $(\mathcal{Y}, \tau_{\mathcal{SR}})$, $(x, y) \in \mathcal{A}$, where $x \in \Re_{P_\star}(\mathcal{A})$ and $y \in \Re_{P^\star}(\mathcal{A})$; then, there exist $\mathfrak{B}_\star \in \beta_\star$ and $\mathfrak{B}^\star \in \beta^\star$ such that $x \in \mathfrak{B}_\star \subseteq \Re_{P_\star}(\mathcal{A})$ and $y \in \mathfrak{B}^\star \subseteq \Re_{P^\star}(\mathcal{A})$. Now, we have to prove that $\beta_{\mathcal{SR}} = (\beta_\star, \beta^\star)$ is an $\mathcal{SR}$-base for $\tau_{\mathcal{SR}}$. Let $C = (\Re_{P_\star}(\mathcal{C}), \Re_{P^\star}(\mathcal{C}))$ be any $\mathcal{SR}$-open subset of the SRTS $(\mathcal{Y}, \tau_{\mathcal{SR}})$. By our assumption, for each $x \in \Re_{P_\star}(\mathcal{C})$, we have $\mathfrak{B}_{x\star} \in \beta_\star$ such that $x \in \mathfrak{B}_{x\star} \subseteq \Re_{P_\star}(\mathcal{C})$. Thus, $\Re_{P_\star}(\mathcal{C}) = \bigcup_{x \in \Re_{P_\star}(\mathcal{C})} \mathfrak{B}_{x\star}$. This implies that $\Re_{P_\star}(\mathcal{C})$ can be expressed as the union of some elements of $\beta_\star$. Since $C = (\Re_{P_\star}(\mathcal{C}), \Re_{P^\star}(\mathcal{C}))$ is taken arbitrarily, $\beta_\star$ is a lower base for $\mathcal{SR}$-topology $\tau_{\mathcal{SR}}$.

Similarly, by the same argument, $\Re_{P^\star}(\mathcal{C})$ can be expressed as the union of some members of $\beta^\star$; therefore, $\beta^\star$ is an upper base for the $\mathcal{SR}$-topology $\tau_{\mathcal{SR}}$. Hence, $\beta_{\mathcal{SR}} = (\beta_\star, \beta^\star)$ is an $\mathcal{SR}$-base for $\tau_{\mathcal{SR}}$. $\square$

Definition 13. *In an SRTS $(\mathcal{Y}, \tau_{\mathcal{SR}})$, where $\mathcal{Y} = (\Re_{P_\star}(\mathcal{Y}), \Re_{P^\star}(\mathcal{Y}))$ and $\tau_{\mathcal{SR}} = (\tau_{\mathcal{SR}\star}, \tau_{\mathcal{SR}}{}^\star)$. The collection $\mathcal{S} = (\mathcal{S}_\star, \mathcal{S}^\star)$ of subsets of $\mathcal{Y}$, where $\mathcal{S}_\star$ and $\mathcal{S}^\star$ are a collection of subsets of $\Re_{P_\star}(\mathcal{Y})$ and $\Re_{P^\star}(\mathcal{Y})$. $\mathcal{S}$ is said to be an $\mathcal{SR}$-subbase for the topology $\tau_{\mathcal{SR}}$ iff the following conditions are satisfied:*
(i) $\mathcal{S}_\star \subset \tau_{\mathcal{SR}\star}$ and $\mathcal{S}^\star \subset \tau_{\mathcal{SR}}{}^\star$.
(ii) Finite intersection of elements of $\mathcal{S}_\star$ gives a base for $\tau_{\mathcal{SR}\star}$ and finite intersection of elements of $\mathcal{S}^\star$ gives a base for $\tau_{\mathcal{SR}}{}^\star$.

Definition 14. *In an SRTS $(\mathcal{Y}, \tau_{\mathcal{SR}})$, where $\mathcal{Y} = (\Re_{P_\star}(\mathcal{Y}), \Re_{P^\star}(\mathcal{Y}))$ and $\tau_{\mathcal{SR}} = (\tau_{\mathcal{SR}\star}, \tau_{\mathcal{SR}}{}^\star)$. Let $\mathcal{A} = (\Re_{P_\star}(\mathcal{A}), \Re_{P^\star}(\mathcal{A}))$ be any $\mathcal{SR}$-subset of $\mathcal{Y}$. Then, the lower closure of $\mathcal{A}$ is the closure of $\Re_{P_\star}(\mathcal{A})$ in $(\Re_{P_\star}(\mathcal{Y}), \tau_{\mathcal{SR}\star})$ and is defined as the intersection of all closed supersets of $\Re_{P_\star}(\mathcal{A})$, and it is denoted by $Cl_{\mathcal{SR}}(\Re_{P_\star}(\mathcal{A}))$. Also, the upper closure of $\Re_{P^\star}(\mathcal{A})$ in $(\Re_{P^\star}(\mathcal{Y}), \tau_{\mathcal{SR}}{}^\star)$ is the intersection of all closed supersets of $\Re_{P^\star}(\mathcal{A})$ and is denoted by $Cl_{\mathcal{SR}}\Re_{P^\star}(\mathcal{A})$. Then, the $\mathcal{SR}$-closure of $\mathcal{A} = (\Re_{P_\star}(\mathcal{A}), \Re_{P^\star}(\mathcal{A}))$ is defined as $Cl_{\mathcal{SR}}(\mathcal{A}) = (Cl_{\mathcal{SR}}(\Re_{P_\star}(\mathcal{A})), Cl_{\mathcal{SR}}(\Re_{P^\star}(\mathcal{A})))$.*

Definition 15. *In an SRTS $(\mathcal{Y}, \tau_{\mathcal{SR}})$, where $\mathcal{Y} = (\Re_{P_\star}(\mathcal{Y}), \Re_{P^\star}(\mathcal{Y}))$ and $\tau_{\mathcal{SR}} = (\tau_{\mathcal{SR}\star}, \tau_{\mathcal{SR}}{}^\star)$. Let $A = (\Re_{P_\star}(\mathcal{A}), \Re_{P^\star}(\mathcal{A}))$ be any $\mathcal{SR}$-subset of $\mathcal{Y}$. Then, the lower interior of $\mathcal{A}$ is the interior of $\Re_{P_\star}(\mathcal{A})$ in $(\Re_{P_\star}(\mathcal{Y}), \tau_{\mathcal{SR}\star})$ and is defined as union of all $\mathcal{SR}$-open subsets of $(\Re_{P_\star}(\mathcal{Y}), \tau_{\mathcal{SR}\star})$ contained in $\Re_{P_\star}(\mathcal{A})$, and it is denoted by $Int_{\mathcal{SR}}(\Re_{P_\star}(\mathcal{A}))$. Also, the upper interior of $\Re_{P^\star}(\mathcal{A})$ in $(\Re_{P^\star}(\mathcal{Y}), \tau_{\mathcal{SR}}{}^\star)$ is the union of all $\mathcal{SR}$-open subsets of $(\Re_{P_\star}(\mathcal{Y}), \tau_{\mathcal{SR}\star})$ contained in $\Re_{P^\star}(\mathcal{A})$ and is denoted by $Int_{\mathcal{SR}}(\Re_{P^\star}(\mathcal{A}))$. Then, the $\mathcal{SR}$-interior of $\mathcal{A} = (\Re_{P_\star}(\mathcal{A}), \Re_{P^\star}(\mathcal{A}))$ and is defined as $Int_{\mathcal{SR}}(\mathcal{A}) = (Int_{\mathcal{SR}}(\Re_{P_\star}(\mathcal{A})), Int_{\mathcal{SR}}(\Re_{P^\star}(\mathcal{A})))$.*

Definition 16. *An $\mathcal{SR}$-subset $\mathcal{A}$ of $(\mathcal{Y}, \tau_{\mathcal{SR}})$ is said to be dense in $\mathcal{Y}$ if $Cl_{\mathcal{SR}}(\mathcal{A}) = \mathcal{Y}$, i.e., an $\mathcal{SR}$-subset $\mathcal{A} = (\Re_{P_\star}(\mathcal{A}), \Re_{P}{}^\star(\mathcal{A}))$ is dense in $\mathcal{Y}$ if $Cl_{\mathcal{SR}}(\Re_{P_\star}(\mathcal{A})) = \Re_{P_\star}(\mathcal{Y})$ and $Cl_{\mathcal{SR}}(\Re_{P}{}^\star(\mathcal{A})) = \Re_{P}{}^\star(\mathcal{Y})$.*

Theorem 3. *An $\mathcal{SR}$-subset $\mathcal{A} = (\Re_{P_\star}(\mathcal{A}), \Re_{P}{}^\star(\mathcal{A}))$ of SRTS $(\mathcal{Y}, \tau_{\mathcal{SR}})$ is dense in $\mathcal{Y}$ iff for every non-empty $\mathcal{SR}$-open set $\mathcal{B} = (\Re_{P_\star}(\mathcal{B}), \Re_{P}{}^\star(\mathcal{B}))$ of $(\mathcal{Y}, \tau_{\mathcal{SR}})$, $\Re_{P_\star}(\mathcal{A}) \cap \Re_{P_\star}(\mathcal{B}) \neq \emptyset$ and $\Re_{P}{}^\star(\mathcal{A}) \cap \Re_{P}{}^\star(\mathcal{B}) \neq \emptyset$.*

Proof. Suppose $\mathcal{A} = (\Re_{P_\star}(\mathcal{A}), \Re_{P}{}^\star(\mathcal{A}))$ is dense in $\mathcal{Y}$. Then, $Cl_{\mathcal{SR}}(\mathcal{A}) = (Cl_{\mathcal{SR}}(\Re_{P_\star}(\mathcal{A})), Cl_{\mathcal{SR}}(\Re_{P}{}^\star(\mathcal{A}))) = (\Re_{P_\star}(\mathcal{Y}), \Re_{P}{}^\star(\mathcal{Y})) = \mathcal{Y}$. Therefore, $Cl_{\mathcal{SR}}(\Re_{P_\star}(\mathcal{A})) = \Re_{P_\star}(\mathcal{Y})$ and $Cl_{\mathcal{SR}}(\Re_{P}{}^\star(\mathcal{A})) = \Re_{P}{}^\star(\mathcal{Y})$. Now, $\mathcal{B} = (\Re_{P_\star}(\mathcal{B}), \Re_{P}{}^\star(\mathcal{B}))$ be any non-empty $\mathcal{SR}$-open subset of $(\mathcal{Y}, \tau_{\mathcal{SR}})$. Then, $\mathcal{A} \cap \mathcal{B} = (\Re_{P_\star}(\mathcal{A}) \cap \Re_{P_\star}(\mathcal{B}), \Re_{P}{}^\star(\mathcal{A}) \cap \Re_{P}{}^\star(\mathcal{B}))$.

Suppose $\Re_{P_\star}(\mathcal{A}) \cap \Re_{P_\star}(\mathcal{B}) = \emptyset$. Then, $\Re_{P_\star}(\mathcal{A}) \subseteq (\Re_{P_\star}(\mathcal{Y}) \setminus \Re_{P_\star}(\mathcal{B}))$, which implies $Cl_{\mathcal{SR}}(\Re_{P_\star}(\mathcal{A})) \subseteq (\Re_{P_\star}(\mathcal{Y}) \setminus \Re_{P_\star}(\mathcal{B}))$, since $\Re_{P_\star}(\mathcal{B}) \in \tau_{\mathcal{SR}\star}$ and, therefore, $(\Re_{P_\star}(\mathcal{Y}) \setminus \Re_{P_\star}(\mathcal{B}))$ is closed. However, $(\Re_{P_\star}(\mathcal{Y}) \setminus \Re_{P_\star}(\mathcal{B}))$ is a proper subset of $\Re_{P_\star}(\mathcal{Y})$, which contradicts $Cl_{\mathcal{SR}}\Re_{P}{}^\star(\mathcal{A}) = \Re_{P}{}^\star(\mathcal{Y})$. Hence, $\Re_{P_\star}(\mathcal{A}) \cap \Re_{P_\star}(\mathcal{B}) \neq \emptyset$. Similarly, $\Re_{P_\star}(\mathcal{A}) \cap \Re_{P_\star}(\mathcal{B}) \neq \emptyset$.

Conversely, suppose $\mathcal{A} = (\Re_{P_\star}(\mathcal{A}), \Re_{P}{}^\star(\mathcal{A}))$ is a $\mathcal{SR}$-subset of $\mathcal{Y}$ such that for every non-empty $\mathcal{SR}$-open set $\mathcal{B} = (\Re_{P_\star}(\mathcal{B}), \Re_{P}{}^\star(\mathcal{B}))$ of $(\mathcal{Y}, \tau_{\mathcal{SR}})$, $\Re_{P_\star}(\mathcal{A}) \cap \Re_{P_\star}(\mathcal{B}) \neq \emptyset$ and $\Re_{P}{}^\star(\mathcal{A}) \cap \Re_{P}{}^\star(\mathcal{B}) \neq \emptyset$. Let $y \in \Re_{P_\star}(\mathcal{Y})$, since $\Re_{P_\star}(\mathcal{A}) \cap \Re_{P_\star}(\mathcal{B}) \neq \emptyset$; so, either $y \in \Re_{P_\star}(\mathcal{A})$ or it is a limit point of $\Re_{P_\star}(\mathcal{A})$. That is, $y \in Cl_{\mathcal{SR}}(\Re_{P_\star}(\mathcal{A}))$. Therefore, $\Re_{P_\star}(\mathcal{Y}) \subseteq Cl_{\mathcal{SR}}(\Re_{P_\star}(\mathcal{A})) \subseteq \Re_{P_\star}(\mathcal{Y})$, which implies $Cl_{\mathcal{SR}}(\Re_{P_\star}(\mathcal{A})) = \Re_{P_\star}(\mathcal{Y})$. By a similar argument, we can prove that $Cl_{\mathcal{SR}}(\Re_{P}{}^\star(\mathcal{A})) = \Re_{P}{}^\star(\mathcal{Y})$. Hence $Cl_{\mathcal{SR}}(\mathcal{A}) = (Cl_{\mathcal{SR}}(\Re_{P}{}^\star(\mathcal{A})), Cl_{\mathcal{SR}}(\Re_{P}{}^\star(\mathcal{A}))) = (\Re_{P_\star}(\mathcal{Y}), \Re_{P}{}^\star(\mathcal{Y})) = \mathcal{Y}$. So, $\mathcal{A}$ is dense in $\mathcal{Y}$. □

Definition 17. *In an SRTS $(\mathcal{Y}, \tau_{\mathcal{SR}})$, where $\mathcal{Y} = (\Re_{P_\star}(\mathcal{Y}), \Re_{P}{}^\star(\mathcal{Y}))$ and $\tau_{\mathcal{SR}} = (\tau_{\mathcal{SR}\star}, \tau_{\mathcal{SR}}{}^\star)$. If for $\gamma \in \mathcal{Y}$ there exist an open set $\mathcal{V}_1$ of $\Re_{P_\star}(\mathcal{Y})$ such that $\gamma \in \mathcal{V}_1 \subseteq \mathcal{N}_\star$, where $\mathcal{N}_\star \subseteq \Re_{P_\star}(\mathcal{Y})$, then the subset $\mathcal{N}_\star$ is called $\tau_{\mathcal{SR}\star} -$ neighborhood. Similarly, if for $\gamma \in \mathcal{Y}$ there exist an open set $\mathcal{V}_2$ of $\Re_{P}{}^\star(\mathcal{Y})$ such that $\gamma \in \mathcal{V}_2 \subseteq \mathcal{N}^\star$, where $\mathcal{N}^\star \subseteq \Re_{P}{}^\star(\mathcal{Y})$, then the subset $\mathcal{N}^\star$ is called $\tau_{\mathcal{SR}}{}^\star -$ neighborhood. If, at the same time, $\mathcal{N}_\star \subseteq \Re_{P_\star}(\mathcal{Y})$ and $\mathcal{N}^\star \subseteq \Re_{P}{}^\star(\mathcal{Y})$, then $\mathcal{N}_{\mathcal{SR}} = (\mathcal{N}_\star, \mathcal{N}^\star)$ is said to be a $\tau_{\mathcal{SR}} -$ neighborhood of $\gamma \in \mathcal{Y}$.*

Proposition 1. *Consider an SRTS $(\mathcal{Y}, \tau_{\mathcal{SR}})$, where $\mathcal{Y} = (\Re_{P_\star}(\mathcal{Y}), \Re_{P}{}^\star(\mathcal{Y}))$ and $\tau_{\mathcal{SR}} = (\tau_{\mathcal{SR}\star}, \tau_{\mathcal{SR}}{}^\star)$. Let $\mathcal{A} = (\Re_{P_\star}(\mathcal{A}), \Re_{P}{}^\star(\mathcal{A}))$ be an $\mathcal{SR}$-subset of $\mathcal{SR}$-set $\mathcal{Y}$ satisfying $\Re_{P_\star}(\mathcal{A}) \subseteq \Re_{P_\star}(\mathcal{Y}) \subseteq \Re_{P}{}^\star(\mathcal{Y})$. Then, $\mathcal{A}$ is $\mathcal{SR}$-open iff it is a neighborhood of each of its points.*

Proof. Suppose that $\mathcal{A} = (\Re_{P_\star}(\mathcal{A}), \Re_{P}{}^\star(\mathcal{A}))$ as an open subset of $\mathcal{SR}$-set $\mathcal{Y} = (\Re_{P_\star}(\mathcal{Y}), \Re_{P}{}^\star(\mathcal{Y}))$. Then, for every $\mu \in \Re_{P_\star}(\mathcal{A})$, $\mu \in \Re_{P_\star}(\mathcal{A}) \subset \Re_{P_\star}(\mathcal{A})$, and for every $\nu \in \Re_{P}{}^\star(\mathcal{A})$, $\nu \in \Re_{P}{}^\star(\mathcal{A}) \subset \Re_{P}{}^\star(\mathcal{A})$. Hence, $\Re_{P_\star}(\mathcal{A})$ and $\Re_{P}{}^\star(\mathcal{A})$ satisfy the neighborhood definition and are neighborhoods of each point, and, hence, $\mathcal{A} = (\Re_{P_\star}(\mathcal{A}), \Re_{P}{}^\star(\mathcal{A}))$ is a neighborhood of each of its points.

Conversely, suppose $\mathcal{A} = (\Re_{P_\star}(\mathcal{A}), \Re_{P}{}^\star(\mathcal{A}))$ is a neighborhood of each of its points. Given the assumption $\Re_{P_\star}(\mathcal{A}) \subseteq \Re_{P_\star}(\mathcal{Y}) \subseteq \Re_{P}{}^\star(\mathcal{Y})$, if $\mathcal{A} = \emptyset$, then it is $\mathcal{SR}$-open. For $\mu \in \mathcal{A}$, then there exists an $\mathcal{SR}$-open set $\mathcal{V} = (\mathcal{V}_{\mu_\star}, \mathcal{V}_{\mu}{}^\star)$ in $\mathcal{Y}$ such that $\mu \in \mathcal{V}_{\mu_\star} \subset \Re_{P_\star}(\mathcal{A})$ and $\mu \in \mathcal{V}_{\mu}{}^\star \subset \Re_{P}{}^\star(\mathcal{A})$. This implies $\Re_{P_\star}(\mathcal{A}) = \bigcup\{\mathcal{V}_{\mu_\star} \,/\, \mu \in \Re_{P_\star}(\mathcal{A})\}$ and $\Re_{P}{}^\star(\mathcal{A}) = \bigcup\{\mathcal{V}_{\mu}{}^\star \,/\, \mu \in \Re_{P}{}^\star(\mathcal{A})\}$. Hence, $\Re_{P_\star}(\mathcal{A})$ and $\Re_{P}{}^\star(\mathcal{A})$ are $\mathcal{SR}$-open, which implies $\mathcal{A}$ is open. □

4. Continuity in $\mathcal{SR}$-Sets

In this section, we discuss the continuity of functions in $\mathcal{SR}$-topological spaces, the continuous image of an $\mathcal{SR}$-closed set. The $\mathcal{SR}$-homeomorphism is the part of the conversation.

Definition 18. *Let $(\mathcal{Y}, \tau_{\mathcal{SR}})$ and $(\mathcal{Z}, \rho_{\mathcal{SR}})$ be topological $\mathcal{SR}$-sets with topologies $\tau_{\mathcal{SR}} = (\tau_{\mathcal{SR}\star}, \tau_{\mathcal{SR}}{}^{\star})$ and $\rho_{\mathcal{SR}} = (\rho_{\mathcal{SR}\star}, \rho_{\mathcal{SR}}{}^{\star})$, respectively. A function $\varphi_1 : \Re_{P_\star}(\mathcal{Y}) \to \Re_{P_\star}(\mathcal{Z})$ is continuous at $\mu \in \mathcal{Y}$ iff every ρ_1-neighborhood $\mathcal{H}_1$ of $\varphi_1(\mu)$ in $\Re_{P_\star}(\mathcal{Z})$ there exists a τ_1-neighborhood $\mathcal{G}_1$ of μ in $\Re_{P_\star}(\mathcal{Y})$ such that $\varphi_1(\mathcal{G}_1) \subset \mathcal{H}_1$ and $\varphi_2 : \Re_P{}^{\star}(\mathcal{Y}) \to \Re_P{}^{\star}(\mathcal{Z})$ is continuous at $\mu \in \mathcal{Z}$ iff every ρ_2-neighborhood $\mathcal{H}_2$ of $\varphi_2(\mu)$ in $\Re_{P_\star}(\mathcal{Z})$ there exists a τ_2-neighborhood $\mathcal{G}_2$ of μ in $\Re_P{}^{\star}(\mathcal{Z})$ such that $\varphi_2(\mathcal{G}_2) \subset \mathcal{H}_2$. Then, the function $\varphi = (\varphi_1, \varphi_2) : \mathcal{Y} \to \mathcal{Z}$ is said to be a continuous function at μ if both φ_1 and φ_2 are continuous functions at μ.*

Example 6. *Assume that $\mathcal{V} = \{\wp_1, \wp_2, \wp_3, \wp_4\}$, $\mathcal{E} = \{\zeta_1, \zeta_2, \zeta_3, \zeta_4\}$, $\mathcal{A} = \{\zeta_1, \zeta_3, \zeta_4\} \subset \mathcal{E}$ and $\mathcal{G} = \{(\zeta_1, (\wp_1, \wp_4), (\zeta_3, \wp_2), (\zeta_4, \wp_3)\}$ is a soft set. Thus, we get $P = (\mathcal{V}, \mathcal{G})$ as a soft approximation space. If we take $\mathcal{Y} \subset \mathcal{V}$, where $\mathcal{Y} = \{\wp_3, \wp_4\}$, then we have $\Re_{P_\star}(\mathcal{Y}) = \{\wp_3\}$, $\Re_P{}^{\star}(\mathcal{Y}) = \{\wp_1, \wp_3, \wp_4\}$ and $Bnd_P = \{\wp_1, \wp_4\}$. Thus, $\tau_{\mathcal{SR}}(\mathcal{Y}) = \{\mathcal{V}, \emptyset, \{\wp_3\}, \{\wp_1, \wp_3, \wp_4\}, \{\wp_1, \wp_4\}\}$ is an $\mathcal{SR}$-topology.*

Let $\mathcal{W} = \{\varpi_1, \varpi_2, \varpi_3, \varpi_4\}$ and $\mathcal{H} = \{(\zeta_1, \{\varpi_1\}), (\zeta_3, \{\varpi_2, \varpi_3\}), (\zeta_4, \{\varpi_4\})\}$ be a soft set; then, we have $P' = (\mathcal{W}, \mathcal{H})$ as a soft approximation space. If we take $\mathcal{Z} \subset \mathcal{W}$, where $\mathcal{Z} = \{\varpi_3, \varpi_4\}$, then $\Re_{P'\star}(\mathcal{Z}) = \{\varpi_4\}$, $\Re_{P'}{}^{\star}(\mathcal{Z}) = \{\varpi_2, \varpi_3, \varpi_4\}$ and $Bnd_{P'} = \{\varpi_2, \varpi_3\}$,
and $\rho_{\mathcal{SR}} = \{\mathcal{W}, \emptyset, \{\varpi_4\}, \{\varpi_2, \varpi_3, \varpi_4\}, \{\varpi_2, \varpi_3\}\}$ is another $\mathcal{SR}$-topology.

Define a function $\varphi = (\varphi_1, \varphi_2) : \mathcal{V} \to \mathcal{W}$ such that $\varphi(\wp_1) = \varphi_2(\wp_1) = \varpi_2$, $\varphi(\wp_2) = \varphi_2(\wp_2) = \varpi_1$, $\varphi(\wp_3) = \varphi_1(\wp_3) = \varphi_1(\wp_1) = \varpi_4$ and $\varphi(\wp_4) = \varphi_2(\wp_4) = \varpi_3$. Then, $\varphi^{-1}(\{\varpi_2, \varpi_3, \varpi_4\}) = \{\wp_1, \wp_3, \wp_4\}$, $\varphi^{-1}(\{\varpi_2, \varpi_3\}) = \{\wp_1, \wp_4\}$ and $\varphi^{-1}(\{\varpi_4\}) = \{\wp_3\}$. Thus, φ is $\mathcal{SR}$-continuous, since the inverse image for each $\mathcal{SR}$-open set in $\mathcal{W}$ is $\mathcal{SR}$-open in $\mathcal{V}$.

Theorem 4. *Consider $(\mathcal{Y}, \tau_{\mathcal{SR}})$ and $(\mathcal{Z}, \rho_{\mathcal{SR}})$ are topological $\mathcal{SR}$-sets and $\varphi = (\varphi_1, \varphi_2) : \mathcal{Y} \to \mathcal{Z}$. For every ρ-$\mathcal{SR}$-open set $\mathcal{V} = (\mathcal{V}_1, \mathcal{V}_2)$, $\varphi_1{}^{-1}(\mathcal{V}_1) \subseteq \Re_{P_\star}(\mathcal{Y}) \subseteq \varphi_2{}^{-1}(\mathcal{V}_2) \subseteq \Re_P{}^{\star}(\mathcal{Y})$. Then, φ is a continuous function if and only if the inverse image of every $\mathcal{SR}$-open set in $\mathcal{Z}$ under φ is $\mathcal{SR}$-open in $\mathcal{Y}$.*

Proof. Suppose $\varphi = (\varphi_1, \varphi_2) : \mathcal{Y} \to \mathcal{Z}$ is a continuous function and $\mathcal{V} = (\mathcal{V}_1, \mathcal{V}_2)$ is an $\mathcal{SR}$-open set in $\mathcal{Z}$. We have to prove that $\varphi^{-1}(\mathcal{V}) = \big(\varphi_1{}^{-1}(\mathcal{V}_1), \varphi_2{}^{-1}(\mathcal{V}_2)\big)$ is an $\mathcal{SR}$-open set in $\mathcal{Y}$. If $\varphi_1{}^{-1}(\mathcal{V}_1)$ and $\varphi_2{}^{-1}(\mathcal{V}_2)$ are empty, then the result is obvious.

Suppose $\mu \in \varphi_1{}^{-1}(\mathcal{V}_1) \Rightarrow \mu \in \varphi_2{}^{-1}(\mathcal{V}_2)$, that is, $\varphi_1(\mu) \in \mathcal{V}_1$ and $\varphi_2(\mu) \in \mathcal{V}_2$. By following the definition of continuity of φ_1, there exists a neighborhood $\mathcal{N}_1$ of μ such that $\varphi_1(\mathcal{N}_1) \subset \mathcal{V}_1$; then, $\mu \in \mathcal{N}_1 = \varphi_1{}^{-1}(\varphi(\mathcal{N}_1)) \subseteq \varphi_1{}^{-1}(\mathcal{V}_1)$, which implies $\varphi_1{}^{-1}(\mathcal{V}_1)$ is $\mathcal{SR}$-open. Similarly, $\varphi_2{}^{-1}(\mathcal{V}_2)$ is also $\mathcal{SR}$-open. Hence, $\varphi^{-1}(\mathcal{V})$ is $\mathcal{SR}$-open.

Conversely, let $\varphi^{-1}(\mathcal{V})$ be $\mathcal{SR}$-open in $\mathcal{Y}$ for every $\mathcal{SR}$-open set $\mathcal{V}$ in $\mathcal{Z}$. We have to prove that φ is a continuous function.

Consider $\mu \in \mathcal{Y}$ as an arbitrary point, and $\varphi_1(\mu) \in \mathcal{V}_1$ implies $\varphi_2(\mu) \in \mathcal{V}_2$ (by hypothesis). Then, $\mu \in \varphi_1{}^{-1}(\mathcal{V}_1)$ and $\mu \in \varphi_2{}^{-1}(\mathcal{V}_2)$, which means $\varphi_1(\varphi_1{}^{-1}(\mathcal{V}_1)) \subset \mathcal{V}_1$ and $\varphi_2(\varphi_2{}^{-1}(\mathcal{V}_2)) \subset \mathcal{V}_2$ implies that φ_1 and φ_2 are continuous at μ. Since we take μ as an arbitrary point, then φ_1 and φ_2 are continuous everywhere. Hence, φ is continuous. $\square$

Corollary 1. *A function $\varphi = (\varphi_1, \varphi_2) : \mathcal{Y} \to \mathcal{Z}$ is continuous if and only if for every $\mathcal{SR}$-closed subset $\mathcal{C}$ in $\mathcal{Z}$, $\varphi^{-1}(\mathcal{C})$ is $\mathcal{SR}$-closed in $\mathcal{Y}$.*

Proof. Consider $\varphi = (\varphi_1, \varphi_2) : \mathcal{Y} = (\Re_{P_\star}(\mathcal{Y}), \Re_P{}^{\star}(\mathcal{Y})) \to \mathcal{Z} = (\Re_{P_\star}(\mathcal{Z}), \Re_P{}^{\star}(\mathcal{Z}))$ is a continuous function and $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2)$ is an arbitrary $\mathcal{SR}$-closed subset of $\mathcal{Z}$. Then, $\Re_{P_\star}(\mathcal{Z}) \setminus \mathcal{C}_1$ and $\Re_P{}^{\star}(\mathcal{Z}) \setminus \mathcal{C}_2$ are $\mathcal{SR}$-open in $\mathcal{Y} = (\Re_{P_\star}(\mathcal{Y}), \Re_P{}^{\star}(\mathcal{Y}))$ and $\varphi_1{}^{-1}(\Re_{P_\star}(\mathcal{Y}) \setminus \mathcal{C}_1) = \Re_{P_\star}(\mathcal{Y}) \setminus \varphi_1{}^{-1}(\mathcal{C}_1)$ and $\varphi_1{}^{-1}(\Re_P{}^{\star}(\mathcal{Y}) \setminus \mathcal{C}_2) = \Re_P{}^{\star}(\mathcal{Y}) \setminus \varphi_1{}^{-1}(\mathcal{C}_2)$, which implies that $\varphi_1{}^{-1}(\mathcal{C}_1)$ and $\varphi_1{}^{-1}(\mathcal{C}_2)$ are closed in $\Re_{P_\star}(\mathcal{Y})$ and $\Re_P{}^{\star}(\mathcal{Y})$, respectively. Hence, $\varphi^{-1}(\mathcal{C})$ is $\mathcal{SR}$-closed in $\mathcal{Y}$.

Conversely, suppose that for any $\mathcal{SR}$-closed subset $\mathcal{C} = (\mathcal{C}_1, \mathcal{C}_2)$ in $\mathcal{Z}$, $\varphi^{-1}(\mathcal{C})$ is $\mathcal{SR}$-closed in $\mathcal{Y}$. Let $\mathcal{V} = (\mathcal{V}_1, \mathcal{V}_2)$ be any $\mathcal{SR}$-open subset of $\mathcal{Z} = (\Re_{P_\star}(\mathcal{Z}), \Re_P{}^\star(\mathcal{Z}))$. Then, $\mathcal{Z} \setminus (\mathcal{V}) = (\Re_{P_\star}(\mathcal{Z}) \setminus (\mathcal{V}_1), \Re_P{}^\star(\mathcal{Z}) \setminus (\mathcal{V}_2))$ is $\mathcal{SR}$-closed and $\varphi^{-1}(\mathcal{Z} \setminus \mathcal{V}) = \left(\varphi_1{}^{-1}(\Re_{P_\star}(\mathcal{Z}) \setminus \mathcal{V}_1), \varphi_2{}^{-1}(\Re_P{}^\star(\mathcal{Z}) \setminus \mathcal{V}_2)\right) = \varphi^{-1}(\mathcal{Z}) \setminus \varphi^{-1}(\mathcal{V}) = \mathcal{Z} \setminus \varphi^{-1}(\mathcal{V})$ is $\mathcal{SR}$-closed in $\mathcal{Y}$, which implies $\varphi^{-1}(\mathcal{V})$ is $\mathcal{SR}$-open in $\mathcal{Y}$. Thus, φ is continuous. □

Remark 2. *1. Every restriction of a continuous mapping is also continuous.*

Let $\psi = (\psi_1, \psi_2) : \mathcal{Y} = (\Re_{P_\star}(\mathcal{Y}), \Re_P{}^\star(\mathcal{Y})) \to \mathcal{Z} = (\Re_{P_\star}(\mathcal{Z}), \Re_P{}^\star(\mathcal{Z}))$ be a continuous function and $\mathcal{A} = (\Re_{P_\star}(\mathcal{A}), \Re_P{}^\star(\mathcal{A}))$ be a $\mathcal{SR}$-subset of $\mathcal{Y}$. Then, the restriction $\psi|_\mathcal{A} = \psi_\mathcal{A} : \mathcal{A} \to \mathcal{Z}$ of ψ to $\mathcal{A}$ is continuous. This is so because for each $\mathcal{SR}$-open subset W in $\mathcal{Z}$, $\psi_\mathcal{A}{}^{-1}(W) = \psi^{-1}(W) \cap \mathcal{A}$, which is $\mathcal{SR}$-open in $\mathcal{A}$.
2. Consider $\beta_{\mathcal{SR}} = (\beta_\star, \beta^\star)$ as a base for a $\mathcal{SR}$-topology on $\mathcal{Z}$. Then, the function $\psi : \mathcal{Y} \to \mathcal{Z}$ is continuous if and only if, for each $\mathcal{SR}$-basic open set in $\mathcal{Z}$, $\psi^{-1}(\beta)$ is $\mathcal{SR}$-open in $\mathcal{Y}$.
3. A function $\psi : \mathcal{Y} \to \mathcal{Z}$ is open if the image of every $\mathcal{SR}$-open set in $\mathcal{Y}$ is $\mathcal{SR}$-open.
4. A function $\psi : \mathcal{Y} \to \mathcal{Y}$ is closed if the image of every $\mathcal{SR}$-closed set in $\mathcal{Y}$ is $\mathcal{SR}$-closed.

Definition 19. *Let $(\mathcal{Y}, \tau_{\mathcal{SR}})$ and $(\mathcal{Z}, \rho_{\mathcal{SR}})$ be topological $\mathcal{SR}$-sets. A function $\varphi = (\varphi_1, \varphi_2) : \mathcal{Y} \to \mathcal{Z}$ is known as $\mathcal{SR}$-homeomorphism if*

(i) φ is $\mathcal{SR}$-bijective.
(ii) φ is $\mathcal{SR}$-continuous.
(iii) φ^{-1} is $\mathcal{SR}$-continuous.

Two soft rough topological spaces (SRTS) are said to be $\mathcal{SR}$-homeomorphic if there is a $\mathcal{SR}$-homeomorphism between $\mathcal{Y}$ and $\mathcal{Z}$.

Definition 20. *Consider $\mathcal{Y} = (\Re_{P_\star}(\mathcal{Y}), \Re_P{}^\star(\mathcal{Y}))$ and $\mathcal{Z} = (\Re_{P_\star}(\mathcal{Z}), \Re_P{}^\star(\mathcal{Z}))$ as two topological $\mathcal{SR}$-sets with topologies $\tau_{\mathcal{SR}} = (\tau_1, \tau_2)$ and $\rho_{\mathcal{SR}} = (\rho_1, \rho_2)$, respectively, and $\mathcal{Y} \times \mathcal{Z} = (\Re_{P_\star}(\mathcal{Y}) \times \Re_{P_\star}(\mathcal{Z}), \Re_P{}^\star(\mathcal{Y}) \times \Re_P{}^\star(\mathcal{Z}))$ is the Cartesian product of $\mathcal{Y}$ and $\mathcal{Z}$. The topology ξ_1 on $\Re_{P_\star}(\mathcal{Y}) \times \Re_{P_\star}(\mathcal{Y})$ containing a gathering of open sets of the form $\mathcal{L}_1 \times \mathcal{M}_1$, where $\mathcal{L}_1$ is a $\tau_1 - \mathcal{SR}$-open and $\mathcal{M}_1$ is a $\rho_1 - \mathcal{SR}$-open, as basis, is known as the product topology. Similarly, the topology ξ_2 on $\Re_P{}^\star(\mathcal{Y}) \times \Re_P{}^\star(\mathcal{Z})$ is the topology containing a gathering of open sets of the form $\mathcal{L}_2 \times \mathcal{M}_2$, where $\mathcal{L}_2$ is a $\tau_2 - \mathcal{SR}$-open and $\mathcal{M}_2$ is a $\rho_2 - \mathcal{SR}$-open, as basis, is known as the product topology. Hence, the topology $\xi = (\xi_1, \xi_2)$ is called thte product topology on $\mathcal{Y} \times \mathcal{Z}$.*

Definition 21. *Consider $\mathcal{Y} = (\Re_{P_\star}(\mathcal{Y}), \Re_P{}^\star(\mathcal{Y}))$ and $\mathcal{Z} = (\Re_{P_\star}(\mathcal{Z}), \Re_P{}^\star(\mathcal{Z}))$ as two topological $\mathcal{SR}$-sets with topologies $\tau_{\mathcal{SR}} = (\tau_1, \tau_2)$ and $\rho_{\mathcal{SR}} = (\rho_1, \rho_2)$, respectively. The mapping $\prod_{\mu_\star} = \Re_{P_\star}(\mathcal{Y}) \times \Re_{P_\star}(\mathcal{Z}) \to \Re_{P_\star}(\mathcal{Y})$ and $\prod_\mu{}^\star = \Re_P{}^\star(\mathcal{Y}) \times \Re_P{}^\star(\mathcal{Z}) \to \Re_P{}^\star(\mathcal{Y})$, defined as $\prod_{\mu_\star}((\mu, \nu)) = \mu$, $\forall\ (\mu, \nu) \in \Re_{P_\star}(\mathcal{Y}) \times \Re_{P_\star}(\mathcal{Z})$ and $\prod_\mu{}^\star((\mu, \nu)) = \mu$, $\forall\ (\mu, \nu) \in \Re_P{}^\star(\mathcal{Y}) \times \Re_P{}^\star(\mathcal{Z})$, respectively, are known as projection mappings. Then, $\prod_\mu = \left(\prod_{\mu_\star}, \prod_\mu{}^\star\right)$ is known as the projection mapping from $\mathcal{Y} \times \mathcal{Z} \to \mathcal{Y}$. Similarly, we can define the projection mapping $\prod_\nu = (\prod_{\nu\star}, \prod_\nu{}^\star)$ from $\mathcal{Y} \times \mathcal{Z} \to \mathcal{Y}$.*

Theorem 5. *Consider $\mathcal{Y}$ and $\mathcal{Z}$ as two topological $\mathcal{SR}$-sets and $\mathcal{Y} \times \mathcal{Z}$ as the product space. Then, the projections $\prod_\mu$ and $\prod_\nu$ are continuous mappings.*

Proof. Suppose $\mathcal{Y} = (\Re_{P_\star}(\mathcal{Y}), \Re_P{}^\star(\mathcal{Y}))$ and $\mathcal{Z} = (\Re_{P_\star}(\mathcal{Z}), \Re_P{}^\star(\mathcal{Z}))$ are two topological $\mathcal{SR}$-sets with topologies $\tau_{\mathcal{SR}} = (\tau_1, \tau_2)$ and $\rho_{\mathcal{SR}} = (\rho_1, \rho_2)$, respectively. Let ξ be the product topology on $\mathcal{Y} \times \mathcal{Y}$ and $\mathcal{L} = (\Re_{P_\star}(\mathcal{L}), \Re_P{}^\star(\mathcal{L}))$ be an $\tau - \mathcal{SR}$-open set. Then, $\prod_{\mu_\star}{}^{-1}(\Re_{P_\star}(\mathcal{L})) = \Re_{P_\star}(\mathcal{L}) \times \Re_{P_\star}(\mathcal{Z})$, where $\Re_{P_\star}(\mathcal{L}) \in \tau_1$ and $\Re_{P_\star}(\mathcal{Z}) \in \rho_1$ imply that $\Re_{P_\star}(\mathcal{L}) \times \Re_{P_\star}(\mathcal{Z})$ belongs to the basis for τ_1. Also, $\prod_\mu{}^{\star-1}(\Re_P{}^\star(\mathcal{L})) = \Re_P{}^\star(\mathcal{L}) \times \Re_P{}^\star(\mathcal{Z})$, where $\Re_P{}^\star(\mathcal{L}) \in \tau_2$ and $\Re_P{}^\star(\mathcal{Y}) \in \rho_2$ imply $\Re_P{}^\star(\mathcal{L}) \times \Re_P{}^\star(\mathcal{Y})$ belongs to the basis for τ_2, which implies that $(\Re_{P_\star}(\mathcal{L}) \times \Re_{P_\star}(\mathcal{Z}), \Re_P{}^\star(\mathcal{L}) \times \Re_P{}^\star(\mathcal{Z}))$. Thus, $\prod_{\alpha\star}$ and $\prod_\alpha{}^\star$ are continuous mappings. Therefore, $\prod_\alpha = \left(\prod_{\mu_\star}, \prod_\mu{}^\star\right)$ is a continuous mapping. Similarly, we can show that $\prod_\nu = (\prod_{\nu\star}, \prod_\nu{}^\star)$ is also a continuous mapping. □

5. Compactness in $\mathcal{SR}$-Set

In this section, we study the compactness of $\mathcal{SR}$-topological spaces, discuss images of $\mathcal{SR}$-compact spaces, and prove some basic results.

Definition 22. *Let $\mathcal{Y} = (\Re_{P_\star}(\mathcal{Y}), \Re_P{}^\star(\mathcal{Y}))$ be a $\mathcal{SR}$-set. For any open covering $\mathfrak{V}_1 = \{V_\mu / i \in \Omega\}$ of $\Re_{P_\star}(\mathcal{Y})$, if we get a finite subcovering $\mathfrak{V}_1{}^F = \{V_\mu \; / i = 1, 2, ..., m\}$, then $\Re_{P_\star}(\mathcal{Y})$ is said to be the compact lower approximation of $\mathcal{Y}$. Similarly, for any open covering $\mathfrak{V}_2 = \{V_j / j \in \Omega\}$ of $\Re_P{}^\star(\mathcal{Y})$, if we get a finite subcovering $\mathfrak{V}_2{}^F = \{V_j \; / j = 1, 2, ..., n\}$, then $\Re_P{}^\star(\mathcal{Y})$ is said to be the compact upper approximation of $\mathcal{Y}$. Then, the $\mathcal{SR}$-set $\mathcal{Y} = (\Re_{P_\star}(\mathcal{Y}), \Re_P{}^\star(\mathcal{Y}))$ is known as a compact $\mathcal{SR}$-set.*

Definition 23. *Suppose $\mathcal{A} = (\Re_{P_\star}(\mathcal{A}), \Re_P{}^\star(\mathcal{A}))$ is an $\mathcal{SR}$ subset of $\mathcal{Y} = (\Re_{P_\star}(\mathcal{Y}), \Re_P{}^\star(\mathcal{Y}))$. If, for any open covering $\mathfrak{W} = \{W_j / j \in \Omega\}$ of $\Re_{P_\star}(\mathcal{A})$, we get a finite subcovering $\mathfrak{V}^F = \{V_j \; / j = 1, 2, ..., n\}$ of $\Re_{P_\star}(\mathcal{A})$, then $\Re_{P_\star}(\mathcal{A})$ as the subset of $\Re_{P_\star}(\mathcal{Y})$ is said to be compact. If, at the same time, $\Re_P{}^\star(\mathcal{A})$ is also compact, then we call $\mathcal{A}$ a compact $\mathcal{SR}$-subset of $\mathcal{Y}$.*

Theorem 6. *The continuous image of a compact topological $\mathcal{SR}$-set is compact.*

Proof. Consider $\mathcal{Y} = (\Re_{P_\star}(\mathcal{Y}), \Re_P{}^\star(\mathcal{Y}))$ as a compact $\mathcal{SR}$-set and suppose that $\psi = (\psi_1, \psi_2)$: $(\Re_{P_\star}(\mathcal{Y}), \Re_P{}^\star(\mathcal{Y})) \to (\Re_{P_\star}(\mathcal{Z}), \Re_P{}^\star(\mathcal{Z}))$ is a continuous mapping. Then, $\psi_1 : \Re_{P_\star}(\mathcal{Y}) \to \Re_{P_\star}(\mathcal{Z})$ and $\psi_2 : \Re_P{}^\star(\mathcal{Y}) \to \Re_P{}^\star(\mathcal{Z})$ individually are continuous mappings. Let $\mathcal{C}_1 = \{W_\nu \; / \nu \in \Omega\}$ be an open covering of $\Re_{P_\star}(\mathcal{Z})$. Then, $\psi_1{}^{-1}(\mathcal{C}_1) = \{\psi_1{}^{-1}(W_\nu) \; / \nu \in \Omega\}$ is an open covering for $\Re_{P_\star}(\mathcal{Y})$. Since $\Re_{P_\star}(\mathcal{Y})$ is compact, then, by definition of compactness, it has a finite subcovering, and there are indices $\nu_1 \nu_2, ..., \nu_m$ such that $\Re_{P_\star}(\mathcal{Y}) = \bigcup\limits_{i=1}^{m} \psi_1{}^{-1}(\mathcal{C}_1)$. $\psi_1(\Re_{P_\star}(\mathcal{Y})) \subseteq \bigcup\limits_{i=1}^{m} (\mathcal{C}_1) \subseteq \psi_1(\Re_{P_\star}(\mathcal{Y}))$. Therefore, $\{W_{\nu_\mu} \; / i = 1, 2, 3, ..., m\}$ is a finite subcovering of $\psi(\Re_{P_\star}(\mathcal{Y})) = \Re_{P_\star}(\mathcal{Z})$. So, $\Re_{P_\star}(\mathcal{Z})$ is also compact. Similarly, we can show that $\Re_P{}^\star(\mathcal{Z})$ is compact and, hence, $\mathcal{Z} = (\Re_{P_\star}(\mathcal{Z}), \Re_P{}^\star(\mathcal{Z}))$ is a compact $\mathcal{SR}$-set. □

Corollary 2. *The homeomorphic image of a compact $\mathcal{SR}$-space is compact.*

Remark 3. *In topological $\mathcal{SR}$-sets, compactness is a topological property.*

Definition 24. *Consider an SRTS $(\mathcal{Y}, \tau_{\mathcal{SR}})$, where $\mathcal{Y} = (\Re_{P_\star}(\mathcal{Y}), \Re_P{}^\star(\mathcal{Y}))$ and $\tau_{\mathcal{SR}} = (\tau_{\mathcal{SR}\star}, \tau_{\mathcal{SR}}{}^\star)$. Let $\Gamma = \{\mathcal{A}_\mu = (\Re_{P_\star}(\mathcal{A}_\mu), \Re_P{}^\star(\mathcal{A}_\mu)) : \mu \in \Lambda\}$ be a collection of $\mathcal{SR}$-subsets of $\mathcal{Y}$. If every finite subcollection of Γ has a non-empty intersection, which means that if we consider any finite subset Λ_1 of Λ, we get $\bigcap_{\nu \in \Lambda_1} \mathcal{A}_\nu \neq \emptyset$, then the finite intersection property holds in collection Γ.*

Theorem 7. *Consider an SRTS $(\mathcal{Y}, \tau_{\mathcal{SR}})$, where $\mathcal{Y} = (\Re_{P_\star}(\mathcal{Y}), \Re_P{}^\star(\mathcal{Y}))$ and $\tau_{\mathcal{SR}} = (\tau_{\mathcal{SR}\star}, \tau_{\mathcal{SR}}{}^\star)$; $\mathcal{Y}$ is $\mathcal{SR}$-compact iff every collection of $\mathcal{SR}$-closed subsets in $\mathcal{Y}$ following the finite intersection property itself has non-empty intersections.*

Proof. First, we suppose $\mathcal{Y}$ is $\mathcal{SR}$-compact and $\Gamma = \{\mathcal{D}_\mu = (\Re_{P_\star}(\mathcal{D}_\mu), \Re_{P^\star}(\mathcal{D}_\mu)) : \mu \in \Lambda\}$ is an arbitrary collection of $\mathcal{SR}$-closed sets satisfying the finite intersection property. We have to prove that the collection $\{\mathcal{D}_\mu = (\Re_{P_\star}(\mathcal{D}_\mu), \Re_{P^\star}(\mathcal{D}_\mu)) : \mu \in \Lambda\}$ itself has non-empty intersection. Suppose, on the contrary, that $\bigcap_{\mu\in\Lambda} \mathcal{D}_\mu = \emptyset$. By taking the complement $(\bigcap_{\mu\in\Lambda} \mathcal{D}_\mu)' = \emptyset'$, we have $\mathcal{Y} = \bigcup_{\mu\in\Lambda} \mathcal{D}'_\mu$, which implies $\{\mathcal{D}_\mu{}' = (\Re_{P_\star}(\mathcal{D}_\mu{}'), \Re_{P^\star}(\mathcal{D}_\mu{}')) : \mu \in \Lambda\}$ is an open cover for $\mathcal{Y} = (\Re_{P_\star}(\mathcal{Y}), \Re_{P^\star}(\mathcal{Y}))$. By our assumption, $\mathcal{Y} = (\Re_{P_\star}(\mathcal{Y}), \Re_{P^\star}(\mathcal{Y}))$ is $\mathcal{SR}$-compact, and there are indices $\mu_1, \mu_2, \mu_3, ..., \mu_k$ such that $\Re_{P_\star}(\mathcal{Y}) = \bigcup_{\iota=1}^{k} \Re_{P_\star}(\mathcal{D}_{\mu_\iota}{}')$ and $\Re_{P^\star}(\mathcal{Y}) = \bigcup_{\iota=1}^{k} \Re_{P^\star}(\mathcal{D}_{\mu_\iota}{}')$. Again, by taking the complement, we get $\bigcap_{\iota=1}^{k} \Re_{P_\star}(\mathcal{D}_{\mu_\iota}) = \emptyset$ and $\bigcap_{\iota=1}^{k} \Re_{P^\star}(\mathcal{D}_{\mu_\iota}) = \emptyset$, that is, $\bigcap_{\iota=1}^{k} \mathcal{D}_{\mu_\iota} = \left(\bigcap_{\iota=1}^{k} \Re_{P_\star}(\mathcal{D}_{\mu_\iota}), \bigcap_{\iota=1}^{k} \Re_{P^\star}(\mathcal{D}_{\mu_\iota})\right) = \emptyset$, which contradicts the finite intersection property. So, our assumption is wrong and $\bigcap_{\mu\in\Lambda} \mathcal{D}_\mu \neq \emptyset$.

Conversely, suppose that every collection of $\mathcal{SR}$-closed sets satisfying the finite intersection property has a non-empty intersection itself. We now have to prove that $\mathcal{Y} = (\Re_{P_\star}(\mathcal{Y}), \Re_{P^\star}(\mathcal{Y}))$ is $\mathcal{SR}$-compact. For this, let us consider $\{\mathcal{V}_\varepsilon = (\Re_{P_\star}(\mathcal{V}_\varepsilon), \Re_{P^\star}(\mathcal{V}_\varepsilon)) : \varepsilon \in \mathrm{Y}\}$ as an open cover of $\mathcal{Y}$, i.e., $\mathcal{Y} = (\Re_{P_\star}(\mathcal{Y}), \Re_{P^\star}(\mathcal{Y})) = \left(\bigcup_{\varepsilon\in\mathrm{Y}} \Re_{P_\star}(\mathcal{V}_\varepsilon), \bigcup_{\varepsilon\in\mathrm{Y}} \Re_{P^\star}(\mathcal{V}_\varepsilon)\right)$. To prove that $\mathcal{Y}$ is $\mathcal{SR}$-compact, we have to show that this open cover has a finite subcover. On the contrary, suppose that there does not exist any finite subcover for this open cover. Then, for any finite subcover Y_1 of Y, $\bigcup_{\epsilon\in\mathrm{Y}_1} \mathcal{V}_\epsilon \neq \mathcal{Y}$, i.e., $\left(\bigcup_{\epsilon\in\mathrm{Y}_1} \Re_{P_\star}(\mathcal{V}_\epsilon), \bigcup_{\epsilon\in\mathrm{Y}_1} \Re_{P^\star}(\mathcal{V}_\epsilon)\right) \neq (\Re_{P_\star}(\mathcal{Y}), \Re_{P^\star}(\mathcal{Y}))$. This implies $\bigcap_{\epsilon\in\mathrm{Y}_1} \mathcal{V}_\epsilon{}' \neq \emptyset$. Now, $\{\mathcal{V}_\varepsilon{}' = (\Re_{P_\star}(\mathcal{V}_\varepsilon{}'), \Re_{P^\star}(\mathcal{V}_\varepsilon)') : \varepsilon \in \mathrm{Y}\}$ is a collection of $\mathcal{SR}$-closed sets satisfying the finite intersection property, so $\bigcap_{\varepsilon\in\mathrm{Y}_1} \mathcal{V}_\varepsilon{}' \neq \emptyset$ i.e., $\left(\bigcap_{\varepsilon\in\mathrm{Y}} \Re_{P_\star}(\mathcal{V}_\varepsilon{}'), \bigcap_{\varepsilon\in\mathrm{Y}} \Re_{P^\star}(\mathcal{V}_\varepsilon{}')\right) \neq \emptyset$. By taking the complement, we get $\left(\bigcup_{\varepsilon\in\mathrm{Y}} \Re_{P_\star}(\mathcal{V}_\varepsilon), \bigcup_{\varepsilon\in\mathrm{Y}} \Re_{P^\star}(\mathcal{V}_\varepsilon)\right) \neq (\Re_{P_\star}(\mathcal{Y}), \Re_{P^\star}(\mathcal{Y}))$, which contradicts our supposition that $\{\mathcal{V}_\varepsilon = (\Re_{P_\star}(\mathcal{V}_\varepsilon), \Re_{P^\star}(\mathcal{V}_\varepsilon)) : \varepsilon \in \mathrm{Y}\}$ is an open cover of $\mathcal{Y}$. Hence, $\{\mathcal{V}_\varepsilon = (\Re_{P_\star}(\mathcal{V}_\varepsilon), \Re_{P^\star}(\mathcal{V}_\varepsilon)) : \varepsilon \in \mathrm{Y}\}$ has a finite subcover, so $\mathcal{Y}$ is $\mathcal{SR}$-compact. □

Theorem 8. *Every $\mathcal{SR}$-closed subset of $\mathcal{SR}$-compact space is $\mathcal{SR}$-compact.*

Proof. Let $\mathcal{Y}$ be an $\mathcal{SR}$-compact space and $\mathcal{D} = (\Re_{P_\star}(\mathcal{D}), \Re_{P^\star}(\mathcal{D}))$ be a $\mathcal{SR}$-closed subset of $\mathcal{Y}$. Let $\{\mathcal{V}_\varepsilon = (\Re_{P_\star}(\mathcal{V}_\varepsilon), \Re_{P^\star}(\mathcal{V}_\varepsilon)) : \varepsilon \in \mathrm{Y}\}$ be an open cover for $\mathcal{D} = (\Re_{P_\star}(\mathcal{D}), \Re_{P^\star}(\mathcal{D}))$; there exist an $\mathcal{SR}$-open set $\mathcal{W}_\varepsilon = (\Re_{P_\star}(\mathcal{W}_\varepsilon), \Re_{P^\star}(\mathcal{W}_\varepsilon))$ in $\mathcal{Y} = (\Re_{P_\star}(\mathcal{Y}), \Re_{P^\star}(\mathcal{Y}))$ such that $\mathcal{V}_\varepsilon = \mathcal{W}_\varepsilon \cap \mathcal{D}$, $\varepsilon \in \mathrm{Y}$, i.e., $\Re_{P_\star}(\mathcal{V}_\varepsilon) = \Re_{P_\star}(\mathcal{W}_\varepsilon) \cap \Re_{P_\star}(\mathcal{D})$ and $\Re_{P^\star}(\mathcal{V}_\varepsilon) = \Re_{P^\star}(\mathcal{W}_\varepsilon) \cap \Re_{P^\star}(\mathcal{D})$. The collection $\{\mathcal{D}', \mathcal{W}_\varepsilon : \varepsilon \in \mathrm{Y}\}$ is an open cover for $\mathcal{Y}$. Since $\mathcal{Y}$ is compact, there exists a finite subcover $\{\mathcal{D}', \mathcal{W}_\varepsilon : \varepsilon \in \mathrm{Y}\}$ of $\mathcal{Y}$, that is, $\mathcal{Y} = \mathcal{D}' \bigcup_{\iota=1}^{k} \mathcal{W}_{\varepsilon\iota}$, which implies $\Re_{P_\star}(\mathcal{Y}) = \Re_{P_\star}(\mathcal{D}') \bigcup_{\iota=1}^{k} \Re_{P_\star}(\mathcal{W}_{\varepsilon\iota})$ and $\Re_{P^\star}(\mathcal{Y}) = \Re_{P^\star}(\mathcal{D}') \bigcup_{\iota=1}^{k} \Re_{P^\star}(\mathcal{W}_{\varepsilon\iota})$. $\mathcal{D} = \mathcal{Y} \cap \mathcal{D} = \left(\Re_{P_\star}(\mathcal{D}') \cap \bigcup_{\iota=1}^{k} \Re_{P_\star}(\mathcal{W}_{\varepsilon\iota}, \Re_{P^\star}(\mathcal{D}') \cap \bigcup_{\iota=1}^{k} \Re_{P^\star}(\mathcal{W}_{\varepsilon\iota}\right)$ $= \left(\bigcup_{\iota=1}^{k} \Re_{P_\star}(\mathcal{V}_{\varepsilon\iota}), \bigcup_{\iota=1}^{k} \Re_{P_\star}(\mathcal{V}_{\varepsilon\iota})\right)$, which indicates $\mathcal{D} = (\Re_{P_\star}(\mathcal{D}), \Re_{P^\star}(\mathcal{D}))$ is $\mathcal{SR}$-compact. □

6. Application of $\mathcal{SR}$-Set in Multi-Attribute Group Decision Making

Decision-making performs a vital role in our daily life, and this process yields the best alternative among different choices. In this section, we present an application of an $\mathcal{SR}$-set in multi-attribute group decision making (MAGDM) for cosmetic brand selection. First, we present Algorithm 1 and its flowchart for multi-attribute group decision making.

Algorithm 1 The scheme of the algorithm is given as.

Step-1: Write the soft set $\mathfrak{G} = (\mathcal{T}, \mathcal{A})$ which describes the given data.
Step-2: Based on initial assessment results of the group of analysts $\mathcal{S}$, define a soft set.
Step-3: Obtain an $\mathcal{SR}$-approximations in the form of soft sets $\Lambda_\star = (\lambda_\star, \mathcal{S})$ and $\Lambda^\star = (\lambda^\star, \mathcal{S})$.
Step-4: Define fuzzy sets $\nu_{\Lambda\star}$, ν_Λ and $\nu_\Lambda{}^\star$ corresponding to the soft sets $\Lambda_\star = (\lambda_\star, \mathcal{S})$, $\Lambda = (\lambda, \mathcal{S})$ and $\Lambda^\star = (\lambda^\star, \mathcal{S})$ defined by the formulas:

$$\nu_{\Lambda\star}(\alpha_k) = \frac{1}{m}\sum_{\iota=1}^{m} C_{\lambda_\star D_\iota}(\alpha_k),$$

$$\nu_\Lambda(\alpha_k) = \frac{1}{m}\sum_{\iota=1}^{m} C_{\lambda D_\iota}(\alpha_k),$$

$$\nu_\Lambda{}^\star(\alpha_k) = \frac{1}{m}\sum_{\iota=1}^{m} C_{\lambda^\star D_\iota}(\alpha_k).$$

Step-6: Find the final decision set by adding $\Lambda_\star$, Λ, and $\Lambda^\star$, calculated as

$$\Lambda_\star + \Lambda + \Lambda^\star = \nu_{\Lambda_\star}(\alpha_k) + \nu_\Lambda(\alpha_k) + \nu_\Lambda{}^\star(\alpha_k) - (\nu_{\Lambda_\star}(\alpha_k) * \nu_\Lambda(\alpha_k) * \nu_\Lambda{}^\star(\alpha_k))$$

Step-7: Finally, the alternative having the maximum decision value can be chosen as the optimal solution.

Now we present flow chart Algorithm 1 as given by Figure 1 and its flowchart for multi-attribute group decision making.

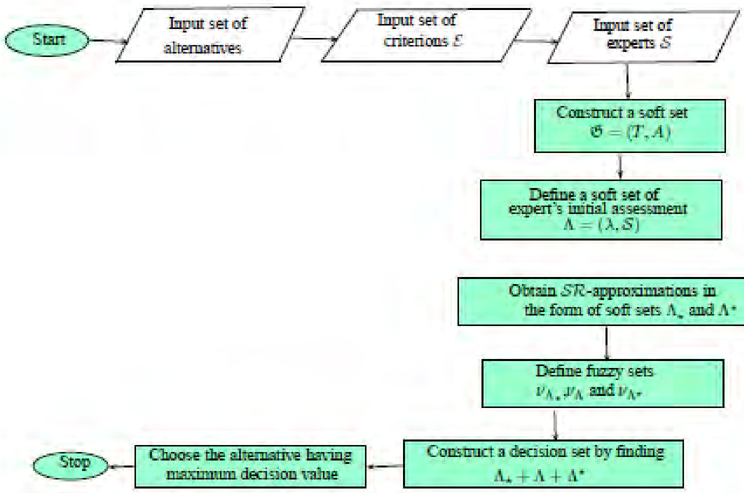

Figure 1. Graphical representation of Algorithm 1.

Example 7. *The trade of quality cosmetics is growing rapidly among the lower-middle class of developing countries like Pakistan and India. Assume that a popular departmental store of the city wants to make a contract with a multinational company for the production of cosmetics. The managing committee of the store consist of three managers,* $\mathcal{S} = \{M_1, M_2, M_3\}$*: the product manager, marketing manager, and accounts manager. The team of these three managers is elected to choose one brand which covers the major production of cosmetics. They consider seven brands:* $\mathcal{V} = \{\hbar_1, \hbar_2, \hbar_3, \hbar_4, \hbar_5, \hbar_6, \hbar_7, \hbar_8\}$*, where*
$\hbar_1$*: Loreal,*
$\hbar_2$*: Maybelline,*
$\hbar_3$*: Remmil,*
$\hbar_4$*: Art Deco,*

$\hbar_5$: Essence,
$\hbar_6$: Color Studio,
$\hbar_7$: Mac,
$\hbar_8$: Sephora.
They define a set of criteria for the selection of a suitable brand for their store as follows, $\mathcal{E} = \{\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \rho_7, \rho_8\}$, where
ρ_1: Product quality
ρ_2: Relationship closeness (customer–brand relationship)
ρ_3: Delivery performance
ρ_4: Price stability
ρ_5: Plans for major events
ρ_6: Distribution plans (in-store furniture)
ρ_7: Recovery services in case of damages
ρ_8: Shopper marketing activities.

We construct a soft set $\mathfrak{G} = (\mathcal{T}, \mathcal{A})$ which explains the qualities of the brands under consideration. The tabular form of the soft set is given in Table 2.

Table 2. Soft set $(\mathcal{T}, \mathcal{A})$.

$(\mathcal{T}, \mathcal{A})$	ρ_1	ρ_2	ρ_3	ρ_4	ρ_5	ρ_6	ρ_7	ρ_8
$\hbar_1$	1	0	1	1	0	1	0	0
$\hbar_2$	1	0	1	1	1	1	0	1
$\hbar_3$	0	1	0	1	0	0	1	0
$\hbar_4$	0	1	0	1	0	0	0	0
$\hbar_5$	1	0	1	0	0	1	0	1
$\hbar_6$	0	1	1	1	0	0	0	0
$\hbar_7$	1	0	1	1	0	1	1	0
$\hbar_8$	0	1	0	1	0	0	1	0

Let X_i be the initial assessment result of the manager team. We represent this evaluation by means of a soft set $\Lambda = (\lambda, \mathcal{S})$ whose tabular representation is given by Table 3.

Table 3. Soft set $(\lambda, \mathcal{S})$.

	D_1	D_2	D_3
$\hbar_1$	1	0	1
$\hbar_2$	1	0	1
$\hbar_3$	0	1	0
$\hbar_4$	0	1	0
$\hbar_5$	0	0	1
$\hbar_6$	1	0	1
$\hbar_7$	1	0	1
$\hbar_8$	0	1	0

From this soft set $\Lambda = (\lambda, \mathcal{S})$, the primary evaluation result of experts is

$$X_1 = \lambda(D_1) = \{\hbar_1, \hbar_2, \hbar_6, \hbar_7\},$$
$$X_2 = \lambda(D_2) = \{\hbar_3, \hbar_4, \hbar_7, \hbar_8\},$$
$$X_3 = \lambda(D_3) = \{\hbar_1, \hbar_2, \hbar_5, \hbar_6\}$$

Now, we find the $\mathcal{SR}$-approximations as

$$\lambda_\star(D_1) = \Re_{P_\star}(X_1) = \{\hbar_1\},$$
$$\lambda_\star(D_2) = \Re_{P_\star}(X_2) = \{\hbar_3, \hbar_7, \hbar_8\},$$
$$\lambda_\star(D_3) = \Re_{P_\star}(X_3) = \{\hbar_2, \hbar_5\},$$

and

$$\lambda^\star(D_1) = \Re_{P}{}^\star(X_1) = \mathcal{V},$$
$$\lambda^\star(D_2) = \Re_{P}{}^\star(X_2) = \mathcal{V},$$
$$\lambda^\star(D_3) = \Re_{P}{}^\star(X_3) = \mathcal{V}.$$

Following these $\mathcal{SR}$-approximations, we get two soft sets, $\Lambda_\star = (\lambda_\star, \mathcal{S})$ and $\Lambda^\star = (\lambda^\star, \mathcal{S})$, where $\lambda_\star(D_i) = \Re_{P_\star}(X_i)$ and $\lambda^\star(D_i) = \Re_{P}{}^\star(X_i)$. Tabular representation of these soft sets are given in Tables 4 and 5.

Table 4. Soft set $\Lambda_\star$.

	D_1	D_2	D_3
$\hbar_1$	0	0	0
$\hbar_2$	1	0	1
$\hbar_3$	0	1	0
$\hbar_4$	0	0	0
$\hbar_5$	0	0	1
$\hbar_6$	0	0	0
$\hbar_7$	0	1	0
$\hbar_8$	0	1	0

Table 5. Soft set $\Lambda^\star$.

	D_1	D_2	D_3
$\hbar_1$	1	1	1
$\hbar_2$	1	1	1
$\hbar_3$	1	1	1
$\hbar_4$	1	1	1
$\hbar_5$	1	1	1
$\hbar_6$	1	1	1
$\hbar_7$	1	1	1
$\hbar_8$	1	1	1

Now, we define a fuzzy set $\nu_{\Lambda_\star}(\hbar_k)$, $\nu_{\Lambda}(\hbar_k)$, and $\nu_{\Lambda}{}^\star(\hbar_k)$ as follows:

$$\nu_{\Lambda_\star}(\hbar_k) = \frac{1}{3}\sum_{i=1}^{3} C_{\lambda_\star D_i}(\hbar_k),$$

$$\nu_{\Lambda}(\hbar_k) = \frac{1}{3}\sum_{i=1}^{3} C_{\lambda D_i}(\hbar_k),$$

$$\nu_{\Lambda}{}^\star(\hbar_k) = \frac{1}{3}\sum_{i=1}^{3} C_{\lambda^\star D_i}(\hbar_k).$$

Thus, we have

$$\nu_{\Lambda_\star}(\hbar_k) = \{(\hbar_1, 0), (\hbar_2, 2/3), (\hbar_3, 1/3), (\hbar_4, 0), (\hbar_5, 1/3), (\hbar_6, 0), (\hbar_7, 1/3), (\hbar_8, 1/3)\},$$
$$\nu_{\Lambda}(\hbar_k) = \{(\hbar_1, 2/3), (\hbar_2, 2/3), (\hbar_3, 1/3), (\hbar_4, 1/3), (\hbar_5, 1/3), (\hbar_6, 2/3), (\hbar_7, 2/3), (\hbar_8, 1/3)\},$$
$$\nu_{\Lambda}{}^\star(\hbar_k) = \{(\hbar_1, 0), (\hbar_2, 2/3), (\hbar_3, 1/3), (\hbar_4, 0), (\hbar_5, 1/3), (\hbar_6, 0), (\hbar_7, 1/3), (\hbar_8, 1/3)\},$$

Now, we find the decision set by adding $\Lambda_\star$, Λ, and $\Lambda^\star$. Then, we have

$$\nu_{\Lambda_\star+\Lambda+\Lambda^\star}(\hbar_k) = \nu_{\Lambda_\star}(\hbar_k) + \nu_{\Lambda}(\hbar_k) + \nu_{\Lambda^\star}(\hbar_k) - [\nu_{\Lambda_\star}(\hbar_k) * \nu_{\Lambda}(\hbar_k) * \nu_{\Lambda^\star}(\hbar_k)].$$

Since $\hbar_2$ is the brand having the maximum decision value in Table 6, then $\hbar_2$ is selected by the the manager team as the major production brand for cosmetics in the departmental store.

In the proposed algorithm, we observe that the use of $\mathcal{SR}$-methodology filters the primary assessment results and permits the experts to choose the optimal alternative in a suitable manner. Particularly, the $\mathcal{SR}$-upper approximation can be used to add optimal objects possibly neglected by the selectors in the primary assessment, while the $\mathcal{SR}$-lower approximation can be used to remove the objects that are irregularly selected as optimal. Hence, $\mathcal{SR}$ reduces the error, to some extent, that is caused by the subjective nature of experts during group decision making.

Table 6. Decision value table.

	Decision Value
$\hbar_1$	1.000
$\hbar_2$	1.889
$\hbar_3$	1.556
$\hbar_4$	1.000
$\hbar_5$	1.556
$\hbar_6$	1.778
$\hbar_7$	1.778
$\hbar_8$	1.556

Now we present bar chart as given by Figure 2 of the decision values.

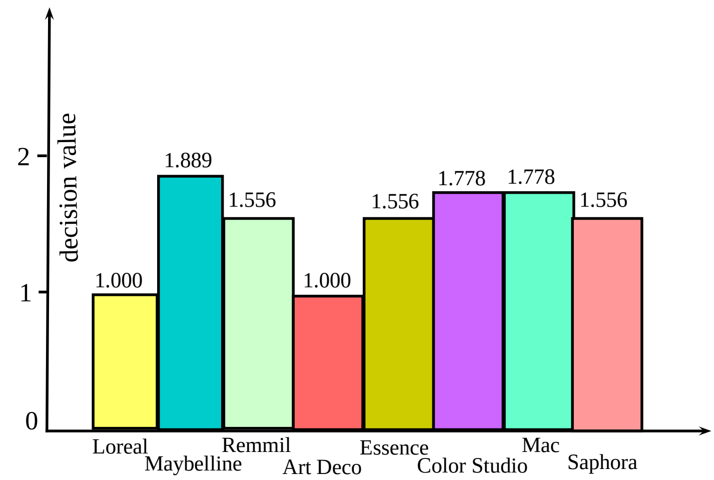

Figure 2. Graphical representation of Decision Values.

7. Applications of the $\mathcal{SR}$-Topological Spaces in Image Processing

We know that a geometrical figure can be obtained by its part information and its topological structure properties. Similarly, according to the $\mathcal{SR}$-topological properties of the $\mathcal{SR}$-space, we can also restore the $\mathcal{SR}$-topological diagram for some incomplete diagram.

Example 8. *One of the best examples of an incomplete image is that of fingerprints. Figure 3 shows a portion of fingerprint information from a person; however, the fingerprint information is incomplete. We can obtain the real fingerprint image on the basis of this Figure 3 by using $\mathcal{SR}$-approximations and $\mathcal{SR}$-topological properties.*

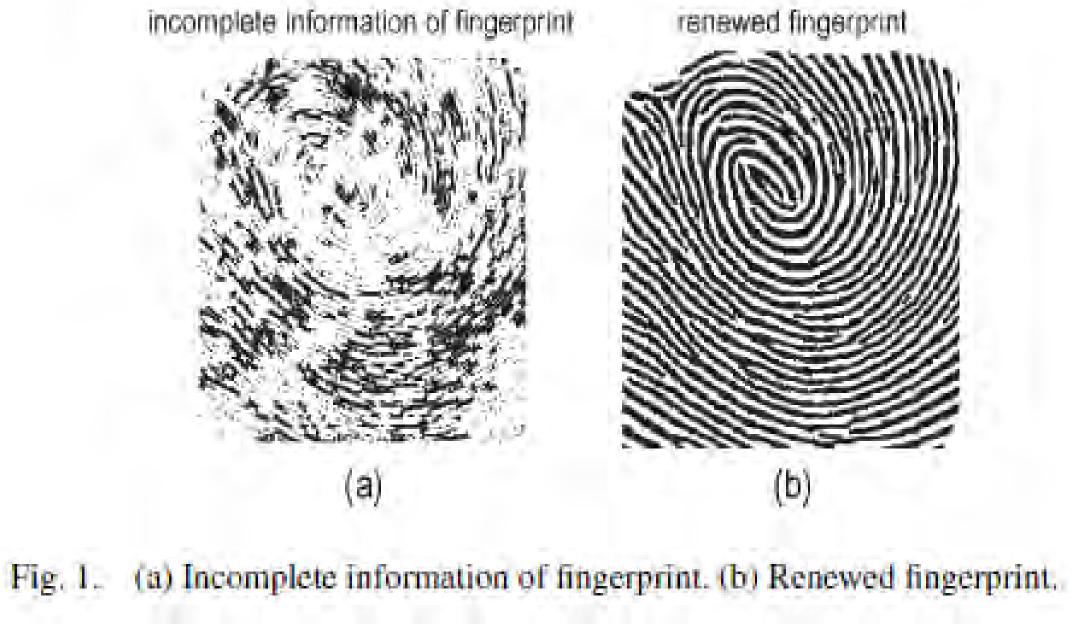

Figure 3. (**a**) Incomplete information of a fingerprint. (**b**) Renewed fingerprint.

The development of these theories can form the theoretical basis for further applications of the $\mathcal{SR}$-set and $\mathcal{SR}$-topology in many science and engineering areas, such as image processing, protein structure prediction, target recognition, and gene structure prediction.

8. Conclusions

We established the topological structure on the $\mathcal{SR}$-set in a new way. We define various topological terms, define $\mathcal{SR}$-continuity, product topology in $\mathcal{SR}$-set, and compactness in $\mathcal{SR}$-sets by taking an $\mathcal{SR}$-set as a pair of sets corresponding to the lower and upper approximations. Furthermore, we present an algorithm to cope with uncertainties in multi-attribute group decision-making problems by utilizing $\mathcal{SR}$-sets. The effectiveness of the algorithm was verified by a case study for cosmetic brand selection. However, under topological transformation, some properties of the $\mathcal{SR}$-set and topological theory, like connectedness and separation axioms on the $\mathcal{SR}$-set, need to be further studied. If we combine the $\mathcal{SR}$-set with other soft computing methods, such as bipolar fuzzy, neutrosophic set, and other hybrid structures, and use them in image processing, expert systems, and cognitive maps, a high machine IQ and hybrid intelligent system can be designed, which will be a productive attempt.

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IoT and Its Impact on the Electronics Market: A Powerful Decision Support System for Helping Customers in Choosing the Best Product

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Abstract: Many companies have observed the significant benefits they can get via using internet. Since then, large companies have been able to develop business transactions with customers at anytime, anywhere, and in relation to anything, so that we now need a more comprehensive concept than the internet. This concept is the Internet of Things (IoT). IoT will influence decision making style in various phases of selling, buying and marketing process. Therefore, every individual and company should know precisely what IoT is, and how and why they should incorporate it in their operations. This motivated us to propose a smart system based on IoT to help companies and marketers make a powerful marketing strategy via utilizing obtained data from IoT devices. Not only this, but the proposed system can also solve the problems which face companies and customers in online shopping. Since there are different types of the same product, and also different criteria for purchasing which can be different between individuals, customers will need a decision support system to recommend them with the best selection. This motivates us to also propose a neutrsophic technique to deal with unclear and conflicting information which exists usually in the purchasing process. Therefore, the smart system and neutrosophic technique is considered as a comprehensive system which links between customers, companies, marketers to achieve satisfaction for each of them.

Keywords: e-marketing; Internet of Things; neutrsophic set; multi-criteria decision making techniques

1. Introduction

Internet of Things (IoT) was presented as a concept in 1999. It has provided a platform to connect to different hardware and mobile devices, so that different people can be connected to each other. The networks can be on the local wide area networks subscribed to each organization, or wireless networks, or both. IoT can also collect data via wireless sensors, and then connect to its central servers for processing and storage. Similarly, it enables people to connect to the internet and other people's mobile devices via central servers and/or wireless sensors. The efficient use of IoT can improve operational efficiency due to its capability to gather and explicate big data, as well as automate connections among machines [1].

The IoT can be applied in several areas such as smart cities, smart homes, education, agriculture, health, wearables, and industrial automation [2]. It provides enormous benefits to the society as a whole. We can see the effect of IoT in cars with built-in sensors, health-monitoring systems, biochip transponders which are used for farm animals, search and deliverance devices, smart washer/dryers which use Wi-Fi for remote monitoring, etc. There will be almost twenty billion devices on the IoT by 2020 according to Gartner [3].

Although more companies and retail stores have adopted IoT, many consumers are still unaware of IoT services.

Using IoT can significantly make users' day to day activities more convenient since many services can be accessed on their mobile devices. It also improves inventory management, tracks product usage, monitors selling rates and locations. Also, the IoT can improve the customer services to allow real-time communications. Additionally, it can allow businesses to forecast possible customers' concerns and cases, and proactively provide solutions [4]. By doing so, it can achieve a better customer satisfaction. As a result, IoT can also save time, reduce costs and also human errors.

Due to the significant role of IoT in enhancing services quality, managing customer demands, and achieving customer satisfaction and loyalty, some studies are presented that highlight this role. Jie et al. [5] illustrated in their study how e-retailers who deal with innovative products in the era of the Internet of Things (IoT) select product delivery service providers to ensure timely and efficient delivery to customers. Additionally, Desai [6] could model IoT services on the basis of service quality dimensions in the electricity distribution center of Bangalore Electricity Supply Company. The researchers in [7] have determined IoT solutions to improve the effectiveness of the service product.

Several research papers are presented to demonstrate discovering the capabilities for IoT adoption in the organization and also studying its effect on customers' experience. The way in which the IoT changes customers' experience while shopping in a retailing context is presented by Balaji and Roy [8]. The theoretical understanding of consumers' adoption and continued use of wearable technology for advanced health and fitness purposes is illustrated by Canhoto and Arp [9]. Wu, Chen, and Dou [10] gave insights into how companies can improve their brand building through the use of IoT technologies. A better understanding of the underlying causes of consumer resistance to smart and related products was developed by Mani and Chouk [11]. Woodside and Sood [12] proposed substantial revisions of the dominant logic of service because of the next take-off phase to adopt new radical innovations in the Internet of Things. Ehret and Wirtz [13] illustrated that the industrial IoT offers new opportunities and harbors threats that companies are not able to address with presented business models. Additionally, a smart framework for a shopping mall based on IoT technologies was presented by Pathan et al. [14].

IoT also has a huge impact on marketers since it provides them with the access to accurate big data. Marketers can track and record products, estimate the number of customers daily, analyze purchasing behaviors and understand the individual uses of products [4]. The analyzed outputs may eliminate the need for surveys or the collection of costly and time-consuming data, where ideas can be collected from the actual use of connected products and related data. It can also improve direct and hyper-local marketing, where personal messages can be sent via a number of connected mediums, for monitoring response and comments from customers. This can directly benefit all forms of marketing, since the information is the first-hand and the dissemination can reach more individuals regardless of their demographic, psychographic, or geographic generalizations. The growth of IoT will influence all marketing companies, particularly those focused on big data analytics. With more big data from consumers and businesses which became quickly available to marketers, analysts can turn raw data into useful insights, recommendations and predicted outcomes.

For understanding how IoT constitutes to marketing, few studies are presented. Balmer and Yen [15] proposed the appearance of what they call, 'The Corporate Marketing Internet Revolution' which calls for a radical rethinking of marketing practice and scholarship. The influence of the IoT on marketing practices was considered by De Cremer, Nguyen, and Simkin [16] via addressing the overlooked area of the dark side of the IoT. A smart marketing system based on IoT was proposed by Rajabi and Hakim [17] to help customers and shopping centers to interact with each other. The vision and challenges of advertising in the Internet of Things era was presented by Aksu et al. [18]. Celik [19], illustrated how IoT will be a great source of future marketing tools. An intelligent retail 4.0 Internet of Things consumer retailer model for smart and strategic marketing of in-store products was presented by Jayaram [20]. The potential applications of Internet of Things technologies and solutions for effective marketing at retail was presented by Bogdanovic [21]. Also, a precise positioning of marketing and behavior intentions of location-based mobile commerce in the Internet of things was presented by Tsai et al. [22].

The previous studies motivated us to design a smart and comprehensive framework for presenting the impact of IoT on customers, companies, and also on constituting marketing strategies. In this smart framework, customers, companies and marketers interact with each other smartly and achieve desired goals easily.

In addition to the proposed framework, we also proposed a novel neutrosophic multi-criteria decision making technique based on The Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) for supporting customers in selecting the best service or product from among several types. The TOPSIS technique is used for evaluating the performance of IoT in organizations [23]. Similarly, an unclear multicriteria group decision making algorithm based on the TOPSIS approach and the concept of similarity measures was developed by Wibowo and Grandhi [24] for evaluating the overall performance of IoT-based supply chains. We used a neutrosophic set in our technique since it considers the truth, indeterminacy and falsity membership degrees, so that it forms the best representation of reality rather than unclear and intuitionistic uncertain results [25].

The residual parts of this paper are presented as follows: Section 2 describes introductory concepts that includes the e-marketing concept and Internet of Things (IoT). A smart e-marketing model for aiding customers, companies, and marketers is suggested in Section 3. Section 4 presents a neutrosophic technique for aiding customers in selecting the best available product or service, and also the case study and experimental results of our suggested technique are presented. Section 5 concludes and identifies future trends of this paper.

2. Foundations of E-Marketing and Internet of Things (IoT)

The conventional marketing is very costly and takes more time to promote products. The development in technology that comprises the internet media and other digital media has led to the emergence of new marketing concepts.

E-marketing (also known as online marketing or internet marketing) refers to any marketing activities which are presented and serviced online via internet technologies. It includes not only advertising that is shown on websites, but also other types of online activities such as email and social networking.

E-marketing has extended and offered more opportunities for companies to reach out their customers and make direct requests served [26]. The popular media to introduce services and products of companies is on websites to blend information and social media. The three cornerstone principles of e-marketing are immediacy, personalization, and relevance.

The popular e-marketing methods are as follows:

- ○ Search Engine Market (SEM): There are three major search engine marketing activities, which correspond to search engine optimization (SEO), pay-per-click (PPC) and trusted feed [27].
- ○ Online Partnerships: It has three types, which include:
    - Link building: Which is a structured activity to comprise high-quality hyperlinks to your website from pertinent sites with a good page rank.
    - Affiliate marketing: Given rewards of affiliate by business as soon as the customer purchases a product through the own marketing efforts of affiliate, and it is a zero-risk advertising strategy since the merchant does not have to pay any fee until the products are purchased.
    - Online sponsorship: It links to a brand with associated content or context to create brand consciousness. The aim is to enhance the attractiveness of brand.
- ○ Interactive advertising: This means placing ad banners on other websites, and in some respects, it is completely similar to a pay-per-click search engine.
- ○ Email marketing: It is divided into two categories which are inbound and outbound email marketing.

- ○ Online PR: Public relations (PR) means maximizing constructive mentions of an organization, its brands, products or websites on websites of third-party which are probably going to be visited by its target crowd.
- ○ Viral marketing: Viral marketing uses e-mail to send a promotional message to another probable customer [28]. Offline campaigns:
- ○ The use of communications tools like advertising and public relations which are delivered by traditional media for directing visitors to an online attendance.

Nowadays, many people depend on e-marketing websites and services for buying products and services. There is a stable increase in the number of buyers who make purchasing decisions over Google or social network searches or the comments of preceding customers regarding the quality and price of the product. This is due to permanent sales of 24 h/7 days/365 days for customers and businesses, access to customers in distant geographical areas, minimum costs, presenting the right products to the right customers, sustained relationships of customers in the future, and free advertising of businesses, products or services. Hence, e-marketing is an important way to build strong relationships with customers.

Despite all these advantages of e-marketing, there are still many disadvantages which are as follows:

(a) If the infrastructure of e-marketing is weak, users will not have many opportunities to access the internet, learn information online, buy online, and participate in online auctions, and so forth.
(b) If the content control is not good, it can easily to affect the brand image.
(c) It is hard to control the target audience due to the diverse methods of e-marketing.
(d) There is a need to synchronize good information, otherwise, it will lead to information disruption in management.
(e) Customer trust.
(f) Security and privacy issues.

Since the main marketing pertinent element is the fact that consumers anticipate businesses to perform transactions with them at anytime, anywhere, and in relation to anything, we need to apply a more widespread concept than the internet. This concept is based on IoT, since it has become a base for connecting things, sensors, actuators, and other smart technologies.

There are many technical solutions for IoT: Radio Frequency Identification (RFID), Near Field Communication (NFC), Bluetooth Low Energy (BLE), Wi-Fi, Z-wave and others [29]. Protocols like RFID and NFC have been used in retail practice for inventory tracking or payments. BLE is a protocol that has attracted the attention of retailers and marketers in recent years. As it has become the standard in most current smart phones, it presents real-time, contextual, personalized communication and activation at or close to point of purchase, identifying micro location. BLE is a modulation to the standard Bluetooth protocol for allowing short range, low bandwidth, low latency, and efficient communication.

IoT systems contain application, network and perception layers and comprise of a number of component modules.

By its nature, IoT generates an enormous amount of data. For making this data generate useful information and create value to the user, they should be connected and enabled via cloud services and big data analytics, ensuring compliance with security and privacy requirements.

Now the key question here is how will IoT affect e-marketing?

One of the primary goals of marketers is marketing data. The obtained data from IoT devices will help marketers to analyze buying behavior of customers and then determine customers' preferences. The IoT will help marketers to target their audience and then make more relevant advertisements. It will also save time on gathering and analyzing data. Real-time data which is obtained from IoT devices will help marketers to respond to their customers quickly and then achieve customer satisfaction. Every smart product will help marketers to connect with their customers and then increase customer engagement.

Then, we can define IoT in e-marketing as “the interconnection of our digital devices which introduces endless chances for brands to listen and react to the requirements of their customers–with the proper message, at the proper time, and on the proper device”.

Despite the great benefits of using IoT in e-marketing, there exists some challenges. The major challenge is security issues. Although gathering data is a very important characteristic of IoT according to marketer's views, it is a very critical part from point of view of customers. The critical part in this process is that the retrieved data from IoT devices about customers, is personal, numerous and includes not only computers and mobile devices but other kinds of house equipment, wearables, etc. Thus, users' behaviors can be tracked at any time and everywhere. This can lead to hackers gaining unauthorized access to customer databases and physical objects which can pose threats to human lives. For example, cars which are based on IoT technology are susceptible to hacker attacks. The hacking of databases is not only dangerous from the point of view of customers but also from the company's view. For example, hacking companies' databases which contain personal information about customers may make the customers sue the company.

3. Model of IoT Application in E-Marketing

In this section we propose a smart model based on IoT technologies which helps customers and companies to interact with each other and meet their needs in the best way. It also helps marketers make the best marketing strategies for their companies through utilizing the significant data which is obtained from IoT devices. By applying this model, we can also solve the most popular problems which exist in e-marketing models.

In this model we focused only on food commodities as example of products, but this model has the potential to be applied on all kinds of products in our lives.

Before we begin to explain the main parts of the proposed model, let us ask ourselves some questions. What if we could know the food commodities that we need to buy, when we are out of the house or at work? For instance, if you run out of milk, a refrigerator can connect to the internet and decide your needs and present a message on its screen or your phone. Additionally, what if you buy these products with the highest quality and lowest prices? How about knowing all the information about products that you need to buy, and the best recommendations according to your purchase criteria, in only one click? What if you can get these products as soon as possible? How about achieving the highest degree of security when buying these products online? These questions are from the point of view of users.

But, from the point of view of companies the important question is: What if companies can achieve online identity verification of consumers, maintain customers' loyalty, solve the problem of product return and refund, and achieve data security?

Finally, from the point of view of marketers the most important question is: What will happen if marketers can get all data and information about customers' behaviors, habits and preferences?

Our proposed model answers the previous questions, and can help customers, companies and marketers to achieve their goals efficiently and effectively.

3.1. *Knowing the Amount of Food Commodities in Our Kitchens When We Are Away from Home or at Work*

The first part of the proposed model is to know the amount of current food commodities in our kitchens when we are away from home, at work or in our car. This helps people to know their needs and buy it quickly.

In order to do this, we first need to design a smart kitchen based on IoT technologies, but in this part we focus only on specific parts of the kitchen which can help us to know the stock of food commodities. These parts of kitchen are the refrigerator and some shelves in the kitchen that contains some goods like rice, cooking oil, coffee, tea, sugar, etc.

The smart refrigerator should contain an IP address that might sustain functions such as control units, sensors, communication modules, but the most important technologies that will help us in our smart system are: (1) Bluetooth/Wi-Fi. (2) RFID technology: a micro-chip in a label used for transmitting data when the label is exposed to radio waves. The RFID will maintain an updated list of the products in the fridge. Now, all items are tagged with RFID cards when entering the fridge for the first time, and every time items are placed in or removed from the fridge, the RFID antenna which installed inside the fridge recognizes the items and registers them as either in or out of the stock.

So using a smart refrigerator with RFID technology and Wi-Fi connectivity can help us know what is inside the fridge as well as what is consumed and what we need to buy by sending an electronic report to the owner's phone.

The second place that contains food commodities in our kitchen are kitchen shelves. We will design these shelves from keen glass with RFID technology and also connect them to Wi-Fi. The RFID reader in these shelves will determine the quantity of items required, and automatically send alert emails to its owner's phone if the product is less than the threshold as seen on the smart refrigerator.

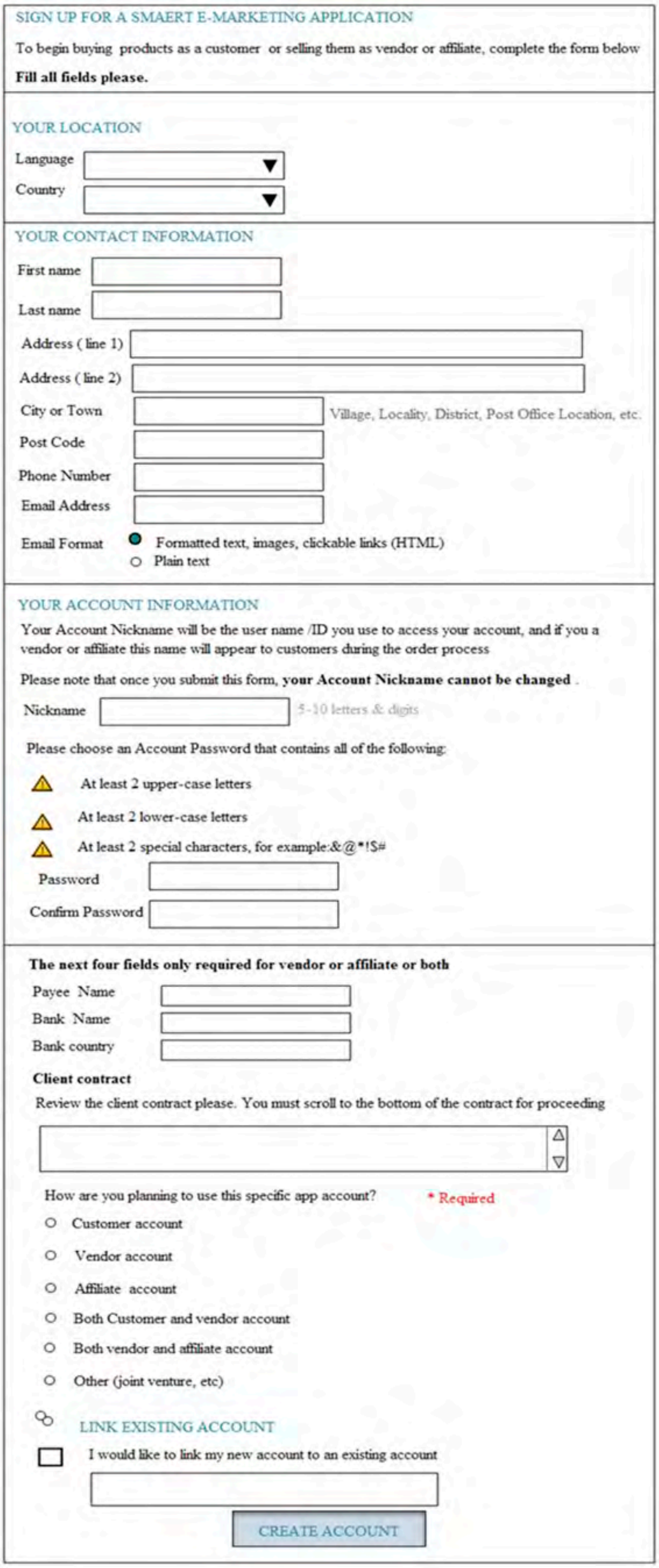
3.2. *Smart E-Marketing Application*

In this sub-section, we suggest a smart app for Android and Apple iOS to help customers make their online shopping in an easy, simple, attractive and secure procedure.

This application saves time for customers who are searching for their needs in various websites. Some websites can deceive customers by not sending products with the required quality, difficulty in shipping and retrieval. But this app supports and advertises trusted sites of companies that are subject to certain specifications.

We will store this application in App Store and Google Play to enable various users to download it easily. Once the program is downloaded and installed, users can register in the application as in Figure 1. In this registration each user will fill his/her details and obtain an ID which will be unique for future prospects. This unique ID will store in our database and through it the user will receive messages on his/her phone as well as emails with all offers and discounts on existing products.

This app supports customers, vendors, and also affiliates. It also helps companies to build their websites and market their products smartly.



SIGN UP FOR A SMAERT E-MARKETING APPLICATION

To begin buying products as a customer or selling them as vendor or affiliate, complete the form below

Fill all fields please.

YOUR LOCATION

Language

Country

YOUR CONTACT INFORMATION

First name

Last name

Address (line 1)

Address (line 2)

City or Town — Village, Locality, District, Post Office Location, etc.

Post Code

Phone Number

Email Address

Email Format — Formatted text, images, clickable links (HTML) / Plain text

YOUR ACCOUNT INFORMATION

Your Account Nickname will be the user name /ID you use to access your account, and if you a vendor or affiliate this name will appear to customers during the order process

Please note that once you submit this form, **your Account Nickname cannot be changed** .

Nickname — 5-10 letters & digits

Please choose an Account Password that contains all of the following:

At least 2 upper-case letters

At least 2 lower-case letters

At least 2 special characters, for example:&@*!$#

Password

Confirm Password

The next four fields only required for vendor or affiliate or both

Payee Name

Bank Name

Bank country

Client contract

Review the client contract please. You must scroll to the bottom of the contract for proceeding

How are you planning to use this specific app account? * Required

Customer account

Vendor account

Affiliate account

Both Customer and vendor account

Both vendor and affiliate account

Other (joint venture, etc)

LINK EXISTING ACCOUNT

I would like to link my new account to an existing account

CREATE ACCOUNT

Figure 1. Registration process in app.

3.2.1. Customer Registration on the Application

In the first part of this section we illustrate how any person will be able to know his/her kitchen's stock of food commodities using IoT technology in the refrigerator and kitchen shelves. The RFID reader in the refrigerator and kitchen shelves will determine the quantity of items in each take, and automatically send alert emails if the product is less than the threshold. Hence, if such circumstances occurred one message will be sent to the owner's phone, then the owner will pass his/her needs to the proposed app.

The first step that the customer must do is the registration process in the proposed app as in Figure 1. After finishing the registration process of user (customer or shopper), this application can perform various processes as follows:

- Firstly, view videos in a unique way to allow the customer to see a full description of the app, its features and benefits.
- Enable the customer to select a category of product that he/she plans to buy either by selecting the category from the app directly or via customer voice since this app supports voice recognition technology.
- After determining products that the customer decides to buy, this app begins to compare prices between various websites and recommend the cheapest and highest quality products to the customer. The app deals only with companies which have an Secure Sockets Layer (SSL) certificate. Usually, SSL is used for securing credit card transactions, transferring data and logins, and more recently, is becoming the norm when securely browsing social media sites [30]. The customer has to choose the suitable product according to his/her purchasing criteria. We will illustrate this part with detail in next section via proposing a new neutrsophic technique for helping the customer in the selection process.
- Customers can also buy products from vendors who have registered in this app, since the vendor in this case is able to advertise his/her product using this app. We illustrate this part with details in the next subsection.
- This application also allows customers to get the best description of products by telling customers about products, clarifying why it is for them, characterizing how the product feels, how it can fix problems, save time, or make them happier, and can complete requests in text with photos, graphics and videos. Entertainment is not just a notification. This helps customers to assess products properly and obtain all the information they need, so that they feel comfortable about their purchasing products.
- If a customer buys his/her products continuously via the app, he/she will obtain large discounts.
- After completing the buying process, all information about customers and products that he/she bought and also the websites that he/she bought from it, will store in our cloud database. This helps marketers capture interactions, conversion metrics, and consumer behavior predictions and link them to purchase-intent data.
- The app also supports direct contacts between customers and vendors via chat, video calls, and customer service. This will help in solving all existing problems, increase customer confidence and satisfaction.
- The app will inform users continuously with special offers via sending messages and e-mails to their smart phones.

This app enables customers to make the payment process as in Figure 2. The development of the international payments market made the payment process very simple [30].

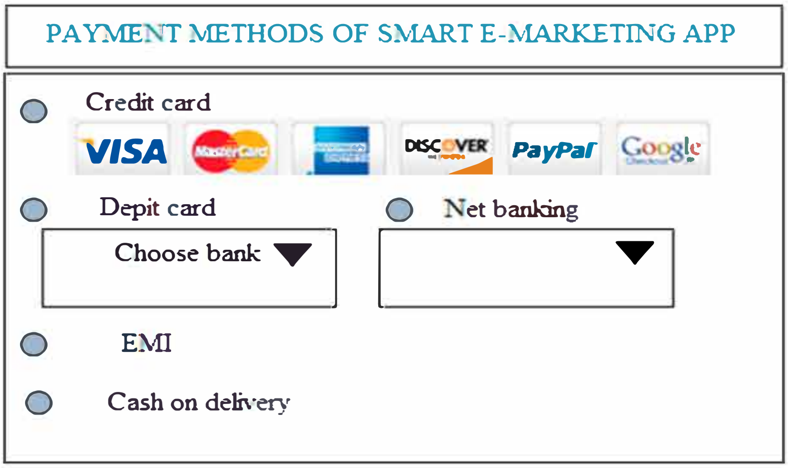


Figure 2. Available options of payment process.

3.2.2. Registering as a Vendor on the Application

This app not only supports customers, but it also supports sellers and affiliates. The seller in this app may be the person who plans to sell a product or a company that aims to advertise its products using this app and already has its own website.

Previously we explained how anyone can remotely access the stock of food commodities in his/her kitchen using IoT technology, and then we enabled them to buy the product easily through the proposed app if his/her stock's state is low.

But, if there exists a large amount of food commodities in our shelves that are about to expire or to dispose, the owner can sell them with special offers by using this app as follows:

- Make the registration process as in Figure 1.
- Next, the app verifies the user's identity in a streamlined manner and then converts it to digital data if it is validated. The verification process of vendor/seller consists of three steps:
    - Registration of vendor identity documents such as identification card, driving license, passports, vital standards as fingerprint and facial recognition.
    - Validate documents in addition to their holder.
    - Create a reliable digital ID. This can combat the fraud, strengthen compliance processes and enhance sensitive security services such as money transfer, etc.
- If the previous step can prove that he/she is an invalid user, then the app will automatically reject his/her registration process and automatically block him/her.
- But if he/she is a valid user, then the app will complete registration and give the vendor a special link. This link will help the vendor in dealing with customers, tracking his/her product status, and also receiving money.
- The vendor then begins to upload the product that he/she wants to sell and supports it with the detailed information, images, videos that explain the product status with detail.
- The product will submit for approval before the vendor is able start selling it within the app.

3.2.3. Registration as an Affiliate on the Application

As we know, the vendor is the person who sells their own products, and the affiliate is the person who promotes products for vendors in order to earn a commission. This smart app also supports the affiliate.

In the first step, the affiliate person must fill in a registration form exactly as the vendor did. A verification process will be performed for validating the affiliate as we explained previously with the vendor case.

After the verification process, the affiliate will have a formal registration in our app and receive special link for helping him/her in the promotion of products and receiving commissions.

After that, the app will recommend various products for the affiliate to select from for the promotion process. Not only this, but the app will also classify products into 'most popular', 'most gravity', and a 'new products' category.

By using the proposed app, any affiliate can obtain advice on how to achieve the greatest earning through the promotion process of products. It also generates a report with the most famous sites that support affiliation. Since, in marketing language, the new products usually achieve the highest earned value of affiliate, then our app will continuously send e-mails and messages to the affiliate with any new product either sold in the app or any site supported by the affiliate.

3.2.4. Use of the App by Companies That Have Their Own Inventory

This app is also designed for supporting companies to create their own website via providing the chance to create and market it smartly.

Before creating the website, the company must use IoT technologies in their inventory's management. Subsequently, products should have either an RFID tag or barcode label for offering the visibility of inventory levels, dates of expiration, item location, and product demand.

Using IoT, it will increase the capability to track and communicate with products. For instance, RFID tags will load information about an object, and communicate with an inventory system. Also, built-in RFID tags can drive information about an object's temperature, weather, damage, and traffic, etc. Moreover, built-in GPS locations permit the vendor company to know precisely where every item is. Every object will have its own unique identifier. As a result, the vendor company will be able to pin-point each and every item or piece of equipment. This will then effectively minimize stock reduction, shortages, and overstocks. The vendor company can identify precisely which areas are efficient and which are not. Therefore, inefficiencies and problems that were not exposed before will become simple to spot with recommendations for further actions.

The vendor company should also insert IoT technologies in their products' shipping services for customers. Our lives depend on transportation, since it is important to travel for work and leisure, as well as the delivery of food and goods to destinations. The growing use of sensors attached to both products and the enclosures (that move them from point A to point B) opens a new window into real-time discovery of actual conditions, with clear ramifications for cost control and accountability. Thus, by adding IoT technologies to shipping services we can track shipments, optimize delivery and shipping routes, minimize costs associated with inefficiencies in logistics, and raise our expectations for goods and services. Additionally, merging data from weather meters and road closure notes makes operations run easily. It can also inform stakeholders of real-time operations - a major win in an era of instant gratification.

After using IoT in inventory management, the shipping service of products and having explained its effective and efficient role, the next step is the creation of a company website for marketing and promotion of their products and services.

By using the proposed app the company can select the creation option of the website which is a free feature in our app. The required steps for creating the company's website are as follows:

- First the vendor company should complete a registration process as in Figure 1, after then, the company must produce all identity documents.
- The validation process of company documents has to be performed.
- For finding new customers and boosting their business, this app enables the company to build a high-quality website with the following prosperities:

    - A high-quality building platform.
    - Simple and attractive designs. The most important thing is to build trust for a business or company and ensure customers can find the content or sales information they are looking for.
    - It can map out company content.
    - It works on most browsers such as Firefox, Chrome, Safari, etc.
    - It is almost effortless to read on mobile platforms.
    - It is quick to load.
    - It supports the use of online social media such as LinkedIn, Facebook.
    - Use of offline channels such as press releases or groups you belong to.
    - Secure: using high level usernames and passwords, up-to date of platform software and any plugging/modules, and considering an external security monitoring software.
    - To assist people with physical and visual disabilities, this site will also provide voice support next to screens where the customer can make the entire purchase process of products by using the voice service.
    - The app also enables companies to create an account with best ePayment gateway.
- After creating the website of a vendor company, this app gives a full update on the related events and content on your sites, as well as customers' average time spent on your site, page views per visitor, percentage of reiterate visitors, and visitors' countries of origins.
- It also compares your large success stories with your less successful endeavors; it is simple to distinguish where your effort should be concentrated to enhance site page rank and draw in more traffic. As soon as you know where to direct your efforts, your expenditures and time can be used more effectively.

Now, let us ask ourselves a question: what is the relation between adding IoT in inventory management and a shipping service with the marketing process of company products via their site?

Large companies spend huge amounts of money on marketing their products, and money that can be used to produce a better product rather than being spent on reaching to the widest possible audience. Thus, why is marketing so expensive? This is because marketing agencies need to gather quite a lot of information to determine their target audience. Once they know their audience a campaign targeted towards them can be created. Since the IoT enables companies to obtain all information about their products, and then the marketer can create their marketing strategy easily and effectively. Moreover, this information can help to understand which products have reached the expiration of its validity and then the marketers can make a marketing strategy for selling this product with various offers and discounts.

By using the proposed app the marketers can obtain a huge amount of data about customers. This data includes customer location, time of buying, a list of purchases, and customer demographics as stored in our cloud database of customers. According to stored information about customers, the marketers can extract and analyze customers' preferences and habits, and build more attractive marketing plans.

In order to avoid any type of risks and misunderstanding, the marketers should inform consumers that their private data is stored and will be used for commercial purposes. The high level of transparency will help companies to minimize or eliminate consumers' dissatisfaction.

The general framework of proposed model is shown in Figure 3.

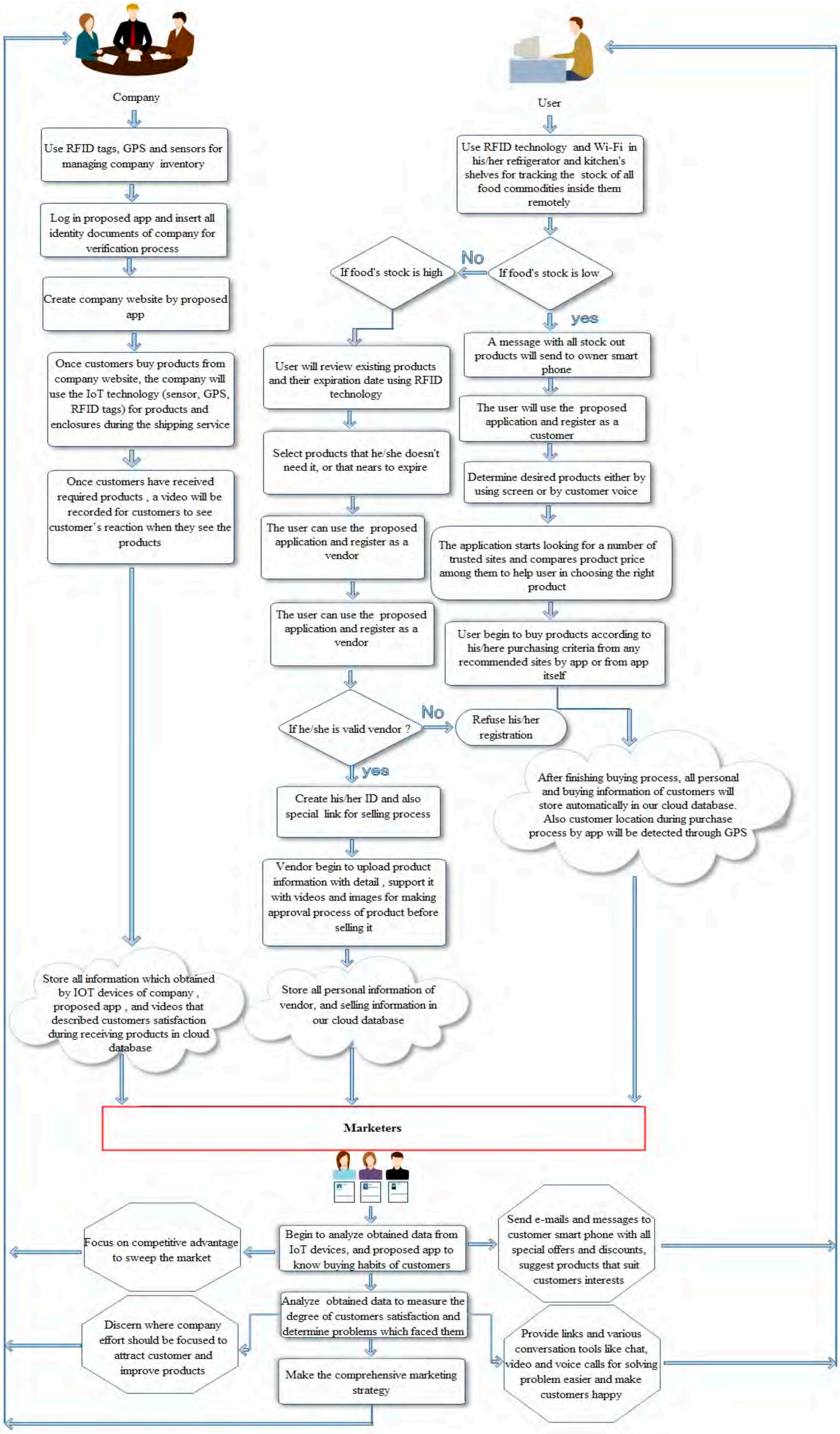


Figure 3. The proposed model.

4. Neutrosophic Technique for Helping the Customer to Select the Best Service or Product from Several Types

To determine the best choice between all of the possible alternatives, the multiple criteria decision making (MCDM) techniques are used widely. The problem of product or service selection on which decision maker has a typically vague and inaccurate result, which is a representative example of an MCDM problem. The traditional techniques have not been very effective for solving MCDM problems due to the inaccurate or unclear nature of the linguistic assessments. Finding the exact values for MCDM problems is complex and not possible in more real world cases. So, it is more rational to consider the values of alternatives regarding to the criteria as neutrosophic numbers (NNs). This part deals with The Technique for Order of Preference by Similarity to Ideal Solution (TOPSIS) method and expands the TOPSIS approach to the MCDM problem with single valued neutrosophic information. Here, the weights of criteria are calculated using the rank order centroids (ROCs) method, and the evaluation matrix for comparing alternatives relating to existing criteria is represented by using triangular neutrosophic numbers (TNNs).

So, in this section, we will explain how the proposed technique can help customers choose the best product among several types.

4.1. Proposed Neutrosophic Technique

As explained in the previous section, the customer will have the opportunity to choose from several products recommended by the proposed application. Next, the customer should choose the best product that suits his/her needs. But the assessment process of existing products is a multifaceted problem owing to several mismatched criteria. These include the interests of different customers, the diversity of products, and the breakdown of dealing with unclear and conflicting information which exists frequently during the selection processes. Therefore, we proposed a neutrosophic technique for helping customers to select the best product or service.

For determining the appropriate product or service, according to the several purchasing criteria of customer like price of product, delivery time to customer, quality of product, etc., let $C = \{C1, C2, \cdots, Cn\}$ be a given set of finite criteria for product or service, and $P = \{P1, P2, \cdots, Pm)$ be given set of finite alternatives (products).

The detailed steps for selecting the best possible product are as follows:

Step 1: Let the customer determine his/her selection criteria and rank them according to their needs.

Step 2: After determining the rank of purchase criteria by customer, the weight of these criteria must be calculated. Here we used the rank order centroids (ROCs) for assigning weights to these criteria [31]. For a set of ranks of order N, the ROC value which corresponds to the kth rank is given by:

$$r_k = (\sum_{i=k}^{N} \left(\frac{1}{i}\right))/N \tag{1}$$

For example, if we have a set of three ranks, associated ROCs values are:

$$r_1 = \left(1 + \frac{1}{2} + \frac{1}{3}\right)/3 = 0.61,$$

$$r_2 = \left(0 + \frac{1}{2} + \frac{1}{3}\right)/3 = 0.28,$$

$$r_3 = \left(0 + 0 + \frac{1}{3}\right)/3 = 0.11.$$

Step 3: After assigning relative weights to the purchase criteria, we begin to build the estimation matrix which consists of m alternatives and n criteria using the following linguistic variables as in Table 1. The crossing of every alternative and criteria indicated as x_{ij}. Then, we have $(x_{ij})_{m\times n}$ matrix.

Table 1. Linguistic variables for comparison matrices.

Linguistic Variables	Neutrosophic Numbers
Very Low/ Bad (VL/VB)	$\langle 0,1,2;0.10,0.85,0.90\rangle$
Low/ Bad (L/B)	$\langle 1,2,3;0.20,0.75,0.80\rangle$
Medium Low/Bad (ML/MB)	$\langle 2,3,4;0.35,0.65,0.60\rangle$
Medium/Fair (M/F)	$\langle 3,4,5;0.50,0.50,0.50\rangle$
Medium High/ Good (MH/MG)	$\langle 4,5,6;0.60,0.35,0.30\rangle$
High/Good (H/G)	$\langle 5,6,7;0.80,0.20,0.15\rangle$
Very High/Good (VH/VG)	$\langle 6,7,8;0.90,0.10,0.05\rangle$
Extremely High/Good (EH/EG)	$\langle 7,8,9;1.00,0.00,0.00\rangle$

Each value in Table 1 is a single valued triangular neutrosophic number which is a special case of single valued neutrosophic set:

Definition 1. *A single valued neutrosophic set A over X, is an object taking the form* $A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in X\}$, *where* $T_A(x)$: $X \to [0, 1]$, $I_A(x)$: $X \to [0, 1]$ *and* $F_A(x)$: $X \to [0, 1]$ *with* $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$ *for all* $x \in X$. *The intervals* $T_A(x)$, $I_A(x)$ *and* $F_A(x)$ *represent the truth-membership degree, the indeterminacy-membership degree and the falsity membership degree of x to A, respectively and X is a universe of discourse. For convenience, a SVN number is represented by A = (a, b, c), where a, b, c ∈ [0, 1] and a + b + c ≤ 3 [32].*

Definition 2. *A single valued triangular neutrosophic number* $\widetilde{a} = \langle (a_1, a_2, a_3); T_a, I_a, F_a\rangle$ *is a special neutrosophic set on the real line set R, whose truth-membership, indeterminacy-membership and falsity-membership functions are defined as follows [32]:*

$$T_a(x) = \begin{cases} T_a\left(\frac{x-a_1}{a_2-a_1}\right) & (a_1 \le x < a_2) \\ T_a & (x = a_2) \\ T_a\left(\frac{a_3-x}{a_3-a_2}\right) & (a_2 < x \le a_3) \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

$$I_a(x) = \begin{cases} \frac{(a_2-x+I_a(x-a_1))}{(a_2-a_1)} & (a_1 \le x < a_2) \\ I_a & (x = a_2) \\ \frac{(x-a_2+I_a(a_3-x))}{(a_3-a_2)} & (a_2 < x \le a_3) \\ 1 & \text{otherwise,} \end{cases} \tag{3}$$

$$F_a(x) = \begin{cases} \frac{(a_2-x+F_a(x-a_1))}{(a_2-a_1)} & (a_1 \le x < a_2) \\ F_a & (x = a_2) \\ \frac{(x-a_2+F_a\,(a_3-x))}{(a_3-a_2)} & (a_2 < x \le a_3) \\ 1 & \text{otherwise.} \end{cases} \tag{4}$$

where T_a, I_a *and* $F_a(x)$, *represent the maximum truth-membership degree, minimum indeterminacy-membership degree and minimum falsity-membership degree respectively. A single valued triangular neutrosophic number* $\widetilde{a}$ = $(a_1, a_2, a_3); T_a, I_a, F_a$ *may express an ill-defined quantity about a, which is approximately equal to a.*

Definition 3. *Let* $\widetilde{a} = \langle (a_1, a_2, a_3); T_a, I_a, F_a\rangle$ *and* $\widetilde{b} = \langle (b_1, b_2, b_3); T_b, I_b, F_b\rangle$ *be two single valued triangular neutrosophic numbers and* $\gamma \neq 0$ *be any real number [32]. Then,*

1. *Addition of two triangular neutrosophic numbers*

$$\widetilde{a} + \widetilde{b} = \langle (a_1 + b_1, a_2 + b_2, a_3 + b_3); T_a \wedge T_b, I_a \vee I_b, F_a \vee F_b\rangle$$

2. *Subtraction of two triangular neutrosophic numbers*

$$\widetilde{a} - \widetilde{b} = \langle (a_1 - b_3\ , a_2 - b_2\ , a_3 - b_1); T_a\ \wedge\ T_b,\ I_a \vee\ I_b, F_a\ \vee\ F_b \rangle$$

3. *Inverse of a triangular neutrosophic number*

$$\widetilde{a}^{-1} = \langle \left(\frac{1}{a_3}, \frac{1}{a_2}, \frac{1}{a_1} \right); T_a, I_a,\ F_a \rangle, \text{ where } (\widetilde{a} \neq 0)$$

4. *Multiplication of triangular neutrosophic number by constant value*

$$\gamma\widetilde{a} = \begin{cases} \langle (\gamma a_1,\ \gamma a_2,\ \gamma a_3); T_a,\ I_a,\ F_a \rangle \text{ if } (\ \gamma > 0) \\ \langle (\gamma a_3,\ \gamma a_2, \gamma a_1); T_a, I_a,\ F_a \rangle \text{ if } (\gamma < 0) \end{cases}$$

5. *Division of triangular neutrosophic number by constant value*

$$\frac{\widetilde{a}}{\gamma} = \begin{cases} \langle \left(\frac{a_1}{\gamma},\ \frac{a_2}{\gamma},\ \frac{a_3}{\gamma} \right); T_a,\ I_a,\ F_a \rangle \text{ if } (\ \gamma > 0) \\ \langle \left(\frac{a_3}{\gamma},\ \frac{a_2}{\gamma},\ \frac{a_1}{\gamma} \right); T_a, I_a,\ F_a \rangle \text{ if } (\gamma < 0) \end{cases}$$

6. *Division of two triangular neutrosophic numbers*

$$\frac{\widetilde{a}}{\widetilde{b}} = \begin{cases} \langle \left(\frac{a_1}{b_3}, \frac{a_2}{b_2}, \frac{a_3}{b_1} \right); T_a\ \wedge\ T_b,\ I_a \vee\ I_b, F_a\ \vee\ F_b \rangle & \text{if}(a_3 > 0, b_3 > 0) \\ \langle \left(\frac{a_3}{b_3}, \frac{a_2}{b_2}, \frac{a_1}{b_1} \right);\ T_a\ \wedge\ T_b,\ I_a \vee\ I_b, F_a\ \vee\ F_b \rangle & \text{if}(a_3 < 0, b_3 > 0\ ) \\ \langle \left(\frac{a_3}{b_1}, \frac{a_2}{b_2}, \frac{a_1}{b_3} \right);\ T_a\ \wedge\ T_b,\ I_a \vee\ I_b, F_a\ \vee\ F_b \rangle & \text{if}(a_3 < 0, b_3 < 0) \end{cases}$$

7. *Multiplication of two triangular neutrosophic numbers*

$$\widetilde{ab} = \begin{cases} \langle (a_1 b_1\ , a_2 b_2, a_3 b_3); T_a\ \wedge\ T_b,\ I_a \vee\ I_b, F_a\ \vee\ F_b \rangle & \text{if } (a_3 > 0, b_3 > 0) \\ \langle (a_1 b_1\ , a_2 b_2, a_3 b_3); T_a\ \wedge\ T_b,\ I_a \vee\ I_b, F_a\ \vee\ F_b \rangle & \text{if } (a_3 < 0, b_3 > 0) \\ \langle (a_3 b_3, a_2 b_2, a_1 b_1); T_a\ \wedge\ T_b,\ I_a \vee\ I_b, F_a\ \vee\ F_b \rangle & \text{if } (a_3 < 0, b_3 < 0) \end{cases}$$

So, the evaluation matrix of alternatives with respect to criteria will take the following form:

$$E = \langle e_{ij} \rangle_{m \times n} = \langle L_{ij}, M_{ij}, U_{ij}; T_{ij}, I_{ij}, F_{ij} \rangle_{m \times n} =$$
$$\begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \\ \begin{pmatrix} \langle L_{11}, M_{11}, U_{11}; T_{11}, I_{11}, F_{11} \rangle & \langle L_{12}, M_{12}, U_{12}; T_{12}, I_{12}, F_{12} \rangle & \cdots & \langle L_{1n}, M_{1n}, U_{1n}; T_{1n}, I_{1n}, F_{1n} \rangle \\ \langle L_{21}, M_{21}, U_{21}; T_{21}, I_{21}, F_{21} \rangle & \langle L_{22}, M_{22}, U_{22}; T_{22}, I_{22}, F_{22} \rangle & \cdots & \langle L_{2n}, M_{2n}, U_{2n}; T_{2n}, I_{2n}, F_{2n} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle L_{m1}, M_{m1}, U_{m1}; T_{m1}, I_{m1}, F_{m1} \rangle & \langle L_{m2}, M_{m2}, U_{m2}; T_{m2}, I_{m2}, F_{m2} \rangle & \cdots & \langle L_{mn}, M_{mn}, U_{mn}; T_{mn}, I_{mn}, F_{mn} \rangle \end{pmatrix} \end{array} \quad (5)$$

where, $e_{ij} = \langle L_{ij}, M, U_{ij}; T_{ij}, I_{ij}, F_{ij} \rangle$ is the triangular neutrosophic element of evaluation matrix E for $i = 1; 2; \ldots; m$ and $j = 1; 2; \ldots; n$. Since L, M, U are the lower, median and upper value of triangular neutrosophic number, and T, I, F are the truth, indeterminacy and falsity degrees of this triangular neutrosophic number.

Step 4: Calculate the neutrosophic weighted evaluation matrix as follows:

$$E^w = E \times w = \langle e_{ij}{}^{wj} \rangle_{m \times n} = \langle L_{ij}{}^{wj}, M_{ij}{}^{wj}, U_{ij}{}^{wj}; T_{ij}^{wj}, I_{ij}{}^{wj}, F_{ij}{}^{wj} \rangle_{m \times n} =$$

$$\begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \\ \begin{pmatrix} \langle L_{11}^{w1}, M_{11}^{w1}, U_{11}^{w1}; T_{11}^{w1}, I_{11}^{w1}, F_{11}^{w1} \rangle & \langle L_{12}^{w2}, M_{12}^{w2}, U_{12}^{w2}; T_{12}^{w2}, I_{12}^{w2}, F_{12}^{w2} \rangle & \cdots & \langle L_{1n}^{wn}, M_{1n}^{wn}, U_{1n}^{wn}; T_{1n}^{wn}, I_{1n}^{wn}, F_{1n}^{wn} \rangle \\ \langle L_{21}^{w1}, M_{21}^{w1}, U_{21}^{w1}; T_{21}^{w1}, I_{21}^{w1}, F_{21}^{w1} \rangle & \langle L_{22}^{w2}, M_{22}^{w2}, U_{22}^{w2}; T_{22}^{w2}, I_{22}^{w2}, F_{22}^{w2} \rangle & \cdots & \langle L_{2n}^{wn}, M_{2n}^{wn}, U_{2n}^{wn}; T_{2n}^{wn}, I_{2n}^{wn}, F_{2n}^{wn} \rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle L_{m1}^{w1}, M_{m1}^{w1}, U_{m1}^{w1}; T_{m1}^{w1}, I_{m1}^{w1}, F_{m1}^{w1} \rangle & \langle L_{m2}^{w2}, M_{m2}^{w2}, U_{m2}^{w2}; T_{m2}^{w2}, I_{m2}^{w2}, F_{m2}^{w2} \rangle & \cdots & \langle L_{mn}^{wn}, M_{mn}^{wn}, U_{mn}^{wn}; T_{mn}^{wn}, I_{mn}^{wn}, F_{mn}^{wn} \rangle \end{pmatrix} \end{array} \quad (6)$$

Here, $\langle e_{ij}{}^{wj} \rangle_{m \times n} = \langle L_{ij}{}^{wj}, M_{ij}{}^{wj}, U_{ij}{}^{wj}; T_{ij}{}^{wj}, I_{ij}{}^{wj}, F_{ij}{}^{wj} \rangle_{m \times n}$ is an element of the weighted neutrosophic evaluation matrix E^w for $i = 1; 2; \cdots; m$ and $j = 1; 2; \cdots; n$.

Step 5: Define the neutrosophic positive and negative ideal solution NPIS and NNIS:

Their exists two types of attributes B_1 and B_2, which are the benefit and cost-type attribute respectively. So, v^+ is the neutrosophic positive ideal solution (NPIS), v^- is the neutrosophic negative ideal solution (NNIS) and have the following formula:

$$v^+ = \left[e_1{}^{w+},\ e_2{}^{w+}, \cdots, e_n{}^{w+} \right]. \quad (7)$$

where $e_j{}^{w+} = \langle L_j{}^{w+}, M_j{}^{w+}, U_j{}^{w+}; T_j{}^{w+}, I_j{}^{w+}, F_j{}^{w+} \rangle$ for $j = 1, 2, \cdots, n$.

$$L_j{}^{w+} = \left\{ \left(\max{}_i \left\{ L_{ij}^{wj} \right\} \middle| j \in B_1 \right), \left(\min{}_i \left\{ L_{ij}^{wj} \right\} \middle| j \in B_2 \right) \right\}, \quad (8)$$

$$M_j{}^{w+} = \left\{ \left(\max{}_i \left\{ M_{ij}^{wj} \right\} \middle| j \in B_1 \right), \left(\min{}_i \left\{ M_{ij}^{wj} \right\} \middle| j \in B_2 \right) \right\}, \quad (9)$$

$$U_j{}^{w+} = \left\{ \left(\max{}_i \left\{ U_{ij}^{wj} \right\} \middle| j \in B_1 \right), \left(\min{}_i \left\{ U_{ij}^{wj} \right\} \middle| j \in B_2 \right) \right\}, \quad (10)$$

$$T_j{}^{w+} = \left\{ \left(\max{}_i \left\{ T_{ij}^{wj} \right\} \middle| j \in B_1 \right), \left(\min{}_i \left\{ T_{ij}^{wj} \right\} \middle| j \in B_2 \right) \right\}, \quad (11)$$

$$I_j{}^{w+} = \left\{ \left(\min{}_i \left\{ I_{ij}^{wj} \right\} \middle| j \in B_1 \right), \left(\max{}_i \left\{ I_{ij}^{wj} \right\} \middle| j \in B_2 \right) \right\}, \quad (12)$$

$$F_j{}^{w+} = \left\{ \left(\min{}_i \left\{ F_{ij}^{wj} \right\} \middle| j \in B_1 \right), \left(\max{}_i \left\{ F_{ij}^{wj} \right\} \middle| j \in B_2 \right) \right\}. \quad (13)$$

Also,

$$v^- = [e_1{}^{w-},\ e_2{}^{w-}, \cdots, e_n{}^{w-}], \quad (14)$$

where $\langle e_j{}^{w-} = L_j{}^{w-}, M_j{}^{w-}, U_j{}^{w-}; T_j{}^{w-}, I_j{}^{w-}, F_j{}^{w-} \rangle$ for $j = 1, 2, \cdots, n$.

$$L_j{}^{w-} = \left\{ \left(\min{}_i \left\{ L_{ij}^{wj} \right\} \middle| j \in B_1 \right), \left(\max{}_i \left\{ L_{ij}^{wj} \right\} \middle| j \in B_2 \right) \right\}, \quad (15)$$

$$M_j{}^{w-} = \left\{ \left(\min{}_i \left\{ M_{ij}^{wj} \right\} \middle| j \in B_1 \right), \left(\max{}_i \left\{ M_{ij}^{wj} \right\} \middle| j \in B_2 \right) \right\}, \quad (16)$$

$$U_j{}^{w-} = \left\{ \left(\min{}_i \left\{ U_{ij}^{wj} \right\} \middle| j \in B_1 \right), \left(\max{}_i \left\{ U_{ij}^{wj} \right\} \middle| j \in B_2 \right) \right\}, \quad (17)$$

$$T_j{}^{w-} = \left\{ \left(\min{}_i \left\{ T_{ij}^{wj} \right\} \middle| j \in B_1 \right), \left(\max{}_i \left\{ T_{ij}^{wj} \right\} \middle| j \in B_2 \right) \right\}, \quad (18)$$

$$I_j{}^{w-} = \left\{ \left(\max{}_i \left\{ I_{ij}^{wj} \right\} \middle| j \in B_1 \right), \left(\min{}_i \left\{ I_{ij}^{wj} \right\} \middle| j \in B_2 \right) \right\}, \quad (19)$$

$$F_j{}^{w-} = \left\{ \left(\max{}_i \left\{ F_{ij}^{wj} \right\} \middle| j \in B_1 \right), \left(\min{}_i \left\{ F_{ij}^{wj} \right\} \middle| j \in B_2 \right) \right\}. \quad (20)$$

Step 6: Measure the Euclidian distance of each alternative from the NPIS and NNIS:

The normalized Euclidian distance of each alternative $\langle L_{ij}{}^{wj}, M_{ij}{}^{wj}, U_{ij}{}^{wj}; T_{ij}{}^{wj}, I_{ij}{}^{wj}, F_{ij}{}^{wj}\rangle$ from the neutrosophic positive ideal solution $\langle L_j{}^{w+}, M_j{}^{w+}, U_j{}^{w+}; T_j{}^{w+}, I_j{}^{w+}, F_j{}^{w+}\rangle$ for $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$ written as follows:

$$D_N(A_i, \text{NPIS}) = D^{i+}{}_N(e_{ij}{}^{wj}, e_j{}^{w+}) = \sqrt{\frac{1}{6n}\sum_n^j \left\{ \begin{array}{c} \left(L_{ij}{}^{wj}(x_j) - L_j{}^{w+}(x_j)\right)^2 + \left(M_{ij}{}^{wj}(x_j) - M_j{}^{w+}(x_j)\right)^2 + \left(U_{ij}{}^{wj}(x_j) - U_j{}^{w+}(x_j)\right)^2 + \\ \left(T_{ij}{}^{wj}(x_j) - T_j{}^{w+}(x_j)\right)^2 + \left(I_{ij}{}^{wj}(x_j) - I_j{}^{w+}(x_j)\right)^2 + \left(F_{ij}{}^{wj}(x_j) - F_j{}^{w+}(x_j)\right)^2 \end{array} \right\}}. \quad (21)$$

Also, the normalized Euclidian distance of each alternative $\langle L_{ij}{}^{wj}, M_{ij}{}^{wj}, U_{ij}{}^{wj}; T_{ij}{}^{wj}, I_{ij}{}^{wj}, F_{ij}{}^{wj}\rangle$ from the neutrosophic negative ideal solution $\langle L_j{}^{w-}, M_j{}^{w-}, U_j{}^{w-}; T_j{}^{w-}, I_j{}^{w-}, F_j{}^{w-}\rangle$ for $i = 1, 2, \cdots, m$ and $j = 1, 2, \cdots, n$ written as follows:

$$D_N(A_i, \text{NNIS}) = D^{i-}{}_N\left(e_{ij}{}^{wj}, e_j{}^{w-}\right) = \sqrt{\frac{1}{6n}\sum_n^j \left\{ \begin{array}{c} \left(L_{ij}{}^{wj}(x_j) - L_j{}^{w-}(x_j)\right)^2 + \left(M_{ij}{}^{wj}(x_j) - M_j{}^{w-}(x_j)\right)^2 + \left(U_{ij}{}^{wj}(x_j) - U_j{}^{w-}(x_j)\right)^2 + \\ \left(T_{ij}{}^{wj}(x_j) - T_j{}^{w-}(x_j)\right)^2 + \left(I_{ij}{}^{wj}(x_j) - I_j{}^{w-}(x_j)\right)^2 + \left(F_{ij}{}^{wj}(x_j) - F_j{}^{w-}(x_j)\right)^2 \end{array} \right\}}. \quad (22)$$

Step 7: Calculate the closeness coefficient of each alternative according to the NPIS:

$$c_i^* = \frac{D^{i-}{}_N\left(e_{ij}{}^{wj}, e_j{}^{w-}\right)}{D^{i+}{}_N\left(e_{ij}{}^{wj}, e_j{}^{w+}\right) + D^{i-}{}_N\left(e_{ij}{}^{wj}, e_j{}^{w-}\right)} \text{ where } 0 \le c_i^* \le 1. \quad (23)$$

Step 8: Rank alternatives according to the largest values of c_i^*.

