Florentin Smarandache
(author and editor)

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(on Neutrosophic Theory and Applications)

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Introductory Note

This eighth volume of *Collected Papers* includes 75 papers comprising 973 pages on (theoretic and applied) neutrosophics, written between 2010-2022 by the author alone or in collaboration with the following 102 co-authors (alphabetically ordered) from 24 countries: Mohamed Abdel-Basset, Abduallah Gamal, Firoz Ahmad, Ahmad Yusuf Adhami, Ahmed B. Al-Nafee, Ali Hassan, Mumtaz Ali, Akbar Rezaei, Assia Bakali, Ayoub Bahnasse, Azeddine Elhassouny, Durga Banerjee, Romualdas Bausys, Mircea Boşcoianu, Traian Alexandru Buda, Bui Cong Cuong, Emilia Calefariu, Ahmet Çevik, Chang Su Kim, Victor Christianto, Dae Wan Kim, Daud Ahmad, Arindam Dey, Partha Pratim Dey, Mamouni Dhar, H. A. Elagamy, Ahmed K. Essa, Sudipta Gayen, Bibhas C. Giri, Daniela Gîfu, Noel Batista Hernández, Hojjatollah Farahani, Huda E. Khalid, Irfan Deli, Saeid Jafari, Tèmítópé Gbóláhàn Jaíyéolá, Sripati Jha, Sudan Jha, Ilanthenral Kandasamy, W.B. Vasantha Kandasamy, Darjan Karabašević, M. Karthika, Kawther F. Alhasan, Giruta Kazakeviciute-Januskeviciene, Qaisar Khan, Kishore Kumar P K, Prem Kumar Singh, Ranjan Kumar, Maikel Leyva-Vázquez, Mahmoud Ismail, Tahir Mahmood, Hafsa Masood Malik, Mohammad Abobala, Mai Mohamed, Gunasekaran Manogaran, Seema Mehra, Kalyan Mondal, Mohamed Talea, Mullai Murugappan, Muhammad Akram, Muhammad Aslam Malik, Muhammad Khalid Mahmood, Nivetha Martin, Durga Nagarajan, Nguyen Van Dinh, Nguyen Xuan Thao, Lewis Nkenyereya, Jagan M. Obbineni, M. Parimala, S. K. Patro, Peide Liu, Pham Hong Phong, Surapati Pramanik, Gyanendra Prasad Joshi, Quek Shio Gai, R. Radha, A.A. Salama, S. Satham Hussain, Mehmet Şahin, Said Broumi, Ganeshsree Selvachandran, Selvaraj Ganesan, Shahbaz Ali, Shouzhen Zeng, Manjeet Singh, A. Stanis Arul Mary, Dragiša Stanujkić, Yusuf Şubaş, Rui-Pu Tan, Mirela Teodorescu, Selçuk Topal, Zenonas Turskis, Vakkas Uluçay, Norberto Valcárcel Izquierdo, V. Venkateswara Rao, Volkan Duran, Ying Li, Young Bae Jun, Wadei F. Al-Omeri, Jian-qiang Wang, Lihshing Leigh Wang, Edmundas Kazimieras Zavadskas.

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Neutrosophic Set - A Generalization of The Intuitionistic Fuzzy Set

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Florentin Smarandache (2010). Neutrosophic Set – A Generalization of The Intuitionistic Fuzzy Set. *Journal of Defense Resources Management* 1(1): 107-116

Abstract: *In this paper one generalizes the intuitionistic fuzzy set (IFS), paraconsistent set, and intuitionistic set to the neutrosophic set (NS). Many examples are presented. Distinctions between NS and IFS are underlined.*

Keywords and Phrases: *Intuitionistic Fuzzy Set, Paraconsistent Set, Intuitionistic Set, Neutrosophic Set, Non-standard Analysis, Philosophy.*
MSC 2000: *03B99, 03E99.*

1. INTRODUCTION

One first presents the evolution of sets from fuzzy set to neutrosophic set. Then one introduces the neutrosophic components T, I, F which represent the membership, indeterminacy, and non-membership values respectively, where $]^-0, 1^+[$ is the non-standard unit interval, and thus one defines the neutrosophic set. One gives examples from mathematics, physics, philosophy, and applications of the neutrosophic set. Afterwards, one introduces the neutrosophic set operations (complement, intersection, union, difference, Cartesian product, inclusion, and n-ary relationship), some generalizations and comments on them, and finally the distinctions between the neutrosophic set and the intuitionistic fuzzy set.

2. SHORT HISTORY

The *fuzzy set* (FS) was introduced by L. Zadeh in 1965, where each element had a degree of membership.

The *intuitionistic fuzzy set* (IFS) on a universe X was introduced by K. Atanassov in 1983 as a generalization of FS, where besides the degree of membership $\mu_A(x) \in [0,1]$ of each element $x \in X$ set A there was considered a degree of non-membership $v_A(x) \in [0,1]$, but such that

$$\forall x \in X, \mu_A(x) + v_A(x) \leq 1 \quad (2.1)$$

According to Deschrijver & Kerre (2003) the *vague set* defined by Gau and Buehrer (1993) was proven by Bustine & Burillo (1996) to be the same as IFS.

Goguen (1967) defined the *L-fuzzy Set* in X as a mapping $X \to L$ such that $(L^*, \leq_{L^*})$ is a complete lattice, Where

$L^* = \{ (x_1, x_2) \in [0,1]^2, x_1 + x_2 \leq 1\}$

and $(x_1, x_2) \leq_{L^*} (y_1, y_2) \Leftrightarrow x_1 \leq y_1$ and $x_2 \geq y_2$.

The *interval-valued fuzzy set* (IVFS) apparently first studied by Sambuc (1975), which were called by Deng (1989) *grey sets*, and IFS are specific kinds of L-fuzzy sets.

According to Cornelis et al. (2003), Gehrke et al. (1996) stated that "Many people believe that assigning an exact number to an expert's opinion is too restrictive, and the assignment of an interval of values is more realistic", which is somehow similar with the imprecise probability theory where instead of a crisp probability one has an interval (upper and lower) probabilities as in Walley (1991).

Atanassov (1999) defined the *interval-valued intuitionistic fuzzy set* (IVIFS) on a universe X as an object A such that:

$L^* = \{ (x_1, x_2) \in [0,1]^2, x_1 + x_2 \leq 1\}$ (2.2)

with M_A:X → Int([0,1]) and

N_A:X → Int([0,1]) and (2.3)

$\forall x \in X \sup M_A(x) + \sup N_A(x) \leq 1$ (2.4)

Belnap (1977) defined a four-valued logic, with truth (T), false (F), unknown (U), and contradiction (C). He used a billatice where the four components were inter-related.

In 1995, starting from philosophy (when I fretted to distinguish between *absolute truth* and *relative truth* or between *absolute falsehood* and *relative falsehood* in logics, and respectively between *absolute membership* and *relative membership* or *absolute non-membership* and *relative non-membership* in set theory) I began to use the non-standard analysis. Also, inspired from the sport games (winning, defeating, or tight scores), from votes (pro, contra, null/black votes), from positive/negative/zero numbers, from yes/no/NA, from decision making and control theory (making a decision, not making, or hesitating), from accepted/rejected/pending, etc. and guided by the fact that the law of excluded middle did not work any longer in the modern logics, I combined the non-standard analysis with a tri-component logic/set/probability theory and with philosophy (I was excited by paradoxism in science and arts and letters, as well as by paraconsistency and incompleteness in knowledge). How to deal with all of them at once, is it possible to unity them?

I proposed the term "neutrosophic" because "neutrosophic" etymologically comes from "neutrosophy" [French *neutre* < Latin *neuter*, neutral, and Greek *sophia*, skill/wisdom] which means knowledge of neutral thought, and this third/neutral represents the main distinction between "fuzzy" and "intuitionistic fuzzy" logic/set, i.e. the *included middle* component (Lupasco-Nicolescu's logic in philosophy), i.e. the neutral/indeterminate/unknown part (besides the "truth"/"membership" and "falsehood"/"non-membership" components that both appear in fuzzy logic/set).

See the Proceedings of the

First International Conference on Neutrosophic Logic, The University of New Mexico, Gallup Campus, 1-3 December 2001, at http://www.gallup.unm.edu/~smarandache/FirstNeutConf.htm.

3. DEFINITION OF NEUTROSOPHIC SET

Let T, I, F be real standard or non-standard subsets of $]^-0, 1^+[$,
with

sup T = t_sup, inf T = t_inf,
sup I = i_sup, inf I = i_inf,
sup F = f_sup, inf F = f_inf,
and n_sup = t_sup+i_sup+f_sup,
n_inf = t_inf+i_inf+f_inf.

T, I, F are called *neutrosophic components*. Let U be a universe of discourse, and M a set included in U. An element x from U is noted with respect to the set M as x(T, I, F) and belongs to M in the following way:

it is t% true in the set, i% indeterminate (unknown if it is) in the set, and f% false, where t varies in T, i varies in I, f varies in F.

4. GENERAL EXAMPLES

Let A, B, and C be three neutrosophic sets.

One can say, by language abuse, that any element neutrosophically belongs to any set, due to the percentages of truth/indeterminacy/falsity involved, which varies between 0 and 1 or even less than 0 or greater than 1.

Thus: x(0.5,0.2,0.3) belongs to A (which means, with a probability of 50% x is in A, with a probability of 30% x is not in A, and the rest is undecidable); or y(0,0,1) belongs to A (which normally means y is not for sure in A); or z(0,1,0) belongs to A (which means one does know absolutely nothing about z's affiliation with A); here 0.5+0.2+0.3=1; thus A is a NS and an IFS too. More general, y((0.20-0.30), (0.40-0.45)4[0.50-0.51], {0.20, 0.24, 0.28}) belongs to the set B, which means:

- with a probability in between 20-30% y is in B (one cannot find an exact approximation because of various sources used);
- with a probability of 20% or 24% or 28% y is not in B;
- the indeterminacy related to the appurtenance of y to B is in between 40-45% or between 50-51% (limits included);

The subsets representing the appurtenance, indeterminacy, and falsity may overlap, and n_sup = 0.30+0.51+0.28 > 1 in this case; then B is a NS but is not an IFS; we can call it *paraconsistent set* (from *paraconsistent logic*, which deals with paraconsistent information). Or, another example, say the element z(0.1, 0.3, 0.4) belongs to the set C, and here 0.1+0.3+0.4<1; then B is a NS but is not an IFS; we can call it *intuitionistic set* (from *intuitionistic logic*, which deals with incomplete information).

Remarkably, in the same NS one can have elements which have paraconsistent information (sum of components >1), others incomplete information (sum of components <

1), others consistent information (in the case when the sum of components = 1), and others interval-valued components (with no restriction on their superior or inferior sums).

5. PHYSICS EXAMPLES

a) For example the Schrödinger's Cat Theory says that the quantum state of a photon can basically be in more than one place in the same time, which translated to the neutrosophic set means that an element (quantum state) belongs and does not belong to a set (one place) in the same time; or an element (quantum state) belongs to two different sets (two different places) in the same time. It is a question of "alternative worlds" theory very well represented by the neutrosophic set theory.

In Schrödinger's Equation on the behavior of electromagnetic waves and "matter waves" in quantum theory, the wave function ψ which describes the superposition of possible states may be simulated by a neutrosophic function, i.e. a function whose values are not unique for each argument from the domain of definition (the vertical line test fails, intersecting the graph in more points).

Don't we better describe, using the attribute "neutrosophic" than "fuzzy" or any others, a quantum particle that neither exists nor non-exists?

b) How to describe a particle ζ in the infinite micro-universe that belongs to two distinct places P1 and P2 in the same time? $\zeta \in P_1$ and $\zeta \notin P_1$ as a true contradiction, or $\zeta \in P_1$ and $\zeta \notin \neg P_1$.

6. PHILOSOPHICAL EXAMPLES

Or, how to calculate the truth-value of Zen (in Japanese) / Chan (in Chinese) doctrine philosophical proposition: the present is eternal and comprises in itself the past and the future?

In Eastern Philosophy the contradictory utterances form the core of the Taoism and Zen/Chan (which emerged from Buddhism and Taoism) doctrines. How to judge the truth-value of a metaphor, or of an ambiguous statement, or of a social phenomenon which is positive from a standpoint and negative from another standpoint? There are many ways to construct them, in terms of the practical problem we need to simulate or approach. Below there are mentioned the easiest ones:

7. APPLICATION

A cloud is a neutrosophic set, because its borders are ambiguous, and each element (water drop) belongs with a neutrosophic probability to the set (e.g. there are a kind of separated water drops, around a compact mass of water drops, that we don't know how to consider them: in or out of the cloud).

Also, we are not sure where the cloud ends nor where it begins, neither if some elements are or are not in the set. That's why the percent of indeterminacy is required and the neutrosophic probability (using subsets - not numbers - as components) should be used for better modeling: it is a more organic, smooth, and especially accurate estimation. Indeterminacy is the zone of ignorance of a proposition's value, between truth and falsehood.

8. OPERATIONS WITH CLASSICAL SETS

We need to present these set operations in order to be able to introduce the neutrosophic connectors.

Let S1 and S2 be two (unidimensional) real standard or non-standard subsets included in the non-standard interval $]^{-}0, \infty)$ then one defines:

8.1 Addition of classical Sets:

$S_1 \oplus S_2 = \{x \mid x = s_1 + s_2,$ where $s_1 \in S_1$ and $s_2 \in S_2\}$,

with

$\inf S_1 \oplus S_2 = \inf S_1 + \inf S_2$,

$\sup S_1 \oplus \sup S_2 = \sup S_1 + \sup S_2$;

and, as some particular cases, we have

$\{a\} \oplus S_2 = \{x \mid x = a + s_2$, where $s_2 \in S_2\}$,

with

$\inf\{a\} \oplus S_2 = a + \inf S_2$,

$\sup\{a\} \oplus S_2 = a + \sup S_2$.

8.2 Subtraction of classical Sets:

$S_1 \Theta S_2 = \{x \mid x = s_1 - s_2$, where $s_1 \in S_1$ and $s_2 \in S_2\}$.

with

$\inf S_1 \Theta S_2 = \inf S_1 - \sup S_2$,

$\sup S_1 \Theta S_2 = \sup S_1 - \inf S_2$;

and, as some particular cases, we have

$\{a\} \Theta S_2 = \{x \mid x = a - s_2$, where $s_2 \in S_2\}$,

with

$\inf\{a\} \Theta S_2 = a - \sup S_2$,

$\sup\{a\} \Theta S_2 = a - \inf S_2$;

also

$\{1^+\} \Theta S_2 = \{x \mid x = 1^+ - s_2$, where $s_2 \in S_2\}$,

with

$\inf\{1^+\} \Theta S_2 = 1^+ - \sup S_2$,

$\sup\{1^+\} \Theta S_2 = 100 - \inf S_2$.

8.3 Multiplication of classical Sets:

$S_1 \otimes S_2 = \{x \mid x = s_1 \cdot s_2$, where $s_1 \in S_1$ and $s_2 \in S_2\}$.

with

$\inf\{a\} \otimes S_2 = a \cdot \inf S_2$,

$\sup\{a\} \otimes S_2 = a \cdot \sup S_2$;

also

$\{1^+\} \otimes S_2 = \{x \mid x = 1 \cdot s_2$, where $s_2 \in S_2\}$

with

$\inf\{1^+\} \otimes S_2 = 1^+ \cdot \inf S_2$

$\sup\{1^+\} \otimes S_2 = 1^+ \cdot \sup S_2$.

8.4 Division of a classical Set by a Number:

Let $k \in \Re^*$,

then

$S_1 O k = \{x \mid x = s_1 / k$, where $s_1 \in S_1\}$.

9. NEUTROSOPHIC SET OPERATIONS

One notes, with respect to the sets A and B over the universe U,
$x = x(T_1, I_1, F_1) \in A$ and
$x = x(T_2, I_2, F_2) \in B$,
by mentioning x's *neutrosophic membership*, *indeterminacy, and non-membership respectively appurtenance*.
And, similarly, $y = y(T', I', F') \in B$.

If, after calculations, in the below operations one obtains values < 0 or > 1, then one replaces them with $^-0$ or 1^+ respectively.

9.1. Complement of A:
If $x(T_1, I_1, F_1) \in A$,
then
$x(\{1^+\}\ominus T_1, \{1^+\}\ominus I_1, \{1^+\}\ominus F_1) \in C(A)$.

9.2. Intersection:
If $x(T_1, I_1, F_1) \in A$,
$x(T_2, I_2, F_2) \in B$,
then
$x(T_1 \otimes T_2, I_1 \otimes I_2, F_1 \otimes F_2) \in A \cap B$.

9.3. Union:
If $x(T_1, I_1, F_1) \in A$,
$x(T_2, I_2, F_2) \in B$,
then
$x(T_1 \oplus T_2 \ominus T_1 \otimes T_2, I_1 \oplus I_2 \ominus I_1 \otimes I_2, F_1 \oplus F_2 \ominus F_1 \otimes F_2) \in A \cup B$.

9.4. Difference:
If $x(T_1, I_1, F_1) \in A$,
$x(T_2, I_2, F_2) \in B$,
then
$x(T_1 \ominus T_1 \otimes T_2, I_1 \ominus I_1 \otimes I_2, F_1 \ominus F_1 \otimes F_2) \in A \setminus B$,
because $A \setminus B = A \cap C(B)$.

9.5. Cartesian Product:
If $x(T_1, I_1, F_1) \in A$,
$y(T', I', F') \in B$,
then
$(x(T_1, I_1, F_1), y(T', I', F')) \in A \times B$.

9.6. M is a subset of N:
If $x(T_1, I_1, F_1) \in M \Rightarrow x(T_2, I_2, F_2) \in N$,
where
$\inf T_1 \leq \inf T_2$, $\sup T_1 \leq \sup T_2$,
and
$\inf F_1 \geq \inf F_2$, $\sup F_1 \geq \sup F_2$.

9.7. Neutrosophic n-ary Relation:
Let $A_1, A_2, \ldots, A_n$ be arbitrary non-empty sets. A Neutrosophic n-ary Relation *R* on $A_1 \times A_2 \times \ldots \times A_n$ is defined as a subset of the Cartesian product $A_1 \times A_2 \times \ldots \times A_n$, such that for each ordered n-tuple $(x_1, x_2, \ldots, x_n)$ (T, I, F), T represents the degree of validity, I the degree of indeterminacy, and F the degree of non-validity respectively of the relation R.

It is related to the definitions for the *Intuitionistic Fuzzy Relation* independently given by Atanassov (1984, 1989), Toader Buhaescu (1989), Darinka Stoyanova (1993), Humberto Bustince Sola and P. Burillo Lopez (1992-1995).

10. GENERALIZATIONS AND COMMENTS

From the intuitionistic logic, paraconsistent logic, dialetheism, faillibilism, paradoxes, pseudoparadoxes, and tautologies we transfer the "adjectives" to the sets, i.e. to intuitionistic set (set incompletely known), paraconsistent set, dialetheist set, faillibilist set (each element has a percenatge of indeterminacy), paradoxist set (an element may belong and may not belong in the same time to the set), pseudoparadoxist set, and tautologic set respectively.

Hence, the neutrosophic set generalizes:

- the *intuitionistic set*, which supports incomplete set theories (for $0 < n < 1$ and $i = 0$, $0 \le t, i, f \le 1$) and incomplete known elements belonging to a set;
- the *fuzzy set* (for $n = 1$ and $i = 0$, and $0 \le t, i, f \le 1$);
- the *intuitionistic fuzzy set* (for $t+i+f=1$ and $0 \le i < 1$);
- the *classical set* (for $n = 1$ and $i = 0$, with t, f either 0 or 1);
- the *paraconsistent set* (for $n > 1$ and $i = 0$, with both $t, f < 1$);

there is at least one element x(T,I,F) of a paraconsistent set M which belongs at the same time to M and to its complement set C(M);
- the *faillibilist set* ($i > 0$);
- the *dialethist set*, which says that the intersection of some disjoint sets is not empty (for $t = f = 1$ and $i = 0$; some paradoxist sets can be denoted this way too); every element x(T,I,F) of a dialethist set M belongs at the same time to M and to its complement set C(M);
- the *paradoxist set*, each element has a part of indeterminacy if it is or not in the set ($i > 1$);
- the *pseudoparadoxist set* ($0 < i < 1$, $t + f > 1$);
- the *tautological set* ($i < 0$).

Compared with all other types of sets, in the neutrosophic set each element has three components which are subsets (not numbers as in fuzzy set) and considers a subset, similarly to intuitionistic fuzzy set, of "indeterminacy" - due to unexpected parameters hidden in some sets, and let the superior limits of the components to even boil *over 1* (overflooded) and the inferior limits of the components to even freeze *under 0* (underdried).

For example: an element in some tautological sets may have $t > 1$, called "overincluded". Similarly, an element in a set may be "overindeterminate" (for $i > 1$, in some paradoxist sets), "overexcluded" (for $f > 1$, in some unconditionally false appurtenances); or "undertrue" (for $t < 0$, in some unconditionally false appurtenances), "underindeterminate" (for $i < 0$, in some unconditionally true or false appurtenances), "underfalse" (for $f < 0$, in some unconditionally true appurtenances).

This is because we should make a distinction between unconditionally true ($t > 1$, and $f < 0$ or $i < 0$) and conditionally true appurtenances ($t \le 1$, and $f \le 1$ or $i \le 1$).

In a *rough set* RS, an element on its boundary-line cannot be classified neither as a member of RS nor of its complement with certainty.

In the neutrosophic set a such element may be characterized by x(T, I, F), with corresponding set-values for $T, I, F \subseteq]^-0, 1^+[$.

Compared to *Belnap's quadruplet logic*, NS and NL do not use restrictions among the components – and that's why the NS/NL have a more general form, while the middle component in NS and NL (the indeterminacy) can be split in more subcomponents if necessarily in various applications.

11. DIFFERENCES BETWEEN NEUTROSOPHIC SET (NS) AND INTUITIONISTIC FUZZY SET (IFS)

a) Neutrosophic Set can distinguish between *absolute membership* (i.e. membership in all possible worlds; we have extended Leibniz's absolute truth to absolute membership) and *relative membership* (membership in at least one world but not in all), because NS (absolute membership element)=1^+ while NS(relative membership element)=1. This has application in philosophy (see the neutrosophy).

That's why the unitary standard interval [0, 1] used in IFS has been extended to the unitary non-standard interval $]^-0, 1^+[$ in NS.

Similar distinctions for *absolute or relative non-membership, and absolute or relative indeterminant appurtenance* are allowed in NS.

b) In NS there is no restriction on T, I, F other than they are subsets of $]^-0, 1^+[$, thus: $^-0 \leq \inf T + \inf I + \inf F \leq \sup T + \sup I + \sup F \leq 3^+$.

The inequalities (2.1) and (2.4) of IFS are relaxed in NS.

This non-restriction allows paraconsistent, dialetheist, and incomplete information to be characterized in NS {i.e. the sum of all three components if they are defined as points, or sum of superior limits of all three components if they are defined as subsets can be >1 (for paraconsistent information coming from different sources), or < 1 for incomplete information}, while that information can not be described in IFS because in IFS the components T (membership), I (indeterminacy), F (non-membership) are restricted either to t+i+f=1 or to $t^2 + f^2 \leq 1$, if T, I, F are all reduced to the points t, i, f respectively, or to sup T + sup I + sup F = 1 if T, I, F are subsets of [0, 1].

Of course, there are cases when paraconsistent and incomplete informations can be normalized to 1, but this procedure is not always suitable.

c) Relation (2.3) from interval-valued intuitionistic fuzzy set is relaxed in NS, i.e. the intervals do not necessarily belong to Int[0,1] but to [0,1], even more general to $]^-0, 1^+[$.

d) In NS the components T, I, F can also be *non-standard* subsets included in the unitary nonstandard interval]-0, 1+[, not only *standard* subsets included in the unitary standard interval [0, 1] as in IFS.

e) NS, like dialetheism, can describe paradoxist elements, NS(paradoxist element) = (1, I, 1), while IFL can not describe a paradox because the sum of components should be 1 in IFS.

f) The connectors in IFS are defined with respect to T and F, i.e. membership and nonmembership only (hence the Indeterminacy is what's left from 1), while in NS they can be defined with respect to any of them (no restriction).

g) Component "I", indeterminacy, can be split into more subcomponents in order to better catch the vague information we work with, and such, for example, one can get more accurate answers to the *Question-Answering Systems* initiated by Zadeh (2003). {In Belnap's four-valued logic (1977) indeterminacy is split into Uncertainty (U) and Contradiction (C), but they were interrelated.}

h) NS has a better and clear name "neutrosophic" (which means the neutral part: i.e. neither true/membership nor false/nonmembership), while IFS's name "intuitionistic" produces confusion with Intuitionistic Logic, which is something different.

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Correlation Coefficient of Interval Neutrosophic Set

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Abstract. In this paper we introduce for the first time the concept of correlation coefficients of interval valued neutrosophic set (INS for short). Respective numerical examples are presented.

1. Introduction

Neutrosophy was pioneered by Smarandache [1]. It is a branch of philosophy which studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra [2]. Neutrosophic set theory is a powerful formal framework which generalizes the concept of the classic set, fuzzy set [3], interval-valued fuzzy set [4], intuitionistic fuzzy set [5], interval-valued intuitionistic fuzzy set [6], and so on. Neutrosophy introduces a new concept called <NeutA> which represents indeterminacy with respect to <A>. It can solve certain problems that cannot be solved by fuzzy logic. For example, a paper is sent to two reviewers, one says it is 90% acceptable and another says it is 90% unacceptable. But the two reviewers may have different backgrounds. One is an expert, and another is a new comer in this field. The impacts on the final decision of the paper by the two reviewers should be different, even though they give the same grade level of the acceptance. There are many similar problems, such as weather forecasting, stock price prediction, and political elections containing indeterminate conditions that fuzzy set theory does not handle well. This theory deals with imprecise and vague situations where exact analysis is either difficult or impossible. After the pioneering work of Smarandache. In 2005, Wang et al. [7] introduced the notion of interval neutrosophic set (INS) which is a particular case of the neutrosophic set (NS) that can be described by a membership interval, a non-membership interval, and an indeterminate interval, thus the NS is flexible and practical, and the NS provides a more reasonable mathematical framework to deal with indeterminate and inconsistent information.

The theories of both neutrosophic set and interval neutrosophic set have achieved great success in various areas such as medical diagnosis [8], database [9,10], topology[11], image processing [12,13,14], and decision making problem [15].

Although several distance measures, similarity measures, and correlation measure of neutrosophic sets have been recently presented in [16, 17], there is a rare investigation on correlation of interval neutrosophic sets.

It is very common in statistical analysis of data to finding the correlation between variables or attributes, where the correlation coefficient is defined on ordinary crisp sets, fuzzy sets [18], intuitionistic fuzzy sets [19,20,21], and neutrosophic set [16,17] respectively. In this paper we first discuss and derive a formula for the correlation coefficient defined on the domain of interval neutrosophic sets. The paper unfolds as follows. The next section briefly introduces some definitions related to the method. Section III presents the correlation and weighted correlation coefficient of the interval neutrosophic set. Conclusions appear in the last section.

2. Preliminaries

In this section, we mainly recall some notions related to neutrosophic sets, and interval neutrosophic sets relevant to the present work. See especially [1, 7, 17] for further details and background.

2.1 Definition ([1]). Let U be an universe of discourse; then the neutrosophic set A is an object having the form A = {< x: $T_A(x)$,$I_A(x)$,$F_A(x)$>,x ∈ U}, where the functions T,I,F : U→]$^-0,1^+$[define respectively the degree of membership, the degree of indeterminacy, and the degree of non-membership of the element x ∈ U to the set A with the condition:

$$^-0 \leq T_A(x)+ I_A(x)+ F_A(x) \leq 3^+. \qquad (1)$$

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of]$^-0,1^+$[.So instead of]$^-0,1^+$[we need to take the interval [0,1] for technical applications, because]$^-0,1^+$[will be difficult to apply in the real applications such as in scientific and engineering problems.

2.2 Definition ([7]). Let X be a space of points (objects) with generic elements in X denoted by x. An interval neutrosophic set A in X is characterized by truth-membership function $T_A(x)$, indeteminacy-membership function $I_A(x)$, and falsity-membership function $F_A(x)$. For each point x in X, we have that $T_A(x)$, $I_A(x)$, $F_A(x)$ ⊂ [0 ,1] .

Remark 1. An INS is clearly a NS.

2.3 Definition ([7]).

- An INS A is empty if $inf T_A(x) = sup T_A(x)$= 0, $inf I_A(x) = sup I_A(x)$= 1, $inf F_A(x) = sup F_A(x)$= 0, for all x in A.
- Let $\underline{0}$ = <0, 1 ,1> and $\underline{1}$ = <1, 0 ,0>

2.4 Correlation Coefficient of Neutrosophic Set ([17]).

Let A and B be two neutrosophic sets in the universe of discourse X = {x_1, x_2, …,x_n}.

The *correlation coefficient* of A and B is given by

$$R(A,B)=\frac{C(A,B)}{(E(A)\cdot E(B))^{\frac{1}{2}}} \qquad (2)$$

where the *correlation* of two NSs A and B is given by

$$C(A,B) = \sum_{1=0}^{n}(T_A(x_i)T_B(x_i) + I_A(x_i)I_B(x_i) + F_A(x_i)F_B(x_i)) \qquad (3)$$

And the *informational energy* of two NSs A and B are given by

$$E(A) = \sum_{1=0}^{n}(T_A^2(x_i) + I_A^2(x_i) + F_A^2(x_i)) \qquad (4)$$

$$E(B) = \sum_{1=0}^{n}(T_B^2(x_i) + I_B^2(x_i) + F_B^2(x_i)) \qquad (5)$$

Respectively, the correlation coefficient of two neutrosophic sets A and B satisfies the following properties:

(1) $0 \leq R(A,B) \leq 1$ (6)
(2) R(A,B)= R(B,A) (7)
(3) R(A,B)= 1 if A=B (8)

3. Correlation of Two Interval Neutrosophic Sets

In this section, following the correlation between two neutrosophic sets defined by A. A. Salama in [17], we extend this definition to interval neutrosophic sets. If we have a random non-crisp set, with a triple membership form, for each of two interval neutrosophic sets, we get the interest in comparing the degree of their relationship. We check if there is any linear relationship between the two interval neutrosophic sets; thus we need a formula for the sample correlation coefficient of two interval neutrosophic sets in order to find the relationship between them.

3.1.Definition

Assume that two interval neutrosophic sets A and B in the universe of discourse $X = \{x_1, x_2, x_3, \ldots, x_n\}$ are denoted by

$$A=\frac{\sum_1^n <[\inf T_A(x_i)\ \sup T_A(x_i)],[\inf I_A(x_i)\ \sup I_A(x_i)],[\inf T_A(x_i)\ \sup F_A(x_i)],>}{x_i},\ x_i \in X, \text{ and} \quad (9)$$

$$B=\frac{\sum_1^n <[\inf T_B(x_i)\ \sup T_B(x_i)],[\inf I_B(x_i)\ \sup I_B(x_i)],[\inf T_B(x_i)\ \sup F_B(x_i)],>}{x_i},\ x_i \in X, \text{ where} \quad (10)$$

$\inf T_A(x_i) \leq \sup T_A(x_i)$,$\inf F_A(x_i) \leq \sup F_A(x_i)$, $\inf I_A(x_i) \leq \sup I_A(x_i)$, $\inf T_B(x_i) \leq \sup T_B(x_i)$, $\inf F_B(x_i) \leq \sup F_B(x_i)$, $\inf I_B(x_i) \leq \sup I_B(x_i)$, and they all belong to [0, 1];

then we define the correlation of the interval neutrosophic sets A and B in X by the formula

$$C_{INS}(A,B)= \sum_1^n\{\inf T_A(x_i)\cdot \inf T_B(x_i) + \sup T_A(x_i)\cdot \sup T_B(x_i) + \inf I_A(x_i)\cdot \inf I_B(x_i) + \sup I_A(x_i)\cdot \sup I_B(x_i) + \inf F_A(x_i)\cdot \inf F_B(x_i) + \sup F_A(x_i)\cdot \sup F_B(x_i)\} \quad (11)$$

Let us notice that this formula coincides with that given by A. A Salama [17] when $\inf T_A(x_i) = \sup T_A(x_i)$, $\inf F_A(x_i) = \sup F_A(x_i)$, $\inf I_A(x_i) = \sup I_A(x_i)$ and $\inf T_B(x_i) = \sup T_B(x_i)$, $\inf F_B(x_i) = \sup F_B(x_i)$, $\inf I_B(x_i) = \sup I_B(x_i)$
and the correlation coefficient of the interval neutrosophic sets A and B given by

$$K_{INS}(A,B) = \frac{C_{INS}(A,B)}{(E(A)\cdot E(B))^{1/2}} \in [0, 1^+[\quad (12)$$

where

$$E(A)=\sum_1^n[T_{AL}{}^2(x_i) + T_{AU}{}^2(x_i) + I_{AL}{}^2(x_i) + I_{AU}{}^2(x_i) + F_{AL}{}^2(x_i) + F_{AU}{}^2(x_i)] \quad (13)$$

$$E(B) = \sum_1^n[T_{BL}{}^2(x_i) + T_{BU}{}^2(x_i) + I_{BL}{}^2(x_i) + I_{BU}{}^2(x_i) + F_{BL}{}^2(x_i) + F_{BU}{}^2(x_i)] \quad (14)$$

express the so-called informational energy of the interval neutrosophic sets A and B respectively.

Remark 2: For the sake of simplicity we shall use the symbols:

$\inf T_A(x_i) = T_{AL}$, $\sup T_A(x_i) = T_{AU}$, (15)
$\inf T_B(x_i) = T_{BL}$, $\sup T_B(x_i) = T_{BU}$, (16)
$\inf I_A(x_i) = I_{AL}$,$\sup I_A(x_i) = I_{AU}$, (17)
$\inf I_B(x_i) = I_{BL}$, $\sup I_B(x_i) = I_{BU}$, (18)
$\inf F_A(x_i) = F_{AL}$, $\sup F_A(x_i) = F_{AU}$, (19)
$\inf F_B(x_i) = F_{BL}$, $\sup F_B(x_i) = F_{BU}$, (20)

For the correlation of interval neutrosophic set, the following proposition is immediate from the definition.

3.2. Proposition

For A, B $\in$ $INSs$ in the universe of discourse X={$x_1,x_2,x_3,\ldots,x_n$} the correlation of interval neutrosophic set have the following properties:

(1) $C_{INS}(A,A) = E(A)$ (21)

(2) $C_{INS}(A,B) = C_{INS}(B,A)$ (22)

3.3. Theorem. For all INSs A, B the correlation coefficient satisfies the following properties:

(3) If $A = B$ *, then* $K_{INS}(A,B) = 1$. (23)

(4) $K_{INS}(A,B) = K_{INS}(B,A)$. (24)

(5) $0 \le K_{INS}(A,B) \le 1$. (25)

Proof. Conditions (1) and (2) are evident; we shall prove condition (3). $K_{INS}(A,B) \ge 0$ is evident. We will prove that $K_{INS}(A,B) \le 1$. From the Schwartz inequality, we obtain

$$K_{INS}(A,B) = (\sum_1^n [T_{AL}(x_i)T_{BL}(x_i) + T_{AU}(x_i)T_{BU}(x_i) + I_{AL}(x_i)I_{BL}(x_i) + I_{AU}(x_i)I_{BU}(x_i) + F_{AL}(x_i)F_{BL}(x_i) + F_{AU}(x_i)F_{BU}(x_i)])$$

$$\left(\sum_1^n T_{AL}{}^2(x_i) + T_{AU}{}^2(x_i) + I_{AL}{}^2(x_i) + I_{AU}{}^2(x_i) + F_{AL}{}^2(x_i) + F_{AU}{}^2(x_i)\right)^{\frac{-1}{2}} \cdot \left(\sum_1^n T_{BL}{}^2(x_i) + T_{BU}{}^2(x_i) + I_{BL}{}^2(x_i) + I_{BU}{}^2(x_i) + F_{BL}{}^2(x_i) + F_{BU}{}^2(x_i)\right)^{\frac{-1}{2}} \le \Big\{(\sum_{i=1}^n T_{AL}{}^2(x_i) \sum_{i=1}^n T_{BL}{}^2(x_i))^{\frac{1}{2}} + (\sum_{i=1}^n T_{AU}{}^2(x_i) \sum_{i=1}^n T_{BU}{}^2(x_i))^{\frac{1}{2}} + (\sum_{i=1}^n I_{AL}{}^2(x_i) \sum_{i=1}^n I_{BL}{}^2(x_i))^{\frac{1}{2}} + (\sum_{i=1}^n I_{AU}{}^2(x_i) \sum_{i=1}^n I_{BU}{}^2(x_i))^{\frac{1}{2}} + (\sum_{i=1}^n F_{AL}{}^2(x_i) \sum_{i=1}^n F_{BL}{}^2(x_i))^{\frac{1}{2}} + (\sum_{i=1}^n F_{AU}{}^2(x_i) \sum_{i=1}^n F_{BU}{}^2(x_i))^{\frac{1}{2}}\Big\} \cdot \{(\sum_{i=1}^n \boldsymbol{T}_{AL}{}^2(\boldsymbol{x}_i) + \sum_{i=1}^n \boldsymbol{T}_{AU}{}^2(\boldsymbol{x}_i) + \sum_{i=1}^n \boldsymbol{I}_{AL}{}^2(\boldsymbol{x}_i) + \sum_{i=1}^n \boldsymbol{I}_{AU}{}^2(\boldsymbol{x}_i) + \sum_{i=1}^n \boldsymbol{F}_{AL}{}^2(\boldsymbol{x}_i) + \sum_{i=1}^n \boldsymbol{F}_{AU}{}^2(\boldsymbol{x}_i)) \cdot (\sum_{i=1}^n \boldsymbol{T}_{BL}{}^2(\boldsymbol{x}_i) +$$

$$\times \quad \sum_{i=1}^n \boldsymbol{T}_{BU}{}^2(\boldsymbol{x}_i) + \sum_{i=1}^n \boldsymbol{I}_{BL}{}^2(\boldsymbol{x}_i) + \sum_{i=1}^n \boldsymbol{I}_{BU}{}^2(\boldsymbol{x}_i) + \sum_{i=1}^n \boldsymbol{F}_{BL}{}^2(\boldsymbol{x}_i) + \sum_{i=1}^n \boldsymbol{F}_{BU}{}^2(\boldsymbol{x}_i))\}^{\frac{-1}{2}} \qquad (26)$$

Let us adopt the following notations:

$$\sum_{i=1}^n T_{AL}{}^2(x_i) = a \quad \sum_{i=1}^n T_{BL}{}^2(x_i) = b \quad \sum_{i=1}^n T_{AU}{}^2(x_i) = c \quad \sum_{i=1}^n T_{BU}{}^2(x_i) = d \qquad (27)$$

$$\sum_{i=1}^n I_{AL}{}^2(x_i) = e \quad \sum_{i=1}^n I_{BL}{}^2(x_i) = f \quad \sum_{i=1}^n I_{AU}{}^2(x_i) = g \quad \sum_{i=1}^n I_{BU}{}^2(x_i) = h \qquad (28)$$

$$\sum_{i=1}^n F_{AL}{}^2(x_i) = i \quad \sum_{i=1}^n F_{BL}{}^2(x_i) = j \quad \sum_{i=1}^n F_{AU}{}^2(x_i) = k \quad \sum_{i=1}^n F_{BU}{}^2(x_i) = l \qquad (29)$$

The above inequality is equivalent to

$$K_{INS}(A,B) \le \frac{\sqrt{ab}+\sqrt{cd}+\sqrt{ef}+\sqrt{gh}+\sqrt{ij}+\sqrt{kl}}{\sqrt{(a+c+e+g+i+k)}\sqrt{b+d+f+h+j+l}} \qquad (30)$$

Then, since $K_{INS}(A,B) \ge 0$ we have

$$K_{INS}^2(A,B) \le \frac{\left(\sqrt{ab}+\sqrt{cd}+\sqrt{ef}+\sqrt{gh}+\sqrt{ij}+\sqrt{kl}\right)^2}{(a+c+e+g+i+k)(b+d+f+h+j+l)}$$

=1-

$$\left\{\left(\sqrt{ad}-\sqrt{bc}\right)^2+\left(\sqrt{af}-\sqrt{be}\right)^2+\left(\sqrt{ah}-\sqrt{bg}\right)^2+\left(\sqrt{aj}-(\sqrt{bi})^2+\sqrt{al}-\sqrt{bk}\right)^2+\right.$$
$$\left(\sqrt{cf}-\sqrt{de}\right)^2+\left(\sqrt{ch}-\sqrt{dg}\right)^2+\left(\sqrt{cj}-\sqrt{di}\right)^2+\left(\sqrt{cl}-(\sqrt{dk})^2+\sqrt{eh}-\sqrt{fg}\right)^2+$$
$$\left.\left(\sqrt{ej}-\sqrt{fi}\right)^2+\left(\sqrt{el}-\sqrt{fk}\right)^2+\left(\sqrt{gi}-\sqrt{fk}\right)^2+\left(\sqrt{gl}-(\sqrt{hk})^2+\sqrt{il}-\sqrt{jk}\right)^2\right\}\times$$
$$\{(a+c+e+g+i+k)(b+d+h+j+l)\}^{-1}\leq$$
1. (31)

And thus we have $0 \leq K_{INS}(A,B) \leq 1.$ (32)

Remark 3: From the following counter-example, we can easily check that

$K_{INS}(A,B) = 1$ but A $\neq B.$ (33)

Remark 4:

Let A and B be two interval neutrosophic set defined on the universe X = {x_1}

A={ x_1: < [0.5, 0.5] [0.5, 0.5] [0.5, 0.5]>}

B={ x_1: < [0.25, 0.25] [0.25, 0.25] [0.25, 0.25]>}

$K_{INS}(A,B) = 1$ but A$\neq B.$

3.4. Weighted Correlation Coefficient of Interval Neutrosophic Sets

In order to investigate the difference of importance considered in the elements in the universe of discourse, we need to take the weights of the elements x_i(i = 1, 2, 3, …, n). In the following we develop a weighted correlation coefficient between the interval neutrosophic sets as follows:

$$W_{INS}(A,B) = \frac{\sum_1^n w_i\{T_{AL}(x_i)\cdot T_{BL}(x_i)+T_{AU}(x_i)\cdot T_{BU}(x_i)+I_{AL}(x_i)\cdot I_{BL}(x_i)+I_{AU}(x_i)\cdot I_{BU}(x_i)+F_{AL}(x_i)\cdot F_{BL}(x_i)+F_{AU}(x_i)\cdot F_{BU}(x_i)\}}{(\sum_1^n w_i\,(T_{AL}{}^2(x_i)+T_{AU}{}^2(x_i)+I_{AL}{}^2(x_i)+I_{AU}{}^2(x_i)+F_{AL}{}^2(x_i)+F_{AU}{}^2(x_i))\cdot\sum_1^n w_i\,(T_{BL}{}^2(x_i)+T_{BU}{}^2(x_i)+I_{BL}{}^2(x_i)+I_{BU}{}^2(x_i)+F_{BL}{}^2(x_i)+F_{BU}{}^2(x_i)))^{1/2}} \in$$
$[0, 1^+[$ (34)

If w = $\{\frac{1}{n},\frac{1}{n},\dots,\frac{1}{n}\}$, equation (34) is reduced to the correlation coefficient (12); it is easy to check that the weighted correlation coefficient $w_{INS}(A,B)$ between INSs A and B also satisfies the properties:

(1) $0 \leq w_{INS}(A,B) \leq 1$ (35)

(2) $w_{INS}(A,B) = w_{INS}(B,A)$ (36)

(3) $w_{INS}(A,B) = 1$ if A = B (37)

3.5. Numerical Illustration.

In this section we present, an example to depict the method defined above, where the data is represented by an interval neutrosophic sets.

Example. For a finite universal set X = {x_1, x_2}, if two interval neutrosophic sets are written, respectively

A = {x_1:<[0.2, 0.3] [0.4, 0.5] [0.1, 0.2]>; x_2:<[0.3, 0.5] [0.1, 0.2] [0.4, 0.5]>}

B = {x_1:<[0.1, 0.2] [0.3, 0.4] [0.1, 0.3]>; x_2:<[0.4, 0.5] [0.2, 0.3] [0.1, 0.2]>}

Therefore, we have

$$K_{INS}(A, B) = \frac{1{,}06}{(1{,}39)^{\frac{1}{2}}(0{,}99)^{1/2}} \in [0, 1^{+}[$$

E(A) = 1,39

E(B) = 0,99

$$K_{INS}(A, B) = 0.90$$

It shows that the interval neutrosophic sets A and B have a good positively correlation.

Conclusion:
In this paper we introduced a method to calculate the correlation coefficient of two interval neutrosophic sets.

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Neutrosophic Modeling of Investment Architectures

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Abstract. Despite numerous attempts to refocus traditional bivalent logic, usually, logic is tributary by keeping fixed limits for truth or false. An alternative answer to this need is the system of Neutrosophy, and its connected logic, Neutrosophic Logic, presented in the work of the authors W. B. Vasantha Kandasamy and Florentin Smarandache. Neutrosophy is a new branch of philosophy that studies the origin, nature and scope of neutralities, as well as their interactions with different ideational spectra. We are surrounded by indeterminacy, which can be pointed in any situation that, from one point of view can be considered true, and from another point of view can be considered false. In Neutrosophic logic, every statement includes a percentage of truth (T), includes a percentage of indeterminacy (I) and a percentage of falsity (F). The major novelty of the work is that investment process is analyzed through the method of Neutrosophic logic, taking into account investment parameters such as: economic return on investment, recovery duration on investment, the resources involved (financial, human and time), the production capacity, the period of execution, the work volume, the market demand, the training of the personnel, etc. Thus, this paper consists of the analysis of interdependence and indeterminacy of these parameters. The model will be calibrated by conducting case studies, with real parameters of input to be analyzed and known output data, which will be used for the research of a present investment process.

Keywords: neutrosophic logic, investment, interdeterminancy, cognitive map.

Introduction

Evaluating quantitative and qualitative aspects of the economic process is an ensemble of instruments and methods which allow a better knowledge and assessment about the financial position and organizational performance. Due to an acceleration of the economic globalization in the last few years, the existing logics with its economic analysis and financial diagnosis have now an alternative to give solutions for elaborating strategies for improving the external relation of the organization (demand, offer, competition, resources), and also the internal strategy of the organization (production, inputs, commercial activity, organizational strategy, decisional system, organizational structure and informational system). This different method is the *Neutrosophic Logic*, as an alternative to the existing logic, which consists of a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, contradiction. In particular, the method solves problems regarding taking decisions about project selection, preparing negotiations for merging of two or more companies, appraisal of the company's position in the group, planning of policies and strategies for long-term development, all of these concerning also financial analysis, strategic analysis and risk analysis.

Reliability, accuracy and availability of data can be a problem for existing organizations or organizations that are moving to new business industry, in fields where innovation is applied, and the lack of information, database and experience is evident. In Neutrosophic logic this is called *indeterminacy* (I). The importance of the decision making needs to be highlighted here, and the expert's opinion is of great importance for the best result to be obtained. This is one of the most important advantages of this method's usage, the *Fuzzy Neutrosophic Matrices* method. This method allows several experts opinion to be used in order to determine the best solution, or option for the problem or innovation that needs to be fixed or implemented.

About the Fuzzy Neutrosophic Matrices Method

A fuzzy neutrosophic matrix can be defined as a matrix which consist in elements from [0,1] and [0,I]. The matrix can consist in elements included in $Z \cup I$ or $R \cup I$ or $Q \cup I$ or $C \cup I$ or $Z_n \cup I$ or $C(Z_n) \cup I$. The indeterminate I has its fundamental application in $I^2 = I$. If the indetermination I is not an element of the matrix, than the matrix is not considered a neutrosophic one. This matrix includes both the fuzzy matrix and the real matrix.

This method represents a generalization of Aristotle classical logic, Lukasiewicz's three-valued logic and Zadeh's fuzzy logic [1]. It allows the performance of revealing studies about the dynamic of the investigated events, events which are not observable by applying bivalent logics or Fuzzy Logic, nor statistical studies, the difference being the existence of the indeterminate I. Neutrosophic Logic is an alternative to the existing logic, and consists of a mathematical model of uncertainty, vagueness, ambiguity, imprecision, undefined, unknown, incompleteness, inconsistency, redundancy, contradiction [2]. Even so, logics work with multiple levels of truth, without allowing the presence of other realities, or states of reality. This is why, in a world that reached saturation regarding traditional logic, scientists have aimed a different system, which could encapsulate a representation of the real world.

Table 1. The causality that can connect two nodes

Value	Consequence	
	Node N_1	Node N_2
0	N_1 – no effect	N_2 – no effect
1	$N_1 - \uparrow$ $\Rightarrow$	$N_2 - \uparrow$
	or	
	$N_1 - \downarrow$ $\Rightarrow$	$N_2 - \downarrow$
-1	$N_1 - \uparrow$ $\Rightarrow$	$N_2 - \downarrow$
	or	
	$N_1 - \downarrow$ $\Rightarrow$	$N_2 - \uparrow$
I	N_1 –indeterminate $\Rightarrow$ N_2-indeterminate	

The expert's opinion is very important in evaluating the values of the neutrosiphics matrix elements. If he/she considers finding even the smallest degree of indeterminacy, than he/she will choose I to the real values. If the expert wishes to disregard the small degrees of indeterminacy, than the real values will be chosen to the detriment of the indeterminate I.

The Neutrosophic Matrices method can be transformed in *Neutrosophic Cognitive Maps*, which are neutrosophic directed graphs that can transpose concepts like processes, events, economic policies, as nodes, and causalities and indeterminate as edges [3]. In this directed graph, which is associated with values for each relation, we can see a representation of the causal relationship between concepts. The nodes of the graph are noted with N_i, and they represent the neutrosophic vector from the neutrosophic vector space V. Each node can have values from the set {-1, 0, 1, I} where the value 1 means that the node is in the on state, I implies that the node is in the indeterminate state at that time, and the value 0 if the node is in the of state, that it has no influence on that node. The causality, as can be given by an expert in the field, which can connect two nodes, is detailed in Tab. 1.

After the construction of the directed neutrosophic graph, the matrix associated with it is written, resulting thereby the adjacency matrix of the neutrosophic cognitive map.

The next step is to threshold and update the state vector Y, which is known as resultant vector. „Updating means keeping the on state of the given vector to remain in the on state. The thresholding is, in the resultant every negative value is made as 0 and positive value is made as 1 and positive coefficient indeterminate value is made as I. The combination of 1 + I is made into I or 1 according to the wishes of the expert depending on the problem.“ [2] The result is interpreted by the specialists in order to improve or solve the problem under discussion.

Investments

Investments represent capital lock-ups. The investors decide to restrain themselves from present consumption, in order to create the possibility of obtaining future benefits. A production system consists in all the natural components and artificial ones like raw materials, energy, tools, devices, technological equipment, buildings labor and relations of production, concepts, work organization and management of manufacturing, aiming to obtain products and services that could be sold [4].

These components must be organized so as to fulfill the main objective of any economic activity: obtaining profit and raising the value of the company. The value of the company is given by the net asset value reflected in the balance sheet, and it can be determined as the net difference between the total assets of the company (fixed and current assets) and the liabilities of the company at a time.

The decision to invest in one company can be difficult and risk involving in the same time. When facing several potential investment possibilities, the analysis of all the involved elements must be a very rigorous one. In order to maximize the value of a portfolio, it is advisable to consult the opinion of many specialists, to select the most opportune balance and mix of projects. The analysis must take into consideration the limited availability of the cash budget, the scarce of the resources, the plough back of the profit, the short-term and long-term development strategy of the organization, the organizational strategies, but also uncertainty and strategic importance.

Table 2. Atlantic Global (2007) categorized projects based on the competitive advantage [5]

Competitive advantage	Explication
Tactical	Deliver competitive advantage today
Strategic	Deliver competitive advantage in the future
Administrative	Deliver concurrently promised service levels and supporting existing strategic projects
Innovation	Smaller and experimental projects, delivering possible competitive advantage tomorrow
Future vision	Contingent upon strategic and innovation projects

The investment can be developed considering the managerial dimension, which can validate both the idea and the financial implementation, the organizational dimension, through reorganization and assuming the risk, the operational dimension regarding project functioning and the strategic dimension by taking in consideration the permanent adaptation on the continuing changing market.

Application

We now study the investment process and analyze it through the method of Neutrosophic logic, taking into account the investment parameters as discussed earlier, and customizing in the situation of a company that exists on the market and is interested in finding the experts opinion regarding reinvestment of profit. The method allows specialists, more freedom of intuition in order to express not only positive, negative and without impact, but also, the indeterminacy of the impact.

We take the following nodes to be related with the company that considers profit reinvestment:

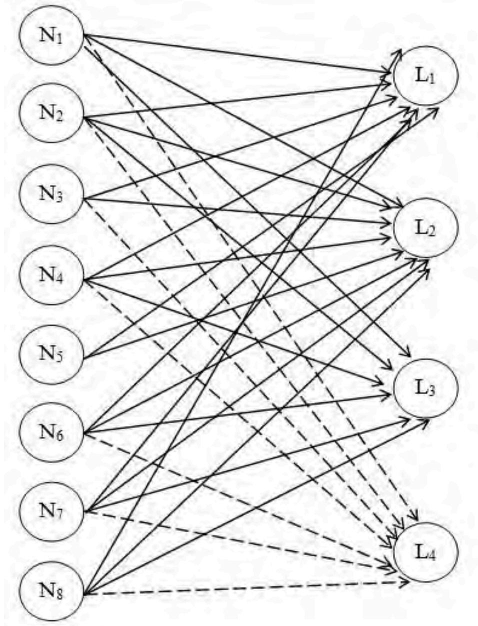
N_1 – Economic return on investment
N_2 – Recovery duration on investment
N_3 – The resources involved (financial, human and time),
N_4 – The production capacity
N_5 – The period of execution
N_6 – The work volume
N_7 – The market demand and N_8 – The training of the personnel

These eight nodes are taken as the domain nodes, as they are the most important aspects for the company's management, when making an investment.

The following four nodes are the range space related with the investment:

L_1 – Good investment; L_2 – Average investment; L_3 – Bad investment; L_4 – Risk involving investment

After consulting a number of five specialists, and they evaluated each factors influence over the investment, the following directed neutrosophic graph with the same set of domain space nodes and range space nodes resulted the graph from Fig. 1. The neutrosophic connection matrix associated with this neutrosophic directed graph is as it is presented in Fig. 1.



$$\Rightarrow \quad N(E) = \begin{array}{c} \\ N_1 \\ N_2 \\ N_3 \\ N_4 \\ N_5 \\ N_6 \\ N_7 \\ N_8 \end{array} \begin{array}{c} \begin{array}{cccc} L_1 & L_2 & L_3 & L_4 \end{array} \\ \begin{bmatrix} 0{,}6 & 0{,}4 & 0{,}1 & 0{,}2I \\ 0{,}2 & 0{,}4 & 0{,}5 & 0{,}7I \\ 0{,}7 & 0{,}3 & 0 & 0{,}8I \\ 0{,}1 & 0{,}2 & 0{,}7 & 0{,}9I \\ 0{,}9 & 0{,}1 & 0 & 0 \\ 0{,}7 & 0{,}6 & 0{,}1 & 0{,}1I \\ 1 & 0{,}8 & 0{,}2 & 0{,}1I \\ 0{,}7 & 0{,}6 & 0{,}3 & 0{,}3I \end{bmatrix} \end{array}$$

Fig. 1. The directed neutrosophic graph and the matrix associated with it

Each specialist evaluated each connection between a domain node and a range space related node with a value, according to his opinion. Equal importance is given to each expert's opinion. The value could consist in elements from [0,1] and [0,I]. After each specialist evaluated all aspects taken into consideration, the arithmetic mean of all five specialists conducted to the neutrosophic connection matrix associated with this neutrosophic directed graph. The straight line arrow indicates that the specialists considered the relation between the nodes as a measurable one, assigning values within the range [0,1], and the dotted arrow indicates that the specialists considered the influence as an indeterminate one (I), assigning each connection values within the range [0,I].

The investor is interested in studying the impact of the state vector A_1 = (1 0 0 0 0 0 0 0) that is the economic return on the investment which is in the on state and all the other nodes are in the off state. When taking the final decision, this is considered to be the most significant factor.

To find the effect of A_1 on the neutrosophic dynamical system of investments N(E), we must perform the operation of multiplication between the vector A_1 and the neutrosophic matrix:

$A_1N(E) = (\ 0.6\ \ 0.4\ \ 0.1\ \ 0.2I\ ) = B_1.$ (1)

$B_1(N(E))^T = (\ 0.53+0.04I\ \ 0.33+0.14I\ \ 0.54+0.16I\ \ 0.21+0.18I\ \ 0.58\ \ 0.67+0.02I\ \ 0.94+0.02I\ \ 0.69+0.06I).$ (2)

$N(E)^T$ is the transposed matrix N(E). The next step is to update and threshold ($\hookrightarrow$), by making every negative value as 0 and positive value as 1 and positive coefficient indeterminate value as I:

$B_1(N(E))^T \hookrightarrow (1\ I\ I\ I\ 0\ 1\ 1\ 1) = A_2\,.$ (3)

$A_2N(E) = (3+I\ \ 2.4+0.9I\ \ 0.7+1.2I\ \ 3.1I).$ (4)

$\hookrightarrow (1\ 1\ I\ I) = B_2.$ (5)

$B_2N(E)^T = (1+0.3I\ \ 0.6+1.2I\ \ 1+0.8I\ \ 0.3+1.6I\ \ 1\ \ 0.13+0.2I\ \ 1.8+0.3I\ \ 1.3+0.6I).$ (6)

$\hookrightarrow (1\ I\ I\ I\ 0\ 1\ 1\ 1) = A_2.$ (7)

The operation of multiplication was performed until a constant result was obtained, in this case four times, applying the same method each time we updated and thresholded the vector.

These operations lead to the hidden pattern of the system, as a fixed point given by {(1 I I I 0 1 1 1) (1 1 I I)}. Since the uncertainty and indeterminacy is involved in these concepts, the usage of a neutrosophic relational map is justified.

The five expert's opinion has conducted to this result:

- the economic return on investment, the work volume, the market demand and the training of the personnel must be of great importance for the investor, and be paid an increased attention, when the aim is obtaining a good investment;
- the period of execution is on the other hand of smaller importance to the investment, as the other investments evaluation parameters mentioned before;
- the recovery duration on investment, the resources involved (financial, human and time) and the production capacity are indeterminate, as this parameters are depending on the investors' interest and willingness to reinvest it's profit;
- the result obtained is interdependent on this evaluation parameters. A bad investment or a risk full one is an indetermination. On the other hand a good or an average investment is based on the fulfillment of the parameters first discussed.

Conclusions

The major novelty of the work is the adaptability of the model and the flexibility of it due to the results that the method can provide. This model can be applied in various spectrums of economic decisions, and can be an alternative or a second opinion for the financial analysis that every company depends on. Another novelty of the work is that investment process is analyzed through the method of Neutrosophic logic, taking into account investment parameters such as: economic return on investment, recovery duration on investment, the resources involved (financial, human and time), the production capacity, the period of execution, the work volume, the market demand, the training of the personnel, etc. Thus, this paper consists of the analysis of interdependence and indeterminacy of these parameters. The model was calibrated by conducting case studies, with real parameters that were used in an investment process.

Future work is various in this method, especially because it is a new method with many application possibilities that have not been used before.

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Neutrosophic Systems and Neutrosophic Dynamic Systems

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Abstract

In this paper, we introduce for the first time the neutrosophic system and neutrosophic dynamic system that represent new per-spectives in science. A neutrosophic system is a quasi- or $(t, i, f)-$classical system, in the sense that the neutrosophic system deals with quasi-terms/concepts/attributes, etc. [or $(t, i, f) -$ terms/ concepts/attributes], which are approximations of the classical terms/concepts/attributes, i.e. they are partially true/membership/probable *(t%),* partially indeterminate *(i%),* and partially false/nonmember-ship/improbable *(f%),* where t, i, f are subsets of the unitary interval [0,1]. {We recall that 'quasi' means relative(ly), approximate(ly), almost, near, partial(ly), etc. or mathematically 'quasi' means (t, i, f) in a neutrophic way.}

Keywords

neutrosophy, neutrosophics, neutrosophic system, neutrosophic patterns, neutrosophic model, neutrosophic synergy, neutrosophic interactions, neutrosophic complexity, neutrosophic process, neutrosophic cognitive science.

1 Introduction

A system $\mathcal{S}$ in general is composed from a space $\mathcal{M}$, together with its elements (concepts) $\{e_j\}, j \in \theta$, and the relationships $\{\mathcal{R}_k\}$, $k \in \psi$, between them, where θ and ψ are countable or uncountable index sets. For a closed system, the space and its elements do not interact with the environment. For an open set, the space or its elements interact with the environment.

2 Definition of the neutrosophic system

A system is called neutrosophic system if at least one of the following occur:

a. The space contains some indeterminacy.
b. At least one of its elements x has some indeterminacy (it is not well-defined or not well-known).

c. At least one of its elements x does not 100% belong to the space; we say $x(t,i,f) \in \mathcal{M}$, with $(t,i,f) \neq (1,0,0)$.
d. At least one of the relationships $\mathcal{R}_o$ between the elements of $\mathcal{M}$ is not 100% well-defined (or well-known); we say $\mathcal{R}_o(t,i,f) \in \mathcal{S}$, with $(t,i,f) \neq (1,0,0)$.
e. For an open system, at least one $[\mathcal{R}_E(t,i,f)]$ of the system's interactions relationships with the environment has some indeterminacy, or it is not well-defined, or not well-known, with $(t,i,f) \neq (1,0,0)$.

2.1 Classical system as particular case of neutrosophic system

By language abuse, a classical system is a neutrosophic system with indeterminacy zero (no indeterminacy) at all system's levels.

2.2 World systems are mostly neutrosophic

In our opinion, most of our world systems are neutrosophic systems, not classical systems, and the dynamicity of the systems is neutrosophic, not classical.

Maybe the mechanical and electronical systems could have a better chance to be classical systems.

3 A simple example of neutrosophic system

Let's consider a university campus Coronado as a whole neutrosophic system $\mathcal{S}$, whose space is a prism having a base the campus land and the altitude such that the prism encloses all campus' buildings, towers, observatories, etc.

The elements of the space are people (administration, faculty, staff, and students) and objects (buildings, vehicles, computers, boards, tables, chairs, etc.).

A part of the campus land is unused. The campus administration has not decided yet what to do with it: either to build a laboratory on it, or to sell it. This is an indeterminate part of the space.

Suppose that a staff (John, from the office of Human Resources) has been fired by the campus director for misconduct. But, according to his co-workers, John was not guilty for anything wrong doing. So, John sues the campus. At this point, we do not know if John belongs to the campus, or not. John's appurtenance to the campus is indeterminate.

Assume the faculty norm of teaching is four courses per semester. But some faculty are part-timers, therefore they teach less number of courses. If an instructor teaches only one class per semester, he belongs to the campus only partially (25%), if he teaches two classes he belongs to the campus 50%, and if he teaches three courses he belongs to the campus 75%.

We may write:

Joe $(0.25, 0, 0.75) \in \mathcal{S}$
George $(0.50, 0, 0.50) \in \mathcal{S}$
and Thom $(0.75, 0.10, 0.25) \in \mathcal{S}$.

Thom has some indeterminacy (0.10) with respect to his work in the campus: it is possible that he might do some administrative work for the campus (but we don't know).

The faculty that are full-time (teaching four courses per semester) may also do overload. Suppose that Laura teaches five courses per semester, therefore Laura $(1.25, 0, 0) \in \mathcal{S}$.

In neutrosophic logic/set/probability it's possible to have the sum of components (t, i, f) different from 1:

$t + i + f > 1$, for paraconsistent (conflicting) information;
$t + i + f = 1$, for complete information;
$t + i + f < 1$, for incomplete information.

Also, there are staff that work only ½ norm for the campus, and many students take fewer classes or more classes than the required full-time norm. Therefore, they belong to the campus Coronado in a percentage different from 100%.

About the objects, suppose that 50 calculators were brought from IBM for one semester only as part of IBM's promotion of their new products. Therefore, these calculators only partially and temporarily belong to the campus.

Thus, not all elements (people or objects) entirely belong to this system, there exist many $e_j(t, i, f) \in \mathcal{S}$, with $(t, i, f) \neq (1, 0, 0)$.

Now, let's take into consideration the relationships. A professor, Frank, may agree with the campus dean with respect to a dean's decision, may disagree with respect to the dean's other decision, or may be ignorant with respect to the dean's various decisions. So, the relationship between Frank and the dean may be, for example:

$$\text{Frank} \xrightarrow{\text{agreement } (0.5, 0.2, 0.3)} \text{dean, i. e. not } (1, 0, 0) \text{ agreement.}$$

This campus, as an open system, cooperates with one Research Laboratory from Nevada, pending some funds allocated by the government to the campus.

Therefore, the relationship (research cooperation) between campus Coronado and the Nevada Research Laboratory is indeterminate at this moment.

4 Neutrosophic patterns

In a neutrosophic system, we may study or discover, in general, neutrosophic patterns, i.e. quasi-patterns, approximated patterns, not totally working; we say: $(t, i, f) -$ patterns, i.e. t% true, i% indeterminate, and f% false, and elucidate (t, i, f) −principles.

The neutrosophic system, through feedback or partial feedback, is (t, i, f) −self-correcting, and (t, i, f) −self-organizing.

5 Neutrosophic holism

From a holistic point of view, the sum of parts of a system may be:

1. Smaller than the whole (when the interactions between parts are unsatisfactory);
2. Equals to the whole (when the interactions between parts are satisfactory);
3. Greater than the whole (when the interactions between parts are super-satisfactory).

The more interactions (interdependance, transdependance, hyper-dependance) between parts, the more complex a system is.

We have positive, neutral, and negative interactions between parts. Actually, an interaction between the parts has a degree of positiveness, degree of neutrality, and degree of negativeness. And these interactions are dynamic, meaning that their degrees of positiveness/neutrality/negativity change in time. They may be partially absolute and partially relative.

6 Neutrosophic model

In order to model such systems, we need a neutrosophic (approximate, partial, incomplete, imperfect) model that would discover the approximate system properties.

7 Neutrosophic successful system

A neutrosophic successful system is a system that is successful with respect to some goals, and partially successful or failing with respect to other goals.

The adaptivity, self-organization, self-reproducing, self-learning, reiteration, recursivity, relationism, complexity and other attributes of a classical system are extended to (t, i, f) −attributes in the neutrosophic system.

8 (t, i, f) −attribute

A (t, i, f) − attribute means an attribute that is t% true (or probable), i% indeterminate (with respect to the true/probable and false/improbable), and f% false/improbable - where t,i,f are subsets of the unitary interval [0,1].

For example, considering the subsets reduced to single numbers, if a neutrosophic system is (0.7, 0.2, 0.3)-adaptable, it means that the system is 70% adaptable, 20% indeterminate regarding adaptability, and 30% inadaptable; we may receive the informations for each attribute phase from different independent sources, that's why the sum of the neutrosophic components is not necessarily 1.

9 Neutrosophic dynamics

While classical dynamics was beset by dialectics, which brought together an entity ⟨A⟩ and its opposite ⟨antiA⟩, the neutrosophic dynamics is beset by tri-alectics, which brings together an entity ⟨A⟩ with its opposite ⟨antiA⟩ and their neutrality ⟨neutA⟩. Instead of duality as in dialectics, we have tri-alities in our world.

Dialectics failed to take into consideration the neutrality between opposites, since the neutrality partially influences both opposites.

Instead of unifying the opposites, the neutrosophic dynamics unifies the triad ⟨A⟩, ⟨antiA⟩, ⟨neutA⟩.

Instead of coupling with continuity as the classical dynamics promise, one has "tripling" with continuity and discontinuity altogether.

All neutrosophic dynamic system's components are interacted in a certain degree, repelling in another degree, and neutral (no interaction) in a different degree.

They comprise the systems whose equilibrium is the disechilibrium - systems that are continuously changing.

The internal structure of the neutrosophic system may increase in complexity and interconnections, or may degrade during the time.

A neutrosophic system is characterized by potential, impotential, and indeterminate developmental outcome, each one of these three in a specific degree.

10 Neutrosophic behavior gradient

In a neutrosophic system, we talk also about neutrosophic structure, which is actually a quasi-structure or structure which manifests into a certain degree; which influences the neutrosophic behavior gradient, that similarly is a behavior quasi-gradient - partially determined by quasi-stimulative effects; one has: discrete systems, continuous systems, hybrid (discrete and continuous) systems.

11 Neutrosophic interactions

Neutrosophic interactions in the system have the form:

$$A \;\xleftarrow[(t,\iota,f)]{\qquad\qquad}\; B$$

Neutrosophic self-organization is a quasi-self-organization. The system's neutrosophic intelligence sets into the neutrosophic patterns formed within the system's elements.

We have a neutrosophic causality between event E_1, that triggers event E_2, and so on. And similarly, neutrosophic structure S_1 (which is an approximate, not clearly know structure) causes the system to turn on neutrosophic structure S_2, and so on. A neutrosophic system has different levels of self-organizations.

12 Potentiality/impotentiality/indeterminacy

Each neutrosophic system has a potentiality/impotentiality/indeterminacy to attain a certain state/stage; we mostly mention herein about the transition from a quasi-pattern to another quasi-pattern. A neutrosophic open system is always transacting with the environment; since always the change is needed.

A neutrosophic system is always oscilating between stability, instability, and ambiguity (indeterminacy). Analysis, synthesis, and neutrosynthesis of existing data are done by the neutrosophic system. They are based on system's principles, antiprinciples, and nonprinciples.

13 Neutrosophic synergy

The Neutrosophic Synergy is referred to partially joined work or partially combined forces, since the participating forces may cooperate in a degree (t), may be antagonist in another degree (f), and may have a neutral interest in joint work in a different degree (i).

14 Neutrosophic complexity

The neutrosophic complex systems produce neutrosophic complex patterns. These patterns result according to the neutrosophic relationships among system's parts. They are well described by the neutrosophic cognitive maps (NCM), neutrosophic relational maps (NRM), and neutrosophic relational equations (NRE), all introduced by W. B. Vasanttha Kandasamy and F. Smarandache in 2003-2004.

The neutrosophic systems represent a new perspective in science. They deal with quasi-terms [or (t, i, f) −terms], quasi-concepts [or (t, i, f) −concepts], and quasi-attributes [or (t, i, f) −attributes], which are approximations of the terms, concepts, attributes, etc., i.e. they are partially true (t%), partially indeterminate (i%), and partially false (f%).

Alike in neutrosophy where there are interactions between ⟨A⟩, ⟨neutA⟩, and ⟨antiA⟩, where ⟨A⟩ is an entity, a system is frequently in one of these general states: equilibrium, indeterminacy (neither equilibrium, nor disequilibrium), and disequilibrium.

They form a neutrosophic complexity with neutrosophically ordered patterns. A neutrosophic order is a quasi or approximate order, which is described by a neutrosophic formalism.

The parts all together are partially homogeneous, partially heterogeneous, and they may combine in finitely and infinitely ways.

15 Neutrosophic processes

The neutrosophic patterns formed are also dynamic, changing in time and space. They are similar, dissimilar, and indeterminate (unknown, hidden, vague, incomplete) processes among the parts.

They are called neutrosophic processes.

16 Neutrosophic system behavior

The neutrosophic system's functionality and behavior are, therefore, coherent, incoherent, and imprevisible (indeterminate). It moves, at a given level, from a neutrosophic simplicity to a neutrosophic complexity, which becomes neutrosophic simplicity at the next level. And so on.

Ambiguity (indeterminacy) at a level propagates at the next level.

17 Classical systems

Although the biologist Bertalanffy is considered the father of general system theory since 1940, it has been found out that the conceptual portion of the system theory was published by Alexander Bogdanov between 1912-1917 in his three volumes of *Tectology*.

18 Classical open systems

A classical open system, in general, cannot be totally deterministic, if the environment is not totally deterministic itself.

Change in energy or in momentum makes a classical system to move from thermodynamic equilibrium to nonequilibrium or reciprocally.

Open classical systems, by infusion of outside energy, may get an unexpected spontaneous structure.

19 Deneutrosophication

In a neutrosophic system, besides the degrees of freedom, one also talk about the degree (grade) of indeterminacy. Indeterminacy can be described by a variable.

Surely, the degrees of freedom should be condensed, and the indetermination reduced (the last action is called "deneutrosophication").

The neutrosophic system has a multi-indeterminate behavior. A neutrosophic operator of many variables, including the variable representing indeterminacy, can approximate and semi-predict the system's behavior.

20 From classical to neutrosophic systems

Of course, in a bigger or more degree, one can consider the neutrosophic cybernetic system (quasi or approximate control mechanism, quasi information processing, and quasi information reaction), and similarly the neutrosophic chaos theory, neutrosophic catastrophe theory, or neutrosophic complexity theory.

In general, when passing from a classical system $\mathcal{S}_c$ in a given field of knowledge $\mathcal{F}$ to a corresponding neutrosophic system $\mathcal{S}_N$ in the same field of knowledge $\mathcal{F}$, one relaxes the restrictions about the system's space, elements, and relationships, i.e. these components of the system (space, elements, relationships) may contain indeterminacy, may be partially (or totally) unknown (or vague, incomplete, contradictory), may only partially belong to the system; they are approximate, quasi.

Scientifically, we write:

$$\mathcal{S}_N = (t, i, f) - \mathcal{S}_c,$$

and we read: *a neutrosophic system is a* (t, i, f)–*classical system*. As mapping, between the neutrosophic algebraic structure systems, we have defined neutrosophic isomorphism.

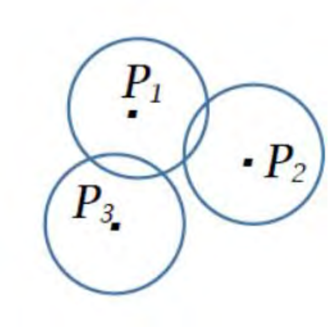
21 Neutrosophic dynamic system

The behavior of a neutrosophic dynamic system is chaotic from a classical point of view. Instead of fixed points, as in classical dynamic systems, one deals with fixed regions (i.e. neighbourhoods of fixed points), as approximate values of the neutrosophic variables [we recall that a neutrosophic variable is, in general, represented by a thick curve – alike a neutrosophic (thick) function].

There may be several fixed regions that are attractive regions in the sense that the neutrosophic system converges towards these regions if it starts out in a nearby neutrosophic state.

And similarly, instead of periodic points, as in classical dynamic systems, one has periodic regions, which are neutrosophic states where the neutrosophic system repeats from time to time.

If two or more periodic regions are non-disjoint (as in a classical dynamic system, where the fixed points lie in the system space too close to each other, such that their corresponding neighbourhoods intersect), one gets double periodic region, triple periodic region:



and so on: n −uple periodic region, for $n \geq 2$.

In a simple/double/triple/…/ $n-$ uple periodic region the neutrosophic system is fluctuating/oscilating from a point to another point.

The smaller is a fixed region, the better is the accuracy.

22 Neutrosophic cognitive science

In the Neutrosophic Cognitive Science, the Indeterminacy “I” led to the definition of the Neutrosophic Graphs (graphs which have: either at least one indeterminate edge, or at least one indeterminate vertex, or both some indeterminate edge and some indeterminate vertex), and Neutrosophic Trees (trees which have: either at least one indeterminate edge, or at least one indeterminate vertex, or both some indeterminate edge and some indeterminate vertex), that have many applications in social sciences.

Another type of neutrosophic graph is when at least one edge has a neutrosophic (t, i, f) truth-value.

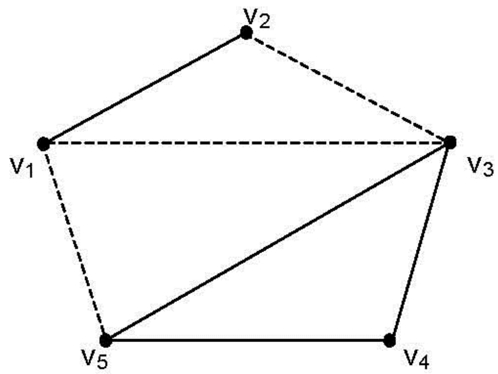
As a consequence, the Neutrosophic Cognitive Maps (Vasantha & Smarandache, 2003) and Neutrosophic Relational Maps (Vasantha & Smarandache, 2004) are generalizations of fuzzy cognitive maps and respectively fuzzy relational maps, Neutrosophic Relational Equations (Vasantha & Smarandache, 2004), Neutrosophic Relational Data (Wang, Smarandache, Sunderraman, Rogatko - 2008), etc.

A Neutrosophic Cognitive Map is a neutrosophic directed graph with concepts like policies, events etc. as vertices, and causalities or indeterminates as edges. It represents the causal relationship between concepts.

An edge is said indeterminate if we don’t know if it is any relationship between the vertices it connects, or for a directed graph we don’t know if it is a directly or inversely proportional relationship. We may write for such edge that (t, i, f) = (0,1,0).

A vertex is indeterminate if we don’t know what kind of vertex it is since we have incomplete information. We may write for such vertex that (t, i, f) = (0,1,0).

Example of Neutrosophic Graph (edges V_1V_3, V_1V_5, V_2V_3 are indeterminate and they are drawn as dotted):



and its neutrosophic adjacency matrix is:

$$\begin{bmatrix} 0 & 1 & I & 0 & I \\ 1 & 0 & I & 0 & 0 \\ I & I & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ I & 0 & 1 & 1 & 0 \end{bmatrix}$$

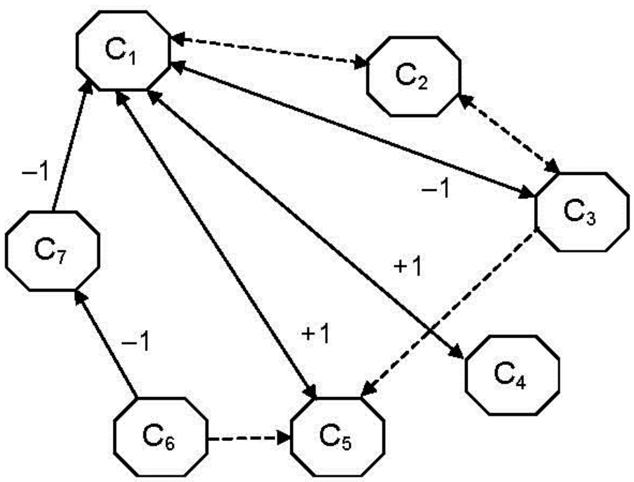
The edges mean: 0 = no connection between vertices, 1 = connection between vertices, I = indeterminate connection (not known if it is, or if it is not).

Such notions are not used in the fuzzy theory.

Let's give an example of Neutrosophic Cognitive Map (NCM), which is a generalization of the Fuzzy Cognitive Maps.

We take the following vertices:

C1 - Child Labor
C2 - Political Leaders
C3 - Good Teachers
C4 - Poverty
C5 - Industrialists
C6 - Public practicing/encouraging Child Labor
C7 - Good Non-Governmental Organizations (NGOs)



The corresponding neutrosophic adjacency matrix related to this neutrosophic cognitive map is:

$$\begin{bmatrix} 0 & I & -1 & 1 & 1 & 0 & 0 \\ I & 0 & I & 0 & 0 & 0 & 0 \\ -1 & I & 0 & 0 & I & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The edges mean: 0 = no connection between vertices, 1 = directly proportional connection, -1 = inversely proportionally connection, and I = indeterminate connection (not knowing what kind of relationship is between the vertices that the edge connects).

Now, we give another type of neutrosophic graphs (and trees): An edge of a graph, let's say from A to B (i.e. how A influences B),

may have a neutrosophic value (t, i, f), where t means the positive influence of A on B, i means the indeterminate/neutral influence of A on B, and f means the negative influence of A on B.

Then, if we have, let's say: $A -> B -> C$ such that $A -> B$ has the neutrosophic value (t_1, i_1, f_1) and $B -> C$ has the neutrosophic value (t_2, i_2, f_2), then $A -> C$ has the neutrosophic value $(t_1, i_1, f_1)/\backslash(t_2, i_2. f_2)$, where $/\backslash$ is the AND_N neutrosophic operator.

Also, again a different type of graph: we can consider a vertex A as: t% belonging/membership to the graph, i% indeterminate membership to the graph, and f% nonmembership to the graph.

Finally, one may consider any of the previous types of graphs (or trees) put together.

23 (t, i, f) –qualitative behavior

We normally study in a neutrosophic dynamic system its long-term (t, i, f) –qualitative behavior, i.e. degree of behavior's good quality *(t)*, degree of behavior's indeterminate (unclear) quality *(i)*, and degree of behavior's bad quality *(f)*.

The questions arise: will the neutrosophic system fluctuate in a fixed region (considered as a neutrosophic steady state of the system)? Will the fluctuation be smooth or sharp? Will the fixed region be large (hence less accuracy) or small (hence bigger accuracy)? How many periodic regions does the neutrosophic system has? Do any of them intersect [i.e. does the neutrosophic system has some n −uple periodic regions (for $n \geq 2$), and for how many]?

24 Neutrosophic state

The more indeterminacy a neutrosophic system has, the more chaotic it is from the classical point of view. A neutrosophic lineal dynamic system still has a degree of chaotic behavior. A collection of numerical sets determines a neutrosophic state, while a classical state is determined by a collection of numbers.

25 Neutrosophic evolution rule

The neutrosophic evolution rule decribes the set of neutrosophic states where the future state (that follows from a given current state) belongs to. If the set of neutrosophic states, that the next neutrosophic state will be in, is known, we have a quasi-deterministic neutrosophic evolution rule, otherwise the neutrosophic evolution rule is called quasi-stochastic.

26 Neutrosophic chaos

As an alternative to the classical Chaos Theory, we have the Neutrosophic Chaos Theory, which is highly sensitive to indeterminacy; we mean that small change in the neutrosophic system's initial indeterminacy produces huge perturbations of the neutrosophic system's behavior.

27 Time quasi-delays and quasi-feedback thick-loops

Similarly, the difficulties in modelling and simulating a Neutrosophic Complex System (also called Science of Neutrosophic Complexity) reside in its degree of indeterminacy at each system's level.

In order to understand the Neutrosophic System Dynamics, one studies the system's time quasi-delays and internal quasi-feedback thick-loops (that are similar to thick functions ad thick curves defined in the neutrosophic precalculus and neutrosophic calculus).

The system may oscillate from linearity to nonlinearity, depending on the neutrosophic time function.

28 Semi-open semi-closed system

Almost all systems are open (exchanging energy with the environment). But, in theory and in laboratory, one may consider closed systems (completely isolated from the environment); such systems can oscillate between closed and open (when they are cut from the environment, or put back in contact with the environment respectively). Therefore, between open systems and closed systems, there also is a semi-open semi-closed system.

29 Neutrosophic system's development

The system's self-learning, self-adapting, self-conscienting, self-developing are parts of the system's dynamicity and the way it moves from a state to another state – as a response to the system internal or external conditions. They are constituents of system's behavior.

The more developed is a neutrosophic system, the more complex it becomes. System's development depends on the internal and external interactions (relationships) as well.

Alike classical systems, the neutrosophic system shifts from a quasi-developmental level to another. Inherent fluctuations are characteristic to neutrosophic complex systems. Around the quasi-steady states, the fluctuations in a neutrosophic system becomes its sources of new quasi-development and quasi-behavior.

In general, a neutrosophic system shows a nonlinear response to its initial conditions. The environment of a neutrosophic system may also be neutrosophic (i.e. having some indeterminacy).

30 Dynamic dimensions of neutrosophic systems

There may be neutrosophic systems whose spaces have dynamic dimensions, i.e. their dimensions change upon the time variable.

Neutrosophic Dimension of a space has the form (t, i, f), where we are $t\%$ sure about the real dimension of the space, $i\%$ indeterminate about the real dimension of the space, and $f\%$ unsure about the real dimension of the space.

31 Noise in a neutrosophic system

A neutrosophic system's noise is part of the system's indeterminacy. A system's pattern may evolve or dissolve over time, as in a classical system.

32 Quasi-stability

A neutrosophic system has a degree of stability, degree of indeterminacy referring to its stability, and degree of instability. Similarly, it has a degree of change, degree of indeterminate change, and degree of non-change at any point in time.

Quasi-stability of a neutrosophic system is its partial resistance to change.

33 (t, i, f) −attractors

Neutrosophic system's quasi-stability is also dependant on the (t, i, f) −attractor, which $t\%$ attracts, $i\%$ its attraction is indeterminate, and $f\%$ rejects. Or we may say that the neutrosophic system $(t\%, i\%, f\%)$ −prefers to reside in a such neutrosophic attractor.

Quasi-stability in a neutrosophic system responds to quasi-perturbations.

When $(t, i, f) \to (1,0,0)$ the quasi-attractors tend to become stable, but if $(t, i, f) \to (0, i, f)$, they tend to become unstable.

Most neutrosophic system are very chaotic and possess many quasi-attractors and anomalous quasi-patterns. The degree of freedom in a neutrosophic complex system increase and get more intricate due to the type of indeterminacies that are specific to that system. For example, the classical system's noise is a sort of indeterminacy.

Various neutrosophic subsystems are assembled into a neutrosophic complex system.

34 (t, i, f) − repellors

Besides attractors, there are systems that have repellors, i.e. states where the system avoids residing. The neutrosophic systems have quasi-repellors, or (t, i, f) −repellors, i.e. states where the neutrosophic system partialy avoid residing.

35 Neutrosophic probability of the system's states

In any (classical or neutrosophic) system, at a given time ρ, for each system state τ one can associate a neutrosophic probability,

$$\mathcal{NP}(\tau) = (\mathrm{t}, \mathrm{i}, \mathrm{f}),$$

where *t, i, f* are subsets of the unit interval [0, 1] such that:

t = the probability that the system resides in τ;
i = the indeterminate probability/improbability about the system residing in τ;
f = the improbability that the system resides in τ;

For a (classical or neutrosophic) dynamic system, the neutrosophic probability of a system's state changes in the time, upon the previous states the system was in, and upon the internal or external conditions.

36 (t, i, f) −reiterative

In Neutrosophic Reiterative System, each state is partially dependent on the previous state. We call this process quasi-reiteration or (t, i, f) −reiteration.

In a more general case, each state is partially dependent on the previous n states, for $n \geq 1$. This is called n-quasi-reiteration, or $n - (t, i, f)$ −reiteration.

Therefore, the previous neutrosophic system history partialy influences the future neutrosophic system's states, which may be different even if the neutrosophic system started under the same initial conditions.

37 Finite and infinite system

A system is finite if its space, the number of its elements, and the number of its relationships are all finite.

If at least one of these three is infinite, the system is considered infinite. An infinite system may be countable (if both the number of its elements and the number of its relationships are countable), or, otherwise, uncountable.

38 Thermodynamic (t, i, f) −equilibrium

The potential energy (the work done for changing the system to its present state from its standard configuration) of the classical system is a minimum if the equilibrium is stable, zero if the equilibrium is neutral, or a maximum if the equilibrium is unstable.

A classical system may be in stable, neutral, or unstable equilibrium. A neutrosophic system may be in quasi-stable, quasi-neutral or quasi-unstable equilibrium, and its potential energy respectively quasi-minimum, quasi-null (i.e. close to zero), or quasi-maximum. {We recall that 'quasi' means relative(ly), approximate(ly), almost, near, partial(ly), etc. or mathematically 'quasi' means (t, i, f) in a neutrophic way.}

In general, we say that a neutrosophic system is in $(t, i, f)-$ equilibrium, or $t\%$ in stable equilibrium, $i\%$ in neutral equilibrium, and $f\%$ in unstable equilibrium (non-equilibrium).

When $f \gg t$ (f is much greater than t), the neutroophic system gets into deep non-equilibrium and the perturbations overtake the system's organization to a new organization.

Thus, similarly to the second law of thermodynamics, the neutrosophic system runs down to a $(t, i, f)-$equilibrium state.

A neutrosophic system is considered at a thermodynamic $(t, i, f)-$equilibrium state when there is not (or insignificant) flow from a region to another region, and the momentum and energy are uninformally at $(t, i, f)-$level.

39 The $(t_1, i_1, f_1)-$cause produces a $(t_2, i_2, f_2)-$effect

The potential energy (the work done for changing the system to its present state from its standard configuration) of the classical system is a minimum if the equilibrium is stable, zero if the equilibrium is neutral, or a maximum if the equilibrium is unstable.

In a neutrosophic system, a (t_1, i_1, f_1)-cause produces a (t_2, i_2, f_2)-effect. We also have cascading (t, i, f)-effects from a given cause, and we have permanent change into the system.

(t, i, f)-principles and (t, i, f)-laws function in a neutrosophic dynamic system. It is endowed with (t, i, f)-invariants and with parameters of (t, i, f)-potential (potentiality, neutrality, impotentiality) control.

40 $(t, i, f)-$holism

A neutrosophic system is a $(t, i, f)-$holism, in the sense that it has a degree of independent entity (t) with respect to its parts, a degree of indeterminate (i) independent-dependent entity with respect to its parts, and a degree of dependent entity (f) with respect to its parts.

41 Neutrosophic soft assembly

Only several ways of assembling (combining and arranging) the neutrosophic system's parts are quasi-stable. The others assemble ways are quasi-transitional.

The neutrosophic system development is viewed as a neutrosophic soft assembly. It is alike an amoeba that changes its shape. In a neutrosophic dynamic system, the space, the elements, the relationships are all flexible, changing, restructuring, reordering, reconnecting and so on, due to heterogeneity, multimodal processes, multi-causalities, multidimensionality, auto-stabilization, auto-hierarchization, auto-embodiement and especially due to synergetism (the neutrosophic system parts cooperating in a (t, i, f) −degree).

42 Neutrosophic collective variable

The neutrosophic system is partially incoherent (because of the indeterminacy), and partially coherent. Its quasi-behavior is given by the neutrosophic collective variable that embeds all neutrosophic variables acting into the (t, i, f) −holism.

43 Conclusion

We have introduced for the first time notions of neutrosophic system and neutrosophic dynamic system. Of course, these proposals and studies are not exhaustive.

Future investigations have to be done about the neutrosophic (dynamic or not) system, regarding: the neutrosophic descriptive methods and neutrosophic experimental methods, developmental and study the neutrosophic differential equations and neutrosophic difference equations, neutrosophic simulations, the extension of the classical A-Not-B Error to the neutrosophic form, the neutrosophic putative control parameters, neutrosophic loops or neutrosophic cyclic alternations within the system, neutrosophic degenerating (dynamic or not) systems, possible programs within the neutrosophic system, from neutrosophic antecedent conditions how to predict the outcome, also how to find the boundary of neutrosophic conditions, when the neutrosophic invariants are innate/genetic, what are the relationships between the neutrosophic attractors and the neutrosophic repellors, etc.

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A Comparison of Combined Overlap Block Fuzzy Cognitive Maps (COBFCM) and Combined Overlap Block Neutrosophic Cognitive Map (COBNCM) in finding the hidden patterns and indeterminacies in Psychological Causal Models:

Case Study of ADHD

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Abstract

In spite of researchers' concerns to find causalities, reviewing the literature of psychological studies one may argue that the classical statistical methods applied in order to find causalities are unable to find uncertainty and indeterminacies of the relationships between concepts.

In this paper, we introduce two methods to find effective solutions by identifying "hidden" patterns in the patients' cognitive maps. Combined Overlap Block Fuzzy Cognitive Map (COBFCM) and Combined Overlap Block Neutrosophic Map (COBNCM) are effective when the number of concepts can be grouped and are large in numbers. In the first section, we introduce COBFCM, COBNCM, their applications, and the advantages of COBNCM over COBFCM in some cases. In the second section, we explain eight overlapped cognitive concepts related to ADHD in children and apply COBNCM and COBFCM to analyze the modeled data, comparing their results. Conclusions, limitations, and implications for applying COBNCM in other psychological areas are also discussed.

Keywords

Fuzzy Cognitive Map, Neutrosophic Cognitive Map, Fuzzy model, Causal model, ADHD, Methodology.

1 Introduction

A portfolio of project is a group of project that share resources creating relation among them of complementarity, incompatibility or synergy [1]. The interdependency modeling and analysis have commonly been ignored in project portfolio management [2].

Identifying causalities is one of the most important concerns of researchers, one may find out reviewing the literature of psychological research. Although there are some statistical methods to investigate this issue, all, or majority, rely on quantitative data. Less attention was directed towards scientific qualitative knowledge and experience. In some methods based on theoretical basics such as structural equation modeling (SEM), there is no chance to find optimal solutions, hidden patterns and indeterminacies (possibilities) of causal relationships between variables, which are common in psychological research. Therefore, for linking quantitative and qualitative knowledge, it seems an urge to use methods as fuzzy cognitive maps or neutrosophic cognitive maps in psychological research. The two methods are rooted in cognitive map (CM). The cognitive maps for representing social scientific knowledge and describing the methods that is used for decision-making were introduced by Axelrod in 1976. The fuzzy cognitive map (FCM) was proposed by Kosko (1986) to present the causal relationship between concepts and analyze inference patterns. Kosko (1986, 1988, 1997) considered fuzzy degree of inter relationships between concepts, its nodes corresponding to a relevant node and the edges stating the relation between two nodes, denoted by a sign. A positive sign implies a positive relation; moreover, any increase in its source value leads to increase in its target value. A negative sign stages a negative relation and any increase or decrease in its source value leads to reverse effect to its target value. If there is no edge between two nodes in a cognitive map, it means that there is no relation between them (Zhang et al., 1998). In a simple fuzzy cognitive map, the relation between two nodes is determined by taking a value in interval [-1, 1].

While -1 corresponds to the strongest negative value, +1 corresponds to strongest positive value. The other values express different levels of influence (Lee, et al., 2003). Fuzzy cognitive maps are important mathematical models representing the structured causality knowledge for quantitative inferences (Carvalho & Tome, 2007). FCM is a soft computing technique that follows an approach similar to the human reasoning and decision-making process (Markinos, et al., 2004). Soft computing is an emerging field that combines and synergies advanced theories and technologies such as Fuzzy Logic, Neural Networks, Probabilistic reasoning and Genetic Algorithms. Soft computing provides a hybrid flexible computing technology that can solve real world problems. Soft computing includes not only the previously mentioned approaches, but also useful combinations of its components, e.g. Neurofuzzy systems, Fuzzy Neural systems, usage of Genetic Algorithms in Neural Networks and Fuzzy Systems, and many other hybrid methodologies (Stylios & Peter, 2000). FCM can successfully represent knowledge and human experiences, introduce concepts to represent the essential elements, cause and effect relationships among the concepts, to model the behavior of a system (Kandasamy, 1999, 2004). This method is a very simple and powerful tool that is used in numerous fields (Thiruppathi, et al. 2010). When dataset is an unsupervised one and there is uncertainty within the concepts, this method is very useful. The FCM give us the hidden patterns; this method is one effective method, providing a tool for unsupervised data. In addition, using this method, one can analyze the data by directed graphs and connection matrices where nodes represent concepts and edges - strength of relationships (Stylios & Groumpos, 2000). FCM works on the opinion of experts or another uncertainty results like the obtained results using structural equation modeling (SEM). FCM clarify optimal solution by using a simple way, while other causal models such as SEM are complicated. They do not perform well to clarify what-if scenario, for example, their results do not clarify what happens to marital satisfaction if Alexithymia is very high and Family intimacy is very low. Another advantage of FCM is its functioning on experts' opinions (Thiruppathi et al. 2010). FCM is a flexible method used in several models to display several types of problems (Vasantha Kandasamy & Devadoss, 2004; Vasantha Kandasamy & Kisho, 1999). Although by using this method we are able to study uncertainty and find hidden patterns, the FCM is unable to investigate indeterminate relationships, which is a limitation in psychological causal models. A solution to overcome this limitation is the Neutrosophic Cognitive Map (NCM).

Vasantha Kandasamy and Smarandache (2003) proposed the neutrosophic cognitive maps, making it possible to mitigate the limitation of fuzzy cognitive maps, which cannot represent the indeterminate relations between variables. The capability of neutrosophic cognitive maps to represent indetermination facilitates the apprehension of systems complexity, and thus elucidates and predicts their behaviors in the absence of complete information.

Neutrosophic Cognitive Map (NCM) relies on Neutrosophy. Neutrosophy is a new branch of philosophy introduced by Smarandache in 1995 as a generalization of dialectics, which studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. Neutrosophic Cognitive Map (NCM) is the generalization and combination of the Fuzzy Cognitive Map in which indeterminacy is included. Fuzzy theory only measures the grade of membership or the non-existence of a membership in a revolutionary way, but failing to attribute the concept when the relationship between concepts in debate are indeterminate (Vasantha Kandasamy & Smarandache, 2007). A Neutrosophic Cognitive Map is a neutrosophic directed graph with concepts like policies, events etc. as nodes and causalities, or indeterminacies as edges. It represents the causal relationship between concepts defined by Smarandache (2001) and Vasantha Kandasamy (2007). Fuzzy cognitive maps deals with the relation / non-relation between two nodes or concepts, but it declines to attribute the relation between two conceptual nodes when the relation is an indeterminate one. In Neutrosophic Logic, each proposition is estimated to have the percentage of truth in a subset T, the percentage of indeterminacy in a subset I, and the percentage of falsity in a subset F. Every logical variable x is described by an ordered triple x = (T, I, F), where T is the degree of truth, F is the degree of false and I - the level of indeterminacy. Neutrosophy means that any proposition has a percentage of truth, a percentage of indeterminacy and a percentage of falsity (some of these percentages may be zero). Neutrosophy also makes distinctions between absolute truth (a proposition true in all possible worlds), which is denoted by 1, and relative truth (a proposition which is true in at least one world, but not in all), which is denoted by I (Smarandache & Liu, 2004). Sometimes, in psychological and educational research, the causality between the two concepts, i.e. the effect of C_i on C_j is indeterminate. Chances of indeterminacy are possible and frequent in case of unsupervised data. Therefore, the NCM is a flexible and effective method based on fuzzy cognitive map for investigating the relations of psychological casual models in which indeterminate relationships are not unusual. We describe the basic components in detail to explain differences between the two methods.

2 Combined Overlap Block Fuzzy Cognitive Maps (COBFCM) and Combined Overlap Block Neutrosophic Cognitive Map (COBNCM)

We can combine arbitrarily FCM and NCM connection matrices $F_1, F_2, \ldots, F_K$ by adding augmented FCM and NCM matrices, $F_1, \ldots, F_K$. Each augmented matrix F_i has *n-rows* and *n-columns*; *n* equals the total number of distinct concepts used by the experts. We permute the rows and columns of the augmented matrices to bring them into mutual coincidence. Then we add the F_i's point wise to yield the combined FCM and NCM matrix F, $F = \Sigma F_i$. We can then use F to construct the combined FCM and NCM directed graph. The combination can be in disjoint or overlapping blocks.

Combined overlap block fuzzy cognitive maps (COBFCM) were introduced and applied in social sciences by Vasantha Kandasamy et al. (2004), and combined overlap block neutrosophic cognitive map (COBNCM) - by Vasantha Kandasamy & Smarandache (2007). In these two methods, finite number of NCM and FCM can be combined together to produce the joint effect of all NCM and FCM. In NCM method, $N(E_1)$, $N(E_2), \ldots, N(E_p)$ are considered the neutrosophic adjacency matrices, with nodes $C_1, C_2, \ldots, C_n$, and $E_1, E_2, \ldots, E_p$ are the adjacency matrices of FCM with nodes $C_1, C_2, \ldots, C_n$. The combined NCM and the combined FCM are obtained by adding all the neutrosophic adjacency matrices $N(E_1) \ldots N(E_p)$ and adjacency matrices by $E_1, \ldots, E_P$ respectively. We denote the Combined NCM adjacency neutrosophic matrix by $N(E) = N(E_1) + N(E_2) + \ldots + N(E_p)$ and the Combined FCM adjacency matrix by $E = E_1 + E_2 + \ldots + E_p$. Both models $\{C_1, C_2, C_3, \ldots C_n\}$ contain *n* concepts associated with *P* (a given problem). We divide the number of concepts $\{C_1, C_2, C_3, \ldots C_n\}$ into *K* classes $S_1, S_2, S_3, \ldots S_K$, where the classes are such that $S_i \cap S_{i+1} \neq \phi$, $\cup S_i = \{C_1, C_2, \ldots, C_n\}$ and $|S_i| \neq |S_j|$, if $i \neq j$ in general. To introduce these methods in detail, we explain their basic components below.

3 Concepts and edges

In Combined Overlap Block Fuzzy Cognitive Maps (COBFCM) and Combined Overlap Block Neutrosophic Cognitive Map (COBNCM), the edges are qualitative concepts considered as nodes and causal influences. Concept nodes possess a numeric state, which denotes qualitative measures of the concepts present in the conceptual domain. When the nodes of FCM are a fuzzy set, they are called fuzzy nodes. Fuzzy means the concepts are not quantitative, they are uncertain, and we have to study them using linguistic variables, such as "very high", "high", "middle", etc. The nodes or concepts are presented by C_1, C_2,

C_3 ,.....,C_n. The state of concepts is portrayed as a vector. In COBNCM, we assume each node is a neutrosophic vector from neutrosophic vector space V. Let C_1, C_2, ..., C_n denote *n* nodes, So a node C_i will be represented by (x_1, ...,x_n), where x_k's - zero or one or *I* (*I* is the indeterminate) and x_k = 1 means that the node C_k is in the *ON* state, and x_k =0 means the node is in the *OFF* state, and x_k = I means the nodes state is an *indeterminate* at that time or in that situation. Let C_1, C_2... C_n be the nodes of COBNCM and let A = (a_1, a_2,..., a_n), where $a_i \in \{0, 1, I\}$. A is called the instantaneous state neutrosophic vector and it denotes the *ON – OFF – indeterminate* state position of the node at an instant:

a_i = 0 if a_i is off (no effect),

a_i = 1 if a_i is on (has effect),

a_i = I if a_i is indeterminate (effect cannot be determined),

for i = 1, 2,..., n.

In COBNCM, the nodes C_1, C_2, ..., C_n are nodes and not indeterminate nodes, because they indicate the concepts which are well known. But the edges connecting C_i and C_j may be indeterminate, i.e. an expert may not be in the position to say that C_i has some causality on C_j, either he will be in the position to state that C_i has no relation with C_j; in such cases, the relation between C_i and C_j, which is indeterminate, is denoted by I. The COBFCM with edge weights or causalities from the set {-1, 0, 1} are called simple, and COBNCM with edge weight from {-1, 0, 1, I} are called simple COBNCM. In COBFCM, the edges (e_{ij}) take values in the fuzzy causal interval [-1, 1], e_{ij} =0, e_{ij}>0 and e_{ij}<0 indicate no causality, positive and negative causality, respectively. In simple FCM, if the causality occurs, it occurs to a maximal positive or negative degree. Every edge in COBNCM is weighted with a number in the set {-1, 0, 1, I}. e_{ij} is the weight of the directed edge C_iC_j, $e_{ij} \in \{-1, 0, 1, I\}$. e_{ij} = 0 if C_i does not have any effect on C_j, e_{ij} = 1 if increase (or decrease) in C_i causes increase (or decrease) in C_j, e_{ij} = –1 if increase (or decrease) in C_i causes decrease (or increase) in C_j . e_{ij} = I if the relation or effect of C_i on C_j is an indeterminate. In such cases, it is denoted by dotted lines in the model.

4 Adjacency Matrix

In COBFCM and COBNCM, the edge weights are presented in a matrix. This matrix is defined by E= (e_{ij}), where e_{ij} indicates the weight of direct edge C_iC_j and $e_{ij} \in \{0, 1,-1\}$, and by N (E) = (e_{ij}), where e_{ij} is the weight of the directed edge C_i C_j, where $e_{ij} \in \{0, 1, -1, I\}$. We denote by N(E) the neutrosophic adjacency matrix of the COBNCM. It is important to note that all matrices used in these methods are always a square matrix with diagonal entries as zeros. All off-diagonal entries are edge weights that link adjacent nodes to each other. A finite number of FCM and NCM can be combined together to produce the joint effect of all FCM and NCM. Suppose $E_1, E_2, E_3 \ldots E_P$ and $N(E_1), N(E_2), N(E_3) \ldots N(E_P)$ are adjacency matrices of FCM and neutrosophic adjacency matrix of NCM, respectively, with nodes C_1, C_2, C_3, ..., C_n. Then combined FCM and NCM are obtained by adding all the adjacency matrices (Vasantha Kandasamy & Smarandache, 2003). In combined overlap FCM and NCM, all entries of all different overlapped matrices are put in a whole matrix and added to each other.

5 Inference process

The states of concepts are rendered as vectors. Therefore, the inference process of FCM and NCM can be represented by an iterative matrix calculation process. Let V_0 be the initial state vector, V_n be the state vector after *n th* iterative calculation, and W be the causal effect degree matrix; then the inference process can be defined as a repeating calculation of Equation 1 until the state vector converges to a stable value or fall in to an infinite loop. Suppose $X_1 = [1\ 0\ 0\ 0 \ldots 0]$ is the input vector and E is the associated adjacency matrix. X_1E is obtained by multiplying X_1 by the matrix E. We obtain $X_1E = [x_1, x_2, x_3, \ldots x_n]$ by replacing x_i by 1, if $x_i > c$, and x_i by 0, if $x_i < c$ (c is a suitable positive integer). After updating the thresholding concept, the concept is included in the updated vector by making the first coordinate as 1 in the resulting vector. Suppose $X_1E \rightarrow X_2$, then X_2E is considered; the same procedure is repeated until it gets limit cycle or a fixed point (Thiruppathi, et al., 2010).

$$V_{n+1} = f\,(V_n \times W + V_n), \qquad (1)$$

where the f is usually simply defined as $f(x) = f_0(x) = 1\ (x \geq 1),\ 0\ (1 > x > -1)$ and $-1\ (-1 \leq x)$.

If the equilibrium state of a dynamical system is a unique state vector, then it is called a fixed point. Consider FCM and NCM with $C_1, C_2 \ldots, C_n$ as nodes. For example, let us start the dynamical system by switching on C_1. Let us assume that NCM and FCM settle down with C_1 and C_n ON, i.e. the state vector remains as (1, 0,..., 1); this state vector (1,0,..., 0, 1) is called the fixed point; if FCM and NCM settle down with a state vector repeating in the form $A_1 \rightarrow A_2 \rightarrow \ldots \rightarrow A_i \rightarrow A_1$, then this equilibrium is called a limit cycle of NCM and FCM (Tabar, 1991).

Let $C_1, C_2, \ldots, C_n$ be the vector of FCM and NSM. Let E be the associated adjacency matrix. Let us find the hidden pattern when x_1 is switched on when an input is given as the vector $A_1 = (1, 0, 0, \ldots, 0)$; the data should pass through the neutrosophic matrix $N_{(E)}$; this is done by multiplying A_1 by the matrix N(E). Let $A_1N_{(E)}$ = (a_1, a_2,..., an) with the threshold operation, by replacing a_i by 1, if a_i > k, and a_i by 0, if a_i < k, and a_i by I, if a_i is not an integer.

$$f(k)\left\{\begin{matrix} a_i < k \rightarrow a_i = 0 \\ a_i > k \rightarrow a_i = 1 \\ a_i = b + c \times I \rightarrow a_i = b \\ a_i = c \times I \rightarrow a_i = I \end{matrix}\right\}$$

(k depends on researcher's opinion, for example K=1 or 0.5).

Note that (a_1, a_2... a_n) and (a'_1, a'_2, ..., a'_n) are two neutrosophic vectors. We say (a_1, a_2, ... , a_n) is equivalent to (a'_1, a'_2, ... , a'_n) denoted by (a_1, a_2, ... , a_n) ~ (a'_1, a'_2, ..., a'_n), if we get (a'_1, a'_2, ... , a'_n) after thresholding and updating the vector (a_1, a_2, ... , a_n), after passing through the neutrosophic adjacency matrix N(E). The initial state vector in FCM and NCM is included 0 and 1 only (OFF and ON states, respectively). But after it passes through the adjacency matrix, the updating resultant vector may have entries from (0 and 1) in FCM and from (0, 1, I) in NCM, respectively. In this case, we cannot confirm the presence of that node (ON state), nor the absence (OFF state). Such possibilities are present only in the case of NCM.

6 Cyclic and acyclic FCM and NCM

If FCM and NCM possess a directed cycle, it is said to be cyclic (to have a feedback) and we call it a dynamical system. FCM and NCM are acyclic if they do not possess any directed cycle.

7 FCM versus NCM

Vasantha Kandasamy and Smarandache (2003) summarize the differences between FCM and NCM:

[1] FCM indicates the existence of causal relation between two concepts, and if no relation exists, it is denoted by 0.

[2] NCM does not indicate only the existence or absence of causal relation between two concepts, but also gives representation to the indeterminacy of relations between any two concepts.

[3] We cannot apply NCM for all unsupervised data. NCM will have meaning only when relation between at least two concepts C_i and C_j are indeterminate.

[4] The class of FCM is strictly contained in the class of NCM. All NCM can be made into FCM by replacing I in the connection matrix by 0.

[5] The directed graphs in case of NCM are called neutrosophic graphs. In the graphs, there are at least two edges, which are related by the dotted lines, meaning the edge between those two vertices is an indeterminate.

[6] All connection matrices of the NCM are neutrosophic matrices. They have in addition to the entries 0, 1, −1, the symbol I.

[7] The resultant vectors, i.e. the hidden pattern resulting in a fixed point or a limit cycle of a NCM, can also be a neutrosophic vector, signifying the state of certain conceptual nodes of the system to be an indeterminate; indeterminate relation is signified by I.

[8] Because NCM measures the indeterminate, the expert of the model can give careful representation while implementing the results of the model.

[9] In case of simple FCM, we have the number of instantaneous state vectors to be the same as the number of resultant vectors, but in the case of NCM the number of instantaneous state vectors is from the set {0,1}, whereas the resultant vectors are from the bigger set {0, 1, I}.

[10] Neutrosophic matrix {N (E)} converts to adjacency matrix (E) by easily recoding I to 0.

8 *Case study:* The comparison of COBFCM and COBNCM to find solution for ADHD

Attention-Deficit/Hyperactivity Disorder (ADHD) is not only the most common neuro-developmental disorder of childhood today, but also the most studied. Literature reviews report very different prevalence estimates. The DSM-IV states that the prevalence of ADHD is about 3–5% among school-age children [American Psychiatric Association, 1994]. Some of consequences of untreated ADHD children are social skills deficits, behavioral disinhibition and emotional skills deficits. Therefore, early diagnosis of ADHD is very important. The purpose of this paper is the comparison of application of COBFCM and COBNCM to identify the risk groups. When data is an unsupervised one and based on experts' opinions and there is uncertainty in the concepts, COBFCM is the best option, and when data is an unsupervised one and there is indeterminacy in the concepts, COBNCM is a preferred method. The comparison of these methods clarifies this fundamental point and the relationship of to-be-determined and not-to-be-determined between the concepts, including the effect on results in casual models in psychological research.

Based on experts' opinions (five child and developmental psychologists) and the corresponding literature, we determined eight cognitive concepts related to ADHD:

[1] C_1: Mother's harmful substance use;
[2] C_2: Mother's low physical self-efficacy;
[3] C_3: Mother's bad nutrition;
[4] C_4: Mother's depression;
[5] C_5: Family conflict;
[6] C_6: Father's addiction;
[7] C_7: Child's emotional problems;
[8] C_8: Child's hyper activity.

9 Combined Overlap Block NCM

We divide these concepts in to 3 equal length classes; each class has just four concepts in the following manner:

$$S_1=\{C_1,C_2,C_3,C_4\},\ S_2=\{C_2,C_4,C_5,C_6\} \text{ and } S_3=\{C_4,C_5,C_7,C_8\}$$

These three classes are offered to experts in order to determine relationships and the strength. In addition, we asked them to delineate edges that have indeterminate effects by dotted lines in the figures and by I in the corresponding matrices. The directed graph and relation matrix for the S_1, S_2 and S_3 given by the expert is as follow:

Figure 1

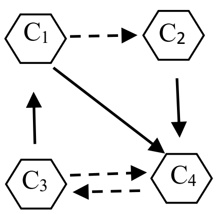


Figure 2

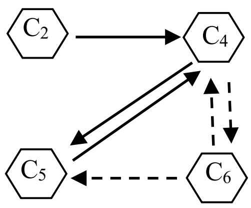
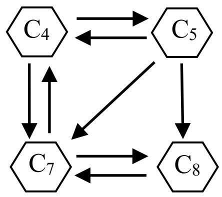


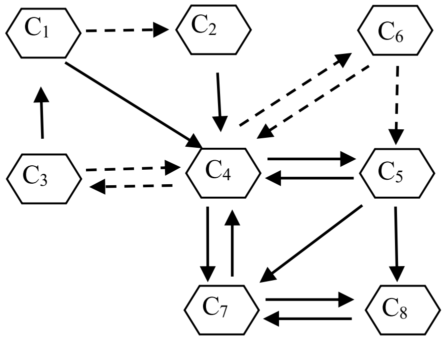
Figure 3



$$\begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ C_4 \end{array} \begin{array}{c} C_1\ C_2\ C_3\ C_4 \\ \begin{pmatrix} 0 & \mathbf{I} & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & \mathbf{I} \\ 0 & 0 & \mathbf{I} & 0 \end{pmatrix} \end{array} \qquad \begin{array}{c} \\ C_2 \\ C_4 \\ C_5 \\ C_6 \end{array} \begin{array}{c} C_2\ C_4\ C_5\ C_6 \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \mathbf{I} \\ 0 & 1 & 0 & 0 \\ 0 & \mathbf{I} & \mathbf{I} & 0 \end{pmatrix} \end{array} \qquad \begin{array}{c} \\ C_4 \\ C_5 \\ C_7 \\ C_8 \end{array} \begin{array}{c} C_4\ C_5\ C_7\ C_8 \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

The combined overlap block connection matrix of NCM is given by E (N).

$$E(N) = \begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \end{array} \begin{array}{c} C_1\ C_2\ C_3\ C_4\ C_5\ C_6\ C_7\ C_8 \\ \begin{pmatrix} 0, I, 0, 1, 0, 0, 0, 0 \\ 0, 0, 0, 2, 0, 0, 0, 0 \\ 1, 0, 0, I, 0, 0, 0, 0 \\ 0, 0, I, 0, 1, I, 1, 0 \\ 0, 0, 0, 2, 0, 0, 1, 1 \\ 0, 0, 0, I, I, 0, 0, 0 \\ 0, 0, 0, 1, 0, 0, 0, 1 \\ 0, 0, 0, 0, 0, 0, 1, 0 \end{pmatrix} \end{array}$$



The combined overlap block connection matrix of FCM is given by E.

$$E = \begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \end{array} \begin{array}{c} C_1\ C_2\ C_3\ C_4\ C_5\ C_6\ C_7\ C_8 \\ \begin{pmatrix} 0, 0, 0, 1, 0, 0, 0, 0 \\ 0, 0, 0, 2, 0, 0, 0, 0 \\ 1, 0, 0, 0, 0, 0, 0, 0 \\ 0, 0, 0, 0, 1, 0, 1, 0 \\ 0, 0, 0, 2, 0, 0, 1, 1 \\ 0, 0, 0, 0, 0, 0, 0, 0 \\ 0, 0, 0, 1, 0, 0, 0, 1 \\ 0, 0, 0, 0, 0, 0, 1, 0 \end{pmatrix} \end{array}$$

10 Hidden Patterns

Now, using the combined matrix E(N), we can determine any hidden patterns embedded in the matrix. Suppose the concept C_4 (Mother's depression) is in the ON state. So, initial vector for studying the effects of these concepts on the dynamical system E is A= [0 0 0 1 0 0 0 0]. Let A state vector depicting the ON state of Mother's depression passing the state vector A in to the dynamical system E (N):

A=[0 0 0 1 0 0 0 0]

AE(N) =[0, 0, I, 0, 1, I, 1, 0] → [0 0 I 1 1 I 1 0]=A_1

A_1E(N)=[I, 0, I, 2*I^2 + 3, I^2 + 1, I, 2, 2] → [I 0 I 1 1 I 1 1]=A_2

A_2E(N)=[I, I^2, I, 2*I^2 + I + 3, I^2 + 1, I, 3, 2] →
[I I I 1 1 I 1 1]=A_3

A_3E(N) =[I, I^2, I, 2*I^2 + 3*I + 3, I^2 + 1, I, 3, 2] →
[I I I 1 1 I 1 1]=A_4=A_3.

Since $A_4=A_3$ (we have reached the fixed point of the dynamical system). A_3 is determined to be a hidden pattern. Now again using the COBFCM we can determine hidden patterns embedded in the matrix (E), such as COBNCM, here initial vector considered A= [0 0 0 1 0 0 0 0], i.e. we suppose the Mother's depression is high. The results obtained are as following:

$AE=[0\ 0\ 0\ 1\ 1\ 0\ 1\ 0] = A_1$

$A_1E=[\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ 1] = A_2$

$A_2E=[\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ 1] = A_3 = A_2$

By $A_3=A_2$ we have reached the fixed point of the dynamical system. A_2 is determined to be a hidden pattern using the COBFCM.

11 Weighted Method

We can use the weighted method to clarify the results, when there is a tie between the concepts inputs. Suppose the resultant vector be A= [10 0 1 1 1 0], i.e., the half of the concepts suggest that the given problem exists, but other three suggest that the problem is not justified on the basis of available concept. In this case, we can adopts a simple weighted approach where in each of the concepts can be assigned weights based on experts' opinions. For example, C_1=20%, C_2=10%, C_3=10%, C_4=60%, C_5=25%, C_6=30%, C_7=20%. The ON - OFF state for each Concept in A vector leads to a weighted average score of the corresponding concepts. Suppose the initial vector is A= [0 0 0 0 0 1 0]; based on the resultant vector and the experts' weights for the concepts, we can find a weighted average score. In this case, Geometric mean is an accurate and appropriate measure for calculating average score, because the data are expressed in percentage terms. The resulting of the example equals to 30% (which tends towards absence of the problem (since this is <50%, the point of no difference).

The results based on the COBNCM indicated when a mother suffering from depression, i.e. the C_4 is in the ON state; there will be family conflict, child's emotional problems, Child's hyper activity and also there *may be* Mother's harmful substance use, Mother's low physical self-efficacy, Mother's bad nutrition and Father's addiction. Based on the results of this study using the COBFCM, when a mother is depressed, there will be child's hyperactivity, emotional problems, and family conflict. Although, based on the results of the two models mother's depression being the main cause of ADHD, based on the COBFCM we cannot determine the occurrence of possibilities of some corresponding concepts in developing ADHD.

12 Discussion

It is important to note that in COBFCM e_{ij} measures only absence or presence of influence of the node C_i on C_j, but untill now any researcher has not contemplated the indeterminacy of any relation between two nodes C_i and C_j. When researchers deal with unsupervised data, there are situations when no relation can be determined between two nodes (Vasantha Kandasamy & Smarandache, 2005). The presence of I in any coordinate implies the expert cannot tell the presence of that node, i.e. *on state* after passing through N (E), nor can we say the absence of the node, i.e. *off state* - the effect on the node after passing through the dynamical system is indeterminate, so it is represented by *I*. Thus, only in case of NCM we can identify that the effect of any node on other nodes can also be indeterminate. Such possibilities and analysis is totally absent in the case of FCM. Therefore, the COBFCM only indicates that what happens for C_j when C_i is in an ON state, but it cannot indicate the effects of the concepts on each other in neutral states. In other words, by using COBFCM, some of the latent layers of the relationships between the concepts are not discovered. Thus, only the COBNCM helps in such conditions.

The core of psychology and education is theoretical. Theories themselves consist of constructs, concepts and variables, which are expressed by linguistic propositions - to describe, explain and predict the phenomena. For these characteristics of theory, Smarandache (2001) believes that no theory is exempted from paradoxes, because of language imprecision, metaphoric expression, various levels or meta-levels of understanding/interpretation, which might overlap. These propositions do not mean a fixed-valued components structure and it is dynamic, i.e. the truth value of a proposition may change from one place to another place and from one time to another time, and it changes with respect to the observer (subjectivity). For example, the proposition "Family conflict leads to divorce" does not mean a fixed-valued components structure; this proposition may be stated 35% true, 45% indeterminate, and 45% false at time t_1; but at time t_2 may change at 55% true, 49% indeterminate, and 32% false (according with new evidences, sources, etc.); or the proposition " Jane is depressed " can be (.76,.56, .30) according to her psychologist, but (.85, .25, .15) according to herself, or (.50, .24, .35) according to her friend, etc. Therefore, considering the indeterminacies in investigating the causal relationships in psychological and educational research is important, and it is closer to the human mind reasoning. A good method in this condition is using the NCM, as seen before, using the FCM leads to ignoring indeterminacies (by converting the e_{ij}=I to e_{ij}=0), and this ignoring itself leads to the covering the latent effects of the concepts of the causal models. It is recommended that in the conditions that indeterminacies are important, researchers use the NCM method.

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A New Type of Group Action Through the Applications of Fuzzy Sets and Neutrosophic Sets

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Abstract: Fuzzy sets are the most significant tools to handle uncertain data while neutrosophic sets are the generalizations of fuzzy sets in the sense to handle uncertain, incomplete, inconsistent, indeterminate, false data. In this paper, we introduced fuzzy subspaces and neutrosophic subspaces (generalization of fuzzy subspaces) by applying group actions.Further, we define fuzzy transitivity and neutrosophic transitivty in this paper. Fuzzy orbits and neutrosophic orbits are introduced as well. We also studied some basic properties of fuzzy subspaces as well as neutrosophic subspaces.

Key Words: Fuzzy set, neutrosophic set, group action, G-space, fuzzy subspace, neutrosophic subspace.

§1. **Introduction**

The theory of fuzzy set was first proposed by Zadeh in the seminal paper [22] in 1965. The concept of fuzzy set is used successfully to modelling uncertain information in several areas of real life. A fuzzy set is defined by a membership function μ with the range in unit interval $[0, 1]$. The theory and applications of fuzzy sets and logics have been studied extensively in several aspects in the last few decades such as control, reasoning, pattern recognition, and computer vision etc. The mathematical framework of fuzzy sets become an important area for the research in several phenomenon such as medical diagnosis, engineering, social sciences etc. Literature on fuzzy sets can be seen in a wide range in $[7, 24, 25, 26]$.

The degree of membership of an element in a fuzzy set is single value between 0 and 1. Thus it may not always be true that the degree of non-membership of an element in a fuzzy set is equal to 1 minus the membership degree because there is some kind of hesitation degree. Therefore, in 1986, Atanassov [1] introduced an extension of fuzzy sets called intuitionistic fuzzy set. An intuitionistic fuzzy sets incorporate the hesitation degree called hesitation margin and this hesitation margin is defining as 1 minus the sum of membership and non-membership degree. Therefore the intuitionistic fuzzy set is defined by a membership degree μ as well as a non-membership function υ with same range $[0,1]$. The concpet of Intuitionistic fuzzy sets have been applied successfully in several fields such as medical diagnosis, sale analysis, product marketing, financial services, psychological investigations, pattern recognition, machine learning decision making etc.

Smarandache [14] in 1980, introduced a new theory called Neutrosophy, which is basically a branch of philosophy that focus on the origin, nature, and scope of neutralities and their interactions with different ideational spectra. On the basis of neutrosophy, he proposed the concept of neutrosophic set which is characterized by a degree of truth membership T , a degree of indeterminacy membership I and a degree falsehood membership F. A neutrosophic set is powerful mathematical tool which generalizes the concept of classical sets, fuzzy sets [22], intuitionistic fuzzy sets [2], interval valued fuzzy sets [15], paraconsistent sets [14], dialetheist sets [14], paradoxist sets [14], and tautological sets [14]. Neutrosophic sets can handle the indeterminate, imprecise and inconsistent information that exists around our daily life. Wang et al. [17] introduced single valued neutrosophic sets in order to use them easily in scientific and engineering areas that gives an extra possibility to represent uncertain, incomplete, imprecise, and inconsistent information. Hanafy *et.al* further studied the correlation coefficient of neutrosophic sets [5,6]. Ye [18] defined the correlation coefficient for single valued neutrosophic sets. Broumi and Smaradache conducted study on the correlation coefficient of interval neutrosophic set in [2]. Salama et al. [12] focused on neutrosophic sets and netrosophic topological spaces. Some more literature about neutrosophic set is presented in [4,8,10,11,13,16,19,20,23].

The notions of a G-spaces [3] were introduced as a consequence of an action of a group on an ordinary set under certain rulers and conditions. Over the passed history of Mathematics and Algebra, the theory of group action [3] has proven to be an applicable and effective mathematical framework for the study of several types of structures to make connection among them. The applications of group action can be found in different areas of science such as physics, chemistry, biology, computer science, game theory, cryptography etc which has been worked out very well. The abstraction provided by group actions is an important one, because it allows geometrical ideas to be applied to more abstract objects. Several objects and things have found in mathematics which have natural group actions defined on them. Specifically, groups can act on other groups, or even on themselves. Despite this important generalization, the theory of group actions comprise a wide-reaching theorems, such as the orbit stabilizer theorem, which can be used to prove deep results in several other fields.

§2. Literature Review and Basic Concepts

Definition 2.1([22]) *Let X be a space of points and let $x \in X$. A fuzzy set A in X is characterized by a membership function μ which is defined by a mapping $\mu : X \to [0,1]$. The fuzzy set can be represented as*

$$A = \{\langle x, \mu(x)\rangle : x \in X\}.$$

Definition 2.2([14]) *Let X be a space of points and let $x \in X$. A neutrosophic set A in X is characterized by a truth membership.*

function T , an indeterminacy membership function I , and a falsity membership function F. T, I, F are real standard or non-standard subsets of $]0^-, 1^+[$, and $T, I, F : X \to]0^-, 1^+[$. The neutrosophic set can be represented as

$$A = \{\langle x, T(x), I(x), F(x)\rangle : x \in X\}.$$

There is no restriction on the sum of T, I, F, so $0^- \leq T + I + F \leq 3^+$.

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]0^-, 1^+[$. Thus it is necessary to take the interval $[0, 1]$ instead of $]0^-, 1^+[$ for technical applications. It is difficult to apply $]0^-, 1^+[$ in the real life applications such as engineering and scientific problems.

Definition 2.3([3]) *Let Ω be a non empty set and G be a group. Let $\upsilon : \Omega \times G \longrightarrow \Omega$ be a mapping. Then υ is called an action of G on Ω if for all $\omega \in \Omega$ and $g, h \in G$, there are*

(1) $\upsilon(\upsilon(\omega, g), h) = \upsilon(\omega, gh)$

(2) $\upsilon(\omega, 1) = \omega$, where 1 is the identity element in G.

Usually we write ω^g instead of $\upsilon(\omega, g)$. Therefore (1) and (2) becomes as

(1) $(\omega^g)^h = (\omega)^{gh}$. For all $\omega \in \Omega$ and $g, h \in G$.

(2) $\omega^1 = \omega$.

A set Ω with an action of some group G on it is called a G-space or a G-set. It basically means a triplet (Ω, G, υ).

Definition 2.4([3]) *Let Ω be a G-space and $\Omega_1 \neq \phi$ be a subset of Ω. Then Ω_1 is called a G-subspace of Ω if $\omega^g \in \Omega_1$ for all $\omega \in \Omega_1$ and $g \in G$.*

Definition 2.5([3]) *Let Ω be a G-space. We say that Ω is transitive G-space if for any $\alpha, \beta \in \Omega$, there exist $g \in G$ such that $\alpha^g = \beta$.*

§3. Fuzzy Subspace

Definition 3.1 *Let Ω be a G-space. Let $\mu : \Omega \to [0, 1]$ be a mapping. Then μ is called a fuzzy subspace of Ω if $\mu(\omega^g) \geq \mu(\omega)$ and $\mu\left(\omega^{g^{-1}}\right) \leq \mu(\omega)$ for all $\omega \in \Omega$ and $g \in G$.*

Example 3.1 Let $\Omega = (\mathbb{Z}_4, +)$ and $G = \{0, 2\} \leq \mathbb{Z}_4$. Let $\upsilon : \Omega \times G \to \Omega$ be an action of G on Ω defined by $\omega^g = \omega + g$ for all $\omega \in \Omega$ and $g \in G$. Then Ω is a G-space. We define $\mu : \Omega \to [0, 1]$ by

$$\mu(0) = \frac{1}{2} \text{ and } \mu(1) = \mu(2) = \mu(3) = 1$$

Then clearly μ is a fuzzy subspace of Ω.

Definition 3.2 *Let Ω_μ be a fuzzy subspace of the G-space Ω. Then μ is called transitive fuzzy subspace if for any α, β from Ω, there exist $g \in G$ such that $\mu(\alpha^g) = \mu(\beta)$.*

Example 3.2 Let $\Omega = G = (\mathbb{Z}_4, +)$. Let $\upsilon : \Omega \times G \to \Omega$ be an action of G on Ω defined by $\omega^g = \omega + g$ for all $\omega \in \Omega$ and $g \in G$. We define $\mu : \Omega \to [0,1]$ by

$$\mu(0) = \frac{1}{2} \text{ and } \mu(1) = \mu(2) = \mu(3) = 1$$

Then clearly μ is a transitive fuzzy subspace of Ω.

Theorem 3.1 *If Ω is transitive G-space, then μ is also transitive fuzzy subspace.*

Proof Suppose that Ω is transitive G-space. Then for any $\alpha, \beta \in \Omega$, there exist $g \in G$ such that $\alpha^g = \beta$. This by taking μ on both sides, we get $\mu(\alpha^g) = \mu(\beta)$ for all $\alpha, \beta \in \Omega$. Hence by definition μ is a transitive fuzzy subspace of Ω. □

Definition 3.3 *A transitive fuzzy subspace of Ω is called fuzzy orbit.*

Example 3.3 Consider above Example, clearly μ is a fuzzy orbit of Ω.

Theorem 3.2 *Every fuzzy orbit is trivially a fuzzy subspace but the converse may not be true.*

For converse, see the following Example.

Example 3.4 Let $\Omega = S_3 = \{e, y, x, x^2, xy, x^2y\}$ and $G = \{e, y\} \leq S_3$. Let $\upsilon : \Omega \times G \to \Omega$ be an action of G on Ω defined by $\rho^\sigma = \rho\sigma$ for all $\rho \in \Omega$ and $\sigma \in G$. Then clearly Ω is a G-space. Let $\mu : \Omega \to [0,1]$ be defined as $\mu(e) = \mu(y) = \mu(x) = \mu(x^2) = \mu(xy) = \mu(x^2y) = \frac{2}{5}$. Thus μ is a fuzzy subspace of Ω but μ is not a transitive fuzzy subspace of Ω as μ has the following fuzzy orbits:

$$\begin{aligned}
\mu_1 &= \left\{\mu(e) = \mu(y) = \frac{2}{5}\right\}, \\
\mu_2 &= \left\{\mu(x) = \mu(x^2) = \frac{2}{5}\right\}, \\
\mu_3 &= \left\{\mu(xy) = \mu(x^2y) = \frac{2}{5}\right\}.
\end{aligned}$$

Definition 3.4 *Let Ω be a G-space and Ω_μ be a fuzzy subspace. Let $\alpha \in \Omega$. The fuzzy stabilizer is denoted by $G_{\mu(\alpha)}$ and is defined to be $G_{\mu(\alpha)} = \{g \in G : \mu(\alpha^g) = \mu(\alpha)\}$.*

Example 3.5 Consider the above Example. Then

$$G_{\mu(e)} = G_{\mu(y)} = G_{\mu(x)} = G_{\mu(x^2)} = G_{\mu(xy)} = G_{\mu(x^2y)} = \{e\}.$$

Theorem 3.3 *If G_α is G-stabilizer, then $G_{\mu(\alpha)}$ is a fuzzy stabilizer.*

Theorem 3.4 *Let $G_{\mu(\alpha)}$ be a fuzzy stabilizer. Then $G_{\mu(\alpha)} \leq G_\alpha$.*

Remark 3.1 *Let $G_{\mu(\alpha)}$ be a fuzzy stabilizer. Then $G_{\mu(\alpha)} \leq G$.*

§4. Neutrosophic Subspaces

Definition 4.1 *Let Ω be a G-space. Let $A : \Omega \to [0,1]^3$ be a mapping. Then A is called a neutrosophic subspace of Ω if The following conditions are hold.*

(1) $T(\omega^g) \geq T(\omega)$ and $T\left(\omega^{g^{-1}}\right) \leq T(\omega)$,

(2) $I(\omega^g) \leq I(\omega)$ and $I\left(\omega^{g^{-1}}\right) \geq I(\omega)$ and

(3) $F(\omega^g) \leq F(\omega)$ and $F\left(\omega^{g^{-1}}\right) \geq F(\omega)$ for all $\omega \in \Omega$ and $g \in G$.

Example 4.1 Let $\Omega = G = (\mathbb{Z}_4, +)$. Let $\upsilon : \Omega \times G \to \Omega$ be an action of G on Ω which is defined by $\omega^g = \omega + g$. Then Ω is a G-space under this action of G. Let $A : \Omega \to [0,1]^3$ be a mapping which is defined by

$$T(0) = 0.5, T(1) = T(2) = T(3) = 1,$$

$$I(0) = 0.3 \text{ and } I(1) = I(2) = I(3) = 0.1,$$

and

$$F(0) = 0.4 \text{ and } F(1) = F(2) = F(3) = 0.2.$$

Thus clearly A is a neutrosophic subspace as A satisfies conditions (1), (2) and (3).

Theorem 4.1 *A neutrosophic subspace is trivially the generalization of fuzzy subspace.*

Definition 4.2 *Let A be a neutrosophic subspace of the G-space Ω. Then A is called fuzzy transitive subspace if for any α, β from Ω, there exist $g \in G$ such that*

$$\begin{aligned} F(\alpha^g) &= F(\beta), \\ F(\alpha^g) &= F(\beta), \\ F(\alpha^g) &= F(\beta). \end{aligned}$$

Example 4.2 Let $\Omega = G = (\mathbb{Z}_4, +)$. Let $\upsilon : \Omega \times G \to \Omega$ be an action of G on Ω defined by $\omega^g = \omega + g$ for all $\omega \in \Omega$ and $g \in G$. We define $A : \Omega \to [0,1]^3$ by

$$\begin{aligned} T(0) &= \frac{1}{2} \text{ and } T(1) = T(2) = T(3) = 1, \\ I(0) &= \frac{1}{3} \text{ and } I(1) = I(2) = I(3) = 1, \\ F(0) &= \frac{1}{4} \text{ and } F(1) = F(2) = F(3) = 1. \end{aligned}$$

Then clearly A is a neutrosophic transitive subspace of Ω.

Theorem 4.2 *If Ω is transitive G-space, then A is also neutrosophic transitive subspace.*

Proof Suppose that Ω is transitive G-space. Then for any $\alpha, \beta \in \Omega$, there exist $g \in G$ such that $\alpha^g = \beta$. This by taking T on both sides, we get $T(\alpha^g) = T(\beta)$ for all $\alpha, \beta \in \Omega$. Similarly, we can prove it for the other two components I and F. Hence by definition A is a neutrosophic transitive subspace of Ω. □

Definition 4.3 *A neutrosophic transitive subspace of Ω is called neutrosophic orbit.*

Example 4.3 Consider above Example 4.2, clearly A is a neutrosophic orbit of Ω.

Theorem 4.3 *All neutrosophic orbits are trivially the generalization of fuzzy orbits.*

Theorem 4.4 *Every neutrosophic orbit is trivially a neutrosophic subspace but the converse may not be true.*

For converse, see the following Example.

Example 4.4 Let $\Omega = S_3 = \{e, y, x, x^2, xy, x^2y\}$ and $G = \{e, y\} \leq S_3$. Let $\upsilon : \Omega \times G \to \Omega$ be an action of G on Ω defined by $\rho^\sigma = \rho\sigma$ for all $\rho \in \Omega$ and $\sigma \in G$. Then clearly Ω is a G-space. Let $A : \Omega \to [0, 1]$ be defined as

$$\begin{aligned} T(e) &= T(y) = T(x) = T\left(x^2\right) = T(xy) = T\left(x^2y\right) = \frac{2}{5}, \\ I(e) &= I(y) = I(x) = I\left(x^2\right) = I(xy) = I\left(x^2y\right) = \frac{3}{7}, \\ F(e) &= F(y) = F(x) = F\left(x^2\right) = F(xy) = F\left(x^2y\right) = \frac{4}{9}. \end{aligned}$$

Thus A is a neutrosophic subspace of Ω but A is not a neutrosophic transitive subspace of Ω as A has the following neutrosophic orbits:

$$\begin{aligned} T_1 &= \left\{\mu(e) = \mu(y) = \frac{2}{5}\right\}, I_1 = \left\{I(e) = I(y) = \frac{3}{7}\right\}, F_1 = \left\{F(e) = F(y) = \frac{4}{9}\right\}, \\ \mu_2 &= \left\{\mu(x) = \mu\left(x^2\right) = \frac{2}{5}\right\}, I_2 = \left\{I(x) = I\left(x^2\right) = \frac{3}{7}\right\}, F_2 = \left\{F(x) = F\left(x^2\right) = \frac{4}{9}\right\}, \\ \mu_3 &= \left\{\mu(xy) = \mu\left(x^2y\right) = \frac{2}{5}\right\}, I_3 = \left\{I(xy) = I\left(x^2y\right) = \frac{3}{7}\right\}, F_3 = \left\{F(xy) = F\left(x^2y\right) = \frac{4}{9}\right\}. \end{aligned}$$

Definition 4.4 *Let Ω be a G-space and A be a neutrosophic subspace. Let $\alpha \in \Omega$. The neutrosophic stablizer is denoted by $G_{A(\alpha)}$ and is defined to be*

$$G_{A(\alpha)} = \{g \in G : T(\alpha^g) = T(\alpha), I(\alpha^g) = I(\alpha), F(\alpha^g) = F(\alpha)\}.$$

Example 4.5 Consider the above Example 4.4. Then

$$G_{A(e)} = G_{A(y)} = G_{A(x)} = G_{A(x^2)} = G_{A(xy)} = G_{A(x^2y)} = \{e\}.$$

Theorem 4.5 *If G_α is G-stabilizer, then $G_{A(\alpha)}$ is a neutrosophic stabilizer.*

Theorem 4.6 *Every neutrosophic stabilizer is a generalization of fuzzy stabilizer.*

Theorem 4.7 *Let* $G_{A(\alpha)}$ *be a neutrosophic stabilizer. Then* $G_{A(\alpha)} \leq G_{\alpha}$.

Remark 4.1 *Let* $G_{A(\alpha)}$ *be a neutrosophic stabilizer. Then* $G_{A(\alpha)} \leq G$.

§5. Conclusion

In this paper, we introduced fuzzy subspaces and neutrosophic subspaces (generalization of fuzzy subspaces) by applying group actions.Further, we define fuzzy transitivity and neutrosophic transitivity in this paper. Fuzzy orbits and neutrosophic orbits are introduced as well. We also studied some basic properties of fuzzy subspaces as well as neutrosophic subspaces. In the near future, we are applying these concepts in the field of physics, chemistry and other related fields to find the uncertainty in symmetries.

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Neutrosophic Quantum Computer

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Abstract. This paper is a theoretical approach for a potential neutrosophic quantum computer to be built in the future, which is an extension of the classical theoretical quantum computer, into which the indeterminacy is inserted.

Keywords: neutrobit, indeterminacy, neutrosophic quantum, neutrosophic polarization, neutrosophic particle, entangled neutrosophic particles, neutrosophic superposition, neutrosophic dynamic system, neutrosophic Turing machine, neutrosophic quantum functions

1. Introduction

Neutrosophic quantum communication is facilitated by the neutrosophic polarization, that favors the use the neutrosophic superposition and neutrosophic entanglement. The neutrosophic superposition can be linear or non-linear. While into the classical presumptive quantum computers there are employed only the coherent superpositions of two states (*0* and *1*), in the neutrosophic quantum computers one supposes the possibilities of using *coherent superpositions amongst three states* (*0, 1,* and *I* = indeterminacy) and one explores the possibility of using the *decoherent superpositions* as well.

2. Neutrosophic polarization

The *neutrosophic polarization* of a photon is referred to as orientation of the oscillation of the photon: oscillation in one direction is interpreted as *0*, oscillation in opposite direction is interpreted as *1*, while the ambiguous or unknown or vague or fluctuating back and forth direction as *I* (indeterminate).

Thus, the neutrosophic polarization of a photon is *0, 1,* or *I*. Since indeterminacy (*I*) does exist independently from *0* and 1, we cannot use fuzzy nor intuitionistic fuzzy logic / set, but neutrosophic logic / set.

These three neutrosophic values are used for *neutrosophically encoding* the data.

3. Refined neutrosophic polarization

In a more detailed development, one may consider the *refined neutrosophic polarization*, where we refine for example *I* as I_1 (ambiguous direction), I_2(unknown direction), I_3(fluctuating direction), etc.

Or we may refine *0* as 0_1(oscillation in one direction at a high angular speed), 0_2 (oscillation in the same direction at a lower angular speed), etc.

Or we may refine *1* as 1_1 (oscillation in opposite direction at a high angular speed), 1_2(oscillation in the same opposite direction at a lower angular speed), etc.

The refinement of the neutrosophic polarization may be given by one or more parameters that influence the oscillation of the photon.

4. Neutrosophic quantum computer

A *Neutrosophic Quantum Computer* uses phenomena of Neutrosophic Quantum Mechanics, such as neutrosophic superposition and neutrosophic entanglement for neutrosophic data operations.

5. Neutrosophic particle

A *particle* is considered *neutrosophic* if it has some indeterminacy with respect to at least one of its attributes (direction of spinning, speed, charge, etc.).

6. Entangled neutrosophic particle

Two *neutrosophic particles* are *entangled* if measuring the indeterminacy of one of them, the other one will automatically have the same indeterminacy.

7. Neutrosophic data

Neutrosophic Data is data with some indeterminacy.

8. Nutrosophic superposition

Neutrosophic Superposition, that we introduce now for the first time, means superpositions only of *0* and *1* as in qubit (=quantum bit), but also involving indeterminacy (*I*), as in neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic measure, and so on.

9. Indeterminate bit

An *indeterminate bit*, that we introduce now for the first time, is a bit that one does not know if it is *0* or *1*, so we note it by *I* (= indeterminacy).

Therefore, neutrosophic superposition means coherent superposition of *0* and *I*, *1* and *I*, or *0* and *1* and *I*:

$$\begin{pmatrix}0\\I\end{pmatrix}, \begin{pmatrix}1\\I\end{pmatrix}, \text{or} \begin{pmatrix}0\\1\\I\end{pmatrix},$$

ordecoherent superposition of classical bits *0* and *1*, or decoherence between *0, 1, I,* such as:

$$\begin{pmatrix}0\\1\end{pmatrix}_{dec}, \begin{pmatrix}0\\I\end{pmatrix}_{dec}, \begin{pmatrix}1\\I\end{pmatrix}_{dec}, \begin{pmatrix}0\\1\\I\end{pmatrix}_{dec}.$$

10. Neutrobit

A *neutrosophic bit* (or “neutrobit”), that we also introduce for the first time, is any of the above neutrosophicsuperpositions:

$$\begin{pmatrix}0\\I\end{pmatrix}, \begin{pmatrix}1\\I\end{pmatrix}, \begin{pmatrix}0\\1\\I\end{pmatrix}, \text{or } \begin{pmatrix}0\\1\end{pmatrix}_{dec}.$$

A neutrobit acts in two or three universes. A neutrobit can exist with, of course, a *(t, i, f)*-neutrosophic probability, simultaneously as *0* and *I*, or *1* and *I*, or *0, 1,* and *I*, where t = percentage of truth, i = percentage of indeterminacy, and f = percentage of falsehood.

11. Refined neutrosophic quantum computer

Thus, we extend the neutrosophic quantum computers to *refined neutrosophic quantum computers.*

12. Neutrosophic filter polarization

The *neutrosophic filter polarization* of the receiver must match the neutrosophic polarization of the transmitter, of course.

13. Neutrosophic quantum parallelism

The *neutosophicquantum parallelism* is referring to the simultaneously calculations done in each universe, but some universe may contain indeterminate bits, or there might be some decoherencesuperpositions.

14. n-Neutrobit quantum computer

Thus, an *n-neutrobit quantum computer*, whose register has *n*neutrobits, requires $3^n- 1$ numbers created from the digits *0, 1,* and *I* (where *I* is considered as an indeterminate digit).

A register of n classical bits represents any number from *0* to 2^n-1. A register of n qubits such that each bit is in superposition or coherent state, can represent simultaneously all numbers from *0* to 2^n-1.

Being in neutrosophic superposition, a neutrosophic quantum computer can simultaneously act on all its possible states.

15. Neutrosophic quantum gates

Moving towards *neutrosophic quantum gates* involves experiments in which one observes quantum phenomena with indeterminacy.

16. Remarks

Building a *Neutrosophic Quantum Computer* requires a neutrosophic technology that enables the “neutrobits”, either with coherent superpositions involving *I,* or with decoherent superpositions.

Since neither classical quantum computers have been built yet, neutrosophic quantum computers would be as today even more difficult to construct.

But we are optimistic that they will gather momentum in practice one time in the future.

17. Reversibility of a neutrosophic quantum computer

The *reversibility of a neutrosophic quantum computer* is more problematic than that of a classical quantum computer, since amongst its neutrosophic inputs that must be entirely deducible from its neutrosophic outputs, there exists *I* (indeterminacy).

This becomes even more complex when one deals with refined neutrosophic polarisations, such as sub-indeterminacies *(I_1, I_2)* and sub-oscillations in one direction, or in another direction.

A loss of neutrosophic information (i.e. information with indeterminacy) results from *irreversible neutrosophic quantum computers* (when its inputs are not entirely deducible from its outputs). The loss of information, which comes from the loss of heat of the photons, means loss of bits, or qubits, or neutrobits.

18. Neutrosophic dynamical system

Any classical dynamical system is, in some degree neutrosophic, since any dynamical system has some indeterminacy because a dynamic system is interconnected with its environment, hence interconnected with other dynamical systems.

We can, in general, take any *neutrosophic dynamical system*, as a neutrosophic quantum computer, and its dynamicity as a *neutrosophic computation*.

19. Neutrosophic Turing machine and neutrosophic Church-Turing principle

We may talk about a *Neutrosophic Turing Machine*, which is a Turing Machine which works approximately (hence it has some indeterminacy), and about a *Neutrosophic Church-Turing Principle*, which deviates and extends the classical Church-Turing Principle to:

"There exists or can be built a universal 'neutrosophic quantum' [NB: *our inserted words*] that can be programmed to perform any computational task that can be performed by any physical object."

20. Human brain as an example of neutrosophic quantum computer

As a particular case, the human brain is a neutrosophic quantum computer (the neutrosophic hardware), since it works with indeterminacy, vagueness, unknown, incomplete and conflicting information from our-world. And because it processes simultaneously information in conscience and sub-conscience (hence netrosophic parallelism). The human mind is neutrosophic software, since works with approximations and indeterminacy.

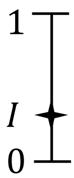
21. Neutrosophic quantum dot

In the classical theoretical quantum computers, a *quantum dot* is represented by one electron contained into a cage of atoms. The electron at the ground state is considered the *0* state of the classical qubit, while the electron at the excited (that is caused by a laser light pulse of a precise duration and wavelength) is considered the *1* state of the classical qubit.

When the laser light pulse that excites the electron is only half of the precise duration, the electron gets in a classical superposition of *0* and *1* states simultaneously.

A right duration-and-wavelength laser light pulse knocks the electron from *0* to *1*, or from *1* to *0*. But, when the laser light pulse is only a fraction of the right duration, then the electron is placed in between the ground state (*0*) and the excited state (*1*), i.e. the electron is placed in indeterminate state (*I*). We denote the indeterminate state by "*I*", as in neeutrosophic logic, and of course $I \in (0, 1)$ in this case.

Hence, one has a *refined neutrosophic logic*, where the indeterminacy is refined infinitely many times, whose values are in the open interval *(0, 1).* Such as



This is a *neutrosophication process*.

22. Neutrosophic NOT function

The *controlled neutrosophic NOT function* is defined by the laser-light application:

$NOT_N: [0,1] \to [0,1]$.

$NOT_N(x) = 1 - x$, where $x \in [0,1]$.

Therefore:

$NOT_N(0) = 1$, $NOT_N(1) = 0$,

and

$NOT_N(I) = 1 - I$.

For example, if indeterminacy $I = 0.3$, then

$NOT_N(0.3) = 1 - 0.3 = 0.7$.

Hence NOT_N (indeterminacy) = indeterminacy.

23. Neutrosophic AND function

The *neutrosophic AND function* is defined as:

$AND_N: [0,1] \times [0,1] \to [0,1]$.

$AND_N(x,y) = min\{x,y\}$, for all $x, y \in [0,1]$.

Therefore:

$AND_N(0,0) = 0$, $AND_N(1,1) = 1$,

$AND_N(0,1) = AND_N(1,0) = 0$.

For indeterminacy,

$AND_N(0,I) = 0$, and $AND_N(1,I) = I$.

Let $I = 0.4$, then:

$AND_N(0,0.4) = 0$, $AND_N(1,0.4) = 0.4$.

Another example with indeterminacies.

$AND_N(0.4,0.6) = 0.4$.

24. Neutrosophic OR function

The *neutrosophic OR function* is defined as:

$OR_N: [0,1] \times [0,1] \to [0,1]$.

$OR_N(x,y) = max\{x,y\}$, for all $x, y \in [0,1]$.

Therefore:

$OR_N(0,0) = 0$, $OR_N(1,1) = 1$,

$OR_N(0,1) = 0, R_N(1,0) = 0$.

For indeterminacy,
$OR_N(0, I) = I$, and $OR_N(1, I) = 1$.
If $I = 0.2$, then $OR_N(0, 0.2) = 0.2$, and $OR_N(1, 0.2) = 0.2$.

25. Neutrosophic IFTHEN function.
The neutrosophic $IFTHEN_N$ function is defined as:
$IFTHEN_N$:$[0,1] \times [0,1] \rightarrow [0,1]$.
$IFTHEN_N(\mathrm{x},\mathrm{y}) = max\{1 - x, y\}$, for all $x, y \in [0,1]$.
$IFTHEN_N$is equivalent to $OR_N(NOT_N(x), y)$, similar to the Boolean logic:
$A \rightarrow B$is equivalent to $non(A)$ or B.
Therefore:
$IFTHEN_N(0,0) = 1$, $IFTHEN_N(1,1) = 1$,
$IFTHEN_N(1,0) = 0$, $IFTHEN_N(0,1) = 1$.
Its neutrosophic value table is:

$IFTHEN_N$

x / ***y***	**0**	**I_α**	**I_β**	**1**
0	1	1- I_α	1- I_β	0
I_α	1	max{1- I_α, I_α}	max{1- I_β, I_α}	I_α
I_β	1	max{1- I_α, I_β}	max{1- I_β, I_β}	I_β
1	1	1	1	1

where I_α, I_β are indeterminacies and they belong to $(0,1)$.
I_α, I_β can be crisp numbers, interval-valued, or in general subsets of [0, 1].

26. Neutrosophic quantum liquids
In classical theoretical quantum computers, there also are used *computing liquids*. In order to store the information, one employs a soup of complex molecules, i.e. molecules with many nuclei. If a molecule is sunk into a magnetic field, each of its nuclei spins either downward (which means state *0*), or upward (which means state *1*).

Precise radio waves bursts change the nuclei spinning from *0* to *1*, and reciprocally. If the radio waves are not at a right amplitude, length and frequency, then the nuclei state is perturbed (which means neither *0*nor *1*, but *I* = indeterminacy). Similarly, this is a *neutrosophication process*.

These spin states (*0*, *1*, or *I*) can be detected with the techniques of NNMR (*Neutrosophic Nuclear Magnetic Resonance*).

The *deneutrosophication* means getting rid of indeterminacy (noise), or at least diminish it as much as possible.

27. Conclusion

This is a theoretical approach and investigation about the possibility of building a quantum computer based on neutrosophic logic. Future investigation in this direction is required. As next research it would be the possibility of extending the Quantum Biocomputer to a potential Neutrosophic Quantum Biocomputer, by taking into consideration the inherent indeterminacy occurring at the microbiological universe.

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On Bipolar Single Valued Neutrosophic Graphs

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Abstract - In this article, we combine the concept of bipolar neutrosophic set and graph theory. We introduce the notions of bipolar single valued neutrosophic graphs, strong bipolar single valued neutrosophic graphs, complete bipolar single valued neutrosophic graphs, regular bipolar single valued neutrosophic graphs and investigate some of their related properties.

Keywords - *Bipolar neutrosophic sets, bipolar single valued neutrosophic graph, strong bipolar single valued neutrosophic graph, complete bipolar single valued neutrosophic graph.*

1. Introduction

Zadeh [32] coined the term ‗degree of membership' and defined the concept of fuzzy set in order to deal with uncertainty. Atanassov [29, 31] incorporated the degree of non-membership in the concept of fuzzy set as an independent component and defined the concept of intuitionistic fuzzy set. Smarandache [12, 13] grounded the term ‗degree of indeterminacy as an independent component and defined the concept of neutrosophic set from the philosophical point of view to deal with incomplete, indeterminate and inconsistent information in real world. The concept of neutrosophic sets is a generalization of the theory of fuzzy sets, intuitionistic fuzzy sets. Each element of a neutrosophic sets has three membership degrees including a truth membership degree, an indeterminacy membership degree, and a falsity membership degree which are within the real standard or nonstandard unit interval]−0, 1+[. Therefore, if their range is restrained within the real standard unit interval [0, 1], the neutrosophic set is easily applied to engineering problems. For this purpose, Wang et al. [17] introduced the concept of a single valued neutrosophic set (SVNS) as a subclass of the neutrosophic set. Recently, Deli et al. [23] defined the concept of bipolar neutrosophic as an extension of the fuzzy sets, bipolar fuzzy sets, intuitionistic fuzzy sets and neutrosophic sets studied some of their related properties including the score, certainty and accuracy functions to compare the bipolar neutrosophic sets. The neutrosophic sets theory of and their extensions have been applied in various part[1, 2, 3, 16, 18, 19, 20, 21, 25, 26, 27, 41, 42, 50, 51, 53].

A graph is a convenient way of representing information involving relationship between objects. The objects are represented by vertices and the relations by edges. When there is vagueness in the description of the objects or in its relationships or in both, it is natural that we need to designe a fuzzy graph Model. The extension of fuzzy graph theory [4, 6, 11] have been developed by several researchers including intuitionistic fuzzy graphs [5, 35, 44] considered the vertex sets and edge sets as intuitionistic fuzzy sets. Interval valued fuzzy graphs [32, 34] considered the vertex sets and edge sets as interval valued fuzzy sets. Interval valued intuitionistic fuzzy graphs [8, 52] considered the vertex sets and edge sets as interval valued intuitionstic fuzzy sets. Bipolar fuzzy graphs [6, 7, 40] considered the vertex sets and edge sets as bipolar fuzzy sets. M-polar fuzzy graphs [39] considered the vertex sets and edge sets as m-polar fuzzy sets. Bipolar intuitionistic fuzzy graphs [9] considered the vertex sets and edge sets as bipolar intuitionistic fuzzy sets. But, when the relations between nodes(or vertices) in problems are indeterminate, the fuzzy graphs and their extensions are failed. For this purpose, Samarandache [10, 11] have defined four main categories of neutrosophic graphs, two based on literal indeterminacy (I), which called them; I-edge neutrosophic graph and I-vertex neutrosophic graph, these concepts are studied deeply and has gained popularity among the researchers due to its applications via real world problems [7, 14, 15, 54, 55, 56]. The two others graphs are based on (t, i, f) components and called them; The (t, i, f)-Edge neutrosophic graph and the (t, i, f)-vertex neutrosophic graph, these concepts are not developed at all. Later on, Broumi et al.[46] introduced a third neutrosophic graph model. This model allows the attachment of truth-membership (t), indeterminacy–membership (i) and falsity- membership degrees (f) both to vertices and edges, and investigated some of their properties. The third neutrosophic graph model is called single valued neutrosophic graph (SVNG for short). The single valued neutrosophic graph is the generalization of fuzzy graph and intuitionistic fuzzy graph. Also the same authors [45] introduced neighborhood degree of a vertex and closed neighborhood degree of vertex in single valued neutrosophic graph as a generalization of neighborhood degree of a vertex and closed neighborhood degree of vertex in fuzzy graph and intuitionistic fuzzy graph. Also, Broumi et al.[47] introduced the concept of interval valued neutrosophic graph as a generalization fuzzy graph, intuitionistic fuzzy graph, interval valued fuzzy graph, interval valued intuitionistic fuzzy graph and single valued neutrosophic graph and have discussed some of their properties with proof and examples. In addition Broumi et al [48] have introduced some operations such as cartesian product, composition, union and join on interval valued neutrosophic graphs and investigate some their properties. On the other hand, Broumi et al [49] have discussed a sub class of interval valued neutrosophic graph called strong interval valued neutrosophic graph, and have introduced some operations such as, cartesian product, composition and join of two strong interval valued neutrosophic graph with proofs. In the literature the study of bipolar single valued neutrosophic graphs (BSVN-graph) is still blank, we shall focus on the study of bipolar single valued neutrosophic graphs in this paper. In the present paper, bipolar neutrosophic sets are employed to study graphs and give rise to a new class of graphs called bipolar single valued neutrosophic graphs. We introduce the notions of bipolar single valued neutrosophic graphs, strong bipolar single valued neutrosophic graphs, complete bipolar single valued neutrosophic graphs, regular bipolar single valued neutrosophic graphs and investigate some of their related properties. This paper is organized as follows;

In section 2, we give all the basic definitions related bipolar fuzzy set, neutrosophic sets, bipolar neutrosophic set, fuzzy graph, intuitionistic fuzzy graph, bipolar fuzzy graph, N-graph and single valued neutrosophic graph which will be employed in later sections. In section 3, we introduce certain notions including bipolar single valued neutrosophic graphs, strong bipolar single valued neutrosophic graphs, complete bipolar single valued neutrosophic graphs, the complement of strong bipolar single valued neutrosophic graphs, regular bipolar single valued neutrosophic graphs and illustrate these notions by several examples, also we described degree of a vertex, order, size of bipolar single valued neutrosophic graphs. In section 4, we give the conclusion.

2. Preliminaries

In this section, we mainly recall some notions related to bipolar fuzzy set, neutrosophic sets, bipolar neutrosophic set, fuzzy graph, intuitionistic fuzzy graph, bipolar fuzzy graph, ***N***-graph and single valued neutrosophic graph relevant to the present work. The readers are referred to [9, 12, 17, 35, 36, 38, 43, 46, 57] for further details and background.

Definition 2.1 [12]. Let U be an universe of discourse; then the neutrosophic set A is an object having the form A = {< x: $T_A(x)$, $I_A(x)$, $F_A(x)$>, x ∈ U}, where the functions T, I, F : U→]$^-$0,1$^+$[define respectively the degree of membership, the degree of indeterminacy, and the degree of non-membership of the element x ∈ U to the set A with the condition:

$^-0 \le$)+ + F)$\le 3^+$. (1)

The functions $T_A(x)$, $I_A(x)$ and are real standard or nonstandard subsets of]−0,1+[. Since it is difficult to apply NSs to practical problems, Wang et al. [16] introduced the concept of a SVNS, which is an instance of a NS and can be used in real scientific and engineering applications.

Definition 2.2 [17]. Let X be a space of points (objects) with generic elements in X denoted by x. A single valued neutrosophic set A (SVNS A) is characterized by truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$. For each point x in X $T_A(x)$, $I_A(x)$, $F_A(x)$ ∈ [0, 1]. A SVNS A can be written as

A = {< x: $T_A(x)$, $I_A(x)$, $F_A(x)$>, x ∈ X} (2)

Definition 2.3 [9]. A bipolar neutrosophic set A in X is defined as an object of the form

A={<x, T^P(x), I^P(x), F^P(x), (x), (x), (x)>: x ∈ X},

where , I^P, :X→ [1, 0] and , I^N, : X→ [-1, 0] .The Positive membership degree (x), (x), F^P(x) denotes the truth membership, indeterminate membership and false membership of an element ∈ X corresponding to a bipolar neutrosophic set A and the negative membership degree T^N (x), I^N (x), F^N (x) denotes the truth membership, indeterminate membership and false membership of an element ∈ X to some implicit counter-property corresponding to a bipolar neutrosophic set A.

Example 2.4 Let X={x_1, x_2, x_3}

$$A=\left\{\begin{matrix} <x_1,0.5,0.3,0.1,-0.6,-0.4,-0.05> \\ <x_2,0.3,0.2,0.7,-0.02,-0.3,-0.02> \\ <x_3,0.8,0.05,0.4,-0.6,-0.6,-0.03> \end{matrix}\right\}$$

is a bipolar neutrosophic subset of X.

Definition 2.5 [9]. Let A_1={<x, T_1^P(x), I_1^P(x), F_1^P(x), T_1^N(x), I_1^N(x), F_1^N(x)>} and A_2={<x, T_2^P(x), I_2^P(x), F_2^P(x), T_2^N(x), I_2^N(x), F_2^N(x) >} be two bipolar neutrosophic sets. Then $A_1 \subseteq A_2$ if and only if

$$T_1^P(x) \leq T_2^P(x)\,,\ I_1^P(x) \leq I_2^P(x),\ F_1^P(x) \geq F_2^P(x)$$

and

$$T_1^N(x) \geq T_2^N(x)\,,\ I_1^N(x) \geq I_2^N(x)\,,\ F_1^N(x) \leq F_2^N(x) \text{ for all } x \in X.$$

Definition 2.6 [9]. Let A_1={<x, T_1^P(x), I_1^P(x), F_1^P(x), T_1^N(x), I_1^N(x), F_1^N(x)>} and A_2={<x, T_2^P(x), I_2^P(x), F_2^P(x), T_2^N(x), I_2^N(x), F_2^N(x) >} be two bipolar neutrosophic sets. Then $A_1 = A_2$ if and only if

$$T_1^P(x) = T_2^P(x)\,,\ I_1^P(x) = I_2^P(x),\ F_1^P(x) = F_2^P(x)$$

and

$$T_1^N(x) = T_2^N(x)\,,\ I_1^N(x) = I_2^N(x)\,,\ F_1^N(x) = F_2^N(x) \text{ for all } x \in X$$

Definition 2.7 [9]. Let A_1={<x, T_1^P(x), I_1^P(x), F_1^P(x), T_1^N(x), I_1^N(x), F_1^N(x)>} and A_2={<x, T_2^P(x), I_2^P(x), F_2^P(x), T_2^N(x), I_2^N(x), F_2^N(x) >} be two bipolar neutrosophic sets. Then their union is defined as:

$$(A_1 \cup A_2)(x) = \begin{pmatrix} \max(T_1^P(x),T_2^P(x)),\frac{I_1^P(x)+I_2^P(x)}{2},\min(T_1^P(x),T_2^P(x)) \\ \min(T_1^N(x),T_2^N(x)),\frac{I_1^N(x)+I_2^N(x)}{2},\max(T_1^N(x),T_2^N(x)) \end{pmatrix}$$

for all x ∈ X.

Definition 2.8 [9]. Let A_1={<x, T_1^P(x), I_1^P(x), F_1^P(x), T_1^N(x), I_1^N(x), F_1^N(x)>} and A_2={<x, T_2^P(x), I_2^P(x), F_2^P(x), T_2^N(x), I_2^N(x), F_2^N(x) >} be two bipolar neutrosophic sets. Then their intersection is defined as:

$$(A_1 \cap A_2)(x) = \begin{pmatrix} \min(T_1^P(x),T_2^P(x)),\frac{I_1^P(x)+I_2^P(x)}{2},\max(T_1^P(x),T_2^P(x)) \\ \max(T_1^N(x),T_2^N(x)),\frac{I_1^N(x)+I_2^N(x)}{2},\min(T_1^N(x),T_2^N(x)) \end{pmatrix}$$

for all x ∈ X.

Definition 2.9 [9]. Let A_1={<x, T_1^P(x), I_1^P(x), F_1^P(x), T_1^N(x), I_1^N(x), F_1^N(x)>: x ∈ X } be a bipolar neutrosophic set in X. Then the complement of A is denoted by A^c and is defined by

$$T_{A^c}^P(x)=\{1^P\}-T_A^P(x),\quad I_{A^c}^P(x)=\{1^P\}-I_A^P(x),\quad F_{A^c}^P(x)=\{1^P\}-F_A^+(x)$$

and

$$T_{A^c}^N(x)=\{1^N\}-T_A^N(x),\quad I_{A^c}^N(x)=\{1^N\}-I_A^N(x),\quad F_{A^c}^N(x)=\{1^N\}-F_A^N(x)$$

Definition 2.10 [43]. A fuzzy graph is a pair of functions G = (σ, μ) where σ is a fuzzy subset of a non empty set V and is a symmetric fuzzy relation on σ. i.e σ : V → [0,1] and μ: VxV→[0,1] such that μ(uv) ≤ σ(u) ∧ σ(v) for all u, v ∈ V where uv denotes the edge between u and v and σ(u) ∧ σ(v) denotes the minimum of σ(u) and σ(v). σ is called the fuzzy vertex set of V and is called the fuzzy edge set of E.

Definition 2.11[38]: By a *N*-graph G of a graph , we mean a pair G= (μ_1, μ_2) where is an *N*-function in V and is an *N*-relation on E such that μ_2(u,v) max (μ (u), μ_1(v)) all u, v ∈ V.

Definition 2.12[35] : An Intuitionistic fuzzy graph is of the form G = (V, E) where

i. V={v_1, v_2,…., v_n} such that μ_1: V→ [0,1] and γ_1: V [0,1] denote the degree of membership and non-membership of the element v_i ∈ V, respectively, and

$$0 \leq \mu_1(v_i) + \gamma_1(v_i)) \leq 1$$

for every ∈ V, (i = 1, 2, ……. n),

ii. E V x V where : VxV→[0,1] and : VxV→ [0,1] are such that

$$(v_i, v_j) \leq \min [\mu\ (v_i), \mu_1(v_j)] \text{ and } \gamma_2(v_i, v_j) \geq \max [\gamma\ (v_i), \gamma_1(v_j)]$$

and 0 ≤ $\mu_2(v_i, v_j) + \gamma_2(v_i, v_j) \leq 1$ for every (v_i, v_j) ∈ E, (i, j = 1,2, ……. n)

Definition 2.13 [57]. Let X be a non-empty set. A bipolar fuzzy set A in X is an object having the form A = {(x, μ_A^P(x), μ_A^N(x)) | x ∈ X}, where μ_A^P(x): X → [0, 1] and μ_A^N(x): X → [−1, 0] are mappings.

Definition 2.14 [57] Let X be a non-empty set. Then we call a mapping

$$A = (\mu_A^P, \mu\ ): X \times X \to [-1, 0] \times [0, 1]$$

a bipolar fuzzy relation on X such that (x, y) ∈ [0, 1] and (x, y) ∈ [−1, 0].

Definition 2.15 [36]. Let *A* = (μ_A^P,) and *B* = (μ_B^P,) be bipolar fuzzy sets on a set *X*. If *A* = (μ_A^P,) is a bipolar fuzzy relation on a set *X*, then *A* =(μ_A^P,) is called a bipolar fuzzy relation on

$$B = (\mu_B^P,\ \mu_B^N) \text{ if } \ (x, y) \leq \min(\mu_A^P(x), \ (y))$$

and

$$(x, y) \geq \max(\mu_A^N(x), \ (y) \text{ or all } x, y \in X.$$

A bipolar fuzzy relation *A* on *X* is called symmetric if (*x, y*) = (*y, x*) and (*x, y*) = (*y, x*) for all *x, y* ∈ *X.*

Definition 2.16 [36]. A bipolar fuzzy graph of a graph = (V, E) is a pair G = (A,B), where A = (μ_A^P,) is a bipolar fuzzy set in V and B = (μ_B^P, μ_B^N) is a bipolar fuzzy set on E V x V such that μ_B^P (xy) ≤ min{ μ_A^P (x), μ_A^P (y)} for all xy ∈ E , μ_B^N (xy) min{μ_A^N(x), μ_A^N(y)} for all xy ∈ E and μ_B^P(xy) = μ_B^N(xy) = 0 for all xy ∈ $\tilde{V}^2$ –E. Here A is called bipolar fuzzy vertex set of V, B the bipolar fuzzy edge set of E.

Definition 2.17 [46] A single valued neutrosophic graph (SVNG) of a graph G^*= (V, E) is a pair G = (A, B), where

1.V= {v_1, v_2,..., v_n} such that T_A:V→[0, 1], I_A:V→[0, 1] and F_A:V→[0, 1] denote the degree of truth-membership, degree of indeterminacy-membership and falsity-membership of the element v_i ∈ V, respectively, and

$$0\leq T_A(v_i) + I_A(v_i) + F_A(v_i) \leq 3$$

for every v_i ∈ V (i=1, 2, …,n)

2. E ⊆ V x V where T_B:V x V →[0, 1], I_B:V x V →[0, 1] and F_B:V x V →[0, 1] are such that

$$T_B(v_i, v_j) \leq \min\,[T_A(v_i), T_A(v_j)], I_B(v_i, v_j) \geq \max\,[I_A(v_i), I_A(v_j)]$$

and

$$F_B(v_i, v_j) \geq \max\,[F_A(v_i), F_A(v_j)]$$

and

$$0\leq T_B(v_i, v_j) + I_B(v_i, v_j) + F_B(v_i, v_j) \leq 3$$

for every (v_i, v_j) ∈ E (i, j = 1, 2,…, n)

Definition 2.18 [46]: Let G=(V, E) be a single valued neutrosophic graph. Then the degree of a vertex v is defined by d(v)= $(d_T(v), d_I(v), d_F(v))$ where

$$d_T(v)=\sum_{u\neq v} T_B(u, v),\ d_I(v)=\sum_{u\neq v} I_B(u, v) \text{ and } d_F(v)=\sum_{u\neq v} F_B(u, v)$$

3. Bipolar Single Valued Neutrosophic Graph

Definition 3.1. Let X be a non-empty set. Then we call a mapping A = (x, T^P(x), I^P(x), F^P(x), T^N(x), I^N(x), F^N(x)):X × X → [−1, 0] × [0, 1] a bipolar single valued neutrosophic relation on X such that T_A^P(x, y) ∈ [0, 1], I_A^P(x, y) ∈ [0, 1], F_A^P(x, y) ∈ [0, 1], and T_A^N(x, y) ∈ [−1, 0], I_A^N(x, y) ∈ [−1, 0], F_A^N(x, y) ∈ [−1, 0].

Definition 3.2. Let A = (T_A^P, I_A^P, F_A^P, T_A^N, I_A^N, F_A^N) and B = (T_B^P, I_B^P, F_B^P, T_B^N, I_B^N, F_B^N) be bipolar single valued neutrosophic graph on a set X. If B = (T_B^P, I_B^P, F_B^P, T_B^N, I_B^N, F_B^N) is a bipolar single valued neutrosophic relation on A = (T_A^P, I_A^P, F_A^P, T_A^N, I_A^N, F_A^N) then

$T_B^P(x, y) \leq \min(T_A^P(x), T_A^P(y))$, $T_B^N(x, y) \geq \max(T_A^N(x), T_A^N(y))$

$I_B^P(x, y) \geq \max(I_A^P(x), I_A^P(y))$, $I_B^N(x, y) \leq \min(I_A^N(x), I_A^N(y))$

$F_B^P(x, y) \geq \max(F_A^P(x), F_A^P(y))$, $F_B^N(x, y) \leq \min(F_A^N(x), F_A^N(y))$ for all $x, y \in X$.

A bipolar single valued neutrosophic relation B on X is called symmetric if

$$T_B^P(x, y) = T_B^P(y, x),\ I_B^P(x, y) = I_B^P(y, x),\ F_B^P(x, y) = F_B^P(y, x)$$

and

$$T_B^N(x, y) = T_B^N(y, x),\ I_B^N(x, y) = I_B^N(y, x),\ F_B^N(x, y) = F_B^N(y, x), \text{ for all } x, y \in X.$$

Definition 3.3. A bipolar single valued neutrosophic graph of a graph $G^*=$ (V, E) is a pair G = (A, B), where A = $(T_A^P, I_A^P, F_A^P, T_A^N, I_A^N, F_A^N)$ is a bipolar single valued neutrosophic set in V and B = $(T_B^P, I_B^P, F_B^P, T_B^N, I_B^N, F_B^N)$ is a bipolar single valued neutrosophic set in $\tilde{V}^2$ such that

$$T_B^P(\boldsymbol{v_i}, \boldsymbol{v_j}) \le \min(T_A^P(\boldsymbol{v_i}), T_A^P(\boldsymbol{v_j}))$$
$$I_B^P(\boldsymbol{v_i}, \boldsymbol{v_j}) \ge \max(I_A^P(\boldsymbol{v_i}), I_A^P(\boldsymbol{v_j}))$$
$$F_B^P(\boldsymbol{v_i}, \boldsymbol{v_j}) \ge \max(F_A^P(\boldsymbol{v_i}), F_A^P(\boldsymbol{v_j}))$$

and

$$T_B^N(\boldsymbol{v_i}, \boldsymbol{v_j}) \ge \max(T_A^N(\boldsymbol{v_i}), T_A^N(\boldsymbol{v_j}))$$
$$I_B^N(\boldsymbol{v_i}, \boldsymbol{v_j}) \le \min(I_A^N(\boldsymbol{v_i}), I_A^N(\boldsymbol{v_j}))$$
$$F_B^N(\boldsymbol{v_i}, \boldsymbol{v_j}) \le \min(F_A^N(\boldsymbol{v_i}), F_A^N(\boldsymbol{v_j}))$$

for all $\boldsymbol{v_i}\boldsymbol{v_j} \in \tilde{V}^2$.

Notation : An edge of BSVNG is denoted by $e_{ij} \in$ E or $\boldsymbol{v_i}\boldsymbol{v_j} \in$ E

Here the sextuple (v_i, T_A^P(v_i) , I_A^P(v_i) , F_A^P(v_i), T_A^N(v_i) , I_A^N(v_i), F_A^N(v_i)) denotes the positive degree of truth-membership, the positive degree of indeterminacy-membership, the positive degree of falsity-membership, the negative degree of truth-membership, the negative degree of indeterminacy-membership, the negative degree of falsity- membership of the vertex vi.

The sextuple (e_{ij} , T_B^P, I_B^P, F_B^P, T_B^N , I_B^N, F_B^N) denotes the positive degree of truth-membership, the positive degree of indeterminacy-membership, the positive degree of falsity-membership, the negative degree of truth-membership, the negative degree of indeterminacy-membership, the negative degree of falsity- membership of the edge relation = $(\boldsymbol{v_i}, \boldsymbol{v}_j)$ on V× V.

Note 1. (i) When $T_A^P = I$ = = 0 and = = = 0 for some i and j, then there is no edge between v_i and v_j .

Otherwise there exists an edge between v_i and .

(ii) If one of the inequalities is not satisfied in (1) and (2), then G is not an BSVNG

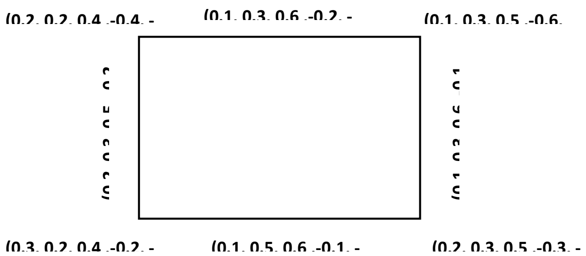

Figure 1: Bipolar single valued neutrosophic graph.

Proposition 3.5: A bipolar single valued neutrosophic graph is the generalization of fuzzy graph

Proof: Suppose G= (A, B) be a bipolar single valued neutrosophic graph. Then by setting the positive indeterminacy-membership, positive falsity-membership and negative truth-membership, negative indeterminacy-membership, negative falsity-membership values of vertex set and edge set equals to zero reduces the bipolar single valued neutrosophic graph to fuzzy graph.

Example 3.6:

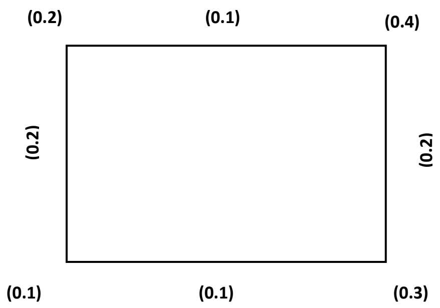

Figure 2: Fuzzy graph

Proposition 3.7: A bipolar single valued neutrosophic graph is the generalization of intuitionistic fuzzy graph

Proof: Suppose G= (A, B) be a bipolar single valued neutrosophic graph. Then by setting the positive indeterminacy-membership, negative truth-membership, negative indeterminacy-membership, negative falsity-membership values of vertex set and edge set equals to zero reduces the bipolar single valued neutrosophic graph to intuitionistic fuzzy graph.

Example 3.8

Figur 3: Intuitionistic fuzzy graph

Proposition 3.9: A bipolar single valued neutrosophic graph is the generalization of single valued neutrosophic graph

Proof: Suppose G= (A, B) be a bipolar single valued neutrosophic graph. Then by setting the negative truth-membership, negative indeterminacy-membership, negative falsity-membership values of vertex set and edge set equals to zero reduces the bipolar single valued neutrosophic graph to single valued neutrosophic graph.

Example 3.10

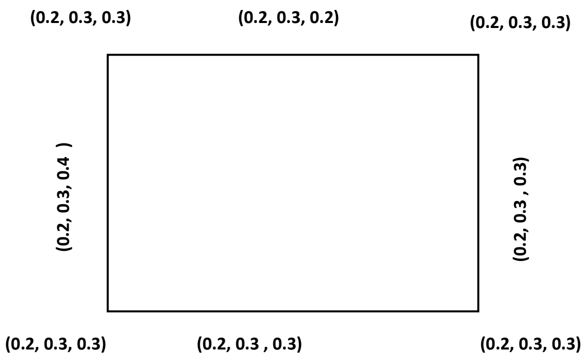

Figure 4: Single valued neutrosophic graph

Proposition 3.11: A bipolar single valued neutrosophic graph is the generalization of bipolar intuitionstic fuzz graph

Proof: Suppose G= (A, B) be a bipolar single valued neutrosophic graph. Then by setting the positive indeterminacy-membership, negative indterminacy-membership values of vertex set and edge set equals to zero reduces the bipolar single valued neutrosophic graph to bipolar intuitionstic fuzzy graph

Example 3.12

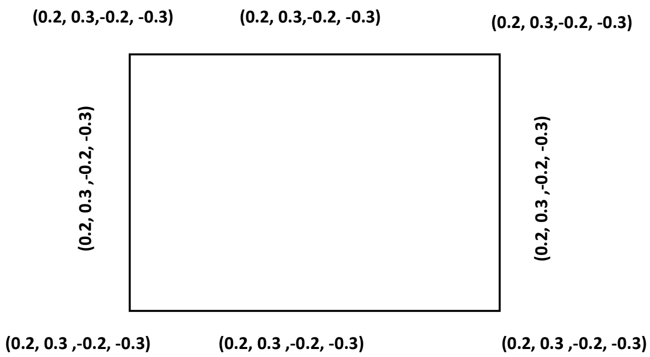

Figure 5: Bipolar intuitionistic fuzzy graph.

Proposition 3.13: A bipolar single valued neutrosophic graph is the generalization of *N*-graph

Proof: Suppose G= (A, B) be a bipolar single valued neutrosophic graph. Then by setting the positive degree membership such truth-membership, indeterminacy- membership, falsity-membership and negative indeterminacy-membership, negative falsity-membership values of vertex set and edge set equals to zero reduces the single valued neutrosophic graph to *N*-graph.

Example 3.14:

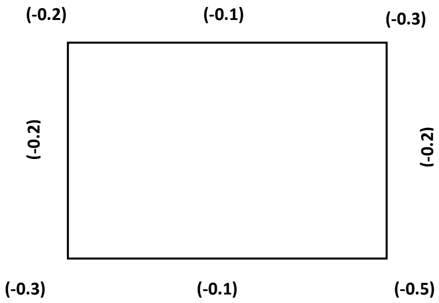

Figure 6: *N*- graph

Definition 3.15. A bipolar single valued neutrosophic graph that has neither self loops nor parallel edge is called simple bipolar single valued neutrosophic graph.

Definition 3.16. A bipolar single valued neutrosophic graph is said to be connected if every pair of vertices has at least one bipolar single valued neutrosophic graph between them, otherwise it is disconnected.

Definition 3.17. When a vertex is end vertex of some edges $(\mathbf{v_i}, \mathbf{v_j})$ of any BSVN-graph G= (A, B). Then and $(\mathbf{v_i},$) are said to be **incident** to each other.

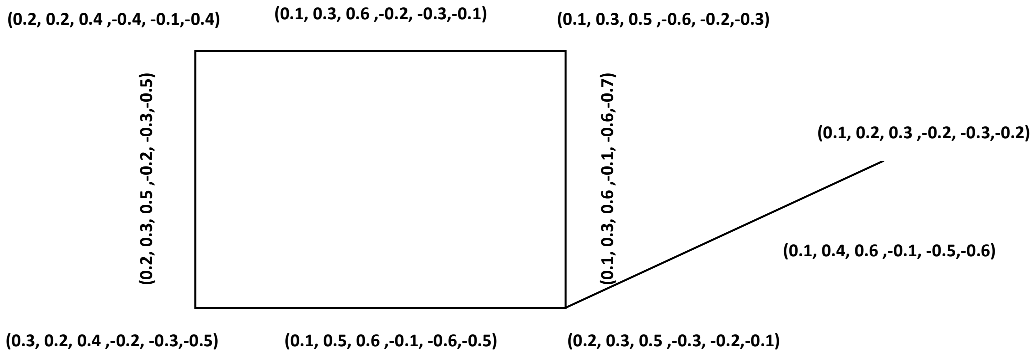

Figure 7 : Incident BSVN-graph

In this graph v_2v_3, v_3v_4 and v_3v_5 are incident on v_3.

Definition 3.18 Let G= (V, E) be a bipolar single valued neutrosophic graph. Then the degree of any vertex **v** is sum of positive degree of truth-membership, positive sum of degree of indeterminacy-membership, positive sum of degree of falsity-membership, negative degree of truth-membership, negative sum of degree of indeterminacy-membership, and negative sum of degree of falsity-membership of all those edges which are incident on vertex **v** denoted by d(v)= $(d_T^P(v), d_I^P(v), d_F^P(v), d_T^N(v), d_I^N(v), d_F^N(v))$ where

$d_T^P(v) = \sum_{u \neq v} T_B^P(u, v)$ denotes the positive T- degree of a vertex v,
$d_I^P(v) = \sum_{u \neq v} I_B^P(u, v)$ denotes the positive I- degree of a vertex v,
$d_F^P(v) = \sum_{u \neq v} F_B^P(u, v)$ denotes the positive F- degree of a vertex v,
$d_T^N(v) = \sum_{u \neq v} T_B^N(u, v)$ denotes the negative T- degree of a vertex v,
$d_I^N(v) = \sum_{u \neq v} I_B^N(u, v)$ denotes the negative I- degree of a vertex v,

$d_F^N(v) = \sum_{u \neq v} F_B^N(u, v)$ denotes the negative F- degree of a vertex v

Definition 3.19: The minimum degree of G is

$$\delta(G) = (\delta_T^P(G), \delta_I^P(G), \delta_F^P(G), \delta_T^N(G), \delta_I^N(G), \delta_F^N(G))$$

where

$\delta_T^P(G) = \wedge \{d_T^P(v) \mid v \in V\}$ denotes the minimum positive T- degree,
$\delta_I^P(G) = \wedge \{d_I^P(v) \mid v \in V\}$ denotes the minimum positive I- degree,
$\delta_F^P(G) = \wedge \{d_F^P(v) \mid v \in V\}$ denotes the minimum positive F- degree,
$\delta_T^N(G) = \wedge \{d_T^N(v) \mid v \in V\}$ denotes the minimum negative T- degree,
$\delta_I^N(G) = \wedge \{d_I^N(v) \mid v \in V\}$ denotes the minimum negative I- degree,
$\delta_F^N(G) = \wedge \{d_F^N(v) \mid v \in V\}$ denotes the minimum negative F- degree

Definition 3.20: The maximum degree of G is

$$\Delta(G) = (\Delta_T^P(G), \Delta_I^P(G), \Delta_F^P(G), \Delta_T^N(G), \Delta_I^N(G), \Delta_F^N(G))$$

where

$\Delta_T^P(G) = \vee \{d_T^P(v) \mid v \in V\}$ denotes the maximum positive T- degree,
$\Delta_I^P(G) = \vee \{d_I^P(v) \mid v \in V\}$ denotes the maximum positive I- degree,
$\Delta_F^P(G) = \vee \{d_F^P(v) \mid v \in V\}$ denotes the maximum positive F- degree,
$\Delta_T^N(G) = \vee \{d_T^N(v) \mid v \in V\}$ denotes the maximum negative T- degree,
$\Delta_I^N(G) = \vee \{d_I^N(v) \mid v \in V\}$ denotes the maximum negative I- degree,
$\Delta_F^N(G) = \vee \{d_F^N(v) \mid v \in V\}$ denotes the maximum negative F- degree

Example 3.21. Let us consider a bipolar single valued neutrosophic graph G= (A, B) of = (V, E), such that V = $\{v_1, \quad, v_3, v_4\}$, E = $\{(v_1, v_2), (v_2, v_3), (v_3, v_4), (v_4, v_1)\}$

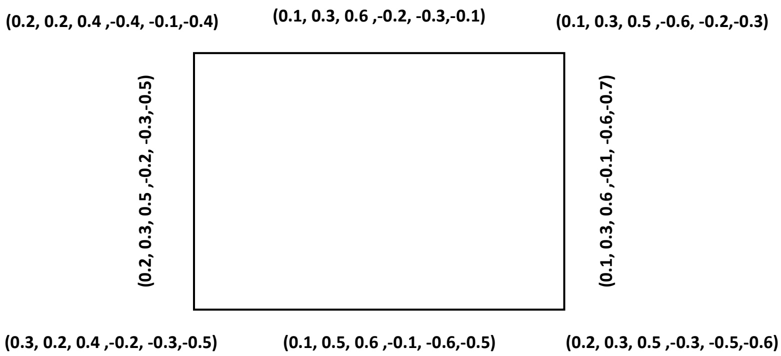

Figure 8: Degree of a bipolar single valued neutrosophic graph G.