4.2. A Numerical Example

If a customer plans to buy a specific product using the proposed app, then the app will search for that product from various websites. After that, the app will return various products with different prices and different qualities. The customer should have to choose the best one according to his/her needs, so the decision in his/her hands. Since the selection of the best product is the customer mission (and it is a complex problem because of vague and incomplete information) we will apply the ROCs method and neutrosophic TOPSIS for the selection process as follows:

Step 1: Ask customer to insert his/her purchase criteria and rank them from the most to the least important. Here the customer ranked the purchase criteria as follows:

1. Quality,
2. Price,
3. Delivery Time.

Step 2: After determining the rank of purchase criteria by the customer, the weight of these criteria must be calculated using the ROCs method. Since the rank of purchase criteria according to customer needs are: Quality, Price and Time respectively. Then, by using the ROCs method, the weight of criteria will be as follows:

$$W_1(\text{Quality}) = 0.61,\ W_2(\text{Price}) = 0.28,\ W_3(\text{Time}) = 0.11.$$

Step 3: Assuming that the customer should choose one from the four initially selected products p_1, p_2, p_3, p_4 with respect to three criteria which determined previously, the decision maker will compare all alternatives according to criteria using the linguistic scale which was presented in Table 1. Since there is no absolute truth and the truth is always relative, the single valued triangular neutrosophic numbers have been used for handling unclear, imperfect and conflicting information which usually exists in actuality.

By comparing the four products with respect to existing criteria, the estimation matrix is as in Table 2.

Step 4: Construct the weighted decision matrix via multiplying weights of criteria by the estimation matrix as in Table 3.

Step 5: Define the neutrosophic positive and negative ideal solution, NPIS and NNIS by using equations from Equation (7) to Equation (20) according to attribute type (i.e., benefit or cost).

Table 2. Estimation matrix of alternatives according to criteria.

P	C_1	C_2	C_3
P_1	G	VH	B
P_2	EG	H	G
P_3	G	L	VG
P_4	VG	H	VG

Table 3. Weighted decision matrix of alternatives.

P	C_1	C_2	C_3
P_1	$\langle 3,4,4;0.80,0.20,0.15\rangle$	$\langle 2,2,2;0.90,0.10,0.05\rangle$	$\langle 0,0,0;0.20,0.75,0.80\rangle$
P_2	$\langle 4,5,5;1.00,0.00,0.00\rangle$	$\langle 1,2,2;0.80,0.20,0.15\rangle$	$\langle 0,1,1;0.80,0.20,0.15\rangle$
P_3	$\langle 3,4,4;0.80,0.20,0.15\rangle$	$\langle 0,1,1;0.20,0.75,0.80\rangle$	$\langle 1,1,1;0.90,0.10,0.05\rangle$
P_4	$\langle 4,4,5;0.90,0.10,0.05\rangle$	$\langle 1,2,2;0.80,0.20,0.15\rangle$	$\langle 0,1,1;0.90,0.10,0.05\rangle$

The NPIS and NNIS are given by:

$$\text{NPIS} = (\langle 4,4,4;1.00,0.00,0.00\rangle, \langle 0,1,1;0.2,0.75,0.80\rangle, \langle 1,0,0;0.9,0.1,0.05\rangle),$$

$$\text{NNIS} = (\langle 3,4,4;0.8,0.20,0.15\rangle, \langle 2,2,2;0.90,0.10,0.05\rangle, \langle 0,0,0;0.20,0.75,0.80\rangle).$$

If obtained values of lower, median, and/or upper are out of order, then reorder them and follow the work.

Step 6: Calculate the normalized Euclidian distance of each alternative from the NPIS as follows:

$$D(P_1, \text{NPIS}) = 0.78,\ D(P_2, \text{NPIS}) = 0.71,$$

$$D(P_3, \text{NPIS}) = 0.41,\ D(P_4, \text{NPIS}) = 0.67.$$

Step 7: Calculate the normalized Euclidian distance of each alternative from the neutrosophic negative ideal solution as follows:

$$D(P_1, \text{NNIS}) = 0.0,\ D(P_2, \text{NNIS}) = 0.63,$$

$$D(P_3, \text{NNIS}) = 0.81,\ D(P_4, \text{NNIS}) = 0.60.$$

Step 8: Calculate the closeness coefficient of each alternative according to the NPIS using Equation (23):

$$c_1{}^* = 0.00,\ c_2{}^* = 0.470,\ c_3{}^* = 0.663,\ c_4{}^* = 0.472.$$

Step 9: Rank alternatives according to the largest values of $c_i{}^*$:

$$P_3 > P_4 > P_2 > P_1$$

Hence, the best product which suits customer needs is P_3 as appears in Figure 4.

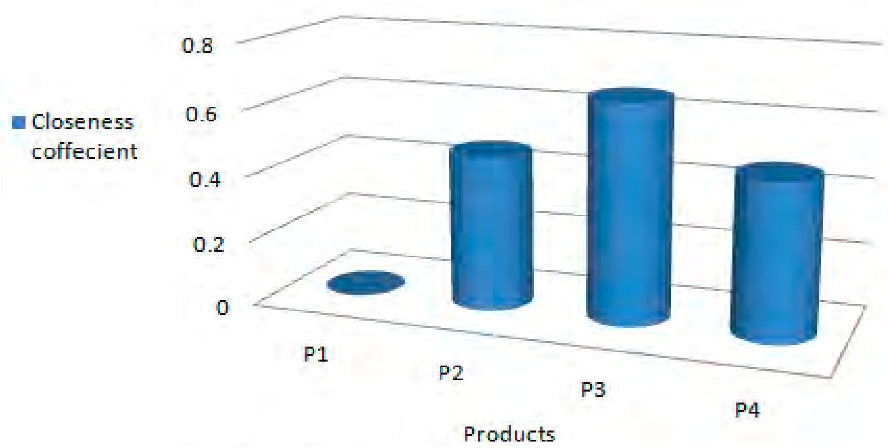


Figure 4. Products and their closeness coefficients.

5. Conclusion and Future Directions

In this paper we proposed a smart e-marketing system based on IoT. Using this system enables marketers to meet customers' expectations for products and services, and then achieve a high degree of satisfaction. It also enables marketers to have new streams of data, and discover consumer preferences and habits. Therefore, it enables marketers to provide customers with what they need with better matches.

By using the proposed system by customers and companies, all problems which face them in online shopping could be solved easily. It also helps customers, sellers, affiliate, companies, and marketers to achieve their goals with a high degree of accuracy.

In this system we focused only on food commodities as an example of products, but this system has the potential to be applied to all kinds of other products in our daily lives.

Although the proposed system can avoid different threats and hacking process, more involvements from governments to create legal basis is very significant and will make the proposed system better.

Since there are different types of the same product and also different criteria of buying, and the main problem of product or service selection is the vague and inaccurate knowledge of the decision maker, we presented a multi-criteria decision making technique based on neutrosophic TOPSIS to deal with unclear and conflicting information. In this technique we calculated the weights of criteria by using the rank order centroids (ROCs) method, and the evaluation matrix for comparing alternatives regarding to existing criteria is represented by using triangular neutrosophic numbers (TNNs). This technique will support customers in selecting the best product or service.

For our future work, we will expand our IoT research outputs, applications and services, aiming to apply them in different domains in agriculture, health, and industry. Furthermore, we will apply the proposed neutrosophic technique in various situations, not only for supporting customers, but also supporting marketers and companies.

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Group Decision Making Based on Triangular Neutrosophic Cubic Fuzzy Einstein Hybrid Weighted Averaging Operators

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Abstract: In this paper, a new concept of the triangular neutrosophic cubic fuzzy numbers (TNCFNs), their score and accuracy functions are introduced. Based on TNCFNs, some new Einstein aggregation operators, such as the triangular neutrosophic cubic fuzzy Einstein weighted averaging (TNCFEWA), triangular neutrosophic cubic fuzzy Einstein ordered weighted averaging (TNCFEOWA) and triangular neutrosophic cubic fuzzy Einstein hybrid weighted averaging (TNCFEHWA) operators are developed. Furthermore, their application to multiple-attribute decision-making with triangular neutrosophic cubic fuzzy (TNCF) information is discussed. Finally, a practical example is given to verify the developed approach and to demonstrate its practicality and effectiveness.

Keywords: triangular neutrosophic cubic fuzzy number; Einstein t-norm; arithmetic averaging operator; Multi-attribute decision making; numerical application

1. Introduction

Atanassov [1] introduced the IFS, which is a generalization of FS. Atanassov [2] introduced operations and relations over IFSs taking as a point of departure respective definitions of relations and operations over fuzzy sets. Bustince et al. [3] introduced the characterization of certain structures of intuitionistic relations according to the structures of two concrete fuzzy relations. Deschrijver et al. [4] established the relationships between intuitionistic fuzzy sets (Atanassov, VII ITKR's Session, Sofia, June 1983 (Deposed in Central Sci.-Techn. Library of Bulg. Acad. of Sci., 1697/84) (in Bulgarian)), L-fuzzy sets. Deschrijver et al. [5] defined the mathematical relationship between intuitionistic fuzzy sets and other models of imprecision. Jun et al. [6] introduced the cubic set. Mohiuddin et al. [7] showed that the union of two internal cubic soft sets might not be internal. Turksen [8] showed that the proposed representation (1) exists for certain families of the conjugate pairs of t-norms and t-norms, and (2) resolves some of the difficulties associated with particular interpretations of conjunction, disjunction, and implication in fuzzy set theories.

Xu [9] developed some aggregation operators, such as the intuitionistic fuzzy weighted averaging operator, intuitionistic fuzzy ordered weighted averaging operator, and intuitionistic fuzzy hybrid aggregation operator, to aggregate intuitionistic fuzzy values. Xu et al. [10] developed some new geometric aggregation operators, such as the intuitionistic fuzzy weighted geometric (IFWG) operator and the intuitionistic fuzzy ordered weighted geometric (IFOWG) operator. Xu et al. [11] provided a survey of the aggregation techniques of intuitionistic fuzzy information and their applications in various fields, such as decision making, cluster analysis, medical diagnosis, forecasting, and manufacturing grid. Liu et al. [12] introduced and discussed the concept of intuitionistic fuzzy point operators. Zeng et al. [13] defined the situation with intuitionistic fuzzy information and developed an intuitionistic fuzzy ordered weighted distance (IFOWD) operator. The fuzzy set was introduced by Zadeh [14]. Zadeh [15] introduced the interval-valued fuzzy set Li et al. [16] proposed group decision-making methods of the interval-valued intuitionistic uncertain linguistic variable based on Archimedean t-norm and Choquet integral. Zhao et al. [17] developed some hesitant triangular fuzzy aggregation operators based on the Einstein operation: the hesitant triangular fuzzy Einstein weighted averaging (HTFEWA) operator. Xu et al. [18] introduced two new aggregation operators: dynamic intuitionistic fuzzy weighted averaging (DIFWA) operator and uncertain dynamic intuitionistic fuzzy weighted averaging (UDIFWA) operator.

The Neutrosophic Set (NS) was projected by Smarandache [19,20]. Neutrosophic sets are characterized by fact participation, an indeterminacy-enrollment work and misrepresentation participation, which are inside the ordinary or nonstandard unit interim $]^{-}0, 1^{+}[$ in order to apply NS to genuine applications. In order to apply NS to real-world applications, Aliya et al. [21] introduced the concept of the triangular cubic fuzzy number. Aliya et al. [22] introduced the triangular cubic hesitant fuzzy Einstein weighted averaging (TCHFEWA) operator, triangular cubic hesitant fuzzy Einstein ordered weighted averaging (TCHFEOWA) operator and triangular cubic hesitant fuzzy Einstein hybrid weighted averaging (TCHFEHWA) operator.

Beg et al. [23] introduced a computational means to manage situations in which experts assess alternatives in possible membership and non-membership values. Przemyslaw et al. [24] introduced a simple test that sometimes might be helpful in detecting non-separability at a glance.

The differences between Reference 21, 22 and the current paper are as Table 1:

Table 1. Difference between references 21, 22 and current paper.

Reference 21	Reference 22	Current Paper
Defines a new extension of the triangular cubic fuzzy number by using a cubic set.	Defines a new extension of the triangular cubic hesitant fuzzy number by using a cubic set.	Defines a new extension of the triangular neutrosophic cubic fuzzy number by using a neutrosophic set.
Introduced the triangular cubic fuzzy number, operational laws, and their score and accuracy functions.	Introduced the triangular cubic hesitant fuzzy number, operational laws, and their score, accuracy functions, membership uncertainty index and hesitation index.	Introduced the triangular cubic fuzzy number, operational laws, and their score and accuracy functions, membership uncertainty index and hesitation index.
Introduced the triangular cubic fuzzy hybrid aggregation operator.	Introduced three Einstein aggregation operators, such as the triangular cubic fuzzy hybrid aggregation operator, and the TCHFEWA, TCHFEOWA and TCHFEHWA operators	Introduced three Einstein aggregation operators, such as the triangular neutrosophic cubic fuzzy hybrid aggregation operator, and the TNCFEWA, TNCFEOWA and TNCFEHWA operators

Based on the above analysis, in this paper we develop TNCFNs, which is the generalization of the triangular neutrosophic intuitionistic fuzzy number and triangular neutrosophic interval fuzzy number. We perform some operations based on Einstein T-norm and Einstein T-conorm for TNCFNs. We also develop score and accuracy functions to compare two TNCFNs. Due to the developed operation, we propose the TNCFEWA operator, TNCFEOWA operator, and TNCFEHWA operator, to aggregate a collection of TNCFNs.

This paper is organized as follows. In Section 2, we define some concepts of FS, CS, and TNCFNs. In Section 3, we discuss some Einstein operations on TNCFNs and their properties. In Section 4, we first develop some novel arithmetic averaging operators, such as the TNCFEWA operator, TNCFEOWA operator, and TNCFEHWA operator, for aggregating a group of TNCFNs. In Section 5, we apply the

TNCFEHWA operator to MADM with TNCFNs material. In Section 6, we offer a numerical example consistent with our approach. In Section 7, we discuss comparison analysis. In Section 8, we present a conclusion.

2. Preliminaries

Definition 1. **[15].** *Let H be a fixed set, a FS F in H is defined as:* $F = \{(h, \Gamma_F(h)|h \in H\}$ *where* $\Gamma_F(h)$ *is a mapping from h to the closed interval* $[0, 1]$ *and for each* $h \in H, \Gamma_F(h)$ *is called the degree of membership of h in H.*

Definition 2. *Let H is a fixed set and an interval-valued fuzzy set I in H is defined as* $I = \{h, \mathrm{R}_I^-(h), \mathrm{R}_I^+(h)|h \in H\}$, *where* $\mathrm{R}_I^- : H \to [0, 1]$ *and* $\mathrm{R}_I^+ : H \to [0, 1]$. *The* $\mathrm{R}_I^-(h)$ *is lower membership and* $\mathrm{R}_I^+(h)$ *is upper membership such that* $0 \leq \mathrm{R}_I^-(h) \leq \mathrm{R}_I^+(h) \leq 1$.

Definition 3. *[1]. An IFS Đ in H* is given by $Đ = \{(h, \mathrm{R}_Đ(h), \Omega_Đ(h)|h \in H\}$, *where* $\mathrm{R}_Đ : H \to [0, 1]$ *and* $\Omega_Đ : H \to [0, 1]$, *with the condition* $0 \leq \mathrm{R}_Đ(h) + \Omega_Đ(h) \leq 1$.

The numbers $\mathrm{R}_Đ(h)$ *and* $\Omega_Đ(h)$ *represent, respectively, the membership degree and non-membership degree of the element h to the set Đ.*

Triangular Neutrosophic Cubic Fuzzy Number

Definition 4. *Let* $A_1 = \left\{\begin{array}{c}[p_1(h), q_1(h),\\ r_1(h)],\\ \langle[\mathrm{Y}_1^-(h),\\ \mathrm{R}_1^-(h), \delta_1^-(h)],\\ [\mathrm{Y}_1^+(h),\\ \mathrm{R}_1^+(h), \delta_1^+(h)],\\ [\mathrm{Y}_1(h),\\ \mathrm{R}_1(h), \delta_1(h)]\rangle\\ |h \in \mathrm{H}\end{array}\right\}$ and $A_2 = \left\{\begin{array}{c}[p_2(h), q_2(h),\\ r_2(h)],\\ \langle[\mathrm{Y}_2^-(h),\\ \mathrm{R}_2^-(h), \delta_2^-(h)],\\ [\mathrm{Y}_2^+(h),\\ \mathrm{R}_2^+(h), \delta_2^+(h)],\\ [\mathrm{Y}_2(h),\\ \mathrm{R}_2(h), \delta_2(h)]\rangle\\ |h \in \mathrm{H}\end{array}\right\}$ *are two TNCFNs, some operations on TNCFNs are defined as follows:*

(a) $A_1 \subseteq A_2$ *iff* $\forall h \in H$, $p_1(h)) \geq p_2(h)$ $q_1(h) \geq q_2(h)$, $r_1(h) \geq r_2(h)$, $\mathrm{Y}_1^-(h) \geq \mathrm{Y}_2^-(h)$, $\mathrm{R}_1^-(h) \geq \mathrm{R}_2^-(h)$, $\delta_1^-(h) \geq \delta_1^-(h)$, $\mathrm{Y}_1^+(h) \geq \mathrm{Y}_2^+(h)$ $\mathrm{R}_1^+(h) \geq \mathrm{R}_2^+(h)$,, $\delta_1^+(h) \geq \delta_2^+(h)$ and $\mathrm{Y}_1(h) \geq \mathrm{Y}_2(h)$ $\delta_1(h) \leq \delta_2(h)$.

(b) $A_1 \cap_{T,S} A_2 = T[p_1(h), p_2(h)]$, $T[q_1(h), q_2(h)]$, $T[r_1(h), r_2(h)]$, $\langle T[\mathrm{Y}_1^-(h), \mathrm{Y}_2^-(h)]$, $T[\mathrm{R}_1^-(h), \mathrm{R}_2^-(h)]$, $T[\delta_1^-(h), \delta_2^-(h)]$, $T[\mathrm{Y}_1^+(h), \mathrm{Y}_2^+(h)]$, $T[\mathrm{R}_1^+(h), \mathrm{R}_2^+(h)]$, $T[\delta_1^+(h), \delta_2^+(h)]$, $S[\mathrm{Y}_1(h), \mathrm{Y}_2(h)]$, $S[\mathrm{R}_1(h), \mathrm{R}_2(h)]$, $S[\delta_1(h), \delta_2(h)]\rangle$

Example 1. *Let* $\ddot{A}_1 = \left\{\begin{array}{c}\langle[0.1, 0.2, 0.3],\\ [0.2, 0.4, 0.6],\\ [0.4, 0.6, 0.8],\\ [0.3, 0.5, 0.7]\rangle\end{array}\right\}$ *and* $\ddot{A}_2 = \left\{\begin{array}{c}\langle[0.103, 0.104, 0.105],\\ [0.100, 0.102, 0.104],\\ [0.102, 0.104, 0.106],\\ [0.101, 0.103, 0.105]\rangle\end{array}\right\}$ *be two TNCFSs*

(a) $\ddot{A}_1 \subseteq \ddot{A}_2$, *if* $\forall \ddot{z} \in Z$, $0.1 \geq 0.103$, $0.2 \geq 0.104$, $0.3 \geq 0.105$, $0.2 \geq 0.100$, $0.4 \geq 0.102$, $0.6 \geq 0.104$, $0.4 \geq 0.102$, $0.6 \geq 0.104$, $0.8 \geq 0.106$ *and* $0.3 \leq 0.101$, $0.5 \leq 0.103$, $0.7 \leq 0.105$.

(b) $\ddot{A}_1 \cap_{T,S} \ddot{A}_2 = T[0.1, 0.103]$, $T[0.2, 0.104]$, $T[0.3, 0.105]$, $[T[0.2, 0.10]$, $T[0.4, 0.12]$, $T[0.6, 0.14]$, $T[0.4, 0.12]$, $T[0.6, 0.14]$, $T[0.8, 0.16]$ *and* $S[0.3, 0.11]$, $S[0.7, 0.15]$.

Definition 5. *Let* $C = \left\{ \begin{array}{c} [p_C(h),\, q_C(h),\, r_C(h)] \\ \left\langle [A_C^-(h), \mathrm{R}_C^-(h),\, \widetilde{U}_C^-(h)], \right. \\ [A_C^+(h), \mathrm{R}_C^+(h),\, \widetilde{U}_C^+(h)], \\ \left. [A_C(h), \mathrm{R}_C(h),\, \widetilde{U}_C(h)] \right\rangle \Big| h \in \mathrm{H} \end{array} \right\}$ *be a TNCFN and then the score function* $S(C)$*, accuracy function* $N(C)$*, membership uncertainty index* $T(C)$ *and hesitation uncertainty index* $G(C)$ *of a TNCFN* C *are defined by*

$$S(C) = \frac{\begin{array}{c} \left\langle [p_C(h) + q_C(h) + r_C(h)][[A_C^-(h) + \mathrm{R}_C^-(h) + \widetilde{U}_C^-(h)] \right. \\ +[A_C^+(h) + \mathrm{R}_C^+(h) + \widetilde{U}_C^+(h)]] \\ \left. -[A_C(h) + \mathrm{R}_C(h) + \widetilde{U}_C(h)] \right\rangle \end{array}}{27},$$

$$N(C) = \frac{\begin{array}{c} \left\langle [p_C(h) + q_C(h) + r_C(h)][[A_C^-(h) + \mathrm{R}_C^-(h) + \widetilde{U}_C^-(h)] \right. \\ \left. +[A_C^+(h) + \mathrm{R}_C^+(h) + \widetilde{U}_C^+(h)]] + [A_C(h) + \mathrm{R}_C(h) + \widetilde{U}_C(h)] \right\rangle \end{array}}{27}$$

$$T(C) = \Big\langle [p_C(h) + q_C(h) + r_C(h)][[A_C^+(h) + \mathrm{R}_C^+(h) + \widetilde{U}_C^+(h)]+$$
$$[A_C(h) + \mathrm{R}_C(h) + \widetilde{U}_C(h)] - [A_C^-(h) + \mathrm{R}_C^-(h) + \widetilde{U}_C^-(h)] \Big\rangle,$$

$$G(C) = \Big\langle [p_C(h) + q_C(h) + r_C(h)][[A_C^+(h) + \mathrm{R}_C^+(h) + \widetilde{U}_C^+(h)]$$
$$+[A_C^-(h) + \mathrm{R}_C^-(h) + \widetilde{U}_C^-(h)] - [A_C(h) + \mathrm{R}_C(h) + \widetilde{U}_C(h)] \Big\rangle.$$

Example 2. *Let* $C = \left\{ \begin{array}{c} \langle [0.101,\, 0.102, \\ 0.103], [0.5,\, 0.7,\, 0.9], \\ [0.7,\, 0.9,\, 0.11], [0.6, \\ 0.8,\, 0.10] \rangle \end{array} \right\}$ *be a TNCFN. Then the score function* $S(C)$*, accuracy function* $H(C)$*, membership uncertainty index* $T(C)$ *and hesitation uncertainty index* $G(C)$ *of a TNCFN* C *are defined by*

$$S(C) = \frac{\begin{array}{c} \langle [0.101 + 0.102 + 0.103][0.5 + 0.7 + 0.9]] \\ +[0.7 + 0.9 + 0.11] - [0.6 + 0.8 + 0.10] \rangle \end{array}}{27} = \frac{0.306(3.81 - 1.5)}{27} = \frac{0.7068}{27} = 0.0261,$$

$$H(C) = \frac{\begin{array}{c} \langle [0.101 + 0.102 + 0.103][0.5 + 0.7 + 0.9]] \\ +[0.7 + 0.9 + 0.11] + [0.6 + 0.8 + 0.10] \rangle \end{array}}{27} = \frac{0.306(3.81 + 1.5)}{27} = \frac{1.6248}{27} = 0.0601,$$

$$T(C) = \left\{ \begin{array}{c} [0.101 + 0.102 + 0.103] \langle [0.7 + 0.9 + 0.11] \\ +[0.6 + 0.8 + 0.10] - [0.5 + 0.7 + 0.9] \rangle \end{array} \right\} = 0.6(1.71 + 1.5 - 2.1) = 0.306(3.21 - 2.1) = 0.3396,$$

$$G(C) = \left\{ \begin{array}{c} \langle [0.101 + 0.102 + 0.103][0.7 + 0.9 + 0.11] \\ +[0.5 + 0.7 + 0.9] - [0.6 + 0.8 + 0.10] \rangle \end{array} \right\} = 0.306(1.71 + 2.1 - 1.5) = 0.7068.$$

See Figure 1.

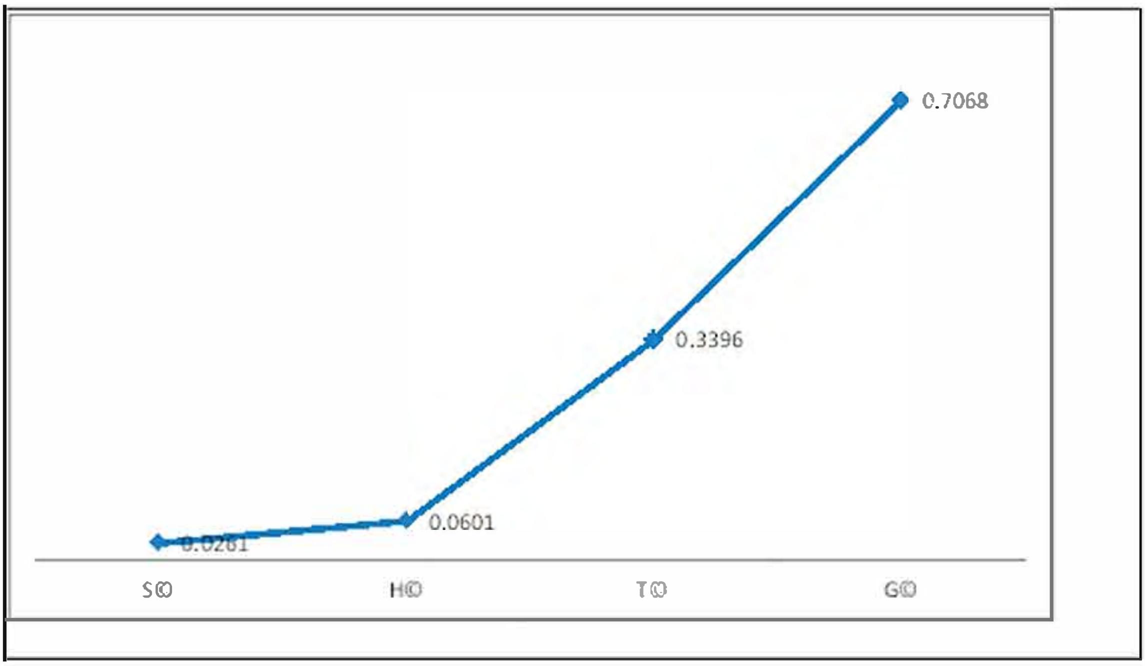


Figure 1. The score function, accuracy function, membership uncertainty index and hesitation uncertainty index are ranking of TNCFN.

3. Some Einstein Operations on TNCFNs

Definition 6. *Let* $C = \left\{ \begin{array}{c} [a(h),\, e(h), \\ G(h)],\, \langle [Y=^{-}(h), \\ k^{-}(h),\, \Gamma^{-}(h)], \\ [Y=^{+}(h),\, k^{+}(h), \\ \Gamma^{+}(h)], [Y=(h), \\ k(h),\, \Gamma(h)]\rangle \\ |h \in H \end{array} \right\}$, $C_1 = \left\{ \begin{array}{c} [a_1(h),\, e_1(h), \\ G_1(h)]\langle [Y=_1^{-}(h), \\ k_1^{-}(h),\, \Gamma_1^{-}(h)], \\ [Y=_1^{+}(h),\, k_1^{+}(h), \\ \Gamma_1^{+}(h)], [Y=_1(h), \\ k_1(h),\, \Gamma_1(h)]\rangle \\ |h \in H \end{array} \right\}$ *and*
$C_2 = \left\{ \begin{array}{c} [a_2(h),\, e_2(h), \\ G_2(h)]\langle [Y=_2^{-}(h), \\ k_2^{-}(h),\, \Gamma_2^{-}(h)], \\ [Y=_2^{+}(h),\, k_2^{+}(h), \\ \Gamma_2^{+}(h)], [Y=_2(h), \\ k_2(h),\, \Gamma_2(h)]\rangle \\ |h \in H \end{array} \right\}$ *be any three TNCFNs. Then some Einstein operations of* C_1 *and* C_2 *can be defined as:* $C_1 + C_2 = \left\langle \left[\begin{array}{c} \frac{a_1(h)+a_2(h)}{1+a_1(h)(1-a_2(h))}, \\ \frac{e_1(h)+e_2(h)}{1+e_1(h)(1-e_2(h))}, \\ \frac{G_1(h)+G_2(h)}{1+G_1(h)(1-G_2(h))} \end{array} \right], \left[\begin{array}{c} \frac{Y=_1^{-}(h)+Y=_2^{-}(h)}{1+Y=_1^{-}(h)(1-Y=_2^{-}(h))}, \\ \frac{Y=_1^{+}(h)+Y=_2^{+}(h)}{1+Y=_1^{+}(h)(1-Y=_2^{+}(h))}, \\ \frac{k_1^{-}(h)+k_2^{-}(h)}{1+k_1^{-}(h)(1-k_2^{-}(h))}, \\ \frac{k_1^{+}(h)+k_2^{+}(h)}{1+k_1^{+}(h)(1-k_2^{+}(h))}, \\ \frac{\Gamma_1^{-}(h)+\Gamma_2^{-}(h)}{1+\Gamma_1^{-}(h)(1-\Gamma_2^{-}(h))}, \\ \frac{\Gamma_1^{+}(h)+\Gamma_2^{+}(h)}{1+\Gamma_1^{+}(h)(1-\Gamma_2^{+}(h))} \end{array} \right], \right.$

$$\left[\begin{array}{c}\frac{Y=_1(h)Y=_2(h)}{(1+(1-Y=_1(h))(1-Y=_2(h)))}, \\ \frac{k_1(h)k_2(h)}{(1+(1-k_1(h))(1-k_2(h)))}, \\ \frac{\Gamma_1(h)\Gamma_2(h)}{(1+((1-\Gamma_1(h))(1-\Gamma_2(h)))}\end{array}\right]\Bigg\rangle, \lambda C = \left\langle \left[\begin{array}{c}\frac{[1+a_C(h)]^\lambda-[1-a_C(h)]^\lambda}{[1+a_C(h)]^\lambda+[1-a(h)]^\lambda}, \\ \frac{[1+e_C(h)]^\lambda-[1-e_C(h)]^\lambda}{[1+e_C(h)]^\lambda+[1-e(h)]^\lambda}, \\ \frac{[1+G_C(h)]^\lambda-[1-G_C(h)]^\lambda}{[1+G_C(h)]^\lambda+[1-G(h)]^\lambda}\end{array}\right], \left[\begin{array}{c}\frac{[1+Y=^-_C(h)]^\lambda-[1-Y=^-_C(h)]^\lambda}{[1+Y=^-_C(h)]^\lambda+[1-Y=^-_C(h)]^\lambda}, \\ \frac{[1+Y=^+_C(h)]^\lambda-[1-Y=^+_C(h)]^\lambda}{[1+Y=^+_C(h)]^\lambda+[1-Y=^+_C(h)]^\lambda}, \\ \frac{[1+k^-_C(h)]^\lambda-[1-k^-_C(h)]^\lambda}{[1+k^-_C(h)]^\lambda+[1-k^-_C(h)]^\lambda}, \\ \frac{[1+k^+_C(h)]^\lambda-[1-k^+_C(h)]^\lambda}{[1+k^+_C(h)]^\lambda+[1-k^+_C(h)]^\lambda}, \\ \frac{[1+\Gamma^-_C(h)]^\lambda-[1-\Gamma^-_C(h)]^\lambda}{[1+\Gamma^-_C(h)]^\lambda+[1-\Gamma^-_C(h)]^\lambda}, \\ \frac{[1+\Gamma^+_C(h)]^\lambda-[1-\Gamma^+_C(h)]^\lambda}{[1+\Gamma^+_C(h)]^\lambda+[1-\Gamma^+_C(h)]^\lambda}\end{array}\right],$$

$$\left[\begin{array}{c}\frac{2[Y=_C(h)]^\lambda}{[(2-Y=_C(h)]^\lambda+[Y=_C(h)]^\lambda}, \\ \frac{2[k_C(h)]^\lambda}{[(2-k_C(h)]^\lambda+[k_C(h)]^\lambda}, \\ \frac{2[\Gamma_C(h)]^\lambda}{[(2-\Gamma_C(h)]^\lambda+[\Gamma_C(h)]^\lambda}\end{array}\right]\Bigg\rangle.$$

Proposition 1. *Let $\ddot{A}$, $\ddot{A}_1$ and $\ddot{A}_2$ be three TNCFNs, λ, λ_1, $\lambda_2 > 0$, then we have:*

(1) $\ddot{A}_1 + \ddot{A}_2 = \ddot{A}_2 + \ddot{A}_1$,

(2) $\lambda(\ddot{A}_1 + \ddot{A}_2) = \lambda\ddot{A}_2 + \lambda\ddot{A}_1$,

(3) $\lambda_1\ddot{A} + \lambda_2\ddot{A} = (\lambda_1 + \lambda_2)\ddot{A}$.

Proof. The proof of these propositions is provided in Appendix A. □

Remark 1. *If $\alpha_1 \leq_{L_{TNCFN}} \alpha_2$, then $\alpha_1 \leq \alpha_2$, the total order is the partial order on L_{TNCFN}, see Figure 2.*

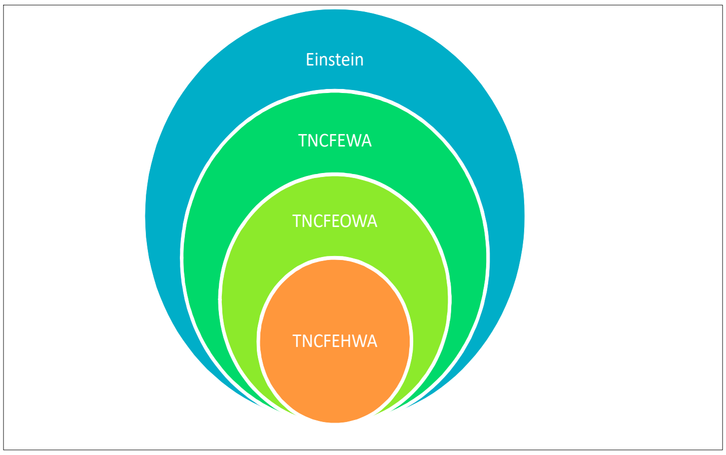


Figure 2. New extend aggregation operators, such as TNCFEWA, TNCFEOWA and TNCFEHWA operators.

4. Triangular Neutrosophic Cubic Fuzzy Averaging Operators Based on Einstein Operations

In this section, we define the aggregation operators.

4.1. Triangular Neutrosophic Cubic Fuzzy Einstein Weighted Averaging Operator

Definition 7. Let $\ddot{A} = \left\{ \begin{array}{c} [\alpha(h),\, \beta(h), \\ \Delta(h)]\langle[\xi_1^-(h), \\ \xi_2^-(h),\, \xi_3^-(h)], \\ [\xi_1^+(h),\, \xi_2^+(h), \\ \xi_3^+(h)], [\xi_1(h), \\ \xi_2(h),\, \xi_3(h)]\rangle \\ |h \in H \end{array} \right\}$ be a collection of TNCFNs in L_{TNCFN} and $\ddot{\omega} = (\ddot{\omega}_1, \ddot{\omega}_2, \ldots, \ddot{\omega}_n)^T$ be the weight vector, with $\ddot{\omega}_j \in [0, 1]$, $\sum\limits_{j=1}^{n} \ddot{\omega}_j = 1$. Hence TNCFEWA operator of dimension n is a mapping TNCFEWA : $L^n_{\mathrm{TNCFN}} \to L_{\mathrm{TNCFN}}$ and defined by $\mathrm{TNCFEWA}(\ddot{A}_1, \ddot{A}_2, \ldots, \ddot{A}_n) = \ddot{\omega}_1\ddot{A}_1 + \ddot{\omega}_2\ddot{A}_2, \ldots, \ddot{\omega}_n\ddot{A}_n$.

If $\ddot{\omega} = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$. *Hence the TNCFEWA operator is reduced to TNCFEA operator of dimension n. It can be defined as follows:* $\mathrm{TNCFEA}(\ddot{A}_1, \ddot{A}_2, \ldots, \ddot{A}_n) = \frac{1}{n}(\ddot{A}_1, \ddot{A}_2, \ldots, \ddot{A}_n)$.

Theorem 1. *Let* $\ddot{A} = \left\{ \begin{array}{c} [\alpha_1(h),\, \beta_1(h), \\ \Delta_1(h)],\, \langle[p_1^-(h), \\ q_1^-(h),\, r_1^-(h)], \\ [p_1^+(h),\, q_1^+(h), \\ r_1^+(h)], [p_1(h), \\ q_1(h),\, r_1(h)]\rangle \\ |h \in H \end{array} \right\}$ *be a collection of TNCFNs in* L_{TNCFN}. *The amassed an incentive by utilizing the TNCFEWA operator is additionally a TNCFN and TNCFEWA.*

$$(\ddot{A}_1, \ddot{A}_2, \ldots, \ddot{A}_n) = \left\langle \begin{bmatrix} \dfrac{\prod_{j=1}^{n}[1+\alpha_1(h)]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-\alpha_1(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+\alpha_1(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-\alpha_1(h)]^{\ddot{\omega}}}, \\ \dfrac{\prod_{j=1}^{n}[1+\beta_1(h)]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-\beta_1(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+\beta_1(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-\beta_1(h)]^{\ddot{\omega}}}, \\ \dfrac{\prod_{j=1}^{n}[1+\Delta_1(h)]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-\Delta_1(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+\Delta_1(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-\Delta_1(h)]^{\ddot{\omega}}}, \end{bmatrix}, \begin{bmatrix} \dfrac{\prod_{j=1}^{n}[1+p_1^-(h)]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-p_1^-(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+p_1^-(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-p_1^-(h)]^{\ddot{\omega}}}, \\ \dfrac{\prod_{j=1}^{n}[1+q_2^-(h)]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-q_2^-(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+q_2^-(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-q_2^-(h)]^{\ddot{\omega}}}, \\ \dfrac{\prod_{j=1}^{n}[1+r_3^-(h)]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-r_3^-(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+r_3^-(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-r_3^-(h)]^{\ddot{\omega}}} \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} \dfrac{\prod_{j=1}^{n}[1+p_1^+(h)]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-p_1^+(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+p_1^+(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-p_1^+(h)]^{\ddot{\omega}}}, \dfrac{\prod_{j=1}^{n}[1+q_2^+(h)]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-q_2^+(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+q_2^+(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-q_2^+(h)]^{\ddot{\omega}}}, \\ \dfrac{\prod_{j=1}^{n}[1+r_3^+(h)]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-r_3^+(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+r_3^+(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-r_3^+(h)]^{\ddot{\omega}}} \end{bmatrix}, \begin{bmatrix} \dfrac{2\prod_{j=1}^{n}[p_1(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[(2-p_1(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[p_1(h)]^{\ddot{\omega}}}, \dfrac{2\prod_{j=1}^{n}[q_2(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[(2-q_2(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[q_2(h)]^{\ddot{\omega}}}, \\ , \dfrac{2\prod_{j=1}^{n}[r_3(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[(2-r_3(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[r_3(h)]^{\ddot{\omega}}} \end{bmatrix} \right\rangle$$

where $\ddot{\omega} = (\ddot{\omega}_1, \ddot{\omega}_2, \ldots, \ddot{\omega}_n)^T$ *be the weight vector of* $\ddot{A}_j (j = 1, 2, \ldots, n)$ *such that* $\ddot{\omega}_j \in [0, 1]$ *and* $\sum\limits_{j=1}^{n} \ddot{\omega}_j = 1$. *If* $\alpha_1(h) = \alpha_1(h)$, $\beta_1(h) = \beta_1(h)$, $\Delta_1(h) = \Delta_1(h)$, $p_1^-(h) = p_1^-(h)$, $q_2^-(h) = q_2^-(h)$, $r_3^-(h) = r_3^-(h)$, $p_1^+(h) = p_1^+(h)$, $q_2^+(h) = q_2^+(h)$, $r_3^+(h) = r_3^+(h)$ *and* $p_1(h) = p_1(h)$, $q_2(h) = q_2(h)$,

$r_3(h) = r_3(h)$. *Then the TNCFN* $\ddot{A} = \left\{ \begin{array}{c} [\alpha_1(h),\ \beta_1(h), \\ \Delta_1(h)],\ \langle [p_1^-(h), \\ q_1^-(h),\ r_1^-(h)], \\ [p_1^+(h),\ q_1^+(h), \\ r_1^+(h)], [p_1(h), \\ q_1(h),\ r_1(h)] \rangle \\ |h \in H \end{array} \right\}$ *are reduced to the triangular neutrosophic cubic fuzzy numbers* $\ddot{A} = \left\{ \begin{array}{c} [\alpha_1(h),\ \beta_1(h), \\ \Delta_1(h)],\ \langle [p_1^-(h), \\ q_1^-(h),\ r_1^-(h)], \\ [p_1^+(h),\ q_1^+(h), \\ r_1^+(h)], [p_1(h), \\ q_1(h),\ r_1(h)] \rangle \\ |h \in H \end{array} \right\}$ *and the TNCFEWA operator is reduced to the TNCFEWA operator.*

Proof. The proof of this theorem is provided in Appendix B. □

Example 3. *Let* $C_1 = \left\{ \begin{array}{c} \langle [0.02,\ 0.03,\ 0.04], \\ [0.02,\ 0.04,\ 0.06], \\ [0.04,\ 0.06,\ 0.08], \\ [0.03,\ 0.05,\ 0.07] \rangle \end{array} \right\}$, $C_2 = \left\{ \begin{array}{c} \langle [0.205,\ 0.207,\ 0.209], \\ [0.211,\ 0.213,\ 0.215], \\ [0.213,\ 0.215,\ 0.217], \\ [0.212,\ 0.214,\ 0.216] \rangle \end{array} \right\}$ *and* $C_3 = \left\{ \begin{array}{c} \langle [0.004,\ 0.005,\ 0.006], \\ [0.102,\ 0.104,\ 0.106], \\ [0.104,\ 0.106,\ 0.108], \\ [0.103,\ 0.105,\ 0.107] \rangle \end{array} \right\}$ *be a TNCFN. Then the score function is defined by* $S(C_1) = \frac{\left\{ \begin{array}{c} \langle [0.02 + 0.03 + 0.04], \\ [0.02 + 0.04 + 0.06] + \\ [0.04 + 0.06 + 0.08] - \\ [0.03 + 0.05 + 0.07] \rangle \end{array} \right\}}{27} = 0.0005$, $S(C_2) = \frac{\left\{ \begin{array}{c} \langle [0.205 + 0.207 + 0.209], \\ [0.211 + 0.213 + 0.215] + \\ [0.213 + 0.215 + 0.217] - \\ [0.212 + 0.214 + 0.216] \rangle \end{array} \right\}}{27} = 0.0147$, $S(C_3) = \frac{\left\{ \begin{array}{c} \langle [0.004,\ 0.005,\ 0.006], \\ [0.102,\ 0.104,\ 0.106], \\ [0.104,\ 0.106,\ 0.108], \\ [0.103,\ 0.105,\ 0.107] \rangle \end{array} \right\}}{27} = \frac{0.015(0.63-0.315)}{27} = 0.0001$.

See Figure 3.

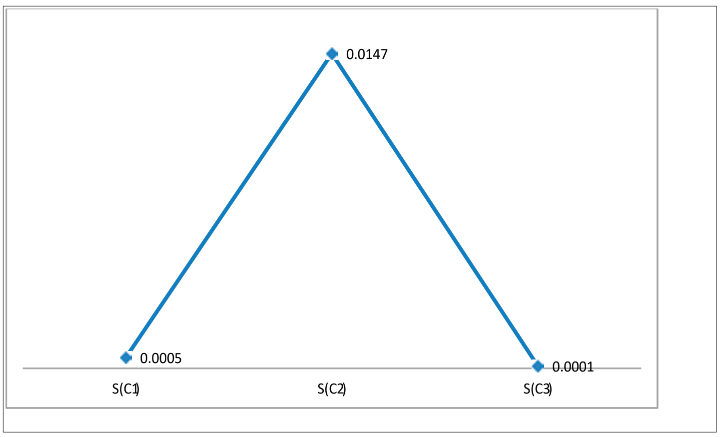


Figure 3. S(C2) is the first score value, S(C1) is the second score value and S(C3) is the third score value.

Example 4. *Let* $C_1 = \left\{ \begin{array}{l} \langle [0.04, 0.05, 0.06], \\ [0.01, 0.03, 0.05], \\ [0.03, 0.05, 0.07], \\ [0.02, 0.04, 0.06] \rangle \end{array} \right\}$, $C_2 = \left\{ \begin{array}{l} \langle [0.05, 0.07, 0.09], \\ [0.11, 0.13, 0.15], \\ [0.13, 0.15, 0.17], \\ [0.12, 0.14, 0.16] \rangle \end{array} \right\}$ *and* $C_3 = \left\{ \begin{array}{l} \langle [0.2004, 0.2005, 0.2006], \\ [0.2102, 0.2104, 0.2106], \\ [0.2104, 0.2106, 0.2108], \\ [0.2103, 0.2105, 0.2107] \rangle \end{array} \right\}$ *be a TNCFN. Then the accuracy function is defined by* $H(C_1) = \dfrac{\left\{ \begin{array}{l} \langle [0.04 + 0.05 + 0.06], \\ [0.01 + 0.03 + 0.05] + \\ [0.03 + 0.05 + 0.07] + \\ [0.02 + 0.04 + 0.06] \rangle \end{array} \right\}}{27} = 0.002$, $H(C_2) = \dfrac{\left\{ \begin{array}{l} \langle [0 + 05 + 0.07 + 0.09], \\ [0.11 + 0.13 + 0.15] + \\ [0.13 + 0.15 + 0.17] + \\ [0.12 + 0.14 + 0.16] \rangle \end{array} \right\}}{27} = 0.0098$, $H(C_3) = \dfrac{\left\{ \begin{array}{l} \langle [0.2004 + 0.2005 + 0.2006], \\ [0.2102 + 0.2104 + 0.2106] + \\ [0.2104 + 0.2106 + 0.2108] + \\ [0.2103 + 0.2105 + 0.2107] \rangle \end{array} \right\}}{27} = 0.0422.$

See Figure 4.

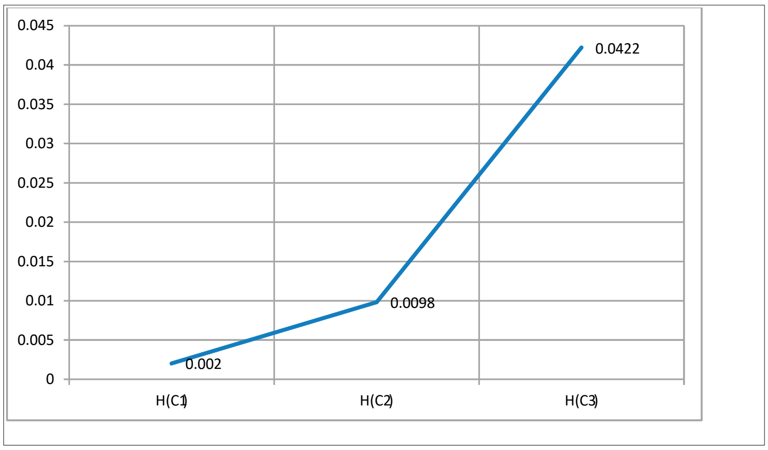


Figure 4. H(C2) is the first score value, H(C1) is the second score value and H(C3) is the third score value.

Proposition 2. *Let* $\ddot{A} = \left\{ \begin{array}{c} [\alpha_1(h), \beta_1(h), \Delta_1(h)] \\ \langle [p_1^-(h), q_1^-(h), r_1^-(h)], \\ [p_1^+(h), q_1^+(h), r_1^+(h)], \\ [p_1(h), q_1(h), r_1(h)] \rangle \\ |h \in H \end{array} \right\}$ *be a collection of TNCFNs in* L_{TNCFN} *and where* $\ddot{\omega} = (\ddot{\omega}_1, \ddot{\omega}_2, \ldots, \ddot{\omega}_n)^T$ *is the weight vector of* $\ddot{A}_j (j = 1, 2, \ldots, n)$ *with* $\ddot{\omega}_j \in [0, 1]$ *and* $\sum_{j=1}^{n} \ddot{\omega}_j = 1$.

Then (1) (Idempotency): If all A_j, $j = 1, 2, \ldots, n$ *are equal, i.e.,* $A_j = A$, *for all* $j = 1, 2, \ldots, n$, *then* $TNCFEWA\ (A_1, A_2, \ldots, A_n) = A$.

(2) (Boundary): If $\alpha_{\min} = \min_{1 \le j \le n} \alpha_j$, $\beta_{\min} = \min_{1 \le j \le n} \beta_j$, $\Delta_{\min} = \min_{1 \le j \le n} \Delta_j$, $p^-_{\min} = \min_{1 \le j \le n} p^-_j$, $q^-_{\min} = \min_{1 \le j \le n} q^-_j$, $r^-_{\min} = \min_{1 \le j \le n} r^-_j$, $p^+_{\min} = \min_{1 \le j \le n} p^+_j$, $q^+_{\min} = \min_{1 \le j \le n} q^+_j$, $r^+_{\min} = \min_{1 \le j \le n} r^+_j$, $p_{\max} = \max_{1 \le j \le n} p_j$, $q_{\max} = \max_{1 \le j \le n} q_j$, $r_{\max} = \max_{1 \le j \le n} r_j$, $\mu_{\max} = \max_{1 \le j \le n} \mu_j$, $p^-_{\max} = \max_{1 \le j \le n} p^-_j$, $q^-_{\max} = \max_{1 \le j \le n} q^-_j$, $r^-_{\max} = \max_{1 \le j \le n} r^-_j$, $\mu^-_{\max} = \max_{1 \le j \le n} \mu^-_j$, $p^+_{\max} = \max_{1 \le j \le n} p^+_j$, $q^+_{\max} = \max_{1 \le j \le n} q^+_j$, $r^+_{\max} = \max_{1 \le j \le n} r^+_j$, $p_{\min} = \min_{1 \le j \le n} p_j$,

$q_{\min} = \min_{1 \le j \le n} q_j$, $r_{\min} = \min_{1 \le j \le n} r_j$, $\mu_{\min} = \min_{1 \le j \le n} \mu_j$ *for all* $j = 1, 2, .., n$, *we can determine that*

$$\left\{\begin{array}{c} [\alpha_{\min}(h), \\ \beta_{\min}(h), \Delta_{\min}(h)] \\ \langle [p^-_{\min}(h), \\ q^-_{\min}(h), r^-_{\min}(h)] \\ [p^+_{\min}(h), \\ q^+_{\min}(h), r^+_{\min}(h)] \\ [p_{\max}(h), q_{\max}(h), \\ r_{\max}(h), s_{\max}(h)] \rangle | \\ h \in H \end{array}\right\} \le TNCFEWA(A_1, A_2, \ldots, A_n) \le \left\{\begin{array}{c} [\alpha_{\max}(h), \beta_{\max}(h), \\ \Delta_{\max}(h)], \langle [p^-_{\max}(h), \\ q^-_{\max}(h), r^-_{\max}(h)], \\ [p^+_{\max}(h), q^+_{\max}(h), \\ r^+_{\max}(h)], [p_{\min}(h), \\ q_{\min}(h), r_{\min}(h)] \rangle | \\ h \in H \end{array}\right\}$$

(3) (Monotonicity): $A = \left\{\begin{array}{c} [\alpha_A(h), \\ \beta_A(h), \Delta_A(h)], \\ \langle [\{p^-_A(h), \\ q^-_A(h), r^-_A(h)] \\ , [p^+_A(h), \\ q^+_A(h), r^+_A(h)] \\ , [p_A(h), \\ q_A(h), r_A(h)] \rangle | \\ h \in H \end{array}\right\}$ *and* $B = \left\{\begin{array}{c} [\alpha_B(h), \\ \beta_B(h), \Delta_B(h)], \\ \langle [p^-_B(h), \\ q^-_B(h), r^-_B(h)] \\ [p^+_B(h), \\ q^+_B(h), r^+_B(h)] \\ [p_B(h), \\ q_B(h), r_B(h) \rangle \\ | h \in H \end{array}\right\}$ *be two collection of TNCFNs in* L_{TNCFN} *and* $A_j \le L_{\text{TNCFN}} B_j$ *i.e.,* $\alpha_A(h) \le \alpha_B(h)$, $\beta_A(h) \le \beta_B(h)$, $\Delta_A(h) \le \Delta_B(h)$, $p^-_A(h) \le p^-_B(h)$, $q^-_A(h) \le q^-_B(h)$, $r^-_A(h) \le r^-_B(h)$, $p^+_A(h) \le p^+_B(h)$, $q^+_A(h) \le q^+_B(h)$, $r^+_A(h) \le r^+_B(h)$ *and* $p_A(h) \le p_B(h)$, $q_A(h) \le q_B(h)$, $r_A(h) \le r_B(h)$ *then TNCFEWA* $(A_1, A_2, \ldots, A_n) \le$ *TNCFEWA* $(B_1, B_2, \ldots, B_n)$.

4.2. Triangular Neutrosophic Cubic Fuzzy Einstein Ordered Weighted Averaging Operator

Definition 8. *Let* $\ddot{A} = \left\{\begin{array}{c} [\alpha_{\ddot{A}}(h), \beta_{\ddot{A}}(h), \\ \Delta_{\ddot{A}}(h)], \Big\langle [p^-_{\ddot{A}}(h), \\ q^-_{\ddot{A}}(h), r^-_{\ddot{A}}(h)], \\ [p^+_{\ddot{A}}(h), q^-_{\ddot{A}}(h), \\ r^-_{\ddot{A}}(h)], p_{\ddot{A}}(h), \\ q_{\ddot{A}}(h), r_{\ddot{A}}(h) \Big\rangle \\ | h \in H \end{array}\right\}$ *be a collection of TNCFNs in* L_{TNCFN}, *a TNCFEOWA operator of dimension n is a mapping TNCFEOWA:* $L^n_{\text{TNCFN}} \to L_{\text{TNCFN}}$, *that has an associated vector* $w = (w_1, w_2, \ldots, w_n)^T$ *such that* $w_j \in [0, 1]$ *and* $\sum_{j=1}^{n} w_j = 1$. *TNCFEOWA* $(\ddot{A}_1, \ddot{A}_2, \ldots, \ddot{A}_n) = \eth_1 \ddot{A}_{(\sigma)1} + \eth_2 \ddot{A}_{(\sigma)2}, \ldots, + \eth_n \ddot{A}_{(\sigma)n}$, *where* $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ *is a permutation of* $(1, 2, \ldots, n)$ *such that* $\ddot{A}_{\sigma(1)} \le \ddot{A}_{\sigma(j-1)}$ *for all* $j = 2, 3, \ldots, n$ *(i.e.,* $\ddot{A}_{\sigma(j)}$ *is the j the largest value in the collection* $(\ddot{A}_1, \ddot{A}_2, \ldots, \ddot{A}_n)$. *If* $w = (w_1, w_2, \ldots, w_n)^T = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$. *Then the TNCFEOWA operator is reduced to the TCFA operator (2) of dimension n.*

Theorem 2. *Let* $\ddot{A} = \left\{ \begin{array}{c} [\alpha_{\ddot{A}}(h), \beta_{\ddot{A}}(h), \\ \Delta_{\ddot{A}}(h)], \Big\langle [p^-_{\ddot{A}}(h), \\ q^-_{\ddot{A}}(h), r^-_{\ddot{A}}(h)], \\ [p^+_{\ddot{A}}(h), q^-_{\ddot{A}}(h), \\ r^-_{\ddot{A}}(h)], p_{\ddot{A}}(h), \\ q_{\ddot{A}}(h), r_{\ddot{A}}(h) \Big\rangle \\ |h \in H \end{array} \right\}$ *be a collection of TNCFNs in* L_{TNCFN}*. Then their aggregated value by using the TNCFEOWA operator is also a TNCFN and TNCFEOWA*

$$(\ddot{A}_1, \ddot{A}_2, \ldots, \ddot{A}_n) = \Bigg\langle \left[\begin{array}{c} \frac{\prod_{j=1}^{n}[1+\alpha_1(h)]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-\alpha_1(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+\alpha_1(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-\alpha_1(h)]^{\ddot{\omega}}}, \\ \frac{\prod_{j=1}^{n}[1+\beta_1(h)]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-\beta_1(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+\beta_1(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-\beta_1(h)]^{\ddot{\omega}}}, \\ \frac{\prod_{j=1}^{n}[1+\Delta_1(h)]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-\Delta_1(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+\Delta_1(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-\Delta_1(h)]^{\ddot{\omega}}}, \end{array} \right], \left[\begin{array}{c} \frac{\prod_{j=1}^{n}[1+p^-_{\sigma(j)}(h)]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-p^-_{\sigma(j)}(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+p^-_{\sigma(j)}(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-p^-_{\sigma(j)}(h)]^{\ddot{\omega}}}, \\ \frac{\prod_{j=1}^{n}[1+q^-_{\sigma(j)}(h)]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-q^-_{\sigma(j)}(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+q^-_{\sigma(j)}(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-q^-_{\sigma(j)}(h)]^{\ddot{\omega}}}, \\ \frac{\prod_{j=1}^{n}[1+r^-_{\sigma(j)}(h)]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-r^-_{\sigma(j)}(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+r^-_{\sigma(j)}(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-r^-_{\sigma(j)}(h)]^{\ddot{\omega}}} \end{array} \right],$$

$$\left[\begin{array}{c} \frac{\prod_{j=1}^{n}[1+p^+_{\sigma(j)}(h)]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-p^+_{\sigma(j)}(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+p^+_{\sigma(j)}(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-p^+_{\sigma(j)}(h)]^{\ddot{\omega}}}, \frac{\prod_{j=1}^{n}[1+q^+_{\sigma(j)}(h)]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-q^+_{\sigma(j)}(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+q^+_{\sigma(j)}(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-q^+_{\sigma(j)}(h)]^{\ddot{\omega}}}, \\ \frac{\prod_{j=1}^{n}[1+r^+_{\sigma(j)}(h)]^{\ddot{\omega}} - \prod_{j=1}^{n}[1-r^+_{\sigma(j)}(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[1+r^+_{\sigma(j)}(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[1-r^+_{\sigma(j)}(h)]^{\ddot{\omega}}} \end{array} \right],$$

$$\left[\begin{array}{c} \frac{2\prod_{j=1}^{n}[p_{\sigma(j)}(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[(2-p_{\sigma(j)}(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[p_{\sigma(j)}(h)]^{\ddot{\omega}}}, \frac{2\prod_{j=1}^{n}[q_{\sigma(j)}(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[(2-q_{\sigma(j)}(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[q_{\sigma(j)}(h)]^{\ddot{\omega}}}, \\ \frac{2\prod_{j=1}^{n}[r_{\sigma(j)}(h)]^{\ddot{\omega}}}{\prod_{j=1}^{n}[(2-r_{\sigma(j)}(h)]^{\ddot{\omega}} + \prod_{j=1}^{n}[r_{\sigma(j)}(h)]^{\ddot{\omega}}} \end{array} \right] \Bigg\rangle$$

where $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ *is a permutation of* $(1, 2, \ldots, n)$ *with* $\ddot{A}_{\sigma(1)} \leq \ddot{A}_{\sigma(j-1)}$ *for all* $j = 2, 3, \ldots, n$, $\ddot{\omega} = (\ddot{\omega}_1, \ddot{\omega}_2, \ldots, \ddot{\omega}_n)^T$ *is the weight vector of* $\ddot{A}_j (j = 1, 2, \ldots, n)$ *such that* $\ddot{\omega}_j \in [0, 1]$, *and* $\sum_{j=1}^{n} \ddot{\omega}_j = 1$. *If* $\ddot{\omega} = (\ddot{\omega}_1, \ddot{\omega}_2, \ldots, \ddot{\omega}_n)^T = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})^T$. *Then the TNCFEOWA operator is reduced to the TNCFA operator of dimension n. Where* $\ddot{\omega} = (\ddot{\omega}_1, \ddot{\omega}_2, \ldots, \ddot{\omega}_n)^T$ *is the weight vector of* $\ddot{A}_j (j = 1, 2, \ldots, n)$ *such that* $\ddot{\omega}_j \in [0, 1]$ *and* $\sum_{j=1}^{n} \ddot{\omega}_j = 1$. *If* $\alpha_1(h) = \alpha_1(h)$, $\beta_1(h) = \beta_1(h)$, $\Delta_1(h) = \Delta_1(h)$, $p^-_{\ddot{A}}(h) = p^-_{\ddot{A}}(h)$, $q^-_{\ddot{A}}(h) = q^-_{\ddot{A}}(h)$, $r^-_{\ddot{A}}(h) = r^-_{\ddot{A}}(h)$, $p^+_{\ddot{A}}(h) = p^+_{\ddot{A}}(h)$, $q^+_{\ddot{A}}(h) = q^+_{\ddot{A}}(h)$, $r^+_{\ddot{A}}(h) = r^+_{\ddot{A}}(h)$ *and* $p_{\ddot{A}}(h) = p_{\ddot{A}}(h)$,

$q_{\ddot{A}}(h) = q_{\ddot{A}}(h)$, $r_{\ddot{A}}(h) = r_{\ddot{A}}(h)$. *The TNCFN* $\ddot{A} = \left\{ \begin{array}{c} [\alpha_{\ddot{A}}(h), \beta_{\ddot{A}}(h), \\ \Delta_{\ddot{A}}(h)] \Big\langle [p^{-}_{\ddot{A}}(h), \\ q^{-}_{\ddot{A}}(h), r^{-}_{\ddot{A}}(h)], \\ [p^{+}_{\ddot{A}}(h), q^{-}_{\ddot{A}}(h), \\ r^{-}_{\ddot{A}}(h)], p_{\ddot{A}}(h), \\ q_{\ddot{A}}(h), r_{\ddot{A}}(h) \Big\rangle \\ |h \in H \end{array} \right\}$ *are reduced to the triangular neutrosophic cubic fuzzy numbers* $\ddot{A} = \left\{ \begin{array}{c} [\alpha_{\ddot{A}}(h), \beta_{\ddot{A}}(h), \\ \Delta_{\ddot{A}}(h)], \Big\langle [p^{-}_{\ddot{A}}(h), \\ q^{-}_{\ddot{A}}(h), r^{-}_{\ddot{A}}(h)], \\ [p^{+}_{\ddot{A}}(h), q^{-}_{\ddot{A}}(h), \\ r^{-}_{\ddot{A}}(h)], p_{\ddot{A}}(h), \\ q_{\ddot{A}}(h), r_{\ddot{A}}(h) \Big\rangle \\ |h \in H \end{array} \right\}$. *Then the TNCFEWA operator is reduced to the triangular neutrosophic cubic fuzzy Einstein ordered weighted averaging operator.*

Proof. The process of this proof is the same as Theorem 1. □

Example 5. *Let* $C_1 = \left\{ \begin{array}{c} \langle [0.01, 0.02, 0.03], \\ [0.103, 0.105, 0.107], \\ [0.105, 0.107, 0.109], \\ [0.104, 0.106, 0.108] \rangle \end{array} \right\}$, $C_2 = \left\{ \begin{array}{c} \langle [0.306, 0.308, 0.310], \\ [0.310, 0.313, 0.315], \\ [0.313, 0.315, 0.317], \\ [0.312, 0.314, 0.316] \rangle \end{array} \right\}$ *and* $C_3 = \left\{ \begin{array}{c} \langle [0.44, 0.55, 0.66], \\ [0.122, 0.124, 0.126], \\ [0.124, 0.126, 0.128], \\ [0.123, 0.125, 0.127] \rangle \end{array} \right\}$ *be a TNCFN. Then the score function is defined by* $S(C_1) = \frac{\left\{ \begin{array}{c} \langle [0.01 + 0.02 + 0.03], \\ [0.103 + 0.105 + 0.107]+ \\ [0.105 + 0.107 + 0.109]- \\ [0.104 + 0.106 + 0.108] \rangle \end{array} \right\}}{27} = 0.00004$, $S(C_2) = \frac{\left\{ \begin{array}{c} \langle [0.306 + 0.308 + 0.310], \\ [0.310 + 0.313 + 0.315]+ \\ [0.313 + 0.315 + 0.317]- \\ [0.312 + 0.314 + 0.316] \rangle \end{array} \right\}}{27} = 0.0322$, $S(C_3) = \frac{\left\{ \begin{array}{c} \langle [0.44 + 0.55 + 0.66], \\ [0.122 + 0.124 + 0.126]+ \\ [0.124 + 0.126 + 0.128]- \\ [0.123 + 0.125 + 0.127] \rangle \end{array} \right\}}{27} = 0.0229$.

See Figure 5.

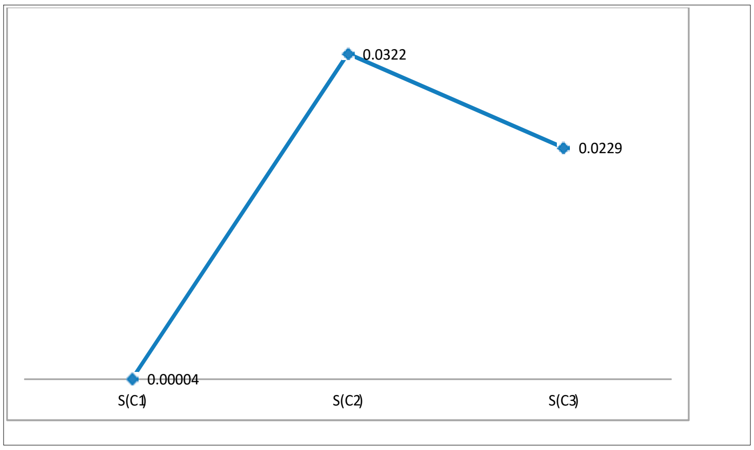


Figure 7. Different score ranking of TNCFEHWA operator.

Example 8. *Let* $C_1 = \left\{ \begin{array}{c} \langle [0.112, 0.113, 0.114], \\ [0.21, 0.24, 0.28], \\ [0.24, 0.28, 0.32], \\ [0.23, 0.26, 0.30] \rangle \end{array} \right\}$, $C_2 = \left\{ \begin{array}{c} \langle [0.0019, 0.0021, 0.0034], \\ [0.1231, 0.1233, 0.1235], \\ [0.1233, 0.1235, 0.1237], \\ [0.1232, 0.1234, 0.1236] \rangle \end{array} \right\}$ *and* $C_3 = \left\{ \begin{array}{c} \langle [0.2554, 0.2555, 0.2556], \\ [0.2662, 0.2664, 0.2666], \\ [0.2664, 0.2666, 0.2668], \\ [0.2663, 0.2665, 0.2667] \rangle \end{array} \right\}$ *be a TNCFN. Then the accuracy function is defined by* $H(C_1) = \frac{\left\{ \begin{array}{c} \langle [0.112 + 0.113 + 0.114], \\ [0.21 + 0.24 + 0.28] + \\ [0.24 + 0.28 + 0.32] - \\ [0.23 + 0.26 + 0.30] \rangle \end{array} \right\}}{27} = 0.0111$, $H(C_2) = \frac{\left\{ \begin{array}{c} \langle [0.0019 + 0.0021 + 0.0034] \\ [0.1231 + 0.1233 + 0.1235] + \\ [0.1233 + 0.1235 + 0.1237] - \\ [0.1232 + 0.1234 + 0.1236] \rangle \end{array} \right\}}{27} = 0.1124$, $H(C_3) = \frac{\left\{ \begin{array}{c} \langle [0.2554, 0.2555, 0.2556], \\ [0.2662, 0.2664, 0.2666], \\ [0.2664, 0.2666, 0.2668], \\ [0.2663, 0.2665, 0.2667] \rangle \end{array} \right\}}{27} = 0.0226$.

See Figure 8.

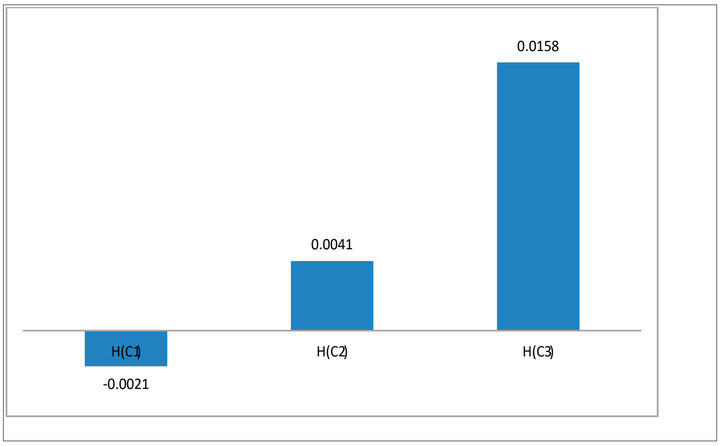


Figure 6. Different accuracy ranking of TNCFEOWA operator.

4.3. Triangular Neutrosophic Cubic Fuzzy Einstein Hybrid Weighted Averaging Operator

Definition 9. *Let* $\ddot{A} = \left\{ \begin{array}{c} [\Gamma_{\ddot{A}}(h), \Omega_{\ddot{A}}(h), \\ _{\ddot{A}}(h)], \left\langle [p^{-}_{\ddot{A}}(h), \right. \\ q^{-}_{\ddot{A}}(h), r^{-}_{\ddot{A}}(h)], \\ [p^{+}_{\ddot{A}}(h), q^{-}_{\ddot{A}}(h), \\ r^{-}_{\ddot{A}}(h)], p_{\ddot{A}}(h), \\ \left. q_{\ddot{A}}(h), r_{\ddot{A}}(h) \right\rangle \\ |h \in H \end{array} \right\}$ *be a collection of TNCFNs in* L_{TNCFN} *and* $\ddot{\omega} = (\ddot{\omega}_1, \ddot{\omega}_2, \ldots, \ddot{\omega}_n)^T$ *is the weight vector of* $\ddot{A}_j (j = 1, 2, \ldots, n)$ *such that* $\ddot{\omega}_j \in [0, 1]$ *and* $\sum_{j=1}^{n} \ddot{\omega}_j = 1$. *Then TNCFEHWA operator of dimension n is a mapping* $TNCFEHWA : L^{n}_{\text{TNCFN}} \to L_{\text{TNCFN}}$, *that is an associated vector* $w = (w_1, w_2, \ldots, w_n)^T$ *such that* $w_j \in [0, 1]$ *and* $\sum_{j=1}^{n} w_j = 1$. *TNCFEHWA* $(\ddot{A}_1, \ddot{A}_2, \ldots, \ddot{A}_n) = p_1 \ddot{A}_{\sigma(1)} + p_2 \ddot{A}_{\sigma(1)}, \ldots, p_n \ddot{A}_{\sigma(1)}$. *If* $p = \theta \ddot{\omega}_{\sigma(j)} + (1 - \theta) w_{\sigma(j)}$ *with a balancing coefficient* $\theta \in [0, 1]$, $(\sigma(1), \sigma(2), \ldots, \sigma(n))$ *is a permutation of* $(1, 2, \ldots, n)$ *such that* $\ddot{A}_{\sigma(j)} \leq \ddot{A}_{\sigma(j-1)}$ *for all* $j = 2, 3, \ldots, n$ *(i.e.,* $\ddot{A}_{\sigma(j)}$ *is the j th largest value in the collection* $(\ddot{A}_1, \ddot{A}_2, \ldots, \ddot{A}_n)$*).*

Example 7. *Let* $C_1 = \left\{ \begin{array}{c} \langle [0.11, 0.13, 0.14], \\ [0.62, 0.64, 0.66], \\ [0.64, 0.66, 0.68], \\ [0.63, 0.65, 0.67] \rangle \end{array} \right\}$, $C_2 = \left\{ \begin{array}{c} \langle [0.51, 0.52, 0.53], \\ [0.311, 0.313, 0.315], \\ [0.313, 0.315, 0.317], \\ [0.312, 0.314, 0.316] \rangle \end{array} \right\}$ *and* $C_3 = \left\{ \begin{array}{c} \langle [0.1004, 0.1005, 0.1006], \\ [0.3102, 0.3104, 0.3106], \\ [0.3104, 0.3106, 0.3108], \\ [0.3103, 0.3105, 0.3107] \rangle \end{array} \right\}$ *be a TNCFN. Then the score function is defined by* $S(C_1) = \dfrac{\left\{ \begin{array}{c} \langle [0.11 + 0.13 + 0.14], \\ [0.62 + 0.64 + 0.66]+ \\ [0.64 + 0.66 + 0.68]- \\ [0.63 + 0.65 + 0.67] \rangle \end{array} \right\}}{27} = 0.0281$, $S(C_2) = \dfrac{\left\{ \begin{array}{c} \langle [0.51 + 0.52 + 0.53], \\ [0.311 + 0.313 + 0.315]+ \\ [0.313 + 0.315 + 0.317]- \\ [0.312 + 0.314 + 0.316] \rangle \end{array} \right\}}{27} = 0.0543$, $S(C_3) = \dfrac{\left\{ \begin{array}{c} \langle [0.1004, 0.1005, 0.1006], \\ [0.3102, 0.3104, 0.3106], \\ [0.3104, 0.3106, 0.3108], \\ [0.3103, 0.3105, 0.3107] \rangle \end{array} \right\}}{27} = 0.0104$.

See Figure 7.

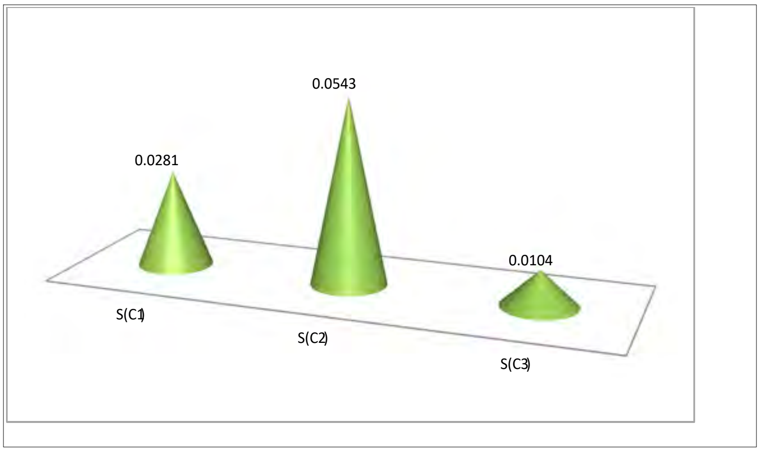


Figure 7. Different score ranking of TNCFEHWA operator.

Example 8. *Let* $C_1 = \left\{ \begin{array}{c} \langle [0.112, 0.113, 0.114], \\ [0.21, 0.24, 0.28], \\ [0.24, 0.28, 0.32], \\ [0.23, 0.26, 0.30] \rangle \end{array} \right\}$, $C_2 = \left\{ \begin{array}{c} \langle [0.0019, 0.0021, 0.0034], \\ [0.1231, 0.1233, 0.1235], \\ [0.1233, 0.1235, 0.1237], \\ [0.1232, 0.1234, 0.1236] \rangle \end{array} \right\}$ *and* $C_3 = \left\{ \begin{array}{c} \langle [0.2554, 0.2555, 0.2556], \\ [0.2662, 0.2664, 0.2666], \\ [0.2664, 0.2666, 0.2668], \\ [0.2663, 0.2665, 0.2667] \rangle \end{array} \right\}$ *be a TNCFN. Then the accuracy function is defined by* $H(C_1) = \dfrac{\left\{ \begin{array}{c} \langle [0.112 + 0.113 + 0.114], \\ [0.21 + 0.24 + 0.28]+ \\ [0.24 + 0.28 + 0.32]- \\ [0.23 + 0.26 + 0.30] \rangle \end{array} \right\}}{27} = 0.0111$, $H(C_2) = \dfrac{\left\{ \begin{array}{c} \langle [0.0019 + 0.0021 + 0.0034] \\ [0.1231 + 0.1233 + 0.1235]+ \\ [0.1233 + 0.1235 + 0.1237]- \\ [0.1232 + 0.1234 + 0.1236] \rangle \end{array} \right\}}{27} = 0.1124$, $H(C_3) = \dfrac{\left\{ \begin{array}{c} \langle [0.2554, 0.2555, 0.2556], \\ [0.2662, 0.2664, 0.2666], \\ [0.2664, 0.2666, 0.2668], \\ [0.2663, 0.2665, 0.2667] \rangle \end{array} \right\}}{27} = 0.0226.$

See Figure 8.

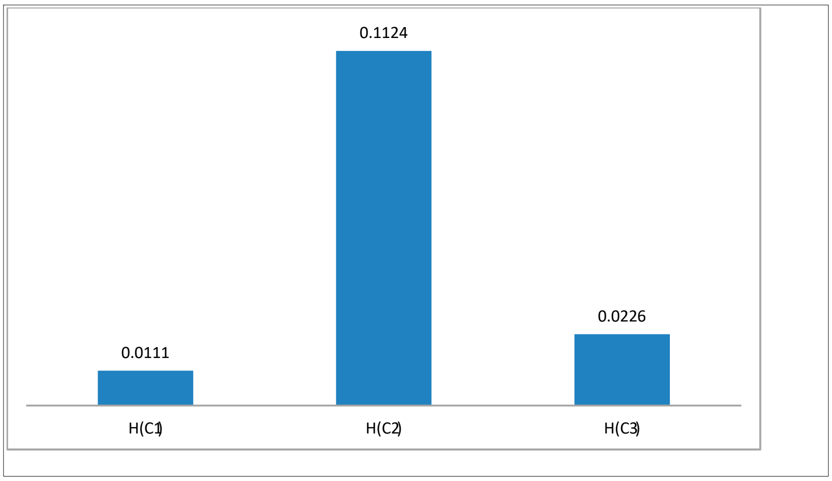


Figure 8. Different accuracy ranking of TNCFEHWA operator.

5. An Approach to MADM with TNCF Data

Let us suppose the discrete set is $h = \{h_1, h_2, \ldots, h_n\}$ and $G = \{g_1, g_2, \ldots, g_n\}$ are the attributes. Consider that the value of alternatives h_i $(i = 1, 2, \ldots, n)$ on attributes g_j $(j = 1, 2, \ldots, m)$ given by decision maker are TNCFNs in L_{TNCFN}: $\ddot{A} = \left\{ \begin{array}{c} [\Gamma_{\ddot{A}}(h), \Omega_{\ddot{A}}(h), \\ {}_{\ddot{A}}(h)], \left\langle [p^-_{\ddot{A}}(h), \right. \\ q^-_{\ddot{A}}(h), r^-_{\ddot{A}}(h)], \\ [p^+_{\ddot{A}}(h), q^+_{\ddot{A}}(h), \\ r^+_{\ddot{A}}(h)], p_{\ddot{A}}(h), \\ \left. q_{\ddot{A}}(h), r_{\ddot{A}}(h) \right\rangle \\ |h \in H \end{array} \right\}$, a MADM problem is expressed in the TNCF-decision matrix $\ddot{D} = (\ddot{A}_{ij})_{m \times n} = \left\{ \begin{array}{c} [\Gamma_{\ddot{A}}(h), \Omega_{\ddot{A}}(h), {}_{\ddot{A}}(h)], \\ \left\langle [p^-_{\ddot{A}}(h), q^-_{\ddot{A}}(h), r^-_{\ddot{A}}(h)], \right. \\ [p^+_{\ddot{A}}(h), q^+_{\ddot{A}}(h), r^+_{\ddot{A}}(h)], \\ \left. p_{\ddot{A}}(h), q_{\ddot{A}}(h), r_{\ddot{A}}(h) \right\rangle \\ |h \in H \end{array} \right\}$.

Step 1: Calculate the TNCF decision matrix.

Step 2: Utilize the TNCFEWA operator to mix all values $\ddot{\beta}_{ij}$ $(j = 1, 2, \ldots, m)$ and $\ddot{\omega} = (\ddot{\omega}_1, \ddot{\omega}_2, \ldots, \ddot{\omega}_n)^T$ is the weight vector.

Step 3: Calculate the score function.

Step 4: Find the ranking.

See Figure 9.

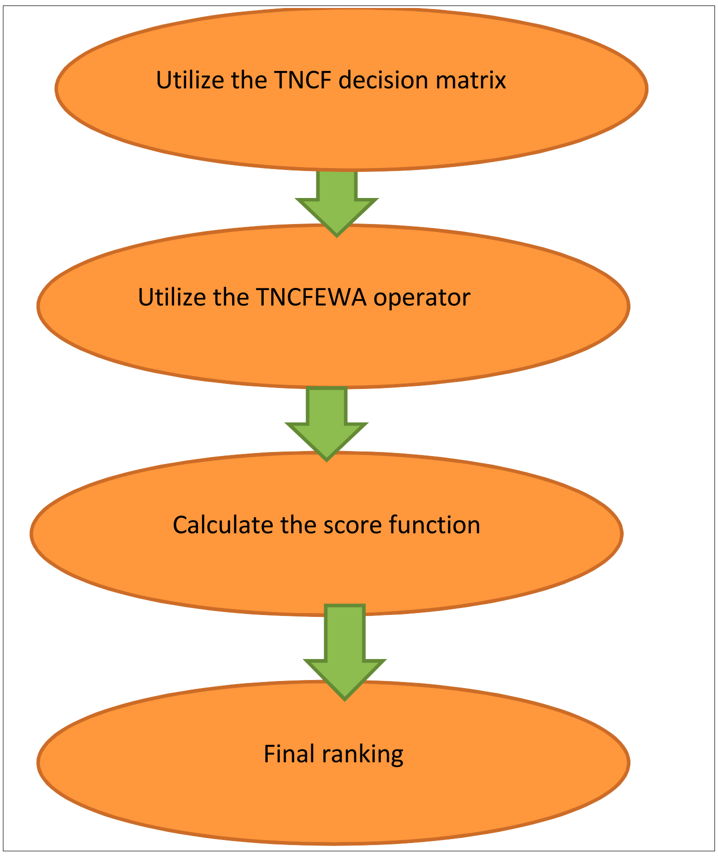


Figure 9. Proposed method.

6. Numerical Application

The inspiration structure is designed to be dependent upon an assessment that has been devised for the purpose of a stimulus/influencing technique of a twofold entire traveler dispersion to work over the Lahore in Faisalabad by lessening the adventure stage in extraordinarily brimful waterway movement. Inspiration structure choices are sure the settled of options $A = \{A_1, A_2, A_3, A_4\}$

A_1 : Old-style propeller and high trundle
A_2 : Get-up-and-go,
A_3 : Cyclonical propeller,
A_4 : Outmoded
See Figure 10.

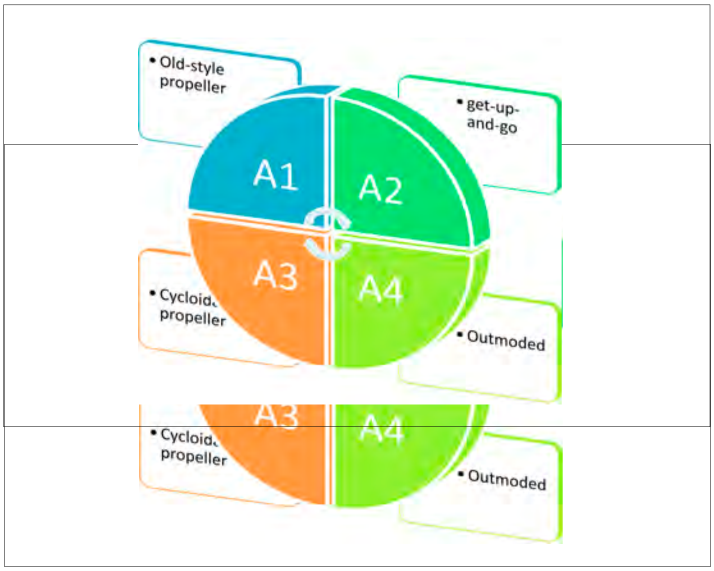


Figure 10. Four alternatives.

The ideal is prepared on the possibility of lone zone and four issue characteristics, which are as follows:
c_1 : Theory rate
c_2 : Reparation and support uses
c_3 : Agility
c_4 : Tremor and unrest.
See Figure 11.

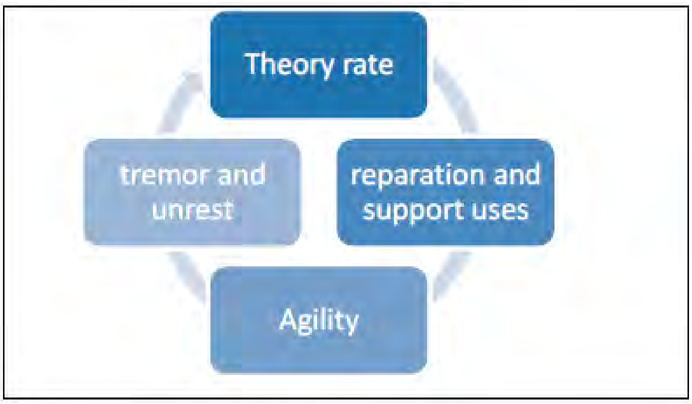


Figure 11. Different criteria.

The weight vector is $\ddot{\omega} = (0.25, 0.50, 0.25)$. So, the triangular neutrosophic cubic fuzzy MADM issue is intended to choose the appropriate energy structure from between 3 choices.

Step 1: Calculate the TNCF decision matrix.
The TNCF decision matrix is as Table 2

Table 2. Triangular Neutrosophic Cubic Fuzzy Decision Matrix.

	c_1	c_2
A_1	$\left\langle \begin{array}{c} [0.1, 0.2, 0.3], \\ [0.4, 0.6, 0.8], \\ [0.6, 0.8, 0.10], \\ [0.5, 0.7, 0.9] \end{array} \right\rangle$	$\left\langle \begin{array}{c} [0.21, 0.22, 0.23], \\ [0.5, 0.7, 0.9], \\ [0.7, 0.9, 0.11], \\ [0.6, 0.8, 0.10] \end{array} \right\rangle$
A_2	$\left\langle \begin{array}{c} [0.21, 0.22, 0.23], \\ [0.5, 0.7, 0.9], \\ [0.7, 0.9, 0.11], \\ [0.6, 0.8, 0.10] \end{array} \right\rangle$	$\left\langle \begin{array}{c} [0.1, 0.2, 0.3], \\ [0.4, 0.6, 0.8], \\ [0.6, 0.8, 0.10], \\ [0.5, 0.7, 0.9] \end{array} \right\rangle$
A_3	$\left\langle \begin{array}{c} [0.15, 0.16, 0.17], \\ [0.12, 0.14, 0.16], \\ [0.14, 0.16, 0.18], \\ [0.13, 0.15, 0.17] \end{array} \right\rangle$	$\left\langle \begin{array}{c} [0.3, 0.4, 0.5], \\ [0.2, 0.4, 0.6], \\ [0.4, 0.6, 0.8], \\ [0.3, 0.5, 0.7] \end{array} \right\rangle$
A_4	$\left\langle \begin{array}{c} [0.3, 0.4, 0.5], \\ [0.2, 0.4, 0.6], \\ [0.4, 0.6, 0.8], \\ [0.3, 0.5, 0.7] \end{array} \right\rangle$	$\left\langle \begin{array}{c} [0.15, 0.16, 0.17], \\ [0.12, 0.14, 0.16], \\ [0.14, 0.16, 0.18], \\ [0.13, 0.15, 0.17] \end{array} \right\rangle$
	c_3	c_4
A_1	$\left\langle \begin{array}{c} [0.15, 0.16, 0.17], \\ [0.12, 0.14, 0.16], \\ [0.14, 0.16, 0.18], \\ [0.13, 0.15, 0.17] \end{array} \right\rangle$	$\left\langle \begin{array}{c} [0.21, 0.22, 0.23], \\ [0.5, 0.7, 0.9], \\ [0.7, 0.9, 0.11], \\ [0.6, 0.8, 0.10] \end{array} \right\rangle$
A_2	$\left\langle \begin{array}{c} [0.3, 0.4, 0.5], \\ [0.4, 0.6, 0.8], \\ [0.6, 0.8, 0.10], \\ [0.5, 0.7, 0.9] \end{array} \right\rangle$	$\left\langle \begin{array}{c} [0.15, 0.16, 0.17], \\ [0.12, 0.14, 0.16], \\ [0.14, 0.16, 0.18], \\ [0.13, 0.15, 0.17] \end{array} \right\rangle$
A_3	$\left\langle \begin{array}{c} [0.1, 0.2, 0.3], \\ [0.4, 0.6, 0.8], \\ [0.6, 0.8, 0.10], \\ [0.5, 0.7, 0.9] \end{array} \right\rangle$	$\left\langle \begin{array}{c} [0.3, 0.4, 0.5], \\ [0.2, 0.4, 0.6], \\ [0.4, 0.6, 0.8], \\ [0.3, 0.5, 0.7] \end{array} \right\rangle$
A_4	$\left\langle \begin{array}{c} [0.21, 0.22, 0.23], \\ [0.5, 0.7, 0.9], \\ [0.7, 0.9, 0.11], \\ [0.6, 0.8, 0.10] \end{array} \right\rangle$	$\left\langle \begin{array}{c} [0.1, 0.2, 0.3], \\ [0.4, 0.6, 0.8], \\ [0.6, 0.8, 0.10], \\ [0.5, 0.7, 0.9] \end{array} \right\rangle$

Step 2: Calculate the TNCFEWA operator to total all the rating values and $w = (0.1,\ 0.2,\ 0.4,\ 0.3)^T$.
The TNCFEWA operator are defined in Table 3.

Table 3. TNCFEWA Operator.

A_1	$\left\{\left\langle \begin{array}{c} [0.2539, 0.2751, 0.2965], [0.1628, 0.2513, 0.3973], \\ [0.2513, 0.3973, 0.0503], [0.7335, 0.8054, 0.6003] \end{array} \right\rangle\right\}$
A_2	$\left\{\left\langle \begin{array}{c} [0.1536, 0.1995, 0.2477], [0.2944, 0.3597, 0.6447], \\ [0.3597, 0.6447, 0.0988], [0.4838, 0.6067, 0.4831] \end{array} \right\rangle\right\}$
A_3	$\left\{\left\langle \begin{array}{c} [0.3481, 0.4499, 0.5594], [0.3626, 0.5867, 0.7852], \\ [0.5867, 0.7852, 0.7582], [0.1049, 0.2122, 0.3571] \end{array} \right\rangle\right\}$
A_4	$\left\{\left\langle \begin{array}{c} [0.2282, 0.2945, 0.3622], [0.3704, 0.5631, 0.7729], \\ [0.5631, 0.7729, 0.4197], [0.2593, 0.3985, 0.2735] \end{array} \right\rangle\right\}$

Step 3: The score value are calculated as $s_1 = -0.0192$, $s_2 = 0.0184$, $s_3 = 0.1603$, $s_4 = 0.0829$.

Step 4: Ranking $\ddot{s}_3 > \ddot{s}_4 > \ddot{s}_2 > \ddot{s}_1$.
See Figure 12.

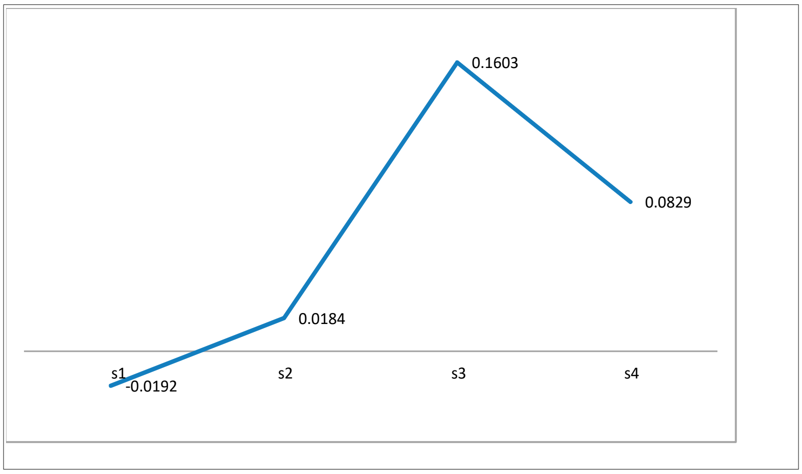


Figure 12. Rating value different range of values.

7. Comparsion Analysis

So as to check the legitimacy and viability of the proposed methodology, a near report is led utilizing the techniques triangular cubic fuzzy number [21], which are unique instances of TNCFNs, to the equivalent illustrative model.

A Comparison Analysis with the Existing MCDM Method Triangular Cubic Fuzzy Number

Aliya et al [21] after transformation, the triangular cubic fuzzy information is given in Table 4.

Table 4. Triangular cubic fuzzy decision matrix.

	c_1	c_2	c_3	c_4
A_1	$\left\{ [0.1, 0.2, 0.3], \langle [0.4, 0.8], 0.7 \rangle \right\}$	$\left\{ [0.21, 0.22, 0.23], \langle [0.5, 0.9], 0.8 \rangle \right\}$	$\left\{ [0.15, 0.16, 0.17], \langle [0.12, 0.16], 0.15 \rangle \right\}$	$\left\{ [0.21, 0.22, 0.23], \langle [0.5, 0.9], 0.8 \rangle \right\}$
A_2	$\left\{ [0.21, 0.22, 0.23], \langle [0.5, 0.9], 0.8 \rangle \right\}$	$\left\{ [0.1, 0.2, 0.3], \langle [0.4, 0.8], 0.7 \rangle \right\}$	$\left\{ [0.3, 0.4, 0.5], \langle [0.2, 0.6], 0.4 \rangle \right\}$	$\left\{ [0.15, 0.16, 0.17], \langle [0.12, 0.16], 0.15 \rangle \right\}$
A_3	$\left\{ [0.15, 0.16, 0.17], \langle [0.12, 0.16], 0.15 \rangle \right\}$	$\left\{ [0.3, 0.4, 0.5], \langle [0.2, 0.6], 0.4 \rangle \right\}$	$\left\{ [0.1, 0.2, 0.3], \langle [0.4, 0.8], 0.7 \rangle \right\}$	$\left\{ [0.3, 0.4, 0.5], \langle [0.2, 0.6], 0.5 \rangle \right\}$
A_4	$\left\{ [0.3, 0.4, 0.5], \langle [0.2, 0.6], 0.4 \rangle \right\}$	$\left\{ [0.15, 0.16, 0.17], \langle [0.12, 0.16], 0.15 \rangle \right\}$	$\left\{ [0.21, 0.22, 0.23], \langle [0.5, 0.9], 0.8 \rangle \right\}$	$\left\{ [0.1, 0.2, 0.3], \langle [0.4, 0.8], 0.7 \rangle \right\}$

Calculate the TCFA operator and $w = (0.1, 0.2, 0.4, 0.3)^T$.
The TCFA operator is presented in Table 5.

Table 5. TCFA operator.

A_1	$\langle [0.067, 0.081, 0.093], [0.1833, 0.4721], 0.1384 \rangle$
A_2	$\langle [0.152, 0.196, 0.24], [0.2672, 0.6329], 0.5073 \rangle$
A_3	$\langle [0.42, 0.464, 0.588], [0.4631, 0.7646], 0.2132 \rangle$
A_4	$\langle [0.228, 0.294, 0.36], [0.3727, 0.7771], 0.3613 \rangle$

Calculate the score function $s_1 = 0.0138$, $s_2 = 0.0256$, $s_3 = 0.1659$, $s_4 = 0.0772$.
See Figure 13.

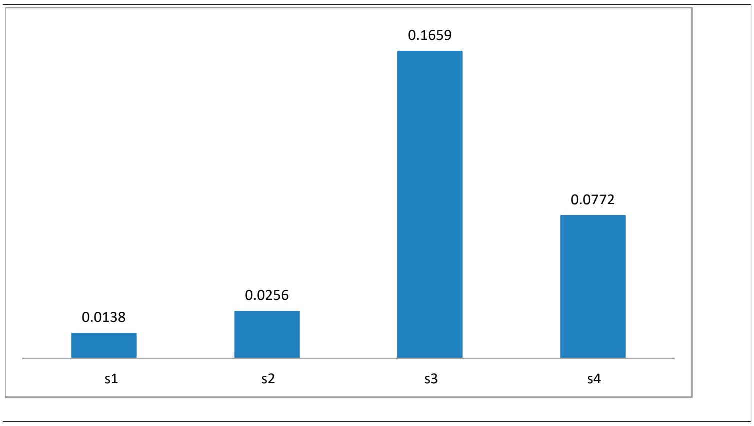


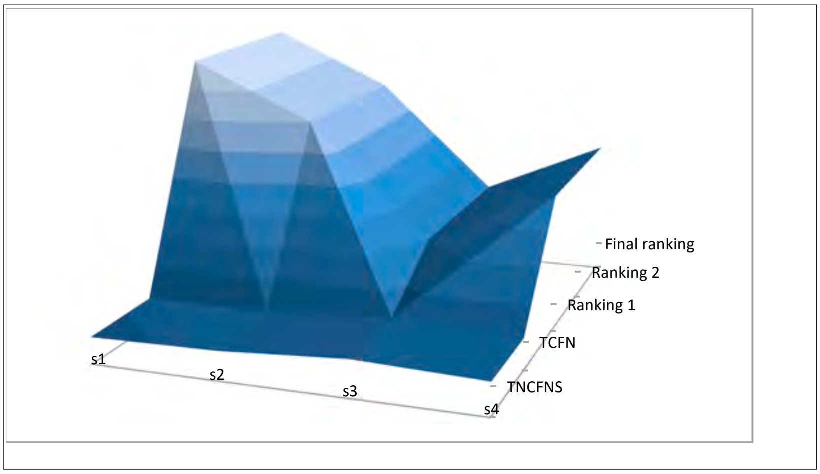
Figure 14. Comparison analysis with existing method.

The comparison method of score function is presented in Table 7.

Table 7. Comparison method with score function.

Score function	Ranking
TNCFEWA operator	$S(C_3) > S(C_2) > S(C_1)$
TNCFEOWA operator	$S(C_1) > S(C_2) > S(C_3)$
TNCFEHWA operator	$S(C_2) > S(C_1) > S(C_3)$

See Figure 15.



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TNCFEHWA operator	$S(C_2) > S(C_1) > S(C_3)$

See Figure 15.

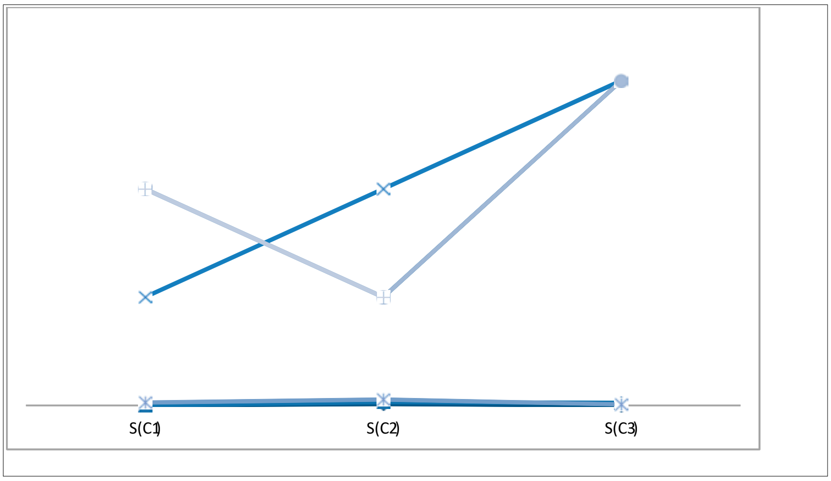

Figure 15. Different score value.

8. Conclusions

In this paper, we introduce a new concept of TNCFNs and operational laws. We introduce three aggregation operators, namely, the TNCFEWA operator, TNCFEOWA operator and TNCFEWA operator. We introduce group decision making under TNCFNs. Finally, a numerical example is provided to demonstrate the utility of the established approach. In cluster decision-making issues, consultants sometimes return from completely different specialty fields and have different backgrounds and levels of data; as such, they sometimes have branching opinions. These operators may be applied to several different fields, like data fusion, data processing, and pattern recognition, triangular neutrosophic cube like linguistic fuzzy Vikor methodology and quadrangle neutrosophic cube linguistic fuzzy Vikor methodology, which may be a suitable topic for longer term analysis, see Figure 16.

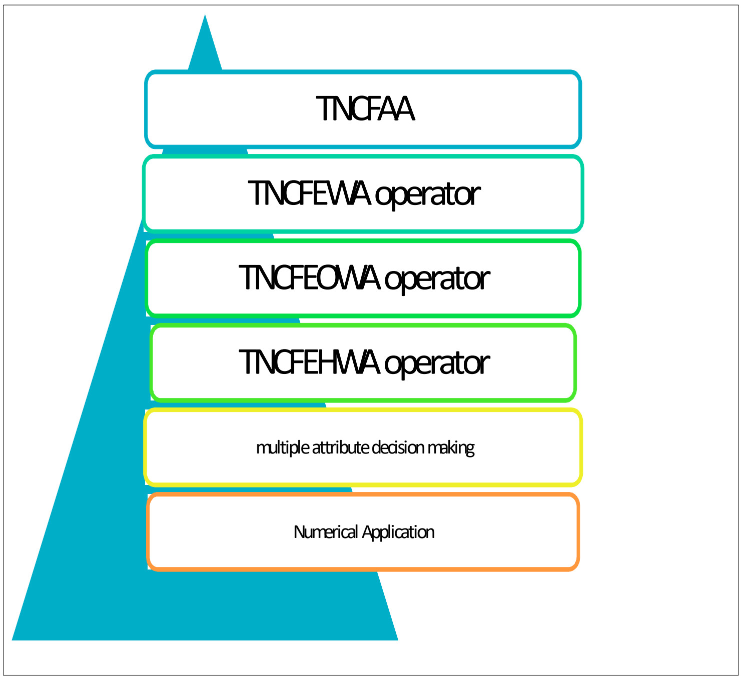

Figure 16. Flowcharts of whole papers.

Appendix A. Proof of Proposition 1

(1) $A_1 + A_2 = A_2 + A_1$;

$$A_1 + A_2 = \left\{ \begin{array}{c} \Big\langle \big[\frac{\alpha_1(h)+\alpha_2(h)}{(1+\alpha_1(h))(1-\alpha_2(h))}, \frac{\beta_1(h)+\beta_2(h)}{(1+\beta_1(h))(1-\beta_2(h))}, \frac{\Delta_1(h)+\Delta_2(h)}{(1+\Delta_1(h))(1-\Delta_2(h))} \big], \\ \left[\begin{array}{c} \big[\frac{p_1^-(h)+p_2^-(h)}{(1+p_1^-(h))(1-p_2^-(h))}, \frac{q_1^-(h)+q_2^-(h)}{(1+q_1^-(h))(1-q_2^-(h))}, \frac{r_1^-(h)+r_2^-(h)}{(1+r_1^-(h))(1-r_2^-(h))} \big], \\ \big[\frac{p_1^+(h)+p_2^+(h)}{(1+p_1^+(h))(1-p_2^+(h))}, \frac{q_1^+(h)+q_2^+(h)}{(1+q_1^+(h))(1-q_2^+(h))}, \frac{r_1^+(h)+r_2^+(h)}{(1+r_1^+(h))(1-r_2^+(h))} \big] \end{array} \right], \\ \Big[\big[\frac{p_1(h)p_2(h)}{1+((1-p_1(h))\,(1-p_2(h)))}, \frac{q_1(h)q_2(h)}{1+((1-q_1(h))\,(1-q_2(h)))}, \frac{r_1(h)r_2(h)}{1+((1-r_1(h))\,(1-r_2(h)))} \big] \Big] \Big\rangle \end{array} \right\} =$$

$$\left\{ \begin{array}{c} \Big\langle \big[\frac{\alpha_2(h)+\alpha_1(h)}{(1+\alpha_2(h))(1-\alpha_1(h))}, \frac{\beta_2(h)+\beta_1(h)}{(1+\beta_2(h))(1-\beta_1(h))}, \frac{\Delta_2(h)+\Delta_1(h)}{(1+\Delta_2(h))(1-\Delta_1(h))} \big], \\ \left[\begin{array}{c} \big[\frac{p_2^-(h)+p_1^-(h)}{(1+p_2^-(h))(1-p_1^-(h))}, \frac{q_2^-(h)+q_1^-(h)}{(1+q_2^-(h))(1-q_1^-(h))}, \frac{r_2^-(h)+r_1^-(h)}{(1+r_2^-(h))(1-r_1^-(h))} \big], \\ \big[\frac{p_2^+(h)+p_1^+(h)}{(1+p_2^+(h))(1-p_1^+(h))}, \frac{q_2^+(h)+q_1^+(h)}{(1+q_2^+(h))(1-q_1^+(h))}, \frac{r_2^+(h)+r_1^+(h)}{(1+r_2^+(h))(1-r_1^+(h))} \big], \end{array} \right], \\ \Big[\big[\frac{p_2(h)p_1(h)}{1+((1-p_2(h))\,(1-p_1(h)))}, \frac{q_2(h)q_1(h)}{1+((1-q_2(h))\,(1-q_1(h)))}, \frac{r_2(h).r_1(h)}{1+((1-r_2(h))\,(1-r_1(h)))} \big] \Big] \Big\rangle \end{array} \right\} = A_2 + A_1$$

Hence $A_1 + A_2 = A_2 + A_1$.
(2) $\lambda(A_1 + A_2) = \lambda A_2 + \lambda A_1$

$$\lambda(A_1 + A_2) = \left\langle \left[\begin{array}{c} \frac{[(1+\alpha_1(h))(1-\alpha_1(h))]^\lambda [((1+\alpha_2(h))(1-\alpha_2(h))]^\lambda}{[(1+\alpha_1(h))(1-\alpha_1(h))]^\lambda [(1+\alpha_2(h))(1-\alpha_2(h))]^\lambda}, \\ \frac{[(1+\beta_1(h))(1-\beta_1(h))]^\lambda [((1+\beta_2(h))(1-\beta_2(h))]^\lambda}{[(1+\beta_1(h))(1-\beta_1(h))]^\lambda [(1+\beta_2(h))(1-\beta_2(h))]^\lambda}, \\ \frac{[(1+\Delta_1(h))(1-\Delta_1(h))]^\lambda [((1+\Delta_2(h))(1-\Delta_2(h))]^\lambda}{[(1+\Delta_1(h))(1-\Delta_1(h))]^\lambda [(1+\Delta_2(h))(1-\Delta_2(h))]^\lambda} \end{array} \right], \left[\begin{array}{c} \big[\frac{[(1+p_1^-(h))(1-p_1^-(h))]^\lambda [(1+p_2^-(h))(1-p_2^-(h))]^\lambda}{[(1+p_1^-(h))(1-p_1^-(h))]^\lambda [(1+p_2^-(h))(1-p_2^-(h))]^\lambda}, \\ \frac{[(1+q_1^-(h))(1-q_1^-(h))]^\lambda [(1+q_2^-(h))(1-q_2^-(h))]^\lambda}{[(1+q_1^-(h))(1-q_1^-(h))]^\lambda [(1+q_2^-(h))(1-q_2^-(h))]^\lambda}, \\ \frac{[(1+r_1^-(h))(1-r_1^-(h))]^\lambda [(1+r_2^-(h))(1-r_2^-(h))]^\lambda}{[(1+r_1^-(h))(1-r_1^-(h))]^\lambda [(1+r_2^-(h))(1-r_2^-(h))]^\lambda} \big] \\ \big[\frac{[(1+p_1^+(h))(1-p_1^+(h))]^\lambda [(1+p_2^+(h))(1-p_2^+(h))]^\lambda}{[(1+p_1^+(h))(1-p_1^+(h))]^\lambda [(1+p_2^+(h))(1-p_2^+(h))]^\lambda}, \\ \frac{[(1+q_1^+(h))(1-q_1^+(h))]^\lambda [(1+q_2^+(h))(1-q_2^+(h))]^\lambda}{[(1+q_1^+(h))(1-q_1^+(h))]^\lambda [(1+q_2^+(h))(1-q_2^+(h))]^\lambda}, \\ \frac{[(1+r_1^+(h))(1-r_1^+(h))]^\lambda [(1+r_2^+(h))(1-r_2^+(h))]^\lambda}{[(1+r_1^+(h))(1-r_1^+(h))]^\lambda [(1+r_2^+(h))(1-r_2^+(h))]^\lambda} \big] \end{array} \right], \\ \left[\begin{array}{c} \big[\frac{2[p_1(h)p_2(h)]^\lambda}{[(4-2p_1(h)-2p_2(h)-p_1(h)p_2(h)]^\lambda + [p_1(h)p_2(h)]^\lambda}, \\ \frac{2[q_1(h)q_2(h)]^\lambda}{[(4-2q_1(h)-2q_2(h)-q_1(h)q_2(h)]^\lambda + [q_1(h)q_2(h)]^\lambda}, \\ \frac{2[r_1(h)r_2(h)]^\lambda}{[(4-2r_1(h)-2r_2(h)-r_1(h)r_2(h)]^\lambda + [r_1(h)r_2(h)]^\lambda} \big] \end{array} \right] \right\rangle$$

and we have

$$\lambda A_1 = \left\langle \left[\begin{array}{c} \frac{[(1+\alpha_1(h))]^\lambda - [(1-\alpha_1(h))]^\lambda}{[(1+\alpha_1(h))]^\lambda + [(1-\alpha_1(h))]^\lambda}, \\ \frac{[(1+\beta_1(h))]^\lambda - [(1-\beta_1(h))]^\lambda}{[(1+\beta_1(h))]^\lambda + [(1-\beta_1(h))]^\lambda}, \\ \frac{[(1+\Delta_1(h))]^\lambda - [(1-\Delta_1(h))]^\lambda}{[(1+\Delta_1(h))]^\lambda + [(1-\Delta_1(h))]^\lambda} \end{array} \right], \left[\begin{array}{c} \big[\frac{[(1+p_1^-(h))^\lambda - (1-p_1^-(h))^\lambda]}{[(1+p_1^-(h))^\lambda + (1-p_1^-(h))^\lambda]}, \\ \frac{[(1+q_1^-(h))^\lambda - (1-q_1^-(h))^\lambda]}{[(1+q_1^-(h))^\lambda + (1-q_1^-(h))^\lambda]}, \frac{[(1+r_1^-(h))^\lambda - (1-r_1^-(h))^\lambda]}{[(1+r_1^-(h))^\lambda + (1-r_1^-(h))^\lambda]} \big] \\ \big[\frac{[(1+p_1^+(h))^\lambda - (1-p_1^+(h))^\lambda]}{[(1+p_1^+(h))^\lambda + (1-p_1^+(h))^\lambda]}, \\ \frac{[(1+q_1^+(h))^\lambda - (1-q_1^+(h))^\lambda]}{[(1+q_1^+(h))^\lambda + (1-q_1^+(h))^\lambda]}, \frac{[(1+r_1^+(h))^\lambda - (1-r_1^+(h))^\lambda]}{[(1+r_1^+(h))^\lambda + (1-r_1^+(h))^\lambda]} \big] \end{array} \right], \\ \left[\begin{array}{c} \big[\frac{2p_1^\lambda(h)}{[(2-p_1(h)]^\lambda + [p_1(h)]^\lambda}, \\ \frac{2q_1^\lambda(h)}{[(2-q_1(h)]^\lambda + [q_1(h)]^\lambda}, \frac{2r_1^\lambda(h)}{[(2-r_1(h)]^\lambda + [r_1(h)]^\lambda} \big] \end{array} \right] \right\rangle$$

$$\lambda A_2 = \left\langle \left[\begin{array}{c} \frac{[(1+\alpha_2(h))]^{\lambda}-[(1-\alpha_2(h))]^{\lambda}}{[(1+\alpha_2(h))]^{\lambda}+[(1-\alpha_2(h))]^{\lambda}}, \\ \frac{[(1+\beta_2(h))]^{\lambda}-[(1-\beta_2(h))]^{\lambda}}{[(1+\beta_2(h))]^{\lambda}+[(1-\beta_2(h))]^{\lambda}}, \\ \frac{[(1+\Delta_2(h))]^{\lambda}-[(1-\Delta_2(h))]^{\lambda}}{[(1+\Delta_2(h))]^{\lambda}+[(1-\Delta_2(h))]^{\lambda}} \end{array}\right], \left[\begin{array}{c} [\frac{[(1+p_2^-(h))^{\lambda}-(1-p_2^-(h))^{\lambda}]}{[(1+p_2^-(h))^{\lambda}+(1-p_2^-(h))^{\lambda}]}, \\ \frac{[(1+q_2^-(h))^{\lambda}-(1-q_2^-(h))^{\lambda}]}{[(1+q_2^-(h))^{\lambda}+(1-q_2^-(h))^{\lambda}]}, \frac{[(1+r_2^-(h))^{\lambda}-(1-r_2^-(h))^{\lambda}]}{[(1+r_2^-(h))^{\lambda}+(1-r_2^-(h))^{\lambda}]}], \\ [\frac{[(1+p_2^+(h))^{\lambda}-(1-p_2^+(h))^{\lambda}]}{[(1+p_2^+(h))^{\lambda}+(1-p_2^+(h))^{\lambda}]}, \\ \frac{[(1+q_2^+(h))^{\lambda}-(1-q_2^+(h))^{\lambda}]}{[(1+q_2^+(h))^{\lambda}+(1-q_2^+(h))^{\lambda}]}, \frac{[(1+r_2^+(h))^{\lambda}-(1-r_2^+(h))^{\lambda}]}{[(1+r_2^+(h))^{\lambda}+(1-r_2^+(h))^{\lambda}]}] \end{array}\right], \left[\begin{array}{c} [\frac{2p_2^{\lambda}(h)}{[(2-p_2(h)]^{\lambda}+[p_2(h)]^{\lambda}}, \\ \frac{2q_2^{\lambda}(h)}{[(2-q_2(h)]^{\lambda}+[q_2(h)]^{\lambda}}, \frac{2r_2^{\lambda}(h)}{[(2-r_2(h)]^{\lambda}+[r_2(h)]^{\lambda}}] \end{array}\right] \right\rangle$$

$$\lambda A_2 + \lambda A_1 = \left\langle \left[\begin{array}{c} \frac{[(1+\alpha_2(h))(1-\alpha_2(h))]^{\lambda}[((1+\alpha_1(h))(1-\alpha_1(h))]^{\lambda}}{[(1+\alpha_2(h))(1-\alpha_2(h))]^{\lambda}[(1+\alpha_1(h))(1-\alpha_1(h))]^{\lambda}}, \\ \frac{[(1+\beta_2(h))(1-\beta_2(h))]^{\lambda}[((1+\beta_1(h))(1-\beta_1(h))]^{\lambda}}{[(1+\beta_2(h))(1-\beta_2(h))]^{\lambda}[(1+\beta_1(h))(1-\beta_1(h))]^{\lambda}}, \\ \frac{[(1+\Delta_2(h))(1-\Delta_2(h))]^{\lambda}[((1+\Delta_1(h))(1-\Delta_1(h))]^{\lambda}}{[(1+\Delta_2(h))(1-\Delta_2(h))]^{\lambda}[(1+\Delta_1(h))(1-\Delta_1(h))]^{\lambda}} \end{array}\right], \left[\begin{array}{c} [\frac{[(1+p_2^-(h))(1-p_2^-(h))]^{\lambda}[(1+p_1^-(h))(1-p_1^-(h))]^{\lambda}}{[(1+p_2^-(h))(1-p_2^-(h))]^{\lambda}[(1+p_1^-(h))(1-p_1^-(h))]^{\lambda}}, \\ \frac{[(1+q_2^-(h))(1-q_2^-(h))]^{\lambda}[(1+q_1^-(h))(1-q_1^-(h))]^{\lambda}}{[(1+q_2^-(h))(1-q_2^-(h))]^{\lambda}[(1+q_1^-(h))(1-q_1^-(h))]^{\lambda}}, \\ \frac{[(1+r_2^-(h))(1-r_2^-(h))]^{\lambda}[(1+r_1^-(h))(1-r_1^-(h))]^{\lambda}}{[(1+r_2^-(h))(1-r_2^-(h))]^{\lambda}[(1+r_1^-(h))(1-r_1^-(h))]^{\lambda}}] \\ [\frac{[(1+p_2^+(h))(1-p_2^+(h))]^{\lambda}[(1+p_1^+(h))(1-p_1^+(h))]^{\lambda}}{[(1+p_2^+(h))(1-p_2^+(h))]^{\lambda}[(1+p_1^+(h))(1-p_1^+(h))]^{\lambda}}, \\ \frac{[(1+q_2^+(h))(1-q_2^+(h))]^{\lambda}[(1+q_1^+(h))(1-q_1^+(h))]^{\lambda}}{[(1+q_2^+(h))(1-q_2^+(h))]^{\lambda}[(1+q_1^+(h))(1-q_1^+(h))]^{\lambda}}, \\ \frac{[(1+r_2^+(h))(1-r_2^+(h))]^{\lambda}[(1+r_1^+(h))(1-r_1^+(h))]^{\lambda}}{[(1+r_2^+(h))(1-r_2^+(h))]^{\lambda}[(1+r_1^+(h))(1-r_1^+(h))]^{\lambda}}] \end{array}\right], \left[\begin{array}{c} [\frac{2[p_2(h)p_1(h)]^{\lambda}}{[(4-2p_2(h)-2p_1(h)-p_2(h)p_1(h)]^{\lambda}+[p_2(h)p_1(h)]^{\lambda}}, \\ \frac{2[q_2(h)q_1(h)]^{\lambda}}{[(4-2q_2(h)-2q_1(h)-q_2(h)q_1(h)]^{\lambda}+[q_2(h)q_1(h)]^{\lambda}}, \\ \frac{2[r_2(h)r_1(h)]^{\lambda}}{[(4-2r_2(h)-2r_1(h)-r_2(h)r_1(h)]^{\lambda}+[r_2(h)r_1(h)]^{\lambda}}] \end{array}\right] \right\rangle$$

so, we have $\lambda(A_1+A_2) = \lambda A_2 + \lambda A_1$.