In this example, the degree of is (0.3, 0.6, 1.1, -0.4, -0.6, -0.6). the degree of is (0.2, 0.6, 1.2, -0.3, -0.9, -0.8). the degree of is (0.2, 0.8, 1.2, -0.2, -1.2, -1.2). the degree of is (0.3, 0.8, 1.1, -0.3, -0.9, -1)

Order and size of a bipolar single valued neutrosophic graph is an important term in bipolar single valued neutrosophic graph theory. They are defined below.

Definition 3.22: Let G =(V, E) be a BSVNG. The order of G, denoted O(G) is defined as O(G)= $(O_T(G), O_I(G), O_F(G), O_T^N(G), O_I^N(G), O_F^N(G))$, where

$O_T^p(G) = \sum_{v \in V} T_1^p(v)$ denotes the positive T- order of a vertex v,
$O_I^p(G) = \sum_{v \in V} I_1^p(v)$ denotes the positive I- order of a vertex v,
$O_F^p(G) = \sum_{v \in V} F_1^p(v)$ denotes the positive F- order of a vertex v,
$O_T^N(G) = \sum_{v \in V} T_1^N(v)$ denotes the negative T- order of a vertex v,
$O_I^N(G) = \sum_{v \in V} I_1^N(v)$ denotes the negative I- order of a vertex v,
$O_F^N(G) = \sum_{v \in V} F_1^N(v)$ denotes the negative F- order of a vertex v.

Definition 3.23: Let G =(V, E) be a BSVNG. The size of G, denoted S(G) is defined as $S(G) = (S_T^p(G), S_I^p(G), S_F^p(G), S_T^N(G), S_I^N(G), S_F^N(G))$, where

$S_T^p(G) = \sum_{u \neq v} T_2^p(u, v)$ denotes the positive T- size of a vertex v,
$S_I^p(G) = \sum_{u \neq v} I_2^p(u, v)$ denotes the positive I- size of a vertex v,
$S_F^p(G) = \sum_{u \neq v} F_2^p(u, v)$ denotes the positive F- size of a vertex v,
$S_T^N(G) = \sum_{u \neq v} T_2^N(u, v)$ denotes the negative T- size of a vertex v,
$S_I^N(G) = \sum_{u \neq v} I_2^N(u, v)$ denotes the negative I- size of a vertex v,
$S_F^N(G) = \sum_{u \neq v} F_2^N(u, v)$ denotes the negative F- size of a vertex v.

Definition 3.24 A bipolar single valued neutrosophic graph G=(V, E) is called constant if degree of each vertex is $k = (k_1, k_2, k_3, k_4, k_5, k_6)$. That is, $d(v) = (k_1, k_2, k_3, k_4, k_5, k_6)$ for all $v \in V$.

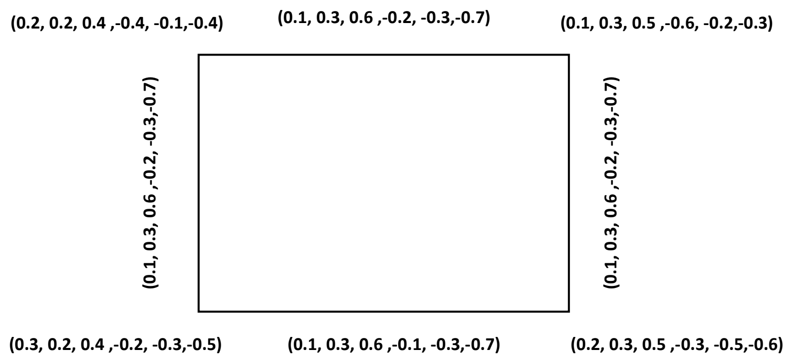

Figure 9: Constant bipolar single valued neutrosophic graph G.

In this example, the degree of v_1, v_2, v_3, v_4 is (0.2, 0.6, 1.2, -0.4, -0.6, -1.4).

O(G)=(0.8, 1, 1.8, -1.5, -1.1, -1.8)
S(G) =(0.4, 1.2, 2.4 ,-0.7, -1.2, -2.8)

Remark 3.25. G is a $(k_i, k_j, k_l, k_m, k_n, k_o)$-constant BSVNG iff $\delta = \Delta = k$, where $k = k_i + k_j + k_l + k_m + k_n + k_o$.

Definition 3.26. A bipolar single valued neutrosophic graph G= (A, B) is called strong bipolar single valued neutrosophic graph if

$T_B^P(u, v) = \min(T_A^P(u), T_A^P(v))$,
$I_B^P(u, v) = \max(I_A^P(u), I_A^P(v))$,
$F_B^P(u, v) = \max(F_A^P(u), F_A^P(v))$,
$T_B^N(u, v) = \max(T_A^N(u), T_A^N(v))$,

$$I_B^N(u,v) = \min(I_A^N(u), I_A^N(v)),$$
$$F_B^N(u,v) = \min(F_A^N(u), F_A^N(v))$$

for all (u, v) ∈ E

Example 3.27. Consider a strong BSVN-graph G such that V = $\{v_1, v_2, v_3, v_4\}$and E = $\{(v_1, v_2),(v_2, v_3), (v_3, v_4), (v_4, v_1)\}$

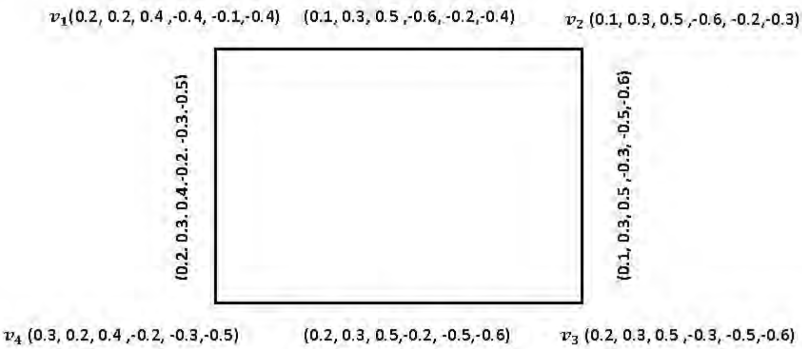

Figure 10: Strong bipolar single valued neutrosophic graph G.

Definition 3.28. A bipolar single valued neutrosophic graph G= (A, B) is called complete if

$$T_B^P(u,v) = \min(T_A^P(u), T_A^P(v)),$$
$$I_B^P(u,v) = \max(I_A^P(u), I_A^P(v)),$$
$$F_B^P(u,v) = \max(F_A^P(u), F_A^P(v)),$$
$$T_B^N(u,v) = \max(T_A^N(u), T_A^N(v)),$$
$$I_B^N(u,v) = \min(I_A^N(u), I_A^N(v)),$$
$$F_B^N(u,v) = \min(F_A^N(u), F_A^N(v))$$

for all u, v ∈ V.

Example 3.29. Consider a complete BSVN-graph G such that V = $\{v_1, v_2, v_3, v_4\}$and E = $\{(v_1, v_2),(v_2, v_3), (v_3, v_4), (v_4, v_1), (v_1, v_3), (v_2, v_4)\}$

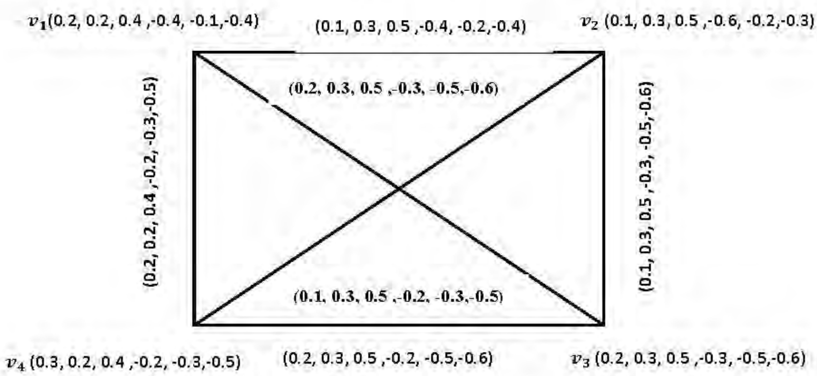

Figure 11: Complete bipolar single valued neutrosophic graph G.

$$d(v_1)= (0.5, 0.8, 1.4, -0.9, -1, -1.5)$$
$$d(v_2)= (0.4, 0.9, 1.5, -1.2, -1, -1.6)$$
$$d(v_3)= (0.4, 0.9, 1.5, -0.7, -1.3, -1.7)$$

$d(v_4)$= (0.5, 0.8, 1.4, -0 .6, -1.1, -1.6)

Definition 3.30. The complement of a bipolar single valued neutrosophic graph G = (A, B) of a graph G^*= (V, E) is a bipolar single valued neutrosophic graph $\bar{G}$ = ($\bar{A}$, $\bar{B}$) of $\overline{G^*}$ = (V,V ×V), where $\bar{A}$ = A = (T_A^P, I_A^P, F_A^P, T_A^N, I_A^N, F_A^N) and $\bar{B}$ = ($\overline{T_B^P}$, $\overline{I_B^P}$, $\overline{F_B^P}$, $\overline{T_B^N}$, $\overline{I_B^N}$, $\overline{F_B^N}$) is defined by

$$\bar{T}_B^P(u,v) = \min(T_A^P(u), T_A^P(v)) - T_B^P(u,v) \text{ for all } u, v \in V, uv \in \tilde{V}^2$$
$$\bar{I}_B^P(u,v) = \max(I_A^P(u), I_A^P(v)) - I_B^P(u,v) \text{ for all } u, v \in V, uv \in \tilde{V}^2$$
$$\bar{F}_B^P(u,v) = \max(F_A^P(u), F_A^P(v)) - F_B^P(u,v) \text{ for all } u, v \in V, uv \in \tilde{V}^2$$
$$\bar{T}_B^N(u,v) = \max(T_A^N(u), T_A^N(v)) - T_B^N(u,v) \text{ for all } u, v \in V, uv \in \tilde{V}^2$$
$$\bar{I}_B^N(u,v) = \min(I_A^N(u), I_A^N(v)) - I_B^N(u,v) \text{ for all } u, v \in V, uv \in \tilde{V}^2$$
$$\bar{F}_B^N(u,v) = \min(F_A^N(u), F_A^N(v)) - F_B^N(u,v) \text{ for all } u, v \in V, uv \in \tilde{V}^2$$

Proposition 3.31: The complement of complete BSVN-graph is a BSVN-graph with no edge. Or if G is a complete then in $\bar{G}$ the edge is empty.

Proof. Let G= (V, E) be a complete BSVN-graph . $T_B^P(u,v)$ =min($T_A^P(u)$, $T_A^P(v)$), So $T_B^P(u,v)$ =min($T_A^P(u)$, $T_A^P(v)$), $T_B^N(u,v)$ =max($T_A^N(u)$, $T_A^N(v)$),

$I_B^P(u,v)$ =max($T_A^P(u)$, $T_A^P(v)$), $I_B^N(u,v)$ =min($I_A^N(u)$, $I_A^N(v)$),
$F_B^P(u,v)$ =max($T_A^P(u)$, $T_A^P(v)$), $F_B^N(u,v)$ =min($F_A^N(u)$, $F_A^N(v)$)

for all $u, v \in$ V. Hence in $\bar{G}$,

$$\begin{aligned}\bar{T}_B^P &= \min(T_A^P(u), T_A^P(v)) - T_B^P(u,v) \text{ for all } u, v \in V\\ &= \min(T_A^P(u), T_A^P(v)) - \min(T_A^P(u), T_A^P(v)) \text{ for all } u, v \in V\\ &= 0 \quad \text{for all } u, v \in V\end{aligned}$$

and

$$\begin{aligned}\bar{I}_B^P &= \max(I_A^P(u), I_A^P(v)) - I_B^P(u,v) \text{ for all } u, v \in V\\ &= \max(I_A^P(u), I_A^P(v)) - \max(I_A^P(u), I_A^P(v)) \text{ for all } u, v \in V\\ &= 0 \quad \text{for all } u, v \in V\end{aligned}$$

Also

$$\begin{aligned}\bar{F}_B^P &= \max(F_A^P(u), F_A^P(v)) - F_B^P(u,v) \text{ for all } u, v \in V\\ &= \max(F_A^P(u), F_A^P(v)) - \max(F_A^P(u), F_A^P(v)) \text{ for all } u, v \in V\\ &= 0 \quad \text{for all } u, v \in V\end{aligned}$$

Similarly

$$\begin{aligned}\bar{T}_B^N &= \max(T_A^N(u), T_A^N(v)) - T_B^N(u,v) \text{ for all } u, v \in V\\ &= \max(T_A^N(u), T_A^N(v)) - \max(T_A^N(u), T_A^N(v)) \text{ for all } u, v \in V\\ &= 0 \quad \text{for all } u, v \in V\end{aligned}$$

and

$$\bar{I}_B^P = \min(I_A^N(u), I_A^N(v)) - I_B^N(u,v) \text{ for all } u, v \in V$$
$$= \min(I_A^N(u), I_A^N(v)) - \min(I_A^N(u), I_A^N(v)) \text{ for all } u, v \in V$$
$$= 0 \quad \text{for all } u, v \in V$$

Also

$$\bar{F}_B^N = \min(F_A^N(u), F_A^N(v)) - F_B^N(u,v) \text{ for all } u, v \in V$$
$$= \min(F_A^N(u), F_A^N(v)) - \min(F_A^N(u), F_A^N(v)) \text{ for all } u, v \in V$$
$$= 0 \quad \text{for all } u, v \in V$$

$(\bar{T}_B^P, \bar{I}_B^P, \bar{F}_B^P, \bar{T}_B^N, \bar{I}_B^N, \bar{F}_B^N)$. Thus $(\bar{T}_B^P, \bar{I}_B^P, \bar{F}_B^P, \bar{T}_B^N, \bar{I}_B^N, \bar{F}_B^N)$= (0 , 0, 0, 0, 0). Hence the edge set of $\bar{G}$ is empty if G is a complete BSVNG.

Definition 3.32: A regular BSVN-graph is a BSVN-graph where each vertex has the same number of open neighbors degree. d_N(v)= $(d_{NT}^P(v),\ d_{NI}^P(v),\ d_{NF}^P(v),\ d_{NT}^N(v),\ d_I^N(v),\ d_{NF}^N(v))$.

The following example shows that there is no relationship between regular BSVN-graph and a constant BSVN-graph

Example 3.33. Consider a graph G^* such that V= $\{v_1, v_2, v_3, v_4\}$, E=$\{v_1v_2, v_2v_3, v_3v_4, v_4v_1\}$. Let A be a single valued neutrosophic subset of V and le B a single valued neutrosophic subset of E denoted by

	0.2	0.2	0.2	0.2
	0.2	0.2	0.2	0.2
	0.4	0.4	0.4	0.4
	-0.4	-0.4	-0.4	-0.4
	-0.1	-0.4	-0.1	-0.1
	-0.4	-0.4	-0.4	-0.4

	0.1	0.1	0.1	0.2
	0.3	0.3	0.5	0.3
	0.6	0.6	0.6	0.5
	-0.2	-0.1	-0.1	-0.2
	-0.3	-0.6	-0.6	-0.3
	-0.5	-0.7	-0.7	-0.5

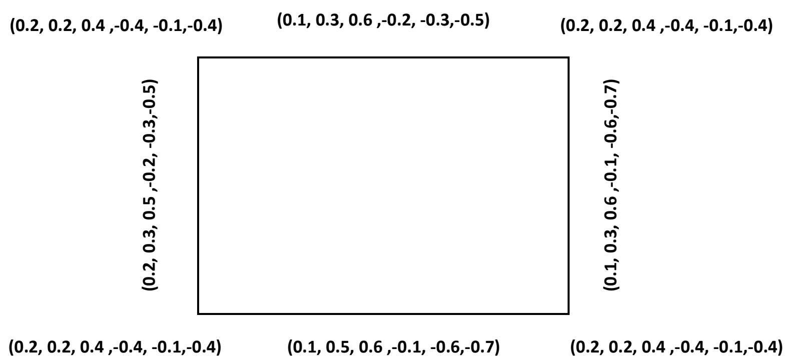

Figure 12: Regular bipolar single valued neutrosophic graph G.

By routing calculations show that G is regular BSVN-graph since each open neighbors degree is same , that is (0.4, 0.4, 0.8 ,-0.8, -0.2,-0.8). But it is not constant BSVN-graph since degree of each vertex is not same.

Definition 3.34 :Let G=(V, E) be a bipolar single valued neutrosophic graph. Then the totally degree of a vertex v ∈ V is defined by

td(v)= $(td_T^P(v), td_I^P(v), td_F^P(v), td_T^N(v), td_I^N(v), td_F^N(v))$ where

$td_T^P(v) = \sum_{u \neq v} T_B^P(u,v) + T_A^P(v)$ denotes the totally positive T- degree of a vertex v,

$td_I^P(v) = \sum_{u \neq v} I_B^P(u,v) + I_A^P(v)$ denotes the totally positive I- degree of a vertex v,

$td_F^P(v) = \sum_{u \neq v} F_B^P(u,v) + F_A^P(v)$ denotes the totally positive F- degree of a vertex v,

$td_T^N(v) = \sum_{u \neq v} T_B^N(u,v) + T_A^N(v)$ denotes the totally negative T- degree of a vertex v,

$td_I^N(v) = \sum_{u \neq v} I_B^N(u,v) + I_A^N(v)$ denotes the totally negative I- degree of a vertex v,

$td_F^N(v) = \sum_{u \neq v} F_B^N(u,v) + F_A^N(v)$ denotes the totally negative F- degree of a vertex v

If each vertex of G has totally same degree **m**=(m_1, m_2, m_3, m_4, m_5, m_6), then G is called a **m**-totally constant BSVN-Graph.

Example 3.35. Let us consider a bipolar single valued neutrosophic graph G= (A, B) of G^* = (V, E), such that V = {v_1, v_2, v_3, v_4}, E = {(v_1, v_2),(v_2, v_3), (v_3, v_4), (v_4, v_1)}

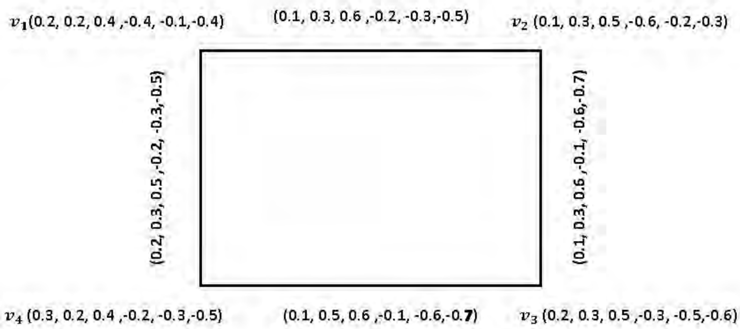

Figure 13: Totally degree of a bipolar single valued neutrosophic graph G.

In this example, the totally degree of v_1 is (0.5, 0.8, 1.4, -0.8, -0.7, -1.4). The totally degree of v_2 is (0.3, 0.9, 1.7, -0.9, -1.1, -1.5). The totally degree of v_3 is (0.4, 1.1, 1.7, -0.5, -1.7, -2). The totally degree of v_4 is (0.6, 1, 1.5, -0.5, -1.1, -1,7).

Definition 3.36: A totally regular BSVN-graph is a BSVN-graph where each vertex has the same number of closed neighbors degree, it is noted d[v].

Example 3.37. Let us consider a BSVN-graph G= (A, B) of G^* = (V, E), such that V = {v_1, v_2, v_3, v_4} and E = {(v_1, v_2),(v_2, v_3), (v_3, v_4), (v_4, v_1)}

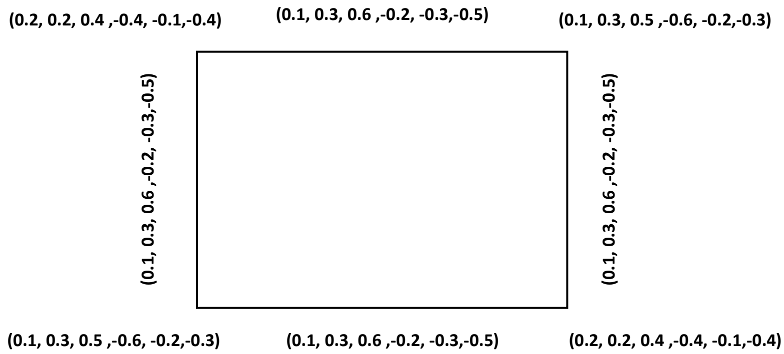

Figure 14: Degree of a bipolar single valued neutrosophic graph G.

By routing calculations we show that G is regular BSVN-graph since the degree of v , v_2, v_3, and v_4 is (0.2, 0.6, 1.2 , -0.4, -0.6, -1). It is neither totally regular BSVN-graph not constant BSVN-graph.

4. Conclusion

In this paper, we have introduced the concept of bipolar single valued neutrosophic graphs and described degree of a vertex, order, size of bipolar single valued neutrosophic graphs, also we have introduced the notion of complement of a bipolar single valued neutrosophic graph, strong bipolar single valued neutrosophic graph, complete bipolar single valued neutrosophic graph, regular bipolar single valued neutrosophic graph. Further, we are going to study some types of single valued neutrosophic graphs such irregular and totally irregular single valued neutrosophic graphs and bipolar single valued neutrosophic graphs.

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Standard Neutrosophic Soft Theory: Some First Results

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Abstract. The traditional soft set is a mapping from a parameter set to family of all crisp subsets of a universe. Molodtsov introduced the soft set as a generalized tool for modelling complex systems involving uncertain or not clearly defined objects. In this paper, the notion of neutrosophic soft set is reanalysed. The novel theory is a combination of neutrosophic set theory and soft set theory. The complement, "and", "or", intersection and union operations are defined on the neutrosophic soft sets. The neutrosophic soft relations accompanied with their compositions are also defined. The basic properties of the neutrosophic soft sets, neutrosophic soft relations and neutrosophic soft compositions are also discussed.

Keywords: Soft sets, Fuzzy soft sets, Intuitionistic fuzzy soft sets, Neutrosophic soft sets, Neutrosophic soft relations

1 Introduction

Uncertain data modelling is a complex problem appearing in many areas such as economics, engineering, environmental science, sociology and medical science. Some mathematical theories such as probability, fuzzy set [1], [2], intuitionistic fuzzy set [3], [4], rough set [5], [6], and the interval mathematics [7], [8] are useful approaches to describing uncertainty. However each of these theories has its inherent difficulties as mentioned by Molodtsov [9]. Soft set theory developed by Molodtsov [9] has become a new useful approach for handling vagueness and uncertainty.

Later, Maji et al. [10] introduced several basic operations of soft set theory and proved some related propositions on soft set operations. Ali et al. [11] analysed the incorrectness of some theorems in [10]. Then they proposed some new soft set operations and proved that De Morgan's laws hold with these new definitions. Maji et al. also [12] gave an application of soft set theory in a decision making problem.

Above works are based on classical soft set. However, in practice, the objects may not precisely satisfy the problems' parameters, thus Maji et al. [13] put forward the concept of fuzzy soft set by combining the fuzzy set and the soft set, then they [14] presented a theoretical approach of the fuzzy soft set in decision making problem. In [15], they considered the concept of intuitionistic fuzzy soft set. By combining the interval-valued fuzzy set and soft set, Yang et al. [16] proposed the interval-valued fuzzy soft set and then analyzed a decision making problem in the interval-valued fuzzy soft set. Yang et al [17] presented the concept of interval-valued intuitionistic fuzzy soft sets which is an interval-valued fuzzy extension of the intuitionistic fuzzy soft set theory.

From philosophical point of view, Smarandache's neutrosophic set [26] generalizes fuzzy set and intuitionistic fuzzy set. However, it is difficult to apply it to the real applications and needs to be specified. Wang et al. [27] proposed interval neutrosophic sets and some operators of then. Wang et al. [28] proposed a single valued neutrosophic set as an instance of the neutrosophic set accompanied with various set theoretic operators and properties. Ye [29] defined the concept of simplified neutrosophic sets, which can be described by three real numbers in the real unit interval $[0,1]$, and some operational laws for simplified neutrosophic sets and to propose two aggregation operators, including a simplified neutrosophic weighted arithmetic average operator and a simplified neutrosophic weighted geometric average operator. In 2013 [18], we presented the definition of picture fuzzy sets, which is a generalization of the Zadeh's fuzzy sets and Atanassov's intuitionistic fuzzy sets, and some basic operations on picture fuzzy sets. In [18] we also discussed some properties of these operations, then the definition of the Cartesian product of picture fuzzy sets and the definition of picture fuzzy relations were given. Our picture fuzzy set turns out a special case of neutrosophic set. Thus, from now on, we also regard picture fuzzy set as standard neutrosophic set.

The purpose of this paper is to combine the standard neutrosophic sets and soft models, from which we can obtain neutrosophic soft sets. Intuitively, the neutrosophic soft set presented in this paper is an extension of the intuitionistic fuzzy soft sets [13][15].

The rest of this paper is organized as follows. Section 2 briefly reviews some background on soft sets, fuzzy soft sets, intuitionistic soft sets as well as neutrosophic set. In Section 3, we recall the concept of the standard neutrosophic sets (SNSs) with some operations on SNSs, then we present the concept of neutrosophic soft sets (NSSs) with some operations. Some properties of these operations are discussed in the Sub-section 3.3. Sub-section 3.4 is devoted to the Cartesian product of NSSs. The neutrosophic soft relations are presented in Section 4. Finally, in Section 5, we draw the conclusion and present some topics for future research.

2 Preliminaries

In this section, we briefly recall the notions of soft sets, fuzzy soft sets, intuitionistic fuzzy soft sets as well as neutrosophic sets. See especially [9][10][13][15] for further details and background.

2.1 Soft sets and some extensions

Molodtsov [8] defined the soft set in the following way. Let U be an initial universe of objects and E be the set of related parameters of objects in U. Parameters are often attributes, characteristics, or properties of objects. Let $P(U)$ denotes the power set of U and $A \subseteq E$.

Definition 2.1. [8] A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to P(U)$.

In other words, the soft set is not a kind of set, but a parameterized family of subsets of U [9][10][16]. For any parameter $e \in E$, $F(e) \subseteq U$ is considered as the set of e-approximate elements of the soft set (F, A).

Maji et al. [13] initiated the study on hybrid structures involving both fuzzy sets and soft sets. They introduced the notion of fuzzy soft sets, which can be seen as a fuzzy generalization of (crisp) soft set.

Definition 2.2 [13] Let $\mathcal{F}(U)$ be the set of all fuzzy subsets of U, E be the set of parameters and $A \subseteq E$. A pair (F, A) is called a fuzzy soft set over U, where F is a mapping given by $F : A \to \mathcal{F}(U)$.

It is easy to see that every (crisp) soft set can be considered as a fuzzy soft set. Generally speaking, for any parameter $e \in E$, $F(e)$ is a fuzzy subset of U and it is called fuzzy value set of parameter e. If for any parameter $e \in A$, $F(e)$ is a subset of U, then (F, A) is degenerated to the standard soft set. For all $x \in U$ and $e \in E$, let us denote by $\mu_{F(e)}(x)$ the membership degree that the object x holds parameter e. So then $F(e)$ can be written as

$$F(e) = \left\{ \left\langle x, \mu_{F(e)}(x) \right\rangle \middle| x \in U \right\}.$$

Before introduce the notion of the intuitionistic fuzzy soft set, let us recall the concept of intuitionistic fuzzy set [3], [4].

Let X be a fixed set. An intuitionistic fuzzy set (IFS) in X is an object having the form

$$A = \left\{ \left\langle x, \mu_A(x), \nu_A(x) \right\rangle \middle| x \in X \right\},$$

where $\mu_A(x) \in [0,1]$ and $\nu_A(x) \in [0,1]$ respectively define the degree of membership and the degree of non-membership of the element x to the set A such that $\mu_A(x) + \nu_A(x) \le 1$ for all $x \in X$. The set of all IFSs on X is denoted by $IFS(X)$.

In [15] Maji et al. proposed the concept of intuitionistic fuzzy soft set as follows.

Definition 2.3 [15] Let E the set of parameters and $A \subseteq E$. A pair (F, A) is called a intuitionistic fuzzy soft set over U, where F is a mapping $F : A \to IFS(U)$.

Clearly, for any parameter $e \in E$, $F(e)$ is an IFS

$$F(e) = \left\{ \left\langle x, \mu_{F(e)}(x), \nu_{F(e)}(x) \right\rangle \middle| x \in U \right\},$$

where $\mu_{F(e)}$ and $\nu_{F(e)}$ are the membership and non-membership functions, respectively. If for any parameter $e \in A$, $\nu_{F(e)}(x) = 1 - \mu_{F(e)}(x)$, then $F(e)$ is a fuzzy set and (F, A) is reduced to a fuzzy soft set.

2.2 Neutrosophic sets

Definition 2.4 [26] A neutrosophic set A in a on a universe X is characterized by a truth-membership function T_A, an indeterminacy-membership function I_A and a falsity-membership function F_A. For each $x \in X$, $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or nonstandard subsets of $\left]0^-, 1^+\right[$, that is T_A, I_A and F_A : $X \to \left]0^-, 1^+\right[$.

There is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$, so $0^- \le \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \le 3^+$, for all $x \in X$.

Definition 2.5 [26] The complement of a neutrosophic set A is denoted by A^c and is defined as $T_{A^c}(x) = \{1^+\} \ominus T_A(x)$, $I_{A^c}(x) = \{1^+\} \ominus I_A(x)$, and $F_{A^c}(x) = \{1^+\} \ominus F_A(x)$ for every x in X.

Definition 2.6 [26] A neutrosophic set A is contained in the other neutrosophic set B, $A \subseteq B$ if and only if

$\inf T_A(x) \le \inf T_B(x)$, $\sup T_A(x) \le \sup T_B(x)$,

$\inf I_A(x) \ge \inf I_B(x)$, $\sup I_A(x) \ge \sup I_B(x)$,

$\inf F_A(x) \ge \inf F_B(x)$, and $\sup F_A(x) \ge \sup F_B(x)$ for every x in X.

Definition 2.7 [26] The union of two neutrosophic sets A and B is a neutrosophic set C, written as $C = A \cup B$, whose truth-membership, indeterminacy membership and false-membership functions are related to those of A and B by

$T_C(x) = T_A(x) \oplus T_B(x) \ominus T_A(x) \odot T_B(x)$,

$I_C(x) = I_A(x) \oplus I_B(x) \ominus I_A(x) \odot I_B(x)$,and

$F_C(x) = F_A(x) \oplus F_B(x) \ominus F_A(x) \odot F_B(x)$ for any x in X.

Definition 2.8 [1] The intersection of two neutrosophic sets A and B is a neutrosophic set C, written as $C = A \cap B$, whose truth-membership, indeterminacy-membership and false-membership functions are related to those of A and B by $T_C(x) = T_A(x) \odot T_B(x)$, $I_C(x) = I_A(x) \odot I_B(x)$,and $F_C(x) = F_A(x) \odot F_B(x)$ for any x in X.

Definition 2.9 [29] Consider a neutrosophic set A in X characterized by a truth-membership function T_A, a indeterminacy-membership function I_A and a falsity - membership function F_A. If $T_A(x)$, $I_A(x)$ and $F_A(x)$ are singleton values in the real standard $[0,1]$ for every x in X, that is T_A, I_A and F_A : $X \to [0,1]$. Then, a simplification of the neutrosophic set A is denoted by

$$A = \left\{ \langle x, T_A(x), I_A(x), F_A(x) \rangle \middle| x \in X \right\},$$

which is called a simplified neutrosophic set.

3 Neutrosophic soft sets

In this section, first we recall the definition of the standard neutrosophic sets (SNSs), some basic operations with their properties, then we will present the neutrosophic soft set theory which is a combination of neutrosophic set theory and a soft set theory.

3.1 Standard neutrosophic sets

Intuitionistic fuzzy sets introduced by Atanassov in 1983 constitute a generalization of fuzzy sets (FS) [3]. While fuzzy sets give the degree of membership of an element in a given set, intuitionistic fuzzy sets give a degree of membership and a degree of non-membership of an element in a given set.

A generalization of fuzzy sets and intuitionistic fuzzy sets are the following notion of standard neutrosophic set (SNS) .

Definition 3.1 [18] A SNS A on a universe X is an object of the form

$$A = \left\{ \left(x, \mu_A(x), \eta_A(x), \nu_A(x)\right) \middle| x \in X \right\},$$

where $\mu_A(x) \in [0,1]$ is called the "degree of positive membership of x in A", $\eta_A(x) \in [0,1]$ is called the "degree of neutral membership of x in A" and $\nu_A(x) \in [0,1]$ is called the "degree of negative membership of x in A", and μ_A, η_A and ν_A satisfy the following condition:

$$\mu_A(x) + \eta_A(x) + \nu_A(x) \le 1, \ \forall x \in X.$$

The expression $\left(1 - \left(\mu_A(x) + \eta_A(x) + \nu_A(x)\right)\right)$ is termed as "degree of refusal membership" of x in A.

Basically, SNSs based models may be adequate in situations when we face human opinions involving more answers of type: *yes*, *abstain*, *no* and *refusal*. Voting can be a good example of such a situation as the voters are divided into four groups: *vote for*, *abstain*, *vote against* and *refusal of the voting*.

Let $SNS(X)$ denote the set of all the standard neutrosophic set SNSs on a universe X.

Definition 3.2 [18] For A, $B \in SNS(X)$, the union, intersection and complement are defined as follows:

- $A \subseteq B \Leftrightarrow \begin{cases} \mu_A(x) \le \mu_B(x) \\ \eta_A(x) \le \eta_B(x) \\ \nu_A(x) \ge \nu_B(x) \end{cases}$, $\forall x \in X$;
- $A = B \Leftrightarrow \begin{cases} A \subseteq B \\ B \subseteq A \end{cases}$;
- $A \cup B \in SNS(X)$ with

  $\mu_{A \cup B}(x) = \max\left(\mu_A(x), \mu_B(x)\right)$,

  $\eta_{A \cup B}(x) = \min\left(\eta_A(x), \eta_B(x)\right)$, and

  $\nu_{A \cup B}(x) = \min\left(\nu_A(x), \nu_B(x)\right)$, $\forall x \in X$;
- $A \cap B \in SNS(X)$ with

  $\mu_{A \cap B}(x) = \min\left(\mu_A(x), \mu_B(x)\right)$,

  $\eta_{A \cap B}(x) = \min\left(\eta_A(x), \eta_B(x)\right)$, and

  $\nu_{A \cap B}(x) = \max\left(\nu_A(x), \nu_B(x)\right)$, $\forall x \in X$;
- $CoA = A^c = \left\{ \left(x, \nu_A(x), \eta_A(x), \mu_A(x)\right) \middle| x \in X \right\}$.

In this paper, we denote $a \wedge b = \min(a,b)$ and $a \vee b = \max(a,b)$, for every a, $b \in \mathbb{R}$.

Definition 3.3 [18] Let X, Y be two universes and $A \in SNS(X)$, $B \in SNS(Y)$. We define the Cartesian product of these two SNSs by $A \times B \in SNS(X \times Y)$ such that

$\mu_{A \times B}(x, y) = \mu_A(x) \wedge \mu_B(y)$,

$\eta_{A \times B}(x, y) = \eta_A(x) \wedge \eta_B(y)$, and

$\nu_{A\times B}(x,y)=\nu_A(x)\vee\nu_B(y)$, $\forall(x,y)\in X\times Y$.

The validation of Definition 3.3 was shown in [18]. Now we consider some properties of the defined operations on SNSs.

Proposition 3.4 [18] For every A, B, $C\in SNS(X)$:

(a) If $A\subseteq B$ and $B\subseteq C$, then $A\subseteq C$;

(b) $\left(A^c\right)^c=A$;

(c) Operations $\cap$ and $\cup$ are commutative, associative and distributive;

(d) Operations $\cap$, *Co* and $\cup$ satisfy the law of De Morgan.

Proof. See [19][20] for detail proof. □

Convex combination is an important operation in mathematics, which is a useful tool on convex analysis, linear spaces and convex optimization. In this sub-section convex combination firstly is defined with some simple propositions.

Definition 3.5 [18] Let A, $B\in SNS(X)$. For each $\theta\in[0,1]$, the convex combination of A and B is defined as follows:

$$C_\theta(A,B)=\left\{\left(x,\mu_{C_\theta}(x),\eta_{C_\theta}(x),\nu_{C_\theta}(x)\right)\middle|x\in X\right\},$$

where

$\mu_{C_\theta}(x)=\theta\mu_A(x)+(1-\theta)\mu_B(x)$,

$\eta_{C_\theta}(x)=\theta\eta_A(x)+(1-\theta)\eta_B(x)$, and

$\nu_{C_\theta}(x)=\theta\nu_A(x)+(1-\theta)\nu_B(x)$, $\forall x\in X$.

Proposition 3.6 [18] Let A, $B\in SNS(X)$ and θ, θ_1, $\theta_2\in[0,1]$, then

- If $\theta=1$, then $C_\theta(A,B)=A$; and if $\theta=0$, then $C_\theta(A,B)=B$;
- If $A\subseteq B$, then $A\subseteq C_\theta(A,B)\subseteq B$;
- If $B\subseteq A$ and $\theta_1\le\theta_2$, then $C_{\theta_1}(A,B)\subseteq C_{\theta_2}(A,B)$.

3.2 Neutrosophic soft sets

Definition 3.7 Let $SNS(U)$ be the set of all standard neutrosophic sets of U, E be the set of parameters and $A\subseteq E$. A pair (F,A) is called a standard neutrosophic soft set (or neutrosophic soft set for short) over U, where F is a mapping given by $F:A\to SNS(U)$.

Clearly, for any parameter $e\in E$, $F(e)$ is a SNS:

$$F(e)=\left\{\left(x,\mu_{F(e)}(x),\eta_{F(e)}(x),\nu_{F(e)}(x)\right)\middle|x\in U\right\},$$

where $\mu_{F(e)}$, $\eta_{F(e)}$ and $\nu_{F(e)}$ are positive membership, neutral membership and negative membership functions respectively. If for all parameter $e\in A$ and for all $x\in U$, $\eta_{F(e)}(x)=0$, then $F(e)$ will degenerated to be an intuitionistic fuzzy set and then (F,A) is degenerated to an intuitionistic fuzzy soft set.

We denote the set of all standard neutrosophic soft sets over U by $SNS(U)$.

Example 1. We consider the situation which involves four economic projects evaluated by a decision committee according to five parameters: *good finance indicator* (e_1), *average finance indicator* (e_2), *good social contribution* (e_3), *average social contribution* (e_4) and *good environment indicator* (e_5). The set of economic projects and the set of parameters are denoted $U=\{p_1,p_2,p_3,p_4\}$ and $A=\{e_1,e_2,e_3,e_4,e_5\}$, respectively. So, the attractiveness of the projects to the decision committee can be represented by a SNS (F,A):

$$F(e_1)=\begin{Bmatrix}(p_1,0.8,0.12,0.05),(p_2,0.6,0.18,0.16),\\(p_3,0.55,0.20,0.21),(p_4,0.50,0.20,0.24)\end{Bmatrix},$$

$$F(e_2)=\begin{Bmatrix}(p_1,0.82,0.05,0.10),(p_2,0.7,0.12,0.10),\\(p_3,0.60,0.14,0.10),(p_4,0.51,0.10,0.24)\end{Bmatrix},$$

$$F(e_3)=\begin{Bmatrix}(p_1,0.60,0.14,0.16),(p_2,0.55,0.20,0.16),\\(p_3,0.70,0.15,0.11),(p_4,0.63,0.12,0.18)\end{Bmatrix},$$

$$F(e_4)=\begin{Bmatrix}(p_1,0.7,0.12,0.07),(p_2,0.75,0.05,0.16),\\(p_3,0.60,0.17,0.18),(p_4,0.55,0.10,0.22)\end{Bmatrix},$$

$$F(e_5)=\begin{Bmatrix}(p_1,0.60,0.12,0.07),(p_2,0.62,0.14,0.16),\\(p_3,0.55,0.10,0.21),(p_4,0.70,0.20,0.05)\end{Bmatrix}.$$

The standard neutrosophic soft set (F,A) is a parameterized family $\left\{F(e_i)\middle|i=1,\ldots,5\right\}$ of standard neutrosophic sets over U.

Definition 3.8 1) For (F,A), $(G,B)\in SNS(U)$ over a common universe U, we say that (F,A) is a subset of (G,B), $(F,A)\subseteq(G,B)$, if the following conditions are satisfied:

(a) $A\subseteq B$;

(b) For all $e\in A$, $F(e)$ and $G(e)$ are identical approximations.

2) (F,A) is termed as a superset of (G,B), $(F,A)\supseteq(G,B)$, if (G,B) is a subset of (F,A).

3) (F,A) and (G,B) are called to be equal, $(F,A)=(G,B)$, if $(F,A)\subseteq(G,B)$ and $(G,B)\subseteq(F,A)$.

It is easy to show that $(F,A)=(G,B)$ iff $A=B$ and $F(e)=G(e)$ for all $e\in A$.

3.3 Some operations and properties

Now we define some operations on standard neutrosophic soft sets and present some properties.

Definition 3.9 The complement of a NSS (F,A), $(F,A)^c$, is defined by $(F,A)^c=(F^c,A)$, where $F^c:A\to P(U)$ is a mapping given by $F^c(e)=(F(e))^c$, for all $e\in A$.

Definition 3.10 If (F,A), $(G,B)\in NSS(U)$, then "(F,A) *and* (G,B)" is a NSS denoted by $(F,A)\wedge(G,B)$ and defined by $(F,A)\wedge(G,B)=(H,A\times B)$, where $H(\alpha,\beta)=F(\alpha)\cap G(\beta)$ for all $(\alpha,\beta)\in A\times B$, that is

$$\mu_{H(\alpha,\beta)}(x)=\min\left(\mu_{F(\alpha)}(x),\mu_{G(\beta)}(x)\right),$$

$$\eta_{H(\alpha,\beta)}(x)=\min\left(\eta_{F(\alpha)}(x),\eta_{G(\beta)}(x)\right),\text{ and}$$

$$\nu_{H(\alpha,\beta)}(x)=\max\left(\nu_{F(\alpha)}(x),\nu_{G(\beta)}(x)\right),\ \forall x\in U.$$

Definition 3.11 If (F,A), $(G,B)\in NSS(U)$, then "(F,A) *or* (G,B)" is a NSS denoted by $(F,A)\vee(G,B)$ and defined by $(F,A)\vee(G,B)=(H,A\times B)$, where $H(\alpha,\beta)=F(\alpha)\cup G(\beta)$ for all $(\alpha,\beta)\in A\times B$, that is

$$\mu_{H(\alpha,\beta)}(x)=\max\left(\mu_{F(\alpha)}(x),\mu_{G(\beta)}(x)\right),$$

$$\eta_{H(\alpha,\beta)}(x)=\min\left(\eta_{F(\alpha)}(x),\eta_{G(\beta)}(x)\right),\text{ and}$$

$$\nu_{H(\alpha,\beta)}(x)=\min\left(\nu_{F(\alpha)}(x),\nu_{G(\beta)}(x)\right),\ \forall x\in U.$$

Theorem 3.1 Let (F,A), $(G,B)\in NSS(U)$, then we have the following properties:

(1) $\left((F,A)\wedge(G,B)\right)^c=(F,A)^c\vee(G,B)^c$;

(2) $\left((F,A)\vee(G,B)\right)^c=(F,A)^c\wedge(G,B)^c$.

Proof. (1) Assume that $(F,A)\wedge(G,B)=(H,A\times B)$. Then

$$\left((F,A)\wedge(G,B)\right)^c=(H,A\times B)^c=(H^c,A\times B).$$

For any $(\alpha,\beta)\in A\times B$, $x\in U$, we have

$$\begin{aligned}H(\alpha,\beta)(x) \;=\; &\Big(\min\left(\mu_{F(\alpha)}(x),\mu_{G(\beta)}(x)\right),\\ &\min\left(\eta_{F(\alpha)}(x),\eta_{G(\beta)}(x)\right)\\ &\max\left(\nu_{F(\alpha)}(x),\nu_{G(\beta)}(x)\right)\Big),\end{aligned}$$

which implies

$$\begin{aligned}H^c(\alpha,\beta)(x) \;=\; &\Big(\max\left(\nu_{F(\alpha)}(x),\nu_{G(\beta)}(x)\right),\\ &\min\left(\eta_{F(\alpha)}(x),\eta_{G(\beta)}(x)\right)\\ &\min\left(\mu_{F(\alpha)}(x),\mu_{G(\beta)}(x)\right)\Big).\end{aligned}\qquad(1)$$

On the other hand,

$$(F,A)^c\vee(G,B)^c=(F^c,A)\vee(G^c,B).$$

Let us assume that $(F^c,A)\vee(G^c,B)=(K,A\times B)$. We obtain

$$\begin{aligned}K(\alpha,\beta)(x) \;=\; &\Big(\max\left(\mu_{F^c(\alpha)}(x),\mu_{G\ (\beta)}(x)\right),\\ &\min\left(\eta_{F^c(\alpha)}(x),\eta_{G^c(\beta)}(x)\right)\\ &\min\left(\nu_{F^c(\alpha)}(x),\nu_{G^c(\beta)}(x)\right)\Big).\end{aligned}$$

Since $\mu_{F^c(\alpha)}=\nu_{F(\alpha)}$, $\eta_{F^c(\alpha)}=\eta_{F(\alpha)}$, $\nu_{F^c(\alpha)}=\mu_{F(\alpha)}$, $\mu_{G^c(\beta)}=\nu_{G(\beta)}$, $\eta_{G^c(\beta)}=\eta_{G(\beta)}$, $\nu_{G^c(\beta)}=\mu_{G(\beta)}$,

$$\begin{aligned}K(\alpha,\beta)(x) \;=\; &\Big(\max\left(\nu_{F(\alpha)}(x),\nu_{G(\beta)}(x)\right),\\ &\min\left(\eta_{F(\alpha)}(x),\eta_{G(\beta)}(x)\right)\\ &\min\left(\mu_{F(\alpha)}(x),\mu_{G(\beta)}(x)\right)\Big).\end{aligned}\qquad(2)$$

Combining (1) and (2), the proof is completed.

(2) The proof is similar to (1). □

Theorem 3.2 Let (F,A), (G,B), $(H,C)\in NSS(U)$, then we have the following properties:

a) $(F,A)\wedge((G,B)\wedge(H,C))=((F,A)\wedge(G,B))\wedge(H,C)$;

b) $(F,A)\vee((G,B)\vee(H,C))=((F,A)\vee(G,B))\vee(H,C)$.

Proof. (1) Assume that

$$(G,B)\wedge(H,C)=(I,B\times C),$$

We have

$$\begin{aligned} I(\beta,\gamma)(x) \;=\; & \Big(\min\big(\mu_{G(\beta)}(x),\mu_{H(\gamma)}(x)\big), \\ & \min\big(\eta_{G(\beta)}(x),\eta_{H(\gamma)}(x)\big) \\ & \max\big(\nu_{G(\beta)}(x),\nu_{H(\gamma)}(x)\big)\Big), \end{aligned}$$

$\forall(\beta,\gamma)\in B\times C$, $x\in U$.

We assume that

$$(F,A)\wedge((G,B)\wedge(H,C))=(K,A\times B\times C).$$

In other words,

$$(K,A\times B\times C)=(F,A)\wedge(I,B\times C).$$

By definition of $\wedge$ operator for two NSSs,

$$\begin{aligned} K(\alpha,\beta,\gamma)(x) \;=\; & \Big(\min\Big(\mu_{F(\alpha)}(x),\min\big(\mu_{G(\beta)}(x),\mu_{H(\gamma)}(x)\big)\Big), \\ & \min\Big(\eta_{F(\alpha)}(x),\min\big(\eta_{G(\beta)}(x),\eta_{H(\gamma)}(x)\big)\Big) \\ & \max\Big(\nu_{F(\alpha)}(x),\max\big(\nu_{G(\beta)}(x),\nu_{H(\gamma)}(x)\big)\Big)\Big), \end{aligned}$$

or

$$\begin{aligned} K(\alpha,\beta,\gamma)(x) \;=\; & \Big(\min\big(\mu_{F(\alpha)}(x),\mu_{G(\beta)}(x),\mu_{H(\gamma)}(x)\big), \\ & \min\big(\eta_{F(\alpha)}(x),\eta_{G(\beta)}(x),\eta_{H(\gamma)}(x)\big) \\ & \max\big(\nu_{F(\alpha)}(x),\nu_{G(\beta)}(x),\nu_{H(\gamma)}(x)\big)\Big). \end{aligned}$$

By a similar argument, we get

$$((F,A)\wedge(G,B))\wedge(H,C)=(K,A\times B\times C).$$

This concludes the proof of a).

The proof of b) is analogous. □

Definition 3.12 The intersection of two NSSs (F,A), $(G,B)\in NSS(U)$, denoted by $(F,A)\cap(G,B)$, is a NSSs (H,C), where $C=A\cup B$ and for all $e\in C$,

$$H(e)=\begin{cases} F(e) & \text{if } e\in A\setminus B \\ G(e) & \text{if } e\in B\setminus A \\ F(e)\cap G(e) & \text{if } e\in A\cap B \end{cases}. \qquad (3)$$

Definition 3.13 The union of two NSSs (F,A), $(G,B)\in NSS(U)$, denoted by $(F,A)\cup(G,B)$, is a NSSs (H,C), where $C=A\cup B$ and for all $e\in C$,

$$H(e)=\begin{cases} F(e) & \text{if } e\in A\setminus B \\ G(e) & \text{if } e\in B\setminus A \\ F(e)\cup G(e) & \text{if } e\in A\cap B \end{cases}. \qquad (3)$$

Theorem 3.3. Let (F,A), $(G,B)\in NSS(U)$, then we have the following properties:

a) $((F,A)\cap(G,B))^c=(F,A)^c\cup(G,B)^c$;

b) $((F,A)\cup(G,B))^c=(F,A)^c\cap(G,B)^c$.

Proof. a) Assume that $(F,A)\cap(G,B)=(H,C)$, with $C=A\cup B$, then

$$((F,A)\cap(G,B))^c=(H,C)^c=(H^c,C).$$

By Definition 3.13,

$$H(e)=\begin{cases} F(e) & \text{if } e\in A\setminus B \\ G(e) & \text{if } e\in B\setminus A \\ F(e)\cap G(e) & \text{if } e\in A\cap B \end{cases}.$$

It implies

$$H^c(e)=\begin{cases} F^c(e) & \text{if } e\in A\setminus B \\ G^c(e) & \text{if } e\in B\setminus A \\ F^c(e)\cup G^c(e) & \text{if } e\in A\cap B \end{cases}. \qquad (5)$$

Similarly, we denote $(F,A)^c\cup(G,B)^c=(K,C)$ with $C=A\cup B$. Since $(K,C)=(F^c,A)\cup(G^c,B)$,

$$K(e)=\begin{cases} F^c(e) & \text{if } e\in A\setminus B \\ G^c(e) & \text{if } e\in B\setminus A \\ F^c(e)\cup G^c(e) & \text{if } e\in A\cap B \end{cases}. \qquad (6)$$

From (5) and (6), we get $H^c=K$. Hence,

$$\left((F,A)\cap(G,B)\right)^c=(F,A)^c\cup(G,B)^c.$$

b) Similarly, we have b). □

3.4 Cartesian product of neutrosophic soft sets

Definition 3.14 Let $O_1\in SNS(X_1)$ and $O_2\in SNS(X_2)$. The Cartesian product of these two NSSs is $O_1\times O_2\in SNS(X_1\times X_2)$ defined as

$$\mu_{O_1\times O_2}(x,y)=\mu_{O_1}(x)\wedge\mu_{O_2}(y),$$

$$\eta_{O_1\times O_2}(x,y)=\eta_{O_1}(x)\wedge\eta_{O_2}(y), \text{ and}$$

$$\nu_{O_1\times O_2}(x,y)=\nu_{O_1}(x)\vee\nu_{O_2}(y),\ \forall(x,y)\in X_1\times X_2.$$

It is easy to check the validation of Definition 3.15.

Theorem 3.4 For O_1, $O_2\in SNS(X_1)$, $O_3\in SNS(X_2)$, and $O_4\in SNS(X_3)$:

a) $O_1\times O_3=O_3\times O_1$;

b) $(O_1\times O_3)\times O_4=O_1\times(O_3\times O_4)$;

c) $(O_1\cup O_2)\times O_3=(O_1\times O_3)\cup(O_2\times O_3)$;

d) $(O_1\cap O_2)\times O_3=(O_1\times O_3)\cap(O_2\times O_3)$.

Proof. a) and b) are straightforward. We consider c) and d).

c) We have

$$\mu_{O_1\cup O_2}(x)=\mu_{O_1}(x)\vee\mu_{O_2}(x),$$

$$\eta_{O_1\cup O_2}(x)=\eta_{O_1}(x)\wedge\eta_{O_2}(x), \text{ and}$$

$$\nu_{O_1\cup O_2}(x)=\nu_{O_1}(x)\wedge\nu_{O_2}(x),\ \forall x\in X_1.$$

Thus,

$$\mu_{(O_1\cup O_2)\times O_3}(x,y)=\left(\mu_{O_1}(x)\vee\mu_{O_2}(x)\right)\wedge\mu_{O_3}(y),$$

$$\eta_{(O_1\cup O_2)\times O_3}(x,y)=\left(\eta_{O_1}(x)\wedge\eta_{O_2}(x)\right)\wedge\eta_{O_3}(y), \text{ and}$$

$$\nu_{(O_1\cup O_2)\times O_3}(x,y)=\left(\nu_{O_1}(x)\wedge\nu_{O_2}(x)\right)\vee\nu_{O_3}(y),$$

$$\forall(x,y)\in X_1\times X_2.$$

Using the properties of the operations $\wedge$ and $\vee$ we obtain

$$\begin{aligned}\mu_{(O_1\cup O_2)\times O_3}(x,y) &= \left(\mu_{O_1}(x)\wedge\mu_{O_3}(y)\right)\vee\left(\mu_{O_2}(x)\wedge\mu_{O_3}(y)\right)\\ &= \mu_{O_1\times O_3}(x,y)\vee\mu_{O_2\times O_3}(x,y)\\ &= \mu_{(O_1\times O_3)\cup(O_2\times O_3)}(x,y),\end{aligned}$$

$$\begin{aligned}\eta_{(O_1\cup O_2)\times O_3}(x,y) &= \left(\eta_{O_1}(x)\wedge\eta_{O_3}(y)\right)\wedge\left(\eta_{O_2}(x)\wedge\eta_{O_3}(y)\right)\\ &= \eta_{O_1\times O_3}(x,y)\wedge\eta_{O_2\times O_3}(x,y)\\ &= \eta_{(O_1\times O_3)\cup(O_2\times O_3)}(x,y),\end{aligned}$$

$$\begin{aligned}\nu_{(O_1\cup O_2)\times O_3}(x,y) &= \left(\nu_{O_1}(x)\vee\nu_{O_3}(y)\right)\wedge\left(\nu_{O_2}(x)\vee\nu_{O_3}(y)\right)\\ &= \nu_{O_1\times O_3}(x,y)\wedge\nu_{O_2\times O_3}(x,y)\\ &= \nu_{(O_1\times O_3)\cup(O_2\times O_3)}(x,y),\forall(x,y)\in X_1\times X_2.\end{aligned}$$

The proof is given.

d) The proof of d) is analogous. □

Now we give the definition of the Cartesian product of neutrosophic soft sets.

Definition 3.15 Let X_1, X_2 be two universes, E be the set of parameters, A, $B\subseteq E$. Then the Cartesian product of $\langle F,A\rangle\in NSS(X_1)$ and $\langle G,B\rangle\in NSS(X_2)$ is denoted by $\langle F,A\rangle\times\langle G,B\rangle$ and defined by $\langle H,A\times B\rangle$, where

$$\begin{aligned}H(\alpha,\beta)(x,y) = &\Big(\min\left(\mu_{F(\alpha)}(x),\mu_{G(\beta)}(y)\right),\\ &\min\left(\eta_{F(\alpha)}(x),\eta_{G(\beta)}(y)\right),\\ &\max\left(\nu_{F(\alpha)}(x),\nu_{G(\beta)}(y)\right)\Big),\end{aligned}$$

$\forall(\alpha,\beta)\in A\times B$, $\forall(x,y)\in X_1\times X_2$.

Theorem 3.5 Let X_1, X_2, X_3 be three universes, E be the set of parameters, A_1, A_2, B, $D\subseteq E$. For $\langle F_1,A_1\rangle$, $\langle F_2,A_2\rangle\in NSS(X_1)$, $\langle G,B\rangle\in NSS(X_2)$ and $\langle H,D\rangle\in NSS(X_3)$, we have:

a) $\langle F_1,A_1\rangle\times\langle G,B\rangle=\langle G,B\rangle\times\langle F_1,A_1\rangle$;

b) $\left(\langle F_1,A_1\rangle\times\langle G,B\rangle\right)\times\langle H,D\rangle$

$=\langle F_1,A_1\rangle\times\left(\langle G,B\rangle\times\langle H,D\rangle\right)$;

c) $\left(\langle F_1,A_1\rangle\cup\langle F_2,A_2\rangle\right)\times\langle G,B\rangle$

$=\left(\langle F_1,A_1\rangle\times\langle G,B\rangle\right)\cup\left(\langle F_2,A_2\rangle\times\langle G,B\rangle\right)$;

d) $\left(\langle F_1,A_1\rangle\cap\langle F_2,A_2\rangle\right)\times\langle G,B\rangle$

$$=\left(\langle F_1,A_1\rangle\times\langle G,B\rangle\right)\cap\left(\langle F_2,A_2\rangle\times\langle G,B\rangle\right).$$

Proof. The proof of a) and b) is omitted.

c) Use Definition 3.14, if $\langle F',A_1\cup A_2\rangle=\langle F_1,A_1\rangle\cup\langle F_2,A_2\rangle$, then for all $\alpha\in A_1\cup A_2$:

$$H'(\alpha)=\begin{cases}F_1(\alpha) & \text{if} \quad \alpha\in A_1\setminus A_2\\ F_2(\alpha) & \text{if} \quad \alpha\in A_2\setminus A_1\\ F_1(\alpha)\cup F_2(\alpha) & \text{if} \quad \alpha\in A_1\cap A_2\end{cases}.$$

Let assume that $\langle K,(A_1\cup A_2)\times B\rangle=\langle F',A_1\cup A_2\rangle\times\langle G,B\rangle$. For all $(x,y)\in X_1\times X_2$, there are following three cases :

* Case 1: $(\alpha,\beta)\in(A_1\setminus A_2)\times B$.

$$\mu_{K(\alpha,\beta)}(x,y)=\min\left(\mu_{F_1(\alpha)}(x),\mu_{G(\beta)}(y)\right),$$

$$\eta_{K(\alpha,\beta)}(x,y)=\min\left(\eta_{F_1(\alpha)}(x),\eta_{G(\beta)}(y)\right), \text{ and}$$

$$\nu_{K(\alpha,\beta)}(x,y)=\max\left(\nu_{F_1(\alpha)}(x),\nu_{G(\beta)}(y)\right).$$

* Case 2: $(\alpha,\beta)\in(A_2\setminus A_1)\times B$.

$$\mu_{K(\alpha,\beta)}(x,y)=\min\left(\mu_{F_2(\alpha)}(x),\mu_{G(\beta)}(y)\right),$$

$$\eta_{K(\alpha,\beta)}(x,y)=\min\left(\eta_{F_2(\alpha)}(x),\eta_{G(\beta)}(y)\right), \text{ and}$$

$$\nu_{K(\alpha,\beta)}(x,y)=\max\left(\nu_{F_2(\alpha)}(x),\nu_{G(\beta)}(y)\right).$$

* Case 3: $(\alpha,\beta)\in(A_2\cap A_1)\times B$.

$$\mu_{K(\alpha,\beta)}(x,y)=\min\left(\mu_{F_1(\alpha)\cup F_2(\alpha)}(x),\mu_{G(\beta)}(y)\right)$$

$$=\min\left(\max\left(\mu_{F_1(\alpha)}(x),\mu_{F_2(\alpha)}(x)\right),\mu_{G(\beta)}(y)\right)$$

$$=\max\left(\min\left(\mu_{F_1(\alpha)}(x),\mu_{G(\beta)}(y)\right),\min\left(\mu_{F_2(\alpha)}(x),\mu_{G(\beta)}(y)\right)\right),$$

$$\eta_{K(\alpha,\beta)}(x,y)=\min\left(\eta_{F_1(\alpha)\cup F_2(\alpha)}(x),\eta_{G(\beta)}(y)\right)$$

$$=\min\left(\min\left(\eta_{F_1(\alpha)}(x),\eta_{F_2(\alpha)}(x)\right),\eta_{G(\beta)}(y)\right)$$

$$=\min\left(\eta_{F_1(\alpha)}(x),\eta_{F_2(\alpha)}(x),\eta_{G(\beta)}(y)\right), \text{ and}$$

$$\nu_{K(\alpha,\beta)}(x,y)=\max\left(\nu_{F_1(\alpha)\cup F_2(\alpha)}(x),\nu_{G(\beta)}(y)\right)$$

$$=\max\left(\min\left(\nu_{F_1(\alpha)}(x),\nu_{F_2(\alpha)}(x)\right),\nu_{G(\beta)}(y)\right)$$

$$=\min\left(\max\left(\nu_{F_1(\alpha)}(x),\nu_{G(\beta)}(y)\right),\max\left(\nu_{F_2(\alpha)}(x),\nu_{G(\beta)}(y)\right)\right).$$

Let us denote $\langle H_1,A_1\times B\rangle=\langle F_1,A_1\rangle\times\langle G,B\rangle$ and $\langle H_2,A_2\times B\rangle=\langle F_2,A_2\rangle\times\langle G,B\rangle$. We have:

$$\mu_{H_1(\alpha,\beta)}(x,y)=\min\left(\mu_{F_1(\alpha)}(x),\mu_{G(\alpha)}(y)\right),$$

$$\eta_{H_1(\alpha,\beta)}(x,y)=\min\left(\eta_{F_1(\alpha)}(x),\eta_{G(\alpha)}(y)\right),$$

$$\nu_{H_1(\alpha,\beta)}(x,y)=\max\left(\nu_{F_1(\alpha)}(x),\nu_{G(\alpha)}(y)\right),$$

$$\forall(\alpha,\beta)\in A_1\times B \text{ and } (x,y)\in X_1\times X_2;$$

$$\mu_{H_2(\alpha,\beta)}(x,y)=\min\left(\mu_{F_2(\alpha)}(x),\mu_{G(\alpha)}(y)\right),$$

$$\eta_{H_2(\alpha,\beta)}(x,y)=\min\left(\eta_{F_2(\alpha)}(x),\eta_{G(\alpha)}(y)\right),$$

$$\nu_{H_2(\alpha,\beta)}(x,y)=\max\left(\nu_{F_2(\alpha)}(x),\nu_{G(\alpha)}(y)\right),$$

$$\forall(\alpha,\beta)\in A_2\times B \text{ and } (x,y)\in X_1\times X_2.$$

We consider,

$$\langle K',(A_1\times B)\cup(A_2\times B)\rangle=\langle H_1,A_1\times B\rangle\cup\langle H_1,A_1\times B\rangle.$$

Again, we have following three cases:

* Case 1: $(\alpha,\beta)\in(A_1\times B)\setminus(A_2\times B)=(A_1\setminus A_2)\times B$. We have:

$$\mu_{K'(\alpha,\beta)}(x,y)=\mu_{H_1(\alpha,\beta)}(x,y)$$

$$=\min\left(\mu_{F_1(\alpha)}(x),\mu_{G(\alpha)}(y)\right);$$

$$\eta_{K'(\alpha,\beta)}(x,y)=\eta_{H_1(\alpha,\beta)}(x,y)$$

$$=\min\left(\eta_{F_1(\alpha)}(x),\eta_{G(\alpha)}(y)\right);$$

$$\nu_{K'(\alpha,\beta)}(x,y)=\nu_{H_1(\alpha,\beta)}(x,y)$$

$$=\max\left(\nu_{F_1(\alpha)}(x),\nu_{G(\alpha)}(y)\right).$$

* Case 2: $(\alpha,\beta)\in(A_2\times B)\setminus(A_1\times B)=(A_2\setminus A_1)\times B$.

$$\mu_{K'(\alpha,\beta)}(x,y)=\mu_{H_2(\alpha,\beta)}(x,y)$$

$$=\min\left(\mu_{F_2(\alpha)}(x),\mu_{G(\alpha)}(y)\right);$$

$$\eta_{K'(\alpha,\beta)}(x,y)=\eta_{H_2(\alpha,\beta)}(x,y)$$

$$= \min\left(\eta_{F_2(\alpha)}(x), \eta_{G(\alpha)}(y)\right);$$

$$\nu_{K'(\alpha,\beta)}(x,y) = \nu_{H_2(\alpha,\beta)}(x,y)$$

$$= \max\left(\nu_{F_2(\alpha)}(x), \nu_{G(\alpha)}(y)\right).$$

* Case 3: $(\alpha,\beta) \in (A_1 \times B) \cap (A_2 \times B) = (A_2 \cap A_1) \times B$.

$$\mu_{K'(\alpha,\beta)}(x,y) = \mu_{(H_1(\alpha,\beta)) \cup (H_2(\alpha,\beta))}(x,y)$$

$$= \max\left(\mu_{H_1(\alpha,\beta)}(x,y), \mu_{H_2(\alpha,\beta)}(x,y)\right)$$

$$= \max\left(\min\left(\mu_{F_1(\alpha)}(x), \mu_{G(\alpha)}(y)\right), \min\left(\mu_{F_2(\alpha)}(x), \mu_{G(\alpha)}(y)\right)\right);$$

$$\eta_{K'(\alpha,\beta)}(x,y) = \eta_{(H_1(\alpha,\beta)) \cup (H_2(\alpha,\beta))}(x,y)$$

$$= \min\left(\eta_{H_1(\alpha,\beta)}(x,y), \eta_{H_2(\alpha,\beta)}(x,y)\right)$$

$$= \min\left(\min\left(\eta_{F_1(\alpha)}(x), \eta_{G(\alpha)}(y)\right), \min\left(\eta_{F_2(\alpha)}(x), \eta_{G(\alpha)}(y)\right)\right)$$

$$= \min\left(\eta_{F_1(\alpha)}(x), \eta_{F_2(\alpha)}(x), \eta_{G(\alpha)}(y)\right); \text{ and}$$

$$\nu_{K'(\alpha,\beta)}(x,y) = \nu_{(H_1(\alpha,\beta)) \cup (H_2(\alpha,\beta))}(x,y)$$

$$= \min\left(\nu_{H_1(\alpha,\beta)}(x,y), \nu_{H_2(\alpha,\beta)}(x,y)\right)$$

$$= \min\left(\max\left(\nu_{F_1(\alpha)}(x), \nu_{G(\alpha)}(y)\right), \max\left(\nu_{F_2(\alpha)}(x), \nu_{G(\alpha)}(y)\right)\right).$$

We then obtain $K = K'$ which completes the proof of c). The proof of d) is analogous. □

4 Standard neutrosophic soft relations

4.1 Standard neutrosophic relations

Fuzzy relations are one of the most important notions of fuzzy set theory and fuzzy system theory. The Zadeh's composition rule of inference [2] is a well-known method in approximation theory and inference methods in fuzzy control theory. Intuitionistic fuzzy relations were received many results [21][22]. Xu [24] defined some new intuitionistic preference relations, such as the consistent intuitionistic preference relation, incomplete intuitionistic preference relation and studied their properties. Thus, it is necessary to develop new approaches to issues, such as multi-period investment decision making, medical diagnosis, personnel dynamic examination, and military system efficiency dynamic evaluation. In this section we shall present some preliminary results on standard neutrosophic relations.

4.1.1 Standard neutrosophic relations

Let X, Y and Z be ordinary non-empty sets. A standard neutrosophic relation is defined as follows.

Definition 4.1 [18] A standard neutrosophic relation (SNR) R between X and Y is a SNS on $X \times Y$, i.e.

$$R = \left\{\left((x,y), \mu_R(x,y), \eta_R(x,y), \nu_R(x,y)\right) \middle| (x,y) \in X \times Y\right\},$$

where μ_R, η_R, $\nu_R : X \times Y \to [0,1]$ satisfy the condition

$$\mu_R(x,y) + \eta_R(x,y) + \nu_R(x,y) \le 1, \ (x,y) \in X \times Y.$$

We will denote by $SNR(X \times Y)$ the set of all SNRs between X and Y.

Definition 4.2 [18] Let $R \in SNR(X \times Y)$, the inverse relation R^{-1} of R is a SNR between Y and X defined as

$$\mu_{R^{-1}}(y,x) = \mu_R(x,y), \ \eta_{R^{-1}}(y,x) = \eta_R(x,y), \text{ and}$$

$$\nu_{R^{-1}}(y,x) = \nu_R(x,y), \forall (y,x) \in Y \times X.$$

Now we will consider some simple properties of SNRs.

Definition 4.3 [18] Let R, $P \in SNR(X \times Y)$, for every, we define**:**

a) $R \le P \Leftrightarrow \begin{cases} \mu_R(x,y) \le \mu_P(x,y) \\ \eta_R(x,y) \le \eta_P(x,y) \\ \nu_R(x,y) \ge \nu_P(x,y) \end{cases}$;

b) $R \vee P = \{((x,y), \mu_R(x,y) \vee \mu_P(x,y), \eta_R(x,y) \wedge \eta_P(x,y), \nu_R(x,y) \wedge \nu_P(x,y)) | (x,y) \in X \times Y\}$;

c) $R \wedge P = \{((x,y), \mu_R(x,y) \wedge \mu_P(x,y), \eta_R(x,y) \wedge \eta_P(x,y), \nu_R(x,y) \vee \nu_P(x,y)) | (x,y) \in X \times Y\}$;

d) $R^c = \{((x,y), \nu_R(x,y), \eta_R(x,y), \mu_R(x,y)) | (x,y) \in X \times Y\}$

Proposition 4.1 [18] Let R, P, $Q \in SNS(X \times Y)$. Then

a) $\left(R^{-1}\right)^{-1} = R$;

b) $R \le P \Rightarrow R^{-1} \le P^{-1}$;

c1) $(R\vee P)^{-1}=R^{-1}\vee P^{-1}$; c2) $(R\wedge P)^{-1}=R^{-1}\wedge P^{-1}$;

d1) $R\wedge(P\vee Q)=(R\wedge P)\vee(R\wedge Q)$;

d2) $R\vee(P\wedge Q)=(R\vee P)\wedge(R\vee Q)$;

e) $R\wedge P\le R$, $R\wedge P\le P$;

f1) If $R\ge P$ and $R\ge Q$ then $R\ge P\vee Q$;

f2) If $R\le P$ and $R\le Q$ then $R\le P\wedge Q$.

Proof. For the detail proof of this proposition, see [20].

4.1.2 Composition of standard neutrosophic relations

In this sub-section we present some compositions of SNRs.

Definition 4.4 [20] Let $R\in SNR(X\times Y)$ and $P\in SNR(Y\times Z)$. We will call max - min composed relation $P\circ_1 R\in SNR(X\times Z)$ to the one defined by

$$\mu_{P\circ_1 R}(x,z)=\vee_y\{\mu_R(x,y)\wedge\mu_P(y,z)\},$$

$$\eta_{P\circ_1 R}(x,z)=\wedge_y\{\eta_R(x,y)\wedge\eta_P(y,z)\}, \text{ and}$$

$$\nu_{P\circ_1 R}(x,z)=\wedge_y\{\nu_R(x,y)\vee\nu_P(y,z)\},\ \forall(x,z)\in X\times Z.$$

Definition 4.5 [20] Let $R\in SNR(X\times Y)$ and $P\in SNR(Y\times Z)$. We will call max-prod composed relation $P\circ_2 R\in SNR(X\times Z)$ to the one defined by

$$\mu_{P\circ_2 R}(x,z)=\vee_y\{\mu_R(x,y)\cdot\mu_P(y,z)\},$$

$$\eta_{P\circ_2 R}(x,z)=\wedge_y[\eta_R(x,y)\cdot\eta_P(y,z)], \text{ and}$$

$$\nu_{P\circ_2 R}(x,z)=\wedge_y\{\nu_R(x,y)+\nu_P(y,z)-\nu_R(x,y)\cdot\nu_P(y,z)\},$$

$\forall(x,z)\in X\times Z$.

Definition 4.6 [20] Let β be a t-norm, ρ be a t-conorm, $R\in SNR(X\times Y)$ and $P\in SNR(Y\times Z)$. We will call max-t composed relation $R\circ_3 P\in PFR(X\times Z)$ to the one defined by

$$\mu_{R\circ_3 P}(x,z)=\vee_y\left(\beta(\mu_R(x,y),\mu_P(y,z))\right),$$

$$\eta_{R\circ_3 P}(x,z)=\wedge_y\{\beta(\eta_R(x,y),\eta_P(y,z))\}, \text{ and}$$

$$\nu_{R\circ_3 P}(x,z)=\wedge_y\{\rho(\nu_R(x,y),\nu_P(y,z))\},$$

$\forall(x,z)\in X\times Z$.

The validation of Definitions 4.5-4.7 were given in [30].

4.2 Neutrosophic soft relations

4.2.1 Some operations on neutrosophic soft relations

In this sub-section, we give the definition of standard neutrosophic soft relation (SNSR) as a generalization of fuzzy soft relation and intuitionistic fuzzy soft relation. The novel concept is actually a parameterized family of standard neutrosophic relations (SNRs).

In following definitions, X, Y are ordinary non-empty sets and E is a set of parameters.

Definition 4.7 Let $A\subseteq E$. A pair (R,A) is called a *standard neutrosophic soft relation* (SNSR) over $X\times Y$ if R assigns to each parameter e in E a SNR $R(e)$ in $SNR(X\times Y)$, that is

$$R:A\to SNR(X\times Y).$$

The set of all SNSRs between X and Y is denoted by $SNSR(X\times Y)$.

Definition 4.8 Let A, $B\subseteq E$. The *intersection* of two SNSRs (R_1,A) and (R_2,B) over $X\times Y$ is a SNSR (R_3,C) over $X\times Y$ such that $C=A\cup B$ and for all $e\in C$,

$$R_3(e)=\begin{cases}R_1(e) & \text{if } e\in A\setminus B,\\ R_2(e) & \text{if } e\in B\setminus A,\\ R_1(e)\wedge R_2(e) & \text{if } e\in A\cap B.\end{cases}$$

This relation is denoted by $(R_1,A)\cap(R_1,B)$.

Definition 4.9 Let $A,B\subseteq E$ .The *union* of two SNSRs (R_1,A) and (R_2,B) over $X\times Y$ is a SNSR (R_3,C) over $X\times Y$, where $C=A\cup B$ and for all $e\in C$,

$$R_3(e)=\begin{cases}R_1(e) & \text{if } e\in A\setminus B,\\ R_2(e) & \text{if } e\in B\setminus A,\\ R_1(e)\vee R_2(e) & \text{if } e\in A\cap B.\end{cases}$$

This relation is denoted by $(R_1,A)\cup(R_2,B)$.

4.2.2 Composition of neutrosophic soft relations

We denote by $SNSR_{E_1}(X\times Y)$ the set of all SNSRs on $X\times Y$ with the corresponding parameter set E_1. Similarly, $SNSR_{E_2}(Y\times Z)$ denotes the set of all SNSRs on $Y\times Z$ with the corresponding parameter set E_2.

Definition 4.10 Let $R\in SNSR_{E_1}(X\times Y)$ and $P\in SNSR_{E_2}(Y\times Z)$. We will call max - min composed relation $P\bullet_1 R\in SNSR_{E_1\times E_2}(X\times Z)$ to the one defined by

$$P\bullet_1 R(e_1,e_2)=\Big\{(x,z),\mu_{P\bullet_1 R}(x,y)(e_1,e_2),\ \eta_{P\bullet_1 R}(x,z)(e_1,e_2)\ \nu_{P\bullet_1 R}(x,z)(e_1,e_2)\Big|(x,z)\in X\times Z\Big\},$$

$\forall(e_1,e_2)\in A_1\times A_2$. Where

$$\mu_{P\bullet_1 R}(x,z)(e_1,e_2)=\vee_y\left\{\mu_{R(e_1)}(x,y)\wedge\mu_{P(e_2)}(y,z)\right\},$$

$$\eta_{P\bullet_1 R}(x,z)(e_1,e_2)=\wedge_y\left\{\eta_{R(e_1)}(x,y)\wedge\eta_{P(e_2)}(y,z)\right\},$$

$$\nu_{P\bullet_1 R}(x,z)(e_1,e_2)=\wedge_y\left\{\nu_{R(e_1)}(x,y)\vee\nu_{P(e_2)}(y,z)\right\},$$

for all $(x,z)\in X\times Z$, $(e_1,e_2)\in A_1\times A_2$.

Definition 4.11 Let $R\in SNSR_{E_1}(X\times Y)$ and $P\in SNSR_{E_2}(Y\times Z)$. We will call max - prod composed relation $P\bullet_2 R\in SNSR_{E_1\times E_2}(X\times Z)$ to the one defined by

$$P\bullet_2 R(e_1,e_2)=\Big\{(x,z),\mu_{P\bullet_2 R}(x,y)(e_1,e_2),\ \eta_{P\bullet_2 R}(x,z)(e_1,e_2)\ \nu_{P\bullet_2 R}(x,z)(e_1,e_2)\Big|(x,z)\in X\times Z\Big\},$$

$\forall(e_1,e_2)\in A_1\times A_2$. Where

$$\mu_{P\bullet_2 R}(x,z)(e_1,e_2)=\vee_y\left\{\mu_{R(e_1)}(x,y)\cdot\mu_{P(e_2)}(y,z)\right\},$$

$$\eta_{P\bullet_2 R}(x,z)(e_1,e_2)=\wedge_y\left\{\eta_{R(e_1)}(x,y)\cdot\eta_{P(e_2)}(y,z)\right\},$$

$$\nu_{P\bullet_2 R}(x,z)(e_1,e_2)=\wedge_y\Big\{\nu_{R(e_1)}(x,y)+\nu_{P(e_2)}(y,z)-\nu_{R(e_1)}(x,y)\cdot\nu_{P(e_2)}(y,z)\Big\},$$

for all $(x,z)\in X\times Z$, $(e_1,e_2)\in A_1\times A_2$.

Definition 4.12 Let $R\in SNSR_{E_1}(X\times Y)$, $P\in SNSR_{E_2}(Y\times Z)$, β is a t-norm and ρ is a t-conorm. We will call max - t composed relation $P\bullet_3 R\in SNSR_{E_1\times E_2}(X\times Z)$ to the one defined by

$$P\bullet_3 R(e_1,e_2)=\Big\{(x,z),\mu_{P\bullet_3 R}(x,y)(e_1,e_2),\ \eta_{P\bullet_3 R}(x,z)(e_1,e_2)\ \nu_{P\bullet_3 R}(x,z)(e_1,e_2)\Big|(x,z)\in X\times Z\Big\},$$

$\forall(e_1,e_2)\in A_1\times A_2$. Where

$$\mu_{P\bullet_3 R}(x,z)(e_1,e_2)=\vee_y\left\{\beta\left(\mu_{R(e_1)}(x,y),\mu_{P(e_2)}(y,z)\right)\right\},$$

$$\eta_{P\bullet_3 R}(x,z)(e_1,e_2)=\wedge_y\left\{\beta\left(\eta_{R(e_1)}(x,y),\eta_{P(e_2)}(y,z)\right)\right\},$$

$$\nu_{P\bullet_2 R}(x,z)(e_1,e_2)=\wedge_y\left\{\rho\left(\nu_{R(e_1)}(x,y),\nu_{P(e_2)}(y,z)\right)\right\},$$

for all $(x,z)\in X\times Z$, $(e_1,e_2)\in A_1\times A_2$.

The validation of Definitions 4.11-4.13 is trivial by following arguments. For each pair $(e_1,e_2)\in A_1\times A_2$, $P\bullet_1 R(e_1,e_2)$ is max - min composition of two SNRs $R(e_1)$ and $P(e_2)$, i.e.

$$P\bullet_1 R(e_1,e_2)=P(e_2)\circ_1 P(e_2).$$

By the validation of $\circ_1$, $P\bullet_1 R(e_1,e_2)\in SNR(X\times Z)$ which yields $P\bullet_1 R\in SNSR_{E_1\times E_2}(X\times Z)$. The validation of $\bullet_2$ and $\bullet_3$ are also obtained by analogous calculations.

Conclusion

In 2013, the new notion of picture fuzzy sets was introduced. The novel concept, which is also termed as standard neutrosophic set (SNS), constitutes an importance case of neutrosophic set. Our neutrosophic soft set (NSS) theory is a combination of the standard neutrosophic theory and the soft set theory. In other words, neutrosophic soft set theory is a neutrosophic extension of the intuitionistic fuzzy soft set theory. The complement, "and", "or", union and intersection operations are defined on the NSSs. The standard neutrosophic soft relations (SNSR) are also considered. The basic properties of the NSSs and the SNSRs are also discussed. Some future work may be concerned interval- valued neutrosophic soft sets and interval-valued neutrosophic relations should be considered.

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The Concept of Neutrosophic Less Than or Equal To: A New Insight in Unconstrained Geometric Programming

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Abstract

In this paper, we introduce the concept of *neutrosophic less than* or *equal to*. The neutrosophy considers every idea < A > together with its opposite or negation < antiA > and with their spectrum of neutralities < neutA > in between them (i.e. notions or ideas supporting neither < A > nor < antiA >). The < neutA > and < antiA > ideas together are referred to as < nonA >. Neutrosophic Set and Neutrosophic Logic are generalizations of the fuzzy set and respectively fuzzy logic (especially of intuitionistic fuzzy set and respectively intuitionistic fuzzy logic) [5]. In neutrosophic logic, a proposition has a degree of truth (T), a degree of indeterminacy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^{-}0, 1^{+}[$. Another purpose of this article is to explain the mathematical theory of *neutrosophic geometric programming* (the unconstrained posynomial case). It is necessary
to work in fuzzy neutrosophic space $FN_s = [0,1] \cup [0, nI]$, $n \in [0,1]$. The theory stated in this article aims to be a complementary theory of *neutrosophic geometric programming*.

Keywords

Neutrosophic Less Than or Equal To, Geometric Programming (GP), Signomial Geometric Programming (SGP), Fuzzy Geometric Programming (FGP), Neutrosophic Geometric Programming (NGP), Neutrosophic Function in Geometric Programming.

1 Introduction

The classical Geometric Programming (GP) is an optimization technique developed for solving a class of non-linear optimization problems in engineering design. GP technique has its origins in Zener's work (1961). Zener tried a new approach to solve a class of unconstrained non-linear optimization problems, where the terms of the objective function were posynomials. To solve these problems, he used the well-known arithmetic-geometric mean inequality (i.e. the arithmetic mean is greater than or equal to the geometric mean). Because of this, the approach came to be known as GP technique. Zener used this technique to solve only problems where the number of posynomial terms of the objective function was one more than the number of variables, and the function was not subject to any constraints. Later on (1962), Duffin extended the use of this technique to solve problems where the number of posynomial terms in the objective function is arbitrary. Peterson (1967), together with Zener and Duffin, extended the use of this technique to solve problems which also include the inequality constraints in the form of posynomials. As well, Passy and Wilde (1967) extended this technique further to solve problems in which some of the posynomial terms have negative coefficients. Duffin (1970) condensed the posynomial functions to a monomial form (by a logarithmic transformation, it became linear), and particularly showed that a "duality gap" function could not occur in geometric programming. Further, Duffin and Peterson (1972) pointed out that each of those posynomial programs GP can be reformulated so that every constraint function becomes posy-/bi-nomial, including at most two posynomial terms, where posynomial programming - with posy-/mo-nomial objective and constraint functions - is synonymous with linear programming.

As geometric programming became a widely used optimization technique, it was desirable that an efficient and highly flexible method of solutions were available. As the complexity of prototype geometric programs to be solved increased, several considerations became important. Canonically, the degree of problem difficulty and the inactive constraints reported an algorithm capable of dealing with these considerations. Consequently, McNamara (1976) proposed a solution procedure for geometric programming involving the formulation of an augmented problem that possessed zero degree of difficulty.

Accordingly, several algorithms have been proposed for solving GP (1980's). Such algorithms are somewhat more effective and reliable when they are applied to a convex problem, and also avoid difficulties with derivative singularities, as variables raised to fractional powers approach zero, since logs of such variables will approach $-\infty$, and large negative lower bounds should be placed on those variables.

In the 1990's, a strong interest in interior point (IP) algorithms has spawned several (IP) algorithms for GP. Rajgopal and Bricker (2002) produced an efficient procedure for solving posynomial geometric programming. The procedure, which used the concept of condensation, was embedded within an algorithm for a more general (signomial) GP problem. The constraint structure of the reformulation provides insight into why this algorithm is successful in avoiding all of the computational problems, traditionally associated with dual-based algorithms.

Li and Tsai (2005) proposed a technique for treating (positive, zero or negative) variables in SGP. Most existing methods of global optimization for SGP actually compute an approximate optimal solution of a linear or convex relaxation of the original problem. However, these approaches may sometimes provide an infeasible solution, or might form the true optimum to overcome these limitations.

A robust solution algorithm is proposed for global algorithm optimization of SGP by Shen, Ma and Chen (2008). This algorithm guarantees adequately to obtain a robust optimal solution which is feasible and close to the actual optimal solution, and is also stable under small perturbations of the constraints [6].

In the past 20 years, FGP has developed extensively. In 2002, B. Y. Cao published the first monography of fuzzy geometric programming as applied optimization. A large number of FGP applications have been discovered in a wide variety of scientific and non-scientific fields, since FGP is superior to classical GP in dealing with issues in fields like power system, environmental engineering, postal services, economical analysis, transportation, inventory theory; and so more to be discovered.

Arguably, fuzzy geometric programming potentially becomes a ubiquitous optimization technology, the same as fuzzy linear programming, fuzzy objective programming, and fuzzy quadratic programming [2].

This work is the first attempt to formulate the neutrosophic posynomial geometric programming (the simplest case, i.e. the unconstrained case). A previous work investigated the maximum and the minimum solutions to the neutrosophic relational GP [7,8].

2 Neutrosophic Less than or Equal To

In order to understand the concept of neutrosophic less than or equal to in optimization, we begin with some preliminaries which serve the subject.

Definition 2.1

Let X be the set of all fuzzy neutrosophic variable vectors x_i, $i = 1,2, \dots, m$, i.e. $X = \{(x_1, x_2, \dots, x_m)^T \mid x_i \in FN_s\}$. The function $g(x)\colon X \to R \cup I$ is said to be the neutrosophic GP function of x, where $g(x) = \sum_{k=1}^{J} c_k \prod_{l=1}^{m} x_l^{\gamma_{kl}}$, $c_k \geq 0$ are constants, γ_{kl} - are arbitrary real numbers.

Definition 2.2

Let $g(x)$ be any linear or non-linear neutrosophic function, and let A_0 be the neutrosophic set for all functions $g(x)$ that are neutrosophically less than or equal to 1.

$$A_0 = \{ g(x) <_N 1, x_i \in FN_s\}$$
$$= \{ g(x) < 1, \quad anti(g(x)) > 1, \quad neut(g(x)) = 1, x_i \in FN_s\}.$$

Definition 2.3

Let $g(x)$ be any linear or non-linear neutrosophic function, where $x_i \in [0,1] \cup [0, nI]$ and $x = (x_1, x_2, \dots, x_m)^T$ a m-dimensional fuzzy neutrosophic variable vector.

We have the inequality

$$g(x) <_N 1 \tag{1}$$

where " $<_N$" denotes the neutrosophied version for " $\leq$ " with the linguistic interpretation being "less than (the original claimed), greater than (the anti-claim of the original less than), equal (neither the original claim, nor the anti-claim)".

The inequality (1) can be redefined as follows:

$$\left.\begin{array}{r} g(x) < 1 \\ anti\,(g(x)) > 1 \\ neut(\,g(x)) = 1 \end{array}\right\} \tag{2}$$

Definition 2.4

Let A_0 be the set of all neutrosophic non-linear functions that are neutrosophically less than or equal to 1.

$$A_0 = \{ g(x) <_N 1, x_i \in FN_s\}$$
$$= \{ g(x) < 1, \quad anti(g(x)) > 1, \quad neut(g(x)) = 1, x_i \in FN_s\}.$$

It is significant to define the following membership functions:

$$\mu_{A_o}(g(x)) = \begin{cases} 1 & 0 \leq g(x) \leq 1 \\ \left(e^{\frac{-1}{d_o}(g(x)-1)} + e^{\frac{-1}{d_o}(anti(g(x))-1)} - 1\right), & 1 < g(x) \leq 1 - d_o \ln 0.5 \end{cases} \tag{3}$$

$$\mu_{A_o}(\text{anti}(\,g(x))) = \begin{cases} 0 & 0 \le g(x) \le 1 \\ \left(1 - e^{\frac{-1}{d_o}(\text{anti}(\,g(x))-1)} - e^{\frac{-1}{d_o}(g(x)-1)}\right), & 1 - d_o \ln 0.5 \le g(x) \le 1 + d_o \end{cases} \quad (4)$$

It is clear that $\mu_{A_o}(\text{neut}(\,g(x)))$ consists of intersection the following functions:

$$e^{\frac{-1}{d_o}(g(x)-1)}, \quad 1 - e^{\frac{-1}{d_o}(\text{anti}(\,g(x))-1)}$$

i.e.

$$\mu_{A_o}(\text{neut}(\,g(x))) = \begin{cases} 1 - e^{\frac{-1}{d_o}(\text{anti}(\,g(x))-1)} & 1 \le g(x) \le 1 - d_o \ln 0.5 \\ e^{\frac{-1}{d_o}(g(x)-1)} & 1 - d_o \ln 0.5 < g(x) \le 1 + d_o \end{cases} \quad (5)$$

Note that $d_o > 0$ is a constant expressing a limit of the admissible violation of the neutrosophic non-linear function $g(x)$ [3].

2.1 The relationship between $g(x)$, anti $g(x)$ in NGP

1. At

$$1 < g(x) \le 1 - d_o \ln 0.5$$

$$\mu_{A_o}(\,g(x)\,) > \mu_{A_o}(\text{anti}(\,g(x)\,) \qquad \text{(see } \textit{Figure 1}\text{)}$$

$$e^{\frac{-1}{d_o}(g(x)-1)} > 1 - e^{\frac{-1}{d_o}(\text{anti}(\,g(x)\,)-1)}$$

$$e^{\frac{-1}{d_o}(\text{anti}(\,g(x)\,)-1)} > 1 - e^{\frac{-1}{d_o}(g(x)-1)}$$

$$\frac{-1}{d_o}(\text{anti}(\,g(x)\,) - 1) > \ln(1 - e^{\frac{-1}{d_o}(g(x)-1)})$$

$$\text{anti}(\,g(x)\,) < 1 - d_o \ \ln(1 - e^{\frac{-1}{d_o}(g(x)-1)})$$

2. Again at

$$1 - d_o \ln 0.5 < g(x) \le 1 + d_o$$

$$\mu_{A_o}(\,g(x)) < \mu_{A_o}(\text{anti}(\,g(x)))$$

$$\therefore \ \text{anti}(\,g(x)\,) > 1 - d_o \ \ln(1 - e^{\frac{-1}{d_o}(g(x)-1)})$$

3 Neutrosophic Geometric Programming (the unconstrained case)

Geometric programming is a relative method for solving a class of non-linear programming problems. It was developed by Duffin, Peterson, and Zener (1967) [4]. It is used to minimize functions that are in the form of posynomials, subject to constraints of the same type.

Inspired by Zadeh's fuzzy sets theory, fuzzy geometric programming emerged from the combination of fuzzy sets theory with geometric programming.

Fuzzy geometric programming was originated by B.Y. Cao in the Proceedings of the second IFSA conferences (Tokyo, 1987) [1].

In this work, the neutrosophic geometric programming (the unconstrained case) was established where the models were built in the form of posynomials.

Definition 3.1

Let

$$\left.\begin{array}{ll} \overset{N}{(P)} & \overset{N}{\min}\ g(x) \\ & x_i \in FN_s \end{array}\right\}. \quad (6)$$

The neutrosophic unconstrained posynomial geometric programming, where $x = (x_1, x_2, \dots, x_m)^{T}$ is a m-dimensional fuzzy neutrosophic variable vector, "T" represents a transpose symbol, and $g(x) = \sum_{k=1}^{J} c_k \prod_{l=1}^{m} x_l^{\gamma_{kl}}$ is a neutrosophic posynomial GP function of x, $c_k \geq 0$ a constant, γ_{kl} an arbitrary real number, $g(x) < ₦\, z \rightarrow \overset{N}{\min} g(x)$; the objective function $g(x)$ can be written as a minimizing goal in order to consider z as an upper bound; z is an expectation value of the objective function $g(x)$, " $<₦$ " denotes the neutrosophied version of " $\leq$ " with the linguistic interpretation (see Definition 2.3), and $d_o > 0$ denotes a flexible index of $g(x)$.

Note that the above program is undefined and has no solution in the case of $\gamma_{kl} < 0$ with some x_l's taking indeterminacy value, for example,

$$\overset{N}{\min} \quad g(x) = 2x_1^{-.2} x_2^{.3} x_4^{1.5} + 7x_1^{3} x_2^{-.5} x_3,$$

where $x_i \in FN_s, i = 1,2,3,4.$

This program is not defined at $x = (.2I, .3, .25, I)^{T}$, $g(x) = 2(.2I)^{-.2}(.3)^{.3} I^{1.5} + 7(.2I)^{3}(.3)^{-.5}(.25)$ is undefined at $x_1 = .2I$ with $\gamma_1 = -0.2$.

Definition 3.2

Let A_0 be the set of all neutrosophic non-linear functions $g(x)$ that are neutrosophically less than or equal to z, i.e.

$$A_0 = \{\ g(x) < ₦\, z, x_i \in FN_s\}.$$

The membership functions of $g(x)$ and anti$(g(x))$ are:

$$\mu_{A_o}(g(x)) = \begin{cases} 1 & 0 \leq g(x) \leq z \\ \left(e^{\frac{-1}{d_o}(g(x)-z)} + e^{\frac{-1}{d_o}(\text{anti}\,(g(x))-z)} - 1 \right), & z < g(x) \leq z - d_o \ln 0.5 \end{cases} \quad (7)$$

$$\mu_{A_o}(\text{anti}(g(x))) = \begin{cases} 0 & 0 \leq g(x) \leq z \\ \left(1 - e^{\frac{-1}{d_o}(\text{anti}\,(g(x))-z)} - e^{\frac{-1}{d_o}(g(x)-z)} \right), & z - d_o \ln 0.5 \leq g(x) \leq z + d_o \end{cases} \quad (8)$$

Eq. (6) can be changed into

$$g(x) < ₦\ z, \quad x = (x_1, x_2, \dots, x_m), x_i \in FN_s \quad (9)$$

The above program can be redefined as follow:

$$\left.\begin{array}{l} g(x) < z \\ \text{anti}(\ g(x)) > z \\ \text{neut}(\ g(x)) = z \\ x = (x_1, x_2, \dots, x_m), x_i \in FN_s \end{array}\right\} \qquad (10)$$

It is clear that $\mu_{A_o}(\text{neut}(\ g(x)))$ consists from the intersection of the following functions:

$$e^{\frac{-1}{d_o}(g(x)-z)} \quad \& \ 1 - e^{\frac{-1}{d_o}(\text{anti}(g(x))-z)}$$

$$\mu_{A_o}(\text{neut}(\ g(x))) = \begin{cases} 1 - e^{\frac{-1}{d_o}(\text{anti}(\ g(x))-z)} & z \le g(x) \le z - d_o \ln 0.5 \\ e^{\frac{-1}{d_o}(g(x)-z)} & z - d_o \ln 0.5 < g(x) \le z + d_o \end{cases} \qquad (11)$$

Definition 3.3

Let $\tilde{N}$ be a fuzzy neutrosophic set defined on $[0,1] \cup [0, nI], n \in [0,1]$; if there exists a fuzzy neutrosophic optimal point set A_o^* of $g(x)$ such that

$$\tilde{N}(x) = \begin{array}{c} \min\{\mu(\text{neut}\ g(x))\} \\ x = (x_1, x_2, \dots, x_m), x_i \in FN_s \end{array} \qquad (12)$$

$$\tilde{N}(x) = e^{\frac{-1}{d_o}\left(\sum_{k=1}^{J} c_k \prod_{l=1}^{m} x_l^{\gamma_{kl}} - z\right)} \wedge \ 1 - e^{\frac{-1}{d_o}\left(\text{anti}\left(\ \sum_{k=1}^{J} c_k \prod_{l=1}^{m} x_l^{\gamma_{kl}}\right) - z\right)},$$

then $\max \tilde{N}(x)$ is said to be a neutrosophic geometric programming (the unconstrained case) with respect to $\tilde{N}(x)$ of $g(x)$.

Definition 3.4

Let x^* be an optimal solution to $\tilde{N}(x)$, i.e.

$$\tilde{N}(x^*) = \max \tilde{N}(x) \ , x = (x_1, x_2, \dots, x_m), x_i \in FN_s \ , \qquad (13)$$

and the fuzzy neutrosophic set $\tilde{N}$ satisfying (12) is a fuzzy neutrosophic decision in (9).

Theorem 3.1

The maximum of $\tilde{N}(x)$ is equivalent to the program:

$$\left.\begin{array}{l} \max \alpha \\ g(x) < z - d_o \ln \alpha \\ \text{anti}\ g(x) > z - d_o \ln(1-\alpha) \\ x = (x_1, x_2, \dots, x_m), x_i \in FN_s\ , d_o > 0 \end{array}\right\} \qquad (14)$$

Proof

It is known by definition (3.4) that x^* satisfied eq. (12), called an optimal solution to (9). Again, x^* bears the similar level for $g(x)$, $\text{anti}(g(x))$ & $\text{neut}(g(x))$. Particularly, x^* is a solution to neutrosophic posynomial geometric programming (6) at $\tilde{N}(x^*) = 1$. However, when $g(x) < z$ and $\text{anti}(g(x)) > z$, there exists

$$\tilde{N}(x) = e^{\frac{-1}{d_o}\left(\sum_{k=1}^{J} c_k \prod_{l=1}^{m} x_l^{\gamma_{kl}} - z\right)} \wedge \ 1 - e^{\frac{-1}{d_o}\left(\text{anti}\,\left(\sum_{k=1}^{J} c_k \prod_{l=1}^{m} x_l^{\gamma_{kl}}\right) - z\right)},$$

given $\alpha = \tilde{N}(x)$. Now, $\forall\ \alpha \in FN_S$; it is clear that

$$e^{\frac{-1}{d_o}\left(\sum_{k=1}^{J} c_k \prod_{l=1}^{m} x_l^{\gamma_{kl}} - z\right)} \geq \alpha \qquad (15)$$

$$1 - e^{\frac{-1}{d_o}\left(\text{anti}\,\left(\sum_{k=1}^{J} c_k \prod_{l=1}^{m} x_l^{\gamma_{kl}}\right) - z\right)} \geq \alpha \qquad (16)$$

From (15), we have

$$\frac{-1}{d_o}\left(\sum_{k=1}^{J} c_k \prod_{l=1}^{m} x_l^{\gamma_{kl}} - z\right) \geq \ln \alpha$$

$$g(x) = \left(\sum_{k=1}^{J} c_k \prod_{l=1}^{m} x_l^{\gamma_{kl}}\right) \leq z - d_o \ln \alpha\,. \qquad (17)$$

From (16), we have

$$1 - \alpha \geq e^{\frac{-1}{d_o}\left(\text{anti}\,\left(\sum_{k=1}^{J} c_k \prod_{l=1}^{m} x_l^{\gamma_{kl}}\right) - z\right)}$$

$$\rightarrow \text{anti}\,\left(\sum_{k=1}^{J} c_k \prod_{l=1}^{m} x_l^{\gamma_{kl}}\right) - z \geq -d_o \ln(1 - \alpha) \qquad (18)$$

$$\text{anti}\,\left(g(x)\right) \geq z - d_o \ln(1 - \alpha).$$

Note that, for the equality in (17) & (18), it is exactly equal to neut $g(x)$.

Therefore, the maximization of $\tilde{N}(x)$ is equivalent to (14) for arbitrary $\alpha \in FN_S$, and the theorem holds.

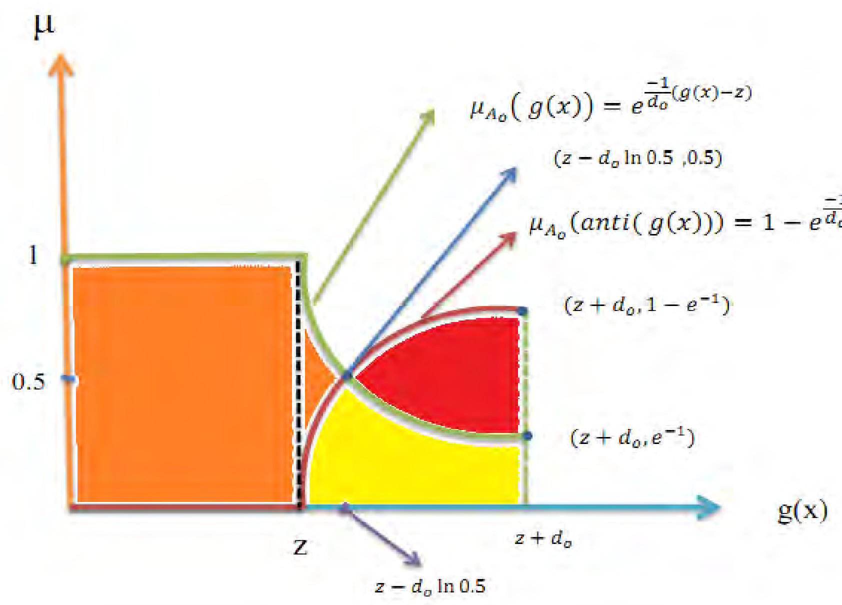

Figure 1. The orange color means the region covered by $\mu_{A_o}(\,g(x))$, the red color means the region covered by $\mu_{A_o}(\text{anti}(\,g(x)))$, and the yellow color means the region covered by $\mu_{A_o}(\text{neut}(\,g(x)))$.

4 Conclusion

The innovative concept and procedure explained in this article suit to the neutrosophic GP. A neutrosophic less than or equal to form can be completely turned into classical less than, greater than and equal forms. The feasible region for unconstrained neutrosophic GP can be determined by a fuzzy neutrosophic optimal point set in the fuzzy neutrosophic decision region $\tilde{N}(x^*)$.

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The Neutrosophic Statistical Distribution: More Problems, More Solutions

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Abstract: In this paper , the authors explore neutrosophic statistics, that was initiated by Florentin Smarandache in 1998 and developed in 2014, by presenting various examples of several statistical distributions, from the work **[1]**. The paper is presented with more case studies, by means of which this neutrosophic version of statistical distribution becomes more pronounced.

Key words: Neutrosophy, Binomial & Normal distributions, Neutrosophic logic etc.

I.Introduction: Neutrosophy was first proposed by Prof. Florentin Smarandache in 1995 . It is a new branch of philosophy , where one can study origin , nature and scope of neutralities . According to Prof. Dr.Huang, this gives advantages to break the mechanical understanding of human culture. For example, according to mechanical theory ,existence and non-existence couldn't be simultaneously , due to some indeterminacy [2].

This theory considers every notion or idea <A> together with its opposite or negation <Anti-A>. The <neut-A> and <Anti-A> ideas together called as a <non-A>. Neutrosophic logic is a general framework for unification of many existing logics, intutionstic logic, paraconsistent logic etc. The focal objective of neutrosophic logic is to characterize each logical statements in a 3D-neutrosophic space, where each dimension of space represents respectively the truth(T) , falsehood(F) and indeterminacies of the statements under consideration . Where T,I,F are standard or non-standard real subset of (-0,1+) without necessary connection between them. **[3]**

The classical distribution is extended neutrosophically. That means that there is some indeterminacy related to the probabilistic experiment. Each experimental observation of each trial can result in an outcome of each trial can result in an outcome labelled failure (F) or some indeterminacy(I).Neutrosophic statistics is an extended form of classical statistics, dealing with crisp values. In this paper, we will discuss about one discrete random distribution such as Binomial distribution and a continuousone by approaching neutrosophically. Before focusing the light on this context, we should familiar with the following notions.

Neutrosophic statistical number 'N' has the form

$$N = d + I;$$

Where, d: Determinate part

I: Indeterminate part of N.

For example, a = 5 + I ; where I ∈ [0 , 0.4] is equivalent to a∈ [5 , 5.4]. So for sure a ≥ 5, where I ∈[O , O.4] .

I.A. Preliminaries: In this context, we are going to discuss about the classical distributions**[4] .**

A). Binomial distribution,

B). Normal distribution.

I. A. a). Binomial distribution:

I. A.a.i. Definition: A random variable X is said to follow Binomial distribution, if it assumes only non-negative values and its probability mass function is given by,

$$p(X=x)=p(x)=\frac{n!}{x!(n-x)!}p^{x}q^{n}\text{6.}$$

otherwise equal to zero .

I.A.a.ii. Physical conditions: We get Binomial distribution under the following conditions–

1. Each trials results in two exhaustive and mutually disjoint outcomes termed as success and failure.

2. The number of trials ‘n’ is finite.

3. The trials are independent on each other.

4. The probability of success ‘p’ is constant for each trial.

I.A.b. Normal Distribution:

I.A.b.i. Definition: A random variable is said to have a normal distribution with parameters μ and σ^2 , ifitsp.d.f is given by the probability law ,

$$f(x;\mu;\sigma)=\frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\frac{x-\mu}{\sigma})^2}=\frac{1}{\sigma\sqrt{2\pi}}e^{\frac{x-\mu)^2}{2\sigma_x^2}}$$

$-\infty < x < \infty$ and $-\infty<\mu<\infty, \sigma_X > 0$.

A.I.b.ii. Chief characteristics of Normal Distribution and normal probability curve:

The normal probability curve is given by the equation

$$f(x)=\frac{1}{\sigma_X\sqrt{2\pi}}e^{\frac{-(x-\mu_X)^2}{2\sigma_X^2}} \quad ;-\infty < x < \infty$$

I.A.b.iii.Properties:

1.The point of inflexion of the curve are:

$$x=\mu_X \pm \sigma_X \ , \ f(x)=\frac{1}{\sigma_X\sqrt{2\pi}}e^{-1/2}$$

2. The curve is symmetrical and bell shaped about the line x = μ.

3. Mean, Median, Mode of distribution coincide.

4. X-axis is an asymptote to the curve.

5. Quartiles, $Q_1 = \mu - 0.6745\sigma$

$Q_3 = \mu + 0.6745\sigma$.

II. Neutrosophic Statistical Distribution:

II.i. Neutrosophic Binomial Distribution: The neutrosophic binomial random variable ‘x’ is then defined as the number of success when we perform the experiment n ≥ 1 times. The neutrosophic probability distribution of ‘x’ is also called neutrosophic binomial probability distribution.

II.i.a.Definitions:

1. **Neutrosophic Binomial Random Variable**: It is defined as the number of success when we perform the experiment n ≥ 1 times, and is denoted as ‘x’.
2. **Neutrosophic Binomial Probability Distribution:** The neutrosophic probability distribution of ‘x’ is called n.p.d.
3. **Indeterminacy:** It is not confined to experimental results (either success or failures).
4. **Indeterminacy Threshold**: It is the number of trials whose outcome is indeterminate. Where

th∈{0,1,2…n}

Let P(S) = The chance of a particular trial results in a success.

P(F) = The chance of a particular trial results in a failure , for both S and f different from indeterminacy .

P(I) = The chance of a particular trial results in an indeterminacy .

For example: for x∈ {0,1,2,…,n} , NP =(TX,IX,FX) with

TX : Chances of 'x' success and (n-x) failures and indeterminacy but such that the no. of indeterminacy is less than or equal to indeterminacy threshold.

FX : Chances of 'y' success , with y≠x and (n-y) failures and indeterminacy is less than the indeterminacy threshold.

IX : Chances of 'z' indeterminacy , where 'z' is strictly greater than thee indeterminacy threshold.

$$TX + FX + IX = (P(S) + P(I) + P(F))^n$$

For complete probability, P(S) + P(I) + P(F) = 1 ;

For incomplete probability ,

$$0 \le P(S) + P(I) + P(F) < 1 ;$$

For paraconsistent probability ,

$$1< P(S) + P(I) + P(F) \le 3 .$$

Now ,

$$Tx = \frac{n!}{x!(n-x)!}[P(S)^x \sum_{k=0}^{th} \frac{k!}{(n-x)!(k-n+x)!} P(I)^K P(F)^{n-x-k}$$

$$= \frac{n!}{x!(n-x)!} P(S)^x \sum_{k=0}^{th} \frac{(n-x)!}{(n-x-k)!} P(I)^k P(F)^{n-x-k}$$

$$= \frac{n!}{x!} P(S)^x . \sum_{k=0}^{th} \frac{P(I)^k . P(F)^{n-x-k}}{k!(n-x-k)!}$$

$$Fx = \sum_{y=0}^{n} T_y = \sum_{y=0, y\neq x} \frac{n!}{y!} P(S)^y \sum_{k=0}^{th} \frac{P(S)^k . P(F)^{n-y-k}}{k!(n-y-k)!}$$

$$Ix = \sum_{z=th+1}^{n} \frac{n!}{z!(n-z)!} P(I)^z . \sum_{k=0}^{n-z} \frac{(n-z)!}{(n-z)!(n-z-k)!} P(S)^k . P(F)^{n-z-k}$$

$$= \sum_{z=th+1}^{n} \frac{n!}{z!} P(I)^z . \sum_{k=0}^{n-z} \frac{P(S)^k . P(F)^{n-z-k}}{k!(n-z-k)!}$$

Where ,

Tx , Ix , Fx ,P(S) ,P(I) , P(F) have their usual meaning. Now we are going to discuss several cases.

II.i.b.1. Case studies :

1. Two friends Asish and Rajesh are going to throw 5 coins simultaneously. There are 60% of chance to get head and 30% of chance to get tail. Independent on the view of Asish ,Rajes said that the probability of the result that are neither Head nor Tail is 20% . Then find the probability of getting 3 Heads when indeterminacy threshold is 2.

Solution:

$$Tx = \frac{5!}{3!(5-3)!}[(0.6)^3 \sum_{k=0}^{2} \frac{k!}{2!(k-2)!}(0.2)^k (0.3)^{2-k}]$$

$$= \frac{5!}{3!(5-3)!}[(0.6)^3 \{\frac{2!}{2!}(0.2)^2\}]$$

$$= 10[(0.6)^3 \{(0.2)^2\}] = 0.0864$$

$$I_x = \sum_{z=th+1}^{n} \frac{n!}{z!} P(I)^z . \sum_{k=0}^{n-z} \frac{P(S)^k . P(F)^{n-z-k}}{k!(n-z-k)!}$$

$$\therefore \ I_3 = \sum_{z=3}^{5} \frac{5!}{3!}(0.2)^z \sum_{k=0}^{2} \frac{(0.6)^k (0.3)^{2-k}}{k!(2-k)!}$$

$$= \sum_{z=3}^{5} \frac{5!}{3!}(0.2)^z \{\frac{(0.3)^2}{2!} + (0.6)(0.3) + \frac{(0.6)^2}{2!}\}$$

$$= \sum_{z=3}^{5} \frac{5!}{3!}(0.2)^z \{\frac{(0.3)^2}{2!} + (0.6)(0.3) + \frac{(0.6)^2}{2!}\}$$

$$= \{0.324+0.072+0.1008\} = 0.496$$

$$F_x = (P(S)+P(I)+P(F))^n \quad Tx - Fx$$

$$\therefore F_3 = (0.6+0.3+0.2)^5 - 0.0864 - 0.496$$

$$= 1.02811$$

2. Five coins are thrown simultaneously , the probability of success is 1/3 and the indeterminacy (the surface is very rough , so the coins may stand up) is 1/3 . Then find the probability of getting 3 Heads when the indeterminacy threshold is 2.

Solution:

Let x be no. of chances of getting heads in 5 trials .

$$T_x = \frac{n!}{x!}P(S)^x \sum_{k=0}^{th} \frac{P(I)^k P(F)^{n-x-k}}{k!(n-x-k)!}$$

$$\therefore T_3 = \frac{5!}{3!}(0.33)^3 \sum_{k=0}^{2} \frac{(0.33)^k (0.33)^{2-k}}{k!(2-k)!}$$

$$= \frac{5!}{3!}(0.33)^3 \{\frac{(0.33)^2}{2!} + (0.33)^2 + \frac{(0.33)^2}{2!}\}$$

$$=40.(0.33)^5 = 0.15654$$

$$I_x = \sum_{z=th+1}^{n} \frac{n!}{z!}P(I)^z . \sum_{k=0}^{n-z} \frac{P(S)^k P(F)^{n-z-k}}{k!(2-k)!}$$

$$\therefore I_3 = \sum_{z=3}^{5} \frac{5!}{3!}(0.33)^3 . \sum_{k=0}^{2} \frac{(0.33)^k (0.33)^{2-k}}{k!(2-k)!}$$

$$=(0.33)^5[40 + (7.5)(0.33) + 1] = 0.17014$$

$$F_x = (P(S) + P(I) + P(F))^n - T_x - I_x$$

$$\therefore F_3 = (0.33 + 0.33 + 0.33)^5 - T_x - I_x$$

$$so, (T_x, I_x, F_x) = (0.15654, 0.17014, 0.67332)$$

3. Two friends Liza and Laxmi play a game in which their chance of winning is 2:3 . The chances of dismissing game is 30% . Then find the probability of Liza's chances of winning at least 3 games out of 5 games played when the indeterminacy threshold is 2.

solution:

x is the no. of chances of winning the game

Let th = 2

$$T_X = \frac{n!}{x!}P(S)^X . \sum_{k=0}^{th} \frac{P(I)^k . P(F)^{n-x-k}}{k!(n-x-k)!}$$

$$\therefore T_3 = \frac{5!}{3!}(0.4)^3 . \sum_{k=0}^{th} \frac{(0.3)^k (0.6)^{2-k}}{k!(2-k)!}$$

$$=20(0.405)^4 = 0.53808$$

$$I_x = \sum_{z=th+1}^{n} \frac{n!}{z!}P(I)^Z . \sum_{k=0}^{n-z} \frac{P(S)^k P(F)^{n-z-k}}{k!(n-z-k)!}$$

$$I_3 = \sum_{Z=}^{5} \frac{5!}{z!}(0.3)^z . \sum_{k=0}^{n-z} \frac{(0.4)^k (0.6)^{5-z-k}}{k!(5-z-k)!}$$

$$=20(0.3)^3(0.5) + 5(0.3)^4(0.42)$$

$$=0.28701$$

and $F_3 = 2.88785$, *this is* a paraconsistent probability which is ≤ 3 .

4. In a precision bombing attack there is a 50% chance that any one bomb will strike the target . Two direct hits are required to destroy the target . If the chance of failure of mission is 30% , then find how many bombs are required to give a 99% chance with th=2 .

Solution:

Let x is the no. of chances of hitting bomb

$$T_x = \frac{n!}{x!}P(S)^x . \sum_{k=0}^{th} \frac{P(I)^k P(F)^{n-x-k}}{k!(n-x-k)!}$$

$$\therefore T_3 = \frac{5!}{3!}(0.5)^3 . \sum_{k=0}^{2} \frac{(0.3)^k (0.3)^{2-k}}{k!(2-k)!}$$

$$=40(0.5)^5 = 0.0972$$

$$I_x = \sum_{z=th+1}^{n} \frac{n!}{z!}P(I)^Z . \sum_{k=0}^{n-z} \frac{P(S)^k P(F)^{n-z-k}}{k!(n-z-k)!}$$

$$\therefore I_3 = \sum_{z=3}^{5} \frac{5!}{z!}(0.3)^z . \sum_{k=0}^{n-z} \frac{(0.5)^k (0.3)^{5-z-k}}{k!(5-z-k)!}$$

$$=0.1728 + 0.0078975 = 0.18069$$

$$\text{therefore } F_3 = (P(S) + P(F) + P(I))^5 - T_3 - I_3$$

$$=(1.1)^5 - 0.27789 = 1.33262$$

$$so\ (T_{x=3}, F_{x=3}, I_{x=3}) = (0.0972, 1.3326, 0.1806).$$

It is an example of paraconsistent probability.

5. It is decided that a cricket player , Jagadiswar has to appear 4 times for a physical test . If the possibility of passing the test is 2/3 ; and one referee guess that the chance of dismiss of game is 30% , then what is the probability of that the player passes the test at least 3 times, provided th=2?

Solution:

Let x is no. of chances that the player passes the test

$$T_x = \frac{n!}{x!}P(S)^x . \sum_{k=0}^{th} \frac{P(I)^k P(F)^{n-x-k}}{k!(n-x-k)!}$$

$$\therefore T_3 = \frac{4!}{3!}(0.66)^3 . \sum_{k=0}^{2} \frac{(0.3)^k (0.33)^{1-k}}{k!(1-k)!}$$

$$=8(0.66)^3(0.33) = 0.7589$$

$$I_x = \sum_{z=th+1}^{n} \frac{n!}{z!}P(I)^z . \sum_{k=0}^{n-z} \frac{P(S)^k P(F)^{n-z-k}}{k!(n-z-k)!}$$

$$\therefore I_3 = \sum_{z=3}^{4} 4(0.33)^z . \sum_{k=0}^{1} \{(0.66)^k (0.33)^{1-k}\} / \{k!(1-k)!\}$$

$$=(0.33)^3[3.96 + 0.33] = 0.15416$$

$$\textit{therefore } F_3 = (0.66 + 0.33 + 0.3)^4 - T_3 - I_3$$

$$=2.76922 - 0.91306 = 1.85616$$

$$SO\ (T_3, I_3, F_3) = (0.7589, 0.1541, 1.8561).$$

II.i.b.2.Exercises:

1. In a B.Sccourse , suppose that a student has to pass a minimum of 4 tests out of 8 conducted tests during the year to get promoted to next academic year . One student, Sarmistha says that his chance of winning is 80% , another student, Baisakhi

says that his chance of winning is 0.3 . Then find the probability of the promotion of Sarmistha , when the indeterminacy (either illegal paper correction or system error) is 20% , provided th=2.

2. If a car agency sells 40% of its inventory of a certain foreign cars equipped with air bags , the asst. manager says that the cars which are neither equipped with air bags nor a general one is 30% , then find theprobability distribution of the 2 cars with airbags among the next 4 cars sold with th=2?

3. A question paper contain 5 questions and a candidate will be declared to have passed the exam. If he/she answered at least one question correctly, considering the uncertainty as 33% (may be improper paper correction or system error etc.). What is the probability that the candidate passes the examination?

II.ii.Neutrosophic Normal Distribution:

Neutrosophic normal distribution of a continuous variable X is a classical normal distribution of X, but such that its mean μ or its standard deviation σ or variance σ^2 or both are imprecise. For example , μ or σ or both can be set with two or more elements . The most common such distribution are when μ, σ or both are intervals .

$$X_N \sim N_N(\mu_N,\sigma_N) = \frac{1}{\sigma_N\sqrt{2\pi}} e^{\left(\frac{(X-\mu_N)^2}{2\sigma_N^2}\right)}$$

N_N : *Normal* distribution may be neutrosophic

X_N : X may be neutrosophic

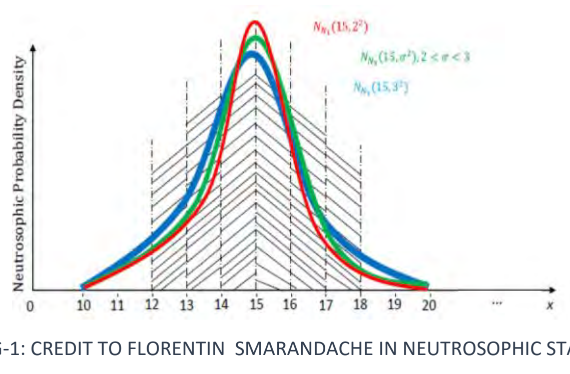

FIG-1: CREDIT TO FLORENTIN SMARANDACHE IN NEUTROSOPHIC STATISTICS

II.ii.a. Case studies:

1. In a college examination of a particular year, 60% of the Student failed when the mean of marks was 50% and the standard deviation is 5% with uncertainty I∈ [0,0.4] .The college decided to relax the condition of passing by lowering the passing marks to show its result as 80% passed , find the minimum marks to be kept for passing when marks are distributed normally .

Solution : Let μ = 50 , σ = 5 with indeterminacy I∈[0,0.4] ,so σ = 5 + [0,0.4] = [5,5.4] . therefore,μ±σ =50 ± [5,5.4] = [50-5.4,50+5] = [44.6 ,55] . Thus, 66.04 % of values lies in [44.6,55] .

$$\begin{aligned}
0.8 &= P(X_N \ge a_N)\\
&=1\text{-P}(X_N \le \alpha_N)\\
&=1\text{-P}\left(\frac{X_N-\mu_N}{\sigma_N} \le \frac{\alpha_N-\mu_N}{\sigma_N}\right) = 1-P\left(Z_N \le \frac{\alpha_N-50}{[5,5.4]}\right)\\
\therefore P\left(Z \le \frac{\alpha\ \ -50}{[5,5.4]}\right) &= 0.2, clearly \frac{a_N-50}{[5,5.4]} < 0, Let\ so, P(Z_N \le Z_{0.2}) = 0.8\\
Z_{O.2} &= -\left(\frac{\alpha_N-50}{[5,5.4]}\right) = 48.45\%\ approx.
\end{aligned}$$

2. If the monthly machine repair and maintenance cost X in a certain factory is known to be neutrosophically normal with mean 1000 and standard deviation 10000 , find the followings-

μ±σ,μ±2σ,when I∈[0,0.3].

Solution: Let μ=10000 ,σ=1000+[0,0.3],then μ±σ=10000±[1000,1000.03] . Thus 66.06% of values lies in [9000.03,11000]. And μ±2σ=10000±2[1000,1000.03]=[7999.97,12000]. Thus 75.04% of values lies in [7999.97,12000].

II.ii.b. Exercises:

1. A machine fills boxes weighting B kg with A kg of salt , where A and B are neutrosophically normal with mean 200kg and 10kg respectively and standard deviation of 2kg and 1kg respectively , what percentage of filled boxes weighting between 110kg an 120kg are to be expected when I∈[0,0.5].

2. The average life of a bulb is 2000 hours and the standard deviation is 400 hours .If X_N is the life period of a bulb which is distributed

normally in a neutrosophic plane. Find the probability that a randomly picked bulb will lasts ≤600 hrs. , considering the distribution is neutrosophically normal with indeterminacy I∈[0,0.2].

Till now, we have discussed various types of practical cases in statistical approach. Now we review the general formula for fusioning classical subjective probability provided by 2 sources.

The principle of redistributing the conflicting chances for ex. t and I are same as in PCR5 rule for the DSmT used in information fusion if 2 sources of information S_1 *and* S_2 give the subjective probability P_1 and P_2 about 't' to combining by PCR5 rule **,[5]**

$$(P_1 \wedge P_2)E = P_1(E)P_2(E) + \sum_{x \in E \wedge x \cap E = \phi} [\frac{P_1(E)^2 P_2(X)}{P_1(E)+P_2(X)} + \frac{P_2(E)\ P_1(X)}{P_2(E)+P_2(X)}]$$

It helps to the generalization of classical probability theory, fuzzy set, fuzzy logic to their respective domains. They are useful in artificial intelligence, neutrosophic dynamic system, quantum mechanics **[6].**

This theory can be used for topical communication study **[7].** It may also be applied to neutrosophic cognitive map study **[8].**

Thus we have presented our discussion with certain essential area of neutrosophy in a synchronized manner. Now we are going to explore some open challenges as follows.

Which are designed for inquiring minds.

Open Problems:

1. Can this Neutrosophic Statistics be applied to Industrial Management study?
2. Can we apply it with the study of Digital Signal Processing?
3. Can we merge the Representation theory [9]with Neutrosophy for a new theory ?
4. Is the uncertain theory, K-theory [10]solve the recent intriguing statistical problems by the power of this Neutrosophic logic ?
5. Can we construct a special master-space by the fusion of manifold concepts [11], soft topology [12], Ergodic theory [13],with Neutrosophic distribution ?
6. Is it possible for the construction of Neutrosophic manifold?
7. Is it possible for the construction of neutrosophic algebraic geometry[14] ?

III. **Conclusion:**

The actual motto of this short paper is to present the theory of Neutrosophic probability distribution in a more lucid and clear-cut way .The author presents various solved and unsolved problems, which are existed in reference to Neutrosophic 3D- space .Various practical situations are described and were tried to solve by Neutrosophic logic. The spectra of this theory may be applied to Quantum physics [15]and M-theory [16]. It may be said that it can also be applied to Human psychology as well as Behavioral study. I hope that the more extended version (with large no. of case studies) with the area of application of this theory will see the light of the day in recent future. Here we limited our discussion of problem analysis to some extent due to limited scope of presentation. And lastly but important that if some unmatched/contradicted idea will occur in this paper, then it is surely unintentional. Finally I hope that the idea on the advanced version of this theory,which is already raised in my brain, will change their abstractskeleton into a paper, in coming future.

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δ-equalities of Neutrosophic Sets

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***Abstract*— Fuzzy sets and intuitionistic fuzzy sets can't handle imprecise, indeterminate, inconsistent, and incomplete information. Neutrosophic sets play an important role to overcome this difficulty. A neutrosophic set has a truth membership function, indeterminate membership function, and a falsehood membership functions that can handle all types of ambiguous information. New type of union and intersection has been proposed in this paper. In this paper, δ -equalities of neutrosophic sets have been introduced. Further, some basic properties of δ -equalities have been discussed. Moreover, these δ -equalities have been applied to set theoretic operations of neutrosophic set such as union, intersection, complement, product, probabilistic sum, bold sum, bold intersection, bounded difference, symmetrical difference, and convex linear sum of min and max. These δ -equalities of neutrosophic sets have been further extended to neutrosophic relations and neutrosophic norms respectively. In this paper, δ -equalities also applied in the composition of neutrosophic relations, Cartesian product and neutrosophic triangular norms. The applications and utilizations of δ -equalities have been presented in this paper. In this regards, δ -equalities have been successfully applied in Fault Tree Analysis and Neutrosophic Reliability (generalization of Profust Reliability).**

Keywords*— *Fuzzy set, intuitionistic fuzzy set, neutrosophic set, neutrosophic relation, δ -equalities.

I. Introduction

Fuzzy sets were introduced by Zadeh [31] in 1965. This novel mathematical framework is used to handle uncertainty in several areas of our real life. The characterization of a fuzzy set is made by a membership function μ which has the range $[0,1]$. The applications and utilization of fuzzy sets have been extensively found in different aspects from the last few decades such as control [31], pattern recognition[13], and computer vision[32] etc. This theory also become an important area for the researchers in medical diagnosis [32], engineering [13], social sciences [32] etc. A huge amount of literature on fuzzy set theory can be found in [13,22,32]. In fuzzy set, the membership function μ is a single value between in the unit interval [0,1]. Therefore it is not always true that the non-membership function ν of an element is equal to $1-\mu$, because there is some kind of hesitation. Thus in 1986, Atanassov [1] introduced intuitionistic fuzzy sets to explain this situation by incorporating the hesitation degree called hesitation margin. The hesitation margin is defined as $1-\mu-\nu$. Thus an intuitionistic fuzzy set has a membership function μ and non-membership function ν which has range $[0,1]$ with an extra condition that $0 \le \mu+\nu \le 1$. In this way, the intuitionistic fuzzy set theory became the generalization of fuzzy set theory. As an application point of view, the intuitionistic fuzzy set theory have been successfully applied in medical diagnosis [21], pattern recognition [22], social sciences [6] and decision making [11] etc.

Fuzzy sets and intuitionistic fuzzy sets can't handle imprecise, indeterminate, inconsistent, and incomplete information. Therefore, Smarandache [20] in 1998, introduced neutrosophic logic and set inspired from Neutrosophy, a branch of philosophy that deals with the origin, nature, and scope of neutralities and their interactions with different ideational spectra. A neutrosophic set has a truth membership function T , an indeterminate membership function I and a falsehood membership function F . The indeterminacy degree I plays a very important role of mediocrity. Thus, neutrosophic set theory generalizes the concept of classical set theory [20], fuzzy sets theory [31], intuitionistic fuzzy sets theory [1], interval valued fuzzy set theory [22], paraconsistent theory [20], dialetheist theory [20], paradoxist theory [20], and tautological theory [20]. Neutrosophic set is a powerful tool to handle the indeterminate and inconsistent information that exists in our real world. The researchers have been successfully applied neutrosophic set theory in several areas. In this regard, Wang et al. [26] introduced single valued neutrosophic sets in order to use them in scientific and engineering that gives some additional possibility to represent uncertain, incomplete, imprecise, and inconsistent data. Hanafy et.al discussed the correlation coefficient of neutrosophic set [8,9]. Ye [27] conducted study on the correlation coefficient of single valued neutrosophic sets. Broumi and Smaradache studied the correlation coefficient of interval neutrosophic set in [2]. Salama et al. [18] discussed neutrosophic topological spaces. Some more literature on neutrosophic set can be seen in [5, 12, 14, 17, 19, 25, 28, 29, 30]. Neutrosophic set have been applied successfully in decision making theory [5, 27-30], data base [25], medical diagnosis [30], pattern recognition [7,15] and so on.

The notion of proximity measure was introduced by Pappis [16] to show that values of precise membership has no practical significance. He believed that the maxmin compositional rule of inference is preserved with approximately equal fuzzy sets. Hong and Hwang [10] proposed another important generalization of the work of Pappis [16] which is mainly based that the maxmin compositional rule of inference is preserved with respect to 'approximately equal fuzzy sets' and 'approximately equal' fuzzy relation respectively. But, Cai [3, 4] felt that both the Pappis [16] and Hong and Hwang [10] tactics were limited to fixed . Therefore, Cai [3, 4] took a different methodology to introduced δ -equalities of fuzzy sets. Cai [3, 4] proposed that if two fuzzy sets are equal to a degree of δ , then they are said to be δ -equal. The approach of δ -equalities have significances in the fuzzy statistics as well as fuzzy reasoning. Virant [23] applied δ -equalities of fuzzy sets in synthesis of real-time fuzzy systems while Cai [3,4] used them for assessing the robustness of fuzzy reasoning. Cai [3,4] also explain several reliability examples of δ -equalities.

This paper extends the theory of δ -equalities to neutrosophic sets. Basically we followed the philosophy of Cai [3, 4] to studied δ -equalities of neutrosophic sets. The organization of the rest of the paper is followed. In section 2, some basic and fundamental concepts of neutrosophic sets were presented. New type of union and intersection has been introduced. δ -equalities on neutrosophic sets were introduced in section 3. Moreover, these δ -equalities have been applied to set theoretic operations of neutrosophic set such as union, intersection, complement, product, probabilistic sum, bold sum, bold intersection, bounded difference, symmetrical difference, and convex linear sum of min and max. In section 4, these δ -equalities of neutrosophic sets have been further extended to neutrosophic relations and neutrosophic norms respectively. The applications and utilizations of δ -equalities have been presented in section 5. In this regards, δ -equalities have been successfully applied in Fault Tree Analysis and Neutrosophic Reliability (generalization of Profust Reliability). Conclusion is given in section 6.

We now review some basic concepts of neutrosophic sets and other related notion which will be used in this paper.

II. Literature Review

In this section, some basic concepts of neutrosophic sets and other related notions have been presented. These notions and definitions have been taken from [3], [4], [20], and [26].

Definition 2.1 [20]. Neutrosophic Set

Let U be a space of points and let $u \in U$. A neutrosophic set S in U is characterized by a truth membership function T_S, an indeterminacy membership function I_S, and a falsity membership function F_S . $T_S(u)$, $I_S(u)$ and $F_S(u)$ are real standard or non-standard subsets of $]0^-,1^+[$, that is $T_S, I_S, F_S : X \to]0^-,1^+[$. The neutrosophic set can be represented as

$$S = \{(u, T_S(u), I_S(u), F_S(u)) : u \in U\}$$

There is no restriction on the sum of $T_S(u), I_S(u)$ and $F_S(u)$, so $0^- \le T_S(u) + I_S(u) + F_S(u) \le 3^+$. From philosophical point view, the neutrosophic set takes the value from real standard or non-standard subsets of $]0^-,1^+[$. Thus it is necessary to take the interval $[0,1]$ instead of $]0^-,1^+[$ for technical applications. It is difficult to apply $]0^-,1^+[$ in the real life applications such as engineering and scientific problems. We now give some set theoretic operations of neutrosophic sets.

Definition 2.2 [26]. Complement of Neutrosophic Set

The complement of a neutrosophic set S is denoted by S^c and is defined by

$$T_{S^c}(u) = 1 - T_S(u),\ I_{S^c}(u) = 1 - I_S(u),$$
$$F_{S^c}(u) = 1 - F_S(u) \text{ for all } u \in U.$$

Definition 2.3 [26]. Union of Neutrosophic Sets

Let A and B be two neutrosophic sets in a universe of discourse X . Then the union of A and B is denoted by $A \cup B$, which is defined by

$$A \cup B = \{(x, T_A(x) \vee T_B(x), I_A(x) \wedge I_B(x), F_A(x) \wedge F_B(x)) : x \in X\}$$

for all $x \in X$, and $\vee$ denote the max-operator and $\wedge$ denote the min-operator respectively.

Definition 2.4 [26]. Intersection of Neutrosophic Sets

Let A and B be two neutrosophic sets in a universe of discourse X . Then the intersection of A and B is denoted as $A \cap B$, which is defined by

$$A \cap B = \{(x, T_A(x) \wedge T_B(x), I_A(x) \vee I_B(x), F_A(x) \vee F_B(x)) : x \in X\}$$

Lemma 2.1 [3,4]: Let $\delta_1 * \delta_2 = \max(0, \delta_1 + \delta_2 - 1)$, where $0 \le \delta_1, \delta_2 \le 1$. Then

1. $0 * \delta_1 = 0$; for all $\delta_1 \in [0,1]$,
2. $1 * \delta_1 = \delta_1$; for all $\delta_1 \in [0,1]$,
3. $0 \le \delta_1 * \delta_2 \le 1$; for all $\delta_1, \delta_2 \in [0,1]$,

4. $\delta_1 \le \delta_1' \Rightarrow \delta_1 * \delta_2 \le \delta_1' * \delta_2$; for all $\delta_1, \delta_1', \delta_2 \in [0,1]$,
5. $\delta_1 * \delta_2 = \delta_2 * \delta_1$; for all $\delta_1, \delta_2 \in [0,1]$,
6. $(\delta_1 * \delta_2) * \delta_3 = \delta_1 * (\delta_2 * \delta_3$; for all $\delta_1, \delta_2, \delta_3 \in [0,1]$.

III. δ-Equalities of Neutrosophic Sets

In this section, new type of union and intersection of neutrosophic set introduced which will be used in this paper. Further, δ-equalities of neutrosophic sets introduced and discussed some of their properties. Moreover, these δ-equalities utilized in set theoretic operations of neutrosophic set such as union, intersection, complement, product, probabilistic sum, bold sum, bold intersection, bounded difference, symmetrical difference, and convex linear sum of min and max.

Definition 3.1: Let A and B be two complex neutrosophic sets in a universe of discourse U. Then the union of A and B is denoted by $A \cup B$, which is defined by

$$A \cup B = \{(u, T_A(u) \vee T_B(u), I_A(u) \vee I_B(u), F_A(u) \vee F_B(u)) : u \in U\}$$

for all $u \in U$, and $\vee$ denote the max-operator.

Definition 3.2: Let A and B be two complex neutrosophic sets in a universe of discourse U. Then the intersection of A and B is denoted as $A \cap B$, which is defined by

$$A \cap B = \{(u, T_A(u) \wedge T_B(u), I_A(u) \wedge I_B(u), F_A(u) \wedge F_B(u)) : u \in U\}$$

for all $u \in U$, and $\wedge$ denote the min-operator.

Definition 3.3: Let U be a universe of discourse. Let A and B be two neutrosophic sets on U, and $T_A(u)$, $I_A(u)$, $F_A(u)$ and $T_B(u)$, $I_B(u)$, $F_B(u)$, their truth membership functions, indeterminate membership functions and falsehood membership functions respectively. Then A and B are said to be δ-equal if and only if

$\sup_{u\in U}|T_A(u) - T_B(u)| \le 1-\delta$, $\sup_{u\in U}|I_A(u) - I_B(u)| \le 1-\delta$, $\sup_{u\in U}|F_A(u) - F_B(u)| \le 1-\delta$, for all $u \in U$ and $0 \le \delta \le 1$.

We denote it as $A = (\delta)B$. From the definition it is clear that $1-\delta$ is the maximum difference or proximity measure between A and B and δ is the degree of equality between them. It is customary to be noted that δ-equality of neutrosophic sets construct the class of neutrosophic relations.

Proposition 3.1: For two neutrosophic sets A and B, defined on U. The following assertions hold.

1. $A = (0)B$,
2. $A = (1)B$ if and only if $A = B$,
3. $A = (\delta)B$ if and only if $B = (\delta)A$,
4. $A = (\delta_1)B$ and if $\delta_1 \ge \delta_2$, then $A = (\delta_2)B$,
5. If $A = (\delta_\alpha)B$ for all $\alpha \in J$, where J is an index set, then $A = \left(\sup_{\alpha\in J}\delta_\alpha\right)B$,
6. For all A, B, there exist a unique δ such that $A = (\delta)B$ and if $A = (\delta')B$, then $\delta' \le \delta$.

Proposition 3.2: If $A = (\delta_1)B$ and $B = (\delta_2)C$, then $A = (\delta)C$ where $\delta = \delta_1 * \delta_2$.

Proposition 3.3: Let $A_1 = (\delta_1)B_1$ and $A_2 = (\delta_2)B_2$. Then

$$A_1 \cup A_2 = (\min(\delta_1, \delta_2 \quad B_1 \cup B_2.$$

Proposition 3.4: Let $A_\alpha = (\delta_\alpha)B_\alpha$, for all $\alpha \in J$, where J is an index set. Let $\bigcup_{\alpha\in J} A_\alpha$ represents the union of $\{A_\alpha : \alpha \in J\}$ and $\bigcup_{\alpha\in J} B_\alpha$ represents the union of $\{B_\alpha : \alpha \in J\}$, and $T_{\bigcup_{\alpha\in J} A_\alpha}(u) = \sup_{\alpha\in J} T_{A_\alpha}(u)$, $I_{\bigcup_{\alpha\in J} A_\alpha}(u) = \inf_{\alpha\in J} I_{A_\alpha}(u)$, $F_{\bigcup_{\alpha\in J} A_\alpha}(u) = \inf_{\alpha\in J} F_{A_\alpha}(u)$ and $T_{\bigcup_{\alpha\in J} B_\alpha}(u) = \sup_{\alpha\in J} T_{B_\alpha}(u)$, $I_{\bigcup_{\alpha\in J} B_\alpha}(u) = \inf_{\alpha\in J} I_{B_\alpha}(u)$, $F_{\bigcup_{\alpha\in J} B_\alpha}(u) = \inf_{\alpha\in J} F_{B_\alpha}(u)$ their truth membership functions, indeterminacy membership functions and falsity membership functions, respectively. Then

$$\bigcup_{\alpha\in J} A_\alpha = \left(\inf_{\alpha\in J} \delta_\alpha \quad \bigcup_{\alpha\in J} B_\alpha.$$

Proposition 3.5: Let A^c be the complement of A and B^c be the complement of B. Further let $A = (\delta)B$. Then

$$A^c = (\delta)B^c.$$

Proposition 3.6: Let $A_\alpha = (\delta_\alpha)B_\alpha$, for all $\alpha \in J$, where J is an index set. Let $\bigcap_{\alpha\in J} A_\alpha$ represents the intersection of $\{A_\alpha : \alpha \in J\}$ and $\bigcap_{\alpha\in J} B_\alpha$ represents the intersection of $\{B_\alpha : \alpha \in J\}$, and $T_{\bigcap_{\alpha\in J} A_\alpha}(u) = \inf_{\alpha\in J} T_{A_\alpha}(u)$, $I_{\bigcap_{\alpha\in J} A_\alpha}(u) = \sup_{\alpha\in J} I_{A_\alpha}(u)$, $F_{\bigcap_{\alpha\in J} A_\alpha}(u) = \sup_{\alpha\in J} F_{A_\alpha}(u)$ and $T_{\bigcap_{\alpha\in J} B_\alpha}(u) = \inf_{\alpha\in J} T_{B_\alpha}(u)$, $I_{\bigcap_{\alpha\in J} B_\alpha}(u) = \sup_{\alpha\in J} I_{B_\alpha}(u)$, $F_{\bigcap_{\alpha\in J} B_\alpha}(u) = \sup_{\alpha\in J} F_{B_\alpha}(u)$ their truth membership functions, indeterminacy membership functions and falsity membership functions, respectively. Then

$$\bigcap_{\alpha\in J} A_\alpha = \left(\inf_{\alpha\in J} \delta_\alpha \quad \bigcap_{\alpha\in J} B_\alpha.$$

Corollary 3.1: Let $A_{\alpha\beta} = (\delta_{\alpha\beta})B_{\alpha\beta}$; $\alpha \in J_1$ and $\beta \in J_2$ where J_1 and J_2 are index sets. Then

$$\bigcup_{\alpha\in J_1}\bigcap_{\beta\in J_2} A_{\alpha\beta} = \left(\inf_{\alpha\in J_1}\inf_{\beta\in J_2} \delta_\alpha \quad \bigcup_{\alpha\in J_1}\bigcap_{\beta\in J_2} B_{\alpha\beta},$$ and

$$\bigcap_{\alpha\in J_1}\bigcup_{\beta\in J_2} A_{\alpha\beta} = \left(\inf_{\alpha\in J_1}\inf_{\beta\in J_2} \delta_\alpha\right)\bigcap_{\alpha\in J_1}\bigcup_{\beta\in J_2} B_{\alpha\beta}.$$

Corollary 3.2: Let $A_k = \delta_k \ B_k$, where $k = 1,2,3,\cdots$ and let $\limsup_{n\to\infty} A_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} A_k$, $\liminf_{n\to\infty} A_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} A_k$;

$$\limsup_{n\to\infty} \mathrm{B}_n = \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} B_k, \ \liminf_{n\to\infty} B_n = \bigcup_{n=1}^{\infty} \bigcap_{k=n}^{\infty} B_k.$$

Then

$$\limsup_{n\to\infty} A_n = \left(\inf_{n\geq 1} \delta_n\right) \limsup_{n\to\infty} B_n,$$

$$\liminf_{n\to\infty} A_n = \left(\inf_{n\geq 1} \delta_n\right) \liminf_{n\to\infty} B_n.$$

Proposition 3.7: Let $A_1 = (\delta_1) B_1$, $A_2 = (\delta_2) B_2$. Let $A_1 A_2$ represent the product of A_1, A_2 ,and $B_1 B_2$ represent the product of B_1, B_2. Let $T_{A_1A_2}(u) = T_{A_1}(u) T_{A_2}(u)$, $I_{A_1A_2}(u) = I_{A_1}(u) I_{A_2}(u)$, $F_{A_1A_2}(u) = F_{A_1}(u) F_{A_2}(u)$ and $T_{B_1B_2}(u) = T_{B_1}(u) T_{B_2}(u)$, $I_{B_1B_2}(u) = I_{B_1}(u) I_{B_2}(u)$, $F_{B_1B_2}(u) = F_{B_1}(u) F_{B_2}(u)$ their truth membership functions, indeterminacy membership functions and falsity membership functions, respectively. Then

$$A_1 A_2 = (\delta_1 * \delta_2) B_1 B_2.$$

Proposition 3.8: Let $A_j = (\delta_j) B_j$, where $j = 1,2,3,\cdots,n$. Then

$$A_1 \cdots A_n = (\delta_1 * \cdots \delta_n \ \ B_1 \cdots B_n.$$

Proposition 3.9: Let $A_1 = (\delta_1) B_1$ and $A_2 = (\delta_2) B_2$. Let $A_1 \hat{+} A_2$ represent the probabilistic sum of A_1 and A_2, and $B_1 \hat{+} B_2$ represent the probablisitc sum of B_1 and B_2 respectively. Let $T_{A_1 \hat{+} A_2}(u)$, $I_{A_1 \hat{+} A_2}(u)$, $F_{A_1 \hat{+} A_2}(u)$ and $T_{B_1 \hat{+} B_2}(u)$, $I_{B_1 \hat{+} B_2}(u)$, $F_{B_1 \hat{+} B_2}(u)$ be their truth membership functions, indeterminacy membership functions and falsity membership functions respectively, where

$$T_{A_1 \hat{+} A_2}(u) = T_{A_1}(u) + T_{A_2}(u) - T_{A_1}(u) T_{A_2}(u),$$

$$I_{A_1 \hat{+} A_2}(u) = I_{A_1}(u) + I_{A_2}(u) - I_{A_1}(u) I_{A_2}(u),$$

$$F_{A_1 \hat{+} A_2}(u) = F_{A_1}(u) + F_{A_2}(u) - F_{A_1}(u) F_{A_2}(u), \text{ and}$$

$$T_{B_1 \hat{+} B_2}(u) = T_{B_1}(u) + T_{B_2}(u) - T_{B_1}(u) T_{B_2}(u),$$

$$I_{B_1 \hat{+} B_2}(u) = I_{B_1}(u) + I_{B_2}(u) - I_{B_1}(u) I_{B_2}(u),$$

$$F_{B_1 \hat{+} B_2}(u) = F_{B_1}(u) + F_{B_2}(u) - F_{B_1}(u) F_{B_2}(u). \text{ Then}$$

$$A_1 \hat{+} A_2 = (\delta_1 * \delta_2) B_1 \hat{+} B_2.$$

Corollary 3.3: Suppose $A_j = (\delta_j) B_j$, where $j = 1,2,3,\cdots,n$. Then

$$A_1 \hat{+} \cdots \hat{+} A_n = (\delta_1 * \cdots * \delta_n \ \ B_1 \hat{+} \cdots \hat{+} B_n.$$

Proposition 3.10: Suppose $A_1 = (\delta_1) B_1$ and $A_2 = (\delta_2) B_2$ and let $A_1 \overset{\&}{\bigcup} A_2$ represent the bold sum of A_1 and A_2, and $B_1 \overset{\&}{\bigcup} B_2$ represent the bold sum of B_1 and B_2 respectively. Let $T_{A_1 \overset{\&}{\bigcup} A_2}(u)$, $I_{A_1 \overset{\&}{\bigcup} A_2}(u)$, $F_{A_1 \overset{\&}{\bigcup} A_2}(u)$ and $T_{B_1 \overset{\&}{\bigcup} B_2}(u)$, $I_{B_1 \overset{\&}{\bigcup} B_2}(u)$, $F_{B_1 \overset{\&}{\bigcup} B_2}(u)$ their truth membership functions, indeterminacy functions and falsity membership functions respectively, where

$$T_{A_1 \overset{\&}{\bigcup} A_2}(u) = \min\left(1, T_{A_1}(u) + T_{A_2}(u)\right),$$

$$I_{A_1 \overset{\&}{\bigcup} A_2}(u) = \min\left(1, \mathrm{I}_{A_1} \ u \ + I_{A_2}(u)\right),$$

$$F_{A_1 \overset{\&}{\bigcup} A_2}(u) = \min\left(1, \mathrm{F}_{A_1} \ u \ + F_{A_2}(u)\right) \text{ and}$$

$$T_{B_1 \overset{\&}{\bigcup} B_2}(u) = \min\left(1, T_{B_1}(u) + T_{B_2}(u)\right),$$

$$I_{B_1 \overset{\&}{\bigcup} B_2}(u) = \min\left(1, I_{B_1}(u) + I_{B_2}(u)\right),$$

$$F_{B_1 \overset{\&}{\bigcup} B_2}(u) = \min\left(1, \mathrm{F}_{B_1} \ u \ + F_{B_2}(u)\right). \text{ Then}$$

$$A_1 \overset{\&}{\bigcup} A_2 = (\delta_1 * \delta_2 \ \ B_1 \overset{\circ}{\bigcup} B_2.$$

Proposition 3.11: Suppose $A_1 = (\delta_1) B_1$ and $A_2 = (\delta_2) B_2$ and let $A_1 \underset{\&}{\bigcap} A_2$ represent the bold intersection of A_1 and A_2, and $B_1 \underset{\&}{\bigcap} B_2$ represent the bold intersection of B_1 and B_2 respectively. Let $T_{A_1 \underset{\&}{\bigcap} A_2}(u)$, $I_{A_1 \underset{\&}{\bigcap} A_2}(u)$, $F_{A_1 \underset{\&}{\bigcap} A_2}(u)$ and $T_{B_1 \underset{\&}{\bigcap} B_2}(u)$, $I_{B_1 \underset{\&}{\bigcap} B_2}(u)$, $F_{B_1 \underset{\&}{\bigcap} B_2}(u)$ their truth membership functions, indeterminacy functions and falsity membership functions respectively, where

$$T_{A_1 \underset{\&}{\bigcap} A_2}(u) = \max\left(0, T_{A_1}(u) + T_{A_2}(u) - 1\right.,$$

$$I_{A_1 \underset{\&}{\bigcap} A_2}(u) = \max\left(0, \mathrm{I}_{A_1} \ u \ + I_{A_2}(u) - 1\right.,$$

$$F_{A_1 \underset{\&}{\bigcap} A_2}(u) = \max\left(0, \mathrm{F}_{A_1} \ u \ + F_{A_2}(u) - 1\right. \text{ and}$$

$$T_{B_1 \underset{\&}{\bigcap} B_2}(u) = \max\left(0, T_{B_1}(u) + T_{B_2}(u) - 1\right.,$$

$$I_{B_1 \underset{\&}{\bigcap} B_2}(u) = \max\left(0, \mathrm{I}_{B_1} \ u \ + I_{B_2}(u) - 1\right.,$$

$$F_{B_1 \underset{\&}{\bigcap} B_2}(u) = \max\left(0, \mathrm{F}_{B_1} \ u \ + F_{B_2}(u) - 1\right.. \text{ Then}$$

$$A_1 \underset{\&}{\bigcap} A_2 = (\delta_1 * \delta_2 \ \ B_1 \underset{\&}{\bigcap} B_2.$$

Proposition 3.12: Suppose $A_1 = (\delta_1) B_1$ and $A_2 = (\delta_2) B_2$ and let $A_1 |-| A_2$ represent the bounded difference of A_1 and A_2, and $B_1 |-| B_2$ represent the bounded difference of B_1 and B_2 respectively. Let $T_{A_1|-|A_2}(u)$, $I_{A_1|-|A_2}(u)$, $F_{A_1|-|A_2}(u)$

and $T_{B_1|-|B_2}(u)$, $I_{B_1|-|B_2}(u)$, $F_{B_1|-|B_2}(u)$ their truth membership functions, indeterminacy functions and falsity membership functions respectively, where

$$T_{A_1|-|A_2}(u)=\max\left(0,T_{A_1}(u)-T_{A_2}(u)\right),$$
$$I_{A_1|-|A_2}(u)=\max\left(0,\mathrm{I}_{A_1}\ \ u\ \ -I_{A_2}(u)\right),$$
$$F_{A_1|-|A_2}(u)=\max\left(0,\mathrm{F}_{A_1}\ \ u\ \ -F_{A_2}(u)\right) \text{ and}$$
$$T_{B_1|-|B_2}(u)=\max\left(0,T_{B_1}(u)-T_{B_2}(u)\right),$$
$$I_{B_1|-|B_2}(u)=\max\left(0,\mathrm{I}_{B_1}\ \ u\ \ -I_{B_2}(u)\right),$$
$$F_{B_1|-|B_2}(u)=\max\left(0,\mathrm{F}_{B_1}\ \ u\ \ -F_{B_2}(u)\right). \text{ Then}$$
$$A_1|-|A_2=(\delta_1*\delta_2)\,B_1|-|B_2.$$

Proposition 3.13: Suppose $A_1=(\delta_1)\,B_1$ and $A_2=(\delta_2)\,B_2$ and let $A_1\nabla A_2$ represent the symmetrical difference of A_1 and A_2, and $B_1\nabla B_2$ represent the symmetrical difference of B_1 and B_2 respectively. Let $T_{A_1\nabla A_2}(u)$, $I_{A_1\nabla A_2}(u)$, $F_{A_1\nabla A_2}(u)$ and $T_{B_1\nabla B_2}(u)$, $I_{B_1\nabla B_2}(u)$, $F_{B_1\nabla B_2}(u)$ their truth membership functions, indeterminacy functions and falsity membership functions respectively, where

$T_{A_1\nabla A_2}(u)=\left|T_{A_1}(x)-T_{A_2}(x)\right|$, $I_{A_1\nabla A_2}(u)=\left|I_{A_1}(x)-I_{A_2}(x)\right|$,
$F_{A_1\nabla A_2}(u)=\left|F_{A_1}(x)-F_{A_2}(x)\right|$ an $T_{B_1\nabla B_2}(u)=\left|T_{B_1}(x)-T_{B_2}(x)\right|$,
$I_{B_1\nabla B_2}(u)=\left|I_{B_1}(x)-I_{B_2}(x)\right|, F_{B_1\nabla B_2}(u)=\left|F_{B_1}(x)-F_{B_2}(x)\right|$.
Then $A_1\nabla A_2=(\delta_1*\delta_2)\,B_1\nabla B_2$.

Proposition 3.14: Suppose $A_1=(\delta_1)\,B_1$ and $A_2=(\delta_2)\,B_2$ and let $A_1\|_{(} A_2$ represent the convex linear sum of min and max of A_1 and A_2, and $B_1\|_{(} B_2$ represent the convex linear sum of B_1 and B_2 respectively. Let $T_{A_1\|_{(} A_2}(u)$, $I_{A_1\|_{(} A_2}(u)$, $F_{A_1\|_{(} A_2}(u)$ and $T_{B_1\|_{(} B_2}(u)$, $I_{B_1\|_{(} B_2}(u)$, $F_{B_1\|_{(} B_2}(u)$ their truth membership functions, indeterminacy functions and falsity membership functions respectively, where

$$T_{A_1\|_{(} A_2}(u)=(\ \min\left(T_{A_1}(u),T_{A_2}(u)\right)+(1-(\ )\max\left(T_{A_1}(u\ \ ,T_{A_2}(u\ \ \right),$$
$$I_{A_1\|_{(} A_2}(u)=(\ \min\left(I_{A_1}(u),\mathrm{I}_{A_2}(u)\right)+(1-(\ )\max\left(I_{A_1}(u\ \ ,I_{A_2}(u\ \ \right),$$
$$F_{A_1\|_{(} A_2}(u)=(\ \min\left(F_{A_1}(u),F_{A_2}(u)\right)+(1-(\ )\max\left(F_{A_1}(u\ \ ,F_{A_2}(u\ \ \right)$$

and

$$T_{B_1\|_{(} B_2}(u)=(\ \min\left(T_{B_1}(u),T_{B_2}(u)\right)+(1-(\ )\max\left(T_{B_1}(u\ \ ,T_{B_2}(u\ \ \right),$$
$$I_{B_1\|_{(} B_2}(u)=(\ \min\left(I_{B_1}(u),I_{B_2}(u)\right)+(1-(\ )\max\left(I_{B_1}(u\ \ ,I_{B_2}(u\ \ \right),$$
$$F_{B_1\|_{(} B_2}(u)=(\ \min\left(F_{B_1}(u),F_{B_2}(u)\right)+(1-(\ )\max\left(F_{B_1}(u\ \ ,F_{B_2}(u\ \ \right),$$

where $(\ \in[0,1]$. Then $A_1\|_{(} A_2=(\delta_1*\delta_2)\,B_1\|_{(} B_2$.

IV. δ-Equalities with respect to Neutrosophic Relations and Norms

In this section, δ-equalities of neutrosophic sets have been further extended to neutrosophic relations and neutrosophic norms respectively. These δ-equalities applied in the composition of neutrosophic relations, Cartesian product and neutrosophic triangular norms.

Proposition 4.1: Let X,Y and Z be initial universes, and) be the collection of all neutrosophic sets defined on $X\times Y$ and + denote the collection of all neutrosophic sets defined on $Y\times Z$ respectively. Let $R,R'\in\Sigma$ and $S,S'\in\Pi$, i.e., R,R',S and S' are neutrosophic relations on $X\times Y$ and $Y\times Z$ respectively. Further, let $R\circ S$ and $R'\circ S'$ be their composition, and $T_{R\circ S}(x,y)$, $I_{R\circ S}(x,y)$, $F_{R\circ S}(x,y)$ and $T_{R'\circ S'}(x,y)$, $I_{R'\circ S'}(x,y)$ and $F_{R'\circ S'}(x,y)$ their truth membership functions, indeterminate membership functions and falsehood membership functions respectively, where

$$T_{R\circ S}(x,z)=\sup_{y\in Y}\min\left(T_R(x,y\ \ ,T_S(y,z\ \ \right),$$
$$I_{R\circ S}(x,z)=\sup_{y\in Y}\min\left(I_R(x,y\ \ ,I_S(y,z\ \ \right),$$
$$F_{R\circ S}(x,z)=\sup_{y\in Y}\min\left(F_R(x,y\ \ ,F_S(y,z\ \ \right),$$

and

$$T_{R'\circ S'}(x,z)=\sup_{y\in Y}\min\left(T_R(x,y\ \ ,T_S(y,z\ \ \right),$$
$$I_{R'\circ S'}(x,z)=\sup_{y\in Y}\min\left(I_R(x,y\ \ ,I_S(y,z\ \ \right),$$
$$F_{R'\circ S'}(x,z)=\sup_{y\in Y}\min\left(F_R(x,y\ \ ,F_S(y,z\ \ \right),$$

for all $x\in X$ and $z\in Z$. Suppose $R=(\delta_1)\,R'$ and $S=(\delta_2)\,S'$. Then

$$R\circ S=\left(\min(\delta_1,\delta_2\ \ \ R'\circ S'.\right.$$

Proposition 4.2: Let $U_1,U_2,\cdots U_n$ be universes and A_j,B_j be neutrosophic sets defined on U_j, $j=1,2,\cdots,n$. Let $A_j=(\delta_j)\,B_j$, where $j=1,2,\cdots,n$.

Let $A=A_1\times A_2\times\cdots\times A_n$ and $B=B_1\times B_2\times\cdots\times B_n$ and $T_A(u_1,u_2,\cdots,u_n$, $I_A(u_1,u_2,\cdots,u_n$, $F_A(u_1,u_2,\cdots,u_n$ and $T_B(u_1,u_2,\cdots,u_n$, $I_B(u_1,u_2,\cdots,u_n$, $F_B(u_1,u_2,\cdots,u_n$, be their truth membership functions, indeterminacy membership functions and falsity membership functions respectively, where

$$T_A(u_1,u_2,\cdots,u_n\ \ =\min\left(T_{A_1}(u\ \ ,T_{A_2}(u\ \ ,\cdots,T_{A_n}(u\ \ \ \right.,$$
$$I_A(u_1,u_2,\cdots,u_n\ \ =\min\left(I_{A_1}(u\ \ ,I_{A_2}(u\ \ ,\cdots,I_{A_n}(u\ \ \ \right.,$$
$$F_A(u_1,u_2,\cdots,u_n\ \ =\min\left(F_{A_1}(u\ \ ,F_{A_2}(u\ \ ,\cdots,F_{A_n}(u\ \ \ \right.,$$

and

$$T_B(u_1,u_2,\cdots,u_n = \min(T_{B_1}(u ,T_{B_2}(u ,\cdots,T_{B_n}(u ,$$

$$I_B(u_1,u_2,\cdots,u_n = \min(I_{B_1}(u ,I_{B_2}(u ,\cdots,I_{B_n}(u ,$$

$$F_B(u_1,u_2,\cdots,u_n = \min(F_{B_1}(u ,F_{B_2}(u ,\cdots,F_{B_n}(u .$$

Then $A = (\inf_{1\le j\le n}\delta_j \ B$.

Proposition 4.3: Let A_1,B_1,C_1 and A_2,B_2,C_2 be neutrosophic sets on U such that $T_{A_1}(u)\le T_{B_1}(u)\le T_{C_1}(u)$, $I_{A_1}(u)\le I_{B_1}(u)\le I_{C_1}(u)$, $F_{A_1}(u)\le F_{B_1}(u)\le F_{C_1}(u)$, and $T_{A_2}(u)\le T_{B_2}(u)\le T_{C_2}(u)$, $I_{A_2}(u)\le I_{B_2}(u)\le I_{C_2}(u)$, $F_{A_2}(u)\le F_{B_2}(u)\le F_{C_2}(u)$, for all $u\in U$. Also, let $A_1=(\delta_a)A_2$, $C_1=(\delta_c)C_2$ and $A_1=(\delta_{ac})C_1$. Then $B_1=(\delta_{ac}*\min(\delta_a,\delta_c))B_2$.

Definition 4.1: A neutrosophic triangular norm N_t is a function

$T_N:\ ]0^-,1^+[\times]0^-,1^+[\times]0^-,1^+[\ ^2\rightarrow\ ]0^-,1^+[\times]0^-,1^+[\times]0^-,1^+[$

defined by

$N_t(x(T_1,I_1,F_1),y(T_2,I_2,F_2\ )=(N_tT(x,y),N_tI(x,y),N_tF(x,y))$,

where $N_tT(x,y),N_tI(x,y),N_tF(x,y)$ are truth membership, indeterminacy membership and falsity membership components respectively. N_t satisfies the following conditions:

i. $N_t(0,0)=0;\ N_t(x,1)=N_t(1,x)=x$;

ii. $N_t(x,y)\le N_t(w,z$, whenever $x\le w$ and $y\le z$;

iii. $N_t(x,y)=N_t(y,x)$;

iv. $N_t(N_t(x,y),z)=N_t(x,N_t(y,z))$.

Proposition 4.4: Let U be a universe of discourse, and A,A' and B,B' be neutrosophic sets on U. Let N_t be a neutrosophic triangular norm and let C,C' be neutrosophic sets define on U via N_t,

$T_C(u)=N_t(T_A(u),T_B(u))$, $I_C(u)=N_t(I_A(u),I_B(u))$, $F_C(u)=N_t(F_A(u),F_B(u))$, and $T_{C'}(u)=N_t(T_{A'}(u),T_{B'}(u))$, $I_{C'}(u)=N_t(I_{A'}(u),I_{B'}(u))$, $F_{C'}(u)=N_t(F_{A'}(u),F_{B'}(u))$.

Suppose that $A=(\delta_1)B$ and $A'=(\delta_2)B'$. Then $C=(\delta*\min(\delta_1,\delta_2))C'$, where

$$\delta = 1-\sup_{u:\max(T_A(u),T_{B(u)}\ ,1}\min(T_A(u),T_B(u)),$$

$$\delta = 1-\sup_{u:\max(I_A(u),I_{B(u)}\ ,1}\min(I_A(u),I_B(u)),$$

$$\delta = 1-\sup_{u:\max(F_A(u),F_{B(u)}\ ,1}\min(F_A(u),F_B(u)).$$

V. APPLICATIONS OF δ-EQUALITIES

In this section, the applications and utilizations of δ-equalities have been presented. In this regards, δ-equalities have been successfully applied in Fault Tree Analysis, Canonical Computer Reliability and Neutrosophic Reliability (generalization of Profust Reliability).

Fault Tree Analysis.

A fault tree can be seen as following in the Fig. 1 [3,4]. We mainly concerned with the relationship between the probability of top events and bottom events in fault tree analysis. Suppose that $P_1,P_2,\cdots,P_5$ denote the occurrence probabilities of bottom events $e_1,e_2,\cdots,e_5$ and P_t represent of top event. Further, suppose that $e_1,e_2,\cdots,e_5$ are independent events. Therefore,

$$P_t = 1-(1-P_1P_2)(1-P_3)(1-P_4P_5).$$

On the other hand, in conventional fault tree analysis $P_1,P_2,\cdots,P_5$ are assumed to accurate numbers, but in [3,4], $P_1,P_2,\cdots,P_5$ are treated as fuzzy numbers. Cai [3,4] utilized the concept of δ-equalities by applying it in fault tree analysis by considering $P_1,P_2,\cdots,P_5$ as fuzzy numbers.

Here, we consider $P_1,P_2,\cdots,P_5$ as neutrosophic numbers instead of fuzzy numbers. In engineering, an expert presents his judgement on P_j as $a_j\le P_j\le a'_j$ for truth membership, $b_j\le P_j\le b'_j$ for indeterminacy membership and $c_j\le P_j\le c'_j$ for falsehood membership respectively. Since we suppose that P_j's can be treated as neutrosophic number, therefore, we can define the following truth membership functions $T_{P_j}(u)$, $I_{P_j}(u)$ and $F_{P_j}(u)$ can be defined as following.

$$T_{P_j}(u)=\begin{cases}\dfrac{u-a_j}{a_{j_1}-a_j}\mathrm{T}_{P_j}, & a_j\le u<a_{j_1},\\ \mathrm{T}_{P_j}, & a_{j_1}\le u<a_{j_2},\\ \dfrac{a'_j-u}{a'_j-a_{j_2}}\mathrm{T}_{P_j}, & a_{j_2}\le u<a'_j\\ 0, & \text{otherwise}.\end{cases}$$

$$I_{P_j}(u)=\begin{cases}\dfrac{b_{j_1}-u+I_{P_j}(u-b_j)}{b_{j_1}-b_j}, & b_j\le u<b_{j_1},\\ I_{P_j}, & b_{j_1}\le u<b_{j_2},\\ \dfrac{u-b_{j_2}+I_{P_j}(b'_j-u)}{b'_j-b_{j_2}}, & b_{j_2}\le u<b'_j\\ 1, & \text{otherwise}.\end{cases}$$

$$F_{P_j}(u)=\begin{cases}\dfrac{c_{j_1}-u+F_{P_j}(u-c_j)}{c_{j_1}-c_j}, & c_j\le u<c_{j_1},\\ F_{P_j}, & c_{j_1}\le u<c_{j_2},\\ \dfrac{u-c_{j_2}+F_{P_j}(c'_j-u)}{c'_j-c_{j_2}}, & c_{j_2}\le u<c'_j\\ 1, & \text{otherwise}.\end{cases}$$

Suppose there is another expert who presents his judgement for P_j as $a''_j\le P_j\le a'''_j$ for truth membership, $b''_j\le P_j\le b'''_j$ for indeterminacy membership and $c''_j\le P_j\le c'''_j$ for falsehood membership respectively.

$$T'_{P_j}(u)=\begin{cases}\dfrac{u-a''_j}{a''_{j_1}-a''_j}\mathrm{T}'_{P_j}, & a''_j\le u<a''_{j_1},\\ \mathrm{T}'_{P_j}, & a''_{j_1}\le u<a''_{j_2},\\ \dfrac{a'''_j-u}{a'''_j-a''_{j_2}}\mathrm{T}'_{P_j}, & a''_{j_2}\le u<a'''_j\\ 0, & \text{otherwise}.\end{cases}$$

$$I'_{P_j}(u)=\begin{cases}\dfrac{b''_{j_1}-u+I'_{P_j}(u-b''_j)}{b''_{j_1}-b''_j}, & b''_j\le u<b''_{j_1},\\ I'_{P_j}, & b''_{j_1}\le u<b''_{j_2},\\ \dfrac{u-b''_{j_2}+I'_{P_j}(b'''_j-u)}{b'''_j-b''_{j_2}}, & b''_{j_2}\le u<b'''_j\\ 1, & \text{otherwise}.\end{cases}$$

$$F'_{P_j}(u)=\begin{cases}\dfrac{c''_{j_1}-u+F'_{P_j}(u-c''_j)}{c''_{j_1}-c''_j}, & c''_j\le u<c''_{j_1},\\ F'_{P_j}, & c''_{j_1}\le u<c''_{j_2},\\ \dfrac{u-c''_{j_2}+F'_{P_j}(c'''_j-u)}{c'''_j-c''_{j_2}}, & c''_{j_2}\le u<c'''_j\\ 1, & \text{otherwise}.\end{cases}$$

Since $P_1,P_2,\cdots,P_5$ are treated as neutrosophic numbers, therefore, P_t is a neutrosophic set. The effect of estimation errors of in truth membership functions, indeterminacy membership functions and falsity membership functions of $P_1,P_2,\cdots,P_5$ on P_t can be formulated as following.

Let $\delta_j=1-\sup\limits_{0\le u\le 1}\left|T_{P_j}(u)-T'_{P_j}(u)\right|$, $\delta_j=1-\sup\limits_{0\le u\le 1}\left|I_{P_j}(u)-I'_{P_j}(u)\right|$ and $\delta_j=1-\sup\limits_{0\le u\le 1}\left|F_{P_j}(u)-F'_{P_j}(u)\right|$. Let $T_{P_t}(u)$, $I_{P_t}(u)$ and $F_{P_t}(u)$ be the truth membership function, indeterminacy membership function and falsity membership function of P_t corresponding to $\left\{T_{P_j}(u),I_{P_j}(u),F_{P_j}(u)\right\}$ and $T'_{P_t}(u)$, $I'_{P_t}(u)$ and $F'_{P_t}(u)$ be the truth membership function, indeterminacy membership function and falsity membership function of P_t corresponding to $\left\{T'_{P_j}(u),I'_{P_j}(u),F'_{P_j}(u)\right\}$ respectively. Then from Proposition 4.3, we have

$$\sup_{0\le u\le 1}\left|T_{P_t}(u)-T'_{P_t}(u)\right|\le 1-\min_{1\le j\le 5}(\delta_j),$$

$$\sup_{0\le u\le 1}\left|I_{P_t}(u)-I'_{P_t}(u)\right|\le 1-\min_{1\le j\le 5}(\delta_j),$$

$$\sup_{0\le u\le 1}\left|F_{P_t}(u)-F'_{P_t}(u)\right|\le 1-\min_{1\le j\le 5}(\delta_j)\cdot$$

It is to be noted that δ_j is a function of $a_j,b_j,c_j,a'_j,b'_j,c'_j,a''_j,b''_j,c''_j,a'''_j,b'''_j,c'''_j$.

Neutrosophic Reliability (Generalization of Profust Reliability).

Cai [3,4] uses successfully the concept of δ-equalties in profust reliability which is based on the probability assumption and fuzzy-state assumption. For more detail, we refer the readers to [3,4]. But there is another state called neutrosophic-state assumption. In neutrosophic-state assumption, we have three state, i.e. the membership state, indeterminate membership state and non-membership state. In other words, state of success, state of failure and a state of neither failure nor success. From this we can say that neutrosophic Reliability is more general framework than the profust reliability. A neutrosophic reliability is based on neutrosophic probability and neutrosophic-state assumption. Now we can say that a system of order n (n-component) is referred to as a neutrosophic system if it satisfies the following conditions.

$\mu_F=\prod_{j=1}^{n}\mu^{(j)}{}_F$, $\mu_S=1-\prod_{j=1}^{n}\left(1-\mu^{(j)}{}_S\right)$ and $\mu_I=\prod_{j=1}^{n}\left(\mu^{(j)}{}_F+\mu^{(j)}{}_S\right)$,

where μ_F, μ_S and μ_I are the false membership function of neutrosophic failure, truth membership function of neutrosophic success and indeterminate membership function of neutrosophic indeterminacy (both failure and success at the same time) of the neutrosophic system, respectively, and $\mu^{(j)}{}_F$, $\mu^{(j)}{}_S$ and $\mu^{(j)}{}_I=\mu^{(j)}{}_F+\mu^{(j)}{}_S$ are false membership, truth membership and indeterminate membership functions of failure, success and indeterminate of the component j.

From here it is clear that neutrosophic system is the generalization of the parallel system as by setting the indeterminate component μ_I equals to 0, the neutrosophic system reduced to parallel system.

Now suppose that there are estimations errors in $\mu^{(j)}{}_F$, $\mu^{(j)}{}_S$, $\mu^{(j)}{}_I$, or we have $\mu'^{(j)}{}_F$, $\mu'^{(j)}{}_S$, $\mu'^{(j)}{}_I$ and suppose that

$$\sup_u\left|\mu^{(j)}{}_F(u)-\mu'^{(j)}{}_F(u)\right|\le 1-\delta_j{}^F,$$

$$\sup_u\left|\mu^{(j)}{}_S(u)-\mu'^{(j)}{}_S(u)\right|\le 1-\delta_j{}^S,$$

$$\sup_u\left|\mu^{(j)}{}_I(u)-\mu'^{(j)}{}_I(u)\right|\le 1-\delta_j{}^I.$$

Then by using Proposition 3.3 and 3.5, we have

$$\sup_u \left| \mu_F(u) - \mu'_F(u) \right| \le 1 - (\delta_1^F * \delta_2^F * \cdots * \delta_n^F \quad ,$$

$$\sup_u \left| \mu_S(u) - \mu'_S(u) \right| \le 1 - (\delta_1^S * \delta_2^S * \cdots * \delta_n^S \quad ,$$

$$\sup_u \left| \mu_I(u) - \mu'_I(u) \right| \le 1 - (\delta_1^I * \delta_2^I * \cdots * \delta_n^I \quad ,$$

where $\mu'_F(u)$ corresponds to $\mu'^{(1)}_F(u), \cdots, \mu'^{(n}_F(u$, $\mu'_S(u)$ corresponds to $\mu'^{(1)}_S(u), \cdots, \mu'^{(n}_S(u)$ and $\mu'_I(u)$ corresponds to $\mu'^{(1)}_I(u), \cdots, \mu'^{(n}_I(u)$.

vi. CONCLUSION

New type of union, intersection and δ -equalities of neutrosophic sets presented in this paper. Two neutrosophic sets are said to be δ-equal if they are equal to some degree of δ . Later, these δ -equalities applied in several set theoretic operations such as union, intersection, complement, product, probabilistic sum, bold sum, bold intersection, bounded difference, symmetrical difference, and convex linear sum of min and max. It has also shown that how δ varies with different operation. These δ-equalities of neutrosophic sets have been further extended to neutrosophic relations and neutrosophic norms respectively. In this paper, δ -equalities also applied in the composition of neutrosophic relations, Cartesian product and neutrosophic triangular norms. The applications and utilizations of δ -equalities have been presented in this paper. In this regards, δ -equalities have been successfully applied in Fault Tree Analysis and Neutrosophic Reliability (generalization of Profust Reliability).

The significance of δ-equality can be justified in theory as well as in practice. On the one hand, this paper shows that the δ -equalities of neutrosophic sets can be defined and investigated in a general framework by introducing δ which is basically a difference between the truth membership functions T , a difference between the indeterminate membership functions I and a difference between the falsehood membership functions F of two neutrosophic sets respectively. On the other hand, as shown in the application section 5, the concept of δ-equalities of neutrosophic sets may be useful in various applications where errors of membership functions, non-membership functions and indeterminate membership function of neutrosophic sets are of concern.

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R-intersections and R-unions of Neutrosophic Cubic Sets

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***Abstract*—R-unions and R-intersections of T-external (I-external, F-external) neutrosophic cubic sets are considered. Examples to show that the R-intersection and R-union of T-external (I-external, F-external) neutrosophic cubic sets may not be a T-external (I-external, F-external) neutrosophic cubic set are provided. Conditions for the R-union and R-intersection of T-external (I-external, F-external) neutrosophic cubic sets to be a T-external (I-external, F-external) neutrosophic cubic set are discussed.**

***Index Terms*—Truth-internal (indeterminacy-internal, falsity-internal) neutrosophic cubic set, truth-external (indeterminacy-external, falsity-external) neutrosophic cubic set, R-union, R-intersection.**

I. Introduction

SMARANDACHE ([5], [6]) developed the concept of neutrosophic set as a more general platform which extends the concepts of the classic set and fuzzy set, intuitionistic fuzzy set and interval valued intuitionistic fuzzy set. We know that neutrosophic set theory is applied to various part (refer to the site http:// fs.gallup.unm.edu/neutrosophy.htm). Ali and Smarandache [1] introduced complex neutrosophic sets to handle imprecise, indeterminate, inconsistent, and incomplete information of periodic nature. Deli et al. [2] introduced the concept of bipolar neutrosophic set and its some operations.

In [3], Jun et al. introduced the notion of (internal, external) cubic sets, and investigated several properties. Jun et al. [4] extended the concept of cubic sets to the neutrosophic sets, and introduced/investigated the notions/properties of T-internal (I-internal, F-internal) neutrosophic cubic sets and T-external (I-external, F-external) neutrosophic cubic sets. As a continuation of the paper [4], we consider R-unions and R-intersections of T-external (I-external, F-external) neutrosophic cubic sets. We provide examples to show that the R-intersection and the R-union of T-external (resp. I-external and F-external) neutrosophic cubic sets may not be a T-external (resp. I-external and F-external) neutrosophic cubic set. We discuss conditions for the R-union of T-external (resp. I-external and F-external) neutrosophic cubic sets to be a T-external (resp. I-external and F-external) neutrosophic cubic set. We consider a condition for the R-intersection of T-external (resp. I-external and F-external) neutrosophic cubic sets to be a T-external (resp. I-external and F-external) neutrosophic cubic set.

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II. R-intersections and R-unions of neutrosophic cubic sets

Jun et al. [4] considered the notion of neutrosophic cubic sets as an extension of cubic sets.

Let X be a non-empty set. A neutrosophic cubic set (NCS) in X is a pair $\mathcal{A} = (\mathbf{A}, \Lambda)$ where

$$\mathbf{A} := \{\langle x; A_T(x), A_I(x), A_F(x)\rangle \mid x \in X\}$$

is an interval neutrosophic set in X and

$$\Lambda := \{\langle x; \lambda_T(x), \lambda_I(x), \lambda_F(x)\rangle \mid x \in X\}$$

is a single-valued neutrosophic set in X.

For further particulars on the notions of T (resp., I, F)-internal neutrosophic cubic sets, T (resp., I, F)-external neutrosophic cubic sets, R-union and R-intersection of neutrosophic cubic sets, we refer the reader to the the paper [4].

We know that R-intersection and R-union of T-external (resp., I-external and F-external) neutrosophic cubic sets may not be a T-external (resp., I-external and F-external) neutrosophic cubic sets as seen in the following example.

Example 2.1: Let $\mathcal{A} = (\mathbf{A}, \Lambda)$ and $\mathcal{B} = (\mathbf{B}, \Psi)$ be neutrosophic cubic sets in $X = [0, 1]$ where

$\mathbf{A} = \{\langle x; [0.5, 0.7], [0.2, 0.4], [0.3, 0.5]\rangle \mid x \in [0, 1]\}$,
$\Lambda = \{\langle x; 0.6, 0.7, 0.8\rangle \mid x \in [0, 1]\}$,
$\mathbf{B} = \{\langle x; [0.6, 0.7], [0.6, 0.8], [0.7, 0.9]\rangle \mid x \in [0, 1]\}$,
$\Psi = \{\langle x; 0.5, 0.9, 0.8\rangle \mid x \in [0, 1]\}$.

Then $\mathcal{A} = (\mathbf{A}, \Lambda)$ and $\mathcal{B} = (\mathbf{B}, \Psi)$ are I-external neutrosophic cubic sets in $X = [0, 1]$. The R-union $\mathcal{A} \cup_R \mathcal{B} = (\mathbf{A} \cup \mathbf{B}, \Lambda \wedge \Psi)$ of $\mathcal{A} = (\mathbf{A}, \Lambda)$ and $\mathcal{B} = (\mathbf{B}, \Psi)$ is given as follows:

$\mathbf{A} \cup \mathbf{B} = \{\langle x; [0.6, 0.7], [0.6, 0.8], [0.7, 0.9]\rangle \mid x \in [0, 1]\}$,
$\Lambda \wedge \Psi = \{\langle x; 0.5, 0.7, 0.8\rangle \mid x \in [0, 1]\}$,

and it is not an I-external neutrosophic cubic set in $X = [0, 1]$.

We provide a condition for the R-union of two T-external (resp., I-external, F-external) neutrosophic cubic sets to be T-external (resp., I-external, F-external).

Theorem 2.2: Let $\mathcal{A} = (\mathbf{A}, \Lambda)$ and $\mathcal{B} = (\mathbf{B}, \Psi)$ be I-external neutrosophic cubic sets in X such that

$$\max\left\{\min\{A_I^+(x), B_I^-(x)\}, \min\{A_I^-(x), B_I^+(x)\}\right\} \le (\lambda_I \wedge \psi_I)(x) < \min\left\{\max\{A_I^+(x), B_I^-(x)\}, \max\{A_I^-(x), B_I^+(x)\}\right\}$$

for all $x \in X$. Then the R-union $\mathcal{A} \cup_R \mathcal{B} = (\mathbf{A} \cup \mathbf{B}, \Lambda \wedge \Psi)$ is an I-external neutrosophic cubic set in X.

Proof. For any $x \in X$, let

$a_x := \max\left\{\min\{A_I^+(x), B_I^-(x)\}, \min\{A_I^-(x), B_I^+(x)\}\right\}$

and

$b_x := \min\left\{\max\{A_I^+(x), B_I^-(x)\}, \max\{A_I^-(x), B_I^+(x)\}\right\}$.

Then $b_x = A_I^-(x)$, $b_x = B_I^-(x)$, $b_x = A_I^+(x)$, or

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$b_x = B_I^+(x)$. It is possible to consider the cases $b_x = B_I^-(x)$ and $b_x = B_I^+(x)$ only because the remaining cases are similar to these cases. If $b_x = B_I^-(x)$, then $A_I^+(x) \le B_I^-(x)$ and so

$$A_I^-(x) \le A_I^+(x) \le B_I^-(x) \le B_I^+(x).$$

Thus $a_x = A_I^+(x)$, and so

$$(A_I \cup B_I)^-(x) = B_I^-(x) = b_x > (\lambda_I \wedge \psi_I)(x)$$

Hence

$$(\lambda_I \wedge \psi_I)(x) \notin \left((A_I \cup B_I)^-(x), (A_I \cup B_I)^+(x)\right).$$

If $b_x = B_I^+(x)$, then $A_I^-(x) \le B_I^+(x) \le A_I^+(x)$ and thus $a_x = \max\{A_I^-(x), B_I^-(x)\}$. Suppose that $a_x = A_I^-(x)$. Then

$$\begin{aligned} B_I^-(x) &\le A_I^-(x) = a_x \le (\lambda_I \wedge \psi_I)(x) \\ &< b_x = B_I^+(x) \le A_I^+(x). \end{aligned}$$

It follows that

$$B_I^-(x) \le A_I^-(x) < (\lambda_I \wedge \psi_I)(x) < B_I^+(x) \le A_I^+(x) \quad (1)$$

or

$$B_I^-(x) \le A_I^-(x) = (\lambda_I \wedge \psi_I)(x) < B_I^+(x) \le A_I^+(x). \quad (2)$$

The case (1) induces a contradiction. The case (2) implies that

$$(\lambda_I \wedge \psi_I)(x) \notin \left((A_I \cup B_I)^-(x), (A_I \cup B_I)^+(x)\right)$$

since $(\lambda_I \wedge \psi_I)(x) = A_I^-(x) = (A_I \cup B_I)^-(x)$. Now, if $a_x = B_I^-(x)$, then

$$\begin{aligned} A_I^-(x) &\le B_I^-(x) = a_x \le (\lambda_I \wedge \psi_I)(x) \\ &< b_x = B_I^+(x) \le A_I^+(x). \end{aligned}$$

Hence we have

$$A_I^-(x) \le B_I^-(x) < (\lambda_I \wedge \psi_I)(x) < B_I^+(x) \le A_I^+(x) \quad (3)$$

or

$$A_I^-(x) \le B_I^-(x) = (\lambda_I \wedge \psi_I)(x) < B_I^+(x) \le A_I^+(x). \quad (4)$$

The case (3) induces a contradiction. The case (4) implies that

$$(\lambda_I \wedge \psi_I)(x) \notin \left((A_I \cup B_I)^-(x), (A_I \cup B_I)^+(x)\right).$$

Therefore the R-union $\mathcal{A} \cup_R \mathcal{B} = (\mathbf{A} \cup \mathbf{B}, \Lambda \wedge \Psi)$ is an I-external neutrosophic cubic set in X.

Similarly, we have the following theorems.

Theorem 2.3: Let $\mathcal{A} = (\mathbf{A}, \Lambda)$ and $\mathcal{B} = (\mathbf{B}, \Psi)$ be T-external neutrosophic cubic sets in X such that

$$\begin{aligned} &\max\left\{\min\{A_T^+(x), B_T^-(x)\}, \min\{A_T^-(x), B_T^+(x)\}\right\} \\ &\le (\lambda_T \wedge \psi_T)(x) \\ &< \min\left\{\max\{A_T^+(x), B_T^-(x)\}, \max\{A_T^-(x), B_T^+(x)\}\right\} \end{aligned}$$

for all $x \in X$. Then the R-union $\mathcal{A} \cup_R \mathcal{B} = (\mathbf{A} \cup \mathbf{B}, \Lambda \wedge \Psi)$ is a T-external neutrosophic cubic set in X.

Theorem 2.4: Let $\mathcal{A} = (\mathbf{A}, \Lambda)$ and $\mathcal{B} = (\mathbf{B}, \Psi)$ be F-external neutrosophic cubic sets in X such that

$$\begin{aligned} &\max\left\{\min\{A_F^+(x), B_F^-(x)\}, \min\{A_F^-(x), B_F^+(x)\}\right\} \\ &\le (\lambda_F \wedge \psi_F)(x) \\ &< \min\left\{\max\{A_F^+(x), B_F^-(x)\}, \max\{A_F^-(x), B_F^+(x)\}\right\} \end{aligned}$$

for all $x \in X$. Then the R-union $\mathcal{A} \cup_R \mathcal{B} = (\mathbf{A} \cup \mathbf{B}, \Lambda \wedge \Psi)$ is an F-external neutrosophic cubic set in X.

Corollary 2.5: Let $\mathcal{A} = (\mathbf{A}, \Lambda)$ and $\mathcal{B} = (\mathbf{B}, \Psi)$ be external neutrosophic cubic sets in X. Then the R-union of $\mathcal{A} = (\mathbf{A}, \Lambda)$ and $\mathcal{B} = (\mathbf{B}, \Psi)$ is an external neutrosophic cubic set in X when the conditions in Theorems 2.2 2.3 and 2.4 are valid.

The following examples show that the R-intersection of two T-external (resp., I-external, F-external) neutrosophic cubic sets may not be T-external (resp., I-external, F-external).

Example 2.6: Let $\mathcal{A} = (\mathbf{A}, \Lambda)$ and $\mathcal{B} = (\mathbf{B}, \Psi)$ be neutrosophic cubic sets in $X = [0, 1]$ where

$\mathbf{A} = \{\langle x; [0.2, 0.4], [0.5, 0.7], [0.3, 0.5]\rangle \mid x \in [0, 1]\}$,
$\Lambda = \{\langle x; 0.1, 0.4, 0.8\rangle \mid x \in [0, 1]\}$,
$\mathbf{B} = \{\langle x; [0.6, 0.8], [0.4, 0.7], [0.7, 0.9]\rangle \mid x \in [0, 1]\}$,
$\Psi = \{\langle x; 0.3, 0.3, 0.8\rangle \mid x \in [0, 1]\}$.

Then $\mathcal{A} = (\mathbf{A}, \Lambda)$ and $\mathcal{B} = (\mathbf{B}, \Psi)$ are T-external neutrosophic cubic sets in $X = [0, 1]$. The R-intersection $\mathcal{A} \cap_R \mathcal{B} = (\mathbf{A} \cap \mathbf{B}, \Lambda \vee \Psi)$ of $\mathcal{A} = (\mathbf{A}, \Lambda)$ and $\mathcal{B} = (\mathbf{B}, \Psi)$ is given as follows:

$$\begin{aligned} &\mathbf{A} \cap \mathbf{B} = \{\langle x; [0.2, 0.4], [0.4, 0.7], [0.3, 0.5]\rangle \mid x \in [0, 1]\}, \\ &\Lambda \vee \Psi = \{\langle x; 0.3, 0.4, 0.8\rangle \mid x \in [0, 1]\}, \end{aligned}$$

and it is not a T-external neutrosophic cubic set in $X = [0, 1]$.

We provide a condition for the R-intersection of two T-external (resp., I-external, F-external) neutrosophic cubic sets to be T-external (resp., I-external, F-external).

Theorem 2.7: Let $\mathcal{A} = (\mathbf{A}, \Lambda)$ and $\mathcal{B} = (\mathbf{B}, \Psi)$ be T-external neutrosophic cubic sets in X such that

$$\begin{aligned} &\max\left\{\min\{A_T^+(x), B_T^-(x)\}, \min\{A_T^-(x), B_T^+(x)\}\right\} \\ &< (\lambda_T \vee \psi_T)(x) \\ &\le \min\left\{\max\{A_T^+(x), B_T^-(x)\}, \max\{A_T^-(x), B_T^+(x)\}\right\} \end{aligned}$$

for all $x \in X$. Then the R-intersection $\mathcal{A} \cap_R \mathcal{B} = (\mathbf{A} \cap \mathbf{B}, \Lambda \vee \Psi)$ is a T-external neutrosophic cubic set in X.

Proof. For any $x \in X$, let

$$c_x := \max\left\{\min\{A_T^+(x), B_T^-(x)\}, \min\{A_T^-(x), B_T^+(x)\}\right\}$$

and

$$d_x := \min\left\{\max\{A_T^+(x), B_T^-(x)\}, \max\{A_T^-(x), B_T^+(x)\}\right\}.$$

Then $d_x = A_T^-(x)$, $d_x = B_T^-(x)$, $d_x = A_T^+(x)$, or $d_x = B_T^+(x)$. It is possible to consider the cases $d_x = A_T^-(x)$ and $d_x = A_T^+(x)$ only because the remaining cases are similar to these cases. If $d_x = A_T^-(x)$, then

$$B_T^-(x) \le B_T^+(x) \le A_T^-(x) \le A_T^+(x).$$

Thus $c_x = B_T^+(x)$, and so

$$\begin{aligned} B_T^-(x) &= (A_T \cap B_T)^-(x) \le (A_T \cap B_T)^+(x) \\ &= B_T^+(x) = c_x < (\lambda_T \vee \psi_T)(x). \end{aligned}$$

Hence $(\lambda_T \vee \psi_T)(x) \notin ((A_T \cap B_T)^-(x), (A_T \cap B_T)^+(x))$. If $d_x = A_T^+(x)$, then $B_T^-(x) \le A_T^+(x) \le B_T^+(x)$ and thus $c_x = \max\{A_T^-(x), B_T^-(x)\}$. Suppose that $c_x = A_T^-(x)$. Then

$$\begin{aligned} B_T^-(x) &\le A_T^-(x) = c_x < (\lambda_T \vee \psi_T)(x) \\ &\le d_x = A_T^+(x) \le B_T^+(x). \end{aligned}$$

It follows that

$$B_T^-(x) \le A_T^-(x) < (\lambda_T \vee \psi_T)(x) < A_T^+(x) \le B_T^+(x) \quad (5)$$

or

$$B_T^-(x) \le A_T^-(x) < (\lambda_T \vee \psi_T)(x) = A_T^+(x) \le B_T^+(x). \quad (6)$$

The case (5) induces a contradiction. The case (6) implies that

$$(\lambda_T \vee \psi_T)(x) \notin \left((A_T \cap B_T)^-(x), (A_T \cap B_T)^+(x)\right)$$

since $(\lambda_T \vee \psi_T)(x) = A_T^+(x) = (A_T \cap B_T)^+(x)$. Now, if $c_x = B_T^-(x)$, then

$$A_T^-(x) \le B_T^-(x) = c_x < (\lambda_T \vee \psi_T)(x) \le d_x = A_T^+(x) \le B_T^+(x).$$

Hence we have

$$A_T^-(x) \le B_T^-(x) < (\lambda_T \vee \psi_T)(x) < A_T^+(x) \le B_T^+(x) \quad (7)$$

or

$$A_T^-(x) \le B_T^-(x) < (\lambda_T \vee \psi_T)(x) = A_T^+(x) \le B_T^+(x). \quad (8)$$

The case (7) induces a contradiction. The case (8) induces

$$(\lambda_T \vee \psi_T)(x) \notin \left((A_T \cap B_T)^-(x), (A_T \cap B_T)^+(x)\right).$$

Therefore the R-intersection $\mathcal{A} \cap_R \mathcal{B} = (\mathbf{A} \cap \mathbf{B}, \Lambda \vee \Psi)$ is a T-external neutrosophic cubic set in X.

Similarly, we have the following theorems.

Theorem 2.8: Let $\mathcal{A} = (\mathbf{A}, \Lambda)$ and $\mathcal{B} = (\mathbf{B}, \Psi)$ be I-external neutrosophic cubic sets in X such that

$$\max\left\{\min\{A_I^+(x), B_I^-(x)\}, \min\{A_I^-(x), B_I^+(x)\}\right\} < (\lambda_I \vee \psi_I)(x) \le \min\left\{\max\{A_I^+(x), B_I^-(x)\}, \max\{A_I^-(x), B_I^+(x)\}\right\}$$

for all $x \in X$. Then the R-intersection $\mathcal{A} \cap_R \mathcal{B} = (\mathbf{A} \cap \mathbf{B}, \Lambda \vee \Psi)$ is an I-external neutrosophic cubic set in X.

Theorem 2.9: Let $\mathcal{A} = (\mathbf{A}, \Lambda)$ and $\mathcal{B} = (\mathbf{B}, \Psi)$ be F-external neutrosophic cubic sets in X such that

$$\max\left\{\min\{A_F^+(x), B_F^-(x)\}, \min\{A_F^-(x), B_F^+(x)\}\right\} < (\lambda_F \vee \psi_F)(x) \le \min\left\{\max\{A_F^+(x), B_F^-(x)\}, \max\{A_F^-(x), B_F^+(x)\}\right\}$$

for all $x \in X$. Then the R-intersection $\mathcal{A} \cap_R \mathcal{B} = (\mathbf{A} \cap \mathbf{B}, \Lambda \vee \Psi)$ is an F-external neutrosophic cubic set in X.

Corollary 2.10: Let $\mathcal{A} = (\mathbf{A}, \Lambda)$ and $\mathcal{B} = (\mathbf{B}, \Psi)$ be external neutrosophic cubic sets in X. Then the R-intersection of $\mathcal{A} = (\mathbf{A}, \Lambda)$ and $\mathcal{B} = (\mathbf{B}, \Psi)$ is an external neutrosophic cubic set in X when the conditions in Theorems 2.7, 2.8 and 2.9 are valid.

III. Conclusion

We have considered the R-union and R-intersection of T-external (I-external, F-external) neutrosophic cubic sets. We have provided examples to show that the R-intersection and R-union of T-external (resp. I-external and F-external) neutrosophic cubic sets may not be a T-external (resp. I-external and F-external) neutrosophic cubic set. We have discussed conditions for the R-union and R-intersection of T-external (resp. I-external and F-external) neutrosophic cubic sets to be a T-external (resp. I-external and F-external) neutrosophic cubic set. Based on this paper, we will study conditions for the R-intersection of two neutrosophic cubic sets to be both an α-external neutrosophic cubic set and an α-internal neutrosophic cubic set for $\alpha \in \{T, I, F\}$. Also, we will consider conditions for the R-union and R-intersection of two α-internal neutrosophic cubic sets to be an α-external neutrosophic cubic set for $\alpha \in \{T, I, F\}$.

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Maintenance Operating System Uncertainties Approached through Neutrosophic Theory

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Abstract - This study introduces the concept of uncertainty analysis of Neutrosophic Theory in the sphere of Maintenance Operating System (MOS). The aim of this study is to underline the importance of uncertainties solving in Maintenance Operating System. In maintenance process appear ambiguous states that can't be assimilated neither true, nor false, meaning that the threshold state is a neutral one, being defined as fault in most of cases. In this regard, this uncertaintymaking decision process can be associated as functioning according to rules of Neutrosophy, and can be evaluated using elements of Neutrosophic Structures, Pareto Charts, maintenance metrics. Identification of uncertainties and study of their impact on maintenance, making the right decision, is the main focus of this paper.

The study is useful for business area, especially manufacturing lines endowed with complex equipment and facilities, also for researchers interested to make improvements for maintenance procedures.

I. INTRODUCTION

All advanced systems depend on having a foundation of Maintenance Operating Systems (MOS) in place, even if the owners will not admit that.

A method to improve efficiency and to decrease the downtime in manufacturing process is a goal for everybody.

An efficient maintenance offers a stable manufacturing process assuring reliability of the system.

Vision for MOS is a standardized, proactive and disciplined operating system that engages all team members to maintain the integrity and availability of equipment, facilities and processes.

Desired outcomes mean to engage stakeholders in the development and implementation of a standard MOS that ensures world class manufacturing Throughput to Potential (TTP) of plant facilities and equipment. Uncertainty represents an unsolved situation; it defines a fuzziness state [5].

The aim of this study is to identify uncertainties of MOS that definitely decrease efficiency of the system, to evaluate them through Neutrosophic Theory (NT) and to show the potentiality of the method for uncertainties solving.

The paper is structured as follows: Section 2 briefly describes the preview works related to Neutrosophic Theory, Section 3 discusses about MOS and uncertainties, Section 4 presents some results and statistics interpretation and finally, Section 5 depicts conclusions and directions for future work.

II. PREVIEW WORKS

In this section will be presented some pragmatic areas where neutrosophy is suitable and the basic concept of neutrosophic theory.

Neutrosophy is an available method for uncertainties investigation for any complex manufacturing line that involves advanced technology, many parameters and metrics. It can be applied to evaluate the uncertainty level, to analyze it [8].

Logistics is the field of study focused on the design, control, and implementation of the efficient flow and storage of goods and services. Because of numerous other related information from the point of origin to the point of final consumption with the aim to satisfy the requirements of its existing and prospective customers, the system can generate a lot of uncertainty [1].

A sample of using neutrosophic decision making model on manufacturing line as used for selection quality clay-brick for construction is developed and denoted to be suitable.

Keywords-uncertainty; neutrosophic theory; constraints; throughput to potential; uncertainty making decision;

Neutrosophy set is a tool that can deal with indeterminacy and inconsistent data [13].

According to the neutrosophy theory, the neutral (uncertainty) instances can be analyzed and accordingly, reduced. There are some spectacular results of applying neutrosophy in practical application such as artificial intelligence [3].

Extending these results, neutrosophy theory can be applied for solving uncertainty also on other domains such as Robotics, where are confirmed results of neutrosophics logics applied to make decisions when appear situations of uncertainty [10],[11].

The real-time adaptive networked control of rescue robots is another project that used neutrosophic logic to control the robot movement in a surface with uncertainties for it [12].

Neutrosophy analyzes, evaluates and interprets uncertainties. The specialty literature denotes that Zadeh introduced the degree of membership/truth (t), so the rest would be (1-t) equal to f, their sum being 1, and he defined the fuzzy set in 1965 [6]. Further, Atanassov improved Zadeh's theory by introducing the degree of nonmembership/falsehood (f) and defining the intuitionist fuzzy set [7].

As novelty to previous theory, Smarandache introduced and defined explicitly the degree of indeterminacy/neutrality (i) as independent component. In any field of knowledge, each structure is composed of two parts: a space, and a set of axioms (or laws) acting (governing) on it. If the space, or at least one of its axioms (laws), has some indeterminacy of the form $(t, i, f) \neq (1, 0, 0)$, that structure is a (t, i, f) -Neutrosophic Structure [4].

III. MAINTENANCE OPERATING SYSTEM

Maintenance definition according to Business Dictionary, represents activities required or undertaken to conserve as nearly, and as long as possible the original condition of an asset or resource while compensating for normal wear and tear [2].

Without any operating system, a management system, some metrics to measure the progress, you are trying to build a foundation on the sand.

In maintenance process also intervene some uncertainties regarding failures involving equipment efficiency, process flow (bottlenecks on production line, constraints), operators skills, spare parts management, manufacturing line potential, etc. All of them decrease somehow the manufacturing process, directly or indirectly the efficiency of production flow. It is a challenge with the seconds of cycle time of product manufacturing, with maintenance concept of preventive, predictive or corrective, with people training, equipment spare parts or maintenance costs reducing [9].

Nowadays, when manufacturing processes are very complex, involving a lot of different types of machines and equipment, in which the product in fact, is an intricate one, maintenance is deployed as a system subordinated to manufacturing process. The main principles of maintenance are: safety and quality always in top; using data to make decisions; using standardized tools, practices, procedures; applying prevention through a continuously improvement process; optimization of the resources; maintaining the integrity of the equipment; maximize throughput of installed equipment and facilities to its potential, such as shown in Fig. 1.

There are 3 major inputs that generate faults or uncertainties:

- *People:* poor training, lack of versatility, missing technically competent, poor communication;

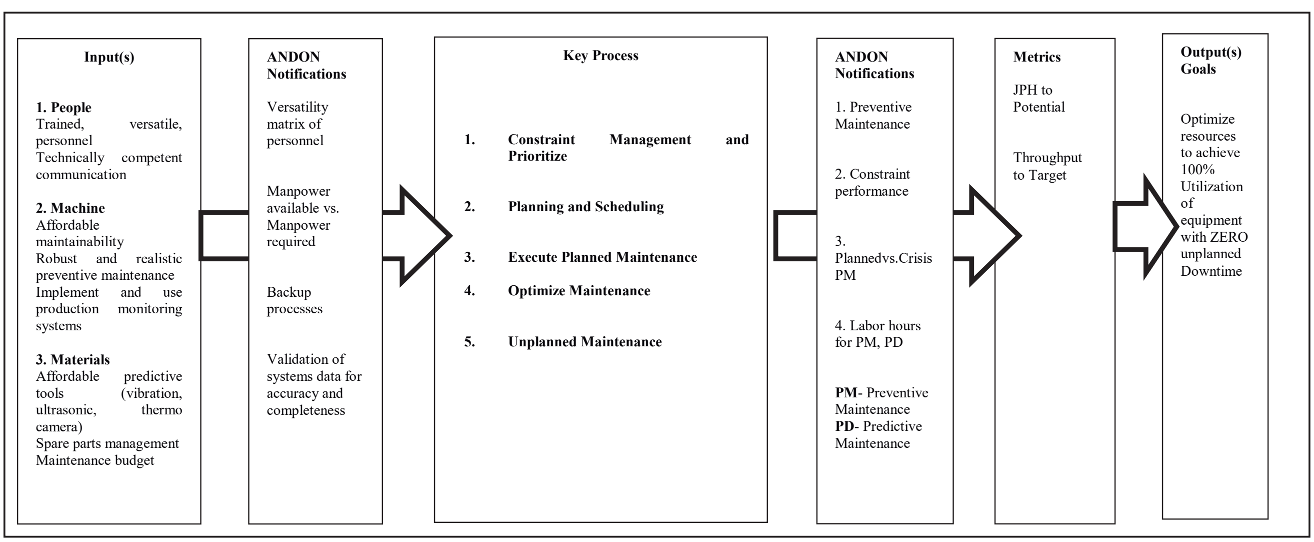


Fig. 1. The structure and elements of MOS

- *Machine*: not affordable maintainability, not realistic preventive maintenance, not existing production monitoring system, not applicable predictive maintenance (thermo camera, ultrasonic, vibration);
- *Materials*: spare parts control with errors, nonstandardization of spare parts, not affordable predictive tools, small maintenance budget .

There are Key Process to control the threats of input: constraint management, planning and scheduling the maintenance, optimize maintenance, do unplanned maintenance.

The efficiency of maintenance is revealed by metrics such as JPH (*Job Per Hour*) and TTP (*Throughput to Potential*).

The goal of each system is ZERO defects, meaning no downtime on manufacturing process, an idealistic situation.

According to our goal, we proposed to analyze the faults that generate downtime of the system. The faults are revealed on TTP, a metric that shows the potentiality of manufacturing line to produce parts to high capacity related to engineering capacity. There are 3 zones of TTP that can be interpreted according to an established level (designed) L = 100% :

$$TTP < L - 2.5\% \quad - \quad \text{red zone}; \qquad (1)$$

$$L - 2.5\% <= TTP < L - 0.5\% \quad - \quad \text{yellow zone}; \qquad (2)$$

$$TTP >= L - 0.5\% \quad - \quad \text{green zone}; \qquad (3)$$

Daily collected data shows the status of manufacturing lines, emphasizing the constraints and bottle-necks, such as in Fig. 2.

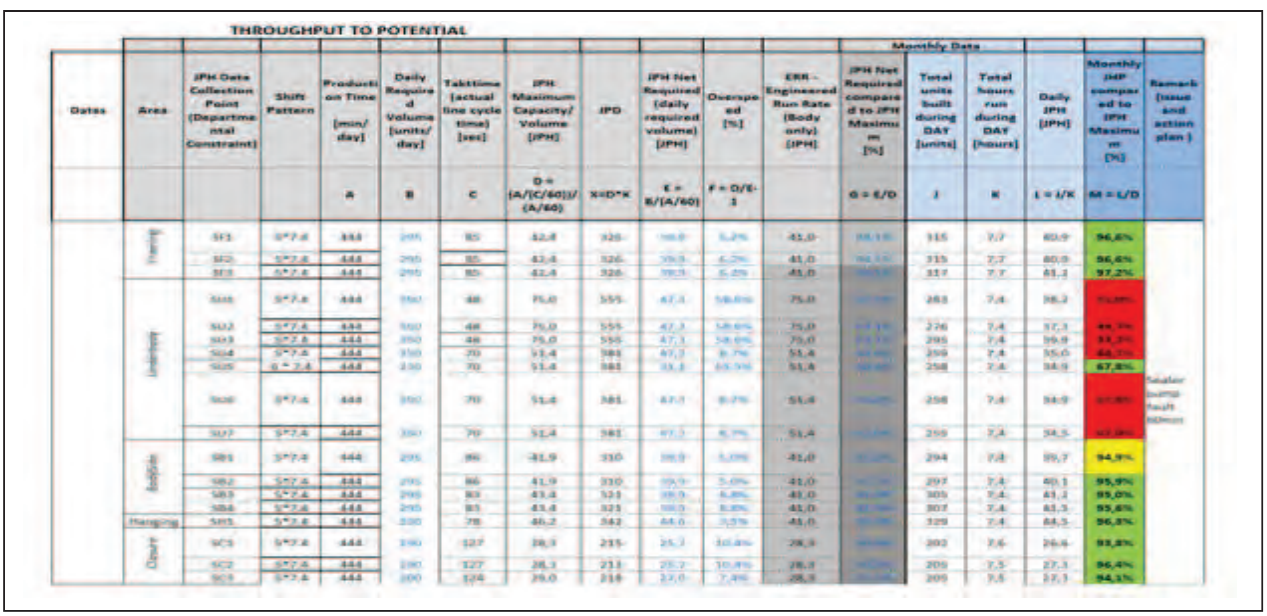
TTP is a maintenance metric that emphasizes the line balancing from cycle time point of view. This means that within manufacturing line the stations have different cycle time to perform a product, according to the assigned operations. The stations with cycle time at the highest limit are called “constraints” and they can supply “bottle-necks” on manufacturing process.

According to these data it can be analyzed monthly productivity trend related to occurred faults, as in Fig. 3. The graphic underlines the evolution of TTP for constraints stations monthly, weekly for the previous month and daily for the current month.

Evaluating and analyzing the faults, they can be grouped such as: people training (operator lack skill), low communication level, missing spare parts, equipment obsolete level part, equipment parameters out of range, operating error, logistic error (wrong part supply)… Each fault type is generator of uncertainty, and involves uncertainty making decision process.

To evaluate correctly the fault rate we will make it in step 1, Pareto analysis for the most frequent faults group, as in Fig. 4.

Looking into Pareto Charts we see that “operator lack skill” induces a percentage of 25% relative frequency. Interpreting this data, we can say that reducing the rate of this item, it can be also reduced the fault rate of whole system. Operator lack skill involves confusion and uncertainty in faults solving making decision.



THROUGHPUT TO POTENTIAL

Datas	Area	JPH Data Collection Point (Departmental Constraint)	Shift Pattern	Production Time [min/day]	Daily Required Volume [units/day]	Takttime (actual line cycle time) [sec]	JPH Maximum Capacity/Volume [JPH]	JPD	JPH Net Required (daily required volume) [JPH]	Overspeed [%]	ERR - Engineered Run Rate (Body only) [JPH]	JPH Net Required compared to JPH Maximum [%]	Monthly Data: Total units built during DAT [units]	Total hours run during DAT [hours]	Daily JPH [JPH]	Monthly JHP compared to JPH Maximum [%]	Remark (issue and action plan)
				A	B	C	D = (A/[C/60])/(A/60)	X=D*K	E = B/(A/60)	F = D/E-1		G = E/D	J	K	L = J/K	M = L/D	
	[illegible]	SF1	5*7.4	444	295	85	42.4	326	39.9	6.2%	41.0	[illegible]	315	7.7	40.9	96.6%	
		SF2	5*7.4	444	295	85	42.4	326	39.9	6.2%	41.0	[illegible]	315	7.7	40.9	96.6%	
		SF3	5*7.4	444	295	85	42.4	326	39.9	6.2%	41.0	[illegible]	317	7.7	41.2	97.2%	
	Underbody	SU1	5*7.4	444	350	48	75.0	555	47.3	58.6%	75.0	[illegible]	283	7.4	38.2	[illegible]	
		SU2	5*7.4	444	350	48	75.0	555	47.3	58.6%	75.0	[illegible]	276	7.4	37.3	[illegible]	
		SU3	5*7.4	444	350	48	75.0	555	47.3	58.6%	75.0	[illegible]	295	7.4	39.9	[illegible]	
		SU4	5*7.4	444	350	70	51.4	381	47.3	8.7%	51.4	[illegible]	259	7.4	35.0	[illegible]	
		SU5	6*7.4	444	330	70	51.4	381	33.3	65.9%	51.4	[illegible]	258	7.4	34.9	67.8%	
		SU6	5*7.4	444	350	70	51.4	381	47.3	8.7%	51.4	[illegible]	258	7.4	34.9	[illegible]	Sealer pump fault 60min
		SU7	5*7.4	444	350	70	51.4	381	47.3	8.7%	51.4	[illegible]	259	7.4	34.5	[illegible]	
	Bodyside	SB1	5*7.4	444	295	86	41.9	310	39.9	5.0%	41.0	[illegible]	294	7.4	39.7	94.9%	
		SB2	5*7.4	444	295	86	41.9	310	39.9	5.0%	41.0	[illegible]	297	7.4	40.1	95.9%	
		SB3	5*7.4	444	295	83	43.4	321	39.9	8.8%	41.0	[illegible]	305	7.4	41.2	95.0%	
		SB4	5*7.4	444	295	83	43.4	321	39.9	8.8%	41.0	[illegible]	307	7.4	41.5	95.6%	
	Hanging	SH1	5*7.4	444	330	78	46.2	342	44.6	3.5%	41.0	[illegible]	329	7.4	44.5	96.3%	
	[illegible]	SC1	5*7.4	444	190	127	28.3	215	25.7	10.4%	28.3	[illegible]	202	7.6	26.6	93.8%	
		SC2	5*7.4	444	190	127	28.3	213	25.7	10.4%	28.3	[illegible]	205	7.5	27.3	96.4%	
		SC3	5*7.4	444	200	124	29.0	218	27.0	7.4%	28.3	[illegible]	205	7.5	27.3	94.1%	

Fig. 2. Throughput to Potential Analysis

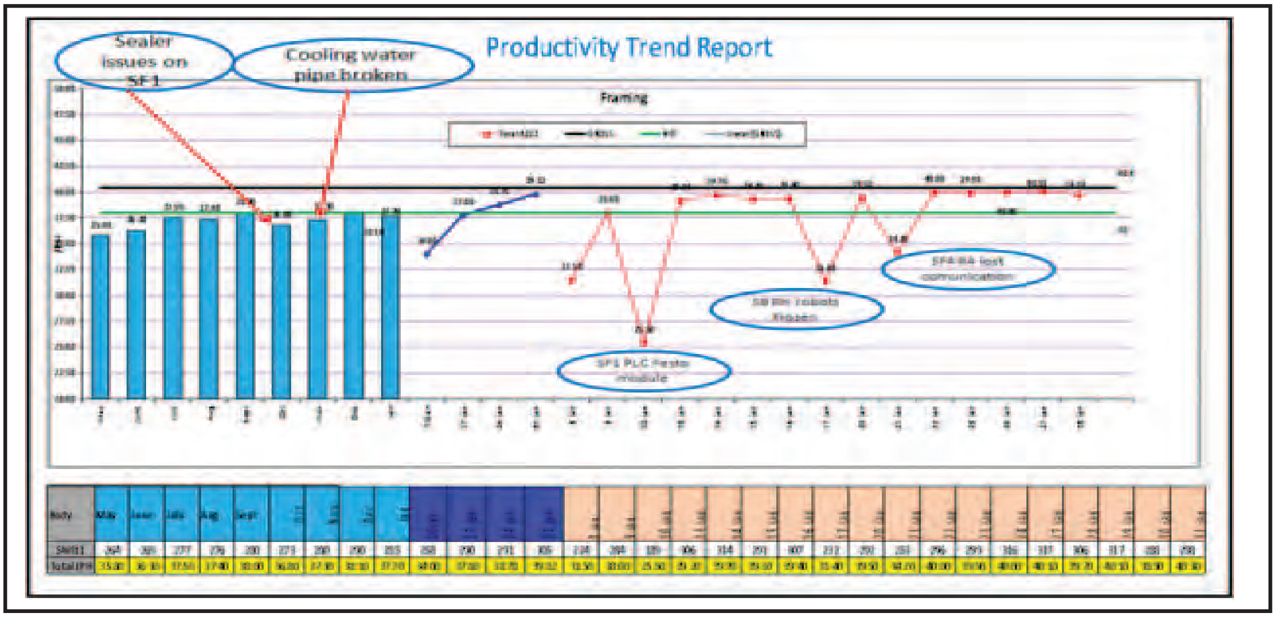
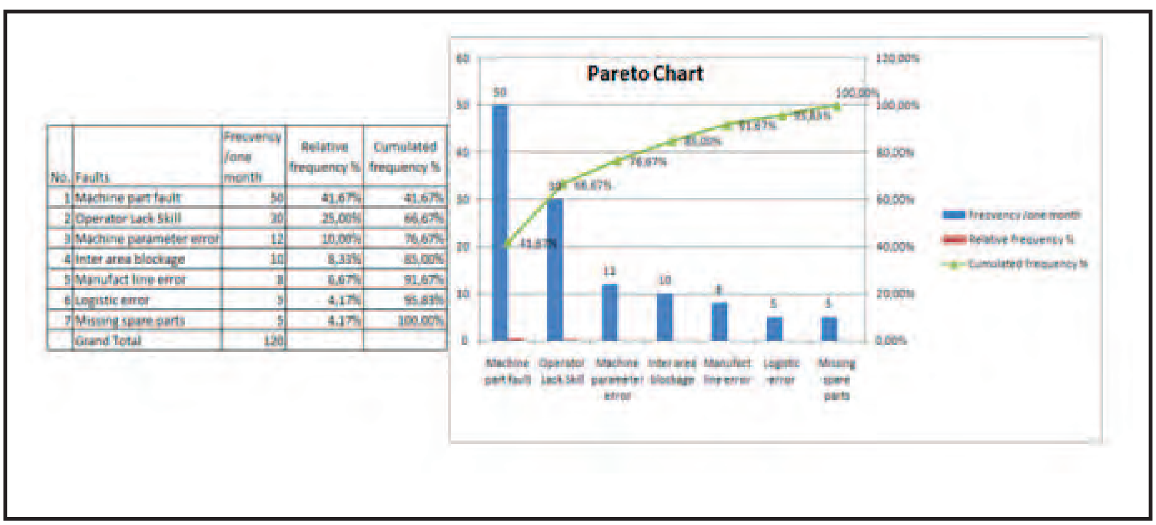


Fig. 3. Evolution in time of TTP

Increasing versatility of maintenance operators, it will cover the solving of wide area of faults making decision, such as quick intervention to equipment, right selection of part replacement, right selection of damaged equipment, avoiding inter-areas blockages, and so on. This is an important moment to make the right decision regarding to choose the appropriate truth degree membership (acceptance), indeterminacy degree membership and falsity degree membership (reject).



No.	Faults	Frecvency /one month	Relative frequency %	Cumulated frequency %
1	Machine part fault	50	41,67%	41,67%
2	Operator Lack Skill	30	25,00%	66,67%
3	Machine parameter error	12	10,00%	76,67%
4	Inter area blockage	10	8,33%	85,00%
5	Manufact line error	8	6,67%	91,67%
6	Logistic error	5	4,17%	95,83%
7	Missing spare parts	5	4,17%	100,00%
	Grand Total	120		

Fig. 4. Pareto Chart Analysis - step 1

To apply the neutrosophic method we have to define the space and the set of axioms for occurred uncertainties. Each operator has skills in different level of specialty knowledge in order to solve a maintenance fault.

To evaluate the level of uncertainty it is important to establish each level of uncertainty related to make the right decision, as is shown in TABLE 1, (as proposal).

The space is represented by operators and the set of axioms, by the skill level of fault solving for each operator. In Fig. 5 is shown the set of axioms for each operator, so we have a (T, I, F), as follows:

a_{11} - operator 1/skill 1
a_{12} - operator 1/skill 2 …

a_{15} - operator 1/ skill 5 and so on to
…
a_{55} - operator 5/ skill 5.

In this context, we can establish the whole space of states True, Indeterminacy, False (T, I, F), and the value of the space. In Fig.6, are presented the spaces and sets of axioms for 5 operators such as a_{ij} (T, I, F).

A complete cycle time to produce a part consists of a sum of specific times, such as: cycle time (effectively), starvation, blockage, wait for auxiliary parts, wait for attention, repair in progress, break, set-up, tool change, no communication. Uncertainties are focused on "wait for attention" when operator is confronted with confusion and ambiguous states, the moment of fault making a decision. The situation is eased byan IT application that monitors and handles the states of equipment in detail, and guides the operator to make more accurate the fault location.

TABLE 1
LEVEL OF UNDERTERMINACY

No.	#L	Level of uncertainty	State		
			T	I	F
1	L1	Solve the fault JIT (Just In Time)	1	0	0
2	L2	Solve the fault + Δt	<1	<0,5	<0,2
3	L3	Solve the fault with help from other operator	<0,5	<0,5<1	<1
4	L4	Nonsolving the fault by operator	0	0	1
5	L5	Nonsolving the fault, nobody in place	0	1	1

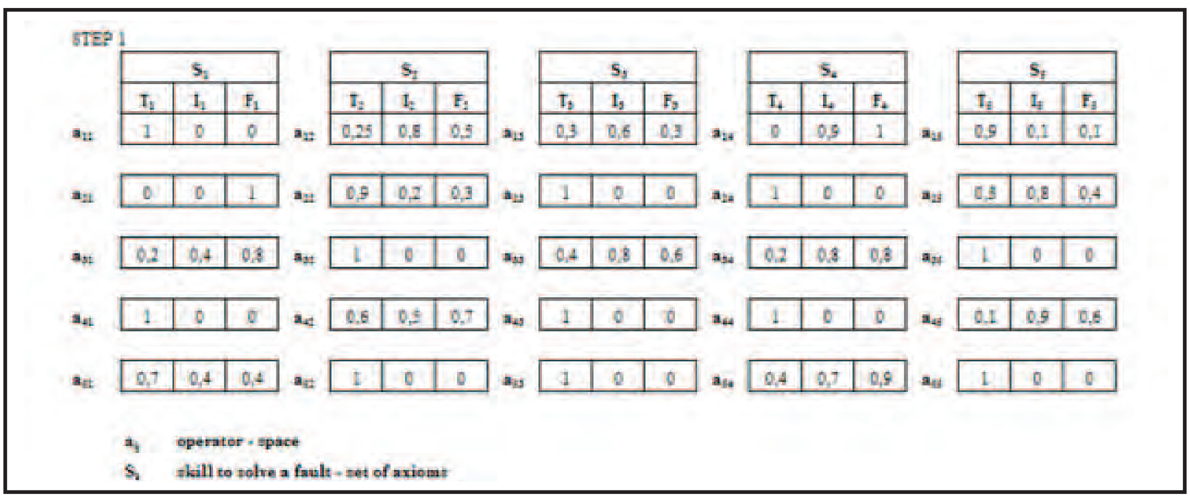


Fig. 5. Relation operator/skill - step 1

IV. RESULTS AND STATISTICS

In these conditions we got a realistic distribution of uncertainties regarding solving faults on the manufacturing line. For the step 2, it was applied training for operators in 5 major types of faults (equipment), as a consequence, the level of solving faults increased and uncertainties degree, decreased. So, we obtained the distribution of states as in Fig. 6.

Evaluating the faults occurred during a month, after applying training courses for operators, by ParetoCharts, we got the data shown in Fig. 7.

The number of solved faults decreased from 120 to 54, related to all types of faults.

The goal of MOS aspirs to an intelligent maintenance system, to achieve and sustain zero breakdown. This is the futere of maintenance, equipment, machines and systems to achieve highest performance including also self maintenance capabilities. The operator is a risk factor in this system that has to be taken into consideration. Such a goal can be achieved transforming raw data to valuable information regarding current and future condition and request of the asset.

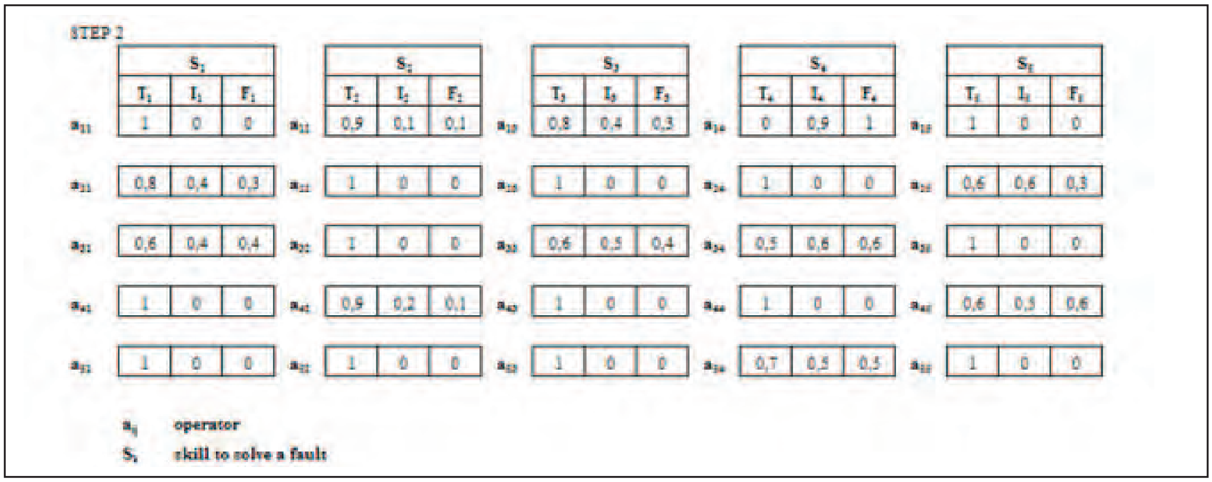


Fig. 6. Relation operator/skill – step 2, after training

No.	Faults	Frecvency /one month	Relative frequency %	Cumulated frequency %
1	Machine part fault	21	38,89%	38,89%
2	Operator Lack Skill	8	14,81%	53,70%
3	Manufact line error	8	14,81%	68,52%
4	Inter area blockage	7	12,96%	81,48%
5	Logistic error	5	9,26%	90,74%
6	Machine parameter error	3	5,56%	96,30%
7	Missing spare parts	2	3,70%	100,00%
	Grand Total	54		

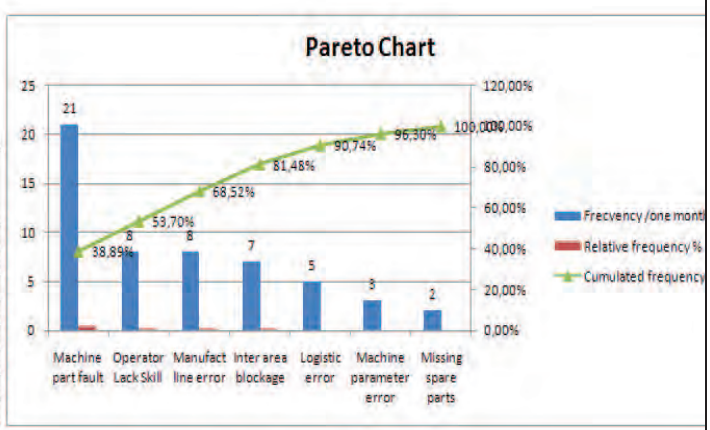


Fig 7. Pareto Chart Analysis - step 2

V. CONCLUSIONS AND FUTURE WORK

The presentedwork in this study is a step towards developing a procedure available to uncertainties emerging in maintenance process of the complex manufacturing lines. Identification, evaluation, proceeding of the specified uncertainties of suggested metrics are incentive and supporting. It is helpful for MOS analysis, to be sustained by IT application monitoring equipment function, classifying faults, downtime calculus, revealing the reliability of the system. It is a real potentiality to make a decision for uncertainties degree towards.

Solving the uncertainties through mentioned method, classifying them into faults (false) or solving state (true),increases the efficiency of the process.

Analyzing the results, we observed that applying NT it can be emphasized the states of neutrality, ambiguously, uncertainty whereby we can act to transform them into stable status, true or false. Finding this applicability in Business, we can get through the next step, to design an algorithm the more inclusive, that reduce the time to make decision regarding the involved status and to decrease the downtime of equipment.

As future research we will extend our work using the complex neutrosophic set to more effectively capture the uncertainties in MOS.

The science of prognostics is based on the analysis of failure modes, detection of early signs of wear and aging, also fault conditions. Uncertainties, as we have seen in above example, are a source of failure. In this regard, we consider that it is a good opportunity to apply the Neutrosophic Theory to evaluate, analyse and make the right decision solving faults in maintenance process.

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Interval Valued Bipolar Fuzzy Weighted Neutrosophic Sets and Their Application

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Abstract—Interval valued bipolar fuzzy weighted neutrosophic set(IVBFWN-set) is a new generalization of fuzzy set, bipolar fuzzy set, neutrosophic set and bipolar neutrosophic set so that it can handle uncertain information more flexibly in the process of decision making. Therefore, in this paper, we propose concept of IVBFWN-set and its operations. Also we give the IVBFWN-set weighted average operator and IVBFWN-set weighted geometric operator to aggregate the IVBFWN-sets, which can be considered as the generalizations of some existing ones under fuzzy, neutrosophic environments and so on. Finally, a decision making algorithm under IVBFWN environment is given based on the given aggregation operators and a real example is used to demonstrate the effectiveness of the method.

Keywords—Neutrosophic set, interval valued neutrosophic set, IVBFWN-set, average and geometric operator, multi-criteria decision making.

I. Introduction

To overcome containing various kinds of uncertainty, the concept of fuzzy sets [18] has been introduced by Zadeh. After Zadeh, many studies on mathematical modeling have been developed. For example; to model indeterminate and inconsistent information Smarandache [13] introduced the concept of neutrosophic set which is independently characterized by three functions called truth-membership function, indeterminacy-membership function and falsity membership function. Recently, studies on neutrosophic sets are made rapidly in [1,2].

Bipolar fuzzy sets, which are a generalization of Zadeh's fuzzy sets [18], were originally proposed by Lee [9]. Bosc and Pivert [4] said that "Bipolarity refers to the propensity of the human mind to reason and make decisions on the basis of positive and negative effects. Positive information states what is possible, satisfactory, permitted, desired, or considered as being acceptable. On the other hand, negative statements express what is impossible, rejected, or forbidden. Negative preferences correspond to constraints, since they specify which values or objects have to be rejected (i.e., those that do not satisfy the constraints), while positive preferences correspond to wishes, as they specify which objects are more desirable than others (i.e., satisfy user wishes) without rejecting those that do not meet the wishes." Presently, works on bipolar fuzzy sets are progressing rapidly in [3,4,8-12,17]. Also, bipolar neutrosophic set(BN-set) and its operations is given in [7].

In this study, to handling some uncertainties in fuzzy sets and neutrosophic sets, the extensions of fuzzy sets[18], bipolar fuzzy sets[9], neutrosophic sets[13] and bipolar neutrosophic sets[7], interval valued bipolar fuzzy weighted neutrosophic sets with application are introduced.

II. Preliminaries

In the section, we give some concepts related to bipolar fuzzy sets, neutrosophic sets, interval valued neutrosophic set, and bipolar neutrosophic sets.

Definition 2.1. [14] Let X be a universe of discourse. Then a single valued neutrosophic set is defined as:

$$A_{NS} = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle : x \in X\}$$

which is characterized by a truth-membership function $T_A(x): X \to [0,1]$, an indeterminacy-membership function $I_A(x): X \to [0,1]$, and a falsity-membership function $F_A(x): X \to [0,1]$. There is not restriction on the sum of $T_A(x), I_A(x)$, and $F_A(x)$ so $0 \le T_A(x) \le I_A(x) \le F_A(x) \le 3$.

Definition 2.2. [15] Let X, be a space of points (objects) with generic elements in X, denoted by x. An interval valued neutrosophic set (for short IVNS) A in X, is characterized by truth-membership function $T_A(x)$, indeteminacy-membership function $I_A(x)$, and falsity-membership function $F_A(x)$. For each point x in X, we have that $T_A(x), I_A(x), F_A(x) \subseteq [0,1]$.

For two IVNS

$A_{IVNS} = \{\langle x, [\inf T_A(x), \sup T_A(x)], [\inf I_A(x), \sup I_A(x)], [\inf F_A(x), \sup F_A(x)]\rangle : x \in X\}$

and

$B_{IVNS} = \{\langle x, [\inf T_B(x), \sup T_B(x)], [\inf I_B(x), \sup I_B(x)], [\inf F_B(x), \sup F_B(x)]\rangle : x \in X\}$

Then,

1. $A_{IVNS} \subseteq B_{IVNS}$ if and only if
$\inf T_A(x) \le \inf T_B(x)$, $\sup T_A(x) \le \sup T_B(x)$,
$\inf I_A(x) \ge \inf I_B(x)$, $\sup I_A(x) \ge \sup I_B(x)$,
$\sup F_A(x) \ge \sup F_B(x)$, $\sup F_A(x) \ge \sup F_B(x)$
for all $x \in X$.

2. $A_{IVNS} = B_{IVNS}$ if and only if
$\inf T_A(x) = \inf T_B(x)$, $\sup T_A(x) = \sup T_B(x)$,
$\inf I_A(x) = \inf I_B(x)$, $\sup I_A(x) = \sup I_B(x)$,
$\sup F_A(x) = \sup F_B(x)$, $\sup F_A(x) = \sup F_B(x)$
for any $x \in X$.

3. ${A_{IVNS}}^C$ if and only if
${A_{IVNS}}^C = \{\langle x, [\inf F_A(x), \sup F_A(x)], [1 - \sup I_A(x), 1 - \inf I_A(x)], [\inf T_A(x), \sup T_A(x)]\rangle : x \in X\}$

4. $A_{IVNS} \cap B_{IVNS}$ if and only if
$A_{IVNS} \cap B_{IVNS} = \{\langle x, [\inf T_A(x) \wedge \inf T_B(x), \sup T_A(x) \wedge \sup T_B(x)], [\inf I_A(x) \vee \inf I_B(x), \sup I_A(x) \vee \sup I_B(x)], [\inf F_A(x) \vee \inf F_B(x), \sup F_A(x) \vee \sup F_B(x)]\rangle : x \in X\}$

5. $A_{IVNS} \cup B_{IVNS}$ if and only if
$A_{IVNS} \cup B_{IVNS} = \{\langle x, [\inf T_A(x) \vee \inf T_B(x), \sup T_A(x) \vee \sup T_B(x)], [\inf I_A(x) \wedge \inf I_B(x), \sup I_A(x) \wedge \sup I_B(x)], [\inf F_A(x) \wedge \inf F_B(x), \sup F_A(x) \wedge \sup F_B(x)]\rangle : x \in X\}$

Definition 2.3. [9] Let X be a non-empty set. Then, a bipolar-valued fuzzy set, denoted by A_{BF} is defined as;

$A_{BF} = \{\langle x, \mu_B^+(x), \mu_B^-(x)\rangle : x \in X\}$

Where $\mu_B^+(x) : X \to [0,1]$ and $\mu_B^-(x) : X \to [0,1]$. The positive membership degree $\mu_B^+(x)$ denotes the satisfaction degree of an element x to the property corresponding to A_{BF} and the negative membership degree $\mu_B^-(x)$ denotes the satisfaction degree of x to some implicit counter property of A_{BF}.

Definition 2.4. [7] A bipolar neutrosophic set A in X is defined as an object of the form

$A = \{\langle x, T^+(x), I^+(x), F^+(x), T^-(x), I^-(x), F^-(x)\rangle : x \in X\}$,

where $T^+, I^+, F^+ : X \to [1,0]$ and $T^-, I^-, F^- : X \to [-1,0]$.

The positive membership degree $T^+(x), I^+(x), F^+(x)$ denotes the truth membership, indeterminate membership and false membership of an element $x \in X$ corresponding to a bipolar neutrosophic set A and the negative membership degree $T^-(x), I^-(x), F^-(x)$ denotes the truth membership, indeterminate membership and false membership of an element $x \in X$ to some implicit counter-property corresponding to a bipolar neutrosophic set A.

Definition 2.5. [7] Let

$A_1 = \{\langle x, T_1^+(x), I_1^+(x), F_1^+(x), T_1^-(x), I_1^-(x), F_1^-(x)\rangle : x \in X\}$

and

$A_2 = \{\langle x, T_2^+(x), I_2^+(x), F_2^+(x), T_2^-(x), I_2^-(x), F_2^-(x)\rangle : x \in X\}$

be two bipolar neutrosophic sets.

I. Then $A_1 \subseteq A_2$ if and only if
$T^+_1(x) \le T_2^+(x)$ $I^+_1(x) \le I_2^+(x)$, $F^+_1(x) \ge F_2^+(x)$,
and
$T^-_1(x) \ge T_2^-(x)$, $I^-_1(x) \ge I^-_2(x)$, $F_1^-(x) \le F_2^-(x)$
for all $x \in X$.

II. Then $A_1 = A_2$ if and only if
$T_1^+(x) = T_2^+(x)$, $I_1^+(x) = I_2^+(x)$, $F_1^+(x) = F_2^+(x)$
and
$T_1^-(x) = T_2^-(x)$, $I_1^-(x) = I_2^-(x)$, $F_1^-(x) = F_2^-(x)$
for all $x \in X$.

III. Then their union is defined as:
$(A_1 \cup A_2)(x) =$
$\left\{\max\{T_1^+(x), T_2^+(x)\}, \frac{I_1^+(x) + I_2^+(x)}{2}, \min\{F_1^+(x), F_2^+(x)\}, \min\{T_1^-(x), T_2^-(x)\}, \frac{I_1^-(x) + I_2^-(x)}{2}, \max\{F_1^-(x), F_2^-(x)\}\right\}$ fo r all $x \in X$.

IV. Then their intersection is defined as:

$(A_1 \cap A_2)(x) =$

$\left\{\min\left\{T_1^+(x), T_2^+(x)\right\}, \dfrac{I_1^+(x)+I_2^+(x)}{2}, \max\left\{F_1^+(x), F_2^+(x)\right\},\right.$

$\left.\max\left\{T_1^-(x), T_2^-(x)\right\}, \dfrac{I_1^-(x)+I_2^-(x)}{2}, \min\left\{F_1^-(x), F_2^-(x)\right\}\right\}$ for all $x \in X$.

V. Then the complement of A_1 is denoted by A_1^c and is defined by

$T_{A_1^c}^+(x) = \{1^+\} - T_{A_1}^+(x)$, $I_{A_1^c}^+(x) = \{1^+\} - I_{A_1}^+(x)$, $F_{A_1^c}^+(x) = \{1^+\} - F_{A_1}^+(x)$

and

$T_{A_1^c}^-(x) = \{1^-\} - T_{A_1}^-(x)$, $I_{A_1^c}^-(x) = \{1^-\} - I_{A_1}^-(x)$, $F_{A_1^c}^-(x) = \{1^-\} - F_{A_1}^-(x)$,

for all $x \in X$.

Definition 2.6. [7] Let

$A_1 = \left\{\left\langle x, T_1^+(x), I_1^+(x), F_1^+(x), T_1^-(x), I_1^-(x), F_1^-(x)\right\rangle : x \in X\right\}$

and

$A_2 = \left\{\left\langle x, T_2^+(x), I_2^+(x), F_2^+(x), T_2^-(x), I_2^-(x), F_2^-(x)\right\rangle : x \in X\right\}$

be two bipolar neutrosophic number. Then the operations for these numbers are defined as below;

a. $\lambda A_1 = \left\langle 1-\left(1-T_1^+\right)^\lambda, \left(I_1^+\right)^\lambda, \left(F_1^+\right)^\lambda, -\left(-T_1^-\right)^\lambda, -\left(-I_1^-\right)^\lambda, -\left(1-\left(1-\left(-F_1^-\right)\right)^\lambda\right)\right\rangle$

b. $A_1^\lambda = \left\langle \left(T_1^+\right)^\lambda, 1-\left(1-I_1^+\right)^\lambda, 1-\left(1-F_1^+\right)^\lambda, -\left(1-\left(1-\left(-T_1^-\right)\right)^\lambda\right), -\left(-I_1^-\right)^\lambda, -\left(-F_1^-\right)^\lambda\right\rangle$

c. $A_1 + A_2 = \left\langle T_1^+ + T_2^+ - T_1^+.T_2^+, I_1^+ I_2^+, F_1^+ F_2^+, -T_1^-.T_2^-, -(-I_1^- - I_2^- - I_1^-.I_2^-), -(-F_1^- - F_2^- - F_{1L}^-.F_2^-)\right\rangle$

d. $A_1 A_2 = \left\langle T_1^+ T_2^+, I_1^+ + I_2^+ - I_1^+.I_2^+, F_1^+ + F_2^+ - F_1^+.F_2^+, -(-T_1^- - T_2^- - T_1^-.T_2^-), I_1^- I_2^-, F_1^- F_2^-\right\rangle$

where $\lambda > 0$.

Definition 2.7. [7] Let

$\tilde{a} = \left\{\left\langle x, T^+(x), I^+(x), F^+(x), T^-(x), I^-(x), F^-(x)\right\rangle : x \in X\right\}$

be a bipolar neutrosophic number. Then, the score function $s(\tilde{a})$, accuracy function $a(\tilde{a})$ and certainty function $c(\tilde{a})$ of an NBN are defined as follows:

$$s(\tilde{a}) = \frac{1}{6}\left(T^+ + 1 - I^+ + 1 - F^+ + 1 + T^- - I^- - F^-\right)$$

$$a(\tilde{a}) = T^+ - F^+ + T^- - F^-$$

$$c(\tilde{a}) = T^+ - F^-$$

Definition 2.8. [7] Let

$\tilde{a}_j = \left\langle T_j^+, I_j^+, F_j^+, T_j^-, I_j^-, F_j^-\right\rangle (j = 1, 2, ..., n)$

be a family of bipolar neutrosophic numbers. Then,

a) $F_W : \Im_n \to \Im$ is called bipolar neutrosophic weighted average operator if it satisfies;

$$F_W\left(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n\right) = \sum_{j=1}^n w_j \tilde{a}_j$$

$$= \left\langle 1-\prod_{j=1}^n\left(1-T_j^+\right)^{w_j}, \prod_{j=1}^n\left(I_j^+\right)^{w_j} \prod_{j=1}^n\left(F_j^+\right)^{w_j}, -\prod_{j=1}^n(-T_j^-)^{w_j}, -\left(1-\prod_{j=1}^n\left(1-\left(-I_j^-\right)\right)^{w_j}\right), -\left(1-\prod_{j=1}^n\left(1-\left(-F_j^-\right)\right)^{w_j}\right)\right\rangle$$

where w_j is the weight of $\tilde{a}_j (j = 1, 2, ..., n)$, $w_j \in [0,1]$ and $\sum_{j=1}^n w_j = 1$.

b) $H_W : \Im_n \to \Im$ is called bipolar neutrosophic weighted geometric operator if it satisfies;

$$H_W\left(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n\right) = \prod_{j=1}^n \tilde{a}_j^{w_j}$$

$$= \left\langle \left[\prod_{j=1}^n\left(T_j^+\right)^{w_j}, 1-\prod_{j=1}^n\left(1-I_j^+\right)^{w_j}, 1-\prod_{j=1}^n\left(1-F_j^+\right)^{w_j}, -\left(1-\prod_{j=1}^n\left(1-(-T_j^-)\right)^{w_j}\right), -\prod_{j=1}^n\left(-I_{jL}^-\right)^{w_j}, -\prod_{j=1}^n\left(-F_j^-\right)^{w_j}\right\rangle$$

where w_j is the weight of $\tilde{a}_j (j = 1, 2, ..., n)$, $w_j \in [0,1]$ and $\sum_{j=1}^n w_j = 1$.

III. Interval Valued Bipolar fuzzy weighted Neutrosophic Set

In this section we give concept of IVBFWN-set and its operations. Also we give the IVBFWN-set weighted average operator and IVBFWN-set weighted geometric operator with properties to aggregate the IVBFWN-sets based on the study given in [7].

Definition 3.1. A interval valued bipolar fuzzy weighted neutrosophic set(IVBFWN-set) A in X is defined as an object of the form

$$A=\{\langle x,[T_L^+(x),T_R^+(x)],[I_L^+(x),I_R^+(x)],[F_L^+(x),F_R^+(x)],$$
$$[T_L^-(x),T_R^-(x)],[I_L^-(x),I_R^-(x)],[F_L^-(x),F_R^-(x)],p(x)\rangle : x\in X\}$$

where $T_L^+,T_R^+,I_L^+,I_R^+,F_L^+,F_R^+ : X\to[0,1]$ and $T_L^-,T_R^-,I_L^-,I_R^-,F_L^-,F_R^- : X\to[-1,0]$. Also $p: X\to[0,1]$ fuzzy weighted index of the element x in X.

Example 3.2. Let $X=\{x_1,x_2,x_3\}$. Then

$$A=\left\{\begin{array}{l}\langle x_1,[0.3,0.9],[0.1,0.8],[0.2,0.5],[-0.8,-0.7],[-0.5,-0.1],[-0.4,-0.3],0.5\rangle,\\ \langle x_2,[0.3,0.8],[0.3,0.9],[0.1,0.2],[-0.7,-0.6],[-0.6,-0.2],[-0.6,-0.2],0.7\rangle,\\ \langle x_3,[0.4,0.7],[0.5,0.7],[0.3,0.4],[-0.9,-0.5],[-0.4,-0.3],[-0.8,-0.1],0.8\rangle\end{array}\right\}$$

is a IVBFWN subset of X.

Theorem 3.3. A IVBFWN-set is the generalization of a bipolar fuzzy set and bipolar neutrosophic set.

Proof: Straightforward.

Definition 3.4. Let

$$A_1=\{x,\langle[T_{1L}^+(x),T_{1R}^+(x)],[I_{1L}^+(x),I_{1R}^+(x)],[F_{1L}^+(x),F_{1R}^+(x)],$$
$$[T_{1L}^+(x),T_{1R}^+(x)],[I_{1L}^+(x),I_{1R}^+(x)],[F_{1L}^+(x),F_{1R}^+(x)],p_1(x)\rangle : x\in X\}$$

and

$$A_2=\{x,\langle[T_{2L}^+(x),T_{2R}^+(x)],[I_{2L}^+(x),I_{2R}^+(x)],[F_{2L}^+(x),F_{2R}^+(x)],$$
$$[T_{2L}^-(x),T_{2R}^-(x)],[I_{2L}^-(x),I_{2R}^-(x)],[F_{2L}^-(x),F_{2R}^-(x)],p_2(x)\rangle : x\in X\}$$

be two IVBFWN-sets.

1. Then $A_1\subseteq A_2$ if and only if

$$T_{1L}^+(x)\le T_{2L}^+(x),\quad T_{1R}^+(x)\le T_{2R}^+(x),\quad I_{1L}^+(x)\ge I_{2L}^+(x),$$
$$I_{1R}^+(x)\ge I_{2R}^+(x),\quad F_{1L}^+(x)\ge F_{2L}^+(x),\quad F_{1R}^+(x)\ge F_{2R}^+(x),$$
$$T_{1L}^-(x)\le T_{2L}^-(x),\quad T_{1R}^-(x)\le T_{2R}^-(x),\quad I_{1L}^-(x)\ge I_{2L}^-(x),$$
$$I_{1R}^-(x)\ge I_{2R}^-(x),\quad F_{1L}^-(x)\ge F_{2L}^-(x),\quad F_{1R}^-(x)\ge F_{2R}^-(x),$$

and

$p_1(x)\le p_2(x)$

for all $x\in X$.

2. Then $A_1=A_2$ if and only if

$$T_{1L}^+(x)=T_{2L}^+(x),\quad T_{1R}^+(x)=T_{2R}^+(x),\quad I_{1L}^+(x)=I_{2L}^+(x),$$
$$I_{1R}^+(x)=I_{2R}^+(x),\quad F_{1L}^+(x)=F_{2L}^+(x),\quad F_{1R}^+(x)=F_{2R}^+(x),$$
$$T_{1L}^-(x)=T_{2L}^-(x),\quad T_{1R}^-(x)=T_{2R}^-(x),\quad I_{1L}^-(x)=I_{2L}^-(x),$$
$$I_{1R}^-(x)=I_{2R}^-(x),\quad F_{1L}^-(x)=F_{2L}^-(x),\quad F_{1R}^-(x)=F_{2R}^-(x),$$

and $p_1(x)=p_2(x)$ for all $x\in X$.

3. Then their union is defined as:

$$(A_1\cup A_2)(x)=$$
$$\left[\max\{T_{1L}^+(x),T_{2L}^+(x)\},\max\{T_{1R}^+(x),T_{2R}^+(x)\}\right],$$
$$\left[\frac{I_{1L}^+(x)+I_{2L}^+(x)}{2},\frac{I_{1R}^+(x)+I_{2R}^+(x)}{2}\right],$$
$$\left[\min\{F_{1L}^+(x),F_{2L}^+(x)\},\min\{F_{1R}^+(x),F_{2R}^+(x)\}\right],$$
$$\left[\min\{T_{1L}^-(x),T_{2L}^-(x)\},\min\{T_{1R}^-(x),T_{2R}^-(x)\}\right],$$
$$\left[\frac{I_{1L}^-(x)+I_{2L}^-(x)}{2},\frac{I_{1R}^-(x)+I_{2R}^-(x)}{2}\right],$$
$$\left[\max\{F_{1L}^-(x),F_{2L}^-(x)\},\max\{F_{1R}^-(x),F_{2R}^-(x)\}\right]$$

for all $x\in X$.

4. Then their intersection is defined as:

$$(A_1\cap A_2)(x)=$$
$$\left[\min\{T_{1L}^+(x),T_{2L}^+(x)\},\min\{T_{1R}^+(x),T_{2R}^+(x)\}\right],$$
$$\left[\frac{I_{1L}^+(x)+I_{2L}^+(x)}{2},\frac{I_{1R}^+(x)+I_{2R}^+(x)}{2}\right],$$
$$\left[\max\{F_{1L}^+(x),F_{2L}^+(x)\},\max\{F_{1R}^+(x),F_{2R}^+(x)\}\right],$$
$$\left[\max\{T_{1L}^-(x),T_{2L}^-(x)\},\max\{T_{1R}^-(x),T_{2R}^-(x)\}\right],$$
$$\left[\frac{I_{1L}^-(x)+I_{2L}^-(x)}{2},\frac{I_{1R}^-(x)+I_{2R}^-(x)}{2}\right],$$
$$\left[\min\{F_{1L}^-(x),F_{2L}^-(x)\},\min\{F_{1R}^-(x),F_{2R}^-(x)\}\right]$$

for all $x\in X$.

5. Then the complement of A_1 is denoted by A_1^c, is defined by

$$A=\{[F_L^+(x),F_R^+(x)],[1-I_L^+(x),1-I_R^+(x)],\langle x,[T_L^+(x),T_R^+(x)],$$
$$[F_L^-(x)F_R^-(x)],[1-I_L^-(x),1-I_R^-(x)],[T_L^-(x),T_R^-(x)],1-p(x)\rangle : x\in X\}$$

Example 3.5. Let $X=\{x_1,x_2,x_3\}$. Then

$$A_1=\left\{\begin{array}{l}\langle x_1,[0.3,0.9],[0.1,0.8],[0.2,0.5],[-0.8,-0.7],[-0.5,-0.1],[-0.4,-0.3]\rangle,\\ \langle x_2,[0.3,0.8],[0.3,0.9],[0.1,0.2],[-0.7,-0.6],[-0.6,-0.2],[-0.6,-0.2]\rangle,\\ \langle x_3,[0.4,0.7],[0.5,0.7],[0.3,0.4],[-0.9,-0.5],[-0.4,-0.3],[-0.8,-0.1]\rangle\end{array}\right\}$$

and

$$A_2=\left\{\begin{array}{l}\langle x_1,[0.2,0.8],[0.3,0.6],[0.3,0.6],[-0.3,-0.2],[-0.6,-0.2],[-0.5,-0.4]\rangle,\\ \langle x_2,[0.4,0.7],[0.5,0.7],[0.2,0.3],[-0.2,-0.1],[-0.8,-0.3],[-0.9,-0.8]\rangle,\\ \langle x_3,[0.5,0.6],[0.4,0.5],[0.1,0.4],[-0.4,-0.2],[-0.9,-0.5],[-0.7,-0.6]\rangle\end{array}\right\}$$

are two IVBFWN-sets in X.

Then their union is given as follows:

$$A_1 \cup A_2 = \left\{ \begin{array}{l} \langle x_1,[0.3,0.9],[0.1,0.6],[0.2,0.5],[-0.8,-0.7],[-0.5,-0.1],[-0.4,-0.3]\rangle, \\ \langle x_2,[0.4,0.8],[0.3,0.7],[0.1,0.2],[-0.7,-0.6],[-0.6,-0.2],[-0.6,-0.2]\rangle, \\ \langle x_3,[0.5,0.7],[0.4,0.5],[0.1,0.4],[-0.9,-0.5],[-0.4,-0.3],[-0.7,-0.1]\rangle \end{array} \right\}$$

Then their intersection is given as follows:

$$A_1 \cap A_2 = \left\{ \begin{array}{l} \langle x_1,[0.2,0.8],[0.3,0.8],[0.3,0.6],[-0.3,-0.2],[-0.6,-0.2],[-0.5,-0.4]\rangle, \\ \langle x_2,[0.3,0.7],[0.5,0.9],[0.2,0.3],[-0.2,-0.1],[-0.8,-0.3],[-0.9,-0.8]\rangle, \\ \langle x_3,[0.4,0.6],[0.5,0.7],[0.3,0.4],[-0.4,-0.2],[-0.9,-0.5],[-0.8,-0.6]\rangle \end{array} \right\}$$

Definition 3.6. Let

$$A_1 = \Big\langle \big[T^+_{1L}(x), T^+_{1R}(x)\big], \big[I^+_{1L}(x), I^+_{1R}(x)\big], \big[F^+_{1L}(x), F^+_{1R}(x)\big], \big[T^+_{1L}(x), T^+_{1R}(x)\big], \big[I^+_{1L}(x), I^+_{1R}(x)\big], \big[F^+_{1L}(x), F^+_{1R}(x)\big], p_1(x)\Big\rangle$$

and

$$A_2 = \Big\langle \big[T^+_{2L}(x), T^+_{2R}(x)\big], \big[I^+_{2L}(x), I^+_{2R}(x)\big], \big[F^+_{2L}(x), F^+_{2R}(x)\big], \big[T^-_{2L}(x), T^-_{2R}(x)\big], \big[I^-_{2L}(x), I^-_{2R}(x)\big], \big[F^-_{2L}(x), F^-_{2R}(x)\big], p_2(x)\Big\rangle$$

be two IVBFWN-numbers.

. Then the operations for IVBFWN-numbers are defined as below;

i. $$\lambda A_1 = \Big\langle \Big[1-\big(1-T^+_L\big)^\lambda, 1-\big(1-T^+_R\big)^\lambda\Big], \Big[\big(I^+_L\big)^\lambda, \big(I^+_R\big)^\lambda\Big], \Big[\big(F^+_L\big)^\lambda, \big(F^+_R\big)^\lambda\Big], \Big[-\big(-T^-_L\big)^\lambda, -\big(-T^-_R\big)^\lambda\Big], \Big[-\big(-I^-_L\big)^\lambda, -\big(-I^-_R\big)^\lambda\Big], \Big[-\Big(1-\big(1-\big(-F^-_L\big)\big)^\lambda\Big), -\Big(1-\big(1-\big(-F^-_R\big)\big)^\lambda\Big)\Big]\Big\rangle$$

ii. $$A_1^\lambda = \Big\langle \Big[\big(T^+_L\big)^\lambda, \big(T^+_R\big)^\lambda\Big], \Big[1-\big(1-I^+_L\big)^\lambda, 1-\big(1-I^+_R\big)^\lambda\Big], \Big[1-\big(1-F^+_L\big)^\lambda, 1-\big(1-F^+_R\big)^\lambda\Big], \Big[-\Big(1-\big(1-\big(-T^-_L\big)\big)^\lambda\Big), -\Big(1-\big(1-\big(-T^-_R\big)\big)^\lambda\Big)\Big], \Big[-\big(-I^-_L\big)^\lambda, -\big(-I^-_R\big)^\lambda\Big], \Big[-\big(-F^-_L\big)^\lambda, -\big(-F^-_R\big)^\lambda\Big]\Big\rangle$$

iii. $$A_1 + A_2 = \Big\langle \big[T^+_{1L} + T^+_{2L} - T^+_{1L}.T^+_{2L}, T^+_{1R} + T^+_{2R} - T^+_{1R}.T^+_{2R}\big], \big[I^+_{1L}I^+_{2L}, I^+_{1R}I^+_{2R}\big], \big[F^+_{1L}F^+_{2L}, F^+_{1R}F^+_{2R}\big], \big[-T^-_{1L}.T^-_{2L}, -T^-_{1R}.T^-_{2R}\big], \big[-(-I^-_{1L} - I^-_{2L} - I^-_{1L}.I^-_{2L}), -(-I^-_{1R} - I^-_{2R} - I^-_{1R}.I^-_{2R})\big], \big[-(-F^-_{1L} - F^-_{2L} - F^-_{1L}.F^-_{2L}), -(-F^-_{1R} - F^-_{2R} - F^-_{1R}.F^-_{2R})\big]\Big\rangle$$

iv. $$A_1 A_2 = \Big\langle \big[T^+_{1L}T^+_{2L}, T^+_{1R}T^+_{2R}\big], \big[I^+_{1L} + I^+_{2L} - I^+_{1L}.I^+_{2L}, I^+_{1R} + I^+_{2R} - I^+_{1R}.I^+_{2R}\big], \big[F^+_{1L} + F^+_{2L} - F^+_{1L}.F^+_{2L}, F^+_{1R} + F^+_{2R} - F^+_{1R}.F^+_{2R}\big], \big[-(-T^-_{1L} - T^-_{2L} - T^-_{1L}.T^-_{2L}), -(-T^-_{1R} - T^-_{2R} - T^-_{1R}.T^-_{2R})\big], \big[I^-_{1L}I^-_{2L}, I^-_{1R}I^-_{2R}\big], \big[F^-_{1L}F^-_{2L}, F^-_{1R}F^-_{2R}\big]\Big\rangle$$

where $\lambda > 0$.

Definition 3.7. Let

$$\tilde{a} = \Big\langle \big[T^+_L, T^+_R\big], \big[I^+_L, I^+_R\big], \big[F^+_L, F^+_R\big], \big[T^-_L, T^-_R\big], \big[I^-_L, I^-_R\big], \big[F^-_L, F^-_R\big]\Big\rangle$$

be a IVBFWN-number. Then, the score function $S(\tilde{a})$ accuracy function $A(\tilde{a})$ and certainty function $C(\tilde{a})$ of an NBN are defined as follows:

$$S(\tilde{a}) = \frac{p(x)}{12}\Big(T^+_L + T^+_R + 1 - I^+_L + 1 - I^+_R + 1 - F^+_L + 1 - F^+_R + 1 + T^-_L + 1 + T^-_R - I^-_L - I^-_R - F^-_L - F^-_R\Big)$$

$$A(\tilde{a}) = p(x)(T^+_L + T^+_R - F^+_L - F^+_R + T^-_L + T^-_R - F^-_L - F^-_R)$$

$$C(\tilde{a}) = p(x)(T^+_L + T^+_R - F^-_L - F^-_R)$$

The comparison method can be defined as follows:

i. If $S(\tilde{a}_1) > S(\tilde{a}_2)$, then $\tilde{a}_1$ is greater than $\tilde{a}_2$, that is, $\tilde{a}_1$ is superior to $\tilde{a}_2$, denoted by $\tilde{a}_1 > \tilde{a}_2$;

ii. If $S(\tilde{a}_1) = S(\tilde{a}_2)$, and $A(\tilde{a}_1) > A(\tilde{a}_2)$, then $\tilde{a}_1$ is greater than $\tilde{a}_2$, that is, $\tilde{a}_1$ is superior to $\tilde{a}_2$, denoted by $\tilde{a}_1 < \tilde{a}_2$;

iii. If $S(\tilde{a}_1) = S(\tilde{a}_2)$, $A(\tilde{a}_1) = A(\tilde{a}_2)$, and $C(\tilde{a}_1) > C(\tilde{a}_2)$, then $\tilde{a}_1$ is greater than $\tilde{a}_2$, that is, $\tilde{a}_1$ is superior to $\tilde{a}_2$, denoted by $\tilde{a}_1 > \tilde{a}_2$;

iv. If $S(\tilde{a}_1) = S(\tilde{a}_2)$, $A(\tilde{a}_1) > A(\tilde{a}_2)$, and $C(\tilde{a}_1) = C(\tilde{a}_2)$, then $\tilde{a}_1$ is equal to $\tilde{a}_2$, that is, $\tilde{a}_1$ is indifferent to $\tilde{a}_2$, denoted by $\tilde{a}_1 > \tilde{a}_2$;

Definition 3.8. Let

$$\tilde{a}_j = \Big\langle \big[T^+_{jL}, T^+_{jR}\big], \big[I^+_{jL}, I^+_{jR}\big], \big[F^+_{jL}, F^+_{jR}\big], \big[T^-_{jL}, T^-_{jR}\big], \big[I^-_{jL}, I^-_{jR}\big], \big[F^-_{jL}, F^-_{jR}\big], p_j\Big\rangle (j = 1, 2, ..., n)$$

be a family of IVBFWN-numbers. A mapping $A_p : \mathfrak{I}_n \to \mathfrak{I}$ is called IVBFWN-weighted average operator if it satisfies

$$A_p(\tilde{a}_1, \tilde{a}_2, ..., \tilde{a}_n) = \sum_{j=1}^{n} p_j \tilde{a}_j$$

$$= \Big\langle \Big[1 - \prod_{j=1}^{n}\big(1 - T^+_{jL}\big)^{w_j}, 1 - \prod_{j=1}^{n}\big(1 - T^+_{jR}\big)^{w_j}\Big],$$

$$\left[\prod_{j=1}^{n}\left(I_{jL}^{+}\right)^{w_j},\prod_{j=1}^{n}\left(I_{jR}^{+}\right)^{w_j}\right],\left[\prod_{j=1}^{n}\left(F_{jL}^{+}\right)^{w_j},\prod_{j=1}^{n}\left(F_{jR}^{+}\right)^{w_j}\right],$$

$$\left[-\prod_{j=1}^{n}(-T_{jL}^{-})^{w_j},-\prod_{j=1}^{n}(-T_{jR}^{-})^{w_j}\right],$$

$$\left[-\left(1-\prod_{j=1}^{n}\left(1-\left(-I_{jL}^{-}\right)\right)^{w_j}\right),-\left(1-\prod_{j=1}^{n}\left(1-\left(-I_{jR}^{-}\right)\right)^{w_j}\right)\right],$$

$$\left[-\left(1-\prod_{j=1}^{n}\left(1-\left(-F_{jL}^{-}\right)\right)^{w_j}\right),-\left(1-\prod_{j=1}^{n}\left(1-\left(-F_{jR}^{-}\right)\right)^{w_j}\right)\right],p^{n}{}_{j}\right\rangle$$

where w_j is the weight of $\tilde{a}_j\,(j=1,2,...,n)$, $w_j\in[0,1]$ and $\sum_{j=1}^{n}w_j=1$.

Theorem 3.9. Let

$\tilde{a}_j=\left\langle\left[T_{jL}^{+},T_{jR}^{+}\right],\left[I_{jL}^{+},I_{jR}^{+}\right],\left[F_{jL}^{+},F_{jR}^{+}\right],\left[T_{jL}^{-},T_{jR}^{-}\right],\left[I_{jL}^{-},I_{jR}^{-}\right],\left[F_{jL}^{-},F_{jR}^{-}\right]\right\rangle (j=1,2,...,n)$

be a family of IVBFWN-numbers. Then,

i. If $\tilde{a}_j=\tilde{a}$ for all $j=1,2,...,n$ then,

$A_W\left(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n\right)=\tilde{a}$

ii. $\min_{j=1,2,...,n}\tilde{a}_j\le A_W\left(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n\right)\le\max_{j=1,2,...,n}\tilde{a}_j$

iii. If $\tilde{a}_j=\tilde{a}_j^{*}$ for all $j=1,2,...,n$ then,

$A_W\left(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n\right)\le A_W\left(\tilde{a}_1^{*},\tilde{a}_2^{*},...,\tilde{a}_n^{*}\right)$

Definition 3.10. Let

$\tilde{a}_j=\left\langle\left[T_{jL}^{+},T_{jR}^{+}\right],\left[I_{jL}^{+},I_{jR}^{+}\right],\left[F_{jL}^{+},F_{jR}^{+}\right],\left[T_{jL}^{-},T_{jR}^{-}\right],\left[I_{jL}^{-},I_{jR}^{-}\right],\left[F_{jL}^{-},F_{jR}^{-}\right]\right\rangle (j=1,2,...,n)$

be a family of IVBFWN-numbers. A mapping $G_W:\Im_n\to\Im$ is called IVBFWN-weighted geometric operator if it satisfies

$$G_W\left(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n\right)=\prod_{j=1}^{n}\tilde{a}_j^{\,w_j}$$

$$=\left\langle\left[\prod_{j=1}^{n}\left(T_{jL}^{+}\right)^{w_j},\prod_{j=1}^{n}\left(T_{jR}^{+}\right)^{w_j}\right],\right.$$

$$\left[1-\prod_{j=1}^{n}\left(1-I_{jL}^{+}\right)^{w_j},1-\prod_{j=1}^{n}\left(1-I_{jR}^{+}\right)^{w_j}\right],$$

$$\left[1-\prod_{j=1}^{n}\left(1-F_{jL}^{+}\right)^{w_j},1-\prod_{j=1}^{n}\left(1-F_{jR}^{+}\right)^{w_j}\right],$$

$$\left[-\left(1-\prod_{j=1}^{n}\left(1-(-T_{jL}^{-})\right)^{w_j}\right),-\left(1-\prod_{j=1}^{n}\left(1-(-T_{jR}^{-})\right)^{w_j}\right)\right],$$

$$\left[-\prod_{j=1}^{n}\left(-I_{jL}^{-}\right)^{w_j},-\prod_{j=1}^{n}\left(-I_{jR}^{-}\right)^{w_j}\right],$$

$$\left.\left[-\prod_{j=1}^{n}\left(-F_{jL}^{-}\right)^{w_j},-\prod_{j=1}^{n}\left(-F_{jR}^{-}\right)^{w_j}\right]\right\rangle$$

where w_j is the weight of $\tilde{a}_j\,(j=1,2,...,n)$, $w_j\in[0,1]$ and $\sum_{j=1}^{n}w_j=1$.

Theorem 3.11. Let

$\tilde{a}_j=\left\langle\left[T_{jL}^{+},T_{jR}^{+}\right],\left[I_{jL}^{+},I_{jR}^{+}\right],\left[F_{jL}^{+},F_{jR}^{+}\right],\left[T_{jL}^{-},T_{jR}^{-}\right],\left[I_{jL}^{-},I_{jR}^{-}\right],\left[F_{jL}^{-},F_{jR}^{-}\right]\right\rangle (j=1,2,...,n)$

be a family of IVBFWN-numbers. Then,

i. If $\tilde{a}_j=\tilde{a}$ for all $j=1,2,...,n$ then,

$G_W\left(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n\right)=\tilde{a}$

ii. $\min_{j=1,2,...,n}\tilde{a}_j\le G_W\left(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n\right)\le\max_{j=1,2,...,n}\tilde{a}_j$

iii. If $\tilde{a}_j=\tilde{a}_j^{*}$ for all $j=1,2,...,n$ then,

$G_W\left(\tilde{a}_1,\tilde{a}_2,...,\tilde{a}_n\right)\le G_W\left(\tilde{a}_1^{*},\tilde{a}_2^{*},...,\tilde{a}_n^{*}\right)$

Note that the aggregation results are still NBNs

IV. NBN- Decision Making Method

In this section, we develop an approach based on the A_W (or G_W) operator and the above ranking method to deal with multiple criteria decision making problems with IVBFWN-information.

Suppose that $A=\{A_1,A_2,...,A_m\}$ and $C=\{C_1,C_2,...,C_n\}$ is the set of alternatives and criterions or attributes, respectively. Let $w=\left(w_1,w_2,...,w_n\right)^T$ be the weight vector of attributes, such that $\sum_{j=1}^{n}w_j=1$, $w_j\ge 0\left(j=1,2,...,n\right)$ and w_j refers to the weight of attribute C_j. An alternative on criterions is evaluated by the decision maker, and the evaluation values are represented by the form of IVBFWN-numbers. Assume that

$\left(\tilde{a}_{ij}\right)_{m\times n}=\left(\left\langle\left[T_{ijL}^{+},T_{ijR}^{+}\right],\left[I_{ijL}^{+},I_{ijR}^{+}\right],\left[F_{ijL}^{+},F_{ijR}^{+}\right],\left[T_{ijL}^{-},T_{ijR}^{-}\right],\left[I_{ijL}^{-},I_{ijR}^{-}\right],\left[F_{ijL}^{-},F_{ijR}^{-}\right]\right\rangle\right)_{m\times n}$

is the decision matrix provided by the decision maker; $\tilde{a}_{ij}$ is a IVBFWN-number for alternative A_i. associated with the criterions C_j. We have the conditions $T^+_{ijL}, T^+_{ijR}, I^+_{ijL}, I^+_{ijR}, F^+_{ijL}, F^+_{ijR}$, and $T^-_{ijL}, T^-_{ijR}, I^-_{ijL}, I^-_{ijR}, F^-_{ijL}, F^-_{ijR} \in [0,1]$ such that

$$0 \le T^+_{ijL} + T^+_{ijR} + I^+_{ijL} + I^+_{ijR} + F^+_{ijL} + F^+_{ijR} - T^-_{ijL} - T^-_{ijR} - I^-_{ijL} - I^-_{ijR} - F^-_{ijL} - F^-_{ijR} \le 12$$

for $(i = 1, 2, ..., m)$ and $(j = 1, 2, ..., n)$.

Now, we can develop an algorithm as follows;

Algorithm

Step 1. Construct the decision matrix provided by the decision maker as;

$$\left(\tilde{a}_{ij}\right)_{m\times n} = \left(\left\langle \left[T^+_{ijL}, T^+_{ijR}\right], \left[I^+_{ijL}, I^+_{ijR}\right], \left[F^+_{ijL}, F^+_{ijR}\right], \left[T^-_{ijL}, T^-_{ijR}\right], \left[I^-_{ijL}, I^-_{ijR}\right], \left[F^-_{ijL}, F^-_{ijR}\right] \right\rangle\right)_{m\times n}$$

Step 2. Compute $\tilde{a}_i = A_W(\tilde{a}_{i1}, \tilde{a}_{i2}, ..., \tilde{a}_{in})$ $(or\ G_W(\tilde{a}_{i1}, \tilde{a}_{i2}, ..., \tilde{a}_{in}))$ for each $\tilde{a}_i\ (i = 1, 2, ..., m)$

Step 3. Calculate the score values of $S(\tilde{a}_i)$ for the $(i = 1, 2, ..., m)$ collective overall IVBFWN-number of $\tilde{a}_i\ (i = 1, 2, ..., m)$

Step 4. Rank all the software systems of $\tilde{a}_i\ (i = 1, 2, ..., m)$ according to the score values

Now, we give a numerical example as follows;

Example 4.1. Let us consider decision making problem adapted from Ye [16]. There is an investment company, which wants to invest a sum of money in the best option. There is a panel with the set of the four alternatives is denoted by C_1 = car company C_2 = food company, C_3 = computer company, C_4 = arms company to invest the money. The investment company must take a decision according to the set of the four attributes is denoted by A_1 = risk, A_2 = growth, A_3 = environmental impact, A_4 = performance. Also, the weight vector of the attributes $C_j\ (j = 1, 2, 3, 4)$ is $w = (0.24, 0.26, 0.26, 024)^T$. Then the according to this algorithm, we have,

Step1. Construct the decision matrix provided by the customer as;

Table 1: Decision matrix given by customer

	C_1
A_1	$\langle[0.5,0.6],[0.2,0.5],[0.1,0.7],[-0.2,-0.1],[-0.6,-0.2],[-0.4,-0.3]\rangle$
A_2	$\langle[0.1,0.2],[0.3,0.8],[0.2,0.4],[-0.5,-0.2],[-0.9,-0.3],[-0.6,-0.1]\rangle$
A_3	$\langle[0.4,0.8],[0.4,0.6],[0.4,0.6],[-0.3-0.2],[-0.7,-0.5],[-0.5,-0.4]\rangle$
A_4	$\langle[0.6,0.9],[0.3,0.8],[0.5,0.6],[-0.8,-0.5],[-0.5,-0.1],[-0.2,-0.1]\rangle$

	C_2
A_1	$\langle[0.3,0.9],[0.1,0.8],[0.2,0.5],[-0.8,-0.7],[-0.5,-0.1],[-0.4,-0.1]\rangle$
A_2	$\langle[0.2,0.8],[0.1,0.4],[0.3,0.4],[-0.5,-0.1],[-0.3,-0.1],[-0.9,-0.4]\rangle$
A_3	$\langle[0.1,0.6],[0.3,0.9],[0.3,0.5],[-0.8,-0.7],[-0.4,-0.3],[-0.7,-0.6]\rangle$
A_4	$\langle[0.1,0.2],[0.8,0.9],[0.2,0.7],[-0.5,-0.4],[-0.6,-0.3],[-0.5,-0.3]\rangle$

	C_3
A_1	$\langle[0.1,0.6],[0.1,0.5],[0.1,0.4],[-0.5,-0.2],[-0.7,-0.3],[-0.4,-0.2]\rangle$
A_2	$\langle[0.3,0.4],[0.1,0.6],[0.5,0.7],[-0.5,-0.1],[-0.8,-0.7],[-0.9,-0.8]\rangle$
A_3	$\langle[0.3,0.9],[0.2,0.8],[0.2,0.3],[-0.5,-0.4],[-0.6,-0.5],[-0.7,-0.6]\rangle$
A_4	$\langle[0.2,0.7],[0.5,0.8],[0.8,0.9],[-0.9,-0.8],[-0.8,-0.5],[-0.5,-0.2]\rangle$

	C_4
A_1	$\langle[0.6,0.8],[0.4,0.6],[0.1,0.3],[-0.4,-0.3],[-0.6,-0.3],[-0.7,-0.5]\rangle$
A_2	$\langle[0.3,0.8],[0.3,0.9],[0.1,0.2],[-0.8,-0.6],[-0.6,-0.4],[-0.4,-0.2]\rangle$
A_3	$\langle[0.7,0.9],[0.1,0.4],[0.2,0.6],[-0.7,-0.6],[-0.9,-0.5],[-0.3,-0.2]\rangle$
A_4	$\langle[0.4,0.6],[0.3,0.5],[0.1,0.7],[-0.3,-0.1],[-0.6,-0.5],[-0.7,-0.3]\rangle$

Step 2. Compute $\tilde{a}_i = A_W(\tilde{a}_{i1}, \tilde{a}_{i2}, \tilde{a}_{i3}, \tilde{a}_{i4})$ for each $(i = 1, 2, 3, 4)$ as;

$\tilde{a}_1$	$\langle[0.4,0.8],[0.2,0.6],[0.1,0.5],[-0.4,-0.3],[-0.6,-0.2],[-0.5,-0.3]\rangle$
$\tilde{a}_2$	$\langle[0.2,0.6],[0.2,0.6],[0.2,0.4],[-0.6,-0.2],[-0.7,-0.4],[-0.8,-0.5]\rangle$
$\tilde{a}_3$	$\langle[0.4,0.8],[0.2,0.7],[0.3,0.5],[-0.5,-0.4],[-0.7,-0.5],[-0.6,-0.5]\rangle$
$\tilde{a}_4$	$\langle[0.3,0.7],[0.4,0.7],[0.3,0.7],[-0.6,-0.4],[-0.6,-0.4],[-0.5,-0.2]\rangle$

Step 3. Calculate the score values of $S(\tilde{a}_i)(i = 1, 2, 3, 4)$ for the collective overall IVBFWN-number of $\tilde{a}_i\ (i = 1, 2, ..., m)$ as;

$$S(\tilde{a}_1) = 0.56 \quad S(\tilde{a}_2) = 0.59 \quad S(\tilde{a}_3) = 0.57 \quad S(\tilde{a}_4) = 0.47$$

Step 4. Rank all the software systems of $A_i\ (i = 1, 2, 3, 4)$ according to the score values as;

$A_2 \succ A_3 \succ A_1 \succ A_4$

and thus A_2 is the most desirable alternative.

Conclusion

This paper presented an interval-valued bipolar neutrosophic set and its score, certainty and accuracy functions. In the future, we shall further study more aggregation operators for interval-valued bipolar neutrosophic set and apply them to solve practical applications in group decision making, expert system, information fusion system, game theory, and so on.

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Neutrosophic-simplified-TOPSIS. Multi-Criteria Decision-Making using combined Simplified-TOPSIS method and Neutrosophics

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Abstract—**In this paper, (1) one simplified the standard TOPSIS to new Multi-Criteria Decision-Making (MCDM) called Simplified-TOPSIS. Simplified-TOPSIS gives the same results and simplifies the calculation of the classical TOPSIS. An example is presented distinctions between Simplified-TOPSIS and classical TOPSIS are underlined. (2) extend the new Simplified-TOPSIS method to Neutrosophic-simplified-TOPSIS using single valued Neutrosophic information. An example showing the interest of Neutrosophic-simplified-TOPSIS to manipulate the uncertainty linked to information presented in Multi-Criteria Decision-Making.**

Keywords—Simplified TOPSIS; Neutrosophic; MCDM; Neutrosophic-simplified-TOPSIS

I. INTRODUCTION

Standard TOPSIS, the Technique for Order Preference by Similarity to Ideal Solution method is a multi-criteria decision-making approach, was introduced by Hwang and Yoon [1]. The classical TOPSIS is one of sophisticated MCDM for solving problems with respect to crisp numbers, often involving complicated steps of calculation algorithms that are difficult to learn and apply.

In the real MCDM problems, the attribute values are always be expressed with imperfect information, however, decision-makers may prefer to use an easy, simple technique and give same result rather than complex algorithm. The objective of this paper, we present, firstly, simplified-TOPSIS, a new MCDM method that simplifies the calculation and gives the same results of traditional TOPSIS. Secondly, we introduce a hybrid method to resolve real MCDM problems with imperfect information based on Neutrosophic and simplified-TOPSIS method (Neutrosophic-simplified-TOPSIS).

Smarandache [2,3] proposed a generalization of the Intuitionistic Fuzzy Set (IFS), called Neutrosophic Set (NS) which based on three values (truth, indeterminacy and falsity) and able to handle incomplete information (such as uncertainty, imprecise, incomplete and inconsistent information)[4].

Wang and Smarandache [5] defined single valued Neutrosophic Set (SVNS). Broumi and Smarandache [4,6,7] offered different operators such as distance and similarity measures over the single valued Neutrosophic Set and their basic properties were studied.

Mumtaz and Smarandache [8] introduced complex Neutrosophic Set. Mumtaz et al. [9] proposed and applied the theory of Neutrosophic cubic Sets in pattern recognition area.

Bahramloo and Hoseini [10] used MCDM method in Intuitionist Fuzzy Sets, which extended by Smarandache [2] to Neutrosophic Set, for raking alternatives.

Biswas [11] summarized the definition given by Wang and Smarandache [5] of single valued Neutrosophic Set as well as the definition of some aggregation operators such as aggregated single valued Neutrosophic, weighted Neutrosophic to solve MCDM problems using extended TOPSIS.

Broumi [7] studied multiple attribute decision making by using interval Neutrosophic uncertain linguistic variables.

Peng [12] also developed a Multi-criteria decision making method based on aggregation operators and TOPSIS in multi hesitant fuzzy environment. Furthermore, Deli et al. [13] applied bipolar Neutrosophic Sets on MCDM problems.

The paper is organized as follows. In the next section we present TOPSIS method. Section 3 will focus on the proposed method Simplified-TOPSIS. Afterwards, the Neutrosophic-TOPSIS in section 4. In section 5 a Neutrosophic-simplified-TOPSIS is introduced and it is shown how it can be applied for ranking preferences. In the final section, conclusions are drawn.

II. TOPSIS METHOD

Let us assume that $\mathrm{C} = \{\mathrm{C}_1, \mathrm{C}_2, \cdots, \mathrm{C}_\mathrm{n}\}$ is a set of Criteria, withn ≥ 2, $\mathrm{A} = \{\mathrm{A}_1, \mathrm{A}_2, \cdots, \mathrm{A}_\mathrm{n}\}$ is the set of Preferences (Alternatives), withm ≥ 1, $\mathrm{a_{ij}}$ the score of preferencei with respect to criterion j, and let ω_i weight of criteria $\mathrm{C_i}$.

Using $\mathrm{a_{ij}}$we construct the decision matrix denoted by

$$D = (a_{ij})_{m\times n} = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}$$

TOPSIS method summarizes as follow :

Step 1: The normalized decision matrix is obtained, which is given here with r_{ij}.

$$r_{ij} = a_{ij} / \left[\sum_{i=1}^{m} a_{ij}^2 \right]^{0.5} ;\ j = 1, 2, \cdots, n;\ i = 1, 2 \cdots, m$$

Step 2: Obtain the weighted normalized decision matrix v_{ij}:

Multiply each column of the normalized decision matrix by its associated weight.

$$v_{ij} = w_j r_{ij};\ j = 1, 2, \cdots, n;\ i = 1, 2 \cdots, m$$

Step 3: Determine the ideal and negative ideal solutions.

$$A^+ = (v_1^+, v_2^+, \cdots, v_n^+) = \left\{ \begin{array}{l} \left(max_i \left\{ v_{ij} \mid j \in B \right\} \right), \\ \left(min_i \left\{ v_{ij} \mid j \in C \right\} \right) \end{array} \right\}$$

$$A^- = (v_1^-, v_2^-, \cdots, v_n^-) = \left\{ \begin{array}{l} \left(min_i \left\{ v_{ij} \mid j \in B \right\} \right), \\ \left(max_i \left\{ v_{ij} \mid j \in C \right\} \right) \end{array} \right\}$$

Where sets B and C are associated with the benefit and cost attribute sets, respectively.

Step 4: Calculate the separation measures for each alternative from the positive (negative) ideal solution.

The separation from the positive ideal alternative is

$$S_i^+ = \left\{ \sum_{j=1}^{n} (v_{ij} - v_j^+)^2 \right\}^{0.5} ;\ i = 1, 2 \cdots, m$$

Similarly, the separation from the negative ideal alternative is

$$S_i^- = \left\{ \sum_{j=1}^{n} (v_{ij} - v_j^-)^2 \right\}^{0.5} ;\ i = 1, 2 \cdots, m$$

Step 5: The relative closeness to the ideal solution of each alternative is calculated as.

$$T_i = \frac{S_i^-}{(S_i^+ + S_i^-)};\ i = 1, 2 \cdots, m$$

A set of alternatives can now be ranked according to the descending order of the value of T_i.

A. Numerical example

In the examples below we used TOPSIS to rank the four alternatives.

The table (Table I) below contains the weights of criteria (three criteria C_1, C_2 and C_3) and the decision matrix summarized by the score of preference A_i (A_1, A_2, A_3 and A_4) with respect to criterion C_i.

TABLE I. DECISION MATRIX

a_{ij}	C_1	C_2	C_3
ω_i	12/16	3/16	1/16
A_1	*7*	*9*	*9*
A_2	*8*	*7*	*8*
A_3	*9*	*6*	*8*
A_4	*6*	*7*	*8*

Calculate $\sum_{i=1}^{n} a_{ij}$ for each column, we get (Table II).

TABLE II. MULTIPLE DECISION MATRIX

a_{ij}^2	C_1	C_2	C_3
ω_i	12/16	3/16	1/16
A_1	49	81	81
A_2	64	49	64
A_3	81	36	64
A_4	36	49	64
$\sum_{i=1}^{n} a_{ij}$	230	215	273

Divide each column by $(\sum_{i=1}^{n} a_{ij}^2)^{1/2}$ to get r_{ij} (Table III).

TABLE III. NORMALIZED DECISION MATRIX

r_{ij}	C_1	C_2	C_3
ω_i	12/16	3/16	1/16
A_1	0.4616	0.6138	0.5447
A_2	0.5275	0.4774	0.4842
A_3	0.5934	0.4092	0.4842
A_4	0.3956	0.4774	0.4842
$\sum_{i=1}^{n} a_{ij}$	230	215	273

Multiply each column by w_j to get v_{ij} (Table IV).

TABLE IV. WEIGHTED DECISION MATRIX

v_{ij}	C_1	C_2	C_3
ω_i	12/16	3/16	1/16
A_1	0.3462	0.1151	0.0340
A_2	0.3956	0.0895	0.0303
A_3	0.4451	0.0767	0.0303
A_4	0.2967	0.0895	0.0303
v_{max}	0.4451	0.1151	0.0340
v_{min}	0.2967	0.0767	0.0303

The distance values from the positive and negative ideal solution and the final rankings for decision matrix are showed in Table V.

TABLE V. DISTANCE MEASURE AND RANKING COEFFICIENT

Alternative	S_i^+	S_i^-	T_i
A_1	0.0989	0.0627	0.3880
A_2	0.0558	0.0997	0.6412
A_3	0.0385	0.1484	**0.7938**
A_4	0.1506	0.0128	0.0783

According to values of ranking measure coefficients, the Table V indicates that better alternative is A_3 and preferences are classified as $A_3 > A_1 > A_4 > A_2$.

III. SIMPLIFIED-TOPSIS METHOD (OUR PROPOSED METHOD)

Let consider $C = \{C_1, C_2, \cdots, C_n\}$ is a set of Criteria, with $n \geq 2$, $A = \{A_1, A_2, \cdots, A_n\}$ is the set of Preferences (Alternatives), with $m \geq 1$, a_{ij} the score of preference i with respect to criterion j, and let ω_i weight of criteria C_i.

$$D = (a_{ij})_{m\times n} = \begin{pmatrix} a_{11} & \cdots & a_{1m} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nm} \end{pmatrix}$$

Our proposed MCDM method called Simplified-TOPSIS can be described in following steps:

Step 1: Calculate weighted decision matrix v_{ij}.

Multiply each column of the normalized decision matrix by its associated weight

$$v_{ij} = w_j a_{ij};\ j = 1, 2, \cdots, n; i = 1, 2 \cdots, m$$

In our method we have not normalized the decision matrix (step1 of classical TOPSIS (section II)), but we calculate directly the weighted decision matrix v_{ij} by multiplying a_{ij} with ω_j.

Step 2: Determine the maximum (largest) ideal solution (LIS) and minimum (smallest) ideal solution (SIS).

$$A^+ = (v_1^+, v_2^+, \cdots, v_m^+) = \left(max_i \left\{v_{ij} \mid j = 1, 2, \cdots, n\right\}\right)$$

$$A^+ = (v_1^-, v_2^-, \cdots, v_m^-) = \left(\min_i \left\{v_{ij} \mid j = 1, 2, \cdots, n\right\}\right)$$

Step 3: Calculate the sums for each line, by subtracting each number from LIS (from SIS).

$$S_i^+ = \left\{ \Big| \sum_{j=1}^{n} (v_{ij} - v_j^+)^2 \right\}^{0.5};\ i = 1, 2 \cdots, m$$

Similarly, we compute the sums for each line, by subtracting each number from SIS.

$$S_i^- = \left\{ \Big| \sum_{j=1}^{n} (v_{ij} - v_j^-)^2 \right\}^{0.5};\ i = 1, 2 \cdots, m$$

Classifying these sums which one is closer to the maximum (or is further from the minimum)

A set of alternatives can now be ranked according to the descending order of the value of sums (S_i^+) or (S_i^-).

Step 4(facultative): We can compute T_i, though the previous steps enough to rank the alternatives.

$$T_i = \frac{S_i^-}{(S_i^+ + S_i^-)};\ i = 1, 2 \cdots, m$$

A. Numerical example

In order to compare the result with classical TOPSIS we use the same numerical examples used in classical TOPSIS.

TABLE VI. DECISION MATRIX

a_{ij}	C_1	C_2	C_3
ω_i	12/16	3/16	1/16
A_1	7	9	9
A_2	8	7	8
A_3	9	6	8
A_4	6	7	8

One multiplies on columns with the weights $12/16$, $3/16$, and $1/16$ respectively, and one gets:

TABLE VII. WEIGHTED DECISION MATRIX

$\omega_j a_{ij}$	C_1	C_2	C_3
ω_i	12/16	3/16	1/16
A_1	84/16	27/16	9/16
A_2	96/16	21/16	8/16
A_3	108/16	18/16	8/16
A_4	72/16	21/16	8/16

We compute the sums for each line, by subtracting each number from the largest one:

$$S_{1+} = |84/16 - 108/16| + |21/16 - 27/16| + |9/16 - 9/16| = 1.5000$$

$$S_{2+} = |96/16 - 108/16| + |27/16 - 27/16| + |9/16 - 9/16| = 1.1875$$

$$S_{3+} = |108/16 - 108/16| + |18/16 - 27/16| + |8/16 - 9/16| = 0.6250$$

$$S_{4+} = |72/16 - 108/16| + |21/16 - 27/16| + |8/16 - 9/16| = 2.6875$$

Classifying these sums we get them on places: S_{3+}, S_{2+}, S_{1+}, S_{4+} in the order of which one is closer to the maximum.

We compute the sums for each line, by subtracting each number from the smaller one:

$$S_{1-} = |84/16 - 72/16| + |27/16 - 18/16| + |9/16 - 8/16| = 1.3750$$

$$S_{2-} = |96/16 - 72/16| + |21/16 - 18/16| + |8/16 - 8/16| = 1.6875$$

$$S_{3-} = |108/16 - 72/16| + |18/16 - 18/16| + |8/16 - 8/16| = 2.2500$$

$$S_{4-} = |72/16 - 72/16| + |21/16 - 18/16| + |8/16 - 8/16| = 0.1875$$

Classifying these sums we get them on places: S_{3-}, S_{2-}, S_{1-}, S_{4-} in the order of which one is further from the minimum.

If we compute T_i, we get the same ordering of classical TOPSIS:

$$T_1 = (S_{1-})/[(S_{1-}) + (S_{1+})] = 0478261$$

$$T_2 = (S_{2-})/[(S_{2-}) + (S_{2+})] = 0.586957$$

$$T_3 = (S_{3-})/[(S_{3-}) + (S_{3+})] = 0.782609$$

$$T_4 = (S_{4-})/[(S_{4-}) + (S_{4+})] = 0.065217$$

The following table (Table VIII) summarized previous calculations

TABLE VIII. DISTANCE MEASURE AND RANKING COEFFICIENT

Alternative	S_i^+	S_i^-	T_i
A_1	1.5000	1.3750	0.478261
A_2	1.1875	1.6875	0.586957
A_3	0.6250	2.2500	**0.782609**
A_4	2.6875	0.1875	0.065217

By applying Simplified-TOPSIS, we get for T_3(0.782609), T_2(0.586957), T_1(0.478261) and T_4(0.065217), and we got with classical TOPSIS T_3(0.7938), T_2(0.6412), T_1(0.3880) and T_4(0.0783).

Hence the order obtained with our approach simplified-TOPSIS is the same of classical TOPSIS: T_3, T_2, T_1 and T_4, with little change in values between both approaches.

IV. NEUTROSOPHIC TOPSIS [11]

The MCDM Neutrosophic TOPSIS approach is explained in the following steps.

Step 1: Construction of the aggregated single valued Neutrosophic decision matrix based on decision makers assessments

$$D = (d_{ij})_{\substack{1\le i\le n \\ 1\le j\le m}} = (T_{ij}, I_{ij}, F_{ij})_{\substack{1\le i\le n \\ 1\le j\le m}}$$

Where T_{ij} denote truth, I_{ij} indeterminacy and F_{ij} falsity membership score of preference i with respect to criterion j in single valued Neutrosophic.

$w = (\omega_1, \omega_2, \cdots, \omega_n)$ with ω_i a single valued Neutrosophic weight of criteria (so $\omega_i = (a_i, b_i, c_i)$).

Example 1: For compare the results obtained by our approach Neutrosophic-simplified-TOPSIS (will be presented afterwards) with those obtained with Neutrosophic-TOPSIS, we use the example introduced by Biswas [11].

Let (DM_1, DM_2, DM_3, DM_4) fours decisions makers aims to select an alternative $A_i(A_1, A_2, A_3, A_4)$ with respect six criteria $(C_1, C_2, C_3, C_4, C_5, C_6)$.

The Neutrosophic weight of each criterion (Table IX) and Neutrosophic decision matrix (Table X) presented respectively.

TABLE IX. CRITERIA WEIGHTS

	C_1	C_2	C_3
ω_i	(0.755,0.222,0.217)	(0.887,0.113,0.107)	(0.765,0.226,0.182)
	C_4	C_5	C_6
ω_i	(0.692,0.277,0.251)	(0.788,0.200,0.180)	(0.700,0.272,0.244)

TABLE X. NEUTROSOPHIC DECISION MATRIX

	C_1	C_2	C_3
A_1	(0.864,0.136,0.081)	(0.853,0.147,0.: 092)	(0.800,0.200,0.150)
A_2	(0.667,0.333,0.277)	(0.727,0.273,0.219)	(0.667,0.333,0.277)
A_3	(0.880,0.120,0.067)	(0.887,0.113,0.064)	(0.834,0.166,0.112)
A_4	(0.667,0.333,0.277)	(0.735,0.265,0.195)	(0.768,0.232,0.180)
	C_4	C_5	C_6
A_1	(0.704,0.296,0.241)	(0.823,0.177,0.123)	(0.864,0.136,0.081)
A_2	(0.744,0.256,0.204)	(0.652,0.348,0.293)	(0.608,0.392,0.336)
A_3	(0.779,0.256,0.204)	(0.811,0.189,0.109)	(0.850,0.150,0.092)
A_4	(0.727,0.273,0.221)	(0.791,0.209,0.148)	(0.808,0.192,0.127)

Step 2: Aggregation of the weighted Neutrosophic decision matrix

$$D^w = D \otimes W = (d_{ij}^w)_{\substack{1\le i\le n \\ 1\le j\le m}} = (T_{ij}^w, I_{ij}^w, F_{ij}^w)_{\substack{1\le i\le n \\ 1\le j\le m}}$$

Step 3: Determination of the relative Neutrosophic positive ideal solution (RNPIS) and the relative negative ideal solution (RNIS) for SVNSs.

$$T_j^{w+} = \left\{\left(max_i\left\{T_{ij}^{w_i} | j \in B\right\}\right), \left(min_i\left\{T_{ij}^{w_i} | j \in C\right\}\right)\right\}$$

$$Q_N^+ = (d_1^{w+}, d_2^{w+}, \cdots, d_n^{w+})$$

$$T_j^{w+} = \left\{\left(max_i\left\{T_{ij}^{w_j} | j \in B\right\}\right), \left(min_i\left\{T_{ij}^{w_j} | j \in C\right\}\right)\right\}$$

$$I_j^{w+} = \left\{\left(min_i\left\{I_{ij}^{w_j} | j \in B\right\}\right), \left(max_i\left\{I_{ij}^{w_j} | j \in C\right\}\right)\right\}$$

$$F_j^{w+} = \left\{\left(min_i\left\{F_{ij}^{w_j} | j \in B\right\}\right), \left(max_i\left\{F_{ij}^{w_j} | j \in C\right\}\right)\right\}$$

$$Q_N^- = (d_1^{w-}, d_2^{w-}, \cdots, d_n^{w-})$$

$$T_j^{w-} = \left\{\left(min_i\left\{T_{ij}^{w_j} | j \in B\right\}\right), \left(max_i\left\{T_{ij}^{w_j} | j \in C\right\}\right)\right\}$$

$$I_j^{w-} = \left\{\left(max_i\left\{I_{ij}^{w_j} | j \in B\right\}\right), \left(min_i\left\{I_{ij}^{w_j} | j \in C\right\}\right)\right\}$$

$$F_j^{w-} = \left\{ \left(max_i \left\{ F_{ij}^{w_j} | j \in B \right\} \right), \left(min_i \left\{ F_{ij}^{w_j} | j \in C \right\} \right) \right\}$$

Where sets B and C are associated with the benefit and cost attribute sets, respectively

Step 4: Determination of the distance measure of each alternative from the RNPIS and the RNNIS for SVNSs.

$$D_{Eu}^{i+}\left(d_{ij}^{wj}, d_{ij}^{w+}\right) = \sqrt{\frac{1}{3n} \sum_{j=1}^{n} \left\{ \begin{array}{l} \left(T_{ij}^{wj}(x) - T_{ij}^{w+}(x)\right)^2 + \\ \left(I_{ij}^{wj}(x) - I_{ij}^{w+}(x)\right)^2 + \\ \left(F_{ij}^{wj}(x) - F_{ij}^{w+}(x)\right)^2 \end{array} \right\}}$$

with $i = 1, 2 \cdots, m$

And

$$D_{Eu}^{i-}\left(d_{ij}^{wj}, d_{ij}^{w-}\right) = \sqrt{\frac{1}{3n} \sum_{j=1}^{n} \left\{ \begin{array}{l} \left(T_{ij}^{wj}(x) - T_{ij}^{w-}(x)\right)^2 + \\ \left(I_{ij}^{wj}(x) - I_{ij}^{w-}(x)\right)^2 + \\ \left(F_{ij}^{wj}(x) - F_{ij}^{w-}(x)\right)^2 \end{array} \right\}}$$

Step 5: Determination of the relative closeness coefficient to the Neutrosophic ideal solution for SVNSs.

$$C_i^* = \frac{NS_i^-}{(NS_i^+ + NS_i^-)};\ i = 1, 2 \cdots, m$$

A set of alternatives can now be ranked according to the descending order of the value of C_i^*.

Table below (Table XI) shows the results obtained by Neutrosophic-TOPSIS.

TABLE XI. CLOSENESSCOEFFICIENT

Alternative	$\mathbf{C_i^*}$
$\mathbf{A_1}$	0.8190
$\mathbf{A_2}$	0.1158
$\mathbf{A_3}$	**0.8605**
$\mathbf{A_4}$	0.4801

Based on the values of closeness coefficient, the four alternatives are classified as $A_3 > A_1 > A_4 > A_2$. Then, the alternative A_3 is the best solution.

V. NEUTROSOPHIC-SIMPLIFIED-TOPSIS (OUR PROPOSED METHOD)

Step 1: Building of the SVNS decision matrix and SVNS weight of each criterion.

$$D = (d_{ij})_{\substack{1\le i\le n \\ 1\le j\le m}} = (T_{ij}, I_{ij}, F_{ij})_{\substack{1\le i\le n \\ 1\le j\le m}}$$

$$\begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & \cdots & C_n \end{array} \\ \begin{bmatrix} d_{11} & d_{12} & \cdots & d_{1n} \\ d_{21} & d_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ d_{m1} & \cdots & \cdots & d_{mn} \end{bmatrix} \end{array}$$

Where T_{ij} denote truth, I_{ij} indeterminacy and N_{ij} falsity membership score of preference i with respect to criterion j in single valued Neutrosophic.

$w = (\omega_1, \omega_2, \cdots, \omega_n)$ with ω_i a single valued Neutrosophic weight of criteria (so $\omega_i = (a_i, b_i, c_i)$)

Step 2: Calculate SVNS weighted decision matrix

$$D^w = D \otimes W = (d_{ij}^w)_{\substack{1\le i\le n \\ 1\le j\le m}} = \omega_j \otimes d_{ij}^w = (T_{ij}^w, I_{ij}^w, F_{ij}^w)_{\substack{1\le i\le n \\ 1\le j\le m}}$$

$$\omega_j \otimes d_{ij} = \left(a_j T_{ij}, b_j + I_{ij} - b_j I_{ij}, c_j + F_{ij} - c_j F_{ij}\right)$$

Step 3: Determine the maximum (larger) Neutrosophic ideal solution (LNIS) and minimum (smaller) Neutrosophic ideal solution (SNIS).

$$A_N^+ = (d_1^{w+}, d_2^{w+}, \cdots, d_n^{w+})$$

$$d_j^{\omega+} = \left(T_j^{\omega+}, I_j^{\omega+}, F_j^{\omega+}\right)$$

$$T_j^{w+} = \left\{ \left(max_i \left\{ T_{ij}^{w_j} | j = 1, \cdots, n \right\} \right) \right\}$$

$$I_j^{w+} = \left\{ \left(min_i \left\{ I_{ij}^{w_j} | j = 1, \cdots, n \right\} \right) \right\}$$

$$F_j^{w+} = \left\{ \left(min_i \left\{ F_{ij}^{w_j} | j = 1, \cdots, n \right\} \right) \right\}$$

$$A_N^- = (d_1^{w-}, d_2^{w-}, \cdots, d_n^{w-})$$

$$d_j^{\omega-} = \left(T_j^{\omega-}, I_j^{\omega-}, F_j^{\omega-}\right)$$

$$T_j^{w-} = \left\{ \left(min_i \left\{ T_{ij}^{w_j} | j = 1, \cdots, n \right\} \right) \right\}$$

$$I_j^{w-} = \left\{ \left(max_i \left\{ I_{ij}^{w_j} | j = 1, \cdots, n \right\} \right) \right\}$$

$$F_j^{w-} = \left\{ \left(max_i \left\{ F_{ij}^{w_j} | j = 1, \cdots, n \right\} \right) \right\}$$

Step 4: Calculate the Neutrosophic separation measures for each alternative from LNIS and from SNIS.

In this case we have introduced à new distance measure (definition 1) between two single-valued Neutrosophic (SVNs) using Manhattan distance [14] instead of the Euclidean distance used to calculate similarity measure between two

SVNs in literature and in Neutrosophic-TOPSIS method, the defined distance is used to calculate distance measure.

Definition 1. Let $X_1 = (x_1, y_1, z_1)$ and $X_2 = (x_2, y_2, z_2)$ be a SVN numbers. Then the separation measure between X_1 and X_2 based on Manhattan distance is defined as follows:

$$D_{Manh}(X_{1,}X_2) = |x_1 - x_2| + |y_1 - y_2| + |z_1 - x_2|$$

The separation from the maximum Neutrosophic ideal solution is :

$$D_{Manh}^{j+}\left(d_{ij}^{wj}, d_{ij}^{w+}\right) = \left\{\begin{array}{l} \left|T_{ij}^{wj}(x) - T_{ij}^{w+}(x)\right| + \\ \left|I_{ij}^{wj}(x) - I_{ij}^{w+}(x)\right| + \\ \left|F_{ij}^{wj}(x) - F_{ij}^{w+}(x)\right| \end{array}\right\}$$

with $j = 1, 2 \cdots, n$

$$NS_i^+ = \sum_{j=1}^{n} D_{Manh}^{j+}\left(d_{ij}^{wj}, d_{ij}^{w+}\right) \text{ with } i = 1, 2 \cdots, m$$

Similarly, the separation from the minimum Neutrosophic ideal solution is:

$$D_{Manh}^{j-}\left(d_{ij}^{wj}, d_{ij}^{w-}\right) = \left\{\begin{array}{l} \left|T_{ij}^{wj}(x) - T_{ij}^{w-}(x)\right| + \\ \left|I_{ij}^{wj}(x) - I_{ij}^{w-}(x)\right| + \\ \left|F_{ij}^{wj}(x) - F_{ij}^{w-}(x)\right| \end{array}\right\}$$

with $j = 1, 2 \cdots, n$

$$NS_i^- = \sum_{j=1}^{n} D_{Manh}^{j-}\left(d_{ij}^{wj}, d_{ij}^{w-}\right) \text{ with } i = 1, 2 \cdots, m$$

Ranking the alternatives according to the values of $\mathrm{NS_i^-}$ or according to $1/\mathrm{NS_i^+}$

Step 5: The measure ranking coefficient is calculated as.

$$NT_i = \frac{NS_i^-}{(NS_i^+ + NS_i^-)}; \; i = 1, 2 \cdots, m$$

A set of alternatives can now be ranked according to the descending order of the value of NT_i

A. Numerical example

Step 1: Building of the SVNS decision matrix and SVNS weight of each criterion.

Let $A_i(A_1, A_2, A_3, A_4)$ a set of alternative and $C_i(C_1, C_2, C_3, C_4, C_5, C_6)$ a set of criteria.

Let considers the following Neutrosophic weights of criteria (Table XII) and Neutrosophic decision matrix (Table XIII) respectively (used in above example 1).

TABLE XII. CRITERIA NEUTROSOPHIC WEIGHTS

	$\mathbf{C_1}$	$\mathbf{C_2}$	$\mathbf{C_3}$
$\boldsymbol{\omega_i}$	(0.755,0.222,0.217)	(0.887,0.113,0.107)	(0.765,0.226,0.182)
	$\mathbf{C_4}$	$\mathbf{C_5}$	$\mathbf{C_6}$
$\boldsymbol{\omega_i}$	(0.692,0.277,0.251)	(0.788,0.200,0.180)	(0.700,0.272,0.244)

TABLE XIII. NEUTROSOPHIC DECISION MATRIX

$\mathbf{d_{ij}}$	$\mathbf{C_1}$	$\mathbf{C_2}$	$\mathbf{C_3}$
$\mathbf{A_1}$	(0.864,0.136,0.081)	(0.853,0.147,0.092)	(0.800,0.200,0.150)
$\mathbf{A_2}$	(0.667,0.333,0.277)	(0.727,0.273,0.219)	(0.667,0.333,0.277)
$\mathbf{A_3}$	(0.880,0.120,0.067)	(0.887,0.113,0.064)	(0.834,0.166,0.112)
$\mathbf{A_4}$	(0.667,0.333,0.277)	(0.735,0.265,0.195)	(0.768,0.232,0.180)
	$\mathbf{C_4}$	$\mathbf{C_5}$	$\mathbf{C_6}$
$\mathbf{A_1}$	(0.704,0.296,0.241)	(0.823,0.177,0.123)	(0.864,0.136,0.081)
$\mathbf{A_2}$	(0.744,0.256,0.204)	(0.652,0.348,0.293)	(0.608,0.392,0.336)
$\mathbf{A_3}$	(0.779,0.256,0.204)	(0.811,0.189,0.109)	(0.850,0.150,0.092)
$\mathbf{A_4}$	(0.727,0.273,0.221)	(0.791,0.209,0.148)	(0.808,0.192,0.127)

Step 2: Calculate SVNs weighted decision matrix

$$D^w = (d_{ij}^w)_{\substack{1 \le i \le n \\ 1 \le j \le m}} = (T_{ij}^w, I_{ij}^w, F_{ij}^w)_{\substack{1 \le i \le n \\ 1 \le j \le m}}$$

$$d_{ij}^w = \left(a_j T_{ij}, b_j + I_{ij} - b_j I_{ij}, c_j + F_{ij} - c_j F_{ij}\right)$$

One multiplies each columns of Neutrosophic decision matrix with the weights of criteria, and one gets:

$$T_{11}^{\omega} = 0.864 \times 0.755 = 0.6523$$

$$I_{11}^{\omega} = 0.136 + 0.222 - 0.136 \times 0.222 = 0.328$$

$$F_{11}^{\omega} = 0.081 + 0.217 - 0.081 \times 0.217 = 0.280$$

TABLE XIV. WEIGHTED DECISION MATRIX

$\mathbf{d_{ij}^w}$	$\mathbf{C_1}$	$\mathbf{C_2}$	$\mathbf{C_3}$
$\mathbf{A_1}$	(0.6523,0.328,0.28)	(0.757,0.243,0.189)	(0.612,0.381,0.305)
$\mathbf{A_2}$	(0.504,0.481,0.434)	(0.645,0.355,0.303)	(0.510,0.484,0.409)
$\mathbf{A_3}$	(0.664,0.315,0.269)	(0.787,0.213,0.164)	(0.638,0.354,0.274)
$\mathbf{A_4}$	(0.504,0.481,0.434)	(0.652,0.348,0.281)	(0.588,0.406,0.329)
	$\mathbf{C_4}$	$\mathbf{C_5}$	$\mathbf{C_6}$
$\mathbf{A_1}$	(0.487,0.491,0.432)	(0.649,0.342,0.281)	(0.605,0.371,0.305)
$\mathbf{A_2}$	(0.515,0.462,0.404)	(0.514,0.478,0.420)	(0.426,0.557,0.498)
$\mathbf{A_3}$	(0.539,0.462,0.404)	(0.639,0.351,0.269)	(0.595,0.381,0.314)
$\mathbf{A_4}$	(0.503,0.474,0.417)	(0.623,0.367,0.301)	(0.566,0.412,0.340)

a. Numbers are rounded to three decimal place.

Step 3: Determine the maximum (larger) Neutrosophic ideal solution (LNIS) and minimum (smaller) Neutrosophic ideal solution (SNIS).

TABLE XV. MAXIMUM (LARGE) NEUTROSOPHIC IDEAL SOLUTION(LNIS)

	C_1	C_2	C_3
$d_j^{\omega+}$	(0.664,0.315,0.269)	(0.887,0.213,0.264)	(0.638,0.354,0.274)
	C_4	C_5	C_6
$d_j^{\omega+}$	(0.539,0.462,0.404)	(0.649,0.341,0.294)	(0.605,0.371,0.305)

TABLE XVI. MINIMUM (SMALLER) NEUTROSOPHICIDEAL SOLUTION (SNIS)

	C_1	C_2	C_3
$d_j^{\omega-}$	(0.504,0.481,0.434)	(0.645,0.355,0.303)	(0.510,0.484,0.409)
	C_4	C_5	C_6
$d_j^{\omega-}$	(0.487,0.491,0.432)	(0.514,0.478,0.420)	(0.426,0.557,0.498)

Step 4: Calculate the Neutrosophic separation measures for each alternative from the LNIS and from SNIS.

We compute the sums for each line, by subtracting each alternative from the larger one and by subtracting each alternative from the smaller one.

TABLE XVII. NEUTROSOPHIC SEPARATION MEASURES AND NEUTROSOPHIC MEASURE RANKING

	NS_i^+	NS_i^-	NT_i
A_1	0,324	2,07	0,86459295
A_2	2,31	0,084	0,03521102
A_3	0,047	2,347	**0,98021972**
A_4	1,293	1,101	0,45987356

Based on the values of coefficients of decreasing rank, four alternatives are ranked as $A_3 > A_1 > A_4 > A_2$ as in Table XVII. Then, the alternative A_3 is also the best solution.

Hence, we get the same rank of Neutrosophic-TOPSIS.

VI. CONCLUSION

In this paper, we have presented two new MCDM methods, the first is simplified-TOPSIS, that simplifies the calculation of classical TOPSIS to a simple formulas easy to applying and a reduced number of steps and give same results of classical TOPSIS. The second is MCDM method in Neutrosophic environment, which is too simplifies the Neutrosophic-TOPSIS, extending the Simplified-TOPSIS using single valued Neutrosophic information. Maximum larger) Neutrosophic Ideal Solution (LNIS) and Minimum (smaller) Neutrosophic Ideal Solution (SNIS) are defined from weighted decision matrix. Manhattan distance Neutrosophic measure is defined and used to determine the distances of each alternative from maximum as well as minimum Neutrosophic ideal solutions, which used to calculate the measure ranking coefficient of each alternative.

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Shortest Path Problem under Bipolar Neutrosophic Setting

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Abstract. This main purpose of this paper is to develop an algorithm to find the shortest path on a network in which the weights of the edges are represented by bipolar neutrosophic numbers. Finally, a numerical example has been provided for illustrating the proposed approach.

Introduction

Smarandache [1, 2] introduced neutrosophic set and neutrosophic logic by considering the non-standard analysis. The concept of neutrosophic sets generalized the concepts of fuzzy sets [3] and intuitionistic fuzzy set [4] by adding an independent indeterminacy-membership. Neutrosophic set is a powerful tool to deal with incomplete, indeterminate and inconsistent information in real world, which have attracted the widespread concerns for researchers. The concept of neutrosophic set is characterized by three independent degrees namely truth-membership degree (T), indeterminacy-membership degree (I), and falsity-membership degree (F). From scientific or engineering point of view, the neutrosophic set and set- theoretic operator will be difficult to apply in the real application. The subclass of the neutrosophic sets called single-valued neutrosophic sets [5] (SVNS for short) was studied deeply by many researchers. The concept of single valued neutrosophic theory has proven to be useful in many different field such as the decision making problem, medical diagnosis and so on. Additional literature on neutrosophic sets can be found in [6]. Recently, Deli et al. [7] introduced the concept of bipolar neutrosophic sets which is an extension of the fuzzy sets, bipolar fuzzy sets, intuitionistic fuzzy sets and neutrosophic sets. The bipolar neutrosophic set (BNS) is an important concept to handle uncertain and vague information exciting in real life, which consists of three membership functions including bipolarity. Also, they give some operations including the score, certainty and accuracy functions to compare the bipolar neutrosophic sets and operators on the bipolar neutrosophic sets. The shortest path problem is a fundamental algorithmic problem, in which a minimum weight path is computed between two nodes of a weighted, directed graph. The shortest path problem has been widely studied in the fields of operations research, computer science, and transportation engineering. In literature, there are many publications which deal with shortest path problems [8-13] that have been studied with different types of input data, including fuzzy set, intuitionstic fuzzy sets, trapezoidal intuitionistic fuzzy sets and vague set. Recently, Broumi et al. [14-17] presented the concept of neutrosophic graphs, interval valued neutrosophic graphs and bipolar single valued neutrosophic graphs. Smarandache [18-19] proposed another variant of neutrosophic graphs based on literal indeterminacy component (I). Also Kandasamy et al. [20] studied the concept of neutrosophic graphs, To do best of our knowledge, few research papers deal with shortest path in neutrosophic environment. Broumi et al. [21] proposed an algorithm for solving neutrosophic shortest path problem based on score function.

neutrosophic number on a network. Till now, there is no study in the literature for computingThe same authors in [22] proposed a study of neutrosphic shortest path with interval valued shortest path problem in bipolar neutrosophic environment. The structure of the paper is as follows. In Section 2, we review some basic concepts about neutrosophic sets, single valued neutrosophic sets and bipolar neutrosophic sets. In section 3, we give the network terminology. In Section 4, an algorithm is proposed for finding the shortest path and shortest distance in bipolar neutrosophic graph. In section 5 an illustrative example is provided to find the shortest path and shortest distance between the source node and destination node. Finally, in Section 6 we provide conclusion and proposal for further research.

Preliminaries

In this section, some basic concepts and definitions on neutrosophic sets, single valued neutrosophic sets and bipolar neutrosophic sets are reviewed from the literature.

Definition 2.1 [1-2]. Let X be a space of points (objects) with generic elements in X denoted by x; then the neutrosophic set A (NS A) is an object having the form A = {< x: $T_A(x)$, $I_A(x)$, $F_A(x)$>, x ∈ X}, where the functions T, I, F: X→]$^-$0,1$^+$[define respectively the truth-membership function, an indeterminacy-membership function, and a falsity-membership function of the element x ∈ X to the set A with the condition:

$$^-0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+. \quad (1)$$

The functions $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or nonstandard subsets of]$^-$0,1$^+$[.

Since it is difficult to apply NSs to practical problems, Wang et al. [14] introduced the concept of a SVNS, which is an instance of a NS and can be used in real scientific and engineering applications.

Definition 2.2 [3]. Let X be a space of points (objects) with generic elements in X denoted by x. A single valued neutrosophic set A (SVNS A) is characterized by truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$. For each point x in X $T_A(x)$, $I_A(x)$, $F_A(x)$ ∈ [0, 1]. A SVNS A can be written as

$$A = \{< x: T_A(x), I_A(x), F_A(x)>, x \in X\} \quad (2)$$

Deli et al. [15] proposed the concept of bipolar neutrosophic set, which is an instance of a neutrosophic set, and introduced the definition of an BNS.

Definition 2.3 [4]. A bipolar neutrosophic set A in X is defined as an object of the form A={<x, $T^p(x)$, $I^p(x)$, $F^p(x)$, $T^n(x)$, $I^n(x)$, $F^n(x)$>: x ∈ X}, where T^p, I^p, F^p :X→ [1, 0] and T^n, I^n, F^n : X→ [-1, 0] .The positive membership degree $T^p(x)$, $I^p(x)$, $F^p(x)$ denotes the truth membership, indeterminate membership and false membership of an element ∈ X corresponding to a bipolar neutrosophic set A and the negative membership degree $T^n(x)$, $I^n(x)$, $F^n(x)$ denotes the truth membership, indeterminate membership and false membership of an element ∈ X to some implicit counter-property corresponding to a bipolar neutrosophic set A.

Definition 2.4 [4]. An empty bipolar neutrosophic set $\tilde{A}_1 =< T_1^p, I_1^p, F_1^p, T_1^n, I_1^n, F_1^n >$ is defined as

$$T_1^p = 0, I_1^p = 0, F_1^p = 1 \text{ and } T_1^n = -1, I_1^n = 0, F_1^n = 0 \quad (4)$$

Definition 2.5 [7]. Let $\tilde{A}_1 =< T_1^p, I_1^p, F_1^p, T_1^n, I_1^n, F_1^n >$ and $\tilde{A}_2 =< T_2^p, I_2^p, F_2^p, T_2^n, I_2^n, F_2^n >$ be two bipolar neutrosophic numbers and $\lambda > 0$. Then, the operations of these numbers defined as below;

(i) $\tilde{A}_1 \oplus \tilde{A}_2 =< T_1^p + T_2^p - T_1^p T_2^p, I_1^p I_2^p, F_1^p F_2^p, -T_1^n T_2^n, -(-I_1^p - I_2^p - I_1^p I_2^p), -(-F_1^p - F_2^p - F_1^p F_2^p) >$ (5)

(ii) $\tilde{A}_1 \otimes \tilde{A}_2 =< T_1^p T_2^p, I_1^p + I_2^p - I_1^p I_2^p, F_1^p + F_2^p - F_1^p F_2^p, -(-T_1^n - T_2^n - T_1^n T_2^n), -I_1^n I_2^n, -F_1^n F_2^n >$ (6)

(iii) $\lambda\tilde{A}_1 =< 1-(1-T_1^p)^\lambda, (I_1^p)^\lambda, (F_1^p)^\lambda, -(-T_1^n)^\lambda, -(-I_1^n)^\lambda, -(1-(1-(-F_1^n))^\lambda) >$ (7)

(iv) $\tilde{A}_1^\lambda =< (T_1^p)^\lambda, 1-(1-I_1^p)^\lambda, 1-(1-F_1^p)^\lambda, -(1-(1-(-T_1^n))^\lambda), -(-I_1^n)^\lambda, -(-F_1^n))^\lambda) >$ where $\lambda > 0$ (8)

Definition 2.6 [7]. In order to make a comparisons between two BNN. Deli et al. [7], introduced a concept of score function. The score function is applied to compare the grades of BNS. This function shows that greater is the value, the greater is the bipolar neutrosphic sets and by using this concept paths can be ranked. Let $\tilde{A} =< T^p, I^p, F^p, T^n, I^n, F^n >$ be a bipolar neutrosophic number. Then, the score function $s(\tilde{A})$, accuracy function $a(\tilde{A})$ and certainty function $c(\tilde{A})$ of an BNN are defined as follows:

(i) $s(\tilde{A}) = \left(\frac{1}{6}\right) \times \left[T^p + 1 - I^p + 1 - F^p + 1 + T^n - I^n - F^n \right]$ (9)

(ii) $a(\tilde{A}) = T^p - F^p + T^n - F^n$ (10)

(iii) $c(\tilde{A}) = T^p - F^n$ (11)

Comparison of bipolar neutrosophic numbers

Let $\tilde{A}_1 =< T_1^p, I_1^p, F_1^p, T_1^n, I_1^n, F_1^n >$ and $\tilde{A}_2 =< T_2^p, I_2^p, F_2^p, T_2^n, I_2^n, F_2^n >$ be two bipolar neutrosophic numbers then

i. If $s(\tilde{A}_1) \succ s(\tilde{A}_2)$, then $\tilde{A}_1$ is greater than $\tilde{A}_2$, that is, $\tilde{A}_1$ is superior to $\tilde{A}_2$, denoted by $\tilde{A}_1 \succ \tilde{A}_2$
ii. If $s(\tilde{A}_1) = s(\tilde{A}_2)$,and $a(\tilde{A}_1) \succ a(\tilde{A}_2)$ then $\tilde{A}_1$ is greater than $\tilde{A}_2$, that is, $\tilde{A}_1$ is superior to $\tilde{A}_2$, denoted by $\tilde{A}_1 \succ \tilde{A}_2$
iii. If $s(\tilde{A}_1) = s(\tilde{A}_2)$, $a(\tilde{A}_1) = a(\tilde{A}_2)$, and $c(\tilde{A}_1) \succ c(\tilde{A}_2)$ then $\tilde{A}_1$ is greater than $\tilde{A}_2$, that is, $\tilde{A}_1$ is superior to $\tilde{A}_2$, denoted by $\tilde{A}_1 \succ \tilde{A}_2$
iv. If $s(\tilde{A}_1) = s(\tilde{A}_2)$, $a(\tilde{A}_1) = a(\tilde{A}_2)$, and $c(\tilde{A}_1) = c(\tilde{A}_2)$ then $\tilde{A}_1$ is equal to $\tilde{A}_2$, that is, $\tilde{A}_1$ is indifferent to $\tilde{A}_2$, denoted by $\tilde{A}_1 = \tilde{A}_2$

Network Terminology

Consider a directed network G(V, E) consisting of a finite set of nodes V={1, 2,…,n} and a set of m directed edges E $\subseteq$ VxV. Each edge is denoted is denoted by an ordered pair (i, j) where i, j $\in$ V and $i \neq j$. In this network, we specify two nodes, denoted by s and t, which are the source node and the destination node, respectively. We define a path P_{ij} ={i= i_1, (i_1, i_2), i_2,…, i_{l-1}, (i_{l-1}, i_l), i_l =j} of alternating nodes and edges. The existence of at least one path P_{si} in G (V, E) is assumed for every i $\in$ V-{s}.

d_{ij} denotes bipolar neutrosophic number associated with the edge (i, j), corresponding to the length necessary to traverse (i, j) from i to j. the bipolar neutrosophic distance along the path P is denoted as d(P) is defined as

$$d(P)=\sum_{(i,j\in P)} d_{ij} \tag{12}$$

Bipolar Neutrosophic Path Problem

In this paper the arc length in a network is considered to be a neutrosophic number, namely, bipolar neutrosophic number.

The algorithm for the shortest path proceeds in 6 steps.

Step 1 Assume $\tilde{d}_1$ =<0, 1, 1, -1, 0, 0>and label the source node (say node1) as [$\tilde{d}_1$ =<0, 1, 1, -1, 0, 0>,-].

Step 2 Find $\tilde{d}_j$ = minimum$\{\tilde{d}_i \oplus \tilde{d}_{ij}\}$;j=2,3,…,n.

Step 3 If minimum occurs corresponding to unique value of i i.e., i= r then label node j as [$\tilde{d}_j$,r]. If minimum occurs corresponding to more than one values of i then it represents that there are more than one bipolar neutrosophic path between source node and node j but bipolar neutrosophic distance along path is $\tilde{d}_j$, so choose any value of i.

Step 4 Let the destination node (node n) be labeled as [$\tilde{d}_n$, *l*], then the bipolar neutrosophic shortest distance between source node is $\tilde{d}_n$.

Step 5 Since destination node is labeled as [$\tilde{d}_n$, *l*], so, to find the bipolar neutrosophic shortest path between source node and destination node, check the label of node *l*. Let it be [$\tilde{d}_l$, p], now check the label of node p and so on. Repeat the same procedure until node 1 is obtained.

Step 6 Now the bipolar neutrosophic shortest path can be obtained by combining all the nodes obtained by the step 5.

Remark: Let $\tilde{A}_i$; i =1, 2,…, n be a set of bipolar neutrosophic numbers, if S($\tilde{A}_k$) < S($\tilde{A}_i$), for all i, the bipolar neutrosophic number is the minimum of $\tilde{A}_k$

Illustrative Example

In order to illustrate the above procedure consider a small example network shown in Fig1, where each arc length is represented as bipolar neutrosophic number as shown in Table 1.The problem is to find the shortest distance and shortest path between source node and destination node on the network.

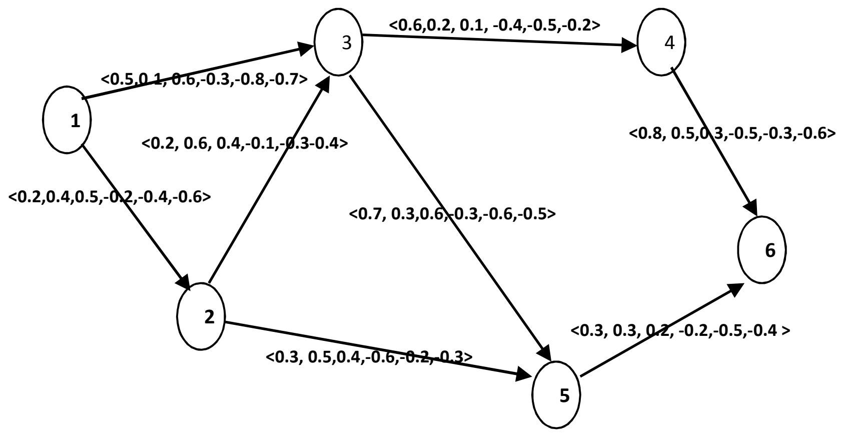

Fig.1. A network with bipolar neutrosophic edges

In this network each edge has been assigned to bipolar neutrosophic number as follows:

Table 1. Weights of the bipolar neutrosophic graphs

Edges	Bipolar Neutrosophic distance
1-2	<0.2, 0.4, 0.5, -0.2, -0.4, -0.6>
1-3	<0.5, 0.1, 0.6, -0.3, -0.8, -0.7>
2-3	<0.2, 0.6, 0.4, -0.1, -0.3, -0.4>
2-5	<0.3, 0.5, 0.4, -0.6, -0.2, -0.3>
3-4	<0.6, 0.2, 0.1, -0.4, -0.5, -0.2>
3-5	<0.7, 0.3, 0.6, -0.3, -0.6, -0.5>
4-6	<0.8, 0.5, 0.3, -0.5, -0.3, -0.6>
5-6	<0.3, 0.3, 0.2, -0.2, -0.5, -0.4>

Solution since node 6 is the destination node, so n= 6.

assume $\tilde{d}_1$ =<0, 1, 1, -1, 0, 0>and label the source node (say node 1) as [<0, 1, 1, -1, 0, 0>,-], the value of $\tilde{d}_j$; j= 2, 3, 4, 5 ,6 can be obtained as follows:

Iteration 1 Since only node 1 is the predecessor node of node 2, so putting i=1 and j= 2 in step2 of the proposed algorithm, the value of $\tilde{d}_2$ is

$\tilde{d}_2$ = minimum{ $\tilde{d}_1 \oplus \tilde{d}_{12}$ }=minimum{<0, 1, 1, -1, 0, 0> $\oplus$ <0.2, 0.4, 0.5, -0.2, -0.4, -0.6>= <0.2, 0.4, 0.5, -0.2, -0.4, -0.6> Since minimum occurs corresponding to i=1, so label node 2 as [<0.2, 0.4, 0.5, -0.2, -0.4, -0.6> , 1]

Iteration 2 The predecessor node of node 3 are node 1 and node 2, so putting i= 1, 2 and j= 3 in step 2 of the proposed algorithm, the value of $\tilde{d}_3$ is $\tilde{d}_3$ =minimum{ $\tilde{d}_1 \oplus \tilde{d}_{13}, \tilde{d}_2 \oplus \tilde{d}_{23}$ }= minimum{<0, 1, 1, -1, 0, 0> $\oplus$ <0.5, 0.1, 0.6, -0.3, -0.8, -0.7>, <0.2, 0.4, 0.5, -0.2, -0.4, -0.6 $\oplus$ <0.2, 0.6, 0.4, -0.1, -0.3, -0.4>}= minimum{<0.5, 0.1, 0.6, -0.3, -0.8, -0.7>, <0.36, 0.24, 0.2, -0.02, -0.58, -0.76>}

S (<0.5, 0.1, 0.6, -0.3, -0.8, -0.7>) = $s(\tilde{A}) = \left(\frac{1}{6}\right) \times \left[T^p + 1 - I^p + 1 - F^p + 1 + T^n - I^n - F^n\right]$ =0.66

S (<0.36, 0.24, 0.2, -0.02, -0.58, -0.76>) =0.70

Since S (<0.5, 0.1, 0.6, -0.3, -0.8, -0.7>) $<$ S (<0.36, 0.24, 0.2, -0.02, -0.58, -0.76>)

So minimum{<0.5, 0.1, 0.6, -0.3, -0.8, -0.7>, <0.36, 0.24, 0.2, -0.02, -0.58, -0.76>}=<0.5, 0.1, 0.6, -0.3, -0.8, -0.7>

Since minimum occurs corresponding to i=1, so label node 3 as [<0.5, 0.1, 0.6, -0.3, -0.8, -0.7>, 1]

Iteration 3. The predecessor node of node 4 is node 3, so putting i= 3 and j= 4 in step 2 of the proposed algorithm, the value of $\tilde{d}_4$ is $\tilde{d}_4$=minimum{$\tilde{d}_3 \oplus \tilde{d}_{34}$}=minimum<0.5, 0.1, 0.6, -0.3, -0.8, -0.7>$\oplus$ <0.6, 0.2, 0.1, -0.4, -0.5, -0.2>}= <0.8, 0.02, 0.06, -0.12, -0.9, -0.76>

So minimum<0.5, 0.1, 0.6, -0.3, -0.8, -0.7>$\oplus$ <0.6, 0.2, 0.1, -0.4, -0.5, -0.2>}= <0.8, 0.02, 0.06, -0.12, -0.9, -0.76>

Since minimum occurs corresponding to i=3, so label node 4 as [<0.8, 0.02, 0.06, -0.12, -0.9, -0.76>,3]

Iteration 4. The predecessor node of node 5 are node 2 and node 3, so putting i= 2, 3and j= 5 in step 2 of the proposed algorithm, the value of $\tilde{d}_5$ is $\tilde{d}_5$ =minimum{ $\tilde{d}_2 \oplus \tilde{d}_{25}, \tilde{d}_3 \oplus \tilde{d}_{35}$ }= minimum{<0.2, 0.4, 0.5, -0.2, -0.4, -0.6> $\oplus$ <0.3, 0.5, 0.4, -0.6, -0.2, -0.3> , <0.5, 0.1, 0.6, -0.3, -0.8, -0.7> $\oplus$ <0.7, 0.3, 0.6, -0.3, -0.6, -0.5>}=

Minimum{<0.44, 0.2, 0.2, -0.12, -0.52, -0.76>, <0.85, 0.03, 0.36, -0.09, -0.92, -0.85>}

S (<0.44, 0.2, 0.2, -0.12, -0.52, -0.76>) =0.69

S (<0.85, 0.03, 0.36, -0.09, -0.92, -0.85>) =0.85

Since S (<0.44, 0.2, 0.2, -0.12, -0.52, -0.76>) $<$ S (<0.85, 0.03, 0.36, -0.09, -0.92, -0.85>)

Minimum{<0.44, 0.2, 0.2, -0.12, -0.52, -0.76>, <0.85, 0.03, 0.36, -0.09, -0.92, -0.85>}

= <0.44, 0.2, 0.2, -0.12, -0.52, -0.76>

$\tilde{d}_5$ = <0.44, 0.2, 0.2, -0.12, -0.52, -0.76>

Since minimum occurs corresponding to i=2, so label node 5 as [<0.44, 0.2, 0.2, -0.12, -0.52, -0.76>, 2]

Iteration 5 The predecessor node of node 6 are node 4 and node 5, so putting i= 4, 5and j= 6 in step 2 of the proposed algorithm, the value of $\tilde{d}_6$ is $\tilde{d}_6$=minimum{$\tilde{d}_4 \oplus \tilde{d}_{46}, \tilde{d}_5 \oplus \tilde{d}_{56}$}=minimum{ <0.8, 0.02, 0.06, -0.12, -0.9, -0.76> $\oplus$ <0.8, 0.5, 0.3, -0.5, -0.3, -0.6>, <0.44, 0.2, 0.2, -0.12, -0.52, -0.76> $\oplus$ <0.3, 0.3, 0.2, -0.2, -0.5, -0.4> }= minimum{ <0.96, 0.01, 0.018, -0.06, -0.93, -0.904>, <0.60, 0.06, 0.04, -0.024, -0.76, -0.85>}

S (<0.96, 0.01, 0.018, -0.06, -0.93, -0.904>) =0.95

S (<0.60, 0.06, 0.04, -0.024, -0.76, -0.85>) =0.85

Since S (<0.60, 0.06, 0.04, -0.024, -0.76, -0.85>) S (<0.96, 0.01, 0.018, -0.06, -0.93, -0.904>)

So minimum { <0.96, 0.01, 0.018, -0.06, -0.93, -0.904>, <0.60, 0.06, 0.04, -0.024, -0.76, -0.85>}

= <0.60, 0.06, 0.04, -0.024, -0.76, -0.85>

$\tilde{d}_6$ =<0.60, 0.06, 0.04, -0.024, -0.76, -0.85>

Since minimum occurs corresponding to i=5, so label node 6 as [<0.60, 0.06, 0.04, -0.024, -0.76, -0.85>, 5]

Since node 6 is the destination node of the given network, so the bipolar neutrosophic shortest distance between node 1 and node 6 is <0.60, 0.06, 0.04, -0.024, -0.76, -0.85>

Now the bipolar neutrosophic shortest path between node 1 and node 6 can be obtained by using the following procedure:

Since node 6 is labeled by [<0.60, 0.06, 0.04, -0.024, -0.76, -0.85>, 5], which represents that we are coming from node 5. Node 5 is labeled by [<0.44, 0.2, 0.2, -0.12, -0.52, -0.76>, 2], which represents that we are coming from node 2. Node 2 is labeled by [<0.2, 0.4, 0.5, -0.2, -0.4, -0.6>,1]which represents that we are coming from node 1. Now the bipolar shortest path between node 1 and node 6 is obtaining by joining all the obtained nodes. Hence the bipolar neutrosophic shortest path is $1 \rightarrow 2 \rightarrow 5 \rightarrow 6$

The bipolar neutrosophic shortest distance and the neutrosophic shortest path of all nodes from node 1 is shown in the table 2 and the labeling of each node is shown in figure 4

Table 2. Tabular representation of different bipolar neutrosophic shortest path

Node No.(j)	$\tilde{d}_i$	Bipolar neutrosophic shortest path between j^th^ and 1st node
2	<0.2, 0.4, 0.5, -0.2, -0.4, -0.6>	$1 \rightarrow 2$
3	<0.5, 0.1, 0.6, -0.3, -0.8, -0.7>	$1 \rightarrow 3$
4	<0.8, 0.02, 0.06, -0.12, -0.9, -0.76>	$1 \rightarrow 3 \rightarrow 4$
5	<0.44, 0.2, 0.2, -0.12, -0.52, -0.76>	$1 \rightarrow 2 \rightarrow 5$
6	<0.60, 0.06, 0.04, -0.024, -0.76, -0.85>	$1 \rightarrow 2 \rightarrow 5 \rightarrow 6$

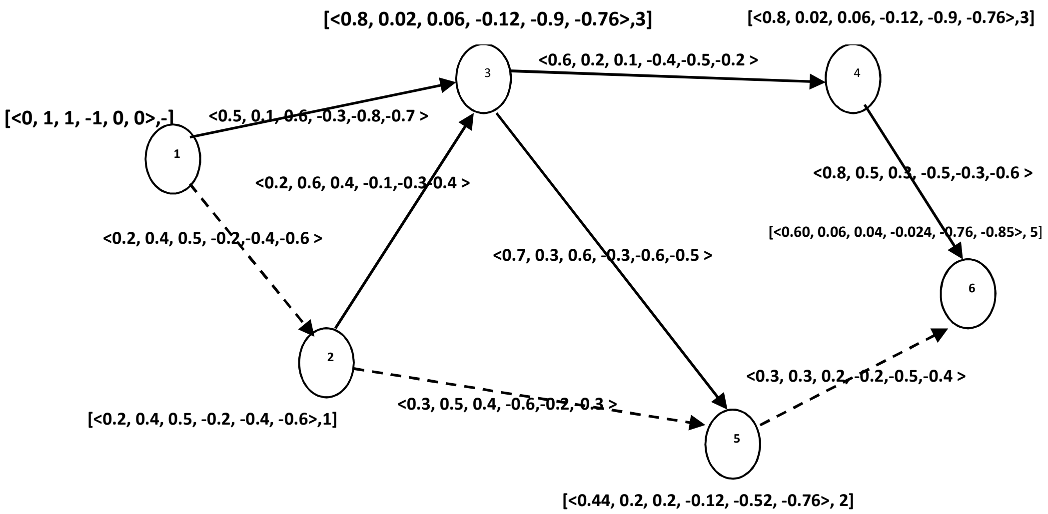

Fig. 2. Network with bipolar neutrosophic shortest distance of each node from node 1

Conclusion

In this paper we developed an algorithm for solving shortest path problem on a network with bipolar neutrosophic arc lengths. The process of ranking the path is very useful to make decisions in choosing the best of all possible path alternatives. We have explained the method by an example with the help of a hypothetical data. Further, we plan to extend the following algorithm of bipolar neutrosophic shortest path problem in an interval valued bipolar fuzzy neutrosophic environment.

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Multicriteria Decision Making Using Double Refined Indeterminacy Neutrosophic Cross Entropy and Indeterminacy Based Cross Entropy

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Abstract. Double Refined Indeterminacy Neutrosophic Set (DRINS) is an inclusive case of the refined neutrosophic set, defined by Smarandache [1], which provides the additional possibility to represent with sensitivity and accuracy the uncertain, imprecise, incomplete, and inconsistent information which are available in real world. More precision is provided in handling indeterminacy; by classifying indeterminacy (I) into two, based on membership; as indeterminacy leaning towards truth membership (I_T) and indeterminacy leaning towards false membership (I_F). This kind of classification of indeterminacy is not feasible with the existing Single Valued Neutrosophic Set (SVNS), but it is a particular case of the refined neutrosophic set (where each T, I, F can be refined into T_1, T_2, ...; I_1, I_2, ...; F_1, F_2, ...). DRINS is better equipped at dealing indeterminate and inconsistent information, with more accuracy than SVNS, which fuzzy sets and Intuitionistic Fuzzy Sets (IFS) are incapable of. Based on the cross entropy of neutrosophic sets, the cross entropy of DRINSs, known as double refined Indeterminacy neutrosophic cross entropy, is proposed in this paper. This proposed cross entropy is used for a multicriteria decision-making problem, where the criteria values for alternatives are considered under a DRINS environment. Similarly, an indeterminacy based cross entropy using DRINS is also proposed. The double refined Indeterminacy neutrosophic weighted cross entropy and indeterminacy based cross entropy between the ideal alternative and an alternative is obtained and utilized to rank the alternatives corresponding to the cross entropy values. The most desirable one(s) in decision making process is selected. An illustrative example is provided to demonstrate the application of the proposed method. A brief comparison of the proposed method with the existing methods is carried out.

Introduction

Fuzzy set theory introduced by Zadeh (1965) [2] provides a constructive analytic tool for soft division of sets. Zadeh's fuzzy set theory [2] was extended to intuitionistic fuzzy set (A-IFS), in which each element is assigned a membership degree and a non-membership degree by Atanassov (1986) [3]. A-IFS is more suitable in dealing with data that has fuzziness and uncertainty than fuzzy set. A-IFS was further generalized into the notion of interval valued intuitionistic fuzzy set (IVIFS) by Atanassov and Gargov (1989) [4].

Entropy is an essential concept for measuring uncertain information. Zadeh introduced the concept of fuzzy entropy [5]. The beginning for the cross entropy approach was founded in information theory by Shannon [6]. A measure of the cross entropy distance between two probability distributions was put forward by Kullback-Leibler [7], later a modified cross entropy measure was proposed by Lin [8]. A fuzzy cross entropy measure and a symmetric discrimination information measure between fuzzy sets was proposed by Shang and Jiang [9]. Since intuitionistic fuzzy set is a generalization of a fuzzy set, an extension of the De-Luca-Termini non probabilistic entropy [10] known as intuitionistic fuzzy cross-entropy was proposed by Vlachos and Sergiadis [11] and it was applied to pattern recognition, image segmentation and also to medical diagnosis. Vague cross-entropy between Vague Sets (VSs) by equivalence with the cross entropy of probability distributions was defined by Zhang and Jiang [12] and its application to the pattern recognition and medical diagnosis was carried out.

The fault diagnosis problem of turbine using the cross entropy of Vague Sets was investigated by Ye [13]. Intuitionistic fuzzy cross entropy was applied to multicriteria fuzzy decision-making problems by Ye [14]. An interval-valued intuitionistic fuzzy cross-entropy based on the generalization of the vague cross-entropy was proposed and applied to multicriteria decision-making problems by Ye [15].

To represent uncertain, imprecise, incomplete, and inconsistent information that are present in real world, the concept of a neutrosophic set from philosophical point of view was proposed by Smarandache [16]. The neutrosophic set is a prevailing framework that generalizes the concept of the classic set, fuzzy set, intuitionistic fuzzy set, interval valued fuzzy set, interval valued intuitionistic fuzzy set, paraconsistent set, paradoxist set, and tautological set. Truth membership, indeterminacy membership, and falsity membership are represented independently in the neutrosophic set. However, the neutrosophic set generalizes the above mentioned sets from the philosophical point of view, and its functions $T_A(x), I_A(x)$, and $F_A(x)$ are real standard or nonstandard subsets of $]^{-}0, 1^{+}[$, that is, $T_A(x) : X \to]^{-}0, 1^{+}[$, $I_A(x) : X \to]^{-}0, 1^{+}[$, and $F_A(x) : X \to]^{-}0, 1^{+}[$, respectively with the condition $^{-}0 \leq supT_A(x) + supI_A(x) + supF_A(x) \leq 3^{+}$.

It is difficult to apply neutrosophic set in this form in real scientific and engineering areas. To overcome this difficulty, Wang et al. [17] introduced a Single Valued Neutrosophic Set (SVNS), which is an instance of a neutrosophic set. SVNS can deal with indeterminate and inconsistent information, which fuzzy sets and intuitionistic fuzzy sets are incapable of. Ye [18, 19, 20] presented the correlation coefficient of SVNSs and its cross-entropy measure and applied them to single-valued neutrosophic decision-making problems. Recently, Ye [21] had proposed a Single Valued Neutrosophic cross entropy to do decision making in multicriteria decision making problems with the data represented by SVNSs.

Owing to the fuzziness, uncertainty and indeterminate nature of many practical problems in the real world, neutrosophy has found application in many fields including Social Network Analysis (Salama et al [22]), Image Processing (Cheng and Guo[23], Sengur and Guo[24], Zhang et al [25]), Social problems (Vasantha and Smarandache [26], [27]) etc.

To provide more accuracy and precision to indeterminacy, the indeterminacy value present in the neutrosophic set has been classified into two; based on membership; as indeterminacy leaning towards truth membership and as indeterminacy leaning towards false membership. When the indeterminacy I can be identified as indeterminacy which is more of truth value than false value, but it cannot be classified as truth it is considered to be indeterminacy leaning towards truth (I_T). When the indeterminacy can be identified to be indeterminacy which is more of the false value than the truth value, but it cannot be classified as false it is considered to be indeterminacy leaning towards false (I_F). Indeterminacy leaning towards truth and indeterminacy leaning towards falsity makes the indeterminacy involved in the scenario to be more accurate and precise. This modified refined neutrosophic set was defined as Double Refined Indeterminacy Neutrosophic Set (DRINS) alias double refined Indeterminacy Neutrosophic Set (DVNS) by Kandasamy [28].

To provide a illustration of real world problem where DRINS can be used to represent the problem; the following scenarios are given: Consider the scenario where the expert's opinion is requested about a particular statement, he/she may state that the possibility in which the statement is true is 0.6 and the statement is false is 0.5, the degree in which he/she is not sure but thinks it is true is 0.2 and the degree in which he/she is not sure but thinks it is false is 0.1. Using a double refined Indeterminacy neutrosophic notation or double refined Indeterminacy neutrosophic representation it can be expressed as x(0.6, 0.2, 0.1, 0.5).

Assume another example, suppose there are 10 voters during a voting process. Two people vote yes, two people vote no, three people are for yes but still undecided and two people are favouring towards a no but still undecided. Using a double refined Indeterminacy neutrosophic notation, it can be expressed as x(0.2, 0.3, 0.3, 0.2). However, these expressions are beyond the scope of representation using the existing SVNS. Therefore, the notion of a Double Refined Indeterminacy neutrosophic set is more general and it overcomes the aforementioned issues.

This paper is organised into seven sections: Section one is introductory in nature. The basic concepts related to the paper is given in section two. Section three of the paper introduces and defines the cross entropy of Double Refined Indeterminacy Neutrosophic Set (DRINS). Section four deals with the solving multi criteria decision making problems using the cross entropy of DRINS under a DRINS based environment. Illustrative examples are provided to demonstrate the proposed approach in section five. Section six provides a brief comparison of the proposed approach with the existing approach. Conclusions and future direction of work is given in the last section.

Preliminaries Basic Concepts

Neutrosophy and Single Valued Neutrosophic Set (SVNS). Neutrosophy is a branch of philosophy, introduced by Smarandache [16], which studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. It considers a proposition, concept, theory, event, or entity, "A" in relation to its opposite, "Anti-A" and that which is not A, "Non-A", and that which is neither "A" nor "Anti-A", denoted by "Neut-A". Neutrosophy is the basis of neutrosophic logic, neutrosophic probability, neutrosophic set, and neutrosophic statistics.

The concept of a neutrosophic set from philosophical point of view, introduced by Smarandache [16], is as follows.

Definition 1. [16] Let X be a space of points (objects), with a generic element in X denoted by x. A neutrosophic set A in X is characterized by a truth membership function $T_A(x)$, an indeterminacy membership function $I_A(x)$, and a falsity membership function $F_A(x)$. The functions $T_A(x)$, $I_A(x)$, and $F_A(x)$ are real standard or nonstandard subsets of $]^-0, 1^+[$, that is, $T_A(x) : X \to]^-0, 1^+[$, $I_A(x) : X \to]^-0, 1^+[$, and $F_A(x) : X \to]^-0, 1^+[$, with the condition $^-0 \leq supT_A(x)+supI_A(x)+supF_A(x) \leq 3^+$.

This definition of neutrosophic set is difficult to apply in real world application of scientific and engineering fields. Therefore, the concept of Single Valued Neutrosophic Set (SVNS), which is an instance of a neutrosophic set was introduced by Wang et al. [17].

Definition 2. [17] Let X be a space of points (objects) with generic elements in X denoted by x. An Single Valued Neutrosophic Set (SVNS) A in X is characterized by truth membership function $T_A(x)$, indeterminacy membership function $I_A(x)$, and falsity membership function $F_A(x)$. For each point x in X, there are $T_A(x), I_A(x), F_A(x) \in [0, 1]$, and $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$. Therefore, an SVNS A can be represented by $A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle \mid x \in X\}$. The following expressions are defined in [17] for SVNSs A, B:

- $A \in B$ if and only if $T_A(x) \leq T_B(x), I_A(x) \geq I_B(x)$, $F_A(x) \geq F_B(x)$ for any x in X.
- $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$.
- $A^c = \{\langle x, F_A(x), 1 - I_A(x), T_A(x)\rangle | x \in X\}$.

The refined neutrosophic logic defined by [1] is as follows:

Definition 3. T can be split into many types of truths: $T_1, T_2, \ldots, T_p$, and I into many types of indeterminacies: I_1, I_2, $\ldots$, I_r, and F into many types of falsities: $F_1, F_2, \ldots, F_s$, where all $p, r, s \geq 1$ are integers, and $p + r + s = n$. In the same way, but all subcomponents T_j, I_k, F_l are not symbols, but subsets of [0, 1], for all $j \in \{1, 2, \ldots, p\}$ all $k \in \{1, 2, \ldots, r\}$ and all $l \in \{1, 2, \ldots, s\}$. If all sources of information that separately provide neutrosophic values for a specific subcomponent are independent sources, then in the general case we consider that each of the subcomponents T_j, I_k, F_l is independent with respect to the others and it is in the non-standard set $]^-0, 1^+[$.

Cross Entropy of SVNSs and Multicriteria Decision Making.

The concepts of cross-entropy and symmetric discrimination information measures between two fuzzy sets proposed by Shang and Jiang [9] and between two SVNSs was proposed by Ye [20].

Definition 4. Assume that $A = (A(x_1), A(x_2), \ldots, A(x_n))$ and $B = (B(x_1), B(x_2), \ldots, B(x_n))$ are two fuzzy sets in the universe of discourse $X = x_1, x_2, \ldots, x_n$. The fuzzy cross entropy of A from B is defined as follows:

$$H(A,B) = \sum_{i=1}^{n} \left\{ A(x_i) log_2 \frac{A(x_i)}{\frac{1}{2}(A(x_i) + B(x_i))} + (1 - A(x_i)) log_2 \frac{(1 - A(x_i))}{1 - \frac{1}{2}(A(x_i) + B(x_i))} \right\} \quad (1)$$

which indicates the degree of discrimination of A from B.

Shang and Jiang [9] proposed a symmetric discrimination information measure $I(A,B) = H(A,B) + H(B,A)$ since $H(A,B)$ is not symmetric with respect to its arguments. Moreover, there are $I(A,B) \geq 0$ and $I(A,B) = 0$ if and only if $A = B$. The cross entropy and symmetric discrimination information measures between two fuzzy sets was extended to SVNSs by Ye [20].

Let A and B be two SVNSs in a universe of discourse $X = \{x_l, x_2, \ldots, x_n\}$, which are denoted by $A = \{\langle x_i, T_A(x_i), I_A(x_i), F_A(x_i)\rangle \mid x_i \in X\}$ and $B = \{\langle x_i, T_B(x_i), I_B(x_i), F_B(x_i)\rangle \mid x_i \in X\}$, where $T_A(x_i), I_A(x_i), F_A(x_i), T_B(x_i), I_B(x_i), F_B(x_i) \in [0,1]$ for every $x_i \in X$.

The information carried by the truth, indeterminacy and falsity memberships in SVNSs, A and B is considered as fuzzy spaces with three elements. Based on Equation 1, the amount of information for discrimination of $T_A(x_i)$ from $T_B(x_i)$ ($i = 1, 2, \ldots, n$) is given as

$$E^T(A,B;x_i) = T_A(x_i) log_2 \frac{T_A(x_i)}{T_B(x_i)} + (1 - T_A(x_i)) log_2 \frac{1 - T_A(x_i)}{1 - \frac{1}{2}(T_A(x_i) + T_B(x_i))}.$$

The expected information based on the single membership for discrimination of A against B is

$$E^T(A,B) = \sum_{i=1}^{n} \left\{ T_A(x_i) log_2 \frac{T_A(x_i)}{T_B(x_i)} + \quad (1 - T_A(x_i)) log_2 \frac{1 - T_A(x_i)}{1 - \frac{1}{2}(T_A(x_i) + T_B(x_i))} \right\}.$$

Similarly, the indeterminacy and the falsity membership function, have the following amounts of information:

$$E^I(A,B) = \sum_{i=1}^{n} \left\{ I_A(x_i) log_2 \frac{I_A(x_i)}{I_B(x_i)} + (1 - I_A(x_i)) log_2 \frac{1 - I_A(x_i)}{1 - \frac{1}{2}(I_A(x_i) + I_B(x_i))} \right\}$$

$$E^F(A,B) = \sum_{i=1}^{n} \left\{ F_A(x_i) log_2 \frac{F_A(x_i)}{F_B(x_i)} + (1 - F_A(x_i)) log_2 \frac{1 - F_A(x_i)}{1 - \frac{1}{2}(F_A(x_i) + F_B(x_i))} \right\}.$$

The single valued neutrosophic cross entropy measure between A and B is obtained as the sum of the three measures:

$$E(A,B) = E^T(A,B) + E^I(A,B) + E^F(A,B)$$

$E(A,B)$ also indicates discrimination degree of A from B.

According to Shannon's inequality [6], it is seen that $E(A,B) \geq 0$, and $E(A,B) = 0$ if and only if $T_A(x_i) = T_B(x_i)$, $I_A(x_i) = I_B(x_i)$, and $F_A(x_i) = F_B(x_i)$ for any $x_i \in X$. Moreover, it is seen that $E(A^c, B^c) = E(A,B)$, where A^c and B^c are the complement of SVNSs of A and B, respectively. Then, $E(A,B)$ is not symmetric, i.e., $E(B,A) \neq E(A,B)$, so it is modified to a symmetric discrimination information measure for SVNSs as

$$D(A,B) = E(A,B) + E(B,A).$$

The larger the difference between A and B is, the larger $D(A,B)$ is. The cross entropy of SVNS was used to handle the multicriteria decision making problem under single valued neutrosophic environment by means of the cross entropy measure of SVNSs.

The weighted cross entropy between an alternative A_i and the ideal alternative A^* is calculated as

$$D(A^*,A_i)=\sum_{i=1}^{n}w_j\left\{log_2\frac{1}{\frac{1}{2}(1+T_{ij})}+log_2\frac{1}{1+\frac{1}{2}(I_{ij})}+log_2\frac{1}{1+\frac{1}{2}(F_{ij})}\right\}$$
$$+\sum_{i=1}^{n}w_j\left\{T_{ij}log_2\frac{T_{ij}}{\frac{1}{2}(1+T_{ij})}+(1-T_{ij})log_2\frac{1-T_{ij}}{1-\frac{1}{2}(1+T_{ij})}\right\}$$
$$+\sum_{i=1}^{n}w_j\left\{I_{ij}+(1-I_{ij})log_2\frac{1-I_{ij}}{1-\frac{1}{2}(I_{ij})}\right\}$$
$$+\sum_{i=1}^{n}w_j\left\{F_{ij}+(1-F_{ij})log_2\frac{1-F_{ij}}{1-\frac{1}{2}(F_{ij})}\right\}.$$

Based on the cross entropy value the ranking is carried out. The best alternative is selected based in the ranking of the cross entropy values.

Double Refined Indeterminacy Neutrosophic Sets (DRINSs) and Their Properties.

Indeterminacy deals with uncertainty that is faced in every sphere of life by everyone. It makes research/science more realistic and sensitive by introducing the indeterminate aspect of life as a concept. There are times in real world where the indeterminacy I can be identified to be indeterminacy which has more of truth value than false value, but it cannot be classified as truth. Similarly in some cases the indeterminacy can be identified to be indeterminacy which has more of false value than truth value, but it cannot be classified as false. To provide more sensitivity to indeterminacy, this kind of indeterminacy is classified into two. When the indeterminacy I can be identified as indeterminacy which is more of truth value than false value, but it cannot be classified as truth, it is considered to be indeterminacy leaning towards truth (I_T). Whereas in case the indeterminacy can be identified to be indeterminacy which is more of false value than truth value, but it cannot be classified as false, it is considered to be indeterminacy leaning towards false (I_F).

Indeterminacy leaning towards truth and indeterminacy leaning towards falsity make the handling of the indeterminacy involved in the scenario to be more meaningful, logical, accurate and precise. It provides a better and detailed view of the existing indeterminacy.

The definition of Double Refined Indeterminacy Neutrosophic Set (DRINS) [28] is as follows:

Definition 5. Let X be a space of points (objects) with generic elements in X denoted by x. A Double Refined Indeterminacy Neutrosophic Set (DRINS) A in X is characterized by truth membership function $T_A(x)$, indeterminacy leaning towards truth membership function $I_{TA}(x)$, indeterminacy leaning towards falsity membership function $I_{FA}(x)$, and falsity membership function $F_A(x)$. For each generic element $x\in X$, there are $T_A(x)$, $I_{TA}(x)$, $I_{FA}(x)$, $F_A(x)\in[0,1]$, and $0\le T_A(x)+I_{TA}(x)+I_{FA}(x)+F_A(x)\le 4$.

Therefore, a DRINS A can be represented by

$$A=\{\langle x,T_A(x),I_{TA}(x),I_{FA}(x),F_A(x)\rangle\mid x\in X\}.$$

A DRINS A is represented as

$$A=\int_X\{\langle T(x),I_T(x),I_F(x),F(x)\rangle/dx,x\in X\}$$

when X is continuous. It is represented as

$$A = \sum_{i=1}^{n} \{\langle T(x_i), I_T(x_i), I_F(x_i), F(x_i)\rangle \mid x_i, x_i \in X\}$$

when X is discrete.

To illustrate the application of DRINS in the real world condsider parameters that are commonly used to define quality of service of semantic web services like capability, trustworthiness and price for illustrative purpose. The evaluation of quality of service of semantic web services [29] is used to illustrate set theoretic operation on Double Refined Indeterminacy Neutrosophic Sets (DRINSs).

Definition 6. The complement of a DRINS A denoted by $c(A)$ is defined as $T_{c(A)}(x) = F_A(x)$, $I_{Tc(A)}(x) = 1 - I_{TA}(x)$, $I_{Fc(A)}(x) = 1 - I_{FA}(x)$ and $F_{c(A)}(x) = T_A(x)$ for all x in X.

Definition 7. A DRINS A is contained in the other DRINS B, that is $A \subseteq B$, if and only if $T_A(x) \leq T_B(x)$, $I_{TA}(x) \leq I_{TB}(x)$, $I_{FA}(x) \leq I_{FB}(x)$ and $F_A(x) \geq F_B(x)$ $\forall x$ in X.

Note that by the definition of containment relation, X is a partially ordered set and not a totally ordered set.

Definition 8. Two DRINSs A and B are equal, denoted as $A = B$, if and only if $A \subseteq B$ and $B \subseteq A$.

The union of two DRINSs A and B is a DRINS C, denoted as $C = A \cup B$, , the intersection of two DRINSs A and B is a DRINS C, denoted as $C = A \cap B$, and the difference of two DRINSs A and B is D, written as $D = A \setminus B$, was defined in [28]. Three operators called as truth favourite ($\triangle$), falsity favourite ($\bigtriangledown$) and indeterminacy neutral (∇) are defined over DRINSs. Two operators truth favourite ($\triangle$) and falsity favourite ($\bigtriangledown$) are defined to remove the indeterminacy in the DRINSs and transform it into intuitionistic fuzzy sets or paraconsistent sets. Similarly the DRINS can be transformed into a SVNS by applying indeterminacy neutral (∇) operator that combines the indeterminacy values of the DRINS. These three operators are unique on DRINSs.

Definition 9. The truth favourite of a DRINS A, written as $B = \triangle A$, whose truth membership and falsity membership functions are related to those of A by $T_B(x) = min(T_A(x) + I_{TA}(x), 1)$, $I_{TB}(x) = 0$, $I_{FB}(x) = 0$ and $F_B(x) = F_A(x)$ for all x in X.

Definition 10. The falsity favourite of a DRINS A, written as $B = \bigtriangledown A$, whose truth membership and falsity membership functions are related to those of A by $T_B(x) = T_A(x)$, $I_{TB}(x) = 0$, $I_{FB}(x) = 0$ and $F_B(x) = min(F_A(x) + I_{FA}(x), 1)$ for all x in X.

Definition 11. The indeterminacy neutral of a DRINS A, written as $B = \nabla A$, whose truth membership, indeterminate membership and falsity membership functions are related to those of A by $T_B(x) = T_A(x)$, $I_{TB}(x) = min(I_{TA}(x) + I_{TB}(x), 1)$, $I_{FB}(x) = 0$ and $F_B(x) = F_A(x)$ for all x in X.

All set theoretic operators like commutativity, Associativity, Distributivity, Idempotency, Absorption and the De Morgan's Laws were defined over DRINSs [28]. The definition of complement, union and intersection of DRINSs and DRINSs itself satisfies most properties of the classical set, fuzzy set, intuitionistic fuzzy set and SNVS. Similar to fuzzy set, intuitionistic fuzzy set and SNVS, it does not satisfy the principle of middle exclude.

Cross Entropy of Double Refined Indeterminacy Neutrosophic Sets (DRINSs)

Consider two DRINSs A and B in a universe of discourse $X = x_l, x_2, \ldots, x_n$, which are denoted by
$A = \{\langle x_i, T_A(x_i), I_{TA}(x_i), I_{FA}(x_i), F_A(x_i)\rangle \mid x_i \in X\}$
and $B = \{\langle x_i, T_B(x_i), I_{TB}(x_i), I_{FB}(x_i), F_B(x_i)\rangle \mid x_i \in X\}$,
where $T_A(x_i), I_{TA}(x_i), I_{FA}(x_i), F_A(x_i), T_B(x_i), I_{TB}(x_i), I_{FB}(x_i), F_B(x_i) \in [0,1]$ for every $x_i \in X$.

The information carried by the truth membership, indeterminacy leaning towards truth membership, indeterminacy leaning towards falsity membership, and the falsity membership in DRINSs A and B are considered as fuzzy spaces with four elements. Thus based on Equation 1, the amount of information for discrimination of $T_A(x_i)$ from $T_B(x_i)$ ($i = 1, 2, \ldots, n$) can be given by

$$E^T(A, B; x_i) = T_A(x_i) log_2 \frac{T_A(x_i)}{T_B(x_i)} + (1 - T_A(x_i)) log_2 \frac{1 - T_A(x_i)}{1 - \frac{1}{2}(T_A(x_i) + T_B(x_i))}$$

Therefore, the expected information based on the single membership for discrimination of A against B is expressed by

$$E^T(A, B) = \sum_{i=1}^{n} \left\{ T_A(x_i) log_2 \frac{T_A(x_i)}{T_B(x_i)} + + (1 - T_A(x_i)) log_2 \frac{1 - T_A(x_i)}{1 - \frac{1}{2}(T_A(x_i) + T_B(x_i))} \right\}$$

Similarly, considering the indeterminacy leaning towards truth membership function, indeterminacy leaning towards falsity membership function and falsity membership function the following amounts of information is given:

$$E^{I_T}(A, B) = \sum_{i=1}^{n} \left\{ I_{TA}(x_i) log_2 \frac{I_{TA}(x_i)}{I_{TB}(x_i)} + (1 - I_{TA}(x_i)) log_2 \frac{1 - I_{TA}(x_i)}{1 - \frac{1}{2}(I_{TA}(x_i) + I_{TB}(x_i))} \right\},$$

$$E^{I_F}(A, B) = \sum_{i=1}^{n} \left\{ I_{FA}(x_i) log_2 \frac{I_{FA}(x_i)}{I_{FB}(x_i)} + (1 - I_{FA}(x_i)) log_2 \frac{1 - I_{FA}(x_i)}{1 - \frac{1}{2}(I_{FA}(x_i) + I_{FB}(x_i))} \right\},$$

$$E^F(A, B) = \sum_{i=1}^{n} \left\{ F_A(x_i) log_2 \frac{F_A(x_i)}{F_B(x_i)} + (1 - F_A(x_i)) log_2 \frac{1 - F_A(x_i)}{1 - \frac{1}{2}(F_A(x_i) + F_B(x_i))} \right\}.$$

The Double Refined Indeterminacy neutrosophic cross entropy measure between A and B is obtained as the sum of the four measures:

$$E(A, B) = E^T(A, B) + E^{I_T}(A, B) + E^{I_F}(A, B) + E^F(A, B)$$

$E(A, B)$ also indicates discrimination degree of A from B. According to Shannon's inequality [5], it can be easily proved that $E(A, B) \geq 0$, and $E(A, B) = 0$ if and only if $T_A(x_i) = T_B(x_i)$, $I_{TA}(x_i) = I_{TB}(x_i)$, $I_{FA}(x_i) = I_{FB}(x_i)$, and $F_A(x_i) = F_B(x_i)$ for any $x_i \in X$. It easily seen that $E(A^c, B^c) = E(A, B)$, where A^c and B^c are the complement of DRINSs A and B, respectively. Since $E(A, B)$ is not symmetric it is modified to a symmetric discrimination information measure for DRINSs as

$$D(A, B) = E(A, B) + E(B, A) \quad (2)$$

The larger $D(A, B)$ is, the larger the difference between A and B is.

A cross entropy measure based only on the indeterminacy involved in the scenario is introduced in this paper. Indeterminacy based cross entropy is defined as the sum of information of indeterminacy leaning towards falsity membership and information of indeterminacy leaning towards truth membership. The indeterminacy based cross entropy $IE(A, B)$ is the Double Refined Indeterminacy neutrosophic cross entropy measure based on indeterminacy between A and B; is obtained as the sum of the two measures:

$$IE(A,B) = \sum_{i=1}^{n} \left\{ I_{TA}(x_i) log_2 \frac{I_{TA}(x_i)}{I_{TB}(x_i)} + (1 - I_{TA}(x_i)) log_2 \frac{1 - I_{TA}(x_i)}{1 - \frac{1}{2}(I_{TA}(x_i) + I_{TB}(x_i))} \right\}$$
$$+ \sum_{i=1}^{n} \left\{ I_{FA}(x_i) log_2 \frac{I_{FA}(x_i)}{I_{FB}(x_i)} + (1 - I_{FA}(x_i)) log_2 \frac{1 - I_{FA}(x_i)}{1 - \frac{1}{2}(I_{FA}(x_i) + I_{FB}(x_i))} \right\}$$

It indicates the discrimination degree of indeterminacy of A from B. According to Shannon's inequality [5], it can be easily proved that $IE(A, B) \geq 0$, and $IE(A, B) = 0$ if and only if $T_A(x_i) = T_B(x_i), I_{TA}(x_i) = I_{TB}(x_i), I_{FA}(x_i) = I_{FB}(x_i)$, and $F_A(x_i) = F_B(x_i)$ for any $x_i \in X$. It is easily seen that $IE(A^c, B^c) = IE(A, B)$, where A^c and B^c are the complement of DRINSs A and B, respectively. Since $IE(A, B)$ is not symmetric, it is modified to a symmetric discrimination information measure for DRINSs as

$$ID(A, B) = IE(A, B) + IE(B, A). \quad (3)$$

The larger the difference in indeterminacy between A and B is, the larger $ID(A, B)$ is.

Multicriteria Decision Making Method Based on the Cross Entropy of DRINS

In a multicriteria decision making problem all the alternatives are evaluated depending on a number of criteria or some attributes, and the best alternative is selected from all the possible alternatives. Mostly multicriteria decision making problem have to be inclusive of uncertain, imprecise, incomplete, and inconsistent information that are present in real world to make it more realistic. DRINS can be used to represent this information with accuracy and precision. In this section, by means of utilizing the cross entropy measure of DRINSs and indeterminacy based cross entropy a method for solving the multi-criteria decision making problem when considered in a Double Refined Indeterminacy neutrosophic environment, is proposed.

Let $A = \{A_1, A_2, \ldots, A_m\}$ be a set of feasible alternatives and $C = \{C_1, C_2, \ldots, C_n\}$ be the set of criteria under consideration. The weight of the criterion $C_j (j = 1, 2, \ldots, n)$, provided by the decision maker, is w_j, $w_j \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. The characteristic of the alternative $A_i (i = 1, 2, \ldots, m)$ is given by DRINS $A_i = \{\langle C_j, T_{A_i}(C_j), I_{TA_i}(C_j), I_{FA_i}(C_j), F_{A_i}(C_j)\rangle \mid C_j \in C\}$ where $T_{A_i}(C_j), I_{TA_i}(C_j), I_{FA_i}(C_j), F_{A_i}(C_j) \in [0, 1]$, $j = 1, 2, \ldots, n$ and $i = 1, 2, \ldots, m$.

Now, $T_{A_i}(C_j)$ specifies the degree to which the alternative A_i fulfils the criterion C_j, $I_{TA_i}(C_j)$ specifies the indeterminacy leaning towards truth degree to which the alternative A_i fulfils or does not fulfil the criterion C_j. Similarly $I_{FA_i}(C_j)$ specifies the indeterminacy leaning towards false degree (or false leaning indeterminacy) to which the alternative A_i fulfils or does not fulfil the criterion C_j, and $F_{A_i}(C_j)$ specifies the degree to which the alternative A_i does not fulfil the criterion C_j.

A criterion value is generally obtained from the calculation of an alternative A_i with respect to a criteria C_j by means of a score law and data processing in practice [12, 17]. It is represented as $\langle C_j, T_{A_i}(C_j), I_{TA_i}(C_j), I_{FA_i}(C_j), F_{A_i}(C_j)\rangle$ in A_i, is denoted by the symbol $a_{ij} = \langle T_{ij}, I_{Tij}, I_{Fij}, F_{ij}\rangle$ $(j = 1, 2, \ldots, n$, and $i = 1, 2, \ldots, m)$,

Therefore, a Double Refined Indeterminacy neutrosophic decision matrix $A = (a_{ij})_{m\times n}$ is obtained.

$$A = \begin{pmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} \langle T_{11}, I_{T11}, I_{F11}, F_{11}\rangle & \langle T_{12}, I_{T12}, I_{F12}, F_{12}\rangle & \dots & \langle T_{1n}, I_{T1n}, I_{F1n}, F_{1n}\rangle \\ \langle T_{21}, I_{T21}, I_{F21}, F_{21}\rangle & \langle T_{22}, I_{T22}, I_{F22}, F_{22}\rangle & \dots & \langle T_{2n}, I_{T2n}, I_{F2n}, F_{2n}\rangle \\ \vdots & \vdots & & \vdots \\ \langle T_{m1}, I_{Tm1}, I_{Fm1}, F_{m1}\rangle & \langle T_{m2}, I_{Tm2}, I_{Fm2}, F_{m2}\rangle & \dots & \langle T_{mn}, I_{Tmn}, I_{Fmn}, F_{mn}\rangle \end{pmatrix}.$$

The concept of ideal point is utilized to aid the identification of the best alternative in the decision set, in multicriteria decision making environments. It is known that an ideal alternative cannot exist in the real world; but it does serves as a useful theoretical construct against which alternatives can be evaluated [12].