(3) $\lambda_1 A + \lambda_2 A = (\lambda_1+\lambda_2)A$

$$\lambda_1 A = \left\langle \left[\begin{array}{c} \frac{[(1+\alpha_A(h))]^{\lambda_1}-[(1-\alpha_A(h))]^{\lambda_1}}{[(1+\alpha_A(h))]^{\lambda_1}+[(1-\alpha_A(h))]^{\lambda_1}}, \\ \frac{[(1+\beta_A(h))]^{\lambda_1}-[(1-\beta_A(h))]^{\lambda_1}}{[(1+\beta_A(h))]^{\lambda_1}+[(1-\beta_A(h))]^{\lambda_1}}, \\ \frac{[(1+\Delta_A(h))]^{\lambda_1}-[(1-\Delta_A(h))]^{\lambda_1}}{[(1+\Delta_A(h))]^{\lambda_1}+[(1-\Delta_A(h))]^{\lambda_1}} \end{array}\right], \left[\begin{array}{c} [\frac{[1+p_A^-(h)]^{\lambda_1}-[1-p_A^-(h)]^{\lambda_1}}{[1+p_A^-(h)]^{\lambda_1}+[1-p_A^-(h)]^{\lambda_1}}, \\ \frac{[1+q_A^-(h)]^{\lambda_1}-[1-q_A^-(h)]^{\lambda_1}}{[1+q_A^-(h)]^{\lambda_1}+[1-q_A^-(h)]^{\lambda_1}}, \frac{[1+r_A^-(h)]^{\lambda_1}-[1-r_A^-(h)]^{\lambda_1}}{[1+r_A^-(h)]^{\lambda_1}+[1-r_A^-(h)]^{\lambda_1}}] \\ [\frac{[1+p_A^+(h)]^{\lambda_1}-[1-p_A^+(h)]^{\lambda_1}}{[1+p_A^+(h)]^{\lambda_1}+[1-p_A^+(h)]^{\lambda_1}}, \\ \frac{[1+q_A^+(h)]^{\lambda_1}-[1-q_A^+(h)]^{\lambda_1}}{[1+q_A^+(h)]^{\lambda_1}+[1-q_A^+(h)]^{\lambda_1}}, \frac{[1+r_A^+(h)]^{\lambda_1}-[1-r_A^+(h)]^{\lambda_1}}{[1+r_A^+(h)]^{\lambda_1}+[1-r_A^+(h)]^{\lambda_1}}] \end{array}\right], \left[\begin{array}{c} [\frac{2[p_A(h)]^{\lambda_1}}{[(2-p_A(h)]^{\lambda_1}+[p_A(h)]^{\lambda_1}}, \\ \frac{2[q_A(h)]^{\lambda_1}}{[(2-q_A(h)]^{\lambda_1}+[q_A(h)]^{\lambda_1}}, \frac{2[r_A(h)]^{\lambda_1}}{[(2-r_A(h)]^{\lambda_1}+[r_A(h)]^{\lambda_1}}] \end{array}\right] \right\rangle$$

$$\text{and } \lambda_2 A = \left\langle \left[\begin{array}{c} \frac{[(1+\alpha_A(h))]^{\lambda_2}-[(1-\alpha_A(h))]^{\lambda_2}}{[(1+\alpha_A(h))]^{\lambda_2}+[(1-\alpha_A(h))]^{\lambda_2}}, \\ \frac{[(1+\beta_A(h))]^{\lambda_2}-[(1-\beta_A(h))]^{\lambda_2}}{[(1+\beta_A(h))]^{\lambda_2}+[(1-\beta_A(h))]^{\lambda_2}}, \\ \frac{[(1+\Delta_A(h))]^{\lambda_2}-[(1-\Delta_A(h))]^{\lambda_2}}{[(1+\Delta_A(h))]^{\lambda_2}+[(1-\Delta_A(h))]^{\lambda_2}} \end{array}\right], \left[\begin{array}{c} [\frac{[1+p_A^-(h)]^{\lambda_2}-[1-p_A^-(h)]^{\lambda_2}}{[1+p_A^-(h)]^{\lambda_2}+[1-p_A^-(h)]^{\lambda_2}}, \frac{[1+q_A^-(h)]^{\lambda_2}-[1-q_A^-(h)]^{\lambda_2}}{[1+q_A^-(h)]^{\lambda_2}+[1-q_A^-(h)]^{\lambda_2}}, \\ \frac{[1+r_A^-(h)]^{\lambda_2}-[1-r_A^-(h)]^{\lambda_2}}{[1+r_A^-(h)]^{\lambda_2}+[1-r_A^-(h)]^{\lambda_2}}], [\frac{[1+p_A^+(h)]^{\lambda_2}-[1-p_A^+(h)]^{\lambda_2}}{[1+p_A^+(h)]^{\lambda_2}+[1-p_A^+(h)]^{\lambda_2}}, \\ \frac{[1+q_A^+(h)]^{\lambda_2}-[1-q_A^+(h)]^{\lambda_2}}{[1+q_A^+(h)]^{\lambda_2}+[1-q_A^+(h)]^{\lambda_2}}, \frac{[1+r_A^+(h)]^{\lambda_2}-[1-r_A^+(h)]^{\lambda_2}}{[1+r_A^+(h)]^{\lambda_2}+[1-r_A^+(h)]^{\lambda_2}}] \end{array}\right], \left[[\frac{2[p_A(h)]^{\lambda_2}}{[(2-p_A(h)]^{\lambda_2}+[p_A(h)]^{\lambda_2}}, \frac{2[q_A(h)]^{\lambda_2}}{[(2-q_A(h)]^{\lambda_2}+[q_A(h)]^{\lambda_2}}, \frac{2[r_A(h)]^{\lambda_2}}{[(2-r_A(h)]^{\lambda_2}+[r_A(h)]^{\lambda_2}}] \right] \right\rangle =$$

$$\left\langle \begin{bmatrix} \frac{[1+\alpha_A(h)]^{\lambda_1+\lambda_2}-[1-\alpha_A(h)]^{\lambda_1+\lambda_2}}{[1+\alpha_A(h)]^{\lambda_1+\lambda_2}+[1-\alpha_A(h)]^{\lambda_1+\lambda_2}}, \frac{[1+\beta_A(h)]^{\lambda_1+\lambda_2}-[1-\beta_A(h)]^{\lambda_1+\lambda_2}}{[1+\beta_A(h)]^{\lambda_1+\lambda_2}+[1-\beta_A(h)]^{\lambda_1+\lambda_2}}, \\ \frac{[1+\Delta_A(h)]^{\lambda_1+\lambda_2}-[1-\Delta_A(h)]^{\lambda_1+\lambda_2}}{[1+\Delta_A(h)]^{\lambda_1+\lambda_2}+[1-\Delta_A(h)]^{\lambda_1+\lambda_2}} \end{bmatrix}, \begin{bmatrix} [\frac{[1+p_A^-(h)]^{\lambda_1+\lambda_2}-[1-p_A^-(h)]^{\lambda_1+\lambda_2}}{[1+p_A^-(h)]^{\lambda_1+\lambda_2}+[1-p_A^-(h)]^{\lambda_1+\lambda_2}}, \frac{[1+q_A^-(h)]^{\lambda_1+\lambda_2}-[1-q_A^-(h)]^{\lambda_1+\lambda_2}}{[1+q_A^-(h)]^{\lambda_1+\lambda_2}+[1-q_A^-(h)]^{\lambda_1+\lambda_2}}, \\ \frac{[1+r_A^-(h)]^{\lambda_1+\lambda_2}-[1-r_A^-(h)]^{\lambda_1+\lambda_2}}{[1+r_A^-(h)]^{\lambda_1+\lambda_2}+[1-r_A^-(h)]^{\lambda_1+\lambda_2}}], [\frac{[1+p_A^+(h)]^{\lambda_1+\lambda_2}-[1-p_A^+(h)]^{\lambda_1+\lambda_2}}{[1+p_A^+(h)]^{\lambda_1+\lambda_2}+[1-p_A^+(h)]^{\lambda_1+\lambda_2}}, \\ \frac{[1+q_A^+(h)]^{\lambda_1+\lambda_2}-[1-q_A^+(h)]^{\lambda_1+\lambda_2}}{[1+q_A^+(h)]^{\lambda_1+\lambda_2}+[1-q_A^+(h)]^{\lambda_1+\lambda_2}}, \frac{[1+r_A^+(h)]^{\lambda_1+\lambda_2}-[1-r_A^+(h)]^{\lambda_1+\lambda_2}}{[1+r_A^+(h)]^{\lambda_1+\lambda_2}+[1-r_A^+(h)]^{\lambda_1+\lambda_2}}] \end{bmatrix}, \begin{bmatrix} [\frac{2[p_A(h)]^{\lambda_1+\lambda_2}}{[(2-p_A(h)]^{\lambda_1+\lambda_2}+[p_A(h)]^{\lambda_1+\lambda_2}}, \\ \frac{2[q_A(h)]^{\lambda_1+\lambda_2}}{[(2-q_A(h)]^{\lambda_1+\lambda_2}+[q_A(h)]^{\lambda_1+\lambda_2}}, \frac{2[r_A(h)]^{\lambda_1+\lambda_2}}{[(2-r_A(h)]^{\lambda_1+\lambda_2}+[r_A(h)]^{\lambda_1+\lambda_2}}] \end{bmatrix} \right\rangle = (\lambda_1+\lambda_2)A.$$

Appendix B. Proof of Theorem 1

Assume that $n = 1$, TCFEWA $(A_1, A_2, \ldots, A_n) = \overset{k}{\underset{j=1}{\oplus}} w_1 A_1 \ \langle(\lambda(A_1 + A_2) = \lambda A_2 + \lambda A_1$

$$\lambda(A_1+A_2) = \begin{bmatrix} \frac{[(1+\alpha_1(h))(1-\alpha_1(h))]^{\lambda}[((1+\alpha_2(h))(1-\alpha_2(h))]^{\lambda}}{[(1+\alpha_1(h))(1-\alpha_1(h))]^{\lambda}[(1+\alpha_2(h))(1-\alpha_2(h))]^{\lambda}}, \\ \frac{[(1+\beta_1(h))(1-\beta_1(h))]^{\lambda}[((1+\beta_2(h))(1-\beta_2(h))]^{\lambda}}{[(1+\beta_1(h))(1-\beta_1(h))]^{\lambda}[(1+\beta_2(h))(1-\beta_2(h))]^{\lambda}}, \\ \frac{[(1+\Delta_1(h))(1-\Delta_1(h))]^{\lambda}[((1+\Delta_2(h))(1-\Delta_2(h))]^{\lambda}}{[(1+\Delta_1(h))(1-\Delta_1(h))]^{\lambda}[(1+\Delta_2(h))(1-\Delta_2(h))]^{\lambda}} \end{bmatrix}, \left\langle \begin{bmatrix} [\frac{[(1+p_1^-(h))(1-p_1^-(h))]^{\lambda}[(1+p_2^-(h))(1-p_2^-(h))]^{\lambda}}{[(1+p_1^-(h))(1-p_1^-(h))]^{\lambda}[(1+p_2^-(h))(1-p_2^-(h))]^{\lambda}}, \\ \frac{[(1+q_1^-(h))(1-q_1^-(h))]^{\lambda}[(1+q_2^-(h))(1-q_2^-(h))]^{\lambda}}{[(1+q_1^-(h))(1-q_1^-(h))]^{\lambda}[(1+q_2^-(h))(1-q_2^-(h))]^{\lambda}}, \\ \frac{[(1+r_1^-(h))(1-r_1^-(h))]^{\lambda}[(1+r_2^-(h))(1-r_2^-(h))]^{\lambda}}{[(1+r_1^-(h))(1-r_1^-(h))]^{\lambda}[(1+r_2^-(h))(1-r_2^-(h))]^{\lambda}}] \\ ,[\frac{[(1+p_1^+(h))(1-p_1^+(h))]^{\lambda}[(1+p_2^+(h))(1-p_2^+(h))]^{\lambda}}{[(1+p_1^+(h))(1-p_1^+(h))]^{\lambda}[(1+p_2^+(h))(1-p_2^+(h))]^{\lambda}}, \\ \frac{[(1+q_1^+(h))(1-q_1^+(h))]^{\lambda}[(1+q_2^+(h))(1-q_2^+(h))]^{\lambda}}{[(1+q_1^+(h))(1-q_1^+(h))]^{\lambda}[(1+q_2^+(h))(1-q_2^+(h))]^{\lambda}}, \\ \frac{[(1+r_1^+(h))(1-r_1^+(h))]^{\lambda}[(1+r_2^+(h))(1-r_2^+(h))]^{\lambda}}{[(1+r_1^+(h))(1-r_1^+(h))]^{\lambda}[(1+r_2^+(h))(1-r_2^+(h))]^{\lambda}}] \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} [\frac{2[p_1(h)p_2(h)]^{\lambda}}{[(4-2p_1(h)-2p_2(h)-p_1(h)p_2(h)]^{\lambda}+[p_1(h)p_2(h)]^{\lambda}}, \\ \frac{2[q_1(h)q_2(h)]^{\lambda}}{[(4-2q_1(h)-2q_2(h)-q_1(h)q_2(h)]^{\lambda}+[q_1(h)q_2(h)]^{\lambda}}, \\ \frac{2[r_1(h)r_2(h)]^{\lambda}}{[(4-2r_1(h)-2r_2(h)-r_1(h)r_2(h)]^{\lambda}+[r_1(h)r_2(h)]^{\lambda}}] \end{bmatrix} \right\rangle$$

and we have

$$\lambda A_1 = \left\langle \begin{bmatrix} \frac{[(1+\alpha_1(h))]^\lambda-[(1-\alpha_1(h))]^\lambda}{[(1+\alpha_1(h))]^\lambda+[(1-\alpha_1(h))]^\lambda}, \\ \frac{[(1+\beta_1(h))]^\lambda-[(1-\beta_1(h))]^\lambda}{[(1+\beta_1(h))]^\lambda+[(1-\beta_1(h))]^\lambda}, \\ \frac{[(1+\Delta_1(h))]^\lambda-[(1-\Delta_1(h))]^\lambda}{[(1+\Delta_1(h))]^\lambda+[(1-\Delta_1(h))]^\lambda} \end{bmatrix}, \begin{bmatrix} [\frac{[(1+p_1^-(h))^\lambda-(1-p_1^-(h))^\lambda]}{[(1+p_1^-(h))^\lambda+(1-p_1^-(h))^\lambda]}, \\ \frac{[(1+q_1^-(h))^\lambda-(1-q_1^-(h))^\lambda]}{[(1+q_1^-(h))^\lambda+(1-q_1^-(h))^\lambda]}, \frac{[(1+r_1^-(h))^\lambda-(1-r_1^-(h))^\lambda]}{[(1+r_1^-(h))^\lambda+(1-r_1^-(h))^\lambda]}] \\ [\frac{[(1+p_1^+(h))^\lambda-(1-p_1^+(h))^\lambda]}{[(1+p_1^+(h))^\lambda+(1-p_1^+(h))^\lambda]}, \\ \frac{[(1+q_1^+(h))^\lambda-(1-q_1^+(h))^\lambda]}{[(1+q_1^+(h))^\lambda+(1-q_1^+(h))^\lambda]}, \frac{[(1+r_1^+(h))^\lambda-(1-r_1^+(h))^\lambda]}{[(1+r_1^+(h))^\lambda+(1-r_1^+(h))^\lambda]}] \end{bmatrix}, \begin{bmatrix} [\frac{2p_1^\lambda(h)}{[(2-p_1(h)]^\lambda+[p_1(h)]^\lambda}, \\ \frac{2q_1^\lambda(h)}{[(2-q_1(h)]^\lambda+[q_1(h)]^\lambda}, \\ \frac{2r_1^\lambda(h)}{[(2-r_1(h)]^\lambda+[r_1(h)]^\lambda}] \end{bmatrix} \right\rangle$$

$$\lambda A_2 = \left\langle \begin{bmatrix} \frac{[(1+\alpha_2(h))]^\lambda-[(1-\alpha_2(h))]^\lambda}{[(1+\alpha_2(h))]^\lambda+[(1-\alpha_2(h))]^\lambda}, \\ \frac{[(1+\beta_2(h))]^\lambda-[(1-\beta_2(h))]^\lambda}{[(1+\beta_2(h))]^\lambda+[(1-\beta_2(h))]^\lambda}, \\ \frac{[(1+\Delta_2(h))]^\lambda-[(1-\Delta_2(h))]^\lambda}{[(1+\Delta_2(h))]^\lambda+[(1-\Delta_2(h))]^\lambda} \end{bmatrix}, \begin{bmatrix} [\frac{[(1+p_2^-(h))^\lambda-(1-p_2^-(h))^\lambda]}{[(1+p_2^-(h))^\lambda+(1-p_2^-(h))^\lambda]}, \\ \frac{[(1+q_2^-(h))^\lambda-(1-q_2^-(h))^\lambda]}{[(1+q_2^-(h))^\lambda+(1-q_2^-(h))^\lambda]}, \frac{[(1+r_2^-(h))^\lambda-(1-r_2^-(h))^\lambda]}{[(1+r_2^-(h))^\lambda+(1-r_2^-(h))^\lambda]}] \\ [\frac{[(1+p_2^+(h))^\lambda-(1-p_2^+(h))^\lambda]}{[(1+p_2^+(h))^\lambda+(1-p_2^+(h))^\lambda]}, \\ \frac{[(1+q_2^+(h))^\lambda-(1-q_2^+(h))^\lambda]}{[(1+q_2^+(h))^\lambda+(1-q_2^+(h))^\lambda]}, \frac{[(1+r_2^+(h))^\lambda-(1-r_2^+(h))^\lambda]}{[(1+r_2^+(h))^\lambda+(1-r_2^+(h))^\lambda]}] \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} [\frac{2p_2^\lambda(h)}{[(2-p_2(h)]^\lambda+[p_2(h)]^\lambda}, \frac{2q_2^\lambda(h)}{[(2-q_2(h)]^\lambda+[q_2(h)]^\lambda}, \\ \frac{2r_2^\lambda(h)}{[(2-r_2(h)]^\lambda+[r_2(h)]^\lambda}] \end{bmatrix} \right\rangle$$

$$\lambda A_2 + \lambda A_1 = \left\langle \begin{bmatrix} \frac{[(1+\alpha_2(h))(1-\alpha_2(h))]^\lambda[((1+\alpha_1(h))(1-\alpha_1(h))]^\lambda}{[(1+\alpha_2(h))(1-\alpha_2(h))]^\lambda[(1+\alpha_1(h))(1-\alpha_1(h))]^\lambda}, \\ \frac{[(1+\beta_2(h))(1-\beta_2(h))]^\lambda[((1+\beta_1(h))(1-\beta_1(h))]^\lambda}{[(1+\beta_2(h))(1-\beta_2(h))]^\lambda[(1+\beta_1(h))(1-\beta_1(h))]^\lambda}, \\ \frac{[(1+\Delta_2(h))(1-\Delta_2(h))]^\lambda[((1+\Delta_1(h))(1-\Delta_1(h))]^\lambda}{[(1+\Delta_2(h))(1-\Delta_2(h))]^\lambda[(1+\Delta_1(h))(1-\Delta_1(h))]^\lambda} \end{bmatrix}, \begin{bmatrix} [\frac{[(1+p_2^-(h))(1-p_2^-(h))]^\lambda[(1+p_1^-(h))(1-p_1^-(h))]^\lambda}{[(1+p_2^-(h))(1-p_2^-(h))]^\lambda[(1+p_1^-(h))(1-p_1^-(h))]^\lambda}, \\ \frac{[(1+q_2^-(h))(1-q_2^-(h))]^\lambda[(1+q_1^-(h))(1-q_1^-(h))]^\lambda}{[(1+q_2^-(h))(1-q_2^-(h))]^\lambda[(1+q_1^-(h))(1-q_1^-(h))]^\lambda}, \\ \frac{[(1+r_2^-(h))(1-r_2^-(h))]^\lambda[(1+r_1^-(h))(1-r_1^-(h))]^\lambda}{[(1+r_2^-(h))(1-r_2^-(h))]^\lambda[(1+r_1^-(h))(1-r_1^-(h))]^\lambda}] \\ [\frac{[(1+p_2^+(h))(1-p_2^+(h))]^\lambda[(1+p_1^+(h))(1-p_1^+(h))]^\lambda}{[(1+p_2^+(h))(1-p_2^+(h))]^\lambda[(1+p_1^+(h))(1-p_1^+(h))]^\lambda}, \\ \frac{[(1+q_2^+(h))(1-q_2^+(h))]^\lambda[(1+q_1^+(h))(1-q_1^+(h))]^\lambda}{[(1+q_2^+(h))(1-q_2^+(h))]^\lambda[(1+q_1^+(h))(1-q_1^+(h))]^\lambda}, \\ \frac{[(1+r_2^+(h))(1-r_2^+(h))]^\lambda[(1+r_1^+(h))(1-r_1^+(h))]^\lambda}{[(1+r_2^+(h))(1-r_2^+(h))]^\lambda[(1+r_1^+(h))(1-r_1^+(h))]^\lambda}] \end{bmatrix}, \right.$$

$$\left. \begin{bmatrix} [\frac{2[p_2(h)p_1(h)]^\lambda}{[(4-2p_2(h)-2p_1(h)-p_2(h)p_1(h)]^\lambda+[p_2(h)p_1(h)]^\lambda}, \\ \frac{2[q_2(h)q_1(h)]^\lambda}{[(4-2q_2(h)-2q_1(h)-q_2(h)q_1(h)]^\lambda+[q_2(h)q_1(h)]^\lambda}, \\ \frac{2[r_2(h)r_1(h)]^\lambda}{[(4-2r_2(h)-2r_1(h)-r_2(h)r_1(h)]^\lambda+[s_2(h)s_1(h)]^\lambda}] \end{bmatrix} \right\rangle$$

so, we have $\lambda(A_1 + A_2) = \lambda A_2 + \lambda A_1$.

$\lambda_1 A + \lambda_2 A = (\lambda_1 + \lambda_2)A$

$$\lambda_1 A = \left\langle \begin{bmatrix} \frac{[1+\alpha_A(h)]^{\lambda_1}-[1-\alpha_A(h)]^{\lambda_1}}{[1+\alpha_A(h)]^{\lambda_1}+[1-\alpha_A(h)]^{\lambda_1}}, \frac{[1+\beta_A(h)]^{\lambda_1}-[1-\beta_A(h)]^{\lambda_1}}{[1+\beta_A(h)]^{\lambda_1}+[1-\beta_A(h)]^{\lambda_1}}, \\ \frac{[1+\Delta_A(h)]^{\lambda_1}-[1-\Delta_A(h)]^{\lambda_1}}{[1+\Delta_A(h)]^{\lambda_1}+[1-\Delta_A(h)]^{\lambda_1}} \end{bmatrix}, \right.$$

$$\begin{bmatrix} [\frac{[1+p_A^-(h)]^{\lambda_1}-[1-p_A^-(h)]^{\lambda_1}}{[1+p_A^-(h)]^{\lambda_1}+[1-p_A^-(h)]^{\lambda_1}}, \frac{[1+q_A^-(h)]^{\lambda_1}-[1-q_A^-(h)]^{\lambda_1}}{[1+q_A^-(h)]^{\lambda_1}+[1-q_A^-(h)]^{\lambda_1}}, \frac{[1+r_A^-(h)]^{\lambda_1}-[1-r_A^-(h)]^{\lambda_1}}{[1+r_A^-(h)]^{\lambda_1}+[1-r_A^-(h)]^{\lambda_1}}] \\ [\frac{[1+p_A^+(h)]^{\lambda_1}-[1-p_A^+(h)]^{\lambda_1}}{[1+p_A^+(h)]^{\lambda_1}+[1-p_A^+(h)]^{\lambda_1}}, \frac{[1+q_A^+(h)]^{\lambda_1}-[1-q_A^+(h)]^{\lambda_1}}{[1+q_A^+(h)]^{\lambda_1}+[1-q_A^+(h)]^{\lambda_1}}, \frac{[1+r_A^+(h)]^{\lambda_1}-[1-r_A^+(h)]^{\lambda_1}}{[1+r_A^+(h)]^{\lambda_1}+[1-r_A^+(h)]^{\lambda_1}}] \end{bmatrix},$$

$$\left. \begin{bmatrix} \frac{2[p_A(h)]^{\lambda_1}}{[(2-p_A(h)]^{\lambda_1}+[p_A(h)]^{\lambda_1}}, \frac{2[q_A(h)]^{\lambda_1}}{[(2-q_A(h)]^{\lambda_1}+[q_A(h)]^{\lambda_1}}, \frac{2[r_A(h)]^{\lambda_1}}{[(2-r_A(h)]^{\lambda_1}+[r_A(h)]^{\lambda_1}} \end{bmatrix} \right\rangle$$

and $$\lambda_2 A = \left\langle \begin{bmatrix} \frac{[1+\alpha_A(h)]^{\lambda_2}-[1-\alpha_A(h)]^{\lambda_2}}{[1+\alpha_A(h)]^{\lambda_2}+[1-\alpha_A(h)]^{\lambda_2}}, \\ \frac{[1+\beta_A(h)]^{\lambda_2}-[1-\beta_A(h)]^{\lambda_2}}{[1+\beta_A(h)]^{\lambda_2}+[1-\beta_A(h)]^{\lambda_2}}, \\ \frac{[1+\Delta_A(h)]^{\lambda_2}-[1-\Delta_A(h)]^{\lambda_2}}{[1+\Delta_A(h)]^{\lambda_2}+[1-\Delta_A(h)]^{\lambda_2}} \end{bmatrix}, \begin{bmatrix} [\frac{[1+p_A^-(h)]^{\lambda_2}-[1-p_A^-(h)]^{\lambda_2}}{[1+p_A^-(h)]^{\lambda_2}+[1-p_A^-(h)]^{\lambda_2}}, \frac{[1+q_A^-(h)]^{\lambda_2}-[1-q_A^-(h)]^{\lambda_2}}{[1+q_A^-(h)]^{\lambda_2}+[1-q_A^-(h)]^{\lambda_2}}, \\ \frac{[1+r_A^-(h)]^{\lambda_2}-[1-r_A^-(h)]^{\lambda_2}}{[1+r_A^-(h)]^{\lambda_2}+[1-r_A^-(h)]^{\lambda_2}}], [\frac{[1+p_A^+(h)]^{\lambda_2}-[1-p_A^+(h)]^{\lambda_2}}{[1+p_A^+(h)]^{\lambda_2}+[1-p_A^+(h)]^{\lambda_2}}, \\ \frac{[1+q_A^+(h)]^{\lambda_2}-[1-q_A^+(h)]^{\lambda_2}}{[1+q_A^+(h)]^{\lambda_2}+[1-q_A^+(h)]^{\lambda_2}}, \frac{[1+r_A^+(h)]^{\lambda_2}-[1-r_A^+(h)]^{\lambda_2}}{[1+r_A^+(h)]^{\lambda_2}+[1-r_A^+(h)]^{\lambda_2}}] \end{bmatrix}, \right.$$

$$\left.\begin{bmatrix} [\frac{2[p_A(h)]^{\lambda_2}}{[(2-p_A(h)]^{\lambda_2}+[p_A(h)]^{\lambda_2}}, \\ \frac{2[q_A(h)]^{\lambda_2}}{[(2-q_A(h)]^{\lambda_2}+[q_A(h)]^{\lambda_2}}, \frac{2[r_A(h)]^{\lambda_2}}{[(2-r_A(h)]^{\lambda_2}+[r_A(h)]^{\lambda_2}}] \end{bmatrix}\right\rangle =$$

$$\left\langle\begin{bmatrix} \frac{[1+\alpha_A(h)]^{\lambda_1+\lambda_2}-[1-\alpha_A(h)]^{\lambda_1+\lambda_2}}{[1+\alpha_A(h)]^{\lambda_1+\lambda_2}+[1-\alpha_A(h)]^{\lambda_1+\lambda_2}}, \frac{[\beta_A(h)]^{\lambda_1+\lambda_2}-[1-\beta_A(h)]^{\lambda_1+\lambda_2}}{[1+\beta_A(h)]^{\lambda_1+\lambda_2}+[1-\beta_A(h)]^{\lambda_1+\lambda_2}}, \\ \frac{[1+\Delta_A(h)]^{\lambda_1+\lambda_2}-[1-\Delta_A(h)]^{\lambda_1+\lambda_2}}{[1+\Delta_A(h)]^{\lambda_1+\lambda_2}+[1-\Delta_A(h)]^{\lambda_1+\lambda_2}} \end{bmatrix}\right.,$$

$$\begin{bmatrix} [\frac{[1+p_A^-(h)]^{\lambda_1+\lambda_2}-[1-p_A^-(h)]^{\lambda_1+\lambda_2}}{[1+p_A^-(h)]^{\lambda_1+\lambda_2}+[1-p_A^-(h)]^{\lambda_1+\lambda_2}}, \frac{[1+q_A^-(h)]^{\lambda_1+\lambda_2}-[1-q_A^-(h)]^{\lambda_1+\lambda_2}}{[1+q_A^-(h)]^{\lambda_1+\lambda_2}+[1-q_A^-(h)]^{\lambda_1+\lambda_2}}, \\ \frac{[1+r_A^-(h)]^{\lambda_1+\lambda_2}-[1-r_A^-(h)]^{\lambda_1+\lambda_2}}{[1+r_A^-(h)]^{\lambda_1+\lambda_2}+[1-r_A^-(h)]^{\lambda_1+\lambda_2}}], [\frac{[1+p_A^+(h)]^{\lambda_1+\lambda_2}-[1-p_A^+(h)]^{\lambda_1+\lambda_2}}{[1+p_A^+(h)]^{\lambda_1+\lambda_2}+[1-p_A^+(h)]^{\lambda_1+\lambda_2}}, \\ \frac{[1+q_A^+(h)]^{\lambda_1+\lambda_2}-[1-q_A^+(h)]^{\lambda_1+\lambda_2}}{[1+q_A^+(h)]^{\lambda_1+\lambda_2}+[1-q_A^+(h)]^{\lambda_1+\lambda_2}}, \frac{[1+r_A^+(h)]^{\lambda_1+\lambda_2}-[1-r_A^+(h)]^{\lambda_1+\lambda_2}}{[1+r_A^+(h)]^{\lambda_1+\lambda_2}+[1-r_A^+(h)]^{\lambda_1+\lambda_2}}] \end{bmatrix},$$

$$\left.\begin{bmatrix} [\frac{2[p_A(h)]^{\lambda_1+\lambda_2}}{[(2-p_A(h)]^{\lambda_1+\lambda_2}+[p_A(h)]^{\lambda_1+\lambda_2}}, \\ \frac{2[q_A(h)]^{\lambda_1+\lambda_2}}{[(2-q_A(h)]^{\lambda_1+\lambda_2}+[q_A(h)]^{\lambda_1+\lambda_2}}, \frac{2[r_A(h)]^{\lambda_1+\lambda_2}}{[(2-r_A(h)]^{\lambda_1+\lambda_2}+[r_A(h)]^{\lambda_1+\lambda_2}}] \end{bmatrix}\right\rangle =$$

$$\begin{bmatrix} \frac{[[1+\alpha_1(h)]^{\lambda_1}-[1-\alpha_1(h)]^{\lambda_1}}{[1+\alpha_1(h)]^{\lambda_1}+[1-\alpha_1(h)]^{\lambda_1}}, \frac{[[1+\beta_1(h)]^{\lambda_1}-[1-\beta_1(h)]^{\lambda_1}}{[1+\beta_1(h)]^{\lambda_1}+[1-\beta_1(h)]^{\lambda_1}}, \frac{[[1+\Delta_1(h)]^{\lambda_1}-[1-\Delta_1(h)]^{\lambda_1}}{[1+\Delta_1(h)]^{\lambda_1}+[1-\Delta_1(h)]^{\lambda_1}} \end{bmatrix}$$

$$\begin{bmatrix} \frac{[[1+p_1^-(h)]^{\lambda_1}-[1-p_1^-(h)]^{\lambda_1}}{[1+p_1^-(h)]^{\lambda_1}+[1-p_1^-(h)]^{\lambda_1}}, \frac{[1+q_1^-(h)]^{\lambda_1}-[1-q_1^-(h)]^{\lambda_1}}{[1+q_1^-(h)]^{\lambda_1}+[1-q_1^-(h)]^{\lambda_1}}, \\ \frac{[1+r_1^-(h)]^{\lambda_1}-[1-r_1^-(h)]^{\lambda_1}}{[1+r_1^-(h)]^{\lambda_1}+[1-r_1^-(h)]^{\lambda_1}} \end{bmatrix}; \begin{bmatrix} \frac{[1+p_1^+(h)]^{\lambda_1}-[1-p_1^+(h)]^{\lambda_1}}{[1+p_1^+(h)]^{\lambda_1}+[1-p_1^+(h)]^{\lambda_1}}, \frac{[1+q_1^+(h)]^{\lambda_1}-[1-q_1^+(h)]^{\lambda_1}}{[1+q_1^+(h)]^{\lambda_1}+[1-q_1^+(h)]^{\lambda_1}}, \\ \frac{[1+r_1^+(h)]^{\lambda_1}-[1-r_1^+(h)]^{\lambda_1}}{[1+r_1^+(h)]^{\lambda_1}+[1-r_1^+(h)]^{\lambda_1}} \end{bmatrix};$$

$$\begin{bmatrix} \frac{2[p_1(h)]^{\lambda_1}}{[(2-p_1(h)]^{\lambda_1}+[p_1(h)]^{\lambda_1}}, \frac{2[q_1(h)]^{\lambda_1}}{[(2-q_1(h)]^{\lambda_1}+[q_1(h)]^{\lambda_1}}, \\ \frac{2[r_1(h)]^{\lambda_1}}{[(2-r_1(h)]^{\lambda_1}+[r_1(h)]^{\lambda_1}} \end{bmatrix}.$$

Assume that $n = k$, TCFEWA $(A_1, A_2, \ldots, A_n) = \overset{k}{\underset{j=1}{\oplus}} w_j A_j$

$$\left\langle\begin{bmatrix} \frac{[\prod_{j=1}^{k}[1+\alpha_1(h)]^{\varpi}-\prod_{j=1}^{k}[1-\alpha_1(h)]^{\varpi}}{\prod_{j=1}^{k}[1+\alpha_1(h)]^{\varpi}+\prod_{j=1}^{k}[1-\alpha_1(h)]^{\varpi}}, \frac{[\prod_{j=1}^{k}[1+\beta_1(h)]^{\varpi}-\prod_{j=1}^{k}[1-\beta_1(h)]^{\varpi}}{\prod_{j=1}^{k}[1+\beta_1(h)]^{\varpi}+\prod_{j=1}^{k}[1-\beta_1(h)]^{\varpi}}, \frac{[\prod_{j=1}^{k}[1+\Delta_1(h)]^{\varpi}-\prod_{j=1}^{k}[1-\Delta_1(h)]^{\varpi}}{\prod_{j=1}^{k}[1+\Delta_1(h)]^{\varpi}+\prod_{j=1}^{k}[1-\Delta_1(h)]^{\varpi}} \end{bmatrix}\right.$$

$$\begin{bmatrix} \frac{[\prod_{j=1}^{k}[1+p_1^-(h)]^{\varpi}-\prod_{j=1}^{k}[1-p_1^-(h)]^{\varpi}}{\prod_{j=1}^{k}[1+p_1^-(h)]^{\varpi}+\prod_{j=1}^{k}[1-p_1^-(h)]^{\varpi}}, \frac{\prod_{j=1}^{k}[1+q_1^-(h)]^{\varpi}-\prod_{j=1}^{k}[1-q_1^-(h)]^{\varpi}}{\prod_{j=1}^{k}[1+q_1^-(h)]^{\varpi}+\prod_{j=1}^{k}[1-q_1^-(h)]^{\varpi}}, \frac{\prod_{j=1}^{k}[1+r_1^-(h)]^{\varpi}-\prod_{j=1}^{k}[1-r_1^-(h)]^{\varpi}}{\prod_{j=1}^{k}[1+r_1^-(h)]^{\varpi}+\prod_{j=1}^{k}[1-r_1^-(h)]^{\varpi}} \end{bmatrix},$$

$$\begin{bmatrix} \frac{\prod_{j=1}^{k}[1+p_1^+(h)]^{\varpi}-\prod_{j=1}^{k}[1-p_1^+(h)]^{\varpi}}{\prod_{j=1}^{k}[1+p_1^+(h)]^{\varpi}+\prod_{j=1}^{k}[1-p_1^+(h)]^{\varpi}}, \frac{\prod_{j=1}^{k}[1+q_1^+(h)]^{\varpi}-\prod_{j=1}^{k}[1-q_1^+(h)]^{\varpi}}{\prod_{j=1}^{k}[1+q_1^+(h)]^{\varpi}+\prod_{j=1}^{k}[1-q_1^+(h)]^{\varpi}}, \frac{\prod_{j=1}^{k}[1+r_1^+(h)]^{\varpi}-\prod_{j=1}^{k}[1-r_1^+(h)]^{\varpi}}{\prod_{j=1}^{k}[1+r_1^+(h)]^{\varpi}+\prod_{j=1}^{k}[1-r_1^+(h)]^{\varpi}} \end{bmatrix};$$

$$\begin{bmatrix} \frac{2\prod_{j=1}^{k}[p_1(h)]^{\varpi}}{\prod_{j=1}^{k}[(2-p_1(h)]^{\varpi}+\prod_{j=1}^{k}[p_1(h)]^{\varpi}}, \frac{2\prod_{j=1}^{k}[q_1(h)]^{\varpi}}{\prod_{j=1}^{k}[(2-q_1(h)]^{\varpi}+\prod_{j=1}^{k}[q_1(h)]^{\varpi}}, \frac{2\prod_{j=1}^{k}[r_1(h)]^{\varpi}}{\prod_{j=1}^{k}[(2-r_1(h)]^{\varpi}+\prod_{j=1}^{k}[r_1(h)]^{\varpi}} \end{bmatrix}.$$

Then when $n = k+1$, we have TCFEWA $(A_1, A_2, \ldots, A_{k+1})$ = TCFEWA $(A_1, A_2, \ldots, A_k) \oplus A_{k+1})$

$$
\begin{aligned}
&\left\langle \left[\frac{[\prod_{j=1}^{k}[1+\alpha_1(h)]^{\varpi}-\prod_{j=1}^{k}[1-\alpha_1(h)]^{\varpi}}{\prod_{j=1}^{k}[1+\alpha_1(h)]^{\varpi}+\prod_{j=1}^{k}[1-\alpha_1(h)]^{\varpi}}, \frac{[\prod_{j=1}^{k}[1+\beta_1(h)]^{\varpi}-\prod_{j=1}^{k}[1-\beta_1(h)]^{\varpi}}{\prod_{j=1}^{k}[1+\beta_1(h)]^{\varpi}+\prod_{j=1}^{k}[1-\beta_1(h)]^{\varpi}}, \frac{[\prod_{j=1}^{k}[1+\Delta_1(h)]^{\varpi}-\prod_{j=1}^{k}[1-\Delta_1(h)]^{\varpi}}{\prod_{j=1}^{k}[1+\Delta_1(h)]^{\varpi}+\prod_{j=1}^{k}[1-\Delta_1(h)]^{\varpi}} \right] \right. \\
&\left[\frac{[\prod_{j=1}^{k}[1+p_1^-(h)]^{\varpi}-\prod_{j=1}^{k}[1-p_1^-(h)]^{\varpi}}{\prod_{j=1}^{k}[1+p_1^-(h)]^{\varpi}+\prod_{j=1}^{k}[1-p_1^-(h)]^{\varpi}}, \frac{\prod_{j=1}^{k}[1+q_1^-(h)]^{\varpi}-\prod_{j=1}^{k}[1-q_1^-(h)]^{\varpi}}{\prod_{j=1}^{k}[1+q_1^-(h)]^{\varpi}+\prod_{j=1}^{k}[1-q_1^-(h)]^{\varpi}}, \frac{\prod_{j=1}^{k}[1+r_1^-(h)]^{\varpi}-\prod_{j=1}^{k}[1-r_1^-(h)]^{\varpi}}{\prod_{j=1}^{k}[1+r_1^-(h)]^{\varpi}+\prod_{j=1}^{k}[1-r_1^-(h)]^{\varpi}} \right]; \\
&\left[\frac{\prod_{j=1}^{k}[1+p_1^+(h)]^{\varpi}-\prod_{j=1}^{k}[1-p_1^+(h)]^{\varpi}}{\prod_{j=1}^{k}[1+p_1^+(h)]^{\varpi}+\prod_{j=1}^{k}[1-p_1^+(h)]^{\varpi}}, \frac{\prod_{j=1}^{k}[1+q_1^+(h)]^{\varpi}-\prod_{j=1}^{k}[1-q_1^+(h)]^{\varpi}}{\prod_{j=1}^{k}[1+q_1^+(h)]^{\varpi}+\prod_{j=1}^{k}[1-q_1^+(h)]^{\varpi}}, \frac{\prod_{j=1}^{k}[1+r_1^+(h)]^{\varpi}-\prod_{j=1}^{k}[1-r_1^+(h)]^{\varpi}}{\prod_{j=1}^{k}[1+r_1^+(h)]^{\varpi}+\prod_{j=1}^{k}[1-r_1^+(h)]^{\varpi}} \right]; \\
&\left[\frac{2\prod_{j=1}^{k}[p_1(h)]^{\varpi}}{\prod_{j=1}^{k}[(2-p_1(h)]^{\varpi}+\prod_{j=1}^{k}[p_1(h)]^{\varpi}}, \frac{2\prod_{j=1}^{k}[q_1(h)]^{\varpi}}{\prod_{j=1}^{k}[(2-q_1(h)]^{\varpi}+\prod_{j=1}^{k}[q_1(h)]^{\varpi}}, \frac{2\prod_{j=1}^{k}[r_1(h)]^{\varpi}}{\prod_{j=1}^{k}[(2-r_1(h)]^{\varpi}+\prod_{j=1}^{k}[r_1(h)]^{\varpi}} \right] \oplus_{k+1} \\
&\left\langle \left[\frac{[\prod_{j=1}^{k+1}[1+\alpha_1(h)]^{\varpi}-\prod_{j=1}^{k+1}[1-\alpha_1(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+\alpha_1(h)]^{\varpi}+\prod_{j=1}^{k+1}[1-\alpha_1(h)]^{\varpi}}, \frac{[\prod_{j=1}^{k+1}[1+\beta_1(h)]^{\varpi}-\prod_{j=1}^{k+1}[1-\beta_1(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+\beta_1(h)]^{\varpi}+\prod_{j=1}^{k+1}[1-\beta_1(h)]^{\varpi}}, \frac{[\prod_{j=1}^{k+1}[1+\Delta_1(h)]^{\varpi}-\prod_{j=1}^{k+1}[1-\Delta_1(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+\Delta_1(h)]^{\varpi}+\prod_{j=1}^{k+1}[1-\Delta_1(h)]^{\varpi}} \right] \right. \\
&\left[\frac{[\prod_{j=1}^{k+1}[1+p_1^-(h)]^{\varpi}-\prod_{j=1}^{k+1}[1-p_1^-(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+p_1^-(h)]^{\varpi}+\prod_{j=1}^{k+1}[1-p_1^-(h)]^{\varpi}}, \frac{\prod_{j=1}^{k+1}[1+q_1^-(h)]^{\varpi}-\prod_{j=1}^{k+1}[1-q_1^-(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+q_1^-(h)]^{\varpi}+\prod_{j=1}^{k+1}[1-q_1^-(h)]^{\varpi}}, \frac{\prod_{j=1}^{k+1}[1+r_1^-(h)]^{\varpi}-\prod_{j=1}^{k+1}[1-r_1^-(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+r_1^-(h)]^{\varpi}+\prod_{j=1}^{k+1}[1-r_1^-(h)]^{\varpi}} \right]; \\
&\left[\frac{\prod_{j=1}^{k+1}[1+p_1^+(h)]^{\varpi}-\prod_{j=1}^{k+1}[1-p_1^+(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+p_1^+(h)]^{\varpi}+\prod_{j=1}^{k+1}[1-p_1^+(h)]^{\varpi}}, \frac{\prod_{j=1}^{k+1}[1+q_1^+(h)]^{\varpi}-\prod_{j=1}^{k+1}[1-q_1^+(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+q_1^+(h)]^{\varpi}+\prod_{j=1}^{k+1}[1-q_1^+(h)]^{\varpi}}, \frac{\prod_{j=1}^{k+1}[1+r_1^+(h)]^{\varpi}-\prod_{j=1}^{k+1}[1-r_1^+(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+r_1^+(h)]^{\varpi}+\prod_{j=1}^{k+1}[1-r_1^+(h)]^{\varpi}} \right]; \\
&\left[\frac{2\prod_{j=1}^{k+1}[p_1(h)]^{\varpi}}{\prod_{j=1}^{k+1}[(2-p_1(h)]^{\varpi}+\prod_{j=1}^{k+1}[p_1(h)]^{\varpi}}, \frac{2\prod_{j=1}^{k+1}[q_1(h)]^{\varpi}}{\prod_{j=1}^{k+1}[(2-q_1(h)]^{\varpi}+\prod_{j=1}^{k+1}[q_1(h)]^{\varpi}}, \frac{2\prod_{j=1}^{k+1}[r_1(h)]^{\varpi}}{\prod_{j=1}^{k+1}[(2-r_1(h)]^{\varpi}+\prod_{j=1}^{k+1}[r_1(h)]^{\varpi}} \right] \\
&= \left[\frac{[\prod_{j=1}^{k+1}[1+p_1^-(h)]^{\varpi}-\prod_{j=1}^{k}[1-p_1^-(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+p_1^-(h)]^{\varpi}+\prod_{j=1}^{k}[1-p_1^-(h)]^{\varpi}}, \frac{\prod_{j=1}^{k+1}[1+q_1^-(h)]^{\varpi}-\prod_{j=1}^{k}[1-q_1^-(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+q_1^-(h)]^{\varpi}+\prod_{j=1}^{k}[1-q_1^-(h)]^{\varpi}}, \frac{\prod_{j=1}^{k+1}[1+r_1^-(h)]^{\varpi}-\prod_{j=1}^{k}[1-r_1^-(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+r_1^-(h)]^{\varpi}+\prod_{j=1}^{k}[1-r_1^-(h)]^{\varpi}} \right], \\
&\left[\frac{\prod_{j=1}^{k+1}[1+p_1^+(h)]^{\varpi}-\prod_{j=1}^{k}[1-p_1^+(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+p_1^+(h)]^{\varpi}+\prod_{j=1}^{k}[1-p_1^+(h)]^{\varpi}}, \frac{\prod_{j=1}^{k+1}[1+q_1^+(h)]^{\varpi}-\prod_{j=1}^{k}[1-q_1^+(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+q_1^+(h)]^{\varpi}+\prod_{j=1}^{k}[1-q_1^+(h)]^{\varpi}}, \frac{\prod_{j=1}^{k+1}[1+r_1^+(h)]^{\varpi}-\prod_{j=1}^{k}[1-r_1^+(h)]^{\varpi}}{\prod_{j=1}^{k+1}[1+r_1^+(h)]^{\varpi}+\prod_{j=1}^{k}[1-r_1^+(h)]^{\varpi}} \right], \\
&\left[\frac{2\prod_{j=1}^{k+1}[p_1(h)]^{\varpi}}{\prod_{j=1}^{k+1}[(2-p_1(h)]^{\varpi}+\prod_{j=1}^{k+1}[p_1(h)]^{\varpi}}, \frac{2\prod_{j=1}^{k+1}[q_1(h)]^{\varpi}}{\prod_{j=1}^{k+1}[(2-q_1(h)]^{\varpi}+\prod_{j=1}^{k+1}[q_1(h)]^{\varpi}}, \frac{2\prod_{j=1}^{k+1}[r_1(h)]^{\varpi}}{\prod_{j=1}^{k+1}[(2-r_1(h)]^{\varpi}+\prod_{j=1}^{k+1}[r_1(h)]^{\varpi}} \right].
\end{aligned}
$$

Especially, if $w = \left(\frac{1}{n}, \frac{1}{n}, ..., \frac{1}{n}\right)^T$, then the the TNCFEWA operator is reduced to the triangular neutrosophic cubic fuzzy einstein averaging operator, which is shown as follows:

$$\left\langle \left[\frac{[\prod_{j=1}^{n}[1+\alpha_1(h)]^{\frac{1}{n}} - \prod_{j=1}^{n}[1-\alpha_1(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[1+\alpha_1(h)]^{\frac{1}{n}} + \prod_{j=1}^{n}[1-\alpha_1(h)]^{\frac{1}{n}}}, \frac{[\prod_{j=1}^{n}[1+\beta_1(h)]^{\frac{1}{n}} - \prod_{j=1}^{n}[1-\beta_1(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[1+\beta_1(h)]^{\frac{1}{n}} + \prod_{j=1}^{n}[1-\beta_1(h)]^{\frac{1}{n}}}, \frac{[\prod_{j=1}^{n}[1+\Delta_1(h)]^{\frac{1}{n}} - \prod_{j=1}^{n}[1-\Delta_1(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[1+\Delta_1(h)]^{\frac{1}{n}} + \prod_{j=1}^{n}[1-\Delta_1(h)]^{\frac{1}{n}}} \right] \right.$$

$$\left[\frac{[\prod_{j=1}^{n}[1+p_1^-(h)]^{\frac{1}{n}} - \prod_{j=1}^{n}[1-p_1^-(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[1+p_1^-(h)]^{\frac{1}{n}} + \prod_{j=1}^{n}[1-p_1^-(h)]^{\frac{1}{n}}}, \frac{\prod_{j=1}^{n}[1+q_1^-(h)]^{\frac{1}{n}} - \prod_{j=1}^{n}[1-q_1^-(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[1+q_1^-(h)]^{\frac{1}{n}} + \prod_{j=1}^{n}[1-q_1^-(h)]^{\frac{1}{n}}}, \frac{\prod_{j=1}^{n}[1+r_1^-(h)]^{\frac{1}{n}} - \prod_{j=1}^{n}[1-r_1^-(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[1+r_1^-(h)]^{\frac{1}{n}} + \prod_{j=1}^{n}[1-r_1^-(h)]^{\frac{1}{n}}} \right];$$

$$\left[\frac{\prod_{j=1}^{n}[1+p_1^+(h)]^{\frac{1}{n}} - \prod_{j=1}^{n}[1-p_1^+(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[1+p_1^+(h)]^{\frac{1}{n}} + \prod_{j=1}^{n}[1-p_1^+(h)]^{\frac{1}{n}}}, \frac{\prod_{j=1}^{n}[1+q_1^+(h)]^{\frac{1}{n}} - \prod_{j=1}^{n}[1-q_1^+(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[1+q_1^+(h)]^{\frac{1}{n}} + \prod_{j=1}^{n}[1-q_1^+(h)]^{\frac{1}{n}}}, \frac{\prod_{j=1}^{n}[1+r_1^+(h)]^{\frac{1}{n}} - \prod_{j=1}^{n}[1-r_1^+(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[1+r_1^+(h)]^{\frac{1}{n}} + \prod_{j=1}^{n}[1-r_1^+(h)]^{\frac{1}{n}}} \right];$$

$$\left. \left[\frac{2\prod_{j=1}^{n}[p_1(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[(2-p_1(h)]^{\frac{1}{n}} + \prod_{j=1}^{n}[p_1(h)]^{\frac{1}{n}}}, \frac{2\prod_{j=1}^{n}[q_1(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[(2-q_1(h)]^{\frac{1}{n}} + \prod_{j=1}^{n}[q_1(h)]^{\frac{1}{n}}}, \frac{2\prod_{j=1}^{n}[r_1(h)]^{\frac{1}{n}}}{\prod_{j=1}^{n}[(2-r_1(h)]^{\frac{1}{n}} + \prod_{j=1}^{n}[r_1(h)]^{\frac{1}{n}}} \right] \right\rangle.$$

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Neutrosophic modifications of Simplified TOPSIS for Imperfect Information (nS-TOPSIS)

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Abstract: Technique for order performance by similarity to ideal solution (TOPSIS) is a Multi-Criteria Decision-Making method (MCDM), that consists on handling real complex problems of decision-making. However, real MCDM problems are often involves imperfect information such as uncertainty and inconsistency. The imperfect information is often manipulated through Neutrosophics theory, using certain degree of truth (T), falsity degree (F) and indeterminacy degree(I). and thus single-valued neutrosophic set (SVNs) had prodded a strong capacity to model such complex information. To overcome that kind of problems, In this paper, first, the authors simplify the popular TOPSIS method to a lite TOPSIS (S-TOPSIS), that gives the same result as standard version. Second, mapping S-TOPSIS to Neutrosophics Environment, investigating SVNS, called nS-TOPSIS, to deal with imperfect information in the real decision-making problems. Numerical examples show the contributions of proposed S-TOPSIS method to get the same results with standard TOPSIS with simple way of calculus, and how Neutrosophic environment manage the uncertain information using SVN.

Keywords: Technique for order performance by similarity to ideal solution (TOPSIS), MCDM, Single-Valued Neutrosophic set(SVNs), Neutrosophic Simplified TOPSIS(nS-TOPSIS).

1 Introduction

Technique for Order Preference by Similarity to Ideal Solution(TOPSIS) is a popular Multicriteria Decision Making (MCDM). TOPSIS was first introduced by Hwang and Yoon ([1]) to deal with structuring Multicriteria issues with crisp numerical values in real situation. However, real MCDM problems are often formulated under as set of indeterminate or inconsistent information. Thus, TOPSIS consists on many complicate steps of calculation. To deal with thoses problems, First, we introduce a lite version of TOPSIS method (S-TOPSIS) with guaranty of obtention of the same results simplifying many complicated steps of calculation. Thus, we introduce single valued neutrosophic set (SVNs) modifications of Simplified TOPSIS (nS-TOPSIS).

To manage information outcome from real problem, that are usually endowed with imperfection such as uncertainty, fuzziness and inconsistency, Smarandache ([2,3]) initiated a new notion, which is a generalization of the Intuitionistic $Fuzzy$ Set (IFS), called Neutrosophics Set (NS), which based on three values (truth (T), indeterminacy (I), and falsity (F) membership degrees). The main propriety of NS is that the sum of three values is 3 instead of 1 in the case of IFS. Although, the NS as introduced by Smarandache was a philosophical concept, unable to be used in real study cases. Many researchers are working on to produce mathematical property, theories, Arithmetic Operations, etc. On the one hand, Wang and al. ([5]) embodied Neutrosophic concept in a metric, called single-valued neutrosophic set (SVNs) as three values in one ($truth-membership \quad degree$, $indeterminacy - membership \quad degree$, and $falsity - membership \quad degree$). In addition, Broumi and al. ([4,6,7]) defined, in Neutrosophic space, similarity mesure and distances metric between SVNS values. the defined SVNS show stronge power to modelize imperfect information, such as uncertainty, imprecise, incomplete, and inconsistent information.

On the other hand, Other researchers are working on deploying Neutrosophic in MCDM field. Biswas ([8]) proposed extended TOPSIS Method to deal with real MCDM problems based on weighted Neutrosophic and aggregated SVNS operators

Ye [9,10] introduced two concepts, single valued neutrosophic cross-entropy of single valued neutrosophic and weighted correlation coefficient of SVNSs into multicriteria decision-making problems. Deli et al. [11] studied deploying Bipolar Neutrosophic Sets in Multi-Criteria Decision Making field

The remainder of the paper presents the preliminaries to build our Method, TOPSIS method and single valued neutrosophic set (SVNs). next Simplified-TOPSIS as first contribution was introduced. Then, hybrid methods Neutrosophic-TOPSIS and Neutrosophic-Simplified-TOPSIS are proposed to deal with real example. Results and discussions are presented at the end of this paper.

2 TOPSIS method

Consider a multi-attribute decision making problem that could be formulated as follow, $A = \{A_1, A_2, \cdots, A_n\}$ a set of m preferences, and $C = \{C_1, C_2, \cdots, C_n\}$ a set of n criteria. The relationships between preferences A_i and criteria C_j quantified by rating a_{ij} provided by decision maker. Weight vector W is a set of weights ω_i associated to criteria C_j. The all details described above could be reshaped on decision matrix bellow, denoted by D.

$$D = (a_{ij})_{m\times n} = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix} \text{(Decision Matrix)} \tag{2.1}$$

Technique for order performance by similarity to ideal solution (TOPSIS) method summarized as follow:

Step 1: Calculate normalized form of decision matrix r_{ij} dividing each element a_{ij} on the sum of whole column.

$$r_{ij} = a_{ij} / \left(\sum_{i=1}^{m} a_{ij}^2 \right)^{0.5} ; \; j = 1, 2, \cdots, n; \; i = 1, 2 \cdots, m \tag{2.2}$$

Step 2: Calculate also weighted form v_{ij} of matrix r_{ij} obtained from previous step, multiplying each element r_{ij} by its associated weight w_j.

$$v_{ij} = w_j r_{ij}; \; j = 1, 2, \cdots, n; \; i = 1, 2 \cdots, m \tag{2.3}$$

Step 3: Based on the weighted decision matrix, we calculate positive ideal solution (POS) and negative ideal solution(NIS).

$$A^+ = (v_1^+, v_2^+, \cdots, v_n^+) = \left\{ \begin{array}{l} (max_i \{v_{ij} | j \in B\}), \\ (min_i \{v_{ij} | j \in C\}) \end{array} \right\} \tag{2.4}$$

$$A^- = (v_1^-, v_2^-, \cdots, v_n^-) = \left\{ \begin{array}{l} (min_i \{v_{ij} | j \in B\}), \\ (max_i \{v_{ij} | j \in C\}) \end{array} \right\} \tag{2.5}$$

B quantify the benefit set, and C is the cost attribute set. **Step 4:** By subtracting each weighted element v_{ij} From POS and NIS, we got tow vectors of separation measures cited below.

$$S_i^+ = \left\{ \sum_{j=1}^{n} (v_{ij} - v_j^+)^2 \right\}^{0.5} ; \; i = 1, 2 \cdots, m \tag{2.6}$$

$$S_i^- = \left\{ \sum_{j=1}^{n} (v_{ij} - v_j^-)^2 \right\}^{0.5} ; \; i = 1, 2 \cdots, m \tag{2.7}$$

Step 5: Using the both measures calculated in the previous step, we calculate the rating metric.

$$T_i = \frac{S_i^-}{(S_i^+ + S_i^-)}; \; i = 1, 2 \cdots, m \tag{2.8}$$

Once we calculate T_i that will be used to rank set of alternatives A_i.

2.1 Numerical example

Let consider the numerical example summarized by table Table-1. below, that contains alternatives with respect of criteria weights.

a_{ij}	C_1	C_2	C_3
ω_i	12/16	3/16	1/16
A_1	7	9	9
A_2	8	7	8
A_3	9	6	8
A_4	6	7	8

Table 1: Decision Matrix.

Table Table-2. is result of application of this formula $\sum_{i=1}^{n} a_{ij}$ on each column.

To determine Normalized matrix r_{ij} Table-3. each value is divide by $(\sum_{i=1}^{n} a_{ij}^2)^{1/2}$:

Weighted Decision matrix v_{ij} Table-4 is the multiplication of each column by w_j.

The table Table-5. below figure out the solution of the above MCDM problem listing furthermore, final rankings for decision matrix, separation metric from POS and NIS.

Preferences, in descending preference order, are ranked as $A_3 > A_1 > A_4 > A_2$ as showed in Table-5.

a_{ij}^2	C_1	C_2	C_3
ω_i	12/16	3/16	1/16
A_1	49	81	81
A_2	64	49	64
A_3	81	36	64
A_4	36	49	64
$\sum_{i=1}^{n} a_{ij}$	230	215	273

Table 2: Multiple decision matrix.

r_{ij}	C_1	C_2	C_3
ω_i	12/16	3/16	1/16
A_1	0.4616	0.6138	0.5447
A_2	0.5275	0.4774	0.4842
A_3	0.5934	0.4092	0.4842
A_4	0.3956	0.4774	0.4842
$\sum_{i=1}^{n} a_{ij}$	230	215	273

Table 3: Normalized decision matrix.

3 Simplified-TOPSIS method (our proposed method)

The Simplified-TOPSIS algorithmic consists on steps bellow :

Step 1: Structure the criteria of the decision-making problem under a hierarchy.

Let considere $C = \{C_1, C_2, \cdots, C_n\}$ is a set of Criteria, with $n \geq 2$, $A = \{A_1, A_2, \cdots, A_n\}$ is the set of Preferences (Alternatives), with $m \geq 1$, a_{ij} the score of preference i with respect to criterion j, and let ω_i weight of criteria C_i.

$$D = (a_{ij})_{m \times n} = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix} \quad \text{(Decision Matrix)} \tag{3.1}$$

Step 2: Calculation of the Weighted Decision Matrix v_{ij}.

Let v_{ij} Weighted Decision Matrix (WDM) that is obtained by multiplication of each column by its weight.

$$v_{ij} = w_j a_{ij};\ j = 1, 2, \cdots, n;\ i = 1, 2 \cdots, m \tag{3.2}$$

The difference between proposed method and standard TOPSIS section 2), the normalized step is ignored and WDM v_{ij} is calculated directly without normalization by multiplying a_{ij} with w_j.

Step 3: Determination of LIS and SIS.

The maximum (largest) ideal solution (LIS), as its name indicate, is the the set of maximums raws and smallest ideal solution (SIS) is the set of minimums raws.

$$A^+ = (v_1^+, v_2^+, \cdots, v_m^+) = (max_i \{v_{ij} | j = 1, 2, \cdots, n\}) \tag{3.3}$$

v_{ij}	$\boldsymbol{C_1}$	$\boldsymbol{C_2}$	$\boldsymbol{C_3}$
ω_i	12/16	3/16	1/16
A_1	0.3462	0.1151	0.0340
A_2	0.3956	0.0895	0.0303
A_3	0.4451	0.0767	0.0303
A_4	0.2967	0.0895	0.0303
v_{max}	0.4451	0.1151	0.0340
v_{min}	0.2967	0.0767	0.0303

Table 4: Weighted decision matrix.

Alternative	S_i^+	S_i^-	T_i
A_1	0.0989	0.0627	0.3880
A_2	0.0558	0.0997	0.6412
A_3	0.0385	0.1484	**0.7938**
A_4	0.1506	0.0128	0.0783

Table 5: Distance measure and ranking coefficient.

$$A^+ = (v_1^-, v_2^-, \cdots, v_m^-) = \left(\min_i \{v_{ij} | j = 1, 2, \cdots, n\}\right) \tag{3.4}$$

Step 4: Calculation of positive and negative solutions.

The positive and negative solution are the entropies of orders two of calculated using the formulas below respectively:

$$S_i^+ = \left\{\sum_{j=1}^{n} (v_{ij} - v_j^+)^2\right\}^{0.5}; \; i = 1, 2 \cdots, m \tag{3.5}$$

$$S_i^- = \left\{\sum_{j=1}^{n} (v_{ij} - v_j^-)^2\right\}^{0.5}; \; i = 1, 2 \cdots, m \tag{3.6}$$

Arrange preferences (set of alternatives A) based on value of sums of either alternative solutions $\left(S_i^+\right)$ or $\left(S_i^-\right)$. The choice of minimum or maximum depend on nature of problem, if the problem to be minimized or maximized

Step 5 (optional): Another step is missed in our Simplified TOPSIS is calculation of ranking measure T_i (relative closeness to the ideal solution), because of many reasons : first preferences can classified according to many aggregated measures calculated before, second, it's a way of normalization that can be changed by any form of normalization dividing by max, or normalized to $[0, 1]$ range, etc.

$$T_i = \frac{S_i^-}{(S_i^+ + S_i^-)}; \; i = 1, 2 \cdots, m \tag{3.7}$$

3.1 Numerical example

In order to check the consistency of our proposed method, the Simplified-TOPSIS method is applied on the same example (Decision Matrix presented in Table-1.) as classical TOPSIS.

a_{ij}	C_1	C_2	C_3
ω_i	12/16	3/16	1/16
A_1	7	9	9
A_2	8	7	8
A_3	9	6	8
A_4	6	7	8

Table 6: Decision matrix.

Weighed Decision Matrix is gotten (Table-2.).

$\omega_j a_{ij}$	C_1	C_2	C_3
ω_i	*12/16*	*3/16*	*1/16*
A_1	*84/16*	*27/16*	*9/16*
A_2	*96/16*	*21/16*	*8/16*
A_3	*108/16*	*18/16*	*8/16*
A_4	*72/16*	*21/16*	*8/16*

Table 7: Weighted decision matrix.

Next, we calculate the positive and negative solutions as follow :
S1+ = |84/16-108/16| + |27/16-27/16| + |9/16-9/16| = 1.5000
S2+ = |96/16-108/16| + |21/16-27/16| + |8/16-9/16| = 1.1875
S3+ = |108/16-108/16| + |18/16-27/16| + |8/16-9/16| = 0.6250
S4+ = |72/16-108/16| + |21/16-27/16| + |8/16-9/16| =2.6875
S1- = |84/16-72/16| + |27/16-18/16| + |9/16-8/16| =1.3750
S2- = |96/16-72/16| + |21/16-18/16| + |8/16-8/16| =1.6875
S3- = |108/16-72/16| + |18/16-18/16| + |8/16-8/16| =2.2500
S4- = |72/16-72/16| + |21/16-18/16| + |8/16-8/16| = 0.1875

By the end we got both sets of negative and positive solutions $(S3-, S2-, S1-, S4-)$ and $(S3+, S2+, S1+, S4+)$, before arranging preferences, we need to determine which solutions to use, that decision tacked based on the nature of problem, if we seek to minimize or maximize. The minimization of the solution, such as cost to pay, consists on the solution closer to the negative solution, while he maximization of the solution, such as price to sale, consists on the solution closer to the positive solution.

The optional ranking measure T_i confirm the same result.

$$T1 = (S1-)/[(S1-) + (S1+)] = 0.478261 \tag{3.8}$$

$$T2 = (S2-)/[(S2-) + (S2+)] = 0.586957 \tag{3.9}$$

$$T3 = (S3-)/[(S3-) + (S3+)] = 0.782609 \tag{3.10}$$

$$T4 = (S4-)/[(S4-) + (S4+)] = 0.065217 \tag{3.11}$$

The table (Table-8.) figure out all calculus did before

Alternative	S_i^+	S_i^-	T_i
A_1	1.5000	1.3750	0.478261
A_2	1.1875	1.6875	0.586957
A_3	0.6250	2.2500	0.782609
A_4	2.6875	0.1875	0.065217

Table 8: Distance measure and ranking coefficient.

By applying Simplified-TOPSIS, we get for T_3 (0.782609), T_2(0.586957), T_1(0.478261) and T_4(0.065217), and we got with classical TOPSIS T_3(0.7938), T_2(0.6412), T_1(0.3880) and T_4(0.0783). Hence the order obtained with our approach simplified-TOPSIS is the same of classical TOPSIS: T_3, T_2, T_1 and T_4, with little change in values between both approaches.

The both methods our simplified-TOPSIS and Standard TOPSIS produce the same results witt the same ranking $(T3, T2, T1 and then T4)$, with a little differences of ranking measures. For example, with Simplified-TOPSIS T_3 is 0.782609, and with TOPSIS T_3 is 0.7938, the same for all others(Simplified-TOPSIS : $T_2(0.586957)$, $T_1(0.478261)$ and $T_4(0.065217)$ and with TOPSIS : $T_2(0.6412)$, $T_1(0.3880)$ and $T_4(0.0783)$.

4 Standard TOPSIS in Neutrosophic [12]

Standard TOPSIS in Neutrosophic procedure can be summarized as follow :

Step 1: In order to apply neutrosophic TOPSIS algorithm, crisp number Decision Matrix need to be mapped to single valued neutrosophic environment, then, we got neutrosophic decision matrix

$$D = (d_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} = (T_{ij}, I_{ij}, F_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} \quad \text{(Neutrosophic Decision Matrix)} \tag{4.1}$$

Where T_{ij}, I_{ij} and F_{ij} are truth, indeterminacy and falsity membership scores respectively. i refer to preference A_i and j to creterion C_j.

And $w = (\omega_1, \omega_2, \cdots, \omega_n)$ with ω_i a single valued neutrosophic weight of criteria (so $\omega_i = (a_i, b_i, c_i)$).

Example 1:

To compare our method Neutrosophic Simplified TOPSIS (nS-TOPSIS : section 5) and standard NeutrosophicTOPSIS proposed by Biswas ([11]). we use Biswas's numercal example.

Let (DM_1, DM_2, DM_3, DM_4) fours decisions makers aims to select an alternative A_i $(A_1, A_2, A_3, A_4)$with respect six criteria$(C_1, C_2, C_3, C_4, C_5, C_6)$. The mapped weights of criteria and decision matrix in Neutrosophic environment are presented in tables Table-9. and Table-10. respectively.

	C_1	C_2	C_3
ω_i	$(0.755, 0.222, 0.217)$	$(0.887, 0.113, 0.107)$	$(0.765, 0.226, 0.182)$
	C_4	C_5	C_6
ω_i	$(0.692, 0.277, 0.251)$	$(0.788, 0.200, 0.180)$	$(0.700, 0.272, 0.244)$

Table 9: Criteria weights.

	$\boldsymbol{C_1}$	$\boldsymbol{C_2}$	$\boldsymbol{C_3}$
$\boldsymbol{A_1}$	$(0.864, 0.136, 0.081)$	$(0.853, 0.147, 0.: 092)$	$(0.800, 0.200, 0.150)$
$\boldsymbol{A_2}$	$(0.667, 0.333, 0.277)$	$(0.727, 0.273, 0.219)$	$(0.667, 0.333, 0.277)$
$\boldsymbol{A_3}$	$(0.880, 0.120, 0.067)$	$(0.887, 0.113, 0.064)$	$(0.834, 0.166, 0.112)$
$\boldsymbol{A_4}$	$(0.667, 0.333, 0.277)$	$(0.735, 0.265, 0.195)$	$(0.768, 0.232, 0.180)$
	$\boldsymbol{C_4}$	$\boldsymbol{C_5}$	$\boldsymbol{C_6}$
$\boldsymbol{A_1}$	$(0.704, 0.296, 0.241)$	$(0.823, 0.177, 0.123)$	$(0.864, 0.136, 0.081)$
$\boldsymbol{A_2}$	$(0.744, 0.256, 0.204)$	$(0.652, 0.348, 0.293)$	$(0.608, 0.392, 0.336)$
$\boldsymbol{A_3}$	$(0.779, 0.256, 0.204)$	$(0.811, 0.189, 0.109)$	$(0.850, 0.150, 0.092)$
$\boldsymbol{A_4}$	$(0.727, 0.273, 0.221)$	$(0.791, 0.209, 0.148)$	$(0.808, 0.192, 0.127)$

Table 10: Neutrosophic Decision Matrix.

Step 2: Weighted decision matrix in neutrosophic is gotten by applying aggregation operator of multiplication i. e. application of generalization of multiplication operator in Neutrosophic space.

$$D^w = D \otimes W = (d_{ij}^w)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} = (T_{ij}^w, I_{ij}^w, F_{ij}^w)_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} \tag{4.2}$$

Step 3: Calculate of POS-SVNs (positive ideal solution in SVNs) and NIS-SVNs (negative ideal solution in SVNS) measures.

$$T_j^{w+} = \left\{ \left(max_i \left\{ T_{ij}^{w_i} \left| j \in B \right. \right\} \right), \left(min_i \left\{ T_{ij}^{w_i} \left| j \in C \right. \right\} \right) \right\} \tag{4.3}$$

$$Q_N^+ = (d_1^{w+}, d_2^{w+}, \cdots, d_n^{w+}) \tag{4.4}$$

$$T_j^{w+} = \left\{ \left(max_i \left\{ T_{ij}^{w_j} \left| j \in B \right. \right\} \right), \left(min_i \left\{ T_{ij}^{w_j} \left| j \in C \right. \right\} \right) \right\} \tag{4.5}$$

$$I_j^{w+} = \left\{ \left(min_i \left\{ I_{ij}^{w_j} \left| j \in B \right. \right\} \right), \left(max_i \left\{ I_{ij}^{w_j} \left| j \in C \right. \right\} \right) \right\} \tag{4.6}$$

$$F_j^{w+} = \left\{ \left(min_i \left\{ F_{ij}^{w_j} \left| j \in B \right. \right\} \right), \left(max_i \left\{ F_{ij}^{w_j} \left| j \in C \right. \right\} \right) \right\} \tag{4.7}$$

$$Q_N^- = (d_1^{w-}, d_2^{w-}, \cdots, d_n^{w-}) \tag{4.8}$$

$$T_j^{w-} = \left\{ \left(min_i \left\{ T_{ij}^{w_j} \left| j \in B \right. \right\} \right), \left(max_i \left\{ T_{ij}^{w_j} \left| j \in C \right. \right\} \right) \right\} \tag{4.9}$$

$$I_j^{w-} = \left\{ \left(max_i \left\{ I_{ij}^{w_j} \left| j \in B \right. \right\} \right), \left(min_i \left\{ I_{ij}^{w_j} \left| j \in C \right. \right\} \right) \right\} \tag{4.10}$$

$$F_j^{w-} = \left\{ \left(max_i \left\{ F_{ij}^{w_j} \left| j \in B \right. \right\} \right), \left(min_i \left\{ F_{ij}^{w_j} \left| j \in C \right. \right\} \right) \right\} \tag{4.11}$$

Where B represents the benefit and C quantify the cost.

Step 4: Calculate length of each alternative from the POS-SVNs and NIS-SVNs calculated in previous step.

$$D_{Eu}^{i+}\left(d_{ij}^{wj}, d_{ij}^{w+}\right) = \sqrt{\frac{1}{3n}\sum_{j=1}^{n}\left\{\begin{array}{l}\left(T_{ij}^{wj}(x) - T_{ij}^{w+}(x)\right)^2 + \\ \left(I_{ij}^{wj}(x) - I_{ij}^{w+}(x)\right)^2 + \\ \left(F_{ij}^{wj}(x) - F_{ij}^{w+}(x)\right)^2\end{array}\right\}} \tag{4.12}$$

$$D_{Eu}^{i-}\left(d_{ij}^{wj}, d_{ij}^{w-}\right) = \sqrt{\frac{1}{3n}\sum_{j=1}^{n}\left\{\begin{array}{l}\left(T_{ij}^{wj}(x) - T_{ij}^{w-}(x)\right)^2 + \\ \left(I_{ij}^{wj}(x) - I_{ij}^{w-}(x)\right)^2 + \\ \left(F_{ij}^{wj}(x) - F_{ij}^{w-}(x)\right)^2\end{array}\right\}} \tag{4.13}$$

With $i = 1, 2 \cdots, m$

Step 5: Calculate the aggregated coefficient of closeness in Neutrosophic.

$$C_i^* = \frac{NS_i^-}{(NS_i^+ + NS_i^-)};\ i = 1, 2 \cdots, m \tag{4.14}$$

All values of aggregated coefficient of closeness are shown in the table Table-11. below.

Alternative	$\boldsymbol{C_i^*}$
A_1	0.8190
A_2	0.1158
A_3	**0.8605**
A_4	0.4801

Table 11: Closeness Coefficient.

Using the associate values of aggregated coefficient of closeness C_i^* to preference A_i, in descending order, to rank alternatives. Hence, preferences could be ordered as follow $A_3 > A_1 > A_4 > A_2$. Then, the alternative A_3 is the best solution.

5 Neutrosophic-Simplified-TOPSIS (our proposed method)

Step 1: Construct Neutrosophic decision matrix.

As made for Standard Neutrosophic TOPSIS, let consider neutrosophic decision matrix and SVNs weighted criteria.

$$D = (d_{ij})_{\substack{1 \le i \le n \\ 1 \le j \le m}} = (T_{ij}, I_{ij}, F_{ij})_{\substack{1 \le i \le n \\ 1 \le j \le m}} \tag{5.1}$$

$$\begin{array}{c c} & \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ \vdots \\ A_m \end{array} & \left(\begin{array}{cccc} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ d_{m1} & \cdots & \cdots & d_{mn} \end{array}\right) \end{array}$$

Where T_{ij} denote truth, I_{ij} indeterminacy and N_{ij} falsity membership score of preference i knowing j in neutrosophic environment.

$w = (\omega_1, \omega_2, \cdots, \omega_n)$ with ω_i a single valued neutrosophic weight of criteria (so $\omega_i = (a_i, b_i, c_i)$).

Step 2: Calculate SVNs weighted decision matrix.

$$D^w = D \otimes W = (d_{ij}^w)_{\substack{1 \le i \le n \\ 1 \le j \le m}} = \omega_j \otimes d_{ij}^w = (T_{ij}^w, I_{ij}^w, F_{ij}^w)_{\substack{1 \le i \le n \\ 1 \le j \le m}} \tag{5.2}$$

$$\omega_j \otimes d_{ij} = \left(a_j T_{ij}, b_j + I_{ij} - b_j I_{ij}, c_j + F_{ij} - c_j F_{ij}\right) \tag{5.3}$$

Step 3: Calculate LNIS and SNIS metrics.

LNIS and SNIS are maximum (larger) and minimum (smaller) neutrosophic ideal solution respectively.

$$A_N^+ = (d_1^{w+}, d_2^{w+}, \cdots, d_n^{w+}) \tag{5.4}$$

$$d_j^{\omega+} = \left(T_j^{\omega+}, I_j^{\omega+}, F_j^{\omega+}\right) \tag{5.5}$$

$$T_j^{w+} = \left\{\left(max_i \left\{T_{ij}^{w_j} \,| j = 1, \cdots, n\right\}\right)\right\} \tag{5.6}$$

$$I_j^{w+} = \left\{\left(min_i \left\{I_{ij}^{w_j} \,| j = 1, \cdots, n\right\}\right)\right\} \tag{5.7}$$

$$F_j^{w+} = \left\{\left(min_i \left\{F_{ij}^{w_j} \,| j = 1, \cdots, n\right\}\right)\right\} \tag{5.8}$$

$$A_N^- = (d_1^{w-}, d_2^{w-}, \cdots, d_n^{w-}) \tag{5.9}$$

$$d_j^{\omega-} = \left(T_j^{\omega-}, I_j^{\omega-}, F_j^{\omega-}\right) \tag{5.10}$$

$$T_j^{w-} = \left\{\left(min_i \left\{T_{ij}^{w_j} \,| j = 1, \cdots, n\right\}\right)\right\} \tag{5.11}$$

$$I_j^{w-} = \left\{\left(max_i \left\{I_{ij}^{w_j} \,| j = 1, \cdots, n\right\}\right)\right\} \tag{5.12}$$

$$F_j^{w-} = \left\{\left(max_i \left\{F_{ij}^{w_j} \,| j = 1, \cdots, n\right\}\right)\right\} \tag{5.13}$$

Step 4: Determination of the distance measure of every alternative from the RNPIS and the RNNIS for SVNSs.

To perform that calculus, we need to introduce a new distance measure, in this paper we mapped Manhattan distance ([13]) to Neutrosophic environment (definition 1). The new proposed distance called Neutrosophic Manhattan distance that perform the difference between two single-valued neutrosophic(SVNs) measures.

Definition 1. Let $X_1 = (x_1, y_1, z_1)$ and $X_2 = (x_2, y_2, z_2)$ be a SVN numbers. Then the separation measure between X_1 and X_2 based on Manhattan distance is defined as follows:

$$D_{Manh} (X_1, X_2) = |x_1 - x_2| + |y_1 - y_2| + |z_1 - x_2| \tag{5.14}$$

The application of Neutrosophic Manhattan distance to calculate the separation from the maximum and minimum Neutrosophic ideal solution respectively are :

$$D_{Manh}^{j+}\left(d_{ij}^{wj}, d_{ij}^{w+}\right) = \left\{ \begin{array}{l} \left|T_{ij}^{wj}(x) - T_{ij}^{w+}(x)\right| + \\ \left|I_{ij}^{wj}(x) - I_{ij}^{w+}(x)\right| + \\ \left|F_{ij}^{wj}(x) - F_{ij}^{w+}(x)\right| \end{array} \right\} \tag{5.15}$$

with $j = 1, 2 \cdots, n$

$$NS_i^+ = \sum_{j=1}^{n} D_{Manh}^{j+}\left(d_{ij}^{wj}, d_{ij}^{w+}\right) \tag{5.16}$$

with $i = 1, 2 \cdots, m$

Similarly, the separation from the minimum neutrosophic ideal solution is:

$$D_{Manh}^{j-}\left(d_{ij}^{wj}, d_{ij}^{w-}\right) = \left\{ \begin{array}{l} \left|T_{ij}^{wj}(x) - T_{ij}^{w-}(x)\right| + \\ \left|I_{ij}^{wj}(x) - I_{ij}^{w-}(x)\right| + \\ \left|F_{ij}^{wj}(x) - F_{ij}^{w-}(x)\right| \end{array} \right\} \tag{5.17}$$

with $j = 1, 2 \cdots, n$

$$NS_i^- = \sum_{j=1}^{n} D_{Manh}^{j-}\left(d_{ij}^{wj}, d_{ij}^{w-}\right) \tag{5.18}$$

with $i = 1, 2 \cdots, m$

Preferences are ordered regarding to the values of NS_i^- or according to $1/NS_i^+$. In other words, the alternatives with the highest appraisal score is the best solution.

Step 5: Rank the alternatives according to Ranking coefficient NT_i.

Ranking coefficient is formulated as :

$$NT_i = \frac{NS_i^-}{\left(NS_i^+ + NS_i^-\right)};\ i = 1, 2 \cdots, m \tag{5.19}$$

A set of alternatives can now be ranked according to the descending order of the value of NT_i

5.1 Numerical example

Step 1. Formulate the MCDM problem in neutrosophic by building Neutrosophic decision matrix decision matrix and SVNs weights of criteria.

Let $A_i\,(A_1, A_2, A_3, A_4)$a set of alternative and $C_i\,(C_1, C_2, C, C_4, C_5, C_6)$ a set of criteria. Let considers the following neutrosophic weights of criteria (Table-12.) and neutrosophic decision matrix (Table-13.) respectively (used in above example 1).

Step 2: Calculation of SVNs Weighted Decision Matrix

$$D^w = \left(d_{ij}^w\right)_{\substack{1 \le i \le n \\ 1 \le j \le m}} = \left(T_{ij}^w, I_{ij}^w, F_{ij}^w\right)_{\substack{1 \le i \le n \\ 1 \le j \le m}} \tag{5.20}$$

	C_1	C_2	C_3
ω_i	$(0.755, 0.222, 0.217)$	$(0.887, 0.113, 0.107)$	$(0.765, 0.226, 0.182)$
	C_4	C_5	C_6
ω_i	$(0.692, 0.277, 0.251)$	$(0.788, 0.200, 0.180)$	$(0.700, 0.272, 0.244)$

Table 12: Criteria neutrosophic weights.

d_{ij}	C_1	C_2	C_3
A_1	$(0.864, 0.136, 0.081)$	$(0.853, 0.147, 0.092)$	$(0.800, 0.200, 0.150)$
A_2	$(0.667, 0.333, 0.277)$	$(0.727, 0.273, 0.219)$	$(0.667, 0.333, 0.277)$
A_3	$(0.880, 0.120, 0.067)$	$(0.887, 0.113, 0.064)$	$(0.834, 0.166, 0.112)$
A_4	$(0.667, 0.333, 0.277)$	$(0.735, 0.265, 0.195)$	$(0.768, 0.232, 0.180)$
	C_4	C_5	C_6
A_1	$(0.704, 0.296, 0.241)$	$(0.823, 0.177, 0.123)$	$(0.864, 0.136, 0.081)$
A_2	$(0.744, 0.256, 0.204)$	$(0.652, 0.348, 0.293)$	$(0.608, 0.392, 0.336)$
A_3	$(0.779, 0.256, 0.204)$	$(0.811, 0.189, 0.109)$	$(0.850, 0.150, 0.092)$
A_4	$(0.727, 0.273, 0.221)$	$(0.791, 0.209, 0.148)$	$(0.808, 0.192, 0.127)$

Table 13: Neutrosophic Decision Matrix.

$$d_{ij}^{w} = \left(a_j T_{ij}, b_j + I_{ij} - b_j I_{ij}, c_j + F_{ij} - c_j F_{ij}\right) \tag{5.21}$$

SVNs Weighted Decision Matrix is obtained by multiplication of weights of criteria with its associated column of neutrosophic decision matrix:

$$T_{11}^{\omega} = 0.864 \times 0.755 = 0.6523$$

$$I_{11}^{\omega} = 0.136 + 0.222 - 0.136 \times 0.222 = 0.328$$

$$F_{11}^{\omega} = 0.081 + 0.217 - 0.081 \times 0.217 = 0.280$$

d_{ij}^{w}	C_1	C_2	C_3
A_1	$(0.6523, 0.328, 0.28)$	$(0.7566, 0.2434, 0.1892)$	$(0.612, 0.381, 0.305)$
A_2	$(0.5036, 0.481, 0.434)$	$(0.6448, 0.3552, 0.3026)$	$(0.510, 0.484, 0.409)$
A_3	$(0.6644, 0.315, 0.269)$	$(0.787, 0.2132, 0.1642)$	$(0.638, 0.354, 0.274)$
A_4	$(0.5036, 0.481, 0.434)$	$(0.6519, 0.3481, 0.2811)$	$(0.588, 0.406, 0.329)$
	C_4	C_5	C_6
A_1	$(0.487, 0.491, 0.432)$	$(0.649, 0.342, 0.281)$	$(0.605, 0.371, 0.305)$
A_2	$(0.515, 0.462, 0.404)$	$(0.514, 0.478, 0.420)$	$(0.426, 0.557, 0.498)$
A_3	$(0.539, 0.462, 0.404)$	$(0.639, 0.351, 0.269)$	$(0.595, 0.381, 0.314)$
A_4	$(0.503, 0.474, 0.417)$	$(0.623, 0.367, 0.301)$	$(0.566, 0.412, 0.34)$

Table 14: Weighted Neutrosophic decision matrix.

Step 3: Determination of LNIS and SNIS.

	C_1	C_2	C_3
$d_j^{\omega+}$	$(0.664, 0.315, 0.269)$	$(0.887, 0.213, 0.264)$	$(0.638, 0.354, 0.274)$
	C_4	C_5	C_6
$d_j^{\omega+}$	$(0.539, 0.462, 0.404)$	$(0.649, 0.341, 0.294)$	$(0.605, 0.371, 0.305)$

Table 15: Maximum (large) Neutrosophic Ideal Solution(LNIS).

	C_1	C_2	C_3
$d_j^{\omega-}$	$(0.504, 0.481, 0.434)$	$(0.645, 0.355, 0.303)$	$(0.510, 0.484, 0.409)$
	C_4	C_5	C_6
$d_j^{\omega-}$	$(0.487, 0.491, 0.432)$	$(0.514, 0.478, 0.420)$	$(0.426, 0.557, 0.498)$

Table 16: Minimum (smaller) Neutrosophic Ideal Solution (SNIS).

	NS_i^+	NS_i^-	NT_i
A_1	*0,324*	*2,07*	*0,86459295*
A_2	*2,31*	*0,084*	*0,03521102*
A_3	*0,047*	*2,347*	***0,98021972***
A_4	*1,293*	*1,101*	*0,45987356*

Table 17: Neutrosophic Separation Measures and Neutrosophic Measure Ranking.

Step 4: Calculation of NS_i^+ and NS_i^- To calculate NS_i^+ and NS_i^-, we calculate sum of each line, and then subtracting from the LNIS and from SNIS respectively.

According to the obtained result (Table-17.), alternatives can be ranked as follow $A_3 > A_1 > A_4 > A_2$. Then the best preference is A_3. Using the same example, our proposed method neutrosophic-simplified-TOPSIS(nTOPSIS), we get similar result as neutrosophic-TOPSIS.

6 Conclusion

This paper aims to present tow new TOPSIS based approaches for MCDM. First one is Simplified TOPSIS (sTOPSIS) that simplify the TOPSIS calculation procedure. Second one, neutrosophic simplified-TOPSIS (nTOPSIS) extend the proposed method to neutrosophic environment, that use, instead of crisp number, the single valued neutrosophic(SVN). To formulate the both proposed method, many measures are defined such as Neutrosophic Manhattan Distance measure, that is used to calculate, distances from Maximum (larger) Neutrosophic Ideal Solution (LNIS) minimum neutrosophic ideal solutions, as two new defined measures.

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Application of the Bipolar Neutrosophic Hamacher Averaging Aggregation Operators to Group Decision Making: An Illustrative Example

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Abstract: The present study aims to introduce the notion of bipolar neutrosophic Hamacher aggregation operators and to also provide its application in real life. Then neutrosophic set (NS) can elaborate the incomplete, inconsistent, and indeterminate information, Hamacher aggregation operators, and extended Einstein aggregation operators to the arithmetic and geometric aggregation operators. First, we give the fundamental definition and operations of the neutrosophic set and the bipolar neutrosophic set. Our main focus is on the Hamacher aggregation operators of bipolar neutrosophic, namely, bipolar neutrosophic Hamacher weighted averaging (BNHWA), bipolar neutrosophic Hamacher ordered weighted averaging (BNHOWA), and bipolar neutrosophic Hamacher hybrid averaging (BNHHA) along with their desirable properties. The prime gain of utilizing the suggested methods is that these operators progressively provide total perspective on the issue necessary for the decision makers. These tools provide generalized, increasingly exact, and precise outcomes when compared to the current methods. Finally, as an application, we propose new methods for the multi-criteria group decision-making issues by using the various kinds of bipolar neutrosophic operators with a numerical model. This demonstrates the usefulness and practicality of this proposed approach in real life.

Keywords: BNHWA aggregation operator; BNHOWA aggregation operator; BNHHA aggregation operator; score function; accuracy function; certainty function; group decision making

1. Introduction

In the recent era of decision making, there is often incomplete, indeterminate, and inconsistent information. Zadeh introduced the notion of fuzzy set [1], which deals with uncertainty and can be applied in many fields. However, it has a shortcoming, i.e., it only expresses membership value and is unable to express non-membership value. At that point, Atanassov [2] introduced the idea of intuitionistic fuzzy set (IFS) to address issues with the fuzzy set. Every component in IFS is shown by a structured pair, and every pair is portrayed by a membership value (truth-membership) $\zeta_A(p)$ and a non-membership value (falsity-membership) $Ю_A(p)$ that satisfy the conditions $\zeta(p), Ю(p) \in [0, 1]$ and $0 \leq \zeta(p), Ю(p) \leq 1$. IFSs can deal with incomplete data but cannot deal with the indeterminate and inconsistent data. Smarandache [3] developed the neutrosophic set (NS) by including an indeterminacy membership value $\Gamma'(p)$, which is a generality of IFS. NS can deal with information very effectively, i.e., incomplete, indeterminate, and inconsistent. When $\zeta(p) + \Gamma'(p) + \text{IO}(p) < 1$, it shows that this information is indeterminate and when $\zeta(p) + \Gamma'(p) + \text{IO}(p) > 1$, it is inconsistent information.

Single valued neutrosophic set (SVNS), as suggested by Wang et al. [4], was applied to decision making with the conditions $\zeta(p), \Gamma'(p), \text{IO}(p) \in [0,1]$ and $0 \leq \zeta(p) + \Gamma'(p) + \text{IO}(p) \leq 3$. Ye [5] introduced the correlation coefficient and also proposed the comparison method for SVNSs. Wang et al. [6] proposed the interval valued SVNSs to extend the truth, indeterminacy, and false membership to interval values. Ye [7] defined the similarity measures between interval valued neutrosophic sets on the basis of the Hamming and Euclidean distances and also proposed a multiple attribute decision-making method.

Aggregation operators are the important research areas, claiming the attention of today's researchers. Since the proposed theory of IFS, many scientists [8–15] have made essential contributions to the advancement of IFS theory. XU and Yager [12] developed the notion of aggregation operators based on IFS. They also applied these aggregation operators to decision making. Wang and Liu [16], proposed the idea of Einstein aggregation operators. Zhao and Wei [17] built up some of the Einstein hybrid aggregation operators. Fahmi et al. [18–21] developed aggregation operators based on triangular and trapezoidal cubic fuzzy numbers with applications to decision making. Rahman et al. [22–25] proposed aggregation operators on different extension of fuzzy numbers. The bipolar fuzzy set (BFS) [26–28] uses a substitute method to deal with uncertainty in decision making. The bipolar fuzzy set consists of a positive as well as negative membership degree. The membership degree of the bipolar fuzzy set ranges from -1 to 1. BFSs have been useful in various research domains and set theory decision analysis and organizational modeling [29], quantum computing [30], physics and philosophy [31], and graph theory [32]. Bipolar averaging and geometric fuzzy aggregations operators were defined by Gul [33]. Irfan et al. [34] presented the bipolar neutrosophic set with basic operations. They also proposed the comparison method for bipolar neutrosophic sets. Fan et al. [35] developed Heronian mean operators in a bipolar neutrosophic environment. Irfan et al. [36] presented the interval valued bipolar neutrosophic set with applications to pattern recognition. Irfan et al. [37] proposed the interval valued neutrosophic soft set with applications to decision making. Zhan et al. [38] proposed Schweizer-Sklar Muirhead mean aggregation operators based on single-valued neutrosophic set. Ashraf et al. introduced some logarithmic aggregation operators on neutrosophic sets [39].

Hamacher t-norm and t-conorm [40], which are the generalization of algebraic and Einstein t-norm and t-conorm, are more general and flexible. There are many researchers who extended the Hamacher operations to solve multiple attribute decision-making problems combined with other fuzzy environments, such as intuitionistic [41], interval valued intuitionistic [42], hesitant fuzzy [43], hesitant Pythagorean fuzzy [44], bipolar fuzzy numbers [45] and neutrosophic numbers [46,47]. Since the development of this field, there has been no significant research on Hamacher operations and its applicability to bipolar neutrosophic numbers. Here, we extended the Hamacher operations to bipolar neutrosophic numbers to develop bipolar neutrosophic Hamacher aggregation operators for multiple attribute decision-making problems.

Decision making plays a key role in present day management. Necessarily, sound decision making is a basic part of administration. Consciously or unconsciously, a manager makes a decision, or decisions, as it is his or her responsibility as a manager. Decision making has a consequential role as organizational and managerial activities are linked with that decision. A decision is explained as a sequence of actions, intentionally taken from a set of alternates, to accomplish managerial or organizational targets. The decision-making process is an incessant and obligatory part of organization management or activities that are carried out in business. Decisions are made to address the events of all activities related to business and organizations. Decision and economic theories are interconnected with the assumption that field experts make a decision to make the best use out of their personal interest and reasonableness. This, though, doesn't take into consideration the probabilities of intervening factors that make decision making dependent upon the situation. The aforementioned factors play a key role in normalizing decision making for the manager to achieve optimal targets.

Although there is much literature available regarding the present study, the points given below, connected to the bipolar neutrosophic set and its aggregation operator, motivated the researcher to construct a detail and deep inquiry in the present study. Our main rationalizations and are as follows:

(1) Single valued neutrosophic sets (SVNSs) help in dealing with uncertain information in a more reliable way. It is a generalization of classical set, fuzzy set, and intuitionistic fuzzy set etc., and adaptable to the framework in comparison with pre-existing fuzzy sets and its versions.
(2) To deal with uncertain real-life problems, bipolar fuzzy sets are of great value and prove to be helpful in dealing with the positive as well as the negative membership values.
(3) We tried to merge these ideas with the Hamacher aggregation operator and strive to develop a more effective tool to deal with uncertainty in the form of bipolar neutrosophic Haymaker averaging aggregation operators.
(4) The objective of the study was to propose bipolar neutrosophic Hamacher operators and also study its properties. Furthermore, we proposed three aggregation operators, namely bipolar neutrosophic Hamacher weighted averaging operators (BNHWA), and bipolar neutrosophic Hamacher ordered weighted averaging operators (BNHOWA) and bipolar neutrosophic Hamacher hybrid averaging operators (BNHHA). Multi-attribute decision making (MADM) program approach is established based on bipolar neutrosophic numbers
(5) So as to affirm the effectiveness of the proposed method, we applied bipolar neutrosophic numbers to the decision-making problem.
(6) The initial decision matrices were composed of bipolar neutrosophic numbers and transformed into a collective bipolar neutrosophic decision matrix.
(7) The proposed operators probably completely elaborate the vagueness of bipolar neutrosophic Hamacher aggregation operator.

The rest of the study is organized as follows:

- Section 2 comprises the fundamental definitions and their related properties, which are required later in paper.
- In Section 3 we introduce the BNHWA operator, BNHWOA operator and BNHHA operator.
- In Section 4, the new aggregation operators are applied to group decision making and we propose a numerical problem.
- In Section 5, there is a comparison of our method in relation to other methods and concluding remarks are also given.

2. Preliminaries

Fundamental definitions of neutrosophic set are provided in the current section for bipolar fuzzy set, bipolar neutrosophic set, score function, accuracy function, certainty function and Hamacher operations.

Definition 1. *[3] Let P be any fixed set. Then a neutrosophic set (NS) is as follows:*

$$B = \left\{ \left(p, \zeta(p), \Gamma'(p), \text{Ю}(p) \right) \middle| p \in P \right\}, \tag{1}$$

where the truth-membership is $\zeta : H \to E$, *the indeterminacy-membership is* $\Gamma' : H \to E$, *and the falsity-membership is* $\text{Ю} : H \to E$, *where* $E =]0^-, 1^+[.\zeta(p), \Gamma'(p)$ *and* $\text{Ю}(p)$ *are real standard or non-standards subsets of* $]0^-, 1^+[$, *which was proposed by Abraham Robinson in 1966 [48]. There is no restriction on the sum of* $\zeta(p), \Gamma'(p)$ *and* $\text{Ю}(p)$, *so* $0^- \leq \zeta(p) + \Gamma'(p) + \text{Ю}(p) \leq 3^+$.

As it is difficult to apply NS in real scientific and engineering areas, Wang et al. [4] proposed the concept of a single valued neutrosophic set (SVNS), which follows.

Definition 2. *[4] Let P be a non-empty set, with an element in P denoted by p, and then the single valued neutrosophic set (SVNS) of A in H is as follows:*

$$A_{NS} = \left\{\left(p, \zeta(p), \text{Ґ}(p), \text{Ю}(p)\right)\middle| p \in P\right\}, \tag{2}$$

where the truth-membership is $\zeta : H \to N$, *the indeterminacy-membership is* $\text{Ґ} : H \to N$, *and the falsity-membership is* $\text{Ю} : H \to N$, *where* $N = [0,1]$. *There is one condition, i.e.,*

$$0 \le \zeta(p) + \text{Ґ}(p) + \text{Ю}(p) \le 3.$$

The basic operations for two SVNSs are:

$$A_{NS} = \left\{\left(p, \zeta_A(p), \text{Ґ}_A(p), \text{Ю}_A(p)\right)\middle| p \in P\right\},\ \mathrm{B}_{NS} = \left\{\left(p, \zeta_B(p), \text{Ґ}_B(p), \text{Ю}_B(p)\right)\middle| p \in P\right\},$$

and are given as follows:

i. The subset $A_{NS} \subseteq B_{NS}$ if, and only if,

$$\zeta_A(p) \le \zeta_B(p), \text{Ґ}_A(p) \ge \text{Ґ}_B(p), \text{Ю}_A(p) \ge \text{Ю}_B(p).$$

ii. $A_{NS} = B_{NS}$ if, and only if,

$$\zeta_A(p) = \zeta_B(p), \text{Ґ}_A(p) = \text{Ґ}_B(p), \text{Ю}_A(p) = \text{Ю}_B(p).$$

iii. the complement A'_{NS} is

$$A'_{NS} = \left\{\left(p, \text{Ю}_A(p), 1 - \text{Ґ}_A(p), \zeta_A(p)\right)\middle| p \in P\right\}.$$

iv. the intersection is defined by

$$A_{NS} \cap B_{NS} = \left\{\left(p, \min\{\zeta_A(p), \zeta_B(p)\}, \max\left\{\text{Ґ}_A(p), \text{Ґ}_B(p)\right\}, \max\{\text{Ю}_A(p), \text{Ю}_B(p)\}\right)\middle| p \in P\right\}, \text{ and}$$

v. the union is defined by

$$A_{NS} \cup B_{NS} = \left\{\left(p, \max\{\zeta_A(p), \zeta_B(p)\}, \min\left\{\text{Ґ}_A(p), \text{Ґ}_B(p)\right\}, \min\{\text{Ю}_A(p), \text{Ю}_B(p)\}\right)\middle| p \in P\right\},$$

Definition 3. *[49] Let* $u_1 = \left(\zeta_1, \text{Ґ}_1, \text{Ю}_1\right)$ *and* $u_2 = \left(\zeta_2, \text{Ґ}_2, \text{Ю}_2\right)$ *be two single-value neutrosophic numbers (SVNNs). Then, the operations for the SVNNs are as follows:*

i. $u_1 + u_2 = \left(\zeta_1 + \zeta_2 - \zeta_1\zeta_2, \text{Ґ}_1\text{Ґ}_2, \text{Ю}_1\text{Ю}_2\right)$,

ii. $u_1 \cdot u_2 = \left(\zeta_1 + \zeta_2, \text{Ґ}_1 + \text{Ґ}_2 - \text{Ґ}_1\text{Ґ}_2, \text{Ю}_1 + \text{Ю}_2 - \text{Ю}_1\text{Ю}_2\right)$,

iii. $\lambda(u_1) = \left(1 - (1 - \zeta_1)^\lambda, \left(\text{Ґ}_1\right)^\lambda, (\text{Ю}_1)^\lambda\right)$ *and*

iv. $(u_1)^\lambda = \left((\zeta_1)^\lambda, 1 - \left(1 - \text{Ґ}_1\right)^\lambda, 1 - (1 - \text{Ю}_1)^\lambda\right)$,

where $\lambda > 0$.

Definition 4. *[49] Let* $u_1 = \left(\zeta_1, \text{Ґ}_1, \text{Ю}_1\right)$ *be a SVNN. Then, the score function* $s(u_1)$ *is as follows:*

$$s(u_1) = \frac{\left(\zeta_1 + 1 - \text{Ґ}_1 + 1 - \text{Ю}_1\right)}{3}.$$

Definition 5. *[49] Let* $u_1 = \left(\zeta_1, \acute{\Gamma}_1, \text{Ю}_1\right)$ *be a SVNN. Then, the accuracy function* $a(u_1)$ *is as follows:*

$$\mathrm{a}(u_1) = (\zeta_1 - \text{Ю}_1).$$

Definition 6. *[49] Let* $u_1 = \left(\zeta_1, \acute{\Gamma}_1, \text{Ю}_1\right)$ *be a SVNN. Then, the certainty function* $c(u_1)$ *is as follows:*

$$c(u_1) = \zeta_1\,.$$

Definition 7. *[49] Let* $u_1 = \left(\zeta_1, \acute{\Gamma}_1, \text{Ю}_1\right)$ *and* $u_2 = \left(\zeta_2, \acute{\Gamma}_2, \text{Ю}_2\right)$ *be two SVNNs. Then, the comparison is as follows:*

i. *if* $s(u_1) > s(u_2)$, *then* u_1 *is greater than* u_2, *denoted by* $u_1 > u_2$,
ii. *if* $s(u_1) = s(u_2)$ *and* $a(u_1) > a(u_2)$, *then* u_1 *is greater than* u_2, *denoted by* $u_1 > u_2$,
iii. *if* $s(u_1) = s(u_2)$, $a(u_1) = a(u_2)$ *and* $c(u_1) > c(u_2)$, *then* u_1 *is greater than* u_2, *denoted by* $u_1 > u_2$ *and*
iv. *if* $s(u_1) = s(u_2)$, $a(u_1) = a(u_2)$ *and* $c(u_1) = c(u_2)$, *then* u_1 *is equal to* u_2, *denoted by* $u_1 = u_2$.

Definition 8. *[28] Let P be a fixed set, and the bipolar fuzzy set is as follows:*

$$F = \left\{\left\langle p, \mu_F^+(p), \eta_F^-(p)\right\rangle \middle| p \in P\right\}, \tag{3}$$

where the positive degree of membership is $\mu_F^+(p) : H \to N$ *and the negative degree of membership is* $\eta_F^-(p) : H \to M$, *where* $N = [1, 0]$ *and* $M = [-1, 0]$.

Definition 9. *[34] A bipolar neutrosophic set (BNS), A in P, is as follows:*

$$A = \left\{\left(p, \zeta^+(p), \acute{\Gamma}^+(p), \text{Ю}^+(p), \zeta^-(p), \acute{\Gamma}^-(p), \text{Ю}^-(p)\right) \middle| p \in P\right\}, \tag{4}$$

Let $\zeta^+(p), \acute{\Gamma}^+(p), \text{Ю}^+(p) = BN^+$ and $\zeta^-(p), \acute{\Gamma}^-(p), \text{Ю}^-(p) = BN^-$, where, $\zeta^+(p), \acute{\Gamma}^+(p), \text{Ю}^+(p)$ is the positive degree of truth, the indeterminate and false membership of $p \in P$ and $\zeta^-(p), \acute{\Gamma}^-(p), \text{Ю}^-(p)$ is the negative degree of truth, the indeterminate and false membership of $p \in P$. Then $BN^+ : H \to N$ and $BN^- : H \to M$, where $N = [1, 0]$ and $M = [-1, 0]$. There are conditions where $0 \le \zeta^+(p) + \acute{\Gamma}^+(p) + \text{Ю}^+(p) + \zeta^-(p) + \acute{\Gamma}^-(p) + \text{Ю}^-(p) \le 6$.

Example 1. *Let* $P = \{p_1, p_2, p_3\}$, *then*

$$A = \left\{\begin{array}{c} (p_1, 0.1, 0.5, 0.3, -0.4, -0.5, -0.6), \\ (p_2, 0.2, 0.7, 0.4, -0.3, -0.6, -0.2), \\ (p_3, 0.4, 0.6, 0.7, -0.2, -0.3, -0.1) \end{array}\right\}$$

is a bipolar neutrosophic subset of P.

Basic operations [34], for two bipolar neutrosophic sets (BNSs), are as follows:
Let

$$A_1 = \left\{\left(p, \zeta_1^+(p), \acute{\Gamma}_1^+(p), \text{Ю}_1^+(p), \zeta_1^-(p), \acute{\Gamma}_1^-(p), \text{Ю}_1^-(p)\right) \middle| p \in P\right\}$$
$$\text{and } A_2 = \left\{\left(p, \zeta_2^+(p), \acute{\Gamma}_2^+(p), \text{Ю}_2^+(p), \zeta_2^-(p), \acute{\Gamma}_2^-(p), \text{Ю}_2^-(p)\right) \middle| p \in P\right\}$$

be two BNSs, then:

i. $A_1 \subseteq A_2$ if, and only if,

$$\zeta_1^+(p) \leq \zeta_2^+(p), \text{Ґ}_1^+(p) \leq \text{Ґ}_2^+(p), \text{Ю}_1^+(p) \geq \text{Ю}_2^+(p)$$

and

$$\zeta_1^-(p) \geq \zeta_2^-(p), \text{Ґ}_1^-(p) \geq \text{Ґ}_2^-(p), \text{Ю}_1^-(p) \leq \text{Ю}_2^-(p),$$

ii. $A_1 = A_2$ if, and only if,

$$\zeta_1^+(p) = \zeta_2^+(p), \text{Ґ}_1^+(p) = \text{Ґ}_2^+(p), \text{Ю}_1^+(p) = \text{Ю}_2^+(p)$$

and

$$\zeta_1^-(p) = \zeta_2^-(p), \text{Ґ}_1^-(p) = \text{Ґ}_2^-(p), \text{Ю}_1^-(p) = \text{Ю}_2^-(p),$$

iii. The union is defined as:

$$(A_1 \cup A_2) = \left\{\left(\max\left(\zeta_1^+(p), \zeta_2^+(p)\right), \frac{\text{Ґ}_1^+(p)+\text{Ґ}_2^+(p)}{2}, \min\left(\text{Ю}_1^+(p), \text{Ю}_2^+(p)\right), \min\left(\zeta_1^-(p), \zeta_2^-(p)\right), \frac{\text{Ґ}_1^-(p)+\text{Ґ}_2^-(p)}{2}, \max\left(\text{Ю}_1^-(p), \text{Ю}_2^-(p)\right)\right)\right\}, \text{ and}$$

iv. The intersection is defined as:

$$(A_1 \cap A_2) = \left\{\left(\min\left(\zeta_1^+(p), \zeta_2^+(p)\right), \frac{\text{Ґ}_1^+(p)+\text{Ґ}_2^+(p)}{2}, \max\left(\text{Ю}_1^+(p), \text{Ю}_2^+(p)\right), \max\left(\zeta_1^-(p), \zeta_2^-(p)\right), \frac{\text{Ґ}_1^-(p)+\text{Ґ}_2^-(p)}{2}, \min\left(\text{Ю}_1^-(p), \text{Ю}_2^-(p)\right)\right)\right\}.$$

Let $A = \left\{\left(p, \zeta^+(p), \text{Ґ}^+(p), \text{Ю}^+(p), \zeta^-(p), \text{Ґ}^-(p), \text{Ю}^-(p)\right) \middle| p \in P\right\}$ and be a BNS. Then the complement A^c is defined as:

$$\zeta_{A^c}^+(p) = \left\{1^+\right\} - \zeta_A^+(p), \text{Ґ}_{A^c}^+(p) = \left\{1^+\right\} - \text{Ґ}_A^+(p), \text{Ю}_{A^c}^+(p) = \left\{1^+\right\} - \text{Ю}_A^+(p)$$

and

$$\zeta_{A^c}^-(p) = \{1^-\} - \zeta_A^-(p), \text{Ґ}_{A^c}^-(p) = \{1^-\} - \text{Ґ}_A^-(p), \text{Ю}_{A^c}^-(p) = \{1^-\} - \text{Ю}_A^-(p).$$

Definition 10. *[34] Let* $u_1 = \left(\zeta_1^+, \text{Ґ}_1^+, \text{Ю}_1^+, \zeta_1^-, \text{Ґ}_1^-, \text{Ю}_1^-\right)$ *and* $u_2 = \left(\zeta_2^+, \text{Ґ}_2^+, \text{Ю}_2^+, \zeta_2^-, \text{Ґ}_2^-, \text{Ю}_2^-\right)$ *be two bipolar neutrosophic numbers (BNNs). Then, the operations for the BNNs are as follows:*

$$u_1 + u_2 = \left(\zeta_1^+ + \zeta_2^+ - \zeta_1^+\zeta_2^+, \text{Ґ}_1^+\text{Ґ}_2^+, \text{Ю}_1^+\text{Ю}_2^+, -\zeta_1^-\zeta_2^-, -\left(-\text{Ґ}_1^- - \text{Ґ}_2^- - \text{Ґ}_1^-\text{Ґ}_2^-\right), -\left(-\text{Ю}_1^- - \text{Ю}_2^- - \text{Ю}_1^-\text{Ю}_2^-\right)\right),$$

$$u_1 \cdot u_2 = \left(\zeta_1^+\zeta_2^+, \text{Ґ}_1^+ + \text{Ґ}_2^+ - \text{Ґ}_1^+\text{Ґ}_2^+, \text{Ю}_1^+ + \text{Ю}_2^+ - \text{Ю}_1^+\text{Ю}_2^+, -\left(-\zeta_1^- - \zeta_2^- - \zeta_1^-\zeta_2^-\right), -\text{Ґ}_1^-\text{Ґ}_2^-, -\text{Ю}_1^-\text{Ю}_2^-\right),$$

$$\lambda(u_1) = \left(1 - \left(1 - \zeta_1^+\right)^\lambda, \left(\text{Ґ}_1^+\right)^\lambda, \left(\text{Ю}_1^+\right)^\lambda, -\left(-\zeta_1^-\right)^\lambda, -\left(-\text{Ґ}_1^-\right)^\lambda, -\left(1 - \left(1 - \left(-\text{Ю}_1^-\right)\right)^\lambda\right)\right),$$

$$(u_1)^\lambda = \left(\left(\zeta_1^+\right)^\lambda, 1 - \left(1 - \text{Ґ}_1^+\right)^\lambda, 1 - \left(1 - \text{Ю}_1^+\right)^\lambda, -\left(1 - \left(1 - \left(-\zeta_1^-\right)\right)^\lambda\right), -\left(-\text{Ґ}_1^-\right)^\lambda, -\left(-\text{Ю}_1^-\right)^\lambda\right).$$

where $\lambda > 0$.

Definition 11. *[34] Let* $u = \left(\zeta^+, \text{Ґ}^+, \text{Ю}^+, \zeta^-, \text{Ґ}^-, \text{Ю}^-\right)$ *be a bipolar neutrosophic number (BNN), then the score function of u is as follows:*

$$S(u) = \frac{1}{6}\left(\zeta^+ + 1 - \text{Ґ}^+ + 1 - \text{Ю}^+ + 1 + \zeta^- - \text{Ґ}^- - \text{Ю}^-\right). \quad (5)$$

Definition 12. *[34] Let* $u = \left(\zeta^{+}, \Gamma^{+}, \text{Ю}^{+}, \zeta^{-}, \Gamma^{-}, \text{Ю}^{-}\right)$ *be a BNN, then the accuracy function of u is as follows:*

$$a(u) = \zeta^{+} - \text{Ю}^{+} + \zeta^{-} - \text{Ю}^{-}. \tag{6}$$

Definition 13. *[34] Let* $u = \left(\zeta^{+}, \Gamma^{+}, \text{Ю}^{+}, \zeta^{-}, \Gamma^{-}, \text{Ю}^{-}\right)$ *be a bipolar neutrosophic value, then the certainty function of u is as follows:*

$$c(u) = \zeta^{+} - \text{Ю}^{-}. \tag{7}$$

Definition 14. *[34] Let* $u_1 = \left(\zeta_1^{+}, \Gamma_1^{+}, \text{Ю}_1^{+}, \zeta_1^{-}, \Gamma_1^{-}, \text{Ю}_1^{-}\right)$ *and* $u_2 = \left(\zeta_2^{+}, \Gamma_2^{+}, \text{Ю}_2^{+}, \zeta_2^{-}, \Gamma_2^{-}, \text{Ю}_2^{-}\right)$ *be two BNNs, then the comparison method is as follows:*

(i) *If* $S(u_1) > S(u_2)$*, then* u_1 *is greater than* u_2*, denoted by* $u_1 > u_2$*,*
(ii) *If* $S(u_1) = S(u_2)$*, and* $a(u_1) > a(u_2)$*, then* u_1 *is superior to* u_2*, denoted by* $u_1 > u_2$*,*
(iii) *If* $S(u_1) = S(u_2)$*,* $a(u_1) = a(u_2)$ *and* $c(u_1) > c(u_2)$*, then* u_1 *is greater than* u_2*, denoted by* $u_1 > u_2$ *and*
(iv) *If* $S(u_1) = S(u_2)$*,* $a(u_1) = a(u_2)$ *and* $c(u_1) = c(u_2)$*, then* u_1 *is equal to* u_2*, denoted by* $u_1 = u_2$

Hamacher [40] proposed a more generalized t-norm and t-conorm. The Hamacher product, $\otimes$, is a t-norm and the Hamacher sum, $\oplus$, is a t-conorm, where:

$$T(a,b) = a \otimes b = \frac{ab}{\ddot{\gamma} + (1-\ddot{\gamma})(a+b-ab)}, \ddot{\gamma} > 0$$
$$T*(a,b) = a \oplus b = \frac{a+b-ab-(1-\ddot{\gamma})ab}{1-(1-\ddot{\gamma})ab}, \ddot{\gamma} > 0$$

when $\ddot{\gamma} = 1$, the Hamacher t-norm and t-conorm will be reduced to algebraic t-norm and *t*-conorm, respectively:

$$T(a,b) = a \otimes b = ab$$
$$T*(a,b) = a \oplus b = a + b - ab$$

when $\ddot{\gamma} = 2$, the Hamacher t-norm and t-conorm will be reduced to the Einstein t-norm and t-conorm, respectively [16]:

$$T(a,b) = a \otimes b = \frac{ab}{1+(1-a)(1-b)}$$
$$T*(a,b) = a \oplus b = \frac{a+b}{1+ab}$$

The following definitions introduce the Hamacher operations of bipolar neutrosophic set, as the notion of the bipolar neutrosophic Hamacher sum, product, scalar multiple and exponential operations are defined.

Definition 15. *Let* $u = \left(\zeta^{+}, \Gamma^{+}, \text{Ю}^{+}, \zeta^{-}, \Gamma^{-}, \text{Ю}^{-}\right)$*,* $u_1 = \left(\zeta_1^{+}, \Gamma_1^{+}, \text{Ю}_1^{+}, \zeta_1^{-}, \Gamma_1^{-}, \text{Ю}_1^{-}\right)$ *and* $u_2 = \left(\zeta_2^{+}, \Gamma_2^{+}, \text{Ю}_2^{+}, \zeta_2^{-}, \Gamma_2^{-}, \text{Ю}_2^{-}\right)$ *be three BNNs values, and* $\lambda > 0$ *be any real number, then we define basic Hamacher operators with* $\ddot{\gamma} > 0$.

$$u_1 \oplus u_2 = \left(\begin{array}{c} \frac{\zeta_1^{+}+\zeta_2^{+}-\zeta_1^{+}\zeta_2^{+}-(1-\ddot{\gamma})\zeta_1^{+}\zeta_2^{+}}{1-(1-\ddot{\gamma})\zeta_1^{+}\zeta_2^{+}}, \frac{\Gamma_1^{+}\Gamma_2^{+}}{\ddot{\gamma}+(1-\ddot{\gamma})\left(\Gamma_1^{+}+\Gamma_2^{+}-\Gamma_1^{+}\Gamma_2^{+}\right)}, \frac{\text{Ю}_1^{+}\text{Ю}_2^{+}}{\ddot{\gamma}+(1-\ddot{\gamma})\left(\text{Ю}_1^{+}+\text{Ю}_2^{+}-\text{Ю}_1^{+}\text{Ю}_2^{+}\right)}, \\ \frac{-\zeta_1^{-}\zeta_2^{-}}{\ddot{\gamma}+(1-\ddot{\gamma})\left(\zeta_1^{-}+\zeta_2^{-}-\zeta_1^{-}\zeta_2^{-}\right)}, \frac{-\left(-\Gamma_1^{-}-\Gamma_2^{-}-\Gamma_1^{-}\Gamma_2^{-}-(1-\ddot{\gamma})\Gamma_1^{-}\Gamma_2^{-}\right)}{1-(1-\ddot{\gamma})\Gamma_1^{-}\Gamma_2^{-}}, \frac{-\left(-\text{Ю}_1^{-}-\text{Ю}_2^{-}-\text{Ю}_1^{-}\text{Ю}_2^{-}-(1-\ddot{\gamma})\text{Ю}_1^{-}\text{Ю}_2^{-}\right)}{1-(1-\ddot{\gamma})\text{Ю}_1^{-}\text{Ю}_2^{-}} \end{array} \right) \tag{8}$$

$$u_1 \otimes u_2 = \left(\begin{array}{c} \frac{\zeta_1^{+}\zeta_2^{+}}{\ddot{\gamma}+(1-\ddot{\gamma})\left(\zeta_1^{+}+\zeta_2^{+}-\zeta_1^{+}\zeta_2^{+}\right)}, \frac{\Gamma_1^{+}+\Gamma_2^{+}-\Gamma_1^{+}\Gamma_2^{+}-(1-\ddot{\gamma})\Gamma_1^{+}\Gamma_2^{+}}{1-(1-\ddot{\gamma})\Gamma_1^{+}\Gamma_2^{+}}, \frac{\text{Ю}_1^{+}+\text{Ю}_2^{+}-\text{Ю}_1^{+}\text{Ю}_2^{+}-(1-\ddot{\gamma})\text{Ю}_1^{+}\text{Ю}_2^{+}}{1-(1-\ddot{\gamma})\text{Ю}_1^{+}\text{Ю}_2^{+}}, \\ \frac{-\left(-\zeta_1^{-}-\zeta_2^{-}-\zeta_1^{-}\zeta_2^{-}-(1-\ddot{\gamma})\zeta_1^{-}\zeta_2^{-}\right)}{1-(1-\ddot{\gamma})\zeta_1^{-}\zeta_2^{-}}, \frac{-\Gamma_1^{-}\Gamma_2^{-}}{\ddot{\gamma}+(1-\ddot{\gamma})\left(\Gamma_1^{-}+\Gamma_2^{-}-\Gamma_1^{-}\Gamma_2^{-}\right)}, \frac{-\text{Ю}_1^{-}\text{Ю}_2^{-}}{\ddot{\gamma}+(1-\ddot{\gamma})\left(\text{Ю}_1^{-}+\text{Ю}_2^{-}-\text{Ю}_1^{-}\text{Ю}_2^{-}\right)} \end{array} \right) \tag{9}$$

$$\lambda(u)=\left(\begin{array}{c}\frac{\left(1+(\ddot{\gamma}-1)\zeta^{+}\right)^{\lambda}-\left(1-\zeta^{+}\right)^{\lambda}}{\left(1+(\ddot{\gamma}-1)\zeta^{+}\right)^{\lambda}+(\ddot{\gamma}-1)\left(1-\zeta^{+}\right)^{\lambda}},\frac{\ddot{\gamma}\left(\Gamma^{+}\right)^{\lambda}}{\left(1+(\ddot{\gamma}-1)\left(1-\Gamma^{+}\right)\right)^{\lambda}+(\ddot{\gamma}-1)\left(\Gamma^{+}\right)^{\lambda}},\frac{\ddot{\gamma}\left(\text{Ю}^{+}\right)^{\lambda}}{\left(1+(\ddot{\gamma}-1)\left(1-\text{Ю}^{+}\right)\right)^{\lambda}+(\ddot{\gamma}-1)\left(\text{Ю}^{+}\right)^{\lambda}},\\ \frac{-\ddot{\gamma}|\zeta^{-}|^{\lambda}}{\left(1+(\ddot{\gamma}-1)(1+\zeta^{-})\right)^{\lambda}+(\ddot{\gamma}-1)|\zeta^{-}|^{\lambda}},\frac{-\ddot{\gamma}\left|\Gamma^{-}\right|^{\lambda}}{\left(1+(\ddot{\gamma}-1)\left(1+\Gamma^{-}\right)\right)^{\lambda}+(\ddot{\gamma}-1)\left|\Gamma^{-}\right|^{\lambda}},\frac{-\left(\left(1+(\ddot{\gamma}-1)|\text{Ю}^{-}|\right)^{\lambda}-(1+\text{Ю}^{-})^{\lambda}\right)}{\left(1+(\ddot{\gamma}-1)|\text{Ю}^{-}|\right)^{\lambda}+(\ddot{\gamma}-1)(1+\text{Ю}^{-})^{\lambda}}\end{array}\right) \tag{10}$$

$$(u)^{\lambda}=\left(\begin{array}{c}\frac{\left(\zeta^{+}\right)^{\lambda}}{\left(1+(\ddot{\gamma}-1)\left(1-\zeta^{+}\right)\right)^{\lambda}+(\ddot{\gamma}-1)\left(\zeta^{+}\right)^{\lambda}},\frac{\left(1+(\ddot{\gamma}-1)\Gamma^{+}\right)^{\lambda}-\left(1-\Gamma^{+}\right)^{\lambda}}{\left(1+(\ddot{\gamma}-1)\Gamma^{+}\right)^{\lambda}+(\ddot{\gamma}-1)\left(1-\Gamma^{+}\right)^{\lambda}},\frac{\left(1+(\ddot{\gamma}-1)\text{Ю}^{+}\right)^{\lambda}-\left(1-\text{Ю}^{+}\right)^{\lambda}}{\left(1+(\ddot{\gamma}-1)\text{Ю}^{+}\right)^{\lambda}+(\ddot{\gamma}-1)\left(1-\text{Ю}^{+}\right)^{\lambda}},\\ \frac{-\left(\left(1+(\ddot{\gamma}-1)|\zeta^{-}|\right)^{\lambda}-(1+\zeta^{-})^{\lambda}\right)}{\left(1+(\ddot{\gamma}-1)\zeta^{-}\right)^{\lambda}+(\gamma-1)(1+\zeta^{-})^{\lambda}},\frac{-\ddot{\gamma}\left|\Gamma^{-}\right|^{\lambda}}{\left(1+(\ddot{\gamma}-1)\left(1+\Gamma^{-}\right)\right)^{\lambda}+(\ddot{\gamma}-1)\left|\Gamma^{-}\right|^{\lambda}},\frac{-\ddot{\gamma}|\text{Ю}^{-}|^{\lambda}}{\left(1+(\ddot{\gamma}-1)(1+\text{Ю}^{-})\right)^{\lambda}+(\ddot{\gamma}-1)|\text{Ю}^{-}|^{\lambda}}\end{array}\right) \tag{11}$$

3. Bipolar Neutrosophic Hamacher Aggregation Operators

We propose some properties of the Hamacher aggregation operators in this part of the paper for bipolar neutrosophic Hamacher weighted averaging (BNHWA), bipolar neutrosophic Hamacher ordered weighted averaging (BNHOWA) and bipolar neutrosophic Hamacher hybrid averaging (BNHHA).

Let $u_{\ell}=\left(\zeta_{\ell}^{+},\Gamma_{\ell}^{+},\text{Ю}_{\ell}^{+},\zeta_{\ell}^{-},\Gamma_{\ell}^{-},\text{Ю}_{\ell}^{-}\right)$ be a family of BNNs, where $\ell\in Z$ and $Z=\{1,2,3,\ldots,n\}$.

3.1. Bipolar Neutrosophic HamacherWeighted Averaging Aggregation Operator

Definition 16. *The bipolar neutrosophic Hamacher weighted averaging (BNHWA) operator can be defined as follows:*

$$BNHWA_{\nu}(u_1,u_2,\ldots..,u_n)=\underset{\ell=1}{\overset{n}{\oplus}}(\nu_{\ell}u_{\ell})=\nu_1u_1\oplus\nu_2u_2\oplus\ldots\ldots..\oplus\nu_nu_n \tag{12}$$

where $\nu=(\nu_1,\nu_2,\ldots,\nu_n)^T$ is the weighted vector of u_{ℓ}, such that $\nu_{\ell}>0$ and $\sum_{\ell=1}^{n}\nu_{\ell}=1,\ddot{\gamma}>0$.

Theorem 1. *The (BNHWA) operator gives a bipolar neutrosophic value when:*

$$BNHWA_{\nu}(u_1,u_2,\ldots..,u_n)=$$
$$\left(\begin{array}{c}\frac{\prod_{\ell=1}^{n}(1+(\ddot{\gamma}-1)\zeta_{\ell}^{+})^{\nu_{\ell}}-\prod_{\ell=1}^{n}(1-\zeta_{\ell}^{+})^{\nu_{\ell}}}{\prod_{\ell=1}^{n}(1+(\ddot{\gamma}-1)\zeta_{\ell}^{+})^{\nu_{\ell}}+(\ddot{\gamma}-1)\prod_{\ell=1}^{n}(1-\zeta_{\ell}^{+})^{\nu_{\ell}}},\frac{\ddot{\gamma}\prod_{\ell=1}^{n}\left(\Gamma_{\ell}^{+}\right)^{\nu_{\ell}}}{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left(1-\Gamma_{\ell}^{+}\right)\right)^{\nu_{\ell}}+(\ddot{\gamma}-1)\prod_{\ell=1}^{n}\left(\Gamma_{\ell}^{+}\right)^{\nu_{\ell}}},\frac{\ddot{\gamma}\prod_{\ell=1}^{n}(\text{Ю}_{\ell}^{+})^{\nu_{\ell}}}{\prod_{\ell=1}^{n}(1+(\ddot{\gamma}-1)(1-\text{Ю}_{\ell}^{+}))^{\nu_{\ell}}+(\ddot{\gamma}-1)\prod_{\ell=1}^{n}(\text{Ю}_{\ell}^{+})^{\nu_{\ell}}},\\ \frac{-\ddot{\gamma}\prod_{\ell=1}^{n}|\zeta_{\ell}^{-}|^{\nu_{\ell}}}{\prod_{\ell=1}^{n}(1+(\ddot{\gamma}-1)(1+\zeta_{\ell}^{-}))^{\nu_{\ell}}+(\ddot{\gamma}-1)\prod_{\ell=1}^{n}|\zeta_{\ell}^{-}|^{\nu_{\ell}}},-\frac{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left|\Gamma_{\ell}^{-}\right|\right)^{\nu_{\ell}}-\prod_{\ell=1}^{n}\left(1+\Gamma_{\ell}^{-}\right)^{\nu_{\ell}}}{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left|\Gamma_{\ell}^{-}\right|\right)^{\nu_{\ell}}+(\ddot{\gamma}-1)\prod_{\ell=1}^{n}\left(1+\Gamma_{\ell}^{-}\right)^{\nu_{\ell}}},-\frac{\prod_{\ell=1}^{n}(1+(\ddot{\gamma}-1)|\text{Ю}_{\ell}^{-}|)^{\nu_{\ell}}-\prod_{\ell=1}^{n}(1+\text{Ю}_{\ell}^{-})^{\nu_{\ell}}}{\prod_{\ell=1}^{n}(1+(\ddot{\gamma}-1)|\text{Ю}_{\ell}^{-}|)^{\nu_{\ell}}+(\ddot{\gamma}-1)\prod_{\ell=1}^{n}(1+\text{Ю}_{\ell}^{-})^{\nu_{\ell}}}\end{array}\right) \tag{13}$$

where $\nu=(\nu_1,\nu_2,\ldots,\nu_n)^T$ *is the weighted vector of* u_{ℓ}*, such that* $\nu_{\ell}>0$ *and* $\sum_{\ell=1}^{n}\nu_{\ell}=1,\ddot{\gamma}>0$.

Proof. This theorem can be proved by mathematical induction as follows:

When $n=2$

$$\nu_1u_1=\left(\begin{array}{c}\frac{\left(1+(\ddot{\gamma}-1)\zeta_1^{+}\right)^{\nu_1}-\left(1-\zeta_1^{+}\right)^{\nu_1}}{\left(1+(\ddot{\gamma}-1)\zeta_1^{+}\right)^{\nu_1}+(\ddot{\gamma}-1)\left(1-\zeta_1^{+}\right)^{\nu_1}},\frac{\ddot{\gamma}\left(\Gamma_1^{+}\right)^{\nu_1}}{\left(1+(\ddot{\gamma}-1)\left(1-\Gamma_1^{+}\right)\right)^{\nu_1}+(\ddot{\gamma}-1)\left(\Gamma_1^{+}\right)^{\nu_1}},\frac{\ddot{\gamma}\left(\text{Ю}_1^{+}\right)^{\nu_1}}{\left(1+(\ddot{\gamma}-1)\left(1-\text{Ю}_1^{+}\right)\right)^{\nu_1}+(\ddot{\gamma}-1)\left(\text{Ю}_1^{+}\right)^{\nu_1}},\\ \frac{-\ddot{\gamma}|\zeta_1^{-}|^{\nu_1}}{\left(1+(\ddot{\gamma}-1)(1+\zeta_1^{-})\right)^{\nu_1}+(\ddot{\gamma}-1)|\zeta_1^{-}|^{\nu_1}},\frac{-\ddot{\gamma}\left|\Gamma_1^{-}\right|^{\nu_1}}{\left(1+(\ddot{\gamma}-1)\left(1+\Gamma_1^{-}\right)\right)^{\nu_1}+(\ddot{\gamma}-1)\left|\Gamma_1^{-}\right|^{\nu_1}},\frac{-((1+((\ddot{\gamma}-1)|\text{Ю}_1^{-}|))^{\nu_1}-(1+\text{Ю}_1^{-})^{\nu_1})}{\left(1+(\ddot{\gamma}-1)|\text{Ю}_1^{-}|\right)^{\nu_1}+(\ddot{\gamma}-1)\left(1+\text{Ю}_1^{-}\right)^{\nu_1}}\end{array}\right)$$

$$\nu_2u_2=\left(\begin{array}{c}\frac{\left(1+(\ddot{\gamma}-1)\zeta_2^{+}\right)^{\nu_2}-\left(1-\zeta_2^{+}\right)^{\nu_2}}{\left(1+(\ddot{\gamma}-1)\zeta_2^{+}\right)^{\nu_2}+(\ddot{\gamma}-1)\left(1-\zeta_2^{+}\right)^{\nu_2}},\frac{\ddot{\gamma}\left(\Gamma_2^{+}\right)^{\nu_2}}{\left(1+(\ddot{\gamma}-1)\left(1-\Gamma_2^{+}\right)\right)^{\nu_2}+(\ddot{\gamma}-1)\left(\Gamma_2^{+}\right)^{\nu_2}},\frac{\ddot{\gamma}\left(\text{Ю}_2^{+}\right)^{\nu_2}}{\left(1+(\ddot{\gamma}-1)\left(1-\text{Ю}_2^{+}\right)\right)^{\nu_2}+(\ddot{\gamma}-1)\left(\text{Ю}_2^{+}\right)^{\nu_2}},\\ \frac{-\ddot{\gamma}|\zeta_2^{-}|^{\nu_2}}{\left(1+(\ddot{\gamma}-1)(1+\zeta_2^{-})\right)^{\nu_2}+(\ddot{\gamma}-1)|\zeta_2^{-}|^{\nu_2}},\frac{-\ddot{\gamma}\left|\Gamma_2^{-}\right|^{\nu_2}}{\left(1+(\ddot{\gamma}-1)\left(1+\Gamma_2^{-}\right)\right)^{\nu_2}+(\ddot{\gamma}-1)\left|\Gamma_2^{-}\right|^{\nu_2}},\frac{-((1+((\ddot{\gamma}-1)|\text{Ю}_2^{-}|))^{\nu_2}-(1+\text{Ю}_2^{-})^{\nu_2})}{\left(1+(\ddot{\gamma}-1)|\text{Ю}_2^{-}|\right)^{\nu_2}+(\ddot{\gamma}-1)\left(1+\text{Ю}_2^{-}\right)^{\nu_2}}\end{array}\right)$$

and for

$$\nu_1 u_1 = \begin{pmatrix} \frac{(1+(\ddot{\gamma}-1)\zeta_1^+)-(1-\zeta_1^+)}{(1+(\ddot{\gamma}-1)\zeta_1^+)+(\ddot{\gamma}-1)(1-\zeta_1^+)}, \frac{\ddot{\gamma}(\Gamma_1^+)}{(1+(\ddot{\gamma}-1)(1-\Gamma_1^+))+(\ddot{\gamma}-1)(\Gamma_1^+)}, \frac{\ddot{\gamma}(\text{Ю}_1^+)}{(1+(\ddot{\gamma}-1)(1-\text{Ю}_1^+))+(\ddot{\gamma}-1)(\text{Ю}_1^+)}, \\ \frac{-\ddot{\gamma}|\zeta_1^-|}{(1+(\ddot{\gamma}-1)(1+\zeta_1^-))+(\ddot{\gamma}-1)|\zeta_1^-|}, -\frac{(1+((\ddot{\gamma}-1)|\Gamma_1^-|)-(1+\Gamma_1^-)}{(1+(\ddot{\gamma}-1)|\Gamma_1^-|)+(\ddot{\gamma}-1)(1+\Gamma_1^-)}, -\frac{(1+((\ddot{\gamma}-1)|\text{Ю}_1^-|)-(1+\text{Ю}_1^-)}{(1+(\ddot{\gamma}-1)|\text{Ю}_1^-|)+(\ddot{\gamma}-1)(1+\text{Ю}_1^-)} \end{pmatrix} \tag{14}$$

So it is proved for $n = 1$.