Therefore an ideal criterion value $a_j^* = \langle T_j^*, I_{Tj}^*, I_{Fj}^*, F_j^*\rangle = \langle 1, 0, 0, 0\rangle (j = 1, 2, \dots, n)$ is defined in the ideal alternative A^*. By applying Equation 2 the weighted cross entropy between an alternative A_i and ideal alternative A^* is obtained to be

$$D(A^*, A_i) = \sum_{j=1}^{n} w_j \left\{ log_2 \frac{1}{\frac{1}{2}(1+T_{ij})} + log_2 \frac{1}{1-\frac{1}{2}(I_{Tij})} + log_2 \frac{1}{1-\frac{1}{2}(I_{Fij})} + log_2 \frac{1}{1-\frac{1}{2}(F_{ij})} \right\} + \sum_{j=1}^{n} w_j \left\{ T_{ij} log_2 \frac{T_{ij}}{\frac{1}{2}(1+T_{ij})} + (1-T_{ij}) log_2 \frac{1-T_{ij}}{1-\frac{1}{2}(1+T_{ij})} \right\} + \sum_{j=1}^{n} w_j \left\{ I_{Tij} + (1-I_{Tij}) log_2 \frac{1-I_{Tij}}{1-\frac{1}{2}(I_{Tij})} \right\} + \sum_{j=1}^{n} w_j \left\{ I_{Fij} + (1-I_{Fij}) log_2 \frac{1-I_{Fij}}{1-\frac{1}{2}(I_{Fij})} \right\} + \sum_{j=1}^{n} w_j \left\{ F_{ij} + (1-F_{ij}) log_2 \frac{1-F_{ij}}{1-\frac{1}{2}(F_{ij})} \right\}. \quad (4)$$

The smaller the value of $D_i(A^*, A_i)$ is, the better the alternative A_i is, it implies that the alternative A_i is close to the ideal alternative A^*. The ranking order of all alternatives is determined and the best one is identified, through the calculation of the weighted cross entropy $D_i(A^*, A_i)$ $(i = 1, 2, \dots, m)$ between each alternative and the ideal alternative.

For calculating the indeterminate based cross entropy measure $I_T D$ between alternative A and the indeterminate ideal alternative $A_{I_T}^*$ the indeterminate ideal alternative $A_{I_T}^*$ is defined as an ideal criterion value

$$a_j^* = \langle T_j^*, I_{Tj}^*, I_{Fj}^*, F_j^*\rangle = \langle 0, 1, 0, 0\rangle (j = 1, 2, \dots, n).$$

By applying Equation 3 the weighted indeterminacy based cross entropy between an alternative A_i and the ideal alternative $A_{I_T}^*$ is obtained to be

$$I_T D(A_{I_T}^*, A_i) = \sum_{j=1}^{n} w_j \left\{ log_2 \frac{1}{\frac{1}{2}(1+I_{Tij})} + log_2 \frac{1}{1-\frac{1}{2}(I_{Fij})} \right\} + \sum_{j=1}^{n} w_j \left\{ I_{Tij} log_2 \frac{I_{Tij}}{\frac{1}{2}(1+I_{Tij})} + (1-I_{Tij}) log_2 \frac{1-I_{Tij}}{1-\frac{1}{2}(1+I_{Tij})} \right\} + \sum_{j=1}^{n} w_j \left\{ I_{Fij} + (1-I_{Fij}) log_2 \frac{1-I_{Fij}}{1-\frac{1}{2}(I_{Fij})} \right\}. \quad (5)$$

To study the indeterminate ideal alternative $A_{I_F}^*$, which based on indeterminacy leaning towards falsity, it is defined using the ideal criterion value $a_j^* = \langle T_j^*, I_{Tj}^*, I_{Fj}^*, F_j^*\rangle = \langle 0, 0, 1, 0\rangle (j = 1, 2, \dots, n)$.

Calculating the indeterminate based cross entropy measure $I_F D$ between alternative A and the indeterminate ideal alternative $A^*_{I_F}$

$$I_F D(A^*_{IF}, A_i) = \sum_{i=1}^{n} w_j \left\{ log_2 \frac{1}{\frac{1}{2}(1 + I_{Fij})} + log_2 \frac{1}{1 - \frac{1}{2}(I_{Tij})} \right\} + \sum_{i=1}^{n} w_j \left\{ I_{Fij} log_2 \frac{I_{Fij}}{\frac{1}{2}(1 + I_{Fij})} + (1 - I_{Fij}) log_2 \frac{1 - I_{Fij}}{1 - \frac{1}{2}(1 + I_{Fij})} \right\} + \sum_{i=1}^{n} w_j \left\{ I_{Tij} + (1 - I_{Tij}) log_2 \frac{1 - I_{Tij}}{1 - \frac{1}{2}(I_{Tij})} \right\} \quad (6)$$

The average of $I_T D(A^*_{IT}, A_i)$ and $I_F D(A^*_{IF}, A_i)$ is taken as $ID(A^*_I, A_i)$.

$$ID(A^*, A_i) = \frac{I_T D(A^*_{IT}, A_i) + I_F D(A^*_{IF}, A_i)}{2}. \quad (7)$$

The larger the value of $ID(A^*, A_i)$ is, the better the alternative A_i is, it implies that the alternative A_i is farther to the ideal alternative A^*. The ranking order of all alternatives is determined and the best one is identified, through the calculation of the indeterminacy based weighted cross entropy $ID(A^*, A_i)$ $(i = 1, 2, \ldots, m)$ between each alternative and the ideal alternative.

Illustrative Examples

To illustrate the application of the proposed method, the multicriteria decision making problem from Tan and Chen [30] and Ye[19] is adapted. It is related with a manufacturing company that wants to select the best global supplier according to the core competencies of suppliers. Suppose that there are four suppliers $A = A_1, A_2, A_3, A_4$ enlisted; whose core competencies are evaluated based of the following four criteria (C_1, C_2, C_3, C_4):

1. (C_1) the level of technology innovation,
2. (C_2) the control ability of flow,
3. (C_3) the ability of management, and
4. (C_4) the level of service.

The weight vector related to the four criteria is w = (0.3, 0.25, 0.25, 0.2).

The proposed multicriteria decision making approach is applied to select the best supplier. From the questionnaire of a domain expert, the evaluation of an alternative A_i (i = 1, 2, 3, 4) with respect to a criterion C_j(j = 1, 2, 3, 4), is obtained. For instance, when the opinion of an expert about an alternative A_1 with respect to a criterion C_1 is asked, he or she may say that the possibility in which the statement is true is 0.5, the degree in which he or she feels it true but is not sure is 0.07, the degree in which he or she feels it is false but is not sure is 0.03 and the possibility the statement is false is 0.3. It can be expressed as $a_{11} = \langle 0.5, 0.07, 0.03, 0.2 \rangle$, using the neutrosophic expression of DRINS. The possible alternatives with respect to the given four criteria is evaluated by the similar method from the expert, the following Double Refined Indeterminacy neutrosophic decision matrix A is obtained.

$$A = \begin{pmatrix} \langle 0.5, 0.07, 0.03, 0.2 \rangle & \langle 0.5, 0.08, 0.02, 0.4 \rangle & \langle 0.7, 0.06, 0.04, 0.2 \rangle & \langle 0.3, 0.4, 0.1, 0.1 \rangle \\ \langle 0.4, 0.12, 0.08, 0.3 \rangle & \langle 0.3, 0.04, 0.16, 0.4 \rangle & \langle 0.9, 0.06, 0.04, 0.1 \rangle & \langle 0.5, 0.1, 0.1, 0.2 \rangle \\ \langle 0.4, 0.17, 0.03, 0.1 \rangle & \langle 0.5, 0.18, 0.02, 0.3 \rangle & \langle 0.5, 0.06, 0.04, 0.4 \rangle & \langle 0.6, 0.14, 0.06, 0.1 \rangle \\ \langle 0.6, 0.07, 0.03, 0.2 \rangle & \langle 0.2, 0.15, 0.05, 0.5 \rangle & \langle 0.4, 0.16, 0.04, 0.2 \rangle & \langle 0.7, 0.11, 0.09, 0.1 \rangle \end{pmatrix}$$

The cross entropy values between an alternative A_i (i = 1, 2, 3, 4) and the ideal alternative A^* is obtained by applying Equation 4 is $D(A^*, A_1)$ = 1.5054, $D(A^*, A_2)$ = 1.1056, $D(A^*, A_3)$ = 1.0821 and $D(A^*, A_4)$ = 1.1849. The ranking order of the four suppliers according to the cross entropy values is

$$D(A^*, A_1) \leq D(A^*, A_3) \leq D(A^*, A_2) \leq D(A^*, A_4)$$

The truth-indeterminacy based cross entropy values between an alternative A_i (i = 1, 2, 3, 4) and the ideal alternative $A^*_{I_T}$ is obtained by applying Equation 5, are $I_T D(A^*, A_1)$ = 1.5054, $I_T D(A^*, A_2)$ = 1.6920, $I_T D(A^*, A_3)$ = 1.4392 and $I_T D(A^*, A_4)$ = 1.5067. The false-indeterminacy based cross entropy values between an alternative A_i (i = 1, 2, 3, 4) and the ideal alternative $A^*_{I_F}$ is obtained by applying Equation 6, are $I_F D(A^*, A_1)$ = 1.9000, $I_F D(A^*, A_2)$ = 1.6348, $I_F D(A^*, A_3)$ = 1.9256 and $I_F D(A^*, A_4)$ = 1.8447. The indeterminacy based cross entropy values based on Equation 7are $ID(A^*, A_1)$ = 1.7027, $ID(A^*, A_2)$ = 1.6634, $ID(A^*, A_3)$ = 1.6824 and $ID(A^*, A_4)$ = 1.6757.

The ranking order of the four suppliers according to the cross entropy values is

$$ID(A^*, A_1) \geq ID(A^*, A_3) \geq ID(A^*, A_4) \geq ID(A^*, A_2).$$

The DRINS cross entropy and indeterminacy based cross entropy results of the different alternatives and the ideal alternatives are tabulated in Table 1.

Table 1: DRIN Cross Entropy and indeterminacy based Cross Entropy Results

Cross Entropy Value A_i	DRIN Cross Entropy $D(A^*, A_i)$	Indeterminate based cross entropy $ID(A^*_I, A_i)$
A_1	1.0793	1.7027
A_2	1.1056	1.6634
A_3	1.0821	1.6824
A_4	1.1845	1.6757
Result	A_1	A_1

An alternative is considered to be best if it has the least DRIN cross entropy value and the maximum indeterminate based DRIN cross entropy. Therefore it is seen that A_1 is the best supplier.

It is clearly seen that the proposed Double Refined Indeterminacy neutrosophic multicriteria decision making method is more preferable and suitable for real scientific and engineering applications because it can handle not only incomplete information but also the indeterminate information and inconsistent information which exist commonly in real situations more logically with much more accuracy and precision that SVNS are incapable of dealing.

Comparison

This paper proposes a technique that extends existing SVNS and fuzzy decision making methods and provides an improvement in dealing indeterminate and inconsistent information with accuracy which is new for decision making problems. For comparative purpose, the results of cross entropy of SVNS [20] and the proposed method are given in Table 2.

From Table 2, it is seen that the results are quite different. The important reason can be obtained by the following comparative analysis of the methods and their capacity to deal indeterminate, inconsistent and incomplete information.

Double Refined Indeterminacy neutrosophic information is a generalization of neutrosophic information. It is observed that neutrosophic information / single valued neutrosophic information is

Table 2: Cross Entropy results of different cross entropy between ideal alternative and alternative

Cross Entropy Value	SVN cross Entropy $D_i(A^*, A_i)$	DRIN Cross Entropy $D(A^*, A_i)$	Indeterminate based cross entropy $ID(A_I^*, A_i)$
A_1	1.1101	1.0793	1.7027
A_2	1.1801	1.1056	1.6634
A_3	0.9962	1.0821	1.6824
A_4	1.2406	1.1850	1.6757
Result	A_3	A_1	A_1

generalization of intuitionistic fuzzy information, and intuitionistic fuzzy information is itself a generalization of fuzzy information.

DRINS is an instance of a neutrosophic set, which approaches the problem more logically with accuracy and precision to represent the existing uncertainty, imprecise, incomplete, and inconsistent information. It has the additional feature of being able to describe with more sensitivity the indeterminate and inconsistent information. While, the SVNS can handle indeterminate information and inconsistent information, it is cannot describe with accuracy about the existing indeterminacy.

It is known that the connector in fuzzy set is defined with respect to T (membership only) so the information of indeterminacy and non membership is lost. The connectors in intuitionistic fuzzy set are defined with respect to truth membership and false membership only; here the indeterminacy is taken as what is left after the truth and false membership.

The intuitionistic fuzzy set cannot deal with the indeterminate and inconsistent information but it has provisions to describe and deal with incomplete information. In SVNS, truth, indeterminacy and falsity membership are represented independently, and they can also be defined with respect to any of them (no restriction) and the approach is more logical. This makes SVNS equipped to deal information better than IFS, whereas in DRINS, more scope is given to describe and deal with the existing indeterminate and inconsistent information because the indeterminacy concept is classified as two distinct values. This provides more accuracy and precision to indeterminacy in DRINS, than SVNS.

It is clearly noted that in the case of the SVN cross entropy based multicriteria decision making method that was proposed in [20], that the indeterminacy concept/ value is not classified into two, but it is represented as a single valued neutrosophic data leading to a loss of accuracy of the indeterminacy. SVNS are incapable of giving this amount of logical approach with accuracy or precision about the indeterminacy concept. Similarly when the intuitionistic fuzzy cross entropy was considered, it was not possible to deal with the indeterminacy membership function independently as it is dealt in SVN or DRIN cross entropy based multicriteria decision making method, leading to a loss of information about the existing indeterminacy. In the fuzzy cross entropy, only the membership degree is considered, details of non membership and indeterminacy are completely lost. It is clearly observed that the DRINs representation and the DRIN-cross entropy based multicriteria decision-making method are better logically equipped to deal with indeterminate, inconsistent and incomplete information.

Conclusions

In this paper a special case of refined neutrosophic set, called as Double Refined Indeterminacy Neutrosophic Set (DRINS), with two distinct indeterminate values was utilized in multicriteria decision making problem. Better logical approach and precision is provided to indeterminacy since the indeterminate concept/value is classified into two based on membership: one as indeterminacy leaning towards truth membership and another as indeterminacy leaning towards false membership. This kind of classification of indeterminacy is not feasible with SVNS. DRINS is better equipped at dealing indeterminate and inconsistent information, with more accuracy than Single Valued Neutrosophic Set (SVNS), which fuzzy sets and Intuitionistic Fuzzy sets are incapable of.

In this paper the cross entropy of DRINS was defined and it was applied solve the multicriteria decision making problem, this approach is called as DRIN cross entropy based multicriteria decision-making method. Through the illustrative computational sample of the DRIN cross entropy based multicriteria decision-making method and other methods, the results have shown that the DRIN-cross entropy based multicriteria decision-making method is more general and more reasonable than the others. Furthermore, in situations that are represented by indeterminate information and inconsistent information, the DRIN cross entropy based multicriteria decision-making method exhibits its great superiority in clustering those Double Refined Indeterminacy neutrosophic data because the DRINSs are a powerful tool to deal with uncertain, imprecise, incomplete, and inconsistent information with accuracy. In the future, DRINS sets and the DRIN cross entropy based multicriteria decision-making method can be applied to many areas such as online social networks, information retrieval, investment decision making, and data mining where fuzzy theory has been used.

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Multi-attribute Decision Making based on Rough Neutrosophic Variational Coefficient Similarity Measure

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Abstract: The purpose of this study is to propose new similarity measures namely rough variational coefficient similarity measure under the rough neutrosophic environment. The weighted rough variational coefficient similarity measure has been also defined. The weighted rough variational coefficient similarity measures between the rough ideal alternative and each alternative are calculated to find the best alternative. The ranking order of all the alternatives can be determined by using the numerical values of similarity measures. Finally, an illustrative example has been provided to show the effectiveness and validity of the proposed approach. Comparisons of decision results of existing rough similarity measures have been provided.

Keywords: Neutrosophic set, Rough neutrosophic set; Rough variation coefficient similarity measure; Decision making.

1 Introduction

In 1965, L. A. Zadeh grounded the concept of degree of membership and defined fuzzy set [1] to represent/manipulate data with non-statistical uncertainty. In 1986, K. T. Atanassov [2] introduced the degree of non-membership as independent component and proposed intuitionistic fuzzy set (IFS). F. Smarandache introduced the degree of indeterminacy as independent component and defined the neutrosophic set [3, 4, 5]. For purpose of solving practical problems, Wang et al. [6] restricted the concept of neutrosophic set to single valued neutrosophic set (SVNS), since single value is an instance of set value. SVNS is a subclass of the neutrosophic set. SVNS consists of the three independent components namely, truth-membership, indeterminacy-membership and falsity-membership functions.

The concept of rough set theory proposed by Z. Pawlak [7] is an extension of the crisp set theory for the study of intelligent systems characterized by inexact, uncertain or insufficient information. The hybridization of rough set theory and neutrosophic set theory produces the rough neutrosophic set theory [8, 9], which was proposed by Broumi, Dhar and Smarandache [8, 9]. Rough neutrosophic set theory is also a powerful mathematical tool to deal with incompleteness.

Literature review reflects that similarity measures play an important role in the analysis and research of clustering analysis, decision making, medical diagnosis, pattern recognition, etc. Various similarity measures [10, 11, 12, 13, 14, 15, 16, 17, 18] of SVNSs and hybrid SVNSs are available in the literature. The concept of similarity measures in rough neutrosophic environment [19, 20, 21] has been recently proposed.

Pramanik and Mondal [19] proposed cotangent similarity measure of rough neutrosophic sets. In the same study [19],Pramanik and Mondal established its basic properties and provided its application to medical diagnosis. Pramanik and Mondal [20] also proposed cosine similarity measure of rough neutrosophic sets and its application in medical diagnosis. The same authors [21] also studied Jaccard similarity measure and Dice similarity measures in rough neutrosophic environment and provided their applications to multi attribute decision making. Mondal and Pramanik [22] presented tri-complex rough neutrosophic similarity measure and its application in multi-attribute decision making. Together with F. Smarandache and S. Pramnik, K. Mondal [23] presented hypercomplex rough neutrosophic similarity measure and its application in multi-attribute decision making. Mondal, Pramanik, and Smarandache [24] presented several trigonometric Hamming similarity measures of rough neutrosophic sets and their applications in multi attribute decision making problems.

Different methods for multiattribute decision making (MADM) and multicriteria decision making (MCDM) problems are available in the literature in different environment such as crisp environment [25, 26, 27, 28, 29], fuzzy environment [30, 31], intuitionistic fuzzy environment [32, 33, 34, 35, 36, 37, 38, 39, 40], neutrosophic environment [41, 42, 43, 44, 45, 46, 47, 48, 49, 50, 51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62], interval neutrosophic environment [63, 65, 66, 67, 68], neutrosophic soft expert environment [69], neutrosophic bipolar environment [70, 71], neutrosophic soft environment [72, 73, 74, 75, 76], neutrosophic hesitant fuzzy environment [77, 78, 79], rough neutrolsophic environment [80, 81], etc. In neutrosophic environment Biswas, Pramanik and Giri [82] studied hybrid vector similarity measure and its application in multi-attribute decision making. Getting motivation from the work of Biswas, Pramanik and Giri [82], for hybrid vector similarity measure in neutrosophic envionment, we extend the concept in rough neutrosophic environment.

In this paper, a new similarity measurement is proposed, namely rough variational coefficient similarity measure under rough neutrosophic environment. A numerical example is also provided.

Rest of the paper is structured as follows. Section 2 presents neutrosophic and rough neutrosophic preliminaries. Section 3 discusses various similarity measures and varional coefficient similarity measure in crisp environment. Section 4 presents various similarity measures and variational similarity measure for single valued neutrosophic sets. Section 5 presents variational coefficient similarity measure and weighted variational coefficient similarity measure for rough neutrosophic sets and establishes their basic properties. Section 6 is devoted to present multi attribute decision making based on rough neutrosophic variational coefficient similarity measure. Section 7 demonstrates the application of rough variational coefficient similarity measures to investment problem Finally, section 8 concludes the paper with stating the future scope of research.

2 Neutrosophic preliminaries

Definition 2.1 [3, 4, 5] Neutrosophic set

Let X be a space of points (objects) with generic element in X denoted by x. Then a neutrosophic set A in X is denoted by $A=\{x(T_A(x),I_A(x),F_A(x)): x\in X\}$ where, $T_A(x)$ is the truth membership function, $I_A(x)$ is the indeterminacy membership function and $F_A(x)$ is the falsity membership function. The functions $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or non-standard subsets of $]^-0, 1^+[$. There is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$ i.e. $^-0 \le T_A(x)+I_A(x)+F_A(x) \le 3^+$.

Definition 2.2 [6] (**Single-valued neutrosophic set**).

Let X be a universal space of points (objects), with a generic element $x\in X$. A single-valued neutrosophic set (SVNS) $N\subset X$ is denoted by

$N=\int_x \langle T_N(x), I_N(x), F_N(x)\rangle / x, \forall x\in X$, when X is continuous;

$N=\sum_{i=1}^{m}\langle T_N(x), I_N(x), F_N(x)\rangle / x, \forall x\in X$, when X is discrete.

SVNS is characterized by a true membership function $T_N(x)$, a falsity membership function $F_N(x)$ and an indeterminacy function $I_N(x)$ ith $T_N(x)$, $F_N(x)$, $I_N(x)$ $\in$ [0, 1] for all $x\in X$. For each $x\in X$, of a SVNS N $0\le T_N(x)+I_N(x)+F_N(x)\le 3$.

2.1 Some operational rules and properties of SVNSs

Let $N_A=\langle T_A, I_A, F_A\rangle$ and $N_B=\langle T_B, I_B, F_B\rangle$ be two SVNSs in *X*. Then the following operations are defined as follows:

I. Complement: $N_{A^c}=\langle F_A, 1\text{-}I_A, T_A\rangle \ \forall x\in X$.

II. Addition: $N_A\oplus N_B=\langle T_A+T_B-T_AT_B, I_AI_B, F_AF_B\rangle$

III. Multiplication:
$N_A\otimes N_B=\langle T_AT_B, I_A+I_B\text{-}I_AI_B, F_A+F_B\text{-}F_AF_B\rangle$

IV. Scalar Multiplication:
$\lambda N_A=\langle 1-(1-T_A)^{\lambda}, I_A^{\lambda}, F_A^{\lambda}\rangle$ for $\lambda>0$.

V. $\langle N_A\rangle^{\lambda}=\langle (T_A)^{\lambda}, 1-(1-I_A)^{\lambda}, 1-(1-F_A)^{\lambda}\rangle$ for $\lambda>0$.

Definition 2.3 [6]

Complement of a SVNS N is denoted by N^c and is defined by
$T_{N^c}(x)=F_N(x)$; $I_{N^c}(x)=1-I_N(x)$; $F_{N^c}(x)=T_N(x)$

Definition 2.4 [6]

A SVNS N_A is contained in the other SVNS N_B, denoted as $N_A\subseteq N_B$, if and only if
$T_{N_A}(x)\le T_{N_B}(x)$; $I_{N_A}(x)\ge I_{N_B}(x)$; $F_{N_A}(x)\ge F_{N_B}(x)\ \forall x\in X$

Definition 2.5 [6]

Two SVNSs N_A and N_B are equal, i.e. $N_A= N_B$, if and only if $N_A\supseteq N_B$ and $N_A\subseteq N_B$

Definition 2.6 [6]

Union of two SVNSs N_A and N_B is a SVNS N_C, written as $N_C=N_A\cup N_B$. Its truth membership, indeterminacy-membership and falsity membership functions are related to those of N_A and N_B by
$T_{N_C}(x)=\max(T_{N_A}(x), T_{N_B}(x))$; $I_{N_C}(x)=\min(I_{N_A}(x), I_{N_B}(x))$; $F_{N_C}(x)=\min(F_{N_A}(x), F_{N_B}(x))$ for all x in X.

Definition 2.7 [6] Intersection of two SVNSs N_A and N_B is a SVNS N_D, written as $N_D=N_A\cap N_B$, whose truth membership, indeterminacy-membership and falsity membership functions are related to those of N_A and N_B by
$T_{N_C}(x)=\min(T_{N_A}(x), T_{N_B}(x))$; $I_{N_C}(x)=\max(I_{N_A}(x), I_{N_B}(x))$; $F_{N_C}(x)=\max(F_{N_A}(x), F_{N_B}(x))$ for all x in X.

Definition 2.8 Rough Neutrosophic Sets [8, 9]

Let Z be a non-null set and R be an equivalence relation on Z. Let P be neutrosophic set in Z with the membership function T_P indeterminacy function I_P and non-membership function F_P. The lower and the upper approximations of P in the approximation (Z, R) denoted by $\underline{N}(P)$ and $\overline{N}(P)$ are respectively defined as follows:

$\underline{N}(P)=\langle <x, T_{\underline{N}(P)}(x), I_{\underline{N}(P)}(x), F_{\underline{N}(P)}(x)> / z \in [x]_R, x \in Z\rangle$,

$\overline{N}(P)=\langle <x, T_{\overline{N}(P)}(x), I_{\overline{N}(P)}(x), F_{\overline{N}(P)}(x)> / z \in [x]_R, x \in Z\rangle$,

Here, $T_{\underline{N}(P)}(x)=\wedge_z \in [x]_R\, T_P(z)$, $I_{\underline{N}(P)}(x)=\wedge_z \in [x]_R\, I_P(z)$, $F_{\underline{N}(P)}(x)=\wedge_z \in [x]_R\, F_P(z)$, $T_{\overline{N}(P)}(x)=\vee_z \in [x]_R\, T_P(z)$, $I_{\overline{N}(P)}(x)=\vee_z \in [x]_R\, I_P(z)$, $F_{\overline{N}(P)}(x)=\vee_z \in [x]_R\, F_P(z)$

So, $0 \le \sup T_{\underline{N}(P)}(x) + \sup I_{\underline{N}(P)}(x) + \sup F_{\underline{N}(P)}(x) \le 3$

$0 \le \sup T_{\overline{N}(P)}(x) + \sup I_{\overline{N}(P)}(x) + \sup F_{\overline{N}(P)}(x) \le 3$

Here $\vee$ and $\wedge$ denote "max" and "min'' operators respectively. $T_P(z)$, $I_P(z)$ and $F_P(z)$ denote respectively the membership, indeterminacy and non-membership function of z with respect to P. It is easy to see that $\underline{N}(P)$ and $\overline{N}(P)$ are two neutrosophic sets in Z.

Thus NS mappings $\underline{N}$, $\overline{N}$: $N(Z) \rightarrow N(Z)$ are, respectively, referred to as the lower and the upper rough NS approximation operators, and the pair $(\underline{N}(P), \overline{N}(P))$ is called the rough neutrosophic set [8, 9] in (Z, R).

From the above definition, it is seen that $\underline{N}(P)$ and $\overline{N}(P)$ have constant membership on the equivalence classes of R. if $\underline{N}(P) = \overline{N}(P)$ i.e. $T_{\underline{N}(P)}(x) = T_{\overline{N}(P)}(x)$, $I_{\underline{N}(P)}(x) = I_{\overline{N}(P)}(x)$ and $F_{\underline{N}(P)}(x) = F_{\overline{N}(P)}(x), \forall x \in Z$.

P is said to be a definable neutrosophic set in the approximation (Z, R). It can be easily proved that zero neutrosophic set (0_N = (0, 1, 1)) and unit neutrosophic sets (1_N = (1, 0, 0)) are definable neutrosophic sets.

Definition 2.9 [8, 9]

If $N(P) = (\underline{N}(P), \overline{N}(P))$ is a rough neutrosophic set in (Z, R) , the rough complement [8, 9] of $N(P)$ is the rough neutrosophic set denoted by $\sim N(P) = (\underline{N}(P)^c, \overline{N}(P)^c)$ where $\underline{N}(P)^c$, $\overline{N}(P)^c$ are the complements of neutrosophic sets of $\underline{N}(P)$, $\overline{N}(P)$ respectively.

$\underline{N}(P)^c = \langle <x, F_{\underline{N}(P)}(x), 1 - I_{\underline{N}(P)}(x), T_{\underline{N}(P)}(x)> / x \in Z\rangle$ and

$\overline{N}(P)^c = \langle <x, F_{\overline{N}(P)}(x), 1 - I_{\overline{N}(P)}(x), T_{\overline{N}(P)}(x)> / x \in Z\rangle$

Definition 2.10 [8, 9]

If $N(P_1)$ and $N(P_2)$ are the two rough neutrosophic sets of the neutrosophic set P respectively in Z, then the following definitions [8, 9] hold:

$N(P_1)=N(P_2) \Leftrightarrow \underline{N}(P_1) = \underline{N}(P_2) \wedge \overline{N}(P_1) = \overline{N}(P_2)$

$N(P_1) \subseteq N(P_2) \Leftrightarrow \underline{N}(P_1) \subseteq \underline{N}(P_2) \wedge \overline{N}(P_1) \subseteq \overline{N}(P_2)$

$N(P_1) \cup N(P_2) = <\underline{N}(P_1) \cup \underline{N}(P_2), \overline{N}(P_1) \cup \overline{N}(P_2)>$

$N(P_1) \cap N(P_2) = <\underline{N}(P_1) \cap \underline{N}(P_2), \overline{N}(P_1) \cap \overline{N}(P_2)>$

$N(P_1) + N(P_2) = <\underline{N}(P_1) + \underline{N}(P_2), \overline{N}(P_1) + \overline{N}(P_2)>$

$N(P_1).N(P_2) = <\underline{N}(P_1).\underline{N}(P_2), \overline{N}(P_1).\overline{N}(P_2)>$

If N, M, L are the rough neutrosophic sets in (Z, R), then the following proposition are stated from definitions [8, 9].

Proposition 1 [8, 9]

1. $\sim(\sim N) = N$
2. $N \cup M = M \cup N, N \cap M = M \cap N$
3. $(L \cup M) \cup N = L \cup (M \cup N)$, $(L \cap M) \cap N = L \cap (M \cap N)$
4. $(L \cup M) \cap N = (L \cup M) \cap (L \cup N)$, $(L \cap M) \cup N = (L \cap M) \cup (L \cap N)$

Proposition 2 [8, 9]

De Morgan's Laws are satisfied for rough neutrosophic sets .

1. $\sim(N(P_1) \cup N(P_2)) = (\sim(N(P_1)) \cap (\sim N(P_2))$
2. $\sim(N(P_1) \cap N(P_2)) = (\sim(N(P_1) \cup (\sim N(P_2))$

Proposition 3 [8, 9]

If P_1 and P_2 are two neutrosophic sets in U such that $P_1 \subseteq P_2$ *then* $N(P_1) \subseteq N(P_2)$

1. $N(P_1 \cap P_2) \subseteq N(P_1) \cap N(P_2)$
2. $N(P_1 \cup P_2) \supseteq N(P_1) \cup N(P_2)$

Proposition 4 [8, 9]

1. $\underline{N}(P) = \sim \overline{N}(\sim P)$
2. $\overline{N}(P) = \sim \underline{N}(\sim P)$
3. $\underline{N}(P) \subseteq \overline{N}(P)$

3 Similarity measures and variational coefficient similarity measure in crisp environment

The vector similarity measure is one of the important tools for the degree of similarity between objects. However, the Jaccard, Dice, and cosine similarity measures are often used for this purpose. Jaccard [83] , Dice [84] , and cosine [85] similarity measures between two vectors are stated below.

Let X = (x_1, x_2, …, x_n) and Y = (y_1, y_2, …, y_n) be two n-dimensional vectors with positive co-ordinates.

Definition 3.1 [83]

Jeccard index of two vectors (measuring the "similarity" of these vectors) can be defined as follows:

$$J(X, Y) = \frac{X.Y}{\|X\|^2 + \|Y\|^2 - X.Y} = \frac{\sum_{i=1}^{n} x_i y_i}{\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2 - \sum_{i=1}^{n} x_i y_i} \quad (1)$$

where $\|X\|^2 = \sum_{i=1}^{n} x_i^2$ and $\|Y\|^2 = \sum_{i=1}^{n} y_i^2$ are the Euclidean norm of X and Y, $X.Y = \sum_{i=1}^{n} x_i y_i$ is the inner product of the vector X and Y.

Proposition 5 [83]

Jaccard index satisfies the following properties:

1. $0 \le J(X, Y) \le 1$

2. J(X, Y) = J(Y, X)
3. J(X, Y) = 1, for $X = Y$ i.e, $x_i = y_i$ (i = 1, 2, …, n) for every $x_i \in X$ and $y_i \in Y$

Definition 3.2 [84]

The Dice similarity measure can be defined as follows:

$$E(X, Y) = \frac{2\,X.Y}{\|X\|^2+\|Y\|^2} = \frac{2\sum_{i=1}^{n} x_i\, y_i}{\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2} \quad (2)$$

Proposition 6 [84]

The Dice similarity measure satisfies the following properties:

1. 0 ≤ E(X, Y) ≤ 1
2. E(X, Y) = E(Y, X)
3. J(X, Y) = 1, for $X = Y$ i.e, $x_i = y_i$ (i = 1, 2, …, n) for every $x_i \in X$ and $y_i \in Y$.

Definition 3.3 [85]

The cosine similarity measure between two vectors X and Y is the inner product of these two vectors divided by the product of their lengths and can be defined as follows:

$$C(X, Y) = \frac{X.Y}{\|X\|.\|Y\|} = \frac{\sum_{i=1}^{n} x_i\, y_i}{\sqrt{\sum_{i=1}^{n} x_i^2}\sqrt{\sum_{i=1}^{n} y_i^2}} \quad (3)$$

Proposition 7 [85]

The cosine similarity measure satisfies the following properties

1. 0 ≤ C(X, Y) ≤ 1
2. C(X, Y) = C(Y, X)
3. C(X, Y) = 1, for $X = Y$ i.e, $x_i = y_i$ (i = 1, 2, …, n) for every $x_i \in X$ and $y_i \in Y$.

These three formulas are similar in the sense that they take values in the interval [0, 1]. Jaccard and Dice similarity measures are undefined when $x_i = 0$, and $y_i = 0$ for i = 1, 2, …, n and cosine similarity measure is undefined when $x_i = 0$ or $y_i = 0$ for i = 1, 2, …, n.

Definition 3.4 [86]

Variational co-efficient similarity measure can be defined as follows:

$$V(X, Y) = \lambda\frac{2\,X.Y}{\|X\|^2+\|Y\|^2} + (1-\lambda)\frac{X.Y}{\|X\|.\|Y\|}$$

$$= \lambda\frac{2\sum_{i=1}^{n} x_i\, y_i}{\sum_{i=1}^{n} x_i^2 + \sum_{i=1}^{n} y_i^2} + (1-\lambda)\frac{\sum_{i=1}^{n} x_i\, y_i}{\sqrt{\sum_{i=1}^{n} x_i^2}\sqrt{\sum_{i=1}^{n} y_i^2}} \quad (4)$$

Proposition 8 [86]

Variational co-efficient similarity measure satisfies the following properties:

1. 0 ≤ V(X, Y) ≤ 1
2. V(X, Y) = V(Y, X)
3. V(X, Y) = 1, for $X = Y$ i.e, $x_i = y_i$ (i = 1, 2, …, n) for every $x_i \in X$ and $y_i \in Y$.

4. Various similarity measures for single valued neutrosophic sets.

Assume $N_A=\langle T_A, I_A, F_A\rangle$ and $N_B=\langle T_B, I_B, F_B\rangle$ be two SVNSs in a universe of discourse $X = (x_1, x_2,…, x_n)$. $T_A, I_A, F_A \in [0,1]$ for any $x_i \in X$ in N_A or $T_B, I_B, F_B \in [0,1]$ for any $x_i \in X$ in N_B can be considered as a vector representation with three elements. Let $w_i \in [0,1]$ be the weight of each element x_i for i = 1, 2, …, n such that $\sum_{i=1}^{n} w_i = 1$, then Jaccard, Dice and cosine similarity measures can be presented as follows:

Definition 4.1[10] Jaccard similarity measure between $N_A=\langle T_A, I_A, F_A\rangle$ and $N_B=\langle T_B, I_B, F_B\rangle$ can be defined as follows:

$Jac(N_A, N_B)$ =

$$\frac{1}{n}\sum_{i=1}^{n}\frac{(T_A(x_i)\,T_B(x_i)+I_A(x_i)\,I_B(x_i)+F_A(x_i)\,F_B(x_i))}{\left\langle\begin{array}{l}\left[(T_A(x_i))^2+(I_A(x_i))^2+(F_A(x_i))^2\right]\\+\left[(T_B(x_i))^2+(I_B(x_i))^2+(F_B(x_i))^2\right]\\-\{T_A(x_i)\,T_B(x_i)+I_A(x_i)\,I_B(x_i)\\+F_A(x_i)\,F_B(x_i)\}\end{array}\right\rangle} \quad (5)$$

Proposition 9 [10]

Jaccard similarity measure satiefies the following properites:

1. $0 \le Jac(N_A, N_B) \le 1$;
2. $Jac(N_A, N_B) = Jac(N_B, N_A)$;
3. $Jac(N_A, N_B) = 1$; if $N_A = N_B$ i.e., $T_A(x_i) = T_B(x_i)$, $I_A(x_i) = I_B(x_i)$, and $F_A(x_i) = F_B(x_i)$, for every x_i (i = 1, 2, …, n) in X.

Definition 4.1.1 [10] Weighted Jaccard similarity measure between $N_A=\langle T_A, I_A, F_A\rangle$ and $N_B=\langle T_B, I_B, F_B\rangle$ can be defined as follows:

$Jac_w(N_A, N_B)$=

$$\sum_{i=1}^{n} w_i\frac{(T_A(x_i)\,T_B(x_i)+I_A(x_i)\,I_B(x_i)+F_A(x_i)\,F_B(x_i))}{\left\langle\begin{array}{l}\left[(T_A(x_i))^2+(I_A(x_i))^2+(F_A(x_i))^2\right]\\+\left[(T_B(x_i))^2+(I_B(x_i))^2+(F_B(x_i))^2\right]\\-\{T_A(x_i)\,T_B(x_i)+I_A(x_i)\,I_B(x_i)\\+F_A(x_i)\,F_B(x_i)\}\end{array}\right\rangle} \quad (6)$$

Proposition 10 [10]

Weighted Jaccard similarity measure satisfies the following properties:

1. $0 \le Jac_w(N_A, N_B) \le 1$;
2. $Jac_w(N_A, N_B) = Jac_w(N_B, N_A)$;
3. $Jac_w(N_A, N_B) = 1$; if $N_A = N_B$ i.e., $T_A(x_i) = T_B(x_i)$, $I_A(x_i) = I_B(x_i)$, and $F_A(x_i) = F_B(x_i)$, for every x_i (i = 1, 2, …, n) in X.

Definition 4.2 [11]

Dice similarity measure between $N_A=\langle T_A, I_A, F_A\rangle$ and $N_B=\langle T_B, I_B, F_B\rangle$ is defined as:

$Dic(N_A, N_B)$ =

$$\frac{1}{n}\sum_{i=1}^{n}\frac{2\begin{bmatrix}T_A(x_i)T_B(x_i)+I_A(x_i)I_B(x_i)\\+F_A(x_i)F_B(x_i)\end{bmatrix}}{\left\langle\begin{matrix}\left[(T_A(x_i))^2+(I_A(x_i))^2+(F_A(x_i))^2\right]\\+\left[(T_B(x_i))^2+(I_B(x_i))^2+(F_B(x_i))^2\right]\end{matrix}\right\rangle} \quad (7)$$

Proposition 11 [11]

Dice similarity measure satisfies the following properties:

1. $0\le Dic(N_A, N_B)\le 1$;
2. $Dic(N_A, N_B) = Dic(N_B, N_A)$;
3. $Dic(N_A, N_B)=1$; if $N_A=N_B$ i.e., $T_A(x_i)=T_B(x_i)$, $I_A(x_i)=I_B(x_i)$, and $F_A(x_i)=F_B(x_i)$, for every x_i (i = 1, 2, …, n) in X.

Definition 4.2.1 [11]

Weighted Dice similarity measure between $N_A=\langle T_A, I_A, F_A\rangle$ and $N_B=\langle T_B, I_B, F_B\rangle$ can be defined as follows:

$Dic_w(N_A, N_B)$ =

$$\sum_{i=1}^{n} w_i\frac{2\begin{bmatrix}T_A(x_i)T_B(x_i)+I_A(x_i)I_B(x_i)\\+F_A(x_i)F_B(x_i)\end{bmatrix}}{\left\langle\begin{matrix}\left[(T_A(x_i))^2+(I_A(x_i))^2+(F_A(x_i))^2\right]\\+\left[(T_B(x_i))^2+(I_B(x_i))^2+(F_B(x_i))^2\right]\end{matrix}\right\rangle} \quad (8)$$

Proposition 12 [11]

Weighted Dice similarity measure

1. $0\le Dic_w(N_A, N_B)\le 1$;
2. $Dic_w(N_A, N_B)= Dic_w(N_B, N_A)$;
3. $Dic_w(N_A, N_B)=1$; if $N_A = N_B$ i.e., $T_A(x_i)=T_B(x_i)$, $I_A(x_i)=I_B(x_i)$, and $F_A(x_i)=F_B(x_i)$, for every x_i (i = 1, 2, …, n) in X.

Definition 4.3 [12]

Cosine similarity measure between $N_A=\langle T_A, I_A, F_A\rangle$ and $N_B=\langle T_B, I_B, F_B\rangle$ can be defined as follows:

$$Cos(N_A, N_B) = \frac{1}{n}\sum_{i=1}^{n}\frac{\begin{matrix}(T_A(x_i)T_B(x_i)+I_A(x_i)I_B(x_i)\\+F_A(x_i)F_B(x_i))\end{matrix}}{\sqrt{(T_A(x_i))^2+(I_A(x_i))^2+(F_A(x_i))^2}\sqrt{(T_B(x_i))^2+(I_B(x_i))^2+(F_B(x_i))^2}} \quad (9)$$

Proposition 13 [12]

Cosine similarity measure satisfies the following properties:

1. $0\le Cos_w(N_A, N_B)\le 1$;
2. $Cos_w(N_A, N_B)= Cos_w(N_B, N_A)$
3. $Cos_w(N_A, N_B)=1$; if $N_A = N_B$ i.e., $T_A(x_i)=T_B(x_i)$, $I_A(x_i)=I_B(x_i)$, and $F_A(x_i)=F_B(x_i)$, for every x_i (i = 1, 2, …, n) in X.

Definition 4.3.1 [12]

Weighted cosine similarity measure between $N_A=\langle T_A, I_A, F_A\rangle$ and $N_B=\langle T_B, I_B, F_B\rangle$ can be defined as follows:

$Cos_w(N_A, N_B)$=

$$\sum_{i=1}^{n} w_i\frac{\begin{matrix}(T_A(x_i)T_B(x_i)+I_A(x_i)I_B(x_i)\\+F_A(x_i)F_B(x_i))\end{matrix}}{\sqrt{(T_A(x_i))^2+(I_A(x_i))^2+(F_A(x_i))^2}\sqrt{(T_B(x_i))^2+(I_B(x_i))^2+(F_B(x_i))^2}} \quad (10)$$

Proposition 14 [12]

Weighted cosine similarity measure satisfies the following properties:

1. $0\le Cos_w(N_A, N_B)\le 1$;
2. $Cos_w(N_A, N_B)= Cos_w(N_B, N_A)$
3. $Cos_w(N_A, N_B)=1$; if $N_A = N_B$ i.e., $T_A(x_i)=T_B(x_i)$, $I_A(x_i)=I_B(x_i)$, and $F_A(x_i)=F_B(x_i)$, for every x_i (i = 1, 2, …, n) in X.

Jaccard and Dice similarity measures between two neutrosophic sets $N_A=\langle T_A, I_A, F_A\rangle$ and $N_B=\langle T_B, I_B, F_B\rangle$ are undefined when $T_A(x_i)=I_A(x_i)=F_A(x_i)=0$ and $T_B(x_i)=I_B(x_i)=F_B(x_i)=0$ for all i = 1, 2, …, n. Similarly the cosine formula for two neutrosophic sets $N_A=\langle T_A, I_A, F_A\rangle$ and $N_B=\langle T_B, I_B, F_B\rangle$ is undefined when $T_A(x_i)=I_A(x_i)=F_A(x_i)=0$ or $T_B(x_i)=I_B(x_i)=F_B(x_i)=0$ for all i = 1, 2, …, n.

5 Variational similarity measures for rough neutrosophic sets

The notion of rough neutrosophic set (RNS) is used as vector representations in 3D-vector space. Assume that $X = (x_1, x_2,\ldots, x_n)$ and $Y = (y_1, y_2, \ldots, y_n)$ be two n-dimensional vectors with positive co-ordinates. Jaccard, Dice, cosine and cotangent similarity measures between two vectors are stated as follows.

Definition 5.1 [21] Jaccard similarity measure under rough neutrosophic environment

Assume that $A=\left\langle(\underline{T}_A(x_i),\underline{I}_A(x_i),\underline{F}_A(x_i)),(\overline{T}_A(x_i),\overline{I}_A(x_i),\overline{F}_A(x_i))\right\rangle$ and $B=\left\langle(\underline{T}_B(x_i),\underline{I}_B(x_i),\underline{F}_B(x_i)),(\overline{T}_B(x_i),\overline{I}_B(x_i),\overline{F}_B(x_i))\right\rangle$ in $X = (x_1, x_2,\ldots, x_n)$ be any two rough neutrosophic sets. Jacard similarity measure [21] between rough neutrosophic sets A and B can be defined as follows:

$Jac_{RNS}(A, B)$ =

$$\frac{1}{n}\sum_{i=1}^{n}\frac{\begin{array}{l}(\delta T_A(x_i)\,\delta T_B(x_i)+\delta I_A(x_i)\,\delta I_B(x_i)\\ +\delta F_A(x_i)\,\delta F_B(x_i))\end{array}}{\left\langle\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\ +\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\\ -[\delta T_A(x_i)\,\delta T_B(x_i)+\delta I_A(x_i)\,\delta I_B(x_i)\\ +\delta F_A(x_i)\,\delta F_B(x_i)]\end{array}\right\rangle} \quad (11)$$

Proposition 15 [21]

Jaccard similarity measure [21] between A and B satisfies the following properties:

1. $0\le Jac_{RNS}(A,B)\le 1$;
2. $Jac_{RNS}(A,B)=Jac_{RNS}(B,A)$;
3. $Jac_{RNS}(A,B)=1$; iff $A = B$
4. If C is a RNS in Y and $A\subset B\subset C$ then, $Jac_{RNS}(A, C)\le Jac_{RNS}(A, B)$, and $Jac_{RNS}(A, C) \le Jac_{RNS}(B, C)$

Definition 5.1.1 [21]

If we consider the weights of each element x_i, weighted rough Jaccard similarity measure [21] between rough neutrosophic sets A and B can be defined as follows:

$Jac_{WRNS}(A, B)$ =

$$\sum_{i=1}^{n}w_i\frac{\begin{array}{l}(\delta T_A(x_i)\,\delta T_B(x_i)+\delta I_A(x_i)\,\delta I_B(x_i)\\ +\delta F_A(x_i)\,\delta F_B(x_i))\end{array}}{\left\langle\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\ +\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\\ -[\delta T_A(x_i)\,\delta T_B(x_i)+\delta I_A(x_i)\,\delta I_B(x_i)\\ +\delta F_A(x_i)\,\delta F_B(x_i)]\end{array}\right\rangle} \quad (12)$$

$w_i\in[0,1]$, $i = 1, 2,\dots, n$ and $\sum_{i=1}^{n}w_i=1$. If we take $w_i=\frac{1}{n}$, $i = 1, 2,\dots, n$, then $Jac_{WRNS}(A, B) = Jac_{RNS}(A,B)$

Proposition 16 [21]

The weighted rough Jaccard similarity [21] measure between two rough neutrosophic sets A and B also satisfies the following properties:

1. $0\le Jac_{WRNS}(A,B)\le 1$;
2. $Jac_{WRNS}(A,B)=Jac_{WRNS}(B,A)$;
3. $Jac_{WRNS}(A,B)=1$; iff $A = B$
4. If C is a WRNS in Y and $A\subset B\subset C$ then, $Jac_{WRNS}(A, C) \le Jac_{WRNS}(A, B)$, and $Jac_{WRNS}(A, C) \le Jac_{WRNS}(B, C)$

Definition 5.2 [21] **Dice similarity measure under rough neutrosophic environment**

In this section, Dice similarity measure and the weighted Dice similarity measure for rough neutrosophic sets have been stated due to Pramanik and Mondal [21].

Suppose that

$A=\left\langle(\underline{T}_A(x_i),\underline{I}_A(x_i),\underline{F}_A(x_i)),(\overline{T}_A(x_i),\overline{I}_A(x_i),\overline{F}_A(x_i))\right\rangle$ and

$B=\left\langle(\underline{T}_B(x_i),\underline{I}_B(x_i),\underline{F}_B(x_i)),(\overline{T}_B(x_i),\overline{I}_B(x_i),\overline{F}_B(x_i))\right\rangle$ be any two rough neutrosophic sets in $X = (x_1, x_2,\dots, x_n)$. Dice similarity measure between rough neutrosophic sets A and B can be defined as follows:

$DIC_{RNS}(A, B)$=

$$\frac{1}{n}\sum_{i=1}^{n}\frac{2\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]}{\left\langle\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\ +\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\end{array}\right\rangle} \quad (13)$$

Proposition 17 [21]

Dice similarity measure [21] satisfies the following properties.

1. $0\le DIC_{RNS}(A,B)\le 1$;
2. $DIC_{RNS}(A,B)=DIC_{RNS}(B,A)$;
3. $DIC_{RNS}(A,B)=1$; iff $A = B$
4. If C is a RNS in Y and $A\subset B\subset C$ then, $DIC_{RNS}(A, C) \le DIC_{RNS}(A, B)$, and $DIC_{RNS}(A, C) \le DIC_{RNS}(B, C)$,

For proofs of the above mentioned four properties, see [21].

Definition 5.2.1

If we consider the weights of each element x_i, a weighted rough Dice similarity measure between rough neutrosophic sets A and B can be defined as follows:

$DIC_{WRNS}(A, B)$ =

$$\sum_{i=1}^{n}w_i\frac{2\left[\begin{array}{l}\delta T_A(x_i)\,\delta T_B(x_i)+\delta I_A(x_i)\,\delta I_B(x_i)\\ +\delta F_A(x_i)\,\delta F_B(x_i)\end{array}\right]}{\left\langle\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\ +\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\end{array}\right\rangle} \quad (14)$$

$w_i\in[0,1]$, $i = 1, 2,\dots, n$ and $\sum_{i=1}^{n}w_i=1$. If we take $w_i=\frac{1}{n}$, $i = 1, 2,\dots, n$, then $DIC_{WRNS}(A, B) = DIC_{RNS}(A,B)$

Proposition 18 [21]

The weighted rough Dice similarity [21] measure between two rough neutrosophic sets A and B also satisfies the following properties:

1. $0\le DIC_{WRNS}(A,B)\le 1$;
2. $DIC_{WRNS}(A,B)=DIC_{WRNS}(B,A)$;
3. $DIC_{WRNS}(A,B)=1$; iff $A = B$
4. If C is a RNS in Y and $A\subset B\subset C$ then, $DIC_{WRNS}(A, C) \le DIC_{WRNS}(A, B)$, and $DIC_{WRNS}(A, C) \le DIC_{WRNS}(B, C)$.

For proofs of the above mentioned four properties, see [21].

Definition 5.3 [20]

Cosine similarity measure can be defined as the inner product of two vectors divided by the product of their lengths. It is the cosine of the angle between the vector representations of two rough neutrosophic sets. The cosine similarity measure is a fundamental measure used in information technology. Pramanik and Mondal [20] defined cosine similarity measure between rough neutrosophic sets in 3-D vector space.

Assume that $A=\langle(\underline{T}_A(x_i),\underline{I}_A(x_i),\underline{F}_A(x_i)),(\overline{T}_A(x_i),\overline{I}_A(x_i),\overline{F}_A(x_i))\rangle$ and $B=\langle(\underline{T}_B(x_i),\underline{I}_B(x_i),\underline{F}_B(x_i)),(\overline{T}_B(x_i),\overline{I}_B(x_i),\overline{F}_B(x_i))\rangle$ in $X = (x_1, x_2,\ldots, x_n)$ be any rough neutrosophic sets. Pramanik and Mondal [20] defined cosine similarity measure between rough neutrosophic sets A and B as follows:

C_{RNS}(A, B) =

$$\frac{1}{n}\sum_{i=1}^{n}\frac{\delta T_A(x_i)\delta T_B(x_i)+\delta I_A(x_i)\delta I_B(x_i)+\delta F_A(x_i)\delta F_B(x_i)}{\sqrt{\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]}} \quad (15)$$

Here, $\delta T_A(x_i)=\frac{\underline{T}_A(x_i)+\overline{T}_A(x_i)}{2}$, $\delta T_B(x_i)=\frac{\underline{T}_B(x_i)+\overline{T}_B(x_i)}{2}$,

$\delta I_A(x_i)=\frac{\underline{I}_A(x_i)+\overline{I}_A(x_i)}{2}$, $\delta I_B(x_i)=\frac{\underline{I}_B(x_i)+\overline{I}_B(x_i)}{2}$,

$\delta F_A(x_i)=\frac{\underline{F}_A(x_i)+\overline{F}_A(x_i)}{2}$, $\delta F_B(x_i)=\frac{\underline{F}_B(x_i)+\overline{F}_B(x_i)}{2}$

Proposition 19 [20]

Let A and B be rough neutrosophic sets. Cosine similarity measure [20] between A and B satisfies the following properties.

1. $0\le C_{RNS}(A,B)\le 1$;
2. $C_{RNS}(A,B)=C_{RNS}(B,A)$;
3. $C_{RNS}(A,B)=1$; iff $A = B$
4. If C is a RNS in Y and $A\subset B\subset C$ then, $C_{RNS}(A, C) \le C_{RNS}(A, B)$, and $C_{RNS}(A, C) \le C_{RNS}(B, C)$.

Definition 5.3.1 [20]

If we consider the weights of each element x_i, a weighted rough cosine similarity measure between rough neutrosophic sets A and B can be defined as follows:

$$C_{WRNS}(A,B)=\sum_{i=1}^{n}w_i\frac{\delta T_A(x_i)\delta T_B(x_i)+\delta I_A(x_i)\delta I_B(x_i)+\delta F_A(x_i)\delta F_B(x_i)}{\sqrt{\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]}} \quad (16)$$

$w_i\in[0,1]$, i = 1, 2,…, n and $\sum_{i=1}^{n}w_i=1$. If we take $w_i=\frac{1}{n}$, i = 1, 2,…, n, then $C_{WRNS}(A, B) = C_{RNS}(A, B)$

Proposition 20 [20]

The weighted rough cosine similarity measure [20] between two rough neutrosophic sets A and B also satisfies the following properties:

1. $0\le C_{WRNS}(A,B)\le 1$;
2. $C_{WRNS}(A,B)=C_{WRNS}(B,A)$;
3. $C_{WRNS}(A,B)=1$; iff $A = B$
4. If C is a WRNS in Y and $A\subset B\subset C$ then, $C_{WRNS}(A, C) \le C_{WRNS}(A, B)$, and $C_{WRNS}(A, C) \le C_{WRNS}(B, C)$.

For proofs of the above mentioned four properties, see [20].

Definition 5.4 [19] **Cotangent similarity measures of rough neutrosophic sets**

Assume that $A=\langle(\underline{T}_A(x_i),\underline{I}_A(x_i),\underline{F}_A(x_i)),(\overline{T}_A(x_i),\overline{I}_A(x_i),\overline{F}_A(x_i))\rangle$ and $B=\langle(\underline{T}_B(x_i),\underline{I}_B(x_i),\underline{F}_B(x_i)),(\overline{T}_B(x_i),\overline{I}_B(x_i),\overline{F}_B(x_i))\rangle$ in $X = (x_1, x_2,\ldots, x_n)$ be any two rough neutrosophic sets. Pramanik and Mondal [19] defined cotangent similarity measure between rough neutrosophic sets A and B as follows:

$COT_{RNS}(A, B)$ =

$$\frac{1}{n}\sum_{i=1}^{n}\left\langle \cot\left(\frac{\pi}{12}\left(\begin{array}{l}3+\left|\delta T_A(x_i)-\delta T_B(x_i)\right|\\+\left|\delta I_A(x_i)-\delta I_B(x_i)\right|\\+\left|\delta I_A(x_i)-\delta I_B(x_i)\right|\end{array}\right)\right)\right\rangle \quad (17)$$

Here, $\delta T_A(x_i)=\frac{\underline{T}_A(x_i)+\overline{T}_A(x_i)}{2}$, $\delta T_B(x_i)=\frac{\underline{T}_B(x_i)+\overline{T}_B(x_i)}{2}$,

$\delta I_A(x_i)=\frac{\underline{I}_A(x_i)+\overline{I}_A(x_i)}{2}$, $\delta I_B(x_i)=\frac{\underline{I}_B(x_i)+\overline{I}_B(x_i)}{2}$,

$\delta F_A(x_i)=\frac{\underline{F}_A(x_i)+\overline{F}_A(x_i)}{2}$, $\delta F_B(x_i)=\frac{\underline{F}_B(x_i)+\overline{F}_B(x_i)}{2}$

Proposition 21 [19]

Cotangent similarity measure satisfies the following properties:

1. $0\le COT_{RNS}(A,B)\le 1$;
2. $COT_{RNS}(A,B)=COT_{RNS}(B,A)$;
3. $COT_{RNS}(A,B)=1$; iff $A = B$
4. If C is a RNS in Y and $A\subset B\subset C$ then, $COT_{RNS}(A, C) \le COT_{RNS}(A, B)$, and $COT_{RNS}(A, C) \le COT_{RNS}(B, C)$.

Definition 5.4.1

If we consider the weights of each element x_i, a weighted rough cotangent similarity measure [19] between rough neutrosophic sets A and B can be defined as follows:

$COT_{WRNS}(A, B)$ =

$$\sum_{i=1}^{n}w_i\left\langle \cot\left(\frac{\pi}{12}\left(\begin{array}{l}3+\left|\delta T_A(x_i)-\delta T_B(x_i)\right|\\+\left|\delta I_A(x_i)-\delta I_B(x_i)\right|\\+\left|\delta I_A(x_i)-\delta I_B(x_i)\right|\end{array}\right)\right)\right\rangle \quad (18)$$

$w_i\in[0,1]$, i = 1, 2,…, n and $\sum_{i=1}^{n}w_i=1$. If we take $w_i=\frac{1}{n}$, i = 1, 2,…, n, then $COT_{WRNS}(A, B) = COT_{RNS}(A, B)$

Proposition 22 [19]

The weighted rough cosine similarity measure between two rough neutrosophic sets A and B also satisfies the following properties:

1. $0\le COT_{WRNS}(A,B)\le 1$;
2. $COT_{WRNS}(A,B)=COT_{WRNS}(B,A)$;
3. $COT_{WRNS}(A,B)=1$; iff $A = B$
4. If C is a WRNS in Y and $A\subset B\subset C$ then, $COT_{WRNS}(A, C) \le COT_{WRNS}(A, B)$, and $COT_{WRNS}(A, C) \le COT_{WRNS}(B, C)$

Definition 5.5 (Variational co-efficient similarity measure between rough neutrosophic sets)

Let $A=\left\langle\left(\underline{T}_A(x_i),\underline{I}_A(x_i),\underline{F}_A(x_i)\right),\left(\overline{T}_A(x_i),\overline{I}_A(x_i),\overline{F}_A(x_i)\right)\right\rangle$ and $B=\left\langle\left(\underline{T}_B(x_i),\underline{I}_B(x_i),\underline{F}_B(x_i)\right),\left(\overline{T}_B(x_i),\overline{I}_B(x_i),\overline{F}_B(x_i)\right)\right\rangle$ be two rough neutrosophic sets. Variational co-efficient similarity measure between rough neutrosophic sets can be presented as follows:

$Var_{RNS}(A, B)=$

$$\frac{1}{n}\left[\begin{array}{l}\lambda\sum_{i=1}^{n}\dfrac{2\left\{\begin{array}{l}\delta T_A(x_i)\delta T_B(x_i)+\delta I_A(x_i)\delta I_B(x_i)\\+\delta F_A(x_i)\delta F_B(x_i)\end{array}\right\}}{\left\{\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\+\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\end{array}\right\}}\\+(1-\lambda)\sum_{i=1}^{n}\dfrac{\delta T_A(x_i)\delta T_B(x_i)+\delta I_A(x_i)\delta I_B(x_i)+\delta F_A(x_i)\delta F_B(x_i)}{\sqrt{\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]}}\end{array}\right] \quad (19)$$

Here, $\delta T_A(x_i)=\dfrac{\underline{T}_A(x_i)+\overline{T}_A(x_i)}{2}$, $\delta T_B(x_i)=\dfrac{\underline{T}_B(x_i)+\overline{T}_B(x_i)}{2}$,

$\delta I_A(x_i)=\dfrac{\underline{I}_A(x_i)+\overline{I}_A(x_i)}{2}$, $\delta I_B(x_i)=\dfrac{\underline{I}_B(x_i)+\overline{I}_B(x_i)}{2}$,

$\delta F_A(x_i)=\dfrac{\underline{F}_A(x_i)+\overline{F}_A(x_i)}{2}$, $\delta F_B(x_i)=\dfrac{\underline{F}_B(x_i)+\overline{F}_B(x_i)}{2}$

Proposition 23

The variational co-efficient similarity measure $Var_{RNS}(A, B)$ between two rough neutrosophic sets A and B, satisfies the following properties:

1. $0\le Var_{RNS}(A,B)\le 1$;
2. $Var_{RNS}(A,B)=Var_{RNS}(B,A)$;
3. $Var_{RNS}(A,B)=1$; if $A = B$ i.e.,

$\delta T_A(x_i)=\delta T_B(x_i)$, $\delta I_A(x_i)=\delta I_B(x_i)$, and $\delta F_A(x_i)=\delta F_B(x_i)$, for every x_i (i = 1, 2, …, n) in X.

Proof.

(1.) It is obvious that $Var_{RNS}(A,B)\ge 0$. Thus it is required to prove that $Var_{RNS}(A,B)\le 1$.

From rough neutrosophic dice similarity measure it can be witten that

$$0\le\frac{1}{n}\sum_{i=1}^{n}\frac{2\left[\begin{array}{l}\delta T_A(x_i)\delta T_B(x_i)+\delta I_A(x_i)\delta I_B(x_i)\\+\delta F_A(x_i)\delta F_B(x_i)\end{array}\right]}{\left\langle\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\+\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\end{array}\right\rangle}\le 1 \quad (20)$$

and from rough neutrosophic cosine similarity measure it can be written that

$$0\le\frac{1}{n}\sum_{i=1}^{n}\frac{\delta T_A(x_i)\delta T_B(x_i)+\delta I_A(x_i)\delta I_B(x_i)+\delta F_A(x_i)\delta F_B(x_i)}{\sqrt{\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]}}\le 1 \quad (21)$$

Combining Eq.(20) and Eq.(21) , we obtain

$Var_{RNS}(A, B)=$

$$\frac{1}{n}\left[\begin{array}{l}\lambda\sum_{i=1}^{n}\dfrac{2\left\{\begin{array}{l}\delta T_A(x_i)\delta T_B(x_i)+\delta I_A(x_i)\delta I_B(x_i)\\+\delta F_A(x_i)\delta F_B(x_i)\end{array}\right\}}{\left\{\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\+\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\end{array}\right\}}\\+(1-\lambda)\sum_{i=1}^{n}\dfrac{\delta T_A(x_i)\delta T_B(x_i)+\delta I_A(x_i)\delta I_B(x_i)+\delta F_A(x_i)\delta F_B(x_i)}{\sqrt{\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\cdot\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]}}\end{array}\right] \quad (22)$$

$\le\lambda+(1-\lambda)=1$

Thus, $0\le Var_{RNS}(A,B)\le 1$;

(2.) $Var_{RNS}(A, B)=$

$$\frac{1}{n}\left[\begin{array}{l}\lambda\sum_{i=1}^{n}\dfrac{2\left\{\begin{array}{l}\delta T_A(x_i)\delta T_B(x_i)+\delta I_A(x_i)\delta I_B(x_i)\\+\delta F_A(x_i)\delta F_B(x_i)\end{array}\right\}}{\left\{\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\+\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\end{array}\right\}}\\+(1-\lambda)\sum_{i=1}^{n}\dfrac{\delta T_A(x_i)\delta T_B(x_i)+\delta I_A(x_i)\delta I_B(x_i)+\delta F_A(x_i)\delta F_B(x_i)}{\sqrt{\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]}}\end{array}\right]$$

$$=\frac{1}{n}\left[\begin{array}{l}\lambda\sum_{i=1}^{n}\dfrac{2\left\{\begin{array}{l}\delta T_B(x_i)\delta T_A(x_i)+\delta I_B(x_i)\delta I_A(x_i)\\+\delta F_B(x_i)\delta F_A(x_i)\end{array}\right\}}{\left\{\begin{array}{l}\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\\+\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\end{array}\right\}}\\+(1-\lambda)\sum_{i=1}^{n}\dfrac{\delta T_B(x_i)\delta T_A(x_i)+\delta I_B(x_i)\delta I_A(x_i)+\delta F_B(x_i)\delta F_A(x_i)}{\sqrt{\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]}}\end{array}\right]$$

$=Var_{RNS}(B,A)$

(3.) If $A = B$ i.e., $\delta T_A(x_i)=\delta T_B(x_i)$, $\delta I_A(x_i)=\delta I_B(x_i)$, and $\delta F_A(x_i)=\delta F_B(x_i)$, for every x_i (i = 1, 2, …, n) in X,

$Var_{RNS}(A, A)=$

$$\frac{1}{n}\left[\begin{array}{l}\lambda\sum_{i=1}^{n}\dfrac{2\left\{\begin{array}{l}\delta T_A(x_i)\delta T_A(x_i)+\delta I_A(x_i)\delta I_A(x_i)\\+\delta F_A(x_i)\delta F_A(x_i)\end{array}\right\}}{\left\{\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\+\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\end{array}\right\}}\\+(1-\lambda)\sum_{i=1}^{n}\dfrac{\begin{array}{l}\delta T_A(x_i)\delta T_A(x_i)+\delta I_A(x_i)\delta I_A(x_i)\\+\delta F_A(x_i)\delta F_A(x_i)\end{array}}{\sqrt{\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\end{array}}}\end{array}\right]$$

$$=\frac{1}{n}\left[n\lambda+n(1-\lambda)\right]=1$$

These results show the completion of the proofs of the three properties.

Definition 5.6 (Weighted variational co-efficient similarity measure between rough neutrosophic sets)

Let $A=\left\langle\left(\underline{T}_A(x_i),\underline{I}_A(x_i),\underline{F}_A(x_i)\right),\left(\overline{T}_A(x_i),\overline{I}_A(x_i),\overline{F}_A(x_i)\right)\right\rangle$ and $B=\left\langle\left(\underline{T}_B(x_i),\underline{I}_B(x_i),\underline{F}_B(x_i)\right),\left(\overline{T}_B(x_i),\overline{I}_B(x_i),\overline{F}_B(x_i)\right)\right\rangle$ be any two rough neutrosophic sets. Rough variational co-efficient similarity measure between rough neutrosophic setsA and B in 3-D vector space can be presented as follows:

Var_{WRNS}(A, B) =

$$\left[\begin{array}{l}\lambda\sum_{i=1}^{n}w_i\dfrac{2\left\{\begin{array}{l}\delta T_A(x_i)\delta T_B(x_i)+\delta I_A(x_i)\delta I_B(x_i)\\+\delta F_A(x_i)\delta F_B(x_i)\end{array}\right\}}{\left\{\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\+\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\end{array}\right\}}\\+(1-\lambda)\sum_{i=1}^{n}w_i\dfrac{\begin{array}{l}\delta T_A(x_i)\delta T_B(x_i)+\delta I_A(x_i)\delta I_B(x_i)\\+\delta F_A(x_i)\delta F_B(x_i)\end{array}}{\sqrt{\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\\cdot\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\end{array}}}\end{array}\right] \quad (23)$$

If $w=\left[\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n}\right]^T$, then Eq.(23) is reduced to Eq.(19).

Proposition 24

The weighted variational co-efficient similarity measure also satisfies the following properties:

1. $0\le Var_{WRNS}(A,B)\le 1$;
2. $Var_{WRNS}(A,B)=Var_{WRNS}(B,A)$;
3. $Var_{WRNS}(A, B) = 1$; if $A = B$ i.e., $\delta T_A(x_i)=\delta T_B(x_i)$, $\delta I_A(x_i)=\delta I_B(x_i)$, and $\delta F_A(x_i)=\delta F_B(x_i)$, for every x_i (i = 1, 2, …, n) in X.

Proof:

(1.) It is obvious that $Var_{WRNS}(A,B)\ge 0$. Thus it is required to prove that $Var_{WRNS}(A,B)\le 1$.

From rough neutrosophic weighted dice similarity measure, it can be written that

$$0\le\frac{1}{n}\sum_{i=1}^{n}w_i\frac{2\left[\begin{array}{l}\delta T_A(x_i)\delta T_B(x_i)+\delta I_A(x_i)\delta I_B(x_i)\\+\delta F_A(x_i)\delta F_B(x_i)\end{array}\right]}{\left\langle\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\+\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\end{array}\right\rangle}\le 1 \quad (24)$$

and from rough neutrosophic weighted cosine similarity measure it can be written that

$$0\le\frac{1}{n}\sum_{i=1}^{n}w_i\frac{\begin{array}{l}\delta T_A(x_i)\delta T_B(x_i)+\delta I_A(x_i)\delta I_B(x_i)\\+\delta F_A(x_i)\delta F_B(x_i)\end{array}}{\sqrt{\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\end{array}}}\le 1 \quad (25)$$

Combining Eq.(24) and Eq.(25), we obtain

Var_{WRNS}(A, B) =

$$\left[\begin{array}{l}\lambda\sum_{i=1}^{n}w_i\dfrac{2\left\{\begin{array}{l}\delta T_A(x_i)\delta T_B(x_i)+\delta I_A(x_i)\delta I_B(x_i)\\+\delta F_A(x_i)\delta F_B(x_i)\end{array}\right\}}{\left\{\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\+\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\end{array}\right\}}\\+(1-\lambda)\sum_{i=1}^{n}w_i\dfrac{\begin{array}{l}\delta T_A(x_i)\delta T_B(x_i)+\delta I_A(x_i)\delta I_B(x_i)\\+\delta F_A(x_i)\delta F_B(x_i)\end{array}}{\sqrt{\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\end{array}}}\end{array}\right] \quad (26)$$

$$\le\lambda+(1-\lambda)=1$$

Thus, $0\le Var_{WRNS}(A,B)\le 1$;

(2.) *Var_{WRNS}(A, B)* =

$$\left[\begin{array}{l}\lambda\sum_{i=1}^{n}w_i\dfrac{2\left\{\begin{array}{l}\delta T_A(x_i)\delta T_B(x_i)+\delta I_A(x_i)\delta I_B(x_i)\\+\delta F_A(x_i)\delta F_B(x_i)\end{array}\right\}}{\left\{\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\+\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\end{array}\right\}}\\+(1-\lambda)\sum_{i=1}^{n}w_i\dfrac{\begin{array}{l}\delta T_A(x_i)\delta T_B(x_i)+\delta I_A(x_i)\delta I_B(x_i)\\+\delta F_A(x_i)\delta F_B(x_i)\end{array}}{\sqrt{\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\end{array}}}\end{array}\right]$$

$$=\left[\begin{array}{l}\lambda\sum_{i=1}^{n}w_i\dfrac{2\left\{\begin{array}{l}\delta T_B(x_i)\delta T_A(x_i)+\delta I_B(x_i)\delta I_A(x_i)\\+\delta F_B(x_i)\delta F_A(x_i)\end{array}\right\}}{\left\{\begin{array}{l}\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\\+\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\end{array}\right\}}\\+(1-\lambda)\sum_{i=1}^{n}w_i\dfrac{\begin{array}{l}\delta T_B(x_i)\delta T_A(x_i)+\delta I_B(x_i)\delta I_A(x_i)\\+\delta F_B(x_i)\delta F_A(x_i)\end{array}}{\sqrt{\begin{array}{l}\left[(\delta T_B(x_i))^2+(\delta I_B(x_i))^2+(\delta F_B(x_i))^2\right]\\\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\end{array}}}\end{array}\right]$$

$$=Var_{WRNS}(B,A)$$

(3.) If $A = B$ i.e.,

$\delta T_A(x_i) = \delta T_B(x_i)$, $\delta I_A(x_i) = \delta I_B(x_i)$, and $\delta F_A(x_i) = \delta F_B(x_i)$, for every x_i (i = 1, 2, …, n) in X,

$Var_{WRNS}(A, A) =$

$$\left[\begin{array}{l} \lambda\sum_{i=1}^{n} w_i \dfrac{2\left\{\begin{array}{l}\delta T_A(x_i)\delta T_A(x_i)+\delta I_A(x_i)\delta I_A(x_i)\\ +\delta F_A(x_i)\delta F_A(x_i)\end{array}\right\}}{\left\{\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\ +\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\end{array}\right\}} \\ +(1-\lambda)\sum_{i=1}^{n} w_i \dfrac{\begin{array}{l}\delta T_A(x_i)\delta T_A(x_i)+\delta I_A(x_i)\delta I_A(x_i)\\ +\delta F_A(x_i)\delta F_A(x_i)\end{array}}{\sqrt{\begin{array}{l}\left[(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2\right]\\ \sqrt{(\delta T_A(x_i))^2+(\delta I_A(x_i))^2+(\delta F_A(x_i))^2}\end{array}}} \end{array}\right]$$

$= \left[\lambda\sum_{i=1}^{n} w_i + (1-\lambda)\sum_{i=1}^{n} w_i\right] = 1$

These results show the completion of the proofs of the three proiperties.

6. Multi attribute decision making based on rough neutrosophic variational coefficient similarity measure

In this section, a rough variational co-efficient similarity measure is employed to multi-attribute decision making in rough neutrosophic environment. Assume that $A = \{A_1, A_2,..., A_m\}$ be the set of alternatives and $C = \{C_1, C_2,..., C_n\}$ be the set of attributes in a multi-attribute decision making problem. Assune that w_j be the weight of the attribute C_j provided by the decision maker such that each $w_i \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$ However, in real situation decision maker may often face difficulty to evaluate alternatives over the attributes due to vague or incomplete information about alternatives in a decision making situation. Rough neutrosophic set can be used in MADM to deal with incomplete information of the alternatives. In this paper, the assessment values of all the alternatives with respect to attributes are considered as the rough neutrosophic values (see Table 1).

Table1: Rough neutrosophic decision matrix

$D_{RNS} = \langle \underline{d}_{ij}, \overline{d}_{ij} \rangle_{m\times n} =$

	C_1	C_2	.	C_n
A_1	$\langle \underline{d}_{11}, \overline{d}_{11} \rangle$	$\langle \underline{d}_{12}, \overline{d}_{12} \rangle$	...	$\langle \underline{d}_{1n}, \overline{d}_{1n} \rangle$
A_2	$\langle \underline{d}_{21}, \overline{d}_{21} \rangle$	$\langle \underline{d}_{22}, \overline{d}_{22} \rangle$	...	$\langle \underline{d}_{2n}, \overline{d}_{2n} \rangle$
.	...	...	...	...
.	...	...	...	...
A_m	$\langle \underline{d}_{m1}, \overline{d}_{m1} \rangle$	$\langle \underline{d}_{m2}, \overline{d}_{m2} \rangle$	...	$\langle \underline{d}_{mn}, \overline{d}_{mn} \rangle$

(27)

Here $\langle \underline{d}_{ij}, \overline{d}_{ij} \rangle$ is the rough neutrosophic number for the i-th alternative and the j-th attribute.

Definition 6.1: Transforming operator for SVNSs [80]

The rough neutrosophic decision matrix (27) can be transformed to single valued neutrosophic decision matrix whose ij-th element α_{ij} can be presented as follows:

$$\alpha_{ij} = \left\langle \frac{\underline{d}_{ij} + \overline{d}_{ij}}{2} \right\rangle_{m\times n}, \text{ for } i = 1, 2, 3,..., m; \; j = 1, 2, 3, ..., n. \quad (28)$$

Step1. Determine the neutrosophic relative positive ideal solution

In multi-criteria decision-making environment, the concept of ideal point has been used to help identify the best alternative in the decision set.

Definition 6.2 [51].

Let H be the collection of two types of attributes, namely, benefit type attribute (P) and cost type attribute (L) in the MADM problems. The relative positive ideal neutrosophic solution (RPINS) $Q_S^+ = [\delta_{q_S}^+, \delta_{q_S}^+, ..., \delta_{q_S}^+]$ is the solution of the decision matrix $D_S = \langle \delta T_{ij}, \delta I_{ij}, \delta F_{ij} \rangle_{m\times n}$ where, every component of Q_S^+ has the following form:

for benefit type attribute, every component of Q_S^+ has the following form:

$$q_S^+ = \langle \delta T_j^+, \delta I_j^+, \delta F_j^+ \rangle = \left\langle \max_i \{\delta T_{ij}\}, \min_i \{\delta I_{ij}\}, \min_i \{\delta F_{ij}\} \right\rangle \; for\; j \in P \quad (29)$$

and for cost type attribute, every component of Q_S^+ has the following form

$$q_S^+ = \langle \delta T_j^+, \delta I_j^+, \delta F_j^+ \rangle = \left\langle \min_i \{\delta T_{ij}\}, \max_i \{\delta I_{ij}\}, \max_i \{\delta F_{ij}\} \right\rangle \; for\; j \in L \quad (30)$$

Step 2. Determine the weighted variational co-efficient similarity measure between ideal alternative and each alternative.

The variational co-efficient similarity measure between ideal alternative Q_S^+ and each alternative A_i for i = 1, 2, …, m can be determined by the following equation as follows:

$Var_{WRNS}(Q_S^+, D_S) =$

$$\left[\begin{array}{l} \lambda\sum_{i=1}^{n} w_i \dfrac{2\left\{\delta T_j^+.\delta T_{ij} + \delta I_j^+.\delta I_{ij} + \delta F_j^+.\delta F_{ij}\right\}}{\left\{\begin{array}{l}\left[(\delta T_j^+)^2+(\delta I_j^+)^2+(\delta F_j^+)^2\right]+\\ \left[(\delta T_{ij})^2+(\delta I_{ij})^2+(\delta F_{ij})^2\right]\end{array}\right\}} \\ +(1-\lambda)\sum_{i=1}^{n} w_i \dfrac{\left\{\delta T_j^+.\delta T_{ij} + \delta I_j^+.\delta I_{ij} + \delta F_j^+.\delta F_{ij}\right\}}{\sqrt{\begin{array}{l}\left[(\delta T_j^+)^2+(\delta I_j^+)^2+(\delta F_j^+)^2\right]\\ \sqrt{(\delta T_{ij})^2+(\delta I_{ij})^2+(\delta F_{ij})^2}\end{array}}} \end{array}\right] \quad (31)$$

Step3. Rank the alternatives.

According to the values obtained from Eq.(31), the ranking order of all the alternatives can be easily determined. Highest value indicates the best alternative.

Step 4. End.

7 Numerical example

In this section, rough neutrosophic MADM regarding investment problem is considered to demonstrate the applicability and the effectiveness of the proposed approach. However, investment problem is not easy to solve. It not only requires oodles of patience and discipline, but also a great deal of research and a sound understanding of the market, mathematical tools, among others. Suppose an investment company wants to invest a sum of money in the best option. Assume that there are four possible alternatives to invest the money: (1) A_1 is a computer company; (2) A_2 is a garment company; (3) A_3 is a telecommunication company; and (4) A_4 is a food company. The investment company must take a decision based on the following three criteria: (1) C_1 is the growth factor; (2) C_2 is the environmental impact; and (3) C_3 is the risk factor. The four possible alternatives are to be evaluated under the attribute by the rough neutrosophic assessments provided by the decision maker. These assessment values are given in the rough neutrosophic decision matrix (see the table 2).

Table2. Rough neutrosophic decision matrix

$$D=\left\langle \underline{N}_{ij}(P), \overline{N}_{ij}(P)\right\rangle_{4\times 3}=$$

$$\begin{array}{c|ccc} & C_1 & C_2 & C_3 \\ \hline A_1 & \left\langle \begin{array}{l}(0.1,0.2,0.2),\\(0.3,0.2,0.2)\end{array}\right\rangle & \left\langle \begin{array}{l}(0.6,0.4,0.3),\\(0.8,0.2,0.3)\end{array}\right\rangle & \left\langle \begin{array}{l}(0.3,0.2,0.3),\\(0.5,0.2,0.1)\end{array}\right\rangle \\ A_2 & \left\langle \begin{array}{l}(0.2,0.4,0.3),\\(0.4,0.2,0.3)\end{array}\right\rangle & \left\langle \begin{array}{l}(0.6,0.3,0.3),\\(0.8,0.1,0.1)\end{array}\right\rangle & \left\langle \begin{array}{l}(0.1,0.4,0.3),\\(0.3,0.2,0.3)\end{array}\right\rangle \\ A_3 & \left\langle \begin{array}{l}(0.3,0.2,0.3),\\(0.5,0.2,0.1)\end{array}\right\rangle & \left\langle \begin{array}{l}(0.5,0.2,0.3),\\(0.7,0.2,0.1)\end{array}\right\rangle & \left\langle \begin{array}{l}(0.0,0.2,0.4),\\(0.2,0.2,0.2)\end{array}\right\rangle \\ A_4 & \left\langle \begin{array}{l}(0.0,0.4,0.4),\\(0.2,0.2,0.2)\end{array}\right\rangle & \left\langle \begin{array}{l}(0.5,0.4,0.4),\\(0.7,0.2,0.2)\end{array}\right\rangle & \left\langle \begin{array}{l}(0.2,0.3,0.3),\\(0.4,0.1,0.1)\end{array}\right\rangle \end{array} \quad (32)$$

The known weight information is given as follows: $W = [w_1, w_2, w_3]^T = [0.3, 0.3, 0.4]$ and $\sum_{i=1}^{3} w_i = 1$.

Step1. Determine the types of criteria.

First two types i.e. C_1 and C_2 of the given criteria are benefit type criteria and the last one criterion i.e. C_3 is the cost type criteria.

Step2. Determine the relative neutrosophic positive ideal solution

Using Eq. (29), Eq.(30), the relative positive ideal neutrosophic solution for the given matrix defined in Eq.(32) can be obtained as:

$$Q_S^+ = [(0.4,0.2,0.2),(0.7,0.2,0.2),(0.1,0.3,0.3)]$$

Step3. Determine the weighted variational similarity measure

The weighted variational co-efficient similarity measure is determined by using Eq.(28), Eq.(31) and Eq.(32). The results obtained for different values of have been shown in the Table-3.

Table-3. Results of rough variational similarity measure for different values of , $0 \le \lambda \le 1$

Similarity measure method	Values of s	Measure values	Ranking order
$Var_{WRNS}(Q_S^+, D_S)$	0.10	0.8769; 0.9741; 0.9917; 0.8107	$A_3 > A_2 > A_1 > A_4$
	0.25	0.8740, 0.9739 0.9905 0.8078	$A_3 > A_2 > A_1 > A_4$
	0.50	0.8692; 0.9735; 0.9887; 0.8028	$A_3 > A_2 > A_1 > A_4$
	0.75	0.8643; 0.9730; 0.9868; 0.7979	$A_3 > A_2 > A_1 > A_4$
	0.90	0.8614; 0.9728; 0.9857; 0.7949	$A_3 > A_2 > A_1 > A_4$

Step 4. Rank the alternatives.

According to the different values of , the results obtained in Table-3 reflects that A_3 is the best alternative.

8. Comparisons of different rough similarity measure with rough variation similarity measure

In this section, four existing rough similarity measures - namely: rough cosine similarity measure, rough dice similarity measure, rough cotangent similarity measure and rough Jaccard similarity measure - have been compared with proposed rough variational co-efficient similarity measure for different values of λ. The comparison results are listed in the Table 3 and Table 4.

Table-4. Results of existing rough neutrosophic similarity measure methods.

Rough similarity measure methods	Values of s	Measure values	Ranking order
$JAC_{WRNS}(Q_S^+, D_S)$ [21]	...	0.7870, 0.9471; 0.9739; 0.6832	$A_3 > A_2 > A_1 > A_4$
$DIC_{WRNS}(Q_S^+, D_S)$ [21]	...	0.8595; 0.9726; 0.9873; 0.7929	$A_3 > A_2 > A_1 > A_4$
$C_{WRNS}(Q_S^+, D_S)$ [20]	...	0.8788; 0.9738; 0.9920; 0.9132	$A_3 > A_2 > A_4 > A_1$
$COT_{WRNS}(Q_S^+, D_S)$ [19]	...	0.8472; 0.9358; 0.9643; 0.8103	$A_3 > A_2 > A_1 > A_4$

Conclusion

In this paper, we have proposed rough variational coefficient similarity measures. We also proved some of their basic properties. We have presented an application of rough neutrosophic variational coefficient similarity measure for a decision making problem on investment. The concept presented in the paper can be applied to deal with other multi attribute decision making problems in rough neutrosophic environment.

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Rough Neutrosophic TOPSIS for Multi-Attribute Group Decision Making

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Abstract: This paper is devoted to present Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) method for multi-attribute group decision making under rough neutrosophic environment. The concept of rough neutrosophic set is a powerful mathematical tool to deal with uncertainty, indeterminacy and inconsistency. In this paper, a new approach for multi-attribute group decision making problems is proposed by extending the TOPSIS method under rough neutrosophic environment. Rough neutrosophic set is characterized by the upper and lower approximation operators and the pair of neutrosophic sets that are characterized by truth-membership degree, indeterminacy membership degree, and falsity membership degree. In the decision situation, ratings of alternatives with respect to each attribute are characterized by rough neutrosophic sets that reflect the decision makers' opinion. Rough neutrosophic weighted averaging operator has been used to aggregate the individual decision maker's opinion into group opinion for rating the importance of attributes and alternatives. Finally, a numerical example has been provided to demonstrate the applicability and effectiveness of the proposed approach.

Keywords: Multi-attribute group decision making; Neutrosophic set; Rough set; Rough neutrosophic set; TOPSIS

1 Introduction

Hwang and Yoon [1] put forward the concept of Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) in 1981 to help select the best alternative with a finite number of criteria. Among numerous multi criteria decision making (MCDM) methods developed to solve real-world decision problems, (TOPSIS) continues to work satisfactorily in diverse application areas such as supply chain management and logistics [2, 3, 4, 5], design, engineering and manufacturing systems [6, 7], business and marketing management [8, 9], health, safety and environment management[10, 11], human resources management [12, 13, 14], energy management [15], chemical engineering [16], water resources management [17, 18], bi-level programming problem [19, 20], multi-level programming problem [21], medical diagnosis [22], military [23], education [24], others topics [25, 26, 27, 28, 29, 30], etc. Behzadian et al. [31] provided a state-of the-art literature survey on TOPSIS applications and methodologies. According to C. T. Chen [32], crisp data are inadequate to model real-life situations because human judgments including preferences are often vague. Preference information of alternatives provided by the decision makers may be poorly defined, partially known and incomplete. The concept of fuzzy set theory grounded by L. A. Zadeh [33] is capable of dealing with impreciseness in a mathematical form. Interval valued fuzzy set [34, 35, 36, 37] was proposed by several authors independently in 1975 as a generalization of fuzzy set. In 1986, K. T. Atanassov [38] introduced the concept of intuitionistic fuzzy set (IFS) by incorporating non-membership degree as independent entity to deal non-statistical impreciseness. In 2003, mathematical equivalence of intuitionistic fuzzy set (IFS) with interval-valued fuzzy sets was proved by Deschrijver and Kerre [39]. C. T. Chen [32] studied the TOPSIS method in fuzzy environment for solving multi-attribute decision making problems. Boran et al. [12] studied TOPSIS method in intuitionistic fuzzy environment and provided an illustrative example of personnel selection in a manufacturing company for a sales manager position. However, fuzzy sets and interval fuzzy sets are not capable of all types of uncertainties in different real physical problems involving indeterminate information.

In order to deal with indeterminate and inconsistent information, the concept of neutrosophic set [40, 41, 42, 43] is useful. In neutrosophic set each element of the universe is characterized by the truth membership degree, indeterminacy membership degree and falsity membership degree lying in the non-standard unit interval$]^-0, 1^+[$. However, it is difficult to apply directly the neutrosophic set in real engineering and scientific applications. Wang et al. [44] introduced single-valued neutrosophic set (SVNS) to face real scientific and engineering fields involving imprecise, incomplete, and inconsistent information. However, the idea was envisioned some years earlier by Smarandache [43] SVNS, a subclass of NS, can also represent each element of universe with the truth membership values, indeterminacy membership values and falsity membership values lying in the real unit interval [0, 1]. SVNS has caught much attention to the researchers on various topics such as, medical diagnosis [45], similarity measure [46, 47, 48, 49, 50], decision making [51, 52, 53, 54, 55, 56, 57, 58, 59, 60, 61, 62, 63, 64, 65, 66, 67, 68, 69, 70], educational problems [71, 72], conflict resolution [73], social problem [74, 75], optimization [76, 77, 78, 79, 80, 81], etc.

Pawlak [82] proposed the notion of rough set theory for the study of intelligent systems characterized by inexact, uncertain or insufficient information. It is a useful mathematical tool for dealing with uncertainty or incomplete information. Broumi et al. [83, 84] proposed new hybrid intelligent structure called rough neutrosophic set by combining the concepts of single valued neutrosophic set and rough set. The theory of rough neutrosophic set [83, 84] is also a powerful mathematical tool to deal with incompleteness. Rough neutrosophic set can be applied in addressing problems with uncertain, imprecise, incomplete and inconsistent information existing in real scientific and engineering applications. In rough neutrosophic environment, Mondal and Pramanik [85] proposed rough neutrosophic multi-attribute decision-making based on grey relational analysis. Mondal and Pramanik [86] also proposed rough neutrosophic multi-attribute decision-making based on rough accuracy score function. Pramanik and Mondal [87] proposed cotangent similarity measure of rough neutrosophic sets and its application to medical diagnosis. Pramanik and Mondal [88] also proposed cosine similarity measure of rough neutrosophic sets and its application in medical diagnosis. Pramanik and Mondal [88] also proposed some similarity measures namely, Dice and Jaccard similarity measures in rough neutrosophic environment and applied them for multi attribute decision making problem. Pramanik and Mondal [90] studied decision making in rough interval neutrosophic environment in 2015. Mondal and Pramanik [91] studied cosine, Dice and Jaccard similarity measures for interval rough neutrosophic sets and presented their applications in decision making problem. So decision making in rough neutrosophic environment appears to be a developing area of study. Mondal et al. [92] proposed rough trigonommetric Hamming similarity measures such as cosine, sine and cotangent rough similarity meaures and proved their basic properties. In the same study Mondal et al. [92] also provided a numerical example of selection of a smart phone for rough use based on the proposed methods.

The objective of the study is to extend the concept of TOPSIS method for multi-attribute group decision making (MAGDM) problems under single valued neutrosophic rough neutrosophic environment. All information provided by different domain experts in MAGDM problems about alternative and attribute values take the form of rough neutrosophic set. In a group decision making process, rough neutrosophic weighted averaging operator is used to aggregate all the decision makers' opinions into a single opinion to select best alternative.

The remaining part of the paper is organized as follows: section 2 presents some preliminaries relating to neutrosophic set, section 3 presents the concept of rough neutrosophic set. In section 4, basics of TOPSIS method are discussed. Section 5 is devoted to present TOPSIS method for MAGDM under rough neutrosophic environment. In section 6, a numerical example is provided to show the effectiveness of the proposed approach. Finally, section 7 presents the concluding remarks and scope of future research.

2 Neutrosophic sets and single valued neutrosophic set [43, 44]

2.1 Definition of Neutrosophic sets [40, 41, 42, 43]

Definition 2.1.1. [43]:

Assume that V be a space of points and v be a generic element in V. Then a neutrosophic set G in V is characterized by a truth membership function T_G, an indeterminacy membership function I_G and a falsity membership function F_G. The functions T_G, I_G and F_G are real standard or non-standard subsets of $]^-0, 1^+[$ i.e. $T_G: V \rightarrow\ ]^-0, 1^+[$, $I_G: V \rightarrow\ ]^-0, 1^+[$, $F_G: V \rightarrow\ ]^-0, 1^+[$, and $^-0 \le T_G(v) + I_G(v) + F_G(v) \le 3^+$.

2.1.2.[43]:

The complement of a neutrosophic set G is denoted by G^c and is defined by

$T_{G^c}(v) = \{1^+\} - T_G(v)$; $I_{G^c}(v) = \{1^+\} - I_G(v)$; $F_{G^c}(v) = \{1^+\} - I_G(v)$

Definition 2.1.3. [43]:

A neutrosophic set G is contained in another neutrosophic set H, $G \subseteq H$ iff the following conditions holds.

$\inf T_G(v) \le \inf T_H(v)$ $\sup T_G(v) \le \sup T_H(v)$

$\inf I_G(v) \ge \inf I_H(v)$, $\sup I_G(v) \ge \sup I_H(v)$

$\inf F_G(v) \ge \inf F_H(v)$, $\sup F_G(v) \ge \sup F_H(v)$

for all v in V.

Definition 2.1.4. [44]:

Assume that V be a universal space of points, and v be a generic element of V. A single-valued neutrosophic set P is characterized by a truth membership function $T_P(v)$, a falsity membership function $I_P(v)$, and an indeterminacy membership function $F_P(v)$. Here, $T_P(v)$, $I_P(v)$, $F_P(v) \in [0, 1]$. When V is continuous, a SVNS P can be written as

$$P = \int_V \langle\langle T_P(v), F_P(v), I_P(v)\rangle / v, v \in V.$$

When V is discrete, a SVNS P can be written as

$$P = \sum\langle T_P(v), F_P(v), I_P(v)\rangle / v, \quad \forall\ v \in V$$

It is obvious that for a SVNS P,

$$0 \le \sup T_P(v) + \sup F_P(v) + \sup I_P(v) \le 3, \ \forall\ v \in V$$

Definition 2.1.5. [44]:
The complement of a SVNS set P is denoted by P^C and is defined as follows:

$$T_P{}^C(v) = F_P(v);\ I_P{}^C(v) = 1 - I_P(v);\ F_P{}^C(v) = T_P(v)$$

Definition 2.1.6. [44]:
A SVNS P_G is contained in another SVNS P_H, denoted as $P_G \subseteq P_H$ if the following conditions hold.

$$T_{P_G}(v) \le T_{P_H}(v)\ ;\ I_{P_G}(v) \ge I_{P_H}(v)\ ;\ F_{P_G}(v) \ge F_{P_H}(v)\ ,\ \forall\ v \in V.$$

Definition 2.1.7. [44]:
Two SVNSs P_G and P_H are equal, i.e., $P_G = P_H$, iff $P_G \subseteq P_H$ and $P_G \supseteq P_H$

Definition 2.1.8. [44]:
The union of two SVNSs P_G and P_H is a SVNS P_Q, written as $P_Q = P_G \cup P_H$.
Its truth, indeterminacy and falsity membership functions are as follows:

$$T_{P_Q}(v) = \max(T_{P_G}(v), T_{P_H}(v))\ ;$$

$$I_{P_Q}(v) = \min(I_{P_G}(v), I_{P_H}(v))\ ;$$

$$F_{P_Q}(v) = \min(F_{P_G}(v), F_{P_H}(v)),\ \forall\ v \in V.$$

Definition 2.1.9. [44]:
The intersection of two SVNSs P_G and P_H is a SVNS P_C written as $P_C = P_G \cap P_H$. Its truth, indeterminacy and falsity membership functions are as follows:

$$T_{P_C}(v) = \min(T_{P_G}(v), T_{P_H}(v))\ ;$$

$$I_{P_C}(v) = \max(I_{P_G}(v), I_{P_H}(v))\ ;$$

$$F_{P_C}(v) = \max(F_{P_G}(v), F_{P_H}(v)),\ \forall\ v \in V.$$

Definition 2.1.10. [44]:
Wang et al. [44] defined the following operation for two SVNS P_G and P_H as follows:

$$P_G \otimes P_H = \left\langle \begin{matrix} T_{P_G}(v).T_{P_H}(v),\ I_{P_G}(v) + I_{P_H}(v) - I_{P_G}(v).I_{P_H}(v), \\ F_{P_G}(v) + F_{P_H}(v) - F_{P_G}(v).F_{P_H}(v) \end{matrix} \right\rangle,$$

$\forall\ v \in V$.

Definition 2.1.11. [93]
Assume that

$$P_G = \begin{Bmatrix} (v_1 / (T_{P_G}(v_1), I_{P_G}(v_1), F_{P_G}(v_1))), \cdots, \\ (v_n / (T_{P_G}(v_n), I_{P_G}(v_n), F_{P_G}(v_n))) \end{Bmatrix}$$

$$P_H = \begin{Bmatrix} (v_1 / (T_{P_H}(v_1), I_{P_H}(v_1), F_{P_H}(v_1))), \cdots, \\ (v_n / (T_{P_H}(v_n), I_{P_H}(v_n), F_{P_H}(v_n))) \end{Bmatrix}$$

be two SVNSs in $v = \{v_1, v_2, v_3, \ldots, v_n\}$

Then the Hamming distance [93] between two SVNSs P_G and P_H is defined as follows:

$$d_P(P_G, P_H) = \sum_{i=1}^{n} \left\langle \begin{matrix} |T_{P_G}(v) - T_{P_H}(v)| + |I_{P_G}(v) - I_{P_H}(v)| \\ + |F_{P_G}(v) - F_{P_H}(v)| \end{matrix} \right\rangle \quad (1)$$

and normalized Hamming distance [93] between two SVNSs P_G and P_H is defined as follow

$$^N d_P(P_G, P_H) = \frac{1}{3n} \sum_{i=1}^{n} \left\langle \begin{matrix} |T_{P_G}(v) - T_{P_H}(v)| + |I_{P_G}(v) - I_{P_H}(v)| \\ + |F_{P_G}(v) - F_{P_H}(v)| \end{matrix} \right\rangle \quad (2)$$

with the following two properties

i. $0 \le d_P(P_G, P_H) \le 3$
ii. $0 \le {}^N d_P(P_G, P_H) \le 1$

Distance between two SVNSs:
Majumder and Samanta [93] studied similarity and entropy measure by incorporating Euclidean distances of SVNSs.

Definition 2.1.12. [93]: (Euclidean distance)

Let $P_G = \begin{Bmatrix} (v_1 | \langle T_{P_G}(v_1), I_{P_G}(v_1), F_{P_G}(v_1)\rangle), \cdots, \\ (v_n | \langle T_{P_G}(v_n), I_{P_G}(v_n), F_{P_G}(v_n)\rangle) \end{Bmatrix}$ and

$P_H = \begin{Bmatrix} (v_1 | \langle T_{P_H}(v_1), I_{P_H}(v_1), F_{P_H}(v_1)\rangle), \cdots, \\ (v_n | \langle T_{P_H}(v_n), I_{P_H}(v_n), F_{P_H}(v_n)\rangle) \end{Bmatrix}$ be two

SVNSs for $v_i \in V$, where i = 1, 2, . . . , n. Then the Euclidean distance between two SVNSs P_G and P_H can be defined as follows:

$$D_{euclid}(P_G, P_H) = \left\langle \sum_{i=1}^{n} \begin{pmatrix} (T_{P_G}(v_i) - T_{P_H}(v_i))^2 + (I_{P_G}(v_i) - I_{P_H}(v_i))^2 \\ + (F_{P_G}(v_i) - F_{P_H}(v_i))^2 \end{pmatrix} \right\rangle^{0.5} \quad (3)$$

and the normalized Euclidean distance [93] between two SVNSs P_G and P_H can be defined as follows:

$$D^N_{euclid}(P_G, P_H) = \frac{1}{3n} \left\langle \sum_{i=1}^{n} \begin{pmatrix} (T_{P_G}(v_i) - T_{P_H}(v_i))^2 + (I_{P_G}(v_i) - I_{P_H}(v_i))^2 \\ + (F_{P_G}(v_i) - F_{P_H}(v_i))^2 \end{pmatrix} \right\rangle^{0.5} \quad (4)$$

Definition 2.1.13. (Deneutrosophication of SVNS) [53]:
Deneutrosophication of SVNS P_G can be defined as a process of mapping P_G into a single crisp output $\theta^* \in V$ i.e. $f : P_G \to \theta^*$ for $v \in V$. If P_G is discrete set then the vector $P_G = \{v \mid \langle T_{P_G}(v), I_{P_G}(v), F_{P_G}(v)\rangle \mid v \in V\}$ is reduced to a single scalar quantity $\theta^* \in V$ by deneutrosophication. The obtained scalar quantity $\theta^* \in V$ best represents the aggregate distribution of three membership degrees of neutrosophic element $\langle T_{P_G}(v), I_{P_G}(v), F_{P_G}(v)\rangle$

3 Rough neutrosophic set [83, 84]

Rough set theory [82] has been developed based on two basic components. The components are crisp set and equivalence relation. The rough set logic is based on the approximation of sets by a couple of sets. These two are known as the lower approximation and the upper approximation of a set. Here, the lower and upper approximation operators are based on equivalence relation. Rough neutrosophic sets [83, 84] are the generalization of rough fuzzy sets [94, 95, 96] and rough intuitionistic fuzzy sets [97].

Definition 3.1. **Rough neutrosophic set [83,84]**

Assume that S be a non-null set and ρ be an equivalence relation on S. Assume that E be neutrosophic set in S with the membership function T_E, indeterminacy function I_E and non-membership function F_E. The lower and the upper approximations of E in the approximation (S, ρ) denoted by $\underline{L}(E)$ and $\overline{U}(E)$ are respectively defined as follows:

$$\underline{L}(E)=\left\langle < v, T_{\underline{L}(E)}(v), I_{\underline{L}(E)}(v), F_{\underline{L}(E)}(v) > / s \in [v]_\rho, v \in S \right\rangle (5)$$

$$\overline{U}(E)=\left\langle < v, T_{\overline{U}(E)}(v), I_{\overline{U}(E)}(v), F_{\overline{U}(E)}(v) > / s \in [v]_\rho, v \in S \right\rangle (6)$$

Here, $T_{\underline{L}(E)}(v) = \wedge_s \in [v]_\rho T_E(s)$, $I_{\underline{L}(E)}(v) = \wedge_s \in [v]_\rho I_E(s)$, $F_{\underline{L}(E)}(v) = \wedge_s \in [v]_\rho F_E(s)$, $T_{\overline{U}(E)}(v) = \vee_s \in [v]_\rho T_E(s)$, $I_{\overline{U}(E)}(v) = \vee_s \in [v]_\rho T_E(s)$, $F_{\overline{U}(E)}(v) = \vee_s \in [v]_\rho I_E(s)$.

So, $0 \le T_{\underline{L}(E)}(v) + I_{\underline{L}(E)}(v) + F_{\underline{L}(E)}(v) \le 3$

$0 \le T_{\overline{U}(E)}(v) + I_{\overline{U}(E)}(v) + F_{\overline{U}(E)}(v) \le 3$

The symbols $\vee$ and $\wedge$ indicate "max" and "min'' operators respectively. $T_E(s)$, $I_E(s)$ and $F_E(s)$ represent the membership , indeterminacy and non-membership of S with respect to E. $\underline{L}(E)$ and $\overline{U}(E)$ are two neutrosophic sets in S.

Thus the mapping $\underline{L}$, $\overline{U}$: N(S) $\rightarrow$ N(S) are, respectively, referred to as the lower and upper rough neutrosophic approximation operators, and the pair $(\underline{L}(E), \overline{U}(E))$ is called the rough neutrosophic set in (S, ρ).

$\underline{L}(E)$ and $\overline{U}(E)$ have constant membership on the equivalence classes of ρ if $\underline{L}(E) = \overline{U}(E)$; i.e. $T_{\underline{L}(E)}(v) = T_{\overline{U}(E)}(v)$, $I_{\underline{L}(E)}(v) = I_{\overline{U}(E)}(v)$, $F_{\underline{L}(E)}(v) = F_{\overline{U}(E)}(v)$ for any v belongs to S.

E is said to be definable neutrosophic set in the approximation (S, ρ). It is obvious that zero neutrosophic set (0_N) and unit neutrosophic sets (1_N) are definable neutrosophic sets.

Definition 3.2 [83, 84].

If N(E) = ($\underline{L}(E), \overline{U}(E)$) be a rough neutrosophic set in (S, ρ), the complement of N(E) is the rough neutrosophic set and is denoted as $\sim N(E) = (\underline{L}(E)^C, \overline{U}(E)^C)$,where $\underline{L}(E)^C, \overline{U}(E)^C$ are the complements of neutrosophic sets of $\underline{L}(E), \overline{U}(E)$ respectively.

$\underline{L}(E)^c = \left\langle < v, T_{\underline{L}(E)}(v), 1 - I_{\underline{L}(E)}(v), F_{\underline{L}(E)}(v) > / \ v \in S \right\rangle$ and

$\overline{U}(E)^c = \left\langle < v, T_{\overline{U}(E)}(v), 1 - I_{\overline{U}(E)}(v), F_{\overline{U}(E)}(v) > / \ v \in S \right\rangle$

Definition 3. 3 [83, 84]

If $N(E_1)$ and $N(E_2)$ be two rough neutrosophic sets in S, then the following definitions hold:

$N(E_1) = N(E_2) \Leftrightarrow \underline{L}(E_1) = \underline{L}(E_2) \wedge \overline{U}(E_1) = \overline{U}(E_2)$

$N(E_1) \subseteq N(E_2) \Leftrightarrow \underline{L}(E_1) \subseteq \underline{L}(E_2) \wedge \overline{U}(E_1) \subseteq \overline{U}(E_2)$

$N(E_1) \bigcup N(E_2) = < \underline{L}(E_1) \bigcup \underline{L}(E_2), \overline{U}(E_1) \bigcup \overline{U}(E_2) >$

$N(E_1) \bigcap N(E_2) = < \underline{L}(E_1) \bigcap \underline{L}(E_2), \overline{U}(E_1) \bigcap \overline{U}(E_2) >$

$N(E_1) + N(E_2) = < \underline{L}(E_1) + \underline{L}(E_2), \overline{U}(E_1) + \overline{U}(E_2) >$

$N(E_1) \,.\, N(E_2) = < \underline{L}(E_1) \,.\, \underline{L}(E_2), \overline{U}(E_1) \,.\, \overline{U}(E_2) >$

If α, β, γ be rough neutrosophic sets in (S, ρ), then the following properties are satisfied.

Properties I:

1. $\sim(\sim \alpha) = \alpha$
2. $\alpha \cup \beta = \beta \cup \alpha, \ \beta \cup \alpha = \alpha \cup \beta$
3. $(\gamma \cup \beta) \cup \alpha = \gamma \cup (\beta \cup \alpha)$, $(\gamma \cap \beta) \cap \alpha = \gamma \cap (\beta \cap \alpha)$
4. $(\gamma \cup \beta) \cap \alpha = (\gamma \cup \beta) \cap (\gamma \cup \alpha)$, $(\gamma \cap \beta) \cup \alpha = (\gamma \cap \beta) \cup (\gamma \cap \alpha)$

Proof. For proofs of the properies, see [83,84].

Properties II:

De Morgan's Laws are satisfied for rough neutrosophic sets

1. $\sim (N(E_1) \cup N(E_2)) = (\sim N(E_1)) \cap (\sim N(E_2))$
2. $\sim (N(E_1) \cap N(E_2)) = (\sim N(E_1)) \cup (\sim N(E_2))$

Proof. For proofs of the properies, see [83,84].

Properties III:

If E_1 and E_2 are two neutrosophic sets of universal collection (U) such that $E_1 \subseteq E_2$, *then* 1. $N(E_1) \subseteq N(E_2)$

2. $N(E_1 \bigcap E_2) \subseteq N(E_2) \bigcap N(E_2)$
3. $N(E_1 \bigcup E_2) \supseteq N(E_2) \bigcup N(E_2)$

Proof. For proofs of the properies, see [83,84].

Properties IV:

1. $\underline{L}(E) = \sim \overline{U}(\sim E)$
2. $\overline{U}(E) = \sim \underline{L}(\sim E)$
3. $\underline{L}(E) \subseteq \overline{U}(E)$

Proof. For proofs of the properies, see [83,84].

4 TOPSIS

The TOPSIS is used to determine the best alternative from the compromise solutions. The best compromise solution should have the shortest Euclidean distance from the positive ideal solution (PIS) and the farthest Euclidean distance from the negative ideal solution (NIS). The TOPSIS method can be described as follows. Assume that K = {K_1, K_2, …,K_m} be the set of alternatives, L = {L_1, L_2, …, L_n} be the set of criteria and $p_{ij}, i=1,2,\ldots,m; j=1,2,\ldots,n$ is the rating of the alternative K_i with respect to the criterion L_j, w_j is the weight of the j- th criterion L_j.

The procedure of TOPSIS method is presented using the following steps:

Step 1. Normalization the decision matrix

Calculation of the normalized value $[\vartheta]_{ij}^{N}$ is as follows:

For benefit criterion, $\vartheta_{ij}=(\vartheta_{ij}-\vartheta_j^-)/(\vartheta_j^+-\vartheta_j^-)$,

where $\vartheta_j^+=\max_i (v_{ij})$ and $\vartheta_j^-=\min_i (v_{ij})$

or setting ϑ_j^+ is the desired level and ϑ_j^- is the worst level.

For cost criterion, $\vartheta_{ij}=(\vartheta_j^- - \vartheta_{ij})/(\vartheta_j^- - \vartheta_j^+)$

Step 2. Weighted normalized decision matrix

In the weighted normalized decision matrix, the upgraded ratings are calculated as follows:

$\eta_{ij}=w_j\times\eta_{ij}$ for i = 1, 2, . . . , m and j = 1, 2, . . . , n. Here w_j is the weight of the j-th criterion such that $w_j \geq 0$ for j = 1, 2, . . . , n and $\sum_{j=1}^{n} w_j=1$

Step 3. The positive and the negative ideal solutions

The positive ideal solution (PIS) and the negative ideal solution (NIS) are calculated as follows:

$$PIS=M^+=\left\langle \eta_1^+,\eta_2^+,\cdots\eta_n^+ \right\rangle =$$

$$\left\langle \left(\max_j \eta_{ij} / j\in C_1\right),\left(\min_j \eta_{ij} / j\in C_2\right): j=1,2,\cdots,n \right\rangle \text{ and}$$

$$NIS=M^-=\left\langle \eta_1^-,\eta_2^-,\cdots\eta_n^- \right\rangle =$$

$$\left\langle \left(\min_j \eta_{ij} / j\in C_1\right),\left(\max_j \eta_{ij} / j\in C_2\right): j=1,2,\cdots,n \right\rangle$$

where $C_1$and C_2 are the benefit and cost type criteria respectively.

Step 4. Calculation of the separation measures for each alternative from the PIS and the NIS

The separation values for the PIS and the separation values for the NIS can be determined by using the n-dimensional Euclidean distance as follows:

$$\delta_i^+=\left\langle \sum_{j=1}^{n}\left(\eta_{ij}-\eta_j^+\right)^2 \right\rangle^{0.5} \text{ for i = 1, 2, . . . , m.}$$

$$\delta_i^-=\left\langle \sum_{j=1}^{n}\left(\eta_{ij}-\eta_j^-\right)^2 \right\rangle^{0.5} \text{ for i = 1, 2, . . . , m.}$$

Step 5. Calculation of the relative closeness coefficient to the PIS

The relative closeness coefficient for the alternative K_i with respect to M^+ is

$$\chi_i = \frac{\delta_i^-}{(\delta_i^+ + \delta_i^-)} \text{ for i = 1, 2, . . . m.}$$

Obviously, $0\leq\chi_i\leq 1$. According to relative closeness coefficient to the ideal alternative, larger value of χ_i indicates the better alternative K_i.

Step 6. Ranking the alternatives

Rank the alternatives according to the descending order of the relative-closeness coefficients to the PIS.

5 Topsis method for multi-attribute decision making under rough neutrosophic environment

Assume that a multi-attribute decision-making problem be characterized by m alternatives and n attributes. Assume that K = (K_1, K_2,..., K_m) be the set of alternatives, and L = (L_1, L_2, ..., L_n) be the set of attributes. The rating measured by the decision maker describes the performance of the alternative K_i against the attribute L_j. Assume that W = {w_1, w_2 . . . , w_n} be the weight vector assigned for the attributes L_1, L_2, ..., L_n by the decision makers. The values associated with the alternatives for multi-attribute decision-making problem (MADM) with respect to the attributes can be presented in rough neutrosophic decision matrix (see Table 1).

Table1: Rough neutrosophic decision matrix

$D=\left\langle \underline{d}_{ij},\overline{d}_{ij} \right\rangle_{m\times n} =$

	L_1	L_2	$\cdots$	L_n
K_1	$\langle \underline{d}_{11},\overline{d}_{11}\rangle$	$\langle \underline{d}_{12},\overline{d}_{12}\rangle$	...	$\langle \underline{d}_{1n},\overline{d}_{1n}\rangle$
K_2	$\langle \underline{d}_{21},\overline{d}_{21}\rangle$	$\langle \underline{d}_{22},\overline{d}_{22}\rangle$	...	$\langle \underline{d}_{2n},\overline{d}_{2n}\rangle$
.	...	...	...	...
.	...	...	...	...
K_m	$\langle \underline{d}_{m1},\overline{d}_{m1}\rangle$	$\langle \underline{d}_{m2},\overline{d}_{m2}\rangle$	...	$\langle \underline{d}_{mn},\overline{d}_{mn}\rangle$

(7)

Here $\left\langle \underline{d}_{ij},\overline{d}_{ij}\right\rangle$ is the rough neutrosophic number according to the i-th alternative and the j-th attribute.

In decision-making situation, there exist many attributes of alternatives. Some of them are important and others may be less important. So it is important to select proper weights of attributes for decision-making situation.

Definition 5.1. Accumulated geometric operator (AGO) [85]

Assume a rough neutrosophic number in the form: $\left\langle \underline{L}_{ij}(\underline{T}_{ij},\underline{I}_{ij},\underline{F}_{ij}),\overline{U}_{ij}(\overline{T}_{ij},\overline{I}_{ij},\overline{F}_{ij})\right\rangle$. We transform the rough neutrosophic number into SVNNs using the accumulated geometric operator (AGO). The operator is expressed as follows.

$$N_{ij}\left\langle T_{ij},I_{ij},F_{ij}\right\rangle = \left\langle \underline{L}_{ij}.\overline{U}_{ij}\right\rangle^{0.5}=$$

$$N_{ij}\left\langle (\underline{T}_{ij}\overline{T}_{ij})^{0.5},(\underline{I}_{ij}\overline{I}_{ij})^{0.5},(\underline{F}_{ij}\overline{F}_{ij})^{0.5}\right\rangle \quad (8)$$

Using AGO operator [85], the rating of each alternative with respect to each attribute is transformed into SVNN for MADM problem. The rough neutrosophic values (transformed as SVNN) associated with the alternatives for

MADM problems can be represented in decision matrix (see Table 2).

Table 2. Tranformed rough neutrosiphic decision matrix

$D=\langle d\rangle_{m\times n}=\langle T_{ij}, I_{ij}, F_{ij}\rangle_{m\times n}=$

$$\begin{array}{c|cccc} & L_1 & L_2 & \cdots & L_n \\ \hline K_1 & \langle T_{11}, I_{11}, F_{11}\rangle & \langle T_{12}, I_{12}, F_{12}\rangle & \cdots & \langle T_{1n}, I_{1n}, F_{1n}\rangle \\ K_2 & \langle T_{21}, I_{21}, F_{21}\rangle & \langle T_{22}, I_{22}, F_{22}\rangle & \cdots & \langle T_{2n}, I_{2n}, F_{2n}\rangle \\ . & \cdots & \cdots & \cdots & \cdots \\ . & \cdots & \cdots & \cdots & \cdots \\ K_m & \langle T_{m1}, I_{m1}, F_{m1}\rangle & \langle T_{m2}, I_{m2}, F_{m2}\rangle & \cdots & \langle T_{mn}, I_{mn}, F_{mn}\rangle \end{array} \quad (9)$$

In the matrix $\langle d\rangle_{m\times n}=\langle T_{ij}, I_{ij}, F_{ij}\rangle_{m\times n}$, T_{ij}, I_{ij} and F_{ij} (i = 1, 2,..., n and j = 1, 2,..., m) denote the degree of truth membership value, indeterminacy membership value and falsity membership value of alternative K_i with respect to attribute L_j.

The ratings of each alternative with respect to the attributes can be explained by the neutrosophic cube [98] proposed by Dezert. The vertices of neutrosophic cube are (0, 0, 0), (1, 0, 0), (1, 0, 1), (0, 0, 1), (0, 1, 0), (1, 1, 0), (1, 1, 1) and (0, 1, 1). The acceptance ratings [53, 99] in neutrosophic cube are classified in three types namely,

I. Highly acceptable neutrosophic ratings,
II. Manageable neutrosophic rating
III. Unacceptable neutrosophic ratings.

Definition 5.2. (Highly acceptable neutrosophic ratings) [99]

In decision making process, the sub cube (Θ) of a neutrosophic cube (Ω) (i.e. $\Theta \subset \Omega$) reflects the field of highly acceptable neutrosophic ratings (Ψ). Vertices of Λ are defined with the eight points (0.5, 0, 0),(1, 0, 0),(1, 0, 0.5), (0.5, 0, 0.5), (0.5, 0, 0.5), (1, 0, 0.5), (1, 0.5, 0.5) and (0.5, 0.5, 0.5). U includes all the ratings of alternative considered with the above average truth membership degree, below average indeterminacy degree and below average falsity membership degree for multi-attribute decision making. So, Ψ has a great role in decision making process and can be defined as follows:

$\Psi = \langle (\underline{T}_{ij}\overline{T}_{ij})^{0.5}, (\underline{I}_{ij}\overline{I}_{ij})^{0.5}, (\underline{F}_{ij}\overline{F}_{ij})^{0.5}\rangle$ where $0.5 < (\underline{T}_{ij}\overline{T}_{ij})^{0.5} < 1$, $0 < (\underline{I}_{ij}\overline{I}_{ij})^{0.5} < 0.5$ and $0 < (\underline{F}_{ij}\overline{F}_{ij})^{0.5} < 0.5$, for i = 1, 2, . . . , m and j = 1, 2, . . . , n.

Definition 5.3. (Unacceptable neutrosophic ratings) [99]

The field Σ of unacceptable neutrosophic ratings Λ is defined by the ratings which are characterized by 0% membership degree, 100% indeterminacy degree and 100% falsity membership degree. Hence, the set of unacceptable ratings Λ can be considered as the set of all ratings whose truth membership value is zero.

$\Lambda = \langle (\underline{T}_{ij}\overline{T}_{ij})^{0.5}, (\underline{I}_{ij}\overline{I}_{ij})^{0.5}, (\underline{F}_{ij}\overline{F}_{ij})^{0.5}\rangle$ where $(\underline{T}_{ij}\overline{T}_{ij})^{0.5} = 0$, $0 < (\underline{I}_{ij}\overline{I}_{ij})^{0.5} \leq 1$ and $0 < (\underline{F}_{ij}\overline{F}_{ij})^{0.5} \leq 1$, for i = 1, 2, . . . , m and j = 1, 2, . . . , n.

In decision making situation, consideration of Λ should be avoided.

Definition 5.4. (Manageable neutrosophic ratings) [99]

Excluding the field of high acceptable ratings and unacceptable ratings from a neutrosophic cube, tolerable neutrosophic rating field Φ $(=\Omega \cap \neg\Theta \cap \neg\Sigma)$ is determined. The tolerable neutrosophic rating (Δ) considered membership degree is taken in decision making process. Δ can be defined by the expression as follows:

$\Delta = \langle (\underline{T}_{ij}\overline{T}_{ij})^{0.5}, (\underline{I}_{ij}\overline{I}_{ij})^{0.5}, (\underline{F}_{ij}\overline{F}_{ij})^{0.5}\rangle$ where $0 < (\underline{T}_{ij}\overline{T}_{ij})^{0.5} < 0.5$, $0.5 < (\underline{I}_{ij}\overline{I}_{ij})^{0.5} < 1$ and $0.5 < (\underline{F}_{ij}\overline{F}_{ij})^{0.5} < 1$.

for i = 1, 2, . . . , m and j = 1, 2, . . . , n.

Definition 5.5 [53].

Fuzzification of transformed rough neutrosophic set $N=\langle T_N(v), I_N(v), F_N(v)\rangle$ for any v ∈ V can be defined as a process of mapping N into fuzzy set F $=\{v/\mu_F(v)/v\in V\}$ i.e. f: N $\rightarrow$ F for v ∈ V. The representative fuzzy membership degree $\mu_F(v)\in[0,1]$ of the vector $\{v/\langle T_N(v), I_N(v), F_N(v)\rangle, v\in V\}$ is defined from the concept of neutrosophic cube. It can be obtained by determining the root mean square of $1-T_N(v)$, $I_N(v)$, and $F_N(v)$ for all v ∈ V. Therefore the equivalent fuzzy membership degree is defined as follows:

$$\mu_F(v)=\begin{cases}1-\left\langle\left\{(1-T_N(v))^2+(I_N(v))^2+(F_N(v))^2\right\}/3\right\rangle^{0.5} & \forall v\in\psi\cup\Delta \\ 0 & \forall v\in\Lambda\end{cases} \quad (10)$$

Now the steps of decision making using TOPSIS method under rough neutrosophic environment are stated as follows.

Step 1. Determination of the weights of decision makers

Assume that a group of k decision makers having their own decision weights involved in the decision making. The importance of the decision makers in a group may not be equal. Assume that the importance of each decision maker is considered with linguistic variables and expressed it by rough neutrosophic numbers.

Assume that $\langle \underline{N}_k(\underline{T}_k, \underline{I}_k, \underline{F}_k), \overline{N}_k(\overline{T}_k, \overline{I}_k, \overline{F}_k)\rangle$ be a rough neutrosophic number for the rating of k−th decision maker. Using AGO operator, we obtain $E_k = \langle T_k, I_k, F_k\rangle$ as a single valued neutrosophic number for the rating of k−th decision maker. Then, according to equation (10) the weight of the k−th decision maker can be written as:

$$\xi_k=\frac{1-\left\langle\left\{(1-T_k(v))^2+(I_k(v))^2+(F_k(v))^2\right\}/3\right\rangle^{0.5}}{\sum_{k=1}^{r}\left\langle 1-\left\{\left((1-T_k(v))^2+(I_k(v))^2+(F_k(v))^2\right)/3\right\}^{0.5}\right\rangle} \quad (11)$$

and $\sum_{k=1}^{r}\xi_k = 1$

Step 2. Construction of the aggregated rough neutrosophic decision matrix based on the assessments of decision makers

Assume that $D^k = \langle \underline{d}^{(k)}_{ij}, \overline{d}^{(k)}_{ij} \rangle_{m\times n}$ be the rough neutrosophic decision matrix of the k−th decision maker. According to equation (11), $D^k = \left(d^{(k)}_{ij}\right)_{m\times n}$ be the single-valued neutrosophic decision matrix corresponding to the rough neutrosophic decision matrix and $\xi = (\xi_1, \xi_2, ..., \xi_k)^T$ be the weight vector of decision maker such that each $\xi_k \in [0, 1]$. In the group decision making process, all the individual assessments need to be accumulated into a group opinion to make an aggregated single valued neutrosophic decision matrix. This aggregated matrix can be obtained by using rough neutrosophic aggregation operator as follows:

$D = (d_{ij})_{m\times n}$ where,

$$(d_{ij})_{m\times n} = RNWA_{\xi}\left(d^1_{ij}, d^2_{ij}, \cdots, d^r_{ij}\right) = \xi_1 d^1_{ij} \oplus \xi_2 d^2_{ij} \oplus \cdots \oplus \xi_r d^r_{ij}$$

$$= \left\langle 1-\prod_{k=1}^{r}(1-T^{(r)}_{ij})^{\xi_k}, \prod_{k=1}^{r}(I^{(r)}_{ij})^{\xi_k}, \prod_{k=1}^{r}(F^{(r)}_{ij})^{\xi_k}, \right\rangle \quad (12)$$

$$\text{Here, } d^r_{ij} = \left\langle \underline{d}^r_{ij}.\overline{d}^r_{ij} \right\rangle^{0.5}$$

Now the aggregated rough neutrosophic decision matrix is defined as follows:

$$(d_{ij})_{m\times n} = \left\langle (\underline{T}_{ij}.\overline{T}_{ij})^{0.5}, (\underline{I}_{ij}.\overline{I}_{ij})^{0.5}, (\underline{F}_{ij}.\overline{F}_{ij})^{0.5} \right\rangle_{m\times n}$$

	L_1	L_2	...	L_n
K_1	$\langle T_{11}, I_{11}, F_{11}\rangle$	$\langle T_{12}, I_{12}, F_{12}\rangle$	...	$\langle T_{1n}, I_{1n}, F_{1n}\rangle$
K_2	$\langle T_{21}, I_{21}, F_{21}\rangle$	$\langle T_{22}, I_{22}, F_{22}\rangle$	...	$\langle T_{2n}, I_{2n}, F_{2n}\rangle$
.	...	...	...	...
.	...	...	...	...
K_m	$\langle T_{m1}, I_{m1}, F_{m1}\rangle$	$\langle T_{m2}, I_{m2}, F_{m2}\rangle$	...	$\langle T_{mn}, I_{mn}, F_{mn}\rangle$

(13)

Here, $d_{ij} = \langle T_{ij}, I_{ij}, F_{ij}\rangle = \left\langle (\underline{T}_{ij}.\overline{T}_{ij})^{0.5}, (\underline{I}_{ij}.\overline{I}_{ij})^{0.5}, (\underline{F}_{ij}.\overline{F}_{ij})^{0.5} \right\rangle$ is the aggregated element of rough neutrosophic decision matrix D for i = 1, 2, . . . m and j = 1, 2, . .. n.

Step 3. Determination of the attribute weights

In the decision-making process, all attributes may not have equal importance. So, every decision maker may have their own opinion regarding attribute weights. To obtain the group opinion of the chosen attributes, all the decision makers' opinions need to be aggregated. Assume that $\left\langle \underline{w}^j_{(k)}, \overline{w}^j_{(k)} \right\rangle$ be rough neutrosophic number (RNN) assigned to the attribute L_j by the k−th decision maker. According to equation (8) w^j_k be the neutrosophic number assigned to the attribute L_j by the k−th decision maker. Then the combined weight W = (w_1, w_2 . . . , w_n) of the attribute can be determined by using rough neutrosophic weighted aggregation (RNWA) operator

$$w_j = RNWA_{\xi}(w^{(1)}_j, w^{(2)}_j, \cdots, w^{(r)}_j) = \xi_1 w^{(1)}_j \oplus \xi_2 w^{(2)}_j \oplus \cdots \oplus \xi_r w^{(r)}_j$$

$$= \left\langle 1-\prod_{k=1}^{r}(1-T^{(r)}_j)^{\xi_k}, \prod_{k=1}^{r}(I^{(r)}_j)^{\xi_k}, \prod_{k=1}^{r}(F^{(r)}_j)^{\xi_k} \right\rangle \quad (14)$$

$\text{Here, } \xi^r_{ij} = \left\langle \underline{d}_{ij}.\overline{d}_{ij} \right\rangle; \; w_j = \left\langle T^{(r)}_j, I^{(r)}_j, F^{(r)}_j \right\rangle =$

$\left\langle (\underline{T}^{(r)}_j.\overline{T}^{(r)}_j)^{0.5}, (\underline{I}^{(r)}_j.\overline{I}^{(r)}_j)^{0.5}, (\underline{F}^{(r)}_j.\overline{F}^{(r)}_j)^{0.5} \right\rangle$ for j = 1, 2, . . . n.

W = (w_1, w_2 . . . , w_n) (15)

Step 4. Aggregation of the weighted rough neutrosophic decision matrix

In this section, the obtained weights of attribute and aggregated rough neutrosophic decision matrix need to be further fused to make the aggregated weighted rough neutrosophic decision matrix. Then, the aggregated weighted rough neutrosophic decision matrix can be defined by using the multiplication properties between two neutrosophic sets as follows:

$$D \otimes W = D^W = \left\langle d^{w_j}_{ij} \right\rangle_{m\times n} = \left\langle T^{w_j}_{ij}, I^{w_j}_{ij}, F^{w_j}_{ij} \right\rangle_{m\times n} =$$

	L_1	L_2	...	L_n
K_1	$\langle T^{w_1}_{11}, I^{w_1}_{11}, F^{w_1}_{11}\rangle$	$\langle T^{w_2}_{12}, I^{w_2}_{12}, F^{w_2}_{12}\rangle$	...	$\langle T^{w_n}_{1n}, I^{w_n}_{1n}, F^{w_n}_{1n}\rangle$
K_2	$\langle T^{w_1}_{21}, I^{w_1}_{21}, F^{w_1}_{21}\rangle$	$\langle T^{w_2}_{22}, I^{w_2}_{22}, F^{w_2}_{22}\rangle$	...	$\langle T^{w_n}_{2n}, I^{w_n}_{2n}, F^{w_n}_{2n}\rangle$
.	...	...	...	...
.	...	...	...	...
K_m	$\langle T^{w_1}_{m1}, I^{w_1}_{m1}, F^{w_1}_{m1}\rangle$	$\langle T^{w_2}_{m2}, I^{w_2}_{m2}, F^{w_2}_{m2}\rangle$	...	$\langle T^{w_n}_{mn}, I^{w_n}_{mn}, F^{w_n}_{mn}\rangle$

(16)

Here, $d^{w_j}_{ij} = \left\langle T^{w_j}_{ij}, I^{w_j}_{ij}, F^{w_j}_{ij} \right\rangle$ is an element of the aggregated weighted rough neutrosophic decision matrix D^W for i = 1, 2, . . . , m and j = 1, 2, . . . , n.

Step 5. Determination of the rough relative positive ideal solution (RRPIS) and the rough relative negative ideal solution (RRNIS)

After transferring RNS decision matrix, assume $D_N = \left\langle d^W_{ij} \right\rangle_{m\times n} = \left\langle T_{ij}, I_{ij}, F_{ij} \right\rangle_{m\times n}$ be a SVNS based decision matrix, where, T_{ij}, I_{ij} and F_{ij} are the membership degree, indeterminacy degree and non-membership degree of evaluation for the attribute L_j with respect to the alternative K_i. In practical siuation, two types of attributes namely, benefit type attribute and cost type attribute are considered in multi-attribute decision making problems.

Definition 5.6.

Assume that $C_1$and C_2 be the benefit type attribute and cost type attribute respectively. Suppose that G^+_N is the relative rough neutrosophic positive ideal solution (RRNPIS) and G^-_N is the relative rough neutrosophic negative ideal solution (RRNNIS).

Then G^+_N can be defined as follows:

$$G^+_N = \left\langle d^{w+}_1, d^{w+}_2, \cdots, d^{w+}_n \right\rangle \quad (17)$$

Here $d^{w+}_j = \left\langle T^{w+}_j, I^{w+}_j, F^{w+}_j \right\rangle$ for j = 1, 2, . . . , n.

$$T^{w+}_j = \{(\max_i\{T^{w_j}_{ij}\} / j \in C_1), (\min_i(T^{w_j}_{ij}) / j \in C_2)\}$$

$$I^{w+}_j = \{(\min_i\{I^{w_j}_{ij}\} / j \in C_1), (\max_i(I^{w_j}_{ij}) / j \in C_2)\}$$

$F_j^{w+} = \{(\min_i \{F_{ij}^{wj}\} / j \in C_1), (\max_i (F_{ij}^{wj}) / j \in C_2)\}$

Then G_N^- can be defined as follows:

$$G_N^- = \langle d_1^{w-}, d_2^{w-}, \cdots, d_n^{w-} \rangle \quad (18)$$

Here $d_j^{w-} = \langle T_j^{w-}, I_j^{w-}, F_j^{w-} \rangle$ for j = 1, 2, . . . , n.

$T_j^{w-} = \{(\min_i \{T_{ij}^{wj}\} / j \in C_1), (\max_i (T_{ij}^{wj}) / j \in C_2)\}$

$I_j^{w-} = \{(\max_i \{I_{ij}^{wj}\} / j \in C_1), (\min_i (I_{ij}^{wj}) / j \in C_2)\}$

$F_j^{w-} = \{(\max_i \{F_{ij}^{wj}\} / j \in C_1), (\min_i (F_{ij}^{wj}) / j \in C_2)\}$

Step 6. Determination of the distance measure of each alternative from the RRNPIS and the RRNNIS

The normalized Euclidean distance measure of all alternative $\langle T_{ij}^{wj}, I_{ij}^{wj}, F_{ij}^{wj} \rangle$ from the RRNPIS $\langle d_1^{w+}, d_2^{w+}, ..., d_n^{w+} \rangle$ for i = 1, 2, …, m and j = 1, 2, …, n can be written as follows:

$$\delta_{euclid}^{i+}(d_{ij}^{wj}, d_{ij}^{w+}) = \frac{1}{3n} \left\langle \sum_{j=1}^{n} \begin{pmatrix} (T_{ij}^{wj}(v_j) - T_j^{w+}(v_j))^2 + (I_{ij}^{wj}(v_j) - I_j^{w+}(v_j))^2 \\ + (F_{ij}^{wj}(v_j) - F_j^{w+}(v_j))^2 \end{pmatrix} \right\rangle^{0.5} \quad (19)$$

The normalized Euclidean distance measure of all alternative $\langle T_{ij}^{wj}, I_{ij}^{wj}, F_{ij}^{wj} \rangle$ from the RRNPIS $\langle d_1^{w-}, d_2^{w-}, ..., d_n^{w-} \rangle$ for i = 1, 2, …, m and j = 1, 2, …, n can be written as follows:

$$\delta_{euclid}^{i-}(d_{ij}^{wj}, d_{ij}^{w-}) = \frac{1}{3n} \left\langle \sum_{j=1}^{n} \begin{pmatrix} (T_{ij}^{wj}(v_j) - T_j^{w-}(v_j))^2 + (I_{ij}^{wj}(v_j) - I_j^{w-}(v_j))^2 \\ + (F_{ij}^{wj}(v_j) - F_j^{w-}(v_j))^2 \end{pmatrix} \right\rangle^{0.5} \quad (20)$$

Step 7. Determination of the relative closeness co-efficient to the rough neutrosophic ideal solution for rough neutrosophic sets

The relative closeness coefficient of each alternative K_i with respect to the neutrosophic positive ideal solution G_N^+ is defined as follows:

$$\chi_i^* = \frac{\langle \delta_{euclid}^{i-}(d_{ij}^{wj}, d_{ij}^{w-}) \rangle}{\langle \delta_{euclid}^{i-}(d_{ij}^{wj}, d_{ij}^{w-}) + \delta_{euclid}^{i+}(d_{ij}^{wj}, d_{ij}^{w+}) \rangle} \quad (21)$$

Here $0 \le \chi_i^* \le 1$. According to the relative closeness coefficient values larger the values of χ_i^* reflects the better alternative K_i for i = 1, 2, …, n.

Step 8. Ranking the alternatives

Rank the alternatives according to the descending order of the relative-closeness coefficients to the RRNPIS.

6 Numerical example

In order to demonstrate the proposed method, logistic center location selection problem is described here. Suppose that a new modern logistic center is required in a town. There are three locations K_1, K_2, K_3. A committee of three decision makers or experts D_1, D_2, D_3 has been formed to select the most appropriate location on the basis of six parameters obtained from expert opinions, namely, cost (L_1), distance to suppliers (L_2), distance to customers (L_3), conformance to government and law (L_4), quality of service (L_5), and environmental impact (L_6).

Based on the proposed approach the considered problem is solved using the following steps:

Step 1. Determination of the weights of decision makers

The importance of three decision makers in a selection committee may be different based on their own status. Their decision values are considered as linguistic terms (seeTable-3). The importance of each decision maker expressed by linguistic term with its corresponding rough neutrosophic values shown in Table-4. The weights of decision makers are determined with the help of equation (11) as:

$\xi_1 = 0.398$, $\xi_2 = 0.359$, $\xi_3 = 0.243$.

We transform rough neutrosophic number (RNN) to neutrosophic number (NN) with the help of AGO operator [85] in Table 3, Table 4 and Table 5.

Step 2. Construction of the aggregated rough neutrosophic decision matrix based on the assessments of decision makers

The linguistic terms along with RNNs are defined in Table-5 to rate each alternative with respect to each attribute. The assessment values of each alternative K_i (i = 1, 2, 3) with respect to each attribute L_j (j = 1, 2, 3, 4, 5, 6) provided by three decision makers are listed in Table-6. Then the aggregated neutrosophic decision matrix can be obtained by fusing all the decision maker opinions with the help of aggregation operator (equation 12) (see Table 7).

Step 3. Determination of the weights of attributes

The linguistic terms shown in Table-3 are used to evaluate each attribute. The importance of each attribute for every decision maker is rated with linguistic terms shown in Table-6. Three decision makers' opinions need to be aggregated to final opinion.

The fused attribute weight vector is determined by using equation (14) as follows:

$$W = \left\{ \begin{matrix} \langle 0.761, 0.205, 0.195 \rangle, \langle 0.800, 0.181, 0.159 \rangle, \langle 0.737, 0.241, 0.196 \rangle, \\ \langle 0.761, 0.223, 0.169 \rangle, \langle 0.774, 0.203, 0.172 \rangle, \langle 0.804, 0.184, 0.172 \rangle \end{matrix} \right\} \quad (23)$$

Step 4. Construction of the aggregated weighted rough neutrosophic decision matrix

Using equation (16) and clculating the combined weights of the attributes and the ratings of the alternatives, the aggregated weighted rough neutrosophic decision matrix is obtained (see Table-8).

Step 5. Determination of the rough neutrosophic relative positive ideal solution and the rough neutrosophic relative negative ideal solution

The RNRPIS can be calculated from the aggregated weighted decision matrix on the basis of attribute types i.e. benefit type or cost type by using equation (17) as

$$G_N^{+}= \begin{bmatrix} \langle 0.670, 0.289, 0.274\rangle, \langle 0.694, 0.284, 0.252\rangle, \langle 0.588, 0.388, 0.309\rangle, \\ \langle 0.607, 0.374, 0.286\rangle, \langle 0.642, 0.331, 0.303\rangle, \langle 0.708, 0.270, 0.253\rangle \end{bmatrix} \quad (25)$$

Here $d_1^{w+} = \langle T_1^{w+}, I_1^{w+}, F_1^{w+} \rangle$ is calculated as:

$T_1^{w+} =$ max [0.670, 0.485, 0.454] = 0.670, $I_1^{w+} =$ min [0.289, 0.449, 0.471] = 0.289,

$F_1^{w+} =$ min [0.274, 0.377, 0.463]= 0.274.

Similarly, other RNRPISs are calculated.

Using equation (18), the RNRNIS are calculated from aggregated weighted decision matrix based on attribute types i.e. benefit type or cost type.

$$G_N^{-}= \begin{bmatrix} \langle 0.454, 0.471, 0.463\rangle, \langle 0.588, 0.377, 0.353\rangle, \langle 0.469, 0.480, 0.309\rangle, \\ \langle 0.522, 0.441, 0.358\rangle, \langle 0.524, 0.429, 0.372\rangle, \langle 0.512, 0.435, 0.414\rangle \end{bmatrix} \quad (26)$$

Here, $d_1^{w-} = \langle T_1^{w-}, I_1^{w-}, F_1^{w-} \rangle$ is calculated as

T_1^{w} = min [0.670, 0.485, 0.454] = 0.454, $I_1^{w-} =$ max [0.289, 0.449, 0.471] = 0.471,

$F_1^{w-} =$ max [0.274, 0.377, 0.463] = 0.463.

Other RNRNISs are calculated in similar way.

Step 6. Determination of the distance measure of each alternative from the RRNPIS and the RRNNIS and relative closeness co-efficient

Normalized Euclidean distance measures defined in equation (19) and equation (20) are used to determine the distances of each alternative from the RRNPIS and the RNNIS.

Step 7. Determination of the relative closeness co-efficient to the rough neutrosophic ideal solution for rough neutrosophic sets

Using equation (21) and distances, relative closeness coefficient of each alternative K_1 , K_2 , K_3 with respect to the rough neutrosophic positive ideal solution G_N^{+} is calculated (see Table 9).

Table 9. Distance measure and relative closeness co-efficient

$Alternatives (K_i)$	δ_{euclid}^{i+}	δ_{euclid}^{i-}	χ_i^*
K_1	0.0078	0.1248	0.9411
K_2	0.1192	0.0682	0.3639
K_3	0.1025	0.0534	0.3425

(27)

Step 9. Ranking the alternatives

According to the values of relative closeness coefficient of each alternative (see Table 9), the ranking order of three alternatives is obtained as follows:

$K_1 \succ K_2 \succ K_3$.

Thus K_1 is the best the logistic center.

7 Conclusion

In general, realistic MAGDM problems adhere to uncertain, imprecise, incomplete, and inconsistent data and rough neutrosophic set theory is adequate to deal with it. In this paper, we have proposed rough neutrosophic TOPSIS method for MAGDM. We have also proposed rough neutrosophic aggregate operator and rough neutrosophic weighted aggregate operator. In the decision-making situation, the ratings of each alternative with respect to each attribute are presented as linguistic variables characterized by rough neutrosophic numbers. Rough neutrosophic aggregation operator has been used to aggregate all the opinions of decision makers. Rough neutrosophic positive ideal and rough neutrosophic negative ideal solution have been defined to form aggregated weighted decision matrix. Euclidean distance measure has been used to calculate the distances of each alternative from positive as well as negative ideal solutions for relative closeness co-efficient of each alternative. The proposed rough neutrosophic TOPSIS approach can be applied in pattern recognition, artificial intelligence, and medical diagnosis in rough neutrosophic environment.

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Table 3. Linguistic terms for rating attributes

Linguistic Terms	Rough neutrosophic numbers	Neutrosophic numbers
Very good / Very important (VG/VI)	$\langle(0.85, 0.05, 0.05), (0.95, 0.15, 0.15)\rangle$	$\langle 0.899, 0.087, 0.087\rangle$
Good / Important(G /I)	$\langle(0.75, 0.15, 0.10), (0.85, 0.25, 0.20)\rangle$	$\langle 0.798, 0.194, 0.141\rangle$
Fair / Medium(F/M)	$\langle(0.45, 0.35, 0.35), (0.55, 0.45, 0.55)\rangle$	$\langle 0.497, 0.397, 0.439\rangle$
Bad / Unimportant (B / UI)	$\langle(0.25, 0.55, 0.65), (0.45, 0.65, 0.75)\rangle$	$\langle 0.335, 0.598, 0.698\rangle$
Very bad/Very Unimportant (VB/VUI)	$\langle(0.05, 0.75, 0.85), (0.15, 0.85, 0.95)\rangle$	$\langle 0.087, 0.798, 0.899\rangle$

Table 4. Importance of decision makers expressed in terms of rough neutrosophic numbers

DM	D_1	D_2	D_3
LT	VI	I	M
RNN	$\langle(0.85, 0.05, 0.05), (0.95, 0.15, 0.15)\rangle$	$\langle(0.75, 0.15, 0.10), (0.85, 0.25, 0.20)\rangle$	$\langle(0.45, 0.35, 0.35), (0.55, 0.45, 0.55)\rangle$
NN	$\langle 0.899, 0.087, 0.087\rangle$	$\langle 0.798, 0.194, 0.141\rangle$	$\langle 0.497, 0.397, 0.439\rangle$

Table 5. Linguistic terms for rating the candidates innterms of rough neutrosophic numbers and neutrosophic numbers

Linguistic terms	RNNs	NNs
Extremely Good/High (EG/EH)	$\langle(1.00,0.00,0.00),(1.00,0.00,0.00)\rangle$	$\langle 1.000,0.000,0.000\rangle$
Very Good/High (VG/VH)	$\langle(0.85,0.05,0.05),(0.95,0.15,0.15)\rangle$	$\langle 0.899, 0.087, 0.087\rangle$
Good/High (G/H)	$\langle(0.75,0.15,0.10),(0.85,0.25,0.20)\rangle$	$\langle 0.798, 0.194, 0.141\rangle$
Medium Good/High (MG/MH)	$\langle(0.55,0.30,0.25),(0.65,0.40,0.35)\rangle$	$\langle 0.598, 0.346, 0.296\rangle$
Medium/Fair (M/F)	$\langle(0.45,0.45,0.35),(0.55,0.55,0.55)\rangle$	$\langle 0.497, 0.497, 0.439\rangle$
MediumBad/MediumLaw(MB/ML)	$\langle(0.30,0.60,0.55),(0.40,0.70,0.65)\rangle$	$\langle 0.346, 0.648, 0.598\rangle$
Bad/Law (G/L)	$\langle(0.15,0.70,0.75),(0.25,0.80,0.85)\rangle$	$\langle 0.194, 0.748, 0.798\rangle$
Very Bad/Low (VB/VL)	$\langle(0.05,0.80,0.85),(0.15,0.90,0.95)\rangle$	$\langle 0.087, 0.849, 0.899\rangle$
VeryVeryBad/low(VVB/VVL)	$\langle(0.05,0.95,0.95),(0.05,0.85,0.95)\rangle$	$\langle 0.050, 0.899, 0.950\rangle$

Table 6. Assessments of alternatives and attribute in terms of linguisterm terms given by three decision makers

Alternatives (K_i)	Decision Makers	L_1	L_2	L_3	L_4	L_5	L_6
K_1	D_1	VG	G	G	G	G	VG
	D_2	VG	VG	G	G	G	VG
	D_3	G	VG	G	G	VG	G
K_2	D_1	M	G	M	G	G	M
	D_2	G	MG	G	G	MG	G
	D_3	M	G	M	MG	M	M
K_3	D_1	M	VG	G	MG	VG	M
	D_2	M	M	G	G	M	G
	D_3	G	M	M	MG	G	VG

Table 7. Aggregated transformed rough neutrosophic decision matrix

	L_1	L_2	L_3	L_4	L_5	L_6
K_1	$\langle 0.881, 0.106, 0.098 \rangle$	$\langle 0.867, 0.126, 0.111 \rangle$	$\langle 0.798, 0.194, 0.141 \rangle$	$\langle 0.798, 0.194, 0.141 \rangle$	$\langle 0.830, 0.160, 0.125 \rangle$	$\langle 0.880, 0.106, 0.098 \rangle$
K_2	$\langle 0.637, 0.307, 0.292 \rangle$	$\langle 0.741, 0.239, 0.184 \rangle$	$\langle 0.637, 0.315, 0.292 \rangle$	$\langle 0.761, 0.223, 0.169 \rangle$	$\langle 0.677, 0.284, 0.242 \rangle$	$\langle 0.637, 0.307, 0.292 \rangle$
K_3	$\langle 0.597, 0.334, 0.333 \rangle$	$\langle 0.735, 0.217, 0.231 \rangle$	$\langle 0.748, 0.231, 0.186 \rangle$	$\langle 0.686, 0.281, 0.227 \rangle$	$\langle 0.787, 0.182, 0.175 \rangle$	$\langle 0.755, 0.212, 0.197 \rangle$

Table 8. Aggregated weighted rough neutrosophic decision matrix

	L_1	L_2	L_3	L_4	L_5	L_6
K_1	$\langle 0.670, 0.289, 0.274 \rangle$	$\langle 0.694, 0.284, 0.252 \rangle$	$\langle 0.588, 0.388, 0.309 \rangle$	$\langle 0.607, 0.374, 0.286 \rangle$	$\langle 0.642, 0.331, 0.303 \rangle$	$\langle 0.708, 0.270, 0.253 \rangle$
K_2	$\langle 0.485, 0.449, 0.377 \rangle$	$\langle 0.593, 0.377, 0.344 \rangle$	$\langle 0.469, 0.480, 0.431 \rangle$	$\langle 0.579, 0.396, 0.309 \rangle$	$\langle 0.524, 0.429, 0.372 \rangle$	$\langle 0.512, 0.435, 0.414 \rangle$
K_3	$\langle 0.454, 0.471, 0.463 \rangle$	$\langle 0.588, 0.359, 0.353 \rangle$	$\langle 0.551, 0.416, 0.346 \rangle$	$\langle 0.522, 0.441, 0.358 \rangle$	$\langle 0.609, 0.348, 0.317 \rangle$	$\langle 0.607, 0.357, 0.335 \rangle$

Multiple Criteria Evaluation Model Based on the Single-Valued Neutrosophic Set

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Abstract. Gathering the attitudes of the examined respondents would be very significant in some evaluation models. Therefore, a multiple criteria approach based on the use of the neutrosophic set is considered in this paper. An example of the evaluation of restaurants is considered at the end of this paper with the aim to present in detail the proposed approach.

Keywords: neutrosophic set, single valued neutrosophic set, multiple criteria evaluation.

1. Introduction

In order to deal with indeterminate and inconsistent information, Smarandache [1] proposed a neutrosophic set (NS), thus simultaneously providing a general framework generalizing the concepts of the classical, fuzzy [2], interval-valued [3, 4], intuitionistic [5] and interval-valued intuitionistic [6] fuzzy sets.

The NS has been applied in different fields, such as: the database [7], image processing [8, 9, 10], the medical diagnosis [11, 12], decision making [13, 14], with a particular emphasis on multiple criteria decision making [15, 16, 17, 18, 19, 20].

In addition to the membership function, or the so-called truth-membership $T_A(x)$, proposed in fuzzy sets, Atanassov [5] introduced the non-membership function, or the so-called falsity-membership $F_A(x)$, which expresses non-membership to a set, thus creating the basis for the solving of a much larger number of decision-making problems.

In intuitionistic fuzzy sets, the indeterminacy $I_A(x)$ is $1-T_A(x)-F_A(x)$ by default.

In the NS, Smarandache [21] introduced independent indeterminacy-membership $I_A(x)$, thus making the NS more flexible and the most suitable for solving some complex decision-making problems, especially decision-making problems related to the use of incomplete and imprecise information, uncertainties and predictions and so on.

Smarandache [1] and Wang *et al.* [22] further proposed the single valued neutrosophic set (SVNS) suitable for solving many real-world decision-making problems.

In multiple criteria evaluation models, where evaluation is based on the ratings generated from respondents, the NS and the SVNS can provide some advantages in relation to the usage of crisp and other forms of fuzzy numbers.

Therefore, the rest of this paper is organized as follows: in Section 2, some basic definitions related to the SVNS are given. In Section 3, an approach to the determining of criteria weights is presented, while Section 4 proposes a multiple criteria evaluation model based on the use of the SVNS. In Section 5, an example is considered with the aim to explain in detail the proposed methodology. The conclusions are presented at the end of the manuscript.

2. The Single Valued Neutrosophic Set

Definition 1. [21] Let X be the universe of discourse, with a generic element in X denoted by x. Then, the Neutrosophic Set (NS) A in X is as follows:

$$A = \{x < T_A(x), I_A(x), F_A(x) > | \, x \in X\}, \qquad (1)$$

where $T_A(x)$, $I_A(x)$ and $F_A(x)$ are the truth-membership function, the indeterminacy-membership function and the falsity-membership function, respectively, $T_A, I_A, F_A : X \to]^-0,1^+[$ and $^-0 \leq T_A(x) + I_A(x) + U_A(x) \leq 3^+$

Definition 2. [1, 22] Let X be the universe of discourse. The Single Valued Neutrosophic Set (SVNS) A over X is an object having the form:

$$A = \{x < T_A(x), I_A(x), F_A(x) > | \, x \in X\}, \qquad (2)$$

where $T_A(x)$, $I_A(x)$ and $F_A(x)$ are the truth-membership function, the intermediacy-membership function and the falsity-membership function, respectively, $T_A, I_A, F_A : X \to [0,1]$ and $0 \le T_A(x) + I_A(x) + U_A(x) \le 3$.

Definition 3. [21] For an SVNS A in X, the triple $< t_A, i_A, f_A >$ is called the single valued neutrosophic number (SVNN).

Definition 4. *SVNNs*. Let $x_1 =< t_1, i_1, f_1 >$ and $x_2 =< t_2, i_2, f_2 >$ be two SVNNs and $\lambda > 0$; then, the basic operations are defined as follows:

$$x_1 + x_2 =< t_1 + t_2 - t_1 t_2, i_1 i_2, f_1 f_2 > . \quad (3)$$

$$x_1 \cdot x_2 =< t_1 t_2, i_1 + i_2 - i_1 i_2, f_1 + f_2 - f_1 f_2 > . \quad (4)$$

$$\lambda x_1 =< 1 - (1 - t_1)^\lambda, i_1^\lambda, f_1^\lambda > . \quad (5)$$

$$x_1^\lambda =< t_1^\lambda, i_1^\lambda, 1 - (1 - f_1)^\lambda > . \quad (6)$$

Definition 5. [23] Let $x =< t_x, i_x, f_x >$ be a SVNN; then the cosine similarity measure $S_{(x)}$ between SVNN x and the ideal alternative (point) <1,0,0> can be defined as follows:

$$S_{(x)} = \frac{t}{\sqrt{t^2 + i^2 + f^2}} . \quad (7)$$

Definition 6. [23] Let $A_j =< t_j, i_j, f_j >$ be a collection of SVNSs and $W = (w_1, w_2, \ldots, w_n)^T$ be an associated weighting vector. Then the Single Valued Neutrosophic Weighted Average (SVNWA) operator of A_j is as follows:

$$SVNWA(A_1, A_2, \ldots, A_n) = \sum_{j=1}^{n} w_j A_j = \left(1 - \prod_{j=1}^{n} (1 - t_j)^{w_j}, \prod_{j=1}^{n} (i_j)^{w_j}, \prod_{j=1}^{n} (f_j)^{w_j} \right), \quad (8)$$

where: w_j is the element j of the weighting vector, $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$.

3. The SWARA Method

The Step-wise Weight Assessment Ratio Analysis (SWARA) technique was proposed by Kersuliene *et al.* [25]. The computational procedure of the adapted SWARA method can be shown through the following steps:

Step 1. Determine the set of the relevant evaluation criteria and sort them in descending order, based on their expected significances.

Step 2. Starting from the second criterion, determine the relative importance s_j of the criterion j in relation to the previous (j-1) criterion, and do so for each particular criterion as follows:

$$s_j = \begin{cases} > 1 & \text{when significance of } C_j \succ C_{j-1} \\ 1 & \text{when significanse of } C_j = C_{j-1} \\ < 1 & \text{when significance of } C_j \prec C_{j-1} \end{cases} . \quad (9)$$

By using Eq. (9), respondents are capable of expressing their opinions more realistically compared to the ordinary SWARA method, proposed by Kersuliene *et al.* [25].

Step 3. The third step in the adapted SWARA method should be performed as follows:

$$k_j = \begin{cases} 1 & j = 1 \\ 2 - s_j & j > 1 \end{cases} . \quad (10)$$

where k_j is a coefficient.

Step 4. Determine the recalculated weight q_j as follows:

$$q_j = \begin{cases} 1 & j = 1 \\ q_{j-1} / k_j & j > 1 \end{cases} . \quad (11)$$

Step 5. Determine the relative weights of the evaluation criteria as follows:

$$w_j = q_j \Big/ \textstyle\sum_{k=1}^{n} q_k , \quad (12)$$

where w_j denotes the relative weight of the criterion j.

4. A Multiple Criteria Evaluation Model Based on the Use of the SVNS

For a multiple criteria evaluation problem involving the m alternatives that should be evaluated by the K respondents based on the n criteria, whereby the performances of alternatives are expressed by using the SVNS, the calculation procedure can be expressed as follows:

The determination of the criteria weights. The determination of the criteria weights can be done by applying various methods, for example by using the AHP method. However, in this approach, it is recommended that the SWARA method should be used due to its simplicity and a smaller number of pairwise comparisons compared with the well-known AHP method.

The determination of the criteria weight is done by using an interactive questionnaire made in a spreadsheet file. By using such an approach, the interviewee can see the calculated weights of the criteria, which enables him/her modify his or her answers if he or she is not satisfied with the calculated weights.

Gathering the ratings of the alternatives in relation to the selected set of the evaluation criteria. Gathering the ratings of the alternatives in relation to the chosen set of criteria is also done by using an interactive questionnaire. In this questionnaire, a declarative sentence is formed for each one of the criteria, thus giving an opportunity to the respondents to fill in their attitudes about the degree of truth, indeterminacy and falsehood of the statement.

The formation of the separated ranking order based on the weights and ratings obtained from each respondent. At this steep, the ranking order is formed for each one of the respondents, based on the respondent's respective weights and ratings, in the following manner:

- the determination of the overall ratings expressed in the form of the SVNN by using Eq. (8), for each respondent;
- the determination of the cosine similarity measure, for each respondent; and
- the determination of the ranking order, for each respondent.

The determination of the most appropriate alternative. Contrary to the commonly used approach in group decision making, no group weights and ratings are used in this approach. As a result of that, there are the K ranking orders of the alternatives and the most appropriate alternative is the one determined on the basis of the theory of dominance [26].

5. A Numerical Illustration

In this numerical illustration, some results adopted from a case study are used. In the said study, four traditional restaurants were evaluated based on the following criteria:

- the interior of the building and the friendly atmosphere,
- the helpfulness and friendliness of the staff,
- the variety of traditional food and drinks,
- the quality and the taste of the food and drinks, including the manner of serving them, and
- the appropriate price for the quality of the services provided.

The survey was conducted via e-mail, using an interactive questionnaire, created in a spreadsheet file. By using such an approach, the interviewee could see the calculated weights of the criteria and was also able to modify his/her answers if he or she was not satisfied with the calculated weights.

In order to explain the proposed approach, three completed surveys have been selected. The attitudes related to the weights of the criteria obtained in the first survey are shown in Table 1. Table 1 also accounts for the weights of the criteria.

Criteria	s_j	k_j	q_j	w_j
C_1		1	1	0.15
C_2	1.00	1.00	1.00	0.15
C_3	1.15	0.85	1.18	0.18
C_4	1.30	0.70	1.68	0.26
C_5	1.00	1.00	1.68	0.26

Table 1. The attitudes and the weights of the criteria obtained on the basis of the first of the three surveys

The attitudes obtained from the three surveys, as well as the appropriate weights, are accounted for in Table 2.

	E_1		E_1		E_1	
	s_j	w_j	s_j	w_j	s_j	w_j
C_1		0.15		0.16		0.19
C_2	1.00	0.15	1.00	0.16	1.00	0.19
C_3	1.15	0.18	1.20	0.20	1.05	0.20
C_4	1.30	0.26	1.10	0.22	1.10	0.22
C_5	1.00	0.26	1.10	0.25	0.95	0.21

Table 2. The attitudes and the weights obtained from the three surveys

The ratings of the alternatives expressed in terms of the SVNS obtained on the basis of the three surveys are given in Tables 3 to 5.

	C_1	C_2	C_3	C_4	C_5
w_j	0.15	0.15	0.18	0.26	0.26
A_1	<0.8,0.1,0.3>	<0.7,0.2,0.2>	<0.8,0.1,0.1>	<1,0.01,0.01>	<0.8,0.1,0.1>
A_2	<0.7,0.1,0.2>	<1.0,0.1,0.1>	<1.0,0.2,0.1>	<1,0.01,0.01>	<0.8,0.1,0.1>
A_3	<0.7,0.1,0.1>	<1.0,0.1,0.1>	<0.7,0.1,0.1>	<0.9,0.2,0.01>	<0.9,0.1,0.1>
A_4	<0.7,0.3,0.3>	<0.7,0.1,0.1>	<0.8,0.1,0.2>	<0.9,0.1,0.1>	<0.9,0.1,0.1>

Table 3. The ratings obtained based on the first survey

	C_1	C_2	C_3	C_4	C_5
w_j	0.16	0.16	0.20	0.22	0.25
A_1	<0.8,0.1,0.4>	<0.9,0.15,0.3>	<0.9,0.2,0.2>	<0.85,0.1,0.25>	<1.0,0.1,0.2>
A_2	<0.9,0.15,0.3>	<0.9,0.15,0.2>	<1.0,0.3,0.2>	<0.7,0.2,0.1>	<0.8,0.2,0.3>
A_3	<0.6,0.15,0.3>	<0.55,0.2,0.3>	<0.55,0.3,0.3>	<0.6,0.3,0.2>	<0.7,0.2,0.3>
A_4	<0.6,0.4,0.5>	<0.6,0.3,0.1>	<0.6,0.1,0.2>	<0.7,0.1,0.3>	<0.5,0.2,0.4>

Table 4. The ratings obtained based on the second survey

	C_1	C_2	C_3	C_4	C_5
w_j	0.19	0.19	0.20	0.22	0.21
A_1	<1.0,0.1,0.1>	<0.9,0.15,0.2>	<1.0,0.2,0.1>	<0.8,0.1,0.1>	<0.9,0.1,0.2>
A_2	<0.8,0.15,0.3>	<0.9,0.15,0.2>	<1,0.2,0.2>	<0.7,0.2,0.1>	<0.8,0.2,0.3>
A_3	<0.6,0.15,0.3>	<0.55,0.2,0.3>	<0.55,0.3,0.3>	<0.6,0.3,0.2>	<0.7,0.2,0.3>
A_4	<0.8,0.4,0.5>	<0.6,0.3,0.1>	<0.6,0.4,0.1>	<0.7,0.1,0.3>	<0.5,0.2,0.4>

Table 5. The ratings obtained from the third of the third survey

The calculated overall ratings obtained on the basis of the first of the three surveys expressed in the form of SVNSs are presented in Table 6. The cosine similarity measures, calculated by using Eq. (7), as well as the ranking order of the alternatives, are accounted for in Table 6.

	Overall ratings	S_i	Rank
A_1	<1.0,0.06,0.07>	0.995	2
A_2	<1.0,0.06,0.06>	0.996	1
A_3	<1.0,0.12,0.06>	0.991	3
A_4	<1.0,0.12,0.13>	0.978	4

Table 6. The ranking orders obtained on the basis of the ratings of the first survey

The ranking orders obtained based on all the three surveys are accounted for in Table 7.

	E_1 S_i	E_2 S_i	E_3 S_i	E_1 Rank	E_2 Rank	E_3 Rank
A1	0.995	0.963	0.985	2	1	1
A2	0.996	0.962	0.966	1	2	2
A3	0.991	0.864	0.867	3	4	4
A4	0.978	0.882	0.894	4	3	3

Table 7. The ranking orders obtained from the three examinees

According to Table 7, the most appropriate alternative based on the theory of dominance is the alternative denoted as A_1.

6. Conclusion

A new multiple criteria evaluation model based on using the single valued neutrosophic set is proposed in this paper. For the purpose of determining criteria weights, the SWARA method is applied due to its simplicity, whereas for the determination of the overall ratings for each respondent, the SVNN is applied. In order to intentionally avoid the group determination of weights and ratings, the final selection of the most appropriate alternative is determined by applying the theory of dominance. In order to form a simple questionnaire and obtain the respondents' real attitudes, a smaller number of the criteria were initially selected. The proposed model has proven to be far more flexible than the other MCDM-based models and is based on the conducted numerical example suitable for the solving of problems related to the selection of restaurants. The usability and efficiency of the proposed model have been demonstrated on the conducted numerical example.

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(I,T)-Standard neutrosophic rough set and its topologies properties

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Abstract. In this paper, we defined $(\mathcal{J},\mathcal{T})-$ standard neutrosophic rough sets based on an implicator $\mathcal{J}$ and a t-norm $\mathcal{T}$ on D^*; lower and upper approximations of standard neutrosophic sets in a standard neutrosophic approximation are defined. Some properties of $(\mathcal{J},\mathcal{T})-$ standard neutrosophic rough sets are investigated. We consider the case when the neutrosophic components (truth, indeterminacy, and falsehood) are totally dependent, single-valued, and hence their sum is ≤ 1.

Keywords: standard neutrosophic, $(\mathcal{J},\mathcal{T})-$ standard neutrosophic rough sets

1. Introduction

Rough set theory was introduced by Z. Pawlak in 1980s [1]. It becomes a useful mathematical tool for data mining, especially for redundant and uncertain data. At first, the establishment of the rough set theory is based on equivalence relation. The set of equivalence classes of the universal set, obtained by an equivalence relation, is the basis for the construction of upper and lower approximation of the subset of the universal set.

Fuzzy set theory was introduced by L.Zadeh since 1965 [2]. Immediately, it became a useful method to study the problems of imprecision and uncertainty. Since, a lot of new theories treating imprecision and uncertainty have been introduced. For instance, Intuitionistic fuzzy sets were introduced in1986, by K. Atanassov [3], which is a generalization of the notion of a fuzzy set. When fuzzy set give the degree of membership of an element in a given set, Intuitionistic fuzzy set give a degree of membership and a degree of non-membership of an element in a given set. In 1998 [22], F. Smarandache gave the concept of neutrosophic set which generalized fuzzy set and intuitionistic fuzzy set. This new concept is difficult to apply in the real appliction. It is a set in which each proposition is estimated to have a degree of truth (T), adegree of indeterminacy (I) and a degree of falsity (F). Over time, the subclass of neutrosophic sets was proposed. They are also more advantageous in the practical application. Wang et al. [11] proposed interval neutrosophic sets and some operators of them. Smarandache [22] and Wang et al. [12] proposed a single valued neutrosophic set as an instance of the neutrosophic set accompanied with various set theoretic operators and properties. Ye [13] defined the concept of simplified neutrosophic sets, it is a set where each element of the universe has a degree of truth, indeterminacy, and falsity respectively and which lie between [0, 1] and some operational laws for simplified neutrosophic sets and to propose two aggregation operators, including a simplified neutrosophic weighted arithmetic average operator and a simplified neutrosophic weighted geometric average operator. In 2013, B.C. Cuong and V. Kreinovich introduced the concept of picture fuzzy set [4,5], and picture fuzzy set is regarded the standard neutrosophic set [6].

More recently, rough set have been developed into the fuzzy environment and obtained many interesting results. The approximation of rough (or fuzzy) sets in fuzzy approximation space gives us the fuzzy rough set [7,8,9]; and the approximation of fuzzy sets in crisp approximation space gives us the rough fuzzy set [8, 9]. In 2014, X.T. Nguyen introduces the rough picture fuzzy set as the result of approximation of a picture fuzzy set with respect to a crisp approximation space [18]. Radzikowska and Kerre defined $(\mathcal{J},\mathcal{T})-$ fuzzy rough sets [19], which determined by an implicator $\mathcal{J}$ and a t-norm $\mathcal{T}$ on [0,1]. In 2008, L. Zhou et al. [20] constructed $(\mathcal{J},\mathcal{T})-$ intuitionistic fuzzy rough sets determined by an implicator $\mathcal{J}$ and a t-norm $\mathcal{T}$ on L^*.

In this paper, we considered the case when the neutrosophic components are single valued numbers in [0, 1] and they are totally dependent [17], which means that their sum is ≤ 1. We defined $(\mathcal{J},\mathcal{T})-$ standard neutrosophic rough sets based on an implicator $\mathcal{J}$ and a t-norm $\mathcal{T}$ on D^*; in which, implicator $\mathcal{J}$ and a t-norm $\mathcal{T}$ on D^* is investigated in [21].

2. Standard neutrosophic logic

We consider the set D^* defined by the following definition.

Definition 1. We denote:

$$D^* = \{x = (x_1, x_2, x_3) | x_1 + x_2 + x_3 \leq 1, x_i \in [0,1], i = 1,2,3\}$$

For $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in D^*$, we define:

$x \leq_{D^*} y$ iff $((x_1 < y_1) \wedge (x_3 \geq y_3)) \vee ((x_1 = y_1) \wedge (x_3 > y_3)) \vee ((x_1 = y_1) \wedge (x_3 = y_3) \wedge (x_2 \leq y_2))$, and $x = y \Leftrightarrow (x \leq_{D^*} y) \wedge (y \leq_{D^*} x)$.

Then $(D^*, \leq_{D^*})$ is a lattice, in which $0_{D^*} = (0,0,1) \leq x \leq 1_{D^*} = (1,0,0), \forall x = (x_1, x_2, x_3) \in D^*$. The meet operator $\wedge$ and the join operator $\vee$ on $(D^*, \leq_{D^*})$ are defined as follows:

For $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in D^*$,

$$x \wedge y = (\min(x_1, y_1), \min(x_2, y_2), \max(x_3, y_3)),$$
$$x \vee y = (\max(x_1, y_1), \min(x_2, y_2), \min(x_3, y_3)).$$

On D^*, we consider logic operators as negation, t-norm, t-conorm, implication.

2.1. Standard neutrosophic negation

Definition 2. A standard neutrosophic negation is any nonincreasing $D^* \to D^*$ mapping n satisfying $n(0_{D^*}) = 1_{D^*}$ và $n(1_{D^*}) = 0_{D^*}$.

Example 1. For all $x = (x_1, x_2, x_3) \in D^*$, we have some standard neutrosophic negations on D^* as follows:
+ $n_0(x) = (x_3, 0, x_1)$
+ $n_1(x) = (x_3, x_4, x_2)$ where $x_4 = 1 - x_1 - x_2 - x_3$.

2.2. Standard neutrosophic t-norm

For $x = (x_1, x_2, x_3) \in D^*$, we denote

$$\Gamma(x) = \{y \in D^* : y = (x_1, y_2, x_3), 0 \leq y_2 \leq x_2\}$$

Obviously, we have $\Gamma(0_{D^*}) = 0_{D^*}$, $\Gamma(1_{D^*}) = 1_{D^*}$.

Definition 3. A standard neutrosophic t-norm is an $(D^*)^2 \to D^*$ mapping $\boldsymbol{\mathcal{T}}$ satisfying the following conditions
(T1) $\mathcal{T}(x,y) = \mathcal{T}(y,x), \forall x, y \in D^*$
(T2) $\mathcal{T}(x, \mathcal{T}(y,z)) = \mathcal{T}(\mathcal{T}(x,y), z)), \forall x, y, z \in D^*$
(T3) $\mathcal{T}(x,y) \leq \mathcal{T}(x,z), \forall x, y, z \in D^*$ and $y \leq_{D^*} z$
(T4) $\mathcal{T}(1_{D^*}, x) \in \Gamma(x)$.

Example 2. Some standard neutrosophic t-norm, for all $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in D^*$
+ t-norm min: $\mathcal{T}_M(x,y) = (x_1 \wedge y_1, x_2 \wedge y_2, x_3 \vee y_3)$
+ t-norm product: $\mathcal{T}_P(x,y) = (x_1 y_1, x_2 y_2, x_3 + y_3 - x_3 y_3)$
+ t-norm Lukasiewicz: $\mathcal{T}_L(x,y) = (\max(0, x_1 + y_1 - 1), \max(0, x_2 + y_2 - 1), \min(1, x_3 + y_3))$.

Remark 1.
+ $\mathcal{T}(0_{D^*}, x) = 0_{D^*}$ for all $x \in D^*$. Indeed, for all $x \in D^*$ we have $\mathcal{T}(0_{D^*}, x) \leq \mathcal{T}(0_{D^*}, 1_{D^*}) = 0_{D^*}$
+$\mathcal{T}(1_{D^*}, 1_{D^*}) = 1_{D^*}$ (obvious)

2.3. Standard neutrosophic t-conorm

Definition 4. A standard neutrosophic t-conorm is an $(D^*)^2 \to D^*$ mapping S satisfying the following conditions
(S1) $S(x,y) = S(y,x), \forall x, y \in D^*$
(S2) $S(x, S(y,z)) = S(S(x,y), z)), \forall x, y, z \in D^*$
(S3) $S(x,y) \leq S(x,z), \forall x, y, z \in D^*$ and $y \leq_{D^*} z$
(S4) $S(0_{D^*}, x) \in \Gamma(x)$

Example 3. Some standard neutrosophic t-norm, for all $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in D^*$
+ t-conorm max: $S_M(x,y) = (x_1 \vee y_1, x_2 \wedge y_2, x_3 \wedge y_3)$
+ t-conorm product: $S_P(x,y) = (x_1 + y_1 - x_1 y_1, x_2 y_2, x_3 y_3)$
+ t-conorm Luksiewicz: $S_L(x,y) = (\min(1, x_1 + y_1), \max(0, x_2 + y_2 - 1), \max(0, x_3 + y_3 - 1))$.

Remark 2.
+ $S(1_{D^*}, x) = 1_{D^*}$ for all $x \in D^*$. Indeed, for all $x \in D^*$ we have $S(0_{D^*}, 1_{D^*}) \in \Gamma(1_{D^*}) = 1_{D^*}$ so that $\leq S(0_{D^*}, 1_{D^*}) \leq S(0_{D^*}, x) \leq 1_{D^*}$.
+ $S(0_{D^*}, 0_{D^*}) = 0_{D^*}$ (obvious).

A standard neutrosophic t-norm $\mathcal{T}$ and a standard neutrosophic t-conorm S on D^* are said to be dual with respect to (w.r.t) a standard neutrosophic negation n if

$$\mathcal{T}(n(x), n(y)) = nS(x,y) \quad \forall x, y \in D^*,$$
$$S(n(x), n(y)) = n\mathcal{T}(x,y) \quad \forall x, y \in D^*.$$

Example 4. With negation $n_0(x) = (x_3, 0, x_1)$ we have some t-norm and t-conorm dual as follows:

a. $\mathcal{T}_M$ and S_M
b. $\mathcal{T}_P$ and S_P
c. $\mathcal{T}_L$ and S_L

Many properties of t-norms, t-conorms, negations should be given in [21].

2.4 Standard neutrosophic implication operators

In this section, we recall two classes of standard neutrosophic implication in [21].

A standard neutrosophic implication off class 1.

Definition 5. A mapping $\mathcal{I}: (D^*)^2 \to D^*$ is referred to as a standard neutrosophic implicator off class 1 on D^* if it satisfying following conditions:

$$\mathcal{I}(0_{D^*}, 0_{D^*}) = 1_{D^*}; \mathcal{I}(0_{D^*}, 1_{D^*}) = 1_{D^*}; \mathcal{I}(1_{D^*}, 1_{D^*}) = 1_{D^*};$$
$$I(1_{D^*}, 0_{D^*}) = 0_{D^*}$$

Proposition 1. Let $\mathcal{T}, S$ and n be standard neutrosophic t-norm $\mathcal{T}$, a standard neutrosophic t-conorm S and a standard neutrosophic negation on D^*, respectively. Then, we have a standard neutrosophic implication on D^*, which defined as following:

$\mathcal{I}_{S,\mathcal{T},n}(x,y) = S(\mathcal{T}(x,y), n(x)), \forall x, y \in D^*$.

Proof.
We consider border conditions in definition 5.

$\mathcal{I}(0_{D^*}, 0_{D^*}) = S(\mathcal{T}(0_{D^*}, 0_{D^*}), n(0_{D^*})) = S(0_{D^*}, 1_{D^*}) = 1_{D^*}$,
$\mathcal{I}(0_{D^*}, 1_{D^*}) = S(\mathcal{T}(0_{D^*}, 1_{D^*}), n(0_{D^*})) = S(0_{D^*}, 1_{D^*}) = 1_{D^*}$,
$\mathcal{I}(1_{D^*}, 1_{D^*}) = S(\mathcal{T}(1_{D^*}, 1_{D^*}), n(1_{D^*})) = S(1_{D^*}, 0_{D^*}) = 1_{D^*}$,

and

$\boldsymbol{\mathcal{I}}(1_{D^*}, 0_{D^*}) = S(\boldsymbol{\mathcal{T}}(1_{D^*}, 0_{D^*}), n(1_{D^*})) = S(0_{D^*}, 0_{D^*}) = 0_{D^*}$.

We have the proof.□

Example 5. For all $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in D$, we have some standard neutrosophic implication of class 1 on D^* based on proposition 1 as follows

a. If $\boldsymbol{\mathcal{T}} = \boldsymbol{\mathcal{T}}_M, S = S_M$ and $n_0(x) = (x_3, 0, x_1)$ then $\boldsymbol{\mathcal{I}}_{S_M,\boldsymbol{\mathcal{T}}_M,n_0}(x, y) = (\max(\min(x_1, y_1), x_3), 0, \min(\max(x_3, y_3), x_1)$.

b. If $\boldsymbol{\mathcal{T}} = \boldsymbol{\mathcal{T}}_P, S = S_P$ and $n_1(x) = (x_3, x_4, x_1)$ then $\boldsymbol{\mathcal{I}}_{S_P,\boldsymbol{\mathcal{T}}_P,n_1}(x, y) = (x_1y_1 + x_3 - x_1y_1x_3, x_2y_2x_4, x_1(x_3 + y_3 - x_3y_3))$.

A standard neutrosophic implication off cals 2.

Definition 6. A mapping $\boldsymbol{\mathcal{I}}: (D^*)^2 \to D^*$ is referred to as a standard neutrosophic implicator off class 2 on D^* if it is decreasing in its first component, increasing in its second component and satisfying following conditions:

$$\boldsymbol{\mathcal{I}}(0_{D^*}, 0_{D^*}) = 1_{D^*};\ \boldsymbol{\mathcal{I}}(1_{D^*}, 1_{D^*}) = 1_{D^*};$$
$$\boldsymbol{\mathcal{I}}(1_{D^*}, 0_{D^*}) = 0_{D^*}$$

Definition 7. A standard neutrosophic implicator $\boldsymbol{\mathcal{I}}$ off class 2 is called boder standard neutrosophic implication if $\boldsymbol{\mathcal{I}}(1_{D^*}, x) = x$ for all $x \in D^*$.

Proposition 2. Let $\boldsymbol{\mathcal{T}}, S$ and n be standard neutrosophic t-norm $\boldsymbol{\mathcal{T}}$, a standard neutrosophic t-conorm S and a standard neutrosophic negation on D^*, respectively. Then, we have a standard neutrosophic implication on D^*, which defined as following:

$$\boldsymbol{\mathcal{I}}_{S,n}(x, y) = S(n(x), y), \forall x, y \in D^*.$$

Example 6. For all $x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in D$, we have some standard neutrosophic implication of class 1 on D^* based on proposition ? as follows

a. If $S = S_M$ and $n_0(x) = (x_3, 0, x_1)$ then $\boldsymbol{\mathcal{I}}_{S_M,n_0}(x, y) = (\max(x_3, y_1), 0, \min(x_1, y_3))$

b. If $S = S_P$ and $n_1(x) = (x_3, x_4, x_1)$ then $\boldsymbol{\mathcal{I}}_{S_P,n_1}(x, y) = (x_3 + y_1 - x_3y_1, x_4y_2, x_1y_3)$

Note that, we can define the negation operators from implication operators, such as, the mapping $n_{\boldsymbol{\mathcal{I}}}(x) = \boldsymbol{\mathcal{I}}(x, 0_{D^*}), \forall x \in D^*$, is a standard negation on D^*. For example, if $\boldsymbol{\mathcal{I}}_{S_P,n_1}(x, y) = (x_3 + y_1 - x_3y_1, x_4y_2, x_1y_3)$ then we obtain $n_{I_{S_P,n_1}}(x) = \boldsymbol{\mathcal{I}}_{S_P,n_1}(x, 0_{D^*}) = (x_3, 0, x_1) = n_0(x)$.

2.5 Standard neutrosophic set

Definition 8. Let U be a universal set. A standard neutrosophic (PF) set A on the universe U is an object of the form $A = \{(x, \mu_A(x), \eta_A(x), \gamma_A(x)) \mid x \in U\}$ where $\mu_A(x) (\in [0,1])$ is called the "degree of positive membership of x in A", $\eta_A(x) (\in [0,1])$ is called the "degree of neutral membership of x in A" and $\gamma_A(x)(\in[0,1])\gamma_A(x) (\in [0,1])$ is called the "degree of negative membership of x in A", and where μ_A, η_A μ_A, γ_A and γ_A η_A satisfy the following condition:

$$\mu_A(x) + \eta_A(x) + \gamma_A(x) \le 1, (\forall x \in X) \mu_A(x) + \gamma_A(x) + \eta_A(x)) \le 1, (\forall x \in X).$$

The family of all standard neutrosophic set in U is denoted by PFS(U).

3. Standard neutrosophic rough set

Definition 9.

Suppose that R is a standard neutrosophic relation on the set of universe U. $\boldsymbol{\mathcal{T}}$ is a t −norm on D^*, $\boldsymbol{\mathcal{I}}$ an implication on D^*, for all $F \in PFS(U)$, we denote $F(v) = (\mu_F(v), \eta_F(v), \gamma_F(v))$. Then (U, R) is a standard neutrosophic approximation space. We define the upper and lower approximation set of F on (U, R) as following

$$\bar{R}^{\mathcal{T}}(F)(u) = \bigvee_{v \in U} \boldsymbol{\mathcal{T}}(R(u, v), F(v)), \forall u \in U$$

and

$$\underline{R}_{\mathcal{I}}(F)(u) = \bigwedge_{v \in U} \boldsymbol{\mathcal{I}}(R(u, v), F(v)), u \in U.$$

Example 7. Let $U = \{a, b, c\}$ be an universe and R is a standard neutrosophic relation on U

$$R = \begin{pmatrix} (0.7,0.2,0.1) & (0.6,0.2,0.1) & (0.5,0.3,0.2) \\ (0.5,0.4,0.1) & (0.6,0.1,0.2) & (0.5,0.1,0.2) \\ (0.3,0.5,0.1) & (0.4,0.2,0.3) & (0.7,0.1,0.1) \end{pmatrix}$$

A standard neutrosophic on U is $F = \{\langle a, 0{,}6, 0.2, 0.2\rangle, \langle b, 0.5, 0.3, 0.1\rangle, \langle c, (0.7, 0.2, 0.1)\rangle\}$. Let $\boldsymbol{\mathcal{T}}_M(x, y) = (x_1 \wedge y_1, x_2 \wedge y_2, x_3 \vee y_3)$ be a t-norm on D^*, and $\boldsymbol{\mathcal{I}}(x, y) = (x_3 \vee y_1, x_2 \wedge y_2, x_1 \wedge y_3)$ be an implication on D^*, forall $x = (x_1, x_2, x_3) \in D^*$ and $y = (y_1, y_2, y_3) \in D^*$, We compute

$$\boldsymbol{\mathcal{T}}(R(a, a), F(a)) = \boldsymbol{\mathcal{T}}((0.7, 0.2, 0.1), (0.6, 0.2, 0.2)) = (0.6, 0.2, 0.2)$$
$$\boldsymbol{\mathcal{T}}(R(a, b), F(b)) = \boldsymbol{\mathcal{T}}((0.6, 0.2, 0.1), (0.5, 0.3, 0.1)) = (0.5, 0.2, 0.1)$$
$$\boldsymbol{\mathcal{T}}(R(a, c), F(c)) = \boldsymbol{\mathcal{T}}((0.5, 0.3, 0.2), (0.7, 0.2, 0.1)) = (0.5, 0.2, 0.2)$$

Hence $\bar{R}^{\mathcal{T}}(F)(a) = \bigvee_{v \in U} \boldsymbol{\mathcal{T}}(R(a, v), F(v)) = (0.6, 0.2, 0.1)$.

And

$$\boldsymbol{\mathcal{T}}(R(b, a), F(a)) = \boldsymbol{\mathcal{T}}((0.5, 0.4, 0.1), (0.6, 0.2, 0.2)) = (0.5, 0.2, 0.2)$$
$$\boldsymbol{\mathcal{T}}(R(b, b), F(b)) = \boldsymbol{\mathcal{T}}((0.6, 0.1, 0.2), (0.5, 0.3, 0.1)) = (0.5, 0.1, 0.3)$$
$$\boldsymbol{\mathcal{T}}(R(b, c), F(c)) = \boldsymbol{\mathcal{T}}((0.5, 0.1, 0.2), (0.7, 0.2, 0.1)) = (0.5, 0.1, 0.2)$$

Hence $\bar{R}^{\mathcal{T}}(F)(b) = \bigvee_{v \in U} \boldsymbol{\mathcal{T}}(R(b, v), F(v)) = (0.5, 0.1, 0.2)$

$$\mathcal{T}\big(R(c,a),F(a)\big)=\mathcal{T}\big((0.3,0.5,0.1),(0.6,0.2,0.2)\big)=(0.3,0.2,0.2)$$
$$\mathcal{T}\big(R(c,b),F(b)\big)=\mathcal{T}\big((0.4,0.2,0.3),(0.5,0.3,0.1)\big)=(0.4,0.2,0.3)$$
$$\mathcal{T}\big(R(c,c),F(c)\big)=\mathcal{T}\big((0.7,0.1,0.1),(0.7,0.2,0.1)\big)=(0.7,0.1,0.1)$$

So that $\bar{R}^{\mathcal{T}}(F)(c)=\bigvee_{v\in U}\mathcal{T}\big(R(c,v),F(v)\big)=(0.7,0.1,0.1)$.

We obtain the upper approximation $\bar{R}^{T}(F)=\frac{(0.6,0.2,0.1)}{a}+\frac{(0.5,0.1,0.2)}{b}+\frac{(0.7,0.1,0.1)}{c}$.