Now when $n = r$ in Equation (14), then

$$BNHWA_\nu(u_1, u_2, \ldots.., u_n) =$$

$$\begin{pmatrix} \frac{\prod_{\ell=1}^{r}(1+(\ddot{\gamma}-1)\zeta_\ell^+)^{\nu_\ell}-\prod_{\ell=1}^{r}(1-\zeta_\ell^+)^{\nu_\ell}}{\prod_{\ell=1}^{r}(1+(\ddot{\gamma}-1)\zeta_\ell^+)^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{r}(1-\zeta_\ell^+)^{\nu_\ell}}, \frac{\ddot{\gamma}\prod_{\ell=1}^{r}(\Gamma_\ell^+)^{\nu_\ell}}{\prod_{\ell=1}^{r}(1+(\ddot{\gamma}-1)(1-\Gamma_\ell^+))^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{r}(\Gamma_\ell^+)^{\nu_\ell}}, \frac{\ddot{\gamma}\prod_{\ell=1}^{r}(\text{Ю}_\ell^+)^{\nu_\ell}}{\prod_{\ell=1}^{r}(1+(\ddot{\gamma}-1)(1-\text{Ю}_\ell^+))^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{r}(\text{Ю}_\ell^+)^{\nu_\ell}}, \\ \frac{-\ddot{\gamma}\prod_{\ell=1}^{r}|\zeta_\ell^-|^{\nu_\ell}}{\prod_{\ell=1}^{r}(1+(\ddot{\gamma}-1)(1+\zeta_\ell^-))^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{r}|\zeta_\ell^-|^{\nu_\ell}}, \frac{-\ddot{\gamma}\prod_{\ell=1}^{r}|\Gamma_\ell^-|}{\prod_{\ell=1}^{r}(1+(\ddot{\gamma}-1)(1+\Gamma_\ell^-))^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{r}|\Gamma_\ell^-|^{\nu_\ell}}, \frac{-\prod_{j=1}^{k}(1-(\ddot{\gamma}-1)|\text{Ю}_\ell^-|)^{\nu_\ell}-\prod_{\ell=1}^{r}(1-\text{Ю}_\ell^-)^{\nu_\ell}}{\prod_{\ell=1}^{r}(1-(\ddot{\gamma}-1)|\text{Ю}_\ell^-|)^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{r}(1+\text{Ю}_\ell^-)^{\nu_\ell}} \end{pmatrix}$$

This proves that it is true for $n = r$

When $n = r + 1$, then

$$\begin{aligned} & BNHWA_\nu(u_1, u_2, \ldots.., u_n) \\ &= \begin{pmatrix} \frac{\prod_{\ell=1}^{r}(1+(\ddot{\gamma}-1)\zeta_\ell^+)^{\nu_\ell}-\prod_{\ell=1}^{r}(1-\zeta_\ell^+)^{\nu_\ell}}{\prod_{\ell=1}^{r}(1+(\ddot{\gamma}-1)\zeta_\ell^+)^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{r}(1-\zeta_\ell^+)^{\nu_\ell}}, \frac{\ddot{\gamma}\prod_{\ell=1}^{r}(\Gamma_\ell^+)^{\nu_\ell}}{\prod_{\ell=1}^{r}(1+(\ddot{\gamma}-1)(1-\Gamma_\ell^+))^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{r}(\Gamma_\ell^+)^{\nu_\ell}}, \\ \frac{\ddot{\gamma}\prod_{\ell=1}^{r}(\text{Ю}_\ell^+)^{\nu_\ell}}{\prod_{\ell=1}^{r}(1+(\ddot{\gamma}-1)(1-\text{Ю}_\ell^+))^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{r}(\text{Ю}_\ell^+)^{\nu_\ell}}, \frac{-\ddot{\gamma}\prod_{\ell=1}^{r}(|\zeta_\ell^-|)^{\nu_\ell}}{\prod_{\ell=1}^{r}(1+(\ddot{\gamma}-1)(1+\zeta_\ell^-))^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{r}|\zeta_\ell^-|^{\nu_\ell}}, \\ \frac{-\ddot{\gamma}\prod_{\ell=1}^{r}|\Gamma_\ell^-|}{\prod_{\ell=1}^{r}(1+(\ddot{\gamma}-1)(1+\Gamma_\ell^-))^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{r}|\Gamma_\ell^-|^{\nu_\ell}}, -\frac{\prod_{\ell=1}^{r}(1+(\ddot{\gamma}-1)|\text{Ю}_\ell^-|)^{\nu_\ell}-\prod_{\ell=1}^{r}(1+\text{Ю}_\ell^-)^{\nu_\ell}}{\prod_{\ell=1}^{r}(1+(\ddot{\gamma}-1)|\text{Ю}_\ell^-|)^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{r}(1+\text{Ю}_\ell^-)^{\nu_\ell}} \end{pmatrix} \\ &\oplus \begin{pmatrix} \frac{(1+(\ddot{\gamma}-1)\zeta_{r+1}^+)^{\nu_{r+1}}-(1-\zeta_{r+1}^+)^{\nu_{r+1}}}{(1+(\ddot{\gamma}-1)\zeta_{r+1}^+)^{\nu_{r+1}}+(\ddot{\gamma}-1)(1-\zeta_{r+1}^+)^{\nu_{r+1}}}, \frac{\ddot{\gamma}(\Gamma_{r+1}^+)^{\nu_{r+1}}}{(1+(\ddot{\gamma}-1)(1-\Gamma_{r+1}^+))^{\nu_{r+1}}+(\ddot{\gamma}-1)(\Gamma_{r+1}^+)^{\nu_{r+1}}}, \\ \frac{\ddot{\gamma}(\text{Ю}_{r+1}^+)^{\nu_{r+1}}}{(1+(\ddot{\gamma}-1)(1-\text{Ю}_{r+1}^+))^{\nu_{r+1}}+(\ddot{\gamma}-1)(\text{Ю}_{r+1}^+)^{\nu_{r+1}}}, \frac{-\ddot{\gamma}|\zeta_{r+1}^-|^{\nu_{r+1}}}{(1+(\ddot{\gamma}-1)(1+\zeta_{r+1}^-))^{\nu_{r+1}}+(\ddot{\gamma}-1)|\zeta_{r+1}^-|^{\nu_{r+1}}}, \\ \frac{-\ddot{\gamma}|\Gamma_{r+1}^-|^{\nu_{r+1}}}{(1+(\ddot{\gamma}-1)(1+\Gamma_{r+1}^-))^{\nu_{r+1}}+(\ddot{\gamma}-1)|\Gamma_{r+1}^-|^{\nu_{r+1}}}, -\frac{(1-(1+(\ddot{\gamma}-1)|\text{Ю}_{r+1}^-|))^{\nu_{r+1}}-(1+\text{Ю}_{r+1}^-)^{\nu_{r+1}})}{(1+(\ddot{\gamma}-1)|\text{Ю}_{r+1}^-|)^{\nu_{r+1}}+(\ddot{\gamma}-1)(1+\text{Ю}_{r+1}^-)^{\nu_{r+1}}} \end{pmatrix} \\ &= \begin{pmatrix} \frac{\prod_{\ell=1}^{r+1}(1+(\ddot{\gamma}-1)\zeta_\ell^+)^{\nu_\ell}-\prod_{\ell=1}^{r+1}(1-\zeta_\ell^+)^{\nu_\ell}}{\prod_{\ell=1}^{r+1}(1+(\ddot{\gamma}-1)\zeta_\ell^+)^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{r+1}(1-\zeta_\ell^+)^{\nu_\ell}}, \frac{\ddot{\gamma}\prod_{\ell=1}^{r+1}(\Gamma_\ell^+)^{\nu_\ell}}{\prod_{\ell=1}^{r+1}(1+(\ddot{\gamma}-1)(1-\Gamma_\ell^+))^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{r+1}(\Gamma_\ell^+)^{\nu_\ell}}, \\ \frac{\ddot{\gamma}\prod_{\ell=1}^{r+1}(\text{Ю}_\ell^+)^{\nu_\ell}}{\prod_{\ell=1}^{r+1}(1+(\ddot{\gamma}-1)(1-\text{Ю}_\ell^+))^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{r+1}(\text{Ю}_\ell^+)^{\nu_\ell}}, \frac{-\ddot{\gamma}\prod_{\ell=1}^{r+1}(|\zeta_\ell^-|)^{\nu_\ell}}{\prod_{\ell=1}^{r+1}(1+(\ddot{\gamma}-1)(1+\zeta_\ell^-))^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{r+1}|\zeta_\ell^-|^{\nu_\ell}}, \\ \frac{-\ddot{\gamma}\prod_{\ell=1}^{r+1}|\Gamma_\ell^-|^{\nu_\ell}}{\prod_{\ell=1}^{r+1}(1+(\ddot{\gamma}-1)(1+\Gamma_\ell^-))^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{r+1}|\Gamma_\ell^-|^{\nu_\ell}}, -\frac{\prod_{\ell=1}^{r+1}(1+(\ddot{\gamma}-1)|\text{Ю}_\ell^-|)^{\nu_\ell}-\prod_{\ell=1}^{r+1}(1+\text{Ю}_\ell^-)^{\nu_\ell}}{\prod_{\ell=1}^{r+1}(1+(\ddot{\gamma}-1)|\text{Ю}_\ell^-|)^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{r+1}(1+\text{Ю}_\ell^-)^{\nu_\ell}} \end{pmatrix}. \end{aligned}$$

Thus, Equation (14) is true for $n = r + 1$, which proves Theorem 1. □

Theorem 2. *(Idempotency) Let* $u_\ell = \left(\zeta_\ell^+, \text{Ґ}_\ell^+, \text{Ю}_\ell^+, \zeta_\ell^-, \text{Ґ}_\ell^-, \text{Ю}_\ell^-\right)$, *where* $\ell \in Z$ *and* $Z = \{1, 2, 3, \ldots., n\}$ *be a collection of BNNs are equal, i.e.,* $u_\ell = u$ *for all* ℓ, *then:*

$$BNHWA_\nu(u_1, u_2, \ldots.., u_n) = u.$$

Theorem 3. *(Boundedness) Let* $u^- = \underset{\ell}{min} u_\ell$, $u^+ = \underset{\ell}{max} u_\ell$, *then:*

$$u^- \leq BNHWA_\nu(u_1, u_2, \ldots.., u_n) \leq u^+.$$

Theorem 4. *(Monotonicity) Let* $u_\ell = \left(\zeta_\ell^+, \text{Ґ}_\ell^+, \text{Ю}_\ell^+, \zeta_\ell^-, \text{Ґ}_\ell^-, \text{Ю}_\ell^-\right)$, *where* $\ell \in Z$ *and* $Z = \{1, 2, 3, \ldots., n\}$, *and* $u'_\ell = \left(\zeta'^+_\ell, \text{Ґ}'^+_\ell, \text{Ю}'^+_\ell, \zeta'^-_\ell, \text{Ґ}'^-_\ell, \text{Ю}'^-_\ell\right)$, *where* $\ell \in Z$ *and* $Z = \{1, 2, 3, \ldots., n\}$ *are two BNNs. If* $u_\ell \leq u'_\ell$, *for all* ℓ, *then:*

$$BNHWA_\nu(u_1, u_2, \ldots.., u_n) \leq BNHWA_\nu\left(u'_1, u'_2, \ldots.., u'_n\right)$$

A discussion of two cases of BNHWA operator follows.

- If $\ddot{\gamma} = 1$, then the BNHWA is converted to the bipolar neutrosophic weighted average (BNWA):

$$BNWA_\nu(u_1, u_2, \ldots.., u_n) = \overset{n}{\underset{\ell=1}{\oplus}} (\nu_\ell u_\ell) = \begin{pmatrix} 1 - \prod_{\ell=1}^{n}\left(1 - \zeta_\ell^+\right)^{\nu_\ell}, \prod_{\ell=1}^{n}\left(\text{Ґ}_\ell^+\right)^{\nu_\ell}, \prod_{\ell=1}^{n}\left(\text{Ю}_\ell^+\right)^{\nu_\ell}, \\ -\prod_{\ell=1}^{n}\left|\zeta_\ell^-\right|^{\nu_\ell}, -\left(1 - \left(\prod_{\ell=1}^{n}\left(1 + \text{Ґ}_\ell^-\right)^{\nu_\ell}\right)\right), -\left(1 - \left(\prod_{\ell=1}^{n}\left(1 + \text{Ю}_\ell^-\right)^{\nu_\ell}\right)\right) \end{pmatrix}.$$

- If $\ddot{\gamma} = 2$, then the BNHWA is converted to the bipolar neutrosophic Einstein weighted average (BNEWA):

$$BNEWA_\nu(u_1, u_2, \ldots.., u_n) = \overset{n}{\underset{\ell=1}{\oplus}} (\nu_\ell u_\ell) =$$
$$\begin{pmatrix} \frac{\prod_{\ell=1}^{n}(1+\zeta_\ell^+)^{\nu_\ell} - \prod_{\ell=1}^{n}(1-\zeta_\ell^+)^{\nu_\ell}}{\prod_{\ell=1}^{n}(1+\zeta_\ell^+)^{\nu_\ell} + \prod_{\ell=1}^{n}(1-\zeta_\ell^+)^{\nu_\ell}}, \frac{\prod_{\ell=1}^{n}\left(\text{Ґ}_\ell^+\right)^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(2-\text{Ґ}_\ell^+\right)^{\nu_\ell} + \prod_{\ell=1}^{n}\left(\text{Ґ}_\ell^+\right)^{\nu_\ell}}, \frac{\prod_{\ell=1}^{n}(\text{Ю}_\ell^+)^{\nu_\ell}}{\prod_{\ell=1}^{n}(2-\text{Ю}_\ell^+)^{\nu_\ell} + \prod_{\ell=1}^{n}(\text{Ю}_\ell^+)^{\nu_\ell}}, \\ \frac{-2\prod_{\ell=1}^{n}\left|\zeta_\ell^-\right|^{\nu_\ell}}{\prod_{\ell=1}^{n}(2+\zeta_\ell^-)^{\nu_\ell} + \prod_{\ell=1}^{n}\left|\zeta_\ell^-\right|^{\nu_\ell}}, -\frac{\prod_{\ell=1}^{n}\left(1+\left|\text{Ґ}_\ell^-\right|\right)^{\nu_\ell} - \prod_{\ell=1}^{n}\left(1+\text{Ґ}_\ell^-\right)^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(1+\left|\text{Ґ}_\ell^-\right|\right)^{\nu_\ell} + \prod_{\ell=1}^{n}\left(1+\text{Ґ}_\ell^-\right)^{\nu_\ell}}, -\frac{\prod_{\ell=1}^{n}(1+|\text{Ю}_\ell^-|)^{\nu_\ell} - \prod_{\ell=1}^{n}(1+\text{Ю}_\ell^-)^{\nu_\ell}}{\prod_{\ell=1}^{n}(1+|\text{Ю}_\ell^-|)^{\nu_\ell} + \prod_{\ell=1}^{n}(1+\text{Ю}_\ell^-)^{\nu_\ell}} \end{pmatrix}.$$

3.2. Bipolar Neutrosophic Hamacher OrderedWeighted Averaging Aggregation Operator

Definition 17. *The bipolar neutrosophic Hamacher ordered weighted averaging (BNHOWA) operator can be defined as follows:*

$$BNHOWA_\nu(u_1, u_2, \ldots\ldots, u_n) = \overset{n}{\underset{\ell=1}{\oplus}}\left(\nu_\ell u_{\rho(\ell)}\right) = \nu_1 u_{\rho(1)} \oplus \nu_2 u_{\rho(2)} \oplus \nu_3 u_{\widetilde{\sigma}(3)} \oplus \ldots\ldots\ldots.. \oplus \nu_n u_{\rho(n)}, \quad (15)$$

where $(\rho(1), \rho(2), \ldots\ldots\ldots, \rho(n))$ *is a permutation with* $u_{\rho(\ell-1)} \geq u_{\rho(\ell)}$, $\forall \ell \in Z$, $Z = \{1, 2, 3, \ldots., n\}$, *and* $\nu = (\nu_1, \nu_2, \ldots, \nu_n)^T$ *is the weighted vector of* u_ℓ *such that* $\nu_\ell > 0$ *and* $\sum_{\ell=1}^{n} \nu_\ell = 1, \ddot{\gamma} > 0$.

Theorem 5. *The (BNHOWA) operator gives a bipolar neutrosophic value:*

$$BNHOWA_{\nu}(u_1, u_2, \ldots.., u_n) = \left(\begin{array}{l} \frac{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\zeta^{+}_{\rho(\ell)}\right)^{\nu_\ell}-\prod_{\ell=1}^{n}\left(1-\zeta^{+}_{\rho(\ell)}\right)^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\zeta^{+}_{\rho(\ell)}\right)^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{n}\left(1-\zeta^{+}_{\rho(\ell)}\right)^{\nu_\ell}}, \frac{\ddot{\gamma}\prod_{\ell=1}^{n}\left(\Gamma^{+}_{\rho(\ell)}\right)^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left(1-\Gamma^{+}_{\rho(\ell)}\right)\right)^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{n}\left(\Gamma^{+}_{\rho(\ell)}\right)^{\nu_\ell}}, \\ \frac{\ddot{\gamma}\prod_{\ell=1}^{n}\left(\text{Ю}^{+}_{\rho(\ell)}\right)^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left(1-\text{Ю}^{+}_{\rho(\ell)}\right)\right)^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{n}\left(\text{Ю}^{+}_{\rho(\ell)}\right)^{\nu_\ell}}, \frac{-\ddot{\gamma}\prod_{\ell=1}^{n}\left|\zeta^{-}_{\rho(\ell)}\right|^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left(1+\zeta^{-}_{\rho(\ell)}\right)\right)^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{n}\left|\zeta^{-}_{\rho(\ell)}\right|^{\nu_\ell}}, \\ -\frac{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left|\Gamma^{-}_{\rho(\ell)}\right|\right)^{\nu_\ell}-\prod_{\ell=1}^{n}\left(1+\Gamma^{-}_{\rho(\ell)}\right)^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left|\Gamma^{-}_{\rho(\ell)}\right|\right)^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{n}\left(1+\Gamma^{-}_{\rho(\ell)}\right)^{\nu_\ell}}, -\frac{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left|\text{Ю}^{-}_{\rho(\ell)}\right|\right)^{\nu_\ell}-\prod_{\ell=1}^{n}\left(1+\text{Ю}^{-}_{\rho(\ell)}\right)^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left|\text{Ю}^{-}_{\rho(\ell)}\right|\right)^{\nu_\ell}+(\ddot{\gamma}-1)\prod_{\ell=1}^{n}\left(1+\text{Ю}^{-}_{\rho(\ell)}\right)^{\nu_\ell}} \end{array} \right) \tag{16}$$

where $(\rho(1), \rho(2), \ldots\ldots\ldots, \rho(n))$ *is a permutation with* $u_{\rho(\ell-1)} \geq u_{\rho(\ell)}$, $\forall \ell \in Z$, $Z = \{1, 2, 3, \ldots., n\}$ *and* $\nu = (\nu_1, \nu_2, \ldots, \nu_n)^T$ *is the weighted vector of* u_ℓ *such that* $\nu_\ell > 0$ *and* $\sum_{\ell=1}^{n} \nu_\ell = 1, \ddot{\gamma} > 0$.

Proof. The theorem is straightforward. □

Theorem 6. *(Idempotency) Let* $u_\ell = \left(\zeta^{+}_{\ell}, \Gamma^{+}_{\ell}, \text{Ю}^{+}_{\ell}, \zeta^{-}_{\ell}, \Gamma^{-}_{\ell}, \text{Ю}^{-}_{\ell}\right)$, *where* $\ell \in Z$ *and* $Z = \{1, 2, 3, \ldots., n\}$ *are a collection of equal BNNs, i.e.,* $u_\ell = u$ *for all* ℓ, *then:*

$$BNHOWA_{\nu}(u_1, u_2, \ldots.., u_n) = u.$$

Theorem 7. *(Boundedness) Let* $u^{-} = \underset{\ell}{min} u_\ell$, $u^{+} = \underset{\ell}{max} u_\ell$, *then:*

$$u^{-} \leq BNHOWA_{\nu}(u_1, u_2, \ldots.., u_n) \leq u^{+}.$$

Theorem 8. *(Monotonicity) Let* $u_\ell = \left(\zeta^{+}_{\ell}, \Gamma^{+}_{\ell}, \text{Ю}^{+}_{\ell}, \zeta^{-}_{\ell}, \Gamma^{-}_{\ell}, \text{Ю}^{-}_{\ell}\right)$, *where* $\ell \in Z$ *and* $Z = \{1, 2, 3, \ldots., n\}$ *and* $u'_\ell = \left(\zeta'^{+}_{\ell}, \Gamma'^{+}_{\ell}, \text{Ю}'^{+}_{\ell}, \zeta'^{-}_{\ell}, \Gamma'^{-}_{\ell}, \text{Ю}'^{-}_{\ell}\right)$, *where* $\ell \in Z$ *and* $Z = \{1, 2, 3, \ldots., n\}$ *are two BNNs. If* $u_\ell \leq u'_\ell$, *for all* ℓ, *then:*

$$BNHOWA_{\varpi}(u_1, u_2, \ldots.., u_n) \leq BNHOWA_{\varpi}\left(u'_1, u'_2, \ldots.., u'_n\right).$$

Now, we discuss two cases of the BNHOWA operator:

- If $\ddot{\gamma} = 1$, the BNHOWA is converted to the bipolar neutrosophic ordered weighted average (BNOWA):

$$BNOWA_{\nu}(u_1, u_2, \ldots.., u_n) = \overset{n}{\underset{\ell=1}{\oplus}}\left(\nu_\ell u_{\rho(\ell)}\right) = \left(\begin{array}{l} 1-\prod_{\ell=1}^{n}\left(1-\zeta^{+}_{\rho(\ell)}\right)^{\nu_\ell}, \prod_{\ell=1}^{n}\left(\Gamma^{+}_{\rho(\ell)}\right)^{\nu_\ell}, \prod_{\ell=1}^{n}\left(\text{Ю}^{+}_{\rho(\ell)}\right)^{\nu_\ell}, \\ -\prod_{\ell=1}^{n}\left|\zeta^{-}_{\rho(\ell)}\right|^{\nu_\ell}, -\left(1-\left(\prod_{\ell=1}^{n}\left(1+\Gamma^{-}_{\rho(\ell)}\right)^{\nu_\ell}\right)\right), -\left(1-\left(\prod_{\ell=1}^{n}\left(1+\text{Ю}^{-}_{\rho(\ell)}\right)^{\nu_\ell}\right)\right) \end{array} \right).$$

- If $\ddot{\gamma} = 2$, the BNHOWA is converted to the bipolar neutrosophic Einstein ordered weighted average (BNEOWA):

$$BNEOWA_{\nu}(u_1, u_2, \ldots.., u_n) = \overset{n}{\underset{\ell=1}{\oplus}} \left(\nu_\ell u_{\rho(\ell)}\right)$$
$$= \left(\begin{array}{c} \frac{\prod_{\ell=1}^{n}\left(1+\zeta^{+}_{\rho(\ell)}\right)^{\nu_\ell} - \prod_{\ell=1}^{n}\left(1-\zeta^{+}_{\rho(\ell)}\right)^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(1+\zeta^{+}_{\rho(\ell)}\right)^{\nu_\ell} + \prod_{\ell=1}^{n}\left(1-\zeta^{+}_{\rho(\ell)}\right)^{\nu_\ell}}, \frac{2\prod_{\ell=1}^{n}\left(\Gamma^{+}_{\rho(\ell)}\right)^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(2-\Gamma^{+}_{\rho(\ell)}\right)^{\nu_\ell} + \prod_{\ell=1}^{n}\left(\Gamma^{+}_{\rho(\ell)}\right)^{\nu_\ell}}, \frac{2\prod_{\ell=1}^{n}\left(\text{Ю}^{+}_{\rho(\ell)}\right)^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(2-\text{Ю}^{+}_{\rho(\ell)}\right)^{\nu_\ell} + \prod_{\ell=1}^{n}\left(\text{Ю}^{+}_{\rho(\ell)}\right)^{\nu_\ell}}, \\ \frac{-2\prod_{\ell=1}^{n}\left|\zeta^{-}_{\rho(\ell)}\right|^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(2+\zeta^{-}_{\rho(\ell)}\right)^{\nu_\ell} + \prod_{\ell=1}^{n}\left|\zeta^{-}_{\rho(\ell)}\right|^{\nu_\ell}}, -\frac{\prod_{\ell=1}^{n}\left(1+\left|\Gamma^{-}_{\rho(\ell)}\right|\right)^{\nu_\ell} - \prod_{\ell=1}^{n}\left(1+\Gamma^{-}_{\rho(\ell)}\right)^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(1+\left|\Gamma^{-}_{\rho(\ell)}\right|\right)^{\nu_\ell} + \prod_{\ell=1}^{n}\left(1+\Gamma^{-}_{\rho(\ell)}\right)^{\nu_\ell}}, -\frac{\prod_{\ell=1}^{n}\left(1+\left|\text{Ю}^{-}_{\rho(\ell)}\right|\right)^{\nu_\ell} - \prod_{\ell=1}^{n}\left(1+\text{Ю}^{-}_{\rho(\ell)}\right)^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(1+\left|\text{Ю}^{-}_{\rho(\ell)}\right|\right)^{\nu_\ell} + \prod_{\ell=1}^{n}\left(1+\text{Ю}^{-}_{\rho(\ell)}\right)^{\nu_\ell}} \end{array} \right).$$

3.3. Bipolar Neutrosophic Hamacher HybridAveraging Aggregation Operator

Definition 18. *The bipolar neutrosophic Hamacher hybrid averaging (BNHHA) operator can be defined as follows:*

$$BNHHA_{w,\nu}(u_1, u_2, \ldots\ldots, u_n) = \overset{n}{\underset{\ell=1}{\oplus}} \left(\nu_\ell \dot{u}_{\rho(\ell)}\right) = \nu_1 \dot{u}_{\rho(1)} \oplus \nu_2 \dot{u}_{\rho(2)} \oplus \nu_3 \dot{u}_{\rho(3)} \oplus \ldots\ldots\ldots.. \oplus \nu_n \dot{u}_{\rho(n)} \quad (17)$$

where $w = (w_1, w_2, \ldots, w_n)$ *is a weighting vector of* $u_\ell(\ell \in Z)$, $Z = \{1, 2, 3, \ldots., n\}$, *such that* $w_\ell \in [0, 1]$, $\sum_{\ell=1}^{n} w_\ell = 1$ *and* $\dot{u}_{\rho(\ell)}$ *is the* ℓ*-th largest element of the bipolar neutrosophic argument,*$\dot{u}_\ell\left(\dot{u}_\ell = (n\nu_\ell)u_\ell, \ell = 1, 2, \ldots, n\right)$ *and also* $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ *are weighting vectors of bipolar neutrosophic arguments* $u_\ell(\ell \in Z)$, $Z = \{1, 2, 3, \ldots., n\}$, *such that* $\nu_\ell \in [0, 1]$, $\sum_{\ell=1}^{n} \nu_\ell = 1$, *where n is the balancing coefficient. Note that BNHHA reduces to BNHWA if* $w = \left(\frac{1}{n}, \frac{1}{n}, \ldots.., \frac{1}{n}\right)^T$ *and BNHOWA operator if:*

$$\nu = \left(\frac{1}{n}, \frac{1}{n}, \ldots.., \frac{1}{n}\right).$$

Theorem 9. *The (BNHHA) operator returns a bipolar neutrosophic value when:*

$$BNHHA_{w,\nu}(u_1, u_2, \ldots.., u_n) =$$
$$\begin{array}{c} \frac{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\dot{\zeta}^{+}_{\rho(\ell)}\right)^{\nu_\ell} - \prod_{\ell=1}^{n}\left(1-\dot{\zeta}^{+}_{\rho(\ell)}\right)^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\dot{\zeta}^{+}_{\rho(\ell)}\right)^{\nu_\ell} + (\ddot{\gamma}-1)\prod_{\ell=1}^{n}\left(1-\dot{\zeta}^{+}_{\rho(\ell)}\right)^{\nu_\ell}}, \frac{\ddot{\gamma}\prod_{\ell=1}^{n}\left(\dot{\Gamma}^{+}_{\rho(\ell)}\right)^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left(1-\dot{\Gamma}^{+}_{\rho(\ell)}\right)\right)^{\nu_\ell} + (\ddot{\gamma}-1)\prod_{\ell=1}^{n}\left(\dot{\Gamma}^{+}_{\rho(\ell)}\right)^{\nu_\ell}}, \\ \frac{\ddot{\gamma}\prod_{\ell=1}^{n}\left(\dot{\text{Ю}}^{+}_{\rho(\ell)}\right)^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left(1-\dot{\text{Ю}}^{+}_{\rho(\ell)}\right)\right)^{\nu_\ell} + (\ddot{\gamma}-1)\prod_{\ell=1}^{n}\left(\dot{\text{Ю}}^{+}_{\rho(\ell)}\right)^{\nu_\ell}}, \frac{-\ddot{\gamma}\prod_{\ell=1}^{n}\left|\dot{\zeta}^{-}_{\rho(\ell)}\right|^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left(1+\dot{\zeta}^{-}_{\rho(\ell)}\right)\right)^{\nu_\ell} + (\ddot{\gamma}-1)\prod_{\ell=1}^{n}\left|\dot{\zeta}^{-}_{\rho(\ell)}\right|^{\nu_\ell}}, \\ -\frac{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left|\dot{\Gamma}^{-}_{\rho(\ell)}\right|\right)^{\nu_\ell} - \prod_{\ell=1}^{n}\left(1+\dot{\Gamma}^{-}_{\rho(\ell)}\right)^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left|\dot{\Gamma}^{-}_{\rho(\ell)}\right|\right)^{\nu_\ell} + (\ddot{\gamma}-1)\prod_{\ell=1}^{n}\left(1+\dot{\Gamma}^{-}_{\rho(\ell)}\right)^{\nu_\ell}}, -\frac{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left|\dot{\text{Ю}}^{-}_{\rho(\ell)}\right|\right)^{\nu_\ell} - \prod_{\ell=1}^{n}\left(1+\dot{\text{Ю}}^{-}_{\rho(\ell)}\right)^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left|\dot{\text{Ю}}^{-}_{\rho(\ell)}\right|\right)^{\nu_\ell} + (\ddot{\gamma}-1)\prod_{\ell=1}^{n}\left(1+\dot{\text{Ю}}^{-}_{\rho(\ell)}\right)^{\nu_\ell}} \end{array} \quad (18)$$

where $w = (w_1, w_2, \ldots, w_n)$ *is the weighting vector of* $u_\ell(\ell \in Z)$, $Z = \{1, 2, 3, \ldots., n\}$, *such that* $w_\ell \in [0, 1]$, $\sum_{\ell=1}^{n} w_\ell = 1$ *and* $\dot{u}_{\rho(\ell)}$ *is the* ℓ *-th largest element of the bipolar neutrosophic arguments,* $\dot{u}_\ell\left(\dot{u}_\ell = (n\nu_\ell)u_\ell, \ell = 1, 2, \ldots, n\right)$ *and also* $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$ *are the weighting vector of bipolar neutrosophic arguments* $u_\ell(\ell \in Z)$, $Z = \{1, 2, 3, \ldots., n\}$, *such that* $\nu_\ell \in [0, 1]$, $\sum_{\ell=1}^{n} \nu_\ell = 1$, *where n is the balancing coefficient,*$\ddot{\gamma} > 0$.

Proof. The theorem is straightforward. □

Now, we discuss two cases of BNHOWA:

- If $\ddot{\gamma} = 1$, the BNHHA is converted to the bipolar neutrosophic hybrid averaging (BNHA):

$$BNHA_{w,\nu}(u_1, u_2, \ldots.., u_n) = \overset{n}{\underset{\ell=1}{\oplus}}\left(\nu_\ell \dot{u}_{\rho(\ell)}\right) = \begin{pmatrix} 1-\prod\limits_{\ell=1}^{n}\left(1-\dot{\zeta}^{+}_{\rho(\ell)}\right)^{\nu_\ell}, \prod\limits_{\ell=1}^{n}\left(\dot{\Gamma}^{+}_{\rho(\ell)}\right)^{\nu_\ell}, \prod\limits_{\ell=1}^{n}\left(\dot{\text{Ю}}^{+}_{\rho(\ell)}\right)^{\nu_\ell}, \\ -\prod\limits_{\ell=1}^{n}\left|\dot{\zeta}^{-}_{\rho(\ell)}\right|^{\nu_\ell}, -\left(1-\left(\prod\limits_{\ell=1}^{n}\left(1+\dot{\Gamma}^{-}_{\rho(\ell)}\right)^{\nu_\ell}\right)\right), -\left(1-\left(\prod\limits_{\ell=1}^{n}\left(1+\dot{\text{Ю}}^{-}_{\rho(\ell)}\right)^{\nu_\ell}\right)\right) \end{pmatrix}.$$

- If $\ddot{\gamma} = 2$, the BNHHA is converted to the bipolar neutrosophic Einstein hybrid averaging (BNEHA):

$$BNEHA_{w,\nu}(u_1, u_2, \ldots.., u_n) = \overset{n}{\underset{\ell=1}{\oplus}}\left(\nu_\ell \dot{u}_{\rho(\ell)}\right)$$
$$= \begin{pmatrix} \dfrac{\prod\limits_{\ell=1}^{n}\left(1+\dot{\zeta}^{+}_{\rho(\ell)}\right)^{\nu_\ell}-\prod\limits_{\ell=1}^{n}\left(1-\dot{\zeta}^{+}_{\rho(\ell)}\right)^{\nu_\ell}}{\prod\limits_{\ell=1}^{n}\left(1+\dot{\zeta}^{+}_{\rho(\ell)}\right)^{\nu_\ell}+\prod\limits_{\ell=1}^{n}\left(1-\dot{\zeta}^{+}_{\rho(\ell)}\right)^{\nu_\ell}}, \dfrac{2\prod\limits_{\ell=1}^{n}\left(\dot{\Gamma}^{+}_{\rho(\ell)}\right)^{\nu_\ell}}{\prod\limits_{\ell=1}^{n}\left(2-\dot{\Gamma}^{+}_{\rho(\ell)}\right)^{\nu_\ell}+\prod\limits_{\ell=1}^{n}\left(\dot{\Gamma}^{+}_{\rho(\ell)}\right)^{\nu_\ell}}, \dfrac{2\prod\limits_{\ell=1}^{n}\left(\dot{\text{Ю}}^{+}_{\rho(\ell)}\right)^{\nu_\ell}}{\prod\limits_{\ell=1}^{n}\left(2-\dot{\text{Ю}}^{+}_{\rho(\ell)}\right)^{\nu_\ell}+\prod\limits_{\ell=1}^{n}\left(\dot{\text{Ю}}^{+}_{\rho(\ell)}\right)^{\nu_\ell}}, \\ \dfrac{-2\prod\limits_{\ell=1}^{n}\left|\dot{\zeta}^{-}_{\rho(\ell)}\right|^{\nu_\ell}}{\prod\limits_{\ell=1}^{n}\left(2+\dot{\zeta}^{-}_{\rho(\ell)}\right)^{\nu_\ell}+\prod\limits_{\ell=1}^{n}\left|\dot{\zeta}^{-}_{\rho(\ell)}\right|^{\nu_\ell}}, \dfrac{\prod\limits_{\ell=1}^{n}\left(1+\left|\dot{\Gamma}^{-}_{\rho(\ell)}\right|\right)^{\nu_\ell}-\prod\limits_{\ell=1}^{n}\left(1+\dot{\Gamma}^{-}_{\rho(\ell)}\right)^{\nu_\ell}}{\prod\limits_{\ell=1}^{n}\left(1+\left|\dot{\Gamma}^{-}_{\rho(\ell)}\right|\right)^{\nu_\ell}+\prod\limits_{\ell=1}^{n}\left(1+\dot{\Gamma}^{-}_{\rho(\ell)}\right)^{\nu_\ell}}, \dfrac{\prod\limits_{\ell=1}^{n}\left(1+\left|\dot{\text{Ю}}^{-}_{\rho(\ell)}\right|\right)^{\nu_\ell}-\prod\limits_{\ell=1}^{n}\left(1+\dot{\text{Ю}}^{-}_{\rho(\ell)}\right)^{\nu_\ell}}{\prod\limits_{\ell=1}^{n}\left(1+\left|\dot{\text{Ю}}^{-}_{\rho(\ell)}\right|\right)^{\nu_\ell}+\prod\limits_{\ell=1}^{n}\left(1+\dot{\text{Ю}}^{-}_{\rho(\ell)}\right)^{\nu_\ell}} \end{pmatrix}.$$

4. An Application of the Bipolar Neutrosophic Hamacher Averaging Aggregation Operators to Group Decision Making

In this section, we apply the bipolar neutrosophic Hamacher averaging aggregation operators to the multiple attribute group decision-making problems in which the attribute weights take the form of crisp numbers and the attribute values take the form of BNNs.

Algorithm 1: Bipolar Neutrosophic Group Decision Making Problems

Let $G = \{G_1, G_2, \ldots, G_m\}$ be the set of m alternatives, $L = \{L_1, L_2, \ldots.., L_n\}$ be the set of n attributes or criterions, and $D = \{D_1, D_2, \ldots, D_k\}$ be the finite k decision makers. Let $\nu = (\nu_1, \nu_2, \ldots, \nu_n)^T$ be the weighted vector of the decision makers $\overline{D^s}(s = 1, 2, \ldots, k)$, such that $\nu_\ell \in [0,1]$ and $\sum\limits_{\ell=1}^{n} \nu_\ell = 1$. Let $w = (w_1, w_2, \ldots, w_n)^T$ be the weighted vector of the attribute set $L = \{L_1, L_2, \ldots.., L_n\}$ such that $w_\ell \in [0,1]$ and $\sum\limits_{\ell=1}^{n} w_\ell = 1$. An alternative of the criterion is assessed by the decision maker and the values are represented by bipolar neutrosophic values, where $u^{(s)}_{\widehat{i}\widehat{j}} = \left[\left(\zeta^{+}_{\widehat{i}\widehat{j}}, \Gamma^{+}_{\widehat{i}\widehat{j}}, \text{Ю}^{+}_{\widehat{i}\widehat{j}}, \zeta^{-}_{\widehat{i}\widehat{j}}, \Gamma^{-}_{\widehat{i}\widehat{j}}, \text{Ю}^{-}_{\widehat{i}\widehat{j}}\right)\right]_{m\times n}$ is the decision matrix provided by the decision maker (Tables 1–3) and $u^{(s)}_{\widehat{i}\widehat{j}}$ is a bipolar neutrosophic number (Table 4) for alternative $G_{\widehat{i}}$, associated with criterion $L_{\widehat{i}}$. The condition $\zeta^{+}_{\widehat{i}\widehat{j}}, \Gamma^{+}_{\widehat{i}\widehat{j}}, \text{Ю}^{+}_{\widehat{i}\widehat{j}}, \zeta^{-}_{\widehat{i}\widehat{j}}, \Gamma^{-}_{\widehat{i}\widehat{j}}$ and $\text{Ю}^{-}_{\widehat{i}\widehat{j}} \in [0,1]$ is such that
$0 \le \zeta^{+}_{\widehat{i}\widehat{j}} + \Gamma^{+}_{\widehat{i}\widehat{j}} + \text{Ю}^{+}_{\widehat{i}\widehat{j}} + \zeta^{-}_{\widehat{i}\widehat{j}} + \Gamma^{-}_{\widehat{i}\widehat{j}} + \text{Ю}^{-}_{\widehat{i}\widehat{j}} \le 6$ for $\widehat{i} = 1, 2, \ldots., m$ and $\widehat{j} = 1, 2, \ldots., n$.

Step 1: Construct the decision matrix $\overline{D^s} = \left[u^{(s)}_{\widehat{i}\widehat{j}}\right]_{m\times n}$ $(s = 1, 2, \ldots, k)$ for the decision.

Step 2: Compute $BNHWA_\nu\left(r_{\widehat{i}1}, r_{\widehat{i}2}, \ldots.., r_{\widehat{i}n}\right)$ for each $\widehat{i} = 1, 2, 3, \ldots., m$:

$$r_{\widehat{i}} = \left(\zeta^{+}_{\widehat{i}}, \Gamma^{+}_{\widehat{i}}, \text{Ю}^{+}_{\widehat{i}}, \zeta^{-}_{\widehat{i}}, \Gamma^{-}_{\widehat{i}}, \text{Ю}^{-}_{\widehat{i}}\right) =$$
$$BNHWA_\nu\left(r_{\widehat{i}1}, r_{\widehat{i}2}, \ldots.., r_{\widehat{i}n}\right) = \overset{n}{\underset{\ell=1}{\oplus}}\left(\nu_\ell r_{i\ell}\right)$$
$$= \begin{pmatrix} \dfrac{\prod\limits_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\zeta^{+}_{\ell}\right)^{\nu_\ell}-\prod\limits_{\ell=1}^{n}\left(1-\zeta^{+}_{\ell}\right)^{\nu_\ell}}{\prod\limits_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\zeta^{+}_{\ell}\right)^{\nu_\ell}+(\ddot{\gamma}-1)\prod\limits_{\ell=1}^{n}\left(1-\zeta^{+}_{\ell}\right)^{\nu_\ell}}, \dfrac{\ddot{\gamma}\prod\limits_{\ell=1}^{n}\left(\Gamma^{+}_{\ell}\right)^{\nu_\ell}}{\prod\limits_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)(1-\Gamma^{+}_{\ell})\right)^{\nu_\ell}+(\ddot{\gamma}-1)\prod\limits_{\ell=1}^{n}\left(\Gamma^{+}_{\ell}\right)^{\nu_\ell}}, \dfrac{\ddot{\gamma}\prod\limits_{\ell=1}^{n}\left(\text{Ю}^{+}_{\ell}\right)^{\nu_\ell}}{\prod\limits_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)(1-\text{Ю}^{+}_{\ell})\right)^{\nu_\ell}+(\ddot{\gamma}-1)\prod\limits_{\ell=1}^{n}\left(\text{Ю}^{+}_{\ell}\right)^{\nu_\ell}}, \\ \dfrac{-\ddot{\gamma}\prod\limits_{\ell=1}^{n}\left|\zeta^{-}_{\ell}\right|^{\nu_\ell}}{\prod\limits_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)(1+\zeta^{-}_{\ell})\right)^{\nu_\ell}+(\ddot{\gamma}-1)\prod\limits_{\ell=1}^{n}\left|\zeta^{-}_{\ell}\right|^{\nu_\ell}}, -\dfrac{\prod\limits_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left|\Gamma^{-}_{\ell}\right|\right)^{\nu_\ell}-\prod\limits_{\ell=1}^{n}\left(1+\Gamma^{-}_{\ell}\right)^{\nu_\ell}}{\prod\limits_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left|\Gamma^{-}_{\ell}\right|\right)^{\nu_\ell}+(\ddot{\gamma}-1)\prod\limits_{\ell=1}^{n}\left(1+\Gamma^{-}_{\ell}\right)^{\nu_\ell}}, -\dfrac{\prod\limits_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left|\text{Ю}^{-}_{\ell}\right|\right)^{\nu_\ell}-\prod\limits_{\ell=1}^{n}\left(1+\text{Ю}^{-}_{\ell}\right)^{\nu_\ell}}{\prod\limits_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left|\text{Ю}^{-}_{\ell}\right|\right)^{\nu_\ell}+(\ddot{\gamma}-1)\prod\limits_{\ell=1}^{n}\left(1+\text{Ю}^{-}_{\ell}\right)^{\nu_\ell}} \end{pmatrix}$$

Step 3: Calculate the scores of $S\left(r_{\widehat{i}}\right)$ for the $\left(\widehat{i} = 1, 2, 3, \ldots., m\right)$.

Step 4: Rank all the software systems of $BNHWA_\nu\left(u_{\widehat{i}1}, u_{\widehat{i}2}, \ldots.., u_{\widehat{i}n}\right)$ according to the scores values.

Step 5: Select the best alternative(s).

Illustrative Example

We considered an issue, taken from Deli [29], as an application for the proposed method in the present paper. The issue given is that an investment company wants to make some investments in the best possible options. There are four types of companies $G_{\widehat{i}}\left(\widehat{i} = 1, 2, 3 \ldots \ldots \ldots m\right)$ that are available as alternatives, namely G_1: computer company, G_2: food company, G_3: car company, and G_4: arms company, to invest money. The investment company takes into account four attributes to evaluate the alternatives: L_1: risk, L_2: growth, L_3: environmental impact, and L_4: performance. We utilized the bipolar neutrosophic numbers to assess the four possible alternatives of $G_{\widehat{i}}\left(\widehat{i} = 1, 2, 3, 4\right)$ under the four criteria. The weight vector of the attributes is $\nu = \left(\frac{1}{4}, \frac{1}{5}, \frac{3}{10}, \frac{1}{4}\right)^T$. There are three experts, i.e., $\overline{D^s}(s = 1, 2, 3)$, from a group of decision makers, whose weight vector is $\nu = \left(\frac{3}{10}, \frac{3}{10}, \frac{2}{5}\right)^T$. The expert opion about the companies based on atrribute are given in Tables 1–3.

Step 1: Decision matrices

Table 1. Bipolar Neutrosophic Decision Matrix, D_1.

	L_1	L_2	L_3	L_4
$\mathbf{G_1}$	(0.5,0.4,0.3, −0.7, −0.5, −0.6)	(0.1, 0.5, 0.4, −0.2, −0.4, −0.7)	(0.3, 0.7, 0.4−0.6, −0.4, −0.6)	(0.2, 0.4, 0.7, −0.3, −0.5, −0.1)
$\mathbf{G_2}$	(0.2, 0.6, 0.4, −0.3, −0.6, −0.8)	(0.5, 0.7, 0.6, −0.3, −0.4, −0.5)	(0.5, 0.5, 0.1, −0.7, −0.4, −0.8)	(0.6, 0.5, 0.4, −0.5, −0.4, −0.6)
$\mathbf{G_3}$	(0.4, 0.5, 0.3, −0.5, −0.6, −0.7)	(0.8, 0.9, 0.2, −0.7, −0.4, −0.6)	(0.2, 0.6, 0.5, −0.5, −0.4, −0.7)	(0.5, 0.7, 0.3, −0.5, −0.4, −0.2)
$\mathbf{G_4}$	(0.7, 0.6, 0.5, −0.6, −0.5, −0.4)	(0.5, 0.7, 0.6, −0.6, −0.3, −0.5)	(0.3, 0.1, 0.8, −0.9, −0.5, −0.6)	(0.2, 0.5, 0.7, −0.4, −0.5, −0.8)

Table 2. Bipolar Neutrosophic Decision Matrix, D_2.

	L_1	L_2	L_3	L_4
$\mathbf{G_1}$	(0.2, 0.5,0.3, −0.4, −0.6, −0.5)	(0.4, 0.3, 0.7, −0.5, −0.4, −0.6)	(0.5, 0.7, 0.3, −0.4, −0.7, −0.6)	(0.1, 0.4, 0.6, −0.3, −0.4, −0.2)
$\mathbf{G_2}$	(0.5, 0.6,0.4, −0.2, −0.4, −0.5)	(0.5, 0.1, 0.6, −0.6, −0.4, −0.2)	(0.3, 0.5, 0.4, −0.1, −0.4, −0.6)	(0.5, 0.3, 0.4−0.7, −0.4, −0.5)
$\mathbf{G_3}$	(0.7, 0.4, 0.5, −0.4, −0.5, −0.6)	(0.7, 0.2, 0.4, −0.3, −0.5, −0.1)	(0.1, 0.7, 0.5, −0.4, −0.3, −0.8)	(0.4, 0.3, 0.5, −0.7, −0.4, −0.3)
$\mathbf{G_4}$	(0.3, 0.4, 0.5, −0.7, −0.1, −0.3)	(0.8, 0.2, 0.1, −0.5, −0.3, −0.4)	(0.5, 0.2, 0.4, −0.1, −0.4, −0.7)	(0.4, 0.3, 0.7, −0.5, −0.2, −0.6)

Table 3. Bipolar Neutrosophic Decision Matrix, D_3.

	L_1	L_2	L_3	L_4
$\mathbf{G_1}$	(0.4, 0.5, 0.1, −0.6, −0.4, −0.5)	(0.4, 0.3, 0.6, −0.1, −0.6, −0.3)	(0.2, 0.6, 0.3, −0.7, −0.2, −0.5)	(0.2, 0.1, 0.8, −0.9, −0.2, −0.3)
$\mathbf{G_2}$	(0.5, 0.3, 0.2, −0.4, −0.1, −0.6)	(0.4, 0.5, 0.3, −0.7, −0.2, −0.3)	(0.5, 0.3, 0.4, −0.5, −0.4, −0.6)	(0.8, 0.6, 0.2, −0.1, −0.5, −0.4)
$\mathbf{G_3}$	(0.3, 0.4, 0.6, −0.7, −0.2, −0.4)	(0.2, 0.9, 0.1, −0.4, −0.5, −0.6)	(0.3, 0.7, 0.2, −0.5, −0.3, −0.4)	(0.4, 0.1, 0.6, −0.3, −0.4, −0.1)
$\mathbf{G_4}$	(0.5, 0.3, 0.6, −0.6, −0.3, −0.5)	(0.5, 0.6, 0.2, −0.5, −0.3, −0.6)	(0.4, 0.3, 0.7, −0.8, −0.4, −0.7)	(0.7, 0.4, 0.5, −0.5, −0.3, −0.4)

Step 2: We computed $BNHWA_{\nu}\left(r_{\widehat{i}1}, r_{\widehat{i}2}, \ldots\ldots, r_{\widehat{i}n}\right)$ for $\ddot{\gamma} = 2$: The collective bipolar neutrosophic decision matrix is given in Table 4.

$$r_{\widehat{i}} = \left(\zeta^{+}_{\widehat{i}}, \Gamma^{+}_{\widehat{i}}, \text{Ю}^{+}_{\widehat{i}}, \zeta^{-}_{\widehat{i}}, \Gamma^{-}_{\widehat{i}}, \text{Ю}^{-}_{\widehat{i}}\right) =$$

$$BNHWA_{\nu}\left(r_{\widehat{i}1}, r_{\widehat{i}2}, \ldots\ldots, r_{\widehat{i}n}\right) = \overset{n}{\underset{\ell=1}{\oplus}}(\nu_\ell r_{i\ell})$$

$$= \left(\begin{array}{c} \frac{\prod_{\ell=1}^{n}(1+(\ddot{\gamma}-1)\zeta^{+}_{\ell})^{\nu_\ell} - \prod_{\ell=1}^{n}(1-\zeta^{+}_{\ell})^{\nu_\ell}}{\prod_{\ell=1}^{n}(1+(\ddot{\gamma}-1)\zeta^{+}_{\ell})^{\nu_\ell} + (\ddot{\gamma}-1)\prod_{\ell=1}^{n}(1-\zeta^{+}_{\ell})^{\nu_\ell}}, \frac{\ddot{\gamma}\prod_{\ell=1}^{n}\left(\Gamma^{+}_{\ell}\right)^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left(1-\Gamma^{+}_{\ell}\right)\right)^{\nu_\ell} + (\ddot{\gamma}-1)\prod_{\ell=1}^{n}\left(\Gamma^{+}_{\ell}\right)^{\nu_\ell}}, \frac{\ddot{\gamma}\prod_{\ell=1}^{n}(\text{Ю}^{+}_{\ell})^{\nu_\ell}}{\prod_{\ell=1}^{n}(1+(\ddot{\gamma}-1)(1-\text{Ю}^{+}_{\ell}))^{\nu_\ell} + (\ddot{\gamma}-1)\prod_{\ell=1}^{n}(\text{Ю}^{+}_{\ell})^{\nu_\ell}}, \\ \frac{-\ddot{\gamma}\prod_{\ell=1}^{n}|\zeta^{-}_{\ell}|^{\nu_\ell}}{\prod_{\ell=1}^{n}(1+(\ddot{\gamma}-1)(1+\zeta^{-}_{\ell}))^{\nu_\ell} + (\ddot{\gamma}-1)\prod_{\ell=1}^{n}|\zeta^{-}_{\ell}|^{\nu_\ell}}, -\frac{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left|\Gamma^{-}_{\ell}\right|\right)^{\nu_\ell} - \prod_{\ell=1}^{n}\left(1+\Gamma^{-}_{\ell}\right)^{\nu_\ell}}{\prod_{\ell=1}^{n}\left(1+(\ddot{\gamma}-1)\left|\Gamma^{-}_{\ell}\right|\right)^{\nu_\ell} + (\ddot{\gamma}-1)\prod_{\ell=1}^{n}\left(1+\Gamma^{-}_{\ell}\right)^{\nu_\ell}}, -\frac{\prod_{\ell=1}^{n}(1+(\ddot{\gamma}-1)|\text{Ю}^{-}_{\ell}|)^{\nu_\ell} - \prod_{\ell=1}^{n}(1+\text{Ю}^{-}_{\ell})^{\nu_\ell}}{\prod_{\ell=1}^{n}(1+(\ddot{\gamma}-1)|\text{Ю}^{-}_{\ell}|)^{\nu_\ell} + (\ddot{\gamma}-1)\prod_{\ell=1}^{n}(1+\text{Ю}^{-}_{\ell})^{\nu_\ell}} \end{array}\right)$$

Step 3: We calculated the score function, for $\ddot{\gamma} = 2$:

$$r_1 = (0.2993, 0.3560, 0.4013, -0.4544, -0.4369, -0.4585),$$
$$r_2 = (0.5060, 0.4265, 0.3217, -0.3607, -0.3854, -0.5668),$$
$$r_3 = (0.4150, 0.4765, 0.3629, -0.4780, -0.4010, -0.4920),$$
$$r_4 = (0.4989, 0.3360, 0.5028, -0.5272, -0.3517, -0.5683),$$

$$S\left(r_{\widehat{i}}\right) = \frac{1}{6}\left(\zeta^{+} + 1 - \Gamma'^{+} + 1 - \mathrm{IO}^{+} + 1 + \zeta^{-} - \Gamma'^{-} - \mathrm{IO}^{-}\right)\text{and}$$

$$S(r_1) = 0.4972, S(r_2) = 0.5582, S(r_3) = 0.4984, S(r_4) = 0.5088.$$

Step 4: We calculated the scores for $\ddot{\gamma} = 2$, which gave:

$$G_2 \succ G_4 \succ G_3 \succ G_1$$

Step 5: Thus, the best option was G_2

Table 4. Collective Bipolar Neutrosophic Decision Matrix R.

	L_1	L_2
G₁	(0.3757,0.4683,0.1962, −0.5615, −0.4946, −0.5317)	(0.3155,0.3520,0.5616, −0.2056, −0.4865, −0.5313)
G₂	(0.4181,0.4622,0.3062, −0.3001, −0.3585, −0.6479)	(0.4614,0.3623,0.4622, −0.5292, −0.3233, −0.3359)
G₃	(0.4708,0.4283,0.4669, −0.5406, −0.4250, −0.5632)	(0.5855,0.6202,0.1902, −0.4417, −0.4712, −0.4741)
G₄	(0.5174,0.4072,0.5386, −0.6291, −0.3083, −0.4134)	(0.6132,0.4687,0.2350, −0.5288, −0.3000, −0.5147)
	L_3	L_4
G₁	(0.3264,0.3931,0.3276, −0.5709, −0.4174, −0.5255)	(0.1703,0.2364,0.7077, −0.4916, −0.3566, −0.2115)
G₂	(0.4441,0.4108,0.2709, −0.3623, −0.4000, −0.6722)	(0.6708,0.4669,0.3061, −0.3107, −0.4413, −0.4946)
G₃	(0.2115,0.6692,0.3536, −0.4683, −0.3308, −0.6406)	(0.4312,0.2639,0.4669, −0.4597, −0.4000, −0.1914)
G₄	(0.4029,0.1931,0.6264, −0.4973, −0.4312, −0.6724)	(0.4891,0.3942,0.6154, −0.4683, −0.3359, −0.6088)

5. Comparison with the Different Methods

There are various tools utilized by researchers so far in decision making. Chen et al. [50] utilized FSs, and, later on, Atanassov [2] utilized intuitionistic FSs, Dubois et al. [51] utilized BFSs, Zavadskas et al. [52] utilized NSs, Ali et al. [53] utilized bipolar neutrosophic soft sets, and Irfan et al. [34] utilized BNSs and so many others have studied decision making. In this paper, we applied the bipolarity to the neutrosophic sets via Hamacher operators. If $\ddot{\gamma} = 1$, then our proposed model corresponded to the same BNSs of the decision making as Irfan et al. [34].

The advantage of our proposed methods was that the decision maker could choose different values of $\ddot{\gamma}$ in accordance with their preferences (Table 5). Generally, when the values of $\ddot{\gamma} = 1, 2$, are used, they form algebraic aggregation operators and Einstein aggregation operators. The aggregation operators suggested in this paper were more general and flexible in accordance with the different values of $\ddot{\gamma}$, keeping in view the above computation and analysis, it is derived that although the overall rating values of the alternatives are varying by using different values of $\ddot{\gamma}$, the ranking orders of the alternatives are slightly contrastive (Table 5). However, the most desirable investment company is G_2.

Table 5. Ranking of the four alternatives for the different values of $\ddot{\gamma}$.

$\ddot{\gamma}$	**Aggregation Operators**	**Ranking**
1	*BNHWA*	$G_2 \succ G_4 \succ G_1 \succ G_3$
1.5	*BNHWA*	$G_2 \succ G_4 \succ G_1 \succ G_3$
2	*BNHWA*	$G_2 \succ G_4 \succ G_3 \succ G_1$
2.5	*BNHWA*	$G_2 \succ G_4 \succ G_3 \succ G_1$
3	*BNHWA*	$G_2 \succ G_4 \succ G_1 \succ G_3$

6. Conclusions

The purpose of this paper was to study the different bipolar neutrosophic aggregation operators, based on Hamacher t-norms and *t*-conorms, and their application to multiple criteria group decision making where the criteria are bipolar neutrosophic values. Motivated by the Hamacher operations, we have proposed bipolar neutrosophic Hamacher aggregation operators. Firstly, we have introduced bipolar neutrosophic Hamacher aggregation operators, as well as their desirable properties. These aggregation operators were bipolar neutrosophic Hamacher weighted averaging (BNHWA), bipolar neutrosophic Hamacher ordered weighted averaging (BNHOWA), and bipolar neutrosophic Hamacher hybrid averaging (BNHHA). When $\ddot{\gamma} = 1$, the bipolar neutrosophic Hamacher averaging operators reduced to the bipolar neutrosophic averaging aggregation operator and for $\ddot{\gamma} = 2$, the bipolar neutrosophic Hamacher averaging operators transformed to the bipolar neutrosophic Einstein averaging aggregation operators. Finally, we have introduced a method for multi-attribute group decision making. A descriptive example of opting for the best company or alternative to investing money was provided. The results in this paper showed that our proposed methods were more effective and practical in real life. In our future study, we are determined to extend the proposed models to other domains and applications, such as risk analysis, pattern recognition, and so on.

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MADM Using m-Generalized q-Neutrosophic Sets

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Abstract: Although the single valued neutrosophic sets (SVNSs) are effective tool to express uncertain information and are superior to the fuzzy sets, intuitionistic fuzzy sets, Pythagorean fuzzy sets and q-rung orthopair fuzzy sets, there is not yet reported an operation which can provide desirable generality and flexibility under single valued neutrosophic environment, although many operations have been developed earlier to meet above such eventualities. So, the primary aim of this paper is to propose the concept of m-generalized q-neutrosophic sets (*mGqNSs*) as a further generalization of fuzzy sets, intuitionistic fuzzy sets, Pythagorean fuzzy sets and q-rung orthopair fuzzy sets, single valued neutrosophic sets, n-hyperspherical neutrosophic sets and single valued spherical neutrosophic sets. Under the m-generalized q-neutrosophic environment, we develop some new operational laws and study their properties. Using these operations, we define m-generalized q-neutrosophic weighted aggregation operators. The distinguished features of these proposed weighted aggregation operators are studied in detail. Furthermore, based on these proposed operators, a MADM (multi-attribute decision making) approach is developed. Finally, an illustrative example is provided to show the feasibility and effectiveness of the proposed approach.

Keywords: Single valued neutrosophic set, m-generalized q-neutrosophic set, m-generalized q-neutrosophic weighted averaging aggregation operator (*mGqNWAA*), m-generalized q-neutrosophic weighted geometric aggregation operator (*mGqNWGA*), score value, decision making.

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1. Introduction

Multi-attribute decision making (MADM) is basically a process of selecting an optimal alternative from a set of chosen ones. In our daily life, we come across various types of multi-attribute decision making problems. Therefore, all of us need to learn the techniques to make decisions. The area of decision making problems has attracted the interest of many researchers. Many authors have worked in this field by utilizing various approaches. All the traditional decision making processes involve crisp data set but in many real life problems, data may not be in crisp form always. Fuzzy set theory is one such extremely useful tool that helps us to deal with non-crisp data. In 1965, Lotfi A. Zadeh **[1]** first published the famous research paper on fuzzy sets that originated due to mainly the inclusion of vague human assessments in computing problems and it can deal with uncertainty, vagueness, partially trueness, impreciseness, Sharpless boundaries etc. Basically, the theory of fuzzy set is founded on the concept of relative graded membership which deals with the partial belongings of an element in a set in order to process inexact information. Later on, fuzzy sets have been generalized to intuitionistic fuzzy sets **[2]** by adding a non-membership function by Atanassov in 1986 in order to deal with problems that possess incomplete information. In the context of fuzzy sets or intuitionistic fuzzy sets, it is known that the membership (or non-membership) value of an element in a set admits a unique value in the closed interval [0,1]. However, the application range of intuitionistic fuzzy set is narrow because it has the constraint that sum of membership degree and non-membership degree of an element is not greater than one. But, in complex decision-making problems, decision makers/experts may choose the preferences in such a way that the above condition gets violated. For instance, if an expert gives his preference with membership degree 0.8 and non-membership degree 0.7, then clearly their sum is 1.5, which is greater than 1. Therefore, this situation can't be not properly handled by the intuitionistic fuzzy sets. To solve this problem, Yager **[3, 4]** introduced the nonstandard fuzzy set named as Pythagorean fuzzy sets with membership degree ζ and non-membership degree ϑ with the condition $\zeta^2 + \vartheta^2 \leq 1$. Obviously, the Pythagorean fuzzy sets accommodate more uncertainties than the intuitionistic fuzzy sets. Yager **[5]** defined q-rung orthopair fuzzy sets (*q-ROFSs*) by enlarging the scope of Pythagorean fuzzy sets. The q-rung orthopair fuzzy sets allows the result of the qth power of the membership grade plus the qth power of the non-membership grade to be limited in interval [0,1]. If q=1, the q-rung orthopair fuzzy set transforms into the intuitionistic fuzzy set; if q=2, the q-rung orthopair fuzzy set transforms into the Pythagorean fuzzy set, which means that the q-rung orthopair fuzzy sets are extensions of intuitionistic fuzzy sets and Pythagorean fuzzy sets.

In 1999, Smarandache **[6]** introduced the notion neutrsophic set as a generalization of the classical set, fuzzy set, intuitionistic fuzzy set, Pythagorean fuzzy set and q-rung orthopair fuzzy set. The characterization of this neutrosophic set is explicitly done by truth-membership function, indeterminacy membership function and falsity membership function. The concept of single valued neutrosophic set was developed by Wang et al. **[7]** as an extension of fuzzy sets, Pythagorean fuzzy sets, q-rung orthopair fuzzy sets, intuitionistic fuzzy sets, single valued spherical neutrosophic sets **[8]**, n-hyperspherical neutrosophic sets **[8]**. The possible applications of neutrosophic sets and single valued neutrosophic sets on image segmentation have been studied in Gou and Cheng **[9]**, Gou and Sensur **[10]**. Also, we find their probable infliction on clustering analysis in Karaaslan **[11]** and on medical diagnosis problems in Ansari et al. **[12]** respectively. Furthermore, the subject of the neutrosophic set theory has been practiced in Wang et al. **[13]**, Gou et al. **[14]**, Ye **[15]**, Sun et al. **[16]**, Ye **[17-19]** and Abdel Basset et al. **[20, 21]**. Some recent studies on this area can be found in **[22-37]**.

The growing capacity of decision complexity induces the real-life decision-making problems that indulge both generality and flexibility of the operations used. Some of the basic operations of single valued spherical neutrosophic sets fail to generalize the basic operations of fuzzy sets, intuitionistic fuzzy sets, Pythagorean fuzzy sets and q-rung orthopair fuzzy sets. Getting inspired and provoked with this fact, in this paper, we have tried to propose a new concept called "m-generalized q-neutrosophic sets (*mGqNSs*)" and develop some aggregation operators in m-generalized q-neutrosophic environment to deal with *MADM* problems. The aims in this article are pursued below:

(1) To propose the concept of m-generalized q-neutrosophic sets (*mGqNSs*) as a further generalization of fuzzy sets, Pythagorean fuzzy sets, q-rung orthopair fuzzy sets, intuitionistic fuzzy sets, single valued neutrosophic sets, n-hyperspherical neutrosophic sets and single valued spherical neutrosophic sets.

(2) To define few operations between the m-generalized q-neutrosophic numbers.

(3) To develop the weighted aggregation operators such as m-generalized q-neutrosophic weighted averaging aggregation operator (*mGqNWAA*) and m-generalized q-neutrosophic weighted geometric aggregation operator (*mGqNWGA*) and study their properties.

(4) To propose a multi-attribute decision making method based on the m-generalized q-neutrosophic weighted aggregation operators.

To do so, the rest of the article is arranged as follows:

In section 2, we review some basic concepts. In Section 3, we first define m-generalized q-neutrosophic sets (*mGqNSs*) and m-generalized q-neutrosophic numbers (*mGqNNs*) and then propose few operations between the mGqNNs. Furthermore, we introduce the score of a *mGqNN* to ranking the *mGqNNs*. In section 4, we propose two types of m-generalized q-neutrosophic weighted aggregation operators to aggregate the m-generalized q-neutrosophic information. In section 5, based on the m-generalized q-neutrosophic weighted aggregation operators and score of *mGqNNs*, we develop a multi attribute decision making approach, in which the evaluation values of alternatives on the attribute are represented in terms of *mGqNNs* and the alternatives are ranked according to the values of the score of *mGqNNs* to select the best (most desirable) one. Also, we present a practical example to demonstrate the application and effectiveness of the proposed method. In final section, we present the conclusion of the study.

2. Preliminaries:

In this section, first we recall some basic notions that are relevant to our study.

2.1 Definition: **[7]** A single-valued neutrosophic set ς on the universe set U is given by

$$\varsigma = \{< x, \xi(x), \vartheta(x), \eta(x) >: x \in U\}$$

where the functions $\xi,\vartheta,\eta:U\rightarrow[0,1]$ satisfy the condition $0\leq\xi(x)+\vartheta(x)+\eta(x)\leq 3$ for every $x\in U$. The functions $\xi(x),\vartheta(x),\eta(x)$ define the degree of truth-membership, indeterminacy-membership and falsity-membership, respectively of $x\in U$.

2.2 Definition: [7] Suppose ς and ς' be two single-valued neutrosophic sets on U and are given by

$\varsigma=\{<x,\xi(x),\vartheta(x),\eta(x)>:x\in U\}$ and $\varsigma'=\{<x,\xi'(x),\vartheta'(x),\eta'(x)>:x\in U\}$. Then

(i) $\varsigma\subseteq\varsigma'$ if and only if $\xi(x)\leq\xi'(x),\vartheta(x)\geq\vartheta'(x),\eta(x)\geq\eta'(x)\ \forall x\in U.$

(ii) $\varsigma^c=\{<x,\eta(x),1-\vartheta(x),\xi(x)>:x\in U\}$

(iii) $\varsigma\cup\varsigma'=\{<x,\max(\xi(x),\xi'(x)),\min(\vartheta(x),\vartheta'(x)),\ \min(\eta(x),\eta'(x))>:\ x\in U\}$.

(iv) $\varsigma\cap\varsigma'=\{<x,\min(\xi(x),\xi'(x)),\max(\vartheta(x),\vartheta'(x)),\ \max(\eta(x),\eta'(x))>:\ x\in U\}$.

3. *m*-GENERALIZED *q*-NEUTROSOPHIC SETS:

In this section first we define a *m*-generalized *q*-neutrosophic set as a further generalization of fuzzy set, Pythagorean fuzzy set, *q*-rung orthopair fuzzy set, intuitionistic fuzzy set, single valued neutrosophic set, single valued *n*-hyperspherical neutrosophic set and single valued spherical neutrosophic set. Then we present few operations between the *m*-generalized *q*-neutrosophic numbers.

3.1 Definition: Suppose *U* is a universe set and $x\in U$. A *m*-generalized *q*-neutrosophic set (*mGqNs*) in *U* is described as:

$$\psi=\{<x,\xi(x),\vartheta(x),\eta(x)>:x\in U\}$$

where $\xi,\vartheta,\eta:U\rightarrow[0,r]\ (o<r\leq 1)$ are functions such that $0\leq\xi(x),\vartheta(x),\eta(x)\leq 1$ and

$$0\leq(\xi(x))^{\frac{qm}{3}}+(\vartheta(x))^{\frac{qm}{3}}+(\eta(x))^{\frac{qm}{3}}\leq\frac{3}{m}\quad(m,q\geq 1)\ .$$

Here $\xi(x),\vartheta(x),\eta(x)$ represent *m*-generalized truth membership, *m*-generalized indeterminacy membership and *m*-generalized falsity membership respectively of $x\in U$. The triplet $\psi=<\xi,\vartheta,\eta>$ is termed as *m*-generalized *q*-neutrosophic number (*mGqNN* for short).

In particular,

(i) when *m*=*r*=1 and *q*=3, ψ reduces to a single valued neutrosophic set **[7]**.

(ii) when *m*=3, *r*=*q*=1 and $\eta(x)=0\ \forall x\in U$, ψ reduces to an intuitionistic fuzzy set **[2]**.

(iii) when *m*=3, *r*=*q*=1 and $\eta(x)=\vartheta(x)=0\ \forall x\in U$, ψ reduces to a fuzzy set **[1]**.

(iv) when *m*=3, *r*= 1 and $\eta(x)=0\ \forall x\in U$, ψ reduces to a q-Rung orthopair fuzzy set **[5]**.

(v) when *m*=3, *r*= 1, *q*=2 and $\eta(x)=0\ \forall x\in U$, ψ reduces to a Pythagorean fuzzy set **[3, 4]**.

(vi) For $r=\sqrt[n]{3}$, *m*=1 and *q*=3*n* $(n\geq 1)$, ψ reduces to a single valued n-hyperspherical neutrosophic set **[8]**.

(vii) For $r=\sqrt{3}$, *m*=1 and *q*=6, ψ reduces to a single valued spherical neutrosophic set **[8]**.

Next we define few operations between *m*-generalized *q*-neutrosophic numbers.

3.2 Definition: Suppose $\psi_1=<\xi_1,\vartheta_1,\eta_1>$ and $\psi_2=<\xi_2,\vartheta_2,\eta_2>$ be two *m*-generalized *q*-neutrosophic numbers defined on *U* and λ be any real number >0. We define

(i) $\psi_1 \oplus \psi_2 = \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}}\right)\left(\frac{3}{m} - \xi_2^{\frac{qm}{3}}\right)\right)^{\frac{3}{qm}}, \vartheta_1\vartheta_2, \eta_1\eta_2 \right\rangle$

(ii) $\psi_1 \otimes \psi_2 = \left\langle \xi_1\xi_2, \left(\frac{3}{m} - \left(\frac{3}{m} - \vartheta_1^{\frac{qm}{3}}\right)\left(\frac{3}{m} - \vartheta_2^{\frac{qm}{3}}\right)\right)^{\frac{3}{qm}}, \left(\frac{3}{m} - \left(\frac{3}{m} - \eta_1^{\frac{qm}{3}}\right)\left(\frac{3}{m} - \eta_2^{\frac{qm}{3}}\right)\right)^{\frac{3}{qm}} \right\rangle$

(iii) $\lambda * \psi_1 = \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}}\right)^{\lambda}\right)^{\frac{3}{qm}}, \vartheta_1^{\lambda}, \eta_1^{\lambda} \right\rangle$

(iv) $\lambda \circ \psi_1 = \left\langle \xi_1^{\lambda}, \left(\frac{3}{m} - \left(\frac{3}{m} - \vartheta_1^{\frac{qm}{3}}\right)^{\lambda}\right)^{\frac{3}{qm}}, \left(\frac{3}{m} - \left(\frac{3}{m} - \eta_1^{\frac{qm}{3}}\right)^{\lambda}\right)^{\frac{3}{qm}} \right\rangle$

3.3 Theorem: Suppose $\psi_1 = <\xi_1, \vartheta_1, \eta_1>$ and $\psi_2 = <\xi_2, \vartheta_2, \eta_2>$ be two *m*-generalized *q*-neutrosophic numbers defined on *U* and $\lambda, \lambda_1, \lambda_2$ be any three real numbers >0. Then

(i) $\psi_1 \oplus \psi_2 = \psi_2 \oplus \psi_1$

(ii) $\psi_1 \otimes \psi_2 = \psi_2 \otimes \psi_1$

(iii) $\lambda * (\psi_1 \oplus \psi_2) = (\lambda * \psi_1) \oplus (\lambda * \psi_2)$

(iv) $\lambda \circ (\psi_1 \otimes \psi_2) = (\lambda \circ \psi_1) \otimes (\lambda \circ \psi_2)$

(v) $(\lambda_1 + \lambda_2) * \psi_1 = (\lambda_1 * \psi_1) \oplus (\lambda_2 * \psi_1)$

(vi) $(\lambda_1 + \lambda_2) \circ \psi_1 = (\lambda_1 \circ \psi_1) \otimes (\lambda_2 \circ \psi_1)$

Proof: (i), (ii) are straight forward.

(iii) We have, $\psi_1 \oplus \psi_2 = \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}}\right)\left(\frac{3}{m} - \xi_2^{\frac{qm}{3}}\right)\right)^{\frac{3}{qm}}, \vartheta_1\vartheta_2, \eta_1\eta_2 \right\rangle$.

$\therefore \lambda*(\psi_1 \oplus \psi_2)$

$$=\left\langle \left(\frac{3}{m}-\left(\frac{3}{m}-\left(\frac{3}{m}-\left(\frac{3}{m}-\xi_1^{\frac{qm}{3}}\right)\left(\frac{3}{m}-\xi_2^{\frac{qm}{3}}\right)\right)\right)^{\lambda}\right)^{\frac{3}{qm}}, (\vartheta_1\vartheta_2)^{\lambda}, (\eta_1\eta_2)^{\lambda} \right\rangle$$

$$=\left\langle \left(\frac{3}{m}-\left(\frac{3}{m}-\xi_1^{\frac{qm}{3}}\right)^{\lambda}\left(\frac{3}{m}-\xi_2^{\frac{qm}{3}}\right)^{\lambda}\right)^{\frac{3}{qm}}, \vartheta_1^{\lambda}\vartheta_2^{\lambda}, \eta_1^{\lambda}\eta_2^{\lambda} \right\rangle$$

On the other hand, we have,

$(\lambda*\psi_1)\oplus(\lambda*\psi_2)$

$$=\left\langle \left(\frac{3}{m}-\left(\frac{3}{m}-\xi_1^{\frac{qm}{3}}\right)^{\lambda}\right)^{\frac{3}{qm}}, \vartheta_1^{\lambda}, \eta_1^{\lambda} \right\rangle \oplus \left\langle \left(\frac{3}{m}-\left(\frac{3}{m}-\xi_2^{\frac{qm}{3}}\right)^{\lambda}\right)^{\frac{3}{qm}}, \vartheta_2^{\lambda}, \eta_2^{\lambda} \right\rangle$$

$$=\left\langle \left(\frac{3}{m}-\left(\frac{3}{m}-\left(\frac{3}{m}-\left(\frac{3}{m}-\xi_1^{\frac{qm}{3}}\right)^{\lambda}\right)\right)\left(\frac{3}{m}-\left(\frac{3}{m}-\left(\frac{3}{m}-\xi_2^{\frac{qm}{3}}\right)^{\lambda}\right)\right)\right)^{\frac{3}{qm}}, \vartheta_1^{\lambda}\vartheta_2^{\lambda}, \eta_1^{\lambda}\eta_2^{\lambda} \right\rangle$$

$$=\left\langle \left(\frac{3}{m}-\left(\frac{3}{m}-\xi_1^{\frac{qm}{3}}\right)^{\lambda}\left(\frac{3}{m}-\xi_2^{\frac{qm}{3}}\right)^{\lambda}\right)^{\frac{3}{qm}}, \vartheta_1^{\lambda}\vartheta_2^{\lambda}, \eta_1^{\lambda}\eta_2^{\lambda} \right\rangle$$

Thus, we get, $\lambda*(\psi_1 \oplus \psi_2)=(\lambda*\psi_1)\oplus(\lambda*\psi_2)$.

(iv) Similar to (iii)

(v) We have,

$$(\lambda_1+\lambda_2)*\psi_1=\left\langle \left(\frac{3}{m}-\left(\frac{3}{m}-\xi_1^{\frac{qm}{3}}\right)^{\lambda_1+\lambda_2}\right)^{\frac{3}{qm}}, \vartheta_1^{\lambda_1+\lambda_2}, \eta_1^{\lambda_1+\lambda_2} \right\rangle$$

On the other hand,

$(\lambda_1 * \psi_1) \oplus (\lambda_2 * \psi_1)$

$$= \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}} \right)^{\lambda_1} \right)^{\frac{3}{qm}}, \vartheta_1^{\lambda_1}, \eta_1^{\lambda_1} \right\rangle \oplus \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}} \right)^{\lambda_2} \right)^{\frac{3}{qm}}, \vartheta_1^{\lambda_2}, \eta_1^{\lambda_2} \right\rangle$$

$$= \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}} \right)^{\lambda_1} \right) \right) \left(\frac{3}{m} - \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}} \right)^{\lambda_2} \right) \right) \right)^{\frac{3}{qm}}, \vartheta_1^{\lambda_1} \vartheta_1^{\lambda_2}, \eta_1^{\lambda_1} \eta_1^{\lambda_2} \right\rangle$$

$$= \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}} \right)^{\lambda_1 + \lambda_2} \right)^{\frac{3}{qm}}, \vartheta_1^{\lambda_1 + \lambda_2}, \eta_1^{\lambda_1 + \lambda_2} \right\rangle$$

Thus we get, $(\lambda_1 + \lambda_2) * \psi_1 = (\lambda_1 * \psi_1) \oplus (\lambda_2 * \psi_1)$.

(vi) Similar to (v).

3.4 Definition: The score of the *mGqNN* $\psi = <\xi, \vartheta, \eta>$ is defined as: $S(\psi) = \dfrac{2 + \xi - \vartheta - \eta}{3}$.

The ranking method for ranking the *mGqNNs* is given below:

If $\psi = <\xi, \vartheta, \eta>$ and $\psi' = <\xi', \vartheta', \eta'>$ be two *mGqNNs*, then

(I) if $S(\psi) > S(\psi')$, then $\psi > \psi'$

(II) if $S(\psi) = S(\psi')$, then $\psi = \psi'$

4. *m*-GENERALIZED *q*-NEUTROSOPHIC WEIGHTED AGGREGATION OPERATORS:

In this section first we define *m*-generalized *q*-neutrosophic weighted averaging aggregation operator (*mGqNWAA*) and *m*-generalized *q*-neutrosophic weighted geometric aggregation operator (*mGqNWGA*) and study their basic properties.

4.1 Definition: Suppose $\psi_k = <\xi_k, \vartheta_k, \eta_k> (k = 1, 2, 3, ..., n)$ be a collection of *mGqNNs* defined on the universe set *U*. Then a *m*-generalized *q*-neutrosophic weighted averaging aggregation operator (*mGqNWAA* for short) is given as $mGqNWAA : \Theta^n \to \Theta$ and is defined as:

$$mGqNWAA(\psi_1, \psi_2, \psi_3, ..., \psi_n) = (w_1 * \psi_1) \oplus (w_2 * \psi_2) \oplus (w_3 * \psi_3) \oplus .... \oplus (w_n * \psi_n)$$

where Θ is the collection of all *mGqNNs* defined on the universe set *U*, $w = (w_1, w_2, w_3, ..., w_n)^T$ is the weight vector of $(\psi_1, \psi_2, \psi_3, ..., \psi_n)$ such that $w_k \geq 0 (k = 1, 2, 3, ..., n)$ and $\sum_{k=1}^{n} w_k = 1$.

On the basis of the operational rules of the *mGqNNs*, we can get the aggregation result as described as Theorem 4.2.

4.2 Theorem: Suppose $\psi_k = <\xi_k, \vartheta_k, \eta_k> (k = 1, 2, 3, ..., n)$ be a collection of *mGqNNs* defined on the universe set *U* and $w = (w_1, w_2, w_3, ..., w_n)^T$ is the weight vector of $(\psi_1, \psi_2, \psi_3, ..., \psi_n)$ such that

$w_k \geq 0 (k = 1,2,3,....,n)$ and $\sum_{k=1}^{n} w_k = 1$. Then $mGqNWAA(\psi_1, \psi_2, \psi_3, ..., \psi_n)$ is also a *mGqNN*.

Moreover, we have,

$$mGqNWAA(\psi_1, \psi_2, \psi_3, ..., \psi_n) = \left\langle \left(\frac{3}{m} - \prod_{k=1}^{n} \left(\frac{3}{m} - \xi_k^{\frac{qm}{3}} \right)^{w_k} \right)^{\frac{3}{qm}}, \prod_{k=1}^{n} \vartheta_k^{w_k}, \prod_{k=1}^{n} \eta_k^{w_k} \right\rangle.$$

Proof:

The first part of the theorem can be proved easily. To show the rest part, let us use the method of mathematical induction on *n*.

<u>Step-1:</u> For *n*=1, the proof is straight forward. So first take *n*=2.

Then,

$$mGqNWAA(\psi_1, \psi_2)$$

$$= (w_1 * \psi_1) \oplus (w_2 * \psi_2)$$

$$= \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}} \right)^{w_1} \right)^{\frac{3}{qm}}, \vartheta_1^{w_1}, \eta_1^{w_1} \right\rangle \oplus \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_2^{\frac{qm}{3}} \right)^{\lambda_2} \right)^{\frac{3}{qm}}, \vartheta_1^{w_2}, \eta_1^{w_2} \right\rangle$$

$$= \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}} \right)^{w_1} \right) \right) \left(\frac{3}{m} - \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_2^{\frac{qm}{3}} \right)^{w_2} \right) \right) \right)^{\frac{3}{qm}}, \vartheta_1^{w_1} \vartheta_1^{w_2}, \eta_1^{w_1} \eta_1^{w_2} \right\rangle$$

$$= \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}} \right)^{w_1} \left(\frac{3}{m} - \xi_2^{\frac{qm}{3}} \right)^{w_2} \right)^{\frac{3}{qm}}, \vartheta_1^{w_1} \vartheta_1^{w_2}, \eta_1^{w_1} \eta_1^{w_2} \right\rangle$$

$$= \left\langle \left(\frac{3}{m} - \prod_{k=1}^{2} \left(\frac{3}{m} - \xi_k^{\frac{qm}{3}} \right)^{w_k} \right)^{\frac{3}{qm}}, \prod_{k=1}^{2} \vartheta_k^{w_k}, \prod_{k=1}^{2} \eta_k^{w_k} \right\rangle.$$

Thus the result is true for *n*=2.

<u>Step-2:</u> Suppose that the result is true for *n*=*p* i.e;

$$mGqNWAA(\psi_1, \psi_2, \psi_3, ..., \psi_p) = \left\langle \left(\frac{3}{m} - \prod_{k=1}^{p} \left(\frac{3}{m} - \xi_k^{\frac{qm}{3}} \right)^{w_k} \right)^{\frac{3}{qm}}, \prod_{k=1}^{p} \vartheta_k^{w_k}, \prod_{k=1}^{p} \eta_k^{w_k} \right\rangle$$

<u>Step-3:</u> Take *n*=*p*+1.Then we have,

$$
\begin{aligned}
& mGqNWAA(\psi_1,\psi_2,\psi_3,........,\psi_{p+1}) \\
& = ((w_1 * \psi_1)\oplus(w_2 * \psi_2)\oplus(w_3 * \psi_3)\oplus........\oplus(w_p * \psi_p))\oplus(w_{p+1} * \psi_{p+1}) \\
& = \left\langle \left(\frac{3}{m}-\prod_{k=1}^{p}\left(\frac{3}{m}-\xi_k^{\frac{qm}{3}}\right)^{w_k}\right)^{\frac{3}{qm}}, \prod_{k=1}^{p}\vartheta_k^{w_k}, \prod_{k=1}^{p}\eta_k^{w_k} \right\rangle \\
& \quad \oplus \left\langle \left(\frac{3}{m}-\left(\frac{3}{m}-\xi_{p+1}^{\frac{qm}{3}}\right)^{w_{p+1}}\right)^{\frac{3}{qm}}, \vartheta_{p+1}^{w_{p+1}}, \eta_{p+1}^{w_{p+1}} \right\rangle \\
& = \left\langle \left(\frac{3}{m}-\left(\frac{3}{m}-\left(\frac{3}{m}-\prod_{k=1}^{p}\left(\frac{3}{m}-\xi_k^{\frac{qm}{3}}\right)^{w_k}\right)\right)\left(\frac{3}{m}-\left(\frac{3}{m}-\left(\frac{3}{m}-\xi_{p+1}^{\frac{qm}{3}}\right)^{w_{p+1}}\right)\right)\right)^{\frac{3}{qm}}, \right. \\
& \quad \left. \vartheta_{p+1}^{w_{p+1}}\prod_{k=1}^{p}\vartheta_k^{w_k}, \eta_{p+1}^{w_{p+1}}\prod_{k=1}^{p}\eta_k^{w_k} \right\rangle \\
& = \left\langle \left(\frac{3}{m}-\prod_{k=1}^{p}\left(\frac{3}{m}-\xi_k^{\frac{qm}{3}}\right)^{w_k}\times\left(\frac{3}{m}-\xi_{p+1}^{\frac{qm}{3}}\right)^{w_{p+1}}\right)^{\frac{3}{qm}}, \vartheta_{p+1}^{w_{p+1}}\prod_{k=1}^{p}\vartheta_k^{w_k}, \eta_{p+1}^{w_{p+1}}\prod_{k=1}^{p}\eta_k^{w_k} \right\rangle \\
& = \left\langle \left(\frac{3}{m}-\prod_{k=1}^{p+1}\left(\frac{3}{m}-\xi_k^{\frac{qm}{3}}\right)^{w_k}\right)^{\frac{3}{qm}}, \prod_{k=1}^{p+1}\vartheta_k^{w_k}, \prod_{k=1}^{p+1}\eta_k^{w_k} \right\rangle
\end{aligned}
$$

Thus the result is true for $n=p+1$ also. Hence, by the method of induction, the result is true for all n.

Let us explore some more results related to $mGqNWAA$ operator in the form of theorems 4.3-4.6.

4.3 Theorem: Suppose $\psi_k = <\xi_k,\vartheta_k,\eta_k> (k=1,2,3,.......,n)$ be a collection of m-Gq-NNs defined on the universe set U and $w=(w_1,w_2,w_3,.......,w_n)^T$ is the weight vector of $(\psi_1,\psi_2,\psi_3,........,\psi_n)$ such that $w_k \geq 0 (k=1,2,3,....,n)$ and $\sum_{k=1}^{n} w_k = 1$. Then for $\psi_0 = <\xi_0,\vartheta_0,\eta_0> \in \Theta$ (where Θ is the collection of all *mGqNNs* defined on the universe set *U*), we have $mGqNWAA(\psi_0\oplus\psi_1,\psi_0\oplus\psi_2,\psi_0\oplus\psi_3,........,\psi_0\oplus\psi_n)=\psi_0\oplus mGqNWAA(\psi_1,\psi_2,\psi_3,........,\psi_n)$.

Proof:

$$\psi_0 \oplus \psi_k = \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_0^{\frac{qm}{3}} \right) \left(\frac{3}{m} - \xi_k^{\frac{qm}{3}} \right) \right)^{\frac{3}{qm}}, \vartheta_0 \vartheta_k, \eta_0 \eta_k \right\rangle \ (k = 1, 2, 3, \dots\dots, n)$$

$\therefore$ $mGqNWAA(\psi_0 \oplus \psi_1, \psi_0 \oplus \psi_2, \psi_0 \oplus \psi_3, \dots\dots, \psi_0 \oplus \psi_n)$

$$= \left\langle \left(\frac{3}{m} - \prod_{k=1}^{n} \left(\frac{3}{m} - \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_0^{\frac{qm}{3}} \right) \left(\frac{3}{m} - \xi_k^{\frac{qm}{3}} \right) \right) \right)^{w_k} \right)^{\frac{3}{qm}}, \prod_{k=1}^{n} (\vartheta_0 \vartheta_k)^{w_k}, \prod_{k=1}^{n} (\eta_0 \eta_k)^{w_k} \right\rangle$$

$$= \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_0^{\frac{qm}{3}} \right)^{\sum_{k=1}^{n} w_k} \prod_{k=1}^{n} \left(\frac{3}{m} - \xi_k^{\frac{qm}{3}} \right)^{w_k} \right)^{\frac{3}{qm}}, \vartheta_0^{\sum_{k=1}^{n} w_k} \prod_{k=1}^{n} \vartheta_k^{w_k}, \eta_0^{\sum_{k=1}^{n} w_k} \prod_{k=1}^{n} \eta_k^{w_k} \right\rangle$$

$$= \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_0^{\frac{qm}{3}} \right) \prod_{k=1}^{n} \left(\frac{3}{m} - \xi_k^{\frac{qm}{3}} \right)^{w_k} \right)^{\frac{3}{qm}}, \vartheta_0 \prod_{k=1}^{n} \vartheta_k^{w_k}, \eta_0 \prod_{k=1}^{n} \eta_k^{w_k} \right\rangle$$

On the other hand, $\psi_0 \oplus mGqNWAA(\psi_1, \psi_2, \psi_3, \dots\dots, \psi_n)$

$$= <\xi_0, \vartheta_0, \eta_0> \oplus \left\langle \left(\frac{3}{m} - \prod_{k=1}^{n} \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}} \right)^{w_k} \right)^{\frac{3}{qm}}, \prod_{k=1}^{n} \vartheta_k^{w_k}, \prod_{k=1}^{n} \eta_k^{w_k} \right\rangle$$

$$= \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_0^{\frac{qm}{3}} \right) \prod_{k=1}^{n} \left(\frac{3}{m} - \left(\frac{3}{m} - \prod_{k=1}^{n} \left(\frac{3}{m} - \xi_1^{\frac{qm}{3}} \right)^{w_k} \right) \right) \right)^{\frac{3}{qm}}, \vartheta_0 \prod_{k=1}^{n} \vartheta_k^{w_k}, \eta_0 \prod_{k=1}^{n} \eta_k^{w_k} \right\rangle$$

$$= \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_0^{\frac{qm}{3}} \right) \prod_{k=1}^{n} \left(\frac{3}{m} - \xi_k^{\frac{qm}{3}} \right)^{w_k} \right)^{\frac{3}{qm}}, \vartheta_0 \prod_{k=1}^{n} \vartheta_k^{w_k}, \eta_0 \prod_{k=1}^{n} \eta_k^{w_k} \right\rangle$$

Hence,

$$mGqNWAA(\psi_0 \oplus \psi_1, \psi_0 \oplus \psi_2, \psi_0 \oplus \psi_3, \dots\dots, \psi_0 \oplus \psi_n) = \psi_0 \oplus mGqNWAA(\psi_1, \psi_2, \psi_3, \dots\dots, \psi_n)$$

.

4.4 Theorem: *(Idempotency)* Suppose $\psi_k = <\xi_k, \vartheta_k, \eta_k> (k = 1, 2, 3, \dots\dots, n)$ be a collection of m-Gq-NNs defined on the universe set U and $w = (w_1, w_2, w_3, \dots\dots, w_n)^T$ is the weight vector of

$(\psi_1,\psi_2,\psi_3,........,\psi_n)$ such that $w_k \geq 0(k=1,2,3,....,n)$ and $\sum_{k=1}^{n} w_k = 1$. If $\psi_0 =< \xi_0,\vartheta_0,\eta_0 > \in \Theta$ (where Θ is the collection of all *mGqNNs* defined on the universe set *U*) such that $\psi_k = \psi_0 \,\forall\, k=1,2,3,.......,n$, then we have $mGqNWAA(\psi_1,\psi_2,\psi_3,........,\psi_n) = \psi_0$.

Proof: We have, $mGqNWAA(\psi_1,\psi_2,\psi_3,........,\psi_n)$

$$= \left\langle \left(\frac{3}{m} - \prod_{k=1}^{n} \left(\frac{3}{m} - \xi_k^{\frac{qm}{3}} \right)^{w_k} \right)^{\frac{3}{qm}}, \prod_{k=1}^{n} \vartheta_k^{\,w_k}, \prod_{k=1}^{n} \eta_k^{\,w_k} \right\rangle$$

$$= \left\langle \left(\frac{3}{m} - \prod_{k=1}^{n} \left(\frac{3}{m} - \xi_0^{\frac{qm}{3}} \right)^{w_k} \right)^{\frac{3}{qm}}, \prod_{k=1}^{n} \vartheta_0^{\,w_k}, \prod_{k=1}^{n} \eta_0^{\,w_k} \right\rangle$$

$$= \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_0^{\frac{qm}{3}} \right)^{\sum_{k=1}^{n} w_k} \right)^{\frac{3}{qm}}, \vartheta_0^{\sum_{k=1}^{n} w_k}, \prod_{k=1}^{n} \eta_0^{\sum_{k=1}^{n} w_k} \right\rangle$$

$$= \left\langle \left(\frac{3}{m} - \left(\frac{3}{m} - \xi_0^{\frac{qm}{3}} \right) \right)^{\frac{3}{qm}}, \vartheta_0, \eta_0 \right\rangle =< \xi_0,\vartheta_0,\eta_0 >= \psi_0$$

4.5 Theorem: *(Monotonocity)* Suppose $\psi_k =< \xi_k,\vartheta_k,\eta_k >$ and $\psi'_k =< \xi'_k,\vartheta'_k,\eta'_k >$ $(k=1,2,3,.......,n)$ be two collections of *mGqNNs* defined on the universe set *U* and $w=(w_1,w_2,w_3,.......,w_n)^T$ is the weight vector of $(\psi_1,\psi_2,\psi_3,........,\psi_n)$ as well as $(\psi'_1,\psi'_2,\psi'_3,........,\psi'_n)$ such that $w_k \geq 0(k=1,2,3,....,n)$ and $\sum_{k=1}^{n} w_k = 1$. If $\xi_k \geq \xi'_k, \vartheta_k \leq \vartheta'_k, \eta_k \leq \eta'_k >$ $(k=1,2,3,.......,n)$, then $mGqNWAA(\psi_1,\psi_2,\psi_3,........,\psi_n) \geq$ $mGqNWAA(\psi'_1,\psi'_2,\psi'_3,........,\psi'_n)$.

Proof: Since $\xi_k \geq \xi'_k, \vartheta_k \leq \vartheta'_k, \eta_k \leq \eta'_k >$ for all k,

so $\prod_{k=1}^{n} \left(\frac{3}{m} - \xi_k^{\frac{qm}{3}} \right)^{w_k} \leq \prod_{k=1}^{n} \left(\frac{3}{m} - \xi_k'^{\frac{qm}{3}} \right)^{w_k}, \prod_{k=1}^{n} \vartheta_k^{\,w_k} \leq \prod_{k=1}^{n} \vartheta_k'^{\,w_k}, \prod_{k=1}^{n} \eta_k^{\,w_k} \leq \prod_{k=1}^{n} \eta_k'^{\,w_k}$

$$\Rightarrow\left(\frac{3}{m}-\prod_{k=1}^{n}\left(\frac{3}{m}-\xi_k^{\frac{qm}{3}}\right)^{w_k}\right)^{\frac{3}{qm}}\geq\left(\frac{3}{m}-\prod_{k=1}^{n}\left(\frac{3}{m}-\xi_k'^{\frac{qm}{3}}\right)^{w_k}\right)^{\frac{3}{qm}},\prod_{k=1}^{n}\vartheta_k^{w_k}\leq\prod_{k=1}^{n}\vartheta_k'^{w_k},$$

$$\prod_{k=1}^{n}\eta_k^{w_k}\leq\prod_{k=1}^{n}\eta_k'^{w_k}$$

$$\Rightarrow\left(\frac{3}{m}-\prod_{k=1}^{n}\left(\frac{3}{m}-\xi_k^{\frac{qm}{3}}\right)^{w_k}\right)^{\frac{3}{qm}}-\left(\frac{3}{m}-\prod_{k=1}^{n}\left(\frac{3}{m}-\xi_k'^{\frac{qm}{3}}\right)^{w_k}\right)^{\frac{3}{qm}}-\left(\prod_{k=1}^{n}\vartheta_k^{w_k}-\prod_{k=1}^{n}\vartheta_k'^{w_k}\right)$$
$$-\left(\prod_{k=1}^{n}\eta_k^{w_k}-\prod_{k=1}^{n}\eta_k'^{w_k}\right)\geq 0$$

$$\Rightarrow\frac{2+\left(\frac{3}{m}-\prod_{k=1}^{n}\left(\frac{3}{m}-\xi_k^{\frac{qm}{3}}\right)^{w_k}\right)^{\frac{3}{qm}}-\prod_{k=1}^{n}\vartheta_k^{w_k}-\prod_{k=1}^{n}\eta_k^{w_k}}{3}$$
$$-\frac{2+\left(\frac{3}{m}-\prod_{k=1}^{n}\left(\frac{3}{m}-\xi_k'^{\frac{qm}{3}}\right)^{w_k}\right)^{\frac{3}{qm}}-\prod_{k=1}^{n}\vartheta_k'^{w_k}-\prod_{k=1}^{n}\eta_k'^{w_k}}{3}\geq 0$$

$$\Rightarrow S(mGqNWAA(\psi_1,\psi_2,\psi_3,........,\psi_n))\geq S(mGqNWAA(\psi_1',\psi_2',\psi_3',........,\psi_n'))$$

Hence by definition of score value and ranking method, we have, $mGqNWAA(\psi_1,\psi_2,\psi_3,........,\psi_n)\geq mGqNWAA(\psi_1',\psi_2',\psi_3',........,\psi_n')$.

4.6 Theorem: *(Boundedness)* Suppose $\psi_k=<\xi_k,\vartheta_k,\eta_k>(k=1,2,3,.......,n)$ be a collection of *mGqNNs* defined on the universe set U and $w=(w_1,w_2,w_3,.......,w_n)^T$ is the weight vector of $(\psi_1,\psi_2,\psi_3,........,\psi_n)$ such that $w_k\geq 0(k=1,2,3,....,n)$ and $\sum_{k=1}^{n}w_k=1$. Let us define two *mGqNNs* by: $\psi^-=<\min_{1\leq k\leq n}\xi_k,\max_{1\leq k\leq n}\vartheta_k,\max_{1\leq k\leq n}\eta_k>,\psi^+=<\max_{1\leq k\leq n}\xi_k,\min_{1\leq k\leq n}\vartheta_k,\min_{1\leq k\leq n}\eta_k>$

Then, $\psi^-\leq mGqNWAA(\psi_1,\psi_2,\psi_3,........,\psi_n)\leq\psi^+$.

Proof: Suppose $mGqNWAA(\psi_1,\psi_2,\psi_3,........,\psi_n)=<\xi,\vartheta,\eta>$.

Then we have, $\frac{3}{m}-\max_{1\leq k\leq n}\xi_k^{\frac{qm}{3}}\leq\frac{3}{m}-\xi_k^{\frac{qm}{3}}\leq\frac{3}{m}-\min_{1\leq k\leq n}\xi_k^{\frac{qm}{3}}$

$$\Rightarrow\left(\frac{3}{m}-\max_{1\leq k\leq n}\xi_k^{\frac{qm}{3}}\right)^{w_k}\leq\left(\frac{3}{m}-\xi_k^{\frac{qm}{3}}\right)^{w_k}\leq\left(\frac{3}{m}-\min_{1\leq k\leq n}\xi_k^{\frac{qm}{3}}\right)^{w_k}$$

$$\Rightarrow\left(\frac{3}{m}-\prod_{k=1}^{n}\left(\frac{3}{m}-\min_{1\le k\le n}\xi_k^{\frac{qm}{3}}\right)^{w_k}\right)^{\frac{3}{qm}}\le\left(\frac{3}{m}-\prod_{k=1}^{n}\left(\frac{3}{m}-\xi_k^{\frac{qm}{3}}\right)^{w_k}\right)^{\frac{3}{qm}}\le\left(\frac{3}{m}-\prod_{k=1}^{n}\left(\frac{3}{m}-\max_{1\le k\le n}\xi_k^{\frac{qm}{3}}\right)^{w_k}\right)^{\frac{3}{qm}}$$

$$\Rightarrow\left(\frac{3}{m}-\left(\frac{3}{m}-\min_{1\le k\le n}\xi_k^{\frac{qm}{3}}\right)^{\sum_{k=1}^{n}w_k}\right)^{\frac{3}{qm}}\le\xi\le\left(\frac{3}{m}-\left(\frac{3}{m}-\max_{1\le k\le n}\xi_k^{\frac{qm}{3}}\right)^{\sum_{k=1}^{n}w_k}\right)^{\frac{3}{qm}}$$

$$\Rightarrow\min_{1\le k\le n}\xi_k\le\xi\le\max_{1\le k\le n}\xi_k$$

Again, $\prod_{k=1}^{n}(\min_{1\le k\le n}\vartheta_k)^{w_k}\le\prod_{k=1}^{n}\vartheta_k^{w_k}\le\prod_{k=1}^{n}(\max_{1\le k\le n}\vartheta_k)^{w_k}$

$$\Rightarrow(\min_{1\le k\le n}\vartheta_k)^{\sum_{k=1}^{n}w_k}\le\prod_{k=1}^{n}\vartheta_k^{w_k}\le(\max_{1\le k\le n}\vartheta_k)^{\sum_{k=1}^{n}w_k}\Rightarrow\min_{1\le k\le n}\vartheta_k\le\vartheta\le\max_{1\le k\le n}\vartheta_k$$

Similarly, we can get, $\min_{1\le k\le n}\eta_k\le\eta\le\max_{1\le k\le n}\eta_k$.

Hence $\min_{1\le k\le n}\xi_k-\max_{1\le k\le n}\vartheta_k-\max_{1\le k\le n}\eta_k\le\xi-\vartheta-\eta\le\max_{1\le k\le n}\xi_k-\min_{1\le k\le n}\vartheta_k-\min_{1\le k\le n}\eta_k$

$$\Rightarrow\frac{2+\min_{1\le k\le n}\xi_k-\max_{1\le k\le n}\vartheta_k-\max_{1\le k\le n}\eta_k}{3}\le\frac{2+\xi-\vartheta-\eta}{3}\le\frac{2+\max_{1\le k\le n}\xi_k-\min_{1\le k\le n}\vartheta_k-\min_{1\le k\le n}\eta_k}{3}$$

$$\Rightarrow S(\psi^-)\le S(mGqNWAA(\psi_1,\psi_2,\psi_3,........,\psi_n))\le S(\psi^+)$$

Therefore by definition of score value and ranking method, we have, $\psi^-\le mGqNWAA(\psi_1,\psi_2,\psi_3,........,\psi_n)\le\psi^+$.

4.7 Definition: Suppose $\psi_k=<\xi_k,\vartheta_k,\eta_k>(k=1,2,3,.......,n)$ be a collection of *mGqNNs* defined on the universe set *U*. Then a *m*-generalized *q*-neutrosophic weighted geometric aggregation operator (*mGqNWGA* for short) is given as $mGqNWGA:\Theta^n\to\Theta$ and is defined as:

$$mGqNWGA(\psi_1,\psi_2,\psi_3,........,\psi_n)=(w_1\circ\psi_1)\otimes(w_2\circ\psi_2)\otimes(w_3\circ\psi_3)\otimes.........\otimes(w_n\circ\psi_n)$$

where Θ is the collection of all *mGqNNs* defined on the universe set *U*, $w=(w_1,w_2,w_3,.......,w_n)^T$ is the weight vector of $(\psi_1,\psi_2,\psi_3,........,\psi_n)$ such that $w_k\ge 0\,(k=1,2,3,....,n)$ and $\sum_{k=1}^{n}w_k=1$.

On the basis of the operational rules of the *mGqNNs*, we can get the aggregation result as described as Theorem 4.2.

4.8 Theorem: Suppose $\psi_k=<\xi_k,\vartheta_k,\eta_k>(k=1,2,3,.......,n)$ be a collection of *mGqNNs* defined on the universe set *U* and $w=(w_1,w_2,w_3,.......,w_n)^T$ is the weight vector of $(\psi_1,\psi_2,\psi_3,........,\psi_n)$ such that $w_k\ge 0\,(k=1,2,3,....,n)$ and $\sum_{k=1}^{n}w_k=1$. Then $mGqNWGA(\psi_1,\psi_2,\psi_3,........,\psi_n)$ is also a *mGqNN*.

Moreover, we have, $mGqNWGA(\psi_1,\psi_2,\psi_3,........,\psi_n)$

$$=\left\langle \prod_{k=1}^{n}\xi_k^{w_k}, \left(\frac{3}{m}-\prod_{k=1}^{n}\left(\frac{3}{m}-\vartheta_k^{\frac{qm}{3}}\right)^{w_k}\right)^{\frac{3}{qm}}, \left(\frac{3}{m}-\prod_{k=1}^{n}\left(\frac{3}{m}-\eta_k^{\frac{qm}{3}}\right)^{w_k}\right)^{\frac{3}{qm}}\right\rangle.$$

Proof: Similar to theorem 4.2.

4.9 Theorem: Suppose $\psi_k =<\xi_k,\vartheta_k,\eta_k> (k=1,2,3,......,n)$ be a collection of *mGqNNs* defined on the universe set *U* and $w=(w_1,w_2,w_3,.......,w_n)^T$ is the weight vector of $(\psi_1,\psi_2,\psi_3,........,\psi_n)$ such that $w_k \geq 0 (k=1,2,3,....,n)$ and $\sum_{k=1}^{n} w_k = 1$. Then for $\psi_0 =<\xi_0,\vartheta_0,\eta_0>\in\Theta$ (where Θ is the collection of all *mGqNNs* defined on the universe set *U*), we have $mGqNWGA(\psi_0\otimes\psi_1,\psi_0\otimes\psi_2,\psi_0\otimes\psi_3,........,\psi_0\otimes\psi_n)=\psi_0\otimes mGqNWGA(\psi_1,\psi_2,\psi_3,........,\psi_n)$.

Proof: Similar to theorem 4.3.

4.10 Theorem: *(Idempotency)* Suppose $\psi_k =<\xi_k,\vartheta_k,\eta_k> (k=1,2,3,......,n)$ be a collection of *mGqNNs* defined on the universe set *U* and $w=(w_1,w_2,w_3,.......,w_n)^T$ is the weight vector of $(\psi_1,\psi_2,\psi_3,........,\psi_n)$ such that $w_k \geq 0 (k=1,2,3,....,n)$ and $\sum_{k=1}^{n} w_k = 1$. If $\psi_0 =<\xi_0,\vartheta_0,\eta_0>\in\Theta$ (where Θ is the collection of all *mGqNNs* defined on the universe set *U*) such that $\psi_k=\psi_0\ \forall\, k=1,2,3,.......,n$, then we have $mGqNWGA(\psi_1,\psi_2,\psi_3,........,\psi_n)=\psi_0$.

Proof: Similar to theorem 4.4.

4.11 Theorem: *(Monotonocity)* Suppose $\psi_k =<\xi_k,\vartheta_k,\eta_k>$ and $\psi'_k =<\xi'_k,\vartheta'_k,\eta'_k>$ $(k=1,2,3,.......,n)$ be two collections of *mGqNNs* defined on the universe set *U* and $w=(w_1,w_2,w_3,.......,w_n)^T$ is the weight vector of $(\psi_1,\psi_2,\psi_3,........,\psi_n)$ as well as $(\psi'_1,\psi'_2,\psi'_3,........,\psi'_n)$ such that $w_k \geq 0 (k=1,2,3,....,n)$ and $\sum_{k=1}^{n} w_k = 1$. If $\xi_k\geq\xi'_k,\vartheta_k\leq\vartheta'_k,\eta_k\leq\eta'_k>$ $(k=1,2,3,.......,n)$, then $mGqNWGA(\psi_1,\psi_2,\psi_3,........,\psi_n)\geq mGqNWGA(\psi'_1,\psi'_2,\psi'_3,........,\psi'_n)$.

Proof: Similar to theorem 4.5.

4.12 Theorem: *(Boundedness)* Suppose $\psi_k =<\xi_k,\vartheta_k,\eta_k> (k=1,2,3,......,n)$ be a collection of *mGqNNs* defined on the universe set *U* and $w=(w_1,w_2,w_3,.......,w_n)^T$ is the weight vector of $(\psi_1,\psi_2,\psi_3,........,\psi_n)$ such that $w_k \geq 0 (k=1,2,3,....,n)$ and $\sum_{k=1}^{n} w_k = 1$. Let us define two *mGqNNs* by: $\psi^- =<\min_{1\leq k\leq n}\xi_k,\max_{1\leq k\leq n}\vartheta_k,\max_{1\leq k\leq n}\eta_k>,\psi^+ =<\max_{1\leq k\leq n}\xi_k,\min_{1\leq k\leq n}\vartheta_k,\min_{1\leq k\leq n}\eta_k>$

Then $\psi^-\leq mGqNWGA(\psi_1,\psi_2,\psi_3,........,\psi_n)\leq\psi^+$.

Proof: Similar to theorem 4.6.

5. MULTI ATTRIBUTE DECISION MAKING:

Consider a multi-attribute decision making problem which consists of m different alternatives A_1, A_2, ..., A_l which are evaluated under the set of n different attributes C_1, C_2, ..., C_n. Assume that an expert evaluates the given alternatives $A_i\,(i=1,2,...,l)$ under the attribute $C_j\,(j=1,2,...,n)$ and the evaluation result is presented by the form of m-generalized q-neutrosophic numbers $\zeta_{ij}=\left\langle \xi_{ij},\vartheta_{ij},\delta_{ij}\right\rangle$ such that $0\le \xi_{ij},\vartheta_{ij},\delta_{ij}\le 1$ and $0\le\left(\xi_{ij}\right)^{\frac{qm}{3}}+\left(\vartheta_{ij}\right)^{\frac{qm}{3}}+\left(\delta_{ij}\right)^{\frac{qm}{3}}\le\frac{3}{m}$ where $i=1,2,...,l;\, j=1,2,...,n$. Further assume that $w_j\,(j=1,2,...,n)$ is the weight of the attribute such C_j such that $w_j>0\,(j=1,2,...,n)$ and $\sum_{j=1}^{n} w_j=1$.

Then to determine the most desirable alternative (s), the proposed operators are utilized to develop a multi-attribute decision making with m-generalized q-neutrosophic information, which involves the following steps:

<u>**Step-1**</u> Arrange the rating values of the expert in the form of decision matrix $\tilde{D}=\left(\zeta_{ij}\right)_{l\times n}=\left(\left\langle \xi_{ij},\vartheta_{ij},\delta_{ij}\right\rangle\right)_{l\times n}$.

<u>**Step-2:**</u> Construct aggregated m-generalized q-neutrosophic decision matrix. In order to do that, the proposed operators can be utilized as follows:

Let $\tilde{R}=\left(\tilde{R}_i\right)_{l\times 1}$ be the aggregated m-generalized q-neutrosophic decision matrix, where

$$\tilde{R}_i = mGqNWAA\left(\zeta_{i1},\zeta_{i2},.....,\zeta_{in}\right)=\left(w_1*\zeta_{i1}\right)\oplus\left(w_2*\zeta_{i2}\right)\oplus..........\oplus\left(w_n*\zeta_{in}\right)$$

OR

$$\tilde{R}_i = mGqNWGA\left(\zeta_{i1},\zeta_{i2},.....,\zeta_{in}\right)=\left(w_1\circ\zeta_{i1}\right)\otimes\left(w_2\circ\zeta_{i2}\right)\otimes..........\otimes\left(w_n\circ\zeta_{in}\right)$$

<u>**Step-3:**</u> Calculate the score values $S\left(\tilde{R}_i\right)\,(i=1,2,...,l)$ of m-generalized q-neutrosophic numbers $\tilde{R}_i\,(i=1,2,...,m)$.

<u>**Step-4:**</u> Rank all the alternatives $A_i\,(i=1,2,...,l)$ and hence select the most desirable alternative(s).

- <u>**CASE STUDY**</u>:

We consider a multi attribute decision making problem adapted from **[15, 17, 18, 19]** to demonstrate the application of the proposed decision making method.

"Suppose there is an investment company that wants to invest a sum of money in the best option available. There is a panel with four possible alternatives in which to invest the money: (i) A_1 is a car company, (ii) A_2 is a food company, (iii) A_3 is a computer company and (iv) A_4 is an arms company. The investment company must take a decision according to the following attributes:

(1) C_1 is the risk,

(2) C_2 is the growth and

(3) C_3 is the environmental impact.

The attribute weight vector is given as: $w=(0.35, 0.25, 0.40)^T$. The four alternatives $A_i\,(i=1,2,3,4)$ are to be evaluated using the m-generalized q-neutrosophic information by some decision makers or experts under the attributes $C_j\,(j=1,2,3)$**"**.

Step-1: The rating values of the expert(s) are given in the form of the following decision matrix $\tilde{D}$:

	c_1	c_2	c_3
A_1	<0.3, 0.1, 0.4>	<0.5, 0.3, 0.4>	<0.3, 0.2, 0.6>
A_2	<0.8, 0.2, 0.3>	<0.7, 0.1, 0.3>	<0.7, 0.2 0.2>
A_3	<0.5, 0.4, 0.3>	<0.6, 0.3, 0.4>	<0.5, 0.1, 0.3>
A_4	<0.6, 0.1, 0.2>	<0.7, 0.1, 0.2>	<0.3, 0.2, 0.3>

Step-2: Using the operator $mGqNWAA$, we construct the aggregated *m*-generalized *q*-neutrosophic decision matrix $\tilde{R}$ given below (taking *m*=3 and *q*=3):

A_1	<0.374405104, 0.173657007, 0.470431609>
A_2	<0.741650663, 0.168179283, 0.2550849>
A_3	<0.529784239, 0.213796854, 0.322237098>
A_4	<0.56691263, 0.131950791, 0.235215805>

Step-3: The score values of the alternatives are calculated as:

$S(A_1)$=0.5767, $S(A_2)$=0.7727, $S(A_3)$=0.6645, $S(A_4)$=0.7332

Step-4: The ranking order of the alternatives are: $A_2 > A_4 > A_3 > A_1$ which coincides with the ranking order determined by Jun Ye **[15, 17, 18, 19]** and hence the most desirable alternative is A_2.

Now if we want to utilize the $mGqNWGA$ operator instead of $mGqNWAA$ operator, then the steps for solving the multi attribute decision making problem are as follows:

Step-1: The rating values of the expert(s) are given in the form of the following decision matrix $\tilde{D}$:

	c_1	c_2	c_3
A_1	<0.3, 0.1, 0.4>	<0.5, 0.3, 0.4>	<0.3, 0.2, 0.6>
A_2	<0.8, 0.2, 0.3>	<0.7, 0.1, 0.3>	<0.7, 0.2 0.2>
A_3	<0.5, 0.4, 0.3>	<0.6, 0.3, 0.4>	<0.5, 0.1, 0.3>
A_4	<0.6, 0.1, 0.2>	<0.7, 0.1, 0.2>	<0.3, 0.2, 0.3>

Step-2: Using the operator $mGqNWGA$, we construct the aggregated *m*-generalized *q*-neutrosophic decision matrix $\tilde{R}$ given below (taking *m*=3 and *q*=3):

A_1	<0.34086581, 0.2179417, 0.504033104>
A_2	<0.73349173, 0.184246871, 0.268903022>
A_3	<0.52331757, 0.310497774, 0.331365356>
A_4	<0.472580665, 0.156129905, 0.250101937>

Step-3: The score values of the alternatives are calculated as:

$S(A_1)=0.5396$, $S(A_2)=0.7601$, $S(A_3)=0.6271$, $S(A_4)=0.6887$

Step-4: The ranking order of the alternatives are: $A_2 > A_4 > A_3 > A_1$ which also coincides with the ranking order determined by Jun Ye **[15, 17, 18, 19]** and hence the most desirable alternative is still A_2.

6. CONCLUSIONS:

In this paper, the notion of *m*-generalized *q*- neutrosophic sets is proposed and the basic properties of *m*-generalized *q*- neutrosophic numbers (*mGqNNs* for short) are presented. Also, various types of operations between the *mGqNNs* are discussed. Then, two types of *m*-generalized *q*- neutrosophic weighted aggregation operators are proposed to aggregate the *m*-generalized *q*- neutrosophic information. Furthermore, score of *a mGqNN* is proposed to ranking the *mGqNNs*. Utilizing the *m*-generalized *q*- neutrosophic weighted aggregation operators and score of *a mGqNN*, a multi attribute decision making method is developed, in which the evaluation values of alternatives on the attribute are represented in terms of *mGqNNs* and the alternatives are ranked according to the values of the score of *mGqNNs* to select the most desirable one. Finally, a practical example for investment decision making is presented to demonstrate the application and effectiveness of the proposed method. The advantage of the proposed method is that it is more suitable for solving multi attribute decision making problems because *m*-generalized *q*-neutrosophic sets (*mGqNS*s) are extensions of fuzzy sets, Pythagorean fuzzy sets, *q*-rung orthopair fuzzy sets, intuitionistic fuzzy sets, single valued neutrosophic sets, *n*-hyperspherical neutrosophic sets and single valued spherical neutrosophic sets.

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A Novel Dynamic Multi-Criteria Decision Making Method Based on Generalized Dynamic Interval-Valued Neutrosophic Set

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Abstract: Dynamic multi-criteria decision-making (DMCDM) models have many meaningful applications in real life in which solving indeterminacy of information in DMCDMs strengthens the potential application of DMCDM. This study introduces an extension of dynamic internal-valued neutrosophic sets namely generalized dynamic internal-valued neutrosophic sets. Based on this extension, we develop some operators and a TOPSIS method to deal with the change of both criteria, alternatives, and decision-makers by time. In addition, this study also applies the proposal model to a real application that facilitates ranking students according to attitude-skill-knowledge evaluation model. This application not only illustrates the correctness of the proposed model but also introduces its high potential appliance in the education domain.

Keywords: generalized dynamic interval-valued neutrosophic set; hesitant fuzzy set; dynamic neutrosophic environment; dynamic TOPSIS method; neutrosophic data analytics

1. Introduction

Multi-criteria decision-making (MCDM) in real world is often dynamic [1]. In the dynamic MCDM (DMCDM) model, neither alternatives nor criteria are constant throughout the whole problem and do not change over time. Besides, the DMCDM model has to cope with both dynamic and indeterminate problems of data. For example, when ranking tertiary students during learning time in a university by the set of criteria based on attitudes-skills-knowledge model (ASK), the criteria, students and lecturers are changing during semesters. The lecturers' evaluations using scores, or other ordered scales, are also subject to indeterminacy because of lecturers' personal experiences and biases. Therefore, a ranking model that can handle these issues is necessary.

In [2], Smarandache introduced neutrosophic set including truth-membership, an indeterminacy-membership and a falsity-membership to well treat the problem of information indeterminacy. Since then, variant forms of MCDM and DMCDM models have been proposed as in [3–15]. In order to consider the time dimension, Wang [16] proposed the interval neutrosophic set and its mathematical operators. Ye [9] proposed MCDM in interval-valued neutrosophic set. Dynamic MCDM for dynamic interval-valued neutrosophic set (DIVNS) was proposed in [14]. The authors have developed mathematical operators for TOPSIS method in DIVNSs.

In some cases, criteria, alternatives and decision-makers are changing by time. This fact requires a new method for DMCDM using TOPSIS method in the interval-valued neutrosophic set [17] with diversion of history data. The TOPSIS method for DIVNS in [14] did not solve the problem with the changing criteria, alternatives, and decision-makers. Liu et al. [13] combined the theory of both interval-valued neutrosophic set and hesitant fuzzy set to solve the MCDM problem. However, this study did not use TOPSIS method, and it did not consider the change of criteria also. In order to take the history data into account, Je [10] proposed two hesitant interval neutrosophic linguistic weighted operators to ranking alternatives in dynamic environment. In short, the DMCDM model in DIVNS based on TOPSIS method has not been addressed before.

The purpose of this paper is to deal with the change of criteria, alternatives, and decision-makers during time. We define generalized dynamic interval-valued neutrosophic set (GDIVNS) and some operators. Based on mathematical operators in GDIVNS (distance and weighted aggregation operators), a framework of dynamic TOPSIS is introduced. The proposed method is applied for ranking students of Thuongmai University, Vietnam on attributes of ASK model. ASK model is applicable for evaluation of tertiary students' performance, and it gives more information that support employers besides a set of university exit benchmark. It also facilitates students to make proper self-adjustments and help them pursue appropriate professional orientation for their future career [18–21]. This application proves the suitability of the proposed model for real ranking problems.

This paper is structured as follows: The Section 1 is an introduction, and the Section 2 provides the brief preliminaries for DMCDM model in both legacy environment and interval-valued neutrosophic set. The Section 3 presents the definition of GDIVNS and some mathematical operators on this set. The Section 4 introduces the framework of dynamic TOPSIS method in GDIVNSs environment. The Section 5 presents the application of dynamic TOPSIS method in the problem of ranking students based on attributes of ASK model. The Section 6 compares the result of proposed model with previous TOPSIS model in DIVNS. The last section mentions the brief summary of this study and intended future works.

2. Preliminary

2.1. Multi-Criteria Decision-Making Model Based on History

A dynamic multi-criteria decision-making model introduced by Campanella and Ribeiro [1] is a DMCDM in which all alternatives and criteria are subject to change. The model gives decisions at all periods or just at the last one. The final rating of alternatives is calculated as:

$$E_t(a) = \begin{cases} R_t(a), & a \in A_t \backslash H_{t-1}^A \\ D_E(E_{t-1}(a), R_t(a)), & a \in A_t \cap H_{t-1}^A \\ E_{t-1}(a), & a \in H_{t-1}^A \backslash A_t \end{cases} \tag{1}$$

where A_t is a set of alternatives at period t, H_{t-1}^A is a historical set of alternatives at period $t-1$ ($H_0^A = \varnothing$), $R_t(a)$ is rating of alternative a at period t, and D_E is an aggregation operator.

2.2. Dynamic Interval-Valued Neutrosophic Set and Hesitant Fuzzy Set

Thong et al. [14] introduced the concept of dynamic interval-valued neutrosophic set (DIVNS).

Definition 1. [14] *Let U be a universe of discourse, and A be a dynamic interval-valued neutrosophic Set (DIVNS) expressed by,*

$$A = \left\{x, \left\langle \left[T_x^L(\tau), T_x^U(\tau)\right], \left[I_x^L(\tau), I_x^U(\tau)\right], \left[F_x^L(\tau), F_x^U(\tau)\right] \right\rangle \middle| x \in U \right\} \tag{2}$$

where T_x, I_x, F_x are the truth-membership, indeterminacy-membership, falsity-membership respectively, $\tau = \{\tau_1, \tau_2, \ldots, \tau_k\}$ is set of time sequence and

$$\left[T_x^L(\tau), T_x^U(\tau)\right] \subseteq [0,1]; \left[I_x^L(\tau), I_x^U(\tau)\right] \subseteq [0,1]; \left[F_x^L(\tau), F_x^U(\tau)\right] \subseteq [0,1]$$

Example 1. *A DIVNS in time sequence $\tau = \{\tau_1, \tau_2\}$ and universal $U = \{x_1, x_2, x_3\}$ is:*

$$A = \left\{ \begin{array}{l} x_1, \langle([0.5, 0.6], [0.1, 0.3], [0.2, 0.4]), ([0.4, 0.55], [0.25, 0.3], [0.3, 0.42])\rangle \\ x_2, \langle([0.7, 0.81], [0.1, 0.2], [0.1, 0.2]), ([0.72, 0.8], [0.11, 0.25], [0.2, 0.4])\rangle \\ x_3, \langle([0.3, 0.5], [0.4, 0.5], [0.6, 0.7]), ([0.4, 0.5], [0.5, 0.6], [0.66, 0.73])\rangle \end{array} \right\}$$

Hesitant fuzzy set (HFS) first introduced by Torra and Narukawa [19] and Torra [20] is defined as follows.

Definition 2. *[20] A hesitant fuzzy set E on U is defined by the function $h_E(x)$. When $h_E(x)$ is applied to U, it returns a finite subset of [0, 1], which can be represented as*

$$E = \{\langle x, h_E(x)\rangle | x \in U\} \tag{3}$$

where $h_E(x)$ is a set of some values in $[0,1]$.

Example 2. *Let $X = \{x_1, x_2, x_3\}$ be the discourse set, and $h_E(x_1) = \{0.1, 0.2\}$, $h_E(x_2) = \{0.3\}$ and $h_E(x_3) = \{0.2, 0.3, 0.5\}$. Then, E can be considered as a HFS:*

$$E = \{\langle x_1, \{0.1, 0.2\}\rangle, \langle x_2, \{0.3\}\rangle, \langle x_3, \{0.2, 0.3, 0.5\}\rangle\}$$

3. Generalized Dynamic Interval-Valued Neutrosophic Set

Extending DIVNS by the concept of HFS is considered how to express the criteria, alternatives, and DMs that are changing during time criteria, alternatives and decision-makers are changing by time.

In this section, we propose the concepts of generalized dynamic interval-valued neutrosophic set (GDIVNS) and generalized dynamic interval-valued neutrosophic element (GDIVNE) including fundamental elements, operational laws as well as the score functions. Then, GDIVNS's theory is applied for the decision-making model in Section 4.

Definition 3. *Let U be a universe of discourse. A generalized dynamic interval-valued neutrosophic set (GDIVNS) in U can be expressed as,*

$$\widetilde{E} = \left\{\left\langle x, \widetilde{h}_{\widetilde{E}}(x(t_r))\right\rangle | x \in U; \forall t_r \in t; \right\} \tag{4}$$

where $\widetilde{h}_{\widetilde{E}}(x(t_r))$ is expressed for importing HFS into DIVNS. $\widetilde{h}_{\widetilde{E}}(x(t_r))$ is a set of DIVNSs at period t_r and $t = \{t_1, t_2, t_3, \ldots, t_s\}$, which denotes the possible DIVNSs of the element $x \in X$ to the set $\widetilde{E}$, $\widetilde{h}_{\widetilde{E}}(x(t_r))$ can be represented by a generalized dynamic interval-valued neutrosophic element (GDIVNE). When $s = 1$ and

$\left|\widetilde{h}_{\widetilde{E}}(x(t_r))\right| = 1$, *GDIVNS simplifies to DIVNS [14]. For convenience, we denote* $\widetilde{h} = \widetilde{h}_{\widetilde{E}}(x(t)) = \left\{\gamma \middle| \gamma \in \widetilde{h}\right\}$, *where*

$$\gamma = \left(\left[T^L(x(\tau)), T^U(x(\tau))\right], \left[I^L(x(\tau)), I^U(x(\tau))\right], \left[F^L(x(\tau)), F^U(x(\tau))\right]\right)$$

is a dynamic interval-valued neutrosophic number.

Example 3. *Let* $t = \{t_1, t_2\}$; $\tau = \{\tau_1, \tau_2\}$ *and an universal* $X = \{x_1, x_2, x_3\}$. *A GDIVNS in X is given as:*

$$\widetilde{E} = \left\{ \begin{array}{l} \left\langle x_1, \left\{ \begin{array}{l} \langle([0.2, 0.33], [0.4, 0.5], [0.6, 0.7]), ([0.24, 0.39], [0.38, 0.47], [0.56, 0.7])\rangle, \\ \langle([0.29, 0.37], [0.3, 0.5], [0.4, 0.58]), ([0.4, 0.5], [0.2, 0.3], [0.35, 0.42])\rangle \end{array} \right\} \right\rangle, \\ \left\langle x_2, \left\{ \begin{array}{l} \langle([0.8, 0.9], [0.1, 0.2], [0.1, 0.2]), ([0.72, 0.8], [0.11, 0.25], [0.23, 0.45])\rangle, \\ \langle([0.4, 0.6], [0.2, 0.4], [0.3, 0.4]), ([0.41, 0.5], [0.26, 0.39], [0.2, 0.3])\rangle \end{array} \right\} \right\rangle, \\ \left\langle x_3, \left\{ \begin{array}{l} \langle([0.6, 0.7], [0.2, 0.3], [0.4, 0.5]), ([0.52, 0.66], [0.34, 0.4], [0.6, 0.77])\rangle, \\ \langle([0.54, 0.62], [0.15, 0.3], [0.2, 0.4]), ([0.4, 0.5], [0.25, 0.32], [0.39, 0.43])\rangle \end{array} \right\} \right\rangle \end{array} \right\}$$

Definition 4. *Let* $\widetilde{h}$, $\widetilde{h}_1$ *and* $\widetilde{h}_2$ *be three GDIVNEs. When* $\lambda > 0$, *the operations of GDIVNEs are defined as follows:*

(i) *Addition*

$$\begin{aligned} \widetilde{h}_1 \oplus \widetilde{h}_2 &= \cup_{\forall \gamma_1 \in \widetilde{h}_1; \forall \gamma_2 \in \widetilde{h}_2} \{\gamma_1 \oplus \gamma_2\} \\ &= \left\{\left\langle \begin{array}{l} \left[T^L_{\gamma_1}(x(\tau)) + T^L_{\gamma_2}(x(\tau)) - T^L_{\gamma_1}(x(\tau)) \times T^L_{\gamma_2}(x(\tau)), T^U_{\gamma_1}(x(\tau)) + T^U_{\gamma_2}(x(\tau)) - T^U_{\gamma_1}(x(\tau)) \times T^U_{\gamma_2}(x(\tau))\right], \\ \left[I^L_{\gamma_1}(x(\tau)) \times I^L_{\gamma_2}(x(\tau)), I^U_{\gamma_1}(x(\tau)) \times I^U_{\gamma_2}(x(\tau))\right], \left[F^L_{\gamma_1}(x(\tau)) \times F^L_{\gamma_2}(x(\tau)), F^U_{\gamma_1}(x(\tau)) \times F^U_{\gamma_2}(x(\tau))\right] \end{array} \right\rangle\right\} \end{aligned}$$

(ii) *Multiplication*

$$\begin{aligned} \widetilde{h}_1 \otimes \widetilde{h}_2 &= \cup_{\forall \gamma_1 \in \widetilde{h}_1; \forall \gamma_2 \in \widetilde{h}_2} \{\gamma_1 \otimes \gamma_2\} \\ &= \left\{\left\langle \begin{array}{l} \left[T^L_{\gamma_1}(x(\tau)) \times T^L_{\gamma_2}(x(\tau)), T^U_{\gamma_1}(x(\tau)) \times T^U_{\gamma_2}(x(\tau))\right], \\ \left[I^L_{\gamma_1}(x(\tau)) + I^L_{\gamma_2}(x(\tau)) - I^L_{\gamma_1}(x(\tau)) \times I^L_{\gamma_2}(x(\tau)), I^U_{\gamma_1}(x(\tau)) + I^U_{\gamma_2}(x(\tau)) - I^U_{\gamma_1}(x(\tau)) \times I^U_{\gamma_2}(x(\tau))\right], \\ \left[F^L_{\gamma_1}(x(\tau)) + F^L_{\gamma_2}(x(\tau)) - F^L_{\gamma_1}(x(\tau)) \times F^L_{\gamma_2}(x(\tau)), F^U_{\gamma_1}(x(\tau)) + F^U_{\gamma_2}(x(\tau)) - F^U_{\gamma_1}(x(\tau)) \times F^U_{\gamma_2}(x(\tau))\right] \end{array} \right\rangle\right\} \end{aligned}$$

(iii) *Scalar Multiplication*

$$\begin{aligned} \lambda\widetilde{h} &= \cup_{\forall \gamma \in \widetilde{h}} \{\lambda\gamma\} \\ &= \cup_{\forall \gamma \in \widetilde{h}} \left\{\left\langle \begin{array}{l} \left[1 - \left(1 - T^L(x(\tau))\right)^\lambda, 1 - \left(1 - T^U(x(\tau))\right)^\lambda\right], \\ \left[\left(I^L(x(\tau))\right)^\lambda, \left(I^U(x(\tau))\right)^\lambda\right], \left[\left(F^L(x(\tau))\right)^\lambda, \left(F^U(x(\tau))\right)^\lambda\right] \end{array} \right\rangle\right\} \end{aligned}$$

(iv) *Power*

$$\begin{aligned} \widetilde{h}^\lambda &= \cup_{\forall \gamma \in \widetilde{h}} \left\{\gamma^\lambda\right\} \\ &= \cup_{\forall \gamma \in \widetilde{h}} \left\{\left\langle \begin{array}{l} \left[\left(T^L(x(\tau))\right)^\lambda, \left(T^U(x(\tau))\right)^\lambda\right], \left[1 - \left(1 - I^L(x(\tau))\right)^\lambda, 1 - \left(1 - I^U(x(\tau))\right)^\lambda\right], \\ \left[1 - \left(1 - F^L(x(\tau))\right)^\lambda, 1 - \left(1 - F^U(x(\tau))\right)^\lambda\right] \end{array} \right\rangle\right\} \end{aligned}$$

Definition 5. *Let* $\widetilde{h}$ *be a GDIVNE. Then, the score functions of the GDIVNE* $\widetilde{h}$ *are defined by,*

$$S\left(\widetilde{h}\right) = \frac{1}{\#\widetilde{h}} \times \frac{1}{k} \sum_{\forall \gamma \in \widetilde{h}} \sum_{l=1}^{k} \left(\left(\frac{T^L(\tau_l) + T^U(\tau_l)}{2} + \left(1 - \frac{I^L(\tau_l) + I^U(\tau_l)}{2}\right) + \left(1 - \frac{F^L(\tau_l) + F^U(\tau_l)}{2}\right)\right)/3\right) \quad (5)$$

where $\tau = \{\tau_1, \tau_2, \ldots, \tau_k\}$, *and* $\#\widetilde{h}$ *is number of elements in* $\widetilde{h}$. *Obviously,* $S\left(\widetilde{h}\right) \in [0,1]$. *If* $S\left(\widetilde{h_1}\right) \geq S\left(\widetilde{h_2}\right)$, *then* $\widetilde{h_1} \geq \widetilde{h_2}$.

Example 4. *Let three GDIVNEs:*

$$\begin{aligned}
\widetilde{h_1} &= \{\langle([1,1],[0,0],[0,0]),([1,1],[0,0],[0,0])\rangle,\langle([1,1],[0,0],[0,0]),([1,1],[0,0],[0,0])\rangle\} \\
\widetilde{h_2} &= \{\langle([0,0],[1,1],[0,0]),([0,0],[1,1],[0,0])\rangle,\langle([0,0],[1,1],[0,0]),([0,0],[1,1],[0,0])\rangle\} \\
\widetilde{h_3} &= \{\langle([0,0],[1,1],[1,1]),([0,0],[1,1],[1,1])\rangle,\langle([0,0],[1,1],[1,1]),([0,0],[1,1],[1,1])\rangle\}
\end{aligned}$$

According to Equation (5), we have $S\left(\widetilde{h_1}\right) = 1;\ S\left(\widetilde{h_2}\right) = \frac{1}{3};\ S\left(\widetilde{h_3}\right) = 0$. Thus, $\widetilde{h_1} > \widetilde{h_2} > \widetilde{h_3}$.