Similarly, computing with the lower approximation set, we have $\mathcal{J}\big((0.7,0.2,0.1),(0.6,0.2,0.2)\big)=(0.1,0.2,0.7)\vee(0.6,0.2,0.2)=(0.6,0.2,0.2)$

$$\mathcal{J}\big(R(a,b),F(b)\big)=\mathcal{J}\big((0.6,0.2,0.1),(0.5,0.3,0.1)\big)=(0.1,0.2,0.6)\vee(0.5,0.3,0.1)=(0.5,0.2,0.1)$$
$$\mathcal{J}\big(R(a,c),F(c)\big)=\mathcal{J}\big((0.5,0.3,0.2),(0.7,0.2,0.1)\big)=(0.2,0.3,0.5)\vee(0.7,0.2,0.1)=(0.7,0.2,0.1)$$

$\underline{R}_{\mathcal{J}}(F)(a)=\bigwedge_{v\in U}\mathcal{J}\big(R(a,v),F(v)\big)=(0.5,0.2,0.2)$.

And

$$\mathcal{J}\big(R(b,a),F(a)\big)=\mathcal{J}\big((0.5,0.4,0.1),(0.6,0.2,0.2)\big)=(0.6,0.2,0.1)$$
$$\mathcal{J}\big(R(b,b),F(b)\big)=\mathcal{J}\big((0.6,0.1,0.2),(0.5,0.3,0.1)\big)=(0.5,0.1,0.1)$$
$$\mathcal{J}\big(R(b,c),F(c)\big)=\mathcal{J}\big((0.5,0.1,0.2),(0.7,0.2,0.1)\big)=(0.7,0.1,0.1)$$

$\underline{R}_{\mathcal{J}}(F)(b)=\bigwedge_{v\in U}\mathcal{J}\big(T(b,v),F(v)\big)=(0.5,0.1,0.1)$.

$$\mathcal{J}\big(R(c,a),F(a)\big)=\mathcal{J}\big((0.3,0.5,0.1),(0.6,0.2,0.2)\big)=(0.6,0.2,0.1)$$
$$\mathcal{J}\big(R(c,b),F(b)\big)=\mathcal{J}\big((0.4,0.2,0.3),(0.5,0.3,0.1)\big)=(0.5,0.2,0.1)$$
$$\mathcal{J}\big(R(c,c),F(c)\big)=\mathcal{J}\big((0.7,0.1,0.1),(0.7,0.2,0.1)\big)=(0.7,0.1,0.1)$$

Hence $\underline{R}_{\mathcal{J}}(F)(c)=\bigwedge_{v\in U}\mathcal{J}\big(R(c,v),F(v)\big)=(0.5,0.1,0.1)$.

So that

$\underline{R}_{\mathcal{J}}(F)=\frac{(0.5,0.2,0.2)}{a}+\frac{(0.5,0.1,0.1)}{b}+\frac{(0.5,0.1,0.1)}{c}$.

Now, we have the upper and lower approximations of $F=\frac{(0,6,0.2,0.2)}{a}+\frac{(0.5,0.3,0.1)}{b}+\frac{(0.7,0.2,0.1)}{c}$ are

$$\bar{R}^{\mathcal{T}}(F)=\frac{(0,6,0.2,0.1)}{a}+\frac{(0.5,0.1,0.2)}{b}+\frac{(0.7,0.1,0.1)}{c}$$

and

$$\underline{R}_{\mathcal{J}}(F)=\frac{(0.5,0.2,0.2)}{a}+\frac{(0.5,0.1,0.1)}{b}+\frac{(0.5,0.1,0.1)}{c}$$

Example 8. Let $U=\{a,b,c\}$ be an universe set. And R is a standard neutrosophic relation on U with

$$R=\begin{pmatrix}(1,0,0) & (0.6,0.3,0) & (0.6,0.3,0)\\ (0.6,0.3,0) & (1,0,0) & (0.6,0.3,0)\\ (0.6,0.3,0) & (0.6,0.3,0) & (1,0,0)\end{pmatrix}$$

Let $F=\frac{(0.4,0.3,0.3)}{a}+\frac{(0.5,0.2,0.3)}{b}+\frac{(0.4,0.4,0.1)}{c}$ be standard neutrosophic set on U. A $t-$ norm $\mathcal{T}(x,y)=(x_1\wedge y_1,x_2\wedge y_2,x_3\vee y_3)$, and an implication operator $\mathcal{J}(x,y)=(x_3\vee y_1,x_2\wedge y_2,x_1\wedge y_3)$ for all $x=(x_1,x_2,x_3)\in D^*$, $y=(y_1,y_2,y_3)\in D^*$, we put

$$\mathcal{T}\big(R(a,a),F(a)\big)=\mathcal{T}\big((1,0,0),(0.7,0.2,0.1)\big)=(0.7,0,0.1)$$
$$\mathcal{T}\big(R(a,b),F(b)\big)=\mathcal{T}\big((0.6,0.3,0),(0.5,0.2,0.3)\big)=(0.5,0.2,0.3)$$
$$\mathcal{T}\big(R(a,c),F(c)\big)=\mathcal{T}\big((0.6,0.3,0),(0.4,0.4,0.1)\big)=(0.4,0.3,0.1)$$

Then $\bar{R}^{\mathcal{T}}(F)(a)=\bigvee_{v\in U}\mathcal{T}\big(R(a,v),F(v)\big)=(0.7,0,0.1)$.

$$\mathcal{T}\big(R(b,a),F(a)\big)=\mathcal{T}\big((0.6,0.3,0),(0.7,0.2,0.1)\big)=(0.6,0.2,0.1)$$
$$\mathcal{T}\big(R(b,b),F(b)\big)=\mathcal{T}\big((1,0,0),(0.5,0.2,0.3)\big)=(0.5,0,0.3)$$
$$\mathcal{T}\big(R(b,c),F(c)\big)=\mathcal{T}\big((0.6,0.3,0),(0.4,0.4,0.1)\big)=(0.4,0.3,0.1)$$

Hence $\bar{R}^{\mathcal{T}}(F)(b)=\bigvee_{v\in U}\mathcal{T}\big(R(b,v),F(v)\big)=(0.6,0,0.1)$.

$$\mathcal{T}\big(R(c,a),F(a)\big)=\mathcal{T}\big((0.6,0.3,0),(0.7,0.2,0.1)\big)=(0.6,0.2,0.1)$$
$$\mathcal{T}\big(R(c,b),F(b)\big)=\mathcal{T}\big((0.6,0.3,0),(0.5,0.2,0.3)\big)=(0.5,0.2,0.3)$$
$$\mathcal{T}\big(R(c,c),F(c)\big)=\mathcal{T}\big((1,0,0),(0.4,0.4,0.1)\big)=(0.4,0,0.1)$$
$$\bar{R}^{\mathcal{T}}(F)(a)=\bigvee_{v\in U}\mathcal{T}\big(R(a,v),F(v)\big)=(0.6,0,0.1).$$

We obtain the upper approximation set $\bar{R}^{\mathcal{T}}(F)=\frac{(0.7,0,0.1)}{a}+\frac{(0.6,0,0.1)}{b}+\frac{(0.6,0,0.1)}{c}$.

Similarly, computing with the lower approximation, we have

$$\mathcal{J}\big(R(a,a),F(a)\big)=\mathcal{J}\big((1,0,0),(0.7,0.2,0.1)\big)=(0,0,1)\vee(0.7,0.2,0.1)=(0.7,0,0.1)$$
$$\mathcal{J}\big(R(a,b),F(b)\big)=\mathcal{J}\big((0.6,0.3,0),(0.5,0.2,0.3)\big)=(0,0.3,0.6)\vee(0.5,0.2,0.3)=(0.5,0.2,0.3)$$
$$\mathcal{J}\big(R(a,c),F(c)\big)=\mathcal{J}\big((0.6,0.3,0),(0.4,0.4,0.1)\big)=(0,0.3,0.6)\vee(0.4,0.4,0.1)=(0.4,0.3,0.1)$$

$\underline{R}_{\mathcal{J}}(F)(a)=\bigwedge_{v\in U}\mathcal{J}\big(T(a,v),F(v)\big)=(0.4,0,0.3)$.

Compute

$$\mathcal{J}\big(R(b,a),F(a)\big)=\mathcal{J}\big((0.6,0.3,0),(0.7,0.2,0.1)\big)=(0,0.3,0.6)\vee(0.7,0.2,0.1)=(0.7,0.2,0.1)$$
$$\mathcal{J}\big(R(b,b),F(b)\big)=\mathcal{J}\big((1,0,0),(0.5,0.2,0.3)\big)=(0,0,1)\vee(0.5,0.2,0.3)=(0.5,0,0.3)$$

$\mathcal{I}(R(b,c),F(c)) = \mathcal{I}((0.6,0.3,0),(0.4,0.4,0.1))$
$= (0,0.3,0.6) \vee (0.4,0.4,0.1)$
$= (0.4,0.3,0.1)$

$\underline{R_{\mathcal{I}}}(F)(b) = \bigwedge_{v\in U} \mathcal{I}(T(b,v),F(v)) = (0.4,0,0.3).$

and

$\mathcal{I}(R(c,a),F(a)) = \mathcal{I}((0.6,0.3,0),(0.7,0.2,0.1))$
$= (0,0.3,0.6) \vee (0.7,0.2,0.1)$
$= (0.7,0.2,0.1)$

$\mathcal{I}(R(c,b),F(b)) = \mathcal{I}((0.6,0.3,0),(0.5,0.2,0.3))$
$= (0,0.3,0.6) \vee (0.5,0.2,0.3)$
$= (0.5,0.2,0.3)$

$\mathcal{I}(R(c,c),F(c)) = \mathcal{I}((1,0,0),(0.4,0.4,0.1))$
$= (0,0,1) \vee (0.4,0.4,0.1) = (0.4,0,0.1)$

$\underline{R_{\mathcal{I}}}(F)(c) = \bigwedge_{v\in U} \mathcal{I}(T(c,v),F(v)) = (0.4,0,0.3).$

Hence

$$\underline{R_{\mathcal{I}}}(F) = \frac{(0.4,0,0.1)}{a} + \frac{(0.4,0,0.3)}{b} + \frac{(0.4,0,0.3)}{c}$$

Now, we have the upper and lower approximation sets of $F = \frac{(0.4,0.3,0.3)}{a} + \frac{(0.5,0.2,0.3)}{b} + \frac{(0.4,0.4,0.1)}{c}$ as following

$$\bar{R}^{\mathcal{T}}(F) = \frac{(0.7,0,0.1)}{a} + \frac{(0.6,0,0.1)}{b} + \frac{(0.6,0,0.1)}{c}$$

and

$$\underline{R_{\mathcal{I}}}(F) = \frac{(0.4,0,0.3)}{a} + \frac{(0.4,0,0.3)}{b} + \frac{(0.4,0,0.3)}{c}.$$

Remark 3. If R is reflexive, symmetric transitive then $\underline{R_{\mathcal{I}}}(F) \subset F \subset \bar{R}^{\mathcal{T}}(F)$.

4. Some properties of standard neutrosophic rough set

Theorem 1. Let (U,R) be the standard neutrosophic approximation space. Let $\mathcal{T}, S$ be the t-norm , and t –conorm D^*, n is a negative on D^*. If S and T are dual w.r.t n then

(i) $\sim_n \underline{R_{\mathcal{I}}}(A) = \bar{R}^{\mathcal{T}}(\sim_n A)$

(ii) $\sim_n \bar{R}^{\mathcal{T}}(A) = \underline{R_{\mathcal{I}}}(\sim A)$

where $\mathcal{I}(x,y) = S(n(x),y), \forall x,y \in D^*$.

Proof.

(i) $\sim_n \bar{R}^{\mathcal{T}}(\sim_n A) = \underline{R_{\mathcal{I}}}(A)$.

Indeed, for all $x \in U$, we have

$$\bar{R}^{\mathcal{T}}(\sim_n A)(x) = \bigvee_{y\in U} \mathcal{T}[R(x,y), \sim_n A(y)]$$
$$= \bigvee_{y\in U} nS[nR(x,y), n(\sim_n A(y))]$$
$$= \bigvee_{y\in U} nS[nR(x,y), A(y)] .$$

Moreover,

$$\underline{R_{\mathcal{I}}}(A)(x) = \bigwedge_{y\in U} \mathcal{I}(R(x,y),A(y))$$
$$= \bigwedge_{y\in U} S[nR(x,y),A(y)]$$

Hence

$$\sim_n \underline{R_{\mathcal{I}}}(A)(x)(x) = n(\bigwedge_{y\in U} S[nR(x,y),A(y)])$$
$$= = \bigvee_{y\in U} nS[nR(x,y),A(y)]$$

and $\bar{R}^{T}(\sim_n A)(x) =\sim_n \underline{R_{\mathcal{I}}}(A)(x), \forall x \in U.$

(ii) $\underline{R_{\mathcal{I}}}(\sim_n A) =\sim_n \bar{R}^{\mathcal{T}}(A)$

Indeed, for all $x \in U$ we have

$\underline{R_{\mathcal{I}}}(\sim_n A)(x) = \bigwedge_{y\in U} \mathcal{I}(R(x,y), \sim_n A(y)), x \in U = \bigwedge_{y\in U} S[nR(x,y), \sim_n A(y)]$

And $\sim_n$

$$\bar{R}^{T}(A)(x) = n(\bigvee_{y\in U} \mathcal{T}[R(x,y),A(y))]) = \bigvee_{y\in U} n\mathcal{T}[R(x,y),A(y)]$$
$$= \bigwedge_{y\in U} S[nR(x,y), \sim_n A(y)]$$

It means that $\underline{R_{\mathcal{I}}}(\sim_n A)(x) =\sim_n \bar{R}^{\mathcal{T}}(A)(x), \forall x \in U.$ □

Theorem 2. a) $\bar{R}^{\mathcal{T}}(\widehat{(\alpha,\beta,\theta)}) \subset \widehat{(\alpha,\beta,\theta)}$, where $\widehat{(\alpha,\beta,\theta)}x = (\alpha,\beta,\theta)$, $\forall x \in U$

b) $\underline{R_{\mathcal{I}}}(\widehat{(\alpha,\beta,\theta)}) \supset \widehat{(\alpha,\beta,\theta)}$, where I is a border implication in class 2.

Proof.

a) We have

$\bar{R}^{\mathcal{T}}(\widehat{(\alpha,\beta,\theta)})(u) = \bigvee_{v\in U} \mathcal{T}\left(R(u,v), \widehat{(\alpha,\beta,\theta)}(v)\right) = \mathcal{T}\left(\bigvee_{v\in U} R(u,v),(\alpha,\beta,\theta)\right) \leq_{D^*} \mathcal{T}(1_{D^*},(\alpha,\beta,\theta))$

$= (\alpha,\beta,\theta) = \widehat{(\alpha,\beta,\theta)}(u), \ \forall u \in U$

b) We have

$\underline{R_{\mathcal{I}}}(\widehat{(\alpha,\beta,\theta)})(u) = \bigwedge_{v\in U} \mathcal{I}\begin{pmatrix} R(u,v), \\ \widehat{(\alpha,\beta,\theta)}(v) \end{pmatrix} = \bigwedge_{v\in U} \mathcal{I}\begin{pmatrix} R(u,v), \\ (\alpha,\beta,\theta) \end{pmatrix} \geq_{D^*} \bigwedge_{v\in U} \mathcal{I}(1_{D^*},(\alpha,\beta,\theta)) = (\alpha,\beta,\theta) = \widehat{(\alpha,\beta,\theta)}(u), \ \forall u \in U$ □

5. Conclusion

In this paper, we introduce the ($\mathcal{I},\mathcal{T}$) $-$ standard neutrosophic rough sets based on an implicator $\mathcal{I}$ and a t-norm $\mathcal{T}$ on D^*, lower and upper approximations of standard neutrosophic sets in a standard neutrosophic approximation are first introduced. We also have some notes on logic operations. Some properties of ($\mathcal{I},\mathcal{T}$) $-$ standard neutrosophic rough sets are investigated. In the feature, we will investigate more properties on ($\mathcal{I},\mathcal{T}$) $-$ standard neutrosophic rough sets.

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From Linked Data Fuzzy to Neutrosophic Data Set Decision Making in Games vs. Real Life

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Abstract

In our lives, reality becomes a game, and in the same way, the game becomes reality, the game is an exercise, simulation of real life on a smaller scale, then it extends itself into reality. This article aims to make a connection between decision making in game which comprises all the issues that intervene in the process and further making a connection with real life. The method for identification involved, detected or induced uncertainties is a jointing process from linked data fuzzy to neutrosophic data set on a case study, EVE Online game. This analysis is useful for psychologists, sociologists, economic analysis, process management, business area, also for researchers of games domain.

Keywords

Game theory, real life, decision making, neutrosophic theory, uncertainty.

1. Introduction

The aim of this study is to offer a method of refining the uncertainties, neutral states appeared in a process being a game reflected in the real life, through neutrosophic theory.

In higher forms concerning us, we can associate the function of play as derived from two basic aspects met by us: "as a contest for something or a representation of something", as asserts Huizinga (Huizinga,1980, p.13).

The games, in their configuration, structure, follow the rules, procedures, concepts defined by game theory. There are three categories of games: games of skills, games of chance and games of strategy.

Games of chance type face uncertainty and risk in decision making process (Janis, Mann, 1977). These decisions are evaluated, analyzed and taken in accordance with game theory, according to the social system involved.

Neutrosophic theory applied in decision making for solving the uncertainties matches with game theory requirements (Von Neumann, Morgenstern, 1944).

From the multitude of games we chose to study neutrosophic making decisions, for the case of the EVE online, a complex game both as structure and players, involving complex criteria of the decisions making mechanism.

We have to take into consideration that Dr. Eyjólfur Guðmundsson as economist of the game EVE Online, applied the concept of Vernon Smith, Nobel Laureate for experimental economics, asserting: "This would be any economist's dream, because this is not just an experiment, this is more like a simulation. More like a fully-fledged system where you can input to see what happens" (http://www.ibtimes.co.uk/eve-online-meet-man-controlling-18-million-space-economy-1447437).

Our opinion is that the game is a precious source of ideas, energy, adrenaline, a simulator, an exercise for real life that promotes success but also decay through addiction, tolerance and thus it can be treated just like drugs. But we want to discuss only the positive side of the game.

This game covers both linked data and social media practices, in this context, social media representing computer-mediated tools that allow people or entities to create, share, or exchange information, emotions, feelings, ideas, pictures/videos in virtual communities and networks and on the other side, to provide linked data as method of publishing structured data, interlinked and to become more useful through semantic queries. It builds upon standard Web technologies (such as HTTP, RDF and URIs). It extends them to share information in a way that can be read automatically by computers.

2. Background

We are surrounded by data characterized by the performance of our activities, the fuel efficiency of our cars, a multitude of products from different vendors, the values of the air parameters, or the way our taxes are spent. It helps us to make better decision; this data is playing an increasingly central role in our lives, driving the emergence of data economy. Increasing numbers of individuals and organizations are contributing to this deluge by choosing to share their data with others. Availability of data is very important in evaluating, analysis, making decision process (Heath, Bizer, 2011).

2.1 Fundamentals of Neutrosophic Theory

Uncertainty represents an unsolved situation, it defines a fuzziness state. Uncertainty is an actant's subjective state related to a phenomenon, or decision making, and it becomes objective when it is inserted in a probability calculus system or into an algorithm.

It is mentioned in specialty literature that Zadeh introduced the degree of membership/truth (t), the rest would be (1-t) equal to f, their sum being 1, so he defined the fuzzy set in 1965 (Zadeh, 1965). Further, Atanassov introduced the degree of non- membership /falsehood (f) and he defined the intuitionistic fuzzy set (Atanassov, 1986), asserting: if $0 <= t + f <= 1$ and $0 <= 1 – t – f$, it would be interpreted as indeterminacy $t + f <= 1$. In this case, the indeterminacy state, as proposition, cannot be described in fuzzy logic, is missing the uncertainty state; the neutrosophic logic helps to make a distinction between a "relative truth" and an "absolute truth", while fuzzy logic does not. As novelty to previous theory, Smarandache introduced and defined explicitly the degree of indeterminacy/ neutrality (i) as independent component $^{-}0 <= t+ i +f <= 3^{+}$. In neutrosophy set, the three components t, i, f are independent because it is possible from a source to get (t), from another independent source to get (i) and from the third source to get (f). Smarandache goes further; he refined the range (Smarandache, 2005).

Neutrosophic Set: Let U be a universe of discourse, and M a set included in U. An element x from U is noted with respect to the set M as x(T, I, F) and belongs to M in the following way: it is t% true in the set, i% indeterminate (unknown if it is) in the set, and f% false, where t varies in T, i varies in I, f varies in F (Smarandache, 2005).

Statically T, I, F are subsets, but dynamically T, I, F are functions/operators depending on many known or unknown parameters. Neutrosophic set generalizes the fuzzy set (especially intuitionistic fuzzy set), paraconsistent set, intuitionistic set, etc.

2.2 Applicability of Neutrosophic Theory

Applicability of neutrosophic theory is large, from social sciences such as sociology, philosophy, literature, arts (Smarandache, Vlăduțescu, 2014; Smarandache, 2015; Păun, Teodorescu, 2014; Opran, Voinea, Teodorescu,2014; Smarandache, Gîfu, Teodorescu, 2015) to sciences such as physics, artificial intelligence, mathematics (Smarandache, Vlădăreanu, 2014).

There are some remarkable results of netrosophic theory applied in practical applications such as artificial intelligence (Gal et al, 2014), in robotics there are confirmed results of neutrosofics logics applied to make decisions for uncertainty situations (Okuyama el al 2013; Smarandache, 2011), also for the real-time adaptive networked control of the robot movement on surface with uncertainties (Smarandache, 2014).

Athar Kharal has also a contribution to multi criteria making decision (MCDM) developing an algorithm of uncertainty criteria selection using neutrosophic sets. The proposed method allows the degree of satisfiability (t), non-satisfiability (f) and indeterminacy (i) mentioning a set of criteria represented by neutrosophic sets (Kharal, 2014).

There is no instant game, or instant action; if they existed, it would involve a very limited time fund, if they were instant, we could calculate the uncertainty, we should not have too many variables. If the time is longer, more variables appear, more uncertainties. We evaluate the situation “1” according to what every social actor wants, it is the sustained decision. The state "0" represents the decision that is rejected by social actors. Between “0” and “1” remain states of uncertainty, neutrality, uncertain decisions. In this manner we extend the fuzzy theory to neutrosophic theory. In fact, the novelty of neutrosophy consists of approaching the indeterminacy status. (Smarandahe, 2005).

Starting of this point, we are confidence that neutrosophic theory can help to analyze, evaluate and make the right decision in process analysis taking into account all sources that can generate uncertainty, from human being (not appropriate skill), logistics concept, lack of information, programming automation process according to requirements, etc.

3. Games, elements of culture, double articulated

Probably, Ludwig Wittgenstein was the first academic philosopher who addressed the definition of the word *game*. In his *work, Philosophical Investigations*, Wittgenstein argued that the “elements of games, such as play, rules, and competition”, all contribute to define what games are, but not totally (Wittgenstein, 1953).

Jean Piaget suggested in his work, *Genetic Epistemology*, “that children think differently than adults and proposed a stage theory of cognitive development”. He was the first one to note that children play an active role in gaining knowledge of the world, playing games; children can be thought of as "little scientists" who are actively constructing their knowledge and understanding of the world (Piaget, 1970; Piaget, 1983).

Argument 1.

Games are culture related, everything is repeated by as many people as possible, and it becomes acceptable to most, spread and cultivated (e.g. internet; it shows a minimal know-how). Culture is a complex principle of behavior, spiritual and material values created by mankind, beliefs, tradition and art, passed down from generation to generation. The sense of culture finds its significance in the life of an individual and society. In this context, "any authentic creation is a gift to the future." – asserts Albert Camus.

For humans, culture is the specific environment of existence. It defines an existential field, characterized by a synthesis between objective and subjective, between real and ideal. Culture defines a synthetic human way of existence and it is the symbol of man's creative force. It represents a real value system.

Argument 2.

Games produce culture, the Internet being unlimited, is a huge catalyst of desire. It is a sublime achievement in economic terms. For example, the Internet can offer so many texts about Kant that you never got around to finish in silence any of his “Critiques”. The time of assimilation is now dedicated to search. More than ever, McLuhan's equation says it all about the Internet: “The medium is the message”, says a voice that is heard beyond any meaning of utterances made. Only the pleasure, the voice, the search on Internet are now authentic. Time is limited, not space. From time only desire can provide the intensity necessary to forget this ontological asymmetry (Luhan, 1967).

The Internet allows many people to discover their identity more easily. Some people who were shy or lonely or feel unattractive, discover that they can socialize more successfully and express themselves more freely in an online environment.

Being able to pretend you are someone else is an important mental skill that the child acquires if he is involved in such games. The same thing is experienced by an adult on Internet games, on Facebook, for example, a doubling of personality, a place where you can be different, without constraints, where you are at your own free will, where decision belongs entirely to you, where only uncertainties hinder you. You can think what you want but you can never think of everything that can be thought. If it were possible for every man to think all that is conceivable and with a consistent content, there would be no freedom of thought or thoughts individualized particular to each topic. Mentally anticipating the future, one can access one’s individual mental states of the distant future; one can get art work depicting thoughts, theories or advanced and complicated scientific technologies, currently unimaginable.

4. Case study

EVE Online is a "massively multiplayer online game set 23,000 years in the future. As an elite pilot of one of the four controlling races, the player will explore, build, and dominate across an universe of over 7,000 star systems"[1], see Figure 1. In EVE Online the possibilities are endless. Eve Online is a peculiar concept, it is a simulation, it is an experiment which mirrors the social interactions and communications of the real world, just like the real world, it has a fully functioning economy. In fact, "it has an economy which could be used and studied in order to help what we do in the real world, according to the man charged with overseeing how the $18 million economy operates"[2].

Figure 1.**Very real implications**

While what is happening in universe environment of EVE planets in the far off star systems may not have much relevance in today's world, but the way the EVE economy functions could have very real implications: "We try to have a relative balance of money coming in and money coming out and the increase per month should represent the net increase in economic value"; "We function as a national economics institute, statistics office and central bank giving advice to government, with the government being the developers and us being the monitoring authority"[3].

Considering data of EVE Online game, the complexity of environment, we can estimate some causes that can generate uncertainties such as: unknown universe; cohesion of the team members; alliance trust; financial system crisis; equipment reliability.

1 https://www.eveonline.com
2 David Gilbert, Eve Online: Meet the Man Controlling the $18 Million Space Economy, International Business Time, May 6, 2014
3 http://www.ibtimes.co.uk/eve-online-meet-man-controlling-18-million-space-economy-1447437

For each of these causes the occurrences in a time unit are analyzed (e.g.: 1 week), the space M assimilated to the environment of universe, the governance is poorly defined in this space: the control of universe, defeat the forces of evil, building a stable system. According to this conditions we can simulate the situation by a Pareto Chart, see Figure 2:

Causes	Frequency	Relative frequency %	Cumulated frequency %
Unknown Universe	18	35,29%	35,29%
Team cohesion	13	25,49%	60,78%
Alliance trust	10	19,61%	80,39%
Financial stability	6	11,76%	92,16%
Equipment Reliability	4	7,84%	100,00%
	51		

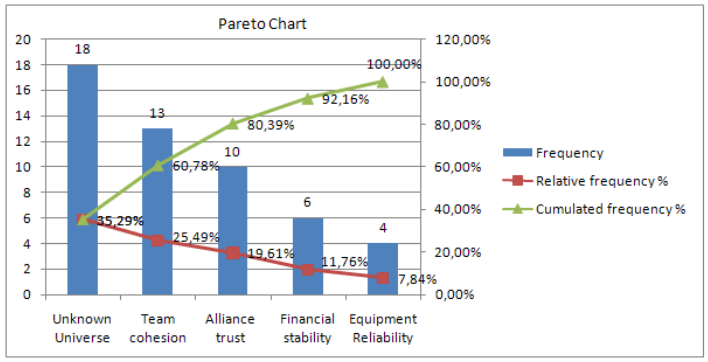

Figure 2. *Pareto Chart Step 1*

Pareto analysis is a creative way of evaluation causes of problems because it helps to stimulate the processes, thinking and organize thoughts, assessing the causes that lead to system instability through neutrosophic theory.

In this context, we define a space M consisting of 5 elements, where **t** means true, **i** means uncertainty and **f** means false:

$$M = \{ a_1 (t_1, i_1, f_1), a_2 (t_2, i_2, f_2), a_3 (t_3, i_3, f_3), a_4 (t_4, i_4, f_4), a_5 (t_5, i_5, f_5)\}$$

- Unknown universe: $a_1(t_1, i_1, f_1)$
- Team cohesion: $a_2(t_2, i_2, f_2)$
- Alliance trust: $a_3(t_3, i_3, f_3)$
- Financial stability: $a_4(t_4, i_4, f_4)$
- Equipment reliability: $a_5 (t_5, i_5, f_5)$

According to Pareto Chart, we established the rate for each space element percentage for the set (t, i, f).

Analyzing the content of elements data, we can establish that relative frequency of events means uncertainty and events solving means true, respectively non solving, false. The process is revealed in Table 1.

Parameter	Space M	Frequency	T /action	T%	I%	F/action	F%
Unknown Universe	a_{11}	18	10	55,56	35,29	8	44,44
Team cohesion	a_{21}	13	7	53,85	25,49	6	46,15
Alliance trust	a_{31}	10	8	80,00	19,61	2	20,00
Financial stability	a_{41}	6	5	83,33	11,76	1	16,67
Equipment Reliability	a_{51}	4	3	75,00	7,84	1	25,00

Table 1. *Determination of T, I, F consistency - Step 1*

Neutrosophic interpretation gives an ordered list of alternatives of uncertainties, depending on us, which is the most preferred element.

We will stop at $a_2$component that generates uncertainty for the cause of "team cohesion". Pareto Chart says that by addressing the cause 20%, it determines 80% uncertainty and can also solve 80% of problems of the system stability. We have to concentrate on uncertainty of a_2 component, to reduce its value,

$a_{21}(t_{21}, i_{21}, f_{21})$ → a_{21}(53,85%, **25,49%,** 46,15%) in the first step of the process.

The refining process, step 2, can be seen in the next set of data presented in Figure 3.

Causes	Frequency	Relative frequency %	Cumulated frequency %
Unknown Universe	16	41,03%	41,03%
Alliance trust	9	23,08%	64,10%
Team cohesion	6	15,38%	79,49%
Equipment Reliability	4	10,26%	89,74%
Financial stability	4	10,26%	100,00%
	39		

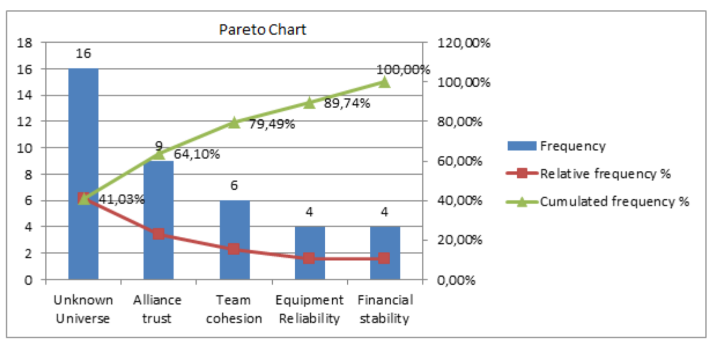

Figure 3. *Pareto Chart Step 2*

The relative dataset for the step 2 is shown in Table 2, where we follow up the element a_{22}.

Parameter	Space M	Frequency	T /action	T%	I%	F/action	F%
Unknown Universe	a_{12}	16	12	75,00	41,03	4	25,00
Alliance trust	a_{32}	9	7	77,78	23,08	2	22,22
Team cohesion	a_{22}	6	4	66,67	15,38	2	33,33
Equipment Reliability	a_{52}	4	3	75,00	10,26	1	25,00
Financial stability	a_{42}	4	3	75,00	10,26	1	25,00

Table 2. *Determination of T, I, F consistency - Step 2*

$a_{22}(t_{22}, i_{22}, f_{22}) \rightarrow a_{22}$(66,67%, **15,38%,** 33,33%) the second step of the process.

On the third step of the refining process, the data set is presented in Table 3.

Parameter	Space M	Frequency	T /action	T%	I%	F/action	F%
Unknown Universe	a_{13}	16	12	75,00	43,24	4	25,00
Alliance trust	a_{33}	9	7	77,78	24,32	2	22,22
Equipment Reliability	a_{53}	4	3	75,00	10,81	1	25,00
Financial stability	a_{43}	4	3	75,00	10,81	1	25,00
Team cohesion	a_{23}	4	3	75,00	10,81	1	25,00

Table 3. *Determination of T, I, F consistency - Step 3*

$a_{23}(t_{23}, i_{23}, f_{23}) \rightarrow a_{23}$(75%, **10.81%,** 25%) the third step of the process.

$a_{24}(t_{24}, i_{24}, f_{24}) \rightarrow a_{24}$(100%, **8,33%,** 0%) the last step of the process.

The data set of last step of the refining process, is shown in Table 4.

Parameter	Space M	Frequency	T /action	T%	I%	F/action	F%
Unknown Universe	a_{14}	16	12	75,00	44,44	4	25,00
Alliance trust	a_{34}	9	7	77,78	25	2	22,22
Equipment Reliability	a_{54}	4	3	75,00	11,11	1	25,00
Financial stability	a_{44}	4	3	75,00	11,11	1	25,00
Team cohesion	a_{24}	3	3	100,00	8,33	0	0,00

Table 4. *Determination of T, I, F consistency - Step 4*

We show below the effects of refining process in 4 steps, until the value of F is zero. The representation of "team cohesion" evolution is shown in Figure 4.

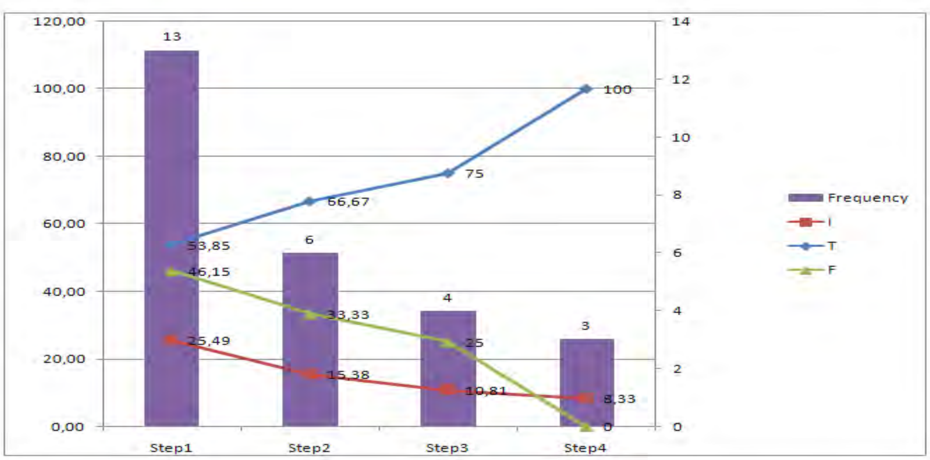

Figure 4. *Refining the process for "Team cohesion", element of space M*

The process of The Uncertainty Risk Management will take into consideration:

- Uncertainty management is a creative process, it involves identifying, evaluation and mitigation of the impact of the uncertainties in the process;
- Uncertainty management can be very formal with defined work process, or informal with no defined processes or methods;
- Uncertainty evaluation prioritizes the identified uncertainties by the likelihood and the potential impact if the event happens;
- Uncertainty mitigation is the development and deployment of a plan to avoid, transferring, sharing and reducing the process uncertainties.

When we try to make a good decision, a person must weigh the positives, negatives and uncertainty of each option, and to consider all the alternatives. For effective decision making, a person must be able to forecast the outcome of each option as well, and based on all these items, to determine which option is the best for that particular situation.

Decision-making is identified as a cognitive process that results in the selection of a belief or an action among several alternative possibilities. Decision-making is a complex process of identifying, analyzing and choosing alternatives based on the values and preferences of the decision maker. Decision-making is one of the central activities of management and it is an important part of any implementation process (Kahneman, Tversky, 2000).

Usually, in our daily lives, we implicitly compare multiple criteria and we want to be comfortable with the consequences of such decisions that are mostly made based only on intuition. On the other hand, when we confront with high stakes, it is important to structure the problem and to evaluate multiple criteria. In decision making process based on multiple criteria (Multi Criteria Decision Making) of whether to do an important issue or not, there are involved not only very complex multiple criteria, there are also inferred multiple parties who are deeply affected from the consequences, because present decisions, act in the future.

Decisions making related to games area, shows its similitude with real life, can be easily transferred to the real world. It is our choice whether to do this or not.

5. Conclusions

Decisions making is a complex act including variables related to uncertainty, with implications for the future work. Uncertainty, in turn, involves classification criteria based on methods that may be applied for determining the degree of uncertainty and settlement. Establishing the types of variables that influence uncertainty, it makes possible the identification of the decisions that we are referring, that will influence, will constrain the process on the one hand, and will be influenced and constrained by a specific decision on the other hand.

Problem solving and decision-making are important skills for business and life. Problem-solving often involves decision-making, and decision-making is especially important for management and leadership. Between them there is the correspondence: identification of the problem vs. frame of the decision; exploring the alternatives vs. improve to address needs and identify alternatives; select an alternative vs. decision and commitment to act; implementation of the solution vs. management of the consequences; evaluation of the situation vs. management of the consequences and frame the related decisions.

What we deduced on the basis of this study is that the game is reality and reality is game. We build reality through the game, we take risks that include uncertainties, the game becomes a training and an experimentation place for many specialists, proving that the school becomes life. Here is how the neutrosophic theory, guide us to be closer to solve uncertainties, transforming them into true or false, stable and controllable states of the systems.

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Operations on Interval Valued Neutrosophic Graphs

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Abstract

Combining the single valued neutrosophic set with graph theory, a new graph model emerges, called single valued neutrosophic graph. This model allows attaching the truth-membership (t), indeterminacy–membership (i) and falsity- membership degrees (f) both to vertices and edges. Combining the interval valued neutrosophic set with graph theory, a new graph model emerges, called interval valued neutrosophic graph. This model generalizes the fuzzy graph, intuitionistic fuzzy graph and single valued neutrosophic graph. In this paper, the authors define operations of Cartesian product, composition, union and join on interval valued neutrosophic graphs, and investigate some of their properties, with proofs and examples.

Keywords

Neutrosophy, neutrosophic set, fuzzy set, fuzzy graph, neutrosophic graph, interval valued neutrosophic set, single valued neutrosophic graph, interval valued neutrosophic graph.

1. Introduction

The neutrosophy was pioneered by F. Smarandache (1995, 1998). It is a branch of philosophy which studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. The neutrosophic set proposed by Smarandache is a powerful tool to deal with incomplete, indeterminate and inconsistent information in real world, being a generalization of fuzzy set (Zadeh 1965; Zimmermann 1985), intuitionistic fuzzy set (Atanassov 1986; Atanassov 1999),interval valued fuzzy set (Turksen 1986) and interval valued intuitionistic fuzzy sets (Atanassov and Gargov 1989).The neutrosophic set is characterized by a truth-membership degree (t), an indeterminacy-membership degree (i) and a falsity-membership degree (f) independently, which are within the real standard or nonstandard unit interval $]^{-}0, 1^{+}[$. If the range is restrained within the real standard unit interval [0, 1], the neutrosophic set easily applies to engineering problems. For this purpose, Wang et al. (2010) introduced the concept of single valued neutrosophic set (SVNS) as a subclass of the neutrosophic set. The same author introduced the notion of interval valued neutrosophic sets (Wang et al. 2005b, 2010) as subclass of neutrosophic sets in which the value of truth-membership, indeterminacy-membership and falsity-membership degrees are intervals of numbers instead of real numbers. The single valued neutrosophic set and the interval valued neutrosophic set have been applied in a wide variety of fields, including computer science, engineering, mathematics, medicine and economics (Ansari 2013a, 2013b, 2013c;Aggarwal 2010;Broumi 2014;Deli 2015;Hai-Long 2015;Liu and Shi 2015;Şahin 2015; Wang et al. 2005b;Ye 2014a, 2014b,2014c).

Graph theory has now become a major branch of applied mathematics and it is generally regarded as a branch of combinatorics. Graph is a widely used tool for solving combinatorial problems in different areas, such as geometry, algebra, number theory, topology, optimization and computer science. To be noted that, when there is uncertainty regarding either the set of vertices or edges, or both, the model becomes a fuzzy graph. Many works on fuzzy graphs, intuitionistic fuzzy graphs and interval valued intuitionistic fuzzy graphs (Antonios K et al. 2014; Bhattacharya 1987; Mishra and Pal 2013; Nagoor Gani and Shajitha Begum 2010; Nagoor Gani and Latha 2012; Nagoor Gani and Basheer Ahamed 2003;Parvathi and Karunambigai 2006; Shannon and Atanassov 1994) have been carried out and all of them have considered the vertex sets and edge sets as fuzzy and /or intuitionistic fuzzy sets. But, when the relations between nodes (or vertices) are indeterminate, the fuzzy graphs and intuitionistic fuzzy graphs fail to work. For this purpose, Smarandache (2015a, 2015b, 2015c) defined four main categories of neutrosophic graphs. Two are based on literal indeterminacy (I): I-edge neutrosophic graph and I-vertex neutrosophic graph. The two categories were deeply studied and gained popularity among the researchers (Garg et al. 2015,Vasantha Kandasamy2004, 2013, 2015) due to their applications via real world problems. The other neutrosophic graph categories are based on (t, i, f) components and are called:(t, i, f)-edge neutrosophic graph and (t, i, f)-vertex neutrosophic graph. These two categories are not developed at all.

Further on, Broumi et al. (2016b) introduced a new neutrosophic graph model, called single valued neutrosophic graph (SVNG), and investigated some of its properties as well. This model allows attaching the membership (t), indeterminacy (i) and non-membership degrees (f) both to vertices and edges. The single valued neutrosophic graph is a generalization of fuzzy graph and intuitionistic fuzzy graph. Broumi et al. (2016a) also introduced neighborhood degree of a vertex and closed neighborhood degree of a vertex in single valued neutrosophic graph, as a generalization of neighborhood degree of a vertex and closed neighborhood degree of a vertex in fuzzy graph and intuitionistic fuzzy graph. Moreover, Broumi et al. (2016c) introduced the concept of interval valued neutrosophic graph, as a generalization of single valued neutrosophic graph, and discussed some properties, with proofs and examples. In addition, Broumi et al.(2016c) introduced the concept of bipolar single valued neutrosophic graph, as a generalization of fuzzy graphs, intuitionistic fuzzy graph, N-graph, bipolar fuzzy graph and single valued neutrosophic graph, and studied some related properties.

In this paper, researchers' objective is to define some operations on interval valued neutrosophic graphs, and to investigate some properties.

2. Preliminaries

In this section, the authors mainly recall some notions related to neutrosophic sets, single valued neutrosophic sets, interval valued neutrosophic sets, fuzzy graphs, intuitionistic fuzzy graphs, interval valued intuitionistic fuzzy graphs, single valued neutrosophic graphs and interval valued neutrosophic graphs, relevant to the present work. The readers are referred for further details to (Broumi et al. 2016b;Mishra and Pal 2013;Nagoor Gani and Basheer Ahamed 2003;Parvathi and Karunambigai 2006;Smarandache 2006;Wang et al. 2010;Wang et al. 2005a).

Definition 1 (Smarandache 2006)

Let X be a space of points (objects) with generic elements in X denoted by x; then the neutrosophic set A (NS A) is an object having the form A = {< x: $T_A(x)$, $I_A(x)$, $F_A(x)$>, x ∈X}, where the functions T, I, F: X→]$^-0,1^+$[define respectively a truth-membership function, an indeterminacy-membership function, and a falsity-membership function of the element x ∈X to the set A with the condition:

$$^-0 \leq T_A(x)+ I_A(x)+ F_A(x)\leq 3^+. \quad (1)$$

The functions $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or nonstandard subsets of]$^-0,1^+$[.

Since it is difficult to apply NSs to practical problems, Wang et al. 2010 introduced the concept of a SVNS, which is an instance of a NS and can be used in real scientific and engineering applications.

Definition 2 (Wang et al. 2010)

Let X be a space of points (objects) with generic elements in X denoted by x. A single valued neutrosophic set A (SVNS A) is characterized by truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$. For each point x in X, $T_A(x)$, $I_A(x)$, $F_A(x)$∈ [0, 1]. A SVNS A can be written as

$$A = \{< x: T_A(x), I_A(x), F_A(x)>, x \in X\} \quad (2)$$

Definition 3 (Wang et al. 2005a)

Let X be a space of points (objects) with generic elements in X denoted by x. An interval valued neutrosophic set (for short IVNS A) A in X is characterized by truth-membership function $T_A(x)$, indeteminacy-membership function $I_A(x)$ and falsity-membership function $F_A(x)$. For each point x in X, one has that

$$T_A(x) = [T_{AL}(x), T_{AU}(x)],$$

$$I_A(x) = [I_{AL}(x), I_{AU}(x)],$$

$$F_A(x) = [F_{AL}(x), F_{AU}(x)] \subseteq [0, 1], \text{ and}$$

$$0 \leq T_A(x)+ I_A(x)+ F_A(x)\leq 3. \quad (3)$$

Definition 4 (Wang et al. 2005a)

An IVNS A is contained in the IVNS B, A ⊆ B, if and only if

$$T_{AL}(x) \le T_{BL}(x),\ T_{AU}(x) \le T_{BU}(x),$$
$$I_{AL}(x) \ge I_{BL}(x), I_{AU}(x) \ge I_{BU}(x),$$
$$F_{AL}(x) \ge F_{BL}(x), F_{AU}(x) \ge F_{BU}(x), \quad \text{for any x in X.} \tag{4}$$

Definition 5 (Wang et al. 2005a)

The union of two interval valued neutrosophic sets A and B is an interval neutrosophic set C, written as C = A ∪ B, whose truth-membership, indeterminacy-membership, and false membership are related to those A and B by

$$T_{CL}(x) = \max(T_{AL}(x),\ T_{BL}(x))$$
$$T_{CU}(x) = \max(T_{AU}(x),\ T_{BU}(x))$$
$$I_{CL}(x) = \min(I_{AL}(x),\ I_{BL}(x))$$
$$I_{CU}(x) = \min(I_{AU}(x),\ I_{BU}(x))$$
$$F_{CL}(x) = \min(F_{AL}(x),\ F_{BL}(x))$$
$$F_{CU}(x) = \min(F_{AU}(x),\ F_{BU}(x)), \text{ for all x in X.} \tag{5}$$

Definition 6 (Wang et al 2005a)

Let X and Y be two non-empty crisp sets. An interval valued neutrosophic relation R(X, Y) is a subset of product space X × Y, and is characterized by the truth membership function T_R(x, y), the indeterminacy membership function I_R(x, y), and the falsity membership function F_R(x, y), where x ∈ X and y ∈ Y and T_R(x, y),I_R(x, y),F_R(x, y) ⊆ [0, 1].

Definition 7 (Nagoor Gani and Basheer Ahamed 2003)

A fuzzy graph is a pair of functions G = (σ, μ), where σ is a fuzzy subset of a non-empty set V and μ is a symmetric fuzzy relation on σ, i.e.σ: V → [0,1] and μ: VxV→[0,1], such that μ(uv) ≤ σ(u) ∧σ(v), for all u, v ∈ V where uv denotes the edge between u and v and σ(u) ∧σ(v) denotes the minimum of σ(u) and σ(v). σ is called the fuzzy vertex set of G andμ is called the fuzzy edge set of G.

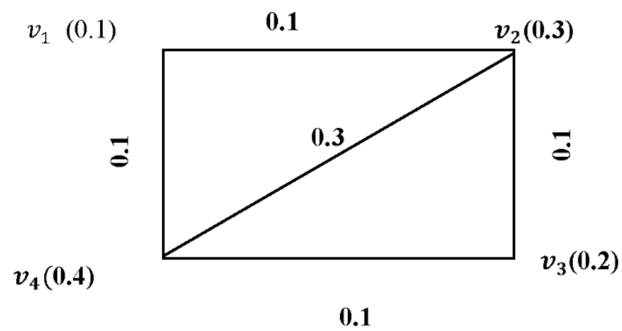

Figure 1: Fuzzy Graph

Definition 8 (Nagoor Gani and Basheer Ahamed 2003)

The fuzzy subgraph H=(τ,ρ) is called a fuzzy subgraph of G =(σ, μ) if τ(u) ≤ σ(u) for all u ∈ V and ρ(u, v) ≤ μ(u, v) for all u, v ∈ V.

Definition 9 (Parvathi and Karunambigai 2006)

An Intuitionistic fuzzy graph is of the form G=<V,E>,where V={v_1,v_2,….,v_n},such that μ_1:V→ [0,1] and γ_1:V→ [0,1] denote the degree of membership and non-membership of the elementv$_i$ ∈ V, respectively, and

$$0\leq \mu_1(v_i)+\gamma_1(v_i))\leq 1,\text{forevery } v_i \in V,(i=1, 2,\ldots\ldots n), \quad (6)$$

E ⊆ VxVwhere μ_2:VxV→[0,1]and γ_2:VxV→ [0,1] are such that $\mu_2(v_i,v_j)\leq \min[\mu_1(v_i),\mu_1(v_j)]$ and $\gamma_2(v_i,v_j)\geq \max[\gamma_1(v_i),\gamma_1(v_j)]$, and

$$0\leq\mu_2(v_i,v_j)+\gamma_2(v_i,v_j)\leq 1 \text{ for every } (v_i,v_j) \in E,(i,j =1,2,\ldots\ldots n) \quad (7)$$

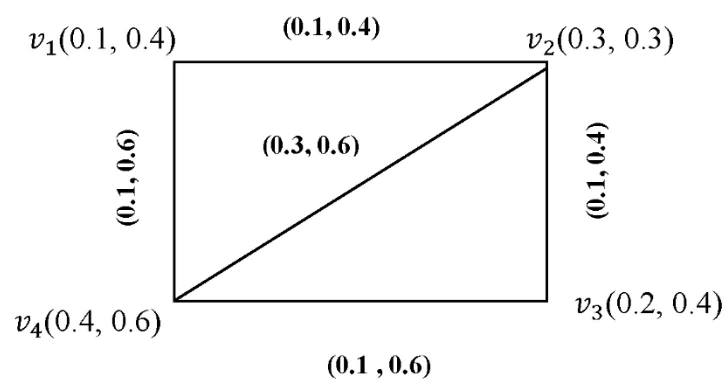

Figure 2: Intuitionistic Fuzzy Graph

Definition 10 (Mishra and Pal 2013)

An interval valued intuitionistic fuzzy graph (IVIFG) G= (A, B) satisfies the following conditions:

1.V= $\{v_1, v_2, \ldots, v_n\}$ such that M_{AL}:V→[0, 1], M_{AU}:V→[0, 1] and N_{AL}:V→[0,1], N_{AU}:V→[0, 1] denote the degree of membership and non-membership of the element $y \in$ V, respectively, and

$$0 \leq M_A(x) + N_A(x) \leq 1 \text{ for every } x \in V \quad (8)$$

2.The functions M_{BL}:V x V →[0, 1], M_{BU}:V x V →[0, 1] and N_{BL}:V x V →[0,1], N_{BU}:V x V →[0, 1] are denoted by

$$M_{BL}(xy) \leq \min [M_{AL}(x), M_{AL}(y)],\ M_{BU}(xy) \leq \min [M_{AU}(x), M_{AU}(y)]$$

$$N_{BL}(xy) \geq \max [N_{BL}(x), N_{BL}(y)],\ N_{BU}(xy) \geq \max[N_{BU}(x), N_{BU}(y)]$$

such that $0 \leq M_B(xy) + N_B(xy) \leq 1$, for every $xy \in$ E (9)

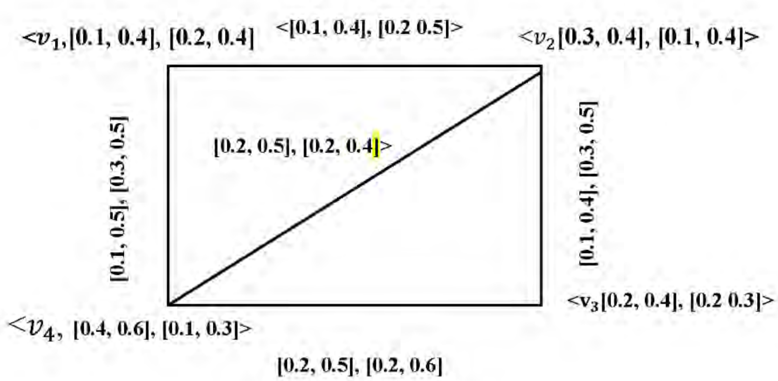

Figure 3: Interval valued intuitionistic graph

Definition 11 (Broumi et al. 2016b)

A single valued neutrosophic graph (SVN-graph) with underlying set V is defined to be a pair G= (A, B), where:

1.The functions T_A:V→[0, 1], I_A:V→[0, 1] and F_A:V→[0, 1] denote the degree of truth-membership, degree of indeterminacy-membership and falsity-membership of the element $v_i \in$ V, respectively, and

$$0 \leq T_A(v_i) + I_A(v_i) + F_A(v_i) \leq 3, \text{ for all } v_i \in V \text{ (i=1, 2, …,n)} \quad (10)$$

2. The functions T_B: E ⊆ V x V →[0, 1], I_B: E ⊆ V x V →[0, 1] and F_B: E ⊆ V x V →[0, 1] are defined by

$T_B(\{v_i, v_j\}) \leq \min [T_A(v_i), T_A(v_j)]$,

$I_B(\{v_i, v_j\}) \geq \max [I_A(v_i), I_A(v_j)]$ and

$F_B(\{v_i, v_j\}) \geq \max [F_A(v_i), F_A(v_j)]$ (11)

and denote the degree of truth-membership, indeterminacy-membership and falsity-membership of the edge $(v_i, v_j) \in$ E respectively, where

$0 \leq T_B(\{v_i, v_j\}) + I_B(\{v_i, v_j\}) + F_B(\{v_i, v_j\}) \leq 3$,

for all $\{v_i, v_j\} \in$ E (i, j = 1, 2,…, n). (12)

"A"is called the single valued neutrosophic vertex set of V, "B" - the single valued neutrosophic edge set of E, respectively. B is a symmetric single valued neutrosophic relation on A. The notation $(v_i, v_j)$is used for an element of E. Thus, G = (A, B) is a single valued neutrosophic graph of G*= (V, E), if :

$T_B(v_i, v_j) \leq \min [T_A(v_i), T_A(v_j)]$,

$I_B(v_i, v_j) \geq \max [I_A(v_i), I_A(v_j)]$ and

$F_B(v_i, v_j) \geq \max [F_A(v_i), F_A(v_j)]$, for all $(v_i, v_j) \in$ E (13)

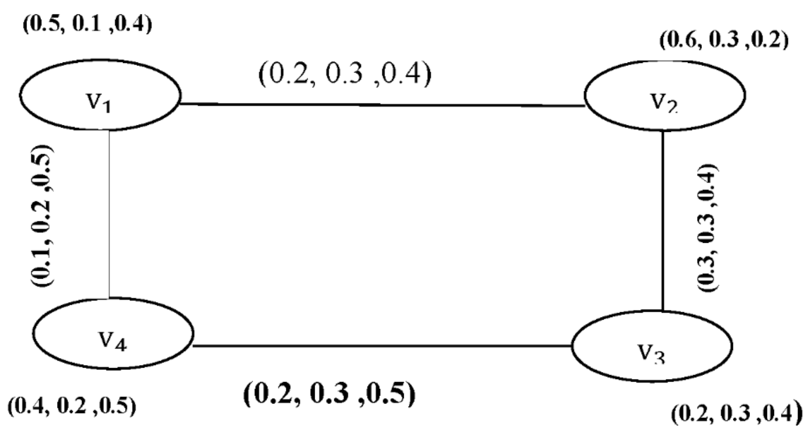

Figure 4: Single valued neutrosophic graph

Definition 12 (Broumi et al. 2016b)

Let G = (A, B) be a single valued neutrosophic graph. Then the degree of a vertex v is defined by d(v)= $(d_T(v), d_I(v), d_F(v))$, where

$d_T(v)=\sum_{u\neq v} T_B(u, v)$, $d_I(v)=\sum_{u\neq v} I_B(u, v)$ and $d_F(v)=\sum_{u\neq v} F_B(u, v)$ (14)

Definition 13 (Broumi et al. 2016b)

A single valued neutrosophic graph G= (A, B) and G^* is called strong neutrosophic graph

$$T_B(v_i, v_j) = \min\,[T_A(v_i), T_A(v_j)]$$

$$I_B(v_i, v_j) = \max\,[I_A(v_i), I_A(v_j)]$$

$$F_B(v_i, v_j) = \max\,[F_A(v_i), F_A(v_j)] \text{ for all } (v_i, v_j) \in E. \quad (15)$$

Definition 14 (Broumi et al. 2016b)

The complement of a strong single valued neutrosophic graph G on G^* is strong single valued neutrosophic graph $\bar{G}$ on G^* where

1.$\bar{V}$ =V

2.$\overline{T_A}(v_i)= T_A(v_i)$,$\overline{I_A}(v_i)= I_A(v_i)$,$\overline{F_A}(v_i)= F_A(v_i)$, $v_j \in$ V.

3.$\overline{T_B}(v_i, v_j)= \min\left[T_A(v_i), T_A(v_j)\right]-T_B(v_i, v_j)$

$\overline{I_B}(v_i, v_j)= \max\left[I_A(v_i), I_A(v_j)\right]-I_B(v_i, v_j)$ and

$\overline{F_B}(v_i, v_j)= \max\left[F_A(v_i), F_A(v_j)\right]-F_B(v_i, v_j)$, for all $(v_i, v_j) \in$ E. (16)

Definition 15 (Broumi et al. 2016b)

A single valued neutrosophic graph G = (A, B) is called complete, if:

$T_B(v_i, v_j)$= min($T_A(v_i)$, $T_A(v_j)$),

$I_B(v_i, v_j)$= max($I_A(v_i)$, $I_A(v_j)$)

and $F_B(v_i, v_j)$= max($F_A(v_i)$, $F_A(v_j)$), for every $v_i, v_j \in$ V. (17)

Example 1

Consider a graph G^*= (V, E) such that V = {a, b, c, d} , E= {ab ,ac ,bc , cd}. Then, G= (A, B) is a single valued neutrosophic complete graph of G^*.

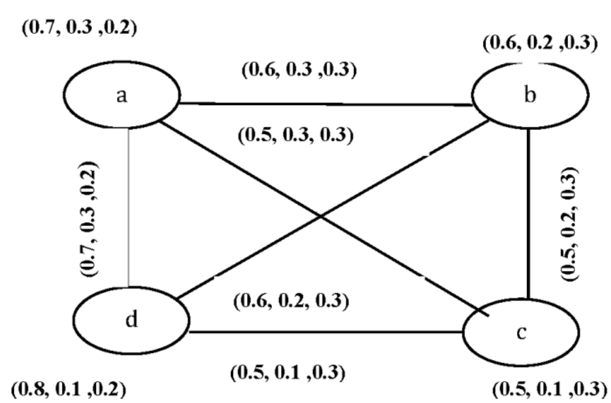

Figure 5: Complete single valued neutrosophic graph

3. Operations on Interval-Valued Neutrosophic Graphs

Throughout this section, G^* = (V, E) denotes a crisp graph, and G - an interval valued neutrosophic graph.

Definition 16

By an interval-valued neutrosophic graph of a graph G^*=(V,E)one means a pair G=(A,B), where A=< [T_{AL}, T_{AU}], [I_{AL}, I_{AU}], [F_{AL}, F_{AU}]>is an interval-valued neutrosophic set on V and B=< [T_{BL}, T_{BU}], [I_{BL}, I_{BU}], [F_{BL}, F_{BU}] > is an interval-valued neutrosophic relation on E satisfying the following condition:

1.V= {v_1,v_2 ,…,v_n} such that T_{AL}:V→[0, 1],T_{AU}:V→[0, 1], I_{AL}:V→[0,1],I_{AU}:V→[0, 1] and F_{AL}:V→[0,1], F_{AU}:V→[0, 1] denote the degree of truth-membership, the degree of indeterminacy-membership and falsity-membership of the element $y \in$ V, respectively, and

$$0\leq T_A(v_i)+ I_A(v_i) +F_A(v_i) \leq 3,$$

for every$v_i \in$ V. (18)

2. The functions T_{BL}:V x V →[0, 1],T_{BU}:V x V →[0, 1],I_{BL}:V x V →[0, 1],I_{BU}:V x V →[0, 1] and F_{BL}:V x V →[0,1], F_{BU}:V x V →[0, 1], such that

$$T_{BL}(v_i, v_j) \leq \min [T_{AL}(v_i), T_{AL}(v_j)]$$

$$T_{BU}(v_i, v_j) \leq \min [T_{AU}(v_i), T_{AU}(v_j)]$$

$$I_{BL}(v_i, v_j) \geq \max [I_{BL}(v_i), I_{BL}(v_j)]$$

$$I_{BU}(v_i, v_j) \geq \max [I_{BU}(v_i), I_{BU}(v_j)]$$

and

$$F_{BL}(v_i, v_j) \geq \max [F_{BL}(v_i), F_{BL}(v_j)]$$

$$F_{BU}(v_i, v_j) \geq \max [F_{BU}(v_i), F_{BU}(v_j)] \quad (19)$$

denote the degree of truth-membership, indeterminacy-membership and falsity-membership of the edge (v_i,v_j) ∈ E respectively, where

$$0\leq T_B(v_i, v_j) + I_B(v_i, v_j)+ F_B(v_i, v_j) \leq 3,$$

for all $(v_i, v_j) \in$ E. (20)

Example 2

Figure 5 is an example for IVNG, G = (A,B) defined on a graph G^*= (V, E)

such that V = {x, y, z}, E = {xy, yz, zx}, A is an interval valued neutrosophic set of V

A={ < x, [0.5, 0.7], [0.2, 0.3], [0.1, 0.3]>, <y, [0.6, 0.7],[0.2, 0.4], [0.1, 0.3]>, <z, [0.4, 0.6],[0.1, 0.3], [0.2, 0.4],>}, and B an interval valued neutrosophic set of E⊆V x V

B={ <xy, [0.3, 0.6], [0.2, 0.4], [0.2, 0.4]>, <yz, [0.3, 0.5],[0.2, 0.5], [0.2, 0.4]>, <xz, [0.3, 0.5],[0.3, 0.5], [0.2, 0.4]>}.

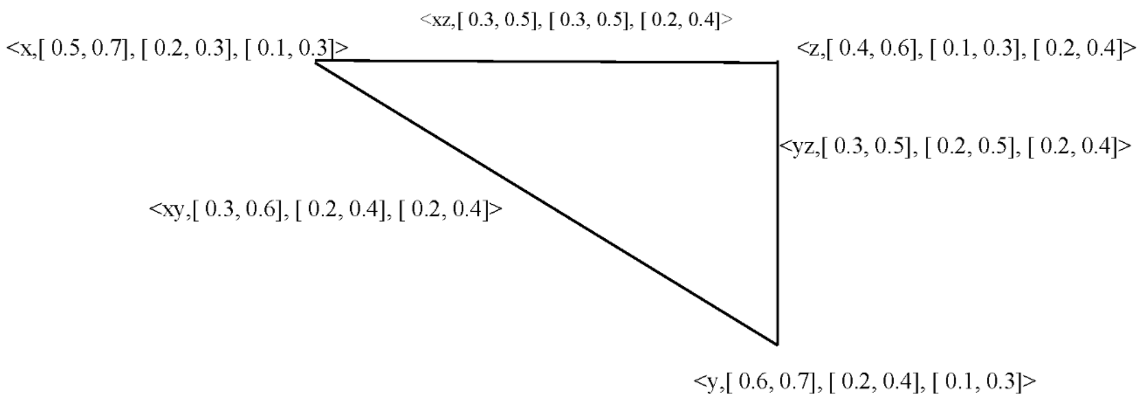

Figure 6: Interval valued neutrosophic graph

By routine computations, it is easy to see that G=(A,B) is an interval valued neutrosophic graph of G^*.

Here, the new concept of Cartesian product is given.

Definition 17

Let $G^* = G_1^* \times G_2^*$=(V, E) be the Cartesian product of two graphs where V = $V_1 \times V_2$and E= $\{(x, x_2)\,(x, y_2)\,/x \in V_1,\, x_2y_2 \in E_2\} \cup \{(x_1,\ z)\,(y_1, z)\,/z \in V_2,\, x_1y_1 \in E_1\}$; then, the Cartesian product G = $G_1 \times G_2$ = ($A_1 \times A_2$, $B_1 \times B_2$) is an interval valued neutrosophic graph defined by

1) $(T_{A_1L} \times T_{A_2L})\,(x_1,x_2) = \min\,(T_{A_1L}(x_1), T_{A_2L}(x_2))$
$(T_{A_1U} \times T_{A_2U})\,(x_1,x_2) = \min\,(T_{A_1U}(x_1), T_{A_2U}(x_2))$
$(I_{A_1L} \times I_{A_2L})\,(x_1,x_2) = \max\,(I_{A_1L}(x_1), I_{A_2L}(x_2))$
$(I_{A_1U} \times I_{A_2U})\,(x_1,x_2) = \max\,(I_{A_1U}(x_1), I_{A_2U}(x_2))$
$(F_{A_1L} \times F_{A_2L})\,(x_1,x_2) = \max\,(F_{A_1L}(x_1), F_{A_2L}(x_2))$
$(F_{A_1U} \times F_{A_2U})\,(x_1,x_2) = \max\,(F_{A_1U}(x_1), F_{A_2U}(x_2))$

for all (x_1,x_2) $\in V$.

(21)

2) $(T_{B_1L} \times T_{B_2L})\,((x,x_2)(x,y_2)) = \min\,(T_{A_1L}(x), T_{B_2L}(x_2y_2))$
$(T_{B_1U} \times T_{B_2U})\,((x,x_2)(x,y_2)) = \min\,(T_{A_1U}(x), T_{B_2U}(x_2y_2))$
$(I_{B_1L} \times I_{B_2L})\,((x,x_2)(x,y_2)) = \max\,(I_{A_1L}(x), I_{B_2L}(x_2y_2))$
$(I_{B_1U} \times I_{B_2U})\,((x,x_2)(x,y_2)) = \max\,(I_{A_1U}(x), I_{B_2U}(x_2y_2))$
$(F_{B_1L} \times F_{B_2L})\,((x,x_2)\,(x,y_2)) = \max\,(F_{A_1L}(x), F_{B_2L}(x_2y_2))$

$(F_{B_1U} \times F_{B_2U})$ $((x,x_2)(x,y_2))$ = max$(F_{A_1U}(x), F_{B_2U}(x_2y_2))$,

$\forall$ x $\in V_1, \forall x_2y_2 \in E_2$.
(22)

3) $(T_{B_1L} \times T_{B_2L})$ $((x_1,z)\ (y_1,z))$ = min $(T_{B_1L}(x_1y_1), T_{A_2L}(z))$
$(T_{B_1U} \times T_{B_2U})$ $((x_1,z)\ (y_1,z))$ = min $(T_{B_1U}(x_1y_1), T_{A_2U}(z))$
$(I_{B_1L} \times I_{B_2L})$ $((x_1\ ,z)\ (y_1,z))$ = max $(I_{B_1L}(x_1y_1), I_{A_2L}(z))$
$(I_{B_1U} \times I_{B_2U})$ $((x_1\ ,z)\ (y_1,z))$ = max $(I_{B_1U}(x_1y_1), I_{A_2U}(z))$
$(F_{B_1L} \times F_{B_2L})$ $((x_1\ ,z)\ (y_1,z))$ = max $(F_{B_1L}(x_1y_1), F_{A_2L}(z))$
$(F_{B_1U} \times F_{B_2U})$ $((x_1\ ,z)\ (y_1,z))$ = max $(F_{B_1U}(x_1y_1), F_{A_2U}(z))$

$\forall$ z $\in V_2$, $\forall x_1y_1 \in E_1$. (23)

Example 3

Let G_1^*= (A_1, B_1) and G_2^*= (A_2, B_2) be two graphs whereV_1 ={a, b}, V_2 ={c, d},E_1 ={a, b} and E_2 ={c, d}.Consider two interval valued neutrosophic graphs:

A_1={ <a, [0.5, 0.7], [0.2, 0.3], [0.1, 0.3]>, <b, [0.6, 0.7],[0.2, 0.4], [0.1, 0.3]>},

B_1={ <ab, [0.3, 0.6], [0.2, 0.4], [0.2, 0.4]>};

A_2={ <c, [0.4, 0.6], [0.2, 0.3], [0.1, 0.3]>, <d, [0.4, 0.7],[0.2, 0.4], [0.1, 0.3}>},

B_2={ <cd, [0.3, 0.5], [0.4, 0.5], [0.3, 0.5]>}.

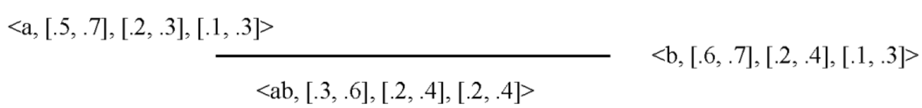

Figure 7: Interval valued neutrosophic graph G1

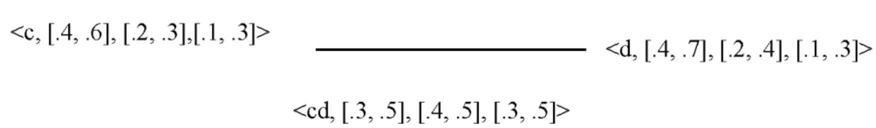

Figure 8: Interval valued neutrosophic graph G2

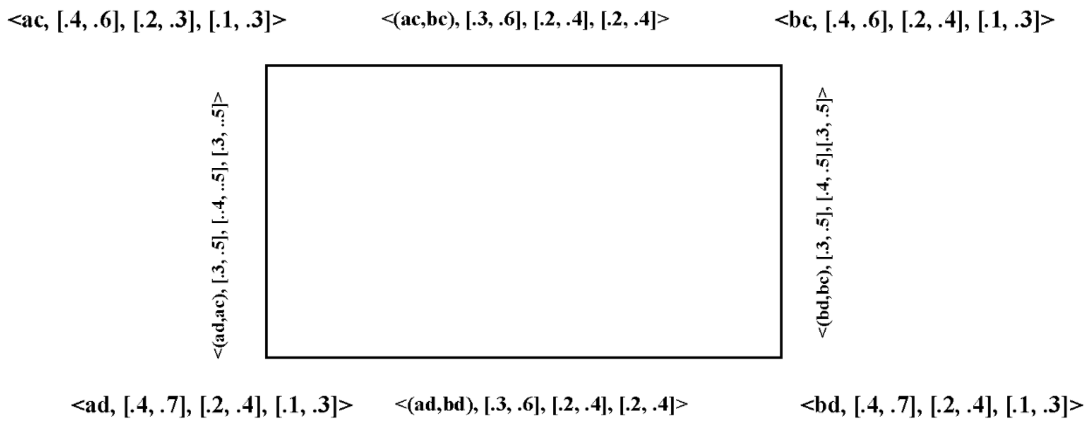

Figure 9: Cartesian product of interval valued neutrosophic graph

By routine computations, It is easy to see that $G_1 \times G_2$ is an interval-valued neutrosophic graph of $G_1^* \times G_2^*$.

Proposition 1

The Cartesian product $G_1 \times G_2$=($A_1 \times A_2$, $B_1 \times B_2$) of two interval valued neutrosophic graphs of the graphs G_1^*andG_2^* is an interval valued neutrosophic graph of $G_1^* \times G_2^*$.

Proof. Verifying only conditions for $B_1 \times B_2$, because conditions for $A_1 \times A_2$ areobvious.

Let E= $\{(x,x_2)\ (x,y_2)\ /x \in V_1, x_2y_2 \in E_2\} \cup \{(x_1, z)\ (y_1, z)\ /z \in V_2, x_1y_1 \in E_1\}$

Considering $(x,x_2)\ (x,y_2) \in E$, one has:

$(T_{B_1L} \times T_{B_2L})\ ((x, x_2)\ (x, y_2)) = \min\ (T_{A_1L}(x),\ T_{B_2L}(x_2y_2)) \leq \min\ (T_{A_1L}(x), \min(T_{A_2L}(x_2), T_{A_2L}(y_2))) = \min(\min\ (T_{A_1L}(x), T_{A_2L}(x_2)), \min\ (T_{A_1L}(x), T_{A_2L}(y_2)))$

$= \min\ ((T_{A_1L} \times T_{A_2L})\ (x,x_2), (T_{A_1L} \times T_{A_2L})\ (x,y_2)),$ (24)

$(T_{B_1U} \times T_{B_2U})\ ((x, x_2)\ (x, y_2)) = \min\ (T_{A_1U}(x),\ T_{B_2U}(x_2y_2)) \leq \min\ (T_{A_1U}(x), \min(T_{A_2U}(x_2), T_{A_2U}(y_2))) = \min(\min\ (T_{A_1U}(x), T_{A_2U}(x_2)), \min\ T_{A_1U}(x), T_{A_2U}(y_2))) = \min\ ((T_{A_1U} \times T_{A_2U})\ (x,x_2), (T_{A_1U} \times T_{A_2U})\ (x,y_2)),$ (25)

$(I_{B_1L} \times I_{B_2L})\ ((x, x_2)\ (x, y_2)) = \max\ (I_{A_1L}(x),\ I_{B_2L}(x_2y_2)) \geq \max\ (I_{A_1L}(x), \max(I_{A_2L}(x_2), I_{A_2L}(y_2))) = \max(\max\ (I_{A_1L}(x), I_{A_2L}(x_2)), \max\ (I_{A_1L}(x), I_{A_2L}(y_2))) = \max\ ((I_{A_1L} \times I_{A_2L})\ (x,x_2), (I_{A_1L} \times I_{A_2L})\ (x,y_2)),$ (26)

$(I_{B_1U} \times I_{B_2U})\ ((x, x_2)\ (x, y_2)) = \max\ (I_{A_1U}(x),\ I_{B_2U}(x_2y_2)) \geq \max\ (I_{A_1U}(x), \max(I_{A_2U}(x_2), I_{A_2U}(y_2))) = \max(\max\ (I_{A_1U}(x), I_{A_2U}(x_2)), \max\ (I_{A_1U}(x), I_{A_2U}(y_2))) = \max\ ((I_{A_1U} \times I_{A_2U})\ (x,x_2), (I_{A_1U} \times I_{A_2U})\ (x,y_2)),$ (27)

$(F_{B_1L} \times F_{B_2L})$ $((x, x_2)$ $(x, y_2))$ = max $(F_{A_1L}(x)$, $F_{B_2L}(x_2y_2))$ ≥ max $(F_{A_1L}(x)$, max$(F_{A_2L}(x_2)$,$F_{A_2L}(y_2)))$ = max(max $(F_{A_1L}(x)$,$F_{A_2L}(x_2))$, max $(F_{A_1L}(x)$,$F_{A_2L}(y_2)))$ = max $((F_{A_1L} \times F_{A_2L})$ (x,x_2),$(F_{A_1L} \times F_{A_2L})$ $(x,y_2))$, (28)

$(F_{B_1U} \times F_{B_2U})$ $((x, x_2)$ $(x, y_2))$ = max $(F_{A_1U}(x)$, $F_{B_2U}(x_2y_2))$ ≥ max $(F_{A_1U}(x)$, max$(F_{A_2U}(x_2)$,$F_{A_2U}(y_2)))$ = max(max $(F_{A_1U}(x)$,$F_{A_2U}(x_2))$, max $(F_{A_1U}(x)$,$F_{A_2U}(y_2)))$= max $((F_{A_1U} \times F_{A_2U})$ (x,x_2),$(F_{A_1U} \times F_{A_2U})$ $(x,y_2))$. (29)

Similarly, for(x_1, z) $(y_1, z) \in E$, one has:

$(T_{B_1L} \times T_{B_2L})$ $((x_1, z)$ $(y_1, z))$ = min $(T_{B_1L}(x_1y_1)$, $T_{A_2L}(z)) \leq$ min $(min(T_{A_1L}(x_1)$, $T_{A_1L}(y_1)))$,$T_{A_2L}(z)))$ = min(min $(T_{A_1L}(x)$,$T_{A_2L}(z))$, min $(T_{A_1L}(y_1)$,$T_{A_2L}(z)))$= min $((T_{A_1L} \times T_{A_2L})$ (x_1,z),$(T_{A_1L} \times T_{A_2L})$ $(y_1,z))$, (30)

$(T_{B_1U} \times T_{B_2U})$ $((x_1, z)$ $(y_1, z))$ = min $(T_{B_1U}(x_1y_1)$, $T_{A_2U}(z)) \leq$ min $(min(T_{A_1U}(x_1)$, $T_{A_1U}(y_1)))$,$T_{A_2U}(z)))$= min(min $(T_{A_1U}(x)$,$T_{A_2U}(z))$, min $(T_{A_1U}(y_1)$,$T_{A_2U}(z)))$ = min $((T_{A_1U} \times T_{A_2U})$ (x_1, z),$(T_{A_1U} \times T_{A_2U})$ $(y_1, z))$, (31)

$(I_{B_1L} \times I_{B_2L})$ $((x_1, z)$ $(y_1, z))$ = max $(I_{B_1L}(x_1y_1)$, $I_{A_2L}(z)) \geq$ max$(max(I_{A_1L}(x_1)$, $I_{A_1L}(y_1)))$,$I_{A_2L}(z)))$ = max(max $(I_{A_1L}(x)$,$I_{A_2L}(z))$, max $(I_{A_1L}(y_1)$,$I_{A_2L}(z)))$ = max $((I_{A_1L} \times I_{A_2L})$ (x_1,z),$(I_{A_1L} \times I_{A_2L})$ $(y_1,z))$, (32)

$(I_{B_1U} \times I_{B_2U})$ $((x_1, z)$ $(y_1, z))$ = max $(I_{B_1U}(x_1y_1)$, $I_{A_2U}(z)) \geq$ max $(max(I_{A_1U}(x_1)$, $I_{A_1U}(y_1)))$,$I_{A_2U}(z)))$ = max(max $(I_{A_1U}(x)$,$I_{A_2U}(z))$, max $(I_{A_1U}(y_1)$,$I_{A_2U}(z)))$ = max $((I_{A_1U} \times I_{A_2U})$ (x_1,z),$(I_{A_1U} \times I_{A_2U})$ $(y_1,z))$, (33)

$(F_{B_1L} \times F_{B_2L})$ $((x_1, z)$ $(y_1, z))$ = max$(F_{B_1L}(x_1y_1)$, $F_{A_2L}(z)) \geq$ max $(max(F_{A_1L}(x_1)$, $F_{A_1L}(y_1)))$,$F_{A_2L}(z)))$ = max(max $(F_{A_1L}(x)$,$F_{A_2L}(z))$, max $(F_{A_1L}(y_1)$,$F_{A_2L}(z)))$ = max $((F_{A_1L} \times F_{A_2L})$ (x_1, z),$(F_{A_1L} \times F_{A_2L})$ $(y_1, z))$, (34)

$(F_{B_1U} \times F_{B_2U})$ $((x_1, z)$ $(y_1, z))$ = max $(F_{A_1U}(x_1y_1)$, $F_{B_2U}(z)) \geq$ max (max$(F_{A_1U}(x_1)$, $F_{A_1U}(y_1)))$,$F_{A_2U}(z)))$ = max(max $(F_{A_1U}(x)$,$F_{A_2U}(z))$, max $(F_{A_1U}(y_1)$,$F_{A_2U}(z)))$ = max $((F_{A_1U} \times F_{A_2U})$ (x_1,z),$(F_{A_1U} \times F_{A_2U})$ $(y_1,z))$. (35)

This completes the proof.

Definition 18

Let $G^* = G_1^* \times G_2^*$=$(V_1 \times V_2$, E) be the composition of two graphs where E= $\{(x, x_2)\ (x, y_2)\ /x \in V_1,\ x_2y_2 \in E_2\} \cup \{(x_1,\ z)\ (y_1, z)\ /z \in V_2,\ x_1y_1 \in E_1\} \cup \{(x_1, x_2)\ (y_1, y_2)\ |x_1y_1 \in E_1,\ x_2 \neq y_2\}$,then the composition of interval valued neutrosophic graphs $G_1[G_2] = (A_1 \circ A_2, B_1 \circ B_2)$ is an interval valued neutrosophic graphs defined by:

1. $(T_{A_1L} \circ T_{A_2L})$ (x_1,x_2) = min $(T_{A_1L}(x_1), T_{A_2L}(x_2))$ (36)
   $(T_{A_1U} \circ T_{A_2U})$ (x_1,x_2) = min $(T_{A_1U}(x_1), T_{A_2U}(x_2))$

$(I_{A_1L} \circ I_{A_2L})$ (x_1,x_2) = max $(I_{A_1L}(x_1), I_{A_2L}(x_2))$
$(I_{A_1U} \circ I_{A_2U})$ (x_1, x_2) = max $(I_{A_1U}(x_1), I_{A_2U}(x_2))$
$(F_{A_1L} \circ F_{A_2L})$ (x_1, x_2) = max $(F_{A_1L}(x_1), F_{A_2L}(x_2))$
$(F_{A_1U} \circ F_{A_2U})$ (x_1, x_2) = max $(F_{A_1U}(x_1), F_{A_2U}(x_2))$ $\forall x_1 \in V_1, x_2 \in V_2$;

2. $(T_{B_1L} \circ T_{B_2L})$ $((x,x_2)(x,y_2))$ = min $(T_{A_1L}(x), T_{B_2L}(x_2y_2))$ (37)
$(T_{B_1U} \circ T_{B_2U})$ $((x, x_2)(x, y_2))$ = min $(T_{A_1U}(x), T_{B_2U}(x_2y_2))$
$(I_{B_1L} \circ I_{B_2L})$ $((x, x_2)(x, y_2))$ = max $(I_{A_1L}(x), I_{B_2L}(x_2y_2))$
$(I_{B_1U} \circ I_{B_2U})$ $((x, x_2)(x, y_2))$ = max $(I_{A_1U}(x), I_{B_2U}(x_2y_2))$
$(F_{B_1L} \circ F_{B_2L})$ $((x,x_2)(x,y_2))$ = max $(F_{A_1L}(x), F_{B_2L}(x_2y_2))$
$(F_{B_1U} \circ F_{B_2U})$ $((x,x_2)(x,y_2))$ = max $(F_{A_1U}(x), F_{B_2U}(x_2y_2))$ $\forall x \in V_1, \forall x_2y_2 \in E_2$;

3. $(T_{B_1L} \circ T_{B_2L})$ $((x_1,z)(y_1,z))$ = min $(T_{B_1L}(x_1y_1), T_{A_2L}(z))$
(38)
$(T_{B_1U} \circ T_{B_2U})$ $((x_1,z)(y_1,z))$ = min $(T_{B_1U}(x_1y_1), T_{A_2U}(z))$
$(I_{B_1L} \circ I_{B_2L})$ $((x_1,z)(y_1,z))$ = max $(I_{B_1L}(x_1y_1), I_{A_2L}(z))$
$(I_{B_1U} \circ I_{B_2U})$ $((x_1,z)(y_1,z))$ = max $(I_{B_1U}(x_1y_1), I_{A_2U}(z))$
$(F_{B_1L} \circ F_{B_2L})$ $((x_1,z)(y_1,z))$ = max $(F_{B_1L}(x_1y_1), F_{A_2L}(z))$
$(F_{B_1U} \circ F_{B_2U})$ $((x_1,z)(y_1,z))$= max $(F_{B_1U}(x_1y_1), F_{A_2U}(z))$ $\forall$ z $\in V_2, \forall x_1y_1 \in E_1$;

4. $(T_{B_1L} \circ T_{B_2L})$ $((x_1,x_2)(y_1,y_2))$ = min $(T_{A_2L}(x_2), T_{A_2L}(y_2), T_{B_1L}(x_1y_1))$
(39)

$(T_{B_1U} \circ T_{B_2U})$ $((x_1,x_2)(y_1,y_2))$ = min $(T_{A_2U}(x_2), T_{A_2U}(y_2), T_{B_1U}(x_1y_1))$

$(I_{B_1L} \circ I_{B_2L})$ $((x_1,x_2)(y_1,y_2))$ = max $(I_{A_2L}(x_2), I_{A_2L}(y_2), I_{B_1L}(x_1y_1))$

$(I_{B_1U} \circ I_{B_2U})$ $((x_1,x_2)(y_1,y_2))$ = max $(I_{A_2U}(x_2), I_{A_2U}(y_2), I_{B_1U}(x_1y_1))$

$(F_{B_1L} \circ F_{B_2L})$ $((x_1,x_2)(y_1,y_2))$ = max $(F_{A_2L}(x_2), F_{A_2L}(y_2), F_{B_1L}(x_1y_1))$

$(F_{B_1U} \circ F_{B_2U})((x_1, x_2)(y_1, y_2))$ = max $(F_{A_2U}(x_2), F_{A_2U}(y_2), F_{B_1U}(x_1y_1))$, $\forall (x_1,x_2)(y_1,y_2) \in E^0$-E,where E^0= E $\cup$ $\{(x_1,x_2)(y_1,y_2) \mid x_1y_1 \in E_1, x_2 \neq y_2\}$.

Example 4

Let G_1^*= (V_1, E_1) and G_2^*= (V_2, E_2) be two graphs such that V_1 ={a, b}, V_2 ={c, d},E_1 ={a, b} and E_2 ={c, d}. Consider two interval-valued neutrosophic graphs:

A_1={ < a, [0.5, 0.7], [0.2, 0.3], [0.1, 0.3]>, <b, [0.6, 0.7],[0.2, 0.4], [0.1, 0.3},

B_1={ < ab, [0.3, 0.6], [0.2, 0.4], [0.2, 0.4]>};

A_2={ < c, [0.4, 0.6], [0.2, 0.3], [0.1, 0.3]>, <d, [0.4, 0.7],[0.2, 0.4], [0.1, 0.3},

B_2={ < cd, [0.3, 0.5], [0.2, 0.5], [0.3, 0.5]>}.

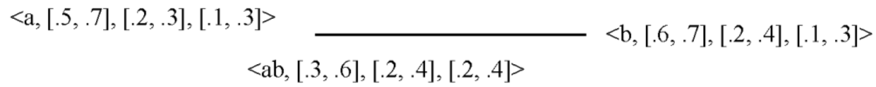

Figure 10: Interval valued neutrosophic graph G₁

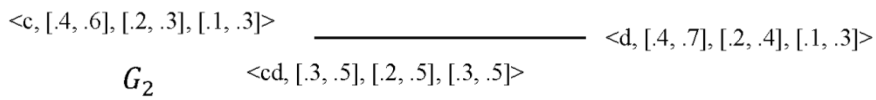

Figure 11: Interval valued neutrosophic graph G₂

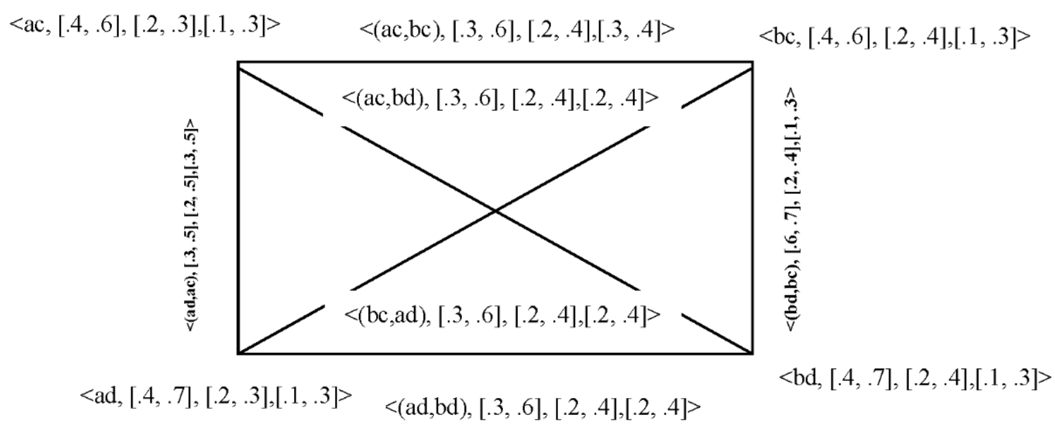

Figure 12: Composition of interval valued neutrosophic graph.

Proposition2

The composition$G_1[\ G_2]$ =($A_1 \circ A_2$, $B_1 \circ B_2$) of two interval valued neutrosophic graphs of the graphs $G_1^*$and G_2^* is an interval valued neutrosophic graph of $G_1^*[G_2^*]$.

Proof**.** Verifying only conditions for $\mathrm{B}_1 \circ \mathrm{B}_2$, because conditions for $\mathrm{A}_1 \circ \mathrm{A}_2$ are obvious. Let E= $\{(x,x_2)\ (x,y_2)\ /x_1 \in V_1, x_2y_2 \in E_2\} \cup \{(x_1, z)\ (y_1, z)\ /z \in V_2, x_1y_1 \in E_1\}$.Considering$(x,x_2)$ $(x,y_2) \in E$, one has:

$(T_{B_1L} \circ T_{B_2L})\ ((x, x_2)\ (x, y_2)) = \min\ (T_{A_1L}(x),\ T_{B_2L}(x_2y_2)) \leq \min\ (T_{A_1L}(x), \min(T_{A_2L}(x_2), T_{A_2L}(y_2))) = \min(\min\ (T_{A_1L}(x), T_{A_2L}(x_2)),\ \min\ (T_{A_1L}(x), T_{A_2L}(y_2))) = \min\ ((T_{A_1L} \circ T_{A_2L})\ (x,x_2), (T_{A_1L} \circ T_{A_2L})\ (x,y_2))$, (40)

$(T_{B_1U} \circ T_{B_2U})$ $((x, x_2)\ (x, y_2))$ = min $(T_{A_1U}(x), T_{B_2U}(x_2y_2)) \leq$ min $(T_{A_1U}(x)$, min$(T_{A_2U}(x_2)$,$T_{A_2U}(y_2)))$= min(min $(T_{A_1U}(x)$,$T_{A_2U}(x_2))$, min $(T_{A_1U}(x)$,$T_{A_2U}(y_2)))$= min $((T_{A_1U} \circ T_{A_2U})\ (x,x_2)$,$(T_{A_1U} \circ T_{A_2U})\ (x,y_2))$, (41)

$(I_{B_1L} \circ I_{B_2L})$ $((x, x_2)\ (x, y_2))$ = max $(I_{A_1L}(x), I_{B_2L}(x_2y_2)) \geq$ max $(I_{A_1L}(x)$, max$(I_{A_2L}(x_2), I_{A_2L}(y_2)))$ = max(max $(I_{A_1L}(x), I_{A_2L}(x_2))$, max $(I_{A_1L}(x), I_{A_2L}(y_2)))$ = max$((I_{A_1L} \circ I_{A_2L})\ (x,x_2)$,$(I_{A_1L} \circ I_{A_2L})\ (x,y_2))$, (42)

$(I_{B_1U} \circ I_{B_2U})$ $((x, x_2)\ (x, y_2))$ = max $(I_{A_1U}(x), I_{B_2U}(x_2y_2)) \geq$ max $(I_{A_1U}(x)$, max$(I_{A_2U}(x_2)$,$I_{A_2U}(y_2)))$ = max(max $(I_{A_1U}(x)$,$I_{A_2U}(x_2))$, max $(I_{A_1U}(x)$,$I_{A_2U}(y_2)))$ = max $((I_{A_1U} \circ I_{A_2U})\ (x,x_2)$,$(I_{A_1U} \circ I_{A_2U})\ (x,y_2))$, (43)

$(F_{B_1L} \circ F_{B_2L})$ $((x, x_2)\ (x, y_2))$ = max $(F_{A_1L}(x), F_{B_2L}(x_2y_2)) \geq$ max $(F_{A_1L}(x)$, max$(F_{A_2L}(x_2)$,$F_{A_2L}(y_2)))$ = max(max $(F_{A_1L}(x)$,$F_{A_2L}(x_2))$, max $(F_{A_1L}(x)$,$F_{A_2L}(y_2)))$ = max $((F_{A_1L} \circ F_{A_2L})\ (x,x_2)$,$(F_{A_1L} \circ F_{A_2L})\ (x,y_2))$, (44)

$(F_{B_1U} \circ F_{B_2U})$ $((x, x_2)\ (x, y_2))$ = max $(F_{A_1U}(x), F_{B_2U}(x_2y_2)) \geq$ max $(F_{A_1U}(x)$, max$(F_{A_2U}(x_2), F_{A_2U}(y_2)))$ = max(max $(F_{A_1U}(x), F_{A_2U}(x_2)$, max $(F_{A_1U}(x), F_{A_2U}(y_2))$ = max$((F_{A_1U} \circ F_{A_2U})\ (x,x_2)$,$(F_{A_1U} \circ F_{A_2U})\ (x,y_2))$. (45)

In the case $(x_1, z)\ (y_1, z) \in E$, the proof is similar.

In the case $(x_1,x_2)\ (y_1,y_2) \in E^0$-E.

$(T_{B_1L} \circ T_{B_2L})((x_1,x_2)\ (y_1,y_2))$ = min $(T_{A_2L}(x_2), T_{A_2L}(y_2), T_{B_1L}(x_1y_1)) \leq$ min $(T_{A_2L}(x_2)$, $T_{A_2L}(y_2)$,min $(T_{A_1L}(x_1), T_{A_1L}(y_1)))$ = min(min $(T_{A_1L}(x_1), T_{A_2L}(x_2))$, min $(T_{A_1L}(y_1)$,$T_{A_2L}(y_2)))$ = min $((T_{A_1L} \circ T_{A_2L})\ (x_1,x_2)$,$(T_{A_1L} \circ T_{A_2L})\ (y_1,y_2))$, (46)

$(T_{B_1U} \circ T_{B_2U})$ $((x_1, x_2)\ (y_1, y_2))$ = min $(T_{A_2U}(x_2), T_{A_2U}(y_2), T_{B_1L}(x_1y_1)) \leq$ min $(T_{A_2U}(x_2), T_{A_2U}(y_2)$,min $(T_{A_1U}(x_1), T_{A_1U}(y_1)))$ = min(min $(T_{A_1U}(x_1)$,$T_{A_2U}(x_2))$, min $(T_{A_1U}(y_1)$,$T_{A_2U}(y_2)))$ = min $((T_{A_1U} \circ T_{A_2U})\ (x_1,x_2)$,$(T_{A_1U} \circ T_{A_2U})\ (y_1,y_2))$,(47)

$(I_{B_1L} \circ I_{B_2L})\ ((x_1, x_2)\ (y_1, y_2))$ = max $(I_{A_2L}(x_2), I_{A_2L}(y_2), I_{B_1L}(x_1y_1)) \geq$ max $(I_{A_2L}(x_2)$, $I_{A_2L}(y_2)$,max $(I_{A_1L}(x_1), I_{A_1L}(y_1)))$ = max(max $(I_{A_1L}(x_1), I_{A_2L}(x_2))$, max $(I_{A_1L}(y_1)$,$I_{A_2L}(y_2)))$ = max $((I_{A_1L} \circ I_{A_2L})\ (x_1,x_2)$,$(I_{A_1L} \circ I_{A_2L})\ (y_1,y_2))$, (48)

$(I_{B_1U} \circ I_{B_2U})\ ((x_1,x_2)\ (y_1,y_2))$ = max $(I_{A_2U}(x_2), I_{A_2U}(y_2), I_{B_1L}(x_1y_1)) \geq$ max $(I_{A_2U}(x_2)$, $I_{A_2U}(y_2)$,max $(I_{A_1U}(x_1), I_{A_1U}(y_1)))$ = max(max $(I_{A_1U}(x), I_{A_2U}(x_2))$, max $(I_{A_1U}(y_1)$,$I_{A_2U}(y_2)))$ = max $((I_{A_1U} \circ I_{A_2U})\ (x_1,x_2)$,$(I_{A_1U} \circ I_{A_2U})\ (y_1,y_2))$, (49)

$(F_{B_1L} \circ F_{B_2L})$ $((x_1, x_2)\ (y_1, y_2))$ = max $(F_{A_2L}(x_2), F_{A_2L}(y_2), F_{B_1L}(x_1y_1)) \geq$ max $(F_{A_2L}(x_2), F_{A_2L}(y_2)$,max $(F_{A_1L}(x_1), F_{A_1L}(y_1)))$ = max(max $(F_{A_1L}(x), F_{A_2L}(x_2))$, max $(F_{A_1L}(y_1)$,$F_{A_2L}(y_2)))$= max $((F_{A_1L} \circ F_{A_2L})\ (x_1,x_2)$,$(F_{A_1L} \circ F_{A_2L})\ (y_1,y_2))$, (50)

$(F_{B_1U} \circ F_{B_2U})$ $((x_1, x_2)$ $(y_1, y_2))$ = max$(F_{A_2U}(x_2)$, $F_{A_2U}(y_2)$, $F_{B_1L}(x_1y_1)) \geq$ max $(F_{A_2U}(x_2)$, $F_{A_2U}(y_2)$,max $(F_{A_1U}(x_1)$, $F_{A_1U}(y_1)))$ = max(max $(F_{A_1U}(x), F_{A_2U}(x_2))$, max $(F_{A_1U}(y_1), F_{A_2U}(y_2)))$ = max $((F_{A_1U} \circ F_{A_2U})$ $(x_1, x_2), (F_{A_1U} \circ F_{A_2U})$ $(y_1, y_2))$. (51)

This completes the proof.

Definition 19

The union $G_1 \cup G_2 = (A_1 \cup A_2, B_1 \cup B_2)$ of two interval valued neutrosophic graphs of the graphs G_1^* and G_2^* is an interval-valued neutrosophic graph of $G_1^* \cup G_2^*$.

1) $(T_{A_1L} \cup T_{A_2L})(x) = T_{A_1L}(x)$ if $x \in V_1$ and $x \notin V_2$,
$(T_{A_1L} \cup T_{A_2L})(x) = T_{A_2L}(x)$ if $x \notin V_1$ and $x \in V_2$,
$(T_{A_1L} \cup T_{A_2L})(x) = \max(T_{A_1L}(x), T_{A_2L}(x))$ if $x \in V_1 \cap V_2$,
(52)
2) $(T_{A_1U} \cup T_{A_2U})(x) = T_{A_1U}(x)$ if $x \in V_1$ and $x \notin V_2$,
$(T_{A_1U} \cup T_{A_2U})(x) = T_{A_2U}(x)$ if $x \notin V_1$ and $x \in V_2$,
$(T_{A_1U} \cup T_{A_2U})(x) = \max(T_{A_1U}(x), T_{A_2U}(x))$ if $x \in V_1 \cap V_2$,
(53)
3) $(I_{A_1L} \cup I_{A_2L})(x) = I_{A_1L}(x)$ if $x \in V_1$ and $x \notin V_2$,
$(I_{A_1L} \cup I_{A_2L})(x) = I_{A_2L}(x)$ if $x \notin V_1$ and $x \in V_2$,
$(I_{A_1L} \cup I_{A_2L})(x) = \min(I_{A_1L}(x), I_{A_2L}(x))$ if $x \in V_1 \cap V_2$, (54)
4) $(I_{A_1U} \cup I_{A_2U})(x) = I_{A_1U}(x)$ if $x \in V_1$ and $x \notin V_2$,
$(I_{A_1U} \cup I_{A_2U})(x) = I_{A_2U}(x)$ if $x \notin V_1$ and $x \in V_2$,
$(I_{A_1U} \cup I_{A_2U})(x) = \min(I_{A_1U}(x), I_{A_2U}(x))$ if $x \in V_1 \cap V_2$,
(55)
5) $(F_{A_1L} \cup F_{A_2L})(x) = F_{A_1L}(x)$ if $x \in V_1$ and $x \notin V_2$,
$(N_{A_1L} \cup N_{A_2L})(x) = F_{A_2L}(x)$ if $x \notin V_1$ and $x \in V_2$,
$(N_{A_1L} \cup N_{A_2L})(x) = \min(F_{A_1L}(x), F_{A_2L}(x))$ if $x \in V_1 \cap V_2$,
(56)
6) $(F_{A_1U} \cup F_{A_2U})(x) = F_{A_1U}(x)$ if $x \in V_1$ and $x \notin V_2$,
$(F_{A_1U} \cup F_{A_2U})(x) = F_{A_2U}(x)$ if $x \notin V_1$ and $x \in V_2$,
$(F_{A_1U} \cup F_{A_2U})(x) = \min(F_{A_1U}(x), F_{A_2U}(x))$ if $x \in V_1 \cap V_2$,
(57)
7) $(T_{B_1L} \cup T_{B_2L})(xy) = T_{B_1L}(xy)$ if $xy \in E_1$ and $xy \notin E_2$,
$(T_{B_1L} \cup T_{B_2L})(xy) = T_{B_2L}(xy)$ if $xy \notin E_1$ and $xy \in E_2$,
$(T_{B_1L} \cup T_{B_2L})(xy) = \max(T_{B_1L}(xy), T_{B_2L}(xy))$ if $xy \in E_1 \cap E_2$, (58)
8) $(T_{B_{1U}} \cup T_{B_2U})(xy) = T_{B_1U}(xy)$ if $xy \in E_1$ and $xy \notin E_2$,
$(T_{B_1U} \cup T_{B_2U})(xy) = T_{B_2U}(xy)$ if $xy \notin E_1$ and $xy \in E_2$,
$(T_{B_1U} \cup T_{B_2U})(xy) = \max(T_{B_1U}(xy), T_{B_2U}(xy))$ if $xy \in E_1 \cap E_2$,
(59)
9) $(I_{B_1L} \cup I_{B_2L})(xy) = I_{B_1L}(xy)$ if $xy \in E_1$ and $xy \notin E_2$,
$(I_{B_1L} \cup M_{B_2L})(xy) = I_{B_2L}(xy)$ if $xy \notin E_1$ and $xy \in E_2$,

$(I_{B_1L} \cup I_{B_2L})\,(xy) = \min\,(I_{B_1L}(xy), I_{B_2L}(xy)\,)$ if xy $\in E_1 \cap E_2$, (60)

10) $(I_{B_1U} \cup I_{B_2U})\,(xy) = I_{B_1U}(xy)$ if xy $\in E_1$ and xy $\notin E_2$,

$(I_{B_1U} \cup I_{B_2U})\,(xy) = I_{B_2U}(xy)$ if xy $\notin E_1$ and xy $\in E_2$,

$(I_{B_1U} \cup I_{B_2U})\,(xy) = \min\,(I_{B_1U}(xy), I_{B_2U}(xy)\,)$ if xy $\in E_1 \cap E_2$, (61)

11) $(F_{B_1L} \cup F_{B_2L})\,(xy) = F_{B_1L}(xy)$ if xy $\in E_1$ and xy $\notin E_2$,

$(F_{B_1L} \cup F_{B_2L})\,(xy) = F_{B_2L}(xy)$ if xy $\notin E_1$ and xy $\in E_2$,

$(F_{B_1L} \cup F_{B_2L})\,(xy) = \min\,(F_{B_1L}(xy), F_{B_2L}(xy)\,)$ if xy $\in E_1 \cap E_2$, (62)

12) $(F_{B_1U} \cup F_{B_2U})\,(xy) = F_{B_1U}(xy)$ if xy $\in E_1$ and xy $\notin E_2$,

$(F_{B_1U} \cup F_{B_2U})\,(xy) = F_{B_2U}(xy)$ if xy $\notin E_1$ and xy $\in E_2$,

$(F_{B_1U} \cup F_{B_2U})\,(xy) = \min\,(F_{B_1U}(xy), F_{B_2U}(xy)\,)$ if xy $\in E_1 \cap E_2$. (63)

Proposition 3

Let G_1 and G_2 are two interval valued neutrosophic graphs, then $G_1 \cup G_2$ is an interval valued neutrosophic graph.

Proof. Verifying only conditions for $B_1 \circ B_2$, because conditions for $A_1 \circ A_2$ are obvious.

Let x y $\in E_1 \cap E_2$.

Then:

$(T_{B_1L} \cup T_{B_2L})(xy) = \max(T_{B_1L}(xy), T_{B_2L}(xy)) \leq \max(\min(T_{A_1L}(x), T_{A_1L}(y)), \min(T_{A_2L}(x), T_{A_2L}(y))) = \min(\max(T_{A_1L}(x), T_{A_2L}(x)), \max(T_{A_1L}(y), T_{A_2L}(y))) = \min((T_{A_1L} \cup T_{A_2L})\,(x), (T_{A_1L} \cup T_{A_2L})\,(y)))$; (64)

$(T_{B_1U} \cup T_{B_2U})(xy) = \max(T_{B_1U}(xy), T_{B_2U}(xy)) \leq \max(\min(T_{A_1U}(x), T_{A_1U}(y)), \min(T_{A_2U}(x), T_{A_2U}(y))) = \min(\max(T_{A_1U}(x), T_{A_2U}(x)), \max(T_{A_1U}(y), T_{A_2U}(y))) = \min((T_{A_1U} \cup T_{A_2U})\,(x), (T_{A_1U} \cup T_{A_2U})\,(y)))$; (65)

$(I_{B_1L} \cup I_{B_2L})(xy) = \min(I_{B_1L}(xy), I_{B_2L}(xy)) \geq \min(\max(I_{A_1L}(x), I_{A_1L}(y)), \max(I_{A_2L}(x), I_{A_2L}(y))) = \min(\min(I_{A_1L}(x), I_{A_2L}(x)), \min(I_{A_1L}(y), I_{A_2L}(y))) = \max((I_{A_1L} \cup I_{A_2L})\,(x), (I_{A_1L} \cup I_{A_2L})\,(y)))$; (66)

$(I_{B_1U} \cup I_{B_2U})(xy) = \min(I_{B_1U}(xy), I_{B_2U}(xy)) \geq \min(\max(I_{A_1U}(x), I_{A_1U}(y)), \max(I_{A_2U}(x), I_{A_2U}(y))) = \max(\min(I_{A_1U}(x), I_{A_2U}(x)), \min(I_{A_1U}(y), I_{A_2U}(y))) = \max((I_{A_1U} \cup I_{A_2U})\,(x), (I_{A_1U} \cup I_{A_2U})\,(y)))$; (67)

$(F_{B_1L} \cup F_{B_2L})(xy) = \min(F_{B_1L}(xy), F_{B_2L}(xy)) \geq \min(\max(F_{A_1L}(x), F_{A_1L}(y)), \max(F_{A_2L}(x), F_{A_2L}(y))) = \min(\min(F_{A_1L}(x), F_{A_2L}(x)), \min(F_{A_1L}(y), F_{A_2L}(y))) = \max((F_{A_1L} \cup F_{A_2L})\,(x), (F_{A_1L} \cup F_{A_2L})\,(y)))$; (68)

$(F_{B_1U} \cup F_{B_2U})(xy) = \min(F_{B_1U}(xy), F_{B_2U}(xy)) \geq \min(\max(F_{A_1U}(x), F_{A_1U}(y)), \max(F_{A_2U}(x), F_{A_2U}(y))) = \max(\min(F_{A_1U}(x), F_{A_2U}(x)), \min(F_{A_1U}(y), F_{A_2U}(y))) = \max((F_{A_1U} \cup F_{A_2U})\,(x), (F_{A_1U} \cup F_{A_2U})\,(y)))$. (69)

This completes the proof.

Example 5

Let $G_1^*= (V_1, E_1)$ and $G_2^*= (V_2, E_2)$ be two graphs such that $V_1 =\{v_1, v_2, v_3, v_4, v_5\}$. $V_2 =\{v_1, v_2, v_3, v_4\}$, $E_1 =\{v_1v_2, v_1v_5, v_2v_3, v_5v_3, v_5v_4, v_4v_3\}$ and $E_2 =\{v_1v_2, v_2v_3, v_2v_4, v_3v_{34}, v_4v_1\}$. Consider two interval valued neutrosophic graphs $G_1= (A_1, B_1)$ and $G_2= (A_2, B_2)$.

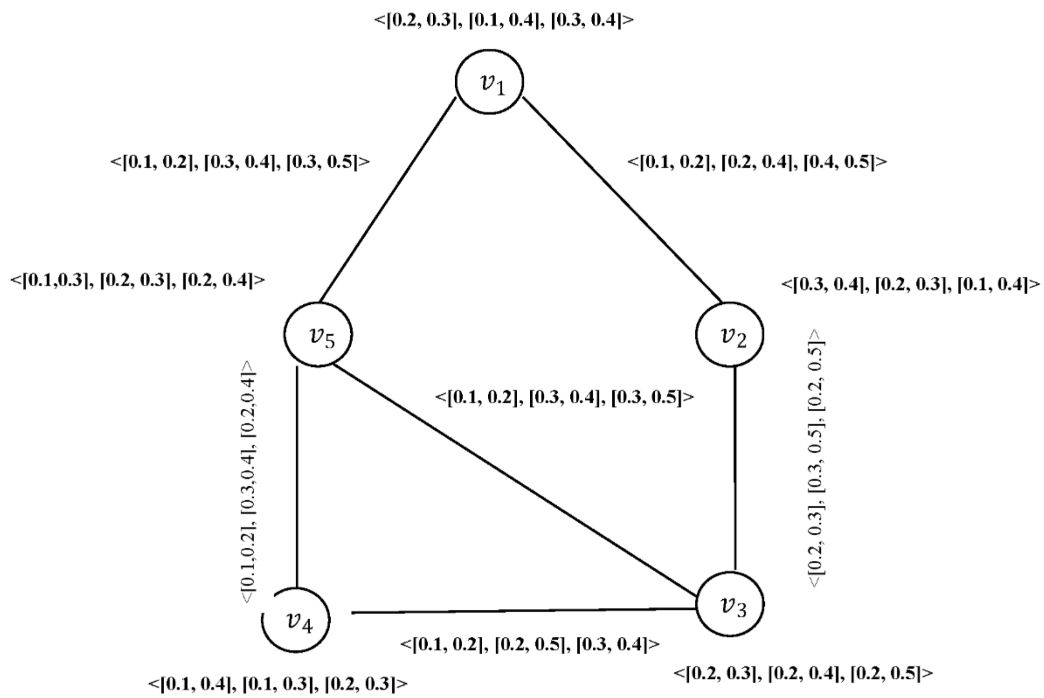

Figure 13: Interval valued neutrosophic graph G1

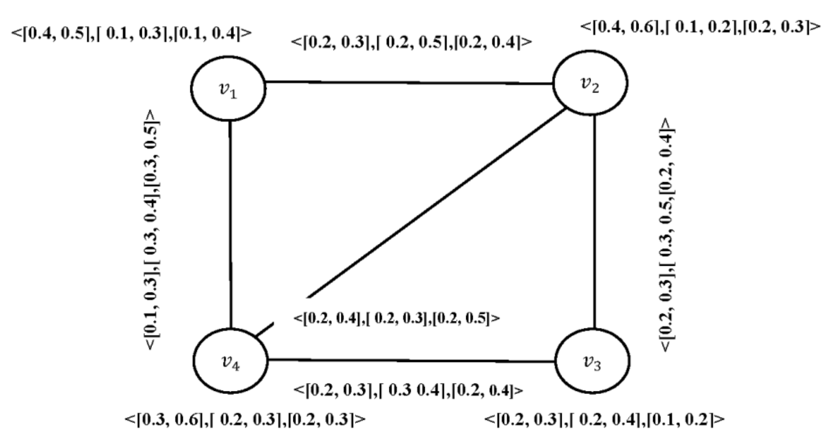

Figure 14: Interval valued neutrosophic graph G1

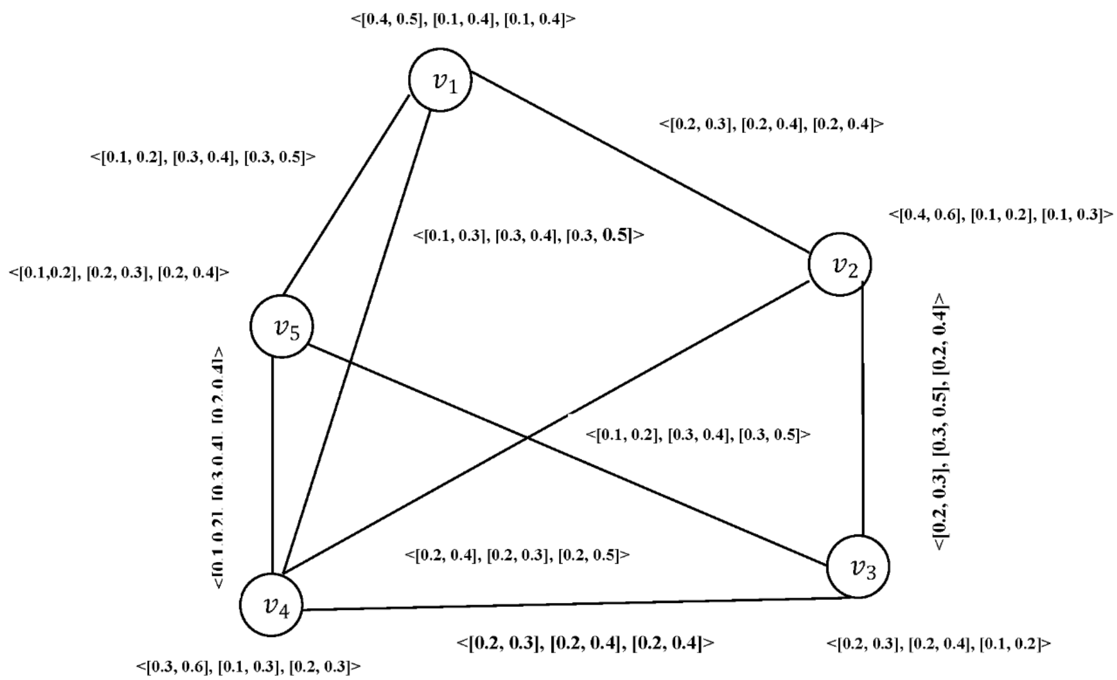

Figure 15: Interval valued neutrosophic graph $G_1 \cup G_2$

Definition 20

The join of $G_1 + G_2 = (A_1 + A_2, B_1 + B_2)$ interval valued neutrosophic graphs G_1 and G_2 of the graphs G_1^* and G_2^* is defined as follows:

1) $$(T_{A_1L} + T_{A_2L})(x) = \begin{cases} (T_{A_1L} \cup T_{A_2L})(x) & \text{if } x \in V_1 \cup V_2 \\ T_{A_1L}(x) & \text{if } x \in V_1 \\ T_{A_2L}(x) & \text{if } x \in V_2 \end{cases} \quad (70)$$

$$(T_{A_1U} + T_{A_2U})(x) = \begin{cases} (T_{A_1U} \cup T_{A_2U})(x) & \text{if } x \in V_1 \cup V_2 \\ T_{A_1U}(x) & \text{if } x \in V_1 \\ T_{A_2U}(x) & \text{if } x \in V_2 \end{cases}$$

$$(I_{A_1L} + I_{A_2L})(x) = \begin{cases} (I_{A_1L} \cap I_{A_2L})(x) & \text{if } x \in V_1 \cup V_2 \\ I_{A_1L}(x) & \text{if } x \in V_1 \\ I_{A_2L}(x) & \text{if } x \in V_2 \end{cases}$$

$$(I_{A_1U} + I_{A_2U})(x) = \begin{cases} (I_{A_1U} \cap I_{A_2U})(x) & \text{if } x \in V_1 \cup V_2 \\ I_{A_1U}(x) & \text{if } x \in V_1 \\ I_{A_2U}(x) & \text{if } x \in V_2 \end{cases}$$

$$(F_{A_1L} + F_{A_2L})\,(x) = \begin{cases} (F_{A_1L} \cap F_{A_2L})(x) & \text{if } x \in V_1 \cup V_2 \\ F_{A_1L}(x) & \text{if } x \in V_1 \\ F_{A_2L}(x) & \text{if } x \in V_2 \end{cases}$$

$$(F_{A_1U} + F_{A_2U})\,(x) = \begin{cases} (F_{A_1U} \cap F_{A_2U})(x) & \text{if } x \in V_1 \cup V_2 \\ F_{A_1U}(x) & \text{if } x \in V_1 \\ F_{A_2U}(x) & \text{if } x \in V_2 \end{cases}$$

2) $$(T_{B_1L} + T_{B_2L})\,(x\text{y}) = \begin{cases} (T_{B_1L} \cup T_{B_2L})(\text{xy}) & \text{if } xy \in E_1 \cup E_2 \\ T_{B_1L}(xy) & \text{if } xy \in E_1 \\ T_{B_2L}(xy) & \text{if } xy \in E_2 \end{cases} \quad (71)$$

$$(T_{B_1U} + T_{B_2U})\,(x\text{y}) = \begin{cases} (T_{B_1U} \cup T_{B_2U})(\text{xy}) & \text{if } xy \in E_1 \cup E_2 \\ T_{B_1U}(xy) & \text{if } xy \in E_1 \\ T_{B_2U}(xy) & \text{if } xy \in E_2 \end{cases}$$

$$(I_{B_1L} + I_{B_2L})\,(x\text{y}) = \begin{cases} (I_{B_1L} \cap I_{B_2L})(\text{xy}) & \text{if } xy \in E_1 \cup E_2 \\ I_{B_1L}(xy) & \text{if } xy \in E_1 \\ I_{B_2L}(xy) & \text{if } xy \in E_2 \end{cases}$$

$$(I_{B_1U} + I_{B_2U})\,(x\text{y}) = \begin{cases} (I_{B_1U} \cap I_{B_2U})(\text{xy}) & \text{if } xy \in E_1 \cup E_2 \\ I_{B_1U}(xy) & \text{if } xy \in E_1 \\ I_{B_2U}(xy) & \text{if } xy \in E_2 \end{cases}$$

$$(F_{B_1L} + F_{B_2L})\,(x\text{y}) = \begin{cases} (F_{B_1L} \cap F_{B_2L})(\text{xy}) & \text{if } xy \in E_1 \cup E_2 \\ F_{B_1L}(xy) & \text{if } xy \in E_1 \\ F_{B_2L}(xy) & \text{if } xy \in E_2 \end{cases}$$

$$(F_{B_1U} + F_{B_2U})\,(x\ \text{y}) = \begin{cases} (F_{B_1U} \cap F_{B_2U})(\text{xy}) & \text{if } xy \in E_1 \cup E_2 \\ F_{B_1U}(xy) & \text{if } xy \in E_1 \\ F_{B_2U}(xy) & \text{if } xy \in E_2 \end{cases}$$

3) $(T_{B_1L} + T_{B_2L})\,(x\ \text{y}) = \min\,(T_{B_1L}(x), T_{B_2L}(x))$ (72)
$(T_{B_1U} + T_{B_2U})\,(xy) = \min\,(T_{B_1U}(x), T_{B_2U}(x))$
$(I_{B_1L} + I_{B_2L})\,(x\text{y}) = \max\,(I_{B_1L}(x), I_{B_2L}(x))$
$(I_{B_1U} + I_{B_2U})\,(x\ \text{y}) = \max\,(I_{B_1U}(x), I_{B_2U}(x)$
$(F_{B_1L} + F_{B_2L})\,(x\text{y}) = \max\,(F_{B_1L}(x), F_{B_2L}(x))$
$(F_{B_1U} + F_{B_2U})\,(x\ \text{y}) = \max\,(F_{B_1U}(x), F_{B_2U}(x)) if xy \ \in\ E'$,

where E' is the set of all edges joining the nodes of V_1 and V_2, assuming$V_1 \cap V_2 = \emptyset$.

Example 6

Let G_1^*= (V_1, E_1) and G_2^*= (V_2, E_2) be two graphs such that V_1 ={u_1,u_2,u_3},V_2 ={v_1,v_2,v_3},E_1 ={u_1u_2, u_2u_3} and E_2 ={v_1v_2, v_2v_3}. Consider two interval valued neutrosophic graphs G_1= (A_1, B_1) and G_2= (A_2, B_2).

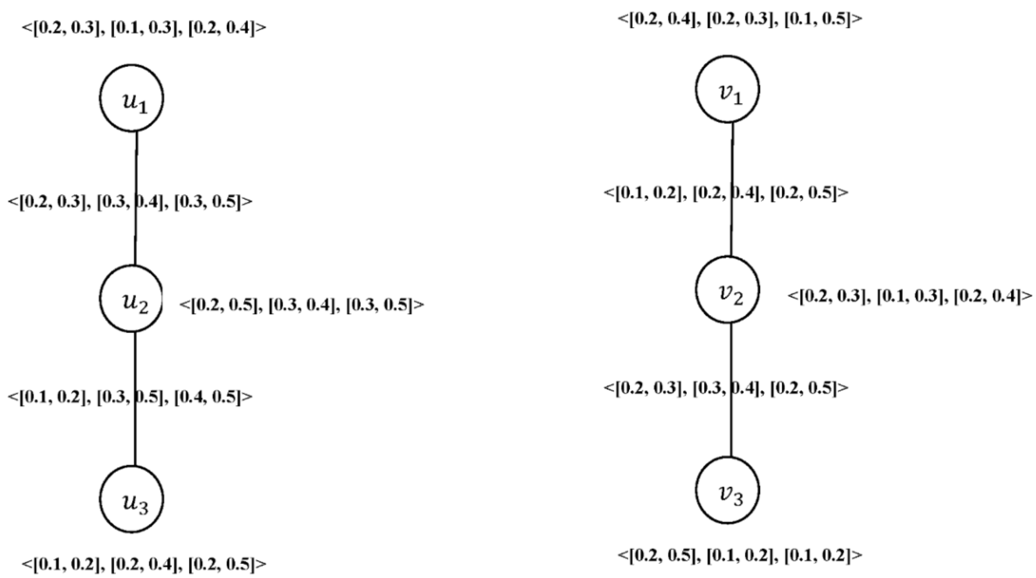

Figure 16: Interval valued neutrosophic graph of G_1 and G_2

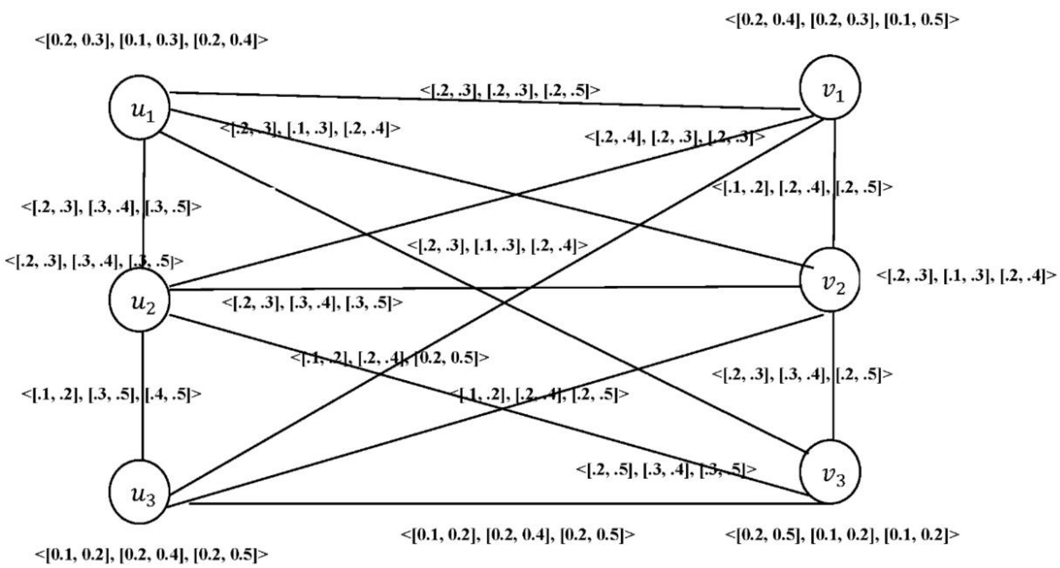

Figure 17: Interval valued neutrosophic graph of $G_1 + G_2$

5. Conclusion

The interval valued neutrosophic models give more precision, flexibility and compatibility to the system as compared to the classical, fuzzy, intuitionistic fuzzy and neutrosophic models. In this paper, the authors introduced some operations: Cartesian product, composition, union and join on interval valued neutrosophic graphs, and investigated some of their properties. In the future, the authors plan to study others operations, such as: tensor product and normal product of two interval valued neutrosophic graphs.

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A New Order Relation on the Set of Neutrosophic Truth Values

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Abstract

In this article, we discuss all possible cases to construct an atom of matter, antimatter, or unmatter, and also the cases of contradiction (i.e. impossible case).

1. Introduction

Anti-particle in physics means a particle which has one or more opposite properties to its "original particle kind". If one property of a particle has the opposite sign to its original state, this particle is anti-particle, and it annihilates with its original particle.

The anti-particles can be electrically charged, color or fragrance (for quarks). Meeting each other, a particle and its anti-particle annihilate into gamma-quanta.

This formulation may be mistaken with the neutrosophic <antiA>, which is strong opposite to the original particle kind. The <antiA> state is the ultimate case of anti-particles [6].

In [7], F. Smarandache discusses the refinement of neutrosophic logic. Hence, <A>, <neutA> and <antiA> can be split into: $<A_1>$, $<A_2>$, ...; $<neutA_1>$, $<neutA_2>$, ...; $<antiA_1>$, $<antiA_2>$, ...; therefore, more types of matter, more types of unmatter, and more types of antimatter.

One may refer to <A>, <neutA>, <anti-A> as "matter", "unmatter" and "anti-matter".

Following this way, in analogy to anti-matter as the ultimate case of anti-particles in physics, the unmatter can be extended to "strong unmatter", where all properties of a substance or a field are unmatter, and to "regular unmatter", where just one of the properties of it satisfies the unmatter.

2. Objective

The aim is to check whether the indeterminacy component I can be split to sub-indeterminacies I_1, I_2, I_3, and then justify that the below are all different:

$$I_1 \cap I_2 \cap I_3,\ I_1 \cap I_3 \cap I_2,\ I_2 \cap I_3 \cap I_1,\ I_2 \cap I_1 \cap I_3,\ I_3 \cap I_1 \cap I_2,\ I_3 \cap I_2 \cap I_1. \qquad (1)$$

3. Cases

Let e , e^+, P, $antiP$, N, $antiN$ be electrons, anti-electrons, protons, anti-protons, neutrons, anti-neutrons respectively, also $\cup$ means union/OR, while $\cap$ means intersection/AND, and suppose:

$$I = (e \ \cup \ e^+) \cap (P \ \cup \ antiP) \cap (N \ \cup \ antiN) \qquad (2)$$

The statement (2) shows indeterminacy, since one cannot decide the result of the interaction if it will produce any of the following cases:

1. $(e \cup e^{+}) \cap (\mathrm{P} \cup \mathrm{antiP}) \cap (\mathrm{N} \cup \mathrm{antiN}) \rightarrow e \cap \mathrm{P} \cap \mathrm{antiN}$,
   which is *unmatter* type (a), see reference [2];
2. $(e \cup e^{+}) \cap (\mathrm{N} \cup \mathrm{antiN}) \cap (\mathrm{P} \cup \mathrm{antiP}) \rightarrow e^{+} \cap \mathrm{N} \cap \mathrm{antiP}$,
   which is *unmatter* type (b), see reference [2];
3. $(\mathrm{P} \cup \mathrm{antiP}) \cap (\mathrm{N} \cup \mathrm{antiN}) \cap (e \cup e^{+}) \rightarrow \mathrm{P} \cap \mathrm{N} \cap e^{+} = contradiction$;
4. $(\mathrm{P} \cup \mathrm{antiP}) \cap (e \cup e^{+}) \cap (\mathrm{N} \cup \mathrm{anti\,N}) \rightarrow \mathrm{antiP} \cap \mathrm{e} \cap anti\mathrm{N} = contradiction$;
5. $(\mathrm{N} \cup \mathrm{antiN}) \cap (e \cup e^{+}) \cap (\mathrm{P} \cup \mathrm{antiP}) \rightarrow \mathrm{N} \cap \mathrm{e} \cap \mathrm{P}$,
   which is a *matter*;
6. $(\mathrm{N} \cup \mathrm{antiN}) \cap (\mathrm{P} \cup \mathrm{antiP}) \cap (e \cup e^{+}) \rightarrow \mathrm{antiN} \cap anti\mathrm{P} \cap e^{+}$,
   which is *antimatter*.

4. Comment

It is obvious that all above six cases are not equal in pairs; suppose:

$e \cup e^{+} = I_1 = uncertainty$,

$\mathrm{P} \cup \mathrm{antiP} = I_2 = uncertainty$,

$\mathrm{N} \cup \mathrm{antiN} = I_3 = uncertainty$.

Consequently, the statement (2) can be rewritten as:

$I = I_1 \cap I_2 \cap I_3$

but we cannot get the equality for any pairs in eq. (1).

5. Remark

This example is a response to the article [4], where Florentin Smarandache stated that "for each application we might have some different order relations on the set of neutrosophic truth values; (…) one can get one such order relation workable for all problems", and also to a commentary in [5], that "It would be very useful to define suitable order relations on the set of neutrosophic truth values".

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Introduction to the Complex Refined Neutrosophic Set

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Abstract

In this paper, one extends the single-valued complex neutrosophic set to the subset-valued complex neutrosophic set, and afterwards to the subset-valued complex refined neutrosophic set.

Keywords

single-valued complex neutrosophic set, subset-valued complex neutrosophic set, subset-valued complex refined neutrosophic set.

1 Introduction

One first recalls the definitions of the single-valued neutrosophic set (SVNS), and of the subset-value neutrosophic set (SSVNS).

Definition 1.1.

Let X be a space of elements, with a generic element in X denoted by x. A *Single-Valued Neutrosophic Set (SVNS) A* is characterized by a truth membership function $T_A(x)$, an indeterminacy membership function $I_A(x)$, and a falsity membership function $F_A(x)$, where for each element $x \in X$, $T_A(x)$, $I_A(x)$, $F_A(x) \in [0, 1]$ and $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$.

Definition 1.2.

Let X be a space of elements, with a generic element in X denoted by x. A *SubSet-Valued Neutrosophic Set (SSVNS)* A [3] is characterized by a truth membership function $T_A(x)$, an indeterminacy membership function $I_A(x)$, and a falsity membership function $F_A(x)$, where for each element $x \in X$, the subsets $T_A(x), I_A(x), F_A(x) \subseteq [0,1]$,

with $0 \le \sup(T_A(x)) + \sup(I_A(x)) + \sup(F_A(x)) \le 3$.

2 Complex Neutrosophic Set

Ali and Smarandache [1] introduced the notion of single-valued complex neutrosophic set (SVCNS) as a generalization of the single-valued neutrosophic set (SVNS) [2].

Definition 2.1.

Let *X* be a space of elements, with a generic element in *X* denoted by *x*. A *Single-Valued Complex Neutrosophic Set (SVCNS) A* [1] is characterized by a truth membership function $T_{1_A}(x)e^{iT_{2A}(x)}$, an indeterminacy membership function $I_{1_A}(x)e^{iI_{2A}(x)}$, and a falsity membership function $F_{1_A}(x)e^{iF_{2A}(x)}$, where for each element $x \in X$, single-valued numbers $T_{1_A}(x), I_{1_A}(x), F_{1_A}(x) \in [0,1]$, $0 \le T_{1_A}(x) + I_{1_A}(x) + F_{1_A}(x) \le 3$, $i = \sqrt{-1}$,

and the single-valued numbers $T_{2_A}(x), I_{2_A}(x), F_{2_A}(x) \in [0, 2\pi]$,

with $0 \le T_{2_A}(x) + I_{2_A}(x) + F_{2_A}(x) \le 6\pi$.

$T_{1_A}(x), I_{1_A}(x), F_{1_A}(x)$ represent the real part (or amplitude) of the truth membership, indeterminacy membership, and falsehood membership respectively; while $T_{2_A}(x), I_{2_A}(x), F_{2_A}(x)$ represent the imaginary part (or phase) of the truth membership, indeterminacy membership, and falsehood membership respectively.

Definition 2.2.

In the previous Definition 2.1., if one replaces the single-valued numbers with subset-values, i.e. the subset-values $T_{1_A}(x), I_{1_A}(x), F_{1_A}(x) \subseteq [0,1]$, $i = \sqrt{-1}$, and the subset-values $T_{2_A}(x), I_{2_A}(x), F_{2_A}(x) \subseteq [0, 2\pi]$,

with $0 \le \sup(T_{1_A}(x)) + \sup(I_{1_A}(x)) + \sup(F_{1_A}(x)) \le 3$,

and $0 \le \sup(T_{2_A}(x)) + \sup(I_{2_A}(x)) + \sup(F_{2_A}(x)) \le 6\pi$,

one obtains the *SubSet-Valued Complex Neutrosophic Set (SSVCNS).*

3 Refined Neutrosophic Set

Smarandache introduced the refined neutrosophic set [4] in 2013.

Definition 3.1.

Let X be a space of elements, with a generic element in X denoted by x. A *Single-Valued Refined Neutrosophic Set (SVRNS) A* is characterized by p sub-truth membership functions $T_{1_A}(x), T_{2_A}(x), \ldots, T_{p_A}(x)$, r sub-indeterminacy membership functions $I_{1_A}(x), I_{2_A}(x), \ldots, I_{r_A}(x)$, and s sub-falsity membership functions $F_{1_A}(x), F_{2_A}(x), \ldots, F_{s_A}(x)$, where for each element $x \in X$, the single-valued numbers

$$T_{1_A}(x), T_{2_A}(x), \ldots, T_{p_A}(x), I_{1_A}(x), I_{2_A}(x), \ldots, I_{r_A}(x), F_{1_A}(x), F_{2_A}(x), \ldots, F_{s_A}(x) \in [0,1],$$

$$0 \le T_{1_A}(x) + T_{2_A}(x) + \ldots + T_{p_A}(x) + I_{1_A}(x) + I_{2_A}(x) + \ldots + I_{r_A}(x) + F_{1_A}(x) + F_{2_A}(x) + \ldots$$
$$\ldots + F_{s_A}(x) \le p + r + s,$$

and the integers $p, r, s \ge 0$, with at least one of p, r, s to be ≥ 2.

In other words, the truth membership function $T_A(x)$ was refined (split) into p sub-truths $T_{1_A}(x), T_{2_A}(x), \ldots, T_{p_A}(x)$, the indeterminacy membership function $I_A(x)$ was refined (split) into r sub-indeterminacies $I_{1_A}(x), I_{2_A}(x), \ldots, I_{r_A}(x)$, and the falsity membership function $F_A(x)$ was refined (split) into s sub-falsities $F_{1_A}(x), F_{2_A}(x), \ldots, F_{s_A}$.

Definition 3.2.

In the previous Definition 3.1., if one replaces the single-valued numbers with subset-values i.e., the subset-values $T_{1_A}(x), T_{2_A}(x), \ldots, T_{p_A}(x), I_{1_A}(x), I_{2_A}(x), \ldots, I_{r_A}(x), F_{1_A}(x), F_{2_A}(x), \ldots, F_{s_A}(x) \subseteq [0,1]$, and

$$0 \le \sup(T_{1_A}(x)) + \sup(T_{2_A}(x)) + \ldots + \sup(T_{p_A}(x)) + \sup(I_{1_A}(x)) + \sup(I_{2_A}(x)) + \ldots$$
$$\ldots + \sup(I_{r_A}(x)) + \sup(F_{1_A}(x)) + \sup(F_{2_A}(x)) + \ldots + \sup(F_{s_A}(x)) \le p + r + s,$$

one obtains the *SubSet-Valued Refined Neutrosophic Set (SSVRNS)*.

4 Complex Refined Neutrosophic Set

Now one combines the complex neutrosophic set with refined neutrosophic set in order to get the complex refined neutrosophic set.

Definition 4.1.

Let X be a space of elements, with a generic element in X denoted by x. A *Single-Valued Complex Refined Neutrosophic Set (SVCRNS) A* is characterized by p sub-truth membership functions

$T_{11_A}(x)e^{iT_{21_A}(x)}, T_{12_A}(x)e^{iT_{22_A}(x)}, \ldots, T_{1p_A}(x)e^{iT_{2p_A}(x)}$, r sub-indeterminacy membership functions $I_{11_A}(x)e^{iI_{21_A}(x)}, I_{12_A}(x)e^{iI_{22_A}(x)}, \ldots, I_{1r_A}(x)e^{iI_{2r_A}(x)}$, and s sub-falsity membership functions $F_{11_A}(x)e^{iF_{21_A}(x)}, F_{12_A}(x)e^{iF_{22_A}(x)}, \ldots, F_{1s_A}(x)e^{iF_{2s_A}(x)}$, and $i = \sqrt{-1}$, where for each element $x \in X$, the single-valued numbers (sub-real parts, or sub-amplitudes)

$$T_{11_A}(x), T_{12_A}(x), \ldots, T_{1p_A}(x), I_{11_A}(x), I_{12_A}(x), \ldots, I_{1r_A}(x), F_{11_A}(x), F_{12_A}(x), \ldots, F_{1s_A}(x) \in [0,1]$$

with

$$0 \le T_{11_A}(x) + T_{12_A}(x) + \ldots + T_{1p_A}(x) + I_{11_A}(x) + I_{12_A}(x) + \ldots + I_{1r_A}(x) + F_{11_A}(x) + F_{12_A}(x) + \ldots + F_{1s_A}(x) \le p + r + s,$$

and the single-valued numbers (sub-imaginary parts, or sub-phases)

$$T_{21_A}(x), T_{22_A}(x), \ldots, T_{2p_A}(x), I_{21_A}(x), I_{22_A}(x), \ldots, I_{2r_A}(x), F_{21_A}(x), F_{22_A}(x), \ldots, F_{2s_A}(x) \in [0, 2\pi]$$

with

$$0 \le T_{21_A}(x) + T_{22_A}(x) + \ldots + T_{2p_A}(x) + I_{21_A}(x) + I_{22_A}(x) + \ldots + I_{2r_A}(x) + F_{21_A}(x) + F_{22_A}(x) + \ldots + F_{2s_A}(x) \le 2(p + r + s)\pi,$$

and the integers *p, r, s ≥ 0*, with at least one of *p, r, s* to be *≥ 2*.

Definition 4.2.

In the previous Definition 4.1., if one replaces the single-valued numbers with subset-values i.e., the subset-values

$$T_{11_A}(x), T_{12_A}(x), \ldots, T_{1p_A}(x), I_{11_A}(x), I_{12_A}(x), \ldots, I_{1r_A}(x), F_{11_A}(x), F_{12_A}(x), \ldots, F_{1s_A}(x) \subseteq [0,1]$$

with

$$0 \le \sup(T_{11_A}(x)) + \sup(T_{12_A}(x)) + \ldots + \sup(T_{1p_A}(x)) + \sup(I_{11_A}(x)) + \sup(I_{12_A}(x)) + \ldots + \sup(I_{1r_A}(x)) + \sup(F_{11_A}(x)) + \sup(F_{12_A}(x)) + \ldots + \sup(F_{1s_A}(x)) \le p + r + s,$$

and

$$T_{21_A}(x), T_{22_A}(x), \ldots, T_{2p_A}(x), I_{21_A}(x), I_{22_A}(x), \ldots, I_{2r_A}(x), F_{21_A}(x), F_{22_A}(x), \ldots, F_{2s_A}(x) \subseteq [0, 2\pi]$$

with

$$0 \le \sup(T_{21_A}(x)) + \sup(T_{22_A}(x)) + \ldots + \sup(T_{2p_A}(x)) + \sup(I_{21_A}(x)) + \sup(I_{22_A}(x)) + \ldots$$
$$\ldots + \sup(I_{2r_A}(x)) + \sup(F_{21_A}(x)) + \sup(F_{22_A}(x)) + \ldots + \sup(F_{2s_A}(x)) \le 2(p+r+s)\pi,$$

one obtains the *SubSet-Valued Complex Refined Neutrosophic Set (SSVCRNS).*

5 Conclusion

After the introduction of the single-valued and subset-valued complex refined neutrosophic sets as future research is the construction of their aggregation operators, the study of their properties, and their applications in various fields.

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GRA for Multi Attribute Decision Making in Neutrosophic Cubic Set Environment

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Abstract. In this paper, multi attribute decision making problem based on grey relational analysis in neutrosophic cubic set environment is investigated. In the decision making situation, the attribute weights are considered as single valued neutrosophic sets. The neutrosophic weights are converted into crisp weights. Both positve and negative GRA coefficients, and weighted GRA coefficients are determined. Hamming distances for weighted GRA coefficients and standard (ideal) GRA coefficients are determined. The relative closeness coefficients are derived in order to rank the alternatives. The relative closeness coefficients are designed in ascending order. Finally, a numerical example is solved to demonstrate the applicability of the proposed approach.

Keywords: Grey relational coefficient, interval valued neutrosophic set, multi attribute decision making, neutrosophic set, neutrosophic cubic set, relative closeness coefficient

1 Introduction

In management section, banking sector, factory, plant multi attribute decision making (MADM) problems are to be extensively encountered. In a MADM situation, the most appropriate alternative is selecting from the set of alternatives based on highest degree of acceptance. In a decision making situation, decision maker (DM) considers the efficiency of each alternative with respect to each attribute. In crisp MADM, there are several approaches [1, 2, 3, 4, 5] in the literature. The weight of each attribute and the elements of decision matrix are presented by crisp numbers. But in real situation, DMs may prefer to use linguistic variables like 'good', 'bad', 'hot', 'cold', 'tall', etc. So, there is an uncertainty in decision making situation which can be mathematically explained by fuzzy set [6]. Zadeh [6] explained uncertainty mathematically by defining fuzzy set (FS). Bellman and Zadeh [7] studied decision making in fuzzy environment. Atanassov [8, 9] narrated uncertainty by introducing non-membership as independent component and defined intuitionistic fuzzy set (IFS). Degree of indeterminacy (hesitency) is not independent .

Later on DMs have recognized that indeterminacy plays an important role in decision making. Smarandache [10] incorporated indeterminacy as independent component and developed neutrosophic set (NS) and together with Wang et a. [11] defined single valued neutrosophic set (SVNS) which is an instance of neutrosophic set. Ye [12] proposed a weighted correlation coefficients for ranking the alternatives for multicriteria decision making (MCDM). Ye [13] established single valued neutrosophic cross entropy for MCDM problem. Sodenkamp [14] studied multiple-criteria decision analysis in neutrosophic environment. Mondal and Pramanik [15] defined neutrosophic tangent similarity measure and presented its application to MADM. Biswas et al. [16] studied cosine similarity measure based MADM with trapezoidal fuzzy neutrosophic numbers. Mondal and Pramanik [17] presented multi-criteria group decision making (MCGDM) approach for teacher recruitment in higher education. Mondal and Pramanik [18] studied neutrosophic decision making model of school choice. Liu and Wang [19] presented MADM method based on single-valued neutrosophic normalized weighted Bonferroni mean. Biswas et al. [20] presented TOPSIS method for MADM under single-valued neutrosophic environment. Chi and Liu [21] presented extended TOPSIS method for MADM on interval neutrosophic set. Broumi et al. [22] presented extended TOPSIS method for MADM based on interval neutrosophic uncertain linguistic variables. Nabdaban and Dzitac [23] presented a very short review of TOPSIS in neutrosophic environment. Pramanik et al. [24] studied hybrid vector similarity measures and their applications to MADM under neutrosophic environment. Biswas et al. [25] presented triangular fuzzy neutrosophic set information and its application to MADM. Sahin and Liu [26] studied maximizing deviation method for neutrosophic MADM with incomplete weight information. Ye [27] studied bidirectional projection method for MADM with neutrosophic numbers of the form $a + bI$, where I is characterized by indeterminacy. Biswas et al. [28] presented value and ambiguity index based ranking method of single-valued trapezoidal neutrosophic numbers and its application to MADM. Dey et al. [29] studied extended projection-based models for solving MADM problems with interval-valued neutrosophic information.

Deng [30, 31] studied grey relational analysis (GRA). Pramanik and Mukhopadhyaya [32] developed GRA based intuitionistic fuzzy multi criteria decision making (MCDM) approach for teacher selection in higher education. Dey et al. [33] established MCDM in intuitionistic fuzzy environment based on GRA for weaver selection in Khadi institution. Rao, and Singh [34] established modified GRA method for decision making in manufacturing situation. Wei [35] presented GRA method for intuitionistic fuzzy MCDM. Biswas et al. [36] studied GRA method for MADM under single valued neutrosophic assessment based on entropy. Dey et al. [37] presented extended GRA based neutrosophic MADM in interval uncertain linguistic setting. Pramanik and K. Mondal [38] employed GRA for interval neutrosophic MADM and presented numerical examples.

Several neutrosophic hybrid sets have been recently proposed in the literature, such as neutrosophic soft set proposed by Maji [39], single valued soft expert set proposed by Broumi and Smarandache [40], rough neutrosophic set proposed by Broumi, et al. [41], neutrosophic bipolar set proposed by Deli et al. [42], rough bipolar neutrosophic set proposed by Pramanik and Mondal [43], neutrosophic cubic set proposed by Jun et al. [44] and Ali et al. [45]. Jun et al. [44] presented the concept of neut-rosophic cubic set by extending the concept of cubic set proposed by Jun et al. [46] and introduced the notions of truth-internal (indeterminacy-internal, falsity-internal) neut-rosophic cubic sets and truth-external (indeterminacy-external, falsity-external) and investigated related properties. Ali et al. [45] presented concept of neutrosophic cubic set by extending the concept of cubic set [46] and defined internal neutrosophic cubic set (INCS) and external neutrosophic cubic set (ENCS). In their study, Ali et al.[45] also introduced an adjustable approach to neutrosophic cubic set based decision making.

GRA based MADM/ MCDM problems have been proposed for various neutrosophic hybrid environments [47, 48, 49, 50]. MADM with neutrosophic cubic set is yet to appear in the literature. It is an open area of research in neutrosophic cubic set environment.

The present paper is devoted to develop GRA method for MADM in neutrosophic cubic set environment. The attribute weights are described by single valued neutrosophic sets. Positive and negative grey relational coefficients are determined. We define ideal grey relational coefficients and relative closeness coefficients in neutrosophic cubic set environment. The ranking of alternatives is made in descending order.

The rest of the paper is designed as follows: In Section 2, some relevant definitions and properties are recalled. Section 3 presents MADM in neutrosophic cubic set environment based on GRA. In Section 4, a numerical example is solved to illustrate the proposed approach. Section 5 presents conclusions and future scope of research.

2 Preliminaries

In this section, we recall some established definitions and properties which are connected in the present article.

2.1 Definition (Fuzzy set) [6]

Let W be a universal set. Then a fuzzy set F over W can be defined by F={<w, $\mu_F(w)$ > : w $\in W$} where $\mu_F(w)$: $W \rightarrow [0, 1]$is called membership function of F and $\mu_F(w)$ is the degree of membership to which w $\in$ F.

2.2 Definition (Interval valued fuzzy set) [52]

Let W be a universal set. Then, an interval valued fuzzy set F over W is defined by F = $\{[F^-(w), F^+(w)]/w : w \in W\}$, where $F^-(w)$ and $F^+(w)$ are referred to as the lower and upper degrees of membership w $\in W$ where

$0 \le F^-(w) + F^+(w) \le 1$, respectively.

2.3 Definition (Cubic set) [46]

Let W be a non-empty set. A cubic set C in W is of the form c = $\{w, F(w), \lambda(w))/w \in W\}$ where F is an interval valued fuzzy set in W and λ is a fuzzy set in W.

2.4 Definition (Neutrosophic set (NS)) [10]

Let W be a space of points (objects) with generic element w in W. A neutrosophic set N in W is denoted by N= {< w: $T_N(w)$, $I_N(w)$, $F_N(w)$>: w$\in W$} where T_N, I_N, F_N represent membership, indeterminacy and non-membership function respectively. T_N, I_N, F_N can be defined as follows:

$T_N : W \rightarrow]^-0, 1^+[$

$I_N : W \rightarrow]^-0, 1^+[$

$F_A : W \rightarrow]^-0, 1^+[$

Here, $T_N(w)$, $I_N(w)$, $F_N(w)$ are the real standard and non-standard subset of $]^-0, 1^+[$ and

$^-0 \le T_N(w)+I_N(w)+F_N(w) \le 3^+$.

2.5 Definition (Complement of neutrosophic set) [10]

The complement of a neutrosophic set N is denoted by N′and defined as

$N' = \{<w: T_{N'}(w), I_{N'}(w), F_{N'}(w)>, w \in W\}$

$T_{N'}(w) = \{1^+\} - T_N(w)$

$I_{N'}(w) = \{1^+\} - I_N(w)$

$F_{N'}(w) = \{1^+\} - F_N(w)$

2.6 Definition (Containment) [10, 20]

A neutrosophic set P is contained in the other neutrosophic set Q, $P \subseteq Q$, if and only if

$\inf(T_P) \leq \inf(T_Q), \sup(T_P)) \leq \sup((T_Q)$,

$\inf(I_P) \geq \inf(I_Q), \sup(I_P)) \geq \sup((I_Q)$.

$\inf(F_P) \geq \inf(F_Q), \sup(F_P)) \geq \sup((F_Q)$.

2.7 Definition (Union) [10]

The union of two neutrosophic sets P and Q is a neutrosophic set R, written as $R = P \cup Q$, whose truth-membership, indeterminacy-membership and falsity membership functions are related to those of P and Q by

$T_R(w) = T_P(w) + T_Q(w) - T_P(w) \times T_Q(w)$,

$I_R(w) = I_P(w) + I_Q(w) - I_P(w) \times I_Q(w)$,

$F_R(w) = F_P(w) + F_Q(w) - F_P(w) \times F_Q(w)$, for all $w \in W$.

2.8 Definition (Intersection) [10]

The intersection of two neutrosophic sets P and Q is a neutrosophic set C, written as $R = P \cup Q$, whose truth-membership, indeterminacy-membership and falsity-membership functions are related to those of P and Q by

$T_R(w) = T_P(w) \times T_Q(w)$,

$I_R(w) = I_P(w) \times I_Q(w)$,

$F_R(w) = F_P(w) \times F_Q(w)$, for all $w \in W$.

2.9 Definition (Hamming distance) [20, 53]

Let $P = \{\langle w_i : T_P(w_i), I_P(w_i), F_P(w_i)\rangle, i = 1, 2, ..., n\}$ and $Q = \{\langle w_i : T_P(w_i), I_Q(w_i), F_Q(w_i)\rangle, i = 1, 2, ..., n\}$ be any two neutrosophgic sets. Then the Hamming distance between P and Q can be defined as follows:

$$d(P,Q) = \sum_{i=1}^{n}(|T_P(w_i) - T_Q(w_i)| + |I_P(w_i) - I_Q(w_i)| + |F_P(w_i) - F_Q(w_i)|) \quad (1)$$

2.10 Definition (Normalized Hamming distance)

The normalized Hamming distance between two SVNSs, A and B can be defined as follows:

$${}_N d(P,Q) = \frac{1}{3n}\sum_{i=1}^{n}(|T_P(w_i) - T_Q(w_i)| + |I_P(w_i) - I_Q(w_i)| + |F_P(w_i) - F_Q(w_i)|) \quad (2)$$

2.11 Definition (Interval neutrosophic set) [51]

Let W be a non-empty set. An interval neutrosophic set (INS) P in W is characterized by the truth-membership function P_T, the indeterminacy-membership function P_I and the falsity-membership function P_F. For each point $w \in W$, $P_T(w)$, $P_I(w), P_F(w)) \subseteq [0,1]$. Here P can be presented as follows:

$P = \{< w, [P_T^L(w), P_T^U(w)], [P_I^L(w), P_I^U(w)], [P_F^L(w), P_F^U(w)] > : w \in W\}$.

2.12 Definition (Neutrosophic cubic set) [44, 45]

Let W be a set. A neutrosophic cubic set (NCS) in W is a pair (P, Λ) where $P = \{\langle w, P_T(w), P_I(w), P_F(w)\rangle / w \in W\}$ is an interval neutrosophic set in W and $\Lambda = \{\langle w, \lambda_T(w), \lambda_I(w), \lambda_F(w)\rangle / w \in W\}$ is a neutrosophic set in W.

3 GRA for MADM in neutrosophic cubic set environment

We consider a MADM problem with r alternatives $\{A_1, A_2, \ldots, A_r\}$ and s attributes $\{C_1, C_2, \ldots, C_s\}$. Every attribute is not equally important to decision maker. Decision maker provides the neutrosophic weights for each attribute. Let $W = \{w_1, w_2, ..., w_s\}^T$ be the neutrosophic weights of the attributes.

Step 1 Construction of decision matrix

Step1.The decision matrix (see Table 1) is constructed as follows:

Table 1: Decision matrix

$$A = (a_{ij})_{r \times s} = \begin{pmatrix} & C_1 & C_2 & \ldots & C_s \\ A_1 & (A_{11}, \Lambda_{11}) & (A_{12}, \Lambda_{12}) & \ldots & (A_{1s}, \Lambda_{1s}) \\ A_2 & (A_{21}, \Lambda_{21}) & (A_{22}, \Lambda_{22}) & \ldots & (A_{2s}, \Lambda_{2s}) \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ A_r & (A_{r1}, \Lambda_{r1}) & (A_{r2}, \Lambda_{r2}) & \ldots & (A_{rs}, \Lambda_{rs}) \end{pmatrix}_{r \times s}$$

Here $a_{ij} = (A_{ij}, \Lambda_{ij})$, $A_{ij} = ([T_{ij}^L, T_{ij}^U], [I_{ij}^L, I_{ij}^U], [F_{ij}^L, F_{ij}^U])$, $\Lambda_{ij} = (T_{ij}, I_{ij}, F_{ij})$, a_{ij} means the rating of alternative A_i with respect to the attribute C_j. Each weight component w_j of attribute C_j has been taken as neutrosophic set and

$w_j = (T_j, I_j, F_j)$, $A_{ij} = ([T_{ij}^L, T_{ij}^U], [I_{ij}^L, I_{ij}^U], [F_{ij}^L, F_{ij}^U])$

are interval neutrosophic set and $\Lambda_{ij} = (T_{ij}, I_{ij}, F_{ij})$ is a neutrosophic set.

Step 2 Crispification of neutrosophic weight set

Let $w_j = (T_j, I_j, F_j)$ be the j – th neutrosophic weight for the attribute C_j. The equivalent crisp weight of C_j is defined as follows:

$$w_j^c = \frac{\sqrt{T_j^2 + I_j^2 + F_j^2}}{\sum_{j=1}^{n} \sqrt{T_j^2 + I_j^2 + F_j^2}} \text{ and } \sum_{j=1}^{s} w_j^c = 1 .$$

Step 3 Conversion of interval neutrosophic set into neutrosophic set decision matrix

In the decision matrix (1), each $A_{ij} = ([T_{ij}^L, T_{ij}^U],[I_{ij}^L, I_{ij}^U],[F_{ij}^L, F_{ij}^U])$ is an INS. Taking mid value of each interval the decision matrix reduces to single valued neutrosophic decision matrix (See Table 2).

Table 2: Neutrosophic decision matrix

$$M=(m_{ij})_{r\times s} = \begin{pmatrix} & C_1 & C_2 & \ldots & C_s \\ A_1 & (M_{11},\Lambda_{11}) & (M_{12},\Lambda_{12}) & \ldots & (M_{1s},\Lambda_{1s}) \\ A_2 & (M_{21},\Lambda_{21}) & (M_{22},\Lambda_{22}) & \ldots & (M_{2s},\Lambda_{2s}) \\ . & . & . & . & . \\ . & . & . & . & . \\ . & . & . & . & . \\ A_r & (M_{r1},\Lambda_{r1}) & (M_{r2},\Lambda_{r2}) & \ldots & (M_{rs},\Lambda_{rs}) \end{pmatrix}_{r\times s}$$

where each $m_{ij} = (M_{ij}, \Lambda_{ij})$ and

$$M_{ij} = \left(\frac{T_{ij}^L + T_{ij}^U}{2}, \frac{I_{ij}^L + I_{ij}^U}{2}, \frac{F_{ij}^L + F_{ij}^U}{2}\right) = (T_{ij}^m, I_{ij}^m, F_{ij}^m).$$

Step 4 Some definitions of GRA method for MADM with NCS

The GRA method for MADM with NCS can be presented in the following steps:

Step 4.1 Definition:

The ideal neutrosophic estimates reliability solution (INERS) can be denoted as

$(M^+, \Lambda^+) = [(M_1^+, \Lambda_1^+), (M_2^+, \Lambda_2^+), \ldots, (M_q^+, \Lambda_q^+)]$

and defined as $M_j^+ = (T_j^+, I_j^+, F_j^+)$, where $T_j^+ = \max_i T_{ij}^m$, $I_j^{m+} = \min_i I_{ij}^m$, $F_j^{m+} = \min_i F_{ij}^m$ and $\Lambda_j^+ = (T_j^+, I_j^+, F_j^+)$ where $T_j^+ = \max_i T_{ij}, I_j^+ = \min_i I_{ij}, F_j^+ = \min_i F_{ij}$ in the neutrosophic cubic decision matrix $M = (m_{ij})_{p\times q}$, i = 1,2,...,r and j = 1, 2, ..., s.

Step 4.2 Definition:

The ideal neutrosophic estimates unreliability solution (INEURS) can be denoted as

$(M^-, \Lambda^-) = [(M_1^-, \Lambda_1^-), (M_2^-, \Lambda_2^-), \ldots, (M_s^-, \Lambda_s^-)]$

and defined as $M_j^- = (T_j^{m-}, I_j^{m-}, F_j^{m-})$ where $T_j^{m-} = \min_i T_{ij}^m$, $I_j^{m-} = \max_i I_{ij}^m$, $F_j^{m-} = \max_i F_{ij}^m$ and $\Lambda_j^- = (T_j^-, I_j^-, F_j^-)$ where $T_j^- = \min_i T_{ij}, I_j^- = \max_i I_{ij}, F_j^- = \max_i F_{ij}$ in the neutrosophic cubic decision matrix $M = (m_{ij})_{r\times s}$, i = 1,2,...,r and j = 1, 2, ...,s.

Step 4.3 Definition:

The grey relational coefficients of each alternative from INERS can be defined as:

$$(\eta_{ij}^+, \xi_{ij}^+) = \left(\frac{\min_i \min_j \delta_{ij}^+ + \lambda \max_i \max_j \delta_{ij}^+}{\delta_{ij}^+ + \lambda \max_i \max_j \delta_{ij}^+}, \frac{\min_i \min_j \Omega_{ij}^+ + \lambda \max_i \max_j \Omega_{ij}^+}{\Omega_{ij}^+ + \lambda \max_i \max_j \Omega_{ij}^+}\right)$$

Here,

$$\delta_{ij}^+ = d(M_j^+, M_{ij}) = \sum_{i=1}^{r} \left(|T_j^{m+} - T_{ij}^m| + |I_j^{m+} - I_{ij}^m| + |F_j^{m+} - F_{ij}^m|\right)$$

and $\Omega_{ij}^+ = d(\Lambda_j^+, \Lambda_{ij}) = \sum_{i=1}^{r} \left(|T_j^+ - T_{ij}| + |I_j^+ - I_{ij}| + |F_j^+ - F_{ij}|\right)$,

i = 1, 2 ,..., r and j = 1, 2, ..., s, $\lambda \in [0,1]$.

We call $(\eta_{ij}^+, \xi_{ij}^+)$ as positive grey relational coefficient.

Step 4.4 Definition:

The grey relational coefficient of each alternative from INEURS can be defined as:

$$(\eta_{ij}^-, \xi_{ij}^-) = \left(\frac{\min_i \min_j \delta_{ij}^- + \lambda \max_i \max_j \delta_{ij}^-}{\delta_{ij}^- + \lambda \max_i \max_j \delta_{ij}^-}, \frac{\min_i \min_j \Omega_{ij}^- + \lambda \max_i \max_j \Omega_{ij}^-}{\Omega_{ij}^- + \lambda \max_i \max_j \Omega_{ij}^-}\right)$$

Here,

$$\delta_{ij}^- = d(M_j^-, M_{ij}) = \sum_{i=1}^{r} \left(|T_j^{m-} - T_{ij}^m| + |I_j^{m-} - I_{ij}^m| + |F_j^{m-} - F_{ij}^m|\right)$$

and:

$\Omega_{ij}^- = d(\Lambda_j^-, \Lambda_{ij}) = \sum_{i=1}^{r} \left(|T_j^- - T_{ij}| + |I_j^- - I_{ij}| + |F_j^- - F_{ij}|\right)$, i = 1, 2,..., r and j = 1, 2, ..., s, $\lambda \in [0,1]$.

We call $(\eta_{ij}^-, \xi_{ij}^-)$ as negative grey relational coefficient.

λ is called distinguishable coefficient or identification coefficient and it is used to reflect the range of comparison environment that controls the level of differences of the grey relational coefficient. $\lambda = 0$ indicates comparison environment disappears and $\lambda = 1$ indicates comparison environment is unaltered. Generally, $\lambda = 0.5$ is assumed for decision making.

Step 4.5 Calculation of weighted grey relational coefficients for MADM with NCS

We can construct two $r \times s$ order matrices namely $M^{+}_{GR} = \left(\eta^{+}_{ij}, \xi^{+}_{ij}\right)_{r \times s}$ and $M^{-}_{GR} = \left(\eta^{-}_{ij}, \xi^{-}_{ij}\right)_{r \times s}$. The crisp weight is to be multiplied with the corresponding elements of M^{+}_{GR} and M^{-}_{GR} to obtain weighted matrices ${}_{w}M^{+}_{GR}$ and ${}_{w}M^{-}_{GR}$ and defined as:

$${}_{w}M^{+}_{GR} = \left(w^{c}_{j}\eta^{+}_{ij}, w^{c}_{j}\xi^{+}_{ij}\right)_{r \times s} = \left(\tilde{\eta}^{+}_{ij}, \tilde{\xi}^{+}_{ij}\right)_{r \times s}$$

and $${}_{w}M^{-}_{GR} = \left(w^{c}_{j}\eta^{-}_{ij}, w^{c}_{j}\xi^{-}_{ij}\right)_{r \times s} = \left(\tilde{\eta}^{-}_{ij}, \tilde{\xi}^{-}_{ij}\right)_{r \times s}$$

Step 4.6

From the definition of grey relational coefficient, it is clear that grey relational coefficients of both types must be less than equal to one. This claim is going to be proved in the following theorems.

Theorem 1
The positive grey relational coefficient is less than unity i.e. $\eta^{+}_{ij} \le 1$, and $\xi^{+}_{ij} \le 1$.

Proof:
From the definition

$$\eta^{+}_{ij} = \frac{\min_i \min_j \delta^{+}_{ij} + \lambda \max_i \max_j \delta^{+}_{ij}}{\delta^{+}_{ij} + \lambda \max_i \max_j \delta^{+}_{ij}}$$

Now, $\min_i \min_j \delta^{+}_{ij} \le \delta^{+}_{ij}$

$$\Rightarrow \min_i \min_j \delta^{+}_{ij} + \lambda \max_i \max_j \delta^{+}_{ij} \le \delta^{+}_{ij} + \lambda \max_i \max_j \delta^{+}_{ij}$$

$$\Rightarrow \frac{\min_i \min_j \delta^{+}_{ij} + \lambda \max_i \max_j \delta^{+}_{ij}}{\delta^{+}_{ij} + \lambda \max_i \max_j \delta^{+}_{ij}} \le 1$$

$\Rightarrow \eta^{+}_{ij} \le 1$

Again, from the definition, we can write:

$$\xi^{+}_{ij} = \frac{\min_i \min_j \Omega^{+}_{ij} + \lambda \max_i \max_j \Omega^{+}_{ij}}{\Omega^{+}_{ij} + \lambda^{+} \max_i \max_j \Omega^{+}_{ij}}$$

Now, $\min_i \min_j \Omega^{+}_{ij} \le \Omega^{+}_{ij}$

$$\Rightarrow \min_i \min_j \Omega^{+}_{ij} + \lambda \max_i \max_j \Omega^{+}_{ij} \le \Omega^{+}_{ij} + \lambda \max_i \max_j \Omega^{+}_{ij}$$

$$\Rightarrow \xi^{+}_{ij} = \frac{\min_i \min_j \Omega^{+}_{ij} + \lambda \max_i \max_j \Omega^{+}_{ij}}{\Omega^{+}_{ij} + \lambda \max_i \max_j \Omega^{+}_{ij}}$$

$\Rightarrow \xi^{+}_{ij} \le 1$.

Theorem 2
The negative grey relational coefficient is less than unity i.e. $\eta^{-}_{ij} \le 1$, $\xi^{-}_{ij} \le 1$.

Proof:

From the definition, we can write

$$\eta^{-}_{ij} = \frac{\min_i \min_j \delta^{-}_{ij} + \lambda \max_i \max_j \delta^{-}_{ij}}{\delta^{-}_{ij} + \lambda \max_i \max_j \delta^{-}_{ij}}$$

Now, $\min_i \min_j \delta^{-}_{ij} \le \delta^{-}_{ij}$

$$\Rightarrow \min_i \min_j \delta^{-}_{ij} + \lambda \max_i \max_j \delta^{-}_{ij} \le \delta^{-}_{ij} + \lambda \max_i \max_j \delta^{-}_{ij}$$

$$\eta^{-}_{ij} = \frac{\min_i \min_j \delta^{-}_{ij} + \lambda \max_i \max_j \delta^{-}_{ij}}{\delta^{-}_{ij} + \lambda \max_i \max_j \delta^{-}_{ij}}$$

$\Rightarrow \eta^{-}_{ij} \le 1$

Again, from the definition

$$\xi^{-}_{ij} = \frac{\min_i \min_j \Omega^{-}_{ij} + \lambda \max_i \max_j \Omega^{-}_{ij}}{\Omega^{-}_{ij} + \lambda \max_i \max_j \Omega^{-}_{ij}}$$

Now, $\min_i \min_j \Omega^{-}_{ij} \le \Omega^{-}_{ij}$

$$\Rightarrow \min_i \min_j \Omega^{-}_{ij} + \lambda \max_i \max_j \Omega^{-}_{ij} \le \Omega^{-}_{ij} + \lambda \max_i \max_j \Omega^{-}_{ij}$$

$$\Rightarrow \xi^{-}_{ij} = \frac{\min_i \min_j \Omega^{-}_{ij} + \lambda \max_i \max_j \Omega^{-}_{ij}}{\Omega^{-}_{ij} + \lambda \max_i \max_j \Omega^{-}_{ij}}$$

$\Rightarrow \xi^{-}_{ij} \le 1$.

Note 1:

i. Since $\eta^{+}_{ij} \le 1$, $w^{c}_{j} \le 1$ then $\eta^{+}_{ij} w^{c}_{j} \le 1 \Rightarrow \tilde{\eta}^{+}_{ij} \le 1$

ii. Since $\eta^{-}_{ij} \le 1$, $w^{c}_{j} \le 1$ then $\eta^{-}_{ij} w^{c}_{j} \le 1 \Rightarrow \tilde{\eta}^{-}_{ij} \le 1$

iii. Since $\xi^{+}_{ij} \le 1$, $w^{c}_{j} \le 1$ then $\xi^{+}_{ij} w^{c}_{j} \le 1 \Rightarrow \tilde{\xi}^{+}_{ij} \le 1$

iv. Since $\xi^{-}_{ij} \le 1$, $w^{c}_{j} \le 1$ then $\xi^{-}_{ij} w^{c}_{j} \le 1 \Rightarrow \tilde{\xi}^{-}_{ij} \le 1$

Step 4.7

We define the ideal or standard grey relational coefficient as (1, 1). Then we construct ideal grey relational coefficient matrix of order $r \times s$ (see Table 3).

Table 3: Ideal grey relational coefficient matrix of order $r \times s$

$$I = \begin{pmatrix} (1,1)\,(1,1)\ldots(1,1) \\ (1,1)\,(1,1)\ldots(1,1) \\ \ldots\ldots\ldots\ldots\ldots \\ (1,1)\,(1,1)\ldots(1,1) \end{pmatrix}_{r \times s}$$

Step 5 Determination of Hamming distances

We find the distance d^{+}_{i} between the corresponding elements of i-th row of I and ${}_{w}M^{+}_{GR}$ by employing Hamming distance. Similarly, d_i^- can be determined between I and $_wM_{GR}^-$ by employing Hamming distance as follows:

$$d_i^+ = \frac{1}{2s}[\sum_{j=1}^{s}\{|1-\tilde{\eta}_{ij}^+| + |1-\tilde{\xi}_{ij}^+|\}], \ i = 1, 2, \ldots, r.$$

$$d_i^- = \frac{1}{2s}[\sum_{j=1}^{s}\{|1-\tilde{\eta}_{ij}^-| + |1-\tilde{\xi}_{ij}^-|\}], \ i = 1, 2, \ldots, r.$$

Step 6 Determination of relative closeness coefficient

The relative closeness coefficient can be calculated as:

$$\Delta_i = \frac{d_i^+}{d_i^+ + d_i^-} \quad i = 1, 2, ..., r.$$

Step 7 Ranking the alternatives

According to the relative closeness coefficient, the ranking order of all alternatives is determined. The ranking order is made according to descending order of relative closeness coefficients.

4 Numerical example

Consider a hypothetical MADM problem. The problem consists of single decision maker, three alternatives with three alternatives $\{A_1, A_2, A_3\}$ and four attributes $\{C_1, C_2, C_3, C_4\}$. The solution of the problem is presented using the following steps:

Step 1. Construction of neutrosophic cubic decision matrix

The decison maker forms the decision matrix which is displayed in the Table 4, at the end of article.

Step 2. Crispification of neutrosophic weight set

The neutrosophic weights of the attributes are taken as:

$$W = \{(0.5, 0.2, 0.1), (0.6, 0.1, 0.1), (0.9, 0.2, 0.1), (0.6, 0.3, 0.4)\}^T$$

The equivalent crisp weights are

$$W^c = \{(0.1907), (0.2146), (0.3228), (0.2719)\}^T$$

Step 3 Conversion of interval neutrosophic set into neutrosophic set in decision matrix

Taking the mid value of INS in the Table 4, the new decision matrix is presented in the following Table 5, at the end of article.

Step 4 Some Definitions of GRA method for MADM with NCS

The ideal neutrosophic estimates reliability solution (INERS) (M^+, Λ^+) and the ideal neutrosophic estimates unreliability solution (INEURS) (M^-, Λ^-) are presented in the Table 6, at the end of article.

$\delta^+ = (\delta_{ij}^+) = (d(M_j^+, M_{ij})) \forall i, j$ is presented as below:

$$\delta^+ = \begin{pmatrix} 0.85 & 0.95 & 0.05 & 0.15 \\ 0.65 & 0 & 0.7 & 0.25 \\ 0.05 & 0.15 & 0.25 & 0.45 \end{pmatrix}$$

The $\Omega^+ = (\Omega_{ij}^+) = (d(\Lambda_j^+, \Lambda_{ij})) \forall i, j$ is presented as below:

$$\Omega^+ = \begin{pmatrix} 0.45 & 1.2 & 0.4 & 0.15 \\ 0.05 & 0.5 & 0.2 & 0.2 \\ 0.25 & 0.3 & 0.2 & 0.5 \end{pmatrix}$$

$\delta^- = (\delta_{ij}^-) = (d(M_j^-, M_{ij})) \forall i, j$ is presented as below:

$$\delta^- = \begin{pmatrix} 0.25 & 0.3 & 0.7 & 0.55 \\ 0.45 & 1.2 & 0 & 0.45 \\ 1.05 & 0.65 & 0.6 & 0.25 \end{pmatrix}$$

The $\Omega^- = (\Omega_{ij}^-) = (d(\Lambda_j^-, \Lambda_{ij})) \forall i, j$ is presented as:

The positive grey relational coefficient $M_{GR}^+ = (\eta_{ij}^+, \xi_{ij}^+)_{3\times4}$ is presented in the Table 7, at the end of article.

The negative grey relational coefficient $M_{GR}^- = (\eta_{ij}^-, \xi_{ij}^-)_{3\times4}$ is presented in the Table 8, at the end of article.

Now, we multiply the crisp weight with the corresponding elements of M_{GR}^+ and M_{GR}^- to get weighted matrices $_wM_{GR}^+$ and $_wM_{GR}^-$ and which are described in the Table 9 and 10 respectively, at the end of article.

Step 5 Determination of Hamming distances

Hamming distances are calculated as follows:

$d_1^+ = 0.84496$, $d_1^- = 0.83845625$,

$d_2^+ = 0.82444375$, $d_2^- = 0.85328875$,

$d_3^+ = 0.82368675$, $d_3^- = 0.85277$.

Step 6 Determination of relative closeness coefficient

The relative closeness coefficients are calculated as:

$$\Delta_1 = \frac{d_1^+}{d_1^+ + d_1^-} = 0.501932$$

$$\Delta_2 = \frac{d_2^+}{d_2^+ + d_2^-} = 0.491403576$$

$$\Delta_3 = \frac{d_3^+}{d_3^+ + d_3^-} = 0.49132$$

Step 7 Ranking the alternatives

The ranking of alternatives is made according to descending order of relative closeness coefficients. The ranking order is shown in the Table 11 below.

Alternatives	Ranking order
A_3	1
A_2	2
A_1	3

Conclusion

This paper develops GRA based MADM in neutrosophic cubic set environment. This is the first approach of GRA in MADM in neutrosophic cubic set environment. The proposed approach can be applied to other decision making problems such as pattern recognition, personnel selection, etc.

The proposed approach can be applied for decision making problem described by internal NCSs and external NCSs. We hope that the proposed approach will open up a new avenue of research in newly developed neutrosophic cubic set environment.

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Table 4: Construction of neutrosophic cubic decision matrix

$A = (a_{ij})_{3\times 4} =$

	C_1	C_2	C_3	C_4
A_1	(([0.2, 0.3],[0.3, 0.5],[0.2, 0.5]), (0.3, 0.2, 0.3))	([0.1, 0.3],[0.2, 0.4],[0.3, 0.6]), (0.2, 0.5, 0.4))	(([0.6, 0.9],[0.1, 0.2],[0, 0.2]), (0.4, 0.5, 0.1))	(([0.4, 0.7],[0.1, 0.3],[0.2, 0.3]), (0.7, 0.3, 0.2))
A_2	([0.6, 0.8],[0.4, 0.6],[0.3, 0.7]), (0.5, 0.2, 0.1))	([0.7, 0.9],[0.2, 0.3],[0.1, 0.3]), (0.7, 0.3, 0.3))	(([0.5, 0.7],[0.4, 0.6],[0.3, 0.5]), (0.4, 0.1, 0.2))	(([0.4, 0.5],[0.1, 0.3],[0.2, 0.3]), (0.6, 0.2, 0.1))
A_3	(([0.4, 0.9],[0.1, 0.4],[0, 0.2]), (0.25, 0.15, 0.1))	(([0.8, 0.9],[0.4, 0.7],[0.4, 0.6]), (0.8, 0.1, 0.2))	(([0.6, 0.9],[0.1, 0.3],[0, 0.3]), (0.5, 0.4, 0.3))	(([0.6, 0.8],[0.5, 0.7],[0.2, 0.4]), (0.5, 0.1, 0.4))

Table 5: Construction of neutrosophic decision matrix

$M = (m_{ij})_{3\times 4} =$

	C_1	C_2	C_3	C_4
A_1	((0.25, 0.4, 0.35), (0.3, 0.2, 0.3))	((0.2, 0.3, 0.45), (0.2, 0.5, 0.4))	((0.75, 0.15, 0.1), (0.4, 0.5, 0.1))	((0.55, 0.2, 0.25), (0.7, 0.3, 0.2))
A_2	((0.7,0.5, 0.5), (0.5, 0.2, 0.1))	((0.8, 0.25, 0.2), (0.7, 0.3, 0.3))	((0.6, 0.5, 0.4), (0.4, 0.1, 0.2))	((0.45, 0.2, 0.25), (0.6, 0.2, 0.1))
A_3	((0.65, 0.25, 0.1), (0.25, 0.15, 0.1))	((0.85, 0.55, 0.5), (0.8, 0.1, 0.2))	((0.75, 0.2, 0.15), (0.5, 0.4, 0.3))	((0.7, 0.6, 0.3), (0.5, 0.1, 0.4))

Table 6: The ideal neutrosophic estimates reliability solution (INERS) (M^+, Λ^+) and the ideal neutrosophic estimates unreliability solution (INEURS) (M^-, Λ^-)

(M^+, Λ^+)	((0.7,0.25,0.1), (0.5,0.15,0.1))	((0.85,0.25,0.2), (0.8,0.1,0.2))	((0.75,0.15,0.1), (0.5,0.1,0.1))	((0.7,0.2,0.25), (0.7,0.1,0.1))
(M^-, Λ^-)	((0.25,0.5,0.5), (0.25,0.2,0.3))	((0.2,0.55,0.5), (0.2,0.5,0.4))	((0.6,0.5,0.4), (0.4,0.5,0.3))	((0.45,0.6,0.3), (0.5,0.3,0.4))

Table 7: The positive grey relational coefficient $M_{GR}^{+} = \left(\eta_{ij}^{+}, \xi_{ij}^{+}\right)_{3\times 4}$

$$M_{GR}^{+} = \begin{pmatrix} (0.3585, 0.6190) & (0.333, 0.3611) & (0.9048, 0.65) & (0.76, 0.7222) \\ (0.4222, \quad 1 \quad) & (1, 0.5909) & (0.4042, 0.8125) & (0.6552, 0.8125) \\ (0.9048, 0.7647) & (0.76, 0.7222) & (0.6552, 0.8125) & (0.5135, 0.5909) \end{pmatrix}$$

Table 8: The negative grey relational coefficient $M_{GR}^{-} = \left(\eta_{ij}^{-}, \xi_{ij}^{-}\right)_{3\times 4}$

$$M_{GR}^{-} = \begin{pmatrix} (0.7059, 0.5454) & (0.6667, 1) & (0.4615, 0.75) & (0.5217, 0.6) \\ (0.5714, 0.5714) & (0.3333, 0.4286) & (1, 0.5454) & (0.5714, 0.5454) \\ (0.3636, 0.7059) & (0.48, 0.3333) & (0.5, 0.75) & (0.7059, 0.75) \end{pmatrix}$$

Table 9: Weighted matrix ${}_{W}M_{GR}^{+}$ ${}_{W}M_{GR}^{+} =$

$$\begin{pmatrix} (0.06836, 0.11804) & (0.07153, 0.07749) & (0.29207, 0.20982) & (0.20664, 0.19637) \\ (0.08051, 0.1907) & (0.2146, 0.12681) & (0.13048, 0.26228) & (0.17815, 0.22092) \\ (0.17252, 0.14583) & (0.163096, 0.15498) & (0.21150, 0.26228) & (0.13962, 0.16066) \end{pmatrix}$$

Table 10: Weighted matrix ${}_{W}M_{GR}^{-}$

$${}_{W}M_{GR}^{-} = \begin{pmatrix} (0.13461, 0.10401) & (0.14307, 0.2146) & (0.14897, 0.2421) & (0.14185, 0.16314) \\ (0.10896, 0.10896) & (0.07153, 0.08173) & (0.3228, 0.17606) & (0.15536, 0.14829) \\ (0.06934, 0.13461) & (0.10301, 0.07153) & (0.1614, 0.2421) & (0.19193, 0.20392) \end{pmatrix}$$

Bipolar Neutrosophic Projection Based Models for Solving Multi-Attribute Decision Making Problems

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Abstract. Bipolar neutrosophic sets are the extension of neutrosophic sets and are based on the idea of positive and negative preferences of information. Projection measure is a useful apparatus for modelling real life decision making problems. In the paper, we define projection, bidirectional projection and hybrid projection measures between bipolar neutrosophic sets. Three new methods based on the proposed projection measures are developed for solving multi-attribute decision making problems. In the solution process, the ratings of performance values of the alternatives with respect to the attributes are expressed in terms of bipolar neutrosophic values. We calculate projection, bidirectional projection, and hybrid projection measures between each alternative and ideal alternative with bipolar neutrosophic information. All the alternatives are ranked to identify the best alternative. Finally, a numerical example is provided to demonstrate the applicability and effectiveness of the developed methods. Comparison analysis with the existing methods in the literature in bipolar neutrosophic environment is also performed.

Keywords: Bipolar neutrosophic sets; projection measure; bidirectional projection measure; hybrid projection measure; multi-attribute decision making.

1 Introduction

For describing and managing indeterminate and inconsistent information, Smarandache [1] introduced neutrosophic set which has three independent components namely truth membership degree (*T*), indeterminacy membership degree (*I*) and falsity membership degree (*F*) where *T*, *I*, and *F* lie in$]^-0, 1^+[$. Later, Wang et al. [2] proposed single valued neutrosophic set (SVNS) to deal real decision making problems where *T*, *I*, and *F* lie in [0, 1].

Zhang [3] grounded the notion of bipolar fuzzy sets by extending the concept of fuzzy sets [4]. The value of membership degree of an element of bipolar fuzzy set belongs to [-1, 1]. With reference to a bipolar fuzzy set, the membership degree zero of an element reflects that the element is irrelevant to the corresponding property, the membership degree belongs to (0, 1] of an element reflects that the element somewhat satisfies the property, and the membership degree belongs to [−1,0) of an element reflects that the element somewhat satisfies the implicit counter-property.

Deli et al. [5] extended the concept of bipolar fuzzy set to bipolar neutrosophic set (BNS). With reference to a bipolar neutrosophic set *Q*, the positive membership degrees $T_Q^+(x)$, $I_Q^+(x)$, and $F_Q^+(x)$ represent respectively the truth membership, indeterminate membership and falsity membership of an element $x \in X$ corresponding to the bipolar neutrosophic set *Q* and the negative membership degrees $T_Q^-(x)$, $I_Q^-(x)$, and $F_Q^-(x)$ denote respectively the truth membership, indeterminate membership and false membership degree of an element $x \in X$ to some implicit counter-property corresponding to the bipolar neutrosophic set *Q*.

Projection measure is a useful decision making device as it takes into account the distance as well as the included angle for measuring the closeness degree between two objects [6, 7]. Yue [6] and Zhang et al. [7] studied projection based multi-attribute decision making (MADM) in crisp environment i.e. projections are defined by ordinary numbers or crisp numbers. Yue [8] further investigated a new multi-attribute group decision making (MAGDM) method based on determining the weights of the decision makers by employing projection technique with interval data. Yue and Jia [9] established a methodology for MAGDM based on a new normalized projection measure, in which the attribute values are provided by decision makers in hybrid form with crisp values and interval data.

Xu and Da [10] and Xu [11] studied projection method for decision making in uncertain environment with preference information. Wei [12] discussed a MADM method based on the projection technique, in which the attribute values are presented in terms of intuitionistic fuzzy numbers. Zhang et al. [13] proposed a grey relational projection method for MADM based on intuitionistic trapezoidal fuzzy number. Zeng et al. [14] investigated projections on interval valued intuitionistic fuzzy numbers and developed algorithm to the MAGDM problems with interval-valued intuitionistic fuzzy information. Xu and Hu [15] developed two projection based models for MADM in intuitionistic fuzzy environment and interval valued intuitionistic fuzzy environment. Sun [16] presented a group decision making method based on projection method and score function under interval valued intuitionistic fuzzy environment. Tsao and Chen [17] developed a novel projection based compromising method for multi-criteria decision making (MCDM) method in interval valued intuitionistic fuzzy environment.

In neutrosophic environment, Chen and Ye [18] developed projection based model of neutrosophic numbers and presented MADM method to select clay-bricks in construction field. Bidirectional projection measure [19, 20] considers the distance and included angle between two vectors *x, y.* Ye [19] defined bidirectional projection measure as an improvement of the general projection measure of SVNSs to overcome the drawback of the general projection measure. In the same study, Ye [19] developed MADM method for selecting problems of mechanical design schemes under a single-valued neutrosophic environment. Ye [20] also presented bidirectional projection method for MAGDM with neutrosophic numbers.

Ye [21] defined credibility – induced interval neutrosophic weighted arithmetic averaging operator and credibility – induced interval neutrosophic weighted geometric averaging operator and developed the projection measure based ranking method for MADM problems with interval neutrosophic information and credibility information. Dey et al. [22] proposed a new approach to neutrosophic soft MADM using grey relational projection method. Dey et al. [23] defined weighted projection measure with interval neutrosophic assessments and applied the proposed concept to solve MADM problems with interval valued neutrosophic information. Pramanik et al. [24] defined projection and bidirectional projection measures between rough neutrosophic sets and proposed two new multi-criteria decision making (MCDM) methods based on projection and bidirectional projection measures in rough neutrosophic set environment.