Definition 6. *Let* $\widetilde{h_j}(j = 1, 2, \ldots, n)$ *be a collection of GDIVNEs. Generalized dynamic interval-valued neutrosophic weighted average (GDIVNWA) operator is defined as*

$$\begin{aligned}
&GDIVNWA\left(\widetilde{h_1}, \widetilde{h_2}, \ldots, \widetilde{h_n}\right) = \sum_{j=1}^{n} w_j \widetilde{h_j} \\
&= \bigcup_{\gamma_1 \in \widetilde{h_1}, \gamma_2 \in \widetilde{h_2}, \ldots, \gamma_n \in \widetilde{h_n}} \left\{\left(\begin{array}{l} \left[1 - \prod_{j=1}^{n}\left(1 - T^L_{\gamma_j}(\tau)\right)^{w_j}, 1 - \prod_{j=1}^{n}\left(1 - T^U_{\gamma_j}(\tau)\right)^{w_j}\right], \\ \left[\prod_{j=1}^{n}\left(I^L_{\gamma_j}(\tau)\right)^{w_j}, \prod_{j=1}^{n}\left(I^U_{\gamma_j}(\tau)\right)^{w_j}\right], \left[\prod_{j=1}^{n}\left(F^L_{\gamma_j}(\tau)\right)^{w_j}, \prod_{j=1}^{n}\left(F^U_{\gamma_j}(\tau)\right)^{w_j}\right] \end{array}\right)\right\}
\end{aligned} \tag{6}$$

Theorem 1. *Let* $\widetilde{h_j}(j = 1, 2, \ldots, n)$ *be the collection of GDIVNEs. The result aggregated from GDIVNWA operator is still a GDIVNE.*

Proof. The Equation (6) is proved by mathematical inductive reasoning method. □

When $n = 1$, Equation (6) holds because it simplifies to the trivial outcome, which is obviously GDIVNE as,

$$GDIVNWA\left(\widetilde{h_1}\right) = \left(\begin{array}{l} \left[1 - \left(1 - T^L_{\gamma_1}(\tau)\right)^{w_1}, 1 - \left(1 - T^U_{\gamma_1}(\tau)\right)^{w_1}\right], \\ \left[\left(I^L_{\gamma_1}(\tau)\right)^{w_1}, \left(I^U_{\gamma_1}(\tau)\right)^{w_1}\right], \left[\left(F^L_{\gamma_1}(\tau)\right)^{w_1}, \left(F^U_{\gamma_1}(\tau)\right)^{w_1}\right] \end{array}\right) \tag{7}$$

Let us assume that (6) is true for $n = z$,

$$\sum_{j=1}^{z} w_j \widetilde{h_j} = \bigcup_{\gamma_1 \in \widetilde{h_1}, \gamma_2 \in \widetilde{h_2}, \ldots, \gamma_z \in \widetilde{h_z}} \left\{\left(\begin{array}{l} \left[1 - \prod_{j=1}^{z}\left(1 - T^L_{\gamma_j}(\tau)\right)^{w_j}, 1 - \prod_{j=1}^{z}\left(1 - T^U_{\gamma_j}(\tau)\right)^{w_j}\right], \\ \left[\prod_{j=1}^{z}\left(I^L_{\gamma_j}(\tau)\right)^{w_j}, \prod_{j=1}^{z}\left(I^U_{\gamma_j}(\tau)\right)^{w_j}\right], \left[\prod_{j=1}^{z}\left(F^L_{\gamma_j}(\tau)\right)^{w_j}, \prod_{j=1}^{z}\left(F^U_{\gamma_j}(\tau)\right)^{w_j}\right] \end{array}\right)\right\} \tag{8}$$

When $n = z + 1$

$$\begin{aligned}
&\sum_{j=1}^{z+1} w_j\widetilde{h}_j = \sum_{j=1}^{z} w_j\widetilde{h}_j \oplus w_{z+1}\widetilde{h}_{z+1} \\
&= \bigcup_{\gamma_1\in\widetilde{h}_1,\gamma_2\in\widetilde{h}_2,\ldots,\gamma_z\in\widetilde{h}_z}\left\{\left(\begin{array}{l}\left[1-\prod_{j=1}^{z}\left(1-T^L_{\gamma_j}(\tau)\right)^{w_j},1-\prod_{j=1}^{z}\left(1-T^U_{\gamma_j}(\tau)\right)^{w_j}\right],\\ \left[\prod_{j=1}^{z}\left(I^L_{\gamma_j}(\tau)\right)^{w_j},\prod_{j=1}^{z}\left(I^U_{\gamma_j}(\tau)\right)^{w_j}\right],\left[\prod_{j=1}^{z}\left(F^L_{\gamma_j}(\tau)\right)^{w_j},\prod_{j=1}^{z}\left(F^U_{\gamma_j}(\tau)\right)^{w_j}\right]\end{array}\right)\right\} \\
&\oplus\left(\begin{array}{l}\left[1-\left(1-T^L_{\gamma_{k+1}}(\tau)\right)^{w_{z+1}},1-\left(1-T^U_{\gamma_{z+1}}(\tau)\right)^{w_{z+1}}\right],\\ \left[\left(I^L_{\gamma_{z+1}}(\tau)\right)^{w_{z+1}},\left(I^U_{\gamma_{z+1}}(\tau)\right)^{w_{z+1}}\right],\left[\left(F^L_{\gamma_{z+1}}(\tau)\right)^{w_{z+1}},\left(F^U_{\gamma_{z+1}}(\tau)\right)^{w_{z+1}}\right]\end{array}\right) \\
&= \bigcup_{\gamma_1\in\widetilde{h}_1,\gamma_2\in\widetilde{h}_2,\ldots,\gamma_{z+1}\in\widetilde{h}_{z+1}}\left\{\left(\begin{array}{l}\left[1-\prod_{j=1}^{z+1}\left(1-T^L_{\gamma_j}(\tau)\right)^{w_j},1-\prod_{j=1}^{z+1}\left(1-T^U_{\gamma_j}(\tau)\right)^{w_j}\right],\\ \left[\prod_{j=1}^{z+1}\left(I^L_{\gamma_j}(\tau)\right)^{w_j},\prod_{j=1}^{z+1}\left(I^U_{\gamma_j}(\tau)\right)^{w_j}\right],\left[\prod_{j=1}^{z+1}\left(F^L_{\gamma_j}(\tau)\right)^{w_j},\prod_{j=1}^{z+1}\left(F^U_{\gamma_j}(\tau)\right)^{w_j}\right]\end{array}\right)\right\}
\end{aligned} \tag{9}$$

It follows that if (6) holds for $n = z$, then it holds for $n = z + 1$. Because it is also true for $n = 1$, according to the method of mathematical inductive reasoning, Equation (6) holds for natural numbers N and Theorem 1 is proven.

Definition 7. *Let $\widetilde{h}_j (j = 1, 2, \ldots, n)$ be a collection of GDIVNEs. Generalized dynamic interval-valued neutrosophic weighted geometric (GDIVNWG) operator is defined as*

$$\begin{aligned}
&GDIVNWG\left(\widetilde{h}_1,\widetilde{h}_2,\ldots,\widetilde{h}_n\right) = \prod_{j=1}^{n}\widetilde{h}_j^{w_j} \\
&= \bigcup_{\gamma_1\in\widetilde{h}_1,\gamma_2\in\widetilde{h}_2,\ldots,\gamma_n\in\widetilde{h}_n}\left\{\left(\begin{array}{l}\left[\prod_{j=1}^{n}\left(T^L_{\gamma_j}(\tau)\right)^{w_j},\prod_{j=1}^{n}\left(T^U_{\gamma_j}(\tau)\right)^{w_j}\right],\left[1-\prod_{j=1}^{n}\left(1-I^L_{\gamma_j}(\tau)\right)^{w_j},1-\prod_{j=1}^{n}\left(1-I^U_{\gamma_j}(\tau)\right)^{w_j}\right],\\ \left[1-\prod_{j=1}^{n}\left(1-F^L_{\gamma_j}(\tau)\right)^{w_j},1-\prod_{j=1}^{n}\left(1-F^U_{\gamma_j}(\tau)\right)^{w_j}\right]\end{array}\right)\right\}
\end{aligned} \tag{10}$$

Theorem 2. *Let $\widetilde{h}_j (j = 1, 2, \ldots, n)$ be the collection of GDIVNEs. The result aggregated from GDIVNWG operator is still a GDIVNE.*

Proof. The Equation (10) is proved by mathematical inductive reasoning method. □

When $n = 1$, Equation (10) is true because it simplifies to the trivial outcome, which is obviously GDIVNE,

$$GDIVNWG\left(\widetilde{h}_1\right) = \left(\begin{array}{l}\left[\left(T^L_{\gamma_1}(\tau)\right)^{w_1},\left(T^U_{\gamma_1}(\tau)\right)^{w_1}\right],\left[1-\left(1-I^L_{\gamma_1}(\tau)\right)^{w_1},1-\left(1-I^U_{\gamma_1}(\tau)\right)^{w_1}\right],\\ \left[1-\left(1-F^L_{\gamma_1}(\tau)\right)^{w_1},1-\left(1-F^U_{\gamma_1}(\tau)\right)^{w_1}\right]\end{array}\right) \tag{11}$$

Let us assume that (10) is true for $n = z$.

$$\prod_{j=1}^{z}\widetilde{h}_j^{w_j} = \bigcup_{\gamma_1\in\widetilde{h}_1,\gamma_2\in\widetilde{h}_2,\ldots,\gamma_z\in\widetilde{h}_z}\left\{\left(\begin{array}{l}\left[\prod_{j=1}^{z}\left(T^L_{\gamma_j}(\tau)\right)^{w_j},\prod_{j=1}^{z}\left(T^U_{\gamma_j}(\tau)\right)^{w_j}\right],\left[1-\prod_{j=1}^{z}\left(1-I^L_{\gamma_j}(\tau)\right)^{w_j},1-\prod_{j=1}^{z}\left(1-I^U_{\gamma_j}(\tau)\right)^{w_j}\right],\\ \left[1-\prod_{j=1}^{z}\left(1-F^L_{\gamma_j}(\tau)\right)^{w_j},1-\prod_{j=1}^{z}\left(1-F^U_{\gamma_j}(\tau)\right)^{w_j}\right]\end{array}\right)\right\} \tag{12}$$

When $n = z + 1$

$$\begin{array}{l}
\prod_{j=1}^{z+1} \widetilde{h}_j^{w_j} = \prod_{j=1}^{z} \widetilde{h}_j^{w_j} \otimes \widetilde{h}_{z+1}^{w_{z+1}} \\
= \bigcup_{\gamma_1 \in \widetilde{h}_1, \gamma_2 \in \widetilde{h}_2, \ldots, \gamma_z \in \widetilde{h}_z} \left\{ \left(\begin{array}{l} \left[\prod_{j=1}^{k} \left(T^L_{\gamma_j}(\tau)\right)^{w_j}, \prod_{j=1}^{k} \left(T^U_{\gamma_j}(\tau)\right)^{w_j} \right], \left[1 - \prod_{j=1}^{k} \left(1 - I^L_{\gamma_j}(\tau)\right)^{w_j}, 1 - \prod_{j=1}^{k} \left(1 - I^U_{\gamma_j}(\tau)\right)^{w_j} \right], \\ \left[1 - \prod_{j=1}^{k} \left(1 - F^L_{\gamma_j}(\tau)\right)^{w_j}, 1 - \prod_{j=1}^{k} \left(1 - F^U_{\gamma_j}(\tau)\right)^{w_j} \right] \end{array} \right) \right\} \\
\otimes \left(\begin{array}{l} \left[\left(T^L_{\gamma_j}(\tau)\right)^{w_{z+1}}, \left(T^U_{\gamma_{z+1}}(\tau)\right)^{w_{z+1}} \right], \left[1 - \left(1 - I^L_{\gamma_{z+1}}(\tau)\right)^{w_{z+1}}, 1 - \left(1 - I^U_{\gamma_{z+1}}(\tau)\right)^{w_{z+1}} \right], \\ \left[1 - \left(1 - F^L_{\gamma_{z+1}}(\tau)\right)^{w_{z+1}}, 1 - \left(1 - F^U_{\gamma_{z+1}}(\tau)\right)^{w_{z+1}} \right] \end{array} \right) \\
= \bigcup_{\gamma_1 \in \widetilde{h}_1, \gamma_2 \in \widetilde{h}_2, \ldots, \gamma_{z+1} \in \widetilde{h}_{z+1}} \left\{ \left(\begin{array}{l} \left[\prod_{j=1}^{z+1} \left(T^L_{\gamma_j}(\tau)\right)^{w_j}, \prod_{j=1}^{z+1} \left(T^U_{\gamma_j}(\tau)\right)^{w_j} \right], \left[1 - \prod_{j=1}^{z+1} \left(1 - I^L_{\gamma_j}(\tau)\right)^{w_j}, 1 - \prod_{j=1}^{z+1} \left(1 - I^U_{\gamma_j}(\tau)\right)^{w_j} \right], \\ \left[1 - \prod_{j=1}^{z+1} \left(1 - F^L_{\gamma_j}(\tau)\right)^{w_j}, 1 - \prod_{j=1}^{z+1} \left(1 - F^U_{\gamma_j}(\tau)\right)^{w_j} \right] \end{array} \right) \right\}
\end{array} \tag{13}$$

It follows that if (10) holds for $n = z$, then it holds for $n = z + 1$. Because it is also true for $n = 1$, according to the method of mathematical inductive reasoning, Equation (10) holds for all natural numbers N and Theorem 2 is proven.

Herein, we define the generalized dynamic interval-valued neutrosophic hybrid weighted averaging (GDIVNHWA) operator to combine the effects of attribute weight vector and the positional weight vector, which are mentioned in Definitions 6 and 7.

Definition 8. *Let $\lambda > 0$ and $\widetilde{h}_j (j = 1, 2, \ldots, n)$ be a collection of GDIVNEs. Generalized dynamic interval-valued neutrosophic hybrid weighted averaging (GDIVNHWA) operator is defined as,*

$$\begin{array}{l}
DIVHNWG\left(\widetilde{h}_1, \widetilde{h}_2, \ldots, \widetilde{h}_n\right) = \left(\sum_{j=1}^{n} w_j \widetilde{h}_j^{\lambda} \right)^{\frac{1}{\lambda}} \\
= \bigcup_{\gamma_1 \in \widetilde{h}_1, \gamma_2 \in \widetilde{h}_2, \ldots, \gamma_n \in \widetilde{h}_n} \left\{ \left(\begin{array}{l} \left[\left(1 - \prod_{j=1}^{n} \left(1 - \left(T^L_{\gamma_j}(\tau)\right)^{\lambda}\right)^{w_j}\right)^{\frac{1}{\lambda}}, \left(1 - \prod_{j=1}^{n} \left(1 - \left(T^U_{\gamma_j}(\tau)\right)^{\lambda}\right)^{w_j}\right)^{\frac{1}{\lambda}} \right], \\ \left[1 - \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - I^L_{\gamma_j}(\tau)\right)^{\lambda}\right)^{w_j}\right)^{\frac{1}{\lambda}}, 1 - \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - I^U_{\gamma_j}(\tau)\right)^{\lambda}\right)^{w_j}\right)^{\frac{1}{\lambda}} \right], \\ \left[1 - \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - F^L_{\gamma_j}(\tau)\right)^{\lambda}\right)^{w_j}\right)^{\frac{1}{\lambda}}, 1 - \left(1 - \prod_{j=1}^{n} \left(1 - \left(1 - F^U_{\gamma_j}(\tau)\right)^{\lambda}\right)^{w_j}\right)^{\frac{1}{\lambda}} \right] \end{array} \right) \right\}
\end{array} \tag{14}$$

Theorem 3. *Let $\widetilde{h}_j (j = 1, 2, \ldots, n)$ be the collection of GDIVNEs. The result aggregated from GDIVNHWA operator is still a GDIVNE.*

Proof. The Equation (14) can be proved by mathematical inductive reasoning method. □

We first prove that (15) is a collection of GDIVNEs,

$$\sum_{j=1}^{n} w_j \widetilde{h}_j^{\lambda} = \bigcup_{\gamma_1 \in \widetilde{h}_1, \gamma_2 \in \widetilde{h}_2, \ldots, \gamma_n \in \widetilde{h}_n} \left\{ \left(\begin{array}{l} \left[1 - \prod_{j=1}^{n} \left(1 - \left(T^L_{\gamma_j}(\tau)\right)^{\lambda}\right)^{w_j}, 1 - \prod_{j=1}^{n} \left(1 - \left(T^U_{\gamma_j}(\tau)\right)^{\lambda}\right)^{w_j} \right], \\ \left[1 - \prod_{j=1}^{n} \left(1 - \left(1 - I^L_{\gamma_j}(\tau)\right)^{\lambda}\right)^{w_j}, 1 - \prod_{j=1}^{n} \left(1 - \left(1 - I^U_{\gamma_j}(\tau)\right)^{\lambda}\right)^{w_j} \right], \\ \left[1 - \prod_{j=1}^{n} \left(1 - \left(1 - F^L_{\gamma_j}(\tau)\right)^{\lambda}\right)^{w_j}, 1 - \prod_{j=1}^{n} \left(1 - \left(1 - F^U_{\gamma_j}(\tau)\right)^{\lambda}\right)^{w_j} \right] \end{array} \right) \right\} \tag{15}$$

When $n = 1$, Equation (15) is true because it simplifies to the trivial outcome, which is obviously GDIVNE,

$$w_1\widetilde{h}_1^{\lambda} = \left(\begin{array}{l} \left[1-\left(1-\left(T_{\gamma_1}^L(\tau)\right)^{\lambda}\right)^{w_1}, 1-\left(1-\left(T_{\gamma_1}^U(\tau)\right)^{\lambda}\right)^{w_1}\right], \\ \left[1-\left(1-\left(1-I_{\gamma_1}^L(\tau)\right)^{\lambda}\right)^{w_1}, 1-\left(1-\left(1-I_{\gamma_1}^U(\tau)\right)^{\lambda}\right)^{w_1}\right], \\ \left[1-\left(1-\left(1-F_{\gamma_1}^L(\tau)\right)^{\lambda}\right)^{w_1}, 1-\left(1-\left(1-F_{\gamma_1}^U(\tau)\right)^{\lambda}\right)^{w_1}\right] \end{array} \right) \tag{16}$$

Let us assume that (15) is true for $n = z$,

$$\sum_{j=1}^{z} w_j\widetilde{h}_j^{\lambda} = \bigcup_{\gamma_1\in\widetilde{h}_1,\gamma_2\in\widetilde{h}_2,\ldots,\gamma_z\in\widetilde{h}_z} \left\{ \left(\begin{array}{l} \left[1-\prod_{j=1}^{z}\left(1-\left(T_{\gamma_j}^L(\tau)\right)^{\lambda}\right)^{w_j}, 1-\prod_{j=1}^{z}\left(1-\left(T_{\gamma_j}^U(\tau)\right)^{\lambda}\right)^{w_j}\right], \\ \left[1-\prod_{j=1}^{z}\left(1-\left(1-I_{\gamma_j}^L(\tau)\right)^{\lambda}\right)^{w_j}, 1-\prod_{j=1}^{z}\left(1-\left(1-I_{\gamma_j}^U(\tau)\right)^{\lambda}\right)^{w_j}\right], \\ \left[1-\prod_{j=1}^{z}\left(1-\left(1-F_{\gamma_j}^L(\tau)\right)^{\lambda}\right)^{w_j}, 1-\prod_{j=1}^{z}\left(1-\left(1-F_{\gamma_j}^U(\tau)\right)^{\lambda}\right)^{w_j}\right] \end{array} \right) \right\} \tag{17}$$

When $n = z + 1$,

$$\begin{array}{l} \sum_{j=1}^{z+1} w_j\widetilde{h}_j^{\lambda} = \sum_{j=1}^{z} w_j\widetilde{h}_j^{\lambda} \oplus w_{z+1}\widetilde{h}_{z+1}^{\lambda} \\ = \bigcup_{\gamma_1\in\widetilde{h}_1,\gamma_2\in\widetilde{h}_2,\ldots,\gamma_z\in\widetilde{h}_z} \left\{ \left(\begin{array}{l} \left[1-\prod_{j=1}^{z}\left(1-\left(T_{\gamma_j}^L(\tau)\right)^{\lambda}\right)^{w_j}, 1-\prod_{j=1}^{z}\left(1-\left(T_{\gamma_j}^U(\tau)\right)^{\lambda}\right)^{w_j}\right], \\ \left[1-\prod_{j=1}^{z}\left(1-\left(1-I_{\gamma_j}^L(\tau)\right)^{\lambda}\right)^{w_j}, 1-\prod_{j=1}^{z}\left(1-\left(1-I_{\gamma_j}^U(\tau)\right)^{\lambda}\right)^{w_j}\right], \\ \left[1-\prod_{j=1}^{z}\left(1-\left(1-F_{\gamma_j}^L(\tau)\right)^{\lambda}\right)^{w_j}, 1-\prod_{j=1}^{z}\left(1-\left(1-F_{\gamma_j}^U(\tau)\right)^{\lambda}\right)^{w_j}\right] \end{array} \right) \right\} \\ \oplus \left\{ \left(\begin{array}{l} \left[1-\left(1-\left(T_{\gamma_{z+1}}^L(\tau)\right)^{\lambda}\right)^{w_{z+1}}, 1-\left(1-\left(T_{\gamma_{z+1}}^U(\tau)\right)^{\lambda}\right)^{w_{z+1}}\right], \\ \left[1-\left(1-\left(1-I_{\gamma_{z+1}}^L(\tau)\right)^{\lambda}\right)^{w_{z+1}}, 1-\left(1-\left(1-I_{\gamma_{z+1}}^U(\tau)\right)^{\lambda}\right)^{w_{z+1}}\right], \\ \left[1-\left(1-\left(1-F_{\gamma_{z+1}}^L(\tau)\right)^{\lambda}\right)^{w_{z+1}}, 1-\left(1-\left(1-F_{\gamma_{z+1}}^U(\tau)\right)^{\lambda}\right)^{w_{z+1}}\right] \end{array} \right) \right\} \\ = \bigcup_{\gamma_1\in\widetilde{h}_1,\gamma_2\in\widetilde{h}_2,\ldots,\gamma_{k+1}\in\widetilde{h}_{k+!}} \left\{ \left(\begin{array}{l} \left[1-\prod_{j=1}^{k+1}\left(1-\left(T_{\gamma_j}^L(\tau)\right)^{\lambda}\right)^{w_j}, 1-\prod_{j=1}^{k+!}\left(1-\left(T_{\gamma_j}^U(\tau)\right)^{\lambda}\right)^{w_j}\right], \\ \left[1-\prod_{j=1}^{k+1}\left(1-\left(1-I_{\gamma_j}^L(\tau)\right)^{\lambda}\right)^{w_j}, 1-\prod_{j=1}^{k+1}\left(1-\left(1-I_{\gamma_j}^U(\tau)\right)^{\lambda}\right)^{w_j}\right], \\ \left[1-\prod_{j=1}^{k+1}\left(1-\left(1-F_{\gamma_j}^L(\tau)\right)^{\lambda}\right)^{w_j}, 1-\prod_{j=1}^{k+1}\left(1-\left(1-F_{\gamma_j}^U(\tau)\right)^{\lambda}\right)^{w_j}\right] \end{array} \right) \right\} \end{array} \tag{18}$$

It follows that if (15) holds for $n = z$, then it holds for $n = z + 1$. Because it is also true for $n = 1$, according to the method of mathematical inductive reasoning, Equation (15) holds for natural numbers N. According to Equation (15) and Definition 4, we have,

$$DIVHNWG\left(\widetilde{h}_1,\widetilde{h}_2,\ldots,\widetilde{h}_n\right)=\left(\sum_{j=1}^{n} w_j\widetilde{h}_j^{\lambda}\right)^{\frac{1}{\lambda}}$$

$$=\bigcup_{\gamma_1\in\widetilde{h}_1,\gamma_2\in\widetilde{h}_2,\ldots,\gamma_n\in\widetilde{h}_n}\left\{\left(\begin{array}{l}\left[\left(1-\prod_{j=1}^{n}\left(1-\left(T^L_{\gamma_j}(\tau)\right)^{\lambda}\right)^{w_j}\right)^{\frac{1}{\lambda}},\left(1-\prod_{j=1}^{n}\left(1-\left(T^U_{\gamma_j}(\tau)\right)^{\lambda}\right)^{w_j}\right)^{\frac{1}{\lambda}}\right],\\ \left[1-\left(1-\prod_{j=1}^{n}\left(1-\left(1-I^L_{\gamma_j}(\tau)\right)^{\lambda}\right)^{w_j}\right)^{\frac{1}{\lambda}},1-\left(1-\prod_{j=1}^{n}\left(1-\left(1-I^U_{\gamma_j}(\tau)\right)^{\lambda}\right)^{w_j}\right)^{\frac{1}{\lambda}}\right],\\ \left[1-\left(1-\prod_{j=1}^{n}\left(1-\left(1-F^L_{\gamma_j}(\tau)\right)^{\lambda}\right)^{w_j}\right)^{\frac{1}{\lambda}},1-\left(1-\prod_{j=1}^{n}\left(1-\left(1-F^U_{\gamma_j}(\tau)\right)^{\lambda}\right)^{w_j}\right)^{\frac{1}{\lambda}}\right]\end{array}\right)\right\}$$

Thus, Theorem 3 is proven.

4. Dynamic TOPSIS Method

Based on the theory of GDVINS, the dynamic decision-making model is proposed to deal with the change of criteria, alternatives, and decision-makers during time.

For each period $t=\{t_1,t_2,\ldots,t_s\}$, assume $\widetilde{A}(t_r)=\{A_1,A_2,\ldots,A_{v_r}\}$ and $\widetilde{C}(t_r)=\left\{C_1,C_2,\ldots,C_{n_r}\right\}$ and $\widetilde{D}(t_r)=\left\{D_1,D_2,\ldots,D_{h_r}\right\}$ being the sets of alternatives, criteria, and decision-makers at period r^{th}, $r=\{1,2,\ldots,s\}$. For a decision-maker $D_q; q=1,\ldots,h_r$, the evaluation of an alternative $A_m; m=1,\ldots,v_r$, on a criteria $C_p; p=1,\ldots,n_r$, in time sequence $\tau=\left\{\tau_1,\tau_2,\ldots,\tau_{k_r}\right\}$ is represented by the Neutrosophic decision matrix $\Re^q(t_r)=\left(\xi^q_{mp}(\tau)\right)_{v_r\times n_r}; l=1,2,\ldots,k_r$. where

$$\xi^q_{mp}(\tau)=\left\langle x^q_{d_{mp}}(\tau),\left(T^q\left(d_{mp},\tau\right),I^q\left(d_{mp},\tau\right),F^q\left(d_{mp},\tau\right)\right)\right\rangle;$$

taken by GDIVNSs evaluated by decision maker D_q.

Step 1. Calculate aggregate ratings at period r^{th}.

Let $x_{mpq}(\tau_l)=\left\{\left[T^L_{mpq}\left(x_{\tau_l}\right),T^U_{mpq}\left(x_{\tau_l}\right)\right],\left[I^L_{mpq}\left(x_{\tau_l}\right),I^U_{mpq}\left(x_{\tau_l}\right)\right],\left[F^L_{mpq}\left(x_{\tau_l}\right),F^U_{mpq}\left(x_{\tau_l}\right)\right]\right\}$ be the appropriateness rating of alternative A_m for criterion C_p by decision-maker D_q in time sequence τ_l, where: $m=1,\ldots,v_r; p=1,\ldots,n_r; q=1,\ldots,h_r; l=1,\ldots,k_r$. The averaged appropriateness rating $\overline{x_{mp}}=\left\{\left[\overline{T^L_{mp}(x)},\overline{T^U_{mp}(x)}\right],\left[\overline{I^L_{mp}(x)},\overline{I^U_{mp}(x)}\right],\left[\overline{F^L_{mp}(x)},\overline{F^U_{mp}(x)}\right]\right\}$ can be evaluated as:

$$\overline{x_{mp}}=\frac{1}{h_r\times k_r}\times\left\langle\begin{array}{l}\left[1-\left\{1-\left(1-\sum_{q=1}^{h_r}T^L_{pmq}\left(x_{\tau_l}\right)\right)^{\frac{1}{h_r}}\right\}^{\frac{1}{k_r}},1-\left\{1-\left(1-\sum_{q=1}^{h}T^U_{pmq}\left(x_{\tau_l}\right)\right)^{\frac{1}{h_r}}\right\}^{\frac{1}{k_r}}\right],\\ \left[\left(\sum_{q=1}^{h}I^L_{pmq}\left(x_{\tau_l}\right)\right)^{\frac{1}{h_r\times k_r}},\left(\sum_{q=1}^{h}I^U_{pmq}\left(x_{\tau_l}\right)\right)^{\frac{1}{h_r\times k_r}}\right],\\ \left[\left(\sum_{q=1}^{h}F^L_{pmq}\left(x_{\tau_l}\right)\right)^{\frac{1}{h_r\times k_r}},\left(\sum_{q=1}^{h}F^U_{pmq}\left(x_{\tau_l}\right)\right)^{\frac{1}{h_r\times k_r}}\right]\end{array}\right\rangle \quad (19)$$

Step 2. Calculate importance weight aggregation at period r^{th}.

Let $y_{pq}(\tau_l)=\left\{\left[T^L_{pq}\left(y_{\tau_l}\right),T^U_{pq}\left(y_{\tau_l}\right)\right],\left[I^L_{pq}\left(y_{\tau_l}\right),I^U_{pq}\left(y_{\tau_l}\right)\right],\left[F^L_{pq}\left(y_{\tau_l}\right),F^U_{pq}\left(y_{\tau_l}\right)\right]\right\}$ be the weight of D_q to criterion C_p in time sequence τ_l, where: $p=1,\ldots,n_r; q=1,\ldots,h_r; l=1,\ldots,k$. The average weight $\overline{w_p}=\left\{\left[\overline{T^L_p(y)},\overline{T^U_p(y)}\right],\left[\overline{I^L_p(y)},\overline{I^U_p(y)}\right],\left[\overline{F^L_p(y)},\overline{F^U_p(y)}\right]\right\}$ can be evaluated as:

$$\overline{w_p} = \frac{1}{h_r \times k_r} \times \left\langle \begin{array}{l} \left[1-\left\{1-\left(1-\sum_{q=1}^{h_r} T_{pq}^{L}\left(y_{\tau_l}\right)\right)^{\frac{1}{h_r}}\right\}^{\frac{1}{k_r}}, 1-\left\{1-\left(1-\sum_{q=1}^{h} T_{pq}^{U}\left(y_{\tau_l}\right)\right)^{\frac{1}{h_r}}\right\}^{\frac{1}{k_r}}\right], \\ \left[\left(\sum_{q=1}^{h_r} I_{pq}^{L}\left(y_{\tau_l}\right)\right)^{\frac{1}{h_r \times k_r}}, \left(\sum_{q=1}^{h_r} I_{pq}^{U}\left(y_{\tau_l}\right)\right)^{\frac{1}{h_r \times k_r}}\right], \\ \left[\left(\sum_{q=1}^{h_r} F_{pq}^{L}\left(y_{\tau_l}\right)\right)^{\frac{1}{h_r \times k_r}}, \left(\sum_{q=1}^{h_r} F_{pq}^{U}\left(y_{\tau_l}\right)\right)^{\frac{1}{h_r \times k_r}}\right] \end{array} \right\rangle, \tag{20}$$

Step 3. Evaluation for aggregate ratings of alternatives with history data.
Using Equation (21), evaluate aggregate ratings and importance weight aggregation.

$$\widetilde{A}(t_r^*) = \{A_1, A_2, \ldots, A_{v_r}\} \cup \widetilde{A}(t_{r-1})$$

$$\overline{x_{mp}^*} = \begin{cases} \overline{x_{mp}^r} \quad if \quad \begin{pmatrix} A_m \in \widetilde{A}(t_r)\backslash\widetilde{A}(t_{r-1})\&C_p \in \widetilde{C}(t_r)\backslash\widetilde{C}(t_{r-1}) \\ or \quad A_m \in \widetilde{A}(t_{r-1})\backslash\widetilde{A}(t_r)\&C_p \in \widetilde{C}(t_r)\backslash\widetilde{C}(t_{r-1}) \\ or \quad A_m \in \widetilde{A}(t_r)\backslash\widetilde{A}(t_{r-1})\&C_p \in \widetilde{C}(t_{r-1})\backslash\widetilde{C}(t_r) \end{pmatrix} \\ \overline{x_{mp}^r} \oplus \overline{x_{mp}^{r-1}} \quad if \quad A_m \in \widetilde{A}(t_r) \cap \widetilde{A}(t_{r-1})\&C_p \in \widetilde{C}(t_r) \cap \widetilde{C}(t_{r-1}) \\ \overline{x_{mp}^{r-1}} \quad if \quad A_m \in \widetilde{A}(t_{r-1})\backslash\widetilde{A}(t_r)\&C_p \in \widetilde{C}(t_{r-1})\backslash\widetilde{C}(t_r) \end{cases} \tag{21}$$

Step 4. Evaluation for importance weight aggregation of criteria with history data.
Using Equation (22), evaluate aggregate ratings and importance weight aggregation.

$$\widetilde{C}(t_r^*) = \left\{C_1, C_2, \ldots, C_{n_r}\right\} \cup \widetilde{C}(t_{r-1})$$

$$\overline{w_p^*} = \begin{cases} \overline{w_p^r} & if \quad C_p \in \widetilde{C}(t_r)\backslash\widetilde{C}(t_{r-1}) \\ \overline{w_p^r} \oplus \overline{w_p^{r-1}} & if \quad C_p \in \widetilde{C}(t_r) \cap \widetilde{C}(t_{r-1}) \\ \overline{w_p^{r-1}} & if \quad C_p \in \widetilde{C}(t_{r-1})\backslash\widetilde{C}(t_r) \end{cases} \tag{22}$$

Step 5. Calculate the average weighted ratings at period r^{th}.
The average weighted ratings of alternatives at period t_r, can be calculated by:

$$\Theta_m = \frac{1}{n_r^*}\sum_{p=1}^{n_r^*} \overline{x_{mp}^*} * \overline{w_p^*}; m = 1, \ldots, v_r^*; p = 1, \ldots, n_r^*; \tag{23}$$

Step 6. Determination of A_r^+, A_r^- and d_r^+, d_r^- at period r^{th}.

Interval-valued neutrosophic positive ideal solution (PIS, A_r^+) and interval-valued neutrosophic negative ideal solution (NIS, A_r^-) are:

$$A_r^+ = \left\{x, \left\{([1,1],[0,0],[0,0])_1, ([1,1],[0,0],[0,0])_2, \ldots, ([1,1],[0,0],[0,0])_{n_r^*}\right\}\right\} \tag{24}$$

$$A_r^- = \left\{x, \left\{([0,0],[1,1],[1,1])_1, ([0,0],[1,1],[1,1])_2, \ldots, ([0,0],[1,1],[1,1])_{n_r^*}\right\}\right\} \tag{25}$$

The distances of each alternative $A_m, m = 1, 2, \ldots, n^*$ from A_r^+ and A_r^- at period t_r, are calculated as:

$$d_m^+ = \sqrt{\left(\Theta_m - A_r^+\right)^2} \tag{26}$$

$$d_m^- = \sqrt{(\Theta_m - A_r^-)^2} \tag{27}$$

where d_m^+ and d_m^- respectively represent the shortest and farthest distances of A_m.

Step 7. Determination the closeness coefficient.

The closeness coefficient at period t_r, is calculated in Equation (28), where an alternative that is close to interval-valued neutrosophic PIS and far from interval-valued neutrosophic NIS, has high value:

$$BC_m = \frac{d_m^-}{d_m^+ + d_m^-} \tag{28}$$

Step 8. Rank the alternatives.

The alternatives are ranked by their closeness coefficient values. See Figure 1 for illustration.

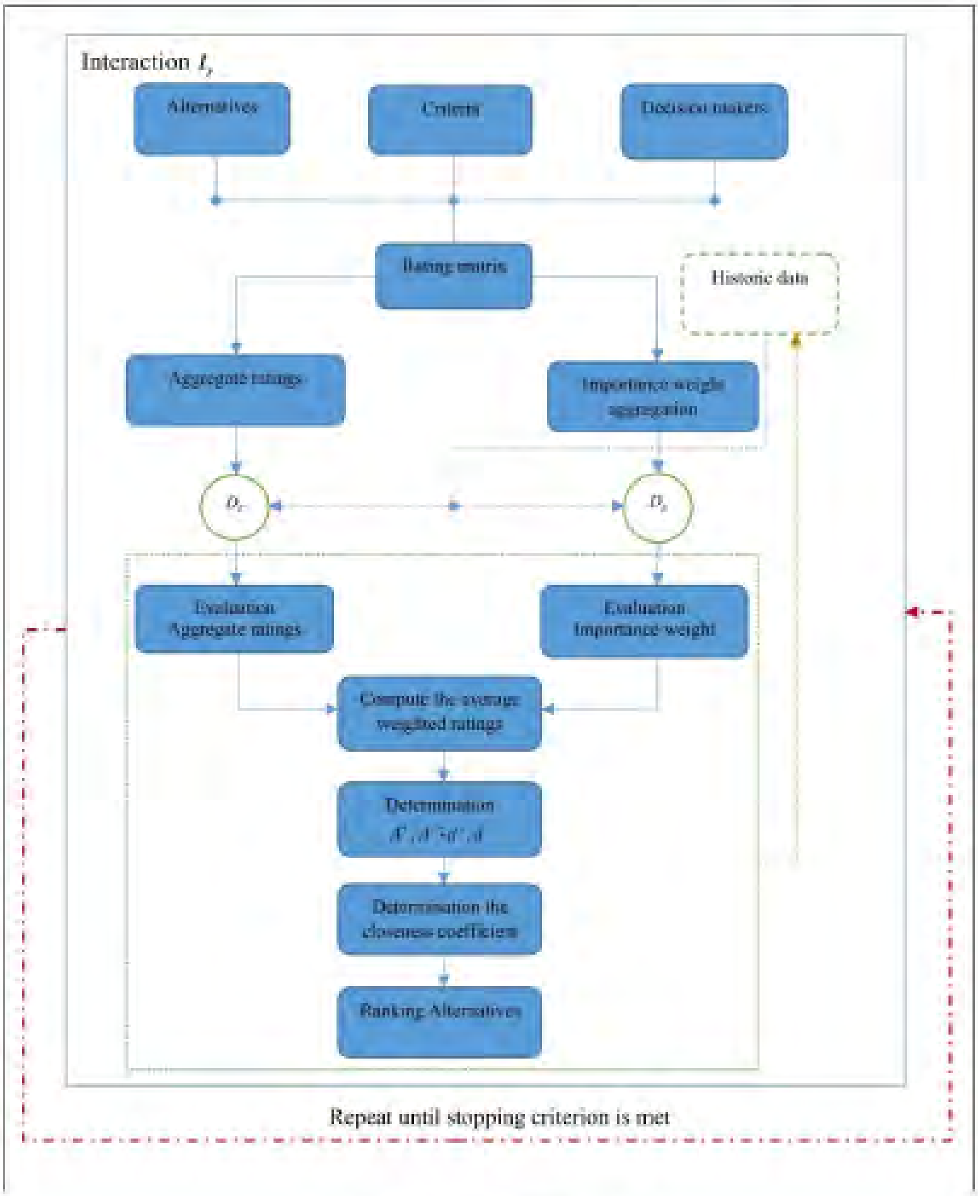

Figure 1. Dynamic TOPSIS method.

5. Applications

5.1. ASK Model for Ranking Students

Human resources recruitment plays a pivotal role in any enterprise as it exerts tremendous impact on its sustainable development. Thus, the selection of competent and job-relevant staff will lay the solid foundation for the successful performance of an enterprise. Notably, every year most of the businesses invest a large sum of money for job vacancy advertisements (on newspapers, websites, and in job fairs) and recruitment activities including application screening and interview. However, to recruit new graduated student the organizations are likely to encounter high potential risks as there are definitely inevitable employee turnovers or the selected candidates fall short of employers' expectation [22]. Mis-assessment of candidate's competence might be rooted from assessors' criteria and model for new graduated student evaluation.

The above problems underline the need for making the right assessment of potential employees. Currently, ASK model (attitude, skill and knowledge) has been widely used by many organizations because of its comprehensive assessment. This model was initially proposed by Bloom [11] with three factors including knowledge which is acquired through education, comprehension, analysis, and application skills which are the ability to process the knowledge to perform activities or tasks, and attitude which is concerned with feeling, emotions, or motivation toward employment. These elements are given divergent weights in the assessment model according to positions and requirements of the job. ASK is applicable to evaluate tertiary students' performance to give more information that support employers besides a set of university exit benchmark. It also facilitates students to make proper self-adjustments and pursue appropriate professional orientation for their future career [23,24]. Ranking students based on attributes of ASK model requires a dynamic multi-criteria decision-making model that is able to combine the estimations of different lecturers in different periods. The proposed DTOPSIS completely fit to this complex task, and the application model is depicted bellow.

5.2. Application Model

As mentioned above, the proposed method is applied to rank students of Thuongmai University, Hanoi, Vietnam. In this research, the datasets were surveyed through three consecutive semesters under three criteria (attitudes-skills-knowledge). Each student will be surveyed at the beginning of semester and by the end of semester. With the model assessing student competence, it will be conducted over semesters and over school years. This is the way of setting the time period in the decision-making model of this research.

Figure 2 shows the ASK model for ranking students where three lectures i.e., D_1, D_2, D_3 are chosen. According to the language labels shown in Tables 1 and 2, rating of five students and criteria' weights are done by the lectures based on fourteenth criteria in three groups: attitude, skill, and knowledge. The attitude group includes five criteria [25], the skill group includes six criteria [26], and the knowledge group includes three criteria [23].

The criteria used for ranking Thuongmai university's students contain 14 criteria divided into three groups (attitudes-skills-knowledge) in the ASK model. In the early stage of each semester, the knowledge criteria will not cause many impacts on student competency assessment so that we only pay attention to 11 criteria in the two remaining groups: attitudes and skills. In the following semesters, the knowledge criterion shall be supplemented that why all 14 criteria in three group shall be conducted.

(1) Period t_1 (the first semester): the decision-maker provides assessments of three students A_1, A_2, A_3 according to 11 criteria in two groups: attitude, skill. Tables 3 and 4 show the steps of the model at time t_1 and Table 5 shows the ranking order as $A_1 \succ A_2 \succ A_3$. Thus, the best student is A_1.

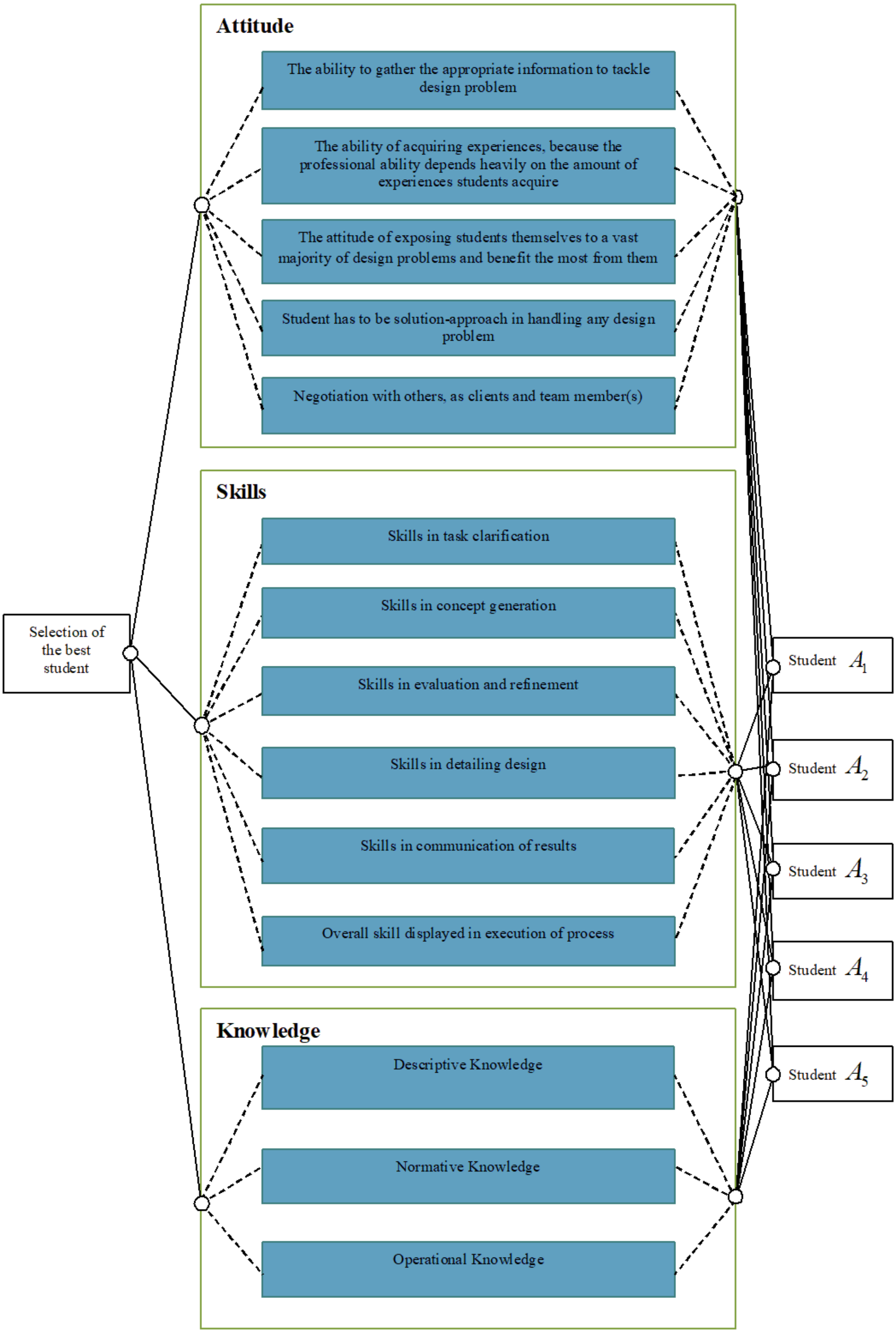

Figure 2. Attitudes-skills-knowledge (ASK) model for recruitment of ternary students.

Table 1. Appropriateness ratings.

Language Labels	Values
Very Poor	([0.1, 0.26], [0.4, 0.5], [0.63, 0.76])
Poor	([0.26, 0.38], [0.47, 0.6], [0.51, 0.6])
Medium	([0.38, 0.5], [0.4, 0.61], [0.44, 0.55])
Good	([0.5, 0.65], [0.36, 0.5], [0.31, 0.48])
Very Good	([0.65, 0.8], [0.1, 0.2], [0.12, 0.2])

Table 2. The importance of criteria.

Language Labels	Values
Unimportant	([0.1, 0.19], [0.32, 0.47], [0.64, 0.8])
Slightly Important	([0.2, 0.38], [0.46, 0.62], [0.36, 0.55])
Important	([0.45, 0.63], [0.41, 0.53], [0.2, 0.42])
Very Important	([0.66, 0.8], [0.3, 0.39], [0.22, 0.32])
Absolutely Important	([0.8, 0.94], [0.18, 0.29], [0.1, 0.2])

Table 3. Aggregated ratings at period t_1.

Criteria	Students		
	A_1	A_2	A_3
C_1	([0.463, 0.606], [0.018, 0.054], [0.014, 0.044])	([0.38, 0.5], [0.022, 0.082], [0.029, 0.059])	([0.43, 0.577], [0.021, 0.053], [0.017, 0.048])
C_2	([0.488, 0.632], [0.005, 0.025], [0.008, 0.021])	([0.419,0.578], [0.011,0.037], [0.011,0.026])	([0.419,0.578], [0.011,0.037], [0.011,0.026])
C_3	([0.463, 0.606], [0.018, 0.054], [0.014, 0.044])	([0.463, 0.606], [0.018, 0.054], [0.014, 0.044])	([0.423, 0.556], [0.02, 0.066], [0.02, 0.051])
C_4	([0.423, 0.556], [0.02, 0.066], [0.02, 0.051])	([0.463, 0.606], [0.018, 0.054], [0.014, 0.044])	([0.388, 0.523], [0.023, 0.065], [0.024, 0.056])
C_5	([0.523, 0.673], [0.005, 0.021], [0.005, 0.018])	([0.423, 0.556], [0.02, 0.066], [0.02, 0.051])	([0.43, 0.577], [0.021, 0.053], [0.017, 0.048])
C_6	([0.43, 0.577], [0.021, 0.053], [0.017, 0.048])	([0.38, 0.5], [0.022, 0.082], [0.029, 0.059])	([0.342, 0.463], [0.026, 0.081], [0.034, 0.065])
C_7	([0.38, 0.5], [0.022, 0.082], [0.029, 0.059])	([0.388, 0.523], [0.023, 0.065], [0.024, 0.056])	([0.342, 0.463], [0.026, 0.081], [0.034, 0.065])
C_8	([0.26, 0.38], [0.036, 0.078], [0.046, 0.078])	([0.38, 0.5], [0.022, 0.082], [0.029, 0.059])	([0.38, 0.5], [0.022, 0.082], [0.029, 0.059])
C_9	([0.463, 0.606], [0.018, 0.054], [0.014, 0.044])	([0.523, 0.673], [0.005, 0.021], [0.005, 0.018])	([0.463, 0.606], [0.018, 0.054], [0.014, 0.044])
C_{10}	([0.5, 0.65], [0.016, 0.044], [0.01, 0.038])	([0.38, 0.5], [0.022, 0.082], [0.029, 0.059])	([0.43, 0.577], [0.021, 0.053], [0.017, 0.048])
C_{11}	([0.463, 0.606], [0.018, 0.054], [0.014, 0.044])	([0.302, 0.423], [0.03, 0.079], [0.04, 0.071])	([0.38, 0.5], [0.022, 0.082], [0.029, 0.059])

Table 4. Aggregated weights at period t_1.

Criteria	Importance Aggregated Weights
C_1	([0.963, 0.996], [0.022, 0.06], [0.004, 0.027])
C_2	([0.908, 0.968], [0.041, 0.094], [0.017, 0.056])
C_3	([0.758, 0.89], [0.077, 0.174], [0.014, 0.097])
C_4	([0.648, 0.816], [0.087, 0.204], [0.026, 0.127])
C_5	([0.604, 0.794], [0.06, 0.154], [0.046, 0.185])
C_6	([0.963, 0.992], [0.022, 0.06], [0.004, 0.027])
C_7	([0.834, 0.925], [0.069, 0.149], [0.008, 0.074])
C_8	([0.758, 0.89], [0.077, 0.174], [0.014, 0.097])
C_9	([0.758, 0.89], [0.077, 0.174], [0.014, 0.097])
C_{10}	([0.936, 0.975], [0.037, 0.081], [0.01, 0.043])
C_{11}	([0.897, 0.959], [0.05, 0.11], [0.009, 0.056])

Table 5. Weighted ratings at period t_1.

Students	Weighted Ratings
A_1	([0.368, 0.409], [0.069, 0.168], [0.03, 0.114])
A_2	([0.34, 0.382], [0.071, 0.181], [0.035, 0.12])
A_3	([0.338, 0.377], [0.072, 0.178], [0.035, 0.121])

(2) Period t_2 (the second semester): At this stage, a new student A_4 is added with new criteria in knowledge group. The steps are shown in Tables 6–12. Finally, the ranking is obtained: $A_1 > A_2 > A_3 > A_4$. Thus, the best student is A_1.

Table 6. The distance of each student from $A^+_{t_1}$ and $A^-_{t_1}$ at period t_1.

Students	$d^+_{t_1}$	$d^-_{t_1}$
A_1	0.364193	0.773329
A_2	0.380989	0.763987
A_3	0.382736	0.763579

Table 7. Closeness coefficient at period t_1.

Students	Closeness Coefficients	Ranking Order
A_1	0.679837	1
A_2	0.667251	2
A_3	0.666116	3

Table 8. Aggregated ratings at period t_2.

Criteria	Students			
	A_1	A_2	A_3	A_4
C_1	([0.699, 0.83], [0.001, 0.005], [0, 0.002])	([0.566, 0.75], [0.001, 0.009], [0.001, 0.003])	([0.637, 0.759], [0.001, 0.007], [0.001, 0.003])	([0.5, 0.6], [0.022, 0.046], [0.009, 0.022])
C_2	([0.707, 0.852], [0.001, 0.007], [0, 0.002])	([0.686, 0.834], [0.001, 0.008], [0, 0.003])	([0.72, 0.862], [0.001, 0.006], [0, 0.002])	([0.498, 0.6], [0.023, 0.049], [0.009, 0.023])
C_3	([0.709, 0.848], [0.003, 0.016], [0, 0.005])	([0.643, 0.783], [0.003, 0.018], [0, 0.006])	([0.603, 0.767], [0.003, 0.019], [0.001, 0.006])	([0.56, 0.669], [0.008, 0.029], [0.004, 0.016])
C_4	([0.598, 0.766], [0.004, 0.022], [0.001, 0.007])	([0.639, 0.782], [0.004, 0.021], [0.001, 0.007])	([0.634, 0.793], [0.004, 0.02], [0.001, 0.008])	([0.506, 0.643], [0.009, 0.034], [0.006, 0.021])
C_5	([0.721, 0.866], [0.002, 0.012], [0.001, 0.015])	([0.651, 0.823], [0.002, 0.013], [0.002, 0.016])	([0.616, 0.765], [0.002, 0.014], [0.002, 0.017])	([0.461, 0.604], [0.013, 0.042], [0.009, 0.035])
C_6	([0.685, 0.81], [0.001, 0.005], [0, 0.002])	([0.623, 0.803], [0.001, 0.007], [0, 0.002])	([0.546, 0.72], [0.001, 0.009], [0.001, 0.004])	([0.3, 0.5], [0.022, 0.08], [0.022, 0.044])
C_7	([0.62, 0.802], [0.002, 0.013], [0, 0.004])	([0.618, 0.769], [0.002, 0.013], [0.001, 0.005])	([0.543, 0.72], [0.002, 0.015], [0.001, 0.006])	([0.438, 0.569], [0.024, 0.061], [0.012, 0.03])
C_8	([0.491, 0.648], [0.005, 0.025], [0.002, 0.013])	([0.686, 0.862], [0.004, 0.02], [0, 0.006])	([0.499, 0.709], [0.005, 0.025], [0.001, 0.009])	([0.43, 0.567], [0.026, 0.071], [0.012, 0.033])
C_9	([0.702, 0.847], [0.004, 0.021], [0, 0.007])	([0.761, 0.891], [0.004, 0.019], [0, 0.006])	([0.682, 0.828], [0.004, 0.022], [0, 0.007])	([0.488, 0.598], [0.026, 0.062], [0.009, 0.027])
C_{10}	([0.687, 0.8], [0.002, 0.01], [0, 0.003])	([0.663, 0.836], [0.001, 0.008], [0, 0.003])	([0.718, 0.842], [0.001, 0.008], [0, 0.003])	([0.534, 0.636], [0.012, 0.032], [0.006, 0.018])
C_{11}	([0.608, 0.751], [0.001, 0.009], [0.001, 0.003])	([0.557, 0.722], [0.001, 0.01], [0.001, 0.006])	([0.565, 0.75], [0.001, 0.011], [0.001, 0.004])	([0.499, 0.6], [0.023, 0.048], [0.009, 0.023])
C_{12}	([0.36, 0.533], [0.043, 0.12], [0.021, 0.06])	([0.4, 0.516], [0.049, 0.11], [0.023, 0.065])	([0.463, 0.606], [0.033, 0.089], [0.012, 0.047])	([0.258, 0.439], [0.049, 0.133], [0.037, 0.087])
C_{13}	([0.229, 0.373], [0.05, 0.119], [0.055, 0.108])	([0.229, 0.373], [0.05, 0.119], [0.055, 0.108])	([0.43, 0.568], [0.038, 0.095], [0.017, 0.047])	([0.43, 0.568], [0.038, 0.095], [0.017, 0.047])
C_{14}	([0.284, 0.408], [0.083, 0.167], [0.046, 0.123])	([0.284, 0.408], [0.083, 0.167], [0.046, 0.123])	([0.269, 0.486], [0.071, 0.179], [0.03, 0.098])	([0.431, 0.592], [0.061, 0.137], [0.017, 0.076])

Table 9. Aggregated weights at period t_2.

Criteria	Importance Aggregated Weights
C_1	([0.999, 1], [0, 0.003], [0, 0.001])
C_2	([0.997, 1], [0.001, 0.006], [0, 0.002])
C_3	([0.985, 0.998], [0.003, 0.014], [0, 0.004])
C_4	([0.978, 0.997], [0.003, 0.016], [0, 0.005])
C_5	([0.959, 0.993], [0.002, 0.011], [0.001, 0.015])
C_6	([0.999, 1], [0, 0.003], [0, 0.001])
C_7	([0.993, 0.999], [0.002, 0.009], [0, 0.002])
C_8	([0.975, 0.997], [0.004, 0.019], [0, 0.005])
C_9	([0.975, 0.997], [0.004, 0.019], [0, 0.005])
C_{10}	([0.996, 1], [0.001, 0.006], [0, 0.002])
C_{11}	([0.998, 1], [0.001, 0.005], [0, 0.001])
C_{12}	([0.963, 0.996], [0.022, 0.06], [0.004, 0.027])
C_{13}	([0.977, 0.998], [0.016, 0.044], [0.005, 0.02])
C_{14}	([0.897, 0.973], [0.05, 0.11], [0.009, 0.056])

Table 10. Weighted ratings at period t_2.

Students	Weighted Ratings
A_1	([0.605, 0.76], [0.004, 0.02], [0.001, 0.009])
A_2	([0.594, 0.761], [0.004, 0.02], [0.001, 0.009])
A_3	([0.581, 0.744], [0.004, 0.021], [0.001, 0.009])
A_4	([0.458, 0.588], [0.022, 0.058], [0.011, 0.031])

Table 11. The distance of each student from $A_{t_2}^{+}$ and $A_{t_2}^{-}$ at period t_2.

Students	$d_{t_2}^{+}$	$d_{t_2}^{-}$
A_1	0.188874	0.901553
A_2	0.192392	0.900405
A_3	0.200641	0.896588
A_4	0.279475	0.848118

Table 12. Closeness coefficient at period t_2.

Students	Closeness Coefficients	Ranking Order
A_1	0.826789	1
A_2	0.823945	2
A_3	0.817138	3
A_4	0.752149	4

(3) Period t_3 (the third semester): At this stage, a new student A_5 is considered and an existing student A_2 is discarded. The criteria remain the same as in the previous period t_2. Tables 13–17 show the steps of this stage. Finally, the ranking is obtained: $A_5 > A_4 > A_2 > A_1 > A_3$. Thus, the best student is A_5.

5.3. Comparison with the Related Methods

The proposed dynamic TOPSIS method has superior features compared to the method in [14]. In Table 18, the ranking order of five students in three periods are presented. We can observe that at period t_1, the results of the both methods are the same i.e., $A_1 > A_2 > A_3$.

Table 13. Aggregated ratings at period t_3.

Criteria	Students				
	A_1	A_2	A_3	A_4	A_5
C_1	([0.794, 0.9], [0, 0], [0, 0])	([0.51, 0.75], [0, 0.006], [0, 0.002])	([0.764, 0.893], [0, 0], [0, 0])	([0.711, 0.822], [0, 0.003], [0, 0.001])	([0.441, 0.569], [0.022, 0.053], [0.012, 0.027])
C_2	([0.871, 0.951], [0, 0], [0, 0])	([0.675, 0.818], [0, 0.002], [0, 0.001])	([0.881, 0.96], [0, 0], [0, 0])	([0.788, 0.891], [0, 0.001], [0, 0])	([0.441, 0.569], [0.022, 0.053], [0.012, 0.027])
C_3	([0.829, 0.918], [0, 0.001], [0, 0])	([0.608, 0.785], [0.001, 0.005], [0, 0.001])	([0.728, 0.884], [0, 0.001], [0, 0])	([0.817, 0.91], [0, 0.001], [0, 0])	([0.569, 0.67], [0.005, 0.016], [0.004, 0.012])
C_4	([0.711, 0.875], [0, 0.001], [0, 0])	([0.608, 0.785], [0.001, 0.005], [0, 0.001])	([0.816, 0.922], [0, 0.001], [0, 0])	([0.663, 0.795], [0, 0.002], [0, 0.001])	([0.569, 0.67], [0.005, 0.016], [0.004, 0.012])
C_5	([0.81, 0.912], [0, 0.001], [0, 0.001])	([0.635, 0.804], [0, 0.003], [0, 0.001])	([0.777, 0.889], [0, 0.001], [0, 0.001])	([0.751, 0.872], [0, 0.001], [0, 0.001])	([0.48, 0.608], [0.011, 0.032], [0.008, 0.021])
C_6	([0.832, 0.923], [0, 0], [0, 0])	([0.608, 0.785], [0, 0.004], [0, 0.001])	([0.744, 0.902], [0, 0], [0, 0])	([0.482, 0.69], [0.001, 0.006], [0.001, 0.003])	([0.536, 0.637], [0.011, 0.026], [0.006, 0.016])
C_7	([0.689, 0.86], [0, 0.001], [0, 0])	([0.591, 0.759], [0.001, 0.004], [0, 0.002])	([0.682, 0.86], [0, 0.001], [0, 0])	([0.586, 0.733], [0.001, 0.004], [0.001, 0.002])	([0.441, 0.569], [0.022, 0.053], [0.012, 0.027])
C_8	([0.751, 0.898], [0, 0.001], [0, 0])	([0.662, 0.822], [0, 0.003], [0, 0.001])	([0.732, 0.89], [0, 0.001], [0, 0])	([0.699, 0.83], [0, 0.004], [0, 0.001])	([0.268, 0.441], [0.027, 0.079], [0.033, 0.062])
C_9	([0.874, 0.95], [0, 0.002], [0, 0])	([0.749, 0.861], [0, 0.002], [0, 0.001])	([0.889, 0.963], [0, 0.002], [0, 0])	([0.743, 0.853], [0, 0.003], [0, 0.001])	([0.418, 0.578], [0.011, 0.039], [0.011, 0.026])
C_{10}	([0.757, 0.891], [0, 0], [0, 0])	([0.636, 0.804], [0, 0.003], [0, 0.001])	([0.837, 0.926], [0, 0], [0, 0])	([0.712, 0.818], [0, 0.002], [0, 0.001])	([0.5, 0.6], [0.022, 0.044], [0.009, 0.022])
C_{11}	([0.753, 0.88], [0, 0], [0, 0])	([0.521, 0.71], [0.001, 0.005], [0.001, 0.003])	([0.696, 0.875], [0, 0.001], [0, 0])	([0.651, 0.769], [0.001, 0.004], [0, 0.002])	([0.569, 0.67], [0.005, 0.015], [0.004, 0.012])
C_{12}	([0.753, 0.884], [0.001, 0.007], [0, 0.002])	([0.53, 0.662], [0.002, 0.011], [0.001, 0.007])	([0.778, 0.903], [0.001, 0.007], [0, 0.002])	([0.544, 0.72], [0.002, 0.013], [0.001, 0.005])	([0.534, 0.636], [0.012, 0.032], [0.006, 0.018])
C_{13}	([0.677, 0.845], [0, 0.002], [0, 0.001])	([0.338, 0.521], [0.001, 0.006], [0.003, 0.009])	([0.759, 0.881], [0, 0.002], [0, 0])	([0.699, 0.83], [0.001, 0.004], [0, 0.001])	([0.374, 0.536], [0.022, 0.065], [0.016, 0.035])
C_{14}	([0.688, 0.837], [0.001, 0.005], [0, 0.001])	([0.407, 0.555], [0.002, 0.008], [0.002, 0.008])	([0.777, 0.916], [0.001, 0.005], [0, 0.001])	([0.699, 0.826], [0.001, 0.007], [0, 0.002])	([0.44, 0.569], [0.023, 0.057], [0.012, 0.029])

Table 14. Aggregated weights at period t_3.

Criteria	Importance Aggregated Weights
C_1	([0.99999, 1], [0, 0.00009], [0, 0.00001])
C_2	([0.99995, 1], [0.00001, 0.00019], [0, 0.00002])
C_3	([0.99964, 1], [0.00005, 0.00062], [0, 0.00009])
C_4	([0.99912, 0.99998], [0.00009, 0.00097], [0, 0.00018])
C_5	([0.99776, 0.99995], [0.00004, 0.00077], [0.00001, 0.00053])
C_6	([0.99999, 1], [0, 0.00006], [0, 0])
C_7	([0.99985, 1], [0.00003, 0.00039], [0, 0.00005])
C_8	([0.99907, 0.99999], [0.00009, 0.00114], [0, 0.00015])
C_9	([0.99842, 0.99996], [0.00014, 0.00154], [0, 0.00024])
C_{10}	([0.99991, 1], [0.00002, 0.00029], [0, 0.00004])
C_{11}	([0.99997, 1], [0.00001, 0.00016], [0, 0.00001])
C_{12}	([0.99615, 0.99988], [0.00112, 0.00657], [0.00004, 0.00152])
C_{13}	([0.99969, 1], [0.00016, 0.00145], [0.00001, 0.00026])
C_{14}	([0.99762, 0.99993], [0.00082, 0.00483], [0.00004, 0.00116])

Table 15. Weighted ratings at period t_3.

Students	Weighted Ratings
A_1	([0.78, 0.901], [0, 0.001], [0, 0])
A_2	([0.589, 0.759], [0.001, 0.004], [0, 0.002])
A_3	([0.785, 0.91], [0, 0.001], [0, 0])
A_4	([0.693, 0.822], [0, 0.003], [0, 0.001])
A_5	([0.476, 0.599], [0.014, 0.037], [0.009, 0.022])

Table 16. The distance of each student from $A_{t_3}^{+}$ and $A_{t_3}^{-}$ at period t_3.

Students	$d_{t_3}^{+}$	$d_{t_3}^{-}$
A_1	0.37844	0.776416
A_2	0.352522	0.752181
A_3	0.381797	0.777005
A_4	0.358066	0.764391
A_5	0.325366	0.738391

Table 17. Closeness coefficient at period t_3.

Students	Closeness Coefficients	Ranking Order
A_1	0.672305	4
A_2	0.680890	3
A_3	0.670525	5
A_4	0.680998	2
A_5	0.694135	1

Table 18. The dynamic rankings obtained at periods.

Time Period	The Method in [14]	The Proposed Method
t_1	$A_1 \succ A_2 \succ A_3$	$A_1 \succ A_2 \succ A_3$
t_2	$A_4 \succ A_2 \succ A_3 \succ A_1$	$A_1 \succ A_2 \succ A_3 \succ A_4$
t_3	$A_5 \succ A_3 \succ A_4 \succ A_1$	$A_5 \succ A_4 \succ A_2 \succ A_1 \succ A_3$

At period t_2, the method in [14] and the proposed method show difference in ranking order of A_1 and A_4. In this period, $A_2 \succ A_3$ according to both methods, and the method in [14] ranks A_4 at the top, meanwhile, A_1 is ranked at the top by the proposed method. The reason is that A_4 is evaluated at the first time and it has not appeared while A_1 has historical data, particularly A_1 were ranked at the top in the previous period. In this circumstance, the proposed model better utilizes the effect of historical data on the alternatives A_1 and A_4. The result of the dynamic TOPSIS model is time-dependent and combines the effect of current and historical data.

At period t_3, the result shows difference in the number of ranked alternatives and in their preferential order. In this period, the alternative A_2 is not evaluated by decision-makers and it has only historical data. The method in [14] could not process alternative A_2, meanwhile the proposed model could. Moreover, the alternative A_5 is highly ranked by both methods. However, there is a change in the relative order of A_3 and A_4. The method in [14] ranks $A_3 \succ A_4$, but the proposed method ranks $A_4 \succ A_3$.

The comparison between the methods again illustrates the effect of historical data over the output of the proposed decision-making model. If an alternative is considered and it has good evaluation in the previous periods, this alternative will have high potential to reach high order. From that point of view, the proposed model presents good compliance with the perceived dynamic rules. It illustrates the advantage and applicability of the model.

6. Conclusions

The proposed dynamic TOPSIS (DTOPSIS) model in dynamic interval-valued neutrosophic sets presents its advantages to cope with dynamic and indeterminate information in decision-making model. DTOPSIS model handles historical data including the change of criteria, alternatives, and decision-makers during periods. The concepts of generalized dynamic interval-valued neutrosophic set, GDIVNS, and their mathematical operators on GDIVNSs have been proposed. Distance and weighted aggregation operators are used to construct a framework of DTOPSIS in DIVNS environment. The proposed DTOPSIS fulfills the requirement of an issue that is evaluates tertiary students' performance based on the attributes of ASK model. Data of Thuongmai University students were used to illustrate the proper of DTOPSIS model which opened the potential application in larger scale also. For the future works, we will extend generalized dynamic interval-valued neutrosophic sets for some other real-world applications [27–35]. Furthermore, we hope to apply GDIVNS for dealing with the unlimited time problems in decision-making model in dynamic neutrosophic environment based on the idea in [36,37].

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A Novel Approach foor Assessing the Reliability of Data Contained in a Single Valued Neutrosophic Number and its Applicattion in Multiple Criteria Decision Making

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Abstract

Multiple criteria decision making is one of the many areas where neutrosophic sets have been successfully applied to solve various problems so far. Compared to a fuzzy set, and similar sets, neutrosophic sets use more membership functions which makes them suitable for using complex evaluation criteria in multiple criteria decision making. On the other hand, the application of three membership functions makes evaluation somewhat more complex compared to evaluation using fuzzy sets. The reliability of the data used to solve a problem can have an impact on the selection of the appropriate solution/alternative. Therefore, this paper discusses an approach that can be used to assess the reliability of information collected by surveying respondents. The usability of the proposed approach is demonstrated in the numerical illustration of the supplier selection.

Keywords: neutrosophy, reliability, single-valued neutrosophic numbers, decision-making.

1. Introduction

Multiple criteria decision making (MCDM) started to emerge about 50 years ago, and until now it is used for solving a number of different decision-making problems in different fields. MCDM can be defined as making choices in the presence of multiple conflicting criteria [1-3]. Solving complex decision-making problems is usually associated with the need to use a larger number of criteria or use of more complex criteria that are later decomposed into sub-criteria [4-5]. However, an increase in the number of criteria, as well as sub-criteria, can be less desirable in cases where data should be collected by the survey [6].

Significant progress in using the MCDM methods for solving complex decision-making problems was made after Zadeh [7] proposed fuzzy sets, on which basis Bellman and Zadeh [8] proposed fuzzy MCDM [9-10]. Since then, many extensions of fuzzy sets theory have been developed, such as: interval-valued fuzzy sets [11], intuitionistic fuzzy sets [12] and bipolar fuzzy sets [13] . In 1999, Smarandache [14] introduced the concept of neutrosophic sets, as a generalization of the fuzzy sets theory and their extensions.

So far, neutrosophic sets are successfully used in the area of multi-criteria decision-making. Many extensions of the MCDM methods are proposed based on the use of neutrosophic numbers, such as: neutrosophic AHP [15]; neutrosophic TOPSIS [1]; neutrosophic MULTIMOORA [16]; neutrosophic WASPAS [17]; neutrosophic PROMETHEE [18]; neutrosophic VIKOR [19]; neutrosophic ARAS [20]; neutrosophic GRA [21]; neutrosophic EDAS [22], and so forth. Besides, it is worth mentioning newly-developed approaches, such as: the importance of neutrosophic soft matrices in decision-making [23], interval-valued neutrosophic soft sets in decision-making [24], as well as ivnpiv-neutrosophic soft sets for decision-making [25]. In general, neutrosophy so far is used in solving a number of decision-making problems [26-32].

Fuzzy sets theory introduces partial membership to a set, expressed by membership function $\mu_{(x)}$, where membership function can have different forms, such as: bell-shaped, triangular, trapezoidal and singleton. Some other extensions of fuzzy sets theory introduced other membership functions such as: a non-membership function $\nu_{(x)}$, a positive membership function $\mu^{+}_{(x)}$ and a negative membership function $\nu^{+}_{(x)}$. Neutrosophic sets theory introduces three membership functions that can be used to describe belonging to a set, that is; truth membership, indeterminacy membership, falsity membership. That is why neutrosophic sets could be more suitable for evaluating complex phenomena, events and problems.

However, the use of three membership functions can make evaluation somewhat more complex compared to evaluation using fuzzy sets. Therefore Smarandache ***et al***. [33] proposed an approach that can be used to assess the reliability of information collected by surveying respondents. This approach is reviewed again in this article, and a new approach for determining the reliability of information contained in single valued neutrosophic numbers is also presented.

Therefore, the remainder of the article is organized as follows: in Section 2 basic elements of neutrosophic sets and single-valued neutrosophic numbers are considered. In Section 3 approaches for ranking single valued neutrosophic numbers are considered, and in Section 4 a numerical illustration is given in order to demonstrate the proposed approach. Finally, conclusions are given.

2. Basic Elements of Neutrosophic Sets and Single Valued Neutrosophic Numbers

Definition 1. Let $\boldsymbol{X}$ be a nonempty set, with a generic element in $\boldsymbol{X}$ denoted by $\boldsymbol{x}$. Then, the Neutrosophic Set (NS) $\boldsymbol{A}$ in $\boldsymbol{X}$ is as follows [14]:

$$\boldsymbol{A}=\left\{\left\langle \boldsymbol{x}, \boldsymbol{T_A}(\boldsymbol{x}), \boldsymbol{I_A}(\boldsymbol{x}), \boldsymbol{F_A}(\boldsymbol{x})\right\rangle \middle| \, \boldsymbol{x}\in \boldsymbol{X}\right\}, \tag{1}$$

with: $\boldsymbol{T_A}:\boldsymbol{X}\rightarrow]^{-}0,1^{+}[$; $\boldsymbol{I_A}:\boldsymbol{X}\rightarrow]^{-}0,1^{+}[$; $\boldsymbol{F_A}:\boldsymbol{X}\rightarrow]^{-}0,1^{+}[$ and $^{-}0\le \boldsymbol{T_A}(\boldsymbol{x})+\boldsymbol{I_A}(\boldsymbol{x})+\boldsymbol{F_A}(\boldsymbol{x})\le 3^{+}$

where: $\boldsymbol{T_A}(\boldsymbol{x})$, $\boldsymbol{I_A}(\boldsymbol{x})$ and $\boldsymbol{F_A}(\boldsymbol{x})$ are the truth-membership function, the indeterminacy-membership function and the falsity-membership function, respectively.

Definition 2. Let $\boldsymbol{X}$ be a nonempty set. The Single Valued Neutrosophic Set (SVNS) $\boldsymbol{A}$ in $\boldsymbol{X}$ is as follows [14, 34]:

$$\boldsymbol{A}=\left\{\left\langle \boldsymbol{x}, \boldsymbol{T_A}(\boldsymbol{x}), \boldsymbol{I_A}(\boldsymbol{x}), \boldsymbol{F_A}(\boldsymbol{x})\right\rangle \middle| \, \boldsymbol{x}\in \boldsymbol{X}\right\}, \tag{2}$$

with: $\boldsymbol{T_A}:\boldsymbol{X}\rightarrow [0,1]$; $\boldsymbol{I_A}:\boldsymbol{X}\rightarrow [0,1]$; $\boldsymbol{F_A}:\boldsymbol{X}\rightarrow [0,1]$ and $0\le \boldsymbol{T_A}(\boldsymbol{x})+\boldsymbol{I_A}(\boldsymbol{x})+\boldsymbol{F_A}(\boldsymbol{x})\le 3$.

Definition 3. For an SVNS $\boldsymbol{A}$ in $\boldsymbol{X}$, the triple $<\boldsymbol{t_A}, \boldsymbol{i_A}, \boldsymbol{f_A}>$ is called the Single Valued Neutrosophic Number (SVNN) [14, 34].

Definition 4. Let $\boldsymbol{X_i}=<\boldsymbol{t_i}, \boldsymbol{i_i}, \boldsymbol{f_i}>$ be a collection of SVNNs and $\boldsymbol{x}=<\boldsymbol{t_x}, \boldsymbol{i_x}, \boldsymbol{f_x}>$ a SVNN; then the Hamming distance $\boldsymbol{h}_{(x)}$ between $\boldsymbol{x}$ and the ideal point $\boldsymbol{x}^{+}=<\boldsymbol{t}^{+}, \boldsymbol{i}^{+}, \boldsymbol{f}^{+}>=<\max\limits_{i}\boldsymbol{t_i}, \min\limits_{i}\boldsymbol{i_i}, \min\limits_{i}\boldsymbol{f_i}>$ is as follows:

$$h_{(x)}=\frac{1}{3}\left(|t^{+}-t_{x}|+|i^{+}-i_{x}|+|f^{+}-f_{x}|\right). \tag{3}$$

Definition 5. Let $A_j=<t_j,i_j,f_j>$ be a collection of SVNNs and $W=(w_1,w_2,...,w_n)^T$ be an associated weighting vector. Then the Single Valued Neutrosophic Weighted Average (SVNWA) operator of A_j is as follows [35]:

$$SVNWA(A_1,A_2,...,A_n)=\sum_{j=1}^{n}w_jA_j=\left(1-\prod_{j=1}^{n}(1-t_j)^{w_j},\prod_{j=1}^{n}(i_j)^{w_j},\prod_{j=1}^{n}(f_j)^{w_j}\right). \tag{4}$$

where: w_j is the element j of the weighting vector, $w_j\in[0,1]$ and $\sum_{j=1}^{n}w_j=1$.

3. Determining the Reliability of the Information Contained in Single Valued Neutrosophic Numbers

Smarandache ***et al***. [33] proposed an approach for accessing the reliability of the information $r_{(x)}$ contained in a SVNN, as follows:

$$r_{(x)}=\frac{t-f}{1+i}, \tag{5}$$

where: t, i, f denote the truth, the intermediacy and the falsity of information contained in SVNN $x=<t,i,f>$, $r\in[-1,1]$.

Example: Assume that x =<0.9, 0.1, 0.3> is a SVNN. Then, the reliability of x is $r_{(x)}=\frac{0.9-0.3}{1+0.1}=0.55$

In this approach, it is proposed to calculate reliability as follows:

$$r_{(x)}=\begin{cases}\frac{|t-f|}{t+i+f} & t+i+f\neq 0\\ 0 & t+i+f=0\end{cases}, \tag{6}$$

where: $r\in[0,1]$.

Example: Assume that x =<0.9, 0.1, 0.3> is a SVNN. Then, the reliability of x is $r_{(x)}=\frac{|0.9-0.3|}{0.9+0.1+0.3}=0.46$

One comparison of the reliability calculated using Eq. (5) and Eq. (6) for some characteristic values of t, i and f is shown in Table 1.

Table 1. The reliability calculated using Eq. (5) and Eq. (6)

t	*i*	*f*	Eq. (5)	Eq. (6)
1	0	0	1	1
0	0	1	-1	1
1	0	1	0	0
1	1	0	0.5	0.5
0	1	1	-0.5	0.5
1	1	1	0	0
0	0	0	0	0

It can be seen from Table 1, Eq. (5) provides values from the interval [-1, 1], with a value of zero being the least desirable. Equation (6) provides values from the interval [0, 1] where a higher value of the reliability function is more desirable.

4. A Numerical illustration

In order to briefly demonstrate the usability of the SVNNs for solving MCDM problems, an example of supplier selection is presented in this section. Assume that one company has to consider engaging with a new supplier. Therefore, a team of three experts if formed with the aim to select the most appropriate supplier from three alternatives, denoted as $\boldsymbol{A}_1$ – $\boldsymbol{A}_3$, on the basis on the following criteria:

- $\boldsymbol{C}_1$ – Delivery,
- $\boldsymbol{C}_2$ – Quality,
- $\boldsymbol{C}_3$ – Flexibility,
- $\boldsymbol{C}_4$ – Service, and
- $\boldsymbol{C}_5$ – Price.

The ratings obtained from three experts are shown in Tables 1 to 3.

Table 2. The ratings obtained from the first of three experts

	$\boldsymbol{C}_1$	$\boldsymbol{C}_2$	$\boldsymbol{C}_3$	$\boldsymbol{C}_4$	$\boldsymbol{C}_5$
$\boldsymbol{A}_1$	<0.9, 0.10, 0.30>	<0.7, 0.2, 0.3>	<0.6, 0.0, 0.0>	<0.7, 0.0, 0.0>	<0.5, 0.0, 0.1>
$\boldsymbol{A}_2$	<0.8, 0.00, 0.00>	<0.8, 0.0, 0.1>	<0.8, 0.0, 0.0>	<0.8, 0.0, 0.0>	<0.8, 0.0, 0.0>
$\boldsymbol{A}_3$	<0.7, 0.00, 0.00>	<0.5, 0.0, 0.0>	<0.6, 0.0, 0.0>	<0.6, 0.0, 0.0>	<0.7, 0.2, 0.0>
$\boldsymbol{A}_4$	<0.8, 0.10, 0.10>	<0.6, 0.0, 0.0>	<0.7, 0.0, 0.3>	<0.5, 0.2, 0.2>	<0.5, 0.0, 0.0>

Table 3. The ratings obtained from the second of three experts

	$\boldsymbol{C}_1$	$\boldsymbol{C}_2$	$\boldsymbol{C}_3$	$\boldsymbol{C}_4$	$\boldsymbol{C}_5$
$\boldsymbol{A}_1$	<0.6, 0.00, 0.10>	<0.7, 0.0, 0.1>	<0.6, 0.0, 0.0>	<0.5, 0.0, 0.0>	<0.2, 0.0, 0.9>
$\boldsymbol{A}_2$	<0.8, 0.00, 0.30>	<0.6, 0.0, 0.1>	<0.7, 0.0, 0.0>	<0.8, 0.0, 0.2>	<0.1, 0.0, 0.8>
$\boldsymbol{A}_3$	<0.7, 0.00, 0.30>	<0.8, 0.0, 0.0>	<0.7, 0.0, 0.0>	<0.6, 0.0, 0.4>	<0.3, 0.0, 0.2>
$\boldsymbol{A}_4$	<0.6, 0.00, 0.20>	<0.7, 0.1, 0.2>	<0.7, 0.0, 0.0>	<0.6, 0.0, 0.4>	<0.5, 0.0, 0.1>

Table 4. The ratings obtained from the third of three experts

	$\boldsymbol{C}_1$	$\boldsymbol{C}_2$	$\boldsymbol{C}_3$	$\boldsymbol{C}_4$	$\boldsymbol{C}_5$
$\boldsymbol{A}_1$	<0.8, 0.20, 0.20>	<0.6, 0.0, 0.4>	<0.5, 0.0, 0.0>	<0.6, 0.0, 0.1>	<0.8, 0.0, 0.4>
$\boldsymbol{A}_2$	<0.6, 0.10, 0.10>	<0.6, 0.2, 0.4>	<0.8, 0.1, 0.0>	<0.5, 0.0, 0.1>	<0.7, 0.0, 0.0>
$\boldsymbol{A}_3$	<0.6, 0.00, 0.00>	<0.7, 0.0, 0.3>	<0.6, 0.1, 0.0>	<0.6, 0.0, 0.0>	<0.5, 0.1, 0.3>
$\boldsymbol{A}_4$	<0.7, 0.00, 0.00>	<0.8, 0.0, 0.2>	<0.7, 0.0, 0.0>	<0.6, 0.0, 0.1>	<0.6, 0.0, 0.0>

The reliability of ratings obtained using Eq. (5) and Eq. (6) are shown in Tables 5 and 6. The average reliability of all ratings are also shown in Tables 5 and 6.

Table 5. The reliability of ratings obtained from the first expert using Eq. (5)

	$\boldsymbol{C}_1$	$\boldsymbol{C}_2$	$\boldsymbol{C}_3$	$\boldsymbol{C}_4$	$\boldsymbol{C}_5$
$\boldsymbol{A}_1$	0.55	0.33	0.60	0.70	0.40
$\boldsymbol{A}_2$	0.80	0.70	0.80	0.80	0.80
$\boldsymbol{A}_3$	0.70	0.50	0.60	0.60	0.58
$\boldsymbol{A}_4$	0.64	0.60	0.40	0.25	0.50
				Avg	0.59

Table 6. The reliability of ratings obtained from the first expert using Eq. (6)

	$\boldsymbol{C}_1$	$\boldsymbol{C}_2$	$\boldsymbol{C}_3$	$\boldsymbol{C}_4$	$\boldsymbol{C}_5$
$\boldsymbol{A}_1$	0.46	0.33	1.00	1.00	0.67
$\boldsymbol{A}_2$	1.00	0.78	1.00	1.00	1.00
$\boldsymbol{A}_3$	1.00	1.00	1.00	1.00	0.78
$\boldsymbol{A}_4$	0.70	1.00	0.40	0.33	1.00
				Avg	0.82

The average reliability of responses obtained from all three decision makers, calculated using Eq. (6), are accounted for in Table 7.

Table 7. The average reliability of ratings obtained from all experts using Eq. (6)

	Reliability
$\boldsymbol{E}_1$	0.82
$\boldsymbol{E}_2$	0.65
$\boldsymbol{E}_3$	0.69

As can be seen from Table 7, all three experts provide relatively consistent responses, and therefore their ratings can be used for further evaluation of alternatives. In contrast, if the average reliability of ratings obtained from a respondent has low value, his or her responses must be rejected from further evaluation of the alternatives or his or her responses must be re-considered again until adequate reliability is achieved.

A possible scenario of the evaluation of alternatives is discussed below. A group decision matrix, shown in Table 8, is constructed using Eq. (4) and the following weights $\boldsymbol{w_j} = \left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$. The overall ratings are calculated using Eq. (4) and the following weighting wector $\boldsymbol{w_j} = (0.19, 0.22, 0.19, 0.18, 0.22)$, as it is shown in Table 9. The ideal point is also shown in Table 9.

Table 8. The group decision matrix

	$\boldsymbol{C}_1$	$\boldsymbol{C}_2$	$\boldsymbol{C}_3$	$\boldsymbol{C}_4$	$\boldsymbol{C}_5$
$\boldsymbol{w_j}$	0.19	0.22	0.19	0.18	0.22
$\boldsymbol{A}_1$	<0.80, 0.1, 0.20>	<0.67, 0.1, 0.28>	<0.57, 0.0, 0.00>	<0.61, 0.0, 0.03>	<0.57, 0.0, 0.62>
$\boldsymbol{A}_2$	<0.75, 0.0, 0.14>	<0.68, 0.1, 0.21>	<0.77, 0.0, 0.00>	<0.73, 0.0, 0.10>	<0.62, 0.0, 0.41>
$\boldsymbol{A}_3$	<0.67, 0.0, 0.11>	<0.69, 0.0, 0.11>	<0.64, 0.0, 0.00>	<0.60, 0.0, 0.16>	<0.53, 0.1, 0.18>
$\boldsymbol{A}_4$	<0.71, 0.0, 0.10>	<0.71, 0.0, 0.14>	<0.70, 0.0, 0.11>	<0.57, 0.1, 0.24>	<0.54, 0.0, 0.03>

Table 9. The overall ratings and ideal point

	Overall ratings
$\boldsymbol{A}_1$	<0.65, 0.00, 0.00>
$\boldsymbol{A}_2$	<0.71, 0.00, 0.00>
$\boldsymbol{A}_3$	<0.63, 0.00, 0.00>
$\boldsymbol{A}_4$	<0.65, 0.00, 0.10>
$\boldsymbol{A}^+$	<0.71, 0.00, 0.00>

Finally, the ranking results, obtained using Eq. (3), are encountered for in Table 10.

Table 10. The ranking results

	$\boldsymbol{h}_{(\boldsymbol{i})}$	Rank
$\boldsymbol{A}_1$	0.0063	2
$\boldsymbol{A}_2$	0.0000	1
$\boldsymbol{A}_3$	0.0092	3
$\boldsymbol{A}_4$	0.0179	4

As can be seen from Table 10, the most appropriate alternative is alternative denoted as $\boldsymbol{A}_2$.

5. Conclusion

Neutrosophic sets theory introduces three membership functions that is why single-valued neutrosophic numbers could be suitable for evaluating alternatives in relation to the complex evaluation criteria in multiple criteria decision making. However, the use of three membership functions can make evaluation somewhat complex especially when the evaluation is based on data collected by the survey.

The reliability of the data used to solve a problem can have an impact on the final selection of the appropriate alternative. In this manuscript, an improved procedure for estimating the reliability of the collected data is proposed.

Therefore, Smarandache et al. [33] has proposed an approach that can be used to assess the reliability of information collected by surveying respondents.

Compared to the previous approach, in the new approach reliability and information belong to the interval [0, 1], unlike the previously proposed approach where reliability belongs to the interval [-1, 1], which makes new application easier for using.

By using the proposed procedure, the reliability of data could be easily determined. In this paper, the usability and efficiency of the proposed approach is successfully demonstrated on an illustrative example of the supplier selection.

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An Innovative Grey Approach for Group Multi-Criteria Decision Analysis Based on the Median of Ratings by Using Python

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Abstract: Some decision-making problems, i.e., multi-criteria decision analysis (MCDA) problems, require taking into account the attitudes of a large number of decision-makers and/or respondents. Therefore, an approach to the transformation of crisp ratings, collected from respondents, in grey interval numbers form based on the median of collected scores, i.e., ratings, is considered in this article. In this way, the simplicity of collecting respondents' attitudes using crisp values, i.e., by applying some form of Likert scale, is combined with the advantages that can be achieved by using grey interval numbers. In this way, a grey extension of MCDA methods is obtained. The application of the proposed approach was considered in the example of evaluating the websites of tourism organizations by using several MCDA methods. Additionally, an analysis of the application of the proposed approach in the case of a large number of respondents, done in Python, is presented. The advantages of the proposed method, as well as its possible limitations, are summarized.

Keywords: MCDA; grey interval numbers; group decision-making; Python

1. Introduction

Multi-criteria decision-making (MCDM), or multi-criteria decision analysis (MCDA), has so far been used for solving a large number of numerous different decision-making problems [1–4]. Therefore, MCDA is dealing with solving complex real-world problems of the greatest interest to the organization that cannot be solved by conventional methods [5–8]. In due course of time, many multi-criteria analysis (MCA) methods were proposed, primarily due to the dynamic and rapid development of the field of operational research. The following can be mentioned as some of the most cited articles from this area: Hajkowicz and Collins [9], Hajkowicz and Higgins [10], Kaklauskas et al. [11], Kostreva et al. [12], and Belton and Vickers [13].

Besides this research, there are many studies in this area, such as: research and development project portfolio selection [14] (Mavrotas and Makryvelios, 2021), assessing national energy sustainability [15], energy consumption analysis of high-speed trains [16], evaluation of transport emissions reduction policies [17], planning renewable energy use and carbon saving [18], and so forth.

MCDA has also been used successfully for solving decision-making problems that are related to uncertainties or require a group decision-making approach for solving them [19–25]. As some examples of such approaches, the following can be mentioned: a grey absolute decision analysis [26], a multiple criteria decision analysis framework for the dispersed group [27], a fuzzy multi-criteria analysis [28–30], and collaborative decision-making in the multi-actor multi-criteria analysis [31].

From the aforestated, it is clear that some decision-making problems can be more adequately solved if a larger number of respondents take part in solving them. In such cases, the question that arises is how to transform the attitudes collected from respondents into group attitudes.

The approach based on the use of a five-point Liker's scale, or similar, can be mentioned as one of the probably simplest approaches for collecting the respondents' attitudes. So far, in numerous articles published in scientific journals, numerous approaches have been proposed for the transformation of individual attitudes acquired in this way into group attitudes. The results obtained, the advantages, as well as the weaknesses of these approaches, are also presented in these journals.

In this article, an approach to the transformation of crisp ratings, collected from respondents, as grey interval numbers form based on the median of collected scores, i.e., ratings, is considered. Therefore, the article proposes the transformation of individual ratings collected from respondents into grey intervals with the aim of performing MCDA with minimal loss of information in relation to cases when crisp ratings are transformed into crisp group ratings. The application of the proposed approach was considered on the example of evaluating the websites of tourism organizations by using several MCDA methods, and also an analysis of the application of the proposed approach in the case of a large number of respondents was done in Python and described. Additionally, the main idea of the article was to propose a simple procedure for gathering respondent's attitudes instead of a complex procedure that is sometimes difficult to understand by ordinary respondents/decision-makers who are not familiar with MCDM and fuzzy logic.

Therefore, the rest of this article is organized as follows: In Section 2, some basic definitions about grey numbers are given, while a new approach is proposed in Section 3. In Section 4, a numerical illustration is presented in order to highlight the basic characteristics of the proposed approach, while in Section 5 an analysis of the obtained results is performed. Finally, conclusions are given at the end of the article.

2. Preliminaries

Definition 1. *Grey number [32]. A grey number $\otimes x$ is such a number whose exact value is unknown, but the range in which value can lie is known.*

Definition 2. *Interval grey number [32]. An interval grey number is a grey number with a known lower bound $\underline{x}$ and upper bound $\overline{x}$, but with the unknown value of x, and it is shown as follows:*

$$\otimes x \in [\underline{x}, \overline{x}] = [\underline{x} \leq x \leq \overline{x}]. \tag{1}$$

Definition 3. *The whitening function [33–35]. The whitening function transforms an interval grey number into a crisp number whose possible values lie between the bounds of the interval grey number. For the given interval grey number, the whitened value $x_{(\lambda)}$ of interval grey number $\otimes x$ is defined as*

$$x_{(\lambda)} = (1 - \lambda)\underline{x} + \lambda\overline{x}, \tag{2}$$

where $\in [0, 1]$ denotes the whitening coefficient.

In the particular case $\lambda = 0.5$, the whitened value becomes the mean of the interval grey number, as follows:

$$x_{(0.5)} = 0.5\left(\underline{x} + \overline{x}\right). \tag{3}$$

3. The Newly Proposed Approach

Suppose that the decision matrix is presented in the form

$$D = [x_{ij}^{k}], \tag{4}$$

where: x_{ij}^{k} denotes the evaluation of alternative i to criterion j stated by the decision-maker k; $i = 1, \ldots, m$, and m denotes the number of alternatives; $j = 1, \ldots, n$, and n denotes the number of criteria; $k = 1 \ldots K$, and K denotes the number of decision-makers.

Such a three-dimensional matrix can be transformed into a group two-dimensional matrix as follows:

$$D = [x'_{ij}], \tag{5}$$

with

$$x'_{ij} = \left(\sum_{k=1}^{K} x_{ij}^{k}\right)/K. \tag{6}$$

Essentially, x'_{ij} denotes rating of alternative i to criterion j. Such defined x'_{ij} is actually the mean value of all assessments of the alternative i in relation to the criterion j.

However, the matrix shown using Equation (4) can be also transformed into a grey group decision matrix, as follows:

$$D = ([\underline{x}_{ij}, \overline{x}_{ij}]), \tag{7}$$

with

$$\underline{x}_{ij} = \left(\sum_{k \in k^{-}} x_{ij}^{k}\right)/n^{-}; \tag{8}$$

$$\overline{x}_{ij} = \left(\sum_{k \in k^{+}} x_{ij}^{k}\right)/n^{+}. \tag{9}$$

In (8), k^{-} denotes the set of elements whose values are less than or equal to the median value of x_{ij}^{k}, and n^{-} denotes the number of elements in this set. Similarly, k^{+} in (9) denotes a set of elements whose values are greater than or equal to the median value of x_{ij}^{k} and n^{+} denotes the number of elements in this set.

Example

Let S be a sequence of 10 integers from interval [1, 5] and S = (1, 2, 3, 1, 5, 3, 3, 1, 4, 5).

Then, the mean and median of S are as follows: *mean* = 2.80 and *median* = 3.00. The mean value of a number which is less than or equal to the median (1, 2, 3, 1, 3, 3, 1) is x_l = 2.00 and the mean value of a number greater or equal to the median (3, 5, 3, 3, 4, 5) is x_u = 3.83.

The mean value of such interval [2.00, 3.83], determined using Equation (3), is 2.915, and the distance between it and the mean is 2.915 − 2.80 = 0.115, that is in percentages 4.11%.

The results obtained based on several sequences of randomly generated numbers from interval [1, 10] are shown in Table 1. The calculation was done in Python using the seed (1).

Table 1. Difference between the mean value of the sequence of numbers and the value obtained by the proposed approach.

Sample	*Mean*	*Median*	x_l	x_u	$x_m = (x_u - x_l)/2$	d = abs(*Mean* $- x_m$)	d (%)
5	5.60	5.00	4.00	7.00	5.50	0.10	1.79
10	7.10	7.50	5.40	8.80	7.10	0.00	0.00
15	5.13	6.00	2.88	7.50	5.19	0.05	1.06
20	6.50	7.00	5.08	8.00	6.54	0.04	0.64
25	4.52	4.00	2.43	6.50	4.46	0.06	1.23
50	5.20	5.00	3.07	7.14	5.11	0.09	1.81
100	5.01	5.00	2.93	7.09	5.01	0.00	0.02
150	5.16	5.00	3.05	6.91	4.98	0.18	3.45

From Table 1, it can be seen that the difference between the mean value of the sequence of numbers and the value obtained by the proposed approach is not large.

4. A Numerical Illustration

In this section, the use of the proposed approach is presented in the case of evaluating websites of tourist organizations from Eastern Serbia. The evaluation was performed on the websites of 5 tourist organizations from the Timok frontier, or more precisely tourist organizations of the Municipalities of Boljevac, Bor, Majdanpek, Negotin and Kladovo (It is important to state that the order of municipalities does not correspond to the order of alternatives, because the aim of this article is not to favor any of the above-mentioned tourist organizations.). The evaluation is performed based on the following criteria: Visual design—C_1, Structure and navigability—C_2, Content—C_3, Innovation—C_4, Personalization—C_5.

The evaluation was performed using ARAS [36], WASPAS [37], CoCoSo [38] and WISP [39] methods. In the first case, the evaluation was performed using ordinary MCDA methods and the mean value of the collected ratings, while in the first case, the evaluation was performed using the proposed approach.

This illustration does not show all the possibilities that the proposed approach provides in terms of analysis. The main goal was to compare the results obtained by applying the mean value of all assessments and the proposed approach, where the transformation of grey numbers was performed using Equation (3) and λ = 0.5.

The rating obtained from 10 respondents is shown in Tables 2–6.

Table 2. Ratings of alternative A_1 in relation to the evaluation criteria obtained from 10 respondents.

A_1	I	II	III	IV	V	VI	VII	VIII	IX	X
C_1	1	2	3	3	5	3	3	4	3	2
C_2	3	3	4	5	3	4	4	4	2	4
C_3	3	3	4	3	3	4	4	5	5	4
C_4	1	1	2	3	4	4	5	2	3	2
C_5	2	1	2	2	1	2	3	4	3	2

Table 3. Ratings of alternative A_2 in relation to the evaluation criteria obtained from 10 respondents.

A_2	I	II	III	IV	V	VI	VII	VIII	IX	X
C_1	3	5	4	4	4	4	5	3	2	2
C_2	5	5	4	5	4	5	5	4	3	4
C_3	2	4	4	4	3	4	5	3	3	3
C_4	4	4	5	4	2	2	5	3	5	3
C_5	4	4	5	3	4	3	4	3	2	4

Table 4. Ratings of alternative A_3 in relation to the evaluation criteria obtained from 10 respondents.

A_3	I	II	III	IV	V	VI	VII	VIII	IX	X
C_1	1	2	2	2	2	2	2	3	2	1
C_2	3	5	5	4	2	2	3	4	4	4
C_3	1	4	4	2	2	2	4	3	2	2
C_4	1	1	3	3	1	1	4	2	1	1
C_5	3	5	5	3	4	4	3	4	4	4

Table 5. Ratings of alternative A_4 in relation to the evaluation criteria obtained from 10 respondents.

A_4	I	II	III	IV	V	VI	VII	VIII	IX	X
C_1	4	4	4	5	4	4	4	4	5	4
C_2	5	5	4	5	3	3	4	5	5	3
C_3	5	5	4	4	3	3	4	5	5	5
C_4	4	4	5	5	5	3	5	5	4	3
C_5	4	4	5	3	4	4	3	5	4	4

Table 6. Ratings of alternative A_5 in relation to the evaluation criteria obtained from 10 respondents.

A_4	I	II	III	IV	V	VI	VII	VIII	IX	X
C_1	4	3	5	5	5	3	5	3	4	3
C_2	4	4	4	5	4	3	5	4	4	4
C_3	5	4	4	4	3	3	4	4	3	5
C_4	4	4	5	3	2	4	3	5	3	3
C_5	3	4	4	4	4	3	4	5	4	4

The group decision matrix, formed on the basis of the responses of all respondents, is shown in Table 7. The elements of this matrix represent the mean value of the ratings obtained from the respondents.

Table 7. Group decision-making matrix.

Criteria Alternatives	C_1	C_2	C_3	C_4	C_5
A_1	2.90	3.60	3.80	2.70	2.20
A_2	3.60	4.40	3.50	3.70	3.60
A_3	1.90	3.60	2.60	1.80	3.90
A_4	4.20	4.20	4.30	4.30	4.00
A_5	4.00	4.10	3.90	3.60	3.90

A similar decision matrix is shown in Table 8, where the elements of that matrix represent the median of ratings obtained from the respondents.

Table 8. The median of ratings obtained from the respondents.

Criteria Alternatives	C_1	C_2	C_3	C_4	C_5
A_1	2.96	3.81	3.92	2.70	2.11
A_2	3.79	4.40	3.50	3.82	3.81
A_3	1.95	3.79	2.31	1.40	3.96
A_4	4.10	4.20	4.30	4.30	4.00
A_5	4.00	4.05	3.96	3.60	3.95

The results of the evaluation performed using ordinary ARAS, WASPAS, CoCoSo and WISP methods, weighting vector w_i = (0.25, 0.24, 0.22, 0.20, 0.10), and the data from Table 7, are shown in Table 9.

Table 9. Ranking of alternatives using ordinary ARAS, WASPAS, CoCoSo and WISP methods.

	ARAS		**WASPAS**		**CoCoSo**		**WISP**	
Alternatives	S_i	Rank	S_i	Rank	S_i	Rank	S_i	Rank
A_1	0.73	4	0.73	4	1.80	4	0.87	4
A_2	0.88	3	0.88	3	2.17	3	0.95	3
A_3	0.61	5	0.60	5	1.49	5	0.81	5
A_4	0.99	1	0.99	1	2.43	1	1.00	1
A_5	0.92	2	0.92	2	2.25	2	0.96	2

In the second case, based on data from Table 8 as well as ratings from Tables 2–6, a grey decision matrix was formed as shown in Table 10.

Table 10. Grey group decision-making matrix.

Criteria Alternatives	C_1	C_2	C_3	C_4	C_5
A_1	[2.50, 3.43]	[3.44, 4.17]	[3.50, 4.33]	[1.60, 3.80]	[1.71, 2.50]
A_2	[3.25, 4.33]	[3.80, 5.00]	[2.80, 4.20]	[3.14, 4.50]	[3.44, 4.17]
A_3	[1.78, 2.13]	[3.25, 4.33]	[1.83, 2.78]	[1.00, 1.80]	[3.62, 4.29]
A_4	[4.00, 4.20]	[3.40, 5.00]	[3.60, 5.00]	[3.60, 5.00]	[3.75, 4.25]
A_5	[3.33, 4.67]	[3.88, 4.22]	[3.63, 4.29]	[2.80, 4.40]	[3.78, 4.13]

The evaluation results generated using the grey ARAS, WASPAS, CoCoSo and WISP methods, and the data from Table 10, are shown in Table 11. It should be noted again that the grey numbers from Table 10 were transformed into crisp values, using Equation (3) and $\lambda = 0.5$, before the evaluation.

Table 11. Ranking of alternatives using grey WS, WP, WASPAS and CoCoSo methods.

	ARAS		**WASPAS**		**CoCoSo**		**WISP**	
Alternatives	S_i	Rank	S_i	Rank	S_i	Rank	S_i	Rank
A_1	0.76	4	0.46	4	1.90	4	0.74	4
A_2	0.91	3	0.55	3	2.29	3	0.89	3
A_3	0.59	5	0.36	5	1.47	5	0.62	5
A_4	0.99	1	0.60	1	2.48	1	0.99	1
A_5	0.92	2	0.56	2	2.32	2	0.92	2

From Tables 9 and 11, it can be seen that differences in ranking orders of alternatives achieved on the basis of the mean value of all assessments and the proposed approach were not observed. Of course, it should be reiterated here that the proposed approach provides significantly greater opportunities in terms of analyzing various scenarios, such as pessimistic or optimistic.

5. Analysis and Discussion

In order to verify the proposed approach, this section presents the results of the evaluation based on the assessments of a number of virtual respondents. For easier evaluation, the scores were generated as random numbers from the interval [1, 10], using a program written in the Python programming language, in which all calculations were also performed. In this analysis, random numbers are generated with the seed (1). The results obtained on the basis of series of 10, 50, 100 and 150 virtual respondents are shown in Tables 12–20. The weighting vector w_i = (0.2, 0.2, 0.2, 0.2, 0.2) is used in this evaluation.

The calculation details obtained on the basis of 10 virtual respondents are shown in Tables 12 and 13. As can be seen from Tables 12 and 13, in this case, the same ranking orders are obtained by applying all methods and approaches.

Table 12. Ranking of alternatives on the basis of 10 virtual respondents and crisp approach.

	WS		WP		WASPAS		CoCoSo		WISP	
Alternatives	S_i	Rank	P_i	Rank	Q_i	Rank	K_i	Rank	S_i	Rank
A_1	0.83	5	0.75	5	0.75	5	1.77	5	0.59	5
A_2	1.00	1	0.91	1	0.91	1	2.16	1	1.00	1
A_3	0.98	2	0.87	2	0.88	2	2.09	2	0.88	2
A_4	0.84	4	0.75	4	0.76	4	1.79	4	0.61	4
A_5	0.89	3	0.80	3	0.81	3	1.91	3	0.71	3

Table 13. Ranking of alternatives on the basis of 10 virtual respondents and the proposed grey approach.

	WS		WP		WASPAS		CoCoSo		WISP	
Alternatives	S_i	Rank	P_i	Rank	Q_i	Rank	K_i	Rank	S_i	Rank
A_1	0.82	5	0.74	5	0.75	5	1.78	5	0.58	5
A_2	1.00	1	0.92	1	0.92	1	2.18	1	1.00	1
A_3	0.95	2	0.84	2	0.86	2	2.04	2	0.80	2
A_4	0.83	4	0.75	4	0.75	4	1.79	4	0.59	4
A_5	0.88	3	0.80	3	0.81	3	1.91	3	0.69	3

Table 14. Ranking of alternatives on the basis of 50 virtual respondents and crisp approach.

	WS		WP		WASPAS		CoCoSo		WISP	
Alternatives	S_i	Rank	P_i	Rank	S_i	S_i	K_i	Rank	S_i	Rank
A_1	0.97	3	0.94	3	0.94	3	1.90	3	0.91	3
A_2	0.92	4	0.89	4	0.89	4	1.80	4	0.78	4
A_3	1.00	1	0.97	1	0.97	1	1.97	1	1.00	1
A_4	0.91	5	0.88	5	0.89	5	1.79	5	0.77	5
A_5	0.97	2	0.94	2	0.95	2	1.91	2	0.91	2

Table 15. Ranking of alternatives on the basis of 50 virtual respondents and the proposed grey approach.

	WS		WP		WASPAS		CoCoSo		WISP	
Alternatives	S_i	Rank	P_i	Rank	Q_i	Rank	K_i	Rank	S_i	Rank
A_1	0.98	2	0.94	2	0.94	2	1.90	2	0.94	2
A_2	0.92	5	0.89	5	0.89	5	1.79	5	0.80	5
A_3	1.00	1	0.96	1	0.96	1	1.94	1	1.00	1
A_4	0.93	4	0.89	4	0.89	4	1.81	4	0.82	4
A_5	0.97	3	0.93	3	0.93	3	1.87	3	0.90	3

Table 16. Ranking orders of alternatives obtained on the basis of 50 virtual respondents.

	Crisp					Grey Approach				
Alternatives	WS	WP	WASPAS	CoCoSo	WISP	WS	WP	WASPAS	CoCoSo	WISP
A_1	3	3	3	3	3	2	2	2	2	2
A_2	4	4	4	4	4	5	5	5	5	5
A_3	1	1	1	1	1	1	1	1	1	1
A_4	5	5	5	5	5	4	4	4	4	4
A_5	2	2	2	2	2	3	3	3	3	3

Table 17. Ranking orders of alternatives obtained on the basis of 100 virtual respondents.

	Crisp					Grey Approach				
Alternatives	WS	WP	WASPAS	CoCoSo	WISP	WS	WP	WASPAS	CoCoSo	WISP
A_1	4	4	4	4	4	4	4	4	4	4
A_2	3	3	3	3	3	2	2	2	2	2
A_3	2	2	2	2	2	3	3	3	3	3
A_4	1	1	1	1	1	1	1	1	1	1
A_5	5	5	5	5	5	5	5	5	5	5

Table 18. Ranking of alternatives on the basis of 150 virtual respondents and crisp methods.

	WS		WP		WASPAS		CoCoSo		WISP	
Alternatives	S_i	Rank	P_i	Rank	S_i	S_i	K_i	Rank	S_i	Rank
A_1	0.993	2	0.960	2	0.961	2	1.840	2	0.977	2
A_2	0.975	3	0.944	3	0.944	3	1.808	3	0.927	3
A_3	0.971	5	0.938	5	0.939	5	1.799	5	0.914	5
A_4	1.000	1	0.968	1	0.968	1	1.854	1	1.000	1
A_5	0.973	4	0.942	4	0.942	4	1.805	4	0.923	4

Table 19. Ranking of alternatives on the basis of 150 virtual respondents and grey methods.

	WS		WP		WASPAS		CoCoSo		WISP	
Alternatives	S_i	Rank	P_i	Rank	Q_i	Rank	K_i	Rank	S_i	Rank
A_1	0.988	2	0.954	2	0.955	2	1.806	2	0.965	2
A_2	0.988	3	0.953	3	0.954	3	1.805	3	0.963	3
A_3	0.987	4	0.952	4	0.953	4	1.804	4	0.960	4
A_4	1.000	1	0.966	1	0.966	1	1.828	1	1.000	1
A_5	0.985	5	0.952	5	0.952	5	1.801	5	0.958	5

Table 20. Ranking orders of alternatives obtained on the basis of 150 virtual respondents.

	Crisp					Grey Approach				
Alternatives	WS	WP	WASPAS	CoCoSo	WISP	WS	WP	WASPAS	CoCoSo	WISP
A_1	2	2	2	2	2	2	2	2	2	2
A_2	3	3	3	3	3	3	3	3	3	3
A_3	5	5	5	5	5	4	4	4	4	4
A_4	1	1	1	1	1	1	1	1	1	1
A_5	4	4	4	4	4	5	5	5	5	5

The calculation details obtained on the basis of 50 virtual respondents are shown in Tables 14–16. In this case, there were some discrepancies in the order of the second and third-placed alternatives, which can be clearly seen in Figure 1.

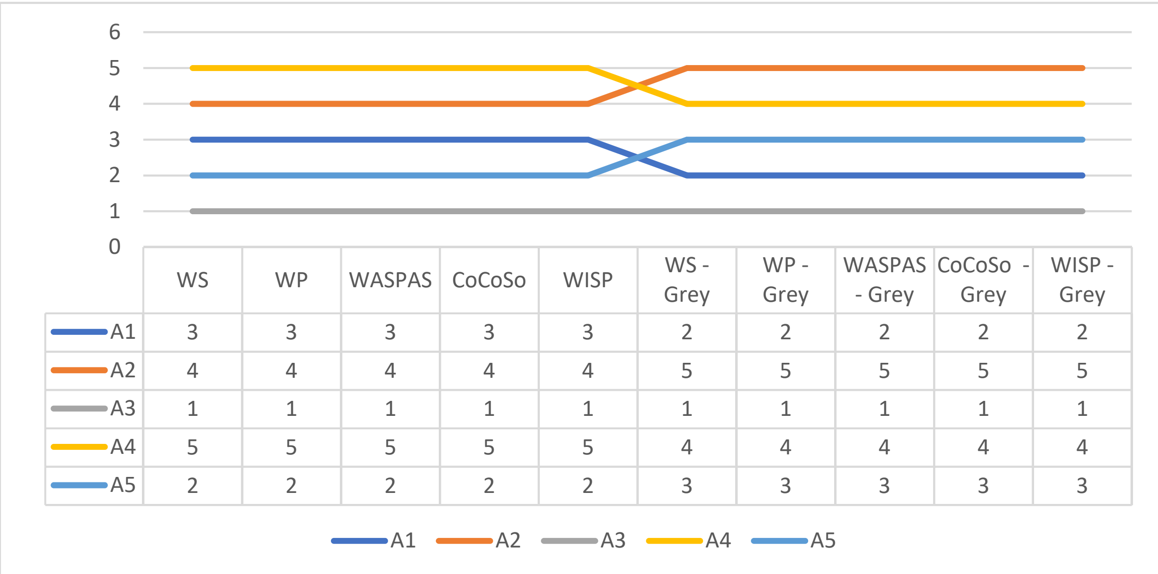
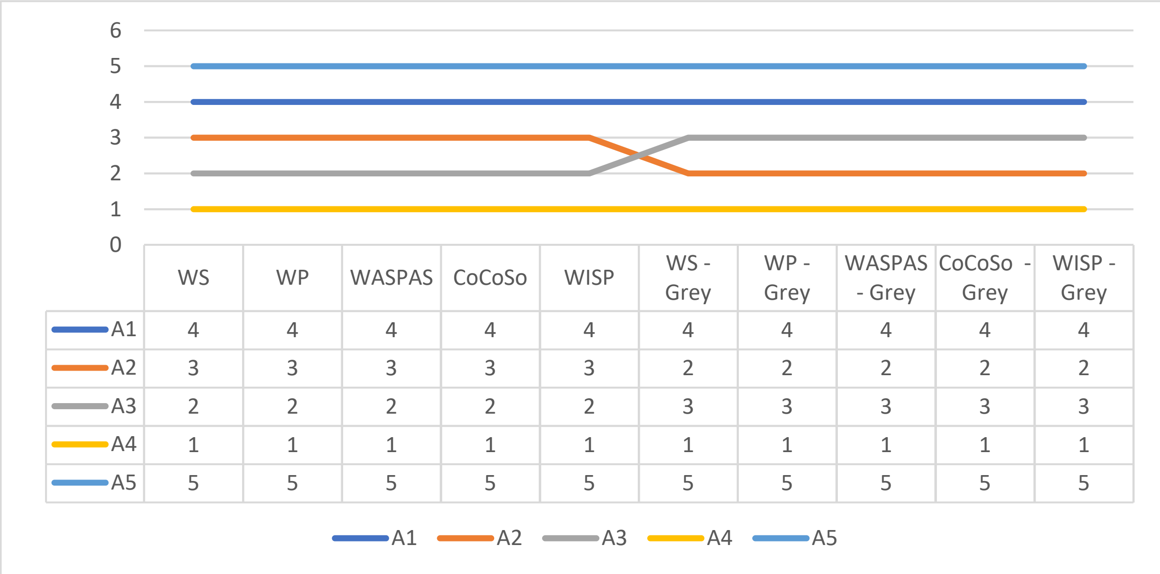


Figure 1. Ranking orders of alternatives obtained on the basis of 50 virtual respondents.

It can be seen from Tables 14 and 15 that differences between second-placed and third-placed alternatives are not high, which is why it can be expected that the same ranking order of alternatives could be obtained by using another weight vector.

Ranking orders of alternatives, obtained on the basis of 100 virtual respondents, are shown in Tabe 17, and presented in Figure 2. This case is similar to the previous one.

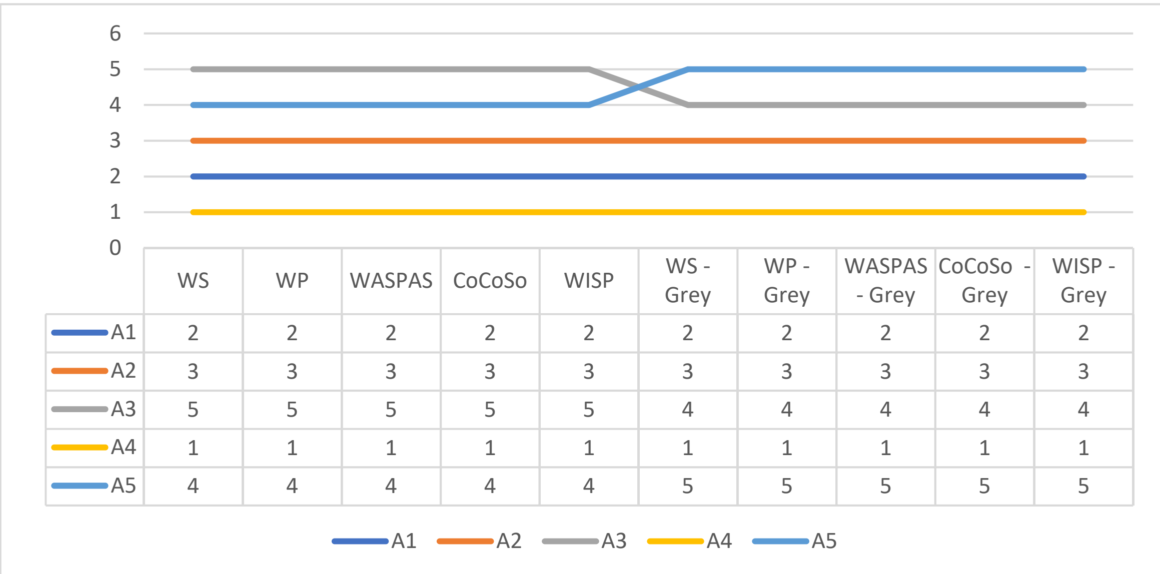


	WS	WP	WASPAS	CoCoSo	WISP	WS - Grey	WP - Grey	WASPAS - Grey	CoCoSo - Grey	WISP - Grey
A1	4	4	4	4	4	4	4	4	4	4
A2	3	3	3	3	3	2	2	2	2	2
A3	2	2	2	2	2	3	3	3	3	3
A4	1	1	1	1	1	1	1	1	1	1
A5	5	5	5	5	5	5	5	5	5	5

Figure 2. Ranking orders of alternatives obtained on the basis of 100 virtual respondents.

From Table 17 and Figure 2 it can be observed that in this case, the differences occur only in the case of the second and third-placed alternatives.

Ranking orders of alternatives that arise from 150 virtual respondents are arranged in Tables 18–20 and presented in Figure 3.



	WS	WP	WASPAS	CoCoSo	WISP	WS - Grey	WP - Grey	WASPAS - Grey	CoCoSo - Grey	WISP - Grey
A1	2	2	2	2	2	2	2	2	2	2
A2	3	3	3	3	3	3	3	3	3	3
A3	5	5	5	5	5	4	4	4	4	4
A4	1	1	1	1	1	1	1	1	1	1
A5	4	4	4	4	4	5	5	5	5	5

Figure 3. Ranking orders of alternatives obtained on the basis of 150 virtual respondents.

Table 20 and Figure 3 clearly show that the alternative A_4 is best ranked according to all methods, with all crisp methods gave the same order of ranking A_4, A_1, A_2, A_5, A_3, while the proposed grey approach gave the following rankings order A_4, A_1, A_2, A_3, A_5. However, from Table 20 it is observable that there are very small differences in overall performance between the second-placed, third-placed and fourth-placed alternatives, which is why it can be expected that different ranking orders of alternatives could be obtained by using another weighting vector.

6. Conclusions

The advantages of using grey instead of crisp numbers in multi-criteria decision analysis have been considered and proven in a number of previously published studies. One of the advantages which should be emphasized using grey numbers is the possibility of considering various scenarios, such as: pessimistic, realistic, and optimistic. The proposed approach allows the transformation of crisp ratings, collected by employing surveys based on the use of the Likert scale, into grey numbers and thus considering different scenarios. The proposed approach may be suitable when it is necessary to collect and analyze the realistic attitudes of a larger number of respondents. Moreover, the proposed transformation enables greater robustness and further possibility of analysis and consideration of different scenarios.

The results of the website evaluation based on the mean value of the ratings obtained from all respondents and the proposed approach did not indicate a difference in the ranking orders of alternatives. However, the results of the conducted analysis indicate that differences may arise between the two approaches, especially in the case of the lower-ranked alternatives.

Some differences in the results are expected because the proposed approach is not a substitute for applying the mean value of the scores obtained from all respondents, but an approach that further allows the possibility of analysis. Certain differences in the ranking results using the newly proposed approach and applying the mean of the scores obtained in all respondents can be cited as a weakness of this approach.

Finally, consideration of the transformation of a larger number of crisp ratings into corresponding triangular fuzzy numbers or interval-valued triangular fuzzy numbers can be mentioned as one of the possible directions for the further development of the proposed approach.

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New Dombi aggregation operators on bipolar neutrosophic set with application in multi-attribute decision-making problems

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Abstract. In this paper, we investigate two new Dombi aggregation operators on bipolar neutrosophic set namely bipolar neutrosophic Dombi prioritized weighted geometric aggregation (BNDPWGA) and bipolar neutrosophic Dombi prioritized ordered weighted geometric aggregation (BNDPOWGA) by means of Dombi t-norm (TN) and Dombi t-conorm (TCN). We discuss their properties along with proofs and multi-attribute decision making (MADM) methods in detail. New algorithms based on proposed models are presented to solve multi-attribute decision-making (MADM) problems. In contrast, with existing techniques a comparison analysis of proposed methods are also demonstrated to test their validity, accuracy and significance.

Keywords: Bipolar neutrosophic set, bipolar neutrosophic Dombi prioritized aggregation operators, decision-making environment

1. Introduction

In fuzzy theory, a newly defined model is not accessible unless it reduce (overcome) drawbacks of previously defined related models. Due to vagueness and uncertainties issues in many daily life problems, routine mathematics is not always available. To deal with such issues, various procedures such as hypothesis of probability, rough set hypothesis, and fuzzy set hypotheses have been considered as alternative models. Unfortunately, most of such alternative mathematics has its own down sides and drawbacks. For instance, most of the words like, genuine, lovely, best, renowned are not measurable and are, in fact, ambiguous. The criteria for words like wonderful, best, renowned etc., fluctuate from individual to individual. To handle such type of ambiguous and uncertain information, Zadeh [1] initiated the study of possibility based on participation function, that assigns an enrollment grade in $[0, 1]$, called fuzzy set (FSs). FSs have only one membership degree and cannot handle complex problems. The concept of intuitionistic fuzzy set (IFS) was introduced by Atanassov [2]. The intuitionistic fuzzy set (IFS) includes both membership degree and non-membership degree. Subsequently, the concept of interval-valued intuitionistic fuzzy set (IVIFS) was introduced by Atanassov and Gargov [3]. Note that IVIFS is a generalization of IFS. In decision-making problems, IFS and IVIFS are not good at solving problems regarding inconsistent information. The concept of neutrosophic set (NS) was introduced by

Smarandache [4]. The neutrosophic set (NS) includes truth, falsity and indeterminacy membership degree which is used to characterize incomplete, inconsistent and uncertain information. Wang et al. [5, 6] introduced the concept of interval neutrosophic set (INS) and the concept of single-valued neutrosophic set (SVNS) to apply NS in daily life decision-making problems.

When it comes to decision-making problems, we think about positive and negative consequences before taking a decision. Positive information explains why a decision is acceptable, permitted, possible, described, or satisfactory . While a negative information explains why a decision is rejected, forbidden, or impossible [7]. Satisfactory or acceptable perceptions are positive preferences, while unsatisfactory or unacceptable perceptions are negative preferences. Positive preferences are related with desires, while negative preferences relate to the constraints [8]. For example, when a decision maker examines an object, he or she need to explain that why he or she considered a satisfaction or an accessible object. Similarly, one has to explain why he or she considered a dissatisfaction or an unaccessible object [9]. In [10], Zhang introduced the concept of bipolar fuzzy set (BFS) consisting of positive membership degree and negative membership degree to describe the words like satisfaction and accessible. Deli et al. introduced the concept of bipolar neutrosophic set (BNS) which describes fuzzy, bipolar, inconsistent and uncertain information in [11]. In [12], Dey et al. proposed the bipolar neutrosophic TOPSIS (BN-TOPSIS) method to solve MADM problems under bipolar neutrosophic fuzzy environments. Zhang et al. proposed the methods based on the Frank Choquet Bonferroni Mean Operators to solve MADM problems under bipolar neutrosophic fuzzy environments in [9]. Many researchers investigated different kind of operators with application in decision-making problems in [14–18].

The MADM model refers to making decisions when there are multiple but a finite list of alternatives and attributes. Dombi [13] introduced new triangular norms which are Dombi TN and Dombi TCN. Dombi TN and Dombi TCN showed good flexibility with operational parameters. Until now, Dombi operations have not been extended to aggregate bipolar neutrosophic fuzzy environments.

The compatibility and validity of a newly defined fuzzy operator over different fuzzy environments is always challenging and worthwhile. Also, a comparative study analysis with perviously defined MADM methods is of great intrust. If a newly defined fuzzy operator can be reduced to MADM method with a batter accuracy then this newly defined operator would be more flexible, charming and useful. The motivation behind this work was to introduce new operators based on Dombi aggregation operator with a batter accuracy in contrast to existing operators.

In this work, we extend Dombi operations to aggregate bipolar neutrosophic fuzzy environments. Also, we establish new aggregation operators based on the combination of bipolar neutrosophic numbers(BNNs) and Dombi operations. We have proposed two new bipolar neutrosophic Dombi aggregation operators to aggregate bipolar neutrosophic fuzzy information, and developed MADM methods based on BNDPWGA, and BNDPOWGA operators to solve MADM problems with bipolar neutrosophic fuzzy information. The flexibility and accuracy over a multi-attributed problem has been demonstrated and verified as an alternative multi-criteria decision making tool. This was the true motivation behind this work.

The rest of the paper is arranged as follows. Section 2 reviews some of the basic definitions and concepts which will be used frequently. Section 3 defines Dombi operations of bipolar neutrosophic numbers (BNNs). Section 4 and Section 5 propose new operators (BNDPWGA and BNDPOWGA) together with their properties and comprehensive MADM methods based on proposed bipolar neutrosophic Dombi aggregation operators. Section 6 provides a numerical example for the selection of cultivating crops. Section 7 confers the effect of parameters and a comparative analysis with existing methods.

2. Preliminaries

In this section, we present a brief survey of few fundamentals of different sorts of sets which will be utilized in sequel.

2.1. Bipolar neutrosophic set and bipolar neutrosophic number

Definition 2.1. [11] Let U be a fixed set. Then, BNS N can be defined as follows: $N(u) = \{< \tau_N^+(u),\ \omega_N^+(u),\ \mho_N^+(u),\ \tau_N^-(u),\ \omega_N^-(u), \mho_N^-(u) > |u \in U\}$. Where $\tau_N^+, \omega_N^+, \mho_N^+ : U \longrightarrow [0, 1]$ and $\tau_N^-, \omega_N^-, \mho_N^- : U \longrightarrow [-1, 0]$. The positive membership degrees $\tau_N^+(u), \omega_N^+(u), \mho_N^+(u)$ are the truth membership, indeterminacy membership degree and falsity membership degree of an element $u \in U$ corresponding to BNS N and the negative membership degrees $\tau_N^-(u), \omega_N^-(u), \mho_N^-(u)$ denote the truth membership degree, indeterminacy membership degree and falsity membership degree of an element $u \in U$ to some implicit counter property corresponding to a BNS N.

In particular, if U has only one element, then $N(u) = < \tau_N^+(u),\ \omega_N^+(u),\ \mho_N^+(u),\ \tau_N^-(u), \omega_N^-(u), \mho_N^-(u) >$ is called bipolar neutrosophic number(BNN). For convenience, BNN $N =< \tau_N^+(u), \omega_N^+(u), \mho_N^+(u), \tau_N^-(u), \omega_N^-(u), \mho_N^-(u) >$ is also denoted as $N =< \tau_N^+, \omega_N^+, \mho_N^+, \tau_N^-, \omega_N^-, \mho_N^- >$.

Deli et al. [11] defined the algebraic operations of BNNs which are as follows:

Definition 2.2. Let $\tilde{\aleph}_1 =< \tau_1^+, \omega_1^+, \mho_1^+, \tau_1^-, \omega_1^-, \mho_1^- >$ and $\tilde{\aleph}_2 =< \tau_2^+, \omega_2^+, \mho_2^+, \tau_2^-, \omega_2^-, \mho_2^- >$ be two BNNs. Then, algebraic operations of BNNs are defined as follows:

(1) $\tilde{\aleph}_1 \oplus \tilde{\aleph}_2 =< \tau_1^+ + \tau_2^+ - \tau_1^+\tau_2^+, \omega_1^+\omega_2^+, \mho_1^+\mho_2^-, -\tau_1^-\tau_2^-, -(-\omega_1^- - \omega_2^- - \omega_1^-\omega_2^-), -(-\mho_1^- - \mho_2^- - \mho_1^-\mho_2^-) >$,
(2) $\tilde{\aleph}_1 \otimes \tilde{\aleph}_2 =< \tau_1^+\tau_2^+, \omega_1^+ + \omega_2^+ - \omega_1^+\omega_2^+, \mho_1^+ + \mho_2^+ - \mho_1^+\mho_2^+, -(-\tau_1^- - \tau_2^- - \tau_1^-\tau_2^-), -\omega_1^-\omega_2^-, -\mho_1^-\mho_2^- >$,
(3) $\ell.\tilde{\aleph}_1 =< 1-(1-\tau_1^+)^\ell, (\omega_1^+)^\ell, (\mho_1^+)^\ell, (-\tau_1^-)^\ell, -(1-(1-(-\omega_1^-)^\ell)), -(1-(1-(-\mho_1^-)^\ell)) > (\ell > 0)$,
(4) $\tilde{\aleph}_1^\ell =< (\tau_1^+)^\ell, 1-(1-\omega_1^+)^\ell, 1-(1-\mho_1^+)^\ell, -(1-(1-(-\tau_1^-)^\ell)), -(-\omega_1^-)^\ell, -(-\mho_1^-)^\ell > (\ell > 0)$.

For comparing two BNNs, Deli et al. [11] developed a comparison method which consists of the score function, accuracy function and certainty function.

Definition 2.3. [11] Let $\tilde{\aleph}_1 =< \tau_1^+, \omega_1^+, \mho_1^+, \tau_1^-, \omega_1^-, \mho_1^- >$ be BNN, then the score function $s(\tilde{\aleph}_1)$, accuracy function $a(\tilde{\aleph}_1)$ and certainty function $c(\tilde{\aleph}_1)$ are defined as:

$$s(\tilde{\aleph}_1) = \frac{\tau_1^+ + 1 - \omega_1^+ + 1 - \mho_1^+ + 1 + \tau_1^- - \omega_1^- - \mho_1^-}{6} \quad (1)$$

$$a(\tilde{\aleph}_1) = (\tau_1^+ - \mho_1^+) + (\tau_1^- - \mho_1^-) \quad (2)$$

$$c(\tilde{\aleph}_1) = 7\tau_1^+ - \mho_1^- \quad (3)$$

The comparison method of BNNs can be obtained based on Equations (1) − (3) as follows.

Definition 2.4. [11] Let $\tilde{\aleph}_1 =< \tau_1^+, \omega_1^+, \mho_1^+, \tau_1^-, \omega_1^-, \mho_1^- >$ and $\tilde{\aleph}_2 =< \tau_2^+, \omega_2^+, \mho_2^+, \tau_2^-, \omega_2^-, \mho_2^- >$ be two BNNs, therefore

(1) if $s(\tilde{\aleph}_1) > s(\tilde{\aleph}_2)$, then $\tilde{\aleph}_1 > \tilde{\aleph}_2$,
(2) if $s(\tilde{\aleph}_1) = s(\tilde{\aleph}_2)$ and $a(\tilde{\aleph}_1) > a(\tilde{\aleph}_2)$, then $\tilde{\aleph}_1 > \tilde{\aleph}_2$,
(3) if $s(\tilde{\aleph}_1) = s(\tilde{\aleph}_2)$, $a(\tilde{\aleph}_1) = a(\tilde{\aleph}_2)$ and $c(\tilde{\aleph}_1) > c(\tilde{\aleph}_2)$, then $\tilde{\aleph}_1 > \tilde{\aleph}_2$,
(4) if $s(\tilde{\aleph}_1) = s(\tilde{\aleph}_2)$, $a(\tilde{\aleph}_1) = a(\tilde{\aleph}_2)$ and $c(\tilde{\aleph}_1) = c(\tilde{\aleph}_2)$, then $\tilde{\aleph}_1 \sim \tilde{\aleph}_2$.

2.2. Dombi operations

The Dombi product and Dombi sum are special cases of TN and TCN respectively, and are given in the following definition.

Definition 2.5. [13] Let $\aleph_1$ and $\aleph_2$ be any two real numbers, then Dombi TN and Dombi TCN are defined in the following expressions:

$$\aleph_1 \otimes_D \aleph_2 = Dom(\aleph_1, \aleph_2) = \frac{1}{1 + \{(\frac{1-\aleph_1}{\aleph_1})^\lambda + (\frac{1-\aleph_2}{\aleph_2})^\lambda\}^{\frac{1}{\lambda}}} \quad (4)$$

and

$$\aleph_1 \oplus_D \aleph_2 = Dom^*(\aleph_1, \aleph_2) = 1 - \frac{1}{1 + \{(\frac{\aleph_1}{1-\aleph_1})^\lambda + (\frac{\aleph_2}{1-\aleph_2})^\lambda\}^{\frac{1}{\lambda}}} \quad (5)$$

where $\lambda \geq 1$ and $(\aleph_1, \aleph_2) \in [0,1] \times [0,1]$. Some special cases can be easily proved.

(1) if $\lambda \longrightarrow 1$, then $\aleph_1 \oplus_D \aleph_2 \longrightarrow \frac{\aleph_1+\aleph_2-2\aleph_1\aleph_2}{1-\aleph_1\aleph_2}$ and $\aleph_1 \otimes_D \aleph_2 \longrightarrow \frac{\aleph_1\aleph_2}{\aleph_1+\aleph_2-\aleph_1\aleph_2}$,

(2) If $\lambda \longrightarrow \infty$, then $\aleph_1 \oplus_D \aleph_2 \longrightarrow max(\aleph_1, \aleph_2)$ and $\aleph_1 \otimes_D \aleph_2 \longrightarrow min(\aleph_1, \aleph_2)$. Dombi sum and Dombi product are reduced to simple max-operator and simple min-operator, respectively.

3. Dombi operations of BNNs

In this section, we discuss the Dombi operations of BNNs and discuss their properties.
Dombi operations of BNNs are defined as follows.

Definition 3.1. Let $\tilde{\aleph}_1 =< \tau_1^+, \omega_1^+, \mho_1^+, \tau_1^-, \omega_1^-, \mho_1^- >$ and $\tilde{\aleph}_2 =< \tau_2^+, \omega_2^+, \mho_2^+, \tau_2^-, \omega_2^-, \mho_2^- >$ be two BNNs and $\lambda \geq 1$. Then, Dombi sum and Dombi product of two BNNs $\tilde{\aleph}_1$ and $\tilde{\aleph}_2$ are denoted by $\tilde{\aleph}_1 \otimes_D \tilde{\aleph}_2$ and $\tilde{\aleph}_1 \oplus_D \tilde{\aleph}_2$ respectively and defined as follows:

$$\textbf{(1)}\ \tilde{\aleph}_1 \oplus_D \tilde{\aleph}_2 = \begin{cases} \langle 1 - \frac{1}{1+\{(\frac{\tau_1^+}{1-\tau_1^+})^\lambda + (\frac{\tau_2^+}{1-\tau_2^+})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{(\frac{1-\omega_1^+}{\omega_1^+})^\lambda + (\frac{1-\omega_2^+}{\omega_2^+})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{(\frac{1-\mho_1^+}{\mho_1^+})^\lambda + (\frac{1-\mho_2^+}{\mho_2^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ \frac{-1}{1+\{(\frac{1+\tau_1^-}{-\tau_1^-})^\lambda + (\frac{1+\tau_2^-}{-\tau_2^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{(\frac{-\omega_1^-}{1+\omega_1^-})^\lambda + (\frac{-\omega_2^-}{1+\omega_2^-})^\lambda\}^{\frac{1}{\lambda}}} - 1, \frac{1}{1+\{(\frac{-\mho_1^-}{1+\mho_1^-})^\lambda + (\frac{-\mho_2^-}{1+\mho_2^-})^\lambda\}^{\frac{1}{\lambda}}} - 1 \rangle \end{cases}$$

$$\textbf{(2)}\ \tilde{\aleph}_1 \otimes_D \tilde{\aleph}_2 = \begin{cases} \langle \frac{1}{1+\{(\frac{1-\tau_1^+}{\tau_1^+})^\lambda + (\frac{1-\tau_2^+}{\tau_2^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{(\frac{\omega_1^+}{1-\omega_1^+})^\lambda + (\frac{\omega_2^+}{1-\omega_2^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{(\frac{\mho_1^+}{1-\mho_1^+})^\lambda + (\frac{\mho_2^+}{1-\mho_2^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ \frac{1}{1+\{(\frac{-\tau_1^-}{1+\tau_1^-})^\lambda + (\frac{-\tau_2^-}{1+\tau_2^-})^\lambda\}^{\frac{1}{\lambda}}} - 1, \frac{-1}{1+\{(\frac{1+\omega_1^-}{-\omega_1^-})^\lambda + (\frac{1+\omega_2^-}{-\omega_2^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1+\{(\frac{1+\mho_1^-}{-\mho_1^-})^\lambda + (\frac{1+\mho_2^-}{-\mho_2^-})^\lambda\}^{\frac{1}{\lambda}}} \rangle \end{cases}$$

$$\textbf{(3)}\ \ell ._D \tilde{\aleph}_1 = \begin{cases} \langle 1 - \frac{1}{1+\{\ell(\frac{\tau_1^+}{1-\tau_1^+})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{\ell(\frac{1-\omega_1^+}{\omega_1^+})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{\ell(\frac{1-\mho_1^+}{\mho_1^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ \frac{-1}{1+\{\ell(\frac{1+\tau_1^-}{-\tau_1^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{\ell(\frac{-\omega_1^-}{1+\omega_1^-})^\lambda\}^{\frac{1}{\lambda}}} - 1, \frac{1}{1+\{\ell(\frac{-\mho_1^-}{1+\mho_1^-})^\lambda\}^{\frac{1}{\lambda}}} - 1 \rangle \qquad (\ell > 0) \end{cases}$$

$$\textbf{(4)}\ \tilde{\aleph}_1^{\bigwedge_D^{\ell}} = \begin{cases} \langle \frac{1}{1+\{\ell(\frac{1-\tau_1^+}{\tau_1^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\ell(\frac{\omega_1^+}{1-\omega_1^+})^\lambda\}^{\frac{1}{q}}}, 1 - \frac{1}{1+\{\ell(\frac{\mho_1^+}{1-\mho_1^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ \frac{1}{1+\{\ell(\frac{-\tau_1^-}{1+\tau_1^-})^\lambda\}^{\frac{1}{\lambda}}} - 1, \frac{-1}{1+\{\ell(\frac{1+\omega_1^-}{-\omega_1^-})^q\}^{\frac{1}{\lambda}}}, \frac{-1}{1+\{\ell(\frac{1+\mho_1^-}{-\mho_1^-})^\lambda\}^{\frac{1}{\lambda}}} \rangle \qquad (\ell > 0). \end{cases}$$

Theorem 3.2. *Let* $\tilde{\aleph}_1 =< \tau_1^+, \omega_1^+, \mho_1^+, \tau_1^-, \omega_1^-, \mho_1^- >$ *and* $\tilde{\aleph}_2 =< \tau_2^+, \omega_2^+, \mho_2^+, \tau_2^-, \omega_2^-, \mho_2^- >$ *be two BNNs and let* $\tilde{e} = \tilde{\aleph}_1 \oplus_D \tilde{\aleph}_2$, $\tilde{f} = \tilde{\aleph}_1 \otimes_D \tilde{\aleph}_2$, $\tilde{g} = \ell ._D \tilde{\aleph}_1$ $(\ell > 0)$ *and* $\tilde{h} = \tilde{\aleph}_1^{\bigwedge_D^{\ell}}$ $(\ell > 0)$. *Therefore,* $\tilde{e}$, $\tilde{f}$, $\tilde{g}$ *and* $\tilde{h}$ *are also BNNs.*

As Theorem 3 can be easily verified. Therefore, the proof is omitted here. The properties of Dombi operations of BNNs are defined as:

Theorem 3.3. *Let* $\tilde{\aleph}_1 =< \tau_1^+, \omega_1^+, \mho_1^+, \tau_1^-, \omega_1^-, \mho_1^- >$ *and* $\tilde{\aleph}_2 =< \tau_2^+, \omega_2^+, \mho_2^+, \tau_2^-, \omega_2^-, \mho_2^- >$ *be two BNNs and* $\ell, \ell_1, \ell_2 > 0$. *Then, the following properties can be proven easily.*

(a) $\tilde{\aleph}_1 \oplus_D \tilde{\aleph}_2 = \tilde{\aleph}_2 \oplus_D \tilde{\aleph}_1$,
(b) $\tilde{\aleph}_1 \otimes_D \tilde{\aleph}_2 = \tilde{\aleph}_2 \otimes_D \tilde{\aleph}_1$,
(c) $\ell ._D (\tilde{\aleph}_1 \oplus_D \tilde{\aleph}_2) = \ell ._D \tilde{\aleph}_2 \oplus_D \ell ._D \tilde{\aleph}_1$,
(d) $(\tilde{\aleph}_1 \otimes_D \tilde{\aleph}_2)^{\bigwedge_D^{\ell}} = \tilde{\aleph}_1^{\bigwedge_D^{\ell}} \otimes_D \tilde{\aleph}_2^{\bigwedge_D^{\ell}}$,
(e) $(\ell_1 + \ell_2) ._D \tilde{\aleph}_1 = \ell_1 ._D \tilde{\aleph}_1 \oplus_D \ell_2 ._D \tilde{\aleph}_1$,
(f) $\tilde{\aleph}_1^{\bigwedge_D^{\ell_1+\ell_2}} = \tilde{\aleph}_1^{\bigwedge_D^{\ell_1}} \otimes_D \tilde{\aleph}_1^{\bigwedge_D^{\ell_2}}$.

4. Bipolar neutrosophic dombi prioritized aggregation operators

In this section, we define bipolar neutrosophic Dombi weighted geometric prioritized aggregation (BNDPWGA) and the bipolar neutrosophic Dombi prioritized ordered weighted geometric aggregation (BNDPOWGA) operators and discussed different properties of these aggregation operators (AOs) in detail.

Definition 4.1. Let $C = \{C_1, C_2, ..., C_n\}$ be a collection of attributes and have a prioritization between the attributes followed by the linear ordering $C_1 \succ C_2 \succ ... \succ C_n$, which imply that C_η has a higher priority than C_ρ, if $\eta < \rho$. The value of $C_\rho(u)$ is the performance of any alternative u under attribute $C_\eta(u)$, which satisfies $C_\eta \in [0, 1]$, If

$$PA(C_\eta(u)) = \Sigma_{\rho=1}^{n} \upsilon_\rho C_\rho(u) \tag{6}$$

where $\upsilon_\rho = \frac{\hbar_\rho}{\Sigma_{\rho=1}^{n}\hbar_\rho}$; $\hbar_\rho = \Pi_{\eta=1}^{\rho-1} C_\eta(u)(\rho = 2, 3, ..., n)$, $\hbar_1 = 1$. Then, PA is called the prioritized averaging operator.

4.1. Bipolar neutrosophic dombi prioritized weighted geometric aggregation operator

Definition 4.2. Let $\tilde{\aleph}_\alpha =< \tau_\alpha^+, \omega_\alpha^+, \mho_\alpha^+, \tau_\alpha^-, \omega_\alpha^-, \mho_\alpha^- > (\alpha = 1, , 2, 3, ..., r)$ be a family of BNNs. A mapping BNDPWGA: $U^r \rightarrow U$ is called BNDPWGA operator, if it satisfies

$$BNDPWGA(\tilde{\aleph}_1, \tilde{\aleph}_2, ..., \tilde{\aleph}_r) = \bigotimes_{\alpha=1}^{r} \tilde{\aleph}_\alpha^{\bigwedge^{\upsilon_\alpha}} = \tilde{\aleph}_1^{\bigwedge_D^{\upsilon_1}} \otimes_D \tilde{\aleph}_2^{\bigwedge_D^{\upsilon_2}} \otimes_D ... \otimes_D \tilde{\aleph}_r^{\bigwedge_D^{\upsilon_r}} \tag{7}$$

where $\upsilon_\alpha = \frac{\hbar_\alpha \psi_\alpha}{\Sigma_{\alpha=1}^{r}\hbar_\alpha\psi_\alpha}$, $\hbar_\alpha = \prod_{k=1}^{\alpha-1} s(\tilde{\aleph}_k)$ $(\alpha = 2, 3, ..., r)$, $\hbar_1 = 1$ and $s(\tilde{\aleph}_k)$ is the value of score for $\tilde{\aleph}_\alpha$. where $\psi = (\psi_1, \psi_2, ..., \psi_r)^T$ is the weight vector of $\tilde{\aleph}_\alpha$ $(\alpha = 1, 2, ..., r)$, $\psi_\alpha \in [0, 1]$ and $\Sigma_{\alpha=1}\psi_\alpha = 1$.

Theorem 4.3. *Let* $\tilde{\aleph}_\alpha =< \tau_\alpha^+, \omega_\alpha^+, \mho_\alpha^+, \tau_\alpha^-, \omega_\alpha^-, \mho_\alpha^- > (\alpha = 1, 2, 3, ..., r)$ *be a family of BNNs and* $\psi = (\psi_1, \psi_2, ..., \psi_r)^T$ *be the weight vector of* $\tilde{\aleph}_\alpha$, $\psi_\alpha \in [0, 1]$ *and* $\Sigma_{\alpha=1}\psi_\alpha = 1$. *Then, the value aggregated by using BNDPWGA operator is still a BNN, which is calculated by using the following formula*
$BNDPWGA(\tilde{\aleph}_1, \tilde{\aleph}_2, ..., \tilde{\aleph}_r) = \tilde{\aleph}_1^{\bigwedge_D^{\upsilon_1}} \otimes_D \tilde{\aleph}_2^{\bigwedge_D^{\upsilon_2}} \otimes_D ... \otimes_D \tilde{\aleph}_r^{\bigwedge_D^{\upsilon_r}}$

$$= \begin{cases} \langle \frac{1}{1+\{\Sigma_{\alpha=1}^{r} \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{r}\hbar_\alpha\psi_\alpha}(\frac{1-\tau_\alpha^+}{\tau_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_{\alpha=1}^{r} \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{j=1}^{r}\hbar_\alpha\psi_\alpha}(\frac{\omega_\alpha^+}{1-\omega_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_{\alpha=1}^{r} \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{r}\hbar_\alpha\psi_\alpha}(\frac{\mho_\alpha^+}{1-\mho_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ \frac{1}{1+\{\Sigma_{\alpha=1}^{r} \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{r}\hbar_\alpha\psi_\alpha}(\frac{-\tau_\alpha^-}{1+\tau_\alpha^-})^\lambda\}^{\frac{1}{\lambda}}} - 1, \frac{-1}{1+\{\Sigma_{\alpha=1}^{r} \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{r}\hbar_\alpha\psi_\alpha}(\frac{1+\omega_\alpha^-}{-\omega_\alpha^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1+\{\Sigma_{\alpha=1}^{r} \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{r}\hbar_\alpha\psi_\alpha}(\frac{1+\mho_\alpha^-}{-\mho_\alpha^-})^\lambda\}^{\frac{1}{\lambda}}} \rangle \end{cases} \tag{8}$$

Proof. If $r = 2$ based on Definition 4.1 for the Dombi operations of BNNs, the following result can be obtained:

BNDPWGA $(\tilde{\aleph}_1, \tilde{\aleph}_2) = \tilde{\aleph}_1^{\bigwedge_D^{\upsilon_1}} \otimes_D \tilde{\aleph}_2^{\bigwedge_D^{\upsilon_2}}$

$$= \begin{cases} \langle \frac{1}{1+\{\frac{\hbar_1\psi_1}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{1-t_1^+}{\tau_1^+})^\lambda + \frac{\hbar_2\psi_2}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{1-\tau_2^+}{\tau_2^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\frac{\hbar_1\psi_1}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{\omega_1^+}{1-\omega_1^+})^\lambda + \frac{\hbar_2\psi_2}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{\omega_2^+}{1-\omega_2^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ 1 - \frac{1}{1+\{\frac{\hbar_1\psi_1}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{\mho_1^+}{1-\mho_1^+})^\lambda + \frac{\hbar_2\psi_2}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{\mho_2^+}{1-\mho_2^+})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{\frac{\hbar_1\psi_1}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{-\tau_1^-}{1+\tau_1^-})^\lambda + \frac{\hbar_2\psi_2}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{-\tau_2^-}{1+\tau_2^-})^\lambda\}^{\frac{1}{\lambda}}} - 1, \\ \frac{-1}{1+\{\frac{\hbar_1\psi_1}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{1+\omega_1^-}{-\omega_1^-})^\lambda + \frac{\hbar_2\psi_2}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{1+\omega_2^-}{-\omega_2^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1+\{\frac{\hbar_1\psi_1}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{1+\mho_1^-}{-\mho_1^-})^\lambda + \frac{\hbar_2\psi_2}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{1+\mho_2^-}{-\mho_2^-})^\lambda\}^{\frac{1}{\lambda}}} \rangle, \end{cases}$$

$$
= \left\{ \begin{array}{l}
\langle \frac{1}{1+\{\Sigma_{\alpha=1}^{2} \frac{\hbar_\alpha \psi_\alpha}{\Sigma_{\alpha=1}^{2} \hbar_\alpha \psi_\alpha} (\frac{1-\tau_\alpha^{+}}{\tau_\alpha^{+}})^{\lambda}\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_{\alpha=1}^{2} \frac{\hbar_\alpha \psi_\alpha}{\Sigma_{\alpha=1}^{2} \hbar_\alpha \psi_\alpha} (\frac{\omega_\alpha^{+}}{1-\omega_\alpha^{+}})^{\lambda}\}^{\frac{1}{\lambda}}}, \\
1 - \frac{1}{1+\{\Sigma_{\alpha=1}^{2} \frac{\hbar_\alpha \psi_\alpha}{\Sigma_{\alpha=1}^{2} \hbar_\alpha \psi_\alpha} (\Sigma_{\alpha=1}^{2} \frac{\mho_\alpha^{+}}{1-\mho_\alpha^{+}})^{\lambda}\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{\Sigma_{\alpha=1}^{2} \frac{\hbar_\alpha \psi_\alpha}{\Sigma_{\alpha=1}^{2} \hbar_\alpha \psi_\alpha} (\frac{-\tau_\alpha^{-}}{1+\tau_\alpha^{-}})^{\lambda}\}^{\frac{1}{\lambda}}} - 1, \\
\frac{-1}{1+\{\Sigma_{\alpha=1}^{2} \frac{\hbar_\alpha \psi_\alpha}{\Sigma_{\alpha=1}^{2} \hbar_\alpha \psi_\alpha} (\frac{1+\omega_\alpha^{-}}{-\omega_\alpha^{-}})^{\lambda}\}^{\frac{1}{\lambda}}}, \frac{-1}{1+\{\Sigma_{\alpha=1}^{2} \frac{\hbar_\alpha \psi_\alpha}{\Sigma_{\alpha=1}^{2} \hbar_\alpha \psi_\alpha} (\frac{1+\mho_\alpha^{-}}{-\mho_\alpha^{-}})^{\lambda}\}^{\frac{1}{\lambda}}} \rangle.
\end{array} \right.
$$

If $r = s$, based on equation (8), then we have got the following equation:

$$
\mathrm{BNDPWGA}(\tilde{\aleph}_1, \tilde{\aleph}_2, ..., \tilde{\aleph}_s) = \bigotimes_{\alpha=1}^{s} \tilde{\aleph}_\alpha^{\bigwedge \upsilon_\alpha}
$$

$$
= \left\{ \begin{array}{l}
\langle \frac{1}{1+\{\Sigma_{\alpha=1}^{s} \frac{\hbar_\alpha \psi_\alpha}{\Sigma_{\alpha=1}^{s} \hbar_\alpha \psi_\alpha} (\frac{1-\tau_\alpha^{+}}{\tau_\alpha^{+}})^{\lambda}\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_{\alpha=1}^{s} \frac{\hbar_\alpha \psi_\alpha}{\Sigma_{\alpha=1}^{s} \hbar_\alpha \psi_\alpha} (\frac{\omega_\alpha^{+}}{1-\omega_\alpha^{+}})^{\lambda}\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_{\alpha=1}^{s} \frac{\hbar_\alpha \psi_\alpha}{\Sigma_{\alpha=1}^{s} \hbar_\alpha \psi_\alpha} (\frac{\mho_\alpha^{+}}{1-\mho_\alpha^{+}})^{\lambda}\}^{\frac{1}{\lambda}}}, \\
\frac{1}{1+\{\Sigma_{\alpha=1}^{s} \frac{\hbar_\alpha \psi_\alpha}{\Sigma_{\alpha=1}^{s} \hbar_\alpha \psi_\alpha} (\frac{-\tau_\alpha^{-}}{1+\tau_\alpha^{-}})^{\lambda}\}^{\frac{1}{\lambda}}} - 1, \frac{-1}{1+\{\Sigma_{\alpha=1}^{s} \frac{\hbar_\alpha \psi_\alpha}{\Sigma_{\alpha=1}^{s} \hbar_\alpha \psi_\alpha} (\frac{1+\omega_\alpha^{-}}{-\omega_\alpha^{-}})^{\lambda}\}^{\frac{1}{\lambda}}}, \frac{-1}{1+\{\Sigma_{\alpha=1}^{s} \frac{\hbar_\alpha \psi_\alpha}{\Sigma_{\alpha=1}^{s} \hbar_\alpha \psi_\alpha} (\frac{1+\mho_\alpha^{-}}{-\mho_\alpha^{-}})^{\lambda}\}^{\frac{1}{\lambda}}} \rangle.
\end{array} \right.
$$

If $r = s + 1$, then there is following result:

$$
\mathrm{BNDPWGA}(\tilde{\aleph}_1, \tilde{\aleph}_2, ..., \tilde{\aleph}_s, \tilde{\aleph}_{s+1}) = \bigotimes_{\alpha=1}^{s} \tilde{\aleph}_\alpha^{\bigwedge \upsilon_\alpha} \bigotimes \tilde{\aleph}_{s+1}^{\bigwedge \upsilon_{s+1}}
$$

$$
= \left\{ \begin{array}{l}
\langle \frac{1}{1+\{\Sigma_{\alpha=1}^{s} \frac{\hbar_\alpha \psi_\alpha}{\Sigma_{\alpha=1}^{s} \hbar_\alpha \psi_\alpha} (\frac{1-\tau_\alpha^{+}}{\tau_\alpha^{+}})^{\lambda}\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_{\alpha=1}^{s} \frac{\hbar_\alpha \psi_\alpha}{\Sigma_{\alpha=1}^{s} \hbar_\alpha \psi_\alpha} (\frac{\omega_\alpha^{+}}{1-\omega_\alpha^{+}})^{\lambda}\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_{\alpha=1}^{s} \frac{\hbar_\alpha \psi_\alpha}{\Sigma_{\alpha=1}^{s} \hbar_\alpha \psi_\alpha} (\frac{\mho_\alpha^{+}}{1-\mho_\alpha^{+}})^{\lambda}\}^{\frac{1}{\lambda}}}, \\
\frac{1}{1+\{\Sigma_{\alpha=1}^{s} \frac{\hbar_\alpha \psi_\alpha}{\Sigma_{\alpha=1}^{s} \hbar_\alpha \psi_\alpha} (\frac{-\tau_\alpha^{-}}{1+\tau_\alpha^{-}})^{\lambda}\}^{\frac{1}{\lambda}}} - 1, \frac{1}{1+\{\Sigma_{\alpha=1}^{s} \frac{\hbar_\alpha \psi_\alpha}{\Sigma_{\alpha=1}^{s} \hbar_\alpha \psi_\alpha} (\frac{1+\omega_\alpha^{-}}{-\omega_\alpha^{-}})^{\lambda}\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{\Sigma_{\alpha=1}^{s} \frac{\hbar_\alpha \psi_\alpha}{\Sigma_{\alpha=1}^{s} \hbar_\alpha \psi_\alpha} (\frac{1+\mho_\alpha^{-}}{-\mho_\alpha^{-}})^{\lambda}\}^{\frac{1}{\lambda}}} \rangle
\end{array} \right.
$$

$$
\otimes_D \left\{ \begin{array}{l}
\langle \frac{1}{1+\{\frac{\hbar_{s+1} \psi_{s+1}}{\Sigma_{\alpha=1}^{s+1} \hbar_\alpha \psi_\alpha} (\frac{1-\tau_{s+1}^{+}}{\tau_{s+1}^{+}})^{\lambda}\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\frac{\hbar_{s+1} \psi_{s+1}}{\Sigma_{\alpha=1}^{s+1} \hbar_\alpha \psi_\alpha} (\frac{\omega_{s+1}^{+}}{1-\omega_{s+1}^{+}})^{\lambda}\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\frac{\hbar_{s+1} \psi_{s+1}}{\Sigma_{\alpha=1}^{s+1} \hbar_\alpha \psi_\alpha} (\frac{\mho_{s+1}^{+}}{1-\mho_{s+1}^{+}})^{\lambda}\}^{\frac{1}{\lambda}}}, \\
\frac{1}{1+\{\frac{\hbar_{s+1} \psi_{s+1}}{\Sigma_{\alpha=1}^{s+1} \hbar_\alpha \psi_\alpha} (\frac{-\tau_\alpha^{-}}{1+\tau_{s+1}^{-}})^{\lambda}\}^{\frac{1}{\lambda}}} - 1, \frac{1}{1+\{\frac{\hbar_{s+1} \psi_{s+1}}{\Sigma_{\alpha=1}^{s+1} \hbar_\alpha \psi_\alpha} (\frac{1+\omega_{s+1}^{-}}{-\omega_{s+1}^{-}})^{\lambda}\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{\frac{\hbar_{s+1} \psi_{s+1}}{\Sigma_{\alpha=1}^{s+1} \hbar_\alpha \psi_\alpha} (\frac{1+\mho_{s+1}^{-}}{-\mho_{s+1}^{-}})^{\lambda}\}^{\frac{1}{\lambda}}} \rangle
\end{array} \right.
$$

$$
= \left\{ \begin{array}{l}
\langle \frac{1}{1+\{\Sigma_{\alpha=1}^{s+1} \frac{\hbar_\alpha \psi_\alpha}{\Sigma_{\alpha=1}^{s+1} \hbar_\alpha \psi_\alpha} (\frac{1-\tau_\alpha^{+}}{\tau_\alpha^{+}})^{\lambda}\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_{\alpha=1}^{s+1} \frac{\hbar_\alpha \psi_\alpha}{\Sigma_{\alpha=1}^{s+1} \hbar_\alpha \psi_\alpha} (\frac{\omega_\alpha^{+}}{1-\omega_\alpha^{+}})^{\lambda}\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_{\alpha=1}^{s+1} \frac{\hbar_\alpha \psi_\alpha}{\Sigma_{\alpha=1}^{s+1} \hbar_\alpha \psi_\alpha} (\frac{\mho_\alpha^{+}}{1-\mho_\alpha^{+}})^{\lambda}\}^{\frac{1}{\lambda}}}, \\
\frac{1}{1+\{\Sigma_{\alpha=1}^{s+1} \frac{\hbar_\alpha \psi_\alpha}{\Sigma_{\alpha=1}^{s+1} \hbar_\alpha \psi_\alpha} (\frac{-\tau_\alpha^{-}}{1+\tau_\alpha^{-}})^{\lambda}\}^{\frac{1}{\lambda}}} - 1, \frac{1}{1+\{\Sigma_{\alpha=1}^{s+1} \frac{\hbar_\alpha \psi_\alpha}{\Sigma_{\alpha=1}^{s+1} \hbar_\alpha \psi_\alpha} (\frac{1+\omega_\alpha^{-}}{-\omega_\alpha^{-}})^{\lambda}\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{\Sigma_{\alpha=1}^{s+1} \frac{\hbar_\alpha \psi_\alpha}{\Sigma_{\alpha=1}^{s+1} \hbar_\alpha \psi_\alpha} (\frac{1+\mho_\alpha^{-}}{-\mho_\alpha^{-}})^{\lambda}\}^{\frac{1}{\lambda}}} \rangle
\end{array} \right.
$$

Hence, Theorem (4.3) is true for $r = s + 1$. Thus, equation (8) holds for all r. □

The BNDPWGA operator also has the following properties:

(1) Idomopotency: Let all the BNNs be $\tilde{b_\alpha} =< \tau_\alpha^{+}, \omega_\alpha^{+}, \mho_\alpha^{+}, \tau_\alpha^{-}, \omega_\alpha^{-}, \mho_\alpha^{-} >= \tilde{\aleph}$ $(\alpha = 1, 2, 3, ..., r)$, where $\psi = (\psi_1, \psi_2, ..., \psi_r)^T$ be the weight vector of $\tilde{\aleph}_\alpha, \psi_\alpha \in [0, 1]$ and $\Sigma_{\alpha=1}\psi_\alpha = 1$. Then, $\mathrm{BNDPWGA}(\tilde{\aleph}_1, \tilde{\aleph}_2, ..., \tilde{\aleph}_r) = \tilde{\aleph}$.

(2) Monotonicity: Let $\tilde{\aleph}_\alpha(\alpha = 1, 2, ..., r)$ and $\tilde{\aleph}_\alpha^{'}(\alpha = 1, 2, ..., r)$ be two families of BNNs, where $\psi = (\psi_1, \psi_2, ..., \psi_r)^T$ be the weight vector of $\tilde{\aleph}_\alpha$ and $\tilde{\aleph}_\alpha^{'}$, $\psi_\alpha \in [0, 1]$ and $\Sigma_{\alpha=1}\psi_\alpha = 1$. For all α ,if $\tilde{\aleph}_\alpha \geq \tilde{\aleph}_\alpha^{'}$, then
$\mathrm{BNDPWGA}(\tilde{\aleph}_1, \tilde{\aleph}_2, ..., \tilde{\aleph}_r) \geq \mathrm{BNDPWGA}(\tilde{\aleph}_1^{'}, \tilde{\aleph}_2^{'}, ..., \tilde{\aleph}_r^{'})$.

(3) Boundedness: Let $\tilde{\aleph}_\alpha =< \tau_\alpha^+, \omega_\alpha^+, \mho_\alpha^+, \tau_\alpha^-, \omega_\alpha^-, \mho_\alpha^- > (\alpha = 1, , 2, 3, ..., r)$ be a family of BNNs, where $\psi = (\psi_1, \psi_2, ..., \psi_r)^T$ be the weight vector of $\tilde{\aleph}_\alpha$, $\psi_\alpha \in [0, 1]$ and $\Sigma_{\alpha=1}\psi_\alpha = 1$. Therefore, we have BNDPWGA$(\tilde{\aleph}^-, \tilde{\aleph}^-, ..., \tilde{\aleph}^-)$

$\leq BNDPWGA(\tilde{\aleph}_1, \tilde{\aleph}_2, ..., \tilde{\aleph}_r) \leq BNDPWGA(\tilde{\aleph}^+, \tilde{\aleph}^+, ..., \tilde{\aleph}^+)$,

where $\tilde{\aleph}^- =< \tau_{\tilde{\aleph}^-}^+, \omega_{\tilde{\aleph}^-}^+, \mho_{\tilde{\aleph}^-}^+, \tau_{\tilde{\aleph}^-}^-, \omega_{\tilde{\aleph}^-}^-, \mho_{\tilde{\aleph}^-}^- >$

$$= \begin{cases} < min(\tau_1^+, \tau_2^+, ..., \tau_r^+), max(\omega_1^+, \omega_2^+, ..., \omega_r^+), max(\mho_1^+, \mho_2^+, ..., \mho_r^+), \\ max(\tau_1^-, \tau_2^-, ..., \tau_r^-), min(\omega_\alpha^-, \omega_\alpha^-, ..., \omega_\alpha^-), min(\mho_1^-, \mho_2^-, ..., \mho_r^-) > \end{cases}$$

and

$\tilde{\aleph}^+ =< \tau_{\tilde{\aleph}^+}^+, \omega_{\tilde{\aleph}^+}^+, \mho_{\tilde{\aleph}^+}^+, \tau_{\tilde{\aleph}^+}^-, \omega_{\tilde{\aleph}^+}^-, \mho_{\tilde{\aleph}^+}^- >$

$$= \begin{cases} < max(\tau_1^+, \tau_2^+, ..., \tau_r^+), min(\omega_1^+, \omega_2^+, ..., \omega_r^+), min(\mho_1^+, \mho_2^+, ..., \mho_r^+), \\ min(\tau_1^-, \tau_2^-, ..., \tau_r^-), max(\omega_\alpha^-, \omega_\alpha^-, ...\omega_\alpha^-), max(\mho_1^-, \mho_2^-, ..., \mho_r^-) > . \end{cases}$$

Proof.

(1) Since $\tilde{\aleph}_\alpha =< \tau_\alpha^+, \omega_\alpha^+, \mho_\alpha^+, \tau_\alpha^-, \omega_\alpha^-, \mho_\alpha^- >= \tilde{\aleph}(\alpha = 1, , 2, 3, ..., r)$. Then, the following result can be obtained by using equation (8).

BNDPWGA$(\tilde{\aleph}_1, \tilde{\aleph}_2, ..., \tilde{\aleph}_r) = \bigotimes_{\alpha=1}^r \tilde{\aleph}_\alpha^{\bigwedge^{\upsilon_\alpha}}$

$$= \begin{cases} \langle \frac{1}{1+\{\Sigma_{\alpha=1}^r \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^r\hbar_\alpha\psi_\alpha}(\frac{1-\tau_\alpha^+}{\tau_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_{\alpha=1}^r \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^r\hbar_\alpha\psi_\alpha}(\frac{\omega_\alpha^+}{1-\omega_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_{\alpha=1}^r \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^r\hbar_\alpha\psi_\alpha}(\frac{\mho_\alpha^+}{1-\mho_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ \frac{1}{1+\{\Sigma_{\alpha=1}^r \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^r\hbar_\alpha\psi_\alpha}(\frac{-\tau_\alpha^-}{1+\tau_\alpha^-})^\lambda\}^{\frac{1}{\lambda}}} - 1, \frac{-1}{1+\{\Sigma_{\alpha=1}^r \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^r\hbar_\alpha\psi_\alpha}(\frac{1+\omega_\alpha^-}{-\omega_\alpha^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1+\{\Sigma_{\alpha=1}^r \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^r\hbar_\alpha\psi_\alpha}(\frac{1+\mho_\alpha^-}{-\mho_\alpha^-})^\lambda\}^{\frac{1}{\lambda}}} \rangle \end{cases}$$

$$= \begin{cases} \langle \frac{1}{1+\{(\frac{1-\tau^+}{\tau^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{(\frac{\omega^+}{1-\omega^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{(\frac{\mho^+}{1-\mho^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ \frac{1}{1+\{(\frac{-\tau^-}{1+\tau^-})^\lambda\}^{\frac{1}{\lambda}}} - 1, \frac{-1}{1+\{(\frac{1+\omega^-}{-\omega^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1+\{(\frac{1+\mho^-}{-\mho^-})^\lambda\}^{\frac{1}{\lambda}}} \rangle \end{cases}$$

$=< \tau^+, \omega^+, \mho^+, \tau^-, \omega^-, \mho^- >= \tilde{\aleph}$.

Hence, BNDPWGA$(\tilde{\aleph}_1, \tilde{\aleph}_2, ..., \tilde{\aleph}_r) = \tilde{\aleph}$ holds.

(2) The property is obvious based on the equation (8).

(3) Let $\tilde{\aleph}^- =< \tau_{\tilde{\aleph}^-}^+, \omega_{\tilde{\aleph}^-}^+, \mho_{\tilde{\aleph}^-}^+, \tau_{\tilde{\aleph}^-}^-, \omega_{\tilde{\aleph}^-}^-, \mho_{\tilde{\aleph}^-}^-$ and $\tilde{\aleph}^+ =< \tau_{\tilde{\aleph}^+}^+, \omega_{\tilde{\aleph}^+}^+, \mho_{\tilde{\aleph}^+}^+, \tau_{\tilde{\aleph}^+}^-, \omega_{\tilde{\aleph}^+}^-, \mho_{\tilde{\aleph}^+}^- >$. There are the following inequalities:

$$\frac{1}{1+\{\Sigma_{\alpha=1}^r \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^r\hbar_\alpha\psi_\alpha}(\frac{1-\tau_{\tilde{\aleph}^-}^+}{1_{\tilde{\aleph}^-}^+})^\lambda\}^{\frac{1}{\lambda}}} \leq \frac{1}{1+\{\Sigma_{\alpha=1}^r \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^r\hbar_\alpha\psi_\alpha}(\frac{1-\tau_\alpha^+}{\tau_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}} \leq \frac{1}{1+\{\Sigma_{\alpha=1}^r \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^r\hbar_\alpha\psi_\alpha}(\frac{1-\tau_{\tilde{\aleph}^+}^+}{\tau_{\tilde{\aleph}^+}^+})^\lambda\}^{\frac{1}{\lambda}}},$$

$$1 - \frac{1}{1+\{\Sigma_{\alpha=1}^r \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^r\hbar_\alpha\psi_\alpha}(\frac{\omega_{\tilde{\aleph}^+}^+}{1-\omega_{\tilde{\aleph}^+}^+})^\lambda\}^{\frac{1}{\lambda}}} \leq 1 - \frac{1}{1+\{\Sigma_{\alpha=1}^r \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^r\hbar_\alpha\psi_\alpha}(\frac{\omega_\alpha^+}{1-\omega_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}} \leq 1 - \frac{1}{1+\{\Sigma_{\alpha=1}^r \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^r\hbar_\alpha\psi_\alpha}(\frac{\omega_{\tilde{\aleph}^-}^+}{1-\omega_{\tilde{\aleph}^-}^+})^\lambda\}^{\frac{1}{\lambda}}},$$

$$1 - \frac{1}{1+\{\Sigma_{\alpha=1}^r \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^r\hbar_\alpha\psi_\alpha}(\frac{\mho_{\tilde{\aleph}^+}^+}{1-\mho_{\tilde{\aleph}^+}^+})^\lambda\}^{\frac{1}{\lambda}}} \leq 1 - \frac{1}{1+\{\Sigma_{\alpha=1}^r \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^r\hbar_\alpha\psi_\alpha}(\frac{\mho_\alpha^+}{1-\mho_\alpha^+})^\lambda\}^{\frac{1}{\lambda}}} \leq 1 - \frac{1}{1+\{\Sigma_{\alpha=1}^r \frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^r\hbar_\alpha\psi_\alpha}(\frac{\mho_{\tilde{\aleph}^-}^+}{1-\mho_{\tilde{\aleph}^-}^+})^\lambda\}^{\frac{1}{\lambda}}},$$

$$\frac{1}{1+\{\Sigma_{\alpha=1}^{r}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{r}\hbar_\alpha\psi_\alpha}(\frac{-\tau^{-}_{\tilde{\aleph}^{+}}}{1+\tau^{-}_{\tilde{\aleph}^{+}}})^{\lambda}\}^{\frac{1}{\lambda}}}-1$$
$$\leq \frac{1}{1+\{\Sigma_{\alpha=1}^{r}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{r}\hbar_\alpha\psi_\alpha}(\frac{-\tau^{-}_{\alpha}}{1+\tau^{-}_{\alpha}})^{\lambda}\}^{\frac{1}{\lambda}}}-1 \leq \frac{1}{1+\{\Sigma_{\alpha=1}^{r}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{r}\hbar_\alpha\psi_\alpha}(\frac{-\tau^{-}_{\tilde{\aleph}^{-}}}{1+\tau^{-}_{\tilde{\aleph}^{-}}})^{\lambda}\}^{\frac{1}{\lambda}}}-1,$$
$$\frac{-1}{1+\{\Sigma_{\alpha=1}^{r}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{r}\hbar_\alpha\psi_\alpha}(\frac{1+\omega^{-}_{\tilde{\aleph}^{-}}}{-\omega^{-}_{\tilde{\aleph}^{-}}})^{\lambda}\}^{\frac{1}{\lambda}}} \leq \frac{-1}{1+\{\Sigma_{\alpha=1}^{r}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{r}\hbar_\alpha\psi_\alpha}(\frac{1+\omega^{-}_{\alpha}}{-\omega^{-}_{\alpha}})^{\lambda}\}^{\frac{1}{\lambda}}} \leq \frac{-1}{1+\{\Sigma_{\alpha=1}^{r}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{r}\hbar_\alpha\psi_\alpha}(\frac{1+\omega^{-}_{\tilde{\aleph}^{+}}}{-\omega^{-}_{\tilde{\aleph}^{+}}})^{\lambda}\}^{\frac{1}{\lambda}}},$$
$$\frac{-1}{1+\{\Sigma_{\alpha=1}^{r}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{r}\hbar_\alpha\psi_\alpha}(\frac{1+\mho^{-}_{\tilde{\aleph}^{-}}}{-\mho^{-}_{\tilde{\aleph}^{-}}})^{\lambda}\}^{\frac{1}{\lambda}}} \leq \frac{-1}{1+\{\Sigma_{\alpha=1}^{r}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{r}\hbar_\alpha\psi_\alpha}(\frac{1+\mho^{-}_{\alpha}}{-\mho^{-}_{\alpha}})^{\lambda}\}^{\frac{1}{\lambda}}} \leq \frac{-1}{1+\{\Sigma_{\alpha=1}^{r}\frac{\hbar_\alpha\psi_\alpha}{\Sigma_{\alpha=1}^{r}\hbar_\alpha\psi_\alpha}(\frac{1+\mho^{-}_{\tilde{\aleph}^{+}}}{-\mho^{-}_{\tilde{\aleph}^{+}}})^{\lambda}\}^{\frac{1}{\lambda}}}.$$

Hence, BNDPWGA$(\tilde{\aleph}^{-}, \tilde{\aleph}^{-}, ..., \tilde{\aleph}^{-}) \leq$ BNDPWGA$(\tilde{\aleph}_1, \tilde{\aleph}_2, ..., \tilde{\aleph}_r) \leq$ BNDPWGA$(\tilde{\aleph}^{+}, \tilde{\aleph}^{+}, ..., \tilde{\aleph}^{+})$ holds.

$\square$

Example 4.4. Let $\tilde{\aleph}_1 =< 0.6, 0.7, 0.3, -0.6, -0.3, -0.5 >$, $\tilde{\aleph}_2 =< 0.5, 0.4, 0.6, -0.6, -0.7, -0.3 >$, $\tilde{\aleph}_3 =< 0.6, 0.7, 0.4, -0.9, -0.7, -0.7 >$ and $\tilde{\aleph}_4 =< 0.2, 0.6, 0.8, -0.6, -0.3, -0.9 >$ be four BNNs and let the weight vector of BNNs $\tilde{\aleph}_\alpha(\alpha = 1, 2, 3, 4)$ be $\psi = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8})^T$. Now, by Definition 4.1, $\hbar_1 = 1$, $\hbar_2 = s(\tilde{\aleph}_1) = 0.4667$, $\hbar_3 = s(\tilde{\aleph}_1)s(\tilde{\aleph}_2) = 0.2256$ and $\hbar_4 = s(\tilde{\aleph}_1)s(\tilde{\aleph}_2)s(\tilde{\aleph}_3) = 0.1128$. $\psi_1 = \frac{1}{2}$, $\psi_2 = \frac{1}{4}$, $\psi_3 = \frac{1}{8}$ and $\psi_4 = \frac{1}{8}$ are the weight of $\tilde{\aleph}_\alpha(\alpha = 1, 2, 3, 4)$ such that $\Sigma_{\alpha=1}\psi_\alpha = 1$. By Definition 4.2, $\frac{\hbar_1\psi_1}{\Sigma_{\alpha=1}^{4}\hbar_\alpha\psi_\alpha} = 0.7588$, $\frac{\hbar_2\psi_2}{\Sigma_{\alpha=1}^{4}\hbar_\alpha\psi_\alpha} = 0.1771$, $\frac{\hbar_3\psi_3}{\Sigma_{\alpha=1}^{4}\hbar_\alpha\psi_\alpha} = 0.0428$, $\frac{\hbar_4\psi_4}{\Sigma_{\alpha=1}^{4}\hbar_\alpha\psi_\alpha} = 0.0214$. Then, by Theorem 4.3, for $\lambda = 3$
$BNDPWGA(\tilde{\aleph}_1, \tilde{\aleph}_2, \tilde{\aleph}_3, \tilde{\aleph}_4) =$

$$\left\{\begin{array}{l}
\langle\frac{1}{1+\{0.7588(\frac{1-0.6}{0.6})^3+0.1771(\frac{1-0.5}{0.5})^3+0.0428(\frac{1-0.6}{0.6})^3+0.0214(\frac{1-0.2}{0.2})^3\}^{\frac{1}{3}}}, \\
1- \\
\frac{1}{1+\{0.7588(\frac{0.7}{1-0.7})^3+0.1771(\frac{0.4}{1-0.4})^3+0.0428(\frac{0.7}{1-0.7})^3+0.0214(\frac{0.6}{1-0.6})^3\}^{\frac{1}{3}}}, \\
1- \\
\frac{1}{1+\{0.7588(\frac{0.3}{1-0.3})^3+0.1771(\frac{0.6}{1-0.6})^3+0.0428(\frac{0.4}{1-0.4})^3+0.0214(\frac{0.8}{1-0.8})^3\}^{\frac{1}{3}}}, \\
\frac{1}{1+\{0.7588(\frac{0.6}{1-0.6})^3+0.1771(\frac{0.6}{1-0.6})^3+0.0428(\frac{0.9}{1-0.9})^3+0.0214(\frac{0.6}{1-0.6})^3\}^{\frac{1}{3}}} \\
-1, \\
\frac{-1}{1+\{0.7588(\frac{1-0.3}{0.3})^3+0.1771(\frac{1-0.7}{0.7})^3+0.0428(\frac{1-0.7}{0.7})^3+0.0214(\frac{1-0.3}{0.3})^3\}^{\frac{1}{3}}}, \\
\frac{-1}{1+\{0.7588(\frac{1-0.5}{0.5})^3+0.1771(\frac{1-0.3}{0.3})^3+0.0428(\frac{1-0.7}{0.7})^3+0.0214(\frac{1-0.9}{0.9})^3\}^{\frac{1}{3}}}\rangle,
\end{array}\right.$$

$BNDPWGA(\tilde{\aleph}_1, \tilde{\aleph}_2, \tilde{\aleph}_3, \tilde{\aleph}_4) = \langle 0.4519, 0.6852, 0.5591, -0.7649, -0.3175, -0.4092\rangle$.

4.2. Bipolar neutrosophic Dombi prioritized ordered weighted geometric aggregation operator

In this section, we propose bipolar neutrosophic Dombi prioritized ordered weighted geometric aggregation operator (BNDPOWGA) which is more useful to pervious defined bipolar neutrosophic Dombi prioritized weighted geometric aggregation operator (BNDPWGA). Because in this operator one more parameter added, called an ordered parameter. This mean that BNDPOWGA is more informative than (BNDPWGA).

Definition 4.5. Let $\tilde{\aleph}_\alpha =< \tau^+_\alpha, \omega^+_\alpha, \mho^+_\alpha, \tau^-_\alpha, \omega^-_\alpha, \mho^-_\alpha >$ $(\alpha = 1, 2, 3, ..., r)$ be a family of BNNs. A mapping BNDPOWGA: $U^r \to U$ is called BNDPOWGA operator, if it satisfies

$$BNDPOWGA(\tilde{\aleph}_1, \tilde{\aleph}_2, ..., \tilde{\aleph}_r) = \bigotimes_{\beta=1}^r \tilde{\aleph}_{\sigma(\beta)}^{\bigwedge^{\upsilon_\beta}} = \tilde{\aleph}_{\sigma(1)}^{\bigwedge_D^{\upsilon_1}} \otimes_D \tilde{\aleph}_{\sigma(2)}^{\bigwedge_D^{\upsilon_2}} \otimes_D ... \otimes_D \tilde{\aleph}_{\sigma(r)}^{\bigwedge_D^{\upsilon_r}} \tag{9}$$

where $\upsilon_\beta = \frac{\hbar_\beta \psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta \psi_\beta}$, $\hbar_\beta = \prod_{\alpha=1}^{\beta-1} s(\tilde{\aleph}_\alpha)$ $(\alpha = 2, 3, ..., r)$, $\hbar_1 = 1$ and $s(\tilde{\aleph}_\alpha)$ is the value of score for $\tilde{\aleph}_\alpha$. where σ is permutation that orders the elements: $\tilde{\aleph}_{\sigma(1)} \geq \tilde{\aleph}_{\sigma(2)} \geq ... \geq \tilde{\aleph}_{\sigma(r)}$. where $\psi = (\psi_1, \psi_2, ..., \psi_r)^T$ is the weight vector of $\tilde{\aleph}_\alpha$ $(\alpha = 1, 2, ..., r)$, $\psi_\alpha \in [0, 1]$ and $\Sigma_{\alpha=1}\psi_\alpha = 1$.

Theorem 4.6. *Let $\tilde{\aleph}_\alpha =< \tau^+_\alpha, \omega^+_\alpha, \mho^+_\alpha, \tau^-_\alpha, \omega^-_\alpha, \mho^-_\alpha >$ $(\alpha = 1, 2, 3, ..., r)$ be a family of BNNs and $\psi = (\psi_1, \psi_2, ..., \psi_r)^T$ be the weight vector of $\tilde{\aleph}_\alpha$, $\psi_\alpha \in [0, 1]$ and $\Sigma_{\alpha=1}\psi_\alpha = 1$. Then, the value aggregated by using BNDPOWGA operator is still a BNN, which is calculated by using the following formula*

$$BNDPOWGA(\tilde{\aleph}_1, \tilde{\aleph}_2, ..., \tilde{\aleph}_r) = \tilde{\aleph}_{\sigma(1)}^{\bigwedge_D^{\upsilon_1}} \otimes_D \tilde{\aleph}_{\sigma(2)}^{\bigwedge_D^{\upsilon_2}} \otimes_D ... \otimes_D \tilde{\aleph}_{\sigma(r)}^{\bigwedge_D^{\upsilon_r}}$$

$$= \begin{cases} \langle \frac{1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{1-\tau^+_\beta}{\tau^+_\beta})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{\omega^+_\beta}{1-\omega^+_\beta})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{\mho^+_\beta}{1-\mho^+_\beta})^\lambda\}^{\frac{1}{\lambda}}}, \\ \frac{1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{-\tau^-_\beta}{1+\tau^-_\beta})^\lambda\}^{\frac{1}{\lambda}}} - 1, \frac{-1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{1+\omega^-_\beta}{-\omega^-_\beta})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{1+\mho^-_\beta}{-\mho^-_\beta})^\lambda\}^{\frac{1}{\lambda}}} \rangle \end{cases} \tag{10}$$

Proof. If $r = 2$ based on Definition 4.2 for the Dombi operations of BNNs , the following result can be obtained:

$$\text{BNDPOWGA } (\tilde{\aleph}_1, \tilde{\aleph}_2) = \tilde{\aleph}_{\sigma(1)}^{\bigwedge_D^{\upsilon_1}} \otimes_D \tilde{\aleph}_{\sigma(2)}^{\bigwedge_D^{\upsilon_2}}$$

$$= \begin{cases} \langle \frac{1}{1+\{\frac{\hbar_1\psi_1}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{1-\tau^+_1}{\tau^+_1})^\lambda + \frac{\hbar_2\psi_2}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{1-\tau^+_2}{\tau^+_2})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\frac{\hbar_1\psi_1}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{\omega^+_1}{1-\omega^+_1})^\lambda + \frac{\hbar_2\psi_2}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{\omega^+_2}{1-\omega^+_2})^\lambda\}^{\frac{1}{\lambda}}}, \\ 1 - \frac{1}{1+\{\frac{\hbar_1\psi_1}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{\mho^+_1}{1-\mho^+_1})^\lambda + \frac{\hbar_2\psi_2}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{\mho^+_2}{1-\mho^+_2})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{\frac{\hbar_1\psi_1}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{-\tau^-_1}{1+\tau^-_1})^\lambda + \frac{\hbar_2\psi_2}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{-\tau^-_2}{1+\tau^-_2})^\lambda\}^{\frac{1}{\lambda}}} - 1, \\ \frac{-1}{1+\{\frac{\hbar_1\psi_1}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{1+\omega^-_1}{-\omega^-_1})^\lambda + \frac{\hbar_2\psi_2}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{1+\omega^-_2}{-\omega^-_2})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1+\{\frac{\hbar_1\psi_1}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{1+\mho^-_1}{-\mho^-_1})^\lambda + \frac{\hbar_2\psi_2}{\hbar_1\psi_1+\hbar_2\psi_2}(\frac{1+\mho^-_2}{-\mho^-_2})^\lambda\}^{\frac{1}{\lambda}}} \rangle, \end{cases}$$

$$= \begin{cases} \langle \frac{1}{1+\{\Sigma_{\beta=1}^2 \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^2 \hbar_\beta\psi_\beta}(\frac{1-\tau^+_\beta}{\tau^+_\beta})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_{\beta=1}^2 \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^2 \hbar_\beta\psi_\beta}(\frac{\omega^+_\beta}{1-\omega^+_\beta})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_{\beta=1}^2 \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^2 \hbar_\beta\psi_\beta}(\Sigma_{\beta=1}^2 \frac{\mho^+_\beta}{1-\mho^+_\beta})^\lambda\}^{\frac{1}{\lambda}}}, \\ \frac{1}{1+\{\Sigma_{\beta=1}^2 \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^2 \hbar_\beta\psi_\beta}(\frac{-\tau^-_\beta}{1+\tau^-_\beta})^\lambda\}^{\frac{1}{\lambda}}} - 1, \frac{-1}{1+\{\Sigma_{\beta=1}^2 \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^2 \hbar_\beta\psi_\beta}(\frac{1+\omega^-_\beta}{-\omega^-_\beta})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1+\{\Sigma_{\beta=1}^2 \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^2 \hbar_\beta\psi_\beta}(\frac{1+\mho^-_\beta}{-\mho^-_\beta})^\lambda\}^{\frac{1}{\lambda}}} \rangle. \end{cases}$$

If $r = s$, based on equation (10), then we have got the following equation:

$$\text{BNDPOWGA}(\tilde{\aleph}_1, \tilde{\aleph}_2, ..., \tilde{\aleph}_s) = \bigotimes_{\beta=1}^s \tilde{\aleph}_\beta^{\bigwedge^{\upsilon_\beta}}$$

$$= \begin{cases} \langle \frac{1}{1+\{\Sigma_{\beta=1}^s \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^s \hbar_\beta\psi_\beta}(\frac{1-\tau^+_\beta}{\tau^+_\beta})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_{\beta=1}^s \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^s \hbar_\beta\psi_\beta}(\frac{\omega^+_\beta}{1-\omega^+_\beta})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_{\beta=1}^s \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^s \hbar_\beta\psi_\beta}(\frac{\mho^+_\beta}{1-\mho^+_\beta})^\lambda\}^{\frac{1}{\lambda}}}, \\ \frac{1}{1+\{\Sigma_{\beta=1}^s \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^s \hbar_\beta\psi_\beta}(\frac{-\tau^-_\beta}{1+\tau^-_\beta})^\lambda\}^{\frac{1}{\lambda}}} - 1, \frac{-1}{1+\{\Sigma_{\beta=1}^s \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^s \hbar_\beta\psi_\beta}(\frac{1+\omega^-_\beta}{-\omega^-_\beta})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1+\{\Sigma_{\beta=1}^s \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^s \hbar_\beta\psi_\beta}(\frac{1+\mho^-_\beta}{-\mho^-_\beta})^\lambda\}^{\frac{1}{\lambda}}} \rangle. \end{cases}$$

If $r = s + 1$, then there is following result:

$$\text{BNDPOWGA}(\tilde{\aleph}_1, \tilde{\aleph}_2, ..., \tilde{\aleph}_s, \tilde{\aleph}_{s+1}) = \bigotimes_{\beta=1}^{s} \tilde{\aleph}_{\sigma(\beta)}^{\bigwedge^{\upsilon_\beta}} \bigotimes \tilde{\aleph}_{\sigma(s+1)}^{\bigwedge^{\upsilon_{s+1}}}$$

$$= \begin{cases} \langle \frac{1}{1+\{\Sigma_{\beta=1}^{s} \frac{\hbar_\beta \psi_\beta}{\Sigma_{\beta=1}^{s} \hbar_\beta \psi_\beta} (\frac{1-\tau_\beta^+}{\tau_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_{\beta=1}^{s} \frac{\hbar_\beta \psi_\beta}{\Sigma_{\beta=1}^{s} \hbar_\beta \psi_\beta} (\frac{\omega_\beta^+}{1-\omega_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_{\beta=1}^{s} \frac{\hbar_\beta \psi_\beta}{\Sigma_{\beta=1}^{s} \hbar_\beta \psi_\beta} (\frac{\mho_\beta^+}{1-\mho_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ \frac{1}{1+\{\Sigma_{\beta=1}^{s} \frac{\hbar_\beta \psi_\beta}{\Sigma_{\beta=1}^{s} \hbar_\beta \psi_\beta} (\frac{-\tau_\beta^-}{1+\tau_\beta^-})^\lambda\}^{\frac{1}{\lambda}}} - 1, \frac{1}{1+\{\Sigma_{\beta=1}^{s} \frac{\hbar_\beta \psi_\beta}{\Sigma_{\beta=1}^{s} \hbar_\beta \psi_\beta} (\frac{1+\omega_\beta^-}{-\omega_\beta^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{\Sigma_{\beta=1}^{s} \frac{\hbar_\beta \psi_\beta}{\Sigma_{\beta=1}^{s} \hbar_\beta \psi_\beta} (\frac{1+\mho_\beta^-}{-\mho_\beta^-})^\lambda\}^{\frac{1}{\lambda}}} \rangle \end{cases}$$

$$\otimes_D \begin{cases} \langle \frac{1}{1+\{\frac{\hbar_{s+1} \psi_{s+1}}{\Sigma_{\beta=1}^{s+1} \hbar_\beta \psi_\beta} (\frac{1-\tau_{s+1}^+}{\tau_{s+1}^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\frac{\hbar_{s+1} \psi_{s+1}}{\Sigma_{\beta=1}^{s+1} \hbar_\beta \psi_\beta} (\frac{\omega_{s+1}^+}{1-\omega_{s+1}^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\frac{\hbar_{s+1} \psi_{s+1}}{\Sigma_{\beta=1}^{s+1} \hbar_\beta \psi_\beta} (\frac{\mho_{s+1}^+}{1-\mho_{s+1}^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ \frac{1}{1+\{\frac{\hbar_{s+1} \psi_{s+1}}{\Sigma_{\beta=1}^{s+1} \hbar_\beta \psi_\beta} (\frac{-\tau_{s+1}^-}{1+\tau_{s+1}^-})^\lambda\}^{\frac{1}{\lambda}}} - 1, \frac{1}{1+\{\frac{\hbar_{s+1} \psi_{s+1}}{\Sigma_{\beta=1}^{s+1} \hbar_\beta \psi_\beta} (\frac{1+\omega_{s+1}^-}{-\omega_{s+1}^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{\frac{\hbar_{s+1} \psi_{s+1}}{\Sigma_{\beta=1}^{s+1} \hbar_\beta \psi_\beta} (\frac{1+\mho_{s+1}^-}{-\mho_{s+1}^-})^\lambda\}^{\frac{1}{\lambda}}} \rangle \end{cases}$$

$$= \begin{cases} \langle \frac{1}{1+\{\Sigma_{\beta=1}^{s+1} \frac{\hbar_\beta \psi_\beta}{\Sigma_{\beta=1}^{s+1} \hbar_\beta \psi_\beta} (\frac{1-\tau_\beta^+}{\tau_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_{\beta=1}^{s+1} \frac{\hbar_\beta \psi_\beta}{\Sigma_{\beta=1}^{s+1} \hbar_\beta \psi_\beta} (\frac{\omega_\beta^+}{1-\omega_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_{\beta=1}^{s+1} \frac{\hbar_\beta \psi_\beta}{\Sigma_{\beta=1}^{s+1} \hbar_\beta \psi_\beta} (\frac{\mho_\beta^+}{1-\mho_\beta^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ \frac{1}{1+\{\Sigma_{\beta=1}^{s+1} \frac{\hbar_\beta \psi_\beta}{\Sigma_{\beta=1}^{s+1} \hbar_\beta \psi_\beta} (\frac{-\tau_\beta^-}{1+\tau_\beta^-})^\lambda\}^{\frac{1}{\lambda}}} - 1, \frac{1}{1+\{\Sigma_{\beta=1}^{s+1} \frac{\hbar_\beta \psi_\beta}{\Sigma_{\beta=1}^{s+1} \hbar_\beta \psi_\beta} (\frac{1+\omega_\beta^-}{-\omega_\beta^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{1}{1+\{\Sigma_{\beta=1}^{s+1} \frac{\hbar_\beta \psi_\beta}{\Sigma_{\beta=1}^{s+1} \hbar_\beta \psi_\beta} (\frac{1+\mho_\beta^-}{-\mho_\beta^-})^\lambda\}^{\frac{1}{\lambda}}} \rangle \end{cases}$$

Hence, Theorem 4.6 is true for $r = s + 1$. Thus, Equation (10) holds for all r. □

The BNDPOWGA operator also has the following properties:

(1) Idompotency: Let all the BNNs be $\tilde{\aleph}_\alpha = < \tau_\alpha^+, \omega_\alpha^+, \mho_\alpha^+, \tau_\alpha^-, \omega_\alpha^-, \mho_\alpha^- > = \tilde{\aleph}$ $(\alpha = 1, 2, 3, ..., r)$, where $\psi = (\psi_1, \psi_2, ..., \psi_r)^T$ be the weight vector of $\tilde{\aleph}_\alpha$, $\psi_\alpha \in [0, 1]$ and $\Sigma_{\alpha=1}\psi_\alpha = 1$.
Then, $\text{BNDPOWGA}(\tilde{\aleph}_1, \tilde{\aleph}_2, ..., \tilde{\aleph}_r) = \tilde{\aleph}$.

(2) Monotonicity: Let $\tilde{\aleph}_\alpha (\alpha = 1, 2, ..., r)$ and $\tilde{\aleph}^{,}_\alpha (\alpha = 1, 2, ..., r)$ be two families of BNNs, where $\psi = (\psi_1, \psi_2, ..., \psi_r)^T$ be the weight vector of $\tilde{\aleph}_\alpha$ and $\tilde{\aleph}^{,}_\alpha$, $\psi_\alpha \in [0, 1]$ and $\Sigma_{\alpha=1}\psi_\alpha = 1$. For all α ,if $\tilde{\aleph}_\alpha \geq \tilde{\aleph}^{,}_\alpha$, then $\text{BNDPOWGA}(\tilde{\aleph}_1, \tilde{\aleph}_2, ..., \tilde{\aleph}_r) \geq \text{BNDPOWGA}(\tilde{\aleph}^{,}_1, \tilde{\aleph}^{,}_2, ..., \tilde{\aleph}^{,}_r)$.

(3) Boundedness: Let $\tilde{\aleph}_\alpha = < \tau_\alpha^+, \omega_\alpha^+, \mho_\alpha^+, \tau_\alpha^-, \omega_\alpha^-, \mho_\alpha^- >$ $(\alpha = 1, , 2, 3, ..., r)$ be a family of BNNs, where $\psi = (\psi_1, \psi_2, ..., \psi_r)^T$ be the weight vector of $\tilde{\aleph}_\alpha$, $\psi_\alpha \in [0, 1]$ and $\Sigma_{\alpha=1}\psi_\alpha = 1$. Therefore, we have $\text{BNDPOWGA}(\tilde{\aleph}^-, \tilde{\aleph}^-, ..., \tilde{\aleph}^-) \leq \text{BNDPOWGA}(\tilde{\aleph}_1, \tilde{\aleph}_2, ..., \tilde{\aleph}_r) \leq \text{BNDPOWGA}(\tilde{\aleph}^+, \tilde{\aleph}^+, ..., \tilde{\aleph}^+)$, where $\tilde{\aleph}^- = < \tau_{\tilde{\aleph}^-}^+, \omega_{\tilde{\aleph}^-}^+, \mho_{\tilde{\aleph}^-}^+, \tau_{\tilde{\aleph}^-}^-, \omega_{\tilde{\aleph}^-}^-, \mho_{\tilde{\aleph}^-}^- >$

$$= \begin{cases} < min(\tau_1^+, \tau_2^+, ..., \tau_r^+), max(\omega_1^+, \omega_2^+, ..., \omega_r^+), max(\mho_1^+, \mho_2^+, ..., \mho_r^+), \\ max(\tau_1^-, \tau_2^-, ..., \tau_r^-), min(\omega_\alpha^-, \omega_\alpha^-, ..., \omega_\alpha^-), min(\mho_1^-, \mho_2^-, ..., \mho_r^-) > \end{cases}$$

and

$$\tilde{\aleph}^+ = < \tau_{\tilde{\aleph}^+}^+, \omega_{\tilde{\aleph}^+}^+, \mho_{\tilde{\aleph}^+}^+, t_{\tilde{\aleph}^+}^-, \omega_{\tilde{\aleph}^+}^-, \mho_{\tilde{\aleph}^+}^- >$$

$$= \begin{cases} < max(\tau_1^+, \tau_2^+, ..., \tau_r^+), min(\omega_1^+, \omega_2^+, ..., \omega_r^+), min(\mho_1^+, \mho_2^+, ..., \mho_r^+), \\ min(\tau_1^-, \tau_2^-, ..., \tau_r^-), max(\omega_\alpha^-, \omega_\alpha^-, ..., \omega_\alpha^-), max(\mho_1^-, \mho_2^-, ..., \mho_r^-) > . \end{cases}$$

Proof.

(1) Since $\tilde{\aleph}_\alpha =< \tau^+_\alpha, \omega^+_\alpha, \mho^+_\alpha, \tau^-_\alpha, \omega^-_\alpha, \mho^-_\alpha >= \tilde{\aleph}(\alpha = 1, , 2, 3, ..., r)$. Then, the following result can be obtained by using Equation (10):

$$\text{BNDPOWGA}(\tilde{\aleph}_1, \tilde{\aleph}_2, ..., \tilde{\aleph}_r) = \bigotimes_{\beta=1}^r \tilde{\aleph}_\beta^{\bigwedge^{\upsilon_\beta}}$$
$$= \begin{cases} \langle \frac{1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{1-\tau^+_\beta}{\tau^+_\beta})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{\omega^+_\beta}{1-\omega^+_\beta})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{\mho^+_\beta}{1-\mho^+_\beta})^\lambda\}^{\frac{1}{\lambda}}}, \\ \frac{1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{-\tau^-_\beta}{1+\tau^-_\beta})^\lambda\}^{\frac{1}{\lambda}}} - 1, \frac{-1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{1+\omega^-_\beta}{-\omega^-_\beta})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{1+\mho^-_\beta}{-\mho^-_\beta})^\lambda\}^{\frac{1}{\lambda}}} \rangle \end{cases}$$
$$= \begin{cases} \langle \frac{1}{1+\{(\frac{1-\tau^+}{\tau^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{(\frac{\omega^+}{1-\omega^+})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{(\frac{\mho^+}{1-\mho^+})^\lambda\}^{\frac{1}{\lambda}}}, \\ \frac{1}{1+\{(\frac{-\tau^-}{1+\tau^-})^\lambda\}^{\frac{1}{\lambda}}} - 1, \frac{-1}{1+\{(\frac{1+\omega^-}{-\omega^-})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1+\{(\frac{1+\mho^-}{-\mho^-})^\lambda\}^{\frac{1}{\lambda}}} \rangle \end{cases} =< \tau^+, \omega^+, \mho^+, \tau^-, \omega^-, \mho^- >= \tilde{\aleph}.$$

Hence, $\text{BNDPOWGA}(\tilde{\aleph}_1, \tilde{\aleph}_2, ..., \tilde{\aleph}_r) = \tilde{\aleph}$ holds.

(2) The property is obvious based on the equation (10).

(3) Let $\tilde{\aleph}^- =< \tau^+_{\tilde{\aleph}^-}, \omega^+_{\tilde{\aleph}^-}, \mho^+_{\tilde{\aleph}^-}, \tau^-_{\tilde{\aleph}^-}, \omega^-_{\tilde{\aleph}^-}, \mho^-_{\tilde{\aleph}^-} >$ and $\tilde{\aleph}^+ =< \tau^+_{\tilde{\aleph}^+}, \omega^+_{\tilde{\aleph}^+}, \mho^+_{\tilde{\aleph}^+}, \tau^-_{\tilde{\aleph}^+}, \omega^-_{\tilde{\aleph}^+}, \mho^-_{\tilde{\aleph}^+} >$. There are the following inequalities:

$$\frac{1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{1-\tau^+_{\tilde{\aleph}^-}}{1^+_{\tilde{\aleph}^-}})^\lambda\}^{\frac{1}{\lambda}}} \leq \frac{1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{1-\tau^+_\beta}{\tau^+_\beta})^\lambda\}^{\frac{1}{\lambda}}} \leq \frac{1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{1-\tau^+_{\tilde{\aleph}^+}}{\tau^+_{\tilde{\aleph}^+}})^\lambda\}^{\frac{1}{\lambda}}},$$

$$1 - \frac{1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{\omega^+_{\tilde{\aleph}^+}}{1-\omega^+_{\tilde{\aleph}^+}})^\lambda\}^{\frac{1}{\lambda}}} \leq 1 - \frac{1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{\omega^+_\beta}{1-\omega^+_\beta})^\lambda\}^{\frac{1}{\lambda}}} \leq 1 - \frac{1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{\omega^+_{\tilde{\aleph}^-}}{1-\omega^+_{\tilde{\aleph}^-}})^\lambda\}^{\frac{1}{\lambda}}},$$

$$1 - \frac{1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{\mho^+_{\tilde{\aleph}^+}}{1-\mho^+_{\tilde{\aleph}^+}})^\lambda\}^{\frac{1}{\lambda}}} \leq 1 - \frac{1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{\mho^+_\beta}{1-\mho^+_\beta})^\lambda\}^{\frac{1}{\lambda}}} \leq 1 - \frac{1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{\mho^+_{\tilde{\aleph}^-}}{1-\mho^+_{\tilde{\aleph}^-}})^\lambda\}^{\frac{1}{\lambda}}},$$

$$\frac{1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{-\tau^-_{\tilde{\aleph}^+}}{1+\tau^-_{\tilde{\aleph}^+}})^\lambda\}^{\frac{1}{\lambda}}} - 1 \leq \frac{1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{-\tau^-_\beta}{1+\tau^-_\beta})^\lambda\}^{\frac{1}{\lambda}}} - 1 \leq \frac{1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{-\tau^-_{\tilde{\aleph}^-}}{1+\tau^-_{\tilde{\aleph}^-}})^\lambda\}^{\frac{1}{\lambda}}} - 1,$$

$$\frac{-1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{1+\omega^-_{\tilde{\aleph}^-}}{-\omega^-_{\tilde{\aleph}^-}})^\lambda\}^{\frac{1}{\lambda}}} \leq \frac{-1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{1+\omega^-_\beta}{-\omega^-_\beta})^\lambda\}^{\frac{1}{\lambda}}}$$
$$\leq \frac{-1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{1+\omega^-_{\tilde{\aleph}^+}}{-\omega^-_{\tilde{\aleph}^+}})^\lambda\}^{\frac{1}{\lambda}}},$$

$$\frac{-1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{1+\mho^-_{\tilde{\aleph}^-}}{-\mho^-_{\tilde{\aleph}^-}})^\lambda\}^{\frac{1}{\lambda}}} \leq \frac{-1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{1+\mho^-_\beta}{-\mho^-_\beta})^\lambda\}^{\frac{1}{\lambda}}} \leq \frac{-1}{1+\{\Sigma_{\beta=1}^r \frac{\hbar_\beta\psi_\beta}{\Sigma_{\beta=1}^r \hbar_\beta\psi_\beta}(\frac{1+\mho^-_{\tilde{\aleph}^+}}{-\mho^-_{\tilde{\aleph}^+}})^\lambda\}^{\frac{1}{\lambda}}}.$$

Hence, $\text{BNDPOWGA}(\tilde{\aleph}^-, \tilde{\aleph}^-, ..., \tilde{\aleph}^-) \leq \text{BNDPOWGA}(\tilde{\aleph}_1, \tilde{\aleph}_2, ..., \tilde{\aleph}_r) \leq \text{BNDPOWGA}(\tilde{\aleph}^+, \tilde{\aleph}^+, ..., \tilde{\aleph}^+)$ holds. □

Example 4.7 Let $\tilde{\aleph}_1 =< 0.6, 0.7, 0.3, -0.6, -0.3, -0.5 >$, $\tilde{\aleph}_2 =< 0.5, 0.4, 0.6, -0.6, -0.7, -0.3 >$, $\tilde{\aleph}_3 =< 0.6, 0.7, 0.4, -0.9, -0.7, -0.7 >$ and $\tilde{\aleph}_4 =< 0.2, 0.6, 0.8, -0.6, -0.3, -0.9 >$ be four BNNs and let the weight vector of BNNs $\tilde{\aleph}_\alpha(\alpha = 1, 2, 3, 4)$ be $\psi = (\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{8})^T$. Now, by Definition 4.2, $\hbar_1 = 1$, $\hbar_2 = s(\tilde{\aleph}_1) = 0.4667$, $\hbar_3 = s(\tilde{\aleph}_1)s(\tilde{\aleph}_2) = 0.2256$ and $\hbar_4 = s(\tilde{\aleph}_1)s(\tilde{\aleph}_2)s(\tilde{\aleph}_3) = 0.1128$. $\psi_1 = \frac{1}{2}$, $\psi_2 = \frac{1}{4}$, $\psi_3 = \frac{1}{8}$ and $\psi_4 = \frac{1}{8}$ are the weights of $\tilde{\aleph}_\alpha(\alpha = 1, 2, 3, 4)$ such that $\Sigma_{\alpha=1}\psi_\alpha = 1$. By Definition 4.2, $\frac{\hbar_1\psi_1}{\Sigma_{\beta=1}^4\hbar_\beta\psi_\beta} = 0.7588$, $\frac{\hbar_2\psi_2}{\Sigma_{\beta=1}^4\hbar_\beta\psi_\beta} = 0.1771$, $\frac{\hbar_3\psi_3}{\Sigma_{\beta=1}^4\hbar_\beta\psi_\beta} = 0.0428$, $\frac{\hbar_4\psi_4}{\Sigma_{\beta=1}^4\hbar_\beta\psi_\beta} = 0.0214$. Then, by Theorem 4.2, for $\lambda = 3$

$BNDPOWGA(\tilde{\aleph}_1, \tilde{\aleph}_2, \tilde{\aleph}_3, \tilde{\aleph}_4) =$

$$
\begin{cases}
\langle \frac{1}{1+\{0.7588(\frac{1-0.6}{0.6})^3+0.1771(\frac{1-0.5}{0.5})^3+0.0428(\frac{1-0.6}{0.6})^3+0.0214(\frac{1-0.2}{0.2})^3\}^{\frac{1}{3}}}, \\
1- \\
\frac{1}{1+\{0.7588(\frac{0.7}{1-0.7})^3+0.1771(\frac{0.4}{1-0.4})^3+0.0428(\frac{0.7}{1-0.7})^3+0.0214(\frac{0.6}{1-0.6})^3\}^{\frac{1}{3}}}, \\
1- \\
\frac{1}{1+\{0.7588(\frac{0.4}{1-0.4})^3+0.1771(\frac{0.6}{1-0.6})^3+0.0428(\frac{0.3}{1-0.3})^3+0.0214(\frac{0.8}{1-0.8})^3\}^{\frac{1}{3}}}, \\
\frac{1}{1+\{0.7588(\frac{0.9}{1+0.9})^3+0.1771(\frac{0.6}{1+0.6})^3+0.0428(\frac{0.6}{1+0.6})^3+0.0214(\frac{0.6}{1+0.6})^3\}^{\frac{1}{3}}} \\
-1, \\
\frac{-1}{1+\{0.7588(\frac{1-0.7}{0.7})^3+0.1771(\frac{1-0.7}{0.7})^3+0.0428(\frac{1-0.3}{0.3})^3+0.0214(\frac{1-0.3}{0.3})^3\}^{\frac{1}{3}}}, \\
\frac{-1}{1+\{0.7588(\frac{1-0.7}{0.7})^3+0.1771(\frac{1-0.3}{0.3})^3+0.0428(\frac{1-0.5}{0.5})^3+0.0214(\frac{1-0.9}{0.9})^3\}^{\frac{1}{3}}} \rangle,
\end{cases}
$$

$BNDPOWGA(\tilde{\aleph}_1, \tilde{\aleph}_2, \tilde{\aleph}_3, \tilde{\aleph}_4) = \langle 0.4519, 0.6852, 0.5651, -0.8915, -0.5098, -0.4292 \rangle$.

5. Model for MADM using bipolar neutrosophic information

In this section, three comprehensive MADM methods are extended based on the proposed BNDPWGA and BNDPOWGA operators.

For MADM model with bipolar neutrosophic fuzzy information, let $A = \{A_1, A_2, ..., A_r\}$ be a set of alternatives and $C = \{C_1, C_2, ..., C_r\}$ be a set of attributes. For BNDPWGA and BNDPOWGA operators, there is a prioritization between the attributes expressed by the linear ordering $C_1 \succ C_2 \succ ... \succ C_r$, indicates attribute C_η has a higher priority than C_ρ, if $\eta < \rho$. Let $\psi = (\psi_1, \psi_2, ..., \psi_r)^T$ be the weight vector of attributes such that $\psi_\beta > 0$, $\Sigma_{\beta=1}\psi_\beta = 1(\beta = 1, 2, ..., r)$ and ψ_β refers to the weight of attribute C_β. Suppose that $N = (\tilde{\aleph}_{\alpha\beta})_{s\times r} = (\tau^+_{\alpha\beta}, \omega^+_{\alpha\beta}, \mho^+_{\alpha\beta}, \tau^-_{\alpha\beta}, \omega^-_{\alpha\beta}, \mho^-_{\alpha\beta})_{s\times r}(\alpha = 1, 2, ..., s)(\beta = 1, 2, ..., r)$ is BNN decision matrix, where $\tau^+_{\alpha\beta}, \omega^+_{\alpha\beta}, \mho^+_{\alpha\beta}$ indicates the truth membership degree, indeterminacy membership degree and falsity membership degree of alternative A_α under attribute C_β with respect to positive preferences and $\tau^-_{\alpha\beta}, \omega^-_{\alpha\beta}, \mho^-_{\alpha\beta}$ indicates the truth membership degree, indeterminacy membership degree and falsity membership degree of alternative A_α under attribute C_β with respect to negative preferences. We have conditions $\tau^+_{\alpha\beta}, \omega^+_{\alpha\beta}, \mho^+_{\alpha\beta} \in [0, 1]$ and $\tau^-_{\alpha\beta}, \omega^-_{\alpha\beta}, \mho^-_{\alpha\beta} \in [-1, 0]$ such that $0 \leq \tau^+_{\alpha\beta} + \omega^+_{\alpha\beta} + \mho^+_{\alpha\beta} - \tau^-_{\alpha\beta} - \omega^-_{\alpha\beta} - \mho^-_{\alpha\beta} \leq 6$ for $(\alpha = 1, 2, ..., s)$ $(\beta = 1, 2, ..., r)$.

An algorithm is constructed based on proposed bipolar neutrosophic Dombi aggregation operators which solves MADM problems.

Algorithm
Step 1 Collect information on the bipolar neutrosophic evaluation .
Step 2 Calculate score and the accuracy values of collected information.
The score values $s(\tilde{\aleph}_{\alpha\beta})$ and accuracy values $a(\tilde{\aleph}_{\alpha\beta})$ of alternatives A_α can be calculated by using Equations (1) and (2).
Step 3 Reordering the value of each attribute by using the comparison method.
Step 4 Derive the collective BNN $\tilde{\aleph}_\alpha$ $(\alpha = 1, 2, ..., s)$ for the alternative $A_\alpha(\alpha = 1, 2, ..., s)$.
By using method (1), calculate the values of $\hbar_{\alpha\beta}$ $(\alpha = 1, 2, ..., s)$ $(\beta, \delta = 2, 3, ..., r)$ as follows:

$$\hbar_{\alpha\beta} = \prod_{\alpha\delta=1}^{\alpha\beta-1} s(\tilde{\aleph}_{\alpha\delta})(\alpha = 1, 2, ..., s)(\beta = 2, 3, ..., r), \tag{11}$$

$$\hbar_{\alpha 1} = 1(\alpha = 1, 2, ..., s), \tag{12}$$

and utilize BNDPWGA operator to calculate the collective BNN for each alternative, then

$$\tilde{\aleph}_\alpha = \text{BNDPWGA}(\tilde{\aleph}_{\alpha 1}, \tilde{\aleph}_{\alpha 2}, ..., \tilde{\aleph}_{\alpha r}) = \bigotimes_{\beta=1}^{r} \tilde{\aleph}_{\alpha\beta}^{\bigwedge^{\upsilon_{\alpha\beta}}}$$

$$= \begin{cases} \langle \frac{1}{1+\{\Sigma_\beta^r \frac{\hbar_{\alpha\beta}\psi_{\alpha\beta}}{\Sigma_\beta^r \hbar_{\alpha\beta}\psi_{\alpha\beta}}(\frac{1-\tau^+_{\alpha\beta}}{\tau^+_{\alpha\beta}})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_\beta^r \frac{\hbar_{\alpha\beta}\psi_{\alpha\beta}}{\Sigma_\beta^r \hbar_{\alpha\beta}\psi_{\alpha\beta}}(\frac{\omega^+_{\alpha\beta}}{1-\omega^+_{\alpha\beta}})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_\beta^r \frac{\hbar_{\alpha\beta}\psi_{\alpha\beta}}{\Sigma_\beta^r \hbar_{\alpha\beta}\psi_{\alpha\beta}}(\frac{\mho^+_{\alpha\beta}}{1-\mho^+_{\alpha\beta}})^\lambda\}^{\frac{1}{\lambda}}}, \\ \frac{1}{1+\{\Sigma_\beta^r \frac{\hbar_{\alpha\beta}\psi_{\alpha\beta}}{\Sigma_\beta^r \hbar_{\alpha\beta}\psi_{\alpha\beta}}(\frac{-\tau^-_{\alpha\beta}}{1+\tau^-_{\alpha\beta}})^\lambda\}^{\frac{1}{\lambda}}} - 1, \frac{-1}{1+\{\Sigma_\beta^r \frac{\hbar_{\alpha\beta}\psi_{\alpha\beta}}{\Sigma_\beta^r \hbar_{\alpha\beta}\psi_{\alpha\beta}}(\frac{1+\omega^-_{\alpha\beta}}{-\omega^-_{\alpha\beta}})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1+\{\Sigma_\beta^r \frac{\hbar_{\alpha\beta}\psi_{\alpha\beta}}{\Sigma_\beta^r \hbar_{\alpha\beta}\psi_{\alpha\beta}}(\frac{1+\mho^-_{\alpha\beta}}{-\mho^-_{\alpha\beta}})^\lambda\}^{\frac{1}{\lambda}}} \rangle, \end{cases} \tag{13}$$

where $\upsilon_{\alpha\beta} = \frac{\hbar_{\alpha\beta}\psi_{\alpha\beta}}{\Sigma_{j=1}^n \hbar_{\alpha\beta}\psi_{\alpha\beta}}$ and $\psi = (\psi_1, \psi_2, ..., \psi_r)^T$ is the weight vector $(\alpha = 1, 2, ..., s)$ such that $\psi_\beta \in [0, 1]$ and $\Sigma_{\beta=1}\psi_\beta = 1$.
By using method (2) calculate the values of $\hbar_{\alpha\gamma}$ $(\alpha = 1, 2, ..., s)$ $(\gamma = 1, 2, ..., r)$ as follows:

$$\hbar_{\alpha\gamma} = \prod_{\alpha\delta=1}^{\alpha\delta-1} s(\tilde{\aleph}_{\alpha\delta})(\alpha = 1, 2, ..., s)(\beta = 2, 3, ..., r), \tag{14}$$

$$\hbar_{\alpha 1} = 1(\alpha = 1, 2, ..., s), \tag{15}$$

and utilize BNDPOWGA operator to calculate the collective BNN for each alternative, then

$$\tilde{\aleph}_\alpha = \text{BNDPOWGA}(\tilde{\aleph}_{\alpha 1}, \tilde{\aleph}_{\alpha 2}, ..., \tilde{\aleph}_{\alpha r}) = \bigotimes_{\gamma=1}^{r} \tilde{\aleph}_{\alpha\gamma}^{\bigwedge^{\upsilon_{\alpha\gamma}}}$$

$$= \begin{cases} \langle \frac{1}{1+\{\Sigma_{\gamma=1}^r \frac{\hbar_{\alpha\gamma}\psi_{\alpha\gamma}}{\Sigma_{\gamma=1}^r \hbar_{\alpha\gamma}\psi_{\alpha\gamma}}(\frac{1-\tau^+_{\alpha\gamma}}{\tau^+_{\alpha\gamma}})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_{\gamma=1}^r \frac{\hbar_{\alpha\gamma}\psi_{\alpha\gamma}}{\Sigma_{\gamma=1}^r \hbar_{\alpha\gamma}\psi_{\alpha\gamma}}(\frac{\omega^+_{\alpha\gamma}}{1-\omega^+_{\alpha\gamma}})^\lambda\}^{\frac{1}{\lambda}}}, 1 - \frac{1}{1+\{\Sigma_{\gamma=1}^r \frac{\hbar_{\alpha\gamma}\psi_{\alpha\gamma}}{\Sigma_{\gamma=1}^r \hbar_{\alpha\gamma}\psi_{\alpha\gamma}}(\frac{\mho^+_{\alpha\gamma}}{1-\mho^+_{\alpha\gamma}})^\lambda\}^{\frac{1}{\lambda}}}, \\ \frac{1}{1+\{\Sigma_{\gamma=1}^r \frac{\hbar_{\alpha\gamma}\psi_{\alpha\gamma}}{\Sigma_{\gamma=1}^r \hbar_{\alpha\gamma}\psi_{\alpha\gamma}}(\frac{-\tau^-_{\alpha\gamma}}{1+\tau^-_{\alpha\gamma}})^\lambda\}^{\frac{1}{\lambda}}} - 1, \frac{-1}{1+\{\Sigma_{\gamma=1}^r \frac{\hbar_{\alpha\gamma}\psi_{\alpha\gamma}}{\Sigma_{\gamma=1}^r \hbar_{\alpha\gamma}\psi_{\alpha\gamma}}(\frac{1+\omega^-_{\alpha\gamma}}{-\omega^-_{\alpha\gamma}})^\lambda\}^{\frac{1}{\lambda}}}, \frac{-1}{1+\{\Sigma_{\gamma=1}^r \frac{\hbar_{\alpha\gamma}\psi_{\alpha\gamma}}{\Sigma_{\gamma=1}^r \hbar_{\alpha\gamma}\psi_{\alpha\gamma}}(\frac{1+\mho^-_{\alpha\gamma}}{-\mho^-_{\alpha\gamma}})^\lambda\}^{\frac{1}{\lambda}}} \rangle, \end{cases} \tag{16}$$

where $\upsilon_{\alpha\gamma} = \frac{\hbar_{\alpha\gamma}\psi_{\alpha\gamma}}{\Sigma_{\gamma=1}^r \hbar_{\alpha\gamma}\psi_{\alpha\gamma}}$ and σ is permutation that orders the elements: $\tilde{\aleph}_{\sigma(\alpha 1)} \geq \tilde{\aleph}_{\sigma(\alpha 2)} \geq ... \geq \tilde{\aleph}_{\sigma(\alpha r)}$. where $\psi = (\psi_1, \psi_2, ..., \psi_r)^T$ is the weight vector such that $\psi_\gamma \in [0, 1]$ and $\Sigma_{\gamma=1}\psi_\gamma = 1$.
Step 5 Calculate the score values $s(\tilde{\aleph}_\alpha)$ $(\alpha = 1, 2, ..., s)$ of BNNs $\tilde{\aleph}_\alpha$ $(\alpha = 1, 2, ..., s)$ to rank all the alternatives A_α $(\alpha = 1, 2, ..., s)$ and then select favorable one(s). If score values $\tilde{\aleph}_\alpha$ and $\tilde{\aleph}_\beta$ of BNNs are equal, then we calculate accuracy values $a(\tilde{\aleph}_\alpha)$ and $a(\tilde{\aleph}_\beta)$ of BNNs $\tilde{\aleph}_\alpha$ and $\tilde{\aleph}_\beta$, respectively and then rank the alternatives A_α and A_β as accuracy values $a(\tilde{\aleph}_\alpha)$ and $a(\tilde{\aleph}_\beta)$.
Step 6 Rank all the alternatives $A_\alpha(\alpha = 1, 2, ..., s)$ and select favorable one(s).
Step 7 End.

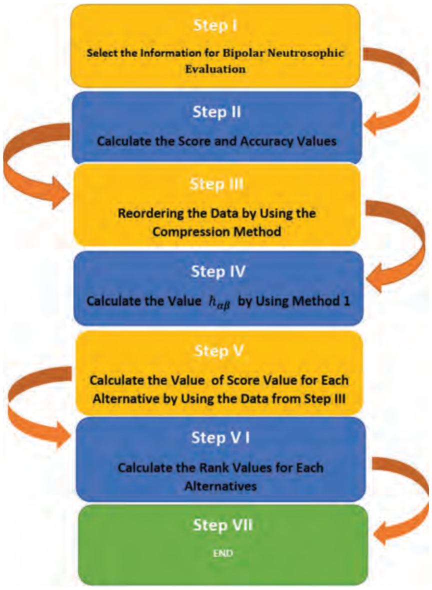

Fig. 1. Flow Chart.

The flow chart of proposed algorithm is given below in Figure 1.

6. Numerical example

A numerical example of the selection of cultivating crops taken from Deli et al. [10] is provided here. Furthermore, parametric analysis and comparative analysis confirm the flexibility and effectiveness of the proposed methods. In order to increase the production of agriculture, an agriculture department considers selecting a crop for cultivating in the farm. Four cultivating crops A_1, A_2, A_3, and A_4 are selected for further evaluation through preliminary screening. The agriculture department decided to invite four group experts to evaluate information. The expert group consists of investment experts, land experts, weather experts and labour experts. The four cultivating crops are evaluated by experts on the basis of four attributes or criteria: cost (C_1), nutrition of land (C_2), effect of weather (C_3), labor (C_4). For the proposed methods BNDPWGA and BNDPOWGA operators, the prioritization relation for the attributes is given as: $C_1 \succ C_2 \succ C_3 \succ C_4$. These attributes are interactive and interlinked.

Step 1 Collect information on bipolar neutrosophic evaluation. The information collected from expert discussion on evaluation is given in Table 1.

Step 2 Calculate score and accuracy values of collected information.

For each alternative A_α under attribute C_β, the score values $s(\tilde{\aleph}_{\alpha\beta})$ and accuracy values $a(\tilde{\aleph}_{\alpha\beta})$ can be calculated based on equations (1) and (2). The score values $s(\tilde{\aleph}_{\alpha\beta})$ and accuracy values $a(\tilde{\aleph}_{\alpha\beta})$ are shown in Tables 2 and 3, respectively.

Step 3 Reordering information on evaluation under each attribute.

Step 4 Derive the collective BNN $\tilde{\aleph}_\alpha$ $(\alpha = 1, 2, ..., s)$ for the alternative $A_\alpha (\alpha = 1, 2, ..., s)$. Method (1) Calculate the values of $\hbar_{\alpha\beta}$ $(\alpha = 1, 2, ..., s)$ $(\beta = 1, 2, ..., r)$ using equations (11) and (12) as follows:

$$\hbar_{\alpha\beta} = \begin{pmatrix} 1.0000 & 0.4667 & 0.2333 & 0.1167 \\ 1.0000 & 0.4667 & 0.1711 & 0.1141 \\ 1.0000 & 0.5167 & 0.3014 & 0.1507 \\ 1.0000 & 0.4833 & 0.2256 & 0.1278 \end{pmatrix}$$

and utilize BNDPWGA operator using equation (13) and supporting $\lambda = 7$ to calculate the collective BNN for each alternative, then

$$\tilde{\aleph}_1 = \langle 0.1608, 0.6933, 0.5760, -0.7305, -0.2840, -0.4601 \rangle;$$

$$\tilde{\aleph}_2 = \langle 0.6325, 0.6923, 0.7578, -0.6983, -0.1610, -0.1007 \rangle;$$

$$\tilde{\aleph}_3 = \langle 0.2387, 0.4263, 0.3979, -0.5917, -0.2807, -0.2631 \rangle;$$

$$\tilde{\aleph}_4 = \langle 0.3533, 0.6914, 0.7013, -0.7934, -0.5494, -0.1037 \rangle.$$

By using method (2) calculate the values of $\hbar_{\alpha\beta}$ $(\alpha = 1, 2, ..., s)$ $(\beta = 1, 2, ..., r)$ using equations (14) and (15) as follows:

$$\hbar_{\alpha\beta} = \begin{pmatrix} 1.0000 & 0.4667 & 0.2333 & 0.1167 \\ 1.0000 & 0.4667 & 0.1711 & 0.1141 \\ 1.0000 & 0.5167 & 0.3014 & 0.1507 \\ 1.0000 & 0.4833 & 0.2256 & 0.1278 \end{pmatrix}$$

and utilize BNDPOWGA operator using equation (16) and supporting $\lambda = 7$ to calculate the collective BNN for each alternative, then

$$\tilde{\aleph}_1 = \langle 0.1608, 0.6933, 0.5788, -0.7937, -0.2999, -0.4601 \rangle;$$

$$\tilde{\aleph}_2 = \langle 0.5612, 0.5900, 0.7046, -0.6067, -0.1244, -0.1441 \rangle;$$

Table 1
Bipolar neutrosophic evaluation information

	C_1	C_2	C_3	C_4
A_1	$\langle 0.5, 0.7, 0.2, -0.7, -0.3, -0.6\rangle$	$\langle 0.4, 0.4, 0.5, -0.7, -0.8, -0.4\rangle$	$\langle 0.7, 0.7, 0.5, -0.8, -0.7, -0.6\rangle$	$\langle 0.1, 0.5, 0.7, -0.5, -0.2, -0.8\rangle$
A_2	$\langle 0.9, 0.7, 0.5, -0.7, -0.7, -0.1\rangle$	$\langle 0.7, 0.6, 0.8, -0.7, -0.5, -0.1\rangle$	$\langle 0.9, 0.4, 0.6, -0.1, -0.7, -0.5\rangle$	$\langle 0.5, 0.2, 0.7, -0.5, -0.1, -0.9\rangle$
A_3	$\langle 0.3, 0.4, 0.2, -0.6, -0.3, -0.7\rangle$	$\langle 0.2, 0.2, 0.2, -0.4, -0.7, -0.4\rangle$	$\langle 0.9, 0.5, 0.5, -0.6, -0.5, -0.2\rangle$	$\langle 0.7, 0.5, 0.3, -0.4, -0.2, -0.2\rangle$
A_4	$\langle 0.9, 0.7, 0.2, -0.8, -0.6, -0.1\rangle$	$\langle 0.3, 0.5, 0.2, -0.5, -0.5, -0.2\rangle$	$\langle 0.5, 0.4, 0.5, -0.1, -0.7, -0.2\rangle$	$\langle 0.4, 0.2, 0.8, -0.5, -0.5, -0.6\rangle$

Table 2
Score values $s(\tilde{\aleph}_{\alpha\beta})$

	C_1	C_2	C_3	C_4
A_1	0.4667	0.5000	0.5000	0.4000
A_2	0.4667	0.3667	0.6667	0.5167
A_3	0.5167	0.5833	0.5000	0.4833
A_4	0.4833	0.4667	0.5667	0.5000

Table 3
Accuracy values $a(\tilde{\aleph}_{\alpha\beta})$

	C_1	C_2	C_3	C_4
A_1	0.2000	−0.4000	0	−0.3000
A_2	−0.2000	−0.7000	0.7000	0.2000
A_3	0.2000	0	0	0.2000
A_4	0	−0.2000	0.1000	−0.3000

$$\tilde{\aleph}_3 = \langle 0.4097, 0.5978, 0.7582, -0.7181, -0.5555, -0.1441\rangle;$$

$$\tilde{\aleph}_4 = \langle 0.2072, 0.4162, 0.3979, -0.5511, -0.2903, -0.2629\rangle.$$

Step 5 Calculate the score values $s(\tilde{\aleph}_\alpha)$ $(\alpha = 1, 2, ..., s)$ of BNNs $\tilde{\aleph}_\alpha$ $(\alpha = 1, 2, ..., s)$ for each alternatives A_α $(\alpha = 1, 2, ..., s)$. The score values $s(\tilde{\aleph}_\alpha)$ is calculated by using equation (1).

Method 1 The following score values are obtained by using the BNDPWGA operator.
$s(\tilde{\aleph}_1) = 0.3175$; $s(\tilde{\aleph}_2) = 0.2910$; $s(\tilde{\aleph}_3) = 0.3944$; $s(\tilde{\aleph}_4) = 0.3034$.

Method 2 The following score values are obtained by using the BNDPOWGA operator.
$s(\tilde{\aleph}_1) = 0.3091$; $s(\tilde{\aleph}_2) = 0.3214$; $s(\tilde{\aleph}_3) = 0.3992$; $s(\tilde{\aleph}_4) = 0.3398$.
Step 6 Rank all the alternatives $A_\alpha(\alpha = 1, 2, ..., s)$ and select favorable one(s).

The alternative can be ranked in descending order based on the comparison method, and favorable alternative can be selected.
Method 1 The ranking order based on score values is obtained by using BNDPWGA operator: $A_3 \succ A_1 \succ A_4 \succ A_2$. Thus, A_3 is favorable.
Method 2 The ranking order based on score values is obtained by using BNDPOWGA operator: $A_3 \succ A_4 \succ A_2 \succ A_1$. Thus, A_3 is favorable.
Step 7 End.

7. Parametric analysis and comparative analysis

This section describes effect of parametric λ on decision making results and comparison between proposed methods and existing methods.

7.1. Analysis on the effect of parameter λ on decision making results

This subsection discusses the effect of parameter λ in detail.
First, effect of parameter λ on the proposed operators is as follows.
Table 5 shows that the corresponding ranking orders with respect to the BNDPWGA operator are changed as the value of λ changing from 1 to 10.
Table 5 shows that ranking order is stable and the corresponding favorable alternative remains identical, when the value of λ is changed for BNDPWGA operator. For, $1 \le \lambda \le 10$ the corresponding ranking order is $A_3 \succ A_1 \succ A_4 \succ A_2$, then favorable one is A_3. As a result, favorable stable alternative is A_3. The behavior of BNDPWGA operator is shown in Figure 2.

Table 4
Reordering bipolar neutrosophic evaluation information

	C_1	C_2	C_3	C_4
A_1	$\langle 0.7, 0.7, 0.5, -0.8, -0.7, -0.6\rangle$	$\langle 0.4, 0.4, 0.5, -0.7, -0.8, -0.4\rangle$	$\langle 0.5, 0.7, 0.2, -0.7, -0.3, -0.6\rangle$	$\langle 0.1, 0.5, 0.7, -0.5, -0.2, -0.8\rangle$
A_2	$\langle 0.9, 0.4, 0.6, -0.1, -0.7, -0.5\rangle$	$\langle 0.5, 0.2, 0.7, -0.5, -0.1, -0.9\rangle$	$\langle 0.9, 0.7, 0.5, -0.7, -0.7, -0.1\rangle$	$\langle 0.7, 0.6, 0.8, -0.7, -0.5, -0.1\rangle$
A_3	$\langle 0.2, 0.2, 0.2, -0.4, -0.7, -0.4\rangle$	$\langle 0.3, 0.4, 0.2, -0.6, -0.3, -0.7\rangle$	$\langle 0.9, 0.5, 0.5, -0.6, -0.5, -0.2\rangle$	$\langle 0.7, 0.5, 0.3, -0.4, -0.2, -0.2\rangle$
A_4	$\langle 0.5, 0.4, 0.5, -0.1, -0.7, -0.2\rangle$	$\langle 0.4, 0.2, 0.8, -0.5, -0.5, -0.6\rangle$	$\langle 0.9, 0.7, 0.2, -0.8, -0.6, -0.1\rangle$	$\langle 0.3, 0.5, 0.2, -0.5, -0.5, -0.2\rangle$

Table 5
Ranking orders with parameter of BNDPWGA operator

λ	$s(\tilde{\aleph}_1)$	$s(\tilde{\aleph}_2)$	$s(\tilde{\aleph}_3)$	$s(\tilde{\aleph}_4)$	$BNDPWGA$
1	0.4426	0.4250	0.4947	0.4380	$A_3 \succ A_1 \succ A_4 \succ A_2$
2	0.4090	0.3784	0.4653	0.3852	$A_3 \succ A_1 \succ A_4 \succ A_2$
3	0.3773	0.3423	0.4422	0.3499	$A_3 \succ A_1 \succ A_4 \succ A_2$
4	0.3547	0.3210	0.4249	0.3299	$A_3 \succ A_1 \succ A_4 \succ A_2$
5	0.3388	0.3073	0.4121	0.3176	$A_3 \succ A_1 \succ A_4 \succ A_2$
6	0.3269	0.2979	0.4023	0.3093	$A_3 \succ A_1 \succ A_4 \succ A_2$
7	0.3175	0.2910	0.3944	0.3034	$A_3 \succ A_1 \succ A_4 \succ A_2$
8	0.3099	0.2857	0.3880	0.2988	$A_3 \succ A_1 \succ A_4 \succ A_2$
9	0.3036	0.2817	0.3826	0.2953	$A_3 \succ A_1 \succ A_4 \succ A_2$
10	0.2984	0.2784	0.3781	0.2924	$A_3 \succ A_1 \succ A_4 \succ A_2$

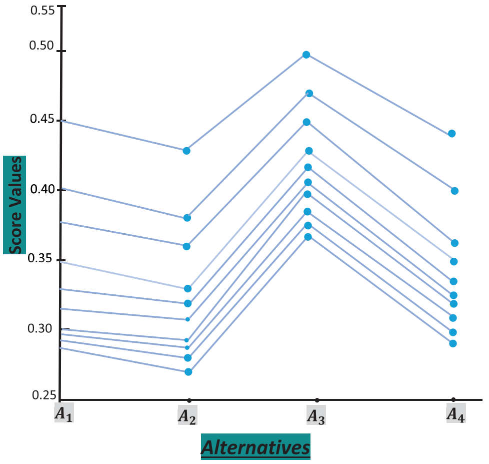

Fig. 2. BNDPWGA operator

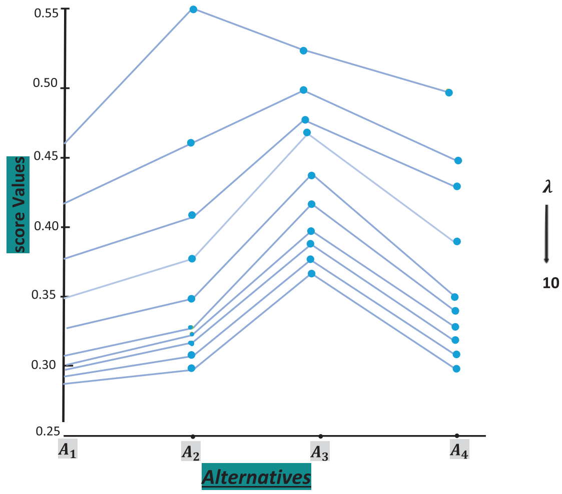

Fig. 3. BNDPOWGA operator

Table 6 shows that the corresponding ranking orders with respect to the BNDPOWGA operator are changed as the value of λ changing from 1 to 10.
Table 6 shows that the ranking order is different when the value of λ is changed for BNDPOWGA operator. For, $1 \leq \lambda \leq 2$, the corresponding ranking orders are $A_2 \succ A_3 \succ A_4 \succ A_1$ and $A_3 \succ A_2 \succ A_4 \succ A_1$. It follows that the favorable alternatives are A_2 and A_3, respectively. For, $3 \leq \lambda \leq 10$, the corresponding ranking order is $A_3 \succ A_4 \succ A_2 \succ A_1$. As a result, favorable one is A_3. The behavior of BNDPOWGA operator is shown in Figure 3.

7.2. Comparative analysis

In this subsection, a comparative analysis of the proposed methods based on proposed bipolar neutro-

Table 6
Ranking orders with parameter of BNDPOWGA operator

λ	$s(\tilde{\aleph}_1)$	$s(\tilde{\aleph}_2)$	$s(\tilde{\aleph}_3)$	$s(\tilde{\aleph}_4)$	$BNDPOWGA$
1	0.4655	0.5416	0.5292	0.5041	$A_2 \succ A_3 \succ A_4 \succ A_1$
2	0.4159	0.4545	0.4947	0.4522	$A_3 \succ A_4 \succ A_2 \succ A_1$
3	0.3747	0.4031	0.4647	0.4140	$A_3 \succ A_4 \succ A_2 \succ A_1$
4	0.3490	0.3710	0.4408	0.3894	$A_3 \succ A_4 \succ A_2 \succ A_1$
5	0.3317	0.3491	0.4229	0.3663	$A_3 \succ A_4 \succ A_2 \succ A_1$
6	0.3189	0.3332	0.4094	0.3512	$A_3 \succ A_4 \succ A_2 \succ A_1$
7	0.3091	0.3214	0.3398	0.3992	$A_3 \succ A_4 \succ A_2 \succ A_1$
8	0.3015	0.3123	0.3912	0.3308	$A_3 \succ A_4 \succ A_2 \succ A_1$
9	0.2955	0.3052	0.3849	0.3237	$A_3 \succ A_4 \succ A_2 \succ A_1$
10	0.2907	0.2994	0.3798	0.3179	$A_3 \succ A_4 \succ A_2 \succ A_1$

Table 7
Ranking orders obtained by different methods

Methods	Rankings
A_ψ operator [11]	$A_3 \succ A_4 \succ A_2 \succ A_1$
G_ψ operator [11]	$A_3 \succ A_4 \succ A_2 \succ A_1$
BN-TOPSIS method(ψ_1) [12]	$A_4 \succ A_2 \succ A_3 \succ A_1$
BN-TOPSIS method(ψ_2) [12]	$A_3 \succ A_2 \succ A_4 \succ A_1$
FBNCWBM operator ($s, t = 1$) [9]	$A_3 \succ A_4 \succ A_2 \succ A_1$
FBNCGBM operator ($s, t = 1$) [9]	$A_3 \succ A_4 \succ A_2 \succ A_1$
BNDPWGA operator ($\lambda = 7$)	$A_3 \succ A_1 \succ A_4 \succ A_2$
BNDPOWGA operator ($\lambda = 7$)	$A_3 \succ A_4 \succ A_2 \succ A_1$

Table 8
Characteristic comparison of different methods

Methods	Flexible measure easier	prioritized
A_ψ operator [11]	No	No
G_ψ operator [11]	No	No
BN-TOPSIS method(ψ_1) [12]	No	No
BN-TOPSIS method(ψ_2) [12]	No	No
FBNCWBM operator ($s, t = 1$) [9]	Yes	No
FBNCGBM operator ($s, t = 1$) [9]	Yes	No
BNDPWGA operator ($\lambda = 7$)	Yes	Yes
BNDPOWGA operator ($\lambda = 7$)	Yes	Yes

sophic Dombi prioritized aggregation operators with existing methods will be discussed.

The ranking orders obtained by the proposed methods show that A_3 is favorable alternative.

In contrast to A_ψ, G_ψ, BN-TOPSIS(ψ_1) and BN-TOPSIS(ψ_2), FBNCWBM ($s, t = 1$) and FBNCGBM ($s, t = 1$) methods, the proposed method based on proposed BNDPWGA operator considers the prioritized relationship among the attributes by establishing the prioritized aggregation operator. In some practical MADM problems, the prioritization relationship exists among attributes. Then, DMs can use the proposed method based on the proposed BNDPWGA operator to solve MADM problems which have prioritization relationship among attributes. In contrast to A_ψ, G_ψ, BN-TOPSIS(ψ_1) and BN-TOPSIS(ψ_2), FBNCWBM ($s, t = 1$) and FBNCGBM ($s, t = 1$) methods, the proposed method based on proposed BNDPOWGA operator considers the prioritized relationship among the attributes by establishing the prioritized aggregation operator, and the interaction and interrelationship among attributes by using ordered weighted geometric aggregation operator. In some practical MADM problems, the prioritization relationship, interaction and interrelationship exist among the attributes. Then, DMs can use the proposed method based on the proposed BNDPOWGA operator to solve the MADM problems which have the prioritization relationship, interaction and interrelationship among the attributes. Thus, the proposed methods based on the proposed bipolar neutrosophic Dombi prioritized aggregation operators are more reliable and flexible. The DMs can use these proposed methods based on the proposed bipolar neutrosophic Dombi prioritized aggregation operators according to their requirements in practical MADM problems.

8. Conclusion

BNSs describe fuzzy, bipolar, inconsistent and uncertain information. BNDPWGA and BNDPOWGA operators were proposed based on Dombi operations to make sure that the bipolar neutrosophic Dombi prioritized aggregation operators are reliable and flexible. Furthermore, a numerical example was given to verify proposed methods. The reliability and flexibility of the proposed methods were further illustrated through a parameter analysis and a comparative analysis with existing methods. The contribution of this paper is as follows. First, BNSs were used to present the decision-making information based on evaluation. Secondly, Dombi operations were put forward to bipolar neutrosophic fuzzy environments. Thirdly, BNDPWGA and BNDPOWGA operators are proposed under the bipolar neutrosophic fuzzy environment. Fourthly, MADM methods based on proposed bipolar neutrosophic Dombi prioritized aggregation operators were developed. Finally, the flexibility and reliability of proposed methods were verified by a numerical example. In future, BNDPWGA and BNDPOWGA operators can be put forward to other fuzzy environments such as bipolar neutrosophic soft expert sets and bipolar interval neutrosophic sets. Also it could be seen for MAGDM with multi-granular hesitant fuzzy linguistic term sets by means of different kinds of fuzzy environments.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

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A New Grey Approach for Using SWARA and PIPRECIA Methods in a Group Decision-Making Environment

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Abstract: The environment in which the decision-making process takes place is often characterized by uncertainty and vagueness and, because of that, sometimes it is very hard to express the criteria weights with crisp numbers. Therefore, the application of the Grey System Theory, i.e., grey numbers, in this case, is very convenient when it comes to determination of the criteria weights with partially known information. Besides, the criteria weights have a significant role in the multiple criteria decision-making process. Many ordinary multiple criteria decision-making methods are adapted for using grey numbers, and this is the case in this article as well. A new grey extension of the certain multiple criteria decision-making methods for the determination of the criteria weights is proposed. Therefore, the article aims to propose a new extension of the Step-wise Weight Assessment Ratio Analysis (SWARA) and PIvot Pairwise Relative Criteria Importance Assessment (PIPRECIA) methods adapted for group decision-making. In the proposed approach, attitudes of decision-makers are transformed into grey group attitudes, which allows taking advantage of the benefit that grey numbers provide over crisp numbers. The main advantage of the proposed approach in relation to the use of crisp numbers is the ability to conduct different analyses, i.e., considering different scenarios, such as pessimistic, optimistic, and so on. By varying the value of the whitening coefficient, different weights of the criteria can be obtained, and it should be emphasized that this approach gives the same weights as in the case of crisp numbers when the whitening coefficient has a value of 0.5. In addition, in this approach, the grey number was formed based on the median value of collected responses because it better maintains the deviation from the normal distribution of the collected responses. The application of the proposed approach was considered through two numerical illustrations, based on which appropriate conclusions were drawn.

Keywords: grey numbers; group decision-making; PIPRECIA; SWARA

1. Introduction

Multiple-Criteria Decision-Making (MCDM) aims to provide the decision-maker with a choice of the best options from the final set of alternatives by analyzing them from several angles, i.e., through several criteria/attributes. Criteria weights can have a significant impact on the ranking of alternatives or the selection of the most acceptable alternative in Multiple-Criteria Decision-Making (MCDM). Therefore, many methods have been proposed so far for determining criteria weights—Entropy method [1], Analytic Hierarchy Process (AHP) introduced by Saaty [2], CRiteria Importance Through Intercriteria Correlation (CRITIC) proposed by Diakoulaki et al. [3], Analytic Network Process (ANP) proposed by Saaty and Vargas [4], Step-wise Weight Assessment Ratio Analysis (SWARA) proposed by Kersuliene et al. [5], Best-Worst Method (BWM) proposed by Rezaei [6], FUll COnsistency Method (FUCOM) proposed by Pamucar et al. [7], etc.

Besides these methods, there is also a new generation of prominent methods, such as the Characteristic Objects Method (COMET) and its extensions [8–12], the preference ranking organization method for enrichment evaluation (PROMETHEE)—partial ranking and PROMETHEE II—complete ranking [13,14]. The success of these methods is attributed to their correct mathematical settings and ease of application in practice.

Besides determining criteria weights, the SWARA method was used to solve many decision-making problems, as well as to determine the weights of criteria in these problems. Although it can be observed as a relatively new MCDM method, the SWARA method has so far been used for—prioritizing sustainability assessment indicators [15], product design from international and local perspectives [16], evaluation of exterior wall insulation [17], evaluation of packaging design [18], technology foresight about R&D project evaluation [19], regional landslide hazard assessment [20], personnel selection [21], copper prospecting mapping [22], risk assessment [23], engine operating parameter optimization [24], evaluation of smart card systems in public transportation [25], marine energy plant location selection problem [26], evaluation of sustainability indicators for renewable energy [27], evaluation of a house plan shape [28], selection of landfill site [29], supplier selection [30,31], and so forth.

Also, many extensions have been proposed for the SWARA method that extends its application to use with grey [32,33], fuzzy [34], and rough sets [35].

Before applying the SWARA method, it is necessary to rank the criteria according to their expected significance, which in some cases may be a limitation for its use. Therefore, Stanujkic et al. [36] extended the SWARA method intending to enable determining criteria weights even in cases when the criteria are not ranked according to the expected significance, and they proposed the name PIvot Pairwise Relative Criteria Importance Assessment (PIPRECIA) method for the mentioned extension.

Unlike the SWARA method, the PIPRECIA method has been used for solving a significantly fewer number of decision-making problems. The following articles can be mentioned as some examples of the use of this method—determining the significance of criteria for the implementation of high-performance computing in Danube region countries [37], green supplier selection [38], assessing the quality of e-learning materials [39,40], safety evaluation of railway traffic [41], assessment of road transportation risks [42], stackers selection [43], evaluation of the ICT adoption [44], and so forth. Besides, the PIPRECIA method has also been extended for using grey [45] and fuzzy [46] numbers.

Many extensions of MCDM methods that enable using of grey, fuzzy and neutrosophic sets have been done to enable solving complex decision-making problems often associated with the inaccuracy and unreliability of the information used to make decisions. However, many decision-making problems often require the participation, or examination of the attitudes, of more than one decision-maker or respondent. Therefore, in this article, a new grey extension of SWARA and PIPRECIA methods adapted for group decision-making is proposed. In this extension, the attitudes of the respondents collected using crisp values are further transformed into grey interval numbers, thus retaining the possibilities of different analyses.

The main contribution is that the proposed approach allows the attitudes of a larger number of respondents, expressed in crisp numbers, to be transformed into gray intervals based on the median value of the collected responses. The proposed approach can be used as a substitute for approaches in which group decision matrices are formed based on the mean value of the answers received from all respondents. Therefore, the main advantage of the proposed approach in relation to the use of crisp numbers is the ability to conduct different analyses, i.e., considering different scenarios, such as pessimistic, optimistic, and so on.

Therefore, the remaining part of this article is organized as follows—in Section 2, research methodology is presented with some basic elements of grey set theory and the calculation procedures of the SWARA and PIPRECIA methods are presented. In Section 3, a new approach for transforming respondents' ratings into appropriate grey interval numbers is presented, while Section 4 presents two numerical illustrations in which the proposed approach is applied. Finally, conclusions are given.

2. Research Methodology

2.1. Grey Numbers

The Grey Systems Theory is an effective methodology that can be used for solving uncertain problems with partially known information [47]. The basic concept of this theory is that all information is classified into three categories, depending on their uncertainty, and the appropriate colors are assigned to them as follows—known information is labeled as white, unknown information is labeled as black, and the uncertainty information is labeled grey. The white, black and grey numbers are shown on Figure 1.

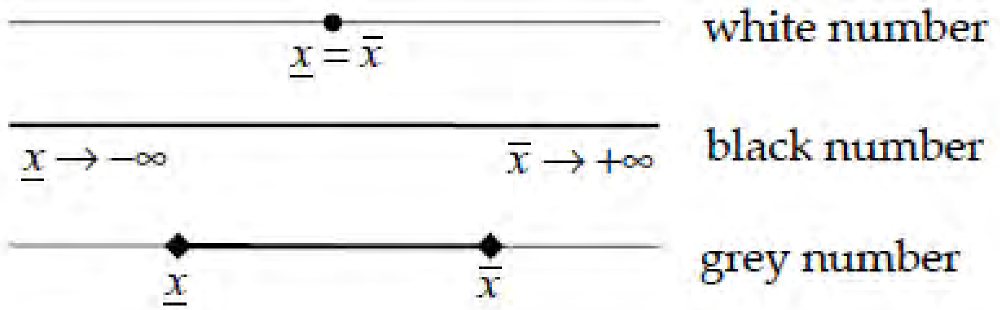


Figure 1. White, black and interval grey numbers.

Definition 1. *Grey number [47]. A grey number $\otimes x$ is such a number whose exact value is unknown, but the range in which value can lies is known.*

Definition 2. *Interval grey number [47]. Interval grey number is a grey number with known lower bound $\underline{x}$ and upper bound $\overline{x}$, but with unknown value of x, and it is shown as follows:*

$$\otimes x \in [\underline{x}, \overline{x}] = [\underline{x} \leq x \leq \overline{x}] \tag{1}$$

Definition 3. *Basic operations of grey numbers [34]. Let $\otimes x_1 = [\underline{x}_1, \overline{x}_1]$ and $\otimes x_2 = [\underline{x}_2, \overline{x}_2]$ be two grey numbers. The basic operations of grey numbers $\otimes x_1$ and $\otimes x_2$ are defined as follows:*

$$\otimes x_1 + \otimes x_2 = [\underline{x}_1 + \underline{x}_2, \overline{x}_1 + \overline{x}_2], \tag{2}$$

$$\otimes x_1 - \otimes x_2 = [\underline{x}_1 - \overline{x}_2, \overline{x}_1 - \underline{x}_2], \tag{3}$$

$$\otimes x_1 \cdot \otimes x_2 = [\underline{x}_1 \cdot \underline{x}_2, \overline{x}_1 \cdot \overline{x}_2], \tag{4}$$

$$\otimes x_1 \div \otimes x_2 = [\underline{x}_1, \overline{x}_1] \cdot \left[\frac{1}{\underline{x}_2}, \frac{1}{\overline{x}_2}\right]. \tag{5}$$

Definition 4. *The whitening function [48]. The whitening function transforms an interval grey number into a crisp number whose possible values lie between the bounds of the interval grey number. For the given interval grey number, the whitened value $x_{(\lambda)}$ of interval grey number $\otimes x$ is defined as*

$$x_{(\lambda)} = (1-\lambda)\underline{x} + \lambda\overline{x}, \tag{6}$$

where λ denotes the whitening coefficient, and $\lambda \in [0,1]$.

In the particular case, when $\lambda = 0.5$, the whitened value becomes the mean of the interval grey number, as follows:

$$x_{(0.5)} = 0.5(\underline{x} + \overline{x}) \tag{7}$$

2.2. The SWARA and the PIPRECIA Methods

The basic steps of the computational procedures of the SWARA and PIPRECIA methods for determining criteria weights based on the attitudes of a respondent, i.e., decision-maker or expert, are presented in this section.

2.2.1. The SWARA Method

The procedure for determining weights of n criteria by applying the SWARA method can be summarized as follows [5,18]:

Step 1. Identification and selection of criteria;

Step 2. Sorting criteria according to their expected significance, in descending order; and

Step 3. Determining the relative significance of criteria. In this step, the relative significance of the first criterion is set to 1, while the relative significance of the others s_j is determined according to the previous one.

In the SWARA method, s_{j-1} has a value 0 if criterion $j-1$ has the same significance as the criterion j, $s_j \in [0,1]$, and the lower value of s_{j-1} indicates a higher significance of criterion j in relation to criterion $j-1$.

Step 4. Calculating the value of the coefficient k_j as:

$$k_j = \begin{cases} 1 & if\ j = 1 \\ s_j + 1 & if\ j > 1 \end{cases} \tag{8}$$

Step 5. Calculating the value of the recalculated significance q_j as follows:

$$q_j = \begin{cases} 1 & if\ j = 1 \\ \frac{q_{j-1}}{k_j} & if\ j > 1 \end{cases} \tag{9}$$

Step 6. Calculating the criteria weights w_j as follows:

$$w_j = \frac{q_j}{\sum_{j=1}^{n} q_k} \tag{10}$$

The calculation procedure of the SWARA method, for determining the criteria weights, is summarized in Figure 2.

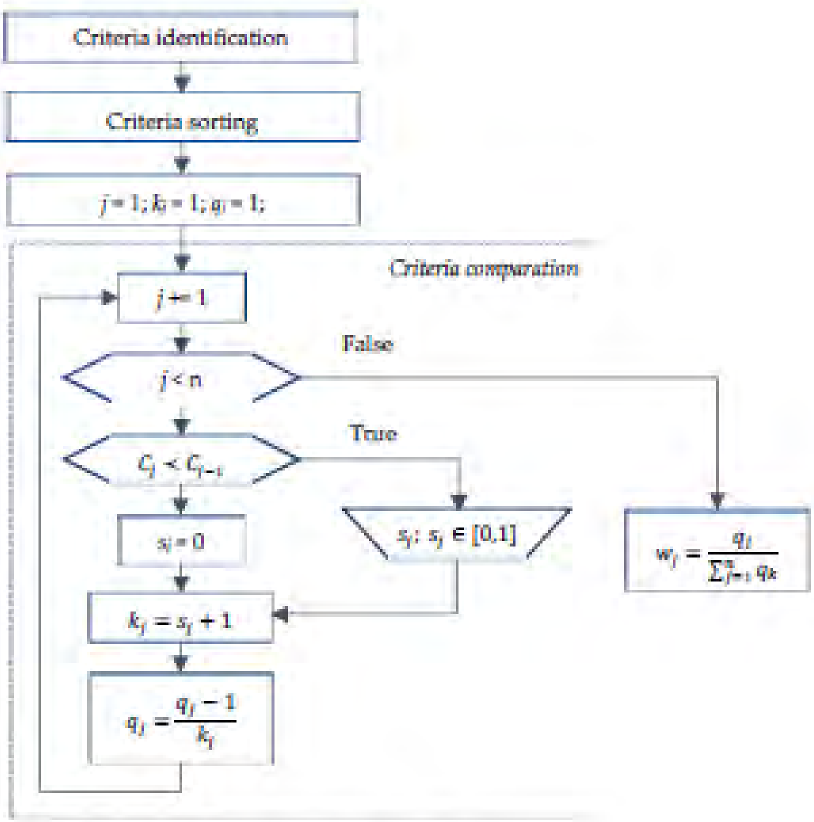


Figure 2. Flowchart of the SWARA method.

2.2.2. The PIPRECIA Method

The procedure for determining criteria weights by applying the PIPRECIA method can be summarized as follows [36]:

Step 1. Identification and selection of criteria;

Step 2. Sorting criteria according to their expected significance, in descending order. This step is not strictly required in the PIPRECIA method. Unlike the SWARA method, the PIPRECIA method can be used even when it is not possible to assess the significance of the criteria in advance; and

Step 3. Determining the relative significance of criteria. In this step, the relative significance of the first criterion is set to 1, while the relative significance of the others s_j is determined concerning the previous one.

Unlike the SWARA method, in the PIPRECIA method, $s_{j-1} = 1$ if criterion $j-1$ has the same significance as the criterion j, $s_j \in (1, 2]$ if criterion $j-1$ has higher significance than criterion j, and $s_j \in (1, 0]$ if criterion $j-1$ has a lower significance than criterion j.

Step 4. Calculating the value of the coefficient as follows:

$$k_j = \begin{cases} 1 & if\ j = 1 \\ 2 - s_j & if\ j > 1 \end{cases} \tag{11}$$

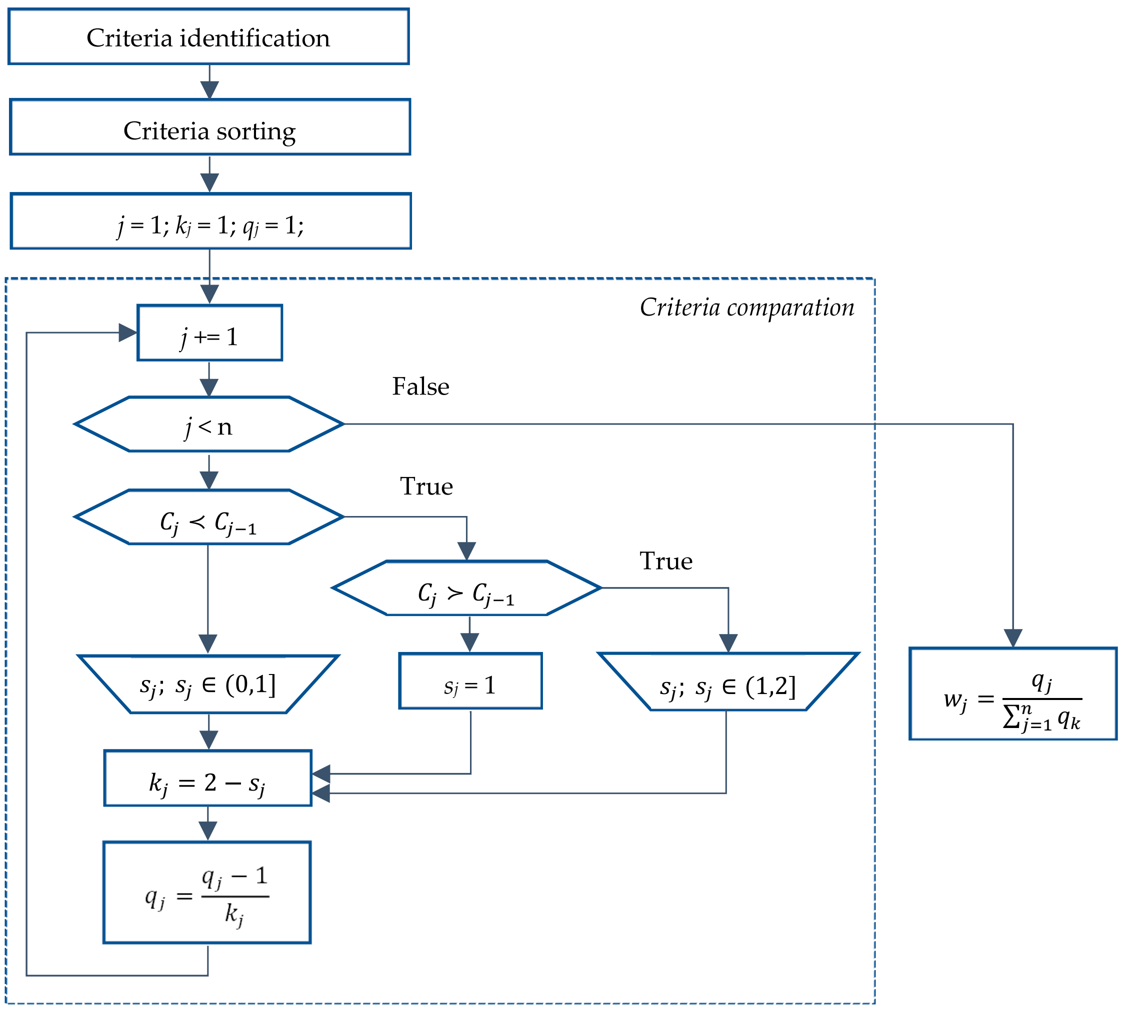
Step 5. Calculating the value of the recalculated significance as:

$$q_j = \begin{cases} 1 & if\ j = 1 \\ \frac{q_{j-1}}{k_j} & if\ j > 1 \end{cases} \tag{12}$$

Step 6. Calculating the criteria weights as:

$$w_j = \frac{q_j}{\sum_{j=1}^{n} q_k} \tag{13}$$

The calculation procedure of the PIPRECIA method is shown in Figure 3.