In the field of bipolar neutrosophic environment, Deli et al. [5] defined score, accuracy, and certainty functions in order to compare BNSs and developed bipolar neutrosophic weighted average (BNWA) and bipolar neutrosophic weighted geometric (BNWG) operators to obtain collective bipolar neutrosophic information. In the same study, Deli et al. [5] also proposed a MCDM approach on the basis of score, accuracy, and certainty functions and BNWA, BNWG operators. Deli and Subas [25] presented a single valued bipolar neutrosophic MCDM through correlation coefficient similarity measure. Şahin et al. [26] provided a MCDM method based on Jaccard similarity measure of BNS. Uluçay et al. [27] defined Dice similarity, weighted Dice similarity, hybrid vector similarity, weighted hybrid vector similarity measures under BNSs and developed MCDM methods based on the proposed similarity measures. Dey et al. [28] defined Hamming and Euclidean distance measures to compute the distance between BNSs and investigated a TOPSIS approach to derive the most desirable alternative.

In this study, we define projection, bidirectional projection and hybrid projection measures under bipolar neutrosophic information. Then, we develop three methods for solving MADM problems with bipolar neutrosophic assessments. We organize the rest of the paper in the following way. In Section 2, we recall several useful definitions concerning SVNSs and BNSs. Section 3 defines projection, bidirectional projection and hybrid projection measures between BNSs. Section 4 is devoted to present three models for solving MADM under bipolar neutrosophic environment. In Section 5, we solve a decision making problem with bipolar neutrosophic information on the basis of the proposed measures. Comparison analysis is provided to demonstrate the feasibility and flexibility of the proposed methods in Section 6. Finally, Section 7 provides conclusions and future scope of research.

2 Basic Concepts Regarding SVNSs and BNSs

In this Section, we provide some basic definitions regarding SVNSs, BNSs which are useful for the construction of the paper.

2.1 Single valued neutrosophic sets [2]

Let X be a universal space of points with a generic element of X denoted by x, then a SVNS P is characterized by a truth membership function $T_P(x)$, an indeterminate membership function $I_P(x)$ and a falsity membership function $F_P(x)$. A SVNS P is expressed in the following way.

$$P = \{x, \langle T_P(x), I_P(x), F_P(x) \rangle \mid x \in X\}$$

where, $T_P(x)$, $I_P(x)$, $F_P(x) : X \to [0, 1]$ and $0 \le T_P(x) + I_P(x) + F_P(x) \le 3$ for each point $x \in X$.

2.2 Bipolar neutrosophic set [5]

Consider X be a universal space of objects, then a BNS Q in X is presented as follows:

$$Q = \{x, \langle T_Q^+(x), I_Q^+(x), F_Q^+(x), T_Q^-(x), I_Q^-(x), F_Q^-(x) \rangle \mid x \in X\},$$

where $T_Q^+(x)$, $I_Q^+(x)$, $F_Q^+(x): X \to [0, 1]$ and $T_Q^-(x)$, $I_Q^-(x)$, $F_Q^-(x)$: $X \to [-1, 0]$.The positive membership degrees $T_Q^+(x)$, $I_Q^+(x)$, $F_Q^+(x)$ denote the truth membership, indeterminate membership, and falsity membership functions of an element $x \in X$ corresponding to a BNS Q and the negative membership degrees $T_Q^-(x)$, $I_Q^-(x)$, $F_Q^-(x)$ denote the truth membership, indeterminate membership, and falsity membership of an element $x \in X$ to several implicit counter property associated with a BNS Q. For convenience, a bipolar neutrosophic value (BNV) is presented as $\tilde{q} = < T_Q^+, I_Q^+, F_Q^+, T_Q^-, I_Q^-, F_Q^- >$.

Definition 1 [5]
Let, $Q_1 = \{x, \langle T_{Q_1}^+(x), I_{Q_1}^+(x), F_{Q_1}^+(x), T_{Q_1}^-(x), I_{Q_1}^-(x), F_{Q_1}^-(x) \rangle \mid x \in X\}$ and $Q_2 = \{x, \langle T_{Q_2}^+(x), I_{Q_2}^+(x), F_{Q_2}^+(x), T_{Q_2}^-(x), I_{Q_2}^-(x), F_{Q_2}^-(x) \rangle \mid x \in X\}$ be any two BNSs. Then $Q_1 \subseteq Q_2$ if and only if $T_{Q_1}^+(x) \le T_{Q_2}^+(x)$, $I_{Q_1}^+(x) \le I_{Q_2}^+(x)$, $F_{Q_1}^+(x) \ge F_{Q_2}^+(x)$; $T_{Q_1}^-(x) \ge T_{Q_2}^-(x)$, $I_{Q_1}^-(x) \ge I_{Q_2}^-(x)$, $F_{Q_1}^-(x) \le F_{Q_2}^-(x)$ for all $x \in X$.

Definition 2 [5]
Let, $Q_1 = \{x, \langle T_{Q_1}^+(x), I_{Q_1}^+(x), F_{Q_1}^+(x), T_{Q_1}^-(x), I_{Q_1}^-(x), F_{Q_1}^-(x) \rangle \mid x \in X\}$ and $Q_2 = \{x, \langle T_{Q_2}^+(x), I_{Q_2}^+(x), F_{Q_2}^+(x), T_{Q_2}^-(x), I_{Q_2}^-(x), F_{Q_2}^-(x) \rangle \mid x \in X\}$ be any two BNSs. Then $Q_1 = Q_2$ if and only if $T_{Q_1}^+(x) = T_{Q_2}^+(x)$, $I_{Q_1}^+(x) = I_{Q_2}^+(x)$, $F_{Q_1}^+(x) = F_{Q_2}^+(x)$; $T_{Q_1}^-(x) = T_{Q_2}^-(x)$, $I_{Q_1}^-(x) = I_{Q_2}^-(x)$, $F_{Q_1}^-(x) = F_{Q_2}^-(x)$ for all $x \in X$.

Definition 3 [5]
Let, $Q = \{x, \langle T_Q^+(x), I_Q^+(x), F_Q^+(x), T_Q^-(x), I_Q^-(x), F_Q^-(x) \rangle \mid x \in X\}$ be a BNS. The complement of Q is represented by Q^c and is defined as follows:

$T_{Q^c}^+(x) = \{1^+\} - T_Q^+(x)$, $I_{Q^c}^+(x) = \{1^+\} - I_Q^+(x)$, $F_{Q^c}^+(x) = \{1^+\} - F_Q^+(x)$;

$T_{Q^c}^-(x) = \{1^-\} - T_Q^-(x)$, $I_{Q^c}^-(x) = \{1^-\} - I_Q^-(x)$, $F_{Q^c}^-(x) = \{1^-\} - F_Q^-(x)$.

Definition 4
Let, $Q_1 = \{x, \langle T_{Q_1}^+(x), I_{Q_1}^+(x), F_{Q_1}^+(x), T_{Q_1}^-(x), I_{Q_1}^-(x), F_{Q_1}^-(x) \rangle \mid x \in X\}$ and $Q_2 = \{x, \langle T_{Q_2}^+(x), I_{Q_2}^+(x), F_{Q_2}^+(x), T_{Q_2}^-(x), I_{Q_2}^-(x), F_{Q_2}^-(x) \rangle \mid x \in X\}$ be any two BNSs. Their union $Q_1 \cup Q_2$ is defined as follows:

$Q_1 \cup Q_2$ = {Max ($T_{Q_1}^+(x)$, $T_{Q_2}^+(x)$), Min ($I_{Q_1}^+(x)$, $I_{Q_2}^+(x)$), Min ($F_{Q_1}^+(x)$, $F_{Q_2}^+(x)$), Min ($T_{Q_1}^-(x)$, $T_{Q_2}^-(x)$), Max ($I_{Q_1}^-(x)$, $I_{Q_2}^-(x)$), Max ($F_{Q_1}^-(x)$, $F_{Q_2}^-(x)$)}, $\forall$ $x \in X$.

Their intersection $Q_1 \cap Q_2$ is defined as follows:

$Q_1 \cap Q_2$ = {Min ($T_{Q_1}^+(x)$, $T_{Q_2}^+(x)$), Max ($I_{Q_1}^+(x)$, $I_{Q_2}^+(x)$), Max ($F_{Q_1}^+(x)$, $F_{Q_2}^+(x)$), Max ($T_{Q_1}^-(x)$, $T_{Q_2}^-(x)$), Min ($I_{Q_1}^-(x)$, $I_{Q_2}^-(x)$), Min ($F_{Q_1}^-(x)$, $F_{Q_2}^-(x)$)}, $\forall$ $x \in X$.

Definition 5 [5]
Let $\tilde{q}_1 = < T_{Q_1}^+, I_{Q_1}^+, F_{Q_1}^+, T_{Q_1}^-, I_{Q_1}^-, F_{Q_1}^- >$ and $\tilde{q}_2 = < T_{Q_2}^+, I_{Q_2}^+, F_{Q_2}^+, T_{Q_2}^-, I_{Q_2}^-, F_{Q_2}^- >$ be any two BNVs, then

i. $\beta . \tilde{q}_1 = < 1 - (1 - T_{Q_1}^+)^\beta, (I_{Q_1}^+)^\beta, (F_{Q_1}^+)^\beta, -(-T_{Q_1}^-)^\beta, -(-I_{Q_1}^-)^\beta, -(1 - (1 - (-F_{Q_1}^-))^\beta) >$;

ii. $(\tilde{q}_1)^\beta = < (T_{Q_1}^+)^\beta, 1 - (1 - I_{Q_1}^+)^\beta, 1 - (1 - F_{Q_1}^+)^\beta, -(1 - (1 - (-T_{Q_1}^-))^\beta), -(-I_{Q_1}^-)^\beta, (-F_{Q_1}^-)^\beta >$;

iii. $\tilde{q}_1 + \tilde{q}_2 = < T_{Q_1}^+ + T_{Q_2}^+ - T_{Q_1}^+ . T_{Q_2}^+, I_{Q_1}^+ . I_{Q_2}^+, F_{Q_1}^+ . F_{Q_2}^+, - T_{Q_1}^- . T_{Q_2}^-, -(-I_{Q_1}^- - I_{Q_2}^- - I_{Q_1}^- . I_{Q_2}^-), -(-F_{Q_1}^- - F_{Q_2}^- - F_{Q_1}^- . F_{Q_2}^-) >$;

iv. $\tilde{q}_1 . \tilde{q}_2 = < T_{Q_1}^+ . T_{Q_2}^+, I_{Q_1}^+ + I_{Q_2}^+ - I_{Q_1}^+ . I_{Q_2}^+, F_{Q_1}^+ + F_{Q_2}^+ - F_{Q_1}^+ . F_{Q_2}^+, -(-T_{Q_1}^- - T_{Q_2}^- - T_{Q_1}^- . T_{Q_2}^-), - I_{Q_1}^- . I_{Q_2}^-, - F_{Q_1}^- . F_{Q_2}^- >$ where $\beta > 0$.

3 Projection, bidirectional projection and hybrid projection measures of BNSs

This Section proposes a general projection, a bidirectional projection and a hybrid projection measures for BNSs.

Definition 6
Assume that $X = (x_1, x_2, \ldots, x_m)$ be a finite universe of discourse and Q be a BNS in X, then modulus of Q is defined as follows:

$$\| Q \| = \sqrt{\sum_{j=1}^{m} \alpha_j^2} = \sqrt{\sum_{j=1}^{m} [(T_{Q_j}^+)^2 + (I_{Q_j}^+)^2 + (F_{Q_j}^+)^2 + (T_{Q_j}^-)^2 + (I_{Q_j}^-)^2 + (F_{Q_j}^-)^2]} \quad (1)$$

where $\alpha_j = \langle T_{Q_j}^+(x), I_{Q_j}^+(x), F_{Q_j}^+(x), T_{Q_j}^-(x), I_{Q_j}^-(x), F_{Q_j}^-(x) \rangle$, j = 1, 2, ..., m.

Definition 7 [10, 29]
Assume that $u = (u_1, u_2, \ldots, u_m)$ and $v = (v_1, v_2, \ldots, v_m)$ be two vectors, then the projection of vector u onto vector v can be defined as follows:

$$Proj\ (u)_v = \| u \| \operatorname{Cos}(u, v) = \sqrt{\sum_{j=1}^{m} u_j^2} \times \frac{\sum_{j=1}^{m}(u_j v_j)}{\sqrt{\sum_{j=1}^{m} u_j^2} \times \sqrt{\sum_{j=1}^{m} v_j^2}} = \frac{\sum_{j=1}^{m}(u_j v_j)}{\sqrt{\sum_{j=1}^{m} v_j^2}} \quad (2)$$

where, $Proj\ (u)_v$ represents that the closeness of u and v in magnitude.

Definition 8
Assume that $X = (x_1, x_2, \ldots, x_m)$ be a finite universe of discourse and R, S be any two BNSs in X, then

$$Proj\ (R)_S = \| R \| \operatorname{Cos}(R, S) = \frac{1}{\| S \|}(R.S) \quad (3)$$

is called the projection of R on S, where

$\|R\| = \sqrt{\sum_{i=1}^{m}[(T_R^+)^2(x_i) + (I_R^+)^2(x_i) + (F_R^+)^2(x_i) + (T_R^-)^2(x_i) + (I_R^-)^2(x_i) + (F_R^-)^2(x_i)]}$,

$\|S\| = \sqrt{\sum_{i=1}^{m}[(T_S^+)^2(x_i) + (I_S^+)^2(x_i) + (F_S^+)^2(x_i) + (T_S^-)^2(x_i) + (I_S^-)^2(x_i) + (F_S^-)^2(x_i)]}$,

and $R.S = \sum_{i=1}^{m}[T_R^+(x_i)T_S^+(x) + I_R^+(x_i)I_S^+(x_i) + F_R^+(x_i)F_S^+(x_i) + T_R^-(x_i)T_S^-(x_i) + I_R^-(x_i)I_S^-(x_i) + F_R^-(x_i)F_S^-(x_i)]$.

Example 1. Suppose that $R = <0.5, 0.3, 0.2, -0.2, -0.1, -0.05>$, $S = <0.7, 0.3, 0.1, -0.4, -0.2, -0.3>$ be the two BNSs in X, then the projection of R on S is obtained as follows:

$$Proj\ (R)_S = \frac{1}{\| S \|}(R.S) = \frac{(0.5)(0.7) + (0.3)(0.3) + (0.2)(0.1) + (-0.2)(-0.4) + (-0.1)(-0.2) + (-0.05)(-0.3)}{\sqrt{(0.7)^2 + (0.3)^2 + (0.1)^2 + (-0.4)^2 + (-0.2)^2 + (-0.3)^2}}$$

$= 0.612952$

The bigger value of $Proj\ (R)_S$ reflects that R and S are closer to each other.

However, in single valued neutrosophic environment, Ye [20] observed that the general projection measure cannot describe accurately the degree of α close to β. We also notice that the general projection incorporated by Xu [11] is not reasonable in several cases under bipolar neutrosophic setting, for example let, $\alpha = \beta = <a, a, a, -a, -a, -a>$ and $\gamma = <2a, 2a, 2a, -2a, -2a, -2a>$, then $Proj\ (\alpha)_\beta = 2.44949\ \|a\|$ and $Proj\ (\gamma)_\beta = 4.898979\ \|a\|$. This shows that β is much closer to γ than α which is not true because $\alpha = \beta$. Ye [20] opined that α is equal to β whenever $Proj\ (\alpha)_\beta$ and $Proj\ (\beta)_\alpha$ should be equal to 1. Therefore, Ye [20] proposed an alternative method called bidirectional projection measure to overcome the limitation of general projection measure as given below.

Definition 9 [20]
Consider x and y be any two vectors, then the bidirectional projection between x and y is defined as follows:

$$B\text{-}proj\ (x, y) = \frac{1}{1 + \left| \frac{x.y}{\| x \|} - \frac{x.y}{\| y \|} \right|} = \frac{\| x \| \| y \|}{\| x \| \| y \| + \| x \| - \| y \| | x.y} \quad (4)$$

where $\|x\|$, $\|y\|$ denote the moduli of x and y respectively, and $x.\ y$ is the inner product between x and y.
Here, $B\text{-}Proj\ (x, y) = 1$ if and only if $x = y$ and $0 \le B\text{-}Proj\ (x, y) \le 1$, i.e. bidirectional projection is a normalized measure.

Definition 10
Consider $R = \left\langle T_R^+(x_i), I_R^+(x_i), F_R^+(x_i), T_R^-(x_i), I_R^-(x_i), F_R^-(x_i) \right\rangle$ and $S = \left\langle T_S^+(x_i), I_S^+(x_i), F_S^+(x_i), T_S^-(x_i), I_S^-(x_i), F_S^-(x_i) \right\rangle$ be any two BNSs in $X = (x_1, x_2, \ldots, x_m)$, then the bidirectional projection measure between R and S is defined as follows:

$$B\text{-}Proj\ (R, S) = \frac{1}{1 + \left| \frac{R.S}{\| R \|} - \frac{R.S}{\| S \|} \right|} = \frac{\| R \| \| S \|}{\| R \| \| S \| + | \| R \| - \| S \| | R.S} \quad (5)$$

where

$\|R\| = \sqrt{\sum_{i=1}^{m}[(T_R^+)^2(x_i) + (I_R^+)^2(x_i) + (F_R^+)^2(x_i) + (T_R^-)^2(x_i) + (I_R^-)^2(x_i) + (F_R^-)^2(x_i)]}$,

$\|S\| = \sqrt{\sum_{i=1}^{m}[(T_S^+)^2(x_i) + (I_S^+)^2(x_i) + (F_S^+)^2(x_i) + (T_S^-)^2(x_i) + (I_S^-)^2(x_i) + (F_S^-)^2(x_i)]}$

and $R.S = \sum_{i=1}^{m}[T_R^+(x_i)T_S^+(x) + I_R^+(x_i)I_S^+(x_i) + F_R^+(x_i)F_S^+(x_i) + T_R^-(x_i)T_S^-(x_i) + I_R^-(x_i)I_S^-(x_i) + F_R^-(x_i)F_S^-(x_i)]$.

Proposition 1. Let $B\text{-}Proj\ (R)_S$ be a bidirectional projection measure between any two BNSs R and S, then
1. $0 \le B\text{-}Proj\ (R, S) \le 1$;
2. $B\text{-}Proj\ (R, S) = B\text{-}Proj\ (S, R)$;
3. $B\text{-}Proj\ (R, S) = 1$ for $R = S$.

Proof.
1. For any two non-zero vectors R and S,

$$\frac{1}{1 + \left| \frac{R.S}{\| R \|} - \frac{R.S}{\| S \|} \right|} > 0, \quad \because \frac{1}{1 + x} > 0, when\ \ x > 0$$

$\therefore$ *B*-Proj (*R*, *S*) > 0, for any two non-zero vectors *R* and *S*.
B-Proj (*R*, *S*) = 0 if and only if either $\| R \| = 0$ or $\| S \| = 0$ i.e. when either *R* = (0, 0, 0, 0, 0, 0) or *S* = (0, 0, 0, 0, 0, 0) which is trivial case.

$\therefore$ *B*-Proj (*R*, *S*) ≥ 0.
For two non-zero vectors *R* and *S*,
$\| R \| \| S \| + | \| R \| - \| S \| | R.S \geq \| R \| \| S \|$
$\therefore \| R \| \| S \| \leq \| R \| \| S \| + | \| R \| - \| S \| | R.S$

$$\therefore \frac{\| R \| \| S \|}{\| R \| \| S \| + | \| R \| - \| S \| | R.S} \leq 1$$

$\therefore$ *B-Proj* (*R*, *S*) ≤ 1.
$\therefore 0 \leq$ *B-Proj* (*R*, *S*) ≤ 1;

2. From definition, *R.S* = *S.R*, therefore,

$$B\text{-}Proj\ (R,\ S) = \frac{\| R \| \| S \|}{\| R \| \| S \| + | \| R \| - \| S \| | R.S} = \frac{\| S \| \| R \|}{\| S \| \| R \| + | \| S \| - \| R \| | S.R} = B\text{-}Proj\ (S, R).$$

Obviously, *B-Proj* (*R*, *S*) = 1, only when $\| R \| = \| S \|$ i. e. when $T_R^+(x_i) = T_S^+(x_i)$, $I_R^+(x_i) = I_S^+(x_i)$, $F_R^+(x_i) = F_S^+(x_i)$, $T_R^-(x_i) = T_S^-(x_i)$, $I_R^-(x_i) = I_S^-(x_i)$, $F_R^-(x_i) = F_S^-.(x_i)$

This completes the proof.

Example 2. Assume that *R* = < 0.5, 0.3, 0.2, -0.2, -0.1, -0.05 >, *S* = < 0.7, 0.3, 0.1, -0.4, -0.2, -0.3 > be the BNSs in *X*, then the bidirectional projection measure between *R* on *S* is computed as given below.
B-Proj (*R*, *S*) =

$$\frac{(0.6576473).(0.9380832)}{(0.6576473).(0.9380832) + | 0.9380832 - 06576473 | (0.575)}$$
= 0.7927845

Definition 11
Let *R* = $\left\langle T_R^+(x_i), I_R^+(x_i), F_R^+(x_i), T_R^-(x_i), I_R^-(x_i), F_R^-(x_i) \right\rangle$ and *S* = $\left\langle T_S^+(x_i), I_S^+(x_i), F_S^+(x_i), T_S^-(x_i), I_S^-(x_i), F_S^-(x_i) \right\rangle$ be any two BNSs in *X* = (x_1, x_2, …, x_m), then hybrid projection measure is defined as the combination of projection measure and bidirectional projection measure. The hybrid projection measure between *R* and *S* is represented as follows:
Hyb-Proj (*R*, *S*) = ρ *Proj* $(R)_S$ + (1 - ρ) *B-Proj* (*R*, *S*)

$$= \rho \frac{R.S}{\| S \|} + (1 - \rho) \frac{\| R \| \| S \|}{\| R \| \| S \| + | \| R \| - \| S \| | R.S} \quad (6)$$

where

$\|R\|$ =
$$\sqrt{\sum_{i=1}^{m} [(T_R^+)^2(x_i) + (I_R^+)^2(x_i) + (F_R^+)^2(x_i) + (T_R^-)^2(x_i) + (I_R^-)^2(x_i) + (F_R^-)^2(x_i)]},$$
$\|S\|$ =
$$\sqrt{\sum_{i=1}^{m} [(T_S^+)^2(x_i) + (I_S^+)^2(x_i) + (F_S^+)^2(x_i) + (T_S^-)^2(x_i) + (I_S^-)^2(x_i) + (F_S^-)^2(x_i)]},$$
and
R.S =
$$\sum_{i=1}^{m} [T_R^+(x_i) T_S^+(x) + I_R^+(x_i) I_S^+(x_i) + F_R^+(x_i) F_S^+(x_i) + T_R^-(x_i) T_S^-(x_i) + I_R^-(x_i) I_S^-(x_i) + F_R^-(x_i) F_S^-(x_i)]$$
where $0 \leq \rho \leq 1$.

Proposition 2
Let *Hyb-Proj* (*R*, *S*) be a hybrid projection measure between any two BNSs *R* and *S*, then
1. $0 \leq$ *Hyb-Proj* (*R*, *S*) ≤ 1;
2. *Hyb-Proj* (*R*, *S*) = *B-Proj* (*S*, *R*);
3. *Hyb-Proj* (*R*, *S*) = 1 for *R* = *S*.

Proof. The proofs of the properties under Proposition 2 are similar as Proposition 1.
Example 3. Assume that *R* = < 0.5, 0.3, 0.2, -0.2, -0.1, -0.05 >, *S* = < 0.7, 0.3, 0.1, -0.4, -0.2, -0.3 > be the two BNSs, then the hybrid projection measure between *R* on *S* with ρ = 0.7 is calculated as given below.
Hyb-Proj (*R*, *S*) = (0.7). (0.612952) + (1 - 0.7). (0.7927845) = 0.6669018.

4 Projection, bidirectional projection and hybrid projection based decision making methods for MADM problems with bipolar neutrosophic information

In this section, we develop projection based decision making models to MADM problems with bipolar neutrosophic assessments. Consider *E* = {E_1, E_2, …, E_m}, ($m \geq 2$) be a discrete set of *m* feasible alternatives, *F* = {F_1, F_2, …, F_n}, ($n \geq 2$) be a set of attributes under consideration and $w = (w_1, w_2, \ldots, w_n)^T$ be the weight vector of the attributes such that $0 \leq w_j \leq 1$ and $\sum_{j=1}^{n} w_j = 1$. Now, we present three algorithms for MADM problems involving bipolar neutrosophic information.

4.1. Method 1

Step 1. The rating of evaluation value of alternative E_i (i = 1, 2, …, *m*) for the predefined attribute F_j (j = 1, 2, …, *n*) is presented by the decision maker in terms of bipolar neutrosophic values and the bipolar neutrosophic decision matrix is constructed as given below.

$$\left\langle q_{ij} \right\rangle_{m \times n} = \begin{bmatrix} q_{11} & q_{12} & \cdots & q_{1n} \\ q_{21} & q_{22} & \cdots & q_{2n} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ q_{m1} & q_{m2} & \cdots & q_{mn} \end{bmatrix}$$

where q_{ij} = < ($T_{ij}^{+}, I_{ij}^{+}, F_{ij}^{+}, T_{ij}^{-}, I_{ij}^{-}, F_{ij}^{-}$) > with $T_{ij}^{+}, I_{ij}^{+}, F_{ij}^{+}$, $-T_{ij}^{-}$, $-I_{ij}^{-}$, $-F_{ij}^{-} \in [0, 1]$ and $0 \le T_{ij}^{+} + I_{ij}^{+} + F_{ij}^{+} - T_{ij}^{-} - I_{ij}^{-} - F_{ij}^{-} \le 6$ for i = 1, 2, …, *m*; j = 1, 2, …, *n*.

Step 2. We formulate the bipolar weighted decision matrix by multiplying weights w_j of the attributes as follows:

$$w_j \otimes \langle q_{ij} \rangle_{m\times n} = \langle z_{ij} \rangle_{m\times n} = \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ z_{m1} & z_{m2} & \cdots & z_{mn} \end{bmatrix}$$

where $z_{ij} = w_j . q_{ij}$ = < $1 - (1 - T_{ij}^{+})^{w_j}$, $(I_{ij}^{+})^{w_j}$, $(F_{ij}^{+})^{w_j}$, $-(-T_{ij}^{-})^{w_j}$, $-(-I_{ij}^{-})^{w_j}$, $-(1 - (1 - (-F_{ij}^{-}))^{w_j})$ > = < $\mu_{ij}^{+}, \nu_{ij}^{+}, \omega_{ij}^{+}, \mu_{ij}^{-}, \nu_{ij}^{-}, \omega_{ij}^{-}$ > with $\mu_{ij}^{+}, \nu_{ij}^{+}, \omega_{ij}^{+}, -\mu_{ij}^{-}, -\nu_{ij}^{-}, -\omega_{ij}^{-} \in [0, 1]$ and $0 \le \mu_{ij}^{+} + \nu_{ij}^{+} + \omega_{ij}^{+} - \mu_{ij}^{-} - \nu_{ij}^{-} - \omega_{ij}^{-} \le 6$ for i = 1, 2, …, *m*; j = 1, 2, …, *n*.

Step 3. We identify the bipolar neutrosophic positive ideal solution (BNPIS) [27, 28] as follows:

$z^{PIS} = \langle e_j^{+}, f_j^{+}, g_j^{+}, e_j^{-}, f_j^{-}, g_j^{-} \rangle$ = < [{ $\underset{i}{\mathrm{Max}}(\mu_{ij}^{+})$ |j ∈ σ }; { $\underset{i}{\mathrm{Min}}(\mu_{ij}^{+})$ |j ∈ ς }], [{ $\underset{i}{\mathrm{Min}}(\nu_{ij}^{+})$ | j ∈ σ }; { $\underset{i}{\mathrm{Max}}(\nu_{ij}^{+})$ |j ∈ ς }], [{ $\underset{i}{\mathrm{Min}}(\omega_{ij}^{+})$ |j ∈ σ }; { $\underset{i}{\mathrm{Max}}(\omega_{ij}^{-})$ |j ∈ ς }], [{ $\underset{i}{\mathrm{Min}}(\mu_{ij}^{-})$ |j ∈ σ }; { $\underset{i}{\mathrm{Max}}(\mu_{ij}^{-})$ |j ∈ ς }], [{ $\underset{i}{\mathrm{Max}}(\nu_{ij}^{-})$ |j ∈ σ }; { $\underset{i}{\mathrm{Min}}(\nu_{ij}^{-})$ |j ∈ ς }], [{ $\underset{i}{\mathrm{Max}}(\omega_{ij}^{-})$ |j ∈ σ }; { $\underset{i}{\mathrm{Min}}(\omega_{ij}^{-})$ |j ∈ ς }] >, j = 1, 2, …, *n*, where σ and ς are benefit and cost type attributes respectively.

Step 4. Determine the projection measure between z^{PIS} and $Z^i = \langle z_{ij} \rangle_{m\times n}$ for all i = 1, 2, …, *m*; j = 1, 2, …, *n* by using the following Eq.

$$Proj\ (Z^i)_{z^{PIS}} = \frac{\sum_{j=1}^{n}[\mu_{ij}^{+}e_j^{+} + \nu_{ij}^{+}f_j^{+} + \omega_{ij}^{+}g_j^{+} + \mu_{ij}^{-}e_j^{-} + \nu_{ij}^{-}f_j^{-} + \omega_{ij}^{-}g_j^{-}]}{\sqrt{\sum_{j=1}^{n}[(e_j^{+})^2 + (f_j^{+})^2 + (g_j^{+})^2 + (e_j^{-})^2 + (f_j^{-})^2 + (g_j^{-})^2]}} \quad (7)$$

Step 5. Rank the alternatives in a descending order based on the projection measure $Proj\ (Z^i)_{z^{PIS}}$ for i = 1, 2, …, *m* and bigger value of $Proj\ (Z^i)_{z^{PIS}}$ determines the best alternative.

4.2. Method 2

Step 1. Give the bipolar neutrosophic decision matrix $\langle q_{ij} \rangle_{m\times n}$, i = 1, 2, …, *m*; j = 1, 2, …, *n*.

Step 2. Construct weighted bipolar neutrosophic decision matrix $\langle z_{ij} \rangle_{m\times n}$, i = 1, 2, …, *m*; j = 1, 2, …, *n*.

Step 3. Determine $z^{PIS} = \langle e_j^{+}, f_j^{+}, g_j^{+}, e_j^{-}, f_j^{-}, g_j^{-} \rangle$; j = 1, 2, …, *n*.

Step 4. Compute the bidirectional projection measure between z^{PIS} and $Z^i = \langle z_{ij} \rangle_{m\times n}$ for all i = 1, 2, …, *m*; j = 1, 2, …, *n* using the Eq. as given below.

$$B\text{-}Proj\ (Z^i, z^{PIS}) = \frac{\| Z^i \| \| z^{PIS} \|}{\| Z^i \| \| z^{PIS} \| + | \| Z^i \| - \| z^{PIS} \| | Z^i . z^{PIS}} \quad (8)$$

where $\| Z^i \| = \sqrt{\sum_{j=1}^{n}[(\mu_{ij}^{+})^2 + (\nu_{ij}^{+})^2 + (\omega_{ij}^{+})^2 + (\mu_{ij}^{-})^2 + (\nu_{ij}^{-})^2 + (\omega_{ij}^{-})^2]}$, i = 1, 2, ..., *m*.

$\| z^{PIS} \| = \sqrt{\sum_{j=1}^{n}[(e_j^{+})^2 + (f_j^{+})^2 + (g_j^{+})^2 + (e_j^{-})^2 + (f_j^{-})^2 + (g_j^{-})^2]}$ and

$Z^i . z^{PIS} = \sum_{j=1}^{n}[\mu_{ij}^{+}e_j^{+} + \nu_{ij}^{+}f_j^{+} + \omega_{ij}^{+}g_j^{+} + \mu_{ij}^{-}e_j^{-} + \nu_{ij}^{-}f_j^{-} + \omega_{ij}^{-}g_j^{-}]$, i = 1, 2, ..., *m*.

Step 5. According to the bidirectional projection measure *B-Proj* (Z^i, z^{PIS}) for i = 1, 2, …, *m* the alternatives are ranked and highest value of *B-Proj* (Z^i, z^{PIS}) reflects the best option.

4.3. Method 3

Step 1. Construct the bipolar neutrosophic decision matrix $\langle q_{ij} \rangle_{m\times n}$, i = 1, 2, …, *m*; j = 1, 2, …, *n*.

Step 2. Formulate the weighted bipolar neutrosophic decision matrix $\langle z_{ij} \rangle_{m\times n}$, i = 1, 2, …, *m*; j = 1, 2, …, *n*.

Step 3. Identify $z^{PIS} = \langle e_j^{+}, f_j^{+}, g_j^{+}, e_j^{-}, f_j^{-}, g_j^{-} \rangle$, j = 1, 2, …, *n*.

Step 4. By combining projection measure $Proj\ (Z^i)_{z^{PIS}}$ and bidirectional projection measure *B-Proj* (Z^i, z^{PIS}), we calculate the hybrid projection measure between z^{PIS} and $Z^i = \langle z_{ij} \rangle_{m\times n}$ for all i = 1, 2, …, *m*; j = 1, 2, …, *n* as follows.

$$Hyb\text{-}Proj\,(Z^i, z^{PIS}) = \rho \; Proj\;(Z^i)_{z^{PIS}} + (1 - \rho)\, B\text{-}Proj\,(Z^i, z^{PIS}) = \rho \frac{Z^i . z^{PIS}}{\| z^{PIS} \|} + (1 - \rho) \frac{\| Z^i \| \| z^{PIS} \|}{\| Z^i \| \| z^{PIS} \| + | \| Z^i \| - \| z^{PIS} \| \,|\, Z^i . z^{PIS}} \quad (9)$$

where $\| Z^i \| = \sqrt{\sum_{j=1}^{n}[(\mu_{ij}^{+})^2 + (\nu_{ij}^{+})^2 + (\omega_{ij}^{+})^2 + (\mu_{ij}^{-})^2 + (\nu_{ij}^{-})^2 + (\omega_{ij}^{-})^2]}$, i = 1, 2, …, m,

$\| z^{PIS} \| = \sqrt{\sum_{j=1}^{n}[(e_j^{+})^2 + (f_j^{+})^2 + (g_j^{+})^2 + (e_j^{-})^2 + (f_j^{-})^2 + (g_j^{-})^2]}$,

$Z^i . z^{PIS} = \sum_{j=1}^{n}[\mu_{ij}^{+} e_j^{+} + \nu_{ij}^{+} f_j^{+} + \omega_{ij}^{+} g_j^{+} + \mu_{ij}^{-} e_j^{-} + \nu_{ij}^{-} f_j^{-} + \omega_{ij}^{-} g_j^{-}]$, i = 1, 2, …, m, with $0 \le \rho \le 1$.

Step 5. We rank all the alternatives in accordance with the hybrid projection measure *Hyb-Proj* (Z^i, z^{PIS}) and greater value of *Hyb-Proj* (Z^i, z^{PIS}) indicates the better alternative.

5 A numerical example

We solve the MADM studied in [5, 28] where a customer desires to purchase a car. Suppose four types of car (alternatives) E_i, (i = 1, 2, 3, 4) are taken into consideration in the decision making situation. Four attributes namely Fuel economy (F_1), Aerod (F_2), Comfort (F_3) and Safety (F_4) are considered to evaluate the alternatives. Assume the weight vector [5] of the attribute is given by $w = (w_1, w_2, w_3, w_4) = (0.5, 0.25, 0.125, 0.125)$.

Method 1: The proposed projection measure based decision making with bipolar neutrosophic information for car selection is presented in the following steps:

Step 1: Construct the bipolar neutrosophic decision matrix

The bipolar neutrosophic decision matrix $\langle q_{ij} \rangle_{m \times n}$ presented by the decision maker as given below (see Table 1)

Table 1. The bipolar neutrosophic decision matrix

	F_1	F_2	F_3	F_4
E_1	<0.5, 0.7, 0.2, -0.7, -0.3, -0.6>	<0.4, 0.5, 0.4, -0.7, -0.8, -0.4>	<0.7, 0.7, 0.5, -0.8, -0.7, -0.6>	<0.1, 0.5, 0.7, -0.5, -0.2, -0.8>
E_2	<0.9, 0.7, 0.5, -0.7, -0.7, -0.1>	<0.7, 0.6, 0.8, -0.7, -0.5, -0.1>	<0.9, 0.4, 0.6, -0.1, -0.7, -0.5>	<0.5, 0.2, 0.7, -0.5, -0.1, -0.9>
E_3	<0.3, 0.4, 0.2, -0.6, -0.3, -0.7>	<0.2, 0.2, 0.2, -0.4, -0.7, -0.4>	<0.9, 0.5, 0.5, -0.6, -0.5, -0.2>	<0.7, 0.5, 0.3, -0.4, -0.2, -0.2>
E_4	<0.9, 0.7, 0.2, -0.8, -0.6, -0.1>	<0.3, 0.5, 0.2, -0.5, -0.5, -0.2>	<0.5, 0.4, 0.5, -0.1, -0.7, -0.2>	<0.2, 0.4, 0.8, -0.5, -0.5, -0.6>

Step 2. Construction of weighted bipolar neutrosophic decision matrix

The weighted decision matrix $\langle z_{ij} \rangle_{m \times n}$ is obtained by multiplying weights of the attributes to the bipolar neutrosophic decision matrix as follows (see Table 2).

Table 2. The weighted bipolar neutrosophic decision matrix

	F_1	F_2	F_3	F_4
E_1	<0.293, 0.837, 0.447,-0.837, -0.818, -0.182 >	<0.120, 0.795, 0.841, 0.915, -0.946, -0.120>	<0.140, 0.956, 0.917, 0.972, -0.956, -0.108>	<0.013, 0.917, 0.956, -0.917, -0.818, -0.182>
E_2	<0.684, 0.837, 0.707, -0.837, -0.837, -0.051>	<0.260, 0.880, 0.946, -0.915, -0.841, -0.026>	<0.250, 0.892, 0.938, -0.750, -0.956, -0.083>	<.083, 0.818, 0.956, 0.917, -0.750, -0.250>
E_3	<0.163, 0.632, 0.447, -0.774, -0.548, -0.452>	<0.054, 0.669, 0.669, - 0.795, -0.915, -0.120>	<0.250, 0.917, 0.917, -0.938, -0.917, -0.028>	<.140, 0.917, 0.860, -0.892, -0.818, -0.028>
E_4	<0.648, 0.837, 0.447, , -0.894,-0.774, -0.051>	<0.085, 0.841, 0.669, -0.841, -0.841, -0.054>	<0.083, 0.892, 0.917, -0.750, -0.956, -0.028>	<0.062, 0.818, 0.972, -0.917, -0.917, -0.108>

Step 3. Selection of BNPIS

The BNRPIS (z^{PIS}) $= \langle e_j^{+}, f_j^{+}, g_j^{+}, e_j^{-}, f_j^{-}, g_j^{-} \rangle$, (j = 1, 2, 3, 4) is computed from the weighted decision matrix as follows:

$\langle e_1^{+}, f_1^{+}, g_1^{+}, e_1^{-}, f_1^{-}, g_1^{-} \rangle$ = < 0.684, 0.632, 0.447, -0.894, -0.548, -0.051 >;

$\langle e_2^{+}, f_2^{+}, g_2^{+}, e_2^{-}, f_2^{-}, g_2^{-} \rangle$ = < 0.26, 0.669, 0.669, -0.915, -0.841, -0.026 >;

$\langle e_3^{+}, f_3^{+}, g_3^{+}, e_3^{-}, f_3^{-}, g_3^{-} \rangle$ = < 0.25, 0.892, 0.917, -0.972, -0.917, -0.028 >;

$\langle e_4^{+}, f_4^{+}, g_4^{+}, e_4^{-}, f_4^{-}, g_4^{-} \rangle$ = < 0.14, 0.818, 0.86, -0.917, -0.75, -0.028 >.

Step 4. Determination of weighted projection measure
The projection measure between positive ideal bipolar neutrosophic solution z^{PIS} and each weighted decision matrix $\langle z_{ij} \rangle_{m\times n}$ can be obtained as follows:

Proj $(Z^1)_{z^{PIS}}$ = 3.4214, *Proj* $(Z^2)_{z^{PIS}}$ = 3.4972, *Proj* $(Z^3)_{z^{PIS}}$ = 3.1821, *Proj* $(Z^4)_{z^{PIS}}$ = 3.3904.

Step 5. Rank the alternatives
We observe that *Proj* $(Z^2)_{z^{PIS}}$ > *Proj* $(Z^1)_{z^{PIS}}$ > *Proj* $(Z^4)_{z^{PIS}}$ > *Proj* $(Z^3)_{z^{PIS}}$. Therefore, the ranking order of the cars is $E_2 \succ E_1 \succ E_4 \succ E_3$. Hence, E_2 is the best alternative for the customer.

Method 2: The proposed bidirectional projection measure based decision making for car selection is presented as follows:
Step 1. Same as Method 1
Step 2. Same as Method 1
Step 3. Same as Method 1
Step 4. Calculation of bidirectional projection measure
The bidirectional projection measure between positive ideal bipolar neutrosophic solution z^{PIS} and each weighted decision matrix $\langle z_{ij} \rangle_{m\times n}$ can be determined as given below.
B-Proj (Z^1, z^{PIS}) = 0.8556, *B-Proj* (Z^2, z^{PIS}) = 0.8101, *B-Proj* (Z^3, z^{PIS}) = 0.9503, *B-Proj* (Z^4, z^{PIS}) = 0.8969.
Step 5. Ranking the alternatives
Here, we notice that *B-Proj* (Z^3, z^{PIS}) > *B-Proj* (Z^4, z^{PIS}) > *B-Proj* (Z^1, z^{PIS}) > *B-Proj* (Z^2, z^{PIS}) and therefore, the ranking order of the alternatives is obtained as $E_3 \succ E_4 \succ E_1 \succ E_2$. Hence, E_3 is the best choice among the alternatives.

Method 3: The proposed hybrid projection measure based MADM with bipolar neutrosophic information is provided as follows:
Step 1. Same as Method 1
Step 2. Same as Method 1
Step 3. Same as Method 1
Step 4. Computation of hybrid projection measure
The hybrid projection measures for different values of $\rho \in [0, 1]$ and the ranking order are shown in the Table 3.

Table 3. Results of hybrid projection measure for different valus of ρ

Similarity measure	ρ	Measure values	Ranking order
Hyb-Proj (Z^i, z^{PIS})	0.25	*Hyb-Proj* (Z^1, z^{PIS}) = 1.4573 *Hyb-Proj* (Z^2, z^{PIS}) = 1.4551 *Hyb-Proj* (Z^3, z^{PIS}) = 1.5297 *Hyb-Proj* (Z^4, z^{PIS}) = 1.5622	$E_4 > E_3 > E_1 > E_2$
Hyb-Proj (Z^i, z^{PIS})	0.50	*Hyb-Proj* (Z^1, z^{PIS}) = 2.1034 *Hyb-Proj* (Z^2, z^{PIS}) = 2.0991 *Hyb-Proj* (Z^3, z^{PIS}) = 2.0740 *Hyb-Proj* (Z^4, z^{PIS}) = 2.1270	$E_4 > E_1 > E_2 > E_3$
Hyb-Proj (Z^i, z^{PIS})	0.75	*Hyb-Proj* (Z^1, z^{PIS}) = 2.4940 *Hyb-Proj* (Z^2, z^{PIS}) = 2.7432 *Hyb-Proj* (Z^3, z^{PIS}) = 2.6182 *Hyb-Proj* (Z^4, z^{PIS}) = 2.6919	$E_2 > E_4 > E_3 > E_1$
Hyb-Proj (Z^i, z^{PIS})	0.90	*Hyb-Proj* (Z^1, z^{PIS}) = 3.1370 *Hyb-Proj* (Z^2, z^{PIS}) = 3.1296 *Hyb-Proj* (Z^3, z^{PIS}) = 2.9448 *Hyb-Proj* (Z^4, z^{PIS}) = 3.0308	$E_1 > E_2 > E_4 > E_3$

6 Comparative analysis

In the Section, we compare the results obtained from the proposed methods with the results derived from other existing methods under bipolar neutrosophic environment to show the effectiveness of the developed methods.

Dey et al. [28] assume that the weights of the attributes are not identical and weights are fully unknown to the decision maker. Dey et al. [28] formulated maximizing deviation model under bipolar neutrosophic assessment to compute unknown weights of the attributes as w = (0.2585, 0.2552, 0.2278, 0.2585). By considering w = (0.2585, 0.2552, 0.2278, 0.2585), the proposed projection measures are shown as follows:

Proj $(Z^1)_{z^{PIS}}$ = 3.3954, *Proj* $(Z^2)_{z^{PIS}}$ = 3.3872, *Proj* $(Z^3)_{z^{PIS}}$ = 3.1625, *Proj* $(Z^4)_{z^{PIS}}$ = 3.2567.

Since, *Proj* $(Z^1)_{z^{PIS}}$ > *Proj* $(Z^2)_{z^{PIS}}$ > *Proj* $(Z^4)_{z^{PIS}}$ > *Proj* $(Z^3)_{z^{PIS}}$, therefore the ranking order of the four alternatives is given by $E_1 \succ E_2 \succ E_4 \succ E_3$. Thus, E_1 is the best choice for the customer.

Now, by taking w = (0.2585, 0.2552, 0.2278, 0.2585), the bidirectional projection measures are calculated as given below.
B-Proj (Z^1, z^{PIS}) = 0.8113, *B-Proj* (Z^2, z^{PIS}) = 0.8111, *B-Proj* (Z^3, z^{PIS}) = 0.9854, *B-Proj* (Z^4, z^{PIS}) = 0.9974.

Since, *B-Proj* (Z^4, z^{PIS}) > *B-Proj* (Z^3, z^{PIS}) > *B-Proj* (Z^1, z^{PIS}) > *B-Proj* (Z^2, z^{PIS}), consequently the ranking order of the four alternatives is given by $E_4 \succ E_3 \succ E_1 \succ E_2$. Hence, E_4 is the best option for the customer.

Also, by taking w = (0.2585, 0.2552, 0.2278, 0.2585), the proposed hybrid projection measures for different values of $\rho \in [0, 1]$ and the ranking order are revealed in the Table 4.

Deli et al. [5] assume the weight vector of the attributes as w = (0.5, 0.25, 0.125, 0.125) and the ranking order based on score values is presented as follows:

$$E_3 \succ E_4 \succ E_2 \succ E_1$$

Thus, E_3 was the most desirable alternative.

Dey et al. [28] employed maximizing deviation method to find unknown attribute weights as w = (0.2585, 0.2552, 0.2278, 0.2585). The ranking order of the alternatives is presented based on the relative closeness coefficient as given below.

$$E_3 \succ E_2 \succ E_4 \succ E_1.$$

Obviously, E_3 is the most suitable option for the customer.

Dey et al. [28] also consider the weight vector of the attributes as w = (0.5, 0.25, 0.125, 0.125), then using TOPSIS method, the ranking order of the cars is represented as follows:

$$E_4 \succ E_2 \succ E_3 \succ E_1.$$

So, E_4 is the most preferable alternative for the buyer. We observe that different projection measure provides different ranking order and the projection measure is weight sensitive. Therefore, decision maker should choose the projection measure and weights of the attributes in the decision making context according to his/her needs, desires and practical situation.

Conclusion

In this paper, we have defined projection, bidirectional projection measures between bipolar neutrosophic sets. Further, we have defined a hybrid projection measure by combining projection and bidirectional projection measures. Through these projection measures we have developed three methods for multi-attribute decision making models under bipolar neutrosophic environment. Finally, a car selection problem has been solved to show the flexibility and applicability of the proposed methods. Furthermore, comparison analysis of the proposed methods with the other existing methods has also been demonstrated.

The proposed methods can be extended to interval bipolar neutrosophic set environment. In future, we shall apply projection, bidirectional projection, and hybrid projection measures of interval bipolar neutrosophic sets for group decision making, medical diagnosis, weaver selection, pattern recognition problems, etc.

Table 4. Results of hybrid projection measure for different values of ρ

Similarity measure	ρ	Measure values	Ranking order
Hyb-Proj (Z^i, Z^{PIS})	0.25	*Hyb-Proj* (Z^1, Z^{PIS}) = 1.4970 *Hyb-Proj* (Z^2, Z^{PIS}) = 1.4819 *Hyb-Proj* (Z^3, Z^{PIS}) = 1.5082 *Hyb-Proj* (Z^4, Z^{PIS}) = 1.5203	$E_4 > E_3 > E_1 > E_2$
Hyb-Proj (Z^i, Z^{PIS})	0.50	*Hyb-Proj* (Z^1, Z^{PIS}) = 2.1385 *Hyb-Proj* (Z^2, Z^{PIS}) = 2.1536 *Hyb-Proj* (Z^3, Z^{PIS}) = 2.0662 *Hyb-Proj* (Z^4, Z^{PIS}) = 2.1436	$E_4 > E_1 > E_2 > E_3$
Hyb-Proj (Z^i, Z^{PIS})	0.75	*Hyb-Proj* (Z^1, Z^{PIS}) = 2.7800 *Hyb-Proj* (Z^2, Z^{PIS}) = 2.8254 *Hyb-Proj* (Z^3, Z^{PIS}) = 2.6241 *Hyb-Proj* (Z^4, Z^{PIS}) = 2.7670	$E_2 > E_4 > E_3 > E_1$
Hyb-Proj (Z^i, Z^{PIS})	0.90	*Hyb-Proj* (Z^1, Z^{PIS}) = 3.1648 *Hyb-Proj* (Z^2, Z^{PIS}) = 3.2285 *Hyb-Proj* (Z^3, Z^{PIS}) = 2.9589 *Hyb-Proj* (Z^4, Z^{PIS}) = 3.1410	$E_2 > E_1 > E_4 > E_3$

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Complex Neutrosophic Set

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Abstract Complex fuzzy sets and complex intuitionistic fuzzy sets cannot handle imprecise, indeterminate, inconsistent, and incomplete information of periodic nature. To overcome this difficulty, we introduce complex neutrosophic set. A complex neutrosophic set is a neutrosophic set whose complex-valued truth membership function, complex-valued indeterminacy membership function, and complex-valued falsehood membership functions are the combination of real-valued truth amplitude term in association with phase term, real-valued indeterminate amplitude term with phase term, and real-valued false amplitude term with phase term, respectively. Complex neutrosophic set is an extension of the neutrosophic set. Further set theoretic operations such as complement, union, intersection, complex neutrosophic product, Cartesian product, distance measure, and δ-equalities of complex neutrosophic sets are studied here. A possible application of complex neutrosophic set is presented in this paper. Drawbacks and failure of the current methods are shown, and we also give a comparison of complex neutrosophic set to all such methods in this paper. We also showed in this paper the dominancy of complex neutrosophic set to all current methods through the graph.

Keywords Fuzzy set · Intuitionistic fuzzy set · Complex fuzzy set · Complex intuitionistic fuzzy set · Neutrosophic set · Complex neutrosophic set

1 Introduction

Fuzzy sets were first proposed by Zadeh in the seminal paper [38]. This novel concept is used successfully in modeling uncertainty in many fields of real life. A fuzzy set is characterized by a membership function μ with the range [0,1]. Fuzzy sets and their applications have been extensively studied in different aspects from the last few decades such as control [19, 38], reasoning [44], pattern recognition [19, 44], and computer vision [44]. Fuzzy sets become an important area for the research in medical diagnosis [29], engineering [19], etc. A large amount of the literature on fuzzy sets can be found in [8, 9, 15, 21, 30, 40–43]. In fuzzy set, the membership degree of an element is single value between 0 and 1. Therefore, it may not always be true that the non-membership degree of an element in a fuzzy set is equal to 1 minus the membership degree because there is some degree of hesitation. Thus, Atanassov [2] introduced intuitionistic fuzzy sets in 1986 which incorporate the hesitation degree called hesitation margin. The hesitation margin is defining as 1 minus the sum of membership and non-membership. Therefore, the intuitionistic fuzzy set is characterized by a membership function μ and non-membership function v with range [0,1]. An intuitionistic fuzzy set is the generalization of fuzzy set. Intuitionistic fuzzy sets can successfully be applied in many fields such as medical diagnosis [29], modeling theories [11], pattern recognition [31], and decision making [17].

Ramot et al. [23] proposed an innovative concept to the extension of fuzzy sets by initiating the complex fuzzy sets where the degree of membership μ is traded by a complex-valued of the form

$$r_s(x) \cdot e^{j\omega_S(x)},\ j = \sqrt{-1}$$

where $r_S(x)$ and $\omega_S(x)$ are both belongs to [0,1] and $r_s(x) . e^{j\omega_S(x)}$ has the range in complex unit disk. Complex fuzzy set is completely a different approach from the fuzzy complex number discussed by Buckley [4–7], Nguyen et al. [21], and Zhang et al. [41]. The complex-valued membership function of the complex fuzzy set has an amplitude term with the combination of a phase term which gives wavelike characteristics to it. Depending on the phase term gives a constructive or destructive interference. Thus, complex fuzzy set is different from conventional fuzzy set [38], fuzzy complex set [23], type 2 fuzzy set [19], etc. due to the character of wavelike. The complex fuzzy set [23] still preserves the characterization of uncertain information through the amplitude term having value in the range of [0,1] with the addition of a phase term. Ramot et al. [23, 24] discussed several properties of complex fuzzy sets such as complement, union, and intersection. with sufficient amount of illustrative examples. Some more theory on complex fuzzy sets can be seen in [10, 35]. Ramot et al. [24] also introduced the concept of complex fuzzy logic which is a novel framework for logical reasoning. The complex fuzzy logic is a generalization of fuzzy logic, based on complex fuzzy set. In complex fuzzy logic [24], the inference rules are constructed and fired in such way that they are closely resembled to traditional fuzzy logic. Complex fuzzy logic [24] is constructed to hold the advantages of fuzzy logic while enjoying the features of complex numbers and complex fuzzy sets. Complex fuzzy logic is not only a linear extension to the conventional fuzzy logic but rather a natural extension to those problems that are very difficult or impossible to describe with one-dimensional grades of membership. Complex fuzzy sets have found their place in signal processing [23], physics [23], stock marketing [23] etc.

The concept of complex intuitionistic fuzzy set in short CIFS is introduced by Alkouri and Saleh in [1]. The complex intuitionistic fuzzy set is an extension of complex fuzzy set by adding complex-valued non-membership grade to the definition of complex fuzzy set. The complex intuitionistic fuzzy sets are used to handle the information of uncertainty and periodicity simultaneously. The complex-valued membership and non-membership function can be used to represent uncertainty in many corporal quantities such as wave function in quantum mechanics, impedance in electrical engineering, complex amplitude, and decision-making problems. The novel concept of phase term is extend in the case of complex intuitionistic fuzzy set which appears in several prominent concepts such as distance measure, Cartesian products, relations, projections, and cylindric extensions. The complex fuzzy set has only one extra phase term, while complex intuitionistic fuzzy set has two additional phase terms. Several properties of complex intuitionistic fuzzy sets have been studied such as complement, union, intersection, T-norm, and S-norm.

Smarandache [28] in 1998 introduced Neutrosophy that studies the origin, nature, and scope of neutralities and their interactions with distinct ideational spectra. A neutrosophic set is characterized by a truth membership function T, an indeterminacy membership function I and a falsehood membership function F. Neutrosophic set is powerful mathematical framework which generalizes the concept of classical sets, fuzzy sets [38], intuitionistic fuzzy sets [2], interval valued fuzzy sets [30], paraconsistent sets [28], dialetheist sets [28], paradoxist sets [28], and tautological sets [28]. Neutrosophic sets handle the indeterminate and inconsistent information that exists commonly in our daily life. Recently neutrosophic sets have been studied by several researchers around the world. Wang et al. [33] studied single-valued neutrosophic sets in order to use them in scientific and engineering fields that give an additional possibility to represent uncertainty, incomplete, imprecise, and inconsistent data. Hanafy et al. [13, 14] studied the correlation coefficient of neutrosophic set. Ye [35] studied the correlation coefficient of single-valued neutrosophic sets. Broumi and Smaradache presented the correlation coefficient of interval neutrosophic set in [3]. Salama et al. [26] studied neutrosophic sets and neutrosophic topological spaces. Some more literature on neutrosophic sets can be found in [12–14, 18, 20, 25, 27, 32, 34, 36, 37, 40].

Pappis [22] studied the notion of "proximity measure," with an attempt to show that "precise membership values should normally be of no practical significance." Pappis observed that the max–min compositional rule of inference is preserved with respect to "approximately equal" fuzzy sets. An important generalization of the work of Pappis proposed by Hong and Hwang [15] which is mainly based that the max–min compositional rule of inference is preserved with respect to "approximately equal fuzzy sets" and "approximately equal" fuzzy relation. But, Cai noticed that both the Pappis and Hong and Hwang approaches were confined to fixed ε. Therefore, Cai [8, 9] takes a different approach and introduced δ-equalities of fuzzy sets. Cai proposed that if two fuzzy sets are equal to an extent of δ, then they are said to be δ-equal. The notions of δ-equality are significance in both the fuzzy statistics and fuzzy reasoning. Cai [8, 9] applied them for assessing the robustness of fuzzy reasoning as well as in synthesis of real-time fuzzy systems. Cai also gave several reliability examples of δ-equalities [8, 9]. Zhang et al. [39] studied the δ-equalities of complex fuzzy set by following the philosophy of Ramot et al. [23, 24] and Cai [8, 9]. They mainly focus on the results of Cai's work [8, 9] to introduce δ-equalities of complex fuzz sets, and thus, they systematically develop distance measure, equality and similarity of complex fuzzy sets. Zhang et al. [39] then applied δ-equalities of complex fuzzy sets in a signal processing application.

This paper is an extension of the work of Ramot et al. [23], Alkouri and Saleh [1], Cai [8, 9], and Zhang et al. [39] to neutrosophic sets. Basically, we follow the philosophy of the work of Ramot et al. [23] to introduce complex neutrosophic set. The complex neutrosophic is characterized by complex-valued truth membership function, complex-valued indeterminate membership function, and complex-valued falsehood membership function. Further, complex neutrosophic set is the mainstream over all because it not only is the generalization of all the current frameworks but also describes the information in a complete and comprehensive way.

1.1 Why complex neutrosophic set can handle the indeterminate information in periodicity

As we can see that uncertainty, indeterminacy, incompleteness, inconsistency, and falsity in data are periodic in nature, to handle these types of problems, the complex neutrosophic set plays an important role. A complex neutrosophic set S is characterized by a complex-valued truth membership function $T_S(x)$, complex-valued indeterminate membership function $I_S(x)$, and complex-valued false membership function $F_S(x)$ whose range is extended from [0,1] to the unit disk in the complex plane. The complex neutrosophic sets can handle the information which is uncertain, indeterminate, inconsistent, incomplete, ambiguous, false because in $T_S(x)$, the truth amplitude term and phase term handle uncertainty and periodicity, in $I_S(x)$, the indeterminate amplitude term and phase term handle indeterminacy and periodicity, and in $F_S(x)$, the false amplitude term and phase term handle the falsity with periodicity. Complex neutrosophic set is an extension of the neutrosophic set with three additional phase terms.

Thus, the complex neutrosophic set deals with the information/data which have uncertainty, indeterminacy, and falsity that are in periodicity while both the complex fuzzy set and complex intuitionistic fuzzy sets cannot deal with indeterminacy, inconsistency, imprecision, vagueness, doubtfulness, error, etc. in periodicity.

The contributions of this paper are:

1. We introduced complex neutrosophic set which deals with uncertainty, indeterminacy, impreciseness, inconsistency, incompleteness, and falsity of periodic nature.
2. Further, we studied set theoretic operations of complex neutrosophic sets such as complement, union, intersection complex neutrosophic product, and Cartesian product.
3. We also introduced the novel concept "the game of winner, neutral, and loser" for phase terms.
4. We studied a distance measure on complex neutrosophic sets which we have used in the application.
5. We introduced δ-equalities of complex neutrosophic set and studied their properties.
6. We also gave an algorithm for signal processing using complex neutrosophic sets.
7. Drawbacks and failures of the current methods presented in this paper.
8. Finally, we gave the comparison of complex neutrosophic sets to the current methods.

The organization of this paper is as follows. In Sect. 2, we presented some basic and fundamental concepts of neutrosophic sets, complex fuzzy sets, and complex intuitionistic fuzzy sets. In the next section, we introduced complex neutrosophic sets and gave some interpretation of complex neutrosophic set for intuition. We also introduced the basic set theoretic operations of complex neutrosophic sets such as complement, union, intersection, complex neutrosophic product, and Cartesian product in the current section. Further, in this section, the game of winner, neutral, and loser is introduced for the phase terms in the case of union and intersection of two complex neutrosophic sets. It is completely an innovative approach for the phase terms. In Sect. 4, we introduced distance measure on complex neutrosophic sets, δ-equality on complex neutrosophic sets and studied some of their properties. An application in signal processing is presented for the possible utilization of complex neutrosophic set in the Sect. 5. In Sect. 6, we give the drawbacks of fuzzy sets, intuitionistic fuzzy sets, neutrosophic sets, complex fuzzy sets, and complex intuitionistic fuzzy sets. We also give a comparison of different current methods to complex neutrosophic set in this section. Further, the dominancy of complex neutrosophic sets over other existing methods is shown in this section.

We now review some basic concepts of neutrosophic sets, single-valued neutrosophic set, complex fuzzy sets, and complex intuitionistic fuzzy sets.

2 Literature review

In this section, we present some basic material which will help in our later pursuit. The definitions and notions are taken from [1, 23, 28, 39].

Definition 2.1 [28] Neutrosophic set.

Let X be a space of points and let $x \in X$. A neutrosophic set S in X is characterized by a truth membership function T_S, an indeterminacy membership function I_S, and a falsity membership function F_S. $T_S(x)$, $I_S(x)$, and $F_S(x)$ are real standard or non-standard subsets of $]0^-, 1^+[$, and $T_S, I_S, F_S : X \to]0^-, 1^+[$. The neutrosophic set can be represented as

$$S = \{(x, T_S(x), I_S(x), F_S(x)) : x \in X\}$$

There is no restriction on the sum of $T_S(x)$, $I_S(x)$, and $F_S(x)$, so $0^- \leq T_S(x) + I_S(x) + F_S(x) \leq 3^+$.

From philosophical point view, the neutrosophic set takes the value from real standard or non-standard subsets of $]0^-, 1^+[$. Thus, it is necessary to take the interval [0,1] instead of $]0^-, 1^+[$ for technical applications. It is difficult to apply $]0^-, 1^+[$ in the real-life applications such as engineering and scientific problems.

A single-valued neutrosophic set [23] is characterized by a truth membership function, $T_S(x)$, an indeterminacy membership function $I_S(x)$ and a falsity membership function $F_S(x)$ with $T_S(x)$, $I_S(x)$, $F_S(x) \in [0, 1]$ for all $x \in X$. If X is continuous, then

$$S = \int_X \frac{(T_S(x), I_S(x), F_S(x))}{x} \quad \text{for all} \quad x \in X.$$

If X is discrete, then

$$S = \sum_X \frac{(T_S(x), I_S(x), F_S(x))}{x} \quad \text{for all} \quad x \in X.$$

Actually, SVNS is an instance of neutrosophic set that can be used in real-life situations such as decision-making, scientific, and engineering applications. We will use single-valued neutrosophic set to define complex neutrosophic set.

We now give some set theoretic operations of neutrosophic sets.

Definition 2.2 [33] Complement of neutrosophic set.

The complement of a neutrosophic set S is denoted by $c(S)$ and is defined by

$$T_{c(S)}(x) = F_S(x), \quad I_{c(S)}(x) = 1 - I_S(x), \quad F_{c(S)}(x) = T_S(x) \quad \text{for all} \quad x \in X.$$

Definition 2.3 [23] Union of neutrosophic sets.

Let A and B be two complex neutrosophic sets in a universe of discourse X. Then, the union of A and B is denoted by $A \cup B$, which is defined by

$$A \cup B = \{(x, T_A(x) \vee T_B(x), I_A(x) \wedge I_B(x), F_A(x) \wedge F_B(x)) : x \in X\}$$

for all $x \in X$, and $\vee$ denote the max operator and $\wedge$ denote the min operator, respectively.

Definition 2.4 [23] Intersection of neutrosophic sets.

Let A and B be two complex neutrosophic sets in a universe of discourse X. Then, the intersection of A and B is denoted as $A \cap B$, which is defined by

$$A \cap B = \{(x, T_A(x) \wedge T_B(x), I_A(x) \vee I_B(x), F_A(x) \vee F_B(x)) : x \in X\}$$

for all $x \in X$.

The definitions and other notions of complex fuzzy sets are given as follows.

Definition 2.5 [23] Complex fuzzy set.

A complex fuzzy set S, defined on a universe of discourse X, is characterized by a membership function $\eta_S(x)$ that assigns any element $x \in X$ a complex-valued grade of membership in S. The values $\eta_S(x)$ all lie within the unit circle in the complex plane and thus all of the form $p_S(x).e^{j \cdot \mu_S(x)}$ where $p_S(x)$ and $\mu_S(x)$ are both real-valued and $p_S(x) \in [0, 1]$. Here, $p_S(x)$ is termed as amplitude term and $e^{j \cdot \mu_S(x)}$ is termed as phase term. The complex fuzzy set may be represented in the set form as

$$S = \{(x, \eta_S(x)) : x \in X\}.$$

The complex fuzzy set is denoted as CFS.

We now present set theoretic operations of complex fuzzy sets.

Definition 2.6 [23] Complement of complex fuzzy set.

Let S be a complex fuzzy set on X, and $\eta_S(x) = p_S(x).e^{j \cdot \mu_S(x)}$ its complex-valued membership function. The complement of S denoted as $c(S)$ and is specified by a function

$$\eta_{c(S)}(x) = p_{c(S)}(x).e^{j \cdot \mu_{c(S)}(x)} = (1 - p_S(x)).e^{j(2\pi - \mu_S(x))}.$$

Definition 2.7 [23] Union of complex fuzzy sets.

Let A and B be two complex fuzzy sets on X, and $\eta_A(x) = r_A(x).e^{j \cdot \mu_A(x)}$ and $\eta_B(x) = r_B(x).e^{j \cdot \mu_B(x)}$ be their membership functions, respectively. The union of A and B is denoted as $A \cup B$, which is specified by a function

$$\eta_{A \cup B}(x) = r_{A \cup B}(x).e^{j \cdot \mu_{A \cup B}(x)} = (r_A(x) \vee r_B(x)).e^{j(\mu_A(x) \vee \mu_B(x))}$$

where $\vee$ denote the max operator.

Definition 2.8 [23] Intersection of complex fuzzy sets.

Let A and B be two complex fuzzy sets on X, and $\eta_A(x) = r_A(x).e^{j \cdot \mu_A(x)}$ and $\eta_B(x) = r_B(x).e^{j \cdot \mu_B(x)}$ be their membership functions, respectively. The intersection of A and B is denoted as $A \cap B$, which is specified by a function

$$\eta_{A \cap B}(x) = r_{A \cap B}(x).e^{j \cdot \mu_{A \cap B}(x)} = (r_A(x) \wedge r_B(x)).e^{j(\mu_A(x) \wedge \mu_B(x))}$$

where $\wedge$ denote the max operator.

We now give some basic definitions and set theoretic operations of complex intuitionistic fuzzy sets.

Definition 2.9 [39] Let A and B be two complex fuzzy sets on X, and $\eta_A(x) = r_A(x).e^{j.\mu_A(x)}$ and $\eta_B(x) = r_B(x).e^{j.\mu_B(x)}$ be their membership functions, respectively. The complex fuzzy product of A and B, denoted as $A \circ B$, and is specified by a function

$$\eta_{A\circ B}(x) = r_{A\circ B}(x).e^{j.\mu_{A\circ B}(x)} = (r_A(x).r_B(x)).e^{j2\pi\left(\frac{\mu_A(x)}{2\pi}.\frac{\mu_B(x)}{2\pi}\right)}.$$

Definition 2.10 [39] Let A and B be two complex fuzzy sets on X, and $\eta_A(x) = r_A(x).e^{j.\mu_A(x)}$ and $\eta_B(x) = r_B(x).e^{j.\mu_B(x)}$ be their membership functions, respectively. Then, A and B are said to be δ equal if and only if $d(A,B) \le 1 - \delta$, where $0 \le \delta \le 1$.

For more literature on δ-equality, we refer to [8, 9] and [35].

Definition 2.11 [1] Complex intuitionistic fuzzy set.

A complex intuitionistic fuzzy set S, defined on a universe of discourse X, is characterized by a membership function $\eta_S(x)$ and non-membership function $\zeta_S(x)$, respectively, that assign an element $x \in X$ a complex-valued grade to both membership and non-membership in S. The values of $\eta_S(x)$ and $\zeta_S(x)$ all lie with in the unit circle in the complex plane and are of the form $\eta_S(x) = p_S(x).e^{j.\mu_S(x)}$ and $\zeta_S(x) = r_S(x).e^{j.\omega_S(x)}$, where $p_S(x), r_S(x)$, $\mu_S(x)$, and $\omega_S(x)$ are all real-valued and $p_S(x)$, $r_S(x) \in [0,1]$ with $j = \sqrt{-1}$. The complex intuitionistic fuzzy set can be represented as

$$S = \{(x, \eta_S(x), \zeta_S(x)) : x \in U\}.$$

It is denoted as CIFS.

Definition 2.12 [1] Complement of complex intuitionistic fuzzy set.

Let S be a complex intuitionistic fuzzy set, and $\eta_S(x) = p_S(x).e^{j.\mu_S(x)}$ and $\zeta_S(x) = r_S(x).e^{j.\omega_S(x)}$ its membership and non-membership functions, respectively. The complement of S, denoted as $c(S)$, is specified by a function

$$\eta_{c(S)}(x) = p_{c(S)}(x).e^{j.\mu_{c(S)}(x)} = r_S(x).e^{j(2\pi-\mu_S(x))} \quad \text{and}$$
$$\zeta_{c(S)}(x) = r_{c(S)}(x).e^{j.\omega_{c(S)}(x)} = p_S(x).e^{j(2\pi-\omega_S(x))}$$

Definition 2.13 [1] Union of complex intuitionistic fuzzy sets.

Let A and B be two complex intuitionistic fuzzy sets on X, and $\eta_A(x) = p_A(x).e^{j.\mu_A(x)}$, $\zeta_A(x) = r_A(x).e^{j.\omega_A(x)}$ and $\eta_B(x) = p_B(x).e^{j.\mu_B(x)}$ and $\zeta_B(x) = r_B(x).e^{j.\omega_B(x)}$ be their membership and non-membership functions, respectively. The union of A and B is denoted as $A \cup B$, which is specified by a function

$$\eta_{A\cup B}(x) = p_{A\cup B}(x).e^{j.\mu_{A\cup B}(x)} = (p_A(x) \vee p_B(x)).e^{j(\mu_A(x)\vee\mu_B(x))}$$
and
$$\zeta_{A\cup B}(x) = r_{A\cup B}(x).e^{j.\omega_{A\cup B}(x)} = (r_A(x) \wedge r_B(x)).e^{j(\omega_A(x)\wedge\omega_B(x))}$$

where $\vee$ and $\wedge$ denote the max and min operator, respectively.

Definition 2.14 [1] Intersection of complex intuitionistic fuzzy sets.

Let A and B be two complex intuitionistic fuzzy sets on X, and $\eta_A(x) = p_A(x).e^{j.\mu_A(x)}$, $\zeta_A(x) = r_A(x).e^{j.\omega_A(x)}$ and $\eta_B(x) = p_B(x).e^{j.\mu_B(x)}$ and $\zeta_B(x) = r_B(x).e^{j.\omega_B(x)}$ be their membership and non-membership functions, respectively. The intersection of A and B is denoted as $A \cap B$, which is specified by a function

$$\eta_{A\cap B}(x) = p_{A\cap B}(x).e^{j.\mu_{A\cap B}(x)} = (p_A(x) \wedge p_B(x)).e^{j(\mu_A(x)\wedge\mu_B(x))} \quad \text{and}$$
$$\zeta_{A\cap B}(x) = r_{A\cap B}(x).e^{j.\omega_{A\cap B}(x)} = (r_A(x) \vee r_B(x)).e^{j(\omega_A(x)\vee\omega_B(x))}$$

where $\wedge$ and $\vee$ denote the min and max operators, respectively.

Next, the notion of complex neutrosophic set is introduced.

3 Complex neutrosophic set

In this section, we introduced the innovative concept of complex neutrosophic set. The definition of complex neutrosophic set is as follows.

Definition 3.1 Complex neutrosophic set.

A complex neutrosophic set S defined on a universe of discourse X, which is characterized by a truth membership function $T_S(x)$, an indeterminacy membership function $I_S(x)$, and a falsity membership function $F_S(x)$ that assigns a complex-valued grade of $T_S(x)$, $I_S(x)$, and $F_S(x)$ in S for any $x \in X$. The values $T_S(x)$, $I_S(x)$, $F_S(x)$ and their sum may all within the unit circle in the complex plane and so is of the following form,

$$T_S(x) = p_S(x).e^{j\mu_S(x)}, \quad I_S(x) = q_S(x).e^{j\nu_S(x)} \text{ and } F_S(x) = r_S(x).e^{j\omega_S(x)}$$

where $p_S(x)$, $q_S(x)$, $r_S(x)$ and $\mu_S(x)$, $\nu_S(x)$, $\omega_S(x)$ are, respectively, real valued and $p_S(x), q_S(x), r_S(x) \in [0,1]$ such that $^-0 \le p_S(x) + q_S(x) + r_S(x) \le 3^+$.

The complex neutrosophic set S can be represented in set form as

$$S = \{(x, T_S(x) = a_T, I_S(x) = a_I, F_S(x) = a_F) : x \in X\},$$

where $T_S{:}X \to \{a_T{:}a_T \in C, |a_T| \le 1\}$, $I_S{:}X \to \{a_I{:}a_I \in C, |a_I| \le 1\}$ and $F_S{:}X \to \{a_F{:}a_F \in C, |a_F| \le 1\}$ and $|T_S(x) + I_S(x) + F_S(x)| \le 3$.

Throughout the paper, complex neutrosophic set refers to a neutrosophic set with complex-valued truth membership function, complex-valued indeterminacy membership function, and complex-valued falsity membership function while the term neutrosophic set with real-valued truth membership function, indeterminacy membership function, and falsity membership function.

3.1 Interpretation of complex neutrosophic set

The concept of complex-valued truth membership function, complex-valued indeterminacy membership function, and complex-valued falsity membership function is not a simple task in understanding. In contrast, real-valued truth membership function, real-valued indeterminacy membership function, and real-valued falsity membership function in the interval [0,1] can be easily intuitive.

The notion of complex neutrosophic set can be easily understood from the form of its truth membership function, indeterminacy membership function, and falsity membership function which appears in above Definition 3.1.

$$T_S(x) = p_S(x).e^{j\mu_S(x)}, \quad I_S(x) = q_S(x).e^{j\nu_S(x)} \quad \text{and} \quad F_S(x) = r_S(x).e^{j\omega_S(x)}$$

It is clear that complex grade of truth membership function is defined by a truth amplitude term $p_S(x)$ and a truth phase term $\mu_S(x)$. Similarly, the complex grade of indeterminacy membership function is defined as an indeterminate amplitude term $q_S(x)$ and an indeterminate phase term $\nu_S(x)$, while the complex grade of falsity membership function is defined by false amplitude term $r_S(x)$ and a false phase term $\omega_S(x)$, respectively. It should be noted that the truth amplitude term $p_S(x)$ is equal to $|T_S(x)|$, the amplitude of $T_S(x)$. Also, the indeterminate amplitude term $q_S(x)$ is equal to $|I_S(x)|$ and the false amplitude term $r_S(x)$ is equal to $|F_S(x)|$.

Complex neutrosophic sets are the generalization of neutrosophic sets. It is a easy task to represent a neutrosophic set in the form of complex neutrosophic set. For instance, the neutrosophic set S is characterized by a real-valued truth membership function $\alpha_S(x)$, indeterminate membership function $\beta_S(x)$, and falsehood membership function $\gamma_S(x)$. By setting the truth amplitude term $p_S(x)$ equal to $\alpha_S(x)$, and the truth phase term $\mu_S(x)$ equal to zero for all x and similarly we can set the indeterminate amplitude term $q_S(x)$ equal to $\beta_S(x)$ and the indeterminate phase term equal to zero, while the false amplitude term $r_S(x)$ equal to $\gamma_S(x)$ with the false phase term equal to zero for all x. Thus, it has seen that a complex neutrosophic set can be easily transformed into a neutrosophic set. From this discussion, it is concluded that the truth amplitude term is equivalent to the real-valued grade of truth membership function, the indeterminate amplitude term is equivalent to the real-valued grade of indeterminate membership function, and the false amplitude term is essentially equivalent to the real-valued grade of false membership function. The only distinguishing factors are truth phase term, indeterminate phase term, and false phase term. This differs the complex neutrosophic set from the ordinary neutrosophic set. In simple words, if we omit all the three phase terms, the complex neutrosophic set will automatically convert into neutrosophic set effectively. All this discussion is supported by the reality that $p_S(x)$, $q_S(x)$, and $r_S(x)$ have range [0,1] which is for real-valued grade of truth membership, real-valued grade of indeterminate membership, and real-valued grade of false membership.

It should also be noted that complex neutrosophic sets are the generalization of complex fuzzy sets, conventional fuzzy sets, complex intuitionistic fuzzy sets and intuitionistic fuzzy sets, classical sets. This means that complex neutrosophic set is an advance generalization to all the existence methods and due to this feature, the concept of complex neutrosophic set is a distinguished and novel one.

3.2 Numerical example of a complex neutrosophic set

Example 3.2 Let $X = \{x_1, x_2, x_3\}$ be a universe of discourse. Then, S be a complex neutrosophic set in X as given below:

$$S = \frac{(0.6e^{j.0.8}, 0.3.e^{j.\frac{3\pi}{4}}, 0.5.e^{j.0.3})}{x_1} + \frac{(0.7e^{j.0}0.2.e^{j.0.9}, 0.1.e^{j.\frac{2\pi}{3}})}{x_2} + \frac{(0.9e^{j.0.1}, 0.4.e^{j.\pi}, 0.7.e^{j.0.7})}{x_3}.$$

3.3 Set theoretic operations on complex neutrosophic set

Ramot et al. [23], calculated the complement of membership phase term $\mu_S(x)$ by several possible method such as $\mu_S^c(x) = \mu_S(x), 2\pi - \mu_S(x)$. Zhang [39] defined the complement of the membership phase term by taking the rotation of $\mu_S(x)$ by π radian as $\mu_S^c(x) = \mu_S(x) + \pi$.

To define the complement of a complex neutrosophic set, we simply take the neutrosophic complement [29] for the truth amplitude term $p_S(x)$, indeterminacy amplitude term $q_S(x)$, and falsehood amplitude term $r_S(x)$. For phase terms, we take the complements defined in [23]. We now proceed to define the complement of complex neutrosophic set.

Definition 3.3 Complement of complex neutrosophic set.

Let $S = \{(x, T_S(x), I_S(x), F_S(x)) : x \in X\}$ be a complex neutrosophic set in X. Then, the complement of a complex neutrosophic set S is denoted as $c(S)$ and is defined by

$$c(S) = \left\{\left(x, T_S^c(x), I_S^c(x), F_S^c(x)\right) : x \in X\right\},$$

where $T_S^c(x) = c\left(p_S(x).e^{j.\mu_S(x)}\right)$, $I_S^c(x) = c\left(q_S(x).e^{j.\nu_S(x)}\right)$, and $F_S^c(x) = c\left(r_S(x).e^{j.\omega_S(x)}\right)$ in which $c\left(p_S(x).e^{j.\mu_S(x)}\right) = c(p_S(x)).e^{j.\mu_s^c(x)}$ is such that $c(p_S(x)) = r_S(x)$ and $\nu_S^c(x) = \nu_S(x), 2\pi - \nu_S(x)$ or $\nu_S(x) + \pi$. Similarly, $c(r_S(x).e^{j.\mu_S(x)}) = c(r_S(x)).e^{j.\omega_s^c(x)}$, where $c(q_S(x)) = 1 - q_S(x)$ and $\nu_S^c(x) = \nu_S(x), 2\pi - \nu_S(x)$ or $\nu_S(x) + \pi$.

Finally, $c\left(r_S(x).e^{j.\mu_S(x)}\right) = c(r_S(x)).e^{j.\omega_s^c(x)}$, where $c(r_S(x)) = p_S(x)$ and $\omega_S^c(x) = \omega_S(x), 2\pi - \omega_S(x)$ or $\omega_S(x) + \pi$.

Proposition 3.4 *Let A be a complex neutrosophic set on X. Then,* $c(c(A)) = A$.

Proof By definition 3.1, we can easily prove it.

Proposition 3.5 *Let A and B be two complex neutrosophic sets on X. Then,* $c(A \cap B) = c(A) \cup c(B)$.

Definition 3.6 Union of complex neutrosophic sets.

Ramot et al. [23] defined the union of two complex fuzzy sets A and B as follows.

Let $\mu_A(x) = r_A(x).e^{j.\omega_A(x)}$ and $\mu_B(x) = r_B(x).e^{j.\omega_B(x)}$ be the complex-valued membership functions of A and B, respectively. Then, the membership union of $A \cup B$ is given by $\mu_{A\cup B}(x) = [r_A(x) \oplus r_B(x)].e^{j.\omega_{A\cup B}(x)}$. Since $r_A(x)$ and $r_B(x)$ are real-valued and belong to $[0, 1]$, the operators max and min can be applied to them. For calculating phase term $\omega_{A\cup B}(x)$, they give several methods.

Now we define the union of two complex neutrosophic sets as follows:

Let A and B be two complex neutrosophic sets in X, where

$$A = \{(x, T_A(x), I_A(x), F_A(X)) : x \in X\} \text{ and}$$
$$B = \{(x, T_B(x), I_B(x), F_B(X)) : x \in X\}.$$

Then the union of A and B is denoted as $A \cup_N B$ and is given as

$$A \cup_N B = \{(x, T_{A\cup B}(x), I_{A\cup B}(x), F_{A\cup B}(x)) : x \in X\}$$

where the truth membership function $T_{A\cup B}(x)$, the indeterminacy membership function $I_{A\cup B}(x)$, and the falsehood membership function $F_{A\cup B}(x)$ are defined by

$$T_{A\cup B}(x) = [(p_A(x) \vee p_B(x))].e^{j.\mu_{T_{A\cup B}}(x)},$$
$$I_{A\cup B}(x) = [(q_A(x) \wedge q_B(x))].e^{j.\nu_{I_{A\cup B}}(x)},$$
$$F_{A\cup B}(x) = [(r_A(x) \wedge r_B(x))].e^{j.\omega_{F_{A\cup B}}(x)},$$

where $\vee$ and $\wedge$ denote the max and min operators, respectively. To calculate phase the terms $e^{j.\mu_{A\cup B}(x)}$, $e^{j.\nu_{A\cup B}(x)}$, and $e^{j.\omega_{A\cup B}(x)}$, we define the following:

Definition 3.7 Let A and B be two complex neutrosophic sets in X with complex-valued truth membership functions $T_A(x)$ and $T_B(x)$, complex-valued indeterminacy membership functions $I_A(x)$ and $I_B(x)$, and complex-valued falsehood membership functions $F_A(x)$ and $F_B(x)$, respectively. The union of the complex neutrosophic sets A and B is denoted by $A \cup_N B$ which is associated with the function:

$$\theta: \{(a_T, a_I, a_F): a_T, a_I, a_F \in C, |a_T + a_I + a_F| \le 3, |a_T|, |a_I|, |a_F| \le 1\}$$
$$\times \{(b_T, b_I, b_F): b_T, b_I, b_F \in C, |b_T + b_I + b_F| \le 3, |b_T|, |b_I|, |b_F| \le 1\}$$
$$\rightarrow \{(d_T, d_I, d_F): d_T, d_I, d_F \in C, |d_T + d_I + d_F| \le 3, |d_T|, |d_I|, |d_F| \le 1\},$$

where a, b, d, a', b', d', and a'', b'', d'' are the complex truth membership, complex indeterminacy membership, and complex falsity membership of A, B, and $A \cup B$, respectively.

A complex value is assigned by θ, that is, for all $x \in X$,

$$\theta(T_A(x), T_B(x)) = T_{A\cup B}(x) = d_T,$$
$$\theta(I_A(x), I_B(x)) = I_{A\cup B}(x) = d_I \quad \text{and}$$
$$\theta(F_A(x), F_B(x)) = F_{A\cup B}(x) = d_F.$$

This function θ must obey at least the following axiomatic conditions.

For any $a, b, c, d, a', b', c', d', a'', b'', c'', d'' \in \{x : x \in C, |x| \le 1\}$:

- Axiom 1: $|\theta_T(a, 0)| = |a|, |\theta_I(a', 1)| = |a'|$ and $|\theta_F(a'', 1)| = |a''|$ (boundary condition).
- Axiom 2: $\theta_T(a, b) = \theta_T(b, a), \theta_I(a', b') = \theta_I(b', a')$ and $\theta_F(a'', b'') = \theta_F(b'', a'')$ (commutativity condition).
- Axiom 3: if $|b| \le |d|$, then $|\theta_T(a, b)| \le |\theta_T(a, d)|$ and if $|b'| \le |d'|$, then $|\theta_I(a', b')| \le |\theta_I(a', d')|$ and if $|b''| \le |d''|$, then $|\theta_F(a'', b'')| \le |\theta_F(a'', d'')|$ (monotonic condition).
- Axiom 4: $\theta_T(\theta_T(a, b), c) = \theta_T(a, \theta_T(b, c))$, $\theta_I(\theta_I(a', b'), c') = \theta_I(a', \theta_I(b', c'))$ and $\theta_F(\theta_F(a'', b''), c'') = \theta_F(a'', \theta_F(b'', c''))$ (associative condition).

It may be possible in some cases that the following are also hold:

- Axiom 5: θ is continuous function (continuity).
- Axiom 6: $|\theta_T(a,a)| > |a|$, $|\theta_I(a',a')| < |a'|$ and $|\theta_F(a'',a'')| < |a''|$ (superidempotency).
- Axiom 7: $|a| \le |c|$ and $|b| \le |d|$, then $|\theta_T(a,b)| \le |\theta_T(c,d)|$, also $|a'| \ge |c'|$ and $|b'| \ge |d'|$, then $|\theta_I(a',b')| \ge |\theta_I(c',d')|$ and $|a''| \ge |c''|$ and $|b''| \le |d''|$, then $|\theta_F(a'',b'')| \ge |\theta_F(c',d')|$ (strict monotonicity).

The phase term of complex truth membership function, complex indeterminacy membership function, and complex falsity membership function belongs to $(0, 2\pi)$. To define the phase terms, we suppose that $\mu_{T_{A\cup B}}(x) = \mu_{A\cup B}(x)$, $\nu_{I_{A\cup B}}(x) = \nu_{A\cup B}(x)$, and $\omega_{F_{A\cup B}}(x) = \omega_{A\cup B}(x)$. Now we take those forms which Ramot et al. presented in [23] to define the phase terms of $T_{A\cup B}(x)$, $I_{A\cup B}(x)$, and $F_{A\cup B}(x)$, respectively.

(a) Sum:

$$\begin{aligned}\mu_{A\cup B}(x) &= \mu_A(x) + \mu_B(x),\\ \nu_{A\cup B}(x) &= \nu_A(x) + \nu_B(x),\\ \omega_{A\cup B}(x) &= \omega_A(x) + \omega_B(x).\end{aligned}$$

(b) Max:

$$\begin{aligned}\mu_{A\cup B}(x) &= \max(\mu_A(x), \mu_B(x)),\\ \nu_{A\cup B}(x) &= \max(\nu_A(x), \nu_B(x)),\\ \omega_{A\cup B}(x) &= \max(\omega_A(x), \omega_B(x)).\end{aligned}$$

(c) Min:

$$\begin{aligned}\mu_{A\cup B}(x) &= \min(\mu_A(x), \mu_B(x)),\\ \nu_{A\cup B}(x) &= \min(\nu_A(x), \nu_B(x)),\\ \omega_{A\cup B}(x) &= \min(\omega_A(x), \omega_B(x)).\end{aligned}$$

(d) "The game of winner, neutral, and loser":

$$\mu_{A\cup B}(x) = \begin{cases} \mu_A(x) & \text{if } p_A > p_B \\ \mu_B(x) & \text{if } p_B > p_A \end{cases},$$

$$\nu_{A\cup B}(x) = \begin{cases} \nu_A(x) & \text{if } q_A < q_B \\ \nu_B(x) & \text{if } q_B < q_A \end{cases},$$

$$\omega_{A\cup B}(x) = \begin{cases} \omega_A(x) & \text{if } r_A < r_B \\ \omega_B(x) & \text{if } r_B < r_A \end{cases}.$$

The game of winner, neutral, and loser is a novel concept, and it is the generalization of the concept "winner take all" introduced by Ramot et al. [23] for the union of phase terms.

Example 3.8 Let $X = \{x_1, x_2, x_3\}$ be a universe of discourse. Let A and B be two complex neutrosophic sets in X as shown below:

$$A = \frac{\left(0.6e^{j.0.8}, 0.3.e^{j\cdot\frac{3\pi}{4}}, 0.5.e^{j.0.3}\right)}{x_1} + \frac{\left(0.7e^{j.0}0.2.e^{j.0.9}, 0.1.e^{j\cdot\frac{2\pi}{3}}\right)}{x_2} + \frac{\left(0.9e^{j.0.1}, 0.4.e^{j.\pi}, 0.7.e^{j.0.7}\right)}{x_3},$$

and

$$B = \frac{\left(0.8e^{j.0.9}, 0.1.e^{j\cdot\frac{\pi}{4}}, 0.4.e^{j.0.5}\right)}{x_1} + \frac{\left(0.6e^{j.0.1}, 1.e^{j.0.6}, 0.01.e^{j\cdot\frac{4\pi}{3}}\right)}{x_2} + \frac{\left(0.2e^{j.0.3}, 0.e^{j.2\pi}, 0.5.e^{j.0.5}\right)}{x_3},$$

Then

$$A \cup_N B = \frac{\left(0.8.e^{j.0.9}, 0.3.e^{j\cdot\frac{3\pi}{4}}, 0.5.e^{j.0.5}\right)}{x_1}, \frac{\left(0.6.e^{j.0.1}, 0.2.e^{j.0.9}, 0.01, e^{j\cdot\frac{4\pi}{3}}\right)}{x_2}, \frac{\left(0.2.e^{j.0.3}, 0.e^{j.2\pi}, 0.5.e^{j.0.7}\right)}{x_3}.$$

Definition 3.9 Intersection of complex neutrosophic sets.

Let A and B be two complex neutrosophic sets in X, where

$$A = \{(x, T_A(x), I_A(x), F_A(X)) : x \in X\} \text{ and}$$
$$B = \{(x, T_B(x), I_B(x), F_B(X)) : x \in X\}.$$

Then, the intersection of A and B is denoted as $A \cap_N B$ and is defined as

$$A \cap_N B = \{(x, T_{A\cap B}(x), I_{A\cap B}(x), F_{A\cap B}(x)) : x \in X\},$$

where the truth membership function $T_{A\cap B}(x)$, the indeterminacy membership function $I_{A\cap B}(x)$, and the falsehood membership function $F_{A\cap B}(x)$ are given as:

$$\begin{aligned}T_{A\cap B}(x) &= [(p_A(x) \wedge p_B(x))].e^{j\cdot\mu_{T_{A\cap B}}(x)},\\ I_{A\cap B}(x) &= [(q_A(x) \vee q_B(x))].e^{j\cdot\nu_{I_{A\cap B}}(x)},\\ F_{A\cap B}(x) &= [(r_A(x) \vee r_B(x))].e^{j\cdot\omega_{F_{A\cap B}}(x)},\end{aligned}$$

where $\vee$ and $\wedge$ denote the max and min operators, respectively. We calculate phase terms $e^{j\cdot\mu_{A\cap B}(x)}$, $e^{j\cdot\nu_{A\cap B}(x)}$, and $e^{j\cdot\omega_{A\cap B}(x)}$ after define the following:

Definition 3.10 Let A and B be two complex neutrosophic sets in X with complex-valued truth membership functions $T_A(x)$ and $T_B(x)$, complex-valued indeterminacy membership functions $I_A(x)$ and $I_B(x)$, and complex-valued falsehood membership functions $F_A(x)$ and $F_B(x)$, respectively. The intersection of the complex neutrosophic sets A and B is denoted by $A \cap_N B$ which is associated with the function:

$$\phi:\{(a_T,a_I,a_F):a_T,a_I,a_F\in C,|a_T+a_I+a_F|\le 3,|a_T|,|a_I|,|a_F|\le 1\}$$
$$\times\{(b_T,b_I,b_F):b_T,b_I,b_F\in C,|b_T+b_I+b_F|\le 3,|b_T|,|b_I|,|b_F|\le 1\}$$
$$\to\{(d_T,d_I,d_F):d_T,d_I,d_F\in C,|d_T+d_I+d_F|\le 3,|d_T|,|d_I|,|d_F|\le 1\},$$

where a,b,d, a',b',d', and a'',b'',d'' are the complex truth membership, complex indeterminacy membership, and complex falsity membership of A, B, and $A\cap B$, respectively. ϕ assigned a complex value, that is, for all $x\in X$,

$$\phi(T_A(x),T_B(x)) = T_{A\cap B}(x) = d_T,$$
$$\phi(I_A(x),I_B(x)) = I_{A\cap B}(x) = d_I \text{ and}$$
$$\phi(F_A(x),F_B(x)) = F_{A\cap B}(x) = d_F.$$

ϕ must satisfy at least the following axiomatic conditions.

For any $a,b,c,d,a',b',c',d',a'',b'',c''$, $d''\in\{x : x\in C, |x|\le 1\}$:

- Axiom 1: If $|b| = 1$, then $|\phi_T(a,b)| = |a|$. If $|b'| = 0$, then $|\phi_I(a',b')| = |a'|$ and if $|\phi_F(a'',b'')| = |a''|$ (boundary condition).
- Axiom 2: $\phi_T(a,b) = \phi_T(b,a)$, $\phi_I(a',b') = \phi_I(b',a')$, and $\phi_F(a'',b'') = \phi_F(b'',a'')$ (commutative condition).
- Axiom 3: if $|b|\le|d|$, then $|\phi_T(a,b)|\le|\phi_T(a,d)|$ and if $|b'|\le|d'|$, then $|\phi_I(a',b')|\le|\phi_I(a',d')|$ and if $|b''|\le|d''|$, then $|\phi_F(a'',b'')|\le|\phi_F(a'',d'')|$ (monotonic condition).
- Axiom 4: $\phi_T(\phi_T(a,b),c) = \phi_T(a,\phi_T(b,c))$, $\phi_I(\phi_I(a',b'),c') = \phi_I(a',\phi_I(b',c'))$, and $\phi_F(\phi_F(a'',b''),c'') = \phi_F(a'',\phi_F(b'',c''))$ (associative condition).

The following axioms also hold in some cases.

- Axiom 5: ϕ is continuous function (continuity).
- Axiom 6: $|\phi_T(a,a)| > |a|$, $|\phi_I(a',a')| < |a'|$, and $|\phi_F(a'',a'')| < |a''|$ (superidempotency).
- Axiom 7: $|a|\le|c|$ and $|b|\le|d|$, then $|\phi_T(a,b)|\le|\phi_T(c,d)|$, also $|a'|\ge|c'|$ and $|b'|\ge|d'|$, then $|\phi_I(a',b')|\ge|\phi_I(c',d')|$ and $|a''|\ge|c''|$ and $|b''|\le|d''|$, then $|\phi_F(a'',b'')|\ge|\phi_F(c'',d'')|$ (strict monotonicity).

We can easily calculate the phase terms $e^{j\cdot\mu_{A\cap B}(x)}$, $e^{j\cdot v_{A\cap B}(x)}$, and $e^{j\cdot\omega_{A\cap B}(x)}$ on the same lines by winner, neutral, and loser game.

Proposition 3.11 *Let A,B,C be three complex neutrosophic sets on X. Then,*

1. $(A\cup B)\cap C = (A\cap C)\cup(A\cap B)$,
2. $(A\cap B)\cup C = (A\cup C)\cap(A\cup B)$.

Proof Here we only prove part 1. Let A,B,C be three complex neutrosophic sets in X and $T_A(x),I_A(x),F_A(x)$, $T_B(x),I_B(x),F_B(x)$ and $T_C(x),I_C(x),F_V(x)$, respectively, be their complex-valued truth membership function, complex-valued indeterminate membership function, and complex-valued falsehood membership functions. Then, we have

$$\begin{aligned}T_{(A\cup B)\cap C}(x) &= p_{(A\cup B)\cap C}(x).e^{j\cdot\mu_{(A\cup B)\cap C}(x)}\\ &= \min(p_{A\cup B}(x),p_C(x)).e^{j\cdot\min(\mu_{A\cup B}(x),\mu_C(x))},\\ &= \min(\max(p_A(x),p_B(x)),p_C(x))\\ &\quad .\,e^{j\cdot\min(\max(\mu_A(x),\mu_B(x)),\mu_C(x))},\\ &= \max(\min(p_A(x),p_c(x)),\min(p_B(x),p_C(x)))\\ &\quad .\,e^{j.max(\min(\mu_A(x),\mu_C(x)),\min(\mu_B(x),\mu_C(x)))},\\ &= \max(p_{A\cap C}(x),p_{B\cap C}(x)).\,e^{j.max(\mu_{A\cap C}(x),\mu_{B\cap C}(x))},\\ &= p_{(A\cap C)\cup(B\cap C)}(x).e^{j\cdot\mu_{(A\cap C)\cup(B\cap C)}(x)} = T_{(A\cap C)\cup(B\cap C)}(x).\end{aligned}$$

Similarly, on the same lines, we can show it for $I_{(A\cup B)\cap C}(x)$ and $F_{(A\cup B)\cap C}(x)$, respectively. □

Proposition 3.12 *Let A and B be two complex neutrosophic sets in X. Then,*

1. $(A\cup B)\cap A = A$,
2. $(A\cap B)\cup A = A$.

Proof We prove it for part 1. Let A and B be two complex neutrosophic sets in X and $T_A(x),I_A(x),F_A(x)$ and $T_B(x),I_B(x),F_B(x)$, respectively, be their complex-valued truth membership function, complex-valued indeterminate membership function, and complex-valued falsehood membership functions. Then,

$$\begin{aligned}T_{(A\cup B)\cap A}(x) &= p_{(A\cup B)\cap A}(x).\,e^{j\cdot\mu_{(A\cup B)\cap A}(x)}\\ &= \min(p_{A\cup B}(x),p_A(x))\\ &\quad .\,e^{i.\min(\mu_{A\cup B}(x),\mu_A(x))},\\ &= \min(\max(p_A(x),p_B(x)),p_A(x))\\ &\quad .\,e^{i.\min(\max(\mu_A(x),\mu_B(x)),\mu_A(x))}\\ &= T_A(x).\end{aligned}$$

Similarly, we can show it for $I_{(A\cup B)\cap A}(x)$ and $F_{(A\cup B)\cap A}(x)$, respectively. □

Definition 3.13 Let A and B be two complex neutrosophic sets on X, and $T_A(x) = p_A(x).e^{j\cdot\mu_A(x)}$, $I_A(x) = q_A(x).e^{j\cdot v_A(x)}$, $F_A(x) = r_A(x).e^{j\cdot\omega_A(x)}$ and $T_B(x) = p_B(x).e^{j\cdot\mu_B(x)}$, $I_B(x) = q_B(x).e^{j\cdot v_B(x)}$, $F_B(x) = r_B(x).e^{j\cdot\omega_B(x)}$, respectively, be their complex-valued truth membership function, complex-valued indeterminacy membership function, and complex-valued falsity membership function.

The complex neutrosophic product of A and B denoted as $A \circ B$ and is specified by the functions,

$$T_{A\circ B}(x) = p_{A\circ B}(x).e^{j.\mu_{A\circ B}(x)} = (p_A(x).p_B(x)).e^{j.2\pi\left(\frac{\mu_A(x)}{2\pi}.\frac{\mu_B(x)}{2\pi}\right)},$$
$$I_{A\circ B}(x) = q_{A\circ B}(x).e^{j.\mu_{A\circ B}(x)} = (q_A(x).q_B(x)).e^{j.2\pi\left(\frac{\nu_A(x)}{2\pi}.\frac{\nu_B(x)}{2\pi}\right)},$$
$$F_{A\circ B}(x) = r_{A\circ B}(x).e^{j.\mu_{A\circ B}(x)} = (r_A(x).r_B(x)).e^{j.2\pi\left(\frac{\omega_A(x)}{2\pi}.\frac{\mu_B(x)}{2\pi}\right)}.$$

Example 3.14 Let $X = \{x_1, x_2, x_3\}$ and let

$$A = \frac{(0.6e^{j1.2\pi}, 0.3e^{j0.5\pi}, 1.0e^{j0.1\pi})}{x_1} + \frac{(1.0e^{j2\pi}, 0.2e^{j.3\pi}, 0.5e^{j0.4\pi})}{x_2} + \frac{(0.8e^{j1.6\pi}, 0.1e^{j1.2}, 0.6e^{j0.1\pi})}{x_3},$$

$$B = \frac{(0.6e^{j1.2\pi}, 0.1e^{j0.4\pi}, 1.0e^{j0.1\pi})}{x_1} + \frac{(1.0e^{j1.2\pi}, 0.3e^{j.2\pi}, 0.7e^{j0.5\pi})}{x_2} + \frac{(0.2e^{j1.6\pi}, 0.2e^{j1.3\pi}, 0.7e^{j0.1\pi})}{x_3}$$

Then

$$A \circ B = \frac{(0.36e^{j0.72\pi}, 0.3e^{j0.1\pi}, 1.0e^{j0.0025\pi})}{x_1}, \frac{(1.0e^{j1.2\pi}, 0.06e^{j3\pi}, 0.35e^{j0.1\pi})}{x_3}, \frac{(0.16e^{j1.28\pi}, 0.02e^{j0.78\pi}, 0.42e^{j0.005\pi})}{x_3}$$

Definition 3.15 Let A_n be N complex neutrosophic sets on X $(n = 1, 2, \ldots, N)$, and $T_{A_n}(x) = p_{A_n}(x).e^{j.\mu_{A_n}(x)}$, $I_{A_n}(x) = q_{A_n}(x).e^{j.\nu_{A_n}(x)}$, and $F_{A_n}(x) = r_{A_n}(x).e^{j.\omega_{A_n}(x)}$ be their complex-valued membership function, complex-valued indeterminacy membership function and complex-valued non-membership function, respectively. The Cartesian product of A_n, denoted as $A_1 \times A_2 \times \cdots \times A_N$, specified by the function

$$T_{A_1\times A_2\times\cdots\times A_N}(x) = p_{A_1\times A_2\times\cdots\times A_N}(x).e^{j.\mu_{A_1\times A_2\times\cdots\times A_N}(x)} = \min(p_{A_1}(x_1), p_{A_2}(x_2), \ldots, p_{A_N}(x_N)) .\, e^{j\min\left(\mu_{A_1}(x_1),\mu_{A_2}(x_2),\ldots,\mu_{A_N}(x_N)\right)},$$

$$I_{A_1\times A_2\times\cdots\times A_N}(x) = q_{A_1\times A_2\times\cdots\times A_N}(x).e^{j.\nu_{A_1\times A_2\times\cdots\times A_N}(x)} = \max(q_{A_1}(x_1), q_{A_2}(x_2), \ldots, q_{A_N}(x_N)) .\, e^{j\max\left(\nu_{A_1}(x_1),\nu_{A_2}(x_2),\ldots,\nu_{A_N}(x_N)\right)},$$

and

$$F_{A_1\times A_2\times\cdots\times A_N}(x) = r_{A_1\times A_2\times\cdots\times A_N}(x).e^{j.\omega_{A_1\times A_2\times\cdots\times A_N}(x)} = \max(r_{A_1}(x_1), r_{A_2}(x_2), \ldots, r_{A_N}(x_N)) .\, e^{j\max\left(\omega_{A_1}(x_1),\omega_{A_2}(x_2),\ldots,\omega_{A_N}(x_N)\right)},$$

where $x = (x_1, x_2, \ldots, x_N) \in \underbrace{X \times X \times \cdots \times X}_{N}$.

4 Distance measure and δ-equalities of complex neutrosophic sets

In this section, we introduced distance measure and other operational properties of complex neutrosophic sets.

Definition 4.1 Let $\mathrm{CN}(X)$ be the collection of all complex neutrosophic sets on X and $A, B \in \mathrm{CN}(X)$. Then, $A \subseteq B$ if and only if $T_A(x) \le T_B(x)$ such that the amplitude terms $p_A(x) \le p_B(x)$ and the phase terms $\mu_A(x) \le \mu_B(x)$, and $I_A(x) \ge I_B(x)$ such that the amplitude terms $q_A(x) \ge q_B(x)$ and the phase terms $\nu_A(x) \ge \nu_B(x)$ whereas $F_A(x) \ge F_B(x)$ such that the amplitude terms $r_A(x) \ge r_B(x)$ and the phase terms $\omega_A(x) \ge \omega_B(x)$.

Definition 4.2 Two complex neutrosophic sets A and B are said to equal if and only if $p_A(x) = p_B(x)$, $q_A(x) = q_B(x)$, and $r_A(x) = r_B(x)$ for amplitude terms and $\mu_A(x) = \mu_B(x)$, $\nu_A(x) = \nu_B(x)$, $\omega_A(x) = \omega_B(x)$ for phase terms (arguments).

Definition 4.3 A distance of complex neutrosophic sets is a function d_{CNS}:$\mathrm{CN}(X) \times \mathrm{CN}(X) \to [0,1]$ such that for any $A, B, C \in \mathrm{CN}(X)$

1. $0 \le d_{\mathrm{CNS}}(A,B) \le 1$,
2. $d_{\mathrm{CNS}}(A,B) = 0$ if and only if $A = B$,
3. $d_{\mathrm{CNS}}(A,B) = d_{\mathrm{CNS}}(B,A)$,
4. $d_{\mathrm{CNS}}(A,B) \le d_{\mathrm{CNS}}(A,C) + d_{\mathrm{CNS}}(C,B)$.

Let d_{CNS}:$\mathrm{CN}(X) \times \mathrm{CN}(X) \to [0,1]$ be a function which is defined as

$$d_{\mathrm{CNS}}(A,B) = \max\begin{pmatrix} \max(\sup_{x\in X}|p_A(x) - p_B(x)|, \sup_{x\in X}|q_A(x) - q_B(x)|, \sup_{x\in X}|r_A(x) - r_B(x)|), \\ \max(\frac{1}{2\pi}\sup_{x\in X}|\mu_A(x) - \mu_B(x)|, \frac{1}{2\pi}\sup_{x\in X}|\nu_A(x) - \nu_B(x)|, \frac{1}{2\pi}\sup_{x\in X}|\omega_A(x) - \omega_B(x)|) \end{pmatrix}$$

Theorem 4.4 *The function $d_{CNS}(A,B)$ defined above is a distance function of complex neutrosophic sets on X.*

Proof The proof is straightforward. □

Definition 4.5 Let A and B be two complex neutrosophic sets on X, and $T_A(x) = p_A(x).e^{j.\mu_A(x)}$, $I_A(x) = q_A(x).e^{j.\nu_A(x)}$, $F_A(x) = r_A(x).e^{j.\omega_A(x)}$ and $T_B(x) = p_A(x).e^{j.\mu_B(x)}$, $I_B(x) = q_B(x).e^{j.\nu_B(x)}$, $F_B(x) = r_B(x).e^{j.\omega_B(x)}$ are their complex-valued truth membership, complex-valued indeterminacy membership, and complex-valued falsity membership functions, respectively. Then, A and B are said to be δ-equal, if and only if $d_{CNS}(A,B) \le 1-\delta$, where $0 \le \delta \le 1$. It is denoted by $A = (\delta)B$.

Proposition 4.6 *For complex neutrosophic sets A, B, and C, the following holds.*

1. $A = (0)B$,
2. $A = (1)B$ if and only if $A = B$,
3. If $A = (\delta)B$ *if and only if* $B = (\delta)A$,
4. $A = (\delta_1)B$ *and* $\delta_2 \le \delta_1$, *then* $A = (\delta_2)B$,
5. If $A = (\delta_\alpha)B$, *then* $A = (\sup_{\alpha\in J}\delta_\alpha)B$ *for all* $\alpha \in J$, *where J is an index set*,
6. If $A = (\delta')B$ *and there exist a unique* δ *such that* $A = (\delta)B$, *then* $\delta' \le \delta$ *for all* A, B
7. *If* $A = (\delta_1)B$ and $B = (\delta_2)C$, *then* $A = (\delta)C$, *where* $\delta = \delta_1 * \delta_2$.

Proof 4.7 Properties 1–4, 6 can be proved easily. We only prove 5 and 7.

5. Since $A = (\delta_\alpha)B$ for all $\alpha \in J$, we have

$$d_{CNS}(A,B) = \max\begin{pmatrix} \max(\sup_{x\in X}|p_A(x)-p_B(x)|, \sup_{x\in X}|q_A(x)-q_B(x)|, \sup_{x\in X}|r_A(x)-r_B(x)|), \\ \max(\frac{1}{2\pi}\sup_{x\in X}|\mu_A(x)-\mu_B(x)|, \frac{1}{2\pi}\sup_{x\in X}|\nu_A(x)-\nu_B(x)|, \frac{1}{2\pi}\sup_{x\in X}|\omega_A(x)-\omega_B(x)|) \end{pmatrix} \le 1-\delta_\alpha$$

Therefore,

$$\sup_{x\in X}|p_A(x)-p_B(x)| \le 1 - \sup_{\alpha\in J}\delta_\alpha,$$
$$\sup_{x\in X}|q_A(x)-q_B(x)| \le 1 - \sup_{\alpha\in J}\delta_\alpha,$$
$$\sup_{x\in X}|r_A(x)-r_B(x)| \le 1 - \sup_{\alpha\in J}\delta_\alpha, \quad \text{and}$$
$$\frac{1}{2\pi}\sup_{x\in X}|\mu_A(x)-\mu_B(x)| \le 1 - \sup_{\alpha\in J}\delta_\alpha,$$
$$\frac{1}{2\pi}\sup_{x\in X}|\nu_A(x)-\nu_B(x)| \le 1 - \sup_{\alpha\in J}\delta_\alpha$$
$$\frac{1}{2\pi}\sup_{x\in X}|\omega_A(x)-\omega_B(x)| \le 1 - \sup_{\alpha\in J}\delta_\alpha.$$

Thus,

$$d_{CNS}(A,B) = \max\begin{pmatrix} \max(\sup_{x\in X}|p_A(x)-p_B(x)|, \sup_{x\in X}|q_A(x)-q_B(x)|, \sup_{x\in X}|r_A(x)-r_B(x)|), \\ \max\left(\frac{1}{2\pi}\sup_{x\in X}|\mu_A(x)-\mu_B(x)|, \frac{1}{2\pi}\sup_{x\in X}|\nu_A(x)-\nu_B(x)|, \frac{1}{2\pi}\sup_{x\in X}|\omega_A(x)-\omega_B(x)|\right) \end{pmatrix}$$
$$\le 1 - \sup_{\alpha\in J}\delta_\alpha$$

Hence, $A = (\sup_{\alpha \in J} \delta_\alpha)B$.

7. Since $A = (\delta_1)B$, we have

$$d_{\text{CNS}}(A,B) = \max\begin{pmatrix} \max(\sup_{x\in X}|p_A(x) - p_B(x)|, \sup_{x\in X}|q_A(x) - q_B(x)|, \sup_{x\in X}|r_A(x) - r_B(x)|), \\ \max(\frac{1}{2\pi}\sup_{x\in X}|\mu_A(x) - \mu_B(x)|, \frac{1}{2\pi}\sup_{x\in X}|\nu_A(x) - \nu_B(x)|, \frac{1}{2\pi}\sup_{x\in X}|\omega_A(x) - \omega_B(x)|) \end{pmatrix} \leq 1 - \delta_1$$

which implies

$$\sup_{x\in X}|p_A(x) - p_B(x)| \leq 1 - \delta_1,$$
$$\sup_{x\in X}|q_A(x) - q_B(x)| \leq 1 - \delta_1,$$
$$\sup_{x\in X}|r_A(x) - r_B(x)| \leq 1 - \delta_1 \text{ and}$$
$$\frac{1}{2\pi}\sup_{x\in X}|\mu_A(x) - \mu_B(x)| \leq 1 - \delta_1, \ \frac{1}{2\pi}\sup_{x\in X}|\nu_A(x) - \nu_B(x)| \leq 1 - \delta_1$$
$$\frac{1}{2\pi}\sup_{x\in X}|\omega_A(x) - \omega_B(x)| \leq 1 - \delta_1.$$

Also we have $B = (\delta_2)C$, so

$$d_{\text{CNS}}(B,C) = \max\begin{pmatrix} \max(\sup_{x\in X}|p_B(x) - p_C(x)|, \sup_{x\in X}|q_B(x) - q_C(x)|, \sup_{x\in X}|r_B(x) - r_C(x)|), \\ \max\left(\frac{1}{2\pi}\sup_{x\in X}|\mu_B(x) - \mu_C(x)|, \frac{1}{2\pi}\sup_{x\in X}|\nu_B(x) - \nu_C(x)|, \frac{1}{2\pi}\sup_{x\in X}|\omega_B(x) - \omega_C(x)|\right) \end{pmatrix} \leq 1 - \delta_2$$

which implies

$$\sup_{x\in X}|p_B(x) - p_C(x)| \leq 1 - \delta_2,$$
$$\sup_{x\in X}|q_B(x) - q_C(x)| \leq 1 - \delta_2,$$
$$\sup_{x\in X}|r_B(x) - r_C(x)| \leq 1 - \delta_1 \text{ and}$$
$$\frac{1}{2\pi}\sup_{x\in X}|\mu_B(x) - \mu_C(x)| \leq 1 - \delta_2, \ \frac{1}{2\pi}\sup_{x\in X}|\nu_B(x) - \nu_C(x)| \leq 1 - \delta_2,$$
$$\frac{1}{2\pi}\sup_{x\in X}|\omega_B(x) - \omega_C(x)| \leq 1 - \delta_2.$$

Now,

$$d_{\mathrm{CNS}}(A,C) = \max\begin{pmatrix} \max(\sup_{x\in X}|p_A(x)-p_C(x)|, \sup_{x\in X}|q_A(x)-q_C(x)|, \sup_{x\in X}|r_A(x)-r_C(x)|), \\ \max(\frac{1}{2\pi}\sup_{x\in X}|\mu_A(x)-\mu_C(x)|, \frac{1}{2\pi}\sup_{x\in X}|\nu_A(x)-\nu_C(x)|, \frac{1}{2\pi}\sup_{x\in X}|\omega_A(x)-\omega_C(x)|) \end{pmatrix}$$
$$\leq \max\begin{pmatrix} \max(\sup_{x\in X}|p_A(x)-p_B(x)|, \sup_{x\in X}|q_A(x)-q_B(x)|, \sup_{x\in X}|r_A(x)-r_B(x)|)+ \\ \max(\sup_{x\in X}|p_B(x)-p_C(x)|, \sup_{x\in X}|q_B(x)-q_C(x)|, \sup_{x\in X}|r_B(x)-r_C(x)|), \\ \max(\frac{1}{2\pi}\sup_{x\in X}|\mu_A(x)-\mu_B(x)|, \frac{1}{2\pi}\sup_{x\in X}|\nu_A(x)-\nu_B(x)|, \frac{1}{2\pi}\sup_{x\in X}|\omega_A(x)-\omega_B(x)|)+ \\ \max(\frac{1}{2\pi}\sup_{x\in X}|\mu_B(x)-\mu_C(x)|, \frac{1}{2\pi}\sup_{x\in X}|\nu_B(x)-\nu_C(x)|, \frac{1}{2\pi}\sup_{x\in X}|\omega_B(x)-\omega_C(x)|) \end{pmatrix}$$
$$\leq \max((1-\delta_1)+(1-\delta_2), (1-\delta_1)+(1-\delta_2)) = (1-\delta_1)+(1-\delta_2) = 1-(\delta_1+\delta_2-1),$$

From Definition 4.3, $d_{\mathrm{CNS}}(A,C) \leq 1$. Therefore, $d_{\mathrm{CNS}}(A,C) \leq 1-\delta_1 * \delta_2 = 1-\delta$, where $\delta = \delta_1 * \delta_2$. Thus, $A = (\delta)C$. □

Theorem 4.8 *If $A = (\delta)B$, then $c(A) = (\delta)c(B)$, where $c(A)$ and $c(B)$ are the complement of the complex neutrosophic sets A and B.*

Proof Since

$$d_{\mathrm{CNS}}(c(A),c(B)) = \max\begin{pmatrix} \max\left(\sup_{x\in X}\left|p_{C(A)}(x)-p_{c(B)}(x)\right|, \sup_{x\in X}\left|q_{c(A)}(x)-q_{c(B)}(x)\right|, \sup_{x\in X}\left|r_{c(A)}(x)-r_{c(B)}(x)\right|\right), \\ \max\begin{pmatrix} \frac{1}{2\pi}\sup_{x\in X}\left|\mu_{c(A)}(x)-\mu_{c(B)}(x)\right|, \frac{1}{2\pi}\sup_{x\in X}\left|\nu_{c(A)}(x)-\nu_{c(B)}(x)\right|, \\ \frac{1}{2\pi}\sup_{x\in X}\left|\omega_{c(A)}(x)-\omega_{c(B)}(x)\right| \end{pmatrix} \end{pmatrix}$$
$$= \max\begin{pmatrix} \max(\sup_{x\in X}|r_A(x)-r_B(x)|, \sup_{x\in X}|(1-q_A(x))-(1-q_B(x))|, \sup_{x\in X}|p_A(x)-p_B(x)|), \\ \max\begin{pmatrix} \frac{1}{2\pi}\sup_{x\in X}|(2\pi-\mu_A(x))-(2\pi-\mu_B(x))|, \frac{1}{2\pi}\sup_{x\in X}|(2\pi-\nu_A(x))-(2\pi-\nu_B(x))|, \\ \frac{1}{2\pi}\sup_{x\in X}|(2\pi-\omega_A(x))-(2\pi-\omega_B(x))| \end{pmatrix} \end{pmatrix}$$
$$= \max\begin{pmatrix} \max(\sup_{x\in X}|p_A(x)-p_B(x)|, \sup_{x\in X}|q_A(x)-q_B(x)|, \sup_{x\in X}|r_A(x)-r_B(x)|), \\ \max\begin{pmatrix} \frac{1}{2\pi}\sup_{x\in X}|\mu_A(x)-\mu_B(x)|, \frac{1}{2\pi}\sup_{x\in X}|\nu_A(x)-\nu_B(x)|, \\ \frac{1}{2\pi}\sup_{x\in X}|\omega_A(x)-\omega_B(x)| \end{pmatrix} \end{pmatrix} = d_{\mathrm{CNS}}(A,B) \leq 1-\delta$$

□

5 Application of complex neutrosophic set in signal processing

The complex neutrosophic set and δ-equalities of complex neutrosophic sets are applied in signal processing application which demonstrates to point out a particular signal of interest out of a large number of signals that are received by a digital receiver. This is the example which Ramot et al. [23] discussed for complex fuzzy set. We now apply complex neutrosophic set to this example.

Suppose that there are L' different signals, $S_1(t), S_2(t), \ldots, S_{L'}(t)$. These signals have been detected and sampled by a digital receiver and each of which is sampled N times. Suppose that $S_{l'}(k')$ denote the k'th of the l'-th signal, where $1 \leq k' \leq N$ and $1 \leq l' \leq L'$. Now we form the following algorithm for this application.

5.1 Algorithm

Step 1. Write the discrete Fourier transforms of the L' signals in the form of complex neutrosophic set,

$$S_{l'}(k') = \frac{1}{N}.\sum_{n=1}^{N}\left(C_{l',n}, D_{l',n}, E_{l',n}\right).e^{\frac{2\pi j(n-1)(k'-1)}{N}} \quad (1)$$

where $C_{l',n}, D_{l',n}, E_{l',n}$ are the complex-valued Fourier coefficients of the signals and $1 \leq n \leq N$.

The above sum may be written as

$$S_{l'}(k') = \frac{1}{N}.\sum_{n=1}^{N}\left(U_{l',n}, V_{l',n}, W_{l',n}\right).e^{\frac{j(2\pi(n-1)(k'-1)+\alpha_{l',n})}{N}} \quad (2)$$

where $C_{l',n} = U_{l',n}.e^{j\alpha_{l',n}}$, $D_{l',n} = V_{l',n}.e^{j\alpha_{l',n}}$, and $E_{l',n} = W_{l',n}.e^{j\alpha_{l',n}}$ with $U_{l',n}$, $V_{l',n}$, $W_{l',n} \geq 0$, and $\alpha_{l',n}$ are real-valued for all n.

The purpose of this application is to point out the reference signals R' of the L' signals. This reference signal R\prime has been also sampled N times $(1 \le n \le N)$.

Step 2. Write the Fourier coefficient of R' in the form of complex neutrosophic set,

$$R'(k') = \frac{1}{N} \cdot \sum_{n=1}^{N} \left(C_{R',n}, D_{R',n}, E_{R',n} \right) . e^{\frac{j2\pi(n-1)(k'-1)}{N}} \tag{3}$$

where $C_{R',n}, D_{R',n}, E_{R',n}$ are the complex Fourier coefficients of the reference signals.

The above expression can be rewritten as

$$R'(k') = \frac{1}{N} \cdot \sum_{n=1}^{N} \left(U_{R',n}, V_{R',n}, W_{R',n} \right) . e^{\frac{j(2\pi(n-1)(k'-1) + \alpha_{R',n})}{N}} \tag{4}$$

where $C_{R',n} = U_{R',n}.e^{j\alpha_{R',n}}$, $D_{R',n} = V_{R',n}.e^{j\alpha_{R',n}}$, and $E_{R',n} = W_{R',n}.e^{j\alpha_{R',n}}$ with $U_{R',n}$, $V_{R',n}$, $W_{R',n} \ge 0$, and $\alpha_{R',n}$ are real-valued for all n.

Step 3 Since the sum of truth amplitude term, indeterminate amplitude term, and falsity amplitude term (in the case when they are crisp numbers, not sets) is not necessarily equal to 1, the normalization is not required and we can keep them un-normalized. But if the normalization is needed, we can normalize the amplitude terms of $S_{l'}(k')$ and $R'(k')$, respectively, as follows:

$$\underset{l',n}{\tilde{U}} = \frac{U_{l',n}}{U_{l',n} + V_{l',n} + W_{l',n}}, \quad \underset{l',n}{\tilde{V}} = \frac{V_{l',n}}{U_{l',n} + V_{l',n} + W_{l',n}},$$
$$\underset{l',n}{\tilde{W}} = \frac{W_{l',n}}{U_{l',n} + V_{l',n} + W_{l',n}} \text{ and } \underset{R',n}{\tilde{U}} = \frac{U_{R',n}}{U_{R',n} + V_{R',n} + W_{R',n}},$$
$$\underset{R',n}{\tilde{V}} = \frac{V_{R',n}}{U_{R',n} + V_{R',n} + W_{R',n}}, \quad \underset{R',n}{\tilde{W}} = \frac{W_{R',n}}{U_{R',n} + V_{R',n} + W_{R',n}}.$$

Step 4 Calculate the similarity/distances between the signals $R'(k')$ and the signals $S_{l'}(k')$ as follows.

$$d_{\mathrm{CNS}}(S_{l'}(k'), R'(k')) = \max \left(\begin{array}{l} \max \left(\sup\limits_{x \in X} \left| U_{l',n} - U_{R',n} \right|, \sup\limits_{x \in X} \left| V_{l',n} - V_{R',n} \right|, \sup\limits_{x \in X} \left| W_{l',n} - W_{R',n} \right| \right), \\ \max \left(\frac{1}{2\pi} \sup\limits_{x \in X} \left| \frac{(2\pi(n-1)(k'-1) + \alpha_{l',n})}{N} - \frac{(2\pi(n-1)(k'-1) + \alpha_{R',n})}{N} \right| \right) \end{array} \right)$$

Step 5 In order to identify $S_{l'}$ as R', compare $1 - d_{CNS}(S_{l'}, R'(k'))$ to a threshold δ, where $1 \le l' \le L'$

If $1 - d_{\mathrm{CNS}}\left(S'_l(k'), R'(k')\right)$ exceeds the threshold, identify $S_{l'}$ as R'.

The similarity between two signals can be measured by this method. By this method, we can find the right signals which have not only uncertain but also indeterminate, inconsistent, false because when the signals are received by a digital receiver, there is a chance for the right signals, chance for the indeterminate signals, and the chance that the signals are not the right one. Thus, by using a complex neutrosophic set, we can find the correct reference signals by taking all the chances, while the complex fuzzy set and complex intuitionistic fuzzy set cannot find the correct reference signals if we take all the chances because they are not able to deal with the chance of indeterminacy.

This method can be effectively used for any application in signal analysis in which the chance of indeterminacy is important.

6 Drawbacks of the current methods

The complex fuzzy sets [23] are used to represents the information with uncertainty and periodicity simultaneously. The novelty of complex fuzzy sets appears in the phase term with membership term (amplitude term). The main problem with complex fuzzy set is that it can only handle the problems of uncertainty with periodicity in the form of amplitude term (real-valued membership function) which handle uncertainty and an additional term called phase term to represent periodicity, but the complex fuzzy set cannot deal with inconsistent, incomplete, indeterminate, false etc. information which appears in a periodic manner in our real life. For example, in quantum mechanics, a wave particle such as an electron can be in two different positions at the same time. Thus, the complex fuzzy set is not able to deal with this phenomenon.

Complex intuitionistic fuzzy set [1] represents the information involving two or more answers of type: yes, no, I do not know, I am not sure, and so on, which is happening repeatedly over a period of time. CIFS can represent the information on people's decision which happens periodically. In CIFS, the novelty also appears in the phase term but for both membership and non-membership functions in some inherent concepts in contrast to CFS which is only characterized by a membership function. The complex fuzzy set [23] has only one additional phase term, but in CIFS [1], we have two additional phase terms. This confers more range values to represent the uncertainty and periodicity semantics simultaneously, and to define the values of belongingness and non-belongingness for any object in these complex-valued functions. The failure of the CIFS appears in the inconsistent, incomplete, indeterminate information which happening repeatedly.

The current research (complex fuzzy set) cannot solve this problem because the complex fuzzy set is not able to deal with indeterminate, incomplete, and inconsistent date which is in periodicity. The weaknesses of complex fuzzy set are that it deals only with uncertainty, but indeterminacy and falsity are far away from the scope of complex fuzzy sets. Similarly, the complex intuitionistic fuzzy set cannot handle the inconsistent, indeterminate, incomplete data in periodicity simultaneously. Thus, both the approaches are unable to deal with inconsistent, indeterminate, and incomplete data of periodic nature. For example, both the methods fail to deal with the information which is true and false at the same time or neither true nor false at the same time.

It is a fundamental fact that some information has not only a certain degree of truth, but also a falsity degree as well as indeterminacy degree that are independent from each other. This indeterminacy exits both in a subjective and an objective sense in a periodic nature. What should we do if we have the following situation? For instance [16], a 20° temperature means a cool day in summer and a warm day in winter. But if we assume this situation as in the following manner, a 20° temperature means cool day in summer and a warm day in winter but neither cool nor warm day in spring. The question is that why we ignore this situation? How we can handle it? Why the current methods fail to handle it? We cannot ignore this kind of situation of daily life. This phenomenon indicates that information is not only of semantic uncertainty and periodicity but also of semantic indeterminacy and periodicity.

7 Discussion

In the Table 1, we showed comparison of different current approaches to complex neutrosophic sets. In the Table 1, from 1, we mean that the corresponding method can handle the uncertain, false, indeterminate, uncertainty with periodicity, falsity with periodicity, and indeterminacy with periodicity, while from 0, we mean the corresponding method fails. It is clear from the Table 1 that how complex neutrosophic sets are dominant over all the current methods.

Consider two voting process for some attribute ρ. In the first voting process, 0.4 voters say "yes," 0.3 say "no," and 0.3 are undecided. Similarly, in second voting process, 0.5 voters say "yes," 0.3 say "no" and 0.2 are undecided for the same attribute ρ. These two voting processes held on two different dates.

We now apply all these mentioned methods in the table one by one to show that which method is suitable to describe the situation of above mentioned voting process best and what is the failure of the rest of the methods. It is clear that the fuzzy set cannot handle this situation because it only represents the membership 0.4 voters while it fails to tell about the non-membership 0.3 and indeterminate membership 0.3 simultaneously in first voting process. Similar is the situation in second voting process. Now when we apply intuitionistic fuzzy set to both the voting process, it tells us only about the membership 0.4 and non-membership 0.3 in first voting process, but cannot tell anything about the 0.3 undecided voters in first process. Thus, intuitionistic fuzzy set also fails to handle this situation. We now apply neutrosophic set. The neutrosophic set tells about the membership 0.4 voters, non-membership 0.3 voters, and indeterminate membership or undecided 0.3 voters in the first round, and similarly, it tells about the second round but neutrosophic set cannot describe both the voting process simultaneously. By applying complex fuzzy set to both the voting process, if we set that the amplitude term represents the membership 0.4 in first voting process and the phase term represents 0.5 voters in second process which form complex-valued membership function to represent in both the voting process for an attribute ρ. But complex fuzzy set remains unsuccessful to describe the non-membership and indeterminacy in both the process. The complex intuitionistic fuzzy set only handle complex-valued membership and complex-valued non-membership in both the process by setting 0.4 and 0.3 as amplitude membership and amplitude non-membership in process one and setting 0.5 and 0.3 as phase terms in second process. But clearly it fails to identify the indeterminacy (undecidedness) in both the voting process. Finally, by applying the complex neutrosophic set to both the voting process by considering the votes in process one as amplitude terms of membership, non-membership and indeterminate membership, and setting the second process vote as phase terms of membership, non-membership, and indeterminacy. Therefore, the amplitude term of membership in first process and the phase term in second process forms complex-valued truth membership function. Similarly, the amplitude term of non-membership in process one and the phase term of non-membership in second process form complex-valued falsity membership function. Also, the amplitude term of undecidedness in first process and the phase term of indeterminacy in second process form the complex-valued indeterminate membership function. Thus, both the voting process forms a complex neutrosophic set as whole which is shown below:

$$S = \left\{ \left(\rho, T_S(\rho) = 0.4.e^{j2\pi(0.5)}, I_S(\rho) = 0.3.e^{j2\pi(0.3)}, F_S(\rho) = 0.3.e^{j2\pi(0.2)} \right) \right\}$$

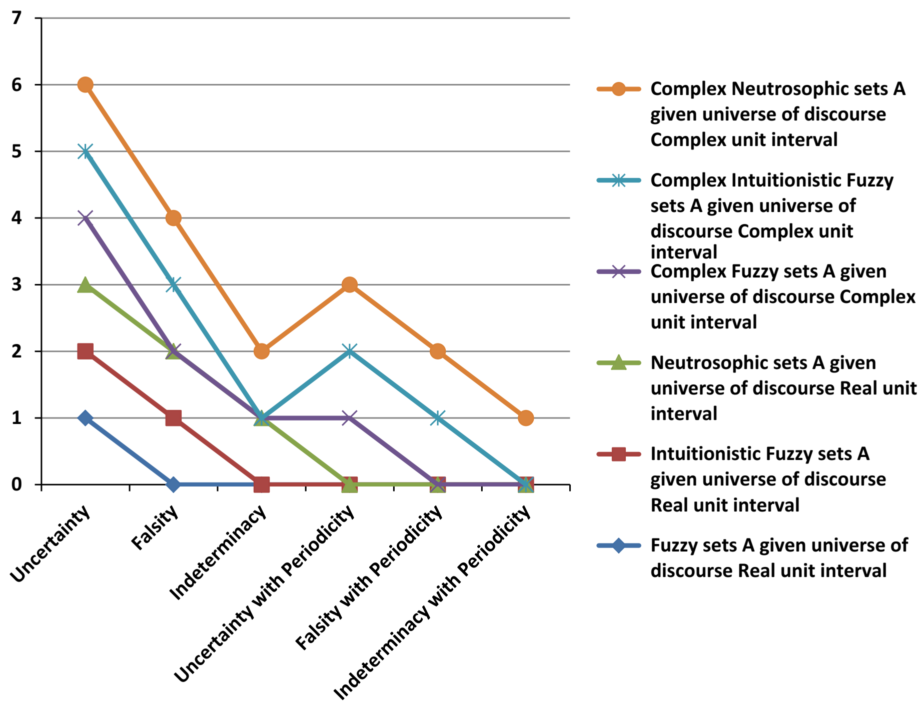

Fig. 1 Dominancy of complex neutrosophic sets to all other current approaches

Therefore, complex neutrosophic set represent both the situations in a single set simultaneously, whereas all the other mentioned methods in the table are not able to handle this situation as whole.

The graphical representation in Fig. 1 shows the dominancy of the complex neutrosophic set to all other existing methods. The highest value indicates the ability of the approach to handle all type of uncertain, incomplete, inconsistent, imprecise information or data in our real-life problems. Each value on the left vertical line shows the value of the ability of the corresponding method on the horizontal line in the graph.

8 Conclusion

An extended form of complex fuzzy set and complex intuitionistic fuzzy set is presented in this paper, so-called complex neutrosophic set. Complex neutrosophic set can handle the redundant nature of uncertainty, incompleteness, indeterminacy, inconsistency, etc. A complex neutrosophic set is defined by a complex-valued truth membership function, complex-valued indeterminate membership function, and a complex-valued falsehood membership function. Therefore, a complex-valued truth membership function is a combination of traditional truth membership function with the addition of an extra term. The traditional truth membership function is called truth amplitude term, and the additional term is called phase term. Thus, in this way, the truth amplitude term represents uncertainty and the phase term represents periodicity in the uncertainty. Thus, a complex-valued truth membership function represents uncertainty with periodicity as a whole. Similarly, complex-valued indeterminate membership function represents indeterminacy with periodicity and complex-valued falsehood membership function represents falsity with periodicity. Further, we presented an interpretation of complex neutrosophic set and also discussed some of the basic set theoretic properties such as complement, union, intersection, complex neutrosophic product, Cartesian product in this paper. We also presented δ-equalities of complex neutrosophic set and then using these δ-equalities in the application of signal processing. Drawbacks of the current methods are discussed and a comparison of all these methods to complex neutrosophic sets was presented in this paper.

This paper is an introductory paper of complex neutrosophic sets, and indeed, much research is still needed for the full comprehension of complex neutrosophic sets. The complex neutrosophic set presented in this paper is an entire general concept which is not limited to a specific application.

Appendix

Comparison of complex neutrosophic sets to fuzzy sets, intuitionistic fuzzy sets, neutrosophic sets, complex fuzzy sets, and complex intuitionistic fuzzy sets is listed below (Table 1).

Table 1 Comparison of complex neutrosophic sets to the current approaches

Sets/logics	Domain	Co-domain	Uncertainty	Falsity	Indeterminacy	Uncertainty with periodicity	Falsity with periodicity	Indeterminacy with periodicity
Fuzzy sets	A given universe of discourse	Real unit interval	1	0	0	0	0	0
Intuitionistic fuzzy sets	A given universe of discourse	Real unit interval	1	1	0	0	0	0
Neutrosophic sets	A given universe of discourse	Real unit interval	1	1	1	0	0	0
Complex fuzzy sets	A given universe of discourse	Complex unit interval	1	0	0	1	0	0
Complex intuitionistic fuzzy sets	A given universe of discourse	Complex unit interval	1	1	0	1	1	0
Complex neutrosophic sets	A given universe of discourse	Complex unit interval	1	1	1	1	1	1

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Support-Neutrosophic Set: A New Concept in Soft Computing

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Abstract. Today, soft computing is a field that is used a lot in solving real-world problems, such as problems in economics, finance, banking... With the aim to serve for solving the real problem, many new theories and/or tools which were proposed, improved to help soft computing used more efficiently. We can mention some theories as fuzzy sets theory (L. Zadeh, 1965), intuitionistic fuzzy set (K Atanasov, 1986), neutrosophic set (F. Smarandache 1999). In this paper, we introduce a new notion of support-neutrosophic set (SNS), which is the combination a neutrosophic set with a fuzzy set. So, SNS set is a direct extension of fuzzy set and neutrosophic sets (F. Smarandache). Then, we define some operators on the support-neutrosophic sets, and investigate some properties of these operators.

Keywords: support-neutrosophic sets, support-neutrosophic fuzzy relations, support- neutrosophic similarity relations

1 Introduction

In 1998, Prof. Smarandache gave the concept of the *neutrosophic set* (NS) [3] which generalized fuzzy set [10] and intuitionistic fuzzy set [1]. It is characterized by a degree of truth (T), a degree of indeterminacy (I) and a degree of falsity (F). Over time, the sub-class of the neutrosophic set were proposed to capture more advantageous in practical applications. Wang et al. [5] proposed the interval neutrosophic set and its operators. Wang et al. [6] proposed a single-valued neutrosophic set as an instance of the neutrosophic set accompanied with various set theoretic operators and properties. Ye [8] defined the concept of simpli-fied neutrosophic set whose elements of the universe have a degree of truth, indeterminacy and falsity respectively that lie between [0, 1]. Some operational laws for the simplified neutrosophic set and two aggregation operators, including a simplified neutrosophic weighted arithmetic average operator and a simplified neutrosophic weighted geometric average operator were presented.

In 2015, Nguyen et al. [2] introduced a Support-intuitionistic fuzzy set, it combines a intuitionistic fuzzy set with a fuzzy set (the support of an intuitionistic). Apter, Young et al [9] applied support – intuitionistic in decision making.

Practically, lets' consider the following case: a customer is interested in two products A and B. The customer has one rating of good (i), indeterminacy (ii) or not good (iii) for each of the products. These ratings (i),(ii) and (iii) (known as neutrosophic ratings) will affect the customer's decision of which product to buy. However, the customer's financial capacity will also affect her decision. This factor is called the support factor, with the value is between 0 and 1. Thus, the decision of which product to buy are determined by truth factors (i), indeterminacy factors (ii), falsity factors (iii) and support factor (iv). If a product is considered good and affordable, it is the best situation for a buying decision. The most unfavorable situation is when a product is considered bad and not affordable (support factor is bad),in this case, it would be easy to refuse to buy the product.

Another example, the business and purchase of cars in the Vietnam market. For customers, they will care about the quality of the car (good, bad and indeterminacy, they are neutrosophic) and prize, which are considered as supporting factors for car buyers. For car dealers, they are also interested in the quality of the car, the price and the government's policy on importing cars such as import duties on cars. Price and government policies can be viewed as supporting components of the car business.

In this paper, we combine a neutrosophic set with a fuzzy set. This raise a new concept called support-neutrosophic set (SNS). In which, there are four membership functions of an element in a given set. The remaining of this paper was structured as follows: In section 2, we introduce the concept of support-neutrosophic set and study some properties of SNS. In section 3, we give some distances between two SNS sets. Finally, we construct the distance of two support-neutrosophic sets.

2 Support-Neutrosophic set

Throughout this paper, U will be a nonempty set called the universe of discourse. First, we recall some the concept about fuzzy set and neutrosophic set. Here, we use mathematical operations on real numbers. Let S_1 and S_2 be two real standard or non-standard subsets, then

$$S_1+S_2=\{x|x=s_1+s_2, s_1\in S_1, s_2\in S_2\}$$

$$S_1-S_2=\{x|x=s_1-s_2, s_1\in S_1, s_2\in S_2\}$$

$$\bar{S}_2=\{1^+\}-S_2=\{x|x=1^+-s_2, s_2\in S_2\}$$

$$S_1\times S_2=\{x|x=s_1\times s_2, s_1\in S_1, s_2\in S_2\}$$

$$S_1\vee S_2=[\max\{infS_1, infS_2\}, \max\{supS_1, supS_2\}]$$

$$S_1\wedge S_2=[\min\{infS_1, infS_2\}, \min\{supS_1, supS_2\}]$$

$$d(S_1,S_2)=\inf_{s_1\in S_1, s_2\in S_2} d(s_1,s_2)$$

Remark: $\overline{S_1\wedge S_2}=\overline{S_1}\vee\overline{S_2}$ and $\overline{S_1\vee S_2}=\overline{S_1}\wedge\overline{S_2}$. Indeed, we consider two cases:

+ if $infS_1\le infS_2$ and $\sup S_1\le\sup S_2$ then $1-infS_2\le 1-infS_1$, $1-supS_2\le 1-supS_1$ and $S_1\wedge S_2=S_1$, $S_1\vee S_2=S_2$. So that $\overline{S_1\wedge S_2}=\overline{S_1}=\overline{S_1}\vee\overline{S_2}$ and $\overline{S_1\vee S_2}=\overline{S_2}=\overline{S_1}\wedge\overline{S_2}$.

+ if $infS_1\le infS_2\le supS_2\le\sup S_1$. Then $S_1\wedge S_2=[infS_1, supS_2]$ and $\overline{S_1}\vee\overline{S_2}=[1-supS_2, 1-infS_1]$. Hence $\overline{S_1\wedge S_2}=\overline{S_1}\vee\overline{S_2}$. Similarly, we have $\overline{S_1\wedge S_2}=\overline{S_1}\vee\overline{S_2}$.

Definition 1. A fuzzy set A on the universe U is an object of the form

$$A=\{(x,\mu_A(x))|x\in U\}$$

where $\mu_A(x)(\in[0,1])$ is called the degree of membership of x in A.

Definition 2. A neutrosophic set A on the universe U is an object of the form

$$A=\{(x,T_A(x),I_A(x),F_A(x))|x\in U\}$$

where T_A is a truth –membership function, I_A is an indeterminacy-membership function, and F_A is falsity – membership function of A. $T_A(x), I_A(x)$ and $F_A(x)$ are real standard or non-standard subsets of $]0^-,1^+[$, that is

$$T_A: U\to]0^-,1^+[$$
$$I_A: U\to]0^-,1^+[$$
$$F_A: U\to]0^-,1^+[$$

In real applications, we usually use

$$T_A: U\to[0,1]$$
$$I_A: U\to[0,1]$$
$$F_A: U\to[0,1]$$

Now, we combine a neutrosophic set with a fuzzy set. That leads to a new concept called support-neutrosophic set (SNS). In which, there are four membership functions of each element in a given set. This new concept is stated as follows:

Definition 3. A support – neutrosophic set (SNS) A on the universe U is characterized by a truth –membership function T_A, an indeterminacy-membership function I_A, a falsity – membership function F_A and support-membership function s_A. For each $x\in U$ we have $T_A(x), I_A(x)$, $F_A(x)$ and $s_A(x)$ are real standard or non-standard subsets of $]0^-,1^+[$, that is

$$T_A: U\to]0^-,1^+[$$
$$I_A: U\to]0^-,1^+[$$
$$F_A: U\to]0^-,1^+[$$
$$s_A: U\to]0^-,1^+[$$

We denote support – neutrosophic set (SNS)

$$A=\{(x,T_A(x),I_A(x),F_A(x),s_A(x))|x\in U\}.$$

There is no restriction on the sum of $T_A(x), I_A(x)$, $F_A(x)$, so $0^-\le supT_A(x)+supI_A(x)+sup\ F_A(x)\le 3^+$, and $0^-\le s_A(x)\le 1^+$.

When U is continuous, a SNS can be written as

$$A=\int_U <T_A(x),I_A(x),F_A(x),s_A(x)>/_x$$

When $U=\{x_1,x_2,..,x_n\}$ is discrete, a SNS can be written as

$$A=\sum_{i=1}^{n}\frac{<T_A(x_i),I_A(x_i),F_A(x_i),s_A(x_i)>}{x_i}$$

We denote $SNS(U)$ is the family of SNS sets on U.

Remarks:

+ The element $x_*\in U$ is called "worst element" in A if $T_A(x_*)=0, I_A(x_*)=0, F_A(x_*)=1, s_A(x_*)=0$. The element $x^*\in U$ is called "best element" in A if

$$T_A(x^*)=1, I_A(x^*)=1, F_A(x_*)=0, s_A(x_*)=1$$

(if there is restriction $sup T_A(x) + sup I_A(x) + sup\ F_A(x) \leq 1$ then the element $x^* \in U$ is called "best element" in A if

$T_A(x^*) = 1, I_A(x^*) = 0, F_A(x_*) = 0, s_A(x_*) = 1$).

+ the support – neutrosophic set A reduce an neutrosophic set if $s_A(x) = c \in [0,1], \forall x \in U$.

+ the support – neutrosophic set A is called a support-standard neutrosophic set if

$T_A(x), I_A(x), F_A(x) \in [0,1]$ and

$T_A(x) + I_A(x) + F_A(x) \leq 1$

for all $x \in U$.

+ the support – neutrosophic set A is a support-intuitionistic fuzzy set if $T_A(x), F_A(x) \in [0,1], I_A(x) = 0$ and $T_A(x) + F_A(x) \leq 1$ for all $x \in U$.

+ A constant SNS set

$(\alpha, \widehat{\beta, \theta}, \gamma) = \{(x, \alpha, \beta, \theta, \gamma) | x \in U$

where $0 \leq \alpha, \beta, \theta, \gamma \leq 1\}$.

+ the SNS universe set is

$U = 1_U = (1, \widehat{1, 0}, 1) = \{(x, 1,1,0,1) | x \in U\}$

+ the SNS empty set is

$U = 0_U = (0, \widehat{0, 1}, 0) = \{(x, 0,0,1,0) | x \in U\}$

Definition 4. The complement of a SNS A is denoted by $c(A)$ and is defined by

$$T_{C(A)}(x) = F_A(x)$$
$$I_{C(A}(x) \quad \{1^+\} - I_A(x)$$
$$F_{C(A)}(x) = T_A(x)$$
$$s_{C(A)}(x) \quad \{1^+\} - s_A(x)$$

for all $x \in U$.

Definition 5. A SNS A is contained in the other SNS B, denote $A \subseteq B$, if and only if

$$inf T_A(x) \leq inf T_B(x), \qquad sup T_A(x) \leq \sup T_B(x)$$
$$inf\ F_A(x) \geq inf\ F_B(x), \qquad sup F_A(x) \geq sub\ F_B(x)$$
$$inf s_A(x) \leq inf s_B(x), \qquad sup s_A(x) \leq \sup s_B(x)$$

for all $x \in U$.

Definition 6. The union of two SNS A and B is a SNS $C = A \cup B$, that is defined by

$$T_C = T_A \vee T_B$$
$$I_C = I_A \vee I_B$$
$$F_C = F_B \wedge F_B$$
$$s_C = s_A \vee s_B$$

Definition 7. The intersection of two SNS A and B is a SNS $D = A \cap B$, that is defined by

$$T_D = T_A \wedge T_B$$
$$I_D = I_A \wedge I_B$$
$$F_D = F_B \vee F_B$$
$$s_D = s_A \wedge s_B$$

Example 1. Let $U = \{x_1, x_2, x_3, x_4\}$ be the universe. Suppose that

$$A = \frac{\langle[0.5,0.8],[0.4,0.6],[0.2,0.7],[0.7,0.9]\rangle}{x_1} + \frac{\langle[0.4,0.5],[0.45,0.6],[0.3,0.6],[0.5,0.8]\rangle}{x_2} + \frac{\langle[0.5,0.9],[0.4,0.5],[0.6,0.7],[0.2,0.6]\rangle}{x_3} + \frac{\langle[0.5,0.9],[0.3,0.6],[0.4,0.8],[0.1,0.6]\rangle}{x_4}$$

and

$$B = \frac{\langle[0.2,0.6],[0.3,0.5],[0.3,0.6],[0.6,0.9]\rangle}{x_1} + \frac{\langle[0.45,0.7],[0.4,0.8],[0.9,1],[0.4,0.9]\rangle}{x_2} + \frac{\langle[0.1,0.7],[0.4,0.8],[0.6,0.9],[0.2,0.7]\rangle}{x_3} + \frac{\langle[0.5,1],[0.2,0.9],[0.3,0.7],[0.1,0.5]\rangle}{x_4}$$

are two support –neutrosophic set on U.

We have

+ complement of A, denote $c(A)$ or $\sim A$, defined by

$$c(A)=\frac{\langle[0.2,0.7],[0.4,0.6],[0.5,0.8],[0.1,0.3]\rangle}{x_1}$$
$$+\frac{\langle[0.3,0.6],[0.4,0.55],[0.4,0.5],[0.2,0.5]\rangle}{x_2}$$
$$+\frac{\langle[0.6,0.7],[0.5,0.6],[0.5,0.9],[0.4,0.8]\rangle}{x_3}$$
$$+\frac{\langle[0.4,0.8],[0.4,0.7],[0.5,0.9],[0.4,0.9]\rangle}{x_4}$$

+ Union $C = A \cup B$:

$$C=\frac{\langle[0.5,0.8],\ [0.4,0.6],\ [0.2,0.6],[0.7,0.9]\rangle}{x_1}$$
$$+\frac{\langle[0.45,0.7],[0.45,0.8],[0.3,0.6],[0.4,0.9]\rangle}{x_2}$$
$$+\frac{\langle[0.5,0.9],[0.4,0.8],[0.6,0.7],[0.2,0.7]\rangle}{x_3}$$
$$+\frac{\langle[0.5,1],[0.3,0.9],[0.3,0.7],[0.1,0.6]\rangle}{x_4}$$

+ the intersection $D = A \cap B$:

$$D=\frac{\langle[0.2,0.6],\ [0.3,0.5],[0.3,0.7],\ [0.6,0.9]\rangle}{x_1}$$
$$+\frac{\langle[0.4,0.5],[0.4,0.6],[0.9,1],[0.4,0.8]\rangle}{x_2}$$
$$+\frac{\langle[0.1,0.7],[0.4,0.5],[0.6,0.9],[0.2,0.6]\rangle}{x_3}$$
$$+\frac{\langle[0.5,0.9],[0.2,0.6],[0.4,0.8],[0.1,0.5]\rangle}{x_4}$$

Proposition 1. For all $A, B, C \in SNS(U)$, we have

(a) If $A \subseteq B$ and $B \subseteq C$ then $A \subseteq C$,
(b) $c(c(A)) = A$,
(c) Operators $\cap$ and $\cup$ are commutative, associative, and distributive,
(d) Operators $\cap, \sim$ and $\cup$ satisfy the law of De Morgan. It means that $\overline{A \cap B} = \overline{A} \cup \overline{B}$ and $\overline{A \cup B} = \overline{A} \cap \overline{B}$

Proof.

It is easy to verify that (a), (b), (c) is truth.

We show that (d) is correct. Indeed, for each

$$T_{\sim(A\cap B)} = F_{A\cap B} = F_A \vee F_B = T_{\sim A} \vee T_{\sim B}$$
$$I_{\sim(A\cap B)} = \{1^+\} - I(A\cap B) = \overline{I(A) \wedge I(B)}$$
$$= \overline{I(A)} \vee \overline{I(B)} = I_{\sim A} \vee I_{\sim B}$$
$$F_{\sim(A\cap B)} = T_{A\cap B} = T_A \wedge T_B = F_{\sim A} \wedge F_{\sim B}$$
$$s_{\sim(A\cap B)} = \{1^+\} - s(A\cap B) = \overline{s(A) \wedge s(B)}$$
$$= \overline{s(A)} \vee \overline{s(B)} = s_{\sim A} \vee s_{\sim B}$$

So that $\overline{A \cap B} = \overline{A} \cup \overline{B}$. By same way, we have $\overline{A \cup B} = \overline{A} \cap \overline{B}$. □

3 The Cartesian product of two SNS

Let U, V be two universe sets.

Definition 8. Let A, B two SNS on U, V, respectively. We define the Cartesian product of these two SNS sets:

a)

$$A\times B=\left\{\left(\begin{matrix}(x,y),T_{A\ B}(x,y),I_{A\times B}(x,y),\\ F_{A\ B}(x,y),s_{A\times B}(x,y)\end{matrix}\right) \mid x\in U, y\in V\right\}$$

where

$$T_{A\times B}(x,y) = T_A(x)T_B(y),$$
$$I_{A\times B}(x,y) = I_A(x)I_B(y),$$
$$F_{A\times B}(x,y) = F_A(x)F_B(y)$$

and

$$s_{A\times B}(x,y) = s_A(x)s_B(y), \forall x \in U, y \in V.$$

$$A\otimes B=\left\{\left(\begin{matrix}(x,y),T_{A\ B}(x,y),I_{A\otimes B}(x,y),\\ F_{A\ B}(x,y),s_{A\otimes B}(x,y)\end{matrix}\right) \mid x\in U, y\in V\right\}$$

Where

$$T_{A\otimes B}(x,y) = T_A(x)\text{ě}\ T_B(y),$$
$$I_{A\otimes B}(x,y) = I_A(x)\text{ě}\ I_B(y),$$
$$F_{A\otimes B}(x,y) = F_A(x)\text{ę}\ F_B(y)$$

and

$$s_{A\otimes B}(x,y) = s_A(x)\text{ě}\ s_B(y), \forall x\in U, y\in V\ .$$

Example 2. Let $U = \{x_1, x_2\}$ be the universe set. Suppose that

$$A = \frac{\langle [0.5, 0.8], [0.4, 0.6], [0.2, 0.7], [0.7, 0.9] \rangle}{x_1} + \frac{\langle [0.4, 0.5], [0.45, 0.6], [0.3, 0.6], [0.5, 0.8] \rangle}{x_2}$$

and

$$B = \frac{\langle [0.2, 0.6], [0.3, 0.5], [0.3, 0.6], [0.6, 0.9] \rangle}{x_1} + \frac{\langle [0.45, 0.7], [0.4, 0.8], [0.9, 1], [0.4, 0.9] \rangle}{x_2}$$

are two SNS on U. Then we have

$$A \times B \quad \frac{\langle [0.25, 0.72], [0.16, 0.3], [0.12, .49], [0.14, 0.54] \rangle}{(x_1, x\ )} + \frac{\langle [0.225, 0.56], [0.16, 0.48], [0.18, 0.7], [0.28, 1] \rangle}{(x_1, x\ )} + \frac{\langle [0.2, 0.45], [0.18, 0.3], [0.18, 0.42], [0.1, 0.48] \rangle}{(x_2, x\ )} + \frac{\langle [0.2, 0.45], [0.135, 0.36], [0.12, 0.48], [0.05, 0.48] \rangle}{(x_2, x\ )}$$

and

$$A \otimes B \quad \frac{\langle [0.5, 0.8], [0.4, 0.5], [0.6, 0.7], [0.2, 0.6] \rangle}{(x_1, x\ )} + \frac{\langle [05, 0.8], [0.3, 0.6], [0.4, 0.8], [0.1, 0.6] \rangle}{(x_1, x\ )} + \frac{\langle [0.4, 0.5], [0.4, 0.5], [0.6, 0.7], [0.2, 0.6] \rangle}{(x_2, x\ )} + \frac{\langle [0.4, 0.5], [0.3, 0.6], [0.4, 0.8], [0.1, 0.6] \rangle}{(x_2, x\ )}$$

Proposition 2. For every three universes U, V, W and three universe sets A on U, B on V, C on W. We have

a) $A \times B = B \times A$ and $A \otimes B = B \otimes A$
b) $(A \times B) \times C = A \times (B \times C)$
and $(A \otimes B) \otimes C = A \otimes (B \otimes C)$

Proof. It is obvious.

4 Distance between support-neutrosophic sets

In this section, we define the distance between two support-neutrosophic sets in the sene of Szmidt and Kacprzyk are presented:

Definition 9. Let $U = \{x_1, x_2, \dots, x_n\}$ be the universe set. Given $A, B \in SNS(U)$, we define

a) The Hamming distance
$d_{SNS}(A, B) = \frac{1}{n} \sum_{i=1}^{n} [d(T_A(x_i), T_B(x_i)) + d(I_A(x_i), I_B(x_i)) + d(F_A(x_i), F_B(x_i)) + d(s_A(x_i), s_B(x_i))]$

b) The Euclidean distance
$e_{SNS}(A, B) = \frac{1}{n} \sum_{i=1}^{n} \left[d^2(T_A(x_i), T_B(x_i)) + d^2(I_A(x_i), I_B(x_i)) + d^2(F_A(x_i), F_B(x_i)) + d^2(s_A(x_i), s_B(x_i))\right]^{\frac{1}{2}}$

Example 3. Let $U = \{x_1, x_2\}$ be the universe set. Two SNS $A, B \in SNS(U)$ as in example 2 we have $d_{SNS}(A, B) = 0.15$; $e_{SNS}(A, B) = 0.15$.

If

$$C = \frac{\langle [0.5, 0.7], [0.4, 0.6], [0.2, 0.7], [0.7, 0.9] \rangle}{x_1} + \frac{\langle [0.4, 0.5], [0.45, 0.6], [0.3, 0.6], [0.5, 0.8] \rangle}{x_2}$$

and

$$D = \frac{\langle [0.2, 0.4], [0.3, 0.5], [0.3, 0.6], [0.6, 0.9] \rangle}{x_1} + \frac{\langle [0.6, 0.7], [0.4, 0.8], [0.9, 1], [0.4, 0.9] \rangle}{x_2}$$

then $d_{SNS}(C, D) = 0{,}25$ and $e_{SNS}(C, D) = 0{,}2081$.

Conclusion

In this paper, we introduce a new concept: support-neutrosophic set. We also study operators on the support-neutrosophic set and their initial properties. We have given the distance and the Cartesian product of two support – neutrosophic sets. In the future, we will study more results on the support-neutrosophic set and their applications.

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Special Types of Bipolar Single Valued Neutrosophic Graphs

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Abstract. Neutrosophic theory has many applications in graph theory, bipolar single valued neutrosophic graphs (BSVNGs) is the generalization of fuzzy graphs and intuitionistic fuzzy graphs, SVNGs. In this paper we introduce some types of BSVNGs, such as subdivision BSVNGs, middle BSVNGs, total BSVNGs and bipolar single valued neutrosophic line graphs (BSVNLGs), also investigate the isomorphism, co weak isomorphism and weak isomorphism properties of subdivision BSVNGs, middle BSVNGs, total BSVNGs and BSVNLGs.

Keywords: Bipolar single valued neutrosophic line graph, Subdivision BSVNG, middle BSVNG, total BSVNG.

1. Introduction

Neutrosophic set theory (NS) is a part of neutrosophy which was introduced by Smarandache [43] from philosophical point of view by incorporating the degree of indeterminacy or neutrality as independent component for dealing problems with indeterminate and inconsistent information. The concept of neutrosophic set theory is a generalization of the theory of fuzzy set [50], intuitionistic fuzzy sets [5], interval-valued fuzzy sets [47] interval-valued intuitionistic fuzzy sets [6]. The concept of neutrosophic set is characterized by a truth-membership degree (T), an indeterminacy-membership degree (I) and a falsity-membership degree (f) independently, which are within the real standard or nonstandard unit interval $]^-0, 1^+[$. Therefore, if their range is restrained within the real standard unit interval $[0, 1]$: Nevertheless, NSs are hard to be apply in practical problems since the values of the functions of truth, indeterminacy and falsity lie in $]^-0, 1^+[$. The single valued neutrosophic set was introduced for the first time by Smarandache [43]. The concept of single valued neutrosophic sets is a subclass of neutrosophic sets in which the value of truth-membership, indeterminacy membership and falsity-membership degrees are intervals of numbers instead of the real numbers. Later on, Wang et al. [49] studied some properties related to single valued neutrosophic sets. The concept of neutrosophic sets and its extensions such as single valued neutrosophic sets, interval neutrosophic sets, bipolar neutrosophic sets and so on have been applied in a wide variety of fields including computer science, engineering, mathematics, medicine and economic and can be found in [9, 15, 16, 30, 31, 32, 33, 34, 35, 36, 37, 51]. Graphs are the most powerful tool used in representing information involving relationship between objects and concepts. In a crisp graphs two vertices are either related or not related to each other, mathematically, the degree of relationship is either 0 or 1. While in fuzzy graphs, the degree of relationship takes values from $[0, 1]$. Atanassov [42] defined the concept of intuitionistic fuzzy graphs (IFGs) using five types of Cartesian products. Theconcept fuzzy graphs, intuitionistic fuzzy graphs and their extensions such interval valued fuzzy graphs, bipolar fuzzy graph, bipolar intuitionitsic fuzzy graphs, interval valued intuitionitic fuzzy graphs, hesitancy fuzzy graphs, vague graphs and so on, have been studied deeply by several researchers in the literature. When description of the object or their relations or both is indeterminate and inconsistent, it cannot be handled by fuzzy intuitionistic fuzzy, bipolar fuzzy, vague and interval valued fuzzy graphs. So, for this purpose, Smaranadache [45] proposed the concept of neutrosophic graphs based on literal indeterminacy (I) to deal with such situations. Later on, Smarandache [44] gave another definition for neutrosphic graph theory using the neutrosophic truth-values (T, I, F) without and constructed three structures of neutrosophic graphs: neutrosophic edge graphs, neutrosophic vertex graphs and neutrosophic vertex-edge graphs. Recently, Smarandache [46] proposed new version of neutrosophic graphs such as neutrosophic offgraph, neutrosophic bipolar/tripola/multipolar graph. Recently several researchers have studied deeply the concept of neutrosophic vertex-edge graphs and presented several extensions neutrosophic graphs. In [1, 2, 3]. Akram et al. introduced the concept of single valued neutrosophic hypergraphs, single valued neutrosophic planar graphs, neutrosophic soft graphs and intuitionstic neutrosophic soft graphs. Then, followed the work of Broumi et al. [7, 8, 9, 10, 11, 12, 13, 14, 15], Malik and Hassan [38] defined the concept of single valued neutrosophic trees and studied some of their properties. Later on, Hassan et Malik [17] introduced some classes of bipolar single valued neutrosophic graphs and studied some of their properties, also the authors generalized the concept of single valued neutrosophic hypergraphs and bipolar single valued neutrosophic hypergraphs in [19, 20]. In [23, 24] Hassan et Malik gave the important types of single (interval) valued neutrosophic graphs, another important classes of single valued neutrosophic graphs have been presented in [22] and in [25] Hassan et Malik introduced the concept of m-Polar single valued neutrosophic graphs and its classes. Hassan et al. [18, 21] studied the concept on regularity and total regularity of single valued neutrosophic hypergraphs and bipolar single valued neutrosophic hypergraphs. Hassan et al. [26, 27, 28] discussed the isomorphism properties on SVNHGs, BSVNHGs and IVNHGs. Nasir et al. [40] introduced a new type of graph called neutrosophic soft graphs and established a link between graphs and neutrosophic soft sets. The authors also studeied some basic operations of neutrosophic soft graphs such as union, intersection and complement. Nasir and Broumi [41] studied the concept of irregular neutrosophic graphs and investigated some of their related properties. Ashraf et al. [4], proposed some novels concepts of edge regular, partially edge regular and full edge regular single valued neutrosophic graphs and investigated some of their properties. Also the authors, introduced the notion of single valued neutrosophic digraphs (SVNDGs) and presented an application of SVNDG in multi-attribute decision making. Mehra and Singh [39] introduced a new concept of neutrosophic graph named single valued neutrosophic Signed graphs (SVNSGs) and examined the properties of this concept with suitable illustration. Ulucay et al. [48] proposed a new extension of neutrosophic graphs called neutrosophic soft expert graphs (NSEGs) and have established a link between graphs and neutrosophic soft expert sets and studies some basic operations of neutrosophic soft experts graphs such as union, intersection and complement. The neutrosophic graphs have many applications in path problems, networks and computer science. Strong BSVNG and complete BSVNG are the types of BSVNG. In this paper, we introduce others types of BSVNGs such as subdivision BSVNGs, middle BSVNGs, total BSVNGs and BSVNLGs and these are all the strong BSVNGs, also we discuss their relations based on isomorphism, co weak isomorphism and weak isomorphism.

2. Preliminaries

In this section we recall some basic concepts on BSVNG. Let G denotes BSVNG and $G^* = (V, E)$ denotes its underlying crisp graph.

Definition 2.1 ([10])**.** Let X be a crisp set, the single valued neutrosophic set (SVNS) Z is characterized by three membership functions $T_Z(x), I_Z(x)$ and $F_Z(x)$ which are truth, indeterminacy and falsity membership functions, $\forall x \in X$

$$T_Z(x), I_Z(x), F_Z(x) \in [0, 1].$$

Definition 2.2 ([10])**.** Let X be a crisp set, the bipolar single valued neutrosophic set (BSVNS) Z is characterized by membership functions $T_Z^+(x)$, $I_Z^+(x)$, $F_Z^+(x)$, $T_Z^-(x)$, $I_Z^-(x)$, and $F_Z^-(x)$. That is $\forall x \in X$

$$T_Z^+(x), I_Z^+(x), F_Z^+(x) \in [0, 1],$$
$$T_Z^-(x), I_Z^-(x), F_Z^-(x) \in [-1, 0].$$

Definition 2.3 ([10])**.** A bipolar single valued neutrosophic graph (BSVNG) is a pair $G = (Y, Z)$ of G^*, where Y is BSVNS on V and Z is BSVNS on E such that

$$T_Z^+(\beta\gamma) \le \min(T_Y^+(\beta), T_Y^+(\gamma)),\ I_Z^+(\beta\gamma) \ge \max(I_Y^+(\beta), I_Y^+(\gamma)),$$
$$I_Z^-(\beta\gamma) \le \min(I_Y^-(\beta), I_Y^-(\gamma)),\ F_Z^-(\beta\gamma) \le \min(F_Y^-(\beta), F_Y^-(\gamma)),$$
$$F_Z^+(\beta\gamma) \ge \max(F_Y^+(\beta), F_Y^+(\gamma)),\ T_Z^-(\beta\gamma) \ge \max(T_Y^-(\beta), T_Y^-(\gamma)),$$

where

$$0 \le T_Z^+(\beta\gamma) + I_Z^+(\beta\gamma) + F_Z^+(\beta\gamma) \le 3$$
$$-3 \le T_Z^-(\beta\gamma) + I_Z^-(\beta\gamma) + F_Z^-(\beta\gamma) \le 0$$

$\forall\ \beta, \gamma \in V$.

In this case, D is bipolar single valued neutrosophic relation (BSVNR) on C. The BSVNG $G=(Y,Z)$ is complete (strong) BSVNG, if

$$T_Z^+(\beta\gamma)=\min(T_Y^+(\beta),T_Y^+(\gamma)),\ I_Z^+(\beta\gamma)=\max(I_Y^+(\beta),I_Y^+(\gamma)),$$

$$I_Z^-(\beta\gamma)=\min(I_Y^-(\beta),I_Y^-(\gamma)),\ F_Z^-(\beta\gamma)=\min(F_Y^-(\beta),F_Y^-(\gamma)),$$

$$F_Z^+(\beta\gamma)=\max(F_Y^+(\beta),F_Y^+(\gamma)),\ T_Z^-(\beta\gamma)=\max(T_Y^-(\beta),T_Y^-(\gamma)),$$

$\forall\ \beta,\gamma\in V(\forall\ \beta\gamma\in E)$. The order of BSVNG $G=(A,B)$ of G^*, denoted by $O(G)$, is defined by

$$O(G)=(O_T^+(G),O_I^+(G),O_F^+(G),O_T^-(G),O_I^-(G),O_F^-(G)),$$

where

$$O_T^+(G)=\sum_{\alpha\in V}T_A^+(\alpha),\ O_I^+(G)=\sum_{\alpha\in V}I_A^+(\alpha),\ O_F^+(G)=\sum_{\alpha\in V}F_A^+(\alpha),$$

$$O_T^-(G)=\sum_{\alpha\in V}T_A^-(\alpha),\ O_I^-(G)=\sum_{\alpha\in V}I_A^-(\alpha),\ O_F^-(G)=\sum_{\alpha\in V}F_A^-(\alpha).$$

The size of BSVNG $G=(A,B)$ of G^*, denoted by $S(G)$, is defined by

$$S(G)=(S_T^+(G),S_I^+(G),S_F^+(G),S_T^-(G),S_I^-(G),S_F^-(G)),$$

where

$$S_T^+(G)=\sum_{\beta\gamma\in E}T_B^+(\beta\gamma),\ S_T^-(G)=\sum_{\beta\gamma\in E}T_B^-(\beta\gamma),$$

$$S_I^+(G)=\sum_{\beta\gamma\in E}I_B^+(\beta\gamma),\ S_I^-(G)=\sum_{\beta\gamma\in E}I_B^-(\beta\gamma),$$

$$S_F^+(G)=\sum_{\beta\gamma\in E}F_B^+(\beta\gamma),\ S_F^-(G)=\sum_{\beta\gamma\in E}F_B^-(\beta\gamma).$$

The degree of a vertex β in BSVNG $G=(A,B)$ of G^*,, denoted by $d_G(\beta)$, is defined by

$$d_G(\beta)=(d_T^+(\beta),d_I^+(\beta),d_F^+(\beta),d_T^-(\beta),d_I^-(\beta),d_F^-(\beta)),$$

where

$$d_T^+(\beta)=\sum_{\beta\gamma\in E}T_B^+(\beta\gamma),\ d_T^-(\beta)=\sum_{\beta\gamma\in E}T_B^-(\beta\gamma),$$

$$d_I^+(\beta)=\sum_{\beta\gamma\in E}I_B^+(\beta\gamma),\ d_I^-(\beta)=\sum_{\beta\gamma\in E}I_B^-(\beta\gamma),$$

$$d_F^+(\beta)=\sum_{\beta\gamma\in E}F_B^+(\beta\gamma),\ d_F^-(\beta)=\sum_{\beta\gamma\in E}F_B^-(\beta\gamma).$$

3. Types of BSVNGs

In this section we introduce the special types of BSVNGs such as subdivision, middle and total and intersection BSVNGs, for this first we give the basic definitions of homomorphism, isomorphism, weak isomorphism and co weak isomorphism of BSVNGs which are very useful to understand the relations among the types of BSVNGs.

Definition 3.1. Let $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ be two BSVNGs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Then the homomorphism $\chi : G_1 \to G_2$ is a mapping $\chi : V_1 \to V_2$ which satisfies the following conditions:

$$T^+_{C_1}(p) \leq T^+_{C_2}(\chi(p)),\ I^+_{C_1}(p) \geq I^+_{C_2}(\chi(p)),\ F^+_{C_1}(p) \geq F^+_{C_2}(\chi(p)),$$

$$T^-_{C_1}(p) \geq T^-_{C_2}(\chi(p)),\ I^-_{C_1}(p) \leq I^-_{C_2}(\chi(p)),\ F^-_{C_1}(p) \leq F^-_{C_2}(\chi(p)),$$

$\forall\ p \in V_1$,

$$T^+_{D_1}(pq) \leq T^+_{D_2}(\chi(p)\chi(q)),\ T^-_{D_1}(pq) \geq T^-_{D_2}(\chi(p)\chi(q)),$$
$$I^+_{D_1}(pq) \geq I^+_{D_2}(\chi(p)\chi(q)),\ I^-_{D_1}(pq) \leq I^-_{D_2}(\chi(p)\chi(q)),$$
$$F^+_{D_1}(pq) \geq F^+_{D_2}(\chi(p)\chi(q)),\ F^-_{D_1}(pq) \leq F^-_{D_2}(\chi(p)\chi(q)),$$

$\forall\ pq \in E_1$.

Definition 3.2. Let $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ be two BSVNGs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Then the weak isomorphism $\upsilon : G_1 \to G_2$ is a bijective mapping $\upsilon : V_1 \to V_2$ which satisfies following conditions:

υ is a homomorphism such that

$$T^+_{C_1}(p) = T^+_{C_2}(\upsilon(p)),\ I^+_{C_1}(p) = I^+_{C_2}(\upsilon(p)),\ F^+_{C_1}(p) = F^+_{C_2}(\upsilon(p)),$$

$$T^-_{C_1}(p) = T^-_{C_2}(\upsilon(p)),\ I^-_{C_1}(p) = I^-_{C_2}(\upsilon(p)),\ F^-_{C_1}(p) = F^-_{C_2}(\upsilon(p)),$$

$\forall\ p \in V_1$.

Remark 3.3. The weak isomorphism between two BSVNGs preserves the orders.

Remark 3.4. The weak isomorphism between BSVNGs is a partial order relation.

Definition 3.5. Let $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ be two BSVNGs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Then the co-weak isomorphism $\kappa : G_1 \to G_2$ is a bijective mapping $\kappa : V_1 \to V_2$ which satisfies following conditions:

κ is a homomorphism such that

$$T^+_{D_1}(pq) = T^+_{D_2}(\kappa(p)\kappa(q)),\ T^-_{D_1}(pq) = T^-_{D_2}(\kappa(p)\kappa(q)),$$
$$I^+_{D_1}(pq) = I^+_{D_2}(\kappa(p)\kappa(q)),\ I^-_{D_1}(pq) = I^-_{D_2}(\kappa(p)\kappa(q)),$$
$$F^+_{D_1}(pq) = F^+_{D_2}(\kappa(p)\kappa(q)),\ F^-_{D_1}(pq) = F^-_{D_2}(\kappa(p)\kappa(q)),$$

$\forall\ pq \in E_1$.

Remark 3.6. The co-weak isomorphism between two BSVNGs preserves the sizes.

Remark 3.7. The co-weak isomorphism between BSVNGs is a partial order relation.

Table 1. BSVNSs of BSVNG.

A	T_A^+	I_A^+	F_A^+	T_A^-	I_A^-	F_A^-
a	0.2	0.1	0.4	-0.3	-0.1	-0.4
b	0.3	0.2	0.5	-0.5	-0.4	-0.6
c	0.4	0.7	0.6	-0.2	-0.6	-0.2
B	T_B^+	I_B^+	F_B^+	T_B^-	I_B^-	F_B^-
p	0.2	0.4	0.5	-0.2	-0.5	-0.6
q	0.3	0.8	0.6	-0.1	-0.7	-0.8
r	0.1	0.7	0.9	-0.1	-0.8	-0.5

Definition 3.8. Let $G_1 = (C_1, D_1)$ and $G_2 = (C_2, D_2)$ be two BSVNGs of $G_1^* = (V_1, E_1)$ and $G_2^* = (V_2, E_2)$, respectively. Then the isomorphism $\psi : G_1 \to G_2$ is a bijective mapping $: V_1 \to V_2$ which satisfies the following conditions:

$$T^+_{C_1}(p) = T^+_{C_2}(\psi(p)),\ I^+_{C_1}(p) = I^+_{C_2}(\psi(p)),\ F^+_{C_1}(p) = F^+_{C_2}(\psi(p)),$$

$$T^-_{C_1}(p) = T^-_{C_2}(\psi(p)),\ I^-_{C_1}(p) = I^-_{C_2}(\psi(p)),\ F^-_{C_1}(p) = F^-_{C_2}(\psi(p)),$$

$\forall\ p \in V_1$,

$$T^+_{D_1}(pq) = T^+_{D_2}(\psi(p)\psi(q)),\ T^-_{D_1}(pq) = T^-_{D_2}(\psi(p)\psi(q)),$$
$$I^+_{D_1}(pq) = I^+_{D_2}(\psi(p)\psi(q)),\ I^-_{D_1}(pq) = I^-_{D_2}(\psi(p)\psi(q)),$$
$$F^+_{D_1}(pq) = F^+_{D_2}(\psi(p)\psi(q)),\ F^-_{D_1}(pq) = F^-_{D_2}(\psi(p)\psi(q)),$$

$\forall\ pq \in E_1$.

Remark 3.9. The isomorphism between two BSVNGs is an equivalence relation.

Remark 3.10. The isomorphism between two BSVNGs preserves the orders and sizes.

Remark 3.11. The isomorphism between two BSVNGs preserves the degrees of their vertices.

Definition 3.12. The subdivision SVNG be $sd(G) = (C, D)$ of $G = (A, B)$, where C is a BSVNS on $V \cup E$ and D is a BSVNR on C such that
(i) $C = A$ on V and $C = B$ on E,
(ii) if $v \in V$ lie on edge $e \in E$, then

$$T^+_D(ve) = \min(T^+_A(v), T^+_B(e)),\ I^+_D(ve) = \max(I^+_A(v), I^+_B(e))$$
$$I^-_D(ve) = \min(I^-_A(v), I^-_B(e)),\ F^-_D(ve) = \min(F^-_A(v), F^-_B(e))$$
$$F^+_D(ve) = \max(F^+_A(v), F^+_B(e)),\ T^-_D(ve) = \max(T^-_A(v), T^-_B(e))$$

else

$$D(ve) = O = (0, 0, 0, 0, 0, 0).$$

Example 3.13. Consider the BSVNG $G = (A, B)$ of a $G^* = (V, E)$, where $V = \{a, b, c\}$ and $E = \{p = ab, q = bc, r = ac\}$, the crisp graph of G is shown in Fig. 1. The BSVNSs A and B are defined on V and E respectively which are defined in Table 1. The SDBSVNG $sd(G) = (C, D)$ of a BSVNG G, the underlying crisp graph of $sd(G)$ is given in Fig. 2. The BSVNSs C and D are defined in Table 2.

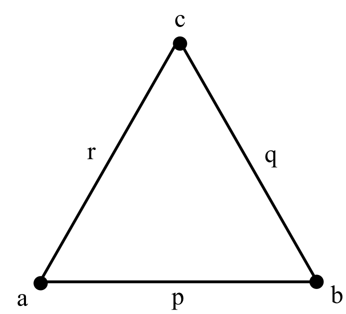

FIGURE 1. Crisp Graph of BSVNG.

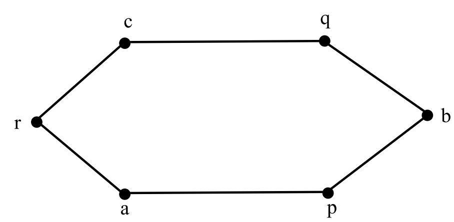

FIGURE 2. Crisp Graph of SDBSVNG.

TABLE 2. BSVNSs of SDBSVNG.

C	T_C^+	I_C^+	F_C^+	T_C^-	I_C^-	F_C^-
a	0.2	0.1	0.4	-0.3	-0.1	-0.4
p	0.2	0.4	0.5	-0.2	-0.5	-0.6
b	0.3	0.2	0.5	-0.5	-0.4	-0.6
q	0.3	0.8	0.6	-0.1	-0.7	-0.8
c	0.4	0.7	0.6	-0.2	-0.6	-0.2
r	0.1	0.7	0.9	-0.1	-0.8	-0.5
D	T_D^+	I_D^+	F_D^+	T_D^-	I_D^-	F_D^-
ap	0.2	0.4	0.5	-0.2	-0.5	-0.6
pb	0.2	0.4	0.5	-0.2	-0.5	-0.6
bq	0.3	0.8	0.6	-0.1	-0.7	-0.8
qc	0.3	0.8	0.6	-0.1	-0.7	-0.8
cr	0.1	0.7	0.9	-0.1	-0.8	-0.5
ra	0.1	0.7	0.9	-0.1	-0.8	-0.5

Proposition 3.14. *Let G be a BSVNG and $sd(G)$ be the SDBSVNG of a BSVNG G, then $O(sd(G)) = O(G) + S(G)$ and $S(sd(G)) = 2S(G)$.*

Remark 3.15. Let G be a complete BSVNG, then $sd(G)$ need not to be complete BSVNG.

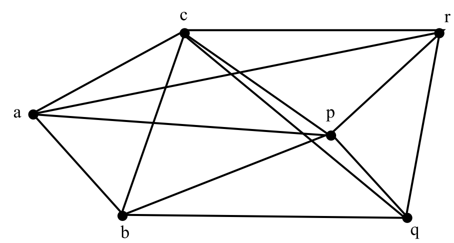

FIGURE 3. Crisp Graph of TSVNG.

Definition 3.16. The total bipolar single valued neutrosophic graph (TBSVNG) is $T(G) = (C, D)$ of $G = (A, B)$, where C is a BSVNS on $V \cup E$ and D is a BSVNR on C such that

(i) $C = A$ on V and $C = B$ on E,

(ii) if $v \in V$ lie on edge $e \in E$, then

$$T^+_D(ve) = \min(T^+_A(v), T^+_B(e)),\ I^+_D(ve) = \max(I^+_A(v), I^+_B(e))$$

$$I^-_D(ve) = \min(I^-_A(v), I^-_B(e)),\ F^-_D(ve) = \min(F^-_A(v), F^-_B(e))$$

$$F^+_D(ve) = \max(F^+_A(v), F^+_B(e)),\ T^-_D(ve) = \max(T^-_A(v), T^-_B(e))$$

else

$$D(ve) = O = (0, 0, 0, 0, 0, 0),$$

(iii) if $\alpha\beta \in E$, then

$$T^+_D(\alpha\beta) = T^+_B(\alpha\beta),\ I^+_D(\alpha\beta) = I^+_B(\alpha\beta),\ F^+_D(\alpha\beta) = F^+_B(\alpha\beta)$$

$$T^-_D(\alpha\beta) = T^-_B(\alpha\beta),\ I^-_D(\alpha\beta) = I^-_B(\alpha\beta),\ F^-_D(\alpha\beta) = F^-_B(\alpha\beta),$$

(iv) if $e, f \in E$ have a common vertex, then

$$T^+_D(ef) = \min(T^+_B(e), T^+_B(f)),\ I^+_D(ef) = \max(I^+_B(e), I^+_B(f))$$

$$I^-_D(ef) = \min(I^-_B(e), I^-_B(f)),\ F^-_D(ef) = \min(F^-_B(e), F^-_B(f))$$

$$F^+_D(ef) = \max(F^+_B(e), F^+_B(f)),\ T^-_D(ef) = \max(T^-_B(e), T^-_B(f))$$

else

$$D(ef) = O = (0, 0, 0, 0, 0, 0).$$

Example 3.17. Consider the Example 3.13 the TBSVNG $T(G) = (C, D)$ of underlying crisp graph as shown in Fig. 3. The BSVNS C is given in Example 3.13. The BSVNS D is given in Table 3.

Proposition 3.18. *Let G be a BSVNG and $T(G)$ be the TBSVNG of a BSVNG G, then $O(T(G)) = O(G) + S(G) = O(sd(G))$ and $S(sd(G)) = 2S(G)$.*

Proposition 3.19. *Let G be a BSVNG, then $sd(G)$ is weak isomorphic to $T(G)$.*

TABLE 3. BSVNS of TBSVNG.

D	T_D^+	I_D^+	F_D^+	T_D^-	I_D^-	F_D^-
ab	0.2	0.4	0.5	-0.2	-0.5	-0.6
bc	0.3	0.8	0.6	-0.1	-0.7	-0.8
ca	0.1	0.7	0.9	-0.1	-0.8	-0.5
pq	0.2	0.8	0.6	-0.1	-0.7	-0.8
qr	0.1	0.8	0.9	-0.1	-0.8	-0.8
rp	0.1	0.7	0.9	-0.1	-0.8	-0.6
ap	0.2	0.4	0.5	-0.2	-0.5	-0.6
pb	0.2	0.4	0.5	-0.2	-0.5	-0.6
bq	0.3	0.8	0.6	-0.1	-0.7	-0.8
qc	0.3	0.8	0.6	-0.1	-0.7	-0.8
cr	0.1	0.7	0.9	-0.1	-0.8	-0.5
ra	0.1	0.7	0.9	-0.1	-0.8	-0.5

Definition 3.20. The middle bipolar single valued neutrosophic graph (MBSVNG) $M(G) = (C, D)$ of G, where C is a BSVNS on $V \cup E$ and D is a BSVNR on C such that

(i) $C = A$ on V and $C = B$ on E, else $C = O = (0, 0, 0, 0, 0, 0)$,

(ii) if $v \in V$ lie on edge $e \in E$, then

$$T_D^+(ve) = T_B^+(e),\ I_D^+(ve) = I_B^+(e),\ F_D^+(ve) = F_B^+(e)$$

$$T_D^-(ve) = T_B^-(e),\ I_D^-(ve) = I_B^-(e),\ F_D^-(ve) = F_B^-(e)$$

else

$$D(ve) = O = (0, 0, 0, 0, 0, 0),$$

(iii) if $u, v \in V$, then

$$D(uv) = O = (0, 0, 0, 0, 0, 0),$$

(iv) if $e, f \in E$ and e and f are adjacent in G, then

$$T_D^+(ef) = T_B^+(uv),\ I_D^+(ef) = I_B^+(uv),\ F_D^+(ef) = F_B^+(uv)$$

$$T_D^-(ef) = T_B^-(uv),\ I_D^-(ef) = I_B^-(uv),\ F_D^-(ef) = F_B^-(uv).$$

Example 3.21. Consider the BSVNG $G = (A, B)$ of a G^*, where $V = \{a, b, c\}$ and $E = \{p = ab, q = bc\}$ the underlaying crisp graph is shown in Fig. 4. The BSVNSs A and B are defined in Table 4. The crisp graph of MBSVNG $M(G) = (C, D)$ is shown in Fig. 5. The BSVNSs C and D are given in Table 5.

Remark 3.22. Let G be a BSVNG and $M(G)$ be the MBSVNG of a BSVNG G, then $O(M(G)) = O(G) + S(G)$.