From the previously described procedures, it can be concluded that the basic difference between SWARA and PIPRECIA methods is in the way of assigning the relative importance of the criteria, which is manifested in Equation (11) of the calculation procedure of the PIPRECIA method.

In the case of applying the SWARA method, s_j shows how much less important a criterion is than the previous one, which is why it is necessary to pre-rank the criteria according to their expected significance. In contrast, the PIPRECIA method allows the relative significance of the criteria to be less than, equal to, or greater than the significance of the previous criterion, which is why in the case of applying this method it is not necessary to rank the criteria according to their expected significance.

The use of the SWARA method has already been proven in numerous published papers. In contrast, the PIPRECIA method is proposed as an extension of the SWARA method that can be used when it is not possible to determine in advance the expected significance of the criteria, as in the case of surveying the attitudes of a large number of respondents using electronic questionnaires.

There are several ways to apply the previously described single-user procedures SWARA and PIPRECIA methods in the case of solving group decision-making problems. Nevertheless, a new approach for transform individual crisp attitudes into a grey group attitude is proposed and discussed below.

3. A New Grey Approach for Using SWARA and PIPRECIA Methods in a Group Decision-Making Environment

As mentioned above, decision-makers or experts involved in group decision-making assess the values of the relative significance of criteria and express these values using crisp numbers. As a result of group evaluation, K lists of relative significances are formed, where K denotes the number of decision-makers or experts.

In such cases, in order to determine the criteria weights, it is possible to:

- Calculate the group significance as follows:

$$s_j = \frac{1}{k}\sum\nolimits_{k=1}^{K} s_j^k, \tag{14}$$

  and then apply the SWARA or PIPRECIA method.
- Calculate the criteria weights for each respondent w_j^k, and then calculate the group criteria weight as follows:

$$w_j = \frac{1}{k}\sum\nolimits_{k=1}^{K} w_j^k \tag{15}$$

However, in order to take advantage of the benefits that grey numbers provide, i.e., the possibility of analysis, a transformation of respondents' attitudes into grey relative group significance is proposed, using the following steps:

Step 1. Determine the median value mdn_j for each criterion, as follows:

$$mdn_j = median(s_j^k) \tag{16}$$

Step 2. Determine the lower $\underline{s}_j$ and upper $\overline{s}_j$ bounds of grey relative group significance for each criterion, as follows:

$$\underline{s}_j = mean(s_j^k | s_j^k \le mdn_j) \tag{17}$$

$$\overline{s}_j = mean(s_j^k | s_j^k \ge mdn_j) \tag{18}$$

Step 3. Using Equation (6) and different values for the coefficient λ, the relative group significance can be determined after which the criteria weights can be determined using the SWARA or PIPRECIA methods.

Using the proposed approach, the attitudes of a larger number of respondents, i.e., a larger number of crisp ratings, can be transformed into a grey interval, i.e., only two crisp numbers, without significant loss of information.

This means that the proposed approach can provide similar results to approaches in which crisp numbers are used, but it also provides an additional possibility to conduct analyses. The word similar is used here because in this approach the edges of the interval are formed based on the median values of the scores collected from all the respondents. Identical weights of the criteria, when the whitened value is 0.5, could be achieved if the limits of the grey number were formed on the basis of the mean value instead of the median value of the collected relative significances. However, in this approach, forming of the grey number based on the median value of collected relative significances was selected because it better maintains the deviation from the normal distribution in the collected relative significances.

4. A Numerical Illustrations

To present the proposed approach in detail, as well as its verification, this section presents the results of two studies conducted by a group of 10 master and doctoral students from the Faculty of Applied Management, Economics and Finance (MEF), Belgrade, Serbia. Before beginning research about evaluation criteria for assessing the quality of websites, students involved in the research are thoroughly introduced using the SWARA and PIPRECIA methods.

4.1. The First Numerical Illustration

To verify and explain in more detail the proposed approach, a study with doctoral and master students, familiar with the use of SWARA and PIPRECIA methods, was conducted. On that occasion, the significance of five criteria for evaluating travel and tourism website design ideas, such as On the Grid (https://onthegrid.city/ (accessed on 1 March 2021) [49]), Visit Asia (https://www.visitasiatravel.com/ (accessed on 1 March 2021) [50]), Visit Australia (https://www.australia.com/en (accessed on 1 March 2021) [51]), Live Africa (http://www.liveafrica.com/ (accessed on 1 March 2021) [52]), Bora Bora (https://www.borabora.com/ (accessed on 1 March 2021) [53]), Visit Serbia (https://www.serbia.travel/en (accessed on 1 March 2021) [54]) and National Park Djerdap (https://npdjerdap.rs/en/ (accessed on 1 March 2021) [55]), was determined.

Based on 7 criteria proposed by the 2020 WebAward Competition (http://www.webaward.org/ (accessed on 1 March 2021) [56]), that are, Design, Innovation, Content, Technology, Interactivity, Copywriting, and Ease of use, the following five criteria are defined for evaluating travel and tourism website design ideas:

- Cr_1, Availability of information;
- Cr_2, Structure and navigability;
- Cr_3, Design;
- Cr_4, Personalization; and
- Cr_5, Innovations.

On that occasion, in interactions with the respondents, their meaning was precisely determined. During this evaluation, we gave the following meaning to the criteria—Criterion Cr_1 refers to the availability of useful information about the tourist destination on the website, while criterion Cr_2, Structure and navigability, refers to the organization of information in such a way that visitors to the website can easily find it. The criterion Cr_3, Design, refers to the visual appearance of the website and its success to convey the beauty of tourist destinations. The Cr_4 criterion indicates the ability to tailor the website to the needs of visitors. This criterion includes the ability to choose the language in which the information is displayed, the use of search bars, as well as responsive web design. Finally, the criterion Cr_5, Innovations, refers to the application of current technologies and innovations in web design.

After choosing the criteria, and considering their significances and impact, during a tour of the above-mentioned websites, respondents gave their attitudes, which are shown in Table 1. It should be noted that the above criteria are listed in order of their expected significance, which is necessary for the application of the SWARA method.

Table 1. Attitudes of respondents as results of the SWARA method.

	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8	E_9	E_{10}	Mean
Cr_1	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Cr_2	0.00	0.05	0.06	0.20	0.15	0.01	0.03	0.18	0.15	0.20	0.10
Cr_3	0.15	0.10	0.10	0.20	0.20	0.05	0.10	0.20	0.15	0.20	0.15
Cr_4	0.25	0.15	0.10	0.25	0.30	0.10	0.15	0.23	0.17	0.25	0.20
Cr_5	0.25	0.20	0.20	0.30	0.40	0.15	0.15	0.25	0.20	0.30	0.24

Table 1 also shows the mean value of the attitudes of all respondents for each criterion. The calculation details and criteria weights obtained based on the mean value, calculated by applying the SWARA method, are encountered in Table 2.

Table 2. Computational details and criteria weights generated by the SWARA method based on mean values.

	Mean	k_j	q_j	$w_j{}'$	**Rank**
Cr_1	1.00	1	1	0.257	1
Cr_2	0.10	1.10	0.91	0.233	2
Cr_3	0.15	1.15	0.79	0.203	3
Cr_4	0.20	1.20	0.66	0.170	4
Cr_5	0.24	1.24	0.53	0.137	5

The individual criteria weights calculated on the attitudes of each respondent are shown in Table 3. Table 3 also shows group criteria weights calculated as the average of individual criteria weights.

Table 3. Individual and group criteria weights calculated by the SWARA method.

	$w_{\mathbf{e1}}$	$w_{\mathbf{e2}}$	$w_{\mathbf{e3}}$	$w_{\mathbf{e4}}$	$w_{\mathbf{e5}}$	$w_{\mathbf{e6}}$	$w_{\mathbf{e7}}$	$w_{\mathbf{e8}}$	$w_{\mathbf{e9}}$	$w_{\mathbf{e10}}$	$w_j{}''$
Cr_1	0.24	0.24	0.24	0.28	0.28	0.22	0.23	0.28	0.26	0.28	0.256
Cr_2	0.24	0.23	0.22	0.24	0.24	0.22	0.23	0.24	0.23	0.24	0.232
Cr_3	0.21	0.21	0.20	0.20	0.20	0.21	0.21	0.20	0.20	0.20	0.203
Cr_4	0.17	0.18	0.18	0.16	0.16	0.19	0.18	0.16	0.17	0.16	0.170
Cr_5	0.14	0.15	0.15	0.12	0.11	0.16	0.16	0.13	0.14	0.12	0.138

From Tables 2 and 3, it can be seen that both approaches give the same criteria weights, as well as the same ranking order of criteria based on their significance, that is $C_{r1} \succ C_{r2} \succ C_{r3} \succ C_{r4} \succ C_{r5}$, which confirms that the expected significance of the criteria was correctly assessed, as well as that all respondents filled in the questionnaires correctly.

It is important to note here that the evaluation was performed by respondents familiar with the methods used and that they entered their views in a questionnaire made in the spreadsheet program which immediately calculated criteria weights and displayed values numerically and graphically after entering the relative significance of each criterion. In this way, the respondents were able to correct their grades if necessary. Respondents were also suggested that they could modify their responses until they gained criteria weights that reflected their real opinion.

Determining the Criteria Weights Based on the Proposed Grey Approach

Determining criteria weights using the proposed approach began by determining the median values for each criterion, after which the lower and upper bounds of the grey numbers were determined. The median, lower and upper bounds of grey numbers are arranged in Table 4. Table 4 also shows the values of relative group significance obtained by applying Equation (6) and $\lambda = 0.5$, as well as its deviations from the mean.

Table 4. Median, left, and right bounds of grey ratings.

	mdn_j	$\underset{-j}{s}$	$\bar{s}_j$	$s_j^{0.5}$	Δ	$\Delta\%$
Cr_1	0.24	0.24	0.24	0.28	0.28	0.22
Cr_2	0.24	0.23	0.22	0.24	0.24	0.22
Cr_3	0.21	0.21	0.20	0.20	0.20	0.21
Cr_4	0.17	0.18	0.18	0.16	0.16	0.19
Cr_5	0.14	0.15	0.15	0.12	0.11	0.16

Due to the complexity of the procedure for determining the left and right bounds of the grey interval, the calculation was performed using a program written in the Python programming language.

The criteria weights determined based on the left and right boundaries of the grey interval are shown in Table 5. The criteria weights calculated using Equation (6) and λ = 0.5 are also shown in Table 5.

Table 5. The criteria weight obtained based on grey ratings.

	$\underline{w}_j$	$\overline{w}_j$	w_j'''
Cr_1	0.279	0.234	0.257
Cr_2	0.237	0.227	0.233
Cr_3	0.201	0.205	0.203
Cr_4	0.160	0.181	0.170
Cr_5	0.123	0.153	0.137

Table 6 shows the criteria weights obtained based on the mean value of relative significance (w_j'), the mean values of the weights obtained based on the respondents' attitudes (w_j''), and the weights obtained by applying the proposed approach (w_j''').

Table 6. The criteria weight obtained using three approaches.

	w_j'	w_j''	w_j'''
Cr_1	0.257	0.256	0.257
Cr_2	0.233	0.232	0.233
Cr_3	0.203	0.203	0.203
Cr_4	0.170	0.170	0.170
Cr_5	0.137	0.138	0.137

It can be noticed from Table 6 that all three considered approaches give almost the same criteria weights.

To verify the applicability of the proposed approach in the case of application of the PIPRECIA method, the values in Table 1 were recalculated as follows:

$$s_j = \begin{cases} 1 & if\ j = 1 \\ 1 - s_j & if\ j > 1 \end{cases} \tag{19}$$

This recalculation was necessary due to differences in the calculation of the coefficient k_j, i.e., in Equations (8) and (11), in the SWARA and PIPRECIA methods. This way of value transformation was chosen because the aim of this research was not to compare the results obtained using the SWARA and PIPRECIA methods but to check the applicability of the proposed procedure. The results achieved using SWARA and PIPRECIA could differ to some extent because the relatively important feature is the assessment of the respondent. Even in the case of re-evaluation using some of the above methods, certain deviations may occur.

The recalculated values from Table 1 are shown in Table 7.

Table 7. Attitudes of respondents adapted for applying the PIPRECIA method.

	E_1	**E_2**	**E_3**	**E_4**	**E_5**	**E_6**	**E_7**	**E_8**	**E_9**	**E_{10}**	**Mean**
Cr_1	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Cr_2	1.00	0.95	0.94	0.80	0.85	0.99	0.97	0.82	0.85	0.80	0.90
Cr_3	0.85	0.90	0.90	0.80	0.80	0.95	0.90	0.80	0.85	0.80	0.86
Cr_4	0.75	0.85	0.90	0.75	0.70	0.90	0.85	0.77	0.83	0.75	0.81
Cr_5	0.75	0.80	0.80	0.70	0.60	0.85	0.85	0.75	0.80	0.70	0.76

The values obtained using the three approaches, and the PIPRECIA method, are given in Tables 8–10, while the comparison of weights obtained using the three approaches is summarized in Table 11.

Table 8. Computational details and the criteria weights obtained by the PIPRECIA method based on mean values.

	Mean	k_j	q_j	$w_j{'}$
Cr_1	1.00	1	1	0.257
Cr_2	0.90	1.10	0.91	0.233
Cr_3	0.86	1.15	0.79	0.203
Cr_4	0.81	1.20	0.66	0.170
Cr_5	0.76	1.24	0.53	0.137

Table 9. Individual and group criteria weights calculated using the PIPRECIA method.

	e1	w_{e2}	w_{e3}	w_{e4}	w_{e5}	w_{e6}	w_{e7}	w_{e8}	w_{e9}	w_{e10}	$w_j{''}$
Cr_1	0.24	0.24	0.24	0.28	0.28	0.22	0.23	0.28	0.26	0.28	0.256
Cr_2	0.24	0.23	0.22	0.24	0.24	0.22	0.23	0.24	0.23	0.24	0.232
Cr_3	0.21	0.21	0.20	0.20	0.20	0.21	0.21	0.20	0.20	0.20	0.203
Cr_4	0.17	0.18	0.18	0.16	0.16	0.19	0.18	0.16	0.17	0.16	0.170
Cr_5	0.14	0.15	0.15	0.12	0.11	0.16	0.16	0.13	0.14	0.12	0.138

Table 10. The criteria weight obtained based on grey ratings.

	mdn_j	$\underline{s}_j$	$\overline{s}_j$	$s_j^{0.5}$	$w_j{'''}$
Cr_1	1.000	1.000	1.000	1.000	0.257
Cr_2	0.895	0.824	0.970	0.897	0.233
Cr_3	0.850	0.817	0.892	0.854	0.203
Cr_4	0.800	0.744	0.866	0.805	0.170
Cr_5	0.775	0.700	0.820	0.760	0.137

Table 11. The criteria weight obtained using three approaches.

	$w_j{'}$	$w_j{''}$	$w_j{'''}$
Cr_1	0.257	0.256	0.257
Cr_2	0.233	0.232	0.233
Cr_3	0.203	0.203	0.203
Cr_4	0.170	0.170	0.170
Cr_5	0.137	0.138	0.137

As in the case of using the SWARA method, it can be seen from Table 11 that all three approaches give almost the same criteria weights.

Finally, Table 12 shows a comparison of criteria weights obtained using the SWARA and PIPRECIA methods, the proposed grey approach, and $\lambda = 0.5$.

Table 12. The comparison of criteria weight obtained using the SWARA and PIPRECIA methods.

	SWARA	**PIPRECIA**
	$w_j{'''}$	$w_j{'''}$
Cr_1	0.257	0.257
Cr_2	0.233	0.233
Cr_3	0.203	0.203
Cr_4	0.170	0.170
Cr_5	0.137	0.137

As can be seen from Table 12, both approaches give the same criteria weights.

4.2. The Second Numerical Illustration

In the second part of the research, the aforementioned group of students determined the significance of the criteria for evaluating e-commerce websites, from the viewpoint of first-time visitors, using the PIPRECIA method. On this occasion, the significance of the following criteria, adopted from Stanujkic and Magdalinovic [57], was determined:

- Cr_1, Design of the website;
- Cr_2, Easy to use;
- Cr_3, Quality and availability of information;
- Cr_4, Ordering process; and
- Cr_5, Security assurance.

In this evaluation, the following meanings were assigned to the above-mentioned criteria. Criterion Cr_1 refers to the website's design, while criterion Cr_2, Easy to use, refers to the possibility of easily finding the desired product and the possibility of sorting and selecting products. Criterion Cr_3 refers to the information about products and organization of information to enable easier decision-making about ordering products. Criterion Cr_4 refers to the complexity, intuitiveness, and flexibility of the ordering process, while criterion Cr_5, Security assurance, refers to the assessment of ordering security from the website in the sense that they will actually receive products of appropriate quality, and the possibility of complaints and the security of their personal information and payment card numbers.

After discussing the meaning of the above criteria, a list that contained five e-commerce websites, that none of the respondents had visited before, was formed. This list of websites was compiled by the following websites—www.eponuda.com (accessed on 21 March 2021) [58], www.gamecentar.rs (accessed on 21 March 2021) [59], www.mi-srbija.rs (accessed on 21 March 2021) [60], www.3gstore.rs (accessed on 21 March 2021) [61], and www.lalafo.rs (accessed on 21 March 2021) [62].

The attitudes of the respondents collected during the evaluation are shown in Table 13, as well as their mean.

Table 13. Respondents' attitudes obtained based on the use of the PIPRECIA method.

	E_1	E_2	E_3	E_4	E_5	E_6	E_7	E_8	E_9	E_{10}	Mean
Cr_1	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00	1.00
Cr_2	1.20	1.10	1.00	1.00	1.05	0.80	0.90	1.20	1.10	1.15	1.05
Cr_3	1.20	1.00	1.00	1.20	1.10	1.20	1.20	1.20	1.20	1.30	1.16
Cr_4	1.00	1.10	0.90	1.00	1.00	0.90	1.10	1.10	1.00	1.00	1.01
Cr_5	1.10	1.10	1.10	1.00	1.00	1.05	0.90	1.00	1.00	1.00	1.03

The calculation details and the criteria weights obtained based on the mean value are encountered in Table 14, while the group criteria weights, calculated based on the average of individual criteria weights, are shown in Table 15.

Table 14. The computational details and the criteria weights obtained using the PIPRECIA method based on mean values.

	Mean	k_j	q_j	$w_j{}'$
Cr_1	1.00	1	1	0.170
Cr_2	1.05	0.95	1.05	0.179
Cr_3	1.16	0.84	1.25	0.213
Cr_4	1.01	0.99	1.27	0.216
Cr_5	1.03	0.98	1.30	0.221

Table 15. Individual and group criteria weights calculated by applying the PIPRECIA method.

	w_{e1}	w_{e2}	w_{e3}	w_{e4}	w_{e5}	w_{e6}	w_{e7}	w_{e8}	w_{e9}	w_{e10}	w_j''
Cr_1	0.14	0.17	0.20	0.17	0.18	0.21	0.18	0.14	0.16	0.14	0.170
Cr_2	0.18	0.19	0.20	0.17	0.19	0.17	0.17	0.17	0.18	0.16	0.179
Cr_3	0.22	0.19	0.20	0.22	0.21	0.22	0.21	0.21	0.22	0.23	0.213
Cr_4	0.22	0.21	0.18	0.22	0.21	0.20	0.23	0.24	0.22	0.23	0.216
Cr_5	0.24	0.24	0.21	0.22	0.21	0.21	0.21	0.24	0.22	0.23	0.221

The criteria weights obtained using the proposed approach are shown in Table 16. Finally, the comparisons of weights obtained using three approaches are presented in Table 17.

Table 16. The criteria weight obtained based on grey ratings.

	mdn_j	$\underline{s}_j$	$\overline{s}_j$	$s_j^{0.5}$	w_j'''
Cr_1	1.000	1.000	1.000	1.000	0.169
Cr_2	0.895	0.824	0.970	0.897	0.178
Cr_3	0.850	0.817	0.892	0.854	0.216
Cr_4	0.800	0.744	0.866	0.805	0.217
Cr_5	0.775	0.700	0.820	0.760	0.220

Table 17. The criteria weight obtained using three approaches.

	w_j'	w_j''	w_j'''
Cr_1	0.170	0.170	0.169
Cr_2	0.179	0.179	0.178
Cr_3	0.213	0.213	0.216
Cr_4	0.216	0.216	0.217
Cr_5	0.221	0.221	0.220

As can be observed from Table 17, all considered approaches gave almost the same criteria weights.

5. Conclusions

An innovative approach for determining criteria weights in case of group decision-making, using SWARA or PIPRECIA methods, is proposed in this article.

In this approach, it is proposed to transform the respondents' attitudes into appropriate grey intervals, instead of transforming them into crisp values. Using grey, instead of crisp, numbers provides much greater possibilities, primarily for performing certain analyses. Therefore, in this article, two numerical examples were conducted and the justification of the proposed approach is confirmed. Also, by varying the value of the whitening coefficient, different weights of the criteria can be obtained, and it should be emphasized that this approach gives the same weights as in the case of crisp numbers when the whitening coefficient has a value of 0.5.

The study was conducted with a group of 10 selected respondents from a group of 25 respondents who were instructed to apply the SWARA and PIPRECIA methods.

The results obtained by applying the proposed approach, in the considered cases, are identical to the results obtained by the used procedures based on the application of crisp numbers. Besides, the results obtained using the SWARA and PIPRECIA methods are the same in all cases considered, which confirms that this approach can be used with the SWARA and PIPRECIA methods.

The advantage of this approach is reflected in the possibility of combining crisp and grey numbers, where crisp numbers are used for collecting the respondents' attitudes, while grey numbers are used for expressing attitudes of all respondents that are group attitudes. In this way, the possibility of a simple collection of respondents' attitudes and the advantages of grey numbers is combined, primarily in terms of conducting analyses.

In addition, in this approach, the grey number was formed based on the median value of collected responses because it better maintains the deviation from the normal distribution of the collected responses.

The proposed approach can be used for determining the criteria weights as well as for completely solving the MCDM problem. Finally, as a direction for future research, the proposed approach can be easily adapted for applying triangular fuzzy numbers, or even interval-valued triangular fuzzy numbers, i.e., new extensions for solving a wider spectrum of complex real-world problems could be proposed.

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Multiple-Criteria Decision-Making Based on The Use of Single-Valued Neutrosophic Sets and Similarity Measures

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Abstract. *As the generalization of the fuzzy and similar sets based on Fuzzy sets, neutrosophic sets provide significant possibilities in the case of solving complex decision problems, often related to uncertainty and unreliability. Neutrosophic sets use three values named the truth degree, the indeterminacy degree and the falsity degree, which allow for a more accurate evaluation of alternatives in relation to complex evaluation criteria. As a result of their application in solving numerous different decision-making problems, several approaches to their ranking have been proposed. Therefore, this paper provides a comprehensive overview of the approaches to the ranking of single-valued neutrosophic numbers and a comparison of the results obtained by using them. Finally, numerical illustrations are given.*

Keywords: *neutrosophy, single-valued neutrosophic set, similarity measures, MCDM.*

1. Introduction

In recent decades, multi-criteria decision-making (MCDM) methods have strongly been developed and have become increasingly popular (Ulutaş et al., 2020). Due to such strong development, a large number of multi-criteria decision-makings techniques have been introduced by scholars throughout the world (Stanujkic et al., 2019; 2017). So far, MCDM has successfully been applied in solving numerous decision-making problems in different areas. Significant progress in MCDM was made after Zadeh (1965) introduced fuzzy set theory, thus providing an efficient approach to solving a much larger number of real-world decision-making problems. An important characteristic of fuzzy sets is that they use the membership function to represent the degree of belonging to a set. Different shapes of membership functions are proposed, such as triangular, trapezoidal and bell-shaped so as to enable the efficient solving of different decision-making problems.

In order to enable solving more complex problems, some extensions of fuzzy set theory have been proposed, such as interval-valued fuzzy sets (Turksen, 1986), intuitionistic fuzzy sets (Atanassov, 1986), interval-valued intuitionistic fuzzy sets (Atanassov & Gargov, 1989), the bipolar fuzzy set (Lee, 2000), Pythagorean fuzzy sets (Yager, 2013) and so on.

In addition to more complex membership functions, these extensions also bring new mathematical operations on sets, as well as new membership functions. In bipolar fuzzy sets theory, Lee (2000) introduced the positive membership function $\mu_A^+(x)$ and the negative membership function $\nu_A^-(x)$, with the following conditions: $\mu_A^+(x) \in [0,1]$ and $\nu_A^-(x) \in [-1,0]$.

In intuitionistic fuzzy sets, Atanassov (1986) introduced the membership $T_A(x)$ and non-membership $F_A(x)$ functions, with the following conditions: $T_A(x), F_A(x) \in [0,1]$ and $T_A(x) + F_A(x) \le 1$. In intuitionistic fuzzy sets, indeterminacy $\pi_A(x)$ is $1 - T_A(x) - F_A(x)$ by default.

The condition $T_A(x) + F_A(x) \le 1$ can be a limitation for using intuitionistic fuzzy numbers. Therefore, in Pythagorean fuzzy sets, this condition is transformed into the following $T_A(x)^2 + F_A(x)^2 \le 1$ in order to facilitate their use in relation to intuitionistic fuzzy sets.

Smarandache (1999) proposed neutrosophic sets, as the generalization of fuzzy sets and their extensions. Neutrosophic sets have three independent membership functions, namely: the truth-membership function $T_A(x)$, the falsity-membership $F_A(x)$ function and the indeterminacy-membership $I_A(x)$ function. Smarandache (1999) and Wang et al. (2010) further proposed a single-valued neutrosophic set by modifying the conditions $T_A(x)$, $I_A(x)$ and $F_A(x)$ $\in[0,1]$ and

$0 \le T_A(x) + I_A(x) + F_A(x) \le 3$, which are more suitable for solving scientific and engineering problems (Li et al., 2016).

Single-valued neutrosophic numbers are used for solving a number of different problems, as a result of which many approaches have been proposed for their ranking, such as ranking based on the score and accuracy functions (Sahin 2014), ranking based on the Euclidean and Hamming distances (Ye, 2015; Pramanik et al., 2015), ranking based on different similarity measures (Ye, 2014).

There are a number of the similarity measures that can be used to rank single-valued neutrosophic numbers. Therefore, an overview of some prominent similarity measures is given below, and the rest of the paper is organized in the following way: in Section 2, some basic concepts of neutrosophic sets and single-valued neutrosophic numbers are presented; in Section 3, some distances and similarity measures between n-dimensional vectors are considered, whereas in Section 3.1, similarity measures between two single-valued neutrosophic numbers are given; in Section 4, approaches for ranking of single-valued neutrosophic numbers are presented; in Section 5, the evaluation of alternatives by using single valued neutrosophic numberss is given; in Section 6, a group multi-criteria decision-making approach based on the use of single-valued neutrosophic numbers and similarity measures is presented, and in Section 7, an example is considered with the aim of explaining the proposed methodology in detail. Section 8 is a presentation of the analysis conducted and the conclusions of the paper are given in the last section.

2. Some Basic Concepts of Neutrosophic Sets and Single-Valued Neutrosophic Numbers

Definition 1. Let X be the universe of discourse, with a generic element in X denoted by x (Smarandache, 1999). The Neutrosophic Set (NS) A in X is an object having the following form:

$$A = \left\{ \langle x, T_A(x), I_A(x), F_A(x) \rangle \middle| x \in X \right\}, \tag{1}$$

with: $T_A : X \to]^-0,1^+[$; $I_A : X \to]^-0,1^+[$; $F_A : X \to]^-0,1^+[$ and $^-0 \le T_A(x) + I_A(x) + F_A(x) \le 3^+$, where: $T_A(x)$, $I_A(x)$ and $F_A(x)$ are the truth-membership function, the indeterminacy-membership function, and the falsity-membership function, respectively.

Definition 2. Let X be a nonempty set. The Single-Valued Neutrosophic Set (SVNS) A in X is as follows (Wang et al., 2010):

$$A = \left\{ \langle x, T_A(x), I_A(x), F_A(x) \rangle \middle| x \in X \right\}, \tag{2}$$

with: $T_A : X \to [0,1]$; $I_A : X \to [0,1]$; $F_A : X \to [0,1]$ and $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$, where: $T_A(x)$, $I_A(x)$ and $F_A(x)$ are the truth-membership function, the intermediacy-membership function, and the falsity-membership function, respectively, $T_A, I_A, F_A : X \to [0,1]$ and $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$.

The truth-membership, indeterminacy-membership, and falsity-membership functions are sometimes referred to as the truth, indeterminacy and falsity degrees, respectively.

Definition 3. A Single-Valued Neutrosophic Number (SVNN) $a =< t_a, i_a, f_a >$ is a special case of an SVNS on the set of real numbers $\Re$ where $t_a, i_a, f_a \in [0,1]$ and $0 \le t_a + i_a + f_a \le 3$ (Wang et al., 2010).

Definition 4. Let $x_1 =< t_1, i_1, f_1 >$ and $x_2 =< t_2, i_2, f_2 >$ be two SVNNs and $\lambda > 0$. The basic operations over the two SVNNs are as follows (Smarandache, 1999):

$$x_1 + x_2 =< t_1 + t_2 - t_1 t_2, i_1 i_2, f_1 f_2 > . \tag{3}$$

$$x_1 \cdot x_2 =< t_1 t_2, i_1 + i_2 - i_1 i_{2,} f_1 + f_2 - f_1 f_2 > . \tag{4}$$

$$\lambda x_1 =< 1 - (1 - t_1)^{\lambda}, i_1^{\lambda}, f_1^{\lambda} > . \tag{5}$$

$$x_1^{\lambda} =< t_1^{\lambda}, i_1^{\lambda}, 1 - (1 - f_1)^{\lambda} > . \tag{6}$$

Definition 5. Let $x =< t_i, i_1, f_i >$ be an SVNN. The score function s_x of x is as follows (Sahin, 2014):

$$s_i = (1 + t_i - 2 i_i - f_i) / 2, \tag{7}$$

where $s_i \in [-1, 1]$.

Definition 6. Let $a_j =< t_j, i_j, f_j >$ be a collection of SVNSs and $W = (w_1, w_2, ..., w_n)^T$ be an associated weighting vector. The Single-Valued Neutrosophic Weighted Average (SVNWA) operator of a_j is as follows (Sahin, 2014):

$$SVNWA(a_1, a_2, ..., a_n) = \sum_{j=1}^{n} w_j a_j = \left(1 - \prod_{j=1}^{n} (1 - t_j)^{w_j}, \prod_{j=1}^{n} (i_j)^{w_j}, \prod_{j=1}^{n} (f_j)^{w_j} \right), \tag{8}$$

where: w_j is the element j of the weighting vector, $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$.

Definition 7. Let $a_j =< t_j, i_j, f_j >$ be a collection of SVNSs and $W = (w_1, w_2, ..., w_n)^T$ be an associated weighting vector. Then, the Single-Valued Neutrosophic Weighted Geometric (SVNWG) operator of a_j is as follows (Sahin, 2014):

$$SVNWG(a_1, a_2, ..., a_n) = \prod_{j=1}^{n} (a_j)^{w_j} = \left(\prod_{j=1}^{n} (t_j)^{w_j}, 1 - \prod_{j=1}^{n} (1 - i_j)^{w_j}, 1 - \prod_{j=1}^{n} (1 - f_j)^{w_j} \right), \tag{9}$$

where: w_j is the element j of the weighting vector, $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$

3. Some Distances and Similarity Measures Between n-Dimensional Vectors

In this section, the definitions of the two distances and the four similarity measures between two *n*-dimensional vectors are presented.

Definition 8. Let $X = (x_1, x_2, ..., x_n)$ and $Y = (y_1, y_2, ..., y_n)$ be two *n*-dimensional vectors, and $x_i, y_i \geq 0$.

The Hamming distance between X and Y is as follows:

$$h_{(X,Y)} = \sum_{i=1}^{n} | x_i - y_i |. \tag{10}$$

Definition 9. Let $X = (x_1, x_2, ..., x_n)$ and $Y = (y_1, y_2, ..., y_n)$ be two *n*-dimensional vectors, and $x_i, y_i \geq 0$. The Euclidean distance between X and Y is as follows:

$$e_{(X,Y)} = \sqrt{\sum_{i=1}^{n} (x_i - y_i)^2}. \tag{11}$$

Definition 10. Let $X = (x_1, x_2, ..., x_n)$ and $Y = (y_1, y_2, ..., y_n)$ be two *n*-dimensional vectors, and $x_i, y_i \geq 0$ (Dice, 1945). The Dice similarity measure of the two vectors X and Y is as follows:

$$d_{(X,Y)} = \frac{2X \cdot Y}{\|X\|_2^2 + \|Y\|_2^2} = \frac{2\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2}. \tag{12}$$

Definition 11. Let $X = (x_1, x_2, ..., x_n)$ and $Y = (y_1, y_2, ..., y_n)$ be two *n*-dimensional vectors, and $x_i, y_i \geq 0$. The Jaccard similarity measure of the two vectors X and Y is as follows:

$$j_{(X,Y)} = \frac{X \cdot Y}{\|X\|_2^2 + \|Y\|_2^2 - X \cdot Y} = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 + \sum_{i=1}^{n} x_i y_i}. \tag{13}$$

Definition 12. Let $X = (x_1, x_2, ..., x_n)$ and $Y = (y_1, y_2, ..., y_n)$ be two *n*-dimensional vectors, and $x_i, y_i \geq 0$ (Salton & McGill, 1983). The cosine similarity measure of the two vectors X and Y is as follows:

$$c_{(X,Y)} = \frac{X \cdot Y}{\|X\|_2 \|Y\|_2} = \frac{\sum_{i=1}^{n} x_i y_i}{\sqrt{\sum_{i=1}^{n} x_i^2} \sqrt{\sum_{i=1}^{n} y_i^2}} . \tag{14}$$

Definition 13. Let $X = (x_1, x_2, ..., x_n)$ and $Y = (y_1, y_2, ..., y_n)$ be two n-dimensional vectors, and $x_i, y_i \geq 0$ (Ye & Zhang, 2014). The similarity measure based on the minimum and maximum operators of the two vectors X and Y is as follows:

$$m_{(X,Y)} = \sum_{i=1}^{n} \frac{\min(x_i, y_i)}{\max(x_i, y_i)} . \tag{15}$$

3.1. The Similarity Measures of Two SVNNs

In this subsection, the similarity measures of two SVNNs are presented. In addition, the similarity measures between an SVNN and an ideal point, defined as <1, 0, 0>, are also considered.

Definition 14. Let $x_1 =< t_1, i_1, f_1 >$ and $x_2 =< t_2, i_2, f_2 >$ be two SVNNs. The similarity measure based on the Hamming distance of the two SVNNs is as follows:

$$h_{(x_1,x_2)} = 1 - \frac{1}{3}\left(| t_1 - t_2 | + | i_1 - i_2 | + | f_1 - f_2 | \right) . \tag{16}$$

Definition 15. Let $x_1 =< t_1, i_1, f_1 >$ and $x_2 =< t_2, i_2, f_2 >$ be two SVNNs. The similarity measure based on the Euclidean distance of the two SVNNs is as follows:

$$e_{(x_1,x_2)} = 1 - \left(\frac{1}{3}(|t_1 - t_2|^2 + | i_1 - i_2 |^2 + | f_1 - f_2 |^2) \right)^{\frac{1}{2}} . \tag{17}$$

Definition 16. Let $x_1 =< t_1, i_1, f_1 >$ and $x_2 =< t_2, i_2, f_2 >$ be two SVNNs (Ye, 2014). The Dice similarity measure of the two SVNNs is as follows:

$$d_{(x_1,x_2)} = \frac{2(t_1 t_2 + i_1 i_2 + f_1 f_2)}{(t_1^2 + i_1^2 + f_1^2) + (t_2^2 + i_2^2 + f_2^2)} . \tag{18}$$

Definition 17. Let $x_1 =< t_1, i_1, f_1 >$ and $x_2 =< t_2, i_2, f_2 >$ be two SVNNs (Ye, 2014). The Jaccard similarity measure of the two SVNNs is as follows:

$$j_{(x_1,x_2)} = \frac{t_1 t_2 + i_1 i_2 + f_1 f_2}{(t_1^2 + i_1^2 + f_1^2) + (t_2^2 + i_2^2 + f_2^2) + (t_1 t_2 + i_1 i_2 + f_1 f_2)} . \tag{19}$$

Definition 18. Let $x_1 =< t_1, i_1, f_1 >$ and $x_2 =< t_2, i_2, f_2 >$ be two SVNNs (Ye, 2014). The cosine similarity measure of the two SVNNs is as follows:

$$c_{(x_1,x_2)} = \frac{t_1 t_2 + i_1 i_2 + f_1 f_2}{\sqrt{t_1^2 + i_1^2 + f_1^2} \cdot \sqrt{t_2^2 + i_2^2 + f_2^2}} . \tag{20}$$

Definition 19. Let $x_1 = < t_1, i_1, f_1 >$ and $x_2 = < t_2, i_2, f_2 >$ be two SVNNs (Ye & Zhang, 2014). The similarity measure based on the minimum and maximum operators of the two SVNNs is as follows:

$$m_{(x_1,x_2)} = \frac{1}{3}\left(\frac{\min(t_1,t_2)}{\max(t_1,t_2)} + \frac{\min(i_1,i_2)}{\max(i_1,i_2)} + \frac{\min(f_1,f_2)}{\max(f_1,f_2)} \right). \tag{21}$$

4. The Ranking of Single-Valued Neutrosophic Numbers

There are several approaches to the comparison and ranking of SVNNs, such as: the ranking based on the use of the score function; the ranking based on distances from the ideal and anti-ideal points; and the ranking based on the use of similarity measures.

4.1. The ranking based on the use of the score function

The approach based on the use of the score function is commonly used. In this approach, the overall single-valued neutrosophic utility (OSVNU) of each alternative is first calculated by using Eq. (8), or Eq. (9), and then the score of each overall utility is determined by using Eq. (7). The alternative with a higher score is the most acceptable.Sahin (2014) and Mondal & Pramanik (2014) can be mentioned as some examples of the ranking of alternatives based on the use of the score function.

4.2. The ranking based on distances from the ideal and anti-ideal points

The ranking of alternatives based on their distances to the ideal point and the anti-ideal point is mainly used in the neutrosophic extensions of the TOPSIS method. The ranking of alternatives in such approaches is based on the application of the following equation:

$$c_i = \frac{d_i^-}{d_i^+ + d_i^-} , \tag{22}$$

where: c_i denotes the relative closeness to the ideal point of the alternative i, d_i^- and d_i^+ denote the distance of the alternative i to the anti-ideal point and the ideal point, respectively.

The alternative with the highest value of relative closeness is the most acceptable.

Ye (2015), Pramanik et al. (2015) and Pouresmaeil et al. (2017) used the neutrosophic extensions of the TOPSIS method to solve different decision-making problems, such as the selection of the best investment alternative, the recruitment of an assistant teacher, and so forth. In these extensions of the use of both, the Euclidean and the Hamming distances are considered.

Biswas et al. (2016) used the neutrosophic extension of the TOPSIS method to select the most suitable tablet. In this approach, the Euclidean distance between two SVNNs was used.

4.3. The ranking based on the use of similarity measures

This approach is based on the similarity between the single-valued neutrosophic overall utility of an alternative and the single-valued neutrosophic ideal point. In a fashion similar to the case of the ranking based on the score function, the overall single-valued neutrosophic utility of each alternative is, first, calculated by using Eq. (8) or Eq. (9). Then, in the next step, the similarity measure between overall single-valued neutrosophic utility and the ideal point should be calculated. The alternative with the highest value of the similarity measure is the most acceptable.

The *single-valued neutrosophic* ideal point $r^* = < t^*, i^*, f^* >$ can be calculated as follows:

$$r^* = \begin{cases} \left\langle \max_i t_i, \min_i i_i, \min_i f_i \right\rangle, & j \in \Omega_{\max} \\ \left\langle \min_i t_i, \min_i i_i, \max_i f_i \right\rangle, & j \in \Omega_{\min} \end{cases}, \tag{23}$$

where $\Omega_{\max}$ and $\Omega_{\min}$ denote a set of beneficial and nonbeneficial criteria, respectively.

In cases when all criteria are beneficial, Eq. (24) is applied as follows:

$$r^* = \left\langle \max_i t_{ij}, \min_i i_{ij}, \min_i f_{ij} \right\rangle, \tag{24}$$

In neutrosophy, higher values for the truth degree and lower values for the falsity and indeterminacy degrees are preferable. Therefore, for the sake of simplicity, r_j^* could be determined as follows:

$$r^* = \langle 1,0,0 \rangle . \tag{25}$$

In the case of using Eq. (25) instead of Eq. (24), the previously defined Eqs. (16)-(21) have the following forms:

Definition 20. Let $x =< t, i, f >$ be an SVNN and let the ideal point be defined as <1, 0, 0>. The similarity measure based on the Hamming distance of the SVNN *x* and the ideal point is as follows:

$$h_{(x)} = 1 - \frac{1}{3}\left(1 - t_x + i_x + f_x\right). \tag{26}$$

Definition 21. Let $x =< t, i, f >$ be an SVNN and let the ideal point be defined as <1, 0, 0>. The similarity measure based on the Euclidean distance between the SVNN *x* and the ideal point is as follows:

$$e_{(x)} = 1 - \left(\frac{1}{3}[(1 - t_x)^2 + i_x{}^2 + f_x{}^2]\right)^{\frac{1}{2}}. \tag{27}$$

Definition 22. Let $x =< t, i, f >$ be an SVNN and let the ideal point be defined as <1 ,0 , 0>. The Dice similarity measure between the SVNN *x* and the ideal point is as follows:

$$d_{(x)} = \frac{2t_x}{1 + t_x^2 + i_x^2 + f_x^2}. \tag{28}$$

Definition 23. Let $x =< t, i, f >$ be an SVNN and let the ideal point be defined as <1 ,0 , 0>. The Jaccard similarity measure between the SVNN *x* and the ideal point is as follows:

$$j_{(x)} = \frac{t_x}{1 + t_x^2 + i_x^2 + f_x^2 + t_x}. \tag{29}$$

Definition 24. Let $x =< t, i, f >$ be an SVNN and let the ideal point be defined as <1 ,0 , 0>. The cosine similarity measure between the SVNN *x* and the ideal point is as follows:

$$c_{(x)} = \frac{t_x}{\sqrt{t_x^2 + i_x^2 + f_x^2}}. \tag{30}$$

Definition 25. Let $x =< t, i, f >$ be an SVNN and let the ideal point be defined as <1 ,0 , 0>. The similarity measure based on the minimum and maximum operators between the SVNN *x* and the ideal point is as follows:

$$m_{(x)} = \frac{1}{3}t_x. \tag{31}$$

The following can be mentioned as some examples of the ranking of alternatives based on the use of similarity measures: Ye (2014) used three similarity measures, i.e. the Jaccard, Dice, and cosine similarity measures, to select the best investment alternative. Ye (2017) also used the cosine similarity measure for the fault diagnosis of a steam turbine. Chen et al. (2017) used similarity measures between two SVNNs to select the most appropriate construction project.

5. The Evaluation of Alternatives by Using SVNNs

The evaluation of alternatives in relation to evaluation criteria is a very significant step, especially so when interviewing the respondents who are unfamiliar with using neutrosophic numbers.

One possible way to perform evaluation more efficiently and more accurately is to form one statement sentence for each criterion, thus enabling the respondent to express his or her opinion regarding its accuracy and/or inaccuracy. For example, the following sentences can be used for the criteria delivery and price: *The supplier's delivery time is timely* and *the supplier provides fair prices*.

The values obtained in such a manner represent the truth degree and/or the falsity degree. In addition, the respondent may also express his or her uncertainty about the given answers. In such a case, the information obtained in that manner belongs to the indeterminacy degree. In some cases, the respondent may only use the truth degree, and then the falsity degree and the indeterminacy degree are set to zero.

Two linguistic scales were suggested in order to facilitate the evaluation of the alternatives. The first linguistic scale, shown in Table 1, can be used to express the level of the accuracy and inaccuracy of a statement sequence, while the second, shown in Table 2, can be used to express uncertainty about the given answers.

Table 1. The linguistic scale for expressing the level of accuracy and inaccuracy

Linguistic variable	0 ÷ 1	0 ÷ 10	0 ÷ 100%
Extremely satisfied	≈ 1.00	≈ 10	≈ 100 %
Very satisfied	≈ 0.75	≈ 7.5	≈ 75 %
Moderately satisfied	≈ 0.50	≈ 5.0	≈ 50 %
Slightly satisfied	≈ 0.25	≈ 2.5	≈ 25 %
Dissatisfied	≈ 0.00	≈ 0.0	≈ 0 %

Table 2. The linguistic scale for expressing the level of uncertainty about the given answers

Linguistic variable	0 ÷ 1	0 ÷ 10	0 ÷ 100%
Unconvinced	≈ 1.00	≈ 10	≈ 100 %
Poorly convinced	≈ 0.75	≈ 7.5	≈ 75 %
Moderately convinced	≈ 0.50	≈ 5.0	≈ 50 %
Convinced	≈ 0.25	≈ 2.5	≈ 25 %
Extremely convinced	≈ 0.00	≈ 0.0	≈ 0 %

The linguistic variables from both tables can be transformed into any of the proposed ranges, depending on the respondents' preferences, but the first interval 0 ÷ 1 does not require normalization, whereas in the case of the second and the third ranges, the values should be transformed into the interval [0, 1]. The use of linguistic variables may make it easier for respondents to perform an evaluation, but in some ways, they can restrict the precise expression of their attitudes, which is why respondents should be informed that the given values are only approximate and that they may use any value from the interval [0, 1], or other scopes.

6. A Group Multi-Criteria Decision-Making Approach Based on Single-Valued Neutrosophic Numbers and Similarity Measures

The procedure for solving a multi-criteria decision-making problem that contains the *m* alternatives that are evaluated based on *n* criteria by *K* experts using SVNNs can precisely be expressed by the following algorithm:

Step 1. Define the goal of the evaluation and identify available alternatives.

Step 2. Form a group of experts who will perform the evaluation.

Step 3. Define a set of evaluation criteria and determine their significance, i.e. criteria weights.

Step 4. Form a questionnaire and define an evaluation scale, as discussed in Section 5.

Step 5. Evaluate the alternatives in relation to the selected criteria.

Step 6. Construct a group decision-making matrix by using Eq. (8) or Eq. (9).

Step 7. Calculate the single-valued neutrosophic overall utility of the alternatives by using Eq. (8) or Eq. (9).

Step 8. Determine the single-valued neutrosophic ideal point by using Eq. (24) or Eq. (25).

Step 9. Calculate a similarity measure for each alternative. Similarity measures can be calculated by applying any of Eqs (16)-(21) if the ideal point is determined by using Eq. (24), or by applying any of Eqs (26)-(31) if the ideal point is determined by using Eq. (25).

Step 10. Rank the alternatives and/or select the best one. The alternative with a higher similarity measure is more preferable.

7. A Numerical Illustration

In order to briefly demonstrate the usability of SVNNs for solving MCDM problems, an example of the supplier selection is presented in this section.

Allow us to assume that one company must consider engaging a new supplier. Therefore, a team of three experts is formed with the aim of selecting the most appropriate supplier from the four alternatives denoted as A_1-A_4 based on the following criteria: C1 – Delivery; C2 – Quality; C3 – Flexibility; C4 – Service; and C5 – Price.

The set of the given criteria is determined based on the paper by Chang et al. (2011), who performed an evaluation of the criteria in order to determine which of the criteria are the most influential in the case of the supplier selection. As a result, the five criteria used in our case are singled out as the most significant.

The ratings obtained from the three experts, after their transformation into SVNNs and corrections, are presented in Tables 3, 4 and 5.

Table 3. The ratings obtained from the first of the three experts

	C_1	C_2	C_3	C_4	C_5
A_1	<0.8, 0.30, 0.10>	<0.7, 0.0, 0.3>	<0.6, 0.0, 0.4>	<0.7, 0.0, 0.3>	<0.5, 0.0, 0.5>
A_2	<0.7, 0.00, 0.20>	<0.8, 0.0, 0.2>	<0.8, 0.0, 0.2>	<0.8, 0.0, 0.2>	<0.8, 0.0, 0.2>
A_3	<0.5, 0.00, 0.20>	<0.5, 0.0, 0.5>	<0.6, 0.0, 0.4>	<0.6, 0.0, 0.4>	<0.7, 0.0, 0.3>
A_4	<0.5, 0.20, 0.20>	<0.5, 0.2, 0.5>	<0.6, 0.2, 0.4>	<0.6, 0.2, 0.4>	<0.7, 0.2, 0.3>

Table 4. The ratings obtained from the second of the three experts

	C_1	C_2	C_3	C_4	C_5
A_1	<0.6, 0.00, 0.40>	<0.7, 0.0, 0.3>	<0.6, 0.0, 0.4>	<0.5, 0.0, 0.0>	<0.6, 0.0, 0.0>
A_2	<0.8, 0.00, 0.20>	<0.6, 0.0, 0.4>	<0.7, 0.0, 0.3>	<0.8, 0.0, 0.2>	<0.6, 0.0, 0.0>
A_3	<0.7, 0.00, 0.30>	<0.8, 0.0, 0.2>	<0.7, 0.0, 0.3>	<0.6, 0.0, 0.4>	<0.7, 0.0, 0.3>
A_4	<0.7, 0.20, 0.30>	<0.8, 0.2, 0.2>	<0.7, 0.2, 0.3>	<0.6, 0.2, 0.4>	<0.7, 0.2, 0.3>

Table 5. The ratings obtained from the third of the three experts

	C_1	C_2	C_3	C_4	C_5
A_1	<0.8, 0.50, 0.20>	<0.6, 0.5, 0.4>	<0.5, 0.5, 0.5>	<0.6, 0.5, 0.4>	<0.8, 0.5, 0.2>
A_2	<0.6, 0.20, 0.10>	<0.6, 0.2, 0.4>	<0.8, 0.2, 0.2>	<0.5, 0.2, 0.5>	<0.7, 0.2, 0.3>
A_3	<0.6, 0.10, 0.40>	<0.7, 0.1, 0.3>	<0.6, 0.1, 0.2>	<0.6, 0.1, 0.4>	<0.5, 0.1, 0.5>
A_4	<0.6, 0.20, 0.40>	<0.7, 0.2, 0.3>	<0.6, 0.2, 0.2>	<0.6, 0.2, 0.4>	<0.5, 0.2, 0.5>

As shown in Step 5 of the previously presented procedure for solving multi-criteria decision-making problems, the group decision-making matrix shown in Table 6 was constructed by using Eq. (8). In this calculation, all the three experts had the same significance, i.e. $w_j = 0.333$; $j = 1, 2$ and 3.

Table 6. The group decision-making matrix

	C_1	C_2	C_3	C_4	C_5
A_1	<0.75, 0.0, 0.24>	<0.67, 0.0, 0.33>	<0.57, 0.0, 0.44>	<0.61, 0.0, 0.25>	<0.66, 0.0, 0.26>
A_2	<0.71, 0.0, 0.17>	<0.68, 0.0, 0.34>	<0.77, 0.0, 0.23>	<0.73, 0.0, 0.32>	<0.71, 0.0, 0.18>
A_3	<0.61, 0.0, 0.30>	<0.69, 0.0, 0.35>	<0.64, 0.0, 0.30>	<0.60, 0.0, 0.40>	<0.64, 0.0, 0.37>
A_4	<0.75, 0.0, 0.24>	<0.67, 0.0, 0.33>	<0.57, 0.0, 0.44>	<0.61, 0.0, 0.25>	<0.66, 0.0, 0.26>

In the next step, Step 6 of the above-proposed procedure, the overall ratings were calculated by using Eq. (7) and the following weights w_j = {0.18, 0.21, 0.20, 0.18, 0.23}. The overall ratings are shown in Table 7.

Table 7. The overall single-valued neutrosophic utility of the alternatives

	Overall utility
A_1	<0.65, 0.00, 0.30>
A_2	<0.72, 0.00, 0.24>
A_3	<0.64, 0.00, 0.34>
A_4	<0.65, 0.00, 0.30>

Finally, the ranking results obtained by using all the approaches considered in Section 3 and the ideal point r^*=<0.72, 0.00, 0.24>, calculated by using Eq. (24), are demonstrated in Table 8.

Table 8. The ranking results obtained by using the considered approaches

	I		II		III		IV		V		VI		VII	
	$h_{(i)}$		$e_{(i)}$		$d_{(i)}$		$j_{(i)}$		$c_{(i)}$		$mm_{(i)}$		$s_{(i)}$	
A_1	0.89	3	0.79	3	0.96	3	0.325	3	0.963	3	0.56	3	0.46	3
A_2	0.99	1	0.96	1	1.00	1	0.333	1	0.999	1	0.67	1	0.67	1
A_3	0.94	2	0.86	2	0.98	2	0.330	2	0.984	2	0.52	4	0.61	2
A_4	0.88	4	0.79	4	0.96	4	0.324	4	0.960	4	0.58	2	0.45	4

As can be seen from Table 8, all the considered approaches provide almost the same ranking orders of the evaluated alternatives. One discrepancy occurs when using the similarity measure based on the minimum and maximum operators of the two SVNNs.

In Column I of Table 8, the ranking results obtained by using the score function calculated by using Eq. (7) are shown. As can be seen, the ranking orders obtained by using the score function and the considered similarity measures are the same, except for the ranking order obtained by using the similarity measure based on the minimum and maximum operators of the two SVNNs. The obtained ranking orders of the alternatives are shown in Table 9.

Table 9. The ranking orders of the alternatives obtained by using the considered approaches

	$h_{(i)}$	$e_{(i)}$	$d_{(i)}$	$j_{(i)}$	$c_{(i)}$	$mm_{(i)}$	$s_{(i)}$
A_1	3	3	3	3	3	3	3
A_2	1	1	1	1	1	1	1
A_3	2	2	2	2	2	4	2
A_4	4	4	4	4	4	2	4

The similar ranking results are obtained in the case when the ideal point r^*=<1.00, 0.00, 0.00> was used. The ranking results obtained in that way are shown in Table 10.

Table 10. The ranking results obtained by using the considered approaches

	I		II		III		IV		V		VI		VII	
	$h_{(i)}$		$e_{(i)}$		$d_{(i)}$		$j_{(i)}$		$c_{(i)}$		$mm_{(i)}$		$s_{(i)}$	
A_1	0.71	3	0.49	3	0.84	3	0.29	3	0.87	3	0.22	2	0.46	3
A_2	0.80	1	0.63	1	0.91	1	0.31	1	0.95	1	0.24	1	0.67	1
A_3	0.75	2	0.50	2	0.84	2	0.29	2	0.88	2	0.21	3	0.61	2
A_4	0.70	4	0.46	4	0.82	4	0.29	4	0.85	4	0.21	3	0.45	4

The obtained ranking orders are the same as those in the previous case, and the only discrepancy occurs when using the similarity measure based on the minimum and maximum operators of the two SVNNs.

8. The Analysis

To determine whether all the considered similarity measures generate the same or similar ranking orders of the alternatives, one theoretical case of the ranking in which the overall utility of the alternatives is represented by certain characteristic SVNNs is considered below, as is shown in Table 11.

Table 11. The overall utility of the alternatives

A_1	<1.00, 0.00, 0.00>
A_2	<1.00, 0.00, 1.00>
A_3	<1.00, 1.00, 0.00>
A_4	<1.00, 1.00, 1.00>
A_5	<0.00, 0.00, 0.00>
A_6	<0.00, 0.00, 1.00>
A_7	<0.00, 1.00, 0.00>
A_8	<0.00, 1.00, 1.00>
r^*	<1.00, 0.00, 0.00>

The reference point determined by using Eq. (24) is also shown in Table 11. The calculation details are presented in Table 12.

Table 12. The ranking results obtained by using the considered approaches

	I		II		III		IV		V		VI		VII	
	$h_{(i)}$		$e_{(i)}$		$d_{(i)}$		$j_{(i)}$		$c_{(i)}$		$mm_{(i)}$		$s_{(i)}$	
A_1	1.00	1	1.00	1	1.00	1	0.33	1	1.00	1	0.33	1	1.00	1
A_2	0.67	2	0.42	2	0.67	2	0.25	2	0.71	2	0.33	1	0.50	2
A_3	0.67	2	0.42	2	0.67	2	0.25	2	0.71	2	0.33	1	0.00	4
A_4	0.33	5	0.18	5	0.50	4	0.20	4	0.58	4	0.33	1	-0.50	6
A_5	0.67	2	0.42	2	0.00	5	0.00	5	0.00	5	0.00	5	0.50	2
A_6	0.33	5	0.18	5	0.00	5	0.00	5	0.00	5	0.00	5	0.00	4
A_7	0.33	5	0.18	5	0.00	5	0.00	5	0.00	5	0.00	5	-0.50	6
A_8	0.00	8	0.00	8	0.00	5	0.00	5	0.00	5	0.00	5	-1.00	8

In order to make a comparison, the value for the alternative A_5 obtained by using the cosine similarity measure is set to zero, because this similarity measure cannot be used when all the three membership functions are equal to zero. The obtained ranking results are shown in Table 13 and in Figure 1.

Table 13. The ranking orders of the alternatives obtained by using the considered approaches

	$h_{(i)}$	$e_{(i)}$	$d_{(i)}$	$j_{(i)}$	$c_{(i)}$	$mm_{(i)}$	$s_{(i)}$
A_1	1	1	1	1	1	1	1
A_2	2	2	2	2	2	1	2
A_3	2	2	2	2	2	1	4
A_4	5	5	4	4	4	1	6
A_5	2	2	5	5	5	5	2
A_6	5	5	5	5	5	5	4
A_7	5	5	5	5	5	5	6
A_8	8	8	5	5	5	5	8
max	8	8	5	5	5	5	8

Table 13 shows that the alternative A_1 is the best-ranked and the alternative A_8 is the worst-ranked in all the approaches. However, the ranking order of the other alternatives is slightly different.

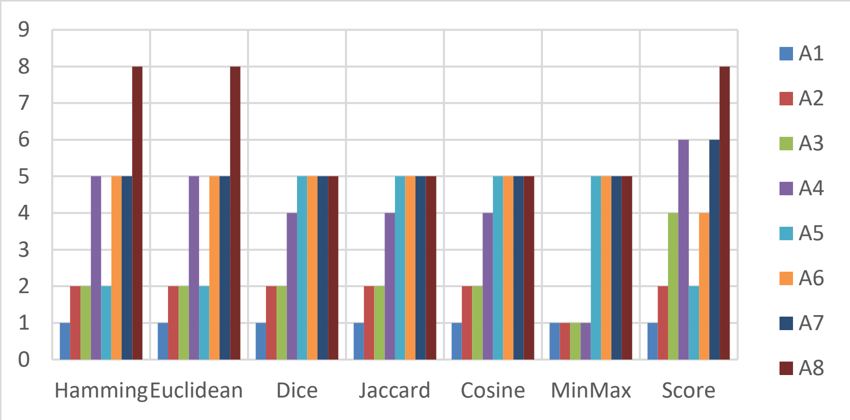

Figure 1. The ranking differences as a result of applying different approaches

The largest discrepancy can be observed in the ranking obtained by using the similarity measure based on the minimum and maximum operators, whereas the highest similarity can be observed in the results obtained by using the Dice, Jaccard and Cosine measures. A high similarity can also be observed in the results obtained by applying the similarity measures based on the Hamming and Euclidean distances.

However, it should be noted that, in this example, the overall significance of the considered alternatives varied from ideally good to ideally bad. In real cases, it can be expected that differences will be less noticeable.

Conclusions

Neutrosophic sets extend the concept proposed in fuzzy and intuitionistic fuzzy sets. These sets enable us to use the three independent membership functions: the truth function, the intermediacy function and the falsity membership function. Therefore, neutrosophic sets can be very useful for solving complex decision-making problems and/or problems related to predictions and uncertainty.

In multi-criteria decision-making, neutrosophic sets enable the evaluation based on the use of a smaller number of more complex evaluation criteria. Despite their complexity, by using appropriate questionnaires, they can be used for an efficient evaluation of alternatives in relation to a set of evaluation criteria.

As a result of applying neutrosophic sets in solving numerous decision-making problems, a few approaches to the ranking of neutrosophic numbers based on similarity measures, as well as a few other, are proposed. Several prominent and less prominent similarity measures, such as the Dice, Jaccard and cosine similarity measures, measure similarity based on the minimum and maximum operators, and the similarity measures derived from the Hamming and Euclidean distances are presented and their use is considered in the case of the two numerical illustrations.

One additional analysis was also carried out so as to determine whether similarity measures always generate the same or similar ranking orders of alternatives.

The obtained results indicate a significant similarity between the results obtained by using certain similarity measures, simultaneously also indicating that in certain cases there are evident differences.

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ECONOMICS

On Rugina's system of thought

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Abstract *This article investigates Rugina's orientation table and gives particular examples for several of its seven models. Leon Walras's* Economics of Stable Equilibrium *and Keynes's* Economics of Disequilibrium *are combined in Rugina's orientation table in systems which are* s *percent stable and* $100 - s$ *percent unstable, where* s *may be 100, 95, 65, 50, 35, 5, and 0. Classical logic and modern logic are united in Rugina's integrated logic, and then generalized in neutrosophic logic.*

I. Introduction

Coming across Rugina's system of thoughts, in his published books and articles (Rugina, 1984, 1989, 1994, 1998), I learned about the connection between classic and modern. It is not a contradiction, but a complementarity from the part of modern with respect to the classic; and always the new "modern" will have something to bring to the old knowledge. In a similar way we may talk on the complementarity between theory and practice, rather than their contradiction.

In economics, Rugina negated Marx's social justice for the mass and Keynes's involuntary unemployment. His methology in science tries to unite all scientific fields, preserving, however, independence in thinking and judgement. Einstein worked in the last period of his life on the unified field theory (a single general theory in physics), but didn't succeed. At the present, his supposition that the speed of light is a barrier in the universe is also being denied.

The economical systems are characterized by free market or centrally-planned and controlled economy. I think each system has a mixture of the previous, where a part of the market is free and another is centrally-planned and controlled.

Rugina's universal hypothesis of duality states that the physical universe is composed of stable and unstable elements arranged in various proportions, may be completed with unknown elements, a strip border between stable and unstable, which are continously changing from the state of equilibrium to disequilibrium and vice-versa, and which therefore are giving the dynamics of the universe.

Unknowns may be:

- anomalies;
- relativities;
- uncertainties;
- revolution risks; and
- hidden parameters.

The internal parameters are involved in Rugina's universal law of natural parameter (NaPa):

> Any system in order to reach and maintain a position of stable equilibrium must have a very strong natural parameter (center of weight).

Whereas the external parameters are involved in Rugina's universal law of general consistency:

> Any system produces and maintains a position of stable equilibrium if there is a suitable space-time frame work.

II. Theory of paradoxes

How did I get to the theory of paradoxes? I have observed that: what's good for someone, may be bad for others – and reciprocally. There are people who are considered terrorists by their enemies, and patriots by their friends. All of them are right and wrong at the same time. If one changes the referential system, the result is different.

Let's see a few nice paradoxes:

Social paradox. In a democracy should the nondemocratic ideas be allowed?

(a) If yes, i.e. the nondemocratic ideas are allowed, then one has not a democracy anymore. (The nondemocratic ideas may overturn the society.)

(b) If no, i.e. other ideas are not allowed – even those nondemocratic – then one has not a democracy either, because the freedom of speech is restricted.

The sets' paradox. The notion of "set of all sets", introduced by Georg Cantor, does not exist.

Let all sets be noted by $\{S_a\}_a$, where a indexes them. But the set of all sets is itself another set, say T_1; and then one constructs again another "set of ⟨all sets⟩", but ⟨all sets⟩ are this time $\{S_a\}$ and T_1, and then the "set of all sets" is now T_2, different from T_1; and so on. Even the notion of "all sets" can not exactly be defined (like the largest number of an open interval, which doesn't exist), as one was just seeing above (we can construct a new set as the "set of all sets") and reunites it to "all sets".

A paradoxist psychological cómplex (with the accent on the first syllable). A collection of fears stemming from previous unsuccessful experience or from unconscious feelings that, wanting to do something ⟨S⟩, the result would be ⟨Anti-S⟩, which give rise to feelings, attitudes, and ideas pushing the subject towards a deviation of action ⟨S⟩ eventually towards an ⟨Anti-S⟩ action. (From the positive and negative brain's electrical activities.) For example, a shy boy, attempting to invite a girl to dance, inhibits himself through fear that she would turn him down. How to manage this phobia? To dote and anti-dote! By transforming it into an opposite one, thinking differently, and being fear in our mind that we would pass our expectancies but we shouldn't. People who do not try for fear of being rejected: they lose by not competing!

Auto-suggestion
If an army leaves for war with anxiety to lose, that army is half-defeated before starting the confrontation.

Paradoxist psychological behavior. How can we explain contrary behaviors of a person: in the same conditions, without any reason, cause? Because our deep unconsciousness is formed of contraries.

Ceaseless anxiety. What you want is, normally, what you don't get. And this is for eternity. Like a chain, because, when you get it (if ever), something else will be your next desire. Man can't live without a new hope.

Inverse desire. The wish to purposely have bad luck, to suffer, to be pessimistic as stimulating factors for more and better creation or work. (Applies to some artists, poets, painters, sculptors, spiritualists.)

All is possible, the impossible too! Is this an optimistic or pessimistic paradox?

(a) It is an optimistic paradox, because it shows that all is possible.

(b) It is a pessimistic paradox, because it shows that the impossible is possible.

Mathematician's paradox. Let *M* be a mathematician who may not be characterized by his mathematical work.

(a) To be a mathematician, *M* should have some mathematical work done, therefore M should be characterized by that work.

(b) The reverse judgement: if *M* may not be characterized by his mathematical work, then *M* is not a mathematician.

Divine paradox (I). Can God commit suicide?

(a) If God cannot, then it appears that there is something God cannot do, therefore God is not omnipotent.

(b) If God can commit suicide, then God dies – because He has to prove it, therefore God is not immortal.

Divine paradox (II). Can God be atheist, governed by scientific laws?

(a) If God can be atheist, then God doesn't believe in Himself, therefore why should we believe in Him?

(b) If God cannot, then again He's not omnipotent.

Religion is full of god-ism and evil-ism. God and evil in the same being. Man is a bearer of good and bad simultaneously. Man is enemy to himself. God and Magog!

Expect the unexpected. If we expect someone to do the unexpected, then:

- Is it possible for him to do the unexpected?
- Is it possible for him to do the expected?

If he does the unexpected, then that's what we expected. If he doesn't do the expected, then he did the unexpected.

The ultimate paradox. Living is the process of dying.

Reciprocally: death of one is the processes of somebody else's life (an animal eating another one).

Exercises for readers:

- If China and Japan are in the Far East, why from the USA do we go west to get there?
- Are humans inhuman, because they committed genocides?

The invisible paradoxes. Our visible world is composed of a totality of invisible particles. Things with mass result from atoms with quasi-null mass. Infinity is formed of finite part(icle)s.

Look at these Sorites paradoxes (associated with Eubulides of Miletus, fourth century BC):

(a) An invisible particle does not form a visible object, nor do two invisible particles, three invisible particles, etc. However, at some point, the collection of invisible particles becomes large enough to form a visible object, but there is apparently no definite point where this occurs.

(b) A similar paradox is developed in an opposite direction. It is always possible to remove an atom from an object in such a way that what is left is still a visible object. However, repeating and repeating this process, at some point, the visible object is decomposed so that the part left becomes invisible, but there is no definite point where this occurs.

Between ⟨A⟩ and ⟨Non-A⟩ there is no clear distinction, no exact frontier. Where does ⟨A⟩ really end and ⟨Non-A⟩ begin? We extend Zadeh's fuzzy set term to fuzzy concept.

Paradoxist existentialism

Life's value consists in its lack of value; life's sense consists in its lack of sense.

Semantic Paradox (I): "I am who I am not". If I am not Socrates, and since I am who I am not, it results that I am Socrates. If I am Socrates, and since I am who I am not, it results that I am not Socrates. Generally speaking: "I am X" if and only if "I am not X". Who am I?

In a similar pattern one constructs the paradoxes: "I am myself when I am not myself"; and "I exist when I don't exist". And, for the most part, "I {verb} when I don't {verb}" (Smarandache, 1997).

What is a dogma? Is it an idea that makes you have no other idea. How can we get rid of such authoritative tenet? (To un-read and un-study it!)

Semantic Paradox (II): "I don't think". This can not be true for, in order to even write this sentence, I needed to think (otherwise I was writing with mistakes, or was not writing it at all). Whence "I don't think" is false, which means "I think".

Unsolved mysteries:

(a) Is it true that for each question there is at least an answer?

(b) Is any statement the result of a question?

(c) Let $P(n)$ be the following assertion: "If $S(n)$ is true, then $S(n+1)$ is false", where $S(n)$ is a sentence relating on parameter n. Can we prove by mathematical induction that $P(n)$ is true?

(d) "$\langle$A$\rangle$ is true if and only if $\langle$A$\rangle$ is false". Is this true or false?

(e) How can this assertion "Living without living" be true? Find a context. Explain.

Other examples of such paradoxes are:

- $\langle$Anti-A$\rangle$ of $\langle$A$\rangle$ (anti-literature of literature);
- $\langle$Non-A$\rangle$ of $\langle$A$\rangle$ (language of non-language); and
- $\langle$A$\rangle$ of $\langle$Non-A$\rangle$ (artistic of the non-artistic).

Tautologies. I want because I want (showing will, ambition) ($\langle$A$\rangle$ because of $\langle$A$\rangle$) (Smarandache, 1997).

Other examples include:

- Our axiom is to break down all axioms.
- Be patient without patience.
- Non-existence exists.
- Culture exists through its non-existence.
- Our culture is our lack of culture.
- Style without style.
- The rule we apply: there is no rule.

Paradox of the paradoxes

Is "this is a paradox" a paradox? I mean is it true or false? Examples are:

- To speak without speaking (without words (body language)).
- To communicate without communicating.
- To do the impossible.
- To know nothing about everything, and everything about nothing.
- I do only what I can't! If I can't do something, of course "I can do" is false. And, if I can do, it's also false because I can do only what I am not able to do.
- I cannot for I can.
- Paradoxal sleep, from a French "*Larousse*" dictionary (1989), is a phase of the sleep when the dreams occur. Sleep, sleep, but why paradoxal? How do the dreams put up with reality?

- Is O.J. Simpson's crime trial an example of: justice of injustice, or injustice of justice? However, his famous release is a victory against the system!
- Corrupt the incorruptible!

Everything which is not paradoxist, is, however, paradoxist. This is the great universal paradox. A superparadox (as a superman in a hyperspace).

Facts exist in isolation from other facts (the analytic philosophy), and in connection as well with each other (Whitehead's and Bergson's thoughts). The neutrosophic philosophy unifies contradictory and noncontradictory ideas in any human field.

The antagonism doesn't exist. Or, if the antagonism does exist, this becomes (by neutrosophic view) a non- (or un-)antagonism: a normal thought. I don't worry about it as well as Wordsworth.

Platonism is the observable of unobservable, the thought of the non-thought.

The essence of a thing may never be reached. It is a symbol, a pure and abstract and absolute notion.

An action may be considered g percent good (or right) and b percent bad (or wrong), where $0 \leq g + b \leq 100$ – the remainder being indeterminacy, not only ⟨good⟩ or only ⟨bad⟩ – with rare exceptions, if its consequence is g percent happiness (pleasure). In this case the action is g percent-useful (in a semi-utilitarian way). Utilitarism shouldn't work with absolute values only!

Verification has a pluri-sense because we have to demonstrate or prove that something is t percent true, and f percent false, where $0 \leq t, f \leq 100$ and $t + f \leq 100$, not only $t = 0$ or 100 – which occurs in rare/absolute exceptions, by means of formal rules of reasoning of this neutrosophic philosophy.

The logical cogitation's structure is discordant. Scientism and empiricism are strongly related. They can't run one without other, because one exists in order to complement the other and to differentiate it from its opponent. Plus doesn't work without minus, and both of them supported by zero. They all are cross-penetrating sometimes up to confusion. The non-understandable is understandable. If vices wouldn't exist, the virtues will not be seen (T. Muşatescu).

Any new born theory (notion, term, event, phenomenon) automatically generates its non-theory – not necessarily anti-(notion, term, event, phenomenon). Generally speaking, for any ⟨A⟩ a ⟨Non-A⟩ (not necessarily ⟨Anti-A⟩) will exist for compensation.

The neutrosophy is a theory of theories, because at any moment new ideas and conceptions are appearing and implicitly their negative and neutral senses are highlighted.

The non-important is important, because the first one is the second one's shadow that makes it grow its value. The important things would not be so without any unimportant comparison.

The neutrosophic philosophy accepts *a priori* and *a posteriori* any philosophical idea, but associates it with adverse and neutral ones, as a *summum*. This is to be neutrosophic without being! Its schemes are related to the neutrality of everything.

III. On Rugina's orientation table

Starting from a new viewpoint in philosophy, the neutrosophy, one extends the classical "probability theory", "fuzzy set" and "fuzzy logic" to ⟨neutrosophic probability⟩, ⟨neutrosophic set⟩ and ⟨neutrosophic logic⟩ respectively.

They are useful in artificial intelligence, neural networks, evolutionary programming, neutrosophic dynamic systems, quantum theory, and decision making in economics.

With the neutrosophic logic help one explores Rugina's orientation table, a remarkable tool of study, at the micro- and macro-level, of problems in all sciences.

(1) Neutrosophy: a new branch of mathematical philosophy

(A) *Etymology*. Neutro-sophy (French *neutre* < Latin *neuter*, neutral, and Greek *sophia*, skill/wisdom) means knowledge of neutral thought.

(B) *Definition*. Neutrosophy is a new branch of philosophy which studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

(C) *Characteristics*. This mode of thinking:

- proposes new philosophical theses, principles, laws, methods, formulas, movements;
- interprets the uninterpretable;
- regards, from many different angles, old concepts, systems: showing that an idea, which is true in a given referential system, may be false in another one, and vice versa;
- measures the stability of unstable systems, and instability of stable systems.

(D) *Methods of neutrosophic study*. The methods of neutrosophic study are mathematization (neutrosophic logic, neutrosophic probability and statistics, duality), generalization, complementarity, contradiction, paradox, tautology, analogy, reinterpretation, combination, interference, aphoristic, linguistic, and multidisciplinarity.

(E) *Formalization*. Let's note by ⟨A⟩ an idea or theory or concept, by ⟨Non-A⟩ what is not ⟨A⟩, and by ⟨Anti-A⟩ the opposite of ⟨A⟩. Also, ⟨Neut-A⟩ means what is neither ⟨A⟩, nor ⟨Anti-A⟩, i.e. neutrality in between the two extremes. And ⟨A′⟩ a version of ⟨A⟩. ⟨Non-A⟩ is different from ⟨Anti-A⟩.

For example, if ⟨A⟩ = white, then ⟨Anti-A⟩ = black (antonym), but ⟨Non-A⟩ = green, red, blue, yellow, black, etc. (any color, except white), while ⟨Neut-A⟩ = green, red, blue, yellow, etc. (any color, except white and black), and ⟨A′⟩ = dark white, etc. (any shade of white). ⟨Neut-A⟩ ≡ ⟨Neut-(Anti-A)⟩, neutralities of ⟨A⟩ are identical with neutralities of ⟨Anti-A⟩.

$\langle$Non-A$\rangle \supset \langle$Anti-A$\rangle$, and $\langle$Non-A$\rangle \supset \langle$Neut-A$\rangle$ as well, also $\langle$A$\rangle \cap \langle$Anti-A$\rangle = \phi$, $\langle$A$\rangle \cap \langle$Non-A$\rangle = \phi$. $\langle$A$\rangle$, $\langle$Neut-A$\rangle$, and $\langle$Anti-A$\rangle$ are disjoint two by two. $\langle$Non-A$\rangle$ is the completitude of $\langle$A$\rangle$ with respect to the universal set.

(F) *Main principle.* Between an idea $\langle$A$\rangle$ and its opposite $\langle$Anti-A$\rangle$, there is a continuum-power spectrum of neutralities $\langle$Neut-A$\rangle$.

(G) *Fundamental thesis.*

Any idea $\langle$A$\rangle$ is t percent true, i percent indeterminate, and f percent false, where $t + i + f = 100$.

(H) *Main laws.* Let $\langle\alpha\rangle$ be an attribute, and $(a, i, b) \varepsilon [0, 100]^3$, with $a + i + b = 100$. Then:

- There is a proposition $\langle$P$\rangle$ and a referential system $\langle R\rangle$, such that $\langle$P$\rangle$ is a percent $\langle\alpha\rangle$, i percent indeterminate or $\langle$Neut-$\alpha\rangle$, and b percent $\langle$Anti-$\alpha\rangle$.
- For any proposition $\langle$P$\rangle$, there is a referential system $\langle R\rangle$, such that $\langle$P$\rangle$ is a percent $\langle\alpha\rangle$, i percent indeterminate or $\langle$Neut-$\alpha\rangle$, and b percent $\langle$Anti-$\alpha\rangle$.
- $\langle\alpha\rangle$ is at some degree $\langle$Anti-$\alpha\rangle$, while $\langle$Anti-$\alpha\rangle$ is at some degree $\langle\alpha\rangle$.

(2) Neutrosophic probability and neutrosophic statistics

Let's first generalize the classical notions of "probability" and "statistics" for practical reasons.

(A) *Definitions.* Neutrosophic probability studies the chance that a particular event E will occur, where that chance is represented by three coordinates (variables): t percent true, i percent indeterminate, and f percent false, with $t + i + f = 100$ and $f, i, t \; \varepsilon \; [0, 100]$. Neutrosophic statistics is the analysis of such events.

(B) *Neutrosophic probability space.* The universal set, endowed with a neutrosophic probability defined for each of its subsets, forms a neutrosophic probability space.

(C) *Applications.*

(1) The probability that candidate C will win an election is say 25 percent true (percentage of people voting for him), 35 percent false (percentage of people voting against him), and 40 percent indeterminate (percentage of people not coming to the ballot box, or giving a blank vote – not selecting anyone, or giving a negative vote – cutting all candidates on the list). Dialectic and dualism don't work in this case anymore.

(2) Another example, the probability that tomorrow it will rain is say 50 percent true according to meteorologists who have investigated the past years' weather, 30 percent false according to today's very sunny and droughty summer, and 20 percent undecided (indeterminate).

(3) Neutrosophic set

Let's second generalize, in the same way, the fuzzy set.

(A) *Definition.* Neutrosophic set is a set such that an element belongs to the set with a neutrosophic probability, i.e. t percent is true that the element is in the set, f percent false, and i percent indeterminate.

(B) *Neutrosophic set operations.* Let M and N be two neutrosophic sets. One can say, by language abuse, that any element neutrosophically belongs to any set, due to the percentage of truth/indeterminacy/falsity which varies between 0 and 100. For example: $x(50, 20, 30)\varepsilon M$ (which means, with a probability of 50 percent x is in M, with a probability of 30 percent x is not in M, and the rest is undecidable), or $y(0, 0, 100)\varepsilon M$ (which normally means y is not for sure in M), or $z(0, 100, 0)\varepsilon M$ (which means one doesn't know absolutely anything about z's affiliation with M).

Let $0 \leq t_1, t_2, t' \leq 1$ represent the truth-probabilities, $0 \leq i_1, i_2, i' \leq 1$ the indeterminacy-probabilities, and $0 \leq f_1, f_2, f' \leq 1$ the falsity-probabilities of an element x to be in the set M and in the set N respectively, and of an element y to be in the set N, where $t_1 + i_1 + f_1 = 1, t_2 + i_2 + f_2 = 1$, and $t' + i' + f' = 1$.

One notes, with respect to the given sets, $x = x(t_1, i_1, f_1)\varepsilon M$ and $x = x(t_2, i_2, f_2)\varepsilon N$, by mentioning x's neutrosophic probability appurtenance. And, similarly, $y = y(t', i', f')\varepsilon N$.

Also, for any $0 \leq x \leq 1$ one notes $1 - x = \bar{x}$. Let $W(a, b, c) = (1 - a)/(b + c)$ and $W(R) = W(R(t), R(i), R(f))$ for any tridimensional vector $R = (R(t), R(i), R(f))$.

Complement of M: Let $N(x) = 1 - x = \bar{x}$. Therefore: if $x(t_1, i_1, f_1)\varepsilon M$, then $x(N(t_1), N(i_1)W(N), N(f_1)W(N))\varepsilon C(M)$.

Intersection. Let $C(x, y) = xy$, and $C(z_1, z_2) = C(z)$ for any bidimensional vector $z = (z_1, z_2)$. Therefore: if $x(t_1, i_1, f_1)\varepsilon M$, $x(t_2, i_2, f_2)\varepsilon N$, then $x(C(t), C(i)W(C), C(f)W(C))\varepsilon M \cap N$.

Union. Let $D1(x, y) = x + y - xy = x + \bar{x}y = y + x\bar{y}$, and $D1(z_1, z_2) = D1(z)$ for any bidimensional vector z = (z_1, z_2). Therefore: if $x(t_1, i_1, f_1)\varepsilon M$, $x(t_2, i_2, f_2)\varepsilon N$, then $x(D1(t), D1(i)W(D1), D1(f)W(D1))\varepsilon M \cup N$.

Cartesian product. If $x(t_1, i_1, f_1)\varepsilon M, y(t', i', f')\varepsilon N$, then $(x(t_1, i_1, f_1), y(t', i', f'))\varepsilon MxN$.

Difference. Let $D(x, y) = x - xy = x\bar{y}$, and $D(z_1, z_2) = D(z)$ for any bidimensional vector $z = (z_1, z_2)$. Therefore: if $x(t_1, i_1, f_1)\varepsilon M$, $x(t_2, i_2, f_2)\varepsilon N$, then $x(D(t), D(i)W(D), D(f)W(D))\varepsilon M/N$, because $M/N = M \cap C(N)$.

(C) *Applications.* From a pool of refugees, waiting in a political refugee camp to get the American visa of emigration, a percent are accepted, r percent rejected, and p percent in pending (not yet decided), $a + r + p = 100$. The chance for someone in the pool to emigrate to USA is not a percent as in classical probability, but a percent true and p percent pending (therefore normally bigger than a percent) – because later, the p percent pending refugees will be distributed into the first two categories, either accepted or rejected.

Another example, a cloud is a neutrosophic set, because its borders are ambiguous, and each element (water drop) belongs with a neutrosophic probability to the set (e.g. there are separated water drops, around a compact mass of water drops, that we don't know how to consider them: in or out of the cloud). We are not sure where the cloud ends nor where it begins, neither if some elements are or are not in the set. That's why the percent of indeterminacy is required: for a more organic, smooth, and especially accurate estimation.

(4) Neutrosophic logic: a generalization of fuzzy logic

(A) *Introduction.* One passes from the classical $\{0, 1\}$ bivalent logic of George Boole, to the three-valued logic of Reichenbach (leader of the logical empiricism), then to the $\{0, a_1, \ldots, a_n, 1\}$ plurivalent one of Lukasiewicz (and Post's m-valued calculus), and finally to the $[0, 1]$ infinite logic as in mathematical analysis and probability: a transcendental logic (with values of the power of continuum), or fuzzy logic.

Falsehood is infinite, and truthhood quite alike; in between, at different degrees, indeterminacy as well. Everything is G percent good, I percent indeterminate, and B percent bad, where $G + I + B = 100$.

Besides Diderot's dialectics on good and bad ("Rameau's Nephew", 1772), any act has its percentage of "good", "indeterminate", and of "bad" as well incorporated.

Rodolph Carnap said:

> Metaphysical propositions are neither true nor false, because they assert nothing, they contain neither knowledge nor error...

Hence, there are infinitely many statuses in between "good" and "bad", and generally speaking in between "A" and "Anti-A", like on the real number segment:

[0,	1]
False	True
Bad	Good
Non-sense	Sense
Anti-A	A

0 is the absolute falsity, 1 the absolute truth. In between each opposite pair, normally in a vicinity of 0.5, are being set up the neutralities.

There exist as many states in between "true" and "false" as in between "good" and "bad". Irrational and transcendental standpoints belong to this interval.

Even if an act apparently looks to be only good, or only bad, the other headed side should be sought. The following ratios vary indefinitely

$$\frac{\text{Anti-A}}{\text{A}}, \frac{\text{Non-A}}{\text{A}}.$$

They are transfinite.

If a statement is 30 percent T (true) and 60 percent I (indeterminate), then it is 10 percent F (false). This is somehow alethic, meaning pertaining to truthhood and falsehood in the same time. In opposition to fuzzy logic, if a statement is 30 percent T doesn't involve it is 70 percent F. We have to study its indeterminacy as well.

(B) *Definition of neutrosophic logic.* This is a generalization (for the case of null indeterminacy) of the fuzzy logic. Neutrosophic logic is useful in the real-world systems for designing control logic, and may work in quantum mechanics.

If a proposition P is t percent true, doesn't necessarily mean it is $100 - t$ percent false as in fuzzy logic. There should also be a percentage of indeterminacy on the values of P. A better approach of the logical value of P is f percent false, i percent indeterminate, and t percent true, where $t + i + f = 100$ and $t, i, f \varepsilon [0, 100]$, called neutrosophic logical value of P, and noted by $n(P) = (t, i, f)$.

Neutrosophic logic means the study of neutrosophic logical values of the propositions. There exist, for each individual event, PRO parameters, CONTRA parameters, and NEUTER parameters which influence the above values. Indeterminacy results from any hazard which may occur, from unknown parameters, or from new arising conditions. This resulted from practice.

(C) *Applications.*

(1) "The candidate C, who runs for election in a metropolis M of p people with right to vote, will win". This proposition is, say, 25 percent true (percentage of people voting for him), 35 percent false (percentage of people voting against him), and 40 percent indeterminate (percentage of people not coming to the ballot box, or giving a blank vote – not selecting anyone, or giving a negative vote – cutting all candidates on the list).

(2) "Tomorrow it will rain". This proposition is, say, 50 percent true according to meteorologists who have investigated the past years' weather, 30 percent false according to today's very sunny and droughty summer, and 20 percent undecided.

(3) "This is a heap". As an application to the sorites paradoxes, we may now say this proposition is t percent true, f percent false, and i percent indeterminate (the neutrality comes for we don't know exactly where is the difference between a heap and a non-heap; and, if we approximate the border, our "accuracy" is subjective).

We are not able to distinguish the difference between yellow and red as well if a continuum spectrum of colors is painted on a wall imperceptibly changing from one into another.

(D) *Definition of neutrosophic logical connectors.* One uses the definitions of neutrosophic probability and neutrosophic set. Let $0 \leq t_1,\ t_2 \leq 1$ represent the truth-probabilities, $0 \leq i_1, i_2 \leq 1$ the indeterminacy-probabilities, and $0 \leq f_1, f_2 \leq 1$ the falsity-probabilities of two events P_1 and P_2 respectively, where $t_1 + i_1 + f_1 = 1$ and $t_2 + i_2 + f_2 = 1$. One notes the neutrosophic logical values of P_1 and P_2 by:

$$n(P_1) = (t_1, i_1, f_1) \text{ and } n(P_2) = (t_2, i_2, f_2).$$

Also, for any $0 \leq x \leq 1$ one notes $1 - x = \bar{x}$. Let $W(a, b, c) = (1 - a)/(b + c)$ and $W(R) = W(R(t), R(i), R(f))$ for any tridimensional vector $R = (R(t), R(i), R(f))$.

Negation. Let $N(x) = 1 - x = \bar{x}$. Then:

$$n(\neg P_1) = (N(t_1), N(i_1)W(N), N(f_1)W(N)).$$

Conjunction. Let $C(x, y) = xy$, and $C(z_1, z_2) = C(z)$ for any bidimensional vector $z = (z_1, z_2)$. Then:

$$n(P_1 \wedge P_2) = (C(t), C(i)W(C), C(f)W(C)).$$

(And, in a similar way, generalized for n propositions.)

Weak or inclusive disjunction. Let $D1(x, y) = x + y - xy = x + \bar{x}y = y + x\bar{y}$, and $D1(z_1, z_2) = D1(z)$ for any bidimensional vector $z = (z_1, z_2)$. Then:

$$n(P_1 \vee (P_2) = (D1(t), D1(i)W(D1), D1(f)W(D1)).$$

(And, in a similar way, generalized for n propositions.)

Strong or exclusive disjunction. Let $D2(x, y) = x(1 - y) + y(1 - x) - xy(1 - x)(1 - y) = x\bar{y} + \bar{x}y - \bar{x}\bar{y}xy$, and $D2(z_1, z_2) = D2(z)$ for any bidimensional vector $z = (z_1, z_2)$. Then:

$$n(P_1 \underline{\vee} P_2) = (D2(t), D2(i)W(D2), D2(f)W(D2)).$$

(And, in a similar way, generalized for n propositions.)

Material conditional (implication). Let $I(x, y) = 1 - x + xy = \bar{x} + xy = 1 - x\bar{y}$, and $I(z_1, z_2) = I(z)$ for any bidimensional vector $z = (z_1, z_2)$. Then:

$$n(P_1 \rightarrow P_2) = (I(t), I(i)W(I), I(f)W(I)).$$

Material biconditional (equivalence). Let $E(x, y) = (1 - x + xy)(1 - y + xy) = (\bar{x} + xy)(\bar{y} + xy) = (1 - x\bar{y})(1 - \bar{x}y)$, and $E(z_1, z_2) = E(z)$ for any bidimensional vector $z = (z_1, z_2)$:

$$n(P \leftrightarrow Q) = (E(t), E(i)W(E), E(f)W(E)).$$

Sheffer's connector. Let $S(x, y) = 1 - xy$, and $S(z_1, z_2) = S(z)$ for any bidimensional vector $z = (z_1, z_2)$:

$$n(P \mid Q) = n(\neg P \vee \neg Q) = (S(t), S(i)W(S), S(f)W(S)).$$

Peirce's connector: Let $P(x, y) = (1 - x)(1 - y) = \overline{x}\overline{y}$, and $P(z_1, z_2) = P(z)$ for any bidimensional vector $z = (z_1, z_2)$.

$$n(P \downarrow Q) = n(\neg P \wedge \neg Q) = (P(t), P(i)W(P), P(f)W(P)).$$

(E) *Properties of neutrosophic logical connectors.*

Let's note by $t(P)$ the truth-component of the neutrosophic value $n(P)$, and $t(P) = p, t(Q) = q$.

(a) *Conjunction.* $t(P \wedge Q) \leq \min\{p, q\}$:

$$\bigwedge_{k=1}^{\infty}(t(P) = 0 \text{ if } t(P) \neq 1.$$

(b) *Weak disjunction.* $t(P \vee Q) \geq \max\{p, q\}$:

$$\bigvee_{k=1}^{\infty}(t(P) = 1 \text{ if } t(P) \neq 0.$$

(c) *Implication.* $t(P \to P) = 1$ if $t(P) = 0$ or 1, and $> p$ otherwise:

$$\lim_{t(P)\to 0} t(P \to Q) = 1$$

$$\lim_{t(Q)\to 1} t(P \to Q) = 1$$

$$\lim_{t(P)\to 1} t(P \to Q) = q$$

$$\lim_{t(Q)\to 0} t(P \to Q) = 1 - p$$

(d) *Equivalence.* $t(P \leftrightarrow Q) = t(Q \leftrightarrow P) = t(\neg P \leftrightarrow \neg Q)$:

$$\lim_{\substack{t(P)\to 0\\ t(Q)\to 0}} t(P \leftrightarrow Q) = 1$$

$$\lim_{\substack{t(P)\to 1\\ t(Q)\to 1}} t(P \leftrightarrow Q) = 1$$

$$\lim_{\substack{t(P)\to 0\\ t(Q)\to 1}} t(P \leftrightarrow Q) = 0$$

$$\lim_{\substack{t(P)\to 1\\ t(Q)\to 0}} t(P \leftrightarrow Q) = 0$$

$$\lim_{t(P)\to 0} t(P \leftrightarrow Q) = 1 - q$$

$$\lim_{t(P)\to 1} t(P \leftrightarrow Q) = q$$

Let $q \neq 0, 1$ be constant, and one notes $p_{\max}(q) = (q^2 - 3q + 1)/(2q^2 - 2q)$. Then $\max t(P \leftrightarrow Q)$ occurs when $0 \leq t(P) \leq 1.\ p = p_{\max}(q)$ if $p_{\max}(q) \varepsilon [0, 1]$, or $p = 0$ if $p_{\max}(q) < 0$, or $p = 1$ if $p_{\max}(q) > 1$, because the equivalence connector is described by a parabola of equation:

$$e_q(p) = (q^2 - q)p^2 + (-q^2 + 3q - 1)p + (1 - q).$$

This equation is concave down.

(5) Neutrosophic topology

(A) *Definition.* Let's construct a neutrosophic topology on $NT = [0, 1]$, considering the associated family of subsets $(0, p)$, for $0 \leq p \leq 1$, the whole set $[0, 1]$, and the empty set $\phi = (0, 0)$, called open sets, which is closed under set union and finite intersection. The union is defined as $(0, p) \cup (0, q) = (0, d)$, where $d = p + q - pq$, and the intersection as $(0, p) \cap (0, q) = (0, c)$, where $c = pq$. The complementary of $(0, p)$ is $(0, n)$, where $n = 1 - p$, which is a closed set.

(B) *Neutrosophic topological space.* The interval NT, endowed with this topology, forms a neutrosophic topological space.

(C) *Isomorphicity.* Neutrosophic logical space, neutrosophic topological space, and neutrosophic probability space are all isomorphic.

A method of neutrosophy is described below.

(6) Transdisciplinarity

(A) *Introduction.* Transdisciplinarity means to find common features to uncommon entities: $\langle \mathrm{A} \rangle \cap \langle \text{Non-A} \rangle \neq \phi$, even if they are disjunct.

(B) *Multi-structure and multi-space.* I consider that life and practice do not deal with "pure" spaces, but with a group of many spaces, with a mixture of structures, a "mongrel", a heterogeneity – the ardently preoccupation is to reunite them, to constitute a multi-structure.

I thought to a multi-space also: fragments (potsherds) of spaces put together, say as an example: Banach, Hausdorff, Tikhonov, compact, paracompact, Fock symmetric, Fock antisymmetric, path-connected, simply connected, discrete metric, indiscrete pseudo-metric, etc. spaces that work together as a whole mechanism. The difficulty is to be the passage over "frontiers" (borders between two disjoint spaces); i.e. how can we organically tie a point $P1$ from a space $S1$ with a point $P2$ from a structurally opposite space $S2$?

Does the problem become more complicated when the spaces' sets are not disjoint?

Let S_1 and S_2 be two distinct structures, induced by the group of laws L which verify the axiom groups A_1 and A_2 respectively, such that A_1 is strictly included in A_2.

One says that the set M, endowed with the properties is called an S_1-structure with respect to the S_2-structure:

- M has an S_1-structure;
- there is a proper subset P (different from the empty set, from the unitary element, and from M) of the initial set M which has an S_2-structure; and
- M doesn't have an S_2-structure.

Let S_1, S_2, …, S_k be distinct space-structures. We define the multi-space (or k-structured-space) as a set M such that for each structure S_i, $1 \leq i \leq k$, there is a proper (different from ϕ and from M) subset M_i of it which has that structure. The $M_1, M_2, \ldots, M_k$ proper subsets are different two by two.

Let's introduce new terms.

(C) *Psychomathematics.* A discipline which studies psychological processes in connection with mathematics.

(D) *Mathematical modeling of psychological processes.* Weber's law and Fechner's law on sensations and stimuli are improved.

(E) *Psychoneutrosophy.* Psychology of neutral thought, action, behavior, sensation, perception, etc. This is a hybrid field deriving from theology, philosophy, economics, psychology, etc. For example, to find the psychological causes and effects of individuals supporting neutral ideologies (neither capitalists, nor communists), politics (not in the left, not in the right), etc.

(F) *Socioneutrosophy.* Sociology of neutralities. For example, the sociological phenomena and reasons which determine a country or group of people or class to remain neutral in a military, political, ideological, cultural, artistic, scientific, economical, etc. international or internal war (dispute).

(G) *Econoneutrosophy.* Economics of non-profit organizations, groups, such as: churches, philanthropic associations, charities, emigrating foundations, artistic or scientific societies, etc. How they function, how they survive, who benefits and who loses, why are they necessary, how they improve, how they interact with for-profit companies.

These terms are in the process of development.

(7) Rugina's orientation table

In order to clarify the anomalies in science, Rugina (1989, 1998) proposes an original method, starting first from an economic point of view but generalizing it to any science, to study the equilibrium and disequilibrium of systems. His table comprises seven basic models:

(1) Model M_1 (which is 100 percent stable)

(2) Model M_2 (which is 95 percent stable, and 5 percent unstable);

(3) Model M_3 (which is 65 percent stable, and 35 percent unstable);

(4) Model M_4 (which is 50 percent stable, and 50 percent unstable);

(5) Model M_5 (which is 35 percent stable, and 65 percent unstable);

(6) Model M_6 (which is 5 percent stable, and 95 percent unstable); and

(7) Model M_7 (which is 100 percent unstable)

He gives orientation tables for physical sciences and mechanics (Rugina, 1989, p. 18), for the theory of probability, for logic, and generally for any natural or social science (Rugina, 1989, pp. 286-88):

> An anomaly can be simply defined as a deviation from a position of stable equilibrium represented by Model M_1 (Rugina, 1989, p. 17).

Rugina proposes the universal hypothesis of duality:

> The physical universe in which we are living, including human society and the world of ideas, all are composed in different and changeable proportions of stable (equilibrium) and unstable (disequilibrium) elements, forces, institutions, behavior and value.

He also proposes the general possibility theorem:

> ...there is an unlimited number of possible combinations or systems in logic and other sciences.

According to the last assertations one can extend Rugina's orientation table in the way that any system in each science is s percent stable and u percent unstable, with $s+u=100$ and both parameters $0 \leq s, u \leq 100$, somehow getting to a fuzzy approach.

But, because each system has hidden features and behaviors, and there would always be unexpected occuring conditions we are not able to control – we mean the indeterminacy plays a role as well, a better approach would be the neutrosophic model:

> Any system in each science is s percent stable, i percent indeterminate, and u percent unstable, with $s+i+u=100$ and all three parameters $0 \leq s, i, u \leq 100$.

Examples of Rugina's orientation table are given in Appendices 1-5.

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Appendix 1. Example of model M_3 in Rugina's orientation table

The paradoxist geometry (actually the percentage of instability is between 20-35)

In 1969, intrigued by geometry, I simultaneously constructed a partially Euclidean and partially non-Euclidean space by a strange replacement of the Euclid's fifth postulate (axiom of parallels) with the following five-statement proposition:

(1) there are at least a straight line and a point exterior to it in this space for which only one line passes through the point and does not intersect the initial line (1 parallel);

(2) there are at least a straight line and a point exterior to it in this space for which only a finite number of lines $l_1, \ldots, l_k (k \geq 2)$ pass through the point and do not intersect the initial line (two or more (in a finite number) parallels);

(3) there are at least a straight line and a point exterior to it in this space for which any line that passes through the point intersects the initial line (0 parallels);

(4) there are at least a straight line and a point exterior to it in this space for which an infinite number of lines that pass through the point (but not all of them) do not intersect the initial line (an infinite number of parallels, but not all lines passing through); and

(5) there are at least a straight line and a point exterior to it in this space for which any line that passes through the point does not intersect the initial line (an infinite number of parallels, all lines passing through the point).

This geometry unites all together: Euclid, Lobachevsky/Bolyai, and Riemann geometries. And separates them as well!

Appendix 2. First example of model M_7 in Rugina's orientation table

The non-geometry (the percentage of instability is 100)

It's a lot easier to deny the Euclid's five postulates than Hilbert's 20 thorough axioms:

(1) It is not always possible to draw a line from an arbitrary point to another arbitrary point. For example this axiom can be denied only if the model's space has at least a discontinuity point (in our model bellow, MD, one takes an isolated point I in between $f1$ and $f2$, the only one which will not verify the axiom).

(2) It is not always possible to extend by continuity a finite line to an infinite line. For example, consider the model bellow, and the segment AB, where both A and B lie on $f1$, A in between P and N, while B on the left side of N; one can not at all extend AB either beyond A or beyond B, because the resulted curve, noted say A'-A-B-B', would not be a geodesic (i.e. line in our Model) anymore.

If A and B lie in delta1-$f1$, both of them closer to $f1$, A in the left side of P, while B in the right side of P, then the segment AB, which is in fact A-P-B, can be extended beyond A and also beyond B only up to $f1$ (therefore one gets a finite line too, A′-A-P-B-B′, where A', B' are the intersections of PA, PB respectively with $f1$). If A, B lie in delta1-$f1$, far enough from $f1$ and P, such that AB is parallel to $f1$, then AB verifies this postulate.

(3) It is not always possible to draw a circle from an arbitrary point and of an arbitrary interval. For example, same as for the first axiom, the isolated point I, and a very small interval not reaching $f1$ neither $f2$, will deny this axiom.

(4) Not all the right angles are congruent. (See example of the anti-geometry, explained below.)

(5) If a line, cutting two other lines, forms the interior angles of the same side of it strictly less than two right angles, then not always the two lines extended towards infinite cut each other in the side where the angles are strictly less than two right angles. For example, let $h1$, $h2$ and l be three lines in delta1-delta2, where $h1$ intersects $f1$ in A, and $h2$ intersects $f1$ in B, with A, B, P different from each other, such that $h1$ and $h2$ do not intersect, but l cuts $h1$ and $h2$ and forms the interior angles of one of its sides (towards

$f1$) strictly less than two right angles; the assumption of the fifth postulate is fulfilled, but the consequence does not hold, because $h1$ and $h2$ do not cut each other (they may not be extended beyond A and B respectively, because the lines would not be geodesics anymore).

Appendix 3. Second example of model M_7 in Rugina's orientation table

The counter-projective geometry (the percentage of instability is 100)

Let P, L be two sets, and r a relation included in $P \times L$. The elements of P are called points, and those of L lines. When (p, l) belongs to r, we say that the line l contains the point p. For these, one imposes the following counter-axioms:

- There exist: either at least two lines, or no line, that contains two given distinct points.
- Let $p1$, $p2$, $p3$ be three non-collinear points, and $q1$, $q2$ two distinct points. Suppose that $\{p1, q1, p3\}$ and $\{p2, q2, p3\}$ are collinear triples. Then the line containing $p1$, $p2$, and the line containing $q1$, $q2$ do not intersect.
- Every line contains at most two distinct points.

Does the duality principle hold in a counter-projective space? What about Desargues's theorem, fundamental theorem of projective geometry/theorem of Pappus, and Staudt algebra? Or Pascal's theorem, Brianchon's theorem? (I think none of them will hold!) However, Rugina's hypothesis of duality does hold (althrough the this geometry is formed by unstable elements only!).

Appendix 4. Third example of model M_7 in Rugina's orientation table

The anti-geometry (the percentage of instability is 100 – even... more, this is the geometry of total chaos!)

It is possible to entirely de-formalize Hilbert's groups of axioms of the Euclidean geometry, and to construct a model such that none of his fixed axioms holds.

Let's consider the following things:

- a set of ⟨points⟩: $A, B, C, \ldots$;
- a set of ⟨lines⟩: $h, k, l, \ldots$; and
- a set of ⟨planes⟩: alpha, beta, gamma, ...

Let us also consider a set of relationships among these elements: "are situated", "between", "parallel", "congruent", "continuous", etc.

Then, we can deny all Hilbert's 20 axioms (see Hilbert, 1950; Binola, 1938).

There exist cases, within a geometric model, when the same axiom is verified by certain points/lines/planes and denied by others.

Group I. Anti-axioms connection

I.1. Two distinct points A and B do not always completely determine a line.

Let's consider the following model MD: get an ordinary plane delta, but with an infinite hole as shown in Fugure A1.

Plane delta is a reunion of two disjoint planar semi-planes; $f1$ lies in MD, but $f2$ does not; P, Q are two extreme points on f that belong to MD.

One defines a LINE l as a geodesic curve: if two points A, B that belong to MD lie in l, then the shortest curve lying in MD between A and B lies in l also. If a line passes twice through the same point, then it is called double point (KNOT).

One defines a PLANE alpha as a surface such that for any two points A, B that lie in alpha and belong to MD there is a geodesic which passes through A, B and lies in alpha also.

Now, let's have two strings of the same length: one ties P and Q with the first string $s1$ such that the curve $s1$ is folded in two or more different planes and $s1$ is under the plane

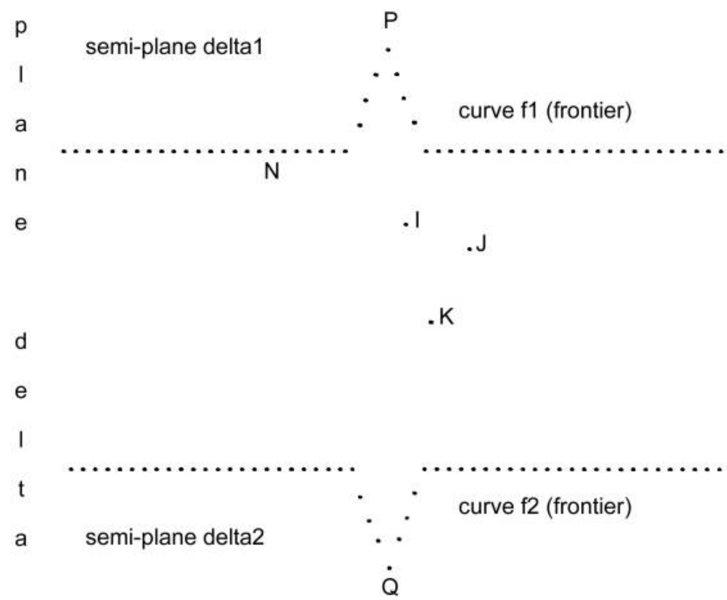

Figure A1.
A model of connection

delta; next, do the same with string $s2$, tie Q with P, but over the plane delta and such that $s2$ has a different form from $s1$; and a third string $s3$, from P to Q, much longer than $s1$. $s1$, $s2$, $s3$ belong to MD.

Let I, J, K be three isolated points – as some islands, i.e. not joined with any other point of MD, exterior to the plane delta. This model has a measure, because the (pseudo-)line is the shortest way (length) to go from a point to another (when possible). Of course, this model is not perfect, and is far from the best. Readers are asked to improve it, or to make up a new one that is better. (Let A, B be two distinct points in delta1-$f1$. P and Q are two points on $s1$, but they do not completely determine a line, referring to the first axiom of Hilbert, because A-P-$s1$-Q are different from B-P-$s1$-Q.)

I.2. There is at least a line l and at least two distinct points A and B of l, such that A and B do not completely determine the line l. (Line A-P-$s1$-Q are not completely determined by P and Q in the previous construction, because CVB-P-$s1$-Q is another line passing through P and Q too.)

I.3. Three points A, B, C not situated in the same line do not always completely determine a plane alpha. (Let A, B be two distinct points in delta1-$f1$, such that A, B, P are not co-linear. There are many planes containing these three points: delta1 extended with any surface s containing $s1$, but not cutting $s2$ in between P and Q, for example.)

I.4. There is at least a plane, alpha, and at least three points A, B, C in it not lying in the same line, such that A, B, C do not completely determine the plane alpha. (See the previous example.)

I.5. If two points A, B of a line l lie in a plane alpha, it doesn't mean that every point of l lies in alpha. (Let A be a point in delta1-$f1$, and B another point on $s1$ in between P and Q. Let alpha be the following plane: delta1 extended with a surface s containing $s1$, but not cutting $s2$ in between P and Q, and tangent to delta2 on a line QC, where C is a point in delta2-$f2$. Let D be point in delta2-$f2$, not lying on the line QC. Now, A, B, D are lying on the same line A-P-$s1$-Q-D, A, B are in the plane alpha, but D is not.)

I.6. If two planes alpha, beta have a point A in common, it doesn't mean they have at least a second point in common. (Construct the following plane alpha: a closed surface containing $s1$ and $s2$, and intersecting delta1 in one point only, P. Then alpha and delta1 have a single point in common.)

I.7. There exist lines where lies only one point, or planes where lie only two points, or space where lie only three points. (Hilbert's I.7 axiom may be contradicted if the model has discontinuities. Let's consider the isolated points area. The point I may be regarded as a line, because it's not possible to add any new point to I to form a line. One constructs a surface that intersects the model only in the points I and J.)

Group II. Anti-axioms of order

II.1. If A, B, C are points of a line and B lies between A and C, it doesn't mean that always B lies also between C and A.
[Let T lie in $s1$, and V lie in $s2$, both of them closer to Q, but different from it. Then: P, T, V are points on the line P-$s1$-Q-$s2$-P (i.e. the closed curve that starts from the point P and lies in $s1$ and passes through the point Q and lies back to $s2$ and ends in P), and T lies between P and V – because PT and TV are both geodesics – but T doesn't lie between V and P because from V the line goes to P and then to T, therefore P lies between V and T.]
[By definition: a segment AB is a system of points lying upon a line between A and B (the extremes are included). Warning, AB may be different from BA; for example: the segment PQ formed by the system of points starting with P, ending with Q, and lying in $s1$, is different from the segment QP formed by the system of points starting with Q, ending with P, but belonging to $s2$. Worse, AB may be sometimes different from AB; for example, the segment PQ formed by the system of points starting with P, ending with Q, and lying in $s1$, is different from the segment PQ formed by the system of points starting with P, ending with Q, but belonging to $s2$.]

II.2. If A and C are two points of a line, then: there does not always exist a point B lying between A and C, or there does not always exist a point D such that C lies between A and D.
[For example, let F be a point on $f1$, F different from P, and G a point in delta1, G doesn't belong to $f1$; draw the line l which passes through G and F; then there exists a point B lying between G and F – because GF is an obvious segment – but there is no point D such that F lies between G and D – because GF is right bounded in F (GF may not be extended to the other side of F, because otherwise the line will not remain a geodesic anymore).]

II.3. There exist at least three points situated on a line such that one point lies between the other two, and another point lies also between the other two.
[For example, let R, T be two distinct points, different from P and Q, situated on the line P-$s1$-Q-$s2$-P, such that the lenghts PR, RT, TP are all equal; then R lies between P and T, and T lies between R and P; also P lies between T and R.]

II.4. Four points A, B, C, D of a line can not always be arranged such that B lies between A and C and also between A and D, and such that C lies between A and D and also between B and D.
[For example:

- let R, T be two distinct points, different from P and Q, situated on the line P-$s1$-Q-$s2$-P such that the lenghts PR, RQ, QT, TP are all equal, therefore R belongs to $s1$, and T belongs to $s2$; then P, R, Q, T are situated on the same line: such that R lies between P and Q, but not between P and T – because the geodesic PT does not pass through R – and such that Q does not lie between P and T – because the geodesic PT does not pass through Q – but lies between R and T.
- Let A, B be two points in delta2-$f2$ such that A, Q, B are colinear, and C, D two points on $s1$, $s2$ respectively, all of the four points being different from P and Q; then A, B, C, D are points situated on the same line A-Q-$s1$-P-$s2$-Q-B, which is the same with line A-Q-

$s2$-P-$s1$-Q-B, therefore we may have two different orders of these four points in the same time: A, C, D, B and A, D, C, B.]

II.5. Let A, B, C be three points not lying in the same line, and l a line lying in the same plane ABC and not passing through any of the points A, B, C. Then, if the line l passes through a point of the segment AB, it doesn't mean that always the line l will pass through either a point of the segment BC or a point of the segment AC.
[For example, let AB be a segment passing through P in the semi-plane delta1, and C a point lying in delta1 too on the left side of the line AB; thus A, B, C do not lie on the same line; now, consider the line Q-$s2$-P-$s1$-Q-D, where D is a point lying in the semi-plane delta2 not on $f2$: therefore this line passes through the point P of the segment AB, but does not pass through any point of the segment BC, nor through any point of the segment AC.]

Group III. Anti-axiom of parallels

In a plane alpha there can be drawn through a point A, lying outside of a line l, either no line, or only one line, or a finite number of lines, or an infinite number of lines which do not intersect the line l. (At least two of these situations should occur.) The line(s) is (are) called the parallel(s) to l through the given point A.
[For example:

- Let $l0$ be the line N-P-$s1$-Q-R, where N is a point lying in delta1 not on $f1$, and R is a similar point lying in delta2 not on $f2$, and let A be a point lying on $s2$, then: no parallel to $l0$ can be drawn through A (because any line passing through A, hence through $s2$, will intersect $s1$, hence $l0$, in P and Q).
- If the line $l1$ lies in delta1 such that $l1$ does not intersect the frontier $f1$, then through any point lying on the left side of $l1$ one and only one parallel will pass.
- Let B be a point lying in $f1$, different from P, and another point C lying in delta1, not on $f1$; let A be a point lying in delta1 outside of BC; then: an infinite number of parallels to the line BC can be drawn through the point A.

Theorem. There are at least two lines $l1$, $l2$ of a plane, which do not meet a third line $l3$ of the same plane, but they meet each other, (i.e. if $l1$ is parallel to $l3$, and $l2$ is parallel to $l3$, and all of them are in the same plane, it's not necessary that $l1$ is parallel to $l2$).
[For example: consider three points A, B, C lying in $f1$, and different from P, and D a point in delta1 not on $f1$; draw the lines AD, BE and CE such that E is a point in delta1 not on $f1$ and both BE and CE do not intersect AD; then: BE is parallel to AD, CE is also parallel to AD, but BE is not parallel to CE because the point E belongs to both of them.]

Group IV. Anti-axioms of congruence

IV.1. If A, B are two points on a line l, and A' is a point upon the same or another line l', then upon a given side of A' on the line l', we can not always find only one point B' so that the segment AB is congruent to the segment $A'B'$.
[For example:

- Let AB be segment lying in delta1 and having no point in common with $f1$, and construct the line C-P-$s1$-Q-$s2$-P (noted by l') which is the same with C-P-$s2$-Q-$s1$-P, where C is a point lying in delta1 not on $f1$ nor on AB; take a point A' on l', in between C and P, such that $A'P$ is smaller than AB; now, there exist two distinct points $B1'$ on $s1$ and $B2'$ on $s2$, such that $A'B1'$ is congruent to AB and $A'B2'$ is congruent to AB, with $A'B1'$ different from $A'B2'$.
- But if we consider a line l' lying in delta1 and limited by the frontier $f1$ on the right side (the limit point being noted by M), and take a point A' on l', close to M, such that $A'M$ is less than $A'B'$, then there is no point B' on the right side of l' so that $A'B'$ is congruent to AB.]

A segment may not be congruent to itself!
[For example, let A be a point on $s1$, closer to P, and B a point on $s2$, closer to P also; A and B are lying on the same line A-Q-B-P-A which is the same with line A-P-B-Q-A, but AB measured on the first representation of the line is strictly greater than AB measured on the second representation of their line.]

IV.2. If a segment AB is congruent to the segment $A'B'$ and also to the segment $A''B''$, then not always the segment $A'B'$ is congruent to the segment $A''B''$.
[For example, let AB be a segment lying in delta1-$f1$, and consider the line C-P-$s1$-Q-$s2$-P-D, where C, D are two distinct points in delta1-$f1$ such that C, P, D are colinear. Suppose that the segment AB is congruent to the segment CD (i.e. C-P-$s1$-Q-$s2$-P-D). Get also an obvious segment $A'B'$ in delta1-$f1$, different from the preceding ones, but congruent to AB. Then the segment $A'B'$ is not congruent to the segment CD (considered as C-P-D, i.e. not passing through Q).

IV.3. If AB, BC are two segments of the same line l which have no points in common aside from the point B, and $A'B'$, $B'C'$ are two segments of the same line or of another line l' having no point other than B' in common, such that AB is congruent to $A'B'$ and BC is congruent to $B'C'$, then not always the segment AC is congruent to $A'C'$.
[For example, let l be a line lying in delta1, not on $f1$, and A, B, C three distinct points on l, such that AC is greater than $s1$; let l' be the following line: A'-P-$s1$-Q-$s2$-P where A' lies in delta1, not on $f1$, and get B' on $s1$ such that $A'B'$ is congruent to AB, get C' on $s2$ such that BC is congruent to $B'C'$ (the points A, B, C are thus chosen); then the segment $A'C'$ which is first seen as A'-P-B'-Q-C' is not congruent to AC, because $A'C'$ is the geodesic A'-P-C' (the shortest way from A' to C' does not pass through B') which is strictly less than AC.]
Definitions. Let h, k be two lines having a point O in common. Then the system (h, O, k) is called the angle of the lines h and k in the point O. (Because some of our lines are curves, we take the angle of the tangents to the curves in their common point.)
The angle formed by the lines h and k situated in the same plane, noted by $<(h, k)$, is equal to the arithmetic mean of the angles formed by h and k in all their common points.

IV.4. Let an angle (h, k) be given in the plane alpha, and let a line h' be given in the plane beta. Suppose that in the plane beta a definite side of the line h' be assigned, and a point O'. Then in the plane beta there are one, or more, or even no half-line(s) k' emanating from the point O' such that the angle (h, k) is congruent to the angle (h', k'), and at the same time the interior points of the angle (h', k') lie upon one or both sides of h'.
For example:

- Let A be a point in delta1-$f1$, and B, C two distinct points in delta2-$f2$; let h be the line A-P-$s1$-Q-B, and k be the line A-P-$s2$-Q-C; because h and k intersect in an infinite number of points (the segment AP), where they normally coincide – i.e. in each such point their angle is congruent to zero, the angle (h, k) is congruent to zero. Now, let A' be a point in delta1-$f1$, different from A, and B' a point in delta2-$f2$, different from B, and draw the line h' as A'-P-$s1$-Q-B'; there exist an infinite number of lines k', of the form A'-P-$s2$-Q-C' (where C' is any point in delta2-$f2$, not on the line QB'), such that the angle (h, k) is congruent to (h', K'), because (h', k') is also congruent to zero, and the line A'-P-$s2$-Q-C' is different from the line A'-P-$s2$-Q-D' if D' is not on the line QC'.
- If h, k, and h' are three lines in delta1-P, which intersect the frontier $f1$ in at most one point, then there exists only one line k' on a given part of h' such that the angle (h, k) is congruent to the angle (h', k').
- Is there any case when, with these hypotheses, no k' exists ?
- Not every angle is congruent to itself; for example, $<(s1, s2)$ is not congruent to $<(s1, s2)$ [because one can construct two distinct lines: P-$s1$-Q-A and P-$s2$-Q-A, where A is a

point in delta2-$f2$, for the first angle, which becomes equal to zero; and P-$s1$-Q-A and P-$s2$-Q-B, where B is another point in delta2-$f2$, B different from A, for the second angle, which becomes strictly greater than zero!].

IV. 5. If the angle (h, k) is congruent to the angle $(h', k',)$ and the angle (h'', k''), then the angle (h', k') is not always congruent to the angle (h'', k''). (A similar construction to the previous one.)

IV. 6. Let ABC and $A'B'C'$ be two triangles such that AB is congruent to $A'B'$, AC is congruent to $A'C'$, $<BAC$ is congruent to $<B'A'C'$. Then not always $<ABC$ is congruent to $<A'B'C'$ and $<ACB$ is congruent to $<A'C'B'$.

[For example, Let M, N be two distinct points in delta2-$f2$, thus obtaining the triangle PMN; Now take three points R, M', N' in delta1-$f1$, such that RM' is congruent to PM, RN' is congruent to RN, and the angle (RM', RN') is congruent to the angle (PM, PN). $RM'N'$ is an obvious triangle. Of course, the two triangles are not congruent, because for example PM and PN cut each other twice – in P and Q – while RM' and RN' only once – in R. (These are geodesical triangles.)]

Definitions. Two angles are called supplementary if they have the same vertex, one side in common, and the other sides not common form a line. A right angle is an angle congruent to its supplementary angle. Two triangles are congruent if its angles are congruent two by two, and its sides are congruent two by two.

Propositions. A right angle is not always congruent to another right angle.

For example: Let A-P-$s1$-Q be a line, with A lying in delta1-$f1$, and B-P-$s1$-Q another line, with B lying in delta1-$f1$ and B not lying in the line AP; we consider the tangent t at $s1$ in P, and B chosen in a way that $<(AP, t)$ is not congruent to $<(BP, t)$; let A', B' be other points lying in delta1-$f1$ such that $<APA'$ is congruent to $<A'P$-$s1$-Q, and $<BPB'$ is congruent to $<B'P$-$s1$-Q.

Then the angle APA' is right, because it is congruent to its supplementary (by construction), and the angle BPB' is also right, because it is congruent to its supplementary (by construction). But $<APA'$ is not congruent to $<BPB'$, because the first one is half of the angle A-P-$s1$-Q, i.e. half of $<(AP, t)$, while the second one is half of the B-P-$s1$-Q, i.e. half of $<(BP, t)$.

The theorems of congruence for triangles [side, side, and angle in between; angle, angle, and common side; side, side, side] may not hold either in the critical zone ($s1$, $s2$, $f1$, $f2$) of the model.

Property: The sum of the angles of a triangle can be:

- 180 degrees, if all its vertexes A, B, C are lying, for example, in delta1-$f1$;
- strictly less than 180 degrees (any value in the interval (0, 180)). For example, let R, T be two points in delta2-$f2$ such that Q does not lie in RT, and S another point on $s2$; then the triangle SRT has $<(SR, ST)$ congruent to 0 because SR and ST have an infinite number of common points (the segment SQ), and $<QTR + <TRQ$ congruent to $180 - <TQR$ [by construction we may vary $<TQR$ in the interval (0, 180)], even 0 degrees! Let A be a point in delta1-$f1$, B a point in delta2-$f2$, and C a point on $s3$, very close to P; then ABC is a non-degenerate triangle (because its vertexes are non-colinear), but $<(A$-P-$s1$-Q-B, A-P-$s3$-$C) = <(B$-Q-$s1$-P-A, B-Q-$s1$-P-$s3$-$C) = <(C$-$s3$-P-A, C-$s3$-P-$s1$-Q-$B) = 0$ (one considers the length C-$s3$-P-$s1$-Q-B strictly less than C-$s3$-B); the area of this triangle is also 0!
- more than 180 degrees. For example; let A, B be two points in delta1-$f1$, such that $<PAB + <PBA + <(s1, s2$; in $Q)$ is strictly greater than 180 degrees; then the triangle ABQ, formed by the intersection of the lines A-P-$s2$-Q, Q-$s1$-P-B, AB will have the sum of its angles strictly greater than 180 degrees.

Definition. A circle of center M is a totality of all points A for which the segments MA are congruent to one another.

For example, if the center is Q, and the length of the segments MA is chosen greater than the length of $s1$, then the circle is formed by the arc of circle centered in Q, of radius MA, and lying in delta2, plus another arc of circle centered in P, of radius MA-length of $s1$, lying in delta1.

Group V. Anti-axiom of continuity (anti-Archimedean axiom)
Let A, B be two points. Take the points $A1$, $A2$, $A3$, $A4$, ... so that $A1$ lies between A and $A2$, $A2$ lies between $A1$ and $A3$, $A3$ lies between $A2$ and $A4$, etc. and the segments $AA1$, $A1A2$, $A2A3$, $A3A4$, ... are congruent to one another.

Then, among this series of points, not always there exists a certain point An such that B lies between A and An.

[For example, let A be a point in delta1-$f1$, and B a point on $f1$, B different from P; on the line AB consider the points $A1$, $A2$, $A3$, $A4$, ... in between A and B, such that $AA1$, $A1A2$, $A2A3$, $A3A4$, etc. are congruent to one another; then we find that there is no point behind B (considering the direction from A to B), because B is a limit point (the line AB ends in B).]

Bolzano's (intermediate value) theorem may not hold in the critical zone of the model.

Appendix 5. Fourth example of model M_7 in Rugina's orientation table

The inconsistent system of axioms, and the contradictory theory (the percentage of instability is 100 – even... more, this is the system of chaos!)
Let ($a1$), ($a2$), ..., (an), (b) be $n+1$ independent axioms, with $n >= 1$; and let (b') be another axiom contradictory to (b). We construct a system of $n+2$ axioms:

$$[I] \quad (a1), (a2), \ldots, (an), (b), (b'),$$

which is inconsistent. But this system may be shared into two consistent systems of independent axioms:

$$[C] \quad (a1), (a2), \ldots, (an), (b),$$

and

$$[C'] \quad (a1), (a2), ..., (an), (b').$$

We also consider the partial system of independent axioms:

$$[P] \quad (a1), (a2), \ldots, (an).$$

Developing $[P]$, we find many propositions (theorems, lemmas):

$$(p1), (p2), \ldots, (pm),$$

by combinations of its axioms.

Developing $[C]$, we find all propositions of $[P]$:

$$(p1), (p2), \ldots, (pm),$$

resulting by combinations of ($a1$), ($a2$), ..., (an), plus other propositions:

$$(r1), (r2), ..., (rt),$$

resulting by combinations of (b) with any of ($a1$), ($a2$), ..., (an).

Similarly for $[C']$, we find the propositions of $[P]$:

$$(p1), (p2), \ldots, (pm),$$

plus other propositions

$$(r'1), (r'2), \ldots, (r't),$$

resulting by combinations of (b') with any of $(a1)$, $(a2)$, …, (an), where $(r'1)$ is an axiom contradictory to $(r1)$, and so on.

Now, developing [I], we'll find all the previous resulted propositions:

$$(p1), (p2), \ldots, (pm),$$
$$(r1), (r2), \ldots, (rt),$$
$$(r'1), (r'2), \ldots, (r't).$$

Therefore, [I] is equivalent to [C] reunited to [C'].

From one pair of contradictory propositions $\{(b)$ and $(b')\}$ in its beginning, [I] adds t more such pairs, where $t \geq 1$, $\{(r1)$ and $(r'1)$, …, (rt) and $(r't)\}$, after a complete step. The further we go, the more pairs of contradictory propositions are accumulating in [I].

It is interesting to study the case when $n = 0$.

Why do people avoid thinking about the contradictory theory?

As you know, nature is not perfect: opposite phenomena occur together, and opposite ideas are simultaneously asserted and, ironically, proved that both of them are true! How is that possible?

A statement may be true in a referential system, but false in another one. The truth is subjective. The proof is relative. (In philosophy there is a theory that "knowledge is relative to the mind, or things can be known only through their effects on the mind, and consequently there can be no knowledge of reality as it is in itself", called "the Relativity of Knowledge"; see *Webster's New World Dictionary of American English*, 1988, p. 1133)

You know? . . .sometimes is good to be wrong!

Fuzziness and funds allocation in portfolio optimization

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Abstract *Each individual investor is different, with different financial goals, levels of risk tolerance and personal preferences. From the point of view of investment management, these characteristics are often defined as objectives and constraints. Objectives can be the type of return being sought, while constraints include factors such as time horizon, how liquid the investor is, any personal tax situation and how risk is handled. It is really a balancing act between risk and return with each investor having unique requirements, as well as a unique financial outlook – essentially a constrained utility maximization objective. To analyze how well a customer fits into a particular investor class, one investment house has even designed a structured questionnaire with about 24 questions that each has to be answered with values from 1 to 5. The questions range from personal background to what the customer expects from an investment. A fuzzy logic system has been designed for the evaluation of the answers to the above questions. The notion of fuzziness with respect to funds allocation is investigated.*

Introduction

In this paper we have designed our fuzzy system so that customers are classified to belong to any one of the following three categories (http://www.fuzzytech.com/e/e_ft4bf6.html):

(1) Conservative and security-oriented (risk shy).

(2) Growth-oriented and dynamic (risk neutral).

(3) Chance-oriented and progressive (risk happy).

Besides being useful for clients, investor classification has benefits for the professional investment consultants as well. Most brokerage houses would value this information as it gives them a way of targeting clients with a range of financial products more effectively – including insurance, saving schemes, mutual funds and so forth. Overall, many responsible brokerage houses realize that if they provide an effective service that is tailored to individual needs, in the long term there is far more chance that they will retain their clients no matter whether the market is up or down.

Yet, although it may be true that investors can be categorized according to a limited number of types based on theories of personality already in the psychological profession's armory, it must be said that these classification systems based on the behavioral sciences are still very much in their infancy and they may still suffer from the problem of their meanings being similar to other related typographies, as well as of greatly oversimplifying the different investor behaviors (http://www.geocities.com/wallstreet/bureau/3486/11.htm).

(I.1) Exploring the implications of utility theory on investor classification

In our present work, we have used the familiar framework of neo-classical utility theory to try to devise a structured system for investor classification according to the utility preferences of individual investors (and also possible re-ordering of such preferences).

The theory of consumer behavior in modern microeconomics is entirely founded on observable utility preferences, rejecting hedonistic and introspective aspects of utility. According to modern utility theory, utility is a representation of a set of mutually consistent choices and not an explanation of a choice. The basic approach is to ask an individual to reveal his or her personal utility preference and not to elicit any numerical measure (Arkes *et al.*, 2000). However, the projections of the consequences of the options that we face and the subsequent choices that we make are shaped by our memories of past experiences – that "mind's eye sees the future through the light filtered by the past". However, this memory often tends to be rather selective Sarin and Wakker, 1997). An investor who allocates a large portion of his/her or funds to the risky asset in period $t - 1$ and makes a significant gain will perhaps be induced to put an even larger portion of the available funds in the risky asset in period t. So this investor may be said to have displayed a very weak risk-aversion attitude up to period t, his or her actions being mainly determined by past happenings one-period back.

There are two interpretations of utility – normative and positive. Normative utility contends that optimal decisions do not always reflect the best decisions, as maximization of instant utility based on selective memory may not necessarily imply maximization of total utility. This is true in many cases, especially in the areas of health economics and social choice theory. However, since we will be applying utility theory to the very specific area of funds allocation between risky and risk-less investments (and investor classification based on such allocation), we will be concerned with positive utility, which considers the optimal decisions as they are, and not as what they should be. We are simply interested in using utility functions to classify an individual investor's attitude toward bearing risk at a given point of time. Given that the neo-classical utility preference approach is an objective one, we feel it is definitely more amenable to formal analysis for our purpose as compared with the philosophical conceptualizations of pure hedonism if we can accept decision utility preferences generated by selective memory.

If u is a given utility function and w is the wealth coefficient, then we have $E[u(w+k)] = u[w + E(k) - p]$, that is, $E[u(w+k)] = u(w-p)$, where k is the outcome of a risky venture given by a known probability distribution whose expected value $E(k)$ is zero. Since the outcome of the risky venture is as likely to be positive as negative, we would be willing to pay a small amount p, the risk premium, to avoid having to undertake the risky venture. Expanding the utilities in Taylor series to second order on the left-hand side and to first order on the right-hand side and subsequent algebraic simplification leads to the general formula $p = -(v/2)\mathrm{u}''(w)/u'(w)$, where $v = E(k^2)$ is the variance of the possible outcomes. This shows that approximate risk premium is proportional to the variance – a notion that carries a similar implication in the mean-variance theorem of classical portfolio theory. The quantity $-\mathrm{u}''(w)/u'(w)$ is termed the "absolute risk aversion" (Korsan, 1994). The nature of this absolute risk aversion depends on the form of a specific utility function. For instance, for a logarithmic utility function, the absolute risk aversion is dependent on the wealth coefficient w, such that it decreases with an increase in w. On the other hand, for an exponential utility function, the absolute risk aversion becomes a constant equal to the reciprocal of the risk premium.

(I.2) The neo-classical utility maximization approach

In its simplest form, we may formally represent an individual investor's utility maximization goal as the following mathematical programming problem:

$$\begin{aligned}&\text{Maximize } U = f(x, y)\\&\text{Subject to } x + y = 1,\\&x \geq 0 \text{ and } y \text{ is unrestricted in sign.}\end{aligned}$$

Here x and y stand for the proportions of investable funds allocated by the investor to the market portfolio and a risk-free asset. The last constraint is to ensure that the investor can never borrow at the market rate to invest in the risk-free asset, as this is clearly unrealistic – the market rate being obviously higher than the risk-free rate. However, an overtly aggressive investor can borrow at the risk-free rate to invest in the market portfolio. In investment parlance this is known as leverage (Kolb, 1995).

As in classical microeconomics, we may solve the above problem using the Lagrangian multiplier technique. The transformed Lagrangian function is as follows:

$$Z = f(x, y) + \lambda(1 - x - y). \tag{1}$$

By the first order (necessary) condition of maximization we derive the following system of linear algebraic equations:

$$Z_x = f_x - \lambda = 0 \tag{2i}$$

$$Z_y = f_y - \lambda = 0 \tag{2ii}$$

$$Z_\lambda = 1 - x - y = 0. \tag{2iii}$$

The investor's equilibrium is then obtained as the condition $f_x = f_y = \lambda * .\lambda*$ may be conventionally interpreted as the marginal utility of money (i.e. the investable funds at the disposal of the individual investor) when the investor's utility is maximized (Chiang, 1984).

The individual investor's indifference curve will be obtained as the locus of all combinations of x and y that will yield a constant level of utility. Mathematically stated, this simply produces the following total differential:

$$\mathrm{dU} = f_x \mathrm{dx} + f_y \mathrm{dy} = 0. \tag{3}$$

The immediate implication of equation (3) is that $\mathrm{dy}/\mathrm{dx} = -f_x/f_y$, i.e. assuming $(\mathrm{fx}, \mathrm{fy}) > 0$; this gives the negative slope of the individual investor's indifference curve and may be equivalently interpreted as the marginal rate of substitution of allocable funds between the market portfolio and the risk-free asset.

A second order (sufficient) condition for maximization of investor utility may be also derived on a similar line as that in economic theory of consumer behavior, using the sign of the bordered Hessian determinant, which is given as follows:

$$\left|\overline{H}\right| = 2\beta_x\beta_y f_{xy} - \beta_y f_{xx} - \beta_x f_{yy}. \tag{4}$$

In the above equation, β_x and β_y stand for the coefficients of x and y in the constraint equation. In this case we have $\beta_x = \beta_y = 1$. Equation (4) therefore reduces to:

$$|H| = \overline{2f}_{xy} - f_{xx} - f_{yy}. \tag{5}$$

If $|\overline{H}| > 0$ then the stationary value of the utility function U^* will be said to have attained its maximum.

To illustrate the application of classical utility theory in investor classification, let the utility function of a rational investor be represented by the following utility function:

$$U(x,y) = ax^2 - by^2;$$

where:

x = proportion of funds invested in the market portfolio; and

y = proportion of funds invested in the risk-free asset.

Quite obviously, $x+y=1$ since the efficient portfolio must consist of a combination of the market portfolio with the risk-free asset. The problem of funds allocation within the efficient portfolio then becomes that of maximizing the given utility function subject to the efficient portfolio constraint. As per Tobin's Separation Theorem; which states that investment is a two-phased process with the problem of portfolio selection which is considered independent of an individual investor's utility preferences (i.e. the first phase) to be treated separately from the problem of funds allocation within the selected portfolio which is dependent on the individual investor's utility function (i.e. the second phase). Using this concept we can mathematically categorize all individual investor attitudes toward bearing risk into any one of the following distinct classes:

- Class A+: "Overtly Aggressive"(no risk aversion attitude).
- Class A: "Aggressive" (weak risk aversion attitude).
- Class B: "Neutral"(balanced risk aversion attitude).
- Class C: "Conservative"(strong risk aversion attitude).

The problem is then to find the general point of maximum investor utility and subsequently derive a mathematical basis to categorize the investors into one of the above classes depending on the optimum values of x and y. The original problem can be stated as a classical non-linear programming with a single equality constraint as follows:

$$\text{Maximize } U(x,y) = ax^2 - by^2$$
$$\text{Subject to :}$$
$$x = y = 1,$$
$$x \geq 0 \text{ and } y \text{ is unrestricted in sign.}$$

We set up the following transformed Lagrangian objective function:

$$\text{Maximize } Z = ax^2 - by^2 + \lambda(1 - x - y)$$
$$\text{Subject to : } x + y = 1,$$
$$x \geq 0 \text{ and } y \text{ is unrestricted in sign (where } \lambda \text{ is the Lagrangian multiplier).}$$

By the usual first-order (necessary) condition we therefore get the following system of linear algebraic equations:

$$Z_x = 2ax - \lambda = 0, \tag{6i}$$

$$Z_y = -2by - \lambda = 0, \tag{6ii}$$

$$Z_\lambda = 1 - x - y = 0. \tag{6iii}$$

Solving the above system we get $x/y = -b/a$. But $x + y = 1$ as per the funds constraint. Therefore $(-b/a)\,y + y = 1$, i.e. $y* = [1 + (-b/a)]^{-1} = [(a - b)/a]^{-1} = a/(a - b)$. Now substituting for y in the constraint equation, we get $x^* = 1 - a/(a - b) = -b/(a - b)$. Therefore the stationary value of the utility function is $U* = a[-b/(a - b)]^2 - b[a/(a - b)]^2 = -ab/(a - b)$.

Now, $f_{xx} = 2a$, $f_{xy} = f_{yx} = 0$ and $f_{yy} = -2b$. Therefore, by the second order (sufficient) condition, we have:

$$\left|\overline{H}\right| = 2f_{xy} - f_{xx} - f_{yy} = 0 - 2a - (-2b) = 2(b - a). \tag{7}$$

Therefore, the bordered Hessian determinant will be positive in this case if, and only if, we have $(a - b) < 0$. That is, given that $a < b$, our chosen utility function will be maximized at $U* = ax*^2 - by*^2$. However, the satisfaction of the non-negativity constraint on $x*$ would require that $b > 0$ so that $-b < 0$; thus yielding $[-b/(a - b)] > 0$.

Classification of investors

The classification of investors can be seen in Table I.

(I.3) Effect of a risk-free asset on investor utility

The possibility to lend or borrow money at a risk-free rate widens the range of investment options for an individual investor. The inclusion of the risk-free asset makes it possible for the investor to select a portfolio that dominates any other portfolio made up of only risky securities. This implies that an individual investor will be able to attain a higher indifference curve than would be possible in the absence of the risk-free asset. The risk-free asset makes it possible to separate the investor's decision-making process into two distinct phases – identifying the market portfolio and funds allocation. The market portfolio is the portfolio of risky assets that includes each and every available risky security. As all investors who hold any risky assets at all will choose to hold the market portfolio, this choice is independent of an individual investor's utility preferences.

Now, the expected return on a two-security portfolio involving a risk-free asset and the market portfolio is given by $E(R_p) = xE(R_m) + yR_f$; where $E(R_p)$ is the expected return on the optimal portfolio, $E(R_m)$ is the expected return on the market portfolio; and Rf is the return on the risk-free asset. Obviously,

Table I. Classification of investors

Class	Basis of determination
A+	$(y* < x*)$ and $(y* \leq 0)$
A	$(y* < x*)$ and $(y* > 0)$
B	$(y* = x*)$
C	$(y* > x*)$

$x + y = 1$. Substituting for x and y with $x*$ and $y*$ from our illustrative case, we therefore get:

$$E(R_p)* = [-b/(a-b)]E(R_m) + [-a/(a-b)]R_f. \quad (8)$$

As may be verified intuitively, if $b = 0$ then of course we have $E(R_p) = R_f$, as in that case the optimal value of the utility function too is reduced to $U* = -a0/(a-0) = 0$.

The equation of the capital market line in the original version of the CAPM may be recalled as $E(R_p) = R_f + [E(R_m) - R_f](S_p/S_m)$; where $E(R_p)$ is expected return on the efficient portfolio, $E(R_m)$ is the expected return on the market portfolio, R_f is the return on the risk-free asset, S_m is the standard deviation of the market portfolio returns; and S_p is the standard deviation of the efficient portfolio returns. Now, equating for $E(R_p)$ with $E(R_p)*$ we therefore get:

$$R_f + [E(R_m) - R_f](S_p/S_m) = [-b/(a-b)]E(R_m) + [a/(a-b)]R_f$$

i.e.

$$\begin{aligned} S_p* &= S_m[R_f\{a/(a-b) - 1\} + \{-b(a-b)\}E(R_m)]/[E(R_m) - R_f] \\ &= S_m[E(R_m) - R_f][-b/(a-b)]/[E(R_m) - R_f] \\ &= S_m[-b(a-b)]. \end{aligned} \quad (9)$$

This mathematically demonstrates that a rational investor having a quadratic utility function of the form $U = ax^2 - by^2$, at his or her point of maximum utility (i.e. affinity to return coupled with aversity to risk), assumes a given efficient portfolio risk (standard deviation of returns) equivalent to $S_p* = S_m[-b/(a-b)]$; when the efficient portfolio consists of the market portfolio coupled with a risk-free asset.

The investor in this case, will be classified within a particular category A, B or C according to whether $-b/(a-b)$ is greater than, equal in value or lesser than $a/(a-b)$, given that $a < b$ and $b > 0$:

(1) *Case I: (*$\mathrm{b} > \mathrm{a}, \mathrm{b} > 0$ *and* $\mathrm{a} > 0$*).* Let $b = 3$ and $a = 2$. Thus, we have $(b > a)$ and $(-b < a)$. Then we have $x* = -3/(2-3) = 3$ and $y* = 2/(2-3) = -2$. Therefore $(x* > y*)$ and $(y* < 0)$. So the investor can be classified as Class A+.

(2) *Case II: (*$\mathrm{b} > \mathrm{a}, \mathrm{b} > 0, \mathrm{a} < 0$ *and* $\mathrm{b} > |\mathrm{a}|$*).* Let $b = 3$ and $a = -2$. Thus, we have $(b \geq a)$ and $(-b < a)$. Then, $x* = -3/(-2-3) = 0.60$ and $y* = -2/(-2-3) = 0.40$. Therefore $(x* > y*)$ and $(y* > 0)$. So the investor can be re-classified as Class A!

(3) *Case III:* ($b > a, b > 0, a < 0$ *and* $b = |a|$). Let $b = 3$ and $a = -3$. Thus, we have ($b > a$) and ($b = |a|$). Then we have $x* = -3/(-3-3) = 0.5$ and $y* = -3/(-3-3) = 0.5$. Therefore we have ($x* = y*$). So now the investor can be re-classified as Class B!

(4) *Case IV:* ($b > a, b > 0, a < 0$ *and* $b < |a|$). Let $b = 3$ and $a = -5$. Thus, we have ($b \geq a$) and ($b < |a|$). Then we have $x* = -3/(-5-3) = 0.375$ and $y* = -5/(-5-3) = 0.625$. Therefore we have ($x* < y*$). So, now the investor can be re-classified as Class C!

So we may see that even for this relatively simple utility function, the final classification of the investor permanently into any one risk-class would be unrealistic as the range of values for the coefficients a and b could be switching dynamically from one range to another as the investor tries to adjust and re-adjust his or her risk-bearing attitude. This makes the neo-classical approach insufficient in itself to arrive at a classification. Here lies the justification to bring in a complimentary fuzzy modeling approach. Moreover, if we bring in time itself as an independent variable into the utility maximization framework, then one choice variable (weighted in favour of risk-avoidance) could be viewed as a controlling factor on the other choice variable (weighted in favour of risk-acceptance). Then the resulting problem could be gainfully explored in the light of optimal control theory.

(II.1) Modeling fuzziness in the funds allocation behavior of an individual investor

The boundary between the preference sets of an individual investor, for funds allocation between a risk-free asset and the risky market portfolio, tends to be rather fuzzy as the investor continually evaluates and shifts his or her position; unless it is a passive buy-and-hold kind of portfolio.

Thus, if the universe of discourse is $U = \mathrm{C}, \mathrm{B}, \mathrm{A}$ and $\mathrm{A}+\}$ where C, B, A and A+ are our four risk classes "conservative", "neutral", "aggressive" and "overtly aggressive" respectively, then the fuzzy subset of U given by $P = \{x_1/\mathrm{C}, x_2/\mathrm{B}, x_3/\mathrm{A}, x_4/\mathrm{A}+\}$ is the *true* preference set for our purposes; where we have $0 \leq (x_1, x_2, x_3, x_4) \leq 1$, all the symbols having their usual meanings. Although theoretically any of the $P(x_i)$ values could be equal to unity, in reality it is far more likely that $P(x_i) < 1$ for $i = 1, 2, 3, 4$ i.e. the fuzzy subset P is most likely to be subnormal. Also, similarly, in most real-life cases it is expected that $P(x_i) > 0$ for $i = 1, 2, 3, 4$ i.e. all the elements of P will be included in its support: $\mathrm{supp}(P) = \mathrm{C}, \mathrm{B}, \mathrm{A}, \mathrm{A}+\} = U$.

The critical point of analysis is definitely the individual investors preference ordering i.e. whether an investor is primarily conservative or primarily aggressive. It is understandable that a primarily conservative investor could behave aggressively at times and vice versa but in general, their behavior will be in line with their classification. So the classification often depends on the height of the fuzzy subset P: height $(P) = \mathrm{Max}_x P(x)$. So one would think that the risk-neutral class becomes largely superfluous, as investors in general will tend to get classified as either primarily conservative or primarily aggressive.

However, as already said, in reality, the element B will also generally have a non-zero degree of membership in the fuzzy subset and hence cannot be dropped.

The fuzziness surrounding investor classification stems from the fuzziness in the preference relations regarding the allocation of funds between the risk-free and the risky assets in the optimal portfolio. It may be mathematically described as follows.

Let M be the set of allocation options open to the investor. Then, the fuzzy preference relation is a fuzzy subset of the $M \times M$ space identifiable by the following membership function:

$$\begin{aligned} &\mu_R(m_i, m_j) = 1; m_i \text{ is definitely preferred to } m_j \\ &c \in (0.5, 1); m_i \text{ is somewhat preferred to } m_j \\ &0.5;\ \text{point of perfect neutrality} \\ &d \in (1, 0.5); m_j \text{ is somewhat preferred to } m_i;\ \text{and} \\ &0; m_j \text{ is defintely preferred to } m_i. \end{aligned} \tag{10}$$

The fuzzy preference relation is assumed to meet the necessary conditions of reciprocity and transitivity. However, owing to substantial confusion regarding acceptable working definition of transitivity in a fuzzy set-up, it is often entirely neglected thereby leaving only the reciprocity property. This property may be succinctly represented as follows:

$$\mu_R(m_i, m_j) - 1 - \mu_R(m_j, m_i), \forall i \neq j. \tag{11}$$

If we are further to assume a reasonable cardinality of the set M, then the preference relation R_v of an individual investor v may also be written in a matrix form as follows (Zadrozny, 1997):

$$[r_{ij}{}^{v}] = [\mu_R(m_i, m_j)], \forall i, j, v. \tag{12}$$

Classically, given the efficient frontier and the risk-free asset, there can be one and only one optimal portfolio corresponding to the point of tangency between the risk-free rate and the convex efficient frontier. Then fuzzy logic modeling framework does not in any way disturbs this bit of the classical framework. The fuzzy modeling, like the classical Lagrangian multiplier method, comes in only after the optimal portfolio has been identified and the problem facing the investor is that of allocating the available funds between the risky and the risk-free assets subject to a governing budget constraint. The investor is theoretically faced with an infinite number of possible combinations of the risk-free asset and the market portfolio but the ultimate allocation depends on the investor's utility function to which we now extend the fuzzy preference relation.

The available choices to the investor given his or her utility preferences determine the universe of discourse. The more uncertain are the investor's utility preferences, the wider is the range of available choices and the greater is the degree fuzziness involved in the preference relation, which would then extend to the investor classification. Also, wider the range of available choices to the investor the higher is the expected information content or entropy of the allocation decision.

(II.2) Entropy as a measure of fuzziness

The term entropy arises in analogy with thermodynamics where the defining expression has the following mathematical form:

$$S = k\log_b \omega. \tag{13}$$

In thermodynamics, entropy is related to the degree of disorder or configuration probability ω of the canonical assembly. Its use involves an analysis of the microstates' distribution in the canonical assembly among the available energy levels for both isothermal reversible and isothermal irreversible (spontaneous) processes (with an attending modification). The physical scale factor k is the Boltzmann constant (Oxford Science Publications, 1984).

However, the thermodynamic form has a different sign and the word "negentropy" is therefore sometimes used to denote expected information. Although Claude Shannon originally conceptualized the entropy measure of expected information, it was DeLuca and Termini (1972) who brought this concept in the realms of fuzzy mathematics when they sought to derive a universal mathematical measure of fuzziness.

Let us consider the fuzzy subset $F = r_1/X, r_2/Y\}, 0 \le (r_1, r_2) \le 1$, where X is the event $(y < x)$ and Y is the event $(y \ge x)$, x being the proportion of funds to be invested in the market portfolio and y being the proportion of funds to be invested in the risk-less security. Then the DeLuca-Termini conditions for measure of fuzziness may be stated as follows (DeLuca and Termini, 1972):

(1) FUZ(F) $= 0$ if F is a crisp set i.e. if the investor classified under a particular risk category *always* invests entire funds either in the risk-free asset (conservative attitude) or in the market portfolio (aggressive attitude).

(2) FUZ(F) $=$ MaxFUZ(F) when $F = (0.5/X, 0.5/Y)$.

(3) FUZ(F) $\ge$ FUZ($F*$) if $F*$ is a sharpened version of F, i.e. if $F*$ is a fuzzy subset satisfying $F*(r_i) \ge F(r_i)$ given that $F(r_i) \ge 0.5$ and $F(r_i) \ge F*(r_i)$ given that $0.5 \ge F(r_i)$.

The second condition is directly derived from the concept of entropy. Shannon's measure of entropy for an n-events case is given as follows (Swarup *et al.*, 1997):

$$H = -k\Sigma(p_i \log p_i), \text{ where we have } \Sigma p_i = 1. \tag{14}$$

The Lagrangian form of the above function is as follows:

$$H_L = -k\Sigma(p_i \log p_i) + \lambda\left(1 - \Sigma p_i\right). \tag{15}$$

Taking partial derivatives w.r.t. p_i and setting equal to zero as per the necessary condition of maximization, we have the following stationary condition:

$$\frac{\partial H_L}{\partial p_i} = -k\left[\log p_i + 1\right] - \lambda = 0. \tag{16}$$

It may be derived from equation (16) that at the point of maximum entropy, $\log p_i = -[(\lambda/k) + 1]$, i.e. $\log p_i$ becomes a constant. This means that at the point of maximum entropy, p_i becomes independent of the i and equalized to a constant value for $i = 1, 2, \ldots, n$. In an n-events case therefore, at the point of maximum entropy we necessarily have:

$$p_1 = p_2 = \ldots = p_n = 1/n. \tag{17}$$

For $n = 2$ therefore, we obviously have the necessary condition for entropy maximization as $p_1 = p_2 = 1/2 = 0.5$. In terms of the fuzzy preference relation, this boils down to exactly the second DeLuca-Termini condition. Keeping this close relation with mathematical information theory in mind, DeLuca and Termini (1972) even went on to incorporate Shannon's entropy measure as their chosen measure of fuzziness. For our portfolio funds allocation model, this measure could simply be stated as follows:

$$\begin{aligned} \mathrm{FUZ}(F) = &- k[\{F(r_1)\log F(r_i) + (1 - F(r_1))\log(1 - F(r_1))\} \\ &+ \{F(r_2)\log F(r_2) + (1 - F(r_2))\log(1 - F(r_2))\}]. \end{aligned} \tag{18}$$

(II.3) Metric measures of fuzziness

Perhaps the best method of measuring fuzziness will be through measurement of the distance between F and F^c, as fuzziness is mathematically equivalent to the lack of distinction between a set and its complement. In terms of our portfolio funds allocation model, this is equivalent to the ambivalence in the mind of the individual investor regarding whether to put a larger or smaller proportion of available funds in the risk-less asset. The higher this ambivalence, the closer F is to F^c and greater is the fuzziness.

This measure may be constructed for our case by considering the fuzzy subset F as a vector with two components. That is, $F(r_i)$ is the ith component of a vector representing the fuzzy subset F and $(1 - F(r_i))$ is the ith component of a vector representing the complementary fuzzy subset F^c. Thus letting D be a metric in two space; we have the distance between F and F^c as follows (Yager and Filev, 1994):

$$D_\rho(F, F^c) = \left[\sum \left|F(r_i) - F^c(r_i)^\rho\right]^{1/\rho}, \text{where } \rho = 1, 2, 3, \ldots \tag{19}$$

For Euclidean Space with $\rho = 2$, this metric becomes very similar to the statistical variance measure root mean square deviation (RMSD). Moreover, as $F^c(ri) = 1 - F(ri)$, the above formula may be written in a simplified manner as follows:

$$D_\rho(F, F^c) = \left[\sum \left|2F(r_i) - 1\right|^\rho\right]^{1/\rho}, \text{where } \rho = 1, 2, 3, \ldots \tag{20}$$

For $\rho = 1$, this becomes the Hamming metric having the following form:

$$D_1(F, F^c) = \sum \left|2F(r_i) - 1\right|. \tag{21}$$

If the investor always puts a greater proportion of funds in either the risk-free asset or the market portfolio, then F is reduced to a crisp set and $|2F(r_i) - 1| = 1$.

Based on the above metrics, a universal measure of fuzziness may now be defined as follows for our portfolio funds allocation model. This is done as follows.

For a crisp set F, F^c is truly complementary, meaning that the metric distance becomes:

$$D_\rho * (F, F^c) = 2^{1/\rho}, \text{where } \rho = 1, 2, 3, \ldots \tag{22}$$

An effective measure of fuzziness could therefore be as follows:

$$\text{FUZ}_\rho(F) = \left[2^{1/\rho} - D_\rho(F, F^c)\right]/2^{1/\rho} = 1 - D_\rho(F, F^c)]/2^{1/\rho}. \tag{23}$$

For the Euclidean metric we would then have:

$$\begin{aligned}\text{FUZ}_2(F) &= 1 - \frac{\left[\sum(2F(r_1) - 1)\right]^{1/2}}{\sqrt{2}} \\ &= 1 - \left(\sqrt{2}\right)(RMSD), \text{where RMSD} = \left(\left[\sum(2\mathrm{F}(\mathrm{r_i}) - 1)^2\right]^{1/2}\right)/2.\end{aligned} \tag{24}$$

For the Hamming metric, the formula will simply be as follows:

$$\mathrm{FUZ}_1(F) = 1 - \frac{\sum|2F(r_1) - 1|}{2}. \tag{25}$$

Having worked on the applicable measure for the degree of fuzziness of our governing preference relation, we devote the next section of our present paper to the incorporation of neurofuzzy control systems to produce and compute risk classification for the homological utility of an investor under different scenarios. In a subsequent section, we also take a passing look at the possible application of optimal control theory to model the dynamics of funds allocation behavior of an individual investor.

(III) Exploring time-dependent funds allocation behavior of individual investor in the light of optimal control theory

If the inter-temporal utility of an individual viewed from time t is recursively defined as $U_t = W[c_t, \mu(U_{t+1}|I_t)]$, then the aggregator function W makes current inter-temporal utility a function of current consumption ct and of a certainty equivalent of next period's random utility It that is computed using information up to t. Then, the individual could choose a control variable x_t in period t to maximize U_t (Haliassos and Hassapis, 2001). In the context of the mean-variance model, a suitable candidate for the control variable could well be the proportion of funds set aside for investment in the risk-free asset. So, the objective function would incorporate the investor's total temporal utility in a given time range $[0, T]$. Given that we include time as a continuous variable in the model, we may effectively formulate the problem applying classical optimal control theory. The plausible methodology for formulating this model is what we shall explore in this section.

The basic optimal control problem can be stated as follows (Rao, 1995).

Find the control vector $\mathbf{u} = (u_1, u_2, \ldots, u_m)$ which optimizes the functional, called the performance index, $J = \int f_0(\mathbf{x}, \mathbf{u}, t)\mathrm{dt}$ over the range $(0, T)$, where $\mathbf{x} = (x_1, x_2, \ldots, x_n)$ is called the state vector, t is the time parameter, T is the terminal time and f_0 is a specified function of $\mathbf{x}, \mathbf{u}$ and t. The state variables x_i and the control variables u_i are related as $\mathrm{dx}_i/\mathrm{dt} = \mathrm{fi}(x_1, x_2, \ldots, x_n; u_1, u_2, \ldots, u_m; t), i = 1, 2, \ldots, n$.

In many control problems, the system is linearly expressible as $\mathbf{x}(.) = [\mathrm{A}]_{\mathrm{nxn}}\mathbf{x} + [\mathrm{B}]_{\mathrm{nxm}}\mathbf{u}$, where all the symbols have their usual connotations. As an illustrative example, we may again consider the quadratic function that we used earlier $f_0(x, y) = ax^2 - by^2$. Then the problem is to find the control vector that makes the performance index $J = \int(ax^2 - by^2)\mathrm{dt}$ stationary with $x = 1 - y$ in the range $(0, T)$.

The Hamiltonian may be expressed as $H = f_0 + \lambda y = (ax^2 - by^2) + \lambda y$. The standard solution technique yields: $-H_{\mathrm{x}} = \lambda(.)$ and $H_{\mathrm{u}} = 0$ whereby we have the following system of equations:

(1) $-2ax = \lambda(.)$; and
(2) $-2y + \lambda = 0$.

Differentiation of (2) leads to $-2y(.) + \lambda(.) = 0$. Solving (1) and this differentiation of (2) simultaneously, we get $2ax = -2y(.) = -\lambda(.)$ i.e. $y(.) = -ax$. Transforming (1) in terms of x and solving the resulting ordinary differential equation would yield the state trajectory $x(t)$ and the optimal control $u(t)$ for the specified quadratic utility function, which can be easily done by most standard mathematical computing software packages.

So, given a particular form of a utility function, we can trace the dynamic time-path of an individual investor's fund allocation behavior (and hence; his or her classification) within the ambit of the mean-variance model by obtaining the state trajectory of x – the proportion of funds invested in the market portfolio and the corresponding control variable y – the proportion of funds invested in the risk-free asset using the standard techniques of optimal control theory.

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STATISTICS

A family of estimators of population mean using multiauxiliary information in presence of measurement errors

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Abstract *This paper proposes a family of estimators of population mean using information on several auxiliary variables and analyzes its properties in the presence of measurement errors.*

Introduction

The discrepancies between the values exactly obtained on the variables under consideration for sampled units and the corresponding true values are termed as measurement errors. In general, standard theory of survey sampling assumes that data collected through surveys are often assumed to be free of measurement or response errors. In reality such a supposition does not hold true and the data may be contaminated with measurement errors due to various reasons (see, for example Cochran (1963) and Sukhatme *et al.* (1984)).

One of the major sources of measurement errors in survey is the nature of variables. This may happen in case of qualitative variables. Simple examples of such variables are intelligence, preference, specific abilities, utility, aggressiveness, tastes, etc. In many sample surveys it is recognized that errors of measurement can also arise from the person being interviewed, from the interviewer, from the supervisor or leader of a team of interviewers, and from the processor who transmits the information from the recorded interview on to the punched cards or tapes that will be analyzed, for instance, see Cochran (1968). Another source of measurement error is when the variable is conceptually well defined but observations can be obtained on some closely related substitutes termed as proxies or surrogates. Such a situation is encountered when one needs to measure the economic status or the level of education of individuals, see Salabh (1997) and Sud and Srivastava (2000). In the presence of measurement errors, inferences may be misleading, see Biermer *et al.* (1991), Fuller (1995) and Manisha and Singh (2001).

There is today a great deal of research on measurement errors in surveys. An attempt has been made to study the impact of measurement errors on a family of estimators of population mean using multiauxiliary information.

The suggested family of estimators

Let Y be the study variate and its population mean μ_0 to be estimated using information on $p(>1)$ auxiliary variates $X_1, X_2, \ldots, X_p$. Further, let the population mean row vector $\underset{\sim}{\mu}' = (\mu_1, \mu_2, \cdots, \mu_p)$ of the vector $\underset{\sim}{X}' = (X_1, X_2, X_p)$. Assume that a simple random sample of size n is drawn from a population, on the study character Y and auxiliary characters $X_1, X_2, \ldots, X_p$. For the sake of simplicity we assume that the population is infinite. The recorded fallible measurements are given by:

$$y_j = Y_j + E_j$$

$$x_{ij} = X_{ij} + \eta_{ij}, i = 1, 2, \cdots, p,$$

$$j = 1, 2, \cdots, n.$$

where Y_j and X_{ij} are correct values of the characteristics Y and $X_i (i = 1, 2, \ldots, p; j = 1, 2, \ldots, n)$.

For the sake of simplicity in exposition, we assume that the error E_js are stochastic with mean "zero" and variance $\sigma_{(0)}^2$ and uncorrelated with Y_js. The errors η_{ij} in x_{ij} are distributed independently of each other and of the X_{ij} with mean "zero" and variance $\sigma_{(i)}^2 (i = 1, 2, \ldots, p)$. Also E_js and η_{ij}s are uncorrelated although Y_js and X_{ij}s are correlated.

Define

$$u_i = \frac{\bar{x}_i}{\mu_i}, (i = 1, 2, \cdots, p),$$

$$u^T = (u_1, u_2, \cdots u_p)_{1 \times p},$$

$$e^T = (1, 1, \cdots, 1)_{1 \times p},$$

$$\bar{y} = \frac{1}{n} \sum_{j=1}^{n} y_j,$$

$$\bar{x}_i = \frac{1}{n}\sum_{j=1}^{n} x_{ij}.$$

With this background we suggest a family of estimators of μ_0 as:

$$\hat{\mu}_g = g\left(\bar{y}, u^T\right). \tag{2.1}$$

where $g\left(\bar{y}, u^T\right)$ is a function of $\bar{y}, u_1, u_2, \cdots, u_p$ such that:

$$g_{\left(u_0, e^T\right)} = \mu_0$$

$$\Rightarrow \left.\frac{\partial g(\cdot)}{\partial \bar{y}}\right|_{\left(\mu_0, e^T\right)} = 1$$

and such that it satisfies the following conditions:

- the function $g\left(\bar{y}, u^T\right)$ is continuous and bounded in Q;
- the first and second order partial derivatives of the function $g\left(\bar{y}, u^T\right)$ exist and are continuous and bounded in Q.

To obtain the mean squared error of $\hat{\mu}_g$, we expand the function $g\left(\bar{y}, u^T\right)$ about the point (μ_0, e^T) in a second order Taylor's series. We get:

$$\hat{\mu}_g = g\left(\mu_0, e^T\right) + \left(\bar{y} - \mu_0\right)\left.\frac{\partial g(\cdot)}{\partial \bar{y}}\right|_{\left(\mu_0, e^T\right)} + (u - e)^T g^{(1)}\left(\mu_0, \mathrm{e}^T\right) \tag{2.2}$$

$$+\frac{1}{2}\left\{\left(\bar{y} - \mu_0\right)^2 \left.\frac{\partial^2 g(\cdot)}{\partial \bar{y}^2}\right|_{\left(\bar{y}^*, u^{*T}\right)} + 2\left(\bar{y} - \mu_0\right)(u - e)^T \left.\frac{\partial g^{(1)}(\cdot)}{\partial \bar{y}}\right|_{\left(\bar{y}^*, u^{?T}\right)}\right.$$

$$\left. + (u - e)^T g^{(2)}\left(\bar{y}*, u*^T\right)(u - e)\right\},$$

where:

$$\bar{y}* = \mu_0 + \theta\left(\bar{y} - \mu_0\right), u* = e + \theta(u - e), 0 < \theta < 1; g^{(1)}(\cdot)$$

denote the p element column vector of first partial derivatives of $g(\cdot)$ and $g^{(2)}(\cdot)$ denotes a $p \times p$ matrix of second partial derivatives of $g(\cdot)$ with respect to u.

Noting that $g(\mu_0, e^T) = \mu_0$, it can be shown that:

$$E\left(\hat{\mu}_g\right) = \mu_0 + O(n^{-1}), \tag{2.3}$$

which follows that the bias of $\hat{\mu}_g$ is of the order of n^{-1}, and hence its contribution to the mean squared error of $\hat{\mu}_g$ will be of the order of n^{-2}. From (2.2), we have to terms of order n^{-1}:

$$
\begin{aligned}
\mathrm{MSE}\left(\hat{\mu}_g\right) &= E\{\left(\bar{y}-\mu_O\right)+(u-e)^T g^{(1)}(\mu_O,e^T)\}^2 \\
&= E\left[\left(\bar{y}=\mu_O\right)^2+2\left(\bar{y}-\mu_O\right)(u-e)^T g^{(1)}(\mu_O,e^T)\right. \\
&\quad \left.+\left(g^{(1)}(\mu_O,e^T)\right)^T(u-e)(u-e)^T\left(g^{(1)}(\mu_O,e^T)\right)\right] \\
&= \frac{1}{n}\left[\mu_0^2\left(C_0^2+C_{(0)}^2\right)+2\mu_0 b^T g^{(1)}(\mu_O,e^T)+\left(g^{(1)}\right)^T A\left(g^{(1)}(\mu_O,e^T)\right)\right]
\end{aligned} \tag{2.4}
$$

where $b^T=(b_1,b_2,\ldots,b_p), b_i,=\rho_{0\mathrm{i}}\mathrm{C}_0 C_i,(i=1,2,\ldots,p)$; $C_i=\sigma_i/\mu_i, C_{(\mathrm{i})}=\sigma_i/\mu_i,(i=1,2,\ldots,p)$ and $C_0=\sigma_0/\mu_0$:

$$
\mathrm{A}=\begin{bmatrix}
C_1^2+C_{(1)}^2 & \rho_{12}C_1C_2 & \rho_{13}C_1C_3 & \cdots & \rho_{1p}C_1C_p \\
\rho_{12}C_1C_2 & C_2^2+C_{(2)}^2 & \rho_{23}C_2C_3 & \cdots & \rho_{2p}C_2C_p \\
\rho_{13}C_1C_3 & \rho_{23}C_2C_3 & C_3^2+C_{(3)}^2 & \cdots & \rho_{3p}C_3C_p \\
\vdots & \vdots & \vdots & \vdots & \\
\rho_{1p}C_1C_p & \rho_{2p}C_2C_p & \rho_{3p}C_3C_p & \cdots & C_p^2+C_{(p)}^2
\end{bmatrix}_{p\times p}
$$

The $\mathrm{MSE}\left(\hat{\mu}_g\right)$ at equation (2.4) is minimized for

$$g^{(1)}(\mu_O,e^T)=-\mu_0 A^{-1}b. \tag{2.5}$$

Thus the resulting minimum MSE of $\hat{\mu}_g$ is given by:

$$\mathrm{min.MSE}\left(\hat{\mu}_g\right)=\left(\mu_0^2/n\right)\left[C_0^2+C_{(0)}^2-b^T A^{-1}b\right]. \tag{2.6}$$

Now we have established the following theorem.

Theorem 2.1=up to terms of order n^{-1},

$$\mathrm{MSE}\left(\hat{\mu}_g\right)\geq\left(\mu_0^2/n\right)\left[C_0^2+C_{(0)}^2-b^T A^{-1}b\right] \tag{2.7}$$

with equality holding if:

$$g^{(1)}=-\mu_0 A^{-1}b.$$

It is to be mentioned that the family of estimators $\hat{\mu}_g$ at equation (2.1) is very large. The following estimators may be identified as particular members of the suggested family of estimators $\hat{\mu}_g$:

- $\hat{\mu}_g^{(1)}=\bar{y}\sum_{i=1}^{p}\omega_i\left(\frac{\mu_i}{\bar{x}_i}\right); \sum_{i=1}^{p}\omega_i=1$, (Olkin, 1958).

- $\hat{\mu}_g^{(2)} = \bar{y}\sum_{i=1}^{p} \omega_i \left(\frac{\bar{x}_i}{\mu_i}\right), \sum_{i=1}^{p} \omega_i - 1$, (Singh, 1967).
- $\hat{\mu}_g^{(3)} = \bar{y}\frac{\sum_{i=1}^{p} \omega_i \mu_i}{\sum_{i=1}^{p} \omega_i \bar{x}_i}, \sum_{i=1}^{p} \omega_i = 1$, (Shukla, 1966; John, 1969).
- $\hat{\mu}_g^{(4)} = \bar{y}\frac{\sum_{i=1}^{p} \omega_i \bar{x}_i}{\sum_{i=1}^{p} \omega_i \mu_i}; \sum_{i=1}^{p} \omega_i = 1$, (Sahai *et al.*, 1980).
- $\hat{\mu}_g^{(5)} = \bar{y}\prod_{i=1}^{p} \left(\frac{\mu_i}{\bar{x}_i}\right)^{\omega_i}, \sum_{i=1}^{p} \omega_i = 1$, (Mohanty and Pattanaik, 1984).
- $\hat{\mu}_g^{(6)} = \bar{y} \left. \sum_{i=1}^{p} \frac{\omega_i \bar{x}_i}{\mu_i} \right)^{-1}, \sum_{i=1}^{p} \omega_i = 1$, (Mohanty and Pattanaik, 1984).
- $\hat{\mu}_g^{(7)} = \bar{y}\prod_{i=1}^{p} \left(\frac{\bar{x}_i}{\mu_i}\right)^{\omega_i}, \sum_{i=1}^{p} \omega_i = 1$, (Tuteja and Bahl, 1991).
- $\hat{\mu}_g^{(8)} = \bar{y}\left[\sum_{i=1}^{p} \frac{\omega_i \mu_i}{\bar{x}_i}\right]^{-1}, \sum_{i=1}^{p} \omega_i = 1$, (Tuteja and Bahl, 1991).
- $\hat{\mu}_g^{(9)} = \bar{y}\left[\omega_{p+1} + \sum_{i=1}^{p} \omega_i \left(\frac{\mu_i}{\bar{x}_i}\right)\right], \sum_{i=1}^{p+1} \omega_i = 1.$
- $\hat{\mu}_g^{(10)} = \bar{y}\left[\omega_{p+1} + \sum_{i=1}^{p} \omega_i \left(\frac{\bar{x}_i}{\mu_i}\right)\right], \sum_{i=1}^{p+1} \omega_i = 1.$
- $\hat{\mu}_g^{(11)} = \bar{y}\left[\sum_{i=1}^{q} \omega_i \left(\frac{\mu_i}{x_i}\right) + \sum_{i=q+1}^{p} \left(\frac{\hat{x}_i}{\mu_i}\right)\right]; \quad \left. \sum_{i=1}^{q} \omega_i + \sum_{i=q+1}^{p} \omega_i \right)^{=1};$

  (Srivastava, 1965; Rao and Mudholkar, 1967).
- $\hat{\mu}_g^{(12)} = \bar{y}\prod_{i=1}^{p} \left(\frac{\bar{x}_i}{\mu_i}\right)^{\alpha_i}$ (α_is are suitably constants) (Srivastava, 1967).

- $\hat{\mu}_g^{(13)} = \bar{y}\prod_{i=1}^{p}\{2 - \left(\frac{\bar{x}_i}{\mu_i}\right)^{\alpha_i}\}$ (Sahai and Ray, 1980).
- $\hat{\mu}_g^{(14)} = \bar{y}\prod_{i=1}^{p}\frac{\bar{x}_i}{\{\mu_i + \alpha_i(\bar{x}_i - \mu_i)\}}$ (Walsh, 1970).
- $\hat{\mu}_g^{(15)} = \bar{y}\exp\{\sum_{i=1}^{p}\theta_i \log u_i\}$ (Srivastava, 1971).
- $\hat{\mu}_g^{(16)} = \bar{y}\exp\{\sum_{i=1}^{p}\theta_i(u_i - 1)\}$ (Srivastava, 1971).
- $\hat{\mu}_g^{(17)} = \bar{y}\sum_{i=1}^{p}\omega_i\exp\{(\theta_i/\omega_i)\log u_i\}; \sum_{i=1}^{p}\omega_i = 1$, (Srivastava, 1971).
- $\hat{\mu}_g^{(18)} = \bar{y} + \sum_{i=1}^{p}\alpha_i(\bar{x}_i - \mu_i)$, etc.

The MSE of these estimators can be obtained from equation (2.4).

It is well known that:

$$\mathrm{V}(\bar{y}) = (\mu_0^2/n)\left(C_0^2 + C_{(0)}^2\right). \tag{2.8}$$

It follows from equation (2.6) and equation (2.8) that the minimum variance of $\hat{\mu}_g$ is no longer than conventional unbiased estimator $\bar{y}$.

On substituting $\sigma_{(0)}^2 = 0$, $\sigma_{(i)}^2 = 0 \forall i = 1, 2, \ldots, p$ in equation (2.4), we obtain the no-measurement error case. In that case, the MSE of $\hat{\mu}_g$, is given by:

$$\begin{aligned}\mathrm{MSE}(\hat{\mu}_g) &= \frac{1}{n}\left[C_0^2\mu_0^2 + 2\mu_0 b^T g *^{(1)}(\mu_O, e^T) + \left(g *^{(1)}(\mu_O, e^T)\right)^T A * \left(g*^{(1)}\right)\right] \\ &= \mathrm{MSE}(\hat{\mu}_g *),\end{aligned} \tag{2.9}$$

where:

$$\begin{aligned}\hat{\mu}_g &= g * \quad \bar{Y}, \frac{\bar{X}_1}{\mu_1}, \frac{\bar{X}_2}{\mu_2}, \cdots, \frac{\bar{X}_p}{\mu_p}\Bigg) \\ &= g * (\bar{Y}, U^T)\end{aligned} \tag{2.10}$$

and $\bar{\mathrm{Y}}\mathrm{and}\bar{\mathrm{X}}_{\mathrm{i}}(i = 1, 2, \cdots, p)$ are the sample means of the characteristics Y and X_i based on true measurements. $(Y_j, X_{ij}, i = 1, 2, \ldots, p; j = 1, 2, \ldots, n)$. The family of estimators $\hat{\mu}_g *$ at equation (2.10) is a generalized version of Srivastava (1971, p. 80).

The MSE of $\hat{\mu}_g*$ is minimized for:

$$g*^{(1)}(\mu_O, e^T) = -A*^{-1}b\mu_0. \tag{2.11}$$

Thus the resulting minimum MSE of $\hat{\mu}_g*$ is given by:

$$\begin{aligned} \text{min.MSE}(\hat{\mu}_g*) &= \frac{\mu_0}{n}\left[C_0^2 - b^T A*^{-1} b\right] \\ &= \frac{\sigma_0}{n}(1-R^2), \end{aligned} \tag{2.12}$$

where $A* = [a*_{\text{ij}}]$ is a $p \times p$ matrix with $a*_{\text{ij}} = \rho_{ij}C_iC_j$ and R stands for the multiple correlation coefficient of Y on $X_1, X_2, \ldots, X_p$.

From equations (2.6) and (2.12) the increase in minimum $\text{MSE}(\hat{\mu}_g)$ due to measurement errors is obtained as:

$$\begin{aligned} \text{min.MSE}(\hat{\mu}_g) - \text{min.MSE}(\hat{\mu}_g*) &= \left(\frac{\mu_0^2}{n}\right)\left[C_{(0)}^2 + b^T A*^{-1} b - b^T A^{-1} b\right] \\ &> 0. \end{aligned}$$

This is due to the fact that the measurement errors introduce the variances fallible measurements of study variate Y and auxiliary variates X_i. Hence there is a need to take the contribution of measurement errors into account.

Biases and mean square errors of some particular estimators in the presence of measurement errors

To obtain the bias of the estimator $\hat{\mu}_g$, we further assume that the third partial derivatives of $g(\bar{y}, u^T)$ also exist and are continuous and bounded. Then expanding $g(\bar{y}, u^T)$ about the point $(\bar{y}, u^T) = (\mu_0, e^T)$ in a third-order Taylor's series we obtain:

$$\begin{aligned} \hat{\mu}_g &= g(\mu_0, e^T) + (\bar{y} - \mu_O)\frac{\partial g(\cdot)}{\partial \bar{y}}\bigg|_{(\mu_0, e^T)} + (u-e)^T g^{(1)}(\mu_O, e^T) \\ &+ \frac{1}{2}\{(\bar{y} - \mu_O)^2 \frac{\partial^2 g(\cdot)}{\partial \bar{y}^2}\bigg|_{(\mu_0, u^T)} + 2(\bar{y} - \mu_O)(u-e)^T g^{(1)}(\mu_O, e^T) \\ &+ (u-e)^T\left(g^{(2)}(\mu_O, e^T)\right)(u-e)\} \\ &+ \frac{1}{6}\{(\bar{y} - \mu_O)\frac{\partial}{\partial \bar{y}} + (u-e)\frac{\partial}{\partial u}\}^3 g(\bar{y}^*, u*^T), \end{aligned} \tag{3.1}$$

where $g^{(12)}(\mu_0, e^T)$ denotes the matrix of second partial derivatives of $g(\bar{y}, u^T)$ at the point $(\bar{y}, u^T) = (\mu_0, e^T)$.

Noting that:

$$g\left(u_0 e^T\right) = \mu_0$$
$$\left.\frac{\partial g(\cdot)}{\partial \bar{y}}\right|_{(\mu_0, e^T)} = 1$$
$$\left.\frac{\partial^2 g(\cdot)}{\partial \bar{y}^2}\right|_{(\mu_0, e^T)} = 0,$$

and taking expectation we obtain the bias of the family of estimators $\hat{\mu}_g$ to the first degree of approximation:

$$B\left(\hat{\mu}_g\right) = \frac{1}{2}\left[E\{(u-e)^T\left(g^{(2)}(\mu_O, e^T)\right)(u-e)\} + 2\left(\frac{\mu_0}{n}\right)b^T g^{(12)}(\mu_O, e^T)\right] \tag{3.2}$$

where $b^T = (b_1, b_2, \ldots, b_p)$ with $bi = \rho_{0i} C_0 C_i; (i = 1, 2, \ldots, p)$. Thus we see that the bias of $\hat{\mu}_g$ depends also on the second order partial derivatives of the function on $g\left(\bar{y}, u^T\right)$ at the point (μ_0, e^T), and hence will be different for different optimum estimators of the family.

The biases and mean square errors of the estimators $\hat{\mu}_g$; $i = 1$ to 18 up to terms of order n^{-1} along with the values of $g^{(1)}(\mu_0, e^T)$, $g^{(2)}(\mu_0, e^T)$ and $g^{(12)}(\mu_0, e^T)$ are given in the Table I.

Estimators based on estimated optimum

It may be noted that the minimum MSE, equation (2.6), is obtained only when the optimum values of constants involved in the estimator, which are functions of the unknown population parameters μ_0, b and A, are known quite accurately.

To use such estimators in practice, one has to use some guessed values of the parameters μ_0, b and A, either through past experience or through a pilot sample survey. Das and Tripathi (1978, sec. 3) have illustrated that even if the values of the parameters used in the estimator are not exactly equal to their optimum values as given by equation (2.5) but are close enough, the resulting estimator will be better than the conventional unbiased estimator $\bar{y}$. For further discussion on this issue, the reader is referred to Murthy (1967), Reddy (1973), Srivenkataramana and Tracy (1984) and Sahai and Sahai (1985).

On the other hand if the experimenter is unable to guess the values of population parameters due to lack of experience, it is advisable to replace the unknown population parameters by their consistent estimators. Let $\hat{\phi}$ be a consistent estimator of $\phi = A^{-1}b$. We then replace ϕ by $\hat{\phi}$ and also μ_0 by $\bar{y}$ if necessary, in the optimum $\hat{\mu}_g$ resulting in the estimator $\hat{\mu}_{g(est)}$, say, which will now be a function of $\bar{y}$, u and ϕ. Thus we define a family of estimators (based on estimated optimum values) of μ_0 as:

Table I. Biases and mean squared errors of various estimators of μ_0

Estimator	$g^{(1)}(\mu 0, eT)$	$g^{(2)}(\mu 0, eT)$	$g^{(12)}(\mu 0, eT)$	Bias	MSE
$\hat{\mu}_g^{(1)}$	$-\mu_0\underset{\sim}{\omega}$	$2\mu_0 \underset{\sim}{W}_{p\times p}$	$-\underset{\sim}{\omega}$	$\left(\frac{\mu_0}{n}\right)\left(C^T \underset{\sim}{W}_{p\times p} - b^T\underset{\sim}{\omega}\right)$	$\left(\frac{\mu_0}{n}\right)\left[C_0^2 + C_{(o)}^2 - 2b^T\underset{\sim}{\omega} + \underset{\sim}{\omega}^T A\underset{\sim}{\omega}\right]$
		where $Wp \times p = \mathrm{dig}(\omega 1, \omega 2, \ldots, \omega \mathrm{p})$		where $C^T = \left(C_1^2 + C_{(1)}^2, C_2^2 + C_{(2)}^2, \cdots, C_p^2 + C_{(p)}^2\right)$	
$\hat{\mu}_g^{(2)}$	$\mu_0\underset{\sim}{\omega}$	$\underset{\sim}{O}_{p\times p}$ (null matrix)	$\underset{\sim}{\omega}$	$\left(\frac{\mu_0}{n}\right) b^T\underset{\sim}{\omega}$	$\left(\frac{\mu_0^2}{n}\right)\left[C_0^2 + C_{(0)}^2 + 2b^T\underset{\sim}{\omega} + \underset{\sim}{\omega}^T A\underset{\sim}{\omega}\right]$
$\hat{\mu}_g^{(3)}$	$-\frac{\mu_0\underset{\sim}{\omega}*}{\underset{\sim}{\omega}^T\underset{\sim}{\mu}}$ where $\underset{\sim}{\omega}*^T = (\omega_1, *\omega_2, *\ldots, \omega_{\mathrm{p}}*)$ with $(\omega i, * = \omega i\mu i)(\mathrm{i} = 1, 2, \ldots, p)$	$\frac{2\mu_0\underset{\sim}{\omega}*\underset{\sim}{\omega}*^T}{\underset{\sim}{\omega}^T\underset{\sim}{\mu}\underset{\sim}{\mu}^T\underset{\sim}{\omega}}$	$-\frac{\underset{\sim}{\omega}*}{\underset{\sim}{\omega}^T\underset{\sim}{\mu}}$	$\left(\frac{\mu_0}{n}\right)\left(\frac{\underset{\sim}{\omega}*^T A\underset{\sim}{\omega}}{\underset{\sim}{\omega}^T\underset{\sim}{\mu}\underset{\sim}{\mu}^T\underset{\sim}{\omega}} - \frac{b^T\underset{\sim}{\omega}*}{\underset{\sim}{\omega}^T\underset{\sim}{\mu}}\right)$	$\left(\frac{\mu_0^2}{n}\right)\left[C_0^2 + C_{(0)}^2 - \frac{2b^T\underset{\sim}{\omega}*}{\underset{\sim}{\omega}^T\underset{\sim}{\mu}} + \frac{\underset{\sim}{\omega}*^T A\underset{\sim}{\omega}*}{\underset{\sim}{\omega}^T\underset{\sim}{\mu}\underset{\sim}{\mu}^T\underset{\sim}{\omega}}\right]$
$\hat{\mu}_g^{(4)}$	$\frac{\mu_0\underset{\sim}{\omega}}{\underset{\sim}{\omega}^T\underset{\sim}{\mu}}$	$\underset{\sim}{O}_{p\times p}$ (null matrix)	$\frac{\underset{\sim}{\omega}}{\underset{\sim}{\omega}^T\underset{\sim}{\mu}}$	$\left(\frac{\mu_0}{n}\right)\frac{b^T\underset{\sim}{\omega}}{\underset{\sim}{\omega}^T\underset{\sim}{\mu}}$	$\left(\frac{\mu_0^2}{n}\right)\left[C_0^2 + C_{(0)}^2 + \frac{2b^T\underset{\sim}{\omega}}{\underset{\sim}{\omega}^T\underset{\sim}{\mu}} + \frac{\underset{\sim}{\omega}^T A\underset{\sim}{\omega}}{\underset{\sim}{\omega}^T\underset{\sim}{\mu}\underset{\sim}{\mu}^T\underset{\sim}{\omega}}\right]$
$\hat{\mu}_g^{(5)}$	$-\mu_0\underset{\sim}{\omega}$	$\mu_0\left(\underset{\sim}{\omega}\underset{\sim}{\omega}^T + \underset{\sim}{W}_{p\times p}\right)$	$-\underset{\sim}{\omega}$	$\left(\frac{\mu_0}{2n}\right)\left[\underset{\sim}{\omega}^T A\underset{\sim}{\omega} + C^T\underset{\sim}{W}_{p\times p} - 2b^T\underset{\sim}{\omega}\right]$	$\left(\frac{\mu_0}{n}\right)\left[C_0^2 + C_{(o)}^2 - 2b^T\underset{\sim}{\omega} + \underset{\sim}{\omega}^T A\underset{\sim}{\omega}\right]$
$\hat{\mu}_g^{(6)}$	$-\mu_0\underset{\sim}{\omega}$	$2\mu_0\underset{\sim}{\omega}^T\underset{\sim}{\omega}$	$-\underset{\sim}{\omega}$	$\left(\frac{\mu_0}{n}\right)\left[\underset{\sim}{\omega}^T A\underset{\sim}{\omega} - b^T\underset{\sim}{\omega}\right]$	$\left(\frac{\mu_0}{n}\right)\left[C_0^2 + C_{(o)}^2 - 2b^T\underset{\sim}{\omega} + \underset{\sim}{\omega}^T A\underset{\sim}{\omega}\right]$
$\hat{\mu}_g^{(7)}$	$\mu_0\underset{\sim}{\omega}$	$\mu_0\left(\underset{\sim}{\omega}\underset{\sim}{\omega}^T - \underset{\sim}{W}_{p\times p}\right)$	$\underset{\sim}{\omega}$	$\left(\frac{\mu_0}{2n}\right)\left[\underset{\sim}{\omega}^T A\underset{\sim}{\omega} - C^T\underset{\sim}{W}_{p\times p} + 2b^T\underset{\sim}{\omega}\right]$	$\left(\frac{\mu_0}{n}\right)\left[C_0^2 + C_{(o)}^2 + 2b^T\underset{\sim}{\omega} + \underset{\sim}{\omega}^T A\underset{\sim}{\omega}\right]$
$\hat{\mu}_g^{(8)}$	$\mu_0\underset{\sim}{\omega}$	$2\mu_0\left(\underset{\sim}{\omega}\underset{\sim}{\omega}^T - \underset{\sim}{W}_{p\times p}\right)$	$\underset{\sim}{\omega}$	$\left(\frac{\mu_0}{n}\right)\left[\underset{\sim}{\omega}^T A\underset{\sim}{\omega} - C^T\underset{\sim}{W}_{p\times p} + b^T\underset{\sim}{\omega}\right]$	$\left(\frac{\mu_0}{n}\right)\left[C_0^2 + C_{(o)}^2 + 2b^T\underset{\sim}{\omega} + \underset{\sim}{\omega}^T A\underset{\sim}{\omega}\right]$

(*continued*)

Table I.

Estimator	$g^{(1)}(\mu 0, eT)$	$g^{(2)}(\mu 0, eT)$	$g^{(12)}(\mu 0, eT)$	Bias	MSE
$\hat{\mu}_g^{(9)}$	$-\mu_0 \underset{\sim}{\omega}$	$2\mu_0 \underset{\sim}{W}_{p\times p}$	$-\underset{\sim}{\omega}$	$\left(\frac{\mu_0}{n}\right) \left(C^T \underset{\sim}{W}_{p\times p} - b^T \underset{\sim}{\omega} \right)$	$\left(\frac{\mu_0^2}{n}\right)\left[C_0^2 + C_{(0)}^2 + 2b^T \underset{\sim}{\omega} + \underset{\sim}{\omega}^T A \underset{\sim}{\omega}\right]$
$\hat{\mu}_g^{(10)}$	$\mu_0 \underset{\sim}{\omega}$	$\underset{\sim}{Q}$	$\underset{\sim}{\omega}$	$\left(\frac{\mu_0}{n}\right) b^T \underset{\sim}{\omega}$	$\left(\frac{\mu_0^2}{n}\right)\left[C_0^2 + C_{(0)}^2 + 2b^T \underset{\sim}{\omega} + \underset{\sim}{\omega}^T A \underset{\sim}{\omega}\right]$
$\hat{\mu}_g^{(11)}$	$\underset{\sim}{\omega}_{(1)} \mu_0$	$2\underset{\sim}{W}_{(1)} \mu_0$	$\underset{\sim}{\omega}_{(1)}$	$\left(\frac{\mu_0}{n}\right) \left(C*^T \underset{\sim}{W}_{(1)} - b^T \underset{\sim}{\omega}_{(1)} \right)$	$\left(\frac{\mu_0^2}{n}\right)\left[C_0^2 + C_{(0)}^2 - 2b^T \underset{\sim}{\omega}_{(1)} + \underset{\sim}{\omega}^T_{(1)} A \underset{\sim}{\omega}_{(1)}\right]$
	where $\underset{\sim}{\omega}_{(1)} = (-\omega_1, -\omega_2, ..., -\omega_q, -\omega_{q+1}, ..., \omega_p)_{1\times p}$				
	$\underset{\sim}{W}_{(1)p\times p} = \begin{bmatrix} \omega_1 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \omega_2 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & 0 & \omega & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & \omega_q & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix}_{p\times p}$,			$C*T = (C12 + C(1)2, \ldots, Cq2 + C(q)2; \ldots 0)$	
$\hat{\mu}_g^{(12)}$	$\underset{\sim}{\alpha} \mu_0$	$\mu_0 \left(\underset{\sim}{\alpha}\underset{\sim}{\alpha}^T - \underset{\sim}{\propto}_{p\times p} \right)$	$-\underset{\sim}{\alpha}$	$\left(\frac{\mu_0}{2n}\right)\left[\underset{\sim}{\alpha}^T A \underset{\sim}{\alpha} - C^T \underset{\sim}{\propto}_{p\times p} + 2b^T \underset{\sim}{\alpha}\right]$	$\left(\frac{\mu_0}{n}\right)\left[C_0^2 + C_{(o)}^2 + 2b^T \underset{\sim}{\alpha} + \underset{\sim}{\alpha}^T A \underset{\sim}{\alpha}\right]$
	where $\underset{\sim}{\alpha}^T = (\alpha_1, \alpha_2, ..., \alpha_p)_{1\times p}$ $\underset{\sim}{\propto} = \mathrm{diag}(\alpha_1, \alpha_2, ..., \alpha_p)$				
$\hat{\mu}_g^{(13)}$	$-\underset{\sim}{\alpha} \mu_0$	$-\mu_0 \left(\underset{\sim}{\alpha}\underset{\sim}{\alpha}^T - \underset{\sim}{\propto}_{p\times p} \right)$	$-\underset{\sim}{\alpha}$	$\left(\frac{\mu_0}{2n}\right)\left[C^T \underset{\sim}{\propto}_{p\times p} - \underset{\sim}{\alpha}^T A \underset{\sim}{\alpha} - 2b^T \underset{\sim}{\alpha}\right]$	$\left(\frac{\mu_0}{n}\right)\left[C_0^2 + C_{(o)}^2 - 2b^T \underset{\sim}{\alpha} + \underset{\sim}{\alpha}^T A \underset{\sim}{\alpha}\right]$

(*continued*)

Estimator	$g^{(1)}(\mu 0, eT)$	$g^{(2)}(\mu 0, eT)$	$g^{(12)}(\mu 0, eT)$	Bias	MSE
$\hat{\mu}_g^{(14)}$	$-\underset{\sim}{\alpha}\mu_0$	$2\mu_0\underset{\sim}{\alpha}\underset{\sim}{\alpha}$	$-\underset{\sim}{\alpha}$	$\left(\frac{\mu_0}{n}\right)\left[\underset{\sim}{\alpha}^T A\underset{\sim}{\alpha} - C^T - b^T\underset{\sim}{\alpha}\right]$	$\left(\frac{\mu_0}{n}\right)\left[C_0^2 + C_{(o)}^2 - 2b^T\underset{\sim}{\alpha} + \underset{\sim}{\alpha}^T A\underset{\sim}{\alpha}\right]$
$\hat{\mu}_g^{(15)}$	$\mu_0\underset{\sim}{\theta}$	$\mu_0 \left(\underset{\sim}{\theta}\underset{\sim}{\theta}^T - \underset{\sim}{\Theta}_{p\times p}\right)$,	$v0^{\underset{\sim}{\theta}}$		
		where $\underset{\sim}{\Theta}_{p\times p} = \mathrm{diag}\{\theta_1, \theta_2, \ldots \theta_p\}$		$\left(\frac{\mu_0}{2n}\right)\left[\underset{\sim}{\theta}^T A\underset{\sim}{\theta} - C^T\underset{\sim}{\Theta}_{p\times p} + 2b^T\underset{\sim}{\theta}\right]$	$\left(\frac{\mu_0^2}{n}\right)\left[C_0^2 + C_{(0)}^2 + 2b^T\underset{\sim}{\theta} + \underset{\sim}{\theta}^T A\underset{\sim}{\theta}\right]$
$\hat{\mu}_g$	$\mu_0\underset{\sim}{\theta}$	$\mu_0\underset{\sim}{\theta}\underset{\sim}{\theta}^T$	$\underset{\sim}{\theta}$	$\left(\frac{\mu_0}{2n}\right)\left[\underset{\sim}{\theta}^T A\underset{\sim}{\theta} + 2b^T\underset{\sim}{\theta}\right]$	$\left(\frac{\mu_0^2}{n}\right)\left[C_0^2 + C_{(0)}^2 + 2b^T\underset{\sim}{\theta} + \underset{\sim}{\theta}^T A\underset{\sim}{\theta}\right]$
$\hat{\mu}_g$	$\mu_0\underset{\sim}{\theta}$	$\underset{\sim}{\Theta} *_{p\times p} \mu_0$,	$\underset{\sim}{\theta}$	$\left(\frac{\mu_0}{2n}\right)\left[C^T\underset{\sim}{\Theta} *_{p\times p} + 2b^T\underset{\sim}{\theta}\right]$	$\left(\frac{\mu_0}{n}\right)\left[C_0^2 + C_{(o)}^2 + 2b^T\underset{\sim}{\theta} + \underset{\sim}{\theta}^T A\underset{\sim}{\theta}\right]$
		where $\underset{\sim}{\Theta}*_{p\times p} = \mathrm{diag}\{\theta_1\left(\frac{\theta_1}{\omega_1} - 1\right)\ldots, \theta_p\left(\frac{\theta_p}{\omega_p} - 1\right)\}$			
$\hat{\mu}_g$	$\underset{\sim}{\alpha}*$	$\underset{\sim}{Q}_{p\times p}$	$\underset{\sim}{Q}_{p\times p}$	Unbiased	$\left(\frac{1}{n}\right)\left[C_0^2 + C_{(o)}^2 + 2\mu_0 b^T\underset{\sim}{\alpha}* + \underset{\sim}{\alpha}*^T A\underset{\sim}{\alpha}*\right]$
	where $\underset{\sim}{\alpha}*^T = (\alpha_{1*}, \alpha_{2*}, \ldots, \alpha_{p*})$ with $\underset{\sim}{\alpha}*_i = (\alpha_i, \mu_i, i = 1, 2, \ldots, p)$				

Table I.

$$\hat{\mu}_{g(est)} = g**\left(\bar{y}, u^T, \hat{\phi}^T\right), \tag{4.1}$$

where $g**(\cdot)$ is a function of $\left(\bar{y}, u^T, \hat{\phi}^T\right)$ such that:

$$\begin{aligned} & g**\left(\mu_0, e^T, \phi^T\right) = \mu_0 \text{ for all } \mu_0, \Rightarrow \left.\frac{\partial g**(\cdot)}{\partial \bar{y}}\right|_{(\mu_0, e^T, \phi^T)} = 1 \\ & \left.\frac{\partial g**(\cdot)}{\partial u}\right|_{(\mu_0, e^T\phi^T)} = \left.\frac{\partial g(\cdot)}{\partial u}\right|_{(\mu_0, e)^T} = -\mu_0 A^{-1} b = -\mu_0 \phi, \end{aligned} \tag{4.2}$$

and:

$$\left.\frac{\partial g**(\cdot)}{\partial \hat{\phi}}\right|_{(\mu_0, e^T, \phi^T)} = 0.$$

With these conditions and following Srivastava and Jhajj (1983), it can be shown to the first degree of approximation that:

$$\mathrm{MSE}\left(\hat{\mu}_{g(est)}\right) = \min.\mathrm{MSE}\left(\hat{\mu}_g\right) = \left(\frac{\mu_0}{n}\right)\left[C_0^2 + C_{(0)}^2 - b^T A^{-1} b\right].$$

Thus if the optimum values of constants involved in the estimator are replaced by their consistent estimators and conditions (4.2) hold true, the resulting estimator $\hat{\mu}_{g(est)}$ will have the same asymptotic mean square error, as that of optimum $\hat{\mu}_g$. Our work needs to be extended and future research will explore the computational aspects of the proposed algorithm.

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Empirical study in finite correlation coefficient in two phase estimation

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Abstract *This paper proposes a class of estimators for population correlation coefficient when information about the population mean and population variance of one of the variables is not available but information about these parameters of another variable (auxiliary) is available, in two phase sampling and analyzes its properties. Optimum estimator in the class is identified with its variance formula. The estimators of the class involve unknown constants whose optimum values depend on unknown population parameters. In earlier research it has been shown that when these population parameters are replaced by their consistent estimates the resulting class of estimators has the same asymptotic variance as that of optimum estimator. An empirical study is carried out to demonstrate the performance of the constructed estimators.*

1. Introduction

Consider a finite population $U = \{1, 2, \ldots, i, \ldots, N\}$. Let y and x be the study and auxiliary variables taking values y_i and x_i, respectively, for the ith unit. The correlation coefficient between y and x is defined by

$$\rho_{yx} = S_{yx}/(S_y S_x) \tag{1.1}$$

where

$$S_{yx} = (N-1)^{-1}\sum_{i=1}^{N}(y_i - \bar{Y})(x_i - \bar{X}),$$

$$S_x^2 = (N-1)^{-1}\sum_{i=1}^{N}(x_i - \bar{X})^2,$$

$$S_y^2 = (N-1)^{-1}\sum_{i=1}^{N}(y_i - \bar{Y})^2,$$

$$\bar{X} = N^{-1}\sum_{i=1}^{N} x_i,$$

$$\bar{Y} = N^{-1}\sum_{i=1}^{N} y_i.$$

Based on a simple random sample of size n drawn without replacement, (x_i, y_i), $i = 1, 2, \ldots, n$; the usual estimator of ρ_{yx} is the corresponding sample correlation coefficient:

$$r = s_{yx}/(s_x s_y) \tag{1.2}$$

where

$$s_{yx} = (n-1)^{-1}\sum_{i=1}^{n} (y_i - \bar{y})(x_i - \bar{x}),$$

$$s_x^2 = (n-1)^{-1}\sum_{i=1}^{n} (x_i - \bar{x})^2$$

$$s_y^2 = (n-1)^{-1}\sum_{i=1}^{n} (y_i - \bar{y})^2,$$

$$\bar{y} = n^{-1}\sum_{i=1}^{n} y_i,$$

$$\bar{x} = n^{-1}\sum_{i=1}^{n} x_i.$$

The problem of estimating ρ_{yx} has been earlier taken up by various authors including Gupta and Singh (1989), Gupta *et al.* (1978, 1979), Koop (1970), Rana (1989), Singh *et al.* (1996) and Wakimoto (1971), in different situations. Srivastava and Jhajj (1986) have further considered the problem of estimating ρ_{yx} in the situations where the information on auxiliary variable x for all units in the population is available. In such situations, they have suggested a class of estimators for ρ_{yx} which utilizes the known values of the population mean $\bar{X}$ and the population variance S_x^2 of the auxiliary variable x.

In this paper, using two-phase sampling mechanism, a class of estimators for ρ_{yx} in the presence of the available knowledge ($\bar{Z}$ and S_z^2) on second auxiliary variable z is considered, when the population mean $\bar{X}$ and population variance S_x^2 of the main auxiliary variable x are not known.

2. The suggested class of estimators

In many situations of practical importance, it may happen that no information is available on the population mean $\bar{X}$ and population variance S_x^2, we seek to estimate the population correlation coefficient ρ_{yx} from a sample "s" obtained through a two-phase selection. Allowing simple random sampling without replacement scheme in each phase, the two-phase sampling scheme will be as follows:

(1) The first phase sample s^* ($s^* \subset U$) of fixed size n_1, is drawn to observe only x in order to furnish a good estimates of $\bar{X}$ and S_x^2.

(2) Given s^*, the second-phase sample s ($s \subset s^*$) of fixed size n is drawn to observe y only.

Let

$$\bar{x} = (1/n)\sum_{i\in s} x_i,$$

$$\bar{y} = (1/n)\sum_{i\in s} y_i,$$

$$\bar{x}^* = (1/n_1)\sum_{i\in s^*} x_i,$$

$$s_x^2 = (n-1)^{-1}\sum_{i\in s}(x_i - \bar{x})^2,$$

$$s_x^{*2} = (n_1-1)^{-1}\sum_{i\in s^*}(x_i - \bar{x}^*)^2.$$

We write $u = \bar{x}/\bar{x}^*$, $v = s_x^2/s_x^{*2}$. Whatever be the sample chosen let (u, v) assume values in a bounded closed convex subset, R, of the two-dimensional real space containing the point (1, 1). Let h (u, v) be a function of u and v such that:

$$h(1,1) = 1 \tag{2.1}$$

and such that it satisfies the following conditions:

(1) The function $h(u, v)$ is continuous and bounded in R.

(2) The first and second partial derivatives of $h(u, v)$ exist and are continuous and bounded in R.

Now one may consider the class of estimators of ρ_{yx}, defined by:

$$\hat{\rho}_{hd} = rh(u,v) \tag{2.2}$$

which is double sampling version of the class of estimators

$$\tilde{r}_t = rf(u^*, v^*)$$

Suggested by Srivastava and Jhajj (1986), where $u^* = \bar{x}/\bar{X}$, $v^* = s_x^2/S_x^2$ and $\left(\bar{X}, S_x^2\right)$ are known.

Sometimes even if the population mean $\bar{X}$ and population variance S_x^2 of x are not known, information on a cheaply ascertainable variable z, closely related to x but compared to x remotely related to y, is available on all units of the population. This type of situation has been briefly discussed by, among others, Chand (1975) and Kiregyera (1980, 1984).

Following Chand (1975) one may define a chain ratio-type estimator for ρ_{yx} as

$$\hat{\rho}_{1d} = r\left(\frac{\bar{x}^*}{\bar{x}}\right)\left(\frac{\bar{Z}}{\bar{z}^*}\right)\left(\frac{s_x^{*2}}{s_x^2}\right)\left(\frac{S_z^2}{s_z^{*2}}\right) \tag{2.3}$$

where the population mean $\bar{Z}$ and population variance S_z^2 of second auxiliary variable z are known, and

$$\bar{z}^* = (1/n_1)\sum_{i\in s^*} z_i,$$

$$s_z^{*2} = (n_1-1)^{-1}\sum_{i\in s^*}(z_i - \bar{z}^*)^2$$

are the sample mean and sample variance of z based on preliminary large sample s^* of size $n_1(> n)$.

The estimator $\hat{\rho}_{1d}$ in equation (2.3) may be generalized as

$$\hat{\rho}_{2d} = r\left(\frac{\bar{x}}{\bar{x}^*}\right)^{\alpha_1}\left(\frac{s_x^2}{s_x^{*2}}\right)^{\alpha_2}\left(\frac{\bar{z}^*}{\bar{Z}}\right)^{\alpha_3}\left(\frac{s_z^{*2}}{S_z^2}\right)^{\alpha_4} \tag{2.4}$$

where $\alpha_i's$ $(i = 1, 2, 3, 4)$ are suitably chosen constants.

Many other generalization of $\hat{\rho}_{1d}$ is possible. We have, therefore, considered a more general class of ρ_{yx}, from which a number of estimators can be generated.

The proposed generalized estimators for population correlation coefficient ρ_{yx}, is defined by

$$\hat{\rho}_{td} = rt(u, v, w, a) \tag{2.5}$$

where $w = \bar{z}^*/\bar{Z}$, $a = s_z^{*2}/S_z^2$ and $t(u, v, w, a)$ is a function of (u, v, w, a) such that

$$t(1, 1, 1, 1) = 1 \tag{2.6}$$

Satisfying the following conditions:

(1) Whatever be the samples (s^* and s) chosen, let (u, v, w, a) assume values in a closed convex subset S, of the four-dimensional real space containing the point $P=(1, 1, 1, 1)$.

(2) In S, the function $t(u, v, w, a)$ is continuous and bounded.

(3) The first and second order partial derivatives of $t(u, v, w, a)$ exist and are continuous and bounded in S.

To find the bias and variance of $\hat{\rho}_{td}$ we write:

$$s_y^2 = S_y^2(1+e_1),$$

$$\bar{x} = \bar{X}(1+e_1),$$

$$\bar{x}* = \bar{X}(1+e_1^*),$$

$$s_x^2 = S_x^2(1+e_2)$$

$$s_x^{*2} = S_x^2(1+e_2^*),$$

$$\bar{z}* = \bar{Z}(1+e_3^*),$$

$$s_z^{*2} = S_z^2(1+e_4^*),$$

$$s_{yx} = S_{yx}(1+e_s)$$

such that

$$E(e_0) = E(e_1) = E(e_2) = E(e_5) = 0$$

and

$$E\left(e_i^*\right) = 0 \ \forall \quad i = 1,2,3,4,$$

and ignoring the finite population correction terms, we write to the first degree of approximation:

$$E\left(e_0^2\right) = (\delta_{400}-1)/n, \quad E\left(e_1^2\right) = C_x^2/n, \quad E\left(e_1^{*2}\right) = C_x^2/n_1, \quad E\left(e_2^2\right) = (\delta_{040}-1)/n,$$

$$E\left(e_2^{*2}\right) = (\delta_{040}-1)/n_1, \quad E\left(e_3^{*2}\right) = C_z^2/n_1, \quad E\left(e_4^{*2}\right) = (\delta_{004}-1)/n_1,$$

$$E\left(e_5^2\right) = \left\{\left(\delta_{220}/\rho_{yx}^2\right)-1\right\}\Big/n, \quad E(e_0e_1) = \delta_{210}C_x/n, \quad E\left(e_0e_1^*\right) = \delta_{210}C_x/n_1,$$

$$E(e_0e_2) = (\delta_{220}-1)/n, \quad E\left(e_0e_2^*\right) = (\delta_{220}-1)/n_1, \quad E\left(e_0e_3^*\right) = \delta_{201}C_z/n_1,$$

$$E\left(e_0e_4^*\right) = (\delta_{202}-1)/n_1, \quad E(e_0e_5) = \{(\delta_{310}/\rho_{yx})-1\}/n,$$

$$E\left(e_1e_1^*\right) = C_x^2/n_1, \quad E(e_1e_2) = \delta_{030}C_x/n, \quad E\left(e_1e_2^*\right) = \delta_{030}C_x/n_1,$$

$$E\left(e_1e_3^*\right) = \rho_{xz}C_xC_z/n_1, \quad E\left(e_1e_4^*\right) = \delta_{012}C_x/n_1, \quad E(e_1e_5) = (\delta_{120}C_x/\rho_{yx})/n,$$

$$E\left(e_1^*e_2\right) = \delta_{030}C_x/n_1, \quad E\left(e_1^*e_2^*\right) = \delta_{030}C_x/n_1, \quad E\left(e_1^*e_3^*\right) = \rho_{xz}C_xC_z/n_1,$$

$$E\left(e_1^* e_4^*\right) = \delta_{012} C_x / n_1, \quad E\left(e_1^* e_5\right) = (\delta_{120} C_x / \rho_{yx}) / n_1,$$

$$E\left(e_2 e_2^*\right) = (\delta_{040} - 1)/n_1, \quad E\left(e_2 e_3^*\right) = \delta_{021} C_z / n_1, \quad E\left(e_2 e_4^*\right) = (\delta_{022} - 1)/n_1,$$

$$E(e_2 e_5) = \{(\delta_{130}/\rho_{yx}) - 1\}/n, \quad E\left(e_2^* e_3^*\right) = \delta_{021} C_z / n_1,$$

$$E\left(e_2^* e_4^*\right) = (\delta_{022} - 1)/n_1, \quad E\left(e_2^* e_5\right) = \{(\delta_{130}/\rho_{yx}) - 1\}/n_1,$$

$$E\left(e_3^* e_4^*\right) = \delta_{003} C_z / n_1, \quad E\left(e_3^* e_5\right) = (\delta_{111} C_z / \rho_{yx}) / n_1,$$

$$E\left(e_4^* e_5\right) = \{(\delta_{112}/\rho_{yx}) - 1\}/n_1.$$

where

$$\delta_{pqm} = \mu_{pqm} / \left(\mu_{200}^{p/2}\, \mu_{020}^{q/2}\, \mu_{002}^{m/2}\right), \quad \mu_{pqm} = (1/N) \sum_{i=1}^{N} (y_i - \bar{Y})^p (x_i - \bar{X})^q (z_i - \bar{Z})^m,$$

(p, q, m) being non-negative integers.

To find the expectation and variance of $\hat{\rho}_{td}$, we expand $t(u, v, w, a)$ about the point $P = (1, 1, 1, 1)$ in a second-order Taylor's series, express this value and the value of r in terms of e's. Expanding in powers of e's and retaining terms up to second power, we have:

$$E(\hat{\rho}_{td}) = \rho_{yx} + o(n^{-1}) \tag{2.7}$$

which shows that the bias of $\hat{\rho}_{td}$ is of the order n^{-1} and so up to order n^{-1}, mean square error and the variance of $\hat{\rho}_{td}$ are same.

Expanding $(\hat{\rho}_{td} - \rho_{yx})^2$, retaining terms up to second power in e's, taking expectation and using the above expected values, we obtain the variance of $\hat{\rho}_{td}$ to the first degree of approximation, as:

$$\begin{aligned} \mathrm{Var}(\hat{\rho}_{td}) = \mathrm{Var}(r) &+ \left(\rho_{yx}^2/n\right)\Big[C_x^2 t_1^2(P) + (\delta_{040} - 1) t_2^2(P) - A t_1(P) \\ &- B t_2(P) + 2\delta_{030} C_x t_1(P) t_2(P)\Big] - \left(\rho_{yx}^2/n_1\right)\Big[C_x^2 t_1^2(P) + (\delta_{040} - 1) t_2^2(P) \\ &- C_z^2 t_3^2(P) - (\delta_{004} - 1) t_4^2(P) - A t_1(P) - B t_2(P) + D t_3(P) + F t_4(P) \\ &+ 2\delta_{030} C_x t_1(P) t_2(P) - 2\delta_{003} C_z t_3(P) t_4(P)\Big] \end{aligned} \tag{2.8}$$

where t_1(P), t_2(P), t_3(P) and t_4(P), respectively, denote the first partial derivatives of $t(u, v, w, a)$ respect to u, v, w and a, respectively, at the point $P = (1, 1, 1, 1)$, with

$$\mathrm{Var}(r) = \left(\rho_{yx}^2/n\right)\left[\left(\delta_{220}/\rho_{yx}^2\right) + (1/4)(\delta_{040} + \delta_{400} + 2\delta_{220}) - \{(\delta_{130} + \delta_{310})/\rho_{yx}\}\right] \tag{2.9}$$

$$A = \{\delta_{210} + \delta_{030} - 2(\delta_{120}/\rho_{yx})\}C_x,$$

$$B = \{\delta_{220} + \delta_{040} - 2(\delta_{130}/\rho_{yx})\},$$

$$D = \{\delta_{201} + \delta_{021} - 2(\delta_{111}/\rho_{yx})\}C_z,$$

$$F = \{\delta_{202} + \delta_{022} - 2(\delta_{112}/\rho_{yx})\}$$

Any parametric function $t(u, v, w, a)$ satisfying equation (2.6) and the conditions 1 and 2 can generate an estimator of the class equation (2.5).

The variance of $\hat{\rho}_{td}$ at equation (2.6) is minimized for:

$$\begin{aligned} t_1(P) &= \frac{[A(\delta_{040} - 1) - B\delta_{030}C_x]}{2C_x^2(\delta_{040} - \delta_{030}^2 - 1)} = \alpha(\text{say}), \\ t_2(P) &= \frac{\left(BC_x^2 - A\delta_{030}C_x\right)}{2C_x^2(\delta_{040} - \delta_{030}^2 - 1)} = \beta(\text{say}), \\ t_3(P) &= \frac{[D(\delta_{004} - 1) - F\delta_{003}C_z]}{2C_z^2(\delta_{004} - \delta_{030}^2 - 1)} = \gamma(\text{say}), \\ t_4(P) &= \frac{\left(C_z^2F - D\delta_{003}C_z\right)}{2C_z^2(\delta_{004} - \delta_{003}^2 - 1)} = \delta(\text{say}), \end{aligned} \tag{2.10}$$

Thus the resulting (minimum) variance of $\hat{\rho}_{td}$ is given by:

$$\begin{aligned} \min.\mathrm{Var}(\hat{\rho}_{td}) = \mathrm{Var}(r) &- \left(\frac{1}{n} - \frac{1}{n_1}\right)\rho_{yx}^2\left[\frac{A^2}{4C_x^2} + \frac{\{(A/C_x)\delta_{030} - B\}^2}{4(\delta_{040} - \delta_{030}^2 - 1)}\right] \\ &- \left(\rho_{yx}^2/n_1\right)\left[\frac{D^2}{4C_z^2} + \frac{\{(D/C_z)\delta_{003} - F\}^2}{4(\delta_{004} - \delta_{003}^2 - 1)}\right] \end{aligned} \tag{2.11}$$

It is observed from equation (2.11) that if optimum values of the parameters given by (2.10) are used, the variance of the estimator $\hat{\rho}_{td}$ is always less than that of r as the last two terms on the right hand sides of (2.11) are non-negative.

Two simple functions $t(u, v, w, a)$ satisfying the required conditions are:

$$t(u, v, w, a) = 1 + \alpha_1(u - 1) + \alpha_2(v - 1) + \alpha_3(w - 1) + \alpha_4(a - 1)$$

$$t(u, v, w, a) = u^{\alpha_1}v^{\alpha_2}w^{\alpha_3}a^{\alpha_4}$$

and for both these functions $t_1(P) = \alpha_1$, $t_2(P) = \alpha_2$, $t_3(P) = \alpha_3$ and $t_4(P) = \alpha_4$. Thus, one should use optimum values of α_1, α_2, α_3 and α_4 in $\hat{\rho}_{td}$ to get the minimum variance.

It is to be noted that the estimated $\hat{\rho}_{td}$ attained the minimum variance only when the optimum values of the constants α_i ($i=1,2,3,4$), which are functions of unknown population parameters, are known. To use such estimators in practice, one has to use some guessed values of population parameters obtained either through past experience or through a pilot sample survey. It may be further noted that even if the values of the constants used in the estimator are not exactly equal to their optimum values as given by equation (2.8) but are close enough, the resulting estimator will be better than the conventional estimator, as illustrated by Das and Tripathi (1978, Section 3).

If no information on second auxiliary variable z is used, then the estimator $\hat{\rho}_{td}$ reduces to $\hat{\rho}_{hd}$ defined in equation (2.2). Taking $z \equiv 1$ in equation (2.8), we get the variance of $\hat{\rho}_{hd}$ to the first degree of approximation, as:

$$\begin{aligned}\mathrm{Var}(\hat{\rho}_{hd}) = \mathrm{Var}(r) + \left(\frac{1}{n} - \frac{1}{n_1}\right)\rho_{yx}^2\Big[C_x^2 h_1^2(1,1) + (\delta_{040} - 1)h_2^2(1,1)\\ - Ah_1(1,1) - Bh_2(1,1) + 2\delta_{030}C_x h_1(1,1)h_2(1,1)\Big]\end{aligned} \tag{2.12}$$

which is minimized for,

$$\begin{aligned}h_1(1,1) &= \frac{[A(\delta_{040} - 1) - B\delta_{030}C_x]}{2C_x^2(\delta_{040} - \delta_{030}^2 - 1)},\\ h_2(1,1) &= \frac{\left(BC_x^2 - A\delta_{030}C_x\right)}{2C_x^2(\delta_{040} - \delta_{030}^2 - 1)}\end{aligned} \tag{2.13}$$

Thus the minimum variance of $\hat{\rho}_{hd}$ is given by:

$$\mathrm{min.Var}(\hat{\rho}_{hd}) = \mathrm{Var}(r) - \left(\frac{1}{n} - \frac{1}{n_1}\right)\rho_{yx}^2\left[\frac{A^2}{4C_x^2} + \frac{\{(A/C_x)\delta_{030} - B\}^2}{4\left(\delta_{040} - \delta_{030}^2 - 1\right)}\right] \tag{2.14}$$

It follows from equations (2.11) and (2.14) that

$$\mathrm{min.Var}(\hat{\rho}_{td}) - \mathrm{min.Var}(\hat{\rho}_{hd}) = \left(\rho_{yx}^2/n_1\right)\left[\frac{D^2}{4C_z^2} + \frac{\{(D/C_z)\delta_{003} - F\}^2}{4\left(\delta_{004} - \delta_{003}^2 - 1\right)}\right] \tag{2.15}$$

which is always positive. Thus, the proposed estimator $\hat{\rho}_{td}$ is always better than $\hat{\rho}_{hd}$.

3. A wider class of estimators

In this section, we consider a class of estimators of ρ_{yx} wider than equation (2.5) given by:

$$\hat{\rho}_{gd} = g(r, u, v, w, a) \tag{3.1}$$

where $g(r, u, v, w, a)$ is a function of r, u, v, w, a and such that

$$g(\rho, 1, 1, 1, 1) = \hat{\rho}_{td} \quad \text{and} \quad \left[\frac{\partial g(\cdot)}{\partial r}\right]_{(\rho,1,1,1)} = 1$$

Proceeding as in Section 2, it can easily be shown, to the first order of approximation, that the minimum variance of $\hat{\rho}_{gd}$ is same as that of $\hat{\rho}_{td}$ given in equation (2.11).

It is to be noted that the difference-type estimator

$$r_{\mathrm{d}} = r + \alpha_1(u-1) + \alpha_2(v-1) + \alpha_3(w-1) + \alpha_4(a-1),$$

is a particular case of $\hat{\rho}_{gd}$, but it is not the member of $\hat{\rho}_{td}$ in equation (2.5).

4. Optimum values and their estimates

The optimum values $t_1(P) = \alpha$, $t_2(P) = \beta$, $t_3(P) = \gamma$ and $t_4(P) = \delta$ given at equation (2.10) involve unknown population parameters. When these optimum values are substituted in equation (2.5), it no longer remains an estimator since it involves unknown $(\alpha, \beta, \gamma, \delta)$, which are functions of unknown population parameters, say, $\delta_{pqm}(p, q, m = 0, 1, 2, 3, 4)$, C_x, C_z and ρ_{yx} itself. Hence, it is advisable to replace them by their consistent estimates from sample values. Let $(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta})$ be consistent estimators of $t_1(P)$, $t_2(P)$, $t_3(P)$ and $t_4(P)$, respectively, where

$$\begin{aligned}
\hat{t}_1(P) = \hat{\alpha} &= \frac{[\hat{A}(\hat{\delta}_{040} - 1) - \hat{B}\hat{\delta}_{030}\hat{C}_x]}{2\hat{C}_x^2\left(\hat{\delta}_{040} - \hat{\delta}_{030}^2 - 1\right)}, \\
\hat{t}_2(P) = \hat{\beta} &= \frac{\left[\hat{B}\hat{C}_x^2 - \hat{A}\hat{\delta}_{030}\hat{C}_x\right]}{2\hat{C}_x^2\left(\hat{\delta}_{040} - \hat{\delta}_{030}^2 - 1\right)}, \\
\hat{t}_3(P) = \hat{\gamma} &= \frac{[\hat{D}(\hat{\delta}_{004} - 1) - \hat{F}\hat{\delta}_{003}\hat{C}_z]}{2\hat{C}_z^2\left(\hat{\delta}_{004} - \hat{\delta}_{003}^2 - 1\right)}, \\
\hat{t}_4(P) = \hat{\delta} &= \frac{\left[\hat{C}_z^2\hat{F} - \hat{D}\hat{\delta}_{003}\hat{C}_z\right]}{2\hat{C}_z^2\left(\hat{\delta}_{004} - \hat{\delta}_{003}^2 - 1\right)},
\end{aligned} \tag{4.1}$$

with

$$\hat{A} = [\hat{\delta}_{210} + \hat{\delta}_{030} - 2(\hat{\delta}_{120}/r)]\hat{C}_x,$$

$$\hat{B} = [\hat{\delta}_{220} + \hat{\delta}_{040} - 2(\hat{\delta}_{130}/r)],$$

$$\hat{D} = [\hat{\delta}_{201} + \hat{\delta}_{021} - 2(\hat{\delta}_{111}/r)]\hat{C}_z,$$

$$\hat{F} = [\hat{\delta}_{202} + \hat{\delta}_{022} - 2(\hat{\delta}_{112}/r)],$$

$$\hat{C}_x = s_x/\bar{x},$$

$$\hat{C}_z = s_z/\bar{z},$$

$$\hat{\delta}_{pqm} = \hat{\mu}_{pqm}/\left(\hat{\mu}_{200}^{p/2}\,\hat{\mu}_{020}^{q/2}\,\hat{\mu}_{002}^{m/2}\right)$$

$$\hat{\mu}_{pqm} = (1/n)\sum_{i=1}^{n}(y_i-\bar{y})^p(x_i-\bar{x})^q(z_i-\bar{z})^m$$

$$\bar{z} = (1/n)\sum_{i=1}^{n} z_i,$$

$$s_x^2 = (n-1)^{-1}\sum_{i=1}^{n}(x_i-\bar{x})^2,$$

$$\bar{x} = (1/n)\sum_{i=1}^{n} x_i,$$

$$r = s_{yx}/(s_y s_x),$$

$$s_y^2 = (n-1)^{-1}\sum_{i=1}^{n}(y_i-\bar{y})^2,$$

$$s_z^2 = (n-1)^{-1}\sum_{i=1}^{n}(x_i-\bar{z})^2.$$

We then replace $(\alpha, \beta, \gamma, \delta)$ by $(\hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta})$ in the optimum $\hat{\rho}_{td}$ resulting in the estimator $\hat{\rho}_{td}^*$ say, which is defined by:

$$\hat{\rho}_{td}^* = rt^*(u, v, w, a, \hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta}), \tag{4.2}$$

where the function $t^*(U)$, $U = (u, v, w, a, \hat{\alpha}, \hat{\beta}, \hat{\gamma}, \hat{\delta})$ is derived from the function $t(u, v, w, a)$ given at equation (2.5) by replacing the unknown constants involved in it by the consistent estimates of optimum values. The condition (2.6) will then imply that

$$t^*(P^*) = 1 \tag{4.3}$$

where $P^* = (1, 1, 1, 1, \alpha, \beta, \gamma, \delta)$.

We further assume that

$$\begin{aligned}
t_1(P^*) &= \left.\frac{\partial t^*(U)}{\partial u}\right]_{U=P^*} = \alpha, \quad t_2(P^*) = \left.\frac{\partial t^*(U)}{\partial v}\right]_{U=P^*} = \beta \\
t_3(P^*) &= \left.\frac{\partial t^*(U)}{\partial w}\right]_{U=P^*} = \gamma, \quad t_4(P^*) = \left.\frac{\partial t^*(U)}{\partial a}\right]_{U=P^*} = \delta \\
t_5(P^*) &= \left.\frac{\partial t^*(U)}{\partial \hat{\alpha}}\right]_{U=P^*} = 0, \quad t_6(P^*) = \left.\frac{\partial t^*(U)}{\partial \hat{\beta}}\right]_{U=P^*} = 0 \\
t_7(P^*) &= \left.\frac{\partial t^*(U)}{\partial \hat{\gamma}}\right]_{U=P^*} = 0, \quad t_8(P^*) = \left.\frac{\partial t^*(U)}{\partial \hat{\delta}}\right]_{U=P^*} = 0
\end{aligned} \tag{4.4}$$

Expanding $t^*(U)$ about $P^* = (1, 1, 1, 1, \alpha, \beta, \gamma, \delta)$, in Taylor's series, we have:

$$\begin{aligned}
\hat{\rho}^*_{td} &= r\big[t^*(P^*) + (u-1)t^*_1(P^*) + (v-1)t^*_2(P^*) + (w-1)t^*_3(P^*) \\
&+ (a-1)t^*_4(P^*) + (\hat{\alpha}-\alpha)t^*_5(P^*) + (\hat{\beta}-\beta)t^*_6(P^*) + (\hat{\gamma}-\gamma)t^*_7(P^*) \\
&+ (\hat{\delta}-\delta)t^*_8(P^*) + \text{second order terms}\big]
\end{aligned} \tag{4.5}$$

Using equation (4.4) in (4.5) we have:

$$\hat{\rho}^*_{td} = r[1 + (u-1)\alpha + (v-1)\beta + (w-1)\gamma + (a-1)\delta + \text{second order terms}] \tag{4.6}$$

Expressing equation (4.6) in term of e's squaring and retaining terms of e's up to second degree, we have:

$$\left(\hat{\rho}^*_{td} - \rho_{yx}\right)^2 = \rho^2_{yx}\left[\frac{1}{2}(2e_5 - e_0 - e_2) + \alpha\left(e_1 - e^*_1\right) + \beta\left(e_2 - e^*_2\right) + \gamma e^*_3 + \delta e^*_4\right]^2 \tag{4.7}$$

Taking expectation of both sides in equation (4.7), we get the variance of $\hat{\rho}^*_{td}$ to the first degree of approximation, as:

$$\begin{aligned}
\mathrm{Var}\left(\hat{\rho}^*_{td}\right) &= \mathrm{Var}(r) - \left(\frac{1}{n} - \frac{1}{n_1}\right)\rho^2_{yx}\left[\frac{A^2}{4C^2_x} + \frac{\{(A/C_x)\delta_{030} - B\}^2}{4\left(\delta_{040} - \delta^2_{030} - 1\right)}\right] \\
&+ \left(\rho^2_{yx}/n_1\right)\left[\frac{D^2}{4C^2_z} + \frac{\{(D/C_z)\delta_{003} - F\}^2}{4\left(\delta_{004} - \delta^2_{003} - 1\right)}\right]
\end{aligned} \tag{4.8}$$

which is same as equation (2.11), we thus have established the following result.

Result 4.1. If optimum values of constants in equation (2.10) are replaced by their consistent estimators and conditions (4.3) and (4.4) hold good, the resulting estimator $\hat{\rho}_{td}^{*}$ has the same variance to the first degree of approximation, as that of optimum $\hat{\rho}_{td}$.

Remark 4.1. It may be easily examined that some special cases:

(1) $\hat{\rho}_{td1}^{*} = ru^{\hat{\alpha}}v^{\hat{\beta}}w^{\hat{\gamma}}a^{\hat{\delta}}$,

(2) $\hat{\rho}_{td2}^{*} = r\{1+\hat{\alpha}(u-1)+\hat{\gamma}(w-1)\}/\{1-\hat{\beta}(v-1)-\hat{\delta}(a-1)\}$

(3) $\hat{\rho}_{td3}^{*} = r[1+\hat{\alpha}(u-1)+\hat{\beta}(u-1)+\hat{\gamma}(w-1)+\hat{\delta}(a-1)]$

(4) $\hat{\rho}_{td4}^{*} = r[1-\hat{\alpha}(u-1)-\hat{\beta}(u-1)-\hat{\gamma}(w-1)-\hat{\delta}(a-1)]^{-1}$

of $\hat{\rho}_{td}^{*}$ satisfy the conditions (4.3) and (4.4) and attain the variance (4.8).

Remark 4.2. The efficiencies of the estimators discussed in this paper can be compared for fixed cost, following the procedure given in Sukhatme *et al.* (1984).

5. Empirical study

To illustrate the performance of various estimators of population correlation coefficient, we consider the data given in Murthy (1967, p. 226). The variates are: $y=$ output, $x=$ number of workers, $z=$ fixed capital, $N=80$, $n=10$, $n_1=25$,

$$\bar{X}=283.875,\quad \bar{Y}=5182.638,\quad \bar{Z}=1126,\quad C_x=0.9430,\quad C_y=0.3520,$$

$$C_z=0.7460,\quad \delta_{003}=1.030,\quad \delta_{004}=2.8664,\quad \delta_{021}=1.1859,\quad \delta_{022}=3.1522,$$

$$\delta_{030}=1.295,\quad \delta_{040}=3.65,\quad \delta_{102}=0.7491,\quad \delta_{120}=0.9145,\quad \delta_{111}=0.8234,$$

$$\delta_{130}=2.8525,\quad \delta_{112}=2.5454,\quad \delta_{210}=0.5475,\quad \delta_{220}=2.3377,\quad \delta_{201}=0.4546,$$

$$\delta_{202}=2.2208,\quad \delta_{300}=0.1301,\quad \delta_{400}=2.2667,\quad \rho_{yx}=0.9136,\quad \rho_{xz}=0.9859,$$

$$\rho_{yz}=0.9413.$$

The percent relative efficiencies (PREs) of $\hat{\rho}_{1d}$, $\hat{\rho}_{hd}$, $\hat{\rho}_{td}$ with respect to conventional estimator r have been computed and compiled in Table I.

Table I clearly shows that the proposed estimator $\hat{\rho}_{td}$ (or $\hat{\rho}_{td}^{*}$) is more efficient than r and $\hat{\rho}_{hd}$.

Estimator	r	$\hat{\rho}_{hd}$	$\hat{\rho}_{td}$ (or $\hat{\rho}_{td}^{*}$)
PRE(., r)	100	129.147	305.441

Table I. The PRE's of different estimators of ρ_{yx}

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Almost Unbiased Estimator for Estimating Population Mean Using Known Value of Some Population Parameter(s)

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Abstract

In this paper we have proposed an almost unbiased estimator using known value of some population parameter(s). Various existing estimators are shown particular members of the proposed estimator. Under simple random sampling without replacement (SRSWOR) scheme the expressions for bias and mean square error (MSE) are derived. The study is extended to the two phase sampling. Empirical study is carried out to demonstrate the superiority of the proposed estimator.

Key words: Auxiliary information, bias, mean square error, unbiased estimator, two phase sampling.

1. Introduction

Consider a finite population U = U_1, U_2, …U_N of N units. Let y and x stand for the variable under study and auxiliary variable respectively. Let (y_i, x_i), i=1, 2,.., n denote the values of the units included in a sample s_n of size n drawn by simple random sampling without replacement (SRSWOR). The auxiliary information has been used in improving the precision of the estimate of a parameter (See Cochran (1977), Sukhatme et. al. (1984) and the references cited there in). Out of many methods, ratio and product methods of estimation are good illustrations in this context.

In order to have a survey estimate of the population mean $\bar{Y}$ of the study character y, assuming the knowledge of the population mean $\bar{X}$ of the auxiliary character x, the well-known ratio estimator is

$$\mathbf{t_r} = \bar{y}\,\frac{\bar{X}}{\bar{x}} \tag{1.1}$$

Bahl and Tuteja (1991) suggested an exponential ratio type estimator as –

$$t_{re} = \bar{y}\exp\left[\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right] \tag{1.2}$$

Several authors have used prior value of certain population parameter(s) to find more precise estimates. Sisodiya and Dwivedi (1981), Sen (1978) and Upadhyaya and Singh (1984) used the known coefficient of variation (CV) of the auxiliary character for estimating population mean of a study character in ratio method of estimation. The use of prior value of coefficient of kurtosis in estimating the population variance of study character y was first made by Singh et. al. (1973). Later used by Singh and Kakaran (1993) in the estimation of population mean of study character. Singh and Tailor (2003) proposed a modified ratio estimator by using the known value of correlation coefficient. Kadilar and Cingi (2006), Khosnevisan et. al. (2007), Singh et. al. (2007) Singh and Kumar (2009) and Singh et. al. (2009) have suggested modified ratio estimators by using different pairs of known value of population parameter(s).

In this paper under SRSWOR, we have proposed almost unbiased estimator for estimating $\bar{Y}$.

2. Almost unbiased ratio type estimator

Suppose

$$t_0 = \bar{y}, \qquad t_{rs} = \bar{y}\left(\frac{a\bar{X}+b}{\bar{x}+b}\right), \qquad t_{rse} = \bar{y}\exp\left\{\frac{(a\bar{X}+b)-(a\bar{x}+b)}{(a\bar{X}+b)+(a\bar{x}+b)}\right\}$$

Such that $t_0, t_{rs}, t_{rse} \in w_r$ where w_r denotes the set of all possible ratio type estimators for estimating the population mean $\bar{Y}$. By definition the set w_r is a linear variety, if

$$t_{wr} = \omega_0\bar{y} + \omega_1 t_{rs} + \omega_2 t_{rse} \in w \tag{2.1}$$

for $\sum_{i=0}^{2}\omega_i = 1, \quad \omega_i \in R$ (2.2)

where ω_i (i=0, 1, 2) denotes the statistical constants and R denotes the set of real numbers.

To obtain the bias and MSE of t_w, we write

$\bar{y} = \bar{Y}(1+e_0), \quad \bar{x} = \bar{X}(1+e_1),$

such that

$E(e_0) = E(e_1) = 0.$

$E(e_0^2) = f_1 C_y^2, \qquad E(e_1^2) = f_1 C_x^2, \qquad E(e_0 e_1) = f_1 \rho C_y C_x.$

where $f_1 = (\frac{1}{n} - \frac{1}{N}), \qquad S_y^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}(y_i - \bar{Y})^2, \qquad S_x^2 = \frac{1}{(N-1)}\sum_{i=1}^{N}(x_i - \bar{X})^2,$

$C_y = \frac{S_y}{\bar{Y}}, \; C_x = \frac{S_x}{\bar{X}}, \qquad K = \rho\left(\frac{C_y}{C_x}\right), \; \rho = \frac{S_{yx}}{(S_y S_x)},$

$S_{yx} = \frac{1}{(N-1)}\sum_{i=1}^{N}(y_i - \bar{Y})(x_i - \bar{X}).$

Expressing t_w we have

$$\mathbf{t_w} = \bar{Y}(1+e_0)\left[\omega_0 + \omega_1(1+\theta e_1)^{-1} + \omega_2 \exp\left\{-\frac{\theta e_1}{2}(1+\theta e_1)^{-1}\right\}\right] \tag{2.3}$$

where $\theta = \frac{a\bar{X}}{a\bar{X}+b}$.

Expanding the right hand side of (2.3) and retaining terms up to second order of e's, we have

$$\mathbf{t_w} \cong \bar{Y}\left[1 + e_0 - \omega\theta e_1 + \theta^2\left(\omega_1 + \frac{3}{8}\right)e_1^2 - \theta\omega e_0 e_1\right] \tag{2.4}$$

where $\omega = (\omega_1 + \frac{\omega_2}{2})$. (2.5)

Taking expectations of both side of (2.4) and then subtracting $\bar{Y}$ from both side, we get the bias of the estimator t_w, up to the first order of approximation as

$$B(t_w) = f_1\bar{Y}\left[\theta^2 C_x^2\left(\omega_1 + \frac{3\omega_2}{8}\right) - \theta\omega\rho C_y C_x\right] \tag{2.6}$$

$$B(t_{rs}) = f_1\bar{Y}\left[\theta^2 C_x^2 - \theta\rho C_y C_x\right] \tag{2.7}$$

$$B(t_{rse}) = f_1\bar{Y}\left[\frac{3\theta^2 C_x^2}{8} - \frac{\theta\rho C_y C_x}{2}\right] \tag{2.8}$$

From (2.4), we have

$$(t_w - \bar{Y}) \cong \bar{Y}\left[e_0 - \theta\omega e_1\right] \tag{2.9}$$

Squaring both sides of (2.9) and then taking expectations, we get MSE of the estimator t_w, up to the first order of approximation, as

$$MSE(t_w) = f_1\bar{Y}\left[C_y^2 + \theta^2\omega^2 C_x^2 - 2\theta\omega\rho C_y C_x\right] \tag{2.10}$$

This is minimum when

$$\omega = k\left(= \rho\frac{C_y}{C_x}\right). \tag{2.11}$$

Putting this value of $\omega(=k)$ in (2.10), we get the minimum MSE of $\mathbf{t_w}$ as

min. $MSE(t_w) = f_1\bar{Y}^2 C_y^2(1-\rho^2)$ (2.12)

which is same as that of traditional linear regression estimator
from (2.5) and (2.11), we have

$$\omega_1 + \frac{\omega_2}{2} = k. \tag{2.13}$$

From (2.2) and (2.13), we have only two equations in three unknowns. It is not possible to find the unique values for $\omega_{i's}, i = 0,1,2.$ In order to get unique values for $\omega_{i's}$, we shall impose the linear restriction

$$\omega_0 B(\bar{y}) + \omega_1 B(t_{rs}) + \omega_2 B(t_{rse}) = 0 \tag{2.14}$$

Equations (2.2), (2.11) and (2.14) can be written as in the matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1/2 \\ 0 & B(\mathbf{t_{rs}}) & B(t_{rse}) \end{bmatrix}\begin{bmatrix} \omega_0 \\ \omega_1 \\ \omega_2 \end{bmatrix} = \begin{bmatrix} 1 \\ k \\ 0 \end{bmatrix} \tag{2.15}$$

Using (2.15) ,we get unique values of $\omega_{i's}$ (i=0,1,2) as

$$\left.\begin{aligned}\omega_0 &= \frac{\Delta_0}{\Delta_r}\\ \omega_1 &= \frac{\Delta_1}{\Delta_r}\\ \omega_2 &= \frac{\Delta_2}{\Delta_r}\end{aligned}\right\} \quad (2.16)$$

where

$$\left.\begin{aligned}\Delta_r &= B(t_{rse}) - \frac{1}{2}B(t_{rs})\\ \Delta_{r0} &= B(t_{rse})\{1-k\} + \frac{1}{2}B(t_{rs})\left\{k-\frac{1}{2}\right\}\\ \Delta_r &= k.B(t_{rse})\\ \Delta_{r2} &= -k\,B(t_{rs})\end{aligned}\right\} \quad (2.17)$$

Use of these $\omega_{i's}$ (i=0,1,2) remove the bias up to terms of order (n^{-1}) at (2.1).

3. Product –type estimators

Suppose $t_0 = \bar{y}$, $t_{ps} = \bar{y}\left(\frac{a\bar{x}+b}{a\bar{X}+b}\right)$, $t_{pse} = \bar{y}\exp\left\{\frac{(a\bar{x}+b)-(a\bar{X}+b)}{(a\bar{x}+b)+(a\bar{X}+b)}\right\}$

such that t_0, t_{ps}, $t_{pse} \in Q$, where Q denotes the set of all possible product –type estimators for estimating the population mean $\bar{Y}$. By definition, the set Q is linear variety if

$$t_q = q_0\bar{y} + q_1 t_{ps} + q_2 t_{pse} \quad \in Q \qquad (3.1)$$

for $\sum_{i=0}^{2} q_i = 1, \quad q_i \in R$ (3.2)

where q_i(i=0,1,2) denotes the statistical constants.

Expressing t_q in terms of e's, we have

$$t_q = \bar{y}(1+e_0)\left[\omega_0 + \omega_1(1+\theta e_1) + \omega_2 \exp\left\{\frac{\theta e_1}{2}(1+\theta e_1)^{-1}\right\}\right] \qquad (3.3)$$

where $\theta = \frac{a\bar{X}}{a\bar{X}+b}$.

Expanding the right hand side of (3.3) and retaining terms up to second power of e's, we have

$$t_q \cong \bar{Y}\left[1 + e_0 + \theta q e_1 - \frac{q_2}{8}e_1^2 + q\theta e_0 e_1\right] \qquad (3.4)$$

where $q = q_1 + \frac{q_2}{2}$ (3.5)

Taking expectations of both sides of (3.4) and then subtracting $\bar{Y}$ from both sides, we get the bias of the estimator t_q, up to the first order of approximation as

$$B(t_q) = f_1\bar{Y}\left[-\frac{q_2}{8}\theta^2 C_x^2 + q\theta\rho C_y C_x\right] \qquad (3.6)$$

Bias expression for the estimators t_{ps} and t_{pse} is given by

$$B(t_{ps}) = f_1\bar{Y}\left[\theta\rho C_y C_x\right] \tag{3.7}$$

$$B(t_{pse}) = f_1\bar{Y}\left[-\frac{1}{8}\theta^2 C_x^2 + \frac{\theta\rho C_y C_x}{2}\right] \tag{3.8}$$

From (3.4), we have

$$(t_q - \bar{Y}) \cong \bar{Y}[e_0 + \theta q e_1] \tag{3.9}$$

Squaring both the sides of (3.9) and then taking expectations, we get MSE of the estimator t_q, up to the first order of approximation, as

$$MSE(t_q) = f_1\bar{Y}^2\left[C_y^2 + \theta^2 q^2 C_x^2 + 2\theta q\rho C_y C_x\right] \tag{3.10}$$

which is minimum for

$$q = -k = -\rho\frac{C_y}{C_x} \tag{3.11}$$

Putting this value of q(=-k) in (3.10),we get the minimum MSE of t_q as

min. $MSE(t_q) = f_1\bar{Y}^2 C_y^2(1-\rho^2)$ (3.12)

which is same as that of traditional linear regression estimator.

From (3.5) and (3.11), we have

$$q_1 + \frac{q_2}{2} = -k \tag{3.13}$$

From (3.2) and (3.13), we have only two equations in three unknowns. It is not possible to find the unique values for q_i's, i=0,1,2. In order to get unique values of q_i's, we shall impose the linear restriction

$$q_0 B(\bar{y}) + q_1 B(t_{ps}) + q_2 B(t_{pse}) = 0 \tag{3.14}$$

Equations (3.2),(3.13) and (3.14) can be written in the matrix form as

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1/2 \\ 0 & B(t_{ps}) & B(t_{pse}) \end{bmatrix}\begin{bmatrix} q_0 \\ q_1 \\ q_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -k \\ 0 \end{bmatrix} \tag{3.15}$$

Solving (3.15), we get the unique values of q_i's (i=0,1,2) as-

$$\left.\begin{aligned} q_0 &= \frac{\Delta_{p0}}{\Delta_p} \\ q_1 &= \frac{\Delta_{p1}}{\Delta_p} \\ q_2 &= \frac{\Delta_{p2}}{\Delta_p} \end{aligned}\right\} \tag{3.16}$$

where

$$\left.\begin{array}{c}\Delta_p = B(t_{pse}) - \frac{1}{2}B(t_{ps}) \\ \Delta_{p0} = B(t_{pse})\{1+k\} + B(t_{ps})\left\{-k-\frac{1}{2}\right\} \\ \Delta_{p1} = -k.B(t_{pse}) \\ \Delta_{p2} = k\,B(t_{ps})\end{array}\right\} \quad (3.17)$$

Use of these q_i's (i=0,1,2) remove the bias up to terms of order $o(n^{-1})$ at (3.1).

In Appendix A we have listed some of the important known estimators of the population mean, which can be obtained by suitable choice of constants $\omega_i, i = 0,1,2$, $q_i, i = 0,1,2$ and a and b.

4. Proposed estimators in two phase sampling

When $\bar{X}$ is unknown, it is sometimes estimated from a preliminary large sample of size n' on which only the characteristic x is measured (for details see Singh et. al. (2007)). Then a second phase sample of size n (n < n') is drawn on which both y and x characteristics are measured. Let $\bar{x} = \frac{1}{n'}\sum_{i=1}^{n'} x_i$ denote the sample mean of x based on first phase sample of size n', $\bar{y} = \frac{1}{n}\sum_{i=1}^{n} y_i$ and $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$, be the sample means of y and x respectively based on second phase of size n.

In two phase sampling the estimator t_{wr} will take the following form

$$t_{wd} = \omega_{0d}\bar{y} + \omega_{1d}t_{rsd} + \omega_{2d}t_{rsed} \in w_d \quad (4.1)$$

for $\sum_{i=0}^{2}\omega_{id} = 1$, $\omega_{id} \in R$ (4.2)

where $t_{rs} = \bar{y}\left(\frac{a\bar{X}'+b}{a\bar{x}+b}\right)$ and $t_{rse} = \bar{y}\exp\left\{\frac{(a\bar{X}'+b)-(a\bar{x}+b)}{(a\bar{X}'+b)+(a\bar{x}+b)}\right\}$

To obtain the bias and MSE of $t_{wd,}$ we write

$\bar{y} = \bar{Y}(1+e_0)$, $\bar{x} = \bar{X}(1+e_1)$, $\bar{x}' = \bar{X}(1+e_1')$

such that

$E(e_0) = E(e_1) = E(e_1') = 0$.

$E(e_0^2) = f_1 C_y^2$, $E(e_1^2) = f_1 C_x^2$, $E(e_1'^2) = f_2 C_x^2$ $E(e_0 e_1) = f_1 \rho C_y C_x$

$E(e_0 e_1') = f_2 \rho C_y C_x$ $E(e_1 e_1') = f_2 C_x^2$

where $f_1 = (\frac{1}{n} - \frac{1}{N})$, $f_2 = (\frac{1}{n'} - \frac{1}{N})$

Following the procedure mentioned in section 2 and 3, we get bias and MSE of t_{wd} as

$$B(t_{\omega d}) = \bar{y}\left[\theta^2 C_x^2 f_3\left(\omega_{1d} + \frac{3\omega_{2d}}{8}\right) - \theta\rho C_y C_x f_3\left(\omega_{1d} + \frac{\omega_{2d}}{2}\right)\right] \quad (4.3)$$

$$MSE(t_{\omega d}) = \bar{Y}^2\left[f_1 C_y^2 + f_3 C_x^2 \omega_d^2 - 2f_3 \rho C_y C_x \omega_d\right] \tag{4.4}$$

where $f_3 = \frac{1}{n} - \frac{1}{n'} = f_1 - f_2$.

MSE ($t_{\omega d}$) is minimum, when

$$\omega_{1d} + \frac{\omega_2 d}{2} = \omega_d = k. \tag{4.5}$$

Putting this value of ω_d in (4.4), we get the minimum MSE of $t_{\omega d}$ as

min. $MSE(t_{\omega d}) = \bar{Y}^2 C_y^2 [f_1 - f_3 \rho^2]$ (4.6)

This is same as that of traditional two phase linear regression estimator.

The bias expression for the estimators t_{rsd} and t_{rsde} is respectively given by

$$B(t_{rsd}) = \bar{Y}\left[\theta^2 C_x^2 f_3 \omega_{1d} - \theta \rho C_y C_x f_3 \omega_{1d}\right] \tag{4.7}$$

$$B(t_{rsde}) = \bar{Y}\left[\theta^2 C_x^2 f_3 \frac{3\omega_{2d}}{8} - \theta \rho C_y C_x f_3 \frac{\omega_{2d}}{2}\right] \tag{4.8}$$

From (4.2) and (4.5), we have only two equations in three unknowns. It is not possible to find the unique values for w_{id}'s i=0,1,2.

In order to get unique values of w_{id}, we shall impose linear restriction

$$\omega_{0d} B(\bar{y}) + \omega_{1d} B(t_{rsd}) + \omega_{2d} B(t_{rsed}) = 0 \tag{4.9}$$

Equations (4.2), (4.5) and (4.9) can be written in matrix form, as

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1/2 \\ 0 & B(t_{rsd}) & B(t_{rsed}) \end{bmatrix} \begin{bmatrix} \omega_{0d} \\ \omega_{1d} \\ \omega_{2d} \end{bmatrix} = \begin{bmatrix} 1 \\ k \\ 0 \end{bmatrix} \tag{4.10}$$

Solving (4.10), we get the unique values of $\omega'_{id}s,\ (i = 0,1,2)$ as

$$\left.\begin{aligned} \omega_{0d} &= \frac{\Delta_{0d}}{\Delta_{rd}} \\ \omega_{1d} &= \frac{\Delta_{1d}}{\Delta_{rd}} \\ \omega_{2d} &= \frac{\Delta_{2d}}{\Delta_{rd}} \end{aligned}\right\} \tag{4.11}$$

where

$$\left.\begin{aligned} \Delta_{rd} &= B(t_{rsed}) - \frac{1}{2} B(t_{rsd}) \\ \Delta_{r0d} &= B(t_{rsed})\{1 - k\} + \frac{1}{2} B(t_{rsd})\left\{k - \frac{1}{2}\right\} \\ \Delta_{r1d} &= k.B(t_{rsed}) \\ \Delta_{r2d} &= -k.B(t_{rsd}) \end{aligned}\right\} \tag{4.12}$$

Use of these w_{id}'s (i=0,1,2) will remove the bias up to terms of order $O(n^{-1})$ at (4.1).

The estimator t_q written in (3.1), in two phase sampling, will take following form

$$t_{qd} = q_{0d}\bar{y} + q_{1d}t_{psd} + q_{2d}t_{psed} \quad \epsilon\, Q_d \tag{4.13}$$

For $\sum_{i=0}^{2} q_{id} = 1$, $q_{id} \epsilon R$. (4.14)

where q_{id}(i=0,1,2) denotes the statistical constants .

The estimators t_{psd} and t_{psde} are

$t_{psd} = \bar{y}\left(\frac{a\bar{x}+b}{a\bar{X'}+b}\right)$ and

$$t_{psed} = \bar{y}\exp\left\{\frac{(a\bar{x}+b)-(a\bar{X}'+b)}{(a\bar{x}+b)+(a\bar{X}'+b)}\right\}$$

Following the procedure of section 4, we get the unique values of q_{id}'s (i=0,1,2) as

$$\left.\begin{aligned} q_{0d} &= \frac{\Delta_{p0d}}{\Delta_{pd}} \\ q_{1d} &= \frac{\Delta_{p1d}}{\Delta_{pd}} \\ q_{2d} &= \frac{\Delta_{p2d}}{\Delta_{pd}} \end{aligned}\right\} \tag{4.15}$$

where

$$\left.\begin{aligned} \Delta_{pd} &= B(t_{rsed}) - \frac{1}{2} B(t_{rsd}) \\ \Delta_{p0d} &= B(t_{rsed})\{1+k\} + \frac{1}{2} B(t_{rsd})\left\{-k - \frac{1}{2}\right\} \\ \Delta_{p1d} &= -k.B(t_{psed}) \\ \Delta_{p2d} &= k.B(t_{psd}) \end{aligned}\right\} \tag{4.16}$$

where

$$B(t_{psd}) = \bar{Y}[\theta q_{1d} f_3 \rho C_y C_x] \tag{4.17}$$

$$B(t_{psed}) = \bar{Y}\left[\theta \frac{q_{2d}}{2} f_3 \rho C_y C_x - \frac{\theta^2}{8} q_{2d} f_3 C_x^2\right] \tag{4.18}$$

The minimum MSE of t_{qd} is given by

$$MSE(t_{qd}) = \bar{Y}^2 C_y^2 [f_1 - f_3\rho^2] .$$

5. Empirical study

For empirical study we use the data sets earlier used by Kadilar and Cingi (2006) (population 1) and Khosnevisan et. al. (2007) (population 2) to verify the theoretical results.

Data statistics

Population	N	n	$\bar{Y}$	$\bar{X}$	C_y	C_x	ρ	$\beta_2(x)$
Population 1	106	20	2212.59	27421.7	5.22	2.10	0.86	34.57
Population 2	20	8	19.55	18.8	0.355	0.394	-0.92	3.06

Table 5.1: **Values of ω_i's and q_i's**

ω_i's	**Population 1**	**Population 2**
ω_0 (q_0)	8.590718 (21.417)	7.892148 (2.919085)
ω_1 (q_1)	11.86615 (16.14158)	5.234461 (3.576773)
ω_2 (q_2)	-19.4569 (-36.5586)	-12.1266 (-5.49586)

The percent relative efficiencies (PRE) of various estimators of $\bar{Y}$ are computed and presented in Table 5.2 below.

Table 5.2: PRE of different estimators of $\bar{Y}$

Estimator	**PRE (Pop I)**	**Estimator**	**PRE (Pop II)**
t_0	100	q_0	100
t_1	212.816	q_1	526
t_2	212.803	q_2	550.261
t_3	212.606	q_3	645.256
t_4	212.815	q_4	534.592
t_5	212.716	q_5	581.732
t_6	212.810	q_6	465.501
t_7	143.992	q_7	384.447
t_8	143.923	q_8	285.920
t_9	143.988	q_9	338.487
t_{10}	143.990	q_{10}	374.584
t_{11}	143.991	q_{11}	345.118
t_{12}	143.959	q_{12}	231.602
t_{13}	143.991	q_{13}	424.194
t_{14}	143.987	q_{14}	360.086
t_{15}	143.992	q_{15}	356.520
t_{16}	143.911	q_{16}	467.051
t_w(opt)	384.025	t_q(opt)	650.263

In order to see the performance of the suggested estimators in two phase sampling we use the data set of Murthy (1967) (Population III) and Steel and Torrie (1960) (Population IV).

Population	C_y	C_x	ρ	N	n'	n
Population 1	0.3542	0.9484	0.9150	80	30	10
Population2	0.4803	0.7493	-0.4996	30	12	4

Table 5.3 : The values of $\omega_{id}{}'s$ and $q_{id}{}'s$

$\omega_{id}{}'s$ ($q_{id}{}'s$)	**Population III**	**Population IV**
ω_{0d} (q_{0d})	-0.241523 (1.808833)	3.011435 (1.089979)
ω_{1d} (q_{1d})	-0.558071 (0.125381)	1.370950 (0.730464)
ω_{2d} (q_{2d})	1.799595 (-0.934214)	-3.382385 (-0.820443)

The percent relative efficiencies of various estimators of $\bar{Y}$ in two phase sampling are computed and presented in Table 5.4 below.

Table 5.4 : PRE of different estimators of $\bar{Y}$ in two phase sampling

Estimator	**PRE (Population I)**	**PRE (Population II)**
$\bar{y}$	100	100
t_{rsd}	36.642	24.562
t_{psd}	4.849	59.770
t_{rsde}	200.420	48.365
t_{psed}	23.628	115.142
$t_{\omega d}$	276.156	63.452
t_{qd}	34.321	123.762
t_{opt}	276.156	123.762

6. Conclusion

From theoretical discussion and empirical study we conclude that the proposed estimators under optimum conditions perform better than other estimators considered in the article. The relative efficiencies of various estimators are listed in Table 5.2 and 5.4.

Appendix A

Table A.1: Some members of the proposed family of estimators -

a	b	ω_0 (q_0)	ω_1 (q_1)	ω_2 (q_2)	Ratio Estimator (corresponding to ω_i,i=0,1,2)	Product Estimator (corresponding to q_i,i=0,1,2)
0	0	1	0	0	$t_0 = \bar{y}$ The mean per unit estimator	$q_0 = \bar{y}$ The mean per unit estimator
1	o	0	1	0	$t_1 = \bar{y}\frac{\bar{X}}{\bar{x}}$ The usual ratio estimator	$q_1 = \bar{y}\frac{\bar{x}}{\bar{X}}$ The usual product estimator
1	C_x	0	1	0	$t_2 = \bar{y}\frac{\bar{X}+C_x}{\bar{x}+C_x}$ Sisodia and Dwivedi (1981) estimator	$q_2 = \bar{y}\frac{\bar{x}+C_x}{\bar{X}+C_x}$ Pandey and Dubey (1988) estimator
1	$\beta_2(x)$	0	1	0	$t_3 = \bar{y}\frac{\bar{X}+\beta_2(x)}{\bar{x}+\beta_2(x)}$ Singh et. al. (2004) estimator	$q_3 = \bar{y}\frac{\bar{x}+\beta_2(x)}{\bar{X}+\beta_2(x)}$ Singh et. al. (2004) estimator
$\beta_2(x)$	C_x	0	1	0	$t_4 = \bar{y}\frac{\bar{X}\beta_2(x)+C_x}{\bar{x}\beta_2(x)+C_x}$ Upadhyaya and Singh (1999) estimator	$q_4 = \bar{y}\frac{\bar{x}\beta_2(x)+C_x}{\bar{X}\beta_2(x)+C_x}$ Upadhyaya and Singh (1999) estimator
C_x	$\beta_2(x)$	0	1	0	$t_5 = \bar{y}\frac{\bar{X}C_x+\beta_2(x)}{\bar{x}C_x+\beta_2(x)}$ Upadhyaya and Singh (1999) estimator	$q_5 = \bar{y}\frac{\bar{x}C_x+\beta_2(x)}{\bar{X}C_x+\beta_2(x)}$ Upadhyaya and Singh (1999) estimator
1	ρ	0	1	0	$t_6 = \bar{y}\left[\frac{\bar{X}+\rho}{\bar{x}+\rho}\right]$ Singh and Tailor (2003) estimator	$q_6 = \bar{y}\left[\frac{\bar{x}+\rho}{\bar{X}+\rho}\right]$ Singh and Tailor (2003) estimator
1	0	0	0	1	$t_7 = \bar{y}\exp\left[\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}}\right]$ Bahl and Tuteja (1991) estimator	$q_7 = \bar{y}\exp\left[\frac{\bar{x}-\bar{X}}{\bar{X}+\bar{x}}\right]$ Bahl and Tuteja (1991) estimator

a	b	ω_0 (q_0)	ω_1 (q_1)	ω_2 (q_2)	Ratio Estimator (corresponding to ω_i, i=0,1,2)	Product Estimator (corresponding to q_i, i=0,1,2)
1	$\beta_2(x)$	0	0	1	$t_8 = \bar{y}\exp\left[\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}+2\beta_2(x)}\right]$ Singh et. al. (2007) estimator	$q_8 = \bar{y}\exp\left[\frac{\bar{x}-\bar{X}}{\bar{X}+\bar{x}+2\beta_2(x)}\right]$ Singh et. al. (2007) estimator
1	C_x	0	0	1	$t_9 = \bar{y}\exp\left[\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}+2C_x}\right]$ Singh et.al. (2007) estimator	$q_9 = \bar{y}\exp\left[\frac{\bar{x}-\bar{X}}{\bar{X}+\bar{x}+2C_x}\right]$ Singh et.al. (2007) estimator
1	ρ	0	0	1	$t_{10} = \bar{y}\exp\left[\frac{\bar{X}-\bar{x}}{\bar{X}+\bar{x}+2\rho}\right]$ Singh et. al.(2007) estimator	$q_{10} = \bar{y}\exp\left[\frac{\bar{x}-\bar{X}}{\bar{X}+\bar{x}+2\rho}\right]$ Singh et. al.(2007) estimator
$\beta_2(x)$	C_x	0	0	1	$t_{11} = \bar{y}\exp\left[\frac{\beta_2(x)(\bar{X}-\bar{x})}{\beta_2(x)(\bar{X}+\bar{x})+2C_x}\right]$ Singh et. al. (2007) estimator	$q_{11} = \bar{y}\exp\left[\frac{\beta_2(x)(\bar{x}-\bar{X})}{\beta_2(x)(\bar{X}+\bar{x})+2C_x}\right]$ Singh et. al. (2007) estimator
C_x	$\beta_2(x)$	0	0	1	$t_{12} = \bar{y}\exp\left[\frac{C_x(\bar{X}-\bar{x})}{C_x(\bar{X}+\bar{x})+2\beta_2(x)}\right]$ Singh et. al. (2007) estimator	$q_{12} = \bar{y}\exp\left[\frac{C_x(\bar{x}-\bar{X})}{C_x(\bar{X}+\bar{x})+2\beta_2(x)}\right]$ Singh et. al. (2007) estimator
C_x	ρ	0	0	1	$t_{13} = \bar{y}\exp\left[\frac{C_x(\bar{X}-\bar{x})}{C_x(\bar{X}+\bar{x})+2\rho}\right]$ Singh et. al. (2007) estimator	$q_{13} = \bar{y}\exp\left[\frac{C_x(\bar{x}-\bar{X})}{C_x(\bar{X}+\bar{x})+2\rho}\right]$ Singh et. al. (2007) estimator
ρ	C_x	0	0	1	$t_{14} = \bar{y}\exp\left[\frac{\rho(\bar{X}-\bar{x})}{\rho(\bar{X}+\bar{x})+2C_x}\right]$ Singh et. al. (2007) estimator	$q_{14} = \bar{y}\exp\left[\frac{\rho(\bar{x}-\bar{X})}{\rho(\bar{X}+\bar{x})+2C_x}\right]$ Singh et. al. (2007) estimator
$\beta_2(x)$	ρ	0	0	1	$t_{15} = \bar{y}\exp\left[\frac{\beta_2(x)(\bar{X}-\bar{x})}{\beta_2(x)(\bar{X}+\bar{x})+2\rho}\right]$ Singh et. al. (2007) estimator	$q_{15} = \bar{y}\exp\left[\frac{\beta_2(x)(\bar{x}-\bar{X})}{\beta_2(x)(\bar{X}+\bar{x})+2\rho}\right]$ Singh et. al. (2007) estimator
ρ	$\beta_2(x)$	0	0	1	$t_{16} = \bar{y}\exp\left[\frac{\rho(\bar{X}-\bar{x})}{\rho(\bar{X}+\bar{x})+2\beta_2(x)}\right]$ Singh et. al. (2007) estimator	$q_{16} = \bar{y}\exp\left[\frac{\rho(\bar{x}-\bar{X})}{\rho(\bar{X}+\bar{x})+2\beta_2(x)}\right]$ Singh et. al. (2007) estimator

In addition to above estimators a large number of estimators can also be generated from the proposed estimators just by putting different values of constants $\omega_i, i = 0,1,2$, $q_i, i = 0,1,2$ and a and b.

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