Remark 3.23. Let G be a BSVNG, then $M(G)$ is a strong BSVNG.

Remark 3.24. Let G be complete BSVNG, then $M(G)$ need not to be complete BSVNG.

Proposition 3.25. *Let G be a BSVNG, then $sd(G)$ is weak isomorphic with $M(G)$.*

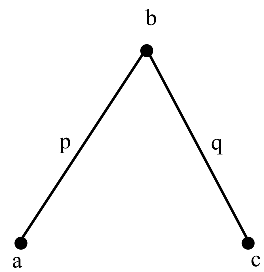

FIGURE 4. Crisp Graph of BSVNG.

TABLE 4. BSVNSs of BSVNG.

A	T_A^+	I_A^+	F_A^+	T_A^-	I_A^-	F_A^-
a	0.3	0.4	0.5	-0.2	-0.1	-0.3
b	0.7	0.6	0.3	-0.3	-0.3	-0.2
c	0.9	0.7	0.2	-0.5	-0.4	-0.6
B	T_B^+	I_B^+	F_B^+	T_B^-	I_B^-	F_B^-
p	0.2	0.6	0.6	-0.1	-0.4	-0.3
q	0.4	0.8	0.7	-0.3	-0.5	-0.6

TABLE 5. BSVNSs of MBSVNG.

C	T_C^+	I_C^+	F_C^+	T_C^-	I_C^-	F_C^-
a	0.3	0.4	0.5	-0.2	-0.1	-0.3
b	0.7	0.6	0.3	-0.3	-0.3	-0.2
c	0.9	0.7	0.2	-0.5	-0.4	-0.6
e_1	0.2	0.6	0.6	-0.1	-0.4	-0.3
e_2	0.4	0.8	0.7	-0.3	-0.5	-0.6
D	T_D^+	I_D^+	F_D^+	T_D^-	I_D^-	F_D^-
pq	0.2	0.8	0.7	-0.1	-0.5	-0.6
ap	0.2	0.6	0.6	-0.1	-0.4	-0.3
bp	0.2	0.6	0.6	-0.1	-0.4	-0.3
bq	0.2	0.6	0.6	-0.3	-0.5	-0.6
cq	0.4	0.8	0.7	-0.3	-0.5	-0.6

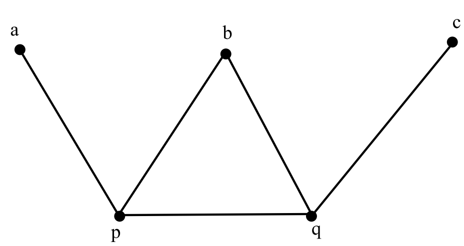

FIGURE 5. Crisp Graph of MBSVNG.

Proposition 3.26. *Let G be a BSVNG, then $M(G)$ is weak isomorphic with $T(G)$.*

Proposition 3.27. *Let G be a BSVNG, then $T(G)$ is isomorphic with $G \cup M(G)$.*

Definition 3.28. Let $P(X) = (X, Y)$ be the intersection graph of a G^*, let C_1 and D_1 be BSVNSs on V and E, respectively and C_2 and D_2 be BSVNSs on X and Y respectively. Then bipolar single valued neutrosophic intersection graph (BSVNIG) of a BSVNG $G = (C_1, D_1)$ is a BSVNG $P(G) = (C_2, D_2)$ such that,

$$T^+_{C_2}(X_i) = T^+_{C_1}(v_i),\ I^+_{C_2}(X_i) = I^+_{C_1}(v_i),\ F^+_{C_2}(X_i) = F^+_{C_1}(v_i)$$
$$T^-_{C_2}(X_i) = T^-_{C_1}(v_i),\ I^-_{C_2}(X_i) = I^-_{C_1}(v_i),\ F^-_{C_2}(X_i) = F^-_{C_1}(v_i)$$
$$T^+_{D_2}(X_iX_j) = T^+_{D_1}(v_iv_j),\ T^-_{D_2}(X_iX_j) = T^-_{D_1}(v_iv_j),$$
$$I^+_{D_2}(X_iX_j) = I^+_{D_1}(v_iv_j),\ I^-_{D_2}(X_iX_j) = I^-_{D_1}(v_iv_j),$$
$$F^+_{D_2}(X_iX_j) = F^+_{D_1}(v_iv_j),\ F^-_{D_2}(X_iX_j) = F^-_{D_1}(v_iv_j)$$

$\forall\ X_i, X_j \in X$ and $X_iX_j \in Y$.

Proposition 3.29. *Let $G = (A_1, B_1)$ be a BSVNG of $G^* = (V, E)$, and let $P(G) = (A_2, B_2)$ be a BSVNIG of $P(S)$. Then BSVNIG is a also BSVNG and BSVNG is always isomorphic to BSVNIG.*

Proof. By the definition of BSVNIG, we have

$$\begin{aligned}
T^+_{B_2}(S_iS_j) &= T^+_{B_1}(v_iv_j) \leq \min(T^+_{A_1}(v_i), T^+_{A_1}(v_j)) = \min(T^+_{A_2}(S_i), T^+_{A_2}(S_j)),\\
I^+_{B_2}(S_iS_j) &= I^+_{B_1}(v_iv_j) \geq \max(I^+_{A_1}(v_i), I^+_{A_1}(v_j)) = \max(I^+_{A_2}(S_i), I^+_{A_2}(S_j)),\\
F^+_{B_2}(S_iS_j) &= F^+_{B_1}(v_iv_j) \geq \max(F^+_{A_1}(v_i), F^+_{A_1}(v_j)) = \max(F^+_{A_2}(S_i), F^+_{A_2}(S_j)),\\
T^-_{B_2}(S_iS_j) &= T^-_{B_1}(v_iv_j) \geq \max(T^-_{A_1}(v_i), T^-_{A_1}(v_j)) = \max(T^-_{A_2}(S_i), T^-_{A_2}(S_j)),\\
I^-_{B_2}(S_iS_j) &= I^-_{B_1}(v_iv_j) \leq \min(I^-_{A_1}(v_i), I^-_{A_1}(v_j)) = \min(I^-_{A_2}(S_i), I^-_{A_2}(S_j)),\\
F^-_{B_2}(S_iS_j) &= F^-_{B_1}(v_iv_j) \leq \min(F^-_{A_1}(v_i), F^-_{A_1}(v_j)) = \min(F^-_{A_2}(S_i), F^-_{A_2}(S_j)).
\end{aligned}$$

This shows that BSVNIG is a BSVNG.

Next define $f : V \to S$ by $f(v_i) = S_i$ for $i = 1, 2, 3, \ldots, n$ clearly f is bijective. Now $v_iv_j \in E$ if and only if $S_iS_j \in T$ and $T = \{f(v_i)f(v_j) : v_iv_j \in E\}$. Also

$$T^+_{A_2}(f(v_i)) = T^+_{A_2}(S_i) = T^+_{A_1}(v_i),\ I^+_{A_2}(f(v_i)) = I^+_{A_2}(S_i) = I^+_{A_1}(v_i),$$
$$F^+_{A_2}(f(v_i)) = F^+_{A_2}(S_i) = F^+_{A_1}(v_i),\ T^-_{A_2}(f(v_i)) = T^-_{A_2}(S_i) = T^-_{A_1}(v_i),$$
$$I^-_{A_2}(f(v_i)) = I^-_{A_2}(S_i) = I^-_{A_1}(v_i),\ F^-_{A_2}(f(v_i)) = F^-_{A_2}(S_i) = F^-_{A_1}(v_i),$$

$\forall\ v_i \in V$,

$$T^+_{B_2}(f(v_i)f(v_j)) = T^+_{B_2}(S_iS_j) = T^+_{B_1}(v_iv_j),$$
$$I^+_{B_2}(f(v_i)f(v_j)) = I^+_{B_2}(S_iS_j) = I^+_{B_1}(v_iv_j),$$
$$F^+_{B_2}(f(v_i)f(v_j)) = F^+_{B_2}(S_iS_j) = F^+_{B_1}(v_iv_j),$$
$$T^-_{B_2}(f(v_i)f(v_j)) = T^-_{B_2}(S_iS_j) = T^-_{B_1}(v_iv_j),$$
$$I^-_{B_2}(f(v_i)f(v_j)) = I^-_{B_2}(S_iS_j) = I^-_{B_1}(v_iv_j),$$
$$F^-_{B_2}(f(v_i)f(v_j)) = F^-_{B_2}(S_iS_j) = F^-_{B_1}(v_iv_j),$$

$\forall\ v_iv_j \in E$. □

Table 6. BSVNSs of BSVNG.

A_1	$T^+_{A_1}$	$I^+_{A_1}$	$F^+_{A_1}$	$T^-_{A_1}$	$I^-_{A_1}$	$F^-_{A_1}$
α_1	0.2	0.5	0.5	-0.1	-0.4	-0.5
α_2	0.4	0.3	0.3	-0.2	-0.3	-0.2
α_3	0.4	0.5	0.5	-0.3	-0.2	-0.6
α_4	0.3	0.2	0.2	-0.4	-0.1	-0.3
B_1	$T^+_{B_1}$	$I^+_{B_1}$	$F^+_{B_1}$	$T^-_{B_1}$	$I^-_{B_1}$	$F^-_{B_1}$
x_1	0.1	0.6	0.7	-0.1	-0.4	-0.5
x_2	0.3	0.6	0.7	-0.2	-0.3	-0.6
x_3	0.2	0.7	0.8	-0.3	-0.2	-0.6
x_4	0.1	0.7	0.8	-0.1	-0.4	-0.5

Definition 3.30. Let $G^* = (V, E)$ and $L(G^*) = (X, Y)$ be its line graph, where A_1 and B_1 be BSVNSs on V and E, respectively. Let A_2 and B_2 be BSVNSs on X and Y, respectively. The bipolar single valued neutrosophic line graph (BSVNLG) of BSVNG $G = (A_1, B_1)$ is BSVNG $L(G) = (A_2, B_2)$ such that,

$$T^+_{A_2}(S_x) = T^+_{B_1}(x) = T^+_{B_1}(u_x v_x),\ I^+_{A_2}(S_x) = I^+_{B_1}(x) = I^+_{B_1}(u_x v_x),$$
$$I^-_{A_2}(S_x) = I^-_{B_1}(x) = I^-_{B_1}(u_x v_x),\ F^-_{A_2}(S_x) = F^-_{B_1}(x) = F^-_{B_1}(u_x v_x),$$
$$F^+_{A_2}(S_x) = F^+_{B_1}(x) = F^+_{B_1}(u_x v_x),\ T^-_{A_2}(S_x) = T^-_{B_1}(x) = T^-_{B_1}(u_x v_x),$$

$\forall\ S_x, S_y \in X$ and

$$T^+_{B_2}(S_x S_y) = \min(T^+_{B_1}(x), T^+_{B_1}(y)),\ I^+_{B_2}(S_x S_y) = \max(I^+_{B_1}(x), I^+_{B_1}(y)),$$
$$I^-_{B_2}(S_x S_y) = \min(I^-_{B_1}(x), I^-_{B_1}(y)),\ F^-_{B_2}(S_x S_y) = \min(F^-_{B_1}(x), F^-_{B_1}(y)),$$
$$F^+_{B_2}(S_x S_y) = \max(F^+_{B_1}(x), F^+_{B_1}(y)),\ T^-_{B_2}(S_x S_y) = \max(T^-_{B_1}(x), T^-_{B_1}(y)),$$

$\forall\ S_x S_y \in Y$.

Remark 3.31. Every BSVNLG is a strong BSVNG.

Remark 3.32. The $L(G) = (A_2, B_2)$ is a BSVNLG corresponding to BSVNG $G = (A_1, B_1)$.

Example 3.33. Consider the $G^* = (V, E)$ where $V = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ and $E = \{x_1 = \alpha_1\alpha_2, x_2 = \alpha_2\alpha_3, x_3 = \alpha_3\alpha_4, x_4 = \alpha_4\alpha_1\}$ and $G = (A_1, B_1)$ is BSVNG of $G^* = (V, E)$ which is defined in Table 6. Consider the $L(G^*) = (X, Y)$ such that $X = \{\Gamma_{x_1}, \Gamma_{x_2}, \Gamma_{x_3}, \Gamma_{x_4}\}$ and $Y = \{\Gamma_{x_1}\Gamma_{x_2}, \Gamma_{x_2}\Gamma_{x_3}, \Gamma_{x_3}\Gamma_{x_4}, \Gamma_{x_4}\Gamma_{x_1}\}$. Let A_2 and B_2 be BSVNSs of X and Y respectively, then BSVNLG $L(G)$ is given in Table 7.

Proposition 3.34. *The $L(G) = (A_2, B_2)$ is a BSVNLG of some BSVNG $G = (A_1, B_1)$ if and only if*

$$T^+_{B_2}(S_x S_y) = \min(T^+_{A_2}(S_x), T^+_{A_2}(S_y)),$$
$$T^-_{B_2}(S_x S_y) = \max(T^-_{A_2}(S_x), T^-_{A_2}(S_y)),$$
$$I^+_{B_2}(S_x S_y) = \max(I^+_{A_2}(S_x), I^+_{A_2}(S_y)),$$
$$F^-_{B_2}(S_x S_y) = \min(F^-_{A_2}(S_x), F^-_{A_2}(S_y)),$$

Table 7. BSVNSs of BSVNLG.

A_1	$T^+_{A_1}$	$I^+_{A_1}$	$F^+_{A_1}$	$T^-_{A_1}$	$I^-_{A_1}$	$F^-_{A_1}$
Γ_{x_1}	0.1	0.6	0.7	-0.1	-0.4	-0.5
Γ_{x_2}	0.3	0.6	0.7	-0.2	-0.3	-0.6
Γ_{x_3}	0.2	0.7	0.8	-0.3	-0.2	-0.6
Γ_{x_4}	0.1	0.7	0.8	-0.1	-0.4	-0.5
B_1	$T^+_{B_1}$	$I^+_{B_1}$	$F^+_{B_1}$	$T^-_{B_1}$	$I^-_{B_1}$	$F^-_{B_1}$
$\Gamma_{x_1}\Gamma_{x_2}$	0.1	0.6	0.7	-0.1	-0.4	-0.6
$\Gamma_{x_2}\Gamma_{x_3}$	0.2	0.7	0.8	-0.2	-0.3	-0.6
$\Gamma_{x_3}\Gamma_{x_4}$	0.1	0.7	0.8	-0.1	-0.4	-0.6
$\Gamma_{x_4}\Gamma_{x_1}$	0.1	0.7	0.8	-0.1	-0.4	-0.5

$$I^-_{B_2}(S_xS_y) = \min(I^-_{A_2}(S_x), I^-_{A_2}(S_y)),$$
$$F^+_{B_2}(S_xS_y) = \max(F^+_{A_2}(S_x), F^+_{A_2}(S_y)),$$

$\forall\ S_xS_y \in Y$.

Proof. Assume that,

$$T^+_{B_2}(S_xS_y) = \min(T^+_{A_2}(S_x), T^+_{A_2}(S_y)),$$
$$T^-_{B_2}(S_xS_y) = \max(T^-_{A_2}(S_x), T^-_{A_2}(S_y)),$$
$$I^+_{B_2}(S_xS_y) = \max(I^+_{A_2}(S_x), I^+_{A_2}(S_y)),$$
$$F^-_{B_2}(S_xS_y) = \min(F^-_{A_2}(S_x), F^-_{A_2}(S_y)),$$
$$I^-_{B_2}(S_xS_y) = \min(I^-_{A_2}(S_x), I^-_{A_2}(S_y)),$$
$$F^+_{B_2}(S_xS_y) = \max(F^+_{A_2}(S_x), F^+_{A_2}(S_y)),$$

$\forall\ S_xS_y \in Y$. Define

$$T^+_{A_1}(x) = T^+_{A_2}(S_x),\ I^+_{A_1}(x) = I^+_{A_2}(S_x),\ F^+_{A_1}(x) = F^+_{A_2}(S_x),$$
$$T^-_{A_1}(x) = T^-_{A_2}(S_x),\ I^-_{A_1}(x) = I^-_{A_2}(S_x),\ F^-_{A_1}(x) = F^-_{A_2}(S_x)$$

$\forall\ x \in E$. Then

$$I^+_{B_2}(S_xS_y) = \max(I^+_{A_2}(S_x), I^+_{A_2}(S_y)) = \max(I^+_{A_2}(x), I^+_{A_2}(y)),$$
$$I^-_{B_2}(S_xS_y) = \min(I^-_{A_2}(S_x), I^-_{A_2}(S_y)) = \min(I^-_{A_2}(x), I^-_{A_2}(y)),$$
$$T^+_{B_2}(S_xS_y) = \min(T^+_{A_2}(S_x), T^+_{A_2}(S_y)) = \min(T^+_{A_2}(x), T^+_{A_2}(y)),$$
$$T^-_{B_2}(S_xS_y) = \max(T^-_{A_2}(S_x), T^-_{A_2}(S_y)) = \max(T^-_{A_2}(x), T^-_{A_2}(y)),$$
$$F^-_{B_2}(S_xS_y) = \min(F^-_{A_2}(S_x), F^-_{A_2}(S_y)) = \min(F^-_{A_2}(x), F^-_{A_2}(y)),$$
$$F^+_{B_2}(S_xS_y) = \max(F^+_{A_2}(S_x), F^+_{A_2}(S_y)) = \max(F^+_{A_2}(x), F^+_{A_2}(y)).$$

A BSVNS A_1 that yields the property

$$T^+_{B_1}(xy) \leq \min(T^+_{A_1}(x), T^+_{A_1}(y)),\ I^+_{B_1}(xy) \geq \max(I^+_{A_1}(x), I^+_{A_1}(y)),$$
$$I^-_{B_1}(xy) \leq \min(I^-_{A_1}(x), I^-_{A_1}(y)),\ F^-_{B_1}(xy) \leq \min(F^-_{A_1}(x), F^-_{A_1}(y)),$$
$$F^+_{B_1}(xy) \geq \max(F^+_{A_1}(x), F^+_{A_1}(y)),\ T^-_{B_1}(xy) \geq \max(T^-_{A_1}(x), T^-_{A_1}(y))$$

will suffice. Converse is straight forward. □

Proposition 3.35. *If $L(G)$ be a BSVNLG of BSVNG G, then $L(G^*) = (X, Y)$ is the crisp line graph of G^*.*

Proof. Since $L(G)$ is a BSVNLG,

$$T^+_{A_2}(S_x) = T^+_{B_1}(x),\ I^+_{A_2}(S_x) = I^+_{B_1}(x),\ F^+_{A_2}(S_x) = F^+_{B_1}(x),$$

$$T^-_{A_2}(S_x) = T^-_{B_1}(x),\ I^-_{A_2}(S_x) = I^-_{B_1}(x),\ F^-_{A_2}(S_x) = F^-_{B_1}(x)$$

$\forall\ x \in E$, $S_x \in X$ if and only if $x \in E$, also

$$T^+_{B_2}(S_xS_y) = \min(T^+_{B_1}(x), T^+_{B_1}(y)),\ I^+_{B_2}(S_xS_y) = \max(I^+_{B_1}(x), I^+_{B_1}(y)),$$

$$I^-_{B_2}(S_xS_y) = \min(I^-_{B_1}(x), I^-_{B_1}(y)),\ F^-_{B_2}(S_xS_y) = \min(F^-_{B_1}(x), F^-_{B_1}(y)),$$

$$F^+_{B_2}(S_xS_y) = \max(F^+_{B_1}(x), F^+_{B_1}(y)),\ T^-_{B_2}(S_xS_y) = \max(T^-_{B_1}(x), T^-_{B_1}(y)),$$

$\forall\ S_xS_y \in Y$. Then $Y = \{S_xS_y : S_x \cap S_y \neq \phi, x, y \in E, x \neq y\}$. □

Proposition 3.36. *The $L(G) = (A_2, B_2)$ be a BSVNLG of BSVNG G if and only if $L(G^*) = (X, Y)$ is the line graph and*

$$T^+_{B_2}(xy) = \min(T^+_{A_2}(x), T^+_{A_2}(y)),\ I^+_{B_2}(xy) = \max(I^+_{A_2}(x), I^+_{A_2}(y)),$$

$$I^-_{B_2}(xy) = \min(I^-_{A_2}(x), I^-_{A_2}(y)),\ F^-_{B_2}(xy) = \min(F^-_{A_2}(x), F^-_{A_2}(y)),$$

$$F^+_{B_2}(xy) = \max(F^+_{A_2}(x), F^+_{A_2}(y)),\ T^-_{B_2}(xy) = \max(T^-_{A_2}(x), T^-_{A_2}(y)),$$

$\forall\ xy \in Y$.

Proof. It follows from propositions 3.34 and 3.35. □

Proposition 3.37. *Let G be a BSVNG, then $M(G)$ is isomorphic with $sd(G) \cup L(G)$.*

Theorem 3.38. *Let $L(G) = (A_2, B_2)$ be BSVNLG corresponding to BSVNG $G = (A_1, B_1)$.*

(1) *If G is weak isomorphic onto $L(G)$ if and only if $\forall\ v \in V$, $x \in E$ and G^* to be a cycle, such that*

$$T^+_{A_1}(v) = T^+_{B_1}(x),\ I^+_{A_1}(v) = T^+_{B_1}(x),\ F^+_{A_1}(v) = T^+_{B_1}(x),$$

$$T^-_{A_1}(v) = T^-_{B_1}(x),\ I^-_{A_1}(v) = T^-_{B_1}(x),\ F^-_{A_1}(v) = T^-_{B_1}(x).$$

(2) *If G is weak isomorphic onto $L(G)$, then G and $L(G)$ are isomorphic.*

Proof. By hypothesis, G^* is a cycle. Let $V = \{v_1, v_2, v_3, \ldots, v_n\}$ and $E = \{x_1 = v_1v_2, x_2 = v_2v_3, \ldots, x_n = v_nv_1\}$, where $P : v_1v_2v_3 \ldots v_n$ is a cycle, characterize a BSVNS A_1 by $A_1(v_i) = (p_i, q_i, r_i, p_i^{'}, q_i^{'}, r_i^{'})$ and B_1 by $B_1(x_i) = (a_i, b_i, c_i, a_i^{'}, b_i^{'}, c_i^{'})$ for $i = 1, 2, 3, \ldots, n$ and $v_{n+1} = v_1$. Then for $p_{n+1} = p_1$, $q_{n+1} = q_1$, $r_{n+1} = r_1$,

$$a_i \leq \min(p_i, p_{i+1}),\ b_i \geq \max(q_i, q_{i+1}),\ c_i \geq \max(r_i, r_{i+1}),$$

$$a_i^{'} \geq \max(p_i^{'}, p_{i+1}^{'}),\ b_i^{'} \leq \min(q_i^{'}, q_{i+1}^{'}),\ c_i^{'} \leq \min(r_i^{'}, r_{i+1}^{'}),$$

for $i = 1, 2, 3, \ldots, n$.

Now let $X = \{\Gamma_{x_1}, \Gamma_{x_2}, \ldots, \Gamma_{x_n}\}$ and $Y = \{\Gamma_{x_1}\Gamma_{x_2}, \Gamma_{x_2}\Gamma_{x_3}, \ldots, \Gamma_{x_n}\Gamma_{x_1}\}$. Then for $a_{n+1} = a_1$, we obtain

$$A_2(\Gamma_{x_i}) = B_1(x_i) = (a_i, b_i, c_i, a_i^{'}, b_i^{'}, c_i^{'})$$

and $B_2(\Gamma_{x_i}\Gamma_{x_{i+1}}) = (\min(a_i, a_{i+1}), \max(b_i, b_{i+1}), \max(c_i, c_{i+1}), \max(a_i^{'}, a_{i+1}^{'}), \min(b_i^{'}, b_{i+1}^{'}), \min(c_i^{'}, c_{i+1}^{'}))$ for $i = 1, 2, 3, \ldots, n$ and $v_{n+1} = v_1$. Since f preserves adjacency, it induce permutation π of $\{1, 2, 3, \ldots, n\}$,

$$f(v_i) = \Gamma_{v_{\pi(i)}v_{\pi(i)+1}}$$

and

$$v_iv_{i+1} \to f(v_i)f(v_{i+1}) = \Gamma_{v_{\pi(i)}v_{\pi(i)+1}}\Gamma_{v_{\pi(i+1)}v_{\pi(i+1)+1}},$$

for $i = 1, 2, 3, \ldots, n-1$. Thus

$$p_i = T^+_{A_1}(v_i) \leq T^+_{A_2}(f(v_i)) = T^+_{A_2}(\Gamma_{v_{\pi(i)}v_{\pi(i)+1}}) = T^+_{B_1}(v_{\pi(i)}v_{\pi(i)+1}) = a_{\pi(i)}.$$

Similarly, $p_i^{'} \geq a_{\pi(i)}^{'}$, $q_i \geq b_{\pi(i)}$, $r_i \geq c_{\pi(i)}$, $q_i^{'} \leq b_{\pi(i)}^{'}$, $r_i^{'} \leq c_{\pi(i)}^{'}$ and

$$\begin{aligned} a_i &= T^+_{B_1}(v_iv_{i+1}) \leq T^+_{B_2}(f(v_i)f(v_{i+1})) \\ &= T^+_{B_2}(\Gamma_{v_{\pi(i)}v_{\pi(i)+1}}\Gamma_{v_{\pi(i+1)}v_{\pi(i+1)+1}}) \\ &= \min(T^+_{B_1}(v_{\pi(i)}v_{\pi(i)+1}), T^+_{B_1}(v_{\pi(i+1)}v_{\pi(i+1)+1})) \\ &= \min(a_{\pi(i)}, a_{\pi(i)+1}). \end{aligned}$$

Similarly, $b_i \geq \max(b_{\pi(i)}, b_{\pi(i)+1})$, $c_i \geq \max(c_{\pi(i)}, c_{\pi(i)+1})$, $a_i^{'} \geq \max(a_{\pi(i)}^{'}, a_{\pi(i)+1}^{'})$, $b_i^{'} \leq \min(b_{\pi(i)}^{'}, b_{\pi(i)+1}^{'})$ and $c_i^{'} \leq \min(c_{\pi(i)}^{'}, c_{\pi(i)+1}^{'})$ for $i = 1, 2, 3, \ldots, n$. Therefore

$$p_i \leq a_{\pi(i)},\ q_i \geq b_{\pi(i)},\ r_i \geq c_{\pi(i)},\ p_i^{'} \geq a_{\pi(i)}^{'},\ q_i^{'} \leq b_{\pi(i)}^{'},\ r_i^{'} \leq c_{\pi(i)}^{'}$$

and

$$a_i \leq \min(a_{\pi(i)}, a_{\pi(i)+1}),\ a_i^{'} \geq \max(a_{\pi(i)}^{'}, a_{\pi(i)+1}^{'}),$$
$$b_i \geq \max(b_{\pi(i)}, b_{\pi(i)+1}),\ b_i^{'} \leq \min(b_{\pi(i)}^{'}, b_{\pi(i)+1}^{'}),$$
$$c_i \geq \max(c_{\pi(i)}, c_{\pi(i)+1}),\ c_i \leq \min(c_{\pi(i)}^{'}, c_{\pi(i)+1}^{'})$$

thus

$$a_i \leq a_{\pi(i)},\ b_i \geq b_{\pi(i)},\ c_i \geq c_{\pi(i)},\ a_i^{'} \geq a_{\pi(i)}^{'},\ b_i^{'} \leq b_{\pi(i)}^{'},\ c_i^{'} \leq c_{\pi(i)}^{'}$$

and so

$$a_{\pi(i)} \leq a_{\pi(\pi(i))},\ b_{\pi(i)} \geq b_{\pi(\pi(i))},\ c_{\pi(i)} \geq c_{\pi(\pi(i))}$$
$$a_{\pi(i)}^{'} \geq a_{\pi(\pi(i))}^{'},\ b_{\pi(i)}^{'} \leq b_{\pi(\pi(i))}^{'},\ c_{\pi(i)}^{'} \leq c_{\pi(\pi(i))}^{'}$$

$\forall\ i = 1, 2, 3, \ldots, n$. Next to extend,

$$a_i \leq a_{\pi(i)} \leq \ldots \leq a_{\pi^j(i)} \leq a_i,\ a_i^{'} \geq a_{\pi(i)}^{'} \geq \ldots \geq a_{\pi^j(i)}^{'} \geq a_i^{'}$$
$$b_i \geq b_{\pi(i)} \geq \ldots \geq b_{\pi^j(i)} \geq b_i,\ b_i^{'} \leq b_{\pi(i)}^{'} \leq \ldots \leq b_{\pi^j(i)}^{'} \leq b_i^{'}$$
$$c_i \geq c_{\pi(i)} \geq \ldots \geq c_{\pi^j(i)} \geq c_i,\ c_i^{'} \leq c_{\pi(i)}^{'} \leq \ldots \leq c_{\pi^j(i)}^{'} \leq c_i^{'}$$

where π^{j+1} identity. Hence

$$a_i = a_{\pi(i)},\ b_i = b_{\pi(i)},\ c_i = c_{\pi(i)},\ a_i^{'} = a_{\pi(i)}^{'},\ b_i^{'} = b_{\pi(i)}^{'},\ c_i^{'} = c_{\pi(i)}^{'}$$

$\forall\ i = 1, 2, 3, \ldots, n$. Thus we conclude that

$$a_i \leq a_{\pi(i+1)} = a_{i+1},\ b_i \geq b_{\pi(i+1)} = b_{i+1},\ c_i \geq c_{\pi(i+1)} = c_{i+1}$$
$$a_i^{'} \geq a_{\pi(i+1)}^{'} = a_{i+1}^{'},\ b_i^{'} \leq b_{\pi(i+1)}^{'} = b_{i+1}^{'},\ c_i^{'} \leq c_{\pi(i+1)}^{'} = c_{i+1}^{'}$$

which together with

$$a_{n+1} = a_1,\ b_{n+1} = b_1,\ c_{n+1} = c_1,\ a'_{n+1} = a'_1,\ b'_{n+1} = b'_1,\ c'_{n+1} = c'_1$$

which implies that

$$a_i = a_1,\ b_i = b_1,\ c_i = c_1,\ a'_i = a'_1,\ b'_i = b'_1,\ c'_i = c'_1$$

$\forall\ i = 1, 2, 3, \ldots, n$. Thus we have

$$\begin{aligned}
a_1 = a_2 = \ldots = a_n = p_1 = p_2 = \ldots = p_n \\
a'_1 = a'_2 = \ldots = a'_n = p'_1 = p'_2 = \ldots = p'_n \\
b_1 = b_2 = \ldots = b_n = q_1 = q_2 = \ldots = q_n \\
b'_1 = b'_2 = \ldots = b'_n = q'_1 = q'_2 = \ldots = q'_n \\
c_1 = c_2 = \ldots = c_n = r_1 = r_2 = \ldots = r_n \\
c'_1 = c'_2 = \ldots = c'_n = r'_1 = r'_2 = \ldots = r'_n
\end{aligned}$$

Therefore (a) and (b) holds, since converse of result (a) is straight forward. □

4. Conclusion

The neutrosophic graphs have many applications in path problems, networks and computer science. Strong BSVNG and complete BSVNG are the types of BSVNG. In this paper, we discussed the special types of BSVNGs, subdivision BSVNGs, middle BSVNGs, total BSVNGs and BSVNLGs of the given BSVNGs. We investigated isomorphism properties of subdivision BSVNGs, middle BSVNGs, total BSVNGs and BSVNLGs.

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An Extended TOPSIS for Multi-Attribute Decision Making Problems with Neutrosophic Cubic Information

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Abstract. The paper proposes a new technique for dealing with multi-attribute decision making problems through an extended TOPSIS method under neutrosophic cubic environment. Neutrosophic cubic set is the generalized form of cubic set and is the hybridization of a neutrosophic set with an interval neutrosophic set. In this study, we have defined some operation rules for neutrosophic cubic sets and proposed the Euclidean distance between neutrosophic cubic sets. In the decision making situation, the rating of alternatives with respect to some predefined attributes are presented in terms of neutrosophic cubic information where weights of the attributes are completely unknown. In the selection process, neutrosophic cubic positive and negative ideal solutions have been defined. An extended TOPSIS method is then proposed for ranking the alternatives and finally choosing the best one. Lastly, an illustrative example is solved to demonstrate the decision making procedure and effectiveness of the developed approach.

Keywords: TOPSIS; neutrosophic sets; interval neutrosophic set; neutrosophic cubic sets; multi-attribute decision making.

1 Introduction

Smarandache [1] proposed neutrosophic set (NS) that assumes values from real standard or non-standard subsets of] $^-0$, 1^+[. Wang et al. [2] defined single valued neutrosophic set (SVNS) that assumes values from the unit interval [0, 1]. Wang et al. [3] also extended the concept of NS to interval neutrosophic set (INS) and presented the set-theoretic operators and different properties of INSs. Multi-attribute decision making (MADM) problems with neutrosophic information caught much attention in recent years due to the fact that the incomplete, indeterminate and inconsistent information about alternatives with regard to predefined attributes are easily described under neutrosophic setting. In interval neutrosophic environment, Chi and Liu [4] at first established an extended technique for order preference by similarity to ideal solution (TOPSIS) method [5] for solving MADM problems with interval neutrosophic information to get the most preferable alternative. Şahin, and Yiğider [6] discussed TOPSIS method for multi-criteria decision making (MCDM) problems with single neutrosophic values for supplier selection problem. Zhang and Wu [7] developed an extended TOPSIS for single valued neutrosophic MCDM problems where the incomplete weights are obtained by maximizing deviation method. Ye [8] proposed an extended TOPSIS method for solving MADM problems under interval neutrosophic uncertain linguistic variables. Biswas et al. [9] studied TOPSIS method for solving multi-attribute group decision making problems with single-valued neutrosophic information where weighted averaging operator is employed to aggregate the individual decision maker's opinion into group opinion.

In 2016, Ali et al. [10] proposed the notion of neutrosophic cubic set (NCS) by extending the concept of cubic set to neutrosophic cubic set. Ali et al. [10] also defined internal neutrosophic cubic set (INCS) and external neutrosophic cubic set (ENCS), $1/3$ -INCS ($2/3$ -ENCS), $2/3$ -INCS ($1/3$ -ENCS) and also proposed some relevant properties. In the same study, Ali et al. [10] proposed Hamming distance between two NCSs and developed a decision making technique via similarity measures of two NCSs in pattern recognition problems. Jun et al. [11] studied the notions of truth-internal (indeterminacy-internal, falsity-internal) neutrosophic cubic sets and truth-external (indeterminacy-external, falsity-external) neutrosophic cubic and investigated related properties. Pramanik et al. [12] defined similarity measure for neutrosophic cubic sets and proved its basic properties. In the same study, Pramanik et al. [12] developed multi criteria group decision making method with linguistic variables in neutrosophic cubic set environment.

In this paper, we develop a new approach for MADM problems with neutrosophic cubic assessments by using TOPSIS method where weights of the attributes are unknown to the decision maker (DM). We define a few operations on NCSs and propose the Euclidean distance between two NCSs. We define accumulated arithmetic operator (AAO) to convert neutrosophic cubic values (NCVs) to single neutrosophic values (SNVs). We also define neutrosophic cubic positive ideal solution (NCPIS) and neutrosophic cubic negative ideal solution (NCNIS) in this study. The rest of the paper is organized in the following way. Section 2 recalls some basic definitions which are useful for the construction of the paper. Subsection 2.1 provides several operational rules of NCSs. Section 3 is devoted to present an extended TOPSIS method for MADM problems in neutrosophic cubic set environment. In Section 4, we solve an illustrative example to demonstrate the applicability and effectiveness of the proposed approach. Finally, the last Section presents concluding remarks and future scope research.

2 The basic definitions

Definition: 1
Fuzzy sets [13]: Consider U be a universe. A fuzzy set Φ over U is defined as follows:

$$\Phi = \{\langle x,\ \mu_{\Phi}(x)\rangle \mid x\in U\}$$

where $\mu_{\Phi}(x)$: $U \rightarrow [0, 1]$ is termed as the membership function of Φ and $\mu_{\Phi}(x)$ represents the degree of membership to which $x\in \Phi$.

Definition: 2
Interval valued fuzzy sets [14]: An interval-valued fuzzy set (IVFS) Θ over U is represented as follows:

$$\Theta = \{\langle x,\ \Theta^{-}(x), \Theta^{+}(x)\rangle \mid x\in U\}$$

where $\Theta^{-}(x), \Theta^{+}(x)$ denote the lower and upper degrees of membership of the element $x\in U$ to the set Θ with $0\leq \Theta^{-}(x) + \Theta^{+}(x) \leq 1$.

Definition: 3
Cubic sets [15]: A cubic set Ψ in a non-empty set U is a structure defined as follows:

$$\Psi = \{\langle x, \Theta(x), \Phi(x)\rangle \mid x\in U\}$$

where Θ and Φ respectively represent an interval valued fuzzy set and a fuzzy set. A cubic set Ψ is denoted by $\Psi = <\Theta, \Phi>$.

Definition: 4
Internal cubic sets [15]: A cubic set $\Psi = <\Theta, \Phi>$ in U is said to be internal cubic set (ICS) if $\Theta^{-}(x) \leq \mu(x) \leq \Theta^{+}(x)$ for all $x\in U$.

Definition: 5
External cubic sets [15]: A cubic set $\Psi = <\Theta, \Phi>$ in U is called external cubic set (ECS) if $\mu(x) \notin (\Theta^{-}(x), \Theta^{+}(x))$ for all $x\in U$.

Definition: 6
Consider $\Psi_1 = <\Theta_1, \Phi_1>$ and $\Psi_2 = <\Theta_2, \Phi_2>$ be two cubic sets in U, then we have the following relations [15].

1. (Equality) $\Psi_1 = \Psi_2$ if and only if $\Theta_1 = \Theta_2$ and $\mu_1 = \mu_2$.
2. (P-order) $\Psi_1 \subseteq_P \Psi_2$ if and only if $\Theta_1 \subseteq \Theta_2$ and $\mu_1 \leq \mu_2$.
3. (R-order) $\Psi_1 \subseteq_R \Psi_2$ if and only if $\Theta_1 \subseteq \Theta_2$ and $\mu_1 \geq \mu_2$.

Definition: 7
Neutrosophic set [1]: Consider U be a space of objects, then a neutrosophic set (NS) χ on U is defined as follows:

$$\chi = \{x, \langle \alpha(x), \beta(x), \gamma(x)\rangle \mid x\in U\}$$

where $\alpha(x), \beta(x), \gamma(x) : U\rightarrow\ ]^{-}0,\ 1^{+}[$ define respectively the degrees of truth-membership, indeterminacy-membership, and falsity-membership of an element $x\in U$ to the set χ with $^{-}0 \leq \sup\ \alpha(x) + \sup\ \beta(x) + \sup\ \gamma(x) \leq 3^{+}$.

Definition: 8
Interval neutrosophic sets [9]: An INS Γ in the universal space U is defined as follows:

$$\Gamma = \{x, \langle [\Gamma^{-}_{\alpha}(x), \Gamma^{+}_{\alpha}(x)], [\Gamma^{-}_{\beta}(x), \Gamma^{+}_{\beta}(x)], [\Gamma^{-}_{\gamma}(x), \Gamma^{+}_{\gamma}(x)]\rangle \mid x\in U\}$$

where, $\Gamma_{\alpha}(x)$, $\Gamma_{\beta}(x)$, $\Gamma_{\gamma}(x)$ are the truth-membership function, indeterminacy-membership function, and falsity-membership function, respectively with $\Gamma_{\alpha}(x)$, $\Gamma_{\beta}(x)$, $\Gamma_{\gamma}(x) \subseteq [0, 1]$ for each point $x\in U$ and $0 \leq \sup \Gamma_{\alpha}(x) + \sup \Gamma_{\beta}(x) + \sup \Gamma_{\gamma}(x) \leq 3$.

Definition: 9
Neutrosophic cubic sets [15]
A neutrosophic cubic set (NCS) Ξ in a universe U is presented in the following form:

$$\Xi = \{ \langle x, \Gamma(x), \chi(x) \rangle \mid x \in U\}$$

where Γ and χ are respectively an interval neutrosophic set and a neutrosophic set in U.
However, NSs take the values from] $^-0$, 1^+[and single-valued neutrosophic set defined by Wang *et al*. [2] is appropriate for dealing with real world problems. Therefore, the definition of NCS should be modified in order to solve practical decision making purposes. Hence, a neutrosophic cubic set (NCS) Ξ in U is defined as follows:

$$\Xi = \{ \langle x, \Gamma(x), \chi(x) \rangle \mid x \in U\}$$

Here, Γ and χ are respectively an INS and a SVNS in U where $0 \le \alpha(x) + \beta(x) + \gamma(x) \le 3$ for each point $x \in U$. Generally, a NCS is denoted by $\Xi = <\Gamma, \chi>$ and sets of all NCS over U will be represented by NCS^U.

Example 1. Assume that $U = \{u_1, u_2, u_3, u_4\}$ be a universal set. An INS A in U is defined as
Γ = {< [0.15, 0.3], [0.25, 0.4], [0.6, 0.75] >/ u_1 + < [0.25, 0.35], [0.35, 0.45], [0.4, 0.65] >/ u_2 + < [0.35, 0.5], [0.25, 0.35], [0.55, 0.85] >/ u_3 + < [0.7, 0.8], [0.15, 0.45], [0.2, 0.3] >/ u_4}
and a SVNS χ in U defined by
χ = {< 0.35, 0.3, 0.15 >/ u_1, < 0.5, 0.1, 0.4 >/ u_2, < 0.25, 0.03, 0.35 >/ u_3, < 0.85, 0.1, 0.15 >/ u_4}
Then $\Xi = <A, \chi>$ is represented as a NCS in U.

Definition: 10
Internal neutrosophic cubic set [10]: Consider $\Xi = <\Gamma, \chi> \in \text{NCS}^U$, if $\Gamma^-_\alpha(x) \le \alpha(x) \le \Gamma^+_\alpha(x)$; $\Gamma^-_\beta(x) \le \beta(x) \le \Gamma^+_\beta(x)$; and $\Gamma^-_\gamma(x) \le \gamma(x) \le \Gamma^+_\gamma(x)$ for all $x \in U$, then Ξ is said to be an internal neutrosophic cubic set (INCS).

Example 2. Consider $\Xi = <\Gamma, \chi> \in \text{NCS}^U$, if $\Gamma(x) =$ < [0.65, 0.8], [0.1, 0.25], [0.2, 0.4] > and $\chi(x) =$ < 0.7, 0.2, 0.3 > for all $x \in U$, then $\Xi = <\Gamma, \chi>$ is an INCS.

Definition: 11
External neutrosophic cubic set [10]: Consider $\Xi = <\Gamma, \chi> \in \text{NCS}^U$, if $\alpha(x) \notin (\Gamma^-_\alpha(x), \Gamma^+_\alpha(x))$; $\beta(x) \notin (\Gamma^-_\beta(x), \Gamma^+_\beta(x))$; and $\gamma(x) \notin (\Gamma^-_\gamma(x), \Gamma^+_\gamma(x))$ for all $x \in U$, then $\Xi = <\Gamma, \chi>$ is said to be an external neutrosophic cubic set (ENCS).

Example 3. Consider $\Xi = <\Gamma, \chi> \in \text{NCS}^U$, if $\Gamma(x) =$ < [0.65, 0.8], [0.1, 0.25], [0.2, 0.4] > and $\chi(x) =$ < 0.85, 0.3, 0.1 > for all $x \in U$, then $\Xi = <\Gamma, \chi>$ is an ENCS.

Theorem 1. [10]
Consider $\Xi = <\Gamma, \chi> \in \text{NCS}^U$, which is not an ENCS, then there exists $x \in U$ such that $\Gamma^-_\alpha(x) \le \alpha(x) \le \Gamma^+_\alpha(x)$; $\Gamma^-_\beta(x) \le \beta(x) \le \Gamma^+_\beta(x)$; or $\Gamma^-_\gamma(x) \le \gamma(x) \le \Gamma^+_\gamma(x)$.

Definition: 12
$\frac{2}{3}$**-INCS(**$\frac{1}{3}$**-ENCS) [10]:** Consider $\Xi = <\Gamma, \chi> \in \text{NCS}^U$, if $\Gamma^-_\alpha(x) \le \alpha(x) \le \Gamma^+_\alpha(x)$; $\Gamma^-_\beta(x) \le \beta(x) \le \Gamma^+_\beta(x)$; and $\gamma(x) \notin (\Gamma^-_\gamma(x), \Gamma^+_\gamma(x))$ or $\Gamma^-_\alpha(x) \le \alpha(x) \le \Gamma^+_\alpha(x)$; $\Gamma^-_\gamma(x) \le \gamma(x) \le \Gamma^+_\gamma(x)$ and $\beta(x) \notin (\Gamma^-_\beta(x), \Gamma^+_\beta(x))$ or $\Gamma^-_\beta(x) \le \beta(x) \le \Gamma^+_\beta(x)$; and $\Gamma^-_\gamma(x) \le \gamma(x) \le \Gamma^+_\gamma(x)$ and $\alpha(x) \notin (\Gamma^-_\alpha(x), \Gamma^+_\alpha(x))$ for all $x \in U$, then $\Xi = <\Gamma, \chi>$ is said to be an $\frac{2}{3}$-INCS or $\frac{1}{3}$-ENCS.

Example 4. Consider $\Xi = <\Gamma, \chi> \in \text{NCS}^U$, if $\Gamma(x) =$ < [0. 5, 0.7], [0.1, 0.2], [0.2, 0.45] > and $\chi(x) =$ < 0.65, 0.3, 0.4 > for all $x \in U$, then $\Xi = <\Gamma, \chi>$ is an $\frac{2}{3}$-INCS or $\frac{1}{3}$-ENCS.

Definition: 13
$\frac{1}{3}$**-INCS (**$\frac{2}{3}$**-ENCS) [10]:** Consider $\Xi = <\Gamma, \chi> \in \text{NCS}^U$, if $\Gamma^-_\alpha(x) \le \alpha(x) \le \Gamma^+_\alpha(x)$; $\beta(x) \notin (\Gamma^-_\beta(x), \Gamma^+_\beta(x))$; and $\gamma(x) \notin (\Gamma^-_\gamma(x), \Gamma^+_\gamma(x))$ or $\Gamma^-_\gamma(x) \le \gamma(x) \le \Gamma^+_\gamma(x)$; $\alpha(x) \notin (\Gamma^-_\alpha(x), \Gamma^+_\alpha(x))$ and $\beta(x) \notin (\Gamma^-_\beta(x), \Gamma^+_\beta(x))$ or $\Gamma^-_\beta(x) \le \beta(x) \le \Gamma^+_\beta(x)$; $\alpha(x) \notin (\Gamma^-_\alpha(x), \Gamma^+_\alpha(x))$ and $\gamma(x) \notin (\Gamma^-_\gamma(x), \Gamma^+_\gamma(x))$ for all $x \in U$, then $\Xi = <\Gamma, \chi>$ is said to be an $\frac{1}{3}$-INCS or $\frac{2}{3}$-ENCS.

Example 5. Consider $\Xi = <\Gamma, \chi> \in \text{NCS}^U$, if $\Gamma(x) =$ < [0. 5, 0.8], [0.15, 0.25], [0.2, 0.35] > and $\chi(x) =$ < 0.55, 0.4, 0.5 > for all $x \in U$, then $\Xi = <\Gamma, \chi>$ is an $\frac{1}{3}$-INCS or $\frac{2}{3}$-ENCS.

Definition: 14 [10]
Consider $\Xi_1 = <\Gamma_1, \chi_1>$ and $\Xi_2 = <\Gamma_2, \chi_2>$ be two NCSs in U, then

1. (Equality) $\Xi_1 = \Xi_2$ if and only if $\Gamma_1 = \Gamma_2$ and $\chi_1 = \chi_2$.
2. (P-order) $\Xi_1 \subseteq_P \Xi_2$ if and only if $\Gamma_1 \widetilde{\subseteq} \Gamma_2$ and $\chi_1 \breve{\subseteq} \chi_2$.
3. (R-order) $\Xi_1 \subseteq_R \Xi_2$ if and only if $\Gamma_1 \widetilde{\subseteq} \Gamma_2$ and $\chi_1 \breve{\supseteq} \chi_2$.

For convenience, $p = <([\Gamma^-_{\alpha}, \Gamma^+_{\alpha_1}], [\Gamma^-_{\beta_1}, \Gamma^+_{\beta_1}], [\Gamma^-_{\gamma}, \Gamma^+_{\gamma}]), (\alpha, \beta, \gamma)>$ is said to represent neutrosophic cubic value (NCV)

Definition: 15
Complement [10]: Consider $\Xi = <\Gamma, \chi>$ be an NCS, then the complement of $\Xi = <\Gamma, \chi>$ is given by

$$\Xi^C = \{\left\langle x, \Gamma^{\tilde{C}}(x), \chi^{\breve{C}}(x)\right\rangle \mid x \in U\}.$$

2.1 Several operational rules of NCVs

Consider $p_1 = <([\Gamma^-_{\alpha_1}, \Gamma^+_{\alpha_1}], [\Gamma^-_{\beta_1}, \Gamma^+_{\beta_1}], [\Gamma^-_{\gamma_1}, \Gamma^+_{\gamma_1}]), (\alpha_1, \beta_1, \gamma_1)>$ and $p_2 = <([\Gamma^-_{\alpha_2}, \Gamma^+_{\alpha_2}], [\Gamma^-_{\beta_2}, \Gamma^+_{\beta_2}], [\Gamma^-_{\gamma_2}, \Gamma^+_{\gamma_2}]), (\alpha_2, \beta_2, \gamma_2)>$ be two NCVs in *U*, then the operational rules are presented as follows:

1. The complement [10] of p_1 is $p_1^C = <([\Gamma^-_{\gamma_1}, \Gamma^+_{\gamma_1}], [1-\Gamma^+_{\beta_1}, 1-\Gamma^-_{\beta_1}], [\Gamma^-_{\alpha_1}, \Gamma^+_{\alpha_1}]), (\gamma_1, 1-\beta_1, \alpha_1)>$.
2. The summation between p_1 and p_2 is defined as follows:

   $p_1 \oplus p_2 = <([\Gamma^-_{\alpha_1} + \Gamma^-_{\alpha_2} - \Gamma^-_{\alpha_1}\Gamma^-_{\alpha_2}, \Gamma^+_{\alpha_1} + \Gamma^+_{\alpha_2} - \Gamma^+_{\alpha_1}\Gamma^+_{\alpha_2}], [\Gamma^-_{\beta_1}\Gamma^-_{\beta_2}, \Gamma^+_{\beta_1}\Gamma^+_{\beta_2}], [\Gamma^-_{\gamma_1}\Gamma^-_{\gamma_2}, \Gamma^+_{\gamma_1}\Gamma^+_{\gamma_2}]), (\alpha_1 + \alpha_2 - \alpha_1\alpha_2, \beta_1\beta_2, \gamma_1\gamma_2)>$.
3. The multiplication between p_1 and p_2 is defined as follows:

   $p_1 \otimes p_2 = <([\Gamma^-_{\alpha_1}\Gamma^-_{\alpha_2}, \Gamma^+_{\alpha_1}\Gamma^+_{\alpha_2}], [\Gamma^-_{\beta_1} + \Gamma^-_{\beta_2} - \Gamma^-_{\beta_1}\Gamma^-_{\beta_2}, \Gamma^+_{\beta_1} + \Gamma^+_{\beta_2} - \Gamma^+_{\beta_1}\Gamma^+_{\beta_2}], [\Gamma^-_{\gamma_1} + \Gamma^-_{\gamma_2} - \Gamma^-_{\gamma_1}\Gamma^-_{\gamma_2}, \Gamma^+_{\gamma_1} + \Gamma^+_{\gamma_2} - \Gamma^+_{\gamma_1}\Gamma^+_{\gamma_2}]), (\alpha_1\alpha_2, \beta_1 + \beta_2 - \beta_1\beta_2, \gamma_1 + \gamma_2 - \gamma_1\gamma_2)>$.
4. Consider $p_1 = <([\Gamma^-_{\alpha_1}, \Gamma^+_{\alpha_1}], [\Gamma^-_{\beta_1}, \Gamma^+_{\beta_1}], [\Gamma^-_{\gamma_1}, \Gamma^+_{\gamma_1}]), (\alpha_1, \beta_1, \gamma_1)>$ be a NCV and κ be an arbitrary positive real number, then $\kappa\, p_1$ and p_1^{κ} are defined as follows:

   (i) $\kappa\, p_1 = <([1-(1-\Gamma^-_{\alpha_1})^{\kappa}, 1-(1-\Gamma^+_{\alpha_1})^{\kappa}], [(\Gamma^-_{\beta_1})^{\kappa}, (\Gamma^+_{\beta_1})^{\kappa}], [(\Gamma^-_{\gamma_1})^{\kappa}, (\Gamma^+_{\gamma_1})^{\kappa}]), (1-(1-\alpha_1)^{\kappa}, (\beta_1)^{\kappa}, (\gamma_1)^{\kappa})>$;

   (ii) $p_1^{\kappa} = <([(\Gamma^-_{\alpha_1})^{\kappa}, (\Gamma^+_{\alpha_1})^{\kappa}], [1-(1-\Gamma^-_{\beta_1})^{\kappa}, 1-(1-\Gamma^+_{\beta_1})^{\kappa}], [1-(1-\Gamma^-_{\gamma_1})^{\kappa}, 1-(1-\Gamma^+_{\gamma_1})^{\kappa}]), ((\alpha_1)^{\kappa}, 1-(1-\beta_1)^{\kappa}, 1-(1-\gamma_1)^{\kappa})>$.

Definition: 16 [10]
Consider $\Xi_1 = <\Gamma_1, \chi_1>$ and $\Xi_2 = <\Gamma_2, \chi_2>$ be two NCSs in *U*, then the Hamming distance between Ξ_1 and Ξ_2 is defined as follows:

$$D_H(\Xi_1, \Xi_2) = \frac{1}{9n}\sum_{i=1}^{n}\big(|\Gamma^-_{1\alpha}(x_i) - \Gamma^-_{2\alpha}(x_i)| + |\Gamma^-_{1\beta}(x_i) - \Gamma^-_{2\beta}(x_i)| + |\Gamma^-_{1\gamma}(x_i) - \Gamma^-_{2\gamma}(x_i)| + |\Gamma^+_{1\alpha}(x_i) - \Gamma^+_{2\alpha}(x_i)| + |\Gamma^+_{1\beta}(x_i) - \Gamma^+_{2\beta}(x_i)| + |\Gamma^+_{1\gamma}(x_i) - \Gamma^+_{2\gamma}(x_i)| + |\alpha_1(x_i) - \alpha_2(x_i)| + |\beta_1(x_i) - \beta_2(x_i)| + |\gamma_1(x_i) - \gamma_2(x_i)|\big).$$

Example 7: Suppose that $\Xi_1 = <\Gamma_1, \chi_1> = <([0.6, 0.75], [0.15, 0.25], [0.25, 0.45]), (0.8, 0.35, 0.15)>$ and $\Xi_2 = <\Gamma_2, \chi_2> = <([0.45, 0.7], [0.1, 0.2], [0.05, 0.2]), (0.3, 0.15, 0.45)>$ be two NCSs in *U*, then $D_H(\Xi_1, \Xi_2) = 0.1944$.

Definition: 17
Consider $\Xi_1 = <\Gamma_1, \chi_1>$ and $\Xi_2 = <\Gamma_2, \chi_2>$ be two NCSs in *U*, then the Euclidean distance between Ξ_1 and Ξ_2 is defined as given below.

$$D_E(\Xi_1, \Xi_2) = \sqrt{\frac{1}{9n}\sum_{i=1}^{n}\begin{pmatrix}(\Gamma^-_{1\alpha}(x_i) - \Gamma^-_{2\alpha}(x_i))^2 + (\Gamma^+_{1\alpha}(x_i) - \Gamma^+_{2\alpha}(x_i))^2 + (\Gamma^-_{1\beta}(x_i) - \Gamma^-_{2\beta}(x_i))^2 + \\ (\Gamma^+_{1\beta}(x_i) - \Gamma^+_{2\beta}(x_i))^2 + (\Gamma^-_{1\gamma}(x_i) - \Gamma^-_{2\gamma}(x_i))^2 + (\Gamma^+_{1\gamma}(x_i) - \Gamma^+_{2\gamma}(x_i))^2 + \\ (\alpha_1(x_i) - \alpha_2(x_i))^2 + (\beta_1(x_i) - \beta_2(x_i))^2 + (\gamma_1(x_i) - \gamma_2(x_i))^2\end{pmatrix}}$$

with the condition $0 \le D_E(\Xi_1, \Xi_2) \le 1$.

Example 8: Suppose that $\Xi_1 = <\Gamma_1, \chi_1> = <([0.4, 0.5], [0.1, 0.2], [0.25, 0.5]), (0.4, 0.3, 0.25)>$ and $\Xi_2 = <\Gamma_2, \chi_2> = <([0.5, 0.9], [0.15, 0.3], [0.05, 0.1]), (0.7, 0.1, 0.15) > be two NCSs in U, then $D_E(\Xi_1, \Xi_2) = 0.2409$.

3 An extended TOPSIS method for MADM problems under neutrosophic cubic set environment

In this Section, we introduce a new extended TOPSIS method to handle MADM problems involving neutrosophic cubic information. Consider $B = \{B_1, B_2, \ldots, B_m\}$, $(m \geq 2)$ be a discrete set of m feasible alternatives and $C = \{C_1, C_2, \ldots, C_n\}$, $(n \geq 2)$ be a set of attributes. Also, let $w = (w_1, w_2, \ldots, w_n)^T$ be the unknown weight vector of the attributes with $0 \leq w_j \leq 1$ such that $\sum_{j=1}^{n} w_j = 1$. Suppose that the rating of alternative B_i (i = 1, 2, …, m) with respect to the attribute C_j (j = 1, 2, …, n) is described by a_{ij} where $a_{ij} = <([\Gamma^-_{\alpha_{ij}}, \Gamma^+_{\alpha_{ij}}], [\Gamma^-_{\beta_{ij}}, \Gamma^+_{\beta_{ij}}], [\Gamma^-_{\gamma_{ij}}, \Gamma^+_{\gamma_{ij}}]), (\alpha_{ij}, \beta_{ij}, \gamma_{ij})>$. The proposed approach for ranking the alternatives under neutrosophic cubic environment is shown using the following steps:

Step 1. Construction and standardization of decision matrix with neutrosophic cubic information

Consider the rating of alternative B_i (i = 1, 2, …, m) with respect to the predefined attribute C_j, (j = 1, 2, …, n) be presented by the decision maker in the neutrosophic cubic decision matrix (See eqn. 1).

$$\langle a_{ij} \rangle_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \quad (1)$$

In general, there are two types of attributes appear in the decision making circumstances namely (i) benefit type attributes $\in J_1$, where the more attribute value denotes better alternative (ii) cost type attributes $\in J_2$, where the less attribute value denotes better alternative. We standardize the above decision matrix $\langle a_{ij} \rangle_{m\times n}$ in order to remove the influence of diverse physical dimensions to decision results.

Consider $\langle s_{ij} \rangle_{m\times n}$ be the standardize decision matrix, where

$$s_{ij} = <([\dot{\Gamma}^-_{\alpha_{ij}}, \dot{\Gamma}^+_{\alpha_{ij}}], [\dot{\Gamma}^-_{\beta_{ij}}, \dot{\Gamma}^+_{\beta_{ij}}], [\dot{\Gamma}^-_{\gamma_{ij}}, \dot{\Gamma}^+_{\gamma_{ij}}]), (\dot{\alpha}_{ij}, \dot{\beta}_{ij}, \dot{\gamma}_{ij})>,$$

where

$s_{ij} = a_{ij}$, if the attribute j is benefit type;

$s_{ij} = a^c_{ij}$, if the attribute j is cost type.

Here a^c_{ij} denotes the complement of a_{ij}.

Step 2. Identify the weights of the attributes

To determine the unknown weight of attribute in the decision making situation is a difficult task for DM. Generally, weights of the attributes are dissimilar and they play a decisive role in finding the ranking order of the alternatives. Pramanik and Mondal [16] defined arithmetic averaging operator (AAO) in order to transform interval neutrosophic numbers to SVNNs. Based on the concept of Pramanik and Mondal [16], we define AAO to transform NCVs to SNVs as follows:

$$NC_{ij} < \dot{\Gamma}_{\alpha_{ij}}, \dot{\Gamma}_{\beta_{ij}}, \dot{\Gamma}_{\gamma_{ij}} > = NC_{ij} \left\langle \frac{\dot{\Gamma}^-_{\alpha_{ij}} + \dot{\Gamma}^+_{\alpha_{ij}} + \dot{\alpha}_{ij}}{3}, \frac{\dot{\Gamma}^-_{\beta_{ij}} + \dot{\Gamma}^+_{\beta_{ij}} + \dot{\beta}_{ij}}{3}, \frac{\dot{\Gamma}^-_{\gamma_{ij}} + \dot{\Gamma}^+_{\gamma_{ij}} + \dot{\gamma}_{ij}}{3} \right\rangle$$

In this paper, we utilize information entropy method to find the weights of the attributes w_j where weihgts of the attributes are unequal and fully unknown to the DM. Majumdar and Samanta [17] investigated some similarity measures and entropy measures for SVNSs where entropy is used to measure uncertain information. Here, we take the following notations:

$$T_{\Omega_P}(x_i) = \left[\frac{\dot{\Gamma}^-_{\alpha_{ij}} + \dot{\Gamma}^+_{\alpha_{ij}} + \dot{\alpha}_{ij}}{3} \right], \; I_{\Omega_P}(x_i) = \left[\frac{\dot{\Gamma}^-_{\beta_{ij}} + \dot{\Gamma}^+_{\beta_{ij}} + \dot{\beta}_{ij}}{3} \right],$$

$$F_{\Omega_P}(x_i) = \left[\frac{\dot{\Gamma}^-_{\gamma_{ij}} + \dot{\Gamma}^+_{\gamma_{ij}} + \dot{\gamma}_{ij}}{3} \right]$$

Then we can write $\Omega_P = \langle T_{\Omega_P}(x_i), I_{\Omega_P}(x_i), F_{\Omega_P}(x_i) \rangle$.

The entropy value is given as follows:

$$E_i(\Omega_P) = 1 - \frac{1}{n} \sum_{i=1}^{m} (T_{\Omega_P}(x_i) + F_{\Omega_P}(x_i)) \left| I_{\Omega_P}(x_i) - I^c_{\Omega_P}(x_i) \right|$$

which has the following properties:

(i). $E_i(\Omega_P) = 0$ if Ω_P is a crisp set and $I_{\Omega_P}(x_i) = 0$, $F_{\Omega_P}(x_i) = 0 \; \forall \, x \in E$.

(ii). $E_i(\Omega_P) = 0$ if $\langle T_{\Omega_P}(x_i), I_{\Omega_P}(x_i), F_{\Omega_P}(x_i) \rangle = < T_{\Omega_P}(x_i)$, 0.5, $F_{\Omega_P}(x_i) >$, $\forall \, x \in E$.

(iii). $E_i(\Omega_P) \geq E_i(\Omega_Q)$ if Ω_P is more uncertain than Ω_Q i.e.

$T_{\Omega_P}(x_i) + F_{\Omega_P}(x_i) \leq T_{\Omega_Q}(x_i) + F_{\Omega_Q}(x_i)$ and $\left| I_{\Omega_P}(x_i) - I^c_{\Omega_P}(x_i) \right| \leq \left| I_{\Omega_Q}(x_i) - I^c_{\Omega_Q}(x_i) \right|$.

(iv). $E_i(\Omega_P) = E_i(\Omega^c_P)$, $\forall \, x \in E$.

Consequently, the entropy value Ev_j of the j-th attribute can be calculated as as follows:.

$Ev_j = 1 - \frac{1}{n}\sum_{i=1}^{m}(T_{ij}(x_i) + F_{ij}(x_i))\left|I_{ij}(x_i) - I_{ij}^{C}(x_i)\right|$, i = 1, 2, …, m; j = 1, 2,…, n.

We observe that $0 \le Ev_j \le 1$. Based on Hwang and Yoon [18] and Wang and Zhang [19] the entropy weight of the j-th attribute is defined as follows:

$w_j = \frac{1 - Ev_j}{\sum_{j=1}^{n}(1 - Ev_j)}$ with $0 \le w_j \le 1$ and $\sum_{j-1}^{n} w_j = 1$.

Step 3. Formulation of weighted decision matrix

The weighted decision matrix is obtained by multiplying weights of the attributes (w_j) and the standardized decision matrix $\langle s_{ij}\rangle_{m\times n}$. Therefore, the weighted neutrosophic cubic decision matrix $\langle z_{ij}\rangle_{m\times n}$ is obtained as:

$$\langle z_{ij}\rangle_{m\times n} = w_j \otimes \langle a_{ij}\rangle_{m\times n} = \begin{bmatrix} w_1 s_{11} & w_2 s_{12} & \cdots & w_n s_{1n} \\ w_1 s_{21} & w_2 s_{22} & \cdots & w_n s_{2n} \\ . & . & \cdots & . \\ w_1 s_{m1} & w_2 s_{m2} & \cdots & w_n s_{mn} \end{bmatrix}$$

$$= \begin{bmatrix} z_{11} & z_{12} & \cdots & z_{1n} \\ z_{21} & z_{22} & \cdots & z_{2n} \\ . & . & \cdots & . \\ . & . & \cdots & . \\ z_{m1} & z_{m2} & \cdots & z_{mn} \end{bmatrix}$$

where

$z_{ij} = <([\ddot{\Gamma}^-_{\alpha_{ij}}, \ddot{\Gamma}^+_{\alpha_{ij}}], [\ddot{\Gamma}^-_{\beta_{ij}}, \ddot{\Gamma}^+_{\beta_{ij}}], [\ddot{\Gamma}^-_{\gamma_{ij}}, \ddot{\Gamma}^+_{\gamma_{ij}}]), (\ddot{\alpha}_{ij}, \ddot{\beta}_{ij}, \ddot{\gamma}_{ij}) >$

$= <([1-(1-\dot{\Gamma}^-_{\alpha_{ij}})^{w_j}, 1-(1-\dot{\Gamma}^+_{\alpha_{ij}})^{w_j}],$

$[(\dot{\Gamma}^-_{\beta_{ij}})^{w_j}, (\dot{\Gamma}^+_{\beta_{ij}})^{w_j}], [(\dot{\Gamma}^-_{\gamma_{ij}})^{w_j}, (\dot{\Gamma}^+_{\gamma_{ij}})^{w_j}]), (1-(1-\dot{\alpha}_{ij})^{w_j},$

$(\dot{\beta}_{ij})^{w_j}, (\dot{\gamma}_{ij})^{w_j}) >$

Step 4. Selection of neutrosophic cubic positive ideal solution (NCPIS) and neutrosophic cubic negative ideal solution (NCNIS)

Consider $z^U = (z_1^U, z_2^U, \ldots, z_n^U)$ and $z^L = (z_1^L, z_2^L, \ldots, z_n^L)$ be the NCPIS and NCNIS respectively, then z_j^U is defined as follows:

$z_j^U = < ([(\ddot{\Gamma}^-_{\alpha_j})^U, (\ddot{\Gamma}^+_{\alpha_j})^U], [(\ddot{\Gamma}^-_{\beta_j})^U, (\ddot{\Gamma}^+_{\beta_j})^U], [(\ddot{\Gamma}^-_{\gamma_j})^U, (\ddot{\Gamma}^+_{\gamma_j})^U]), ((\ddot{\alpha}_j)^U, (\ddot{\beta}_j)^U, (\ddot{\gamma}_j)^U) >$

where

$(\ddot{\Gamma}^-_{\alpha_j})^U = \{(\max_i \{\ddot{\Gamma}^-_{\alpha_{ij}}\} | j \in J_1), (\min_i \{\ddot{\Gamma}^-_{\alpha_{ij}}\} | j \in J_2)\}$,

$(\ddot{\Gamma}^+_{\alpha_j})^U = \{(\max_i \{\ddot{\Gamma}^+_{\alpha_{ij}}\} | j \in J_1), (\min_i \{\ddot{\Gamma}^+_{\alpha_{ij}}\} | j \in J_2)\}$,

$(\ddot{\Gamma}^-_{\beta_j})^U = \{(\min_i \{\ddot{\Gamma}^-_{\beta_{ij}}\} | j \in J_1), (\max_i \{\ddot{\Gamma}^-_{\beta_{ij}}\} | j \in J_2)\}$,

$(\ddot{\Gamma}^+_{\beta_j})^U = \{(\min_i \{\ddot{\Gamma}^+_{\beta_{ij}}\} | j \in J_1), (\max_i \{\ddot{\Gamma}^+_{\beta_{ij}}\} | j \in J_2)\}$,

$(\ddot{\Gamma}^-_{\gamma_j})^U = \{(\min_i \{\ddot{\Gamma}^-_{\gamma_{ij}}\} | j \in J_1), (\max_i \{\ddot{\Gamma}^-_{\gamma_{ij}}\} | j \in J_2)\}$,

$(\ddot{\Gamma}^+_{\gamma_j})^U = \{(\min_i \{\ddot{\Gamma}^+_{\gamma_{ij}}\} | j \in J_1), (\max_i \{\ddot{\Gamma}^+_{\gamma_{ij}}\} | j \in J_2)\}$,

$(\ddot{\alpha}_j)^U = \{(\max_i \{\ddot{\alpha}_{ij}\} | j \in J_1), (\min_i \{\ddot{\alpha}_{ij}\} | j \in J_2)\}$,

$(\ddot{\beta}_j)^U = \{(\min_i \{\ddot{\beta}_{ij}\} | j \in J_1), (\max_i \{\ddot{\beta}_{ij}\} | j \in J_2)\}$,

$(\ddot{\gamma}_j)^U = \{(\min_i \{\ddot{\gamma}_{ij}\} | j \in J_1), (\max_i \{\ddot{\gamma}_{ij}\} | j \in J_2)\}$;

and z_j^L is defined as given below

$z_j^L = < [(\ddot{\Gamma}^-_{\alpha_{ij}})^L, (\ddot{\Gamma}^+_{\alpha_{ij}})^L], [(\ddot{\Gamma}^-_{\beta_{ij}})^L, (\ddot{\Gamma}^+_{\beta_{ij}})^L], [(\ddot{\Gamma}^-_{\gamma_{ij}})^L, (\ddot{\Gamma}^+_{\gamma_{ij}})^L], ((\ddot{\alpha}_{ij})^L, (\ddot{\beta}_{ij})^L, (\ddot{\gamma}_{ij})^L) >$

where $(\ddot{\Gamma}^-_{\alpha_j})^L = \{(\min_i \{\ddot{\Gamma}^-_{\alpha_{ij}}\} | j \in J_1), (\max_i \{\ddot{\Gamma}^-_{\alpha_{ij}}\} | j \in J_2)\}$, $(\ddot{\Gamma}^+_{\alpha_j})^L = \{(\min_i \{\ddot{\Gamma}^+_{\alpha_{ij}}\} | j \in J_1), (\max_i \{\ddot{\Gamma}^+_{\alpha_{ij}}\} | j \in J_2)\}$, $(\ddot{\Gamma}^-_{\beta_j})^L = \{(\max_i \{\ddot{\Gamma}^-_{\beta_{ij}}\} | j \in J_1), (\min_i \{\ddot{\Gamma}^-_{\beta_{ij}}\} | j \in J_2)\}$, $(\ddot{\Gamma}^+_{\beta_j})^L = \{(\max_i \{\ddot{\Gamma}^+_{\beta_{ij}}\} | j \in J_1), (\min_i \{\ddot{\Gamma}^+_{\beta_{ij}}\} | j \in J_2)\}$, $(\ddot{\Gamma}^-_{\gamma_j})^L = \{(\max_i \{\ddot{\Gamma}^-_{\gamma_{ij}}\} | j \in J_1), (\min_i \{\ddot{\Gamma}^-_{\gamma_{ij}}\} | j \in J_2)\}$, $(\ddot{\Gamma}^+_{\gamma_j})^L = \{(\max_i \{\ddot{\Gamma}^+_{\gamma_{ij}}\} | j \in J_1), (\min_i \{\ddot{\Gamma}^+_{\gamma_{ij}}\} | j \in J_2)\}$, $(\ddot{\alpha}_j)^L = \{(\min_i \{\ddot{\alpha}_{ij}\} | j \in J_1), (\max_i \{\ddot{\alpha}_{ij}\} | j \in J_2)\}$, $(\ddot{\beta}_j)^L = \{(\max_i \{\ddot{\beta}_{ij}\} | j \in J_1), (\min_i \{\ddot{\beta}_{ij}\} | j \in J_2)\}$, $(\ddot{\gamma}_j)^L = \{(\max_i \{\ddot{\gamma}_{ij}\} | j \in J_1), (\min_i \{\ddot{\gamma}_{ij}\} | j \in J_2)\}$.

Step 5. Calculate the distance measure of alternatives from NCPIS and NCNIS

The Euclidean distance measure of each alternative B_i, i = 1, 2, …, m from NCPIS can be defined as follows:

$$D^+_{E_i} = \sqrt{\frac{1}{9n}\sum_{j=1}^{n}\begin{pmatrix} (\ddot{\Gamma}^-_{\alpha_{ij}} - (\ddot{\Gamma}^-_{\alpha_j})^U)^2 + (\ddot{\Gamma}^+_{\alpha_{ij}} - (\ddot{\Gamma}^+_{\alpha_j})^U)^2 + (\ddot{\Gamma}^-_{\beta_{ij}} - (\ddot{\Gamma}^-_{\beta_j})^U)^2 + \\ (\ddot{\Gamma}^+_{\beta_{ij}} - (\ddot{\Gamma}^+_{\beta_j})^U)^2 + (\ddot{\Gamma}^-_{\gamma_{ij}} - (\ddot{\Gamma}^-_{\gamma_j})^U)^2 + (\ddot{\Gamma}^+_{\gamma_{ij}} - (\ddot{\Gamma}^+_{\gamma_j})^U)^2 + \\ (\ddot{\alpha}_{ij} - (\ddot{\alpha}_j)^U)^2 + (\ddot{\beta}_{ij} - (\ddot{\beta}_j)^U)^2 + (\ddot{\gamma}_{ij} - (\ddot{\gamma}_j)^U)^2 \end{pmatrix}}$$

Similarly, the Euclidean distance measure of each alternative B_i, i = 1, 2, …, m from NCNIS can be written as follows:

$$D^{-}_{E_i} = \sqrt{\frac{1}{9n}\sum_{j=1}^{n}\left(\begin{array}{l}(\ddot{\Gamma}^{-}_{\alpha_{ij}}\text{-}(\ddot{\Gamma}^{-}_{\alpha_j})^{L})^2+(\ddot{\Gamma}^{+}_{\alpha_{ij}}\text{-}(\ddot{\Gamma}^{+}_{\alpha_j})^{L})^2+(\ddot{\Gamma}^{-}_{\beta_{ij}}\text{-}(\ddot{\Gamma}^{-}_{\beta_j})^{L})^2+\\(\ddot{\Gamma}^{+}_{\beta_{ij}}\text{-}(\ddot{\Gamma}^{+}_{\beta_j})^{L})^2+(\ddot{\Gamma}^{-}_{\gamma_{ij}}\text{-}(\ddot{\Gamma}^{-}_{\gamma_j})^{L})^2+(\ddot{\Gamma}^{+}_{\gamma_{ij}}\text{-}(\ddot{\Gamma}^{+}_{\gamma_j})^{L})^2+\\(\ddot{\alpha}_{ij}\text{-}(\ddot{\alpha}_j)^{L})^2+(\ddot{\beta}_{ij}\text{-}(\ddot{\beta}_j)^{L})^2+(\ddot{\gamma}_{ij}\text{-}(\ddot{\gamma}_j)^{L})^2\end{array}\right)}.$$

Step 6. Evaluate the relative closeness co-efficient to the neutrosophic cubic ideal solution

The relative closeness co-efficient RCC^{*}_{i} of each B_i, i = 1, 2, …, m with respect to NCPIS z^{U}_{j} , j = 1, 2, …, n is defined as follows:

$$RCC^{*}_{i} = \frac{D^{-}_{E_i}}{D^{+}_{E_i} + D^{-}_{E_i}}, \text{ i = 1, 2, …, } m.$$

Step 7. Rank the alternatives

We obtain the ranking order of the alternatives based on the RCC^{*}_{i}. The bigger value of RCC^{*}_{i} reflects the better alternative.

4. Numerical example

In this section, we consider an example of neutrosophic cubic MADM, adapted from Mondal and Pramanik [20] to demonstrate the applicability and the effectiveness of the proposed extended TOPSIS method.

Consider a legal guardian desires to select an appropriate school for his/ her child for basic education [20]. Suppose there are three possible alternatives for his/ her child:

(1) B_1, a Christian missionary school
(2) B_2, a Basic English medium school
(3) B_3, a Bengali medium kindergarten.

He/ She must take a decision based on the following four attributes:

(1) C_1 is the distance and transport,
(2) C_2 is the cost,
(3) C_3 is the staff and curriculum, and
(4) C_4 is the administrative and other facilities

Here C_1 and C_2 are cost type attributes; while C_3 and C_4 are benefit type attributes. Suppose the weights of the four attributes are unknown. Using the the following steps, we solve the problem.

Step 1. The rating of the alternative B_i, i = 1, 2, 3 with respect to the alternative C_j, j = 1, 2, 3, 4 is represented by neutrosophic cubic assessments. The decision matrix $\langle a_{ij} \rangle_{3\times4}$ is shown in Table 1.

Table 1. Neutrosophic cubic decision matrix

	C_1	C_2
B_1	$<([0.3,0.4],[0.1,0.2],[0.2,0.35]), (0.3,0.4,0.25)>$	$<([0.6,0.7],[0.05,0.1],[0.2,0.3]), (0.5,0.1,0.25)>$
B_2	$<([0.8,0.9],[0.1,0.2],[0.15,0.3]), (0.7,0.15,0.3)>$	$<([0.3,0.5],[0.1,0.4],[0.3,0.5]), (0.4,0.3,0.2)>$
B_3	$<([0.6,0.7],[0.2,0.4],[0.25,0.4]), (0.5,0.3,0.3)>$	$<([0.2,0.35],[0.1,0.25],[0.2,0.3]), (0.3,0.3,0.4)>$

	C_3	C_4
B_1	$<([0.5,0.6],[0.2,0.4],[0.1,0.3]), (0.5,0.3,0.4)>$	$<([0.4,0.6],[0.1,0.25],[0.1,0.3]), (0.5,0.2,0.4)>$
B_2	$<([0.4,0.5],[0.2,0.35],[0.05,0.2]), (0.35,0.1,0.1)>$	$<([0.2,0.3],[0.2,0.35],[0.1,0.25]), (0.4,0.1,0.1)>$
B_3	$<([0.4,0.7],[0.1,0.3],[0.15,0.25]), (0.5,0.2,0.2)>$	$<([0.5,0.7],[0.1,0.2],[0.2,0.25]), (0.3,0.1,0.2)>$

Step 2. Standardize the decision matrix.

The standardized decision matrix $\langle s_{ij} \rangle_{3\times4}$ is constructed as follows (see Table 2):

Table 2. The standardized neutrosophic cubic decision matrix

	C_1	C_2
B_1	$<([0.2,0.35],[0.8,0.9,[0.3,0.4]), (0.25,0.6,0.3)>$	$<([0.2,0.3],[0.9,0.95],[0.6,0.7]), (0.25,0.9,0.5)>$
B_2	$<([0.15,0.3],[0.8,0.9],[0.8,0.9]), (0.3,0.85,0.7)>$	$<([0.3,0.5],[0.6,0.9],[0.3,0.5]), (0.2,0.7,0.4)>$
B_3	$<([0.25,0.4],[0.6,0.8],[0.6,0.7]), (0.3,0.7,0.5)>$	$<([0.2,0.3],[0.75,0.9],[0.2,0.35]), (0.4,0.7,0.3)>$

	C_3	C_4
B_1	$<([0.5,0.6],[0.2,0.4],[0.1,0.3]), (0.5,0.3,0.4)>$	$<([0.4,0.6],[0.1,0.25],[0.1,0.3]), (0.5,0.2,0.4)>$
B_2	$<([0.4,0.5],[0.2,0.35],[0.05,0.2]), (0.35,0.1,0.1)>$	$<([0.2,0.3],[0.2,0.35],[0.1,0.25]), (0.4,0.1,0.1)>$
B_3	$<([0.4,0.7],[0.1,0.3],[0.15,0.25]), (0.5,0.2,0.2)>$	$<([0.5,0.7],[0.1,0.2],[0.2,0.25]), (0.3,0.1,0.2)>$

Step 3. Using AAO, we transform NCVs into SNVs. We calculate entropy value E_j of the j-th attribute as follows:

$$Ev_1 = 0.644,\ Ev_2 = 0.655,\ Ev_3 = 0.734,\ Ev_4 = 0.663.$$

The weight vector of the four attributes are obtained as:

$$w_1 = 0.2730,\ w_2 = 0.2646,\ w_3 = 0.2040,\ w_4 = 0.2584.$$

Step 4. After identifying the weight of the attribute (w_j), we multiply the standardized decision matrix with w_j, j = 1, 2, …, n to obtain the weighted decision matrix $\langle z_{ij} \rangle_{3\times4}$ (see Table 3).

Table 3. The weighted neutrosophic cubic decision matrix

	C_1	C_2
B_1	$<([0.059, 0.110],[0.941, 0.972, [0.720, 0.779]), (0.075, 0.87, 0.72)>$	$<([0.057, 0.090, [0.972, 0.986],[0.874, 0.91]), (0.073, 0.972, 0.832)>$
B_2	$<([0.043, 0.093],[0.941, 0.972],[0.941, 0.972]), (0.093, 0.957, 0.907)>$	$<([0.09, 0.168],[0.874, 0.972],[0.727, 0.832]), (0.057, 0.910, 0.785)>$
B_3	$<([0.076, 0.13],[0.87, 0.941],[0.87, 0.907]), (0.093, 0.907, 0.828)>$	$<([0.057, 0.090],[0.928, 0.972],[0.653, 0.757]), (0.126, 0.910, 0.727)>$

	C_3	C_4
B_1	$<([0.132, 0.17],[0.720, 0.830],[0.625, 0.782]), (0.084, 0.625, 0.625)>$	$<([0.124, 0.211],[0.552, 0.699],[0.552, 0.733]), (0.164, 0.660, 0.789)>$
B_2	$<([0.100, 0.132],[0.720, 0.807],[0.543, 0.720]), (0.084, 0.625, 0.625)>$	$<([0.056, 0.088],[0.66, 0.762],[0.552, 0.699]), (0.124, 0.552, 0.552)>$
B_3	$<([0.100, 0.218],[0.625, 0.782],[0.679, 0.754]), (0.132, 0.720, 0.720)>$	$<([0.164, 0.267],[0.552, 0.660],[0.660, 0.699]), (0.088, 0.522, 0.660)>$

Step 5. From Table 3, the NCPIS z_j^U, j = 1, 2, 3, 4 is obtained as follows:

z_1^U = < ([0.043, 0.093], [0.941, 0.972], [0.941, 0.972]), (0.075, 0.957, 0.907) >,

z_2^U = < ([0.057, 0.09], [0.972, 0.986], [0.874, 0.91]), (0.057, 0.972, 0.832) >,

z_3^U = < ([0.132, 0.218], [0.625, 0.782], [0.543, 0.72], (0.132, 0.625, 0.625)>,

z_4^U = < [0.164, 0.267], [0.552, 0.66], [0.552, 0.699], (0.66, 0.552, 0.552)>;

The NCNIS z_j^L, j = 1, 2, 3, 4 is determined from the weighted decision matrix (see Table 3) as follows:

z_1^L = < [0.076, 0.13], [0.87, 0.941], [0.72, 0.779], (0.093, 0.87, 0.72)>,

z_2^L = < [0.09, 0.168], [0.874, 0.972], [0.653, 0.757], (0.126, 0.91, 0.727)>,

z_3^L = < [0.1, 0.132], [0.72, 0.83], [0.679, 0.782], (0.084, 0.782, 0.83)>,

z_4^L = < [0.056, 0.088], [0.66, 0.762], [0.66, 0.733], (0.088, 0.66, 0.789)>.

Step 6. The Euclidean distance measure of each alternative from NCPIS is obtained as follows:

$$D_{E_1}^{+} = 0.1232,\ D_{E_2}^{+} = 0.1110,\ D_{E_3}^{+} = 0.1200.$$

Similarly, the Euclidean distance measure of each alternative from NCNIS is computed as follows:

$$D_{E_1}^{-} = 0.0705,\ D_{E_2}^{-} = 0.0954,\ D_{E_3}^{-} = 0.0736.$$

Step 7. The relative closeness co-efficient RCC_i^*, i = 1, 2, 3 is obtained as follows:

$$RCC_1^* = 0.3640,\ RCC_2^* = 0.4622,\ RCC_3^* = 0.3802.$$

Step 8. The ranking order of the feasible alternative according to the relative closeness co-efficient of the neutrosophic cubic ideal solution is presented as follows:

$$B_2 > B_3 > B_1$$

Therefore, B_2 i.e. a Basic English medium school is the best option for the legal guardian.

5 Conclusions

In the paper, we have presented a new extended TOPSIS method for solving MADM problems with neutrosophic cubic information. We have proposed several operational rules on neutrosophic cubic sets. We have defined Euclidean distance between two neutrosophic cubic sets. We have defined arithmetic average operator for neutrosophic cubic numbers. We have employed information entropy scheme to calculate unknown weights of the attributes. We have also defined neutrosophic cubic positive ideal solution and neutrosophic cubic negative ideal solution in the decision making process. Then, the most desirable alternative is selected based on the proposed extended TOPSIS method under neutrosophic cubic environment. Finally, we have solved a numerical example of MADM problem regarding school selection for a legal guardian to illustarte the proposed TOPSIS method. We hope that the proposed TOPSIS method will be effective in dealing with different MADM problems such as medical diagnosis, pattern recognition, weaver selection, supplier selection, etc in neutrosophic cubic set environment.

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An Isolated Bipolar Single-Valued Neutrosophic Graphs

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Abstract. In this research paper, we propose the graph of the bipolar single-valued neutrosophic set (BSVNS) model. This graph generalized the graphs of single-valued neutrosophic set models. Several results have been proved on complete and isolated graphs for the BSVNS model. Moreover, an essential and satisfactory condition for the graphs of the BSVNS model to become an isolated graph of the BSVNS model has been demonstrated.

Keywords: BSVNGs · Complete BSVNG · Isolated BSVNGs

1 Introduction

Smarandache [1] proposed the concept of neutrosophic sets (in short NSs) as a means of expressing the inconsistencies and indeterminacies that exist in most real-life problems. The proposed concept generalized fuzzy sets and intuitionistic fuzzy sets theory [2, 3]. The notion of NS is described with three functions: truth, an indeterminacy and a falsity, where the functions are totally independent; the three functions are inside the unit interval]$^{-}$0, 1^{+}[. To practice NSs in real-life situations efficiently, a new version of NSs. A new version of NSs named single-valued neutrosophic sets (in short SVNSs) was defined by Smarandache in [1]. Subsequently, Wang et al. [4] defined the various operations and operators for the SVNS model. In [5], Deli et al. introduced the notion of bipolar neutrosophic sets, which combine the bipolar fuzzy sets and SVNS models. Neutrosophic sets and their extensions have been paid great attention recent years [6]. The theory of graphs is the mostly used tool for resolving combinatorial problems in various fields such as computer science, algebra and topology. Smarandache [1, 7] introduced two classes of neutrosophic graphs to deal with situations in which there exist inconsistencies and indeterminacies among the vertices which cannot be dealt with by fuzzy graphs and different hybrid structures [8–10]. The first class is relied on literal indeterminacy (I) component, and the second class of neutrosophic graphs is based on numerical truth values (T, I, F). Subsequently, Broumi et al. [11–13] introduced the concept single-valued neutrosophic graphs (in short SVNGs) and discussed some interesting results. Later on, the same authors [14–17] proposed the concept of bipolar single-valued neutrosophic graphs (BSVNGs) and established some interesting results with proofs and illustrations.

The objective of our article is to demonstrate the essential and satisfactory condition of BSVNGs to be an isolated BSVNG.

2 Background of Research

Some of the important background knowledge for the materials that are presented in this paper is presented in this section. These results can be found in [1, 4, 5, 12, 13].

Definition 2.1 [1]. Let ζ be a universal set. The neutrosophic set A on the universal set ζ is categorized into three membership functions, namely the true $T_A(\mathrm{x})$, indeterminate $I_A(\mathrm{x})$ and false $F_A(\mathrm{x})$ contained in real standard or non-standard subset of $]^-0, 1^+[$, respectively.

$$^-0 \leq \sup \mathrm{T_A(x)} + \sup \mathrm{I_A(x)} + \sup \mathrm{F_A(x)} \leq 3^+ \tag{1}$$

Definition 2.2 [4]. Let ζ be a universal set. The single-valued neutrosophic sets (SVNSs) A on the universal ζ is denoted as following

$$\mathrm{A} = \{<x : \mathrm{T_A(x), I_A(x), F_A(x)} > \mathrm{x} \in \zeta\} \tag{2}$$

The functions $\mathrm{T_A(x)} \in [0, 1]$, $\mathrm{I_A(x)} \in [0, 1]$ and $\mathrm{F_A(x)} \in [0, 1]$ are called "degree of truth, indeterminacy and falsity membership of x in A", which satisfy *the* following condition:

$$0 \leq T_A(\mathrm{x}) + I_A(\mathrm{x}) + F_A(\mathrm{x}) \leq 3 \tag{3}$$

Definition 2.3 [12]. A SVNG of $G^* = (\mathrm{V, E})$ is a graph G = (A, B) where

a. The following memberships: $T_A : \mathrm{V} \to [0, 1]$, $\mathrm{I_A} : \mathrm{V} \to [0, 1]$ and $\mathrm{F_A} : \mathrm{V} \to [0, 1]$ represent the truth, indeterminate and false membership degrees of $\mathrm{x} \in \mathrm{V}$ respectively and

$$0 \leq T_A(w) + I_A(w) + F_A(w) \leq 3 \quad \forall \mathrm{w} \in \mathrm{V} \tag{4}$$

b. The following memberships: $T_B : \mathrm{E} \to [0, 1]$, $I_B : \mathrm{E} \to [0, 1]$ and $\mathrm{F_B} : \mathrm{E} \to [0, 1]$ are defined by

$$\mathrm{T_B}(v, w) \leq \min[\mathrm{T_A}(v), \mathrm{T_A}(w)] \tag{5}$$

$$I_B(v, w) \geq \max[\mathrm{I_A}(v), \mathrm{I_A}(w)] \text{ and} \tag{6}$$

$$F_B(v,\ w) \geq \max[\mathrm{F_A}(v), \mathrm{F_A}(w)] \tag{7}$$

Represent the true, indeterminate and false membership degrees of the arc (*v, w*) ∈ (V x V), where (Fig. 1)

$$0 \leq \mathrm{T_B}(v,\ w) + \mathrm{I_B}(v, w) + \mathrm{F_B}(v, w) \leq 3 \quad \forall (v, w) \in \mathrm{E} \tag{8}$$

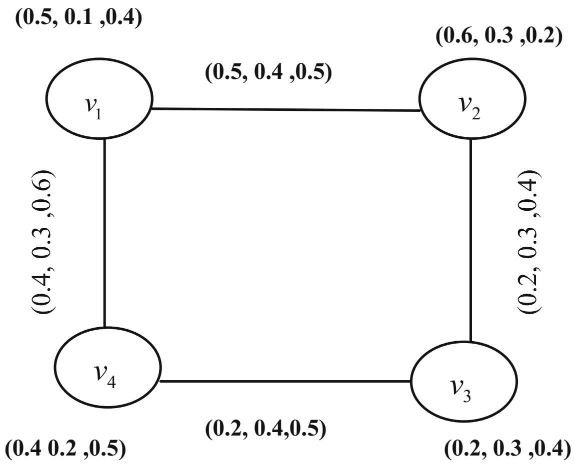

Fig. 1. SVN graph

Definition 2.4 [12]. A SVNG G = (A, B) is named complete SVNG if

$$\mathrm{T_B}(v, w) = min[\mathrm{T_A}(\mathrm{v}), \mathrm{T_A}(w)] \tag{9}$$

$$\mathrm{I_B}(v,\ w) = max[\mathrm{I_A}(\mathrm{v}), \mathrm{I_A}(w)] \tag{10}$$

$$\mathrm{F_B}(v,\ w) = max[\mathrm{F_A}(\mathrm{v}), \mathrm{F_A}(w)] \quad \forall\, v, w \in \mathrm{V} \tag{11}$$

Definition 2.5 [12]. Given a SVNG G = (A, B). Hence, the complement of SVNG on G^* is a SVNG $\bar{G}$ on G^* where

$$a.\bar{A} = \mathrm{A} \tag{12}$$

$$b.\bar{T}_A(w) = T_A(w), \bar{I}_A(w) = I_A(w), \bar{F}_A(w) = F_A(w) \quad \forall\, w \in \ \mathrm{V} \tag{13}$$

$$c.\bar{\mathrm{T}}_\mathrm{B}(v, w) = \min[\mathrm{T_A}(v), \mathrm{T_A}(w)] - \mathrm{T_B}(v, w) \tag{14}$$

$$\bar{\mathrm{I}}_{\mathrm{B}}(v,w) = \max[I_A(v), I_A(w)] - I_B(v,w) \text{ and} \tag{15}$$

$$\bar{\mathrm{F}}_{\mathrm{B}}(v,\ w) = \max[F_A(v), F_A(w)] - F_B(v,w), \forall (v,w) \in \mathrm{E} \tag{16}$$

Definition 2.6 [14]. A BSVNG of $G^* = (\mathrm{V},\mathrm{E})$ is a partner G = (A, B) where $\mathrm{A} = (\mathrm{T}_{\mathrm{A}}^{\mathrm{P}}, \mathrm{I}_{\mathrm{A}}^{\mathrm{P}}, \mathrm{F}_{\mathrm{A}}^{\mathrm{P}}, \mathrm{T}_{\mathrm{A}}^{\mathrm{N}}, \mathrm{I}_{\mathrm{A}}^{\mathrm{N}}, \mathrm{F}_{\mathrm{A}}^{\mathrm{N}})$ is a BSVNS in V and $\mathrm{B} = (\mathrm{T}_{\mathrm{B}}^{\mathrm{P}}, \mathrm{I}_{\mathrm{B}}^{\mathrm{P}}, \mathrm{F}_{\mathrm{B}}^{\mathrm{P}}, \mathrm{T}_{\mathrm{B}}^{\mathrm{N}}, \mathrm{I}_{\mathrm{B}}^{\mathrm{N}}, \mathrm{F}_{\mathrm{B}}^{\mathrm{N}})$ is a BSVNS in $\tilde{\mathrm{V}}^2$ such that (Fig. 2)

$$\mathrm{T}_{\mathrm{B}}^{\mathrm{P}}(v,w) \le min(\mathrm{T}_{\mathrm{A}}^{\mathrm{P}}(v), \mathrm{T}_{\mathrm{A}}^{\mathrm{P}}(w)), \quad \mathrm{T}_{\mathrm{B}}^{\mathrm{N}}(v,w) \ge max(\mathrm{T}_{\mathrm{A}}^{\mathrm{N}}(v), \mathrm{T}_{\mathrm{A}}^{\mathrm{N}}(w)) \tag{17}$$

$$\mathrm{I}_{\mathrm{B}}^{\mathrm{P}}(v,w) \ge max(\mathrm{I}_{\mathrm{A}}^{\mathrm{P}}(v), \mathrm{I}_{\mathrm{A}}^{\mathrm{P}}(w)) \quad \mathrm{I}_{\mathrm{B}}^{\mathrm{N}}(v,\ w) \le \min(\mathrm{I}_{\mathrm{A}}^{\mathrm{N}}(v), \mathrm{I}_{\mathrm{A}}^{\mathrm{N}}(w)) \tag{18}$$

$$\mathrm{F}_{\mathrm{B}}^{\mathrm{P}}(v,w) \ge max(\mathrm{F}_{\mathrm{A}}^{\mathrm{P}}(v), \mathrm{F}_{\mathrm{A}}^{\mathrm{P}}(w)), \quad \mathrm{F}_{\mathrm{B}}^{\mathrm{N}}(v,w) \le \min(\mathrm{F}_{\mathrm{A}}^{\mathrm{N}}(v), \mathrm{F}_{\mathrm{A}}^{\mathrm{N}}(w)) \quad \forall\, \mathrm{v,w} \in \tilde{\mathrm{V}}^2 \tag{19}$$

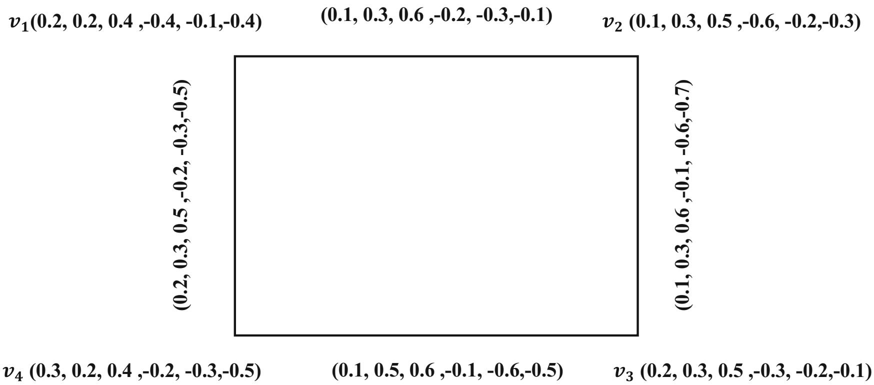

Fig. 2. BSVNG

Definition 2.7 [14]. The complement of BSVNG **G** = $(A,\ B)$ of $G^* = (\mathrm{A},\mathrm{B})$ is a BSVNG $\bar{G} = (\bar{A}, \bar{B})$ of $\bar{G}^* = (\mathrm{V}, \mathrm{V\,x\,V})$ where $\bar{A} = \mathrm{A} = (\mathrm{T}_{\mathrm{A}}^{\mathrm{P}}, \mathrm{I}_{\mathrm{A}}^{\mathrm{P}}, \mathrm{F}_{\mathrm{A}}^{\mathrm{P}}, \mathrm{T}_{\mathrm{A}}^{\mathrm{N}}, \mathrm{I}_{\mathrm{A}}^{\mathrm{N}}, \mathrm{F}_{\mathrm{A}}^{\mathrm{N}})$ and $\bar{\mathrm{B}} = (\bar{\mathrm{T}}_{\mathrm{B}}^{\mathrm{P}}, \bar{\mathrm{I}}_{\mathrm{B}}^{\mathrm{P}}, \bar{\mathrm{F}}_{\mathrm{B}}^{\mathrm{P}}, \bar{\mathrm{T}}_{\mathrm{B}}^{\mathrm{N}}, \bar{\mathrm{I}}_{\mathrm{B}}^{\mathrm{N}}, \bar{\mathrm{F}}_{\mathrm{B}}^{\mathrm{N}})$ is defined as

$$\bar{\mathrm{T}}_{\mathrm{B}}^{\mathrm{P}}(v,\ w) = min(\mathrm{T}_{\mathrm{A}}^{\mathrm{P}}(v), \mathrm{T}_{\mathrm{A}}^{\mathrm{P}}(w)) - \mathrm{T}_{\mathrm{B}}^{\mathrm{P}}(v,w) \tag{20}$$

$$\bar{\mathrm{I}}_{\mathrm{B}}^{\mathrm{P}}(v,\ w) = max(\mathrm{I}_{\mathrm{A}}^{\mathrm{P}}(v), \mathrm{I}_{\mathrm{A}}^{\mathrm{P}}(w)) - \mathrm{I}_{\mathrm{B}}^{\mathrm{P}}(v,w) \tag{21}$$

$$\bar{\mathrm{F}}_{\mathrm{B}}^{\mathrm{P}}(v,w) = max(\mathrm{F}_{\mathrm{A}}^{\mathrm{P}}(v), \mathrm{F}_{\mathrm{A}}^{\mathrm{P}}(w)) - \mathrm{F}_{\mathrm{B}}^{\mathrm{P}}(v,w) \tag{22}$$

$$\bar{T}_B^N(v,w) = max(T_A^N(v), T_A^N(w)) - T_B^N(v,w) \tag{23}$$

$$\bar{I}_B^N(v,w) = min(I_A^N(v), I_A^N(w)) - I_B^N(v,w) \tag{24}$$

$$\bar{F}_B^N(v,w) min(F_A^N(v),(w)) - F_B^N(v,w) \quad \forall v,w \in V, vw \in \tilde{V}^2 \tag{25}$$

Definition 2.8 [14]. A BSVNG $G = (A, B)$ is said to be complete BSVNG if

$$T_B^P(v,\ w) = \min(T_A^P(v), T_A^P(w)), \quad T_B^N(v,w) = max(T_A^N(v), T_A^N(w)), \tag{26}$$

$$I_B^P(v,w) = max(I_A^P(v), I_A^P(w)), \quad I_B^N(v,\ w) = min(I_A^N(v), I_A^N(w)) \tag{27}$$

$$F_B^P(v,w) = max(F_A^P(v), F_A^P(w)), \quad F_B^N(v,w) = min(F_A^N(v), F_A^N(w)) \quad \forall v,w \in V \tag{28}$$

Theorem 2.9 [13]: Let G = (A,B) be a SVNG, then the SVNG is called an isolated SVNG if and only if the complement of G is a complete SVNG.

3 Main Results

Theorem 3.1: A BSVNG = (A,B) is an isolated BSVNG iff the complement of BSVNG is a complete BSVNG.
Proof: Given $G = (A, B)$ be a complete BSVNG.

$$\begin{aligned}
&\text{So } T_B^P(v,w) = min(T_A^P(v), T_A^P(w)), \quad T_B^n(v,\ w) = max(T_A^n(v), T_A^n(w)),\\
&I_B^P(v,w) = max(I_A^P(v), I_A^P(w)), \quad I_B^n(v,w) = min(I_A^n(v), I_A^n(w)),\\
&F_B^P(v,\ w) = max(F_A^P(v), F_A^P(w)), \quad F_B^n(v,\ w) = min(F_A^n(v), F_A^n(w)),\\
&\forall v,w \in V.
\end{aligned}$$

Hence in $\bar{G}$,

$$\begin{aligned}
\bar{T}_B^P(v,\ w) &= min(T_A^P(v), T_A^P(w)) - T_B^P(v,w)\\
&= min(T_A^P(v), T_A^P(w)) - min(T_A^P(v), T_A^P(w))\\
&= 0
\end{aligned}$$

and

$$\begin{aligned}
\bar{I}_B^P(v,w) &= max(I_A^P(v), I_A^p(w)) - I_B^P(v,w)\\
&= max(I_A^P(v), I_A^P(w)) - max(I_A^P(v), I_A^P(w))\\
&= 0
\end{aligned}$$

In addition

$$\begin{aligned}\bar{F}_B^P(v,w) &= max(F_A^P(v), F_A^p(w)) - F_B^P(v,w)\\ &= max(F_A^P(v), F_A^P(w)) - max(F_A^P(v), F_A^P(w))\\ &= 0\end{aligned}$$

We have for the negative membership edges

$$\begin{aligned}\bar{T}_B^N(v,w) &= max(\mathrm{T}_A^N(v), \mathrm{T}_A^N(w)) - T_B^N(v,w)\\ &= max(T_A^N(v), T_A^N(w)) - max(\mathrm{T}_A^N(v), \mathrm{T}_A^N(w))\\ &= 0\end{aligned}$$

and

$$\begin{aligned}\bar{I}_B^N(v,w) &= min(I_A^N(v), I_A^N(w)) - I_B^N(v,w)\\ &= min(I_A^N(v), I_A^N(w)) - min(I_A^N(\mathrm{v}), I_A^N(\mathrm{w}))\\ &= 0\end{aligned}$$

In addition

$$\bar{F}_B^N(v,w) = min(F_A^N(v), F_A^N(w)) - F_B^N(v,w)$$
$$= min(F_A^N(v), F_A^N(w)) - min(F_A^N(v), F_A^N(w))$$
$$= 0$$

So $(\bar{T}_B^P(v,w), \bar{\mathrm{I}}_\mathrm{B}^\mathrm{p}(v,\ w), \bar{F}_B^P(v,w), \bar{T}_B^N(v,w), \bar{I}_B^N(v,w), \bar{F}_B^N(v,\ w)) = (0,0,0,0,0,0)$

Hence $G = (A, B)$ is an isolated BSVNGs

Proposition 3.2: The notion of isolated BSVNGs generalized the notion of isolated fuzzy graphs.

Proof: If the value of $\mathrm{I}_\mathrm{A}^\mathrm{P}(w) = \mathrm{F}_\mathrm{A}^\mathrm{P}(w) = \mathrm{T}_\mathrm{A}^\mathrm{n}(w) = \mathrm{I}_\mathrm{A}^\mathrm{n}(w) = \mathrm{F}_\mathrm{A}^\mathrm{n}(w) = 0$, then the notion of isolated BSVNGs is reduced to isolated fuzzy graphs.

Proposition 3.3: The notion of isolated BSVNGs generalized the notion of isolated SVNGs.

Proof: If the value of $\mathrm{T}_\mathrm{A}^\mathrm{n}(w) = \mathrm{I}_\mathrm{A}^\mathrm{n}(w) = \mathrm{F}_\mathrm{A}^\mathrm{n}(w) = 0\mathrm{I}_\mathrm{A}^\mathrm{n}(w)$, then the concept of isolated BSVNGs is reduced to isolated SVNGs.

4 Conclusion

In this article, we have extended the notion of isolated SVNGs to the notion of isolated BSVNGs. The notion of isolated BSVNGs generalized the isolated SVNGs.

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Minimum Spanning Tree in Trapezoidal Fuzzy Neutrosophic Environment

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Abstract. In this paper, an algorithm for searching the minimum spanning tree (MST) in a network having trapezoidal fuzzy neutrosophic edge weight is presented. The network is an undirected neutrosophic weighted connected graph (UNWCG). The proposed algorithm is based on matrix approach to design the MST of UNWCG. A numerical example is provided to check the validity of the proposed algorithm. Next, a comparison example is made with Mullai's algorithm in neutrosophic graphs.

Keywords: Neutrosophic sets · Trapezoidal fuzzy neutrosophic sets
Score function · Neutrosophic graph · Minimum spanning tree

1 Introduction

In 1998, Smarandache [1] proposed the concept of neutrosophic set (NS) from the philosophical point of view, to represent uncertain, imprecise, incomplete, inconsistent, and indeterminate information that are exist in the real world. The concept of neutrosophic set generalizes the concept of the classic set, fuzzy set, and intuitionistic fuzzy set (IFS). The major differences between the IFS and neutrosophic set (NS) are the structure of the membership functions, the dependence of the membership functions, and the constraints in the values of the membership functions. A NS has a triple-membership structure which consists of three components, namely the truth, falsity and indeterminacy membership functions, as opposed to the IFS in which information is described by a membership and non-membership function only. Another major difference is the constraint between these membership functions. In a NS, the three membership functions are independent of one another and the only constraint is that the sum of these membership functions must not exceed three. This is different from the IFS where the values of the membership and non-membership functions are dependent on one another, and the sum of these must not exceed one. To apply the concept of neutrosophic sets (NS) in science and engineering applications, Smarandache [1] initiated the concept of single-valued neutrosophic set (SVNS). In a subsequent paper, Wang et al. [2], studied some properties related to SVNSs. We refer the readers to [3, 11, 13–15] for more information related to the extensions of NSs and the advances that have been made in the application of NSs and its extensions in various fields. The minimum spanning tree problem is one of well–known problems in combinatorial optimization. When the edge weights assigned to a graph are crisp numbers, the minimum spanning tree problem can be solved by some well-known algorithms such as Prim and Kruskal algorithm. By combining single valued neutrosophic sets theory [1, 2] with graph theory, references [6–9] introduced single valued neutrosophic graph theory (SVNGT for short). The SVNGT is generation of graph theory. In the literature some scholars have studied the minimum spanning tree problem in neutrosophic environment. In [4], Ye introduced a method for finding the minimum spanning tree of a single valued neutrosophic graph where the vertices are represented in the form of SVNS. Mandal and Basu [5] proposed an approach based on similarity measure for searching the optimum spanning tree problems in a neutrosophic environment considering the inconsistency, incompleteness and indeterminacy of the information. In their work, they applied the proposed approach to a network problem with multiple criteria. In another study, Mullai et al. [10] discussed about the minimum spanning tree problem in bipolar neutrosophic environment.

The main purpose of this paper is to propose a neutrosophic version of Kruskal algorithm based on the matrix approach for searching the cost minimum spanning tree in a network having trapezoidal fuzzy neutrosophic edge weight [12].

The rest of the paper is organized as follows. Section 2 briefly introduces the concepts of neutrosophic sets, single valued neutrosophic sets and the score function of trapezoidal neutrosophic number. Section 3 proposes a novel approach for searching the minimum spanning tree in a network having trapezoidal fuzzy neutrosophic edge length. In Sect. 4, a numerical example is presented to illustrate the proposed method. In Sect. 5, a comparative example with other method is provided. Finally, Sect. 6 presents the main conclusions.

2 Preliminaries and Definitions

In this section, the concept of neutrosophic sets single valued neutrosophic sets and trapezoidal fuzzy neutrosophic sets are presented to deal with indeterminate data, which can be defined as follows.

Definition 2.1 [1]. Let ξ be an universal set. The neutrosophic set A on the universal set ξ categorized in to three membership functions called the true T_A(x), indeterminate I_A(x) and false F_A(x) contained in real standard or non-standard subset of $]^-0, 1^+[$ respectively.

$$^-0 \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+ \quad (1)$$

Definition 2.2 [2]. Let ξ be a universal set. The single valued neutrosophic sets (SVNs) A on the universal ξ is denoted as following

$$\mathrm{A} = \left\{< x{:}\, \mathrm{T_A(x), I_A(x), F_A(x)} > \mathrm{x} \in \xi\right\} \tag{2}$$

The functions $\mathrm{T_A(x)} \in [0.1]$, $\mathrm{I_A(x)} \in [0.1]$ and $\mathrm{F_A(x)} \in [0.1]$ are named degree of truth, indeterminacy and falsity membership of x in A, satisfy the following condition:

$$0 \leq T_A(\mathrm{x}) + I_A(\mathrm{x}) + F_A(\mathrm{x}) \leq 3 \tag{3}$$

Definition 2.3 [12]. Let ζ be a universal set and ψ [0, 1] be the sets of all trapezoidal fuzzy numbers on [0, 1]. The trapezoidal fuzzy neutrosophic sets (In short TrFNSs) $\breve{A}$ on the universal is denoted as following:

$$\breve{A} = \left\{< \mathrm{x}{:}\, \breve{T}_A(x), \breve{I}_A(x), \breve{F}_A(x) >, x \in \zeta\right\} \tag{4}$$

Where $\breve{T}_A(\mathrm{x}){:}\, \zeta \rightarrow \psi[0, 1]$, $\breve{I}_A(\mathrm{x}){:}\, \zeta \rightarrow \psi[0, 1]$ and $\breve{F}_A(\mathrm{x}){:}\, \zeta \rightarrow \psi[0, 1]$. The trapezoidal fuzzy numbers

$$\breve{T}_A(\mathrm{x}) = \left(\mathrm{T}_A^1(x), \mathrm{T}_A^2(x), \mathrm{T}_A^3(x), \mathrm{T}_A^4(x)\right) \tag{5}$$

$$\breve{I}_A(x) = \left(I_A^1(\mathrm{x}), I_A^2(\mathrm{x}), I_A^3(\mathrm{x}), I_A^4(\mathrm{x})\right) \tag{6}$$

and

$\breve{F}_A(x) = \left(F_A^1(\mathrm{x}), F_A^2(\mathrm{x}), F_A^3(\mathrm{x}), F_A^4(\mathrm{x})\right)$, respectively denotes degree of truth, indeterminacy and falsity membership of x in $\breve{A}\, \forall x \in \zeta$.

$$0 \leq \mathrm{T}_A^4(x) + I_A^4(x) + F_A^4(x) \leq 3 \tag{7}$$

Definition 2.4. [12]. Let $\breve{A}_1$ be a TrFNV denoted as $\breve{A}_1 = \langle(t_1, t_2, t_3, t_4), (i_1, i_2, i_3, i_4), (f_1, f_2, f_3, f_4)\rangle$ Hence, the score function and the accuracy function of TrFNV are denoted as below:

(i) $$s(\breve{A}_1) = \frac{1}{12}\left[8 + (t_1 + t_2 + t_3 + t_4) - (i_1 + i_2 + i_3 + i_4) - (f_1 + f_2 + f_3 + f_4)\right] \tag{8}$$

(ii) $$H(\breve{A}_1) = \frac{1}{4}\left[(t_1 + t_2 + t_3 + t_4) - (f_1 + f_2 + f_3 + f_4)\right] \tag{9}$$

In order to make a comparisons between two TrFNV, Ye [12], presented the order relations between two TrFNVs.

Definition 2.5 [12]. Let $\breve{A}_1$ and $\breve{A}_2$ be two TrFNV defined on the set of real numbers. Hence, the ranking method is defined as follows:

i. If $s(\breve{A}_1) \succ s(\breve{A}_2)$, then $\breve{A}_1$ is greater than $\breve{A}_2$, that is, $\breve{A}_1$ is superior to $\breve{A}_2$, denoted by $\breve{A}_1 \succ \breve{A}_2$

If $s(\breve{A}_1) = s(\breve{A}_2)$, and $H(\breve{A}_1) \succ H(\breve{A}_2)$ then $\breve{A}_1$ is greater than $\breve{A}_2$, that is, $\breve{A}_1$ is superior to $\breve{A}_2$, denoted by $\breve{A}_1 \succ \breve{A}_2$.

3 Minimum Spannig Tree Algorithm of TrFN- Undirected Graph

In this section, a neutrosophic version of Kruskal's algorithm is proposed to handle Minimum spanning tree in a neutrosophic environment and a trapezoidal fuzzy neutrosophic minimum spanning tree algorithm, whose steps are described below:

Algorithm:

Input: The weight matrix M = $\left[W_{ij}\right]_{n\times n}$ for which is constructed for undirected weighted neutrosophic graph (UWNG).

Step 1: Input trapezoidal fuzzy neutrosophic adjacency matrix A.

Step 2: Construct the TrFN-matrix into a score matrix $\left[S_{ij}\right]_{n\times n}$ by using the score function (8).

Step 3: Repeat step 4 and step 5 up to time that all nonzero elements are marked or in another saying all (n−1) entries matrix of S are either marked or set to zero.

Step 4: There are two ways to find out the weight matrix M that one is columns-wise and the other is row-wise in order to determine the unmarked minimum entries S_{ij}, besides it determines the weight of the corresponding edge e_{ij} in M.

Step 5: Set $S_{ij} = 0$ else mark S_{ij} provided that corresponding edge e_{ij} of selected S_{ij} generate a cycle with the preceding marked entries of the score matrix S.

Step 6: Construct the graph T including the only marked entries from the score matrix S which shall be the desired minimum cost spanning tree of G.

Step 7: Stop.

4 Numerical Example

In this section, a numerical example of TrFNMST is used to demonstrate of the proposed algorithm. Consider the following graph G = (V, E) shown Fig. 1, with fives nodes and fives edges. The various steps involved in the construction of the minimum cost spanning tree are described as follow:

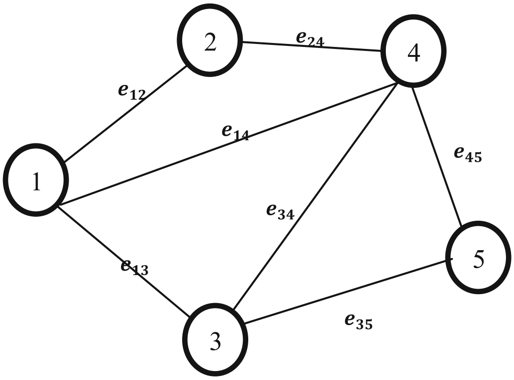

Fig. 1. A neutrosophic graph with TrFN edge weights

The TrFN- adjacency matrix A is written as follows:

$$= \begin{bmatrix} 0 & \boldsymbol{e_{12}} & \boldsymbol{e_{13}} & \boldsymbol{e_{14}} & 0 \\ \boldsymbol{e_{12}} & 0 & 0 & \boldsymbol{e_{24}} & 0 \\ \boldsymbol{e_{13}} & 0 & 0 & \boldsymbol{e_{34}} & \boldsymbol{e_{35}} \\ \boldsymbol{e_{14}} & \boldsymbol{e_{24}} & \boldsymbol{e_{34}} & 0 & \boldsymbol{e_{45}} \\ 0 & 0 & \boldsymbol{e_{35}} & \boldsymbol{e_{45}} & 0 \end{bmatrix}$$

Thus, using the score function, we get the score matrix:

$$S = \begin{bmatrix} 0 & 0.575 & 0.592 & 0.583 & 0 \\ 0.575 & 0 & 0 & 0.542 & 0 \\ 0.592 & 0 & 0 & 0.458 & 0.6 \\ 0.583 & 0.542 & 0.458 & 0 & 0.525 \\ 0 & 0 & 0.6 & 0.525 & 0 \end{bmatrix}$$

Fig. 2. Score matrix

We observe that the minimum record 0.458 according to Fig. 2 is selected and the corresponding edge (3, 4) is marked with red color. Repeat the procedure until the iteration will exist (Table 1).

Table 1. The values of edge weights

e_{ij}	Edge weights
$\boldsymbol{e_{12}}$	< (0.2, 0.3, 0.5, 0.5), (0.1, 0.4, 0.4, 0.6), (0.1, 0.2, 0.3, 0.5) >
$\boldsymbol{e_{13}}$	< (0.3, 0.4, 0.6, 0.7), (0.1, 0.3, 0.5, 0.6), (0.2, 0.3, 0.3, 0.6) >
$\boldsymbol{e_{14}}$	< (0.4, 0.5, 0.7, 0.7), (0.1, 0.4, 0.4, 0.5), (0.3, 0.4, 0.5, 0.7) >
$\boldsymbol{e_{24}}$	< (0.4, 0.5, 0.6, 0.7), (0.3, 0.4, 0.6, 0.7), (0.2, 0.4, 0.5, 0.6) >
$\boldsymbol{e_{34}}$	< (0.1, 0.3, 0.5, 0.6), (0.4, 0.5, 0.6, 0.7), (0.3, 0.4, 0.4, 0.7) >
$\boldsymbol{e_{35}}$	< (0.4, 0.4, 0.5, 0.6), (0.1, 0.3, 0.3, 0.6), (0.1, 0.3, 0.4, 0.6) >
$\boldsymbol{e_{45}}$	< (0.3, 0.5, 0.6, 0.7), (0.1, 0.3, 0.4, 0.7), (0.3, 0.4, 0.8, 0.8) >

According to the Figs. 3 and 4, the next non zero minimum entries 0.525 is marked and corresponding edges (4, 5) are also colored.

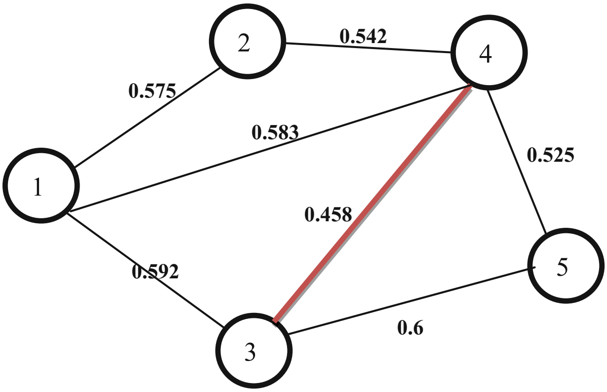

Fig. 3. An illustration of the marked edge

$$S = \begin{bmatrix} 0 & 0.575 & 0.592 & 0.583 & 0 \\ 0.575 & 0 & 0 & 0.542 & 0 \\ 0.592 & 0 & 0 & 0.458 & 0.6 \\ 0.583 & 0.542 & 0.458 & 0 & 0.525 \\ 0 & 0 & 0.6 & 0.525 & 0 \end{bmatrix}$$

Fig. 4. Score matrix

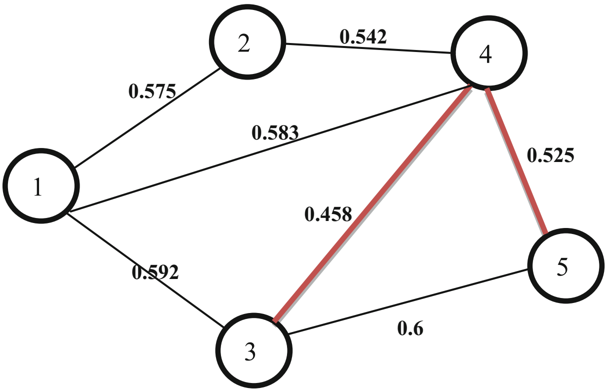

Fig. 5. An illustration of the marked edge (4, 5)

According to the Fig. 6, the next minimum non zero element 0.542 is marked (Figs. 5 and 7).

$$S = \begin{bmatrix} 0 & 0.575 & 0.592 & 0.583 & 0 \\ 0.575 & 0 & 0 & 0.542 & 0 \\ 0.592 & 0 & 0 & 0.458 & 0.6 \\ 0.583 & 0.542 & 0.458 & 0 & 0.525 \\ 0 & 0 & 0.6 & 0.525 & 0 \end{bmatrix}$$

Fig. 6. Score matrix

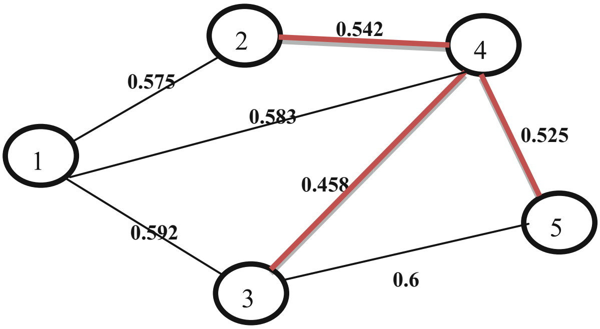

Fig. 7. An illustration of the marked edge (2, 4)

According to the Fig. 8. The next minimum non zero element 0.575 is marked, and corresponding edges (1, 2) are also colored (Fig. 9).

$$S = \begin{bmatrix} 0 & 0.575 & 0.592 & 0.583 & 0 \\ 0.575 & 0 & 0 & 0.542 & 0 \\ 0.592 & 0 & 0 & 0.458 & 0.6 \\ 0.583 & 0.542 & 0.458 & 0 & 0.525 \\ 0 & 0 & 0.6 & 0.525 & 0 \end{bmatrix}$$

Fig. 8. Score matrix

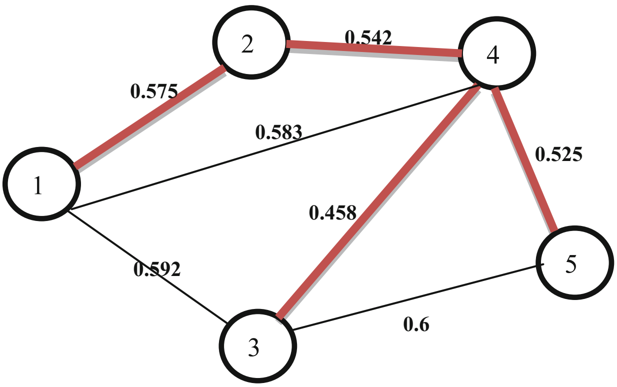

Fig. 9. An illustration of the marked edge (1, 2)

According to the Fig. 10. The next minimum non zero element 0.583 is marked. But while drawing the edges it produces the cycle. So we delete and mark it as 0 instead of 0.583.

$$S = \begin{bmatrix} 0 & 0.575 & 0.592 & 0.583 & 0 \\ 0.575 & 0 & 0 & 0.542 & 0 \\ 0.592 & 0 & 0 & 0.458 & 0.6 \\ \cancel{0.583}\ 0 & 0.542 & 0.458 & 0 & 0.525 \\ 0 & 0 & 0.6 & 0.525 & 0 \end{bmatrix}$$

Fig. 10. Score matrix

The next non zero minimum entries 0.592 is marked it is shown in the Fig. 11. But while drawing the edges it produces the cycle. So we delete and mark it as 0 instead of 0.592.

$$S = \begin{bmatrix} 0 & 0.575 & 0.592 & 0.583 & 0 \\ 0.575 & 0 & 0 & 0.542 & 0 \\ \cancel{0.592}\ 0 & 0 & 0 & 0.458 & 0.6 \\ \cancel{0.583}\ 0 & 0.542 & 0.458 & 0 & 0.525 \\ 0 & 0 & 0.6 & 0.525 & 0 \end{bmatrix}$$

Fig. 11. Score matrix

According to the Fig. 12. The next minimum non zero element 0.6 is marked. But while drawing the edges it produces the cycle so we delete and mark it as 0 instead of 0.6.

$$S = \begin{bmatrix} 0 & 0.575 & 0.592 & 0.583 & 0 \\ 0.575 & 0 & 0 & 0.542 & 0 \\ \cancel{0.592}\ 0 & 0 & 0 & 0.458 & \cancel{0.6}\ 0 \\ \cancel{0.583}\ 0 & 0.542 & 0.458 & 0 & 0.525 \\ 0 & 0 & 0.6 & 0.525 & 0 \end{bmatrix}$$

Fig. 12. Score matrix

After the above steps, the final path of minimum cost of spanning tree of G is portrayed in Fig. 13.

Based on the procedure of matrix approach applied to undirected neutrosophic graph. hence, the crisp minimum cost spanning tree is 2, 1 and the final path of minimum cost of spanning tree is {1, 2}, {2, 4}, {4, 3}, {4, 5}.

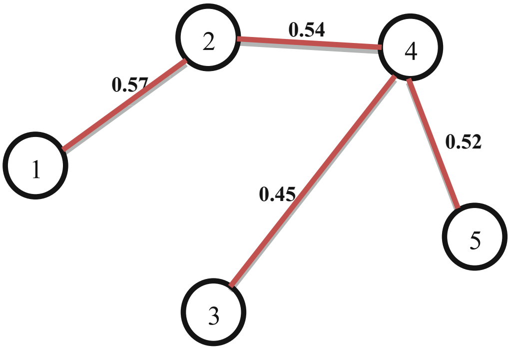

Fig. 13. Final path of minimum cost of spanning tree of G.

5 Comparative Example

To demonstrate the rationality and effectiveness of the proposed method, a comparative example with Mullai's algorithm [10] is provided. Following the step of Mullai's algorithm.

Iteration 1: Let $C_1 = \{1\}$ and $\overline{C_1} = \{2, 3, 4, 5\}$
Iteration 2: Let $C_2 = \{1, 2\}$ and $\overline{C_2} = \{3, 4, 5\}$
Iteration 3: Let $C_3 = \{1, 2, 4\}$ and $\overline{C_3} = \{3, 5\}$
Iteration 4: Let $C_4 = \{1, 2, 4, 3\}$ and $\overline{C_4} = \{5\}$

From the results of the iteration processes, the TrFN minimal spanning tree is:

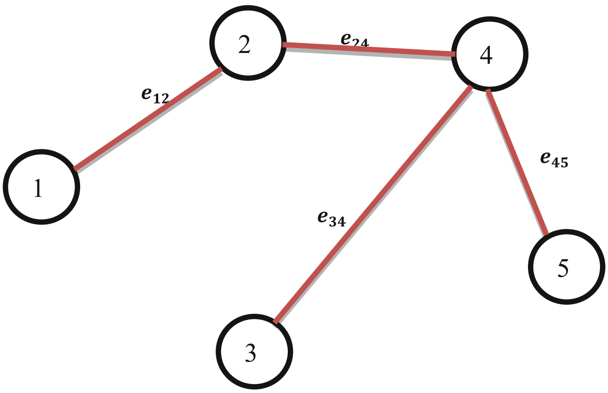

Fig. 14. TrFN minimal spanning tree obtained by Mullai's algorithm.

From the Fig. 14, it can be observed that the TrFN minimal spanning tree {1, 2}, {2, 4}, {4, 3}, {4, 5} obtained by Mullai's algorithm, after deneutrosophication of edges' weight, is the same as the path obtained by the proposed algorithm.

The difference between the proposed algorithm and Mullai's algorithm is that Mullai's algorithm is based on the comparison of edges in each iteration of the algorithm and this leads to high computation whereas the proposed approach based on Matrix approach can be easily implemented in Matlab.

6 Conclusion

In this paper, a new approach for searching the minimum spanning tree in a network having trapezoidal fuzzy neutrosophic edge length is presented. The proposed algorithm use the score function of TrFN number, then a comparative example is worked out to illustrate the applicability of the proposed approach. In the next research paper, we can apply the proposed approach to the case of directed neutrosophic graphs and other kinds of neutrosophic graphs including bipolar neutrosophic graphs, and interval valued neutrosophic graphs.

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A Bipolar Single Valued Neutrosophic Isolated Graphs: Revisited

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ABSTRACT

In this research paper, the graph of the bipolar single-valued neutrosophic set model (BSVNS) is proposed. The graphs of single valued neutrosophic set models is generalized by this graph. For the BSVNS model, several results have been proved on complete and isolated graphs. Adding, an important and suitable condition for the graphs of the BSVNS model to become an isolated graph of the BSVNS model has been demonstrated.

KEYWORDS

Bipolar single valued neutrosophic graphs (BSVNG) , complete-BSVNG, isolated-BSVNGs.

1 Introduction

The concept of 'Neutrosophic logic' was developed by Prof. Dr. F. Smarandache in 1995 and get published in 1998. "It is a branch of philosophy which studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra"[4]. The concepts of fuzzy sets [8] and intuitionistic fuzzy set [6] were generalized by adding an independent indeterminacy-membership. Neutrosophic logic is a powerful tool to deal with incomplete, indeterminate, and inconsistent information, which is the main reason for widespread concerns of researchers. The concept of neutrosophic set(NS for short) is characterized by three independent degrees namely truth-membership degree (T), indeterminacy-membership degree (I), and falsity-membership degree (F).To practice NSs in real life situations efficiently,The subclass of the neutrosophic sets called single-valued neutrosophic set (in short SVNS) was defined by Smarandache in [4]. In another paper [5], Wang et al. defined the various operations and operators for the SVNS model. In [11] Deli et al. proposed a new concept called bipolar neutrosophic sets. This concept appear as a generalization of fuzzy sets, intuitionistic fuzzy sets, bipolar fuzzy sets, bipolar intuitionistic fuzzy sets and single valued neutrosophic set. The benefits of applying the NSs have been addressed in [18].The theory of graphs is the mostly used tool for resolving combinatorial problems in diverse disciplines like computer science, algebra and topology, etc. In [2, 4] Smarandache proposed two kinds of neutrosophic graphs to deal with situations in which there exist inconsistencies and indeterminacies among the vertices which cannot be dealt with by fuzzy graphs and different hybrid structures including bipolar fuzzy graphs, intuitionistic fuzzy graphs, bipolar intuitionsitc fuzzy graphs [1,7,9, 10], The first kind is based on literal indeterminacy (I) component, the second kind of neutrosophic graphs is based on numerical truth-values (T, I, F), Recently, a hybrid study by combining SVNS and classical graph theory was carried out and that concept is called Single valued neutrosophic graph (SVNG) was presented by Broumi et al [12, 13, 14, 17, 20, 22].In addition, the concept of bipolar neutrosophic set was combined with graph theory and new graph model was presented. This concept is called bipolar single valued neutrosophic graph (BSVNGs). In [15,16] Broumi et al. proposed the concept of bipolar single valued neutrosophic graph as a generalized the concept of fuzzy graph, intuitionistic fuzzy graph, bipolar fuzzy graph and single valued neutrosophic graph.

The objective of this article is to demonstrate the essential and satisfactory condition of BSVNGs to be an isolated-BSVNG.

2.Background of research

Some of the important background knowledge in this paper is presented in this section. These results can be found in [4, 5, 12,13,15, 21].

Definition 2.1 [4] Le ζ be a universal set. The neutrosophic set A on the universal set ζ categorized into three membership functions called the true membership function $T_A(x)$, indeterminate membership function $I_A(x)$ and false membership function $F_A(x)$ contained in real standard or non-standard subset of $]^-0, 1^+[$ respectively and satisfy the following condition

$$^-0 \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+ \quad (1)$$

Definition 2.2 [5] Let ζ be a universal set. The single valued neutrosophic sets (SVNs) A on the universal ζ is denoted as following

$$A = \{ < x: T_A(x), I_A(x), F_A(x) > x \in \zeta\} \quad (2)$$

The functions $T_A(x) \in [0. 1]$, $I_A(x) \in [0. 1]$ and $F_A(x) \in [0. 1]$ are called " degree of truth, indeterminacy and falsity membership of x in A", satisfy *the* following condition:

$$0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3 \quad (3)$$

Definition 2.3 [12] A SVNG of $G^* = (V, E)$ is a graph G = (A, B) where

a. The following memberships: $T_A : V \to [0, 1]$, $I_A: V \to [0 ,1]$ and $F_A: V \to [0, 1]$ represent the truth, indeterminate and false membership degrees of x ∈ V respectively and

$$0 \leq T_A(w) + I_A(w) + F_A(w) \leq 3 \quad (4)$$
$\forall$ w ∈ V

b. The following memberships: $T_B: E \to [0 ,1]$, $I_B: E \to [0, 1]$ and $F_B: E \to [0, 1]$are defined by

$$T_B(v, w) \leq \min [T_A(v), T_A(w)] \quad (5)$$

$$I_B(v, w) \geq \max [I_A(v), I_A(w)] \text{ and} \quad (6)$$

$$F_B(v, w) \geq \max [F_A(v), F_A(w)] \quad (7)$$

Represent the true, indeterminate and false membership degrees of the arc (*v*, *w*) ∈ (V x V), where

$$0 \leq T_B(v, w) + I_B(v, w) + F_B(v, w) \leq 3 \quad (8)$$
$\forall (v, w) \in E$

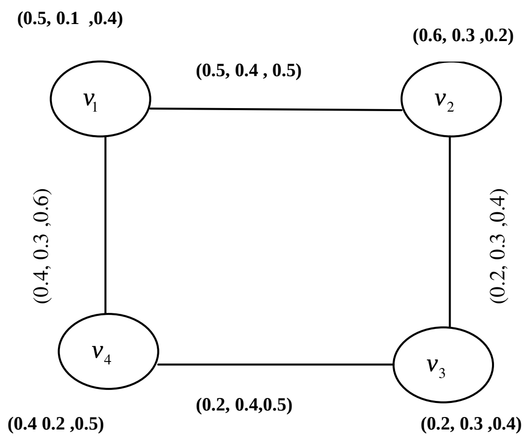

Fig.1.SVN-graph

Definition 2.4 [12]. A SVNG G= (A, B) is named a complete-SVNG if

$$T_B(v, w) = \min [T_A(v), T_A(w)] \quad (9)$$

$$I_B(v, w) = \max [I_A(v), I_A(w)] \quad (10)$$

$$F_B(v, w) = \max [F_A(v), F_A(w)] \quad (11)$$

∀ v, w ∈ V

Definition **2.5[12].** Let G = (A, B) be SVNG. Hence, the complement of SVNG G on G^* is a SVNG $\bar{G}$on G^* where

a. $\bar{A}$ = A (12)

b. $\bar{T}_A(w)=T_A(w)$, $\bar{I}_A(w) = I_A(w)$, $\bar{F}_A(w)=F_A(w)$ ∀ w ∈ V (13)

c. $\bar{T}_B$ $(v, w) = \min[T_A(v), T_A(w)] - T_B(v, w)$ (14)

$\bar{I}_B(v, w) = \max[I_A(v), I_A(w)] - I_B(v, w)$ (15)

$\bar{F}_B(v, w)= \max[F_A(v), F_A(w)] - F_B(v, w)$, ∀ (v, w) ∈ E .

Definition 2.6 [15].A BSVNG G= (A, B)of G^* = (V, E) is a partner such that A= (T_A^P, I_A^P, F_A^P,T_A^N ,I_A^N,F_A^N) is a BSVNS in V and B= (T_B^P,I_B^P, F_B^P,T_B^N ,I_B^N,F_B^N) is a BSVNS in E such that

(i) $T_B^P(v, w) \le min(T_A^P(v), T_A^P(w))$ and $T_B^N(v, w) \ge max(T_A^N(v), T_A^N(w))$ (17)

(ii) $I_B^P(v, w) \ge max(I_A^P(v), I_A^P(w))$ and $I_B^N(v, w) \le \min(I_A^N(v), I_A^N(w))$ (18)

(iii) $F_B^P(v, w) \ge max(F_A^P(v), F_A^P(w))$ and $F_B^N(v, w) \le \min(F_A^N(v), F_A^N(w))$, ∀(v, w) ∈ E (19)

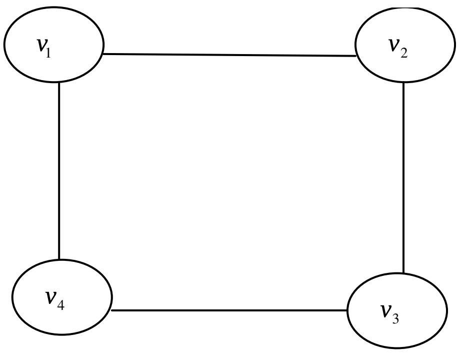

Fig.2 BSVNG

v_i	The values of vertex
v_1	(0.2, 0.2, 0.4 ,-0.4, -0.1,-0.4)
v_2	(0.1, 0.3, 0.5 ,-0.6, -0.2,-0.3)
v_3	(0.2, 0.3, 0.5 ,-0.3, -0.2,-0.1)
v_4	(0.3, 0.2, 0.4 ,-0.2, -0.3,-0.5)

Table1. The values of vertex of BSVNG

	The values of edge
v_{12}	(0.1, 0.3, 0.6 ,-0.2, -0.3,-0.1)
v_{23}	(0.1, 0.3, 0.6 ,-0.1, -0.6,-0.7)
v_{34}	(0.1, 0.5, 0.6 ,-0.1, -0.6,-0.5)
v_{14}	(0.2, 0.3, 0.5 ,-0.2, -0.3,-0.5)

Table2. The values of edge of BSVNG

Definition 2.7 [15]. The complement of BSVNG **G**= (A, B) of G^* =(V, E) is a BSVNG $\bar{G}$ = ($\bar{A}$, $\bar{B}$) of G^* =(V, E) such that

(i) $\bar{A}$ =A=(T_A^P,I_A^P, F_A^P,T_A^N ,I_A^N,F_A^N) and

(ii) $\bar{B}$ = ($T_{\bar{B}}^P$, $I_{\bar{B}}^P$, $F_{\bar{B}}^P$, $T_{\bar{B}}^N$, $I_{\bar{B}}^N$, $F_{\bar{B}}^N$) on E= V× V is defined as

$T_{\bar{B}}^P(v, w) = min\left(T_A^P(v), T_A^P(w)\right) - T_B^P(v, w)$

$T_{\bar{B}}^N(v , w)= max\left(T_A^N(v), T_A^N(w)\right) - T_B^N(v, w)$ (20)

$I_{\bar{B}}^P(v, w) = \max\left(I_A^P(v), I_A^P(w)\right) - I_B^P(v, w)$

$I_{\bar{B}}^N(v, w) = \min\left(I_A^N(v), I_A^N(w)\right) - I_B^N(v, w)$ (21)

$F_{\bar{B}}^P(v, w) = max\left(F_A^P(v), F_A^P(w)\right) - F_B^P(v, w)$

$F_{\bar{B}}^N(v, w)\ min\left(F_A^N(v), (w)\right) - F_B^N(v, w)$, (22)

∀ (v,w)∈ E

Definition 2.8 [15]. A BSVNG G= (A, B) is called a complete-BSVNG if

$T_B^P(v, w) = \min(T_A^P(v), T_A^P(w))$, (23)

$T_B^N(v, w) = max(T_A^N(v), T_A^N(w))$, (24)

$I_B^P(v, w) = max(I_A^P(v), I_A^P(w))$, (25)

$I_B^N(v, w) = min(I_A^N(v), I_A^N(w))$ (26)

$F_B^P(v, w) = max(F_A^P(v), F_A^P(w))$, (27)

$F_B^N(v, w) = min(F_A^N(v), F_A^N(w))$ (28) ∀v, w ∈V

Definition 2.9[7]. The complement of BIFG **G**= (A, B) of G^* =(A, B) is a BIFG $\bar{G}$ = ($\bar{A}$, $\bar{B}$) of $\overline{G^*}$ =(V, V x V) where $\bar{A}$ =A=(T_A^P, F_A^P,T_A^N ,F_A^N) and $\bar{B}$ = ($\bar{T}_B^P$, $\bar{F}_B^P$,$\bar{T}_B^N$,, $\bar{F}_B^N$) are defined as

$\bar{T}_B^P(v, w) = min\left(T_A^P(v), T_A^P(w)\right) - T_B^P(v, w)$ (29)

$\bar{F}_B^P(v, w) = max\left(F_A^P(v), F_A^P(w)\right) - F_B^P(v, w)$ (30)

$\bar{T}_B^N(v , w)= max\left(T_A^N(v), T_A^N(w)\right) - T_B^N(v, w)$ (31)

$\bar{F}_B^N(v, w)\ min\left(F_A^N(v), (w)\right) - F_B^N(v, w)$ ∀v, w∈ V, vw∈ $\tilde{V}^2$ (32)

Theorem 2.10[13]Let G =(A,B) be a SVNG, then the SVNG is called an isolated-SVNG if and only if the complement of G is a complete-SVNG.

Theorem 2.11[21]Let G =(A,B) be a FG, then the FG is called an isolated-FG if and only if the complement of G is a complete- FG

3. MAIN RESULTS

Theorem 3.1: A BSVNG =(A,B) is an isolated-BSVNG iff the complement of BSVNG is a complete- BSVNG.

Proof: Let G =(A, B) be a complete- BSVNG.

Therefore $T_B^P(v, w) = min\,(T_A^P(v), T_A^P(w))$,

$T_B^n(v, w) = max\,(T_A^n(v), T_A^n(w))$,

$I_B^P(v, w) = max\,(I_A^P(v), I_A^P(w))$,

$I_B^n(v, w) = min\,(I_A^n(v), I_A^n(w))$,

$F_B^P(v, w) = max\,(F_A^P(v), F_A^P(w))$,

$F_B^n(v, w) = min\,(F_A^n(v), F_A^n(w)), \forall\, v, w \in V$.

Hence in $\overline{G}$,

$\overline{T}_B^P(v, w) = \min(T_A^P(v), T_A^P(w)) - T_B^P(v, w)$

$= min(T_A^P(v), T_A^P(w)) - min(T_A^P(v), T_A^P(w))$

=0

and

$\overline{I}_B^P(v, w) = max\left(I_A^P(v), I_A^P(w)\right) - I_B^P(v, w)$

$= max(I_A^P(v), I_A^P(w)) - max(I_A^P(v), I_A^P(w))$

= 0

In addition

$\overline{F}_B^P(v, w) = max(F_A^P(v), F_A^P(w)) - F_B^P(v, w)$

$= max(F_A^P(v), F_A^P(w)) - max(F_A^P(v), F_A^P(w))$

= 0

We have for the negative membership edges

$\overline{T}_B^N(v, w) = max(T_A^N(v), T_A^N(w)) - T_B^N(v, w)$

$= max(T_A^N(v), T_A^N(w)) - max(T_A^N(v), T_A^N(w))$

= 0 and

$\overline{I}_B^N(v, w) = min(I_A^N(v), I_A^N(w)) - I_B^N(v, w)$

$= min\,(I_A^N(v), I_A^N(w)) - min\,(I_A^N(v), I_A^N(w))$

= 0

In addition

$\overline{F}_B^N(v, w) = min(F_A^N(v), F_A^N(w)) - F_B^N(v, w)$

$= min(F_A^N(v), F_A^N(w)) - \min\,(F_A^N(v), F_A^N(w))$

= 0

So ($\overline{T}_B^P(v, w)$, $\overline{I}_B^P(v, w)$, $\overline{F}_B^P(v, w)$, $\overline{T}_B^N(v, w)$, $\overline{I}_B^N(v, w)$, $\overline{F}_B^N(v, w)$) = (0, 0, 0, 0, 0, 0)

Hence G =(A, B) is an isolated-BSVNGs

Proposition 3.2: The notion of isolated-BSVNGs generalized the notion of isolated fuzzy graphs.

Proof: If the value of $I_A^P(w) = F_A^P(w) = T_A^n(w) = I_A^n(w) = F_A^n(w) = 0$, then the notion of isolated-BSVNGs is reduced to isolated fuzzy graphs.

Proposition 3.3: The notion of isolated-BSVNGs generalized the notion of isolated-SVNGs.

Proof: If the value of $T_A^n(w) = I_A^n(w) = F_A^n(w) = 0$, then the concept of isolated-BSVNGs is reduced to isolated-SVNGs.

Proposition 3.4: The notion of isolated-BSVNGs generalized the notion of isolated-bipolar intuitionistic fuzzy graph.

Proof: If the value of $I_A^p(w) = I_A^n(w)$, then the concept of isolated-BSVNGs is reduced to isolated-bipolar intutuitionistic fuzzy graphs

IV. COMPARTIVE STUDY

In this section, we present a table showing that the bipolar single valued neutrosophic graph generalized the concept of the crisp graph, fuzzy graph [9], intuitionistic fuzzy graph[1], bipolar fuzzy graph[10], bipolar intuitionistic fuzzy graph[7] and single valued neutrosophic graph[12].
For convenience we denote
F-graph : Fuzzy graphs

IF-graph: Intuitionistic fuzzy graph
BF-graph: Bipolar fuzzy graph
BIF-graph: Bipolar intutionistic fuzzy graph
SVN-graph: Single valued neutrosophic graph
BSVN-graph: Bipolar single valued neutrosophic graph

Type of graphs	The membership values of vertex/ edge					
	$T_A^p(w)$	$I_A^p(w)$	$F_A^p(w)$	$T_A^n(w)$	$I_A^n(w)$	$F_A^n(w)$
crisp graph	1 or 0	0	0	0	0	0
FG	$\in [0,1]$	0	0	0	0	0
IFG	$\in [0,1]$	0	$\in [0,1]$	0	0	0
SVNG	$\in [0,1]$	$\in [0,1]$	$\in [0,1]$	0	0	0
BFG	$\in [0,1]$	0	0	$\in [-1,0]$	0	0
BIFG	$\in [0,1]$	0	$\in [0,1]$	$\in [-1,0]$	0	$\in [-1,0]$
BSVNG	$\in [0,1]$	$\in [0,1]$	$\in [0,1]$	$\in [-1,0]$	$\in [-1,0]$	$\in [-1,0]$

Table3. Different types of graphs

Neutrosophic graph is the generalization of crisp graph, fuzzy graph, intuitionistic fuzzy graph, bi-polar fuzzy graph, bi-polar intuitionistic fuzzy graph and single-valued neutrosophic graph. In this table, we can see that by removing the indeterminacy and non-membership values from neutrosophic graph, the neutrosophic graph reduces to fuzzy graph. By removing the indeterminacy value from neutrosophic graph, the neutrosophic graph reduces to intuitionistic fuzzy graph. Similarly, by removing the positive and negative indeterminacy and non-membership values from bi-polar neutrosophic graph, the bi-polar neutrosophic graph reduces to bi-polar fuzzy graph. By removing the positive and negative indeterminacy values from bi-polar neutrosophic graph, the bi-polar neutrosophic graph reduces to bi-polar intuitionistic fuzzy graph. By the similar way, we can reduce a bi-polar single valued neutrosophic graph to a neutrosophic graph by removing the negative membership, indeterminacy and non-membership values.

5. Conclusion

In this article, we have proved necessary and sufficient condition under which BSVNGs is an isolated-BSVNGs. The notion of isolated-BSVNGs generalized the isolated-fuzzy graph and isolated- SVNGs. In addition, in future research, we shall concentrate on extending the idea of this paper by using the interval valued bipolar neutrosophic graph as a generalized form of bipolar neutrosophic graph.

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Strong Degrees in Single Valued Neutrosophic Graphs

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***Abstract*—The concept of single valued neutrosophic graphs (SVNGs) generalizes the concept of fuzzy graphs and intuitionistic fuzzy graphs. The purpose of this research paper is to define different types of strong degrees in SVNGs and introduce novel concepts, such as the vertex truth-membership, vertex indeterminacy-membership and falsity-membership sequence in SVNG with proof and numerical illustrations.**

Keywords—Single valued neutrosophic graph (SVNG); neutrosophic set; sequence; strong degree

I. Introduction

In [1], [3] Smarandache explored the notion of neutrosophic sets (NS in short) as a powerful tool which extends the concepts of crisp set, fuzzy sets and intuitionistic fuzzy sets [2]-[6]. This concept deals with uncertain, incomplete and indeterminate information that exist in real world. The concept of NS sets associate to each element of the set a degree of membership $T_A(x)$,a degree of indeterminacy $I_A(x)$ and a degree of falsity $F_A(x)$, in which each membership degree is a real standard or non-standard subset of the nonstandard unit $]^-0, 1^+[$. Smaranadache [1], [2] and Wang [7] defined the concept of single valued neutrosophic sets (SVNS), an instance of NS, to deal with real application. In [8], the readers can found a rich literature on SVNS.

In more recent times, combining the concepts of NSs, interval valued neutrosophic sets (IVNSs) and bipolar neutrosophic sets with graph theory, *Broumi et al.* introduced various types of neutrosophic graphs including single valued neutrosophic graphs (SVNGs for short) [9], [11], [14], interval valued neutrosophic graphs [13], [18], [20], bipolar neutrosophic graphs [10], [12], all these graphs are studied deeply. Later on, the same authors presented some papers for solving the shortest path problem on a network having single valued neutrosophic edges length [17], interval valued neutrosophic edge length [32], bipolar neutrosophic edge length [21], trapezoidal neutrosophic numbers [15], SV-trapezoidal neutrosophic numbers [16], triangular fuzzy neutrosophic [19].Our approach of neutrosophic graphs are different from that of Akram et al. [26]-[28] since while Akram considers, for the neutrosophic environment (<=, <=, >=) we do (<=, >=, >=) which is better, since while T is a positive quality, I, F are considered negative qualities. Akram et al. include "I" as a positive quality together with "T". So our papers improve Akram et al.'s papers. After that, several authors are focused on the study of SVNGs and many extensions of SVNGs have been developed. Hamidi and Borumand Saeid [25] defined the notion of accessible-SVNGs and apply it social networks. In [24], Mehra and Manjeet defined the notion of single valued neutrosophic signed graphs. Hassan et al. [30] proposed some kinds of bipolar neutrosophic graphs. Naz et al. [23] studied some basic operations on SVNGs and introduced vertex degree of these operations for SVNGs and provided an application of single valued neutrosophic digraph (SVNDG) in travel time. Ashraf et al. [22] defined new classes of SVNGs and studied some of its important properties. They solved a multi-attribute decision making problem using a SVNDG. Mullai [31] solved the spanning tree problem in bipolar neutrosophic environment and gave a numerical example.

Motivated by the Karunambigai work's [29].The concept of strong degree of intuitionistic fuzzy graphs is extended to strong degree of SVNGs

This paper has been organized in five sections. In Section 2, we firstly review some basic concepts related to neutrosophic set, single valued neutrosophic sets and SVNGs. In Section 3, different strong degree of SVNGs are proposed and studied with proof and example. In Section 4, the concepts of vertex truth-membership, vertex indeterminacy-membership and vertex falsity- membership is discussed. Lastly, Section 5 concludes the paper.

II. PRELIMINAREIS AND DEFINTIONS

In the following, we briefly describe some basic concepts related to neutrosophic sets, single valued neutrosophic sets and SVNGs.

Definition 2.1 [1] Given the universal set . A neutrosophic set A on is characterized by a truth membership function ,an indeterminacy membership function and falsity membership function F_A, where T ,I_A,F_A:$\zeta \rightarrow]^-0,1^+[$. For all x∈ ζ, x=(x,T A is neutrosophic element of A.

The neutrosophic set can be written in the following form:

A = {<x:$T_A(x)$, $_A(x), F_A(x)$>, x∈ ζ} (1)

with the condition

(2)

Definition 2.2 [7] Given the universal set ζ. A single valued neutrosophic set A on ζ is characterized by a truth membership function , an indeterminacy membership function and falsity membership function , where T_A , I_A , F_a : $\zeta \rightarrow [0,1]$. For all x , x=(x, , A is a single valued neutrosophic element of A.

The single valued neutrosophic set can be written in the following form:

A = {<x:$T_A(x)$,I ( )>, x∈ ζ} (3)

with the condition

0 ≤ ≤3 (4)

Definition 2.3 [14] ASVN-graph G is of the form G=(A,B) where A

1. A={$v_1, v_2, \ldots v_n$ }Such that the functions T_A: A→ [0 ,1], $: A \rightarrow [0,\ 1\ ]$, F_A:A→[0, 1]denote the truth-membership function, an indeterminacy-membership function and falsity-membership function of the element v_i ∈ A respectively and

0≤)+i +f ≤3∀

i=1,2,…,n.

2.B={(v_i,v_j); (v_i,v_j) A} and the function :B→[0,1],

:B →[0,1] , F_B:B→ [0, 1] are defined by

≤ min (T)) (5)

≥ max(I)) (6)

≥max(F)) (7)

WhereT_B, I_B, denotes the truth-membership function, indeterminacy membership function and falsity membership function of the edge (v)∈ B respectively where

0≤)+ +F 3 (8)

∈ B, i, j∈{1,2,…,n}

A is called the vertex set of G and B is the edge set of G.

The following Fig. 1 represented a graphical representation of single valued neutrosophic graph.

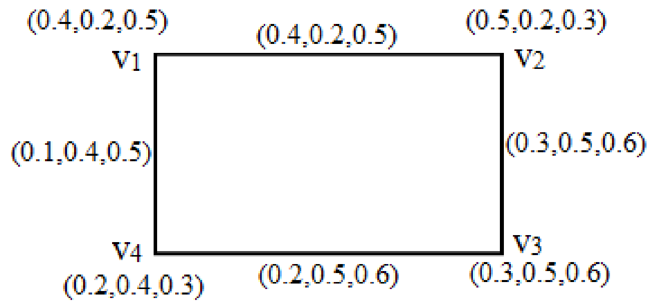

Fig. 1. Single valued neutrosophic graph.

III. STRONG DEGREE IN SINGLE VALUEDNEUTROSOPHIC GRAPH

The following section introduces new concepts and proves their properties.

Definition 3.1 Given the SVN-graph G= (V, E). The T-strong degree of a vertex $v_i \in V$ is defined as $d_{s(T)}(v_i) = \sum_{e_{ij}\in E} T_{ij}$, e_{ij} are strong edges incident at v_i .

Definition 3.2 Given the SVN-graph G=(V, E).The I-strong degree of a vertex $v_i \in V$ is defined as $d_{s(I)}(v_i) = \sum_{e_{ij}\in E} I_{ij}$, e_{ij} are strong edges incident at v_i .

Definition 3.3 Given the SVN-graph G=(V, E).The F-strong degree of a vertex $v_i \in V$ is defined as $d_{s(F)}(v_i) = \sum_{e_{ij}\in E} F_{ij}$, e_{ij} are strong edges incident at v_i .

Definition3.4 Let G = (V,E) be SVNG. The strong degree of a vertex $v_i \in V$ is as follow

$$d_s(v_i) = \left[\sum_{e_{ij}\in E} T_{ij}, \sum_{e_{ij}\in E} I_{ij}, \sum_{e_{ij}\in E} F_{ij}\right]$$, where e_{ij} are strong edge incident at v_i .

Definition 3.5 Let G=(V, E) be a SVNG. The minimum strong degree of G is defined as

$\delta_s(\mathrm{G}) = (\delta_{s(T)}(\mathrm{G}), \delta_{s(I)}(\mathrm{G}), \delta_{s(F)}(\mathrm{G})$, where

$\delta_{s(\mathrm{T})}(\mathrm{G}) = \wedge\{d_{s(\mathrm{T})}(\mathrm{v}_i) / \mathrm{v}_i \in V\}$ is the minimum T-strong degree of G.

$\delta_{s(I)}(G)=\wedge\left\{d_{s(\mathrm{I})}(\mathrm{v}_i)/\mathrm{v}_i\in V\right\}$ is the minimum I-strong degree of G.

$\delta_{s(F)}(G)=\wedge\left\{d_{s(F)}(\mathrm{v}_i)/\mathrm{v}_i\in V\right\}$ is the minimum F-Strong degree of G.

Definition 3.6 Given the SVN-graph G=(V, E). The maximum strong degree of G is defined as

$\Delta_s(G)=(\Delta_{s(\mathrm{T})}(\mathrm{G}),\Delta_{s(\mathrm{I})}(\mathrm{G}),\Delta_{s(\mathrm{F})}(\mathrm{G}))$, where

$\Delta_{s(T)}(G)=\vee\left\{d_{s(\mathrm{T})}(\mathrm{v}_i)/\mathrm{v}_i\in V\right\}$ is the maximum T-strong degree of G.

$\Delta_{s(I)}(G)=\vee\left\{d_{s(I)}(\mathrm{v}_i)/\mathrm{v}_i\in V\right\}$ is the maximum I-strong degree of G.

$\Delta_{s(\mathrm{F})}(\mathrm{G})=\vee\left\{d_{s(\mathrm{F})}(\mathrm{v}_i)/\mathrm{v}_i\in V\right\}$ is the maximum F-Strong degree of G.

Definition 3.7 Let G be a SVNG, the T-total strong degree of a vertex $v_i\in V$ in G is defined as $td_{s(\mathrm{T})}(v_i)=d_{s(\mathrm{T})}(v_i)+T_i$,

Definition 3.8 Let G be a SVNG, the I-total strong degree of a vertex in G is defined as $v_i\in V$ $td_{s(\mathrm{I})}(v_i)=d_{s(\mathrm{I})}(v_i)+I_i$,

Definition 3.9 Let G be a SVNG, the F-total strong degree of a vertex $v_i\in V$ in G is defined $td_{s(F)}(v_i)=d_{s(F)}(v_i)+F_i$,

Definition 3.10 Let G be a SVNG, the total strong degree of a vertex $v_i\in V$ in G is defined as

$$td_s(v_i)=\left[td_{s(T)}(v_i),td_{s(I)}(v_i),td_{s(F)}(v_i)\right]$$

Definition 3.11 Given the SVN-graph G=(V, E). The minimum total strong degree of G is defined as

$\delta_{ts}(G)=(\delta_{ts(T)}(G),\delta_{ts(I)}(G),\delta_{ts(F)}(G))$, where

$\delta_{ts(T)}(G)=\wedge\left\{d_{ts(T)}(\mathrm{v}_i)/\mathrm{v}_i\in V\right\}$ is the minimum T-total strong degree of G.

$\delta_{ts(I)}(G)=\wedge\left\{d_{ts(\mathrm{I})}(\mathrm{v}_i)/\mathrm{v}_i\in V\right\}$ is the minimum I-total strong degree of G.

$\delta_{ts(F)}(G)=\wedge\left\{d_{ts(F)}(\mathrm{v}_i)/\mathrm{v}_i\in V\right\}$ is the minimum F-total strong degree of G.

Definition 3.12 Given the SVN-graph G = (V, E). The maximum total strong degree of G is defined as:

$\Delta_{ts}(G)=(\Delta_{ts(T)}(G),\Delta_{ts(I)}(\mathrm{G}),\Delta_{ts(F)}(G))$, where

$\Delta_{ts(T)}(\mathrm{G})=\vee\left\{d_{ts(\mathrm{T})}(\mathrm{v}_i)/\mathrm{v}_i\in V\right\}$ is the maximum T-total strong degree of G.

$\Delta_{ts(I)}(G)=\vee\left\{d_{ts(I)}(\mathrm{v}_i)/\mathrm{v}_i\in V\right\}$ is the maximum I-total strong degree of G.

$\Delta_{ts(\mathrm{F})}(\mathrm{G})=\vee\left\{d_{ts(\mathrm{F})}(\mathrm{v}_i)/\mathrm{v}_i\in V\right\}$ is the maximum F-total strong degree of G.

Definition 3.13 Given the SVN-graph G=(V,E). The T-strong size of a SVNG is defined as

$S_{s(T)}(G)=\sum_{v_i\neq v_j}T_{ij}$ where T_{ij} is the membership of strong edge $e_{ij}\in E$.

Definition 3.14 Given the SVN-graph G=(V, E). The I-strong size of a SVNG is defined as

$S_{s(I)}(G)=\sum_{v_i\neq v_j}I_{ij}$ where I_{ij} is the indeterminacy-membership of strong edge $e_{ij}\in E$.

Definition 3.15 Given the SVN-graph G=(V, E). The F-strong size of a SVNG is defined as

$S_{s(F)}(G)=\sum_{v_i\neq v_j}F_{ij}$ where F_{ij} is the non-membership of strong edge $e_{ij}\in E$.

Definition 3.16 Given the SVN-graph G=(V, E). The strong size of a SVNG is defined as

$$S_s(G)=\left[S_{s(\mathrm{T})}(\mathrm{G}),S_{s(\mathrm{I})}(\mathrm{G}),S_{s(F)}(\mathrm{G})\right]$$

Definition 3.17 Given the SVN-graph G=(V,E). The T-strong order of a SVNG is defined as

$O_{s(T)}(G)=\sum_{v_i\in V}T_i$ where v_i is the strong vertex in G.

Definition 3.18 Given the SVN-graph G=(V, E). The I-strong order of a SVNG is defined as

$O_{s(\mathrm{I})}(G)=\sum_{v_i\in V}I_i$ where v_i is the strong vertex in G.

Definition 3.19 Given the SVN-graph G=(V, E).The F-strong order of a SVNG is defined as

$O_{s(\mathrm{F})}(G)=\sum_{v_i\in V}F_i$ where v_i is the strong vertex in G.

Definition 3.20 Given the SVN-graph G=(V, E). The strong order of a SVNG is defined as

$$O_s(G)=\left[O_{s(T)}(G),O_{s(I)}(G),O_{s(F)}(G)\right]$$

Definition 3.21 Let G be a SVNG. If $d_{s(T)}(v_i) = k_1$, $d_{s(I)}(v_i) = k_2$ and , $d_{s(F)}(v_i) = k_3$ for all $v_i \in V$, then the SVNG is called as (k_1, k_2, k_3) - strong constant SVNG (or) Strong constant SVNG of degree (k_1, k_2, k_3).

Definition 3.22 Let G be a SVNG. If $td_{s(T)}\ v_i) = r_1$, $td_{s(I)}\ v_i) = r_2$ and , $d_{s(F)}\ v\ ) = r_3$ for all $v_i \in V$, then the SVNG is called as (r_1, r_2, r_3) – totally strong constant SVNG (or) totally strong constant SVNG of degree (r_1, r_2, r_3).

Proposition 3.23 In a SVNG, G

$$2\,S_{s(T}(\ ) = \sum_{i=1}^{n} d_{s(T}\ \ _i), \; 2\,S_{s(I)}(\ ) = \sum_{i=1}^{n} d_{s(I}\ \ _i) \text{ and}$$

$$2\,S_{s(F}(\ ) = \sum_{i=1}^{n} d_{s(F}(\ _i)$$

Proposition 3.24 In a connected SVNG,

1) $d_{S(T)}$(vi) ≤ d_{Ti} , $d_{S(I)}$(vi) ≤ d_{Ii} and d (vi)
2) (vi) , (vi) and $td_{S(F)}$(vi) .

Proposition 3.25 Let G be a SVNG where crisp graph is an odd cycle. Then G is strong constant if f < T_{ij} ,I_{ij} ,F is constant function for every e_{ij} ∈ E.

Proposition 3.26 Let G be a SVNG where crisp graph is an even cycle. Then G is strong constant if f T ,I_{ij} ,F_{ij} > is constant function or alternate edges have same true membership, indeterminate membership and false membership for every e_{ij} ∈ E.

Remark 3.27 The above proposition 3.25 and proposition 3.26 hold for totally strong constant SVNG, if < T_i,I_i ,F_i > is a constant function.

Remark 3.28 A complete SVNG need not be a strong constant SVNG and totally strong constant SVNG.

Remark 3.29 A strong SVNG need not be a strong constant SVNG and totally strong constant SVNG.

Remark 3.30 For a strong vertex V,

1) (v_i) = (v_i) , $d_I(v_i)$ = (v_i) and (v_i) = (v_i)
2) (v_i) = (v_i) , (v_i) = $td_{sI}(v_i)$ and $td_F(v_i)$ = (v_i)

Theorem 3.31 Let G be a complete SVNG with V = $\{v_1, v_2, \ldots, v_n\}$ such that $T_1 \le T_2 \le T_3 \le \ldots \le T_n$, $I_1 \ge I_2 \ge I_3 \ge \ldots \ge I_n$ and $\ge F_2 \ge \ \ge \ldots \ge$ Then

1) is minimum edge truth membership, is the maximum edge indeterminacy membership and is the maximum edge falsity membership of emits from for all j = 2,3,4,…, n.

2) T_{in} is maximum edge truth membership, $_{in}$ is the minimum edge indeterminacy membership and F_{in} is the minimum edge falsity membership of among all edges from emits from v_i to v_n for all i = 1, 2,3,4,…, n-1.

3) $td_T(v_1)$ = (G) = n.T1 , $td_I(v_1)$ = (G) = n.I1 and $td_F(v_1)$ = (G) = n.F1.

4) $td_T(v_n)$ = (G) = ∑ , $td_I(v_n)$ = (G) = ∑ , and $td_F(v_n)$ = (G) = ∑ .

Proof: Throughout the proof, suppose that $\le T_2 \le \ \le \ldots \le T_n$, $I_1 \ge I_2 \ge I_3 \ge \ldots \ge I_n$ and $\ge F_2 \ge F_3 \ge \ldots \ge$.

1) To prove that is minimum edge truth membership, is the maximum edge indeterminacy membership and is the maximum edge falsity membership of e_{ij} emits from v1∀j=2,3,….,n. Assume the contrary i.e. e_{1l} is not an edge of minimum true membership, maximum indeterminate membership and maximum false membership emits from v_l. Also let e_{kl} , 2 ≤ k ≤ n,k≠ 1 be an edge with minimum true membership,, maximum indeterminate membership and maximum false membership emits from v_k.

Being a complete SVNG,

= min { T_1, } , I_{1l}= max { I_1 , I_l } and = max { F_1, }

Then = min { , } , = max { I_k , I_l } and

= max { , }

Since min { T_k, }< min { T_1, }

Thus either or .

Also since max { I_k , I_l } max { I_1, I_l }, so either or .

Since l, k ≠ 1, this is contradiction to our vertex assumption that T is the unique minimum vertex true membership, I is the maximum vertex indeterminate membership and $F_1$is the maximum vertex false membership.

Hence is minimum edge true membership, is the maximum edge indeterminate membership and is the maximum edge false membership of emits from to $v_j$for all j = 2, 3, 4,…, n.

2) On the contrary, assume let is not an edge with maximum true membership, minimum indeterminate membership and minimum false membership emits from for 1 ≤ k ≤ n-1. On the other hand, let be an edge with maximum true membership, minimum indeterminate membership and minimum false membership emits from from 1 ≤ r n-1, k ≠ r.

Then min { , } min{ , } , so ,

⇒max{I_k, } <max{ I_k, } , so and

Similarly $F_{kr} < F_{kn} \Rightarrow \max \{ F_k, F_r \} < \max \{ F_k, F_n \} = F_k$, $\Rightarrow F_r < F_k$

So $T_{kr} = T_k = T_{kn}$, $I_{kr} = I_k = I_{kn}$ and $F_{kr} = F_k = F_{kn}$, which is a contradiction. Hence e_{kn} is an edge with maximum true membership, minimum indeterminate membership and minimum false membership among all edges emits from v_k to v_n.

3) Now

$td_T(v_1) = d_T(v_1) + T_1$

$= \sum_{e_{ij} \in E} T_{1j} + T_1 = \sum_{j=2}^{n} T_{1j} + T_1$

$= (n-1).T_1 + T_1 = nT_1 - T_1 + T_1 = nT_1$,

$td_I(v_1) = d_I(v_1) + I_1$

$= \sum_{e_{ij} \in E} I_{1j} + I_1 = \sum_{j=2}^{n} I_{1j} + I_1$

$= (n-1).I_1 + I_1 = nI_1 - I_1 + I_1 = nI_1$ and

Similarly,

$td_F(v_1) = d_F(v_1) + F_1$

$= \sum_{e_{ij} \in E} F_{1j} + F_1 = \sum_{j=2}^{n} F_{1j} + F_1$

$= (n-1).F_1 + F_1 = nF_1 - F_1 + F_1 = nF_1$

Suppose that $td_T(v_1) \neq \delta_{td_T}(G)$ and let v_k , $k \neq 1$ be a vertex in G with minimum T- total degree.

Then,

$td_T(v_1) > td_T(v_k)$

$$\Rightarrow \sum_{i=2}^{n} T_{1i} + T_1 > \sum_{k \neq 1, k \neq j} T_{kj} + T_k$$

$$\Rightarrow \sum_{i=2}^{n} T_1 \wedge T_i + T_1 > \sum_{k \neq 1, k \neq j} T_k \wedge T_j + T_k$$

Since $T_1 \wedge T_i = T_1$ for i = 1, 2, 3, …, n and for all other indices j, $T_k \wedge T_j > T_1$, it follow that

$(n-1).T_1 + T_1 > \sum_{k \neq 1, k \neq j} T_k \wedge T_j + T_k > (n-1).T_1 + T_1$

Hence, $td_T(v_1) > td_T(v_1)$, a contradiction.

Therefore, $td_T(v_1) = \delta_{td_T}(G)$.

Suppose that $td_I(v_1) \neq \Delta_{td_I}(G)$ and let v_k , $k \neq 1$ be a vertex in G with maximum I- total degree.

Then,

$td_I(v_1) < td_I(v_k)$

$$\Rightarrow \sum_{i=2}^{n} I_{1i} + I_1 < \sum_{k \neq 1, k \neq j} I_{kj} + I_k$$

$$\Rightarrow \sum_{i=2}^{n} I_1 \vee I_i + I_1 < \sum_{k \neq 1, k \neq j} I_k \vee I_j + I_k$$

Since $I_1 \vee I_i = I_1$ for i = 1, 2, 3, …, n and for all other indices j, $I_k \vee I_j < I_1$, it follow that

$(n-1).I_1 + I_1 < \sum_{k \neq 1, k \neq j} I_k \vee I_j + I_k < (n-1).I_1 + I_1$

So that $td_I(v_1) < td_I(v_1)$, a contradiction.

Therefore, $td_I(v_1) = \Delta_{td_I}(G)$.

Also, Suppose that $td_F(v_1) \neq \Delta_{td_F}(G)$ and let v_k , $k \neq 1$ be a vertex in G with maximum F- total degree.

Then

$td_F(v_1) < td_F(v_k)$

$$\Rightarrow \sum_{i=2}^{n} F_{1i} + F_1 < \sum_{k \neq 1, k \neq j} F_{kj} + F_k$$

$$\Rightarrow \sum_{i=2}^{n} F_1 \vee F_i + F_1 < \sum_{k \neq 1, k \neq j} F_k \vee F_j + F_k$$

Since $F_1 \vee F_i = F_1$ for i = 1, 2, 3, …, n and for all other indices j, $F_k \vee F_j < F_1$, it follow that

$(n-1).F_1 + F_1 < \sum_{k \neq 1, k \neq j} F_k \vee F_j + F_k < (n-1).F_1 + F_1$

So that $td_F(v_1) < td_F(v_1)$, a contradiction .

Therefore, $td_F(v_1) = \Delta_{td_F}(G)$.

Hence,

$td_T(v_1) = \delta_{td_T}(G) = n.T_1$,

$td_I(v_1) = \Delta_{td_I}(G) = n.I_1$ and

$td_F(v_1) = \Delta_{td_F}(G) = n.F_1$.

4) Since, $T_n > T_i$, $I_n < I_i$ and $F_n < F_i$, i = 1, 2, 3, …, n-1 and G is complete

$T_{ni} = T_n \wedge T_i = T_i$, $I_{ni} = I_n \vee I_i = I_i$ and $F_{ni} = F_n \vee F_i = F_i$.

Hence, $td_T(v_n) = \sum_{i=1}^{n-1} T_{ni} + T_n$

$= \sum_{i=1}^{n-1} (T_n \wedge T_i) + T_n = \sum_{i=1}^{n-1} T_i + T_n$

$= \sum_{i=1}^{n} T_i$,

$td_I(v_n) = \sum_{i=1}^{n-1} I_{ni} + I_n$

$= \sum_{i=1}^{n-1} (I_n \vee I_i) + I_n = \sum_{i=1}^{n-1} I_i + I_n$

$= \sum_{i=1}^{n} I_i$

And $td_F(v_n) = \sum_{i=1}^{n-1} F_{ni} + F_n$

$= \sum_{i=1}^{n-1} (F_n \vee F_i) + F_n = \sum_{i=1}^{n-1} F_i + F_n$

$= \sum_{i=1}^{n} F_i$.

Suppose that $td_T(v_n) \neq \Delta_{td_T}(G)$. Let v_l , $1 \leq l \leq n-1$ be a vertex in G such that $td_T(v_l) = \Delta_{td_T}(G)$ and

$td_T(v_n) < td_T(v_l)$. In addition,

$td_T(v_l) = [\sum_{i=1}^{l-1} T_{il} + \sum_{i=l+1}^{n-1} T_{il} + T_{nl}] + T_l$

$\leq [\sum_{i=1}^{l-1} T_i + (n-1)T_l + T_l] + T_l$

$\leq \sum_{i=1}^{n-1} T_i + T_l$

$\leq \sum_{i=1}^{n} T_i = td_T(v_n)$. Thus $td_T(v_n) \geq td_T(v_l)$, contradiction. So, $td_T(v_n) = \Delta_{td_T}(G) = \sum_{i=1}^{n} T_i$.

Suppose that $td_I(v_n) \neq \delta_{td_I}(G)$. Let v_l, $1 \leq l \leq n-1$ be a vertex in G such that $td_I(v_l) = \delta_{td_I}(G)$ and $td_I(v_n) > td_I v_l)$.

In addition,

$td_I(v_l) = [\sum_{i=1}^{l-1} I_{il} + \sum_{i=l+1}^{n-1} I_{il} + I_{nl}] + I_l$

$\geq [\sum_{i=1}^{l-1} I_i + (n-1)I_l + I_l] + I_l$

$\geq \sum_{i=1}^{n-1} I_i + I_l$

$\geq \sum_{i=1}^{n} I_i = td_I(v_n)$. Thus $td_I(v_n) \leq td_I(v_l)$, contradiction. So, $td_I(v_n) = \delta_{td_I}(G) = \sum_{i=1}^{n} I_i$.

Also, suppose that $td_F(v_n) \neq \delta_{td_F}(G)$. Let v_l, $1 \leq l \leq n-1$ be a vertex in G such that $td_F(v_l) = \delta_{td_F}(G)$ and $td_F(v_n) > td_F v_l)$. In addition,

$td_F(v_l) = [\sum_{i=1}^{l-1} F_{il} + \sum_{i=l+1}^{n-1} F_{il} + F_{nl}] + F_l$

$\geq [\sum_{i=1}^{l-1} F_i + (n-1)F_l + F_l] + F_l$

$\geq \sum_{i=1}^{n-1} F_i + F_l$

$\geq \sum_{i=1}^{n} F_i = td_F(v_n)$. Thus $td_F(v_n) \leq td_F(v_l)$, contradiction. So, $td_F(v_n) = \delta_{td_F}(G) = \sum_{i=1}^{n} F_i$.

Hence the lemma is proved.

Remark 3.32 In a complete SVNG G,

1) There exists at least one pair of vertices v_i and v_j such that $d_{T_i} = d_{T_j} = \Delta_T(G)$, $d_{I_i} = d_{I_i} = \delta_I(G)$ and $d_{F_i} = d_{F_i} = \delta_F(G)$,

2) $td_T(v_i) = O_T(G) = \Delta_{td_T}(G)$, $td_I(v_i) = O_I(G) = \delta_{td_I}(G)$ and $td_F(v_i) = O_F(G) = \delta_{td_F}(G)$ for a vertex $v_i \in V$,

3) $\sum_{i=1}^{n} td_T(v_i) = 2S_T(G) + O_T(G)$, $\sum_{i=1}^{n} td_I(v_i) = 2S_I(G) + O_I(G)$ and $\sum_{i=1}^{n} td_F(v_i) = 2S_F(G) + O_F(G)$.

IV. VERTEX TRUTH MEMBERSHIP, VERTEX INDTERMINACY MEMBERSHIP AND VERTEX FALSITY MEMEBERSHIP SEQUENCE IN SVNG

In this section, vertex truth membership, vertex indeterminacy membership and vertex falsity membership sequences are defined in SVNGs.

Definition 4.1 Given a SVN-graph G with $|V| = n$. The vertex truth membership sequence of G is defined to be $\{x_i\}_{i=1}^{n}$ with $x_1 \leq x_2 \leq x_3 \leq \ldots \leq x_n$ where x_i, $0 < x_i \leq 1$, is the truth membership value of the vertex v_i when vertices are arranged so that their truth membership values are non-decreasing.

Particular, x_1 is smallest vertex truth membership value and x_n is largest vertex truth membership value in G.

Note 4.2 If vertex truth membership sequence x_i is repeated more than once in G, say $r \neq 1$ times, then it is denoted by x_i^r in the sequence.

Example 4.3 In Fig. 2 the vertex truth membership sequence of G is {0.1, 0.1, 0.3, 0.3, 0.4, 0.8 } or { 0.1^2, 0.3^2, 0.4, 0.8 }.

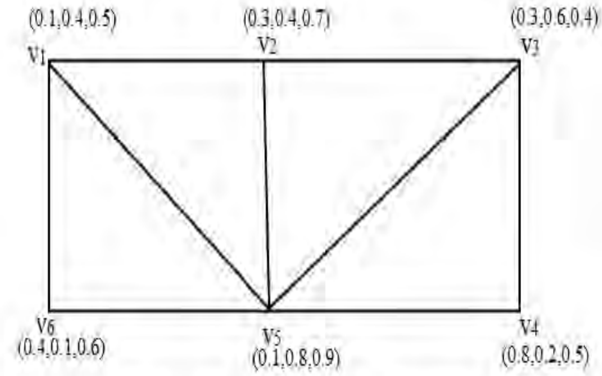

Fig. 2. Vertex truth membership sequence.

Definition 4.4 Let G be a SVNG with $|V| = n$. The vertex indeterminacy membership sequence of G is defined to be $\{y_i\}_{i=1}^{n}$ with $y_1 \leq y_2 \leq y_3 \leq \ldots \leq y_n$ where y_i, $0 < y_i \leq 1$, is the indeterminacy membership value of the vertex v_i when vertices are arranged so that their indeterminacy membership values are non- increasing.

Particular, y_1 is largest vertex indeterminacy membership value and y_n is smallest vertex indeterminacy membership value in G.

Note 4.5 If vertex indeterminacy membership sequence y_i is repeated more than once in G, say $r \neq 1$ times, then it is denoted by y_i^r in the sequence.

Example 4.6 In Fig. 3 the vertex indeterminacy membership sequence of G is {0.7, 0.6, 0.6, 0.5, 0.4, 0.4 } or { 0.7, 0.6^2, 0.5, 0.4^2 }.

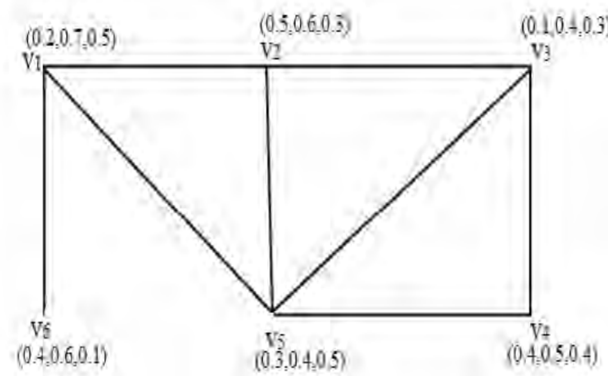

Fig. 3. Vertex indeterminacy membership sequence.

Definition 4.7 Let G be a SVNG with $|V| = n$. The vertex falsity membership sequence of G is defined to be $\{z_i\}_{i=1}^{n}$ with $z_1 \leq z_2 \leq z_3 \leq \ldots \leq z_n$ where z_i, $0 < z_i \leq 1$, is the falsity membership value of the vertex v_i when vertices are arranged so that their falsity membership values are non- increasing. Particular, z_1 is largest vertex falsi Y membership value and z_n is smallest vertex falsity membership value in G.

Note 4.8 If vertex falsity membership sequence z_i is repeated more than once in G, say $r \neq 1$ times, then it is denoted by $z_i{}^r$ in the sequence.

Example 4.9 In Fig. 4 the vertex falsity membership sequence of G is {0.8, 0.8, 0.7, 0.6, 0.6, 0.5} or $\{0.8^2, 0.7, 0.6^2, 0.5\}$.

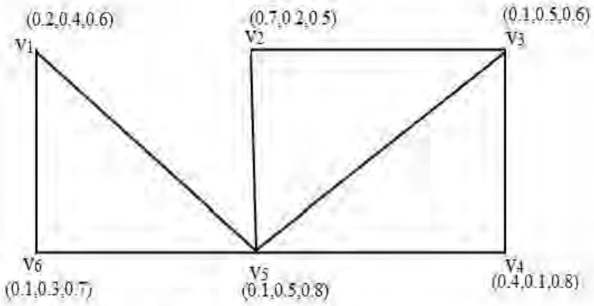

Fig. 4. Vertex falsity membership sequence.

Definition 4.10 If a SVNG with $|V| = n$ has vertex truth membership sequence $\{x_i\}_{i=1}^{n}$, vertex indeterminacy membership sequence$\{y_i\}_{i=1}^{n}$ and vertex falsity membership sequence$\{z_i\}_{i=1}^{n}$ in same order, then it said to have vertex single valued neutrosophic sequence and denoted by $\{< x_i, y_i, z_i >\}_{i=1}^{n}$.

Example 4.11 In Fig. 5 the vertex truth membership, vertex indeterminacy membership and vertex falsity membership sequence of G is{<0.4, 0.4, 0.5>, < 0.2,0.3,0.5>,< 0.1,0.2,0.6> , < 0.5,0.4,0.8>, < 0.4,0.5,0.4>, < 0.3,0.1,0.7> }.

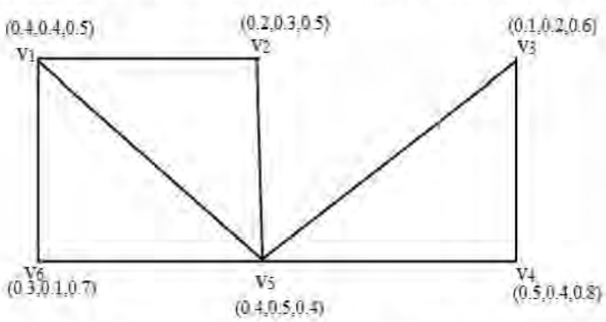

Fig. 5. Vertex single valued neutrosophic sequence.

The properties of vertex truth membership, vertex indeterminacy membership and vertex falsity sequences of complete SVNGs are discussed below:

Theorem 4.12 Let G=(V,E) be a complete SVNG with$|V|$ = n. Then

1) If the vertex truth membership sequence of G is of the form $\{x_1{}^{n-1}, x_2\}$, vertex indeterminacy membership sequence of G is of the form $\{y_1{}^{n-1}, y_2\}$ and vertex falsity membership sequence of G is of the form $\{z_1{}^{n-1}, z_2\}$, then

a.δ_{td_T} (G) = n.T_1 and Δ_{td_T}(G)= $\sum_{i=1}^{n} T_i$

b. Δ_{td_I} (G) = n.I_1 and δ_{td_I}(G)=$\sum_{i=1}^{n} I_i$

c.Δ_{td_F}(G)= n.F_1 and δ_{td_F}(G) = $\sum_{i=1}^{n} F_i$

2) If the vertex truth membership sequence of G is of the form $\{x_1{}^{r_1}, x_2{}^{n-r_1}\}$, vertex indeterminacy membership of G is of the form $\{y_1{}^{r_1}, y_2{}^{n-r_1}\}$ and vertex falsity membership sequence of G is of the form $\{z_1{}^{r_1}, z_2{}^{n-r_1}\}$ with 0< $r_1 \leq$ n-2, then there exists exactly r_1 vertices with minimum T- total degree δ_{td_T} (G), maximum I-total degree $\Delta_{td_I}(G)$ and maximum F-total degree Δ_{td_F} and exactly (n-r_1) vertices with maximum T- total degree Δ_{td_T}(G) , minimum I- total degree δ_{td_I}(G) and minimum F- total degree δ_{td_F}(G).

3) If the vertex truth membership sequence of G is of the form $\{x_1{}^{r_1}, x_2{}^{r_2}, x_3{}^{r_3}, \ldots, x_k{}^{r_k}\}$, vertex indeterminacy membership sequence of G is of the form $\{y_1{}^{r_1}, y_2{}^{r_2}, y_3{}^{r_3}, \ldots, y_k{}^{r_k}\}$ and vertex falsity membership sequence of G is of the form $\{z_1{}^{r_1}, z_2{}^{r_2}, z_3{}^{r_3}, \ldots, z_k{}^{r_k}\}$ with r_k > 1 and k > 2, then there exists exactly r_1 vertices with minimum T- total degree δ_{td_T}(G), maximum I- total degree $\Delta_{td_I}$and maximum F-total degree Δ_{td_F}. Also, there exists exactly r_k vertices with maximum T- total degree Δ_{td_T}(G), minimum I- total degree δ_{td_I}(G) and minimum F- total degree δ_{td_F}(G).

Proof: The proof of (1) and (2) are obvious. 3 Let $v_i^{(j)}$ be the set of vertices in G, for j = 1, 2, 3, . . . , r_i , 1 ≤ i ≤ k. Then by the **Theorem3.31**

$td_T(v_1^{(j)})$=$\delta_{td_T}(G)$=n.T_1=n. x_1,

$td_I(v_1^{(j)})$=$\Delta_{td_I}(G)$= n.I_1=n. y_1, and

$td_F(v_1^{(j)})$=$\delta_{td_F}(G)$= n.F_1=n. z_1, , for j= 1, 2, 3,. . , r_1.

Since T($v_i^{(j)}$,$v_{i+1}^{(l)}$) = T($v_i^{(j)}$)>x_1 for 2 ≤ i ≤ k , j= 1, 2, 3,. .. . , r_i , l = 1, 2, 3, . . . , r_{i+1} , no vertex with truth membership more than x_1 can have degree δ_{td_T}(G),

I($v_i^{(j)}$,$v_{i+1}^{(l)}$) = I($v_i^{(j)}$) < y_1 for 2 ≤ i ≤ k , j = 1, 2, 3, . . . , r_i , l = 1, 2, 3, . . . , r_{i+1} , no vertex with indeterminacy membership less than y_1 can have degree Δ_{td_I}(G)

And F($v_i^{(j)}$,$v_{i+1}^{(l)}$) = F($v_i^{(j)}$) < z_1 for 2 ≤ i ≤ k , j = 1, 2, 3, . . . , r_i , l = 1, 2, 3, . . . , r_{i+1} , no vertex with falsity membership less than z_1 can have degree Δ_{td_F}(G).

Thus, there exist exactly r_1 vertices with degree δ_{td_T}(G), Δ_{td_I}(G), Δ_{td_F}(G).

To prove $td_T(v_k^{(t)})$ =$\Delta_{td_T}(G)$,

$td_I(v_k^{(t)})$ =δ_{td_I} (G) and

$td_F(v_k^{(t)})$ =δ_{td_F} (G), t=1,2,3 . . , r_k.

Since, T($v_k^{(t)}$) is maximum vertex truth membership,

T($v_k^{(t)}$,$v_k^{(j)}$) = x_k , t ≠j, t, j=1, 2, 3,…,r_k

T($v_k^{(t)}$,$v_i^{(j)}$) = min { T($v_k^{(t)}$) , T($v_i^{(j)}$)} = T($v_i^{(j)}$) for t= 1,2,3. . . , r_k , j = 1, 2, 3, . . . , r_i, i=1,2,3,… , k-1

Thus for t= 1,2,3, . . . , r_k,

$td_T(v_k^{(t)}) = \sum_{i=1}^{k}\sum_{j=1}^{r_i} T(v_i^{(j)}) + (r_k - 1)x_k$

$= \sum_{i=1}^{n} T_i$

$= \Delta_{td_T}(G)$ by **Theorem 3.31**

Now, if v_m is vertex such that $T_m = x_{k-1}$, then

$td_T(v_m) = \sum_{i=1}^{k-2}\sum_{j=1}^{r_i} T(v_m, v_i^{(j)}) + (r_{k-1} - 1 + r_k)x_{k-1} + T_m$

$= \sum_{i=1}^{k-2}\sum_{j=1}^{r_i} T(v_i^{(j)}) + \sum_{j=1}^{r_{k-1}} T(v_{k-1}^{(j)}) + (r_k - 1)x_{k-1} + T_m$

$< \sum_{i=1}^{k-2}\sum_{j=1}^{r_i} T(v_i^{(j)}) + \sum_{j=1}^{r_{k-1}} T(v_{k-1}^{(j)}) + (r_k - 1)x_k + T_m$

$= \Delta_{td_T}(G)$

Thus, there exist exactly r_k vertices with degree $\Delta_{td_T}(G)$.

To prove $td_I(v_k^{(t)}) = \delta_{td_I}(G)$, for t = 1, 2, 3, . . . , r_k

Since I($v_k^{(t)}$) is minimum vertex indeterminacy membership,

$I(v_k^{(t)}, v_k^{(j)}) = y_k$, t ≠ j , t, j = 1, 2, 3, . . . , r_k

$I(v_k^{(t)}, v_i^{(j)}) = \max\{ I(v_k^{(t)}) , I(v_i^{(j)})\} = I(v_i^{(j)})$ for t = 1, 2, 3, . . . , r_k , j = 1, 2, 3, . . . , r_i,

i = 1, 2, 3, . . . , k-1.

Thus for t = 1, 2, 3, . . . , r_k,

$td_I(v_k^{(t)}) = \sum_{i=1}^{k}\sum_{j=1}^{r_i} I(v_i^{(j)}) + (r_k - 1)y_k$

$= \sum_{i=1}^{n} I_i$

$= \delta_{td_I}(G)$ by **Theorem 3.31**

Now, if v_m is vertex such that $I_m = y_{k-1}$, then

$td_I(v_m) = \sum_{i=1}^{k-2}\sum_{j=1}^{r_i} I(v_m, v_i^{(j)}) + (r_{k-1} - 1 + r_k)y_{k-1} + I_m$

$= \sum_{i=1}^{k-2}\sum_{j=1}^{r_i} I(v_i^{(j)}) + \sum_{j=1}^{r_{k-1}} I(v_{k-1}^{(j)}) + (r_k - 1)y_{k-1} + I_m$

$< \sum_{i=1}^{k-2}\sum_{j=1}^{r_i} I(v_i^{(j)}) + \sum_{j=1}^{r_{k-1}} I(v_{k-1}^{(j)}) + (r_k - 1)y_k + I_m$

$= \delta_{td_I}(G)$

So, there exist exactly r_k vertices with degree $\delta_{td_I}(G)$.

Similarly, it can be proved that $td_F(v_k^{(t)}) = \delta_{td_F}(G)$, for t = 1, 2, 3, . . . , r_k

Since $F(v_k^{(t)})$ is minimum vertex falsity membership,

$F(v_k^{(t)}, v_k^{(j)}) = z_k$, t ≠ j , t, j = 1, 2, 3, . . . , r_k

$F(v_k^{(t)}, v_i^{(j)}) = \max\{ F(v_k^{(t)}), F(v_i^{(j)})\} = F(v_i^{(j)})$ for t = 1, 2, 3, . . . , r_k , j = 1, 2, 3, . . . , r_i, i = 1, 2, 3, . . . , k-1.

Thus for t = 1, 2, 3, . . . , r_k,

$td_F(v_k^{(t)}) = \sum_{i=1}^{k}\sum_{j=1}^{r_i} F(v_i^{(j)}) + (r_k - 1)z_k$

$= \sum_{i=1}^{n} F_i$

$= \delta_{td_F}(G)$ by **Theorem 3.31**

Now, if v_m is vertex such that $F_m = z_{k-1}$, then

$td_F(v_m) = \sum_{i=1}^{k-2}\sum_{j=1}^{r_i} F(v_m, v_i^{(j)}) + (r_{k-1} - 1 + r_k)F_{k-1} + F_m$

$= \sum_{i=1}^{k-2}\sum_{j=1}^{r_i} F(v_i^{(j)}) + \sum_{j=1}^{r_{k-1}} F(v_{k-1}^{(j)}) + (r_k - 1)z_{k-1} + F_m$

$< \sum_{i=1}^{k-2}\sum_{j=1}^{r_i} F(v_i^{(j)}) + \sum_{j=1}^{r_{k-1}} F(v_{k-1}^{(j)}) + (r_k - 1)z_k + F_m$

$= \delta_{td_F}(G)$

So, there exist exactly r_k vertices with degree $\delta_{td_F}(G)$.

V. CONCLUSION

In this paper, the idea of strong degree is imposed on the existing concepts of degrees in SVNGs. After that, we defined the vertex truth-membership, vertex indeterminacy-membership and vertex falsity membership sequence in SVNG with proofs and suitable examples. In the next research, the proposed concepts can be extended to labeling neutrosophic graph and also characterize the corresponding properties.

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An Approach to Measuring the Website Quality Based on Neutrosophic Sets

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ABSTRACT

Gathering the attitudes of the examined respondents would be very significant in some evaluation models. Therefore, an approach to the evaluation of websites based on the use of the neutrosophic set is proposed in this paper. An example of websites evaluation is considered at the end of this paper with the aim to present in detail the proposed approach.

KEYWORDS: **neutrosophic set; single valued neutrosophic set; website quality; website evaluation; multiple criteria decision making.**

1. INTRODUCTION

A company's website can have a very important role in a competitive environment. It can be used to provide information to its customers, collect new and retain old users and so on.

A website can be visited by various groups of users that could have different requirements, needs and interests. In order to assess the quality of a website, it is necessary to obtain as realistic attitudes of its visitors about the fulfillment of their expectations and the perceived reality as possible.

The evaluation of the quality of websites has been considered in numerous studies, for which reason many approaches have been proposed. Some of them have been devoted to determining the impact of the website quality on customer satisfaction, such as: Al-Manasra *et al*. (2016), Bai *et al*. (2008), Lin (2007) and Kim and Stoel (2004).

Some other studies have been intended to determine the quality of websites and/or define the elements of the website that affect its quality, such as: Canziani and Welsh (2016), Salem and Cavlek (2016), Ting *et al*. (2013), Rocha (2012), Chiou *et al*. (2011) and Kincl and Strach (2012).

In some of them, the evaluation of websites has been considered as a multiple criteria decision making-problem, including the FS theory or its extensions, such as: Stanujkic *et al*. (2015), Chou and Cheng (2012), Kaya and Kahraman (2011), and Kaya (2010).

It is also known that a significant progress in multiple criteria decision making has been made after Zadeh (1965) proposed the Fuzzy Sets (FS) theory, thus introducing partial belonging to a set, expressed by using the membership function.

The FS theory has later been extended in order to provide an effective method for solving many complex decision-making problems, often related to uncertainties and predictions. The Interval-Valued Fuzzy Set (IVFS) Theory, proposed by Turksen (1986; 1996) and Gorzalczany (1987), the Intuitionistic Fuzzy Sets (IFS) Theory, proposed by Atanassov (1986) and the Interval-Valued Intuitionistic Fuzzy Set (IVIFS) Theory, proposed by Atanassov and Gargov (1989), can be mentioned as the prominent and widely used extensions of the FS theory.

In the IFS, Atanassov introduced the non-membership function. Smarandache (1998) proposed the Neutrosophic Set (NS) and so further generalized the IFS by introducing the indeterminacy-membership function, thus providing a general framework generalizing the concepts of the classical, fuzzy, interval-valued fuzzy and intuitionistic fuzzy sets.

Compared with the FS and its extensions, the NS can be identified as more flexible, for which reason they have been chosen in this approach for collecting the respondents' attitudes.

Therefore, this manuscript is organized as follows: in Section 2, the NSs are considered and in Section 3, the SWARA method is presented. In Section 4, a procedure for evaluating companies' websites is considered and in Section 5, its usability is demonstrated. Finally, the conclusion is given.

2. PRELIMINARIES

Definition. *Fuzzy sets* (FS). Let X be the universe of discourse, with a generic element in X denoted by x. Then, the FS $\widetilde{A}$ in X is as follows:

$$\widetilde{A} = \{x(\mu_A(x)) \mid x \in X\}, \tag{1}$$

where: $\mu_A : X \rightarrow [0,1]$ is the membership function and $\mu_A(x)$ denotes the degree of the membership of the element x in the set $\widetilde{A}$ (Zadeh, 1965).

Definition. *Intuitionistic fuzzy set* (IFS). Let X be the universe of discourse, with a generic element in X denoted by x. Then, the IFS $\widetilde{A}$ in X can be defined as follows:

$$\widetilde{A} = \{x < \mu_A(x), \nu_A(x) > \mid x \in X\}, \tag{2}$$

where: $\mu_A(x)$ and $\nu_A(x)$ are the truth-membership and the falsity-membership functions of the element x in the set A, respectively; $\mu_A, \nu_A : X \rightarrow [0,1]$ and $0 \le \mu_A(x) + \nu_A(x) \le 1$.

In intuitionistic fuzzy sets, indeterminacy $\pi_A(x)$ is $1 - \mu_A(x) - \nu_A(x)$ by default (Atanassov, 1986).

Definition. *Neutrosophic set* (NS). Let X be the universe of discourse, with a generic element in X denoted by x. Then, the NS A in X is as follows:

$$A = \{x < T_A(x), I_A(x), F_A(x) > \mid x \in X\}, \tag{3}$$

where $T_A(x)$, $I_A(x)$ and $F_A(x)$ are the truth-membership function, the indeterminacy-membership function and the falsity-membership function, respectively, $T_A, I_A, F_A : X \rightarrow]^-0,1^+[$ and $^-0 \le T_A(x) + I_A(x) + F_A(x) \le 3^+$ (Smarandache, 1999).

Definition. *Single valued neutrosophic set* (SVNS). Let X be the universe of discourse. The SVNS A over X is an object having the form

$$A = \{x < T_A(x), I_A(x), F_A(x) > \mid x \in X\}, \tag{4}$$

where $T_A(x)$, $I_A(x)$ and $F_A(x)$ are the truth-membership function, the intermediacy-membership function and the falsity-membership function, respectively, $T_A, I_A, F_A : X \to [^-0,1^+]$ and $^-0 \le T_A(x) + I_A(x) + F_A(x) \le 3^+$ (Wang *et al*., 2010).

Definition. *Single valued neutrosophic number*. For the SVNS A in X the triple $< t_A, i_A, f_A >$ is called the single valued neutrosophic number (SVNN) (Smarandache, 1999).

Definition. *Basic operations on SVNNs*. Let $x_1 =< t_1, i_1, f_1 >$ and $x_2 =< t_2, i_2, f_2 >$ be two SVNNs, then additive and multiplication operations are defined as follows (Smarandache, 1998):

$$x_1 + x_2 =< t_1 + t_2 - t_1 t_2, i_1 i_2, f_1 f_2 >, \tag{5}$$

$$x_1 \cdot x_2 =< t_1 t_2, i_1 + i_2 - i_1 i_2, f_1 + f_2 - f_1 f_2 >. \tag{6}$$

Definition. *Scalar multiplication*. Let $x =< t_x, i_x, f_x >$ be a SVNN and $\lambda > 0$, then scalar multiplication is defined as follows (Smarandache, 1998):

$$\lambda x_1 =< 1 - (1 - t_1)^\lambda, i_1^\lambda, f_1^\lambda > . \tag{7}$$

Definition. *Power*. Let $x =< t_x, i_x, f_x >$ be a SVNN and $\lambda > 0$, then power is defined as follows:

$$x_1^\lambda =< t_1^\lambda, i_1^\lambda, 1 - (1 - f_1)^\lambda > . \tag{8}$$

Definition. *Score function*. Let $x =< t_x, i_x, f_x >$ be a SVNN, then the score function s_x of x can be as follows:

$$s_x = (1 + t_x - 2i_x - f_x)/2, \tag{9}$$

where $s_x \in [-1,1]$ (Smarandache, 1998).

Definition. *Accuracy function*. Let $x =< t_x, i_x, f_x >$ be a SVNN, then the score function s_x of x can be as follows:

$$h_x = (2 + t_x - i_x - f_x)/3, \tag{10}$$

where $h_x \in [0,1]$ (Smarandache, 1998).

Definition. *Ranking based on score and accuracy functions*. Let x_1 and x_2 be two SVNNs. Then, the ranking method can be defined as follows (Mondal & Pramanik, 2014):

(1) If $s_{x1} > s_{x2}$, then $x_1 > x_2$;
(2) If $s_{x1} = s_{x2}$ and $h_{x1} \ge h_{x2}$, then $x_1 \ge x_2$.

Definition. *Single Valued Neutrosophic Weighted Average Operator*. Let $A_j =< t_j, i_j, f_j >$ be a collection of SVNSs and $W = (w_1, w_2, ..., w_n)^T$ is an associated weighting vector. Then, the Single Valued Neutrosophic Weighted Average (SVNWA) operator of A_j is as follows (Sahin, 2014):

$$SVNWA(A_1, A_2, ..., A_n) = \sum_{j=1}^{n} w_j A_j = \left(1 - \prod_{j=1}^{n} (1 - t_j)^{w_j}, \prod_{j=1}^{n} (i_j)^{w_j}, \prod_{j=1}^{n} (f_j)^{w_j}, \right). \tag{11}$$

where: w_j is the element j of the weighting vector, $w_j \in [0,1]$ and $\sum_{j=1}^{n} w_j = 1$.

3. The SWARA Method

The Step-wise Weight Assessment Ratio Analysis (SWARA) technique was proposed by Kersuliene *et al.* (2010). The computational procedure of the adapted SWARA method can be shown through the following steps (Kersuliene *et al*., 2010; Stanujkic *et al*., 2015):

***Step* 1**. Determine the set of the relevant evaluation criteria and sort them in descending order, based on their expected significances.

***Step* 2**. Starting from the second criterion, determine the relative importance s_j of the criterion j (C_j) in relation to the previous j-1 C_{j-1} criterion, and do so for each particular criterion as follows:

$$s_j = \begin{cases} >1 & \textit{when significance of } C_j \succ C_{j-1} \\ 1 & \textit{when significanse of } C_j = C_{j-1} \\ <1 & \textit{when significance of } C_j \prec C_{j-1} \end{cases}. \tag{12}$$

where C_j and C_{j-1} denote criteria.

Using Eq. (11) respondents can more realistically express their opinions compared to the ordinary SWARA method, proposed by Kersuliene *et al*. (2010).

***Step* 3**. The third step in the adapted SWARA method should be performed as follows:

$$k_j = \begin{cases} 1 & j = 1 \\ 2 - s_j & j > 1 \end{cases}. \tag{13}$$

where k_j is a coefficient.

***Step* 4**. Determine the recalculated weight q_j as follows:

$$q_j = \begin{cases} 1 & j = 1 \\ \dfrac{q_{j-1}}{k_j} & j > 1 \end{cases}. \tag{14}$$

***Step* 5**. Determine the relative weights of the evaluation criteria as follows:

$$w_j = \frac{q_j}{\sum_{k=1}^{n} q_k}, \tag{15}$$

where w_j denotes the relative weight of the criterion j.

4. PROCEDURE FOR EVALUATING WEBSITES BASED ON THE SINGLE VALUED NEUTROSOPHIC SET AND THE SWARA METHOD

In their studies, many authors have identified different phases in the multiple criteria decision-making process. In order to precisely define the procedures for evaluating websites, the below phases have specially been emphasized:

- the selection of evaluation criteria
- the determination of the weights of the criteria
- the evaluation of alternatives in relation to the criteria
- the aggregation and analysis of the results

Selection of Evaluation Criteria

The choice of an appropriate set of the evaluation selection criteria is very important for the successful solving of each MCDM problem.

In many published studies, a number of authors have proposed different criteria for the evaluation of various websites. For example, Kapoun (1998) has proposed the use of the following criteria: Accuracy, Authority, Objectivity, Currency and Coverage. After that, Lydia (2009) has proposed Authority, Accuracy, Objectivity, Currency, Coverage and Appearance for evaluating the quality of a website. For the evaluation of websites at the California State University at Chico (http://www.csuchico.edu/lins/handouts/eval_websites.pdf), the so-called CRAAP test, based on the following criteria: Currency, Relevance, Authority, Accuracy and Purpose, has been proposed.

In this approach, the proven set of the criteria adopted from the Webby Awards (http://webbyawards.com/judging-criteria/) is proposed for the evaluation of the quality of websites. This set of the evaluation criteria is as follows:

- Content (C_1),
- Structure and Navigation (C_2),
- Visual Design (C_3),
- Interactivity (C_4),
- Functionality (C_5) and
- Overall Experience (C_6).

The meaning of the proposed evaluation criteria is as follows:

- ***Content***. The content is the information provided on the website. It is not just a text, but also music, a sound, an animation or a video – anything that communicates the website's body of knowledge.
- ***Structure and Navigation***. The structure and navigation refer to the framework of a website, the organization of the content, the prioritization of information and the method in which you move through the website. Websites with the good structure and navigation are consistent, intuitive, and transparent.
- ***Visual Design***. A visual design is the appearance of a website. It is more than just a pretty homepage and it does not have to be cutting-edge or trendy. A good visual design is high-quality, appropriate and relevant for the audience and the message it is supportive of. It communicates a visual experience and may even take your breath away.
- ***Interactivity***. Interactivity is the way a site allows a user to perform an action. Good interactivity refers to providing opportunities for users to personalize their search and find information or perform some action more easily and efficiently.
- ***Functionality***. Functionality is the use of technology on a website. Good functionality means that a website works well. It loads quickly, has live links and any new technology that has been used is functional and relevant for the intended audience.
- ***Overall Experience***. Demonstrating that websites are frequently more or less than just the sum of their parts, overall experience encompasses the content, a visual design, functionality, interactivity and the structure and navigation, but also includes the intangibles that make one stay on the website or leave it.

Determination of the Weights of the Criteria

In this approach, the SWARA method is used for determining the weights of the criteria. The SWARA method has been chosen because it is relatively simple to use and requires a relatively small number of comparisons in pairs.

The determination of the weights of the criteria is done by using an interactive questionnaire made in a spreadsheet file. By using such an approach, the interviewee can see the calculated weights of the criteria and can also modify his/her answers if he or she is not satisfied with the calculated weights.

Evaluation of Alternatives in Relation to the Evaluation Criteria

In this phase, there are several sub-phases that can be identified.

The evaluation of alternatives in relation to the chosen set of the criteria is also done by using an interactive questionnaire made in a spreadsheet file.

For each criterion, declarative sentences are formed. The respondents have a possibility to fill in their attitudes about the degree of truth, indeterminacy and the falsehood of the statement.

For the sake of simplicity, the respondents fill in their attitudes in the percentage form, which are later transformed into the corresponding numbers in [0,1] intervals.

For completing the questionnaire, it is necessary that between 30 and 90 fields should be filled in, which can be dissuasive for a significant number of respondents. However, this approach can be good because it can distract uninterested respondents from completing the questionnaire, thus reducing the number of the completed questionnaires with incorrect information.

In addition, the Overall Experience criterion has also been used to assess the validity of the data entered.

Aggregation and Analysis of Results

In the Aggregation and Analysis phase, several components, sub-phases, could be identified, such as:

- the determination of the overall ratings and the ranking order of the considered alternatives,
- the assessment of the validity of the data in the completed questionnaire and
- the determination of the overall group ratings and the ranking order of the considered alternatives etc.

The first of them – the determination of the overall ratings – is mandatory, whereas the others are optional.

The determination of the overall ratings and the ranking order of the considered alternatives. The process of assessing the determination of the overall ratings and the ranking order could be shown through the following steps:

- the calculation of the overall single valued neutrosophic ratings of the alternatives by using the SVNWA operator based on the values of the criteria C_1-C_5;
- the calculation of the score function by using Eq. (9) for each alternative; and
- the sorting of the considered alternatives based on the values of the score function and the determination of the best one. The alternative with the highest value of the score function is the best one.

The assessment of the validity of the data in the completed questionnaire. The *Overall Experience* criterion is omitted from the calculation of the overall single valued neutrosophic ratings because it plays a special role in the proposed approach. More precisely, the ratings filled in for this criterion are used to assess the validity of the data in the completed questionnaire.

The process of assessing the validity of the data could be accounted for through the following steps:

- Calculate the value of the score function based on the ratings of the *Overall Experience* criterion, and do so for each alternative.
- Determine the ranking order of the alternatives based on the value of the score function.
- Calculate the correlation coefficient between the ranking order obtained based on C_1-C_5 and the ranking order obtained based on the *Overall Experience* criterion.

Based on the value of the correlation coefficient, the questionnaire could be either accepted or rejected.

The determination of the overall group ratings and the ranking order of the considered alternatives. In the case of real examinations, when more than one respondent is involved in the evaluation, it is necessary to determine the overall group ratings, and based on them the final ranking order of the alternatives.

The process of determining the overall group ratings and the final ranking order of the alternatives is as follows:

- the calculation of the overall group ratings by using the SVNWA operator, based on the overall ratings;
- the calculation of the score function of the overall group rating by using Eq. (9) for each alternative, and
- the sorting of the considered alternatives based on the values of the score function and the determination of the best one. The alternative with the highest value of the score function is the best one.

5. A NUMERICAL ILLUSTRATION

In this numerical illustration, one case of selecting websites is considered. The initial set of the alternatives has been formed based on the keyword “vinarija”, which is the Serbian word for a “winery”, in the Google search engine.

The list of eight top placed websites is as follows:

- Vinarija Zvonko Bogdan - http://www.vinarijazvonkobogdan.com/
- Vinarija Coka - http://www.vinarijacoka.rs/
- Vinarija Dulka - http://www.dulka-vinarija.com/
- Vinarija Milosavljevic - http://www.vinarija-milosavljevic.com/
- Vinarija Kis - http://www.vinarijakis.com/
- Vinarija Vink - http://www.dobrovino.com/
- Vinarija Matalj - http://www.mataljvinarija.rs/
- Vinarija Aleksandrovic - http://www.vinarijaaleksandrovic.rs/

From the above, a set of five alternatives has been formed, denoted A_1 to A_5.

The survey has been conducted by email, with the aim to collect the attitudes from the respondents regarding the significance of the criteria and the ratings of the alternatives.

The interactive questionnaire made in the spreadsheet was used for attitudes gathering, so the participants had an opportunity to see the results and possibly change their own attitudes.

The attitudes obtained from the first of the three examinees are given in Table 1, which also accounts for the weights of the criteria calculated based on the examinees’ responses.

Table 1: The responses and weights of the criteria obtained from one of the evaluated respondents

Criteria		s_j	k_j	q_j	w_j
C_1	Content		1	1	0.22
C_2	Structure and Navigation	0.90	1.10	0.91	0.20
C_3	Visual Design	1.20	0.80	1.14	0.25
C_4	Interactivity	0.60	1.40	0.81	0.18
C_5	Functionality	0.90	1.10	0.74	0.16

The attitudes obtained from the three examinees, as well as the appropriate weights, are presented in Table 2 as well.

Table 2: The attitudes and weights obtained from the three examinees

	E_1		E_1		E_1	
	s_j	w_j	s_j	w_j	s_j	w_j
C_1		0.22		0.20		0.20
C_2	0.90	0.20	1.10	0.22	1.00	0.20
C_3	1.20	0.25	1.10	0.25	1.10	0.22
C_4	0.60	0.18	0.60	0.18	0.90	0.20
C_5	0.90	0.16	0.90	0.16	0.90	0.18

The following are the responses obtained from the three examinees regarding the evaluation of the websites.

Table 3: The ratings obtained from the first of the three examinees

	C_1	C_2	C_3	C_4	C_5	C_6
A_1	<1.0, 0.0, 0.0>	<1.0, 0.2, 0.0>	<1.0, 0.0, 0.0>	<0.7, 0.3, 0.0>	<0.8, 0.2, 0.2>	<0.9, 0.1, 0.1>
A_2	<1.0, 0.0, 0.0>	<1.0, 0.0, 0.0>	<1.0, 0.0, 0.0>	<0.6, 0.0, 0.2>	<1.0, 0.0, 0.0>	<0.7, 0.0, 0.0>
A_3	<0.9, 0.0, 0.0>	<0.9, 0.0, 0.0>	<0.7, 0.2, 0.3>	<0.5, 0.0, 0.0>	<0.9, 0.0, 0.0>	<0.7, 2.0, 2.0>
A_4	<0.7, 0.0, 0.3>	<0.7, 0.3, 0.3>	<0.6, 0.4, 0.2>	<0.4, 0.0, 0.0>	<0.9, 0.0, 0.0>	<0.5, 0.0, 0.2>
A_5	<1.0, 0.0, 0.0>	<1.0, 0.0, 0.0>	<1.0, 0.0, 0.0>	<0.7, 0.0, 0.2>	<1.0, 0.0, 0.0>	<0.9, 0.0, 0.2>

Table 4: The ratings obtained from the second of the three examinees

	C_1	C_2	C_3	C_4	C_5	C_6
A_1	<0.8, 0.2, 0.2>	<1.0, 0.0, 0.0>	<0.7, 0.3, 0.1>	<0.7, 0.3, 0.2>	<1.0, 0.0, 0.0>	<0.8, 0.1, 0.1>
A_2	<1.0, 0.0, 0.0>	<1.0, 0.0, 0.0>	<1.0, 0.0, 0.0>	<0.6, 0.0, 0.2>	<1.0, 0.0, 0.0>	<1.0, 0.1, 0.1>
A_3	<0.7, 0.3, 0.2>	<0.9, 0.0, 0.0>	<0.7, 0.2, 0.3>	<0.5, 0.0, 0.0>	<0.9, 0.0, 0.0>	<0.7, 0.2, 0.2>
A_4	<0.7, 0.0, 0.3>	<0.7, 0.3, 0.3>	<0.6, 0.4, 0.2>	<0.4, 0.0, 0.0>	<0.9, 0.0, 0.0>	<0.5, 0.1, 0.2>
A_5	<1.0, 0.0, 0.0>	<1.0, 0.0, 0.0>	<1.0, 0.0, 0.0>	<0.7, 0.0, 0.2>	<1.0, 0.0, 0.0>	<0.9, 0.0, 0.0>

Table 5: The ratings obtained from the third of the three examinees

	C_1	C_2	C_3	C_4	C_5	C_6
A_1	<0.9, 1.0, 1.0>	<0.9, 0.0, 0.2>	<1.0, 0.0, 1.0>	<0.7, 0.3, 0.2>	<1.0, 0.0, 0.0>	<0.9, 0.0, 0.1>
A_2	<1.0, 0.0, 0.0>	<1.0, 0.0, 0.0>	<1.0, 0.0, 0.0>	<0.6, 0.0, 0.2>	<1.0, 0.0, 0.0>	<1.0, 0.1, 0.1>
A_3	<0.6, 0.3, 0.2>	<0.9, 0.0, 0.0>	<0.5, 0.2, 0.3>	<0.5, 0.3, 0.3>	<0.9, 0.3, 0.4>	<0.7, 0.0, 0.0>
A_4	<0.6, 0.0, 0.3>	<0.5, 0.3, 0.4>	<0.4, 0.4, 0.2>	<0.4, 0.0, 0.0>	<0.9, 0.3, 0.3>	<0.7, 0.0, 0.2>
A_5	<1.0, 0.0, 0.0>	<1.0, 0.0, 0.0>	<1.0, 0.0, 0.0>	<0.7, 0.0, 0.2>	<1.0, 0.0, 0.0>	<0.9, 0.0, 0.0>

The remaining part of the evaluation process is explained on the first of the three examinees.

The overall SVNN ratings calculated by using the SVNWA, i.e. by using Eq. (11), are shown in Table 4. The ranking order obtained based on the values of the score function, calculated by using Eq. (9), is also presented in table 6.

The ranking order obtained based on the Overall Experience criterion is given in table 6, too.

Table 6: The ranking orders obtained on the basis of the ratings of the first of the three examinees

	C_1- C_5	Score	Rank	C_6	Score	Rank
A_1	<1.000, 0.006, 0.000>	0.9936	3	<0.9, 0.1, 0.1>	0.80	3
A_2	<1.000, 0.000, 0.000>	0.9997	1	<0.7, 0.0, 0.0>	0.85	2
A_3	<0.826, 0.001, 0.001>	0.9118	4	<0.7, 2.0, 2.0>	-2.15	5
A_4	<0.695, 0.004, 0.018>	0.8345	5	<0.5, 0.0, 0.2>	0.65	4
A_5	<1.000, 0.000, 0.000>	0.9997	1	<0.9, 0.0, 0.2>	0.85	1

The Pearson correlation coefficient between the two ranking orders, shown in Table 6, is 0.884, which is indicative of the fact that the data in the questionnaire are valid.

The ranking orders obtained from the three examinees obtained based on the ratings of the criteria C_1 to C_5 are shown in Table 7.

Table 7: The ranking orders obtained from the three examinees

	I		II		II	
	Score	Rank	Score	Rank	Score	Rank
A_1	0.99	3	0.98	3	0.93	3
A_2	1.00	1	1.00	1	1.00	1
A_3	0.91	4	0.88	4	0.78	4
A_4	0.83	5	0.83	5	0.75	5
A_5	1.00	1	1.00	1	1.00	1
R		0.884		0.884		0.795

The correlation coefficients are also accounted for in Table 7.

The obtained correlation coefficients indicate that there is no significant difference between the ranking orders obtained based on the criteria C_1 to C_5 and the *Overall Experience* criterion, which is indicative of the fact that the data in the selected questionnaires are valid.

CONCLUSION

Obtaining a realistic attitude by surveying could often be related to some difficulties, when the data collected in such a manner are then further used in multiple criteria decision making.

There are two opposite possibilities. The first one is using a greater number of criteria, often organized into two or more hierarchical levels. Such an approach should lead to the formation of accurate models. However, an increase in the number of criteria could lead to the creation of complex questionnaires, which could have a negative impact on the examinee's response as well as on the verisimilitude of the collected data.

Opposite to the previously said, the usage of a smaller number of criteria could have a positive impact on the collection of data, i.e. respondents' attitudes, on the one hand, but could also lead to the creation of less precise decision-making models, on the other.

The neutrosophic set, or more precisely single valued neutrosophic numbers, could be an adequate basis for collecting the examinee's attitudes by using a smaller number of criteria without losing precision.

By combining the SWARA method, in order to determine the importance of criteria, on the one hand, and Single Valued Neutrosophic Numbers, in order to acquire respondents' attitudes, on the other, effective and easy-to-use multiple criteria decision-making models can be created, as has been shown in the considered numerical illustration.

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Neutrosophic Sets: An Overview

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ABSTRACT

In this study, we give some concepts concerning the neutrosophic sets, single valued neutrosophic sets, interval-valued neutrosophic sets, bipolar neutrosophic sets, neutrosophic hesitant fuzzy sets, inter-valued neutrosophic hesitant fuzzy sets, refined neutrosophic sets, bipolar neutrosophic refined sets, multi-valued neutrosophic sets, simplified neutrosophic linguistic sets, neutrosophic over/off/under sets, rough neutrosophic sets, rough bipolar neutrosophic sets, rough neutrosophic hyper-complex set, and their basic operations. Then we introduce triangular neutrosophic numbers, trapezoidal neutrosophic fuzzy number and their basic operations. Also some comparative studies between the existing neutrosophic sets and neutrosophic number are provided.

KEYWORDS: **Neutrosophic sets (NSs), Single valued neutrosophic sets (SVNSs), Interval-valued neutrosophic sets (IVNSs), Bipolar neutrosophic sets (BNSs), Neutrosophic hesitant fuzzy sets (NHFSs), Interval valued neutrosophic hesitant fuzzy sets (IVNHFSs), Refined neutrosophic sets (RNSs), Bipolar neutrosophic refined sets (BNRSs), Multi-valued neutrosophic sets (MVNSs), Simplified neutrosophic linguistic sets, Neutrosophic numbers, Neutrosophic over/off/under sets, Rough neutrosophic sets, Bipolar rough neutrosophic sets, Rough neutrosophic sets, Bipolar rough neutrosophic sets, Rough neutrosophic hyper-complex set**

1. INTRODUCTION

The concept of fuzzy sets was introduced by L. Zadeh (1965). Since then the fuzzy sets and fuzzy logic are used widely in many applications involving uncertainty. But it is observed that there still remain some situations which cannot be covered by fuzzy sets and so the concept of interval valued fuzzy sets (Zadeh, 1975) came into force to capture those situations, Although Fuzzy set theory is very successful in handling uncertainties arising from vagueness or partial belongingness of an element in a set, it cannot model all sorts of uncertainties prevailing in different real physical problems such as problems involving incomplete information. Further generalization of the fuzzy set was made by Atanassov (1986), which is known as intuitionistic fuzzy sets (IFs). In IFS, instead of one membership grade, there is also a non-membership grade attached with each element. Further there is a restriction that the sum of these two grades is less or equal to unity. The conception of IFS can be viewed as an appropriate/ alternative approach in case where available information is not sufficient to define the impreciseness by the conventional fuzzy sets. Later on intuitionistic fuzzy sets were extended to interval valued intuitionistic fuzzy sets (Atanassov & Gargov, 1989). Neutrosophic sets (NSs) proposed by (Smarandache, 1998, 1999, 2002, 2005, 2006, 2010) which is a generalization of fuzzy sets and intuitionistic fuzzy set, is a powerful tool to deal with incomplete, indeterminate and inconsistent information which exist in the real world. Neutrosophic sets are characterized by truth membership function (T), indeterminacy membership function (I) and falsity membership function (F). This theory is very important in many application areas since indeterminacy is quantified explicitly and the truth membership function, indeterminacy membership function and falsity membership functions are independent. Wang, Smarandache, Zhang, & Sunderraman (2010) introduced the concept of single valued neutrosophic set. The single-valued neutrosophic set can independently express truth-membership degree, indeterminacy-membership degree and falsity-membership degree and deals with incomplete, indeterminate and inconsistent information. All the factors described by the single-valued neutrosophic set are very suitable for human thinking due to the imperfection of knowledge that human receives or observes from the external world.

Single valued neutrosophic set has been developing rapidly due to its wide range of theoretical elegance and application areas; see for examples (Sodenkamp, 2013; Kharal, 2014; Broumi & Smarandache, 2014; Broumi & Smarandache, 2013; Hai-Long, Zhi-Lian, Yanhong, & Xiuwu, 2016; Biswas, Pramanik, & Giri, 2016a, 2016b, 2016c; 2017; Ye, 2014a, 2014b, 2014c, 2015a, 2016).

Wang, Smarandache, Zhang, & Sunderraman (2005) proposed the concept of interval neutrosophic set (INS) which is an extension of neutrosophic set. The interval neutrosophic set (INS) can represent uncertain, imprecise, incomplete and inconsistent information which exists in real world.

Single valued neutrosophic number is an extension of fuzzy numbers and intuitionistic fuzzy numbers. Single valued fuzzy number is a special case of single valued neutrosophic set and is of importance for decision making problems. Ye (2015b) and Biswas, Pramanik, and Giri (2014) studied the concept of trapezoidal neutrosophic fuzzy number as a generalized representation of trapezoidal fuzzy numbers, trapezoidal intuitionistic fuzzy numbers, triangular fuzzy numbers and triangular intuitionistic fuzzy numbers and applied them for dealing with multi-attribute decision making (MADM) problems. Deli & Subas (2017) and Biswas et al. (2016b) studied the ranking of single valued neutrosophic trapezoidal numbers and applied the concept to solve MADM problems. Liang, Wang, & Zhang (2017) presented a multi-criteria decision-making method based on single-valued trapezoidal neutrosophic preference relations with complete weight information.

Ye (2014b) proposed the concept of single valued neutrosophic hesitant fuzzy sets (SVNHFS). As a combination of hesitant fuzzy sets (HFS) and singled valued neutrosophic sets (SVNs), the single valued neutrosophic hesitant fuzzy set (SVNHF) is an important concept to handle uncertainty and vague information existing in real life which consists of three membership functions and encompass the fuzzy set (FS), intuitionistic fuzzy sets (IFS), hesitant fuzzy set (HFs), dual hesitant fuzzy set (DHFs) and single valued neutrosophic set (SVNS). Theoretical development and applications of such concepts can be found in (Wang & Li, 2016; Ye, 2016). Peng, Wang, Wu, Wang, & Chen, 2014; Peng &Wang, 2015) introduced the concept of multi-valued neutrosophic set as a new branch of NSs which is the same concept of neutrosophic hesitant fuzzy set. Multi-valued neutrosophic sets can be applied in addressing problems with uncertain, imprecise, incomplete and inconsistent information existing in real scientific and engineering applications.

Tian, Wang, Zhang, Chen, & Wang (2016) defined the concept of simplified neutrosophic linguistic sets which combine the concept of simplified neutrosophic sets and linguistic term sets. Simplified neutrosophic linguistic sets have enabled great progress in describing linguistic information to some extent. It may be considered to be an innovative construct.

Deli, Ali, and Smarandache (2015a) defined the concept of bipolar neutrosophic set and its score, certainty and accuracy functions. In the same study, Deli et al. (2015a) proposed the A_w and G_w operators to aggregate the bipolar neutrosophic information. Furthermore, based on the A_w and G_w operators and the score, certainty and accuracy functions, Deli et al. (2015a) developed a bipolar neutrosophic multiple criteria decision-making approach, in which the evaluation values of alternatives on the attributes assume the form of bipolar neutrosophic numbers. Some theoretical and applications using bipolar neutrosophic sets are studied by several authors (Uluçay, Deli, & Şahin,2016; Dey, Pramanik, & Giri, 2016a; Pramanik, Dey, Giri, & Smarandache, 2017;).

Maji (2013) defined neutrosophic soft set. The development of decision making algorithms using neutrosophic soft set theory has been reported in the literature (Deli & Broumi, 2015; Dey, Pramanik, & Giri, 2015, 2016b, 2016c; Pramanik & Dalapati (2016), Das, Kumar, Kar, & Pal, 2017).

Broumi, Smarandache, and Dhar (2014a, 2014b) defined rough neutrosophic set and proved its basic properties. Some theoretical advancement and applications have been reported in the literature (Mondal & Pramanik, 2014, 2015a, 2015b, 2015c, 2015d, 2015e, 2015f, 2015g, 2015h); Mondal, Pramanik, and Smarandache (2016a, 2016b, 2016c, 2016d); Pramanik & Mondal (2015a, 2015b, 2015c); Pramanik, Roy, Roy, &Smarandache (2017); Pramanik, Roy, & Roy (2017).

Ali, Deli, and Smarandache (2016) and Jun, Smarandache, and Kim (2017) proposed neutrosophic cubic set by extending the concept of cubic set. Some studies in neutrosophic cubic set environment have been reported in the literature (Banerjee, Giri, Pramanik, & Smarandache (2017); Pramanik, Dey, Giri, & Smarandache (2017b); Pramanik, Dalapati, Alam, & Roy (2017a, 2017b); Pramanik, Dalapati, Alam, Roy & Smarandache (2017); Ye (2017); Lu & Ye (2017).

Another extension of neutrosophic set namely, neutrosophic refined set and its appilication was studied by several researchers (Deli, Broumi, & Smarandache, 2015b; Broumi & Smarandache, 2014b; Broumi,& Deli,2014;. Uluçay, Deli, & Şahin, 2016, Pramanik, S., Banerjee, D., & Giri, 2016a, 2016b; Mondal & Pramanik, 2015h, 2015i.; Ye & Smarandache, 2016., Chen, Ye, & Du, 2017).

Later on, several extensions of neutrosophic set have been proposed in the literature by researchers to deal with different type of problems such as bipolar neutrosophic refined sets (Deli & Şubaş, 2016), tri-complex rough neutrosophic set (Mondal & Pramanik, 2015g), rough neutrosophic hyper-complex set (Mondal, Pramanik & Smarandache, 2016d), rough bipolar neutrosophic set.(Pramanik a& Mondal, 2016) simplified neutrosophic linguistic sets (SNLS) (Tian, Wang, Zhang, Chen, & Wang, 2016), quadripartitioned single valued neutrosophic sets (Chatterjee, Majumdar, Samanta, 2016). Smarandache (2016a. 2016b) proposed new version of neutrosophic sets such as neutrosophic off/under/over sets. To have a glimpse of new trends of neutrosophic theory and applications, readers can see the latest editorial book (Smarandache & Pramanik, 2016). Interested readers can find a variety of applications of single valued neutrosophic sets and their hybrid extensions in the website of the Journal "Neutrosophic Sets and Systems" namely, http://fs.gallup.unm.edu/nss.

BASIC AND FUNDAMENTAL CONCEPTS

2.1. Neutrosophic sets (Smarandache, 1998)

Let ξ be the universe. A neutrosophic set (NS) A in ξ is characterized by a truth membership function T_A, an indeterminacy membership function I_A and a falsity membership function F_A where T_A, I_A and F_A are real standard elements of [0,1]. It can be written as

$A=\{<x,(T_A(x),I_A(x),F_A(x))>: x\in E, T_A, I_A, F_A \in \left]^-0,1^+\right[\}$

There is no restriction on the sum of T_A(x), I_A (x)and F_A(x) and so $0^- \le T_A(x)+I_A(x)+F_A(x)) \le 3^+$

2.2 Single valued neutrosophic sets (Wang et al., 2010)

Let X be a space of points (objects) with generic elements in ξ denoted by x. A single valued neutrosophic set A (SVNS) is characterized by truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$. For each point x in ξ, $T_A(x)$, $I_A(x)$, $F_A(x)$ $\in$[0, 1]. A SVNS A can be written as

$$A = \{< x: T_A(x), I_A(x), F_A(x)>, x \in \xi\}$$

2.3 Interval valued neutrosophic sets (Wang et al., 2005)

Let ξ be a space of points (objects) with generic elements in X denoted by x. An interval valued neutrosophic set A (IVNS A) is characterized by an interval truth-membership function $T_A(x)=\left[T_A^L, T_A^U\right]$, an interval indeterminacy-membership function $I_A(x)=\left[I_A^L, I_A^U\right]$, and an interval falsity-membership function $F_A(x)=\left[F_A^L, F_A^U\right]$. For each point $x\in X$ $T_A(x)$, $I_A(x)$, $F_A(x) \subset$[0, 1]. An IVNS A can be written as

$$A = \{< x: T_A(x), I_A(x), F_A(x)>, x \in \xi\}$$

Numerical Example: Assume that $X=\{x_1,x_2,x_3\}$, x_1 is capability, x_2 trustworthiness, x_3 price. The values of x_1, x_2 and x_3 are in [0,1].They are obtained from questionnaire of some domain experts and the result can be obtained as the degree of good, degree of indeterminacy and the degree of poor. Then an interval neutrosophic set can be obtained as

$$A=\begin{cases} <x_1,[0.5,0.3],[0.1,0.6],[0.4,0.2]>, \\ <x_2,[0.3,0.2],[0.4,0.3],[0.4,0.5]>, \\ <x_3,[0.6,0.3],[0.4,0.1],[0.5,0.4]> \end{cases}$$

2.3 Bipolar neutrosophic sets (Deli et al., 2015)

A bipolar neutrosophic set A in ξ is defined as an object of the form

A={<x, $T^p(x)$, $I^p(x)$, $F^p(x)$, $T^n(x)$, $I^n(x)$, $F^n(x)$>: x $\in\xi$}, where T^p, I^p, $F^p: \xi \to$ [1, 0] and T^n, I^n, $F^n: \xi \to$ [-1, 0] .The positive membership degree $T^p(x)$, $I^p(x)$, $F^p(x)$ denote the truth membership, indeterminate membership and false membership of an element $\in\xi$ corresponding to a bipolar neutrosophic set A and the negative membership degree $T^n(x)$, $I^n(x)$, $F^n(x)$ denotes the truth membership, indeterminate membership and false membership of an element $\in\xi$ to some implicit counter-property corresponding to a bipolar neutrosophic set A.

An empty bipolar neutrosophic set $\tilde{A}_1 =< T_1^p, I_1^p, F_1^p, T_1^n, I_1^n, F_1^n >$ is defined as $T_1^p = 0, I_1^p = 0, F_1^p = 1$ and $T_1^n = -1, I_1^n = 0, F_1^n = 0$.

Numerical Example: Let $X = \{x_1, x_2, x_3\}$ then

$$A = \begin{cases} < x_1, 0.5, 0.3, 0.1, -0.6, -0.4, -0.01 >, \\ < x_2, 0.3, 0.2, 0.4, -0.03, -0.004, -0.05 >, \\ < x_3, 0.6, 0.5, 0.4, -0.1, -0.5, -0.004 > \end{cases}$$

is a bipolar neutrosophic number.

2.4 Neutrosophich hesitant fuzzy set (Ye, 2014)

Let ξ be a non-empty fixed set, a neutrosophic hesitant fuzzy set (NHFS) on X is expressed by: $N = \{\langle x, \tilde{t}(x), \tilde{\imath}(x), \tilde{f}(x)\rangle | x \in \xi\}$, where $\tilde{t}(x) = \{\tilde{\gamma} | \tilde{\gamma} \in \tilde{t}(x)\}, \tilde{\imath}(x) = \{\tilde{\delta} | \tilde{\delta} \in \tilde{\imath}(x)\}$ and $\tilde{f}(x) = \{\tilde{\vartheta} | \tilde{\vartheta} \in \tilde{f}(x)\}$ are three sets with some values in interval [0,1], which represents the possible truth-membership hesitant degrees,indeterminacy-membership hesitant degrees, and falsity-membership hesitant degrees of the element $x \in \xi$ to the set N, and satisfies these limits :

$$\tilde{\gamma} \in [0,1], \tilde{\delta} \in [0,1], \tilde{\vartheta} \in [0,1] \text{and } 0 \leq sup\tilde{\gamma}^+ + sup\tilde{\delta}^+ + sup\tilde{\vartheta}^+ \leq 3$$

where $\tilde{\gamma}^+ = \bigcup_{\tilde{\gamma} \in \tilde{t}(x)} max\{\tilde{\gamma}\}, \tilde{\delta}^+ = \bigcup_{\tilde{\delta} \in \tilde{\imath}(x)} max\{\tilde{\delta}\}$ and $\tilde{\vartheta}^+ = \bigcup_{\tilde{\vartheta} \in \tilde{\imath}(x)} max\{\tilde{\vartheta}\}$ for $x \in X$.

The $\tilde{n} = \{\tilde{t}(x), \tilde{\imath}(x), \tilde{f}(x)\}$ is called a neutrosophic hesitant fuzzy element (NHFE) which is the

basic unit of the NHFS and is denoted by the symbol $\tilde{n} = \{\tilde{t}, \tilde{\imath}, \tilde{f}\}$.

2.5 Interval neutrosophic hesitant fuzzy set (Ye, 2016)

Let ξ be a fixed set, an INHFS on ξ is defined as

$N = \{\langle x, \tilde{t}(x), \tilde{\imath}(x), \tilde{f}(x)\rangle | x \in \xi\}$.

Here $\tilde{t}(x), \tilde{\imath}(x)$ and $\tilde{f}(x)$ are sets of some different interval values in [0, 1], representing the possible truth-membership hesitant degrees, indeterminacy-membership hesitant degrees, and falsity-membership hesitant degrees of the element $x \in \xi$ to the set N, respectively. Then $\tilde{t}(x)$ reads $\tilde{t}(x) = \{\tilde{\gamma} | \tilde{\gamma} \in \tilde{t}(x)\}, where\ \tilde{\gamma} = [\tilde{\gamma}^L, \tilde{\gamma}^U]$ is an interval number, $\tilde{\gamma}^L = inf\tilde{\gamma}$ and $\tilde{\gamma}^U = sup\tilde{\gamma}$ represent the lower and upper limits of $\tilde{\gamma}$, respectively; $\tilde{\imath}(x)$ reads $\tilde{\imath}(x) = \{\tilde{\delta} | \tilde{\delta} \in \tilde{t}(x)\}, where \tilde{\delta} = [\tilde{\delta}^L, \tilde{\delta}^U]$ is an interval number, $\tilde{\delta}^L = inf\tilde{\delta}$ and $\tilde{\delta}^U = sup\tilde{\delta}$ represent the lower and upper limits of $\tilde{\delta}$, respectively; $\tilde{f}(x)$ reads $\tilde{f}(x) = \{\tilde{\vartheta} | \tilde{\vartheta} \in \tilde{t}(x)\}, where \tilde{\vartheta} = [\tilde{\vartheta}^L, \tilde{\vartheta}^U]$ is an interval number, $\tilde{\vartheta}^L = inf\tilde{\vartheta}$ and $\tilde{\vartheta}^U = sup\tilde{\vartheta}$ represent the lower and upper limits of $\tilde{\vartheta}$, respectively. Hence, there is the condition

$$0 \leq sup\tilde{\gamma}^+ + sup\tilde{\delta}^+ + sup\tilde{\vartheta}^+ \leq 3$$

where $\tilde{\gamma}^+ = \bigcup_{\tilde{\gamma} \in \tilde{t}(x)} max\{\tilde{\gamma}\}, \tilde{\delta}^+ = \bigcup_{\tilde{\delta} \in \tilde{\imath}(x)} max\{\tilde{\delta}\}$ and $\tilde{\vartheta}^+ = \bigcup_{\tilde{\vartheta} \in \tilde{\imath}(x)} max\{\tilde{\vartheta}\}$ for $x \in X$.

For convenience, $\tilde{n} = \{\tilde{t}(x), \tilde{\imath}(x), \tilde{f}(x)\}$ is called an interval neutrosophic hesitant fuzzy element (INHFE), which is denoted by the simplified symbol $\tilde{n} = \{\tilde{t}, \tilde{\imath}, \tilde{f}\}$.

2.6 Multi-valued neutrosophic sets (Wang & Li, 2015; Peng & Wang, 2015)

Let X be a space of points (objects) with generic elements in X denoted by x, then multi-valued neutrosophic sets A in X is characterized by a truth-membership function $\tilde{T}_A(x)$, a indeterminacy-membership function $\tilde{I}_A(x)$, and a falsity-membership function $\tilde{F}_A(x)$. Multi-valued neutrosophic sets can be defined as the following form:

$$A = \{\langle x, \tilde{T}_A(x), \tilde{I}_A(x), \tilde{F}_A(x)\rangle | x \in X\},$$

where $\tilde{T}_A(x) \in [0,1]$, $\tilde{I}_A(x) \in [0,1]$, $\tilde{F}_A(x) \in [0,1]$, are sets of finite discrete values, and satisfies the condition $0 \le \gamma, \eta, \xi \le 1$, $0 \le \gamma^+ + \eta^+ + \xi^+ \le 3$, $\gamma \in \tilde{T}_A(x)$, $\eta \in \tilde{I}_A(x)$, $\xi \in \tilde{F}_A(x)$, $\gamma^+ = \sup \tilde{T}_A(x)$, $\eta^+ = \sup \tilde{I}_A(x)$, $\xi^+ = \sup \tilde{F}_A(x)$. For the sake of simplicity, $A = \langle \tilde{T}_A, \tilde{I}_A, \tilde{F}_A \rangle$ is called as multi-valued neutrosophic number.

If $\tilde{T}_A(x)$, $\tilde{I}_A(x)$, $\tilde{F}_A(x)$ has only one value, the multi-valued neutrosophic sets is single valued neutrosophic sets. If $\tilde{T}_A(x) = \varnothing$, the multi-valued neutrosophic sets is double hesitant fuzzy sets. If $\tilde{T}_A(x) = \tilde{F}_A(x) = \varnothing$, the multi-valued neutrosophic sets is hesitant fuzzy sets.

Numerical example: Investment company have four options (to invest): the car company, the food company, the computer company, and the arms company, and it considers three criteria: the risk control capability, the growth potential, and the environmental impact. Then the decision matrix based on the multi-valued neutrosophic numbers is R.

$$R = \begin{bmatrix} \langle \{0.4,0.5\},\{0.2\},\{0.3\}\rangle & \langle \{0.4\},\{0.2,0.3\},\{0.3\}\rangle & \langle \{0.2\},\{0.2\},\{0.5\}\rangle \\ \langle \{0.6\},\{0.1,0.2\},\{0.2\}\rangle & \langle \{0.6\},\{0.1\},\{0.2\}\rangle & \langle \{0.5\},\{0.2\},\{0.1,0.2\}\rangle \\ \langle \{0.3,0.4\},\{0.2\},\{0.3\}\rangle & \langle \{0.5\},\{0.2\},\{0.3\}\rangle & \langle \{0.5\},\{0.2,0.3\},\{0.2\}\rangle \\ \langle \{0.7\},\{0.1,0.2\},\{0.1\}\rangle & \langle \{0.6\},\{0.2\},\{0.3\}\rangle & \langle \{0.4\},\{0.3\},\{0.2\}\rangle \end{bmatrix}.$$

2.7 Neutrosophic overset/ underset/offset (Smarandache, 2016a)

2.7.1. Definition of neutrosophic overset: Let ξ be a universe of discourse and the neutrosophic set $A \subset \xi$. Let $T_A(x)$, $I_A(x)$, $F_A(x)$ be the functions that describe the degree of membership, indterminate membership and non-membership respectively of a generic element $x \in \xi$ with respect to the neutrosophic set A. A neutrosophic overset (NOVs) A on the universe of discourse ξ is defined as:

$$A=\{(x, T_A(x), I_A(x), F_A(x)), x \in \xi \text{ and } T(x), I(x), F(x) \in [0, \Omega]\}, \text{ where}$$

$T(x), I(x), F(x)$: $\xi \to [0, \Omega]$, $0 < 1 < \Omega$ and Ω is called over limit. Then there exist at least one element in A such that it has at least one neutrosophic component > 1, and no element has neutrosophic component < 0.

2.7.2 Definition of neutrosophic underset: Let ξ be a universe of discourse and the neutrosophic set $A \subset \xi$. Let $T_A(x)$, $I_A(x)$, $F_A(x)$ be the functions that describe the degree of membership, indterminate membership and non-membership respectively of a generic element $x \in \xi$ with respect to the neutrosophic set A. A neutrosophic under set (NUs) A on the universe of discourse ξ is defined as:

$$A=\{(x, T_A(x), I_A(x), F_A(x)), x \in \xi \text{ and } T(x), I(x), F(x) \in [\Psi, 1\,]\}$$

Where

$T(x), I(x), F(x)\colon \xi \rightarrow [\Psi, 1\,]$, $\Psi<0<1$ and Ψ is called lowerlimit.Then there exist at least one element in A such that it has at least one neutrosophic component < 0, and no element has neutrosophic component > 1.

2.7.3 Definition of neutrosophic offset: Let ξ be a universe of discourse and the neutrosophic set $A \subset \xi$. Let $T_A(x)$, $I_A(x)$, $F_A(x)$ be the functions that describe the degree of membership, indterminate membership and non-membership respectively of a generic element $x \in \xi$ with respect to the neutrosophic set A. A neutrosophic offset (NOFFs) A on the universe of discourse ξ is defined as:

$$A=\{(x, T_A(x), I_A(x), F_A(x)), x \in \xi \text{ and } T(x), I(x), F(x) \in [\Psi, \Omega\,]\}, \text{ where}$$

$T(x), I(x), F(x)\colon \xi \rightarrow [\Psi, 1\,]$, $\Psi <0< 1< \Omega$ and Ψ is called underlimit while Ω is called overlimit. Then there existe some elments in A such that at least one neutrosophic component > 1, and at least another neutrosophic component < 0.

Numerical example: A={(x_1,<1.2, 0.4,0.1>),(x_2,<0.2, 0.3,-0.7>)}, since $T(x_1)$ = 1.2 >1 , $F(x_2)$ = - 0.7 < 0.

2.7.4 Some operations of neutrosophic over/off/under sets

Definition 1: The complement of a neutrosophic overset/ underset/offset A is denoted by C(A) and is defined by

$$C(A) =\{(x,<F_A(x), \Psi + \Omega\text{-}I_A(x), T_A(x)), x\in \xi\}.$$

Definition 2: The intersection of two neutrosophic overset/ underset/offset A and B is a neutrosophic overset/ underset/offset denoted C and is denoted by

C= A∩ B and is defined by

$$C= A\cap B =\{(x,<\min(T_A(x), T_B(x)), \max(I_A(x), I_B(x)), \max(F_A(x), F_B(x))),x\in \xi\}.$$

Definition 3: The union of two overset/ underset/offset A and B is a neutrosophic overset/ underset/offset denoted C and is denoted by

C= A∪ B and is defined by

C= A∪ B ={(x,<max($T_A(x), T_B(x)$), min($I_A(x), I_B(x)$), min($F_A(x), F_B(x)$)), x∈ ξ}.

Let ξ be a universe of discourse and A the neutrosophic set A⊂U. Let $T_A(x)$, $I_A(x)$,$F_A(x)$ be the functions that describe the degree of membership, indterminate membership and non-membership respectively of a generic element x ∈ $\xi$with respect to the neutrosophic set A. A neutrosophic overset (NOV) A on the universe of discourse U is defined as:

$$A=\{(x, T_A(x), I_A(x), F_A(x)), x \in \xi \text{ and } T(x), I(x), F(x) \in [0, \Omega\,]\}, \text{ where}$$

$T(x), I(x), F(x)$: ξU→ $[0, \Omega\,]$, $0 < 1 < \Omega$ and Ω is called overlimit.Then there exist at least one element in A such that it has at least one neutrosophic component >1, and no element has neutrosophic component <0.

2. OPERATIONS ON SOME NEUTROSOPHIC NUMBERS AND NEUTROSOPHIC SETS

2.1 Single valued neutrosophic number

Let $\tilde{A}_1 = (T_1, I_1, F_1)$ and $\tilde{A}_2 = (T_2, I_2, F_2)$ be two single valued neutrosophic number. Then, the operations for SVNNs are defined as below;

i. $\tilde{A}_1 \oplus \tilde{A}_2 =< T_1 + T_2 - T_1T_2, I_1I_2, F_1F_2 >$

ii. $\tilde{A}_1 \otimes \tilde{A}_2 =< T_1T_2, I_1 + I_2 - I_1I_2, F_1 + F_2 - F_1F_2) >$

iii. $\lambda\tilde{A}_1 =< 1-(1-T_1)^{\lambda}), I_1^{\lambda}, F_1^{\lambda}) >$

iv. $\tilde{A}_1^{\lambda} = (T_1^{\lambda}, 1-(1-I_1)^{\lambda}, 1-(1-F_1)^{\lambda})$ where $\lambda > 0$

It is to be noted here that 0_n may be defined as follow:

$0_n = \{< x, (0,1,1) >: x \in X\}$.

2.2 **Neutrosophic hesitant fuzzy set** (Ye, 2014)

For two NHFEs $\tilde{n}_1 = \{\tilde{t}_1, \tilde{\imath}_1, \tilde{f}_1\}$, $\tilde{n}_2 = \{\tilde{t}_2, \tilde{\imath}_2, \tilde{f}_2\}$ and a positivescale > 0 , the operations operations can be defined as follows:

(1) $\tilde{n}_1 \oplus \tilde{n}_2 = \{\tilde{t}_1 \oplus \tilde{t}_2, \tilde{\imath}_1 \otimes \tilde{\imath}_2, \tilde{f}_1 \otimes \tilde{f}_2\} = \cup_{\tilde{\gamma}_1\in\tilde{t}_1,\tilde{\delta}_1\in\tilde{\imath}_1,\tilde{\vartheta}_1\in\tilde{f}_1,\tilde{\gamma}_2\in\tilde{t}_2,\tilde{\delta}_2\in\tilde{\imath}_2,\tilde{\vartheta}_2\in\tilde{f}_2}\{\tilde{\gamma}_1 + \tilde{\gamma}_2 - \tilde{\gamma}_1.\tilde{\gamma}_2, \tilde{\delta}_1.\tilde{\delta}_2, \tilde{\vartheta}_1.\tilde{\vartheta}_2\}$

(2) $\tilde{n}_1 \otimes \tilde{n}_2 = \{\tilde{t}_1 \otimes \tilde{t}_2, \tilde{\imath}_1 \oplus \tilde{\imath}_2, \tilde{f}_1 \oplus \tilde{f}_2\} = \cup_{\tilde{\gamma}_1\in\tilde{t}_1,\tilde{\delta}_1\in\tilde{\imath}_1,\tilde{\vartheta}_1\in\tilde{f}_1,\tilde{\gamma}_2\in\tilde{t}_2,\tilde{\delta}_2\in\tilde{\imath}_2,\tilde{\vartheta}_2\in\tilde{f}_2}\{\tilde{\gamma}_1.\tilde{\gamma}_2-, \tilde{\delta}_1 + \tilde{\delta}_2 - \tilde{\delta}_1.\tilde{\delta}_2, \tilde{\vartheta}_1 + \tilde{\vartheta}_2 - \tilde{\vartheta}_1.\tilde{\vartheta}_2\}$

(3) $k\tilde{n}_1 = \cup_{\tilde{\gamma}_1\in\tilde{t}_1,\tilde{\delta}_1\in\tilde{\imath}_1,\tilde{\vartheta}_1\in\tilde{f}_1}\left\{1-(1-\tilde{\gamma}_1)^k, \tilde{\delta}_1{}^k, \tilde{\vartheta}_1{}^k\right\}$

(4) $\tilde{n}_1{}^k = \cup_{\tilde{\gamma}_1\in\tilde{t}_1,\tilde{\delta}_1\in\tilde{\imath}_1,\tilde{\vartheta}_1\in\tilde{f}_1}\left\{\tilde{\gamma}_1{}^k, 1-\left(1-\tilde{\delta}_1\right)^k, 1-\left(1-\tilde{\vartheta}_1\right)^k\right\}$.

2.3 Interval neutrosophic hesitant fuzzy set [Ye, 2016]

For two INHFEs$\tilde{n}_1 = \{\tilde{t}_1, \tilde{\imath}_1, \tilde{f}_1\}$, $\tilde{n}_2 = \{\tilde{t}_2, \tilde{\imath}_2, \tilde{f}_2\}$ and a positive scale > 0 , the follwing operations can be given as follows:

(1) $\tilde{n}_1 \oplus \tilde{n}_2 = \{\tilde{t}_1 \oplus \tilde{t}_2, \tilde{\imath}_1 \otimes \tilde{\imath}_2, \tilde{f}_1 \otimes \tilde{f}_2\} = \cup_{\tilde{\gamma}_1 \in \tilde{t}_1, \tilde{\delta}_1 \in \tilde{\imath}_1, \tilde{\vartheta}_1 \in \tilde{f}_1, \tilde{\gamma}_2 \in \tilde{t}_2, \tilde{\delta}_2 \in \tilde{\imath}_2, \tilde{\vartheta}_2 \in \tilde{f}_2} \{[\tilde{\gamma}_1{}^L + \tilde{\gamma}_2{}^L - \tilde{\gamma}_1{}^L.\tilde{\gamma}_2{}^L, \tilde{\gamma}_1{}^U + \tilde{\gamma}_2{}^U - \tilde{\gamma}_1{}^U.\tilde{\gamma}_2{}^U], [\tilde{\delta}_1{}^L.\tilde{\delta}_2{}^L, \tilde{\delta}_1{}^U.\tilde{\delta}_2{}^U], [\tilde{\vartheta}_1{}^L.\tilde{\vartheta}_2{}^L, \tilde{\vartheta}_1{}^U.\tilde{\vartheta}_2{}^U]\}$

(2) $\tilde{n}_1 \otimes \tilde{n}_2 = \{\tilde{t}_1 \otimes \tilde{t}_2, \tilde{\imath}_1 \oplus \tilde{\imath}_2, \tilde{f}_1 \oplus \tilde{f}_2\} = \cup_{\tilde{\gamma}_1 \in \tilde{t}_1, \tilde{\delta}_1 \in \tilde{\imath}_1, \tilde{\vartheta}_1 \in \tilde{f}_1, \tilde{\gamma}_2 \in \tilde{t}_2, \tilde{\delta}_2 \in \tilde{\imath}_2, \tilde{\vartheta}_2 \in \tilde{f}_2} \{[\tilde{\gamma}_1{}^L.\tilde{\gamma}_2{}^L, \tilde{\gamma}_1{}^U.\tilde{\gamma}_2{}^U] -, [\tilde{\delta}_1{}^L + \tilde{\delta}_2{}^L - \tilde{\delta}_1{}^L.\tilde{\delta}_2{}^L, \tilde{\delta}_1{}^U + \tilde{\delta}_2{}^U - \tilde{\delta}_1{}^U.\tilde{\delta}_2{}^U], [\tilde{\vartheta}_1{}^L + \tilde{\vartheta}_2{}^L - \tilde{\vartheta}_1{}^L.\tilde{\vartheta}_2{}^L, \tilde{\vartheta}_1{}^U + \tilde{\vartheta}_2{}^U - \tilde{\vartheta}_1{}^U.\tilde{\vartheta}_2{}^U]\}$

(3) $k\tilde{n}_1 = \cup_{\tilde{\gamma}_1 \in \tilde{t}_1, \tilde{\delta}_1 \in \tilde{\imath}_1, \tilde{\vartheta}_1 \in \tilde{f}_1} \{[1 - (1 - \tilde{\gamma}_1{}^L)^k, 1 - (1 - \tilde{\gamma}_1{}^U)^k], [(\tilde{\delta}_1{}^L)^k, (\tilde{\delta}_1{}^U)^k], [(\tilde{\vartheta}_1{}^L)^k, (\tilde{\vartheta}_1{}^U)^k]\}$

(4) $\tilde{n}_1{}^k = \cup_{\tilde{\gamma}_1 \in \tilde{t}_1, \tilde{\delta}_1 \in \tilde{\imath}_1, \tilde{\vartheta}_1 \in \tilde{f}_1} \{[(\tilde{\gamma}_1{}^L)^k, (\tilde{\gamma}_1{}^U)^k], [1 - (1 - \tilde{\delta}_1{}^L)^k, 1 - (1 - \tilde{\delta}_1{}^U)^k], [1 - (1 - \tilde{\vartheta}_1{}^L)^k, 1 - (1 - \tilde{\vartheta}_1{}^U)^k]\}$.

4. SCORE FUNCTION, ACCURACY FUNCTION AND CERTAINTY FUNCTION OF NEUTROSOPHIC NUMBERS

A convenient method for comparing of single valued neutrosophic number is described as follows:

Let $\tilde{A}_1 = (T_1, I_1, F_1)$ be a single valued neutrosophic number. Then, the score function $s(\tilde{A}_1)$, accuracy function $a(\tilde{A}_1)$ and certainty function $c(\tilde{A}_1)$ of a SVNN are defined as follows:

(i) $s(\tilde{A}_1) = \dfrac{2 + T_1 - I_1 - F_1}{3}$

(ii) $a(\tilde{A}_1) = T_1 - F_1$

(iii) $c(\tilde{A}_1) = T_1$.

5. RANKING OF NEUTROSOPHIC NUMBERS

Suppose that $\tilde{A}_1 = (T_1, I_1, F_1)$ and $\tilde{A}_2 = (T_2, I_2, F_2)$ are two single valued neutrosophic numbers. Then, the ranking method is defiend as follows:

i. If $s(\tilde{A}_1) \succ s(\tilde{A}_2)$, then $\tilde{A}_1$ is greater than $\tilde{A}_2$, that is, $\tilde{A}_1$ is superior to $\tilde{A}_2$, denoted by $\tilde{A}_1 \succ \tilde{A}_2$

ii. If $s(\tilde{A}_1) = s(\tilde{A}_2)$,and $a(\tilde{A}_1) \succ a(\tilde{A}_2)$ then $\tilde{A}_1$ is greater than $\tilde{A}_2$, that is, $\tilde{A}_1$ is superior to $\tilde{A}_2$, denoted by $\tilde{A}_1 \succ \tilde{A}_2$

iii. If $s(\tilde{A}_1) = s(\tilde{A}_2)$, $a(\tilde{A}_1) = a(\tilde{A}_2)$, and $c(\tilde{A}_1) \succ c(\tilde{A}_2)$ then $\tilde{A}_1$ is greater than $\tilde{A}_2$, that is, $\tilde{A}_1$ is superior to $\tilde{A}_2$, denoted by $\tilde{A}_1 \succ \tilde{A}_2$

iv. If $s(\tilde{A}_1) = s(\tilde{A}_2)$, $a(\tilde{A}_1) = a(\tilde{A}_2)$, and $c(\tilde{A}_1) = c(\tilde{A}_2)$ then $\tilde{A}_1$ is equal to $\tilde{A}_2$, that is, $\tilde{A}_1$ is indifferent to $\tilde{A}_2$, denoted by $\tilde{A}_1 = \tilde{A}_2$

6. DIFFERENT TYPES OF NEUTROSOPHIC NUMBERS AND RELATED TERMS ASSOCIATED WITH THEM

6.1 Single valued-triangular neutrosophic numbers (Ye 2015b)

A single valued triangular neutrosophic number (SVTrN-number) $\tilde{a} = <(a_1, b_1, c_1); T_a, I_a, F_a>$ is a special neutrosophic set on the real number set *R*, whose truth membership, indeterminacy-membership, and a falsity-membership are given as follows:

$$T_a(x) = \begin{cases} (x - a_1)T_a \big/ (b_1 - a_1) & (a_1 \le x \le b_1) \\ T_a & (x = b_1) \\ (c_1 - x)T_a \big/ (c_1 - b_1) & (b_1 \le x \le c_1) \\ 0 \quad otherwise \end{cases}$$

$$I_a(x) = \begin{cases} (b_1 - x + I_a(x - a_1)) \big/ (b_1 - a_1) & (a_1 \le x \le b_1) \\ I_a & (x = b_1) \\ (x - b_1 + I_a(c_1 - x)) \big/ (c_1 - b_1) & (b_1 \le x \le c_1) \\ 1 \quad otherwise \end{cases}$$

$$F_a(x) = \begin{cases} (b_1 - x + F_a(x - a_1)) \big/ (b_1 - a_1) & (a_1 \le x \le b_1) \\ F_a & (x = b_1) \\ (x - c_1 + F_a(c_1 - x)) \big/ (c_1 - b_1) & (b_1 \le x \le c_1) \\ 1 \quad otherwise \end{cases}$$

where $0 \le T_a \le 1$; $0 \le I_a \le 1$; $0 \le F_a \le 1$ and $0 \le T_a + I_a + F_a \le 3$; $a_1, b_1, c_1 \in R$

Numerical Example:

Let $\overline{a} = <(2, 4, 6); 0.3, 0.4, 0.5>$ be a single valued triangular neutrosophic number, then the truth membership, indeterminacy membership and falsity membership are expressed as follows

$$T_a(x) = \begin{cases} \frac{0.3(x-2)}{2}, 2 \le x < 4 \\ 0.3, x = 4 \\ 0.3(5 - x), 4 < x \le 5 \\ 0, otherwise \end{cases}$$

$$I_a(x) = \begin{cases} \frac{4 - x + 0.3(x-2)}{2}, 2 \le x < 4 \\ 0.4, x = 4 \\ x - 4 + 0.4(5 - x), 4 < x \le 5 \\ 1, otherwise \end{cases}$$

$$F_a(x) = \begin{cases} \frac{4 - x + 0.5(x-2)}{2}, 2 \le x < 4 \\ 0.5, x = 4 \\ x - 4 + 0.4(5-x), 4 < x \le 5 \\ 1, otherwise \end{cases}$$

6.1.1 Operations on singled valued triangular neutrosophic numbers

Let $\tilde{A}_1 =< (a_1, a_2, a_3); T_1, I_1, F_1 >$ and $\tilde{A}_2 =< (b_1, b_2, b_3); T_2, I_2, F_2 >$ be two single valued triangular neutrosophic numbers. Then, the operations for SVTrN-numbers are defined as below;

(i) $\tilde{A}_1 \oplus \tilde{A}_2 =< (a_1 + b_1, a_2 + b_2, a_3 + b_3); \min(T_1, T_2), \max(I_1, I_2), \max(F_1, F_2) >$

(ii) $\tilde{A}_1 \otimes \tilde{A}_2 =< (a_1 b_1, a_2 b_2, a_3 b_3); \min(T_1, T_2), \max(I_1, I_2), \max(F_1, F_2))$

(iii) $\lambda \tilde{A}_1 =< (\lambda a_1, \lambda a_2, \lambda a_3); \min(T_1, T_2), \max(I_1, I_2), \max(F_1, F_2) >$

6.1.2 Score function and accuracy function of single valued triangular neutrosophic numbers

The convenient method for comparing of two single valued triangular neutrosophic numbers is described as follows:

Let $\tilde{A}_1 =< (a_1, a_2, a_3); T_1, I_1, F_1 >$ be a single valued triangular neutrosophic number. Then, the score function $s(\tilde{A}_1)$ and accuracy function $a(\tilde{A}_1)$ of a SVTrN-numbers are defined as follows:

(i) $s(\tilde{A}_1) = \left(\frac{1}{12}\right)\left[a_1 + 2a_2 + a_3\right] \times \left[2 + T_1 - I_1 - F_1\right]$

(ii) $a(\tilde{A}_1) = \left(\frac{1}{12}\right)\left[a_1 + 2a_2 + a_3\right] \times \left[2 + T_1 - I_1 + F_1\right]$

6.1.3 Ranking of single valued triangular neutrosophic numbers

Let $\tilde{A}_1$ and $\tilde{A}_2$ be two SVTrN-numbers. The ranking of $\tilde{A}_1$ and $\tilde{A}_2$ by score function and accuracy function is defined as follows:

(i) If $s(\tilde{A}_1) \prec s(\tilde{A}_2)$, then $\tilde{A}_1 \prec \tilde{A}_2$

(ii) If $s(\tilde{A}_1) = s(\tilde{A}_2)$ and if

(1) $a(\tilde{A}_1) \prec a(\tilde{A}_2)$, then $\tilde{A}_1 \prec \tilde{A}_2$

(2) $a(\tilde{A}_1) \succ a(\tilde{A}_2)$, then $\tilde{A}_1 \succ \tilde{A}_2$

(3) $a(\tilde{A}_1) = a(\tilde{A}_2)$, then $\tilde{A}_1 = \tilde{A}_2$.

6.2 Single valued-trapezoidal neutrosophic numbers (Deli & Subas, 2017)

A single valued trapezoidal neutrosophic number (SVTN-number) $\tilde{a} = <(a_1, b_1, c_1, d_1); T_a, I_a, F_a>$ is a special neutrosophic set on the real number set *R*, whose truth membership, indeterminacy-membership, and a falsity-membership are given as follows

$$T_a(x) = \begin{cases} (x-a_1)T_a \big/ (b_1-a_1) & (a_1 \le x \le b_1) \\ T_a & (b_1 \le x \le c_1) \\ (d_1-x)T_a \big/ (d_1-c_1) & (c_1 \le x \le d_1) \\ 0 \quad otherwise \end{cases}$$

$$I_a(x) = \begin{cases} (b_1-x+I_a(x-a_1)) \big/ (b_1-a_1) & (a_1 \le x \le b_1) \\ I_a & (b_1 \le x \le c_1) \\ (x-c_1+I_a(d_1-x)) \big/ (d_1-c_1) & (c_1 \le x \le d_1) \\ 1 \quad otherwise \end{cases}$$

$$F_a(x) = \begin{cases} (b_1-x+F_a(x-a_1)) \big/ (b_1-a_1) & (a_1 \le x \le b_1) \\ F_a & (b_1 \le x \le c_1) \\ (x-c_1+F_a(d_1-x)) \big/ (d_1-c_1) & (c_1 \le x \le d_1) \\ 1 \quad otherwise \end{cases}$$

where $0 \le T_a \le 1$; $0 \le I_a \le 1$; $0 \le F_a \le 1$ and $0 \le T_a + I_a + F_a \le 3$; $a_1, b_1, c_1, d_1 \in R$.

Numerical example:

Let $\bar{a} = <(1,2,5,6); 0.8, 0.6, 0.4>$ be a single valued trapezoidal neutrosophic number. Then the truth membership, indeterminacy membership and falsity membership are expressed as follows:

$$T_a(x) = \begin{cases} 0.8(x-1), 1 \le x < 2 \\ 0.8, 2 \le x \le 5 \\ 0.8(6-x), 5 < x \le 6 \\ 0, otherwise \end{cases} \quad I_a(x) = \begin{cases} 1.4-0.4x, 1 \le x < 2 \\ 0.6, 2 \le x \le 5 \\ 0.8x-1.4, 5 < x \le 6 \\ 1, otherwise \end{cases}$$

$$F_a(x) = \begin{cases} 1.6-0.6x, 1 \le x < 2 \\ 0.4, 2 \le x \le 5 \\ 0.6x-2.6, 5 < x \le 6 \\ 1, otherwise \end{cases}.$$

6.2.1 Operation on single valued trapezoidal neutrosophic numbers.

Let $\tilde{A}_1 = <(a_1, a_2, a_3, a_4); T_1, I_1, F_1>$ and $\tilde{A}_2 = <(b_1, b_2, b_3, b_4); T_2, I_2, F_2>$ be two single valued trapezoidal neutrosophic numbers. Then, the operations for SVTN-numbers are defined as below;

(i) $\tilde{A}_1 \oplus \tilde{A}_2 = <(a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4); \min(T_1, T_2), \max(I_1, I_2), \max(F_1, F_2)>$

(ii) $\tilde{A}_1 \otimes \tilde{A}_2 = <(a_1 b_1, a_2 b_2, a_3 b_3, a_4 b_4); \min(T_1, T_2), \max(I_1, I_2), \max(F_1, F_2))$

(iii) $\lambda \tilde{A}_1 = <(\lambda a_1, \lambda a_2, \lambda a_3, \lambda a_4); \min(T_1, T_2), \max(I_1, I_2), \max(F_1, F_2)>$

6.2.2Score function and accuracy function of single valued trapezoidal neutrosophic numbers

The convenient method for comparing of two single valued t**rapezoidal** neutrosophic numbers is described as follows:

Let $\tilde{A}_1 = <(a_1, a_2, a_3, a_4); T_1, I_1, F_1>$ be a single valued trapezoidal neutrosophic number. Then, the score function $s(\tilde{A}_1)$ and accuracy function $a(\tilde{A}_1)$ of a SVTN-numbers are defined as follows:

(i) $s(\tilde{A}_1) = \left(\frac{1}{12}\right)\left[a_1 + a_2 + a_3 + a_4\right] \times \left[2 + T_1 - I_1 - F_1\right]$

(ii) $a(\tilde{A}_1) = \left(\frac{1}{12}\right)\left[a_1 + a_2 + a_3 + a_4\right] \times \left[2 + T_1 - I_1 + F_1\right]$

6.2.3 Ranking of single valued trapezoidal neutrosophic numbers

Let $\tilde{A}_1$ and $\tilde{A}_2$ be two SVTN-numbers. The ranking of $\tilde{A}_1$ and $\tilde{A}_2$ by score function is defined as follows:

(i) If $s(\tilde{A}_1) \prec s(\tilde{A}_2)$ then $\tilde{A}_1 \prec \tilde{A}_2$

(ii) If $s(\tilde{A}_1) = s(\tilde{A}_2)$ and if

(1) $a(\tilde{A}_1) \prec a(\tilde{A}_2)$ then $\tilde{A}_1 \prec \tilde{A}_2$

(2) $a(\tilde{A}_1) \succ a(\tilde{A}_2)$ then $\tilde{A}_1 \succ \tilde{A}_2$

(3) $a(\tilde{A}_1) = a(\tilde{A}_2)$ then $\tilde{A}_1 = \tilde{A}_2$

Later on, Liang et al. (2017) redefined the score function, accuracy function and certainty function as follows:

Let $\tilde{a} = <[a_1, a_2, a_3, a_4], (T_{\tilde{a}}, I_{\tilde{a}}, F_{\tilde{a}})>$ be a SVTNN. Then, the score function, accuracy function, and certainty function of SVTNN ã are defined, respectively, as:

$E(\tilde{a}) = COG(K) \times \frac{(2 + T_{\tilde{a}} - I_{\tilde{a}} - F_{\tilde{a}})}{3}$

$A(\tilde{a}) = COG(K) \times (T_{\tilde{a}} - F_{\tilde{a}})$

$C(\tilde{a}) = COG(K) \times T_{\tilde{a}}$

where (COG) denotes the center of gravity of K and can be defined as follows:

$$COG(K)=\begin{cases} a & \text{if } a_1=a_2=a_3=a_4 \\ \frac{1}{3}\left[a_1+a_2+a_3+a_4-\frac{a_4a_3-a_2a_1}{a_4+a_3-a_2-a_1}\right], & \text{otherwise}\end{cases}$$

6.3 Interval valued neutrosophic number

6.3.1Operations on interval valued neutrosophic number

Let $\tilde{A}_1 = <\left[T_1^L, T_1^U\right], \left[I_1^L, I_1^U\right], \left[F_1^L, F_1^U\right]>$ and $\tilde{A}_2 = <\left[T_2^L, T_2^U\right], \left[I_2^L, I_2^U\right], \left[F_2^L, F_2^U\right]>$ be two interval valued neutrosophic numbers. Then, the operations for IVNNs are defined as below;

(i) $\tilde{A}_1 \oplus \tilde{A}_2 = <\left[T_1^L + T_2^L - T_1^L T_2^L, T_1^U + T_2^U - T_1^U T_2^U\right], \left[I_1^L I_2^L, I_1^U I_2^U\right], \left[F_1^L F_2^L, F_1^U F_2^U\right]>$

(ii) $\tilde{A}_1 \otimes \tilde{A}_2 = <\left[T_1^L T_2^L, T_1^U T_2^U\right], \left[I_1^L + I_2^L - I_1^L I_2^L, I_1^U + I_2^U - I_1^U I_2^U\right],$
$\left[F_1^L + F_2^L - F_1^L F_2^L, F_1^U + F_2^U - F_1^U F_2^U\right]>$

(iii) $\lambda\tilde{A} = <\left[1-(1-T_1^L)^\lambda, 1-(1-T_1^U)^\lambda\right], \left[(I_1^L)^\lambda, (I_1^U)^\lambda\right], \left[(F_1^L)^\lambda, (F_1^U)^\lambda\right]>$

(iv) $\tilde{A}_1^\lambda = <\left[(T_1^L)^\lambda, (T_1^U)^\lambda\right], \left[1-(1-I_1^L)^\lambda, 1-(1-I_1^U)^\lambda\right], \left[1-(1-F_1^L)^\lambda, 1-(1-F_1^U)^\lambda\right]>$ where $\lambda>0$

An interval valued neutrosophic number $\tilde{A}_1 = <\left[T_1^L, T_1^U\right], \left[I_1^L, I_1^U\right], \left[F_1^L, F_1^U\right]>$ is said to be empty if and only if

$T_1^L = 0, T_1^U = 0, I_1^L = 1, I_1^U = 1,$ and $F_1^L = F_1^U$ and is denoted by

$0_n = \{< x, < [0,\ 0],\ [1,\ 1],\ [1,\ 1] >: x \in X\}$

6.3.2 Score function and accuracy functions of interval valued neutrosophic number

The convenient method for comparing of interval valued neutrosophic numbers is described as follows:

Let $\tilde{A}_1 = <\left[T_1^L, T_1^U\right], \left[I_1^L, I_1^U\right], \left[F_1^L, F_1^U\right]>$ be a single valued neutrosophic number. Then, the score function $s(\tilde{A}_1)$ and accuracy function $H(\tilde{A}_1)$ of an IVNN are defined as follows:

(i) $s(\tilde{A}_1) = \left(\frac{1}{4}\right) \times \left[2 + T_1^L + T_1^U - 2I_1^L - 2I_1^U - F_1^L - F_1^U\right]$

(ii) $H(\tilde{A}_1) = \frac{T_1^L + T_1^U - I_1^U(1-T_1^U) - I_1^L(1-T_1^L) - F_1^U(1-I_1^U) - F_1^L(1-I_1^L)}{2}$

6.3.3 Ranking of interval valued neutrosophic numbers

Let $\tilde{A}_1 = <\left[T_1^L, T_1^U\right], \left[I_1^L, I_1^U\right], \left[F_1^L, F_1^U\right]>$ and $\tilde{A}_2 = <\left[T_2^L, T_2^U\right], \left[I_2^L, I_2^U\right], \left[F_2^L, F_2^U\right]>$ are two interval valued neutrosophic numbers. Then, the ranking method for comparing two IVNS is defiend as follows:

v. If $s(\tilde{A}_1) \succ s(\tilde{A}_2)$, then $\tilde{A}_1$ is greater than $\tilde{A}_2$, that is, $\tilde{A}_1$ is superior to $\tilde{A}_2$, denoted by $\tilde{A}_1 \succ \tilde{A}_2$

vi. If $s(\tilde{A}_1) = s(\tilde{A}_2)$,and $H(\tilde{A}_1) \succ H(\tilde{A}_2)$ then $\tilde{A}_1$ is greater than $\tilde{A}_2$, that is, $\tilde{A}_1$ is superior to $\tilde{A}_2$, denoted by $\tilde{A}_1 \succ \tilde{A}_2$.

6.4 Bipolar neutrosophic Number

6.4.1Operation on bipolar neutrosophic numbers

Let $\tilde{A}_1 =< T_1^p, I_1^p, F_1^p, T_1^n, I_1^n, F_1^n >$ and $\tilde{A}_2 =< T_2^p, I_2^p, F_2^p, T_2^n, I_2^n, F_2^n >$ be two bipolar neutrosophic numbers and $\lambda>0$. Then, the operations of these numbers defined as below;

(i) $$\tilde{A}_1 \oplus \tilde{A}_2 =< T_1^p + T_2^p - T_1^p T_2^p, I_1^p I_2^p, F_1^p F_2^p \\ -T_1^n T_2^n, -(-I_1^p - I_2^p - I_1^p I_2^p), -(-F_1^p - F_2^p - F_1^p F_2^p) >$$

(ii) $$\tilde{A}_1 \otimes \tilde{A}_2 =< T_1^p T_2^p, I_1^p + I_2^p - I_1^p I_2^p, F_1^p + F_2^p - F_1^p F_2^p \\ -(-T_1^n - T_2^n - T_1^n T_2^n), -I_1^n I_2^n, -F_1^n F_2^n >$$

(iii) $\lambda \tilde{A}_1 =< 1-(1-T_1^p)^\lambda, (I_1^p)^\lambda, (F_1^p)^\lambda, -(-T_1^n)^\lambda, -(-I_1^n)^\lambda, -(1-(1-(-F_1^n))^\lambda) >$

(iv) $\tilde{A}_1^\lambda =< (T_1^p)^\lambda, 1-(1-I_1^p)^\lambda, 1-(1-F_1^p)^\lambda, -(1-(1-(-T_1^n))^\lambda), -(-I_1^n)^\lambda, -(-F_1^n))^\lambda) >$ where $\lambda>0$.

6.4.2 Score function, accuracy function and certainty function of bipolar neutrosophic number

In order to make comparison between two BNNs. Deli et al. (2015) introduced a concept of score function. The score function is applied to compare the grades of BNS. This function shows that greater is the value, the greater is the bipolar neutrosophic sets and by using this concept paths can be ranked. Let $\tilde{A} =< T^p, I^p, F^p, T^n, I^n, F^n >$ be a bipolar neutrosophic number. Then, the score function $s(\tilde{A})$, accuracy function $a(\tilde{A})$ and certainty function $c(\tilde{A})$ of an BNN are defined as follows:

(i) $s(\tilde{A}) = \left(\frac{1}{6}\right) \times \left[T^p + 1 - I^p + 1 - F^p + 1 + T^n - I^n - F^n \right]$

(ii) $a(\tilde{A}) = T^p - F^p + T^n - F^n$

(iii) $c(\tilde{A}) = T^p - F^n$

6.4.3 Comparison of bipolar neutrosophic numbers

Let $\tilde{A}_1 =< T_1^p, I_1^p, F_1^p, T_1^n, I_1^n, F_1^n >$ and $\tilde{A}_2 =< T_2^p, I_2^p, F_2^p, T_2^n, I_2^n, F_2^n >$ be two bipolar neutrosophic numbers. then

vii. If $s(\tilde{A}_1) \succ s(\tilde{A}_2)$, then $\tilde{A}_1$ is greater than $\tilde{A}_2$, that is, $\tilde{A}_1$ is superior to $\tilde{A}_2$, denoted by $\tilde{A}_1 \succ \tilde{A}_2$

viii. If $s(\tilde{A}_1) = s(\tilde{A}_2)$,and $a(\tilde{A}_1) \succ a(\tilde{A}_2)$ then $\tilde{A}_1$ is greater than $\tilde{A}_2$, that is, $\tilde{A}_1$ is superior to $\tilde{A}_2$, denoted by $\tilde{A}_1 \succ \tilde{A}_2$

ix. If $s(\tilde{A}_1) = s(\tilde{A}_2)$, $a(\tilde{A}_1) = a(\tilde{A}_2)$, and $c(\tilde{A}_1) \succ c(\tilde{A}_2)$ then $\tilde{A}_1$ is greater than $\tilde{A}_2$, that is, $\tilde{A}_1$ is superior to $\tilde{A}_2$, denoted by $\tilde{A}_1 \succ \tilde{A}_2$

x. If $s(\tilde{A}_1) = s(\tilde{A}_2)$, $a(\tilde{A}_1) = a(\tilde{A}_2)$, and $c(\tilde{A}_1) = c(\tilde{A}_2)$ then $\tilde{A}_1$ is equal to $\tilde{A}_2$, that is, $\tilde{A}_1$ is indifferent to $\tilde{A}_2$, denoted by $\tilde{A}_1 = \tilde{A}_2$.

7. TRAPEZOIDAL NEUTROSOPHIC SETS (Ye, 2015b; Biswas et al., 2014)

Assume that X be the finite universe of discourse and F [0, 1] be the set of all trapezoidal fuzzy numbers on [0, 1]. A trapezoidal fuzzy neutrosophic set (TrFNS) $\tilde{A}$ in X is represented as:

$\tilde{A} = \{< x: \tilde{T}_A(x), \tilde{I}_A(x), \tilde{F}_A(x)>, x \in X\}$, where $\tilde{T}_A(x): X \to F[0,1]$, $\tilde{I}_A(x): X \to F[0,1]$ and $\tilde{F}_A(x): X \to F[0,1]$.

The trapezoidal fuzzy numbers $\tilde{T}_A(x) = (T_A^1(x), T_A^2(x), T_A^3(x), T_A^4(x))$, $\tilde{I}_A(x) = (I_A^1(x), I_A^2(x), I_A^3(x), I_A^4(x))$ and $\tilde{F}_A(x) = (F_A^1(x), F_A^2(x), F_A^3(x), F_A^4(x))$, respectively, denote the truth-membership, indeterminacy-membership and a falsity-membership degree of x in $\tilde{A}$ and for every $x \in X$, $0 \le T_A^4(x) + I_A^4(x) + F_A^4(x) \le 3$.

For notational convenience, the trapezoidal fuzzy neutrosophic value (TrFNV) $\tilde{A}$ is denoted by $\tilde{A} = \langle (a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4), (c_1, c_2, c_3, c_4) \rangle$ where,

$(T_A^1(x), T_A^2(x), T_A^3(x), T_A^4(x)) = (a_1, a_2, a_3, a_4)$,

$(I_A^1(x), I_A^2(x), I_A^3(x), I_A^4(x)) = (b_1, b_2, b_3, b_4)$, and

$(F_A^1(x), F_A^2(x), F_A^3(x), F_A^4(x)) = (c_1, c_2, c_3, c_4)$

The parameters satisfy the following relations $a_1 \le a_2 \le a_3 \le a_4$, $b_1 \le b_2 \le b_3 \le b_4$ and $c_1 \le c_2 \le c_3 \le c_4$.

The truth membership function is defined as follows

$$\tilde{T}_A(x) = \begin{cases} \dfrac{x - a_1}{a_2 - a_1}, & a_1 \le x \le a_2 \\ 1, & a_2 \le x \le a_3 \\ \dfrac{x - a_1}{a_2 - a_1}, & a_3 \le x \le a_4 \\ 0, & otherwise \end{cases}$$

The indeterminacy membership function is defined as follows:

$$\tilde{I}_A(x)=\begin{cases}\dfrac{x-b_1}{b_2-b_1}, & b_1\le x\le b_2\\ 1, & b_2\le x\le b_3\\ \dfrac{b_4-x}{b_4-b_3}, & b_3\le x\le b_4\\ 0, & otherwise\end{cases}$$

and the falsity membership function is defined as follows:

$$\tilde{F}_A(x)=\begin{cases}\dfrac{x-c_1}{c_2-c_1}, & c_1\le x\le c_2\\ 1, & c_2\le x\le c_3\\ \dfrac{c_4-x}{c_4-c_3}, & c_3\le x\le c_4\\ 0, & otherwise\end{cases}$$

A trapezoidal neutrosophic number $\tilde{A}=\langle(a_1,a_2,a_3,a_4),(b_1,b_2,\mathrm{b}_3,\mathrm{b}_4),(c_1,c_2,c_3,c_4)\rangle$ is said to be zero triangular fuzzy neutrosophic number if and only if

(a_1,a_2,a_3,a_4) =(0, 0, 0, 0), $(b_1,b_2,\mathrm{b}_3,\mathrm{b}_4)$ =(1, 1, 1, 1) and (c_1,c_2,c_3,c_4) =(1, 1, 1, 1).

Remark: The trapezoidal fuzzy neutrosophic number is a particular case of trapezoidal neutrosophic number when all the three vector are equal: $(a_1,a_2,a_3,a_4)=(b_1,b_2,\mathrm{b}_3,\mathrm{b}_4)=(c_1,c_2,c_3,c_4)$.

7.1 Operation on trapezoidal fuzzy neutrosophic value

Let $\tilde{A}_1=\langle(a_1,\mathrm{a}_2,\mathrm{a}_3,\mathrm{a}_4),(\mathrm{b}_1,\mathrm{b}_2,\mathrm{b}_3,\mathrm{b}_4),(\mathrm{c}_1,\mathrm{c}_2,\mathrm{c}_3,\mathrm{c}_4)\rangle$ and $\tilde{A}_2=\langle(\mathrm{e}_1,\mathrm{e}_2,\mathrm{e}_3,\mathrm{e}_4),(\mathrm{f}_1,\mathrm{f}_2,\mathrm{f}_3,\mathrm{f}_4),(\mathrm{g}_1,\mathrm{g}_2,\mathrm{g}_3,\mathrm{g}_4)\rangle$ be two TrFNVs in the set of real numbers, and $\lambda>0$. Then, the operational rules are defined as follows;

(i) $$\tilde{A}_1\oplus\tilde{A}_2=\left\langle\begin{matrix}\begin{pmatrix}a_1+e_1-a_1e_1,a_2+e_2-a_2e_2,\\ a_3+e_3-a_3e_3,a_4+e_4-a_4e_4\end{pmatrix},\\ (\mathrm{b}_1 f_1,\mathrm{b}_2 f_2,\mathrm{b}_3 f_3,\mathrm{b}_4 f_4),\\ (c_1g_1,c_2g_2,c_3g_3,c_4g_4)\end{matrix}\right\rangle$$

(ii) $$\tilde{A}_1\otimes\tilde{A}_2=\left\langle\begin{matrix}(a_1e_1,a_2e_2,a_3e_3,a_4e_4),\\ \begin{pmatrix}b_1+f_1-b_1f_1,\mathrm{b}_2+f_2-b_2f_2,\\ b_3+f_3-b_3f_3,\mathrm{b}_4+f_4-b_4f_4\end{pmatrix},\\ \begin{pmatrix}c_1+g_1-c_1g_1,\mathrm{c}_2+g_2-c_2g_2,\\ c_3+g_3-c_3g_3,\mathrm{c}_4+g_4-c_4g_4\end{pmatrix}\end{matrix}\right\rangle$$

(iii) $\lambda\tilde{A} = \left\langle \begin{matrix} \left(\begin{matrix} (1-(1-a_1)^{\lambda}, 1-(1-a_2)^{\lambda}, \\ 1-(1-a_3)^{\lambda}), 1-(1-a_4)^{\lambda}) \end{matrix} \right) \\ (b_1^{\lambda}, b_2^{\lambda}, b_3^{\lambda}, b_4^{\lambda}), (c_1^{\lambda}, c_2^{\lambda}, c_3^{\lambda}, c_4^{\lambda}) \end{matrix} \right\rangle$

(iv) $\tilde{A}_1^{\lambda} = \left\langle \begin{matrix} (a_1^{\lambda}, a_2^{\lambda}, a_3^{\lambda}, a_4^{\lambda}), \\ \left((1-(1-b_1)^{\lambda}, 1-(1-b_2)^{\lambda}, 1-(1-b_3)^{\lambda}), 1-(1-b_4)^{\lambda})\right), \\ \left((1-(1-c_1)^{\lambda}, 1-(1-c_2)^{\lambda}, 1-(1-c_3)^{\lambda}), 1-(1-c_4)^{\lambda})\right) \end{matrix} \right\rangle$ where $\lambda > 0$.

Ye (2015b) presented the following definitions of score function and accuracy function. The score function S and the accuracy function H are applied to compare the grades of TrFNSs. These functions show that greater is the value, the greater is the TrFNS.

7.2 Score function and accuracy function of trapezoidal fuzzy neutrosophic value

Let $\tilde{A}_1 = \langle (a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4), (c_1, c_2, c_3, c_4) \rangle$ be a TrFNV. Then, the score function $S(\tilde{A}_1)$ and an accuracy function $H(\tilde{A}_1)$ of TrFNV are defined as follows:

(i) $s(\tilde{A}_1) = \frac{1}{12}\left[8 + (a_1 + a_2 + a_3 + a_4) - (b_1 + b_2 + b_3 + b_4) - (c_1 + c_2 + c_3 + c_4)\right]$

(ii) $H(\tilde{A}_1) = \frac{1}{4}\left[(a_1 + a_2 + a_3 + a_4) - (c_1 + c_2 + c_3 + c_4)\right]$.

In order to make a comparison between two TrFNV, Ye (2015b) presented the order relations between two TrFNVs.

7.3 Ranking of trapezoidal fuzzy neutrosophic value

Let $\tilde{A}_1 = \langle (a_1, a_2, a_3, a_4), (b_1, b_2, b_3, b_4), (c_1, c_2, c_3, c_4) \rangle$ and $\tilde{A}_2 = \langle (e_1, e_2, e_3, e_4), (f_1, f_2, f_3, f_4), (g_1, g_2, g_3, g_4) \rangle$ be two TrFNVs in the set of real numbers. Then, we define a ranking method as follows:

xi. If $s(\tilde{A}_1) \succ s(\tilde{A}_2)$, then $\tilde{A}_1$ is greater than $\tilde{A}_2$, that is, $\tilde{A}_1$ is superior to $\tilde{A}_2$, denoted by $\tilde{A}_1 \succ \tilde{A}_2$

xii. If $s(\tilde{A}_1) = s(\tilde{A}_2)$, and $H(\tilde{A}_1) \succ H(\tilde{A}_2)$ then $\tilde{A}_1$ is greater than $\tilde{A}_2$, that is, $\tilde{A}_1$ is superior to $\tilde{A}_2$, denoted by $\tilde{A}_1 \succ \tilde{A}_2$.

8. TRIANGULAR FUZZY NEUTROSOPHIC SETS (Biswas et al., 2014)

Assume that X be the finite universe of discourse and F [0, 1] be the set of all triangular fuzzy numbers on [0, 1]. A triangular fuzzy neutrosophic set (TFNS) $\tilde{A}$ in X is represented

$\tilde{A} = \{< x: \tilde{T}_A(x), \tilde{I}_A(x), \tilde{F}_A(x) >, x \in X\}$,

where $\tilde{T}_A(x): X \to F[0,1]$, $\tilde{I}_A(x): X \to F[0,1]$ and $\tilde{F}_A(x): X \to F[0,1]$. The triangular fuzzy numbers

$\tilde{T}_A(x) = (T_A^1(x), T_A^2(x), T_A^3(x))$, $\tilde{I}_A(x) = (I_A^1(x), I_A^2(x), I_A^3(x))$ and

$\tilde{F}_A(x) = (F_A^1(x), F_A^2(x), F_A^3(x))$, respectively, denote the truth-membership, indeterminacy-membership and a falsity-membership degree of x in $\tilde{A}$ and for every $x \in X$

$$0 \le T_A^3(x) + I_A^3(x) + F_A^3(x) \le 3.$$

For notational convenience, the triangular fuzzy neutrosophic value (TFNV) $\tilde{A}$ is denoted by $\tilde{A} = \langle (a,b,c),(e,f,g),(r,s,t) \rangle$ where, $(T_A^1(x), T_A^2(x), T_A^3(x)) = (a, b, c)$,

$(I_A^1(x), I_A^2(x), I_A^3(x)) = (e, f, g)$, and $(F_A^1(x), F_A^2(x), F_A^3(x)) = (r, s, t)$.

8.1 Zero triangular fuzzy neutrosophic number

A triangular fuzzy neutrosophic number $\tilde{A} = \langle (a,b,c),(e,f,g),(r,s,t) \rangle$ is said to be zero triangular fuzzy neutrosophic number if and only if

(a, b, c) = (0, 0, 0), (e, f, g) = (1, 1, 1) and (r, s, t) = (1, 1, 1)

8.2 Operation on triangular fuzzy neutrosophic value

Let $\tilde{A}_1 = \langle (a_1,b_1,c_1),(e_1,f_1,g_1),(r_1,s_1,t_1) \rangle$ and $\tilde{A}_2 = \langle (a_2,b_2,c_2),(e_2,f_2,g_2),(r_2,s_2,t_2) \rangle$ be two TFNVs in the set of real numbers, and $\lambda > 0$. Then, the operational rules are defined as follows;

(i) $\tilde{A}_1 \oplus \tilde{A}_2 = \left\langle \begin{array}{l} (a_1 + a_2 - a_1a_2, b_1 + b_2 - b_1b_2, c_1 + c_2 - c_1c_2), \\ (e_1e_2, f_1f_2, g_1g_2), (r_1r_2, s_1s_2, t_1t_2) \end{array} \right\rangle$

(ii) $\tilde{A}_1 \otimes \tilde{A}_2 = \left\langle \begin{array}{l} (a_1a_2, b_1b_2, c_1c_2), \\ (e_1 + e_2 - e_1e_2, f_1 + f_2 - f_1f_2, g_1 + g_2 - g_1g_2), \\ (r_1 + r_2 - r_1r_2, s_1 + s_2 - s_1s_2, t_1 + t_2 - t_1t_2) \end{array} \right\rangle$

(iii) $\lambda \tilde{A} = \left\langle \begin{array}{l} \left((1-(1-a_1)^{\lambda}, 1-(1-b_1)^{\lambda}, 1-(1-c_1)^{\lambda})\right), \\ (e_1^{\lambda}, f_1^{\lambda}, g_1^{\lambda}), (r_1^{\lambda}, s_1^{\lambda}, t_1^{\lambda}) \end{array} \right\rangle$

(iv) $\tilde{A}_1^{\lambda} = \left\langle \begin{array}{l} (a_1^{\lambda}, b_1^{\lambda}, c_1^{\lambda}), \\ \left((1-(1-e_1)^{\lambda}, 1-(1-f_1)^{\lambda}, 1-(1-g_1)^{\lambda})\right), \\ \left((1-(1-r_1)^{\lambda}, 1-(1-s_1)^{\lambda}, 1-(1-t_1)^{\lambda})\right) \end{array} \right\rangle$ where $\lambda > 0$.

Ye (2015b) introduced the concept of score function and accuracy function TFNS. The score function S and the accuracy function H are applied to compare the grades of TFNS. These functions show that greater is the value, the greater is the TFNS.

8.3 Score function and accuracy function of triangular fuzzy neutrosophic value

Let $\tilde{A}_1 = \langle (a_1, b_1, c_1), (e_1, f_1, g_1), (r_1, s_1, t_1) \rangle$ be a TFNV. Then, the score function $S(\tilde{A}_1)$ and an accuracy function $H(\tilde{A}_1)$ of TFNV are defined as follows:

(i) $s(\tilde{A}_1) = \frac{1}{12}\left[8 + (a_1 + 2b_1 + c_1) - (e_1 + 2f_1 + g_1) - (r_1 + 2s_1 + t_1)\right]$

(ii) $H(\tilde{A}_1) = \frac{1}{4}\left[(a_1 + 2b_1 + c_1) - (r_1 + 2s_1 + t_1)\right]$

In order to make a comparison between two TFNVs, Ye (2015b) presented the order relations between two TFNVs.

8.4 Ranking of triangular fuzzy neutrosophic values

Let $\tilde{A}_1 = \langle (a_1, b_1, c_1), (e_1, f_1, g_1), (r_1, s_1, t_1) \rangle$ and $\tilde{A}_2 = \langle (a_2, b_2, c_2), (e_2, f_2, g_2), (r_2, s_2, t_2) \rangle$ be two TFNVs in the set of real numbers. Then, the ranking method is defined as follows:

i. If $s(\tilde{A}_1) \succ s(\tilde{A}_2)$, then $\tilde{A}_1$ is greater than $\tilde{A}_2$, that is, $\tilde{A}_1$ is superior to $\tilde{A}_2$, denoted by $\tilde{A}_1 \succ \tilde{A}_2$
ii. If $s(\tilde{A}_1) = s(\tilde{A}_2)$, and $H(\tilde{A}_1) \succ H(\tilde{A}_2)$ then $\tilde{A}_1$ is greater than $\tilde{A}_2$, that is, $\tilde{A}_1$ is superior to $\tilde{A}_2$, denoted by $\tilde{A}_1 \succ \tilde{A}_2$.

9. DIFFERENCE BETWEEN TRAPEZOIDAL INTUITIONISTIC FUZZY NUMBER AND TRAPEZOIDAL NEUTROSOPHIC FUZZY NUMBER

9.1 Trapezoidal intuitionistic fuzzy number (Nayagam, Jeevaraj,& Sivaraman, 2016)

Definition 1. A trapezoidal intuitionistic fuzzy number $\tilde{a} = <(\underline{a}, a_1, a_2, \overline{a}); w_{\tilde{a}}, u_{\tilde{a}}>$ is a convex intuitionistic fuzzy set on the set $\Re$ of real numbers, whose membership and non-membership functions are follows

$$\mu_{\tilde{a}}(x) = \begin{cases} (x - \underline{a})w_{\tilde{a}} / (a_1 - \underline{a}) & (\underline{a} \le x < a_1) \\ w_{\tilde{a}} & (a_1 \le x \le a_2) \\ (\overline{a} - x)w_{\tilde{a}} / (\overline{a} - a_2) & (a_2 < a \le \overline{a}) \\ 0 & (x < \underline{a}, x > \overline{a}), \end{cases}$$

$$\nu_{\tilde{a}}(x) = \begin{cases} [a_1 - x + u_{\tilde{a}}(x - \underline{a})] / (a_1 - \underline{a}) & (\underline{a} \le x < a_1) \\ u_{\tilde{a}} & (a_1 \le x \le a_2) \\ [x - a_2 + u_{\tilde{a}}(\overline{a} - x)] / (\overline{a} - a_2) & (a_2 < x \le \overline{a}) \\ 1 & (x < \underline{a}, x > \overline{a}). \end{cases}$$

where $0 \le w_{\tilde{a}} \le 1$, $0 \le u_{\tilde{a}} \le 1$ and $0 \le w_{\tilde{a}} + u_{\tilde{a}} \le 1$, $w_{\tilde{a}}$ and $u_{\tilde{a}}$ respectively represent the maximum membership degree and the minimum membership degree of $\tilde{a}$, $\pi_{\tilde{a}}(x) = 1 - \mu_{\tilde{a}}(x) - \nu_{\tilde{a}}(x)$ is called as the intuitionistic fuzzy index of an element x in $\tilde{a}$. a_1 and a_2 respectively represent the minimum and maximum values of the most probable value of the fuzzy number $\tilde{a}$, $\underline{a}$ represents the minimum value of the $\tilde{a}$, and $\overline{a}$ represents the maximum value of the $\tilde{a}$.

9.2 Trapezoidal neutrosophic fuzzy number

Definition 2. Let X be a universe of discourse, then a trapezoidal fuzzy neutrosophic set $\tilde{N}$ in X is defined as the following form:

$$\tilde{N} = \{\langle x, T_{\tilde{N}}(x), I_{\tilde{N}}(x), F_{\tilde{N}}(x) \rangle | x \in X\},$$

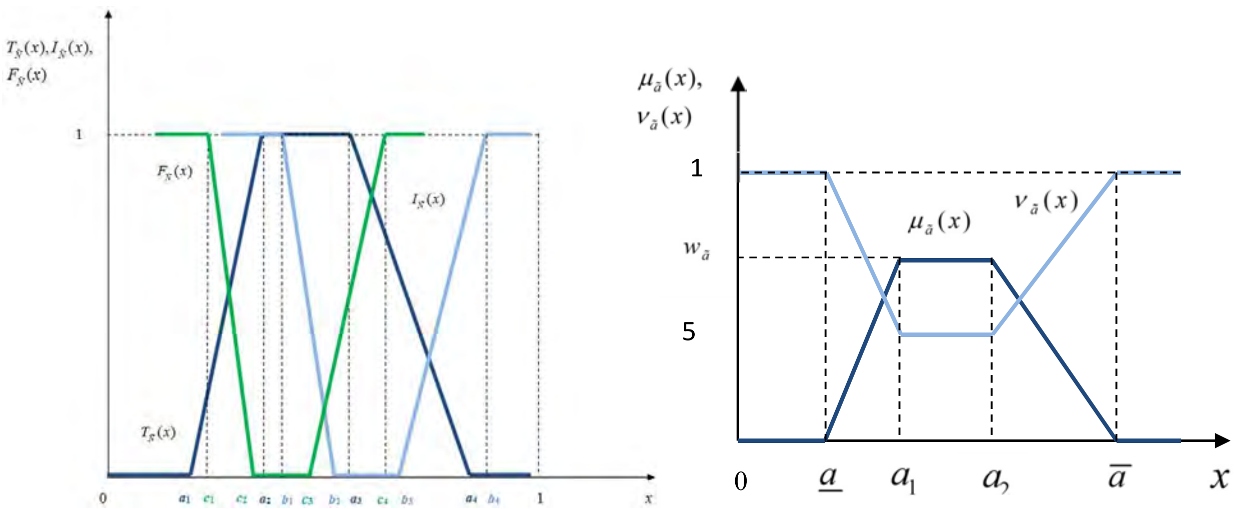
where $T_{\tilde{N}}(x) \subset [0,1]$, $I_{\tilde{N}}(x) \subset [0,1]$ and $F_{\tilde{N}}(x) \subset [0,1]$ are three trapezoidal fuzzy neutrosophic numbers, $T_{\tilde{N}}(x) = \left(t^1_{\tilde{N}}(x), t^2_{\tilde{N}}(x), t^3_{\tilde{N}}(x), t^4_{\tilde{N}}(x)\right) : X \to [0,1]$, $I_{\tilde{N}}(x) = \left(i^1_{\tilde{N}}(x), i^2_{\tilde{N}}(x), i^3_{\tilde{N}}(x), i^4_{\tilde{N}}(x)\right) : X \to [0,1]$,

and $F_{\tilde{N}}(x) = \left(f^1_{\tilde{N}}(x), f^2_{\tilde{N}}(x), f^3_{\tilde{N}}(x), f^4_{\tilde{N}}(x)\right) : X \to [0,1]$ with the condition $0 \le t^4_{\tilde{N}}(x) + i^4_{\tilde{N}}(x) + f^4_{\tilde{N}}(x) \le 3, x \in X$.

9.3 Difference and comparison between trapezoidal intuitionistic fuzzy number and trapezoidal neutrosophic fuzzy number

The difference and comparison between the trapezoidal intuitionistic fuzzy number and trapezoidal neutrosophic fuzzy number are represented in the following way:

First, we give the graphical representation of trapezoidal neutrosophic fuzzy number (TrNFN) and trapezoidal intuitionistic fuzzy number (TrIFN), as shown in Figure 1,



(a) Graphical representation of TrNFN (b) Graphical representation of TrIFN

Fig.1 Graphical representation of trapezoidal neutrosophic fuzzy number and trapezoidal intuitionistic fuzzy number

It can be observed from the Fig. 1, there are some differences between trapezoidal intuitionistic fuzzy number and trapezoidal neutrosophic fuzzy number. On one hand, the membership degree, non-membership degree and hesitancy of trapezoidal intuitionistic fuzzy number are mutually constrained, and the maximum value of the sum of them is not more than 1. However, the truth membership, indeterminacy membership and falsity membership functions of trapezoidal neutrosophic fuzzy number are independent, and their values are between 0 and 3. And the maximum value of their sum is not more than 3. On the other hand, trapezoidal neutrosophic fuzzy number is a generalized representation of trapezoidal fuzzy number and trapezoidal intuitionistic fuzzy number, and trapezoidal intuitionistic fuzzy number is a special case of trapezoidal neutrosophic fuzzy number.

10. DIFFERENCE BETWEEN TRIANGULAR FUZZY NUMBERS, INTUITIONISTIC TRIANGULAR FUZZY NUMBER AND SINGLED VALUED NEUTROSOPHIC SET

Fuzzy sets have been introduced by Zadeh (1965) in order to deal with imprecise numerical quantities in a practical way. A fuzzy number (Kaufmann& Gupta, 1988) is a generalization of a regular, real number in the sense that it does not refer to one single value but rather to a connected set of possible values, where each possible value has its own weight between 0 and 1. This weight is called the membership function. A fuzzy number is thus a special case of a convex, normalized fuzzy set of the real line .

10.1 Triangular fuzzy number (Lee, 2005)

A triangular fuzzy number $A=[a_1,a_2,a_3]$ is expressed by the following membership function

$$\mu_A(x)=\begin{cases}\dfrac{x-a_1}{a_2-a_1}, a_1\le x\le a_2\\ \dfrac{a_3-x}{a_3-a_2}, a_2\le x\le a_3\\ 0, otherwise\end{cases}$$

10.2 Triangular intuitionistic fuzzy number (Li, Nan, & Zhang, 2012)

A TIFN (See Fig. 2) A is a subset of IFS in R with the following membership functions and non-membership function as follows

$$\mu_A(x)=\begin{cases}\dfrac{x-a_1}{a_2-a_1}, a_1\le x\le a_2\\ \dfrac{a_3-x}{a_3-a_2}, a_2\le x\le a_3\\ 0, otherwise\end{cases}\quad \gamma_A(x)=\begin{cases}\dfrac{a_2-x}{a_2-a_1'}, a_1'\le x\le a_2\\ \dfrac{x-a_2}{a_3'-a_2}, a_2\le x\le a_3'\\ 1, otherwise\end{cases}$$

where $a_1' \leq a_1 \leq a_2 \leq a_3 \leq a_3'$

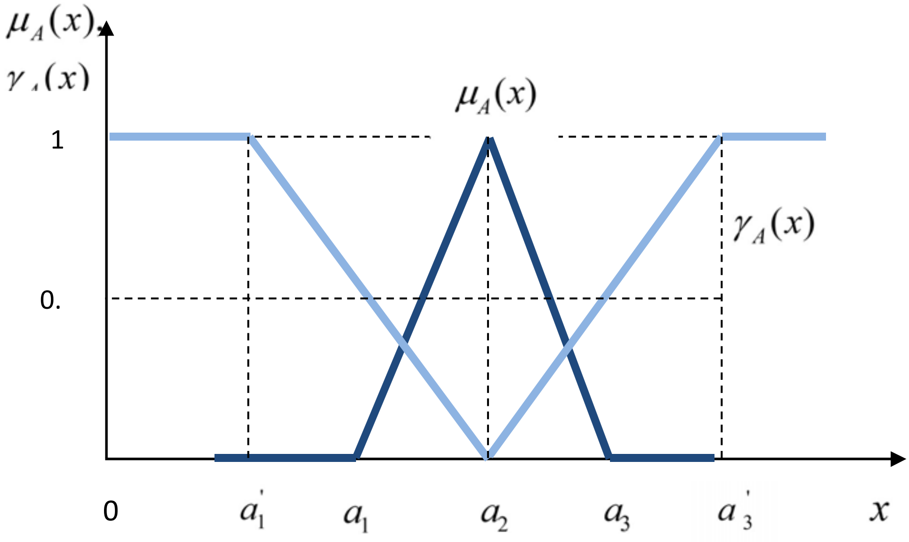


Fig.2. Graphical representation of triangular intuitionistic fuzzy number.

It can be observed from the membership functions that in case of triangular intuitionistic fuzzy number, membership and non-membership degrees are triangular fuzzy numbers. Further it can be noted that the neutrosophic components are best suited in the presentation of indeterminacy and inconsistent information whereas intuitionistic fuzzy sets cannot handle indeterminacy and inconsistent information.
The difference between the fuzzy numbers and singled valued neutrosophic set can be understood clearly with the help of an example. Suppose it is raining continuously for few days in a locality. Then one can guess whether there would be a flood like situation in that area. Observing the rainfall of this year and recalling the incidents of previous years one can only give his judgment on the basis of guess in terms of yes or no but still there remains an indeterminate situation and that indeterminate situation is expressed nicely by the single valued neutrosophic set.

Triangular fuzzy numbers (TFNs) and single valued neutrosophic numbers (SVNNs) are both generalizations of fuzzy numbers that are each characterized by three components. TFNs and SVNNs have been widely used to represent uncertain and vague information in various areas such as engineering, medicine, communication science and decision science. However, SVNNs are far more accurate and convenient to be used to represent the uncertainty and hesitancy that exists in information, as compared to TFNs. SVNNs are characterized by three components, each of which clearly represents the degree of truth membership, indeterminacy membership and falsity membership of a the SVNNs with respect to a an attribute. Therefore, we are able to tell the belongingness of a SVNN to the set of attributes that are being studied, by just looking at the structure of the SVNN. This provides a clear, concise and comprehensive method of representation of the different components of the membership of the number. This is in contrast to the structure of the TFN which only provides us with the maximum, minimum and initial values of the TFN, all of which can only tell us the path of the TFN, but does not tell us anything about the degree of non-belongingness of the TFN with respect to the set of attributes that are being studied. Furthermore, the structure of the TFN is not able to capture the hesitancy that naturally exists within the user in the process of assigning membership values. These reasons clearly show the advantages of SVNNs compared to TFNs.

11. REFINED NEUTROSOPHIC SETS (Smarandache, 2013; Deli et al., 2015b)

Refined neutrosophic sets can be expressed as follows:

Let E be a universe. A neutrosophic refined set (NRS) A on E can be defined as follows

$$A=\left\{\begin{array}{l}< x,(T_A^1(\mathrm{x}),T_A^2(\mathrm{x}),...,T_A^p(\mathrm{x})),(I_A^1(\mathrm{x}),I_A^2(\mathrm{x}),...,I_A^p(\mathrm{x})),\\ (F_A^1(\mathrm{x}),F_A^2(\mathrm{x}),...,F_A^p(\mathrm{x}))\end{array}\right\}$$

Where $T_A^1(\mathrm{x}),T_A^2(\mathrm{x}),...,T_A^p(\mathrm{x})$: E $\to$ [0, 1], $I_A^1(\mathrm{x}),I_A^2(\mathrm{x}),...,I_A^p(\mathrm{x})$: E $\to$ [0, 1] and $F_A^1(\mathrm{x}),F_A^2(\mathrm{x}),...,F_A^p(\mathrm{x})$: E $\to$ [0, 1]

12. BIPOLAR NEUTROSOPHIC REFINED SETS

Bipolar neutrosophic refined sets (Deli et al., 2015a) can be described as follows:

Let E be a universe. A bipolar neutrosophic refined set (BNRS) A on E can be defined as follows:

$$A=\left\{\begin{array}{l}< x,(T_A^{1+}(\mathrm{x}),T_A^{2+}(\mathrm{x}),...,T_A^{p+}(\mathrm{x}),T_A^{1-}(\mathrm{x}),T_A^{2-}(\mathrm{x}),...,T_A^{p-}(\mathrm{x})),\\ (\mathrm{I}_A^{1+}(\mathrm{x}),\mathrm{I}_A^{2+}(\mathrm{x}),...,\mathrm{I}_A^{p+}(\mathrm{x}),\mathrm{I}_A^{1-}(\mathrm{x}),\mathrm{I}_A^{2-}(\mathrm{x}),...,\mathrm{I}_A^{p-}(\mathrm{x})),\\ (\mathrm{F}_A^{1+}(\mathrm{x}),\mathrm{F}_A^{2+}(\mathrm{x}),...,\mathrm{F}_A^{p+}(\mathrm{x}),\mathrm{F}_A^{1-}(\mathrm{x}),\mathrm{F}_A^{2-}(\mathrm{x}),...,\mathrm{F}_A^{p-}(\mathrm{x}))>: x\in X\end{array}\right\}, \text{ where}$$

$(T_A^{1+}(\mathrm{x}),T_A^{2+}(\mathrm{x}),...,T_A^{p+}(\mathrm{x}),T_A^{1-}(\mathrm{x}),T_A^{2-}(\mathrm{x}),...,T_A^{p-}(\mathrm{x}))$: E $\to$ [0, 1],

$(\mathrm{I}_A^{1+}(\mathrm{x}),\mathrm{I}_A^{2+}(\mathrm{x}),...,\mathrm{I}_A^{p+}(\mathrm{x}),\mathrm{I}_A^{1-}(\mathrm{x}),\mathrm{I}_A^{2-}(\mathrm{x}),...,\mathrm{I}_A^{p-}(\mathrm{x}))$: E $\to$ [0, 1] and

$(\mathrm{F}_A^{1+}(\mathrm{x}),\mathrm{F}_A^{2+}(\mathrm{x}),...,\mathrm{F}_A^{p+}(\mathrm{x}),\mathrm{F}_A^{1-}(\mathrm{x}),\mathrm{F}_A^{2-}(\mathrm{x}),...,\mathrm{F}_A^{p-}(\mathrm{x}))$: E $\to$ [0, 1] such that $0\le T_A^i(x)+ I_A^i(x)+ F_A^i(x)\le 3$ (i =1,2,3,…,p)

$(T_A^{1+}(\mathrm{x}),T_A^{2+}(\mathrm{x}),...,T_A^{p+}(\mathrm{x}),T_A^{1-}(\mathrm{x}),T_A^{2-}(\mathrm{x}),...,T_A^{p-}(\mathrm{x}))$ $(\mathrm{I}_A^{1+}(\mathrm{x}),\mathrm{I}_A^{2+}(\mathrm{x}),...,\mathrm{I}_A^{p+}(\mathrm{x}),\mathrm{I}_A^{1-}(\mathrm{x}),\mathrm{I}_A^{2-}(\mathrm{x}),...,\mathrm{I}_A^{p-}(\mathrm{x}))$

$(\mathrm{F}_A^{1+}(\mathrm{x}),\mathrm{F}_A^{2+}(\mathrm{x}),...,\mathrm{F}_A^{p+}(\mathrm{x}),\mathrm{F}_A^{1-}(\mathrm{x}),\mathrm{F}_A^{2-}(\mathrm{x}),...,\mathrm{F}_A^{p-}(\mathrm{x}))$ is respectively the truth membership sequence, indeterminacy membership sequence and falsity membership sequence of the element x. Also, P is called the dimension of BNRS.

The set of all bipolar neutrosophic refined sets on E is denoted by BNRS(E).

13 MULTI-VALUED NEUTROSOPHIC SETS (Peng & Wang, 2015)

13.1 Operation on multi-valued neutrosophic numbers

Let $A=<\tilde{T}_A(x),\tilde{I}_A(x),\tilde{F}_A(x)>$, $B=<\tilde{T}_B(x),\tilde{I}_B(x),\tilde{F}_B(x)>$ are two multi-valued neutrosophic numbers. If $\forall\tilde{T}_A^a\in\tilde{T}_A$, $\forall\tilde{T}_B^b\in\tilde{T}_B$, $\forall\tilde{I}_A^a\in\tilde{I}_A$, $\forall\tilde{I}_B^b\in\tilde{I}_B$, $\forall\tilde{F}_A^a\in\tilde{F}_A$, $\forall\tilde{F}_B^b\in\tilde{F}_B$, and $\tilde{I}_A^a>\tilde{I}_B^b$, $\tilde{F}_A^a>\tilde{F}_B^b$, $\tilde{T}_A^a>\tilde{T}_B^b$, then B is superior to A, denoted as $A\prec B$.

Let $A=<\tilde{T}_A(x),\tilde{I}_A(x),\tilde{F}_A(x)>$, $B=<\tilde{T}_B(x),\tilde{I}_B(x),\tilde{F}_B(x)>$ are any two MVNNs, and $\lambda>0$. The operations for MVNNs are defined as follows.

(1) $\lambda A=\left\langle \cup_{\gamma_A\in\tilde{T}_A}\{1-(1-\gamma_A)^\lambda\},\cup_{\eta_A\in\tilde{I}_A}\{(\eta_A)^\lambda\},\cup_{\xi_A\in\tilde{F}_A}\{(\xi_A)^\lambda\}\right\rangle$;

(2) $A^\lambda=\left\langle \cup_{\gamma_A\in\tilde{T}_A}\{(\gamma_A)^\lambda\},\cup_{\eta_A\in\tilde{I}_A}\{1-(1-\eta_A)^\lambda\},\cup_{\xi_A\in\tilde{F}_A}\{1-(1-\xi_A)^\lambda\}\right\rangle$;

(3) $A+B=\left\langle \cup_{\gamma_A\in\tilde{T}_A,\gamma_B\in\tilde{T}_B}\{\gamma_A+\gamma_B-\gamma_A\cdot\gamma_B\},\cup_{\eta_A\in\tilde{I}_A,\eta_B\in\tilde{I}_B}\{\eta_A\cdot\eta_B\},\cup_{\xi_A\in\tilde{F}_A,\xi_B\in\tilde{F}_B}\{\xi_A\cdot\xi_B\}\right\rangle$;

(4) $A\cdot B=\left\langle \cup_{\gamma_A\in\tilde{T}_A,\gamma_B\in\tilde{T}_B}\{\gamma_A\cdot\gamma_B\},\cup_{\eta_A\in\tilde{I}_A,\eta_B\in\tilde{I}_B}\{\eta_A+\eta_B-\eta_A\cdot\eta_B\},\cup_{\xi_A\in\tilde{F}_A,\xi_B\in\tilde{F}_B}\{\xi_A+\xi_B-\xi_A\cdot\xi_B\}\right\rangle$;

13.2 Score function, accuracy function and certainty function of multi-valued neutrosophic number

(1) $s(A)=\dfrac{1}{l_{\tilde{T}_A}\cdot l_{\tilde{I}_A}\cdot l_{\tilde{F}_A}}\times\sum_{\gamma_i\in\tilde{T}_A,\eta_j\in\tilde{I}_A,\xi_k\in\tilde{F}_A}(\gamma_i+1-\eta_j+1-\xi_k)/3$;

(2) $a(A)=\dfrac{1}{l_{\tilde{T}_A}\cdot l_{\tilde{F}_A}}\times\sum_{\gamma_i\in\tilde{T}_A,\xi_k\in\tilde{F}_A}(\gamma_i-\xi_k)$;

(3) $c(A)=\dfrac{1}{l_{\tilde{T}_A}}\times\sum_{\gamma_i\in\tilde{T}_A}\gamma_i$;

13.3 Comparison of multi-valued neutrosophic numbers

Let $A=<\tilde{T}_A(x),\tilde{I}_A(x),\tilde{F}_A(x)>$, $B=<\tilde{T}_B(x),\tilde{I}_B(x),\tilde{F}_B(x)>$ are two multi-valued neutrosophic numbers. Then the comparision method can be defined as follows:

i. If $s(A)>s(B)$, then A is greater than B, that is, A is superior to B, denoted by $A\succ B$.
ii. If $s(A)=s(B)$ and $a(A)>a(B)$, then A is greater than B, that is, A is superior to B, denoted by $A\succ B$.
iii. If $s(A)=s(B)$, $a(A)=a(B)$ and $c(A)>c(B)$, then A is greater than B, that is, A is superior to B, denoted by $A\succ B$.
iv. If $s(A)=s(B)$, $a(A)=a(B)$ and $c(A)=c(B)$, then A is equal to B, that is, A is indifferent to B, denoted by $A\sim B$.

14. Simplified neutrosophic linguistic sets (SNLSs) (Tian et al., 2016)

14.1 SNLSs

Definition 1. Let *X* be a space of points (objects) with a generic element in *X*, denoted by *x* and $H=\{h_0,h_1,h_2,\cdots,h_{2t}\}$ be a finite and totally ordered discrete term set, where *t* is a nonnegative real number. A SNLS *A* in *X* is characterized as

$$A=\{<x,h_{\theta(x)},(t(x),i(x),f(x))>|x\in X\},$$

where $h_{\theta(x)}\in H$, $t(x)\in[0,1]$, $i(x)\in[0,1]$, $f(x)\in[0,1]$, with the condition $0\le t(x)+i(x)+f(x)\le 3$ for any $x\in X$. And $t_A(x)$, $i_A(x)$ and $f_A(x)$ represent, respectively, the degree of truth-membership, indeterminacy-membership and falsity-membership of the element x in X to the linguistic term $h_{\theta(x)}$. In addition, if $\|X\|=1$, a SNLS will be degenerated to a SNLN, denoted by $A=<h_{\theta},(t,i,f)>$. And A will be degenerated to a linguistic term if $t=1$, $i=0$, and $f=0$.

14.2 Operations of SNLNs

Let $a_i=<h_{\theta_i},(t_i,i_i,f_i)>$ and $a_j=<h_{\theta_j},(t_j,i_j,f_j)>$ be two SNLNs, f^* be a linguistic scale function and $\lambda \geq 0$. Then the following operations of SNLNs can be defined.

(1) $a_i \oplus a_j = \left\langle f^{*-1}(f^*(h_{\theta_i})+f^*(h_{\theta_j})),(\frac{f^*(h_{\theta_i})t_i+f^*(h_{\theta_j})t_j}{f^*(h_{\theta_i})+f^*(h_{\theta_j})},\frac{f^*(h_{\theta_i})i_i+f^*(h_{\theta_j})i_j}{f^*(h_{\theta_i})+f^*(h_{\theta_j})},\frac{f^*(h_{\theta_i})f_i+f^*(h_{\theta_j})f_j}{f^*(h_{\theta_i})+f^*(h_{\theta_j})})\right\rangle$;

(2) $a_i \otimes a_j = \left\langle f^{*-1}(f^*(h_{\theta_i})f^*(h_{\theta_j})),(t_it_j,i_i+i_j-i_ii_j,f_i+f_j-f_if_j)\right\rangle$;

(3) $\lambda a_i = \left\langle f^{*-1}(\lambda f^*(h_{\theta_i})),(t_i,i_i,f_i)\right\rangle$;

(4) $a_i^{\lambda} = \left\langle f^{*-1}((f^*(h_{\theta_i}))^{\lambda}),(t_i^{\lambda},1-(1-i_i)^{\lambda},1-(1-f_i)^{\lambda})\right\rangle$;

(5) $neg(a_i) = \left\langle f^{*-1}(f^*(h_{2t})-f^*(h_{\theta_i})),(f_i,1-i_i,t_i)\right\rangle$;

15. COMPARISON ANALYSIS

Refined neutrosophic set is a generalization of fuzzy set, intuitionistic fuzzy set, neutrosophic set, interval-valued neutrosophic set, neutrosophic hesitant fuzzy set and interval-valued neutrosophic hesitant fuzzy set. Also differences and similarities between these sets are given in Table 1.

Table 1. Comparison of fuzzy set anditsextensive set theory

	Fuzzy	intuitionstic fuzzy	Interval-Valued neutrosophic	Interval- Valued neutrosophic HesitantFuzzy Set	Neutro sophic	Neutrosophic HesitantFuzzy Set	Neutrosophicrefined
Domain	Universe of discourse	Universe of discourse	Universe of discourse	Universe of discourse	Universe of discourse	Universe of discourse	Universe of discourse
Co-domain	Single-value in [0,1]	Two-value in [0,1]	Unipolar interval in [0,1]	Unipolar interval in [0,1]	$[0,1]^3$	$[0,1]^3$	$[0,1]^3$
Number	Yes	Yes	Yes	Yes	Yes	No	No
Membershipfunction	regular	regular	Regular	irregular	regular	irregular	Regular
Uncertainty	Yes	Yes	Yes	Yes	Yes	Yes	Yes
True	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Falsity	No	Yes	Yes	Yes	Yes	Yes	Yes
Indeterminacy	No	No	Yes	Yes	Yes	Yes	Yes
Negativity	No	No	No	No	Yes in [0,1]	No	No
Membership valued	Membershipvalued	Singlevalued	İnterv-valued	Singlevalued	Singlevalued	Singlevalued	Multi- valued

Bosc and Pivert (2013) said that "Bipolarity refers to the propensity of the human mind to reason and make decisions on the basis of positive and negative effects. Positive information states what is possible, satisfactory, permitted, desired, or considered as being acceptable. On the other hand, negative statements express what is impossible, rejected, or forbidden. Negative preferences correspond to constraints, since they specify which values or objects have to be rejected (i.e., those that do not satisfy the constraints), while positive preferences correspond to wishes, as they specify which objects are more desirable than others (i.e., satisfy user wishes) without rejecting those that do not meet the wishes." Therefore, Lee (2000, 2009) introduced the concept of bipolar fuzzy sets which is a generalization of the fuzzy sets. Bipolar neutrosophic refined sets which is an extension of the fuzzy sets, bipolar fuzzy sets, intuitionistic fuzzy sets and bipolar neutrosophic sets. Also differences and similarities between these sets are given in Table 2.

Table 2. Comparison of bipolar fuzzy set and its various extensions

	Bipolar Fuzzy	Bipolar Intuitionistic fuzzy	Bipolar İnterval- Valued neutrosophic	Bipolar Neutrosophic	Bipolar neutrosophic refined
Domain	Universe of discourse	Universe of discourse	Universe of discourse	Universe of discourse	Universe of discourse
Co-domain	Single-value in [-1,1]	Two-value in [-1,1]	Unipolar interval in [-1,1]	Bipolar $[-1,1]^3$	Bipolar $[-1,1]^3$
Number	Yes	Yes	Yes	Yes	Yes
Uncertainty	Yes	Yes	Yes	Yes	Yes
True	Yes	Yes	Yes	Yes	Yes
Falsity	No	Yes	Yes	Yes	Yes
Membership valued	Singlevalued	Singlevalued	Singlevalued	Singlevalued	Multi valued

Table 3. Comparison of different types of neutrosophic sets

	SVNS	IVNS	BNSs	Multi-valued neutrosophic sets	Trapezoidal Neutrosophic sets	Triangular Fuzzy Neutrosophic sets	**SNLSs**
Domain	Universe of discourse	Universe of discourse	Universe of discourse	Universe of discourse	Universe of discourse	Universe of discourse	Universe of iscourse
Co-domain	$[0,1]^3$	Unipolar İnterval in [0,1]	Bipolar $[-1,1]^3$	$[0,1]^3$	$[0,1]^3$	$[0,1]^3$	[0, 2t] or [-t, t]
Number	Yes	Yes	Yes	Yes	Yes	Yes	No
Uncertainty	Yes	Yes	Yes	Yes	Yes	Yes	Yes
True	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Falsity	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Indeterminacy	Yes	Yes	Yes	Yes	Yes	Yes	Yes

CONCLUSIONS

NSs are characterized by truth, indeterminacy, and falsity membership functions which are independent in nature. NSs can handle incomplete, indeterminate, and inconsistent information quite well, whereas IFSs and FSs can only handle incomplete or partial information. However, SVNS, subclass of NSs gain much popularity to apply in concrete areas such as real engineering and scientific problems. Many extensions of NSs have been appeared in the literature. Some of them are discussed in the paper. New hybrid sets derived from neutrosophic sets gain popularity as new research topics. Extensions of neutrosophic sets have been developed by many researchers. This paper presents some of their basic operations. Then, we investigate their properties and the relation between defined numbers and function on neutrosophic sets. We present comparison between bipolar fuzzy sets and its various extensions. We also present the comparison between different types of neutrosophic sets and numbers. The paper can be extended to review different types of neutrosophic hybrid sets and their theoretical development and applications in real world problems.

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A Scientific Decision Framework for Supplier Selection under Neutrosophic MOORA Environment

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Abduallah Gamal, Mahmoud Ismail, Florentin Smarandache (2018). A Scientific Decision Framework for Supplier Selection under Neutrosophic MOORA Environment. *Neutrosophic Operational Research* III: 33-47

Abstract

In this paper, we present a hybrid model of Neutrosophic-MOORA for supplier selection problems. Making a suitable model for supplier selection is an important issue to amelioration competitiveness and capability of the organization, factory, project etc. selecting of the best supplier selection is not decrease delays in any organizations but also maximum profit and saving of material costs. Thus, now days supplier selection is become competitive global environment for any organization to select the best alternative or taking a decision. From a large number of availability alternative suppliers with dissimilar strengths and weaknesses for different objectives or criteria, requiring important rules or steps for supplier selection. In the recent past, the researchers used various multi criteria decision-making (MCDM) methods successfully to solve the problems of supplier selection. In this research, Multi-Objective Optimization based on Ratio Analysis (MOORA) with neutrosophic is applied to solve the real supplier selection problems. We selected a real life example to present the solution of problem that how ranking the alternative based on decreasing cost for each alternative and how formulate the problem in steps by Neutrosophic- MOORA technique.

Keywords

MOORA; Neutrosophic; Supplier selection; MCDM.

1 Introduction

The purpose of this paper is to present a hybrid method between MOORA and Neutrosophic in the framework of neutrosophic for the selection of suppliers with a focus on multi-criteria and multi-group environment. These days, Companies, organizations, factories seek to provide a fast and a good service to meet the requirements of peoples or customers [1, 2].The field of multi criteria decision-making is considered for the selection of suppliers [3]. The selecting of the best supplier increasing the efficiency of any organization whether company, factory according to [4].

Hence, for selecting the best supplier selection there are much of methodologies we presented some of them such as fuzzy sets (FS), Analytic network process (ANP), Analytic hierarchy process (AHP), (TOPSIS) technique for order of preference by similarity to ideal solution, (DSS) Decision support system, (MOORA) multi-objective optimization by ratio analysis. A classification of these methodologies to two group hybrid and individual can reported in [4, 5].

We review that the most methodologies shows the supplier selection Analytic hierarchy process (AHP), Analytic network process (ANP) with neutrosophic in [6].

1.1 Supplier Selection Problem

A Supplier selection is considered one of the most very important components of production and vulgarity management for many organizations service.

The main goal of supplier selection is to identify suppliers with the highest capability for meeting an organization needs consistently and with the minimum cost. Using a set of common criteria and measures for abroad comparison of suppliers.

However, the level of detail used for examining potential suppliers may vary depending on an organization's needs. The main purpose and objective goal of selection is to identify high-potential suppliers. To choose suppliers, the organization present judge of each supplier according to the ability of meeting the organization consistently and cost effective it's needs using selection criteria and appropriate measure.

Criteria and measures are developed to be applicable to all the suppliers being considered and to reflect the firm's needs and its supply and technology strategy.

We show Supplier evaluation and selection process [7].

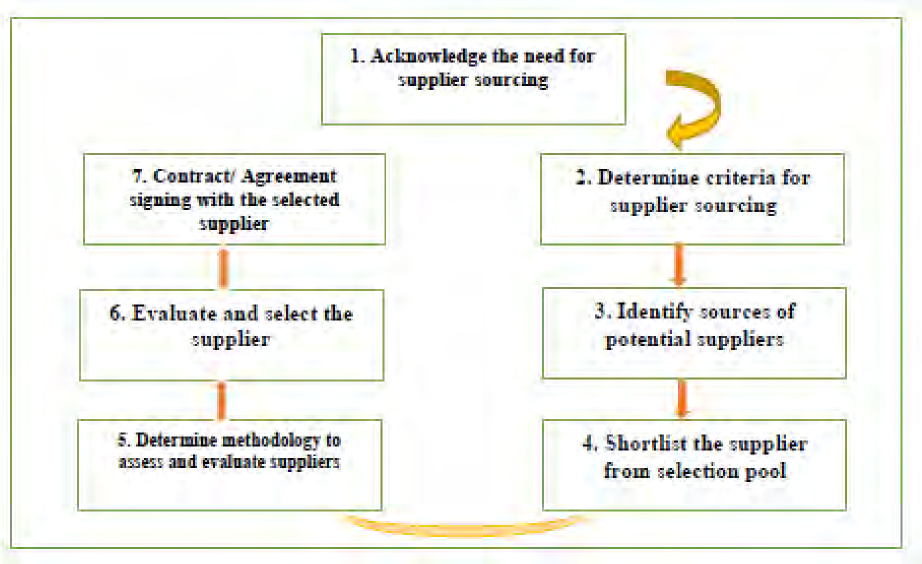

Figure 1. Supplier evaluation and selection process.

1.2 MOORA Technique

Multi-Objective Optimization on the basis of Ratio Analysis (MOORA), also known as multi criteria or multi attribute optimization. (MOORA) method seek to rank or select the best alternative from available option was introduced by Brauers and Zavadskas in 2006 [8].

The (MOORA) method has a large range of applications to make decisions in conflicting and difficult area of supply chain environment. MOORA can be applied in the project selection, process design selection, location selection, product selection etc. the process of defining the decision goals, collecting relevant information and selecting the best optimal alternative is known as decision making process.

The basic idea of the MOORA method is to calculate the overall performance of each alternative as the difference between the sums of its normalized performances which belongs to cost and benefit criteria.

This method applied in various fields successfully such as project management [9].

Table 1. Comparison of MOORA with MADM approaches

MADM method	Computational Time	Simplicity	Mathematical Calculations required
MOORA	Very less	Very simple	Minimum
AHP	Very high	Very critical	Maximum
ANP	Moderate	Moderately critical	Moderate
TOPSIS	Moderate	Moderately critical	Moderate
GRA	Very high	Very critical	Maximum

1.3 Neutrosophic Theory

Smarandache first introduced neutrosophy as a branch of philosophy which studies the origin, nature, and scope of neutralities. Neutrosophic set is an important tool which generalizes the concept of the classical set, fuzzy set, interval-valued fuzzy set, intuitionistic fuzzy set, interval-valued intuitionistic fuzzy set, paraconsistent set, dial theist set, paradoxist set, and tautological set[14-22]**.** Smarandache (1998) defined indeterminacy explicitly and stated that truth, indeterminacy, and falsity-membership are independent and lies within]-*0, 1+[.* which is the non-standard unit interval and an extension of the standard interval]-*0, 1+[.*

We present some of methodologies that it used in the multi criteria decision making and presenting the illustration between supplier selection, MOORA and Neutrosophic. Hence the goal of this paper to present the hybrid of the MOORA (Multi-Objective Optimization on the basis of Ratio Analysis) method with neutrosophic as a methodology for multi criteria decision making (MCDM).

This is ordered as follows: Section 2 gives an insight into some basic definitions on neutrosophic sets and MOORA. Section 3 explains the proposed methodology of neutrosophic MOORA model. In Section 4 a numerical example is presented in order to explain the proposed methodology. Finally, the conclusions.

2 Preliminaries

In this section, the essential definitions involving neutrosophic set, single valued neutrosophic sets, trapezoidal neutrosophic numbers and operations on trapezoidal neutrosophic numbers are defined.

2.1 Definition [10]

Let X be a space of points and $x \in X$. A neutrosophic set A in X is definite by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$ and a falsity-membership function $F_A(x)$, $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or real nonstandard subsets of $]^-0, 1^+[$. That is $T_A(x): X \to]^-0, 1^+[$, $I_A(x): X \to]^-0, 1^+[$ and $F_A(x): X \to]^-0, 1^+[$. There is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$, so $0^- \leq \sup(x) + \sup x + \sup x \leq 3^+$.

2.2 Definition [10, 11]

Let X be a universe of discourse. A single valued neutrosophic set A over X is an object taking the form $A = \{\langle x, T_A(x), I_A(x), F_A(x), \rangle : x \in X\}$, where $T_A(x): X \to [0,1]$, $I_A(x): X \to [0,1]$ and $F_A(x): X \to [0,1]$ with $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$ for all $x \in X$. The intervals $T_A(x)$, $I_A(x)$ and $F_A(x)$ represent the truth-membership degree, the indeterminacy-membership degree and the falsity membership degree of x to A, respectively. For convenience, a SVN number is represented by $A = (a, b, c)$, where $a, b, c \in [0, 1]$ and $a+b+c \leq 3$.

2.3 Definition [12]

Suppose that $\alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}} \in [0,1]$ and $a_1, a_2, a_3, a_4 \in R$ where $a_1 \leq a_2 \leq a_3 \leq a_4$. Then a single valued trapezoidal neutrosophic number, $\tilde{a} = \langle (a_1, a_2, a_3, a_4); \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}} \rangle$ is a special neutrosophic set on the real line set R whose truth-membership, indeterminacy-membership and falsity-membership functions are defined as:

$$T_{\tilde{a}}(x) = \begin{cases} \alpha_{\tilde{a}} \left(\frac{x-a_1}{a_2-a_1}\right) & (a_1 \leq x \leq a_2) \\ \alpha_{\tilde{a}} & (a_2 \leq x \leq a_3) \\ \alpha_{\tilde{a}} \left(\frac{a_4-x}{a_4-a_3}\right) & (a_3 \leq x \leq a_4) \\ 0 & otherwise \end{cases} \quad (1).$$

$$I_{\tilde{a}}(x) = \begin{cases} \frac{(a_2-x+\theta_{\tilde{a}}(x-a_1))}{(a_2-a_1)} & (a_1 \leq x \leq a_2) \\ \alpha_{\tilde{a}} & (a_2 \leq x \leq a_3) \\ \frac{(x-a_3+\theta_{\tilde{a}}(a4-x))}{(a_4-a_3)} & (a_3 \leq x \leq a_4) \\ 1 & otherwise \end{cases} , \quad (2).$$

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$$F_{\tilde{a}}(x) = \begin{cases} \frac{(a_2 - x + \beta_{\tilde{a}}(x - a_1))}{(a_2 - a_1)} & (a_1 \le x \le a_2) \\ \alpha_{\tilde{a}} & (a_2 \le x \le a_3) \\ \frac{(x - a_3 + \beta_{\tilde{a}}(a4 - x))}{(a_4 - a_3)} & (a_3 \le x \le a_4) \\ 1 & otherwise \end{cases} \quad (3).$$

where $\alpha_{\tilde{a}}$, $\theta_{\tilde{a}}$ and $\beta_{\tilde{a}}$and represent the maximum truth-membership degree, minimum indeterminacy-membership degree and minimum falsity-membership degree respectively. A single valued trapezoidal neutrosophic number $\tilde{a}$=$\langle(a_1 , a_2 , a_3 , a_4); \alpha_{\tilde{a}} , \theta_{\tilde{a}} , \beta_{\tilde{a}}\rangle$ may express an ill-defined quantity of the range, which is approximately equal to the interval $[a_2 , a_3]$.

2.4 Definition [11, 10]

Let $\tilde{a}$=$\langle(a_1 , a_2 , a_3 , a_4); \alpha_{\tilde{a}} , \theta_{\tilde{a}} , \beta_{\tilde{a}}\rangle$ and $\tilde{b}$=$\langle(b_1 , b_2 , b_3 , b_4); \alpha_{\tilde{b}} , \theta_{\tilde{b}} , \beta_{\tilde{b}}\rangle$ be two single valued trapezoidal neutrosophic numbers and $\Upsilon \neq 0$ be any real number. Then,

1. Addition of two trapezoidal neutrosophic numbers

$$\tilde{a} + \tilde{b} = \langle(a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}\rangle$$

2. Subtraction of two trapezoidal neutrosophic numbers

$$\tilde{a} - \tilde{b} = \langle(a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}\rangle$$

3. Inverse of trapezoidal neutrosophic number

$$\tilde{a}^{-1} = \langle(\frac{1}{a_4}, \frac{1}{a_3}, \frac{1}{a_2}, \frac{1}{a_1}); \alpha_{\tilde{a}} , \theta_{\tilde{a}} , \beta_{\tilde{a}}\rangle \quad \text{where } (\tilde{a} \neq 0)$$

4. Multiplication of trapezoidal neutrosophic number by constant value

$$\Upsilon\tilde{a} = \begin{cases} \langle(\Upsilon a_1 , \Upsilon a_2 , \Upsilon a_3 , \Upsilon a_4); \alpha_{\tilde{a}} , \theta_{\tilde{a}} , \beta_{\tilde{a}}\rangle & \text{if } (\Upsilon > 0) \\ \langle(\Upsilon a_4 , \Upsilon a_3 , \Upsilon a_2 , \Upsilon a_1); \alpha_{\tilde{a}} , \theta_{\tilde{a}} , \beta_{\tilde{a}}\rangle & \text{if } (\Upsilon < 0) \end{cases}$$

5. Division of two trapezoidal neutrosophic numbers

$$\frac{\tilde{a}}{\tilde{b}} = \begin{cases} \langle(\frac{a_1}{b_4}, \frac{a_2}{b_3}, \frac{a_3}{b_2}, \frac{a_4}{b_1}); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}\rangle & \text{if } (a_4 > 0 , b_4 > 0) \\ \langle(\frac{a_4}{b_4}, \frac{a_3}{b_3}, \frac{a_2}{b_2}, \frac{a_1}{b_1}); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}\rangle & \text{if } (a_4 < 0 , b_4 > 0) \\ \langle(\frac{a_4}{b_1}, \frac{a_3}{b_2}, \frac{a_2}{b_3}, \frac{a_1}{b_4}); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}\rangle & \text{if } (a_4 < 0 , b_4 < 0) \end{cases}$$

6. Multiplication of trapezoidal neutrosophic numbers

$$\tilde{a}\tilde{b} = \begin{cases} \langle(a_1 b_1 , a_2 b_2 , a_3 b_3 , a_4 b_4); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}\rangle & \text{if } (a_4 > 0 , b_4 > 0) \\ \langle(a_1 b_4 , a_2 b_3 , a_3 b_2 , a_4 b_1); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}\rangle & \text{if } (a_4 < 0 , b_4 > 0) \\ \langle(a_4 b_4 , a_3 b_3 , a_2 b_2 , a_1 b_1); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}}\rangle & \text{if } (a_4 < 0 , b_4 < 0) \end{cases}$$

3 Methodology

In this paper, we present the steps of the proposed model MOORA-Neutrosophic, we define the criteria based on the opinions of decision makers (DMs) using neutrosophic trapezoidal numbers to make the judgments on criteria more accuracy, using a scale from 0 to 1 instead of the scale (1-9) that have many drawbacks illustrated by **[13]**. We present a new scale from 0 to 1 to avoid this drawbacks. We use (n-1) judgments to obtain consistent trapezoidal neutrosophic preference relations instead of $\frac{n \times (n-1)}{2}$ to decrease the workload and not tired decision makers. (MOORA-Neutrosophic) method is used for ranking and selecting the alternatives. To do this, we first present the concept of AHP to determine the weight of each criteria based on opinions of decision makers (DMs). Then each alternative is evaluated with other criteria and considering the effects of relationship among criteria.

The steps of our model can be introduced as:

Step - 1. Constructing model and problem structuring.

a. Constitute a group of decision makers (DMs).
b. Formulate the problem based on the opinions of (DMs).

Step - 2. Making the pairwise comparisons matrix and determining the weight based on opinions of (DMs).

a. Identify the criteria and sub criteria C = {C1, C2, C3…Cm}.
b. Making matrix among criteria n × m based on opinions of (DMs).

$$W = \begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ Cn \end{array} \begin{array}{cccc} C_1 & C_2 & \dots & C_m \\ \left[(l_{11}, m_{11l}, m_{11u}, u_{11}) \right. & (l_{11}, m_{11l}, m_{11u}, u_{11}) & \cdots & \left. (l_{1n}, m_{1nl}, m_{1nu}, u_{1n}) \right] \\ (l_{21}, m_{21l}, m_{21u}, u_{21}) & (l_{22}, m_{22l}, m_{22u}, u_{22}) & \cdots & (l_{2n}, m_{2nl}, m_{2nu}, u_{2n}) \\ \cdots & \cdots & \cdots & \cdots \\ (l_{n1}, m_{n1l}, m_{n1u}, u_{n1}) & (l_{n2}, m_{n2l}, m_{n2u}, u_{n2}) & \cdots & (l_{nn}, m_{nnl}, m_{nnu}, u_{nn}) \end{array} \quad (4)$$

Decision makers (DMs) make pairwise comparisons matrix between criteria compared to each criterion focuses only on (n-1) consensus judgments instead of using $\frac{n \times (n-1)}{2}$ that make more workload and Difficult.

c. According to, the opinion of (DMs) should be among from 0 to 1 not negative. Then, we transform neutrosophic matrix to pairwise comparisons deterministic matrix by adding (α, θ, β) and using the following equation to calculate the accuracy and score.

$$S(\tilde{a}_{ij}) = \frac{1}{16}[a_1 + b_1 + c_1 + d_1] \times (2 + \alpha_{\tilde{a}} - \theta_{\tilde{a}} - \beta_{\tilde{a}}) \quad (5)$$

and

$$A(\tilde{a}_{ij}) = \frac{1}{16}[a_1 + b_1 + c_1 + d_1] \times (2 + \alpha_{\tilde{a}} - \theta_{\tilde{a}} + \beta_{\tilde{a}}) \quad (6)$$

d. We obtain the deterministic matrix by using S $(\tilde{a}_{ij})$.

e. From the deterministic matrix we obtain the weighting matrix by dividing each entry on the sum of the column.

Step - 3. Determine the decision-making matrix (DMM). The method begin with define the available alternatives and criteria

$$R = \begin{matrix} \\ A_1 \\ A_2 \\ A_3 \\ An \end{matrix} \begin{matrix} C_1 & C_2 & \dots & C_m \\ \left[\begin{matrix} (l_{11}, m_{11l}, m_{11u}, u_{11}) \\ (l_{21}, m_{21l}, m_{21u}, u_{21}) \\ \dots \\ (l_{n1}, m_{n1l}, m_{n1u}, u_{n1}) \end{matrix}\right. & \begin{matrix} (l_{11}, m_{11l}, m_{11u}, u_{11}) \\ (l_{22}, m_{22l}, m_{22u}, u_{22}) \\ \dots \\ (l_{n2}, m_{n2l}, m_{n2u}, u_{n2}) \end{matrix} & \begin{matrix} \dots \\ \dots \\ \dots \\ \dots \end{matrix} & \left.\begin{matrix} (l_{1n}, m_{1nl}, m_{1nu}, u_{1n}) \\ (l_{2n}, m_{2nl}, m_{2nu}, u_{2n}) \\ \dots \\ (l_{nn}, m_{nnl}, m_{nnu}, u_{nn}) \end{matrix}\right] \end{matrix} \quad (7)$$

where A_i represents the available alternatives where i = 1… n and the C_j represents criteria

a. Decision makers (DMs) make pairwise comparisons matrix between criteria compared to each criterion focuses only on (n-1) consensus judgments instead of using $\frac{n \times (n-1)}{2}$ that make more workload and Difficult.

b. According to, the opinion of (DMs) should be among from 0 to 1 not negative. Then, we transform neutrosophic matrix to pairwise comparisons deterministic matrix by using equations 5 &6 to calculate the accuracy and score.

c. We obtain the deterministic matrix by using S $(\tilde{a}_{ij})$.

Step - 4. Calculate the normalized decision-making matrix from previous matrix (DMM).

a. Thereby, normalization is carried out **[14].** Where the Euclidean norm is obtained according to eq. (8) to the criterionE_j.

i. $$\left|Ey_j\right| = \sqrt{\sum_1^n E_i^2} \quad (8)$$

The normalization of each entry is undertaken according to eq. (9)

ii. $$NE_{ij} = \frac{E_{ij}}{|E_j|} \tag{9}$$

Step - 5. Compute the aggregated weighted neutrosophic decision matrix (AWNDM) as the following:

i. $$\acute{R} = R \times W \tag{10}$$

Step - 6. Compute the contribution of each alternative Ny_i the contribution of each alternative

i. $$Ny_i = \sum_{i=1}^{g} Ny_i \; - \sum_{j=g+1}^{m} Nx_j \tag{11}$$

Step - 7. Rank the alternatives.

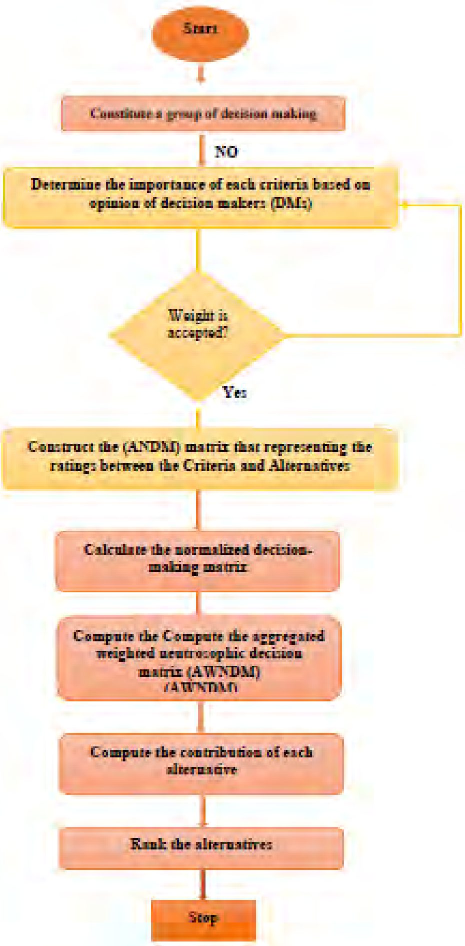

Figure 2 Schematic diagram of MOORA with neutrosophic.

4 Implementation of Neutrosophic – MOORA Technique

In this section, to illustrate the concept of MOORA with Neutrosophic we present an example. An accumulation company dedicated to the production of the computers machines has to aggregate several components in its production line. When failure occurred from suppliers (alternatives), a company ordered from another alternative based on the four criteria C_j (j = 1, 2, 3, and 4), the four criteria are as follows: C_1 for Total Cost, C_2 for Quality, C_3 for Service, C_4 for On-time delivery. The criteria to be considered is the supplier selections are determined by the DMs from a decision group. The team is broken into four groups, namelyDM_1,DM_2,DM_3 and DM_4, formed to select the most suitable alternatives. This example is that the selecting the best alternative from five alternative. A_i (i = 1, 2, 3, 4 and 5). Representing of criteria evaluation:

- Cost (C_1) Minimum values are desired.
- Quality (C_2) Maximum evaluations.
- Service (C_3) maximum evaluation.
- On-time delivery (C_4) maximum evaluation.

Step - 1. Constitute a group of decision makers (DMs) that consist of four (DM).

Step - 2. We determine the importance of each criteria based on opinion of decision makers (DMs).

$$W = \begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ C4 \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{bmatrix} (0.5\,,0.5,0.5,0.5) & (0.6,0.7,0.9,0.1) & (0.7\,,0.2,0.4,0.6) & (0.3\,,0.6,0.4,0.7) \\ (0.6\,,0.3,0.4,0.7) & (0.5\,,0.5,0.5,0.5) & (0.6\,,0.7,0.8,0.9) & (0.3\,,0.5,0.2,0.5) \\ (0.3\,,0.5,0.2,0.5) & (0.3\,,0.7,0.4,0.3) & (0.5\,,0.5,0.5,0.5) & (0.2\,,0.5,0.6,0.8) \\ (0.4\,,0.3,0.1,0.6) & (0.1\,,0.4,0.2,0.8) & (0.5\,,0.3,0.2,0.4) & (0.5\,,0.5,0.5,0.5) \end{bmatrix} \end{array}$$

Then the last matrix appears consistent according to definition 6. And then by ensuring consistency of trapezoidal neutrosophic additive reciprocal preference relations, decision makers (DMs) should determine the maximum truth-membership degree (α), minimum indeterminacy-membership degree (θ) and minimum falsity-membership degree (β) of single valued neutrosophic numbers.

$$W = \begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ C4 \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{bmatrix} (0.5\,,0.5,0.5,0.5) & (0.6,0.7,0.9,0.1;0.4,0.3,0.5) & (0.7,0.2,0.4,0.6;0.8,0.4,0.2) & (0.3,0.6,0.4,0.7;0.4,0.5,0.6) \\ (0.6,0.3,0.4,0.7;0.2,0.5,0.8) & (0.5\,,0.5,0.5,0.5) & (0.6,0.7,0.8,0.9;0.2,0.5,0.7) & (0.3,0.5,0.2,0.5;0.5,0.7,0.8) \\ (0.3,0.5,0.2,0.5;0.4,0.5,0.7) & (0.3,0.7,0.4,0.3;0.2,0.5,0.9) & (0.5\,,0.5,0.5,0.5) & (0.2,0.5,0.6,0.8;0.4,0.3,0.8) \\ (0.4,0.3,0.1,0.6;0.2,0.3,0.5) & (0.1,0.4,0.2,0.8;0.7,0.3,0.6) & (0.5,0.3,0.2,0.4;0.3,0.4,0.7) & (0.5\,,0.5,0.5,0.5) \end{bmatrix} \end{array}$$

From previous matrix we can determine the weight of each criteria by using the following equation of S ($\tilde{a}_{ij}$)

$$S(\tilde{a}_{ij}) = \frac{1}{16}[a_1 + b_1 + c_1 + d_1] \times (2 + \alpha_{\tilde{a}} - \theta_{\tilde{a}} - \beta_{\tilde{a}})$$

and

$$A(\tilde{a}_{ij}) = \frac{1}{16}[a_1 + b_1 + c_1 + d_1] \times (2 + \alpha_{\tilde{a}} - \theta_{\tilde{a}} + \beta_{\tilde{a}})$$

The deterministic matrix can obtain by S ($\tilde{a}_{ij}$) equation in the following step:

$$W = \begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ C4 \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{bmatrix} 0.5 & 0.23 & 0.261 & 0.163 \\ 0.113 & 0.5 & 0.188 & 0.10 \\ 0.113 & 0.085 & 0.5 & 0.17 \\ 0.123 & 0.169 & 0.105 & 0.5 \end{bmatrix} \end{array}$$

From this matrix we can obtain the weight criteria by dividing each entry by the sum of each column.

$$W = \begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ C4 \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{bmatrix} 0.588 & 0.234 & 0.237 & 0.175 \\ 0.133 & 0.508 & 0.171 & 0.107 \\ 0.133 & 0.086 & 0.455 & 0.182 \\ 0.145 & 0.172 & 0.095 & 0.536 \end{bmatrix} \end{array}$$

Step - 3. Construct the (ANDM) matrix that representing the ratings given by every DM between the Criteria and Alternatives.

$\tilde{R} =$

	C_1	C_2	C_3	C_4
A_1	(0.5, 0.3,0.2,0.4)	(0.6, 0.7,0.9,0.1)	(0.7, 0.9,1.0,1.0)	(0.4, 0.7,1.0,1.0)
A_2	(0.0, 0.1, 0.3, 0.4)	(0.7, 0.6,0.8,0.3)	(0.6 , 0.7,0.8,0.9)	(0.3, 0.5,0.9,1.0)
A_3	(0.4, 0.2,0.1,0.3)	(0.3,0.0 ,0.5,0.8)	(0.4, 0.2,0.1,0.3)	(0.2, 0.5,0.6,0.8)
A_4	(0.7, 0.3, 0.3, 0.6)	(0.6, 0.1, 0.7, 1.0)	(0.2, 0.4, 0.5, 0.8)	(0.3, 0.4,0.2,0.5)
A_5	(0.5, 0.4,0.2,0.6)	(0.4, 0.6,0.1,0.2)	(0.6, 0.1,0.3,0.5)	(0.7, 0.1,0.3,0.2)

Then the last matrix appears consistent according to definition 6. And then by ensuring consistency of trapezoidal neutrosophic additive reciprocal preference relations, decision makers (DMs) should determine the maximum truth-membership degree (α), minimum indeterminacy-membership degree (θ) and minimum falsity-membership degree (β) of single valued neutrosophic numbers.

$R=$

	C_1	C_2	C_3	C_4
A_1	(0.5, 0.3,0.2,0.4; 0.3,0.4,0.6)	(0.6, 0.7,0.9,0.1; 0.3,0.4,0.5)	(0.7, 0.9,1.0,1.0; 0.2,0.5,0.3)	(0.4, 0.7,1.0,1.0; 0.1,0.3,0.4)
A_2	(0.0, 0.1, 0.3, 0.4; 0.6,0.1,0.4)	(0.7, 0.6,0.8,0.3; 0.4,0.8,0.1)	(0.6 , 0.7,0.8,0.9; 0.2,0.3,0.5)	(0.3, 0.5,0.9,1.0; 0.2,0.4,0.6)
A_3	(0.4, 0.2,0.1,0.3; 0.3,0.5,0.2)	(0.3,0.0 ,0.5,0.8; 0.5,0.7,0.2)	(0.4, 0.2,0.1,0.3; 0.5,0.7,0.5)	(0.2, 0.5,0.6,0.8; 0.1,0.2,0.5)
A_4	(0.7, 0.3, 0.3, 0.6; 0.5,0.3,0.1)	(0.6, 0.1, 0.7, 1.0; 0.2,0.6,0.3)	(0.2, 0.4, 0.5, 0.8; 0.1,0.4,0.8)	(0.3, 0.4,0.2,0.5; 0.3,0.8,0.7)
A_5	(0.5, 0.4,0.2,0.6; 0.9,0.4,0.6)	(0.4, 0.6,0.1,0.2; 0.1,0.5,0.4)	(0.6, 0.1,0.3,0.5; 0.8,0.6,0.2)	(0.7, 0.1,0.3,0.2; 0.3,0.9,0.6)

From previous matrix we can determine the weight of each criteria by using the following equation of S ($\tilde{a}_{ij}$)

S ($\tilde{a}_{ij}$) = $\frac{1}{16}[a_1 + b_1 + c_1 + d_1] \times (2 + \alpha_{\tilde{a}} - \theta_{\tilde{a}} - \beta_{\tilde{a}})$

and

A ($\tilde{a}_{ij}$) = $\frac{1}{16}[a_1 + b_1 + c_1 + d_1] \times (2 + \alpha_{\tilde{a}} - \theta_{\tilde{a}} + \beta_{\tilde{a}})$

The deterministic matrix can obtain by S ($\tilde{a}_{ij}$) equation in the following step:

$R =$

	C_1	C_2	C_3	C_4
A_1	0.11	0.20	0.32	0.27
A_2	0.11	0.23	0.26	0.20
A_3	0.10	0.16	0.08	0.18
A_4	0.25	0.19	0.11	0.07
A_5	0.20	0.09	0.19	0.07

Step - 4. Calculate the normalized decision-making matrix from previous matrix.

By this equation = $|X_j| = \sqrt{\sum_1^n x_i^2}$,

$NX_{ij} = \frac{x_{ij}}{|x_j|}$

a. Sum of squares and their square roots

	C_1	C_2	C_3	C_4
A_1	0.11	0.20	0.32	0.27
A_2	0.11	0.23	0.26	0.20
A_3	0.10	0.16	0.08	0.18
A_4	0.25	0.19	0.11	0.07
A_5	0.20	0.09	0.19	0.07
Sum of square	0.14	0.16	0.22	0.16
Square root	0.37	0.40	0.47	0.40

b. Objectives divided by their square roots and MOORA

$R =$

	C_1	C_2	C_3	C_4
A_1	0.30	0.50	0.68	0.67
A_2	0.30	0.58	0.55	0.50
A_3	0.27	0.40	0.17	0.45
A_4	0.68	0.48	0.23	0.18
A_5	0.54	0.23	0.40	0.18

Step - 5. Compute the aggregated weighted neutrosophic decision matrix (AWNDM) as the following:

$$\acute{R} = R \times W$$

$$= \begin{bmatrix} 0.30 & 0.50 & 0.68 & 0.67 \\ 0.30 & 0.58 & 0.55 & 0.50 \\ 0.27 & 0.40 & 0.17 & 0.45 \\ 0.68 & 0.48 & 0.23 & 0.18 \\ 0.54 & 0.23 & 0.40 & 0.18 \end{bmatrix} \times \begin{bmatrix} 0.588 & 0.234 & 0.237 & 0.175 \\ 0.133 & 0.508 & 0.171 & 0.107 \\ 0.133 & 0.086 & 0.455 & 0.182 \\ 0.145 & 0.172 & 0.095 & 0.536 \end{bmatrix} =$$

$$= \begin{bmatrix} 0.43 & 0.20 & 0.49 & 0.59 \\ 0.40 & 0.49 & 0.47 & 0.48 \\ 0.29 & 0.59 & 0.25 & 0.36 \\ 0.52 & 0.45 & 0.36 & 0.31 \\ 0.42 & 0.31 & 0.37 & 0.29 \end{bmatrix}$$

Step - 6. Compute the contribution of each alternative Ny_i the contribution of each alternative

$$Ny_i = \sum_{i=1}^{g} Ny_i \ - \ \sum_{j=g+1}^{m} Nx_j$$

	C_1	C_2	C_3	C_4	Y_i	Rank
A_1	0.43	0.20	0.49	0.59	0,85	3
A_2	0.40	0.49	0.47	0.48	0.99	1
A_3	0.29	0.59	0.25	0.36	0.91	2
A_4	0.52	0.45	0.36	0.31	0.60	4
A_5	0.42	0.31	0.37	0.29	0.55	5

Step - 7. Rank the alternatives. The alternatives are ranked according the min cost for alternative as alternative $A_2 > A_3 > A_1 > A_4 > A_5$

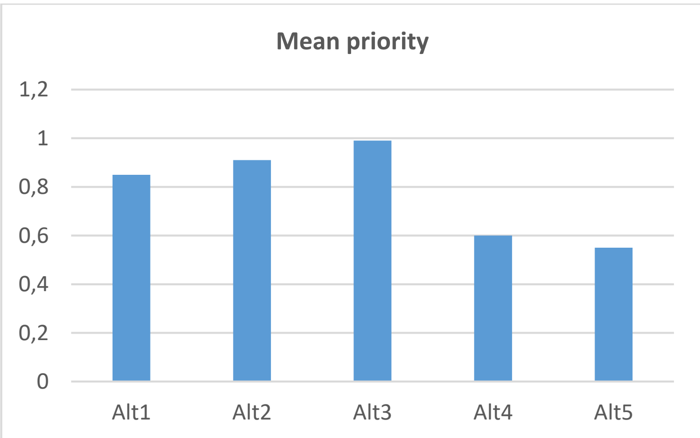

Figure 3. The MOORA- Neutrosophic ranking of alternatives.

5 Conclusion

This research presents a hybrid of the (MOORA) method with Neutrosophic for supplier selection. We presented the steps of the method in seven steps and a numerical case was presented to illustrate it. The proposed methodology provides a good hybrid technique that can facilitate the selecting of the best alternative by decision makers. Then neutrosophic provide better flexibility and the capability of handling subjective information to solve problems in the decision making. As future work, it would be interesting to apply MOORA-Neutrosophic technique in different areas as that is considered one of the decision making for selection of the best alternatives. For example, project selection, production selection, etc. The case study we presented is an example about selecting the alternative that the decision makers (DMs) specify the criteria and how select the best alternatives.

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A New Concept of Matrix Algorithm for MST in Undirected Interval Valued Neutrosophic Graph

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Abstract

In this chapter, we introduce a new algorithm for finding a minimum spanning tree (MST) of an undirected neutrosophic weighted connected graph whose edge weights are represented by an interval valued neutrosophic number. In addition, we compute the cost of MST and compare the de-neutrosophied value with an equivalent MST having the detereministic weights. Finally, a numerical example is provided.

Keywords

Interval valued Neutrosophic Graph, Score function, Minimum Spanning Tree (MST).

1 Introduction

In order to express the inconsistency and indeterminacy that exist in real-life problems reasonably, Smarandache [3] proposed the concept of neutrosophic sets (NSs) from a philosophical standpoint, which is characterized by three totally independent functions, i.e., a truth-function, an indeterminacy function and a falsity function that are inside the real standard or non-standard unit interval]-0, 1+[. Hence, neutrosophic sets can be regarded as many extended forms of classical fuzzy sets [8] such as intuitionistic fuzzy sets [6], interval-valued intuitionistic fuzzy sets [7] etc. Moreover, for the sake of applying neutrosophic sets in real-world problems efficiently, Smarandache [9] put forward the notion of single valued neutrosophic sets (SVNSs for short) firstly, and then various theoretical operators of single valued neutrosophic sets were defined by Wang et al. [4]. Based on single valued neutrosophic sets, Wang et al. [5] further developed the notion of interval valued neutrosophic sets (IVNSs for short), some of their properties were also explored. Since then, studies of neutrosophic sets and their hybrid extensions have been paid great attention by numerous scholars [19]. Many researchers have proposed a frutiful results on interval valued neutrosophic sets [12,14,16,17,18,20,21-31]

MST is most fundamental and well-known optimization problem used in networks in graph theory. The objective of this MST is to find the minimum weighted spanning tree of a weighted connected graph. It has many real time applications, which includes communication system, transportation problems, image processing, logistics, wireless networks, cluster analysis and so on. The classical problems related to MST [1], the arc lengths are taken and it is fixed so that the decision maker use the crisp data to represent the arc lengths of a connected weighted graph. But in the real world scenarios the arch length represents a parameter which may not have a precise value. For example, the demand and supply, cost problems, time constraints, traffic frequencies, capacities etc., For the road networks, even though the geometric distance is fixed, arc length represents the vehicle travel time which fluctuates due to different weather conditions, traffic flow and some other unexpected factors. There are several algorithms for finding the MST in classical graph theory. These are based on most well-known algorithms such as Prims and Kruskals algorithms. Nevertheless, these algorithms cannot handle the cases when the arc length is fuzzy which are taken into consideration [2].

More recently, some scholars have used neutrosophic methods to find minimum spanning tree in neutrosophic environment. Ye [8] defined a method to find minimum spanning tree of a graph where nodes (samples) are represented in the form of NSs and distance between two nodes represents the dissimilarity between the corresponding samples. Mandal and Basu [9] defined a new approach of optimum spanning tree problems considering the inconsistency, incompleteness and indeterminacy of the information. They considered a network problem with multiple criteria represented by weight of each edge in neutrosophic sets. Kandasamy [11] proposed a double-valued Neutrosophic Minimum Spanning Tree (DVN-MST) clustering algorithm to cluster the data represented by double-valued neutrosophic information. Mullai [15] discussed the MST problem on a graph in which a bipolar neutrosophic number is associated to each edge as its edge length, and illustrated it by a numerical example. To the end, no research dealt with the cases of interval valued neutrosophic arc lengths.

The main objective of this work is to find the minimum spanning tree of undirected neutrosophic graphs using the proposed matrix algorithm. It would be very much useful and easy to handle the considered problem of interval valued neutrosophic arc lengths using this algorithm.

The rest of the paper is organized as follows. Section 2 briefly introduces the concepts of neutrosophic sets, single valued neutrosophic sets, interval valued neutrosophic sets and the score function of interval valued neutrosophic number. Section 3 proposes a novel approach for finding the minimum spanning tree of interval valued neutrosophic undirected graph. In Section 4, two illustrative examples are presented to illustrate the proposed method. Finally, Section 5 contains conclusions and future work.

2 Preliminaries

Definition 2.1 [3] Le ξ be an universal set. The neutrosophic set A on the universal set ξ categorized in to three membership functions called the true $T_A(x)$, indeterminate $I_A(x)$ and false $F_A(x)$ contained in real standard or non-standard subset of $]^-0, 1^+[$ respectively.

$$^-0 \le \sup T_A(x) + \sup I_A(x) + \sup I_A(x) \le 3^+ \quad (1)$$

Definition 2.2 [4] Let ξ be a universal set. The single valued neutrosophic sets (SVNs) A on the universal ξ is denoted as following

$$A = \{<x: T_A(x), I_A(x), F_A(x)> x \in \xi\} \quad (2)$$

The functions $T_A(x) \in$ [0. 1], $I_A(x) \in$ [0. 1] and $F_A(x) \in$ [0. 1] are named degree of truth, indeterminacy and falsity membership of x in A, satisfy the following condition:

$$0 \le T_A(x) + I_A(x) + T_A(x) \le 3 \quad (3)$$

Definition 2.3 [5]. An interval valued neutrosophic set A in X is defined as an object of the form

$$\tilde{A} = \{< x, \tilde{t}, \tilde{i}, \tilde{f} >: x \in X\},$$

where $\tilde{t} = [T^L_{\tilde{A}}, T^U_{\tilde{A}}]$, $\tilde{i} = [I^L_{\tilde{A}}, I^U_{\tilde{A}}]$, $\tilde{f} = [F^L_{\tilde{A}}, F^U_{\tilde{A}}]$, and $T_{\tilde{A}}^{L}$, $T_{\tilde{A}}^{U}$ $I_{\tilde{A}}^{L}$, $I_{\tilde{A}}^{U}$,$F_{\tilde{A}}^{L}$, $F_{\tilde{A}}^{U}$:X$\rightarrow$ [0, 1].The interval membership degree where $T_{\tilde{A}}^{L}$, $T_{\tilde{A}}^{U}$ $I_{\tilde{A}}^{L}$, $I_{\tilde{A}}^{U}$,$F_{\tilde{A}}^{L}$, $F_{\tilde{A}}^{U}$ denotes the lower and upper truth membership, lower and upper indeterminate membership and lower and upper false membership of an element $\in$ X corresponding to an interval valued neutrosophic set A where $0 \leq T_{M}^{p} + I_{M}^{p} + F_{M}^{p} \leq 3$

In order to rank the IVNS, TAN [18] defined the following score function.

Definition 2.4 [18]. Let $\widetilde{A} =< \widetilde{t}\ ,\widetilde{i}\ ,\widetilde{f}\ >$ be an interval valued neutrosophic number , where $\widetilde{t} = \left[T_{\tilde{A}}^{L}, T_{\tilde{A}}^{U}\right]$, $\widetilde{i} = \left[I_{\tilde{A}}^{L}, I_{\tilde{A}}^{U}\right]$, $\widetilde{f} = \left[F_{\tilde{A}}^{L}, F_{\tilde{A}}^{U}\right]$,Then, the score function $s(\tilde{A}\ )$, accuracy function $a(\tilde{A}\ )$ and certainty function $c(\tilde{A}\ )$ of an IVNN can be represented as follows:

(i) $$S_{TAN}(\widetilde{A}\ ) = \frac{(2 + T_{\tilde{A}}^{L} - I_{\tilde{A}}^{L} - F_{\tilde{A}}^{L}) + (2 + T_{\tilde{A}}^{U} - I_{\tilde{A}}^{U} - F_{\tilde{A}}^{U})}{6},$$

$$S(\widetilde{A}\ ) \in [0,1] \quad (4)$$

(ii) $$a_{TAN}(\widetilde{A}\ ) = \frac{(T_{\tilde{A}}^{L} - F_{\tilde{A}}^{L}) - (T_{\tilde{A}}^{U} - F_{\tilde{A}}^{U})}{2} \quad a(\widetilde{A}\ ) \in [-1,1] \quad (5)$$

TAN [18] gave an order relation between two IVNNs, which is defined as follows

Let $\widetilde{A}_1 =< \widetilde{t}_1\ ,\widetilde{i}_1\ ,\widetilde{f}_1\ >$ and $\widetilde{A}_2 =< \widetilde{t}_2\ ,\widetilde{i}_2\ ,\widetilde{f}_2\ >$ be two interval valued neutrosophic numbers then

i. If $s(\tilde{A}_1) \succ s(\tilde{A}_2)$, then $\tilde{A}_1$ is greater than $\tilde{A}_2$, that is, $\tilde{A}_1$ is superior to $\tilde{A}_2$, denoted by $\tilde{A}_1 \succ \tilde{A}_2$
ii. If $s(\tilde{A}_1) = s(\tilde{A}_2)$,and $a(\tilde{A}_1) \succ a(\tilde{A}_2)$ then $\tilde{A}_1$ is greater than $\tilde{A}_2$, that is, $\tilde{A}_1$ is superior to $\tilde{A}_2$, denoted by $\tilde{A}_1 \succ \tilde{A}_2$
iii. If $s(\tilde{A}_1) = s(\tilde{A}_2)$, $a(\tilde{A}_1) = a(\tilde{A}_2)$, then $\tilde{A}_1$ is equal to $\tilde{A}_2$, that is, $\tilde{A}_1$ is indifferent to $\tilde{A}_2$, denoted by $\tilde{A}_1 = \tilde{A}_2$

Definition 2.5 [17]: Let $\tilde{A} = \left\langle \left[T_{\tilde{A}}^{L}, T_{\tilde{A}}^{U}\right], \left[I_{\tilde{A}}^{L}, I_{\tilde{A}}^{U}\right], \left[F_{\tilde{A}}^{L}, F_{\tilde{A}}^{U}\right] \right\rangle$ be an IVNN, the score function S of $\tilde{A}$ is defined as follows

$$S_{RIDVAN}\left(\tilde{A}\right) = \frac{1}{4}\left(2 + T_{A}^{L} + T_{A}^{U} - 2I_{A}^{L} - 2I_{A}^{U} - F_{A}^{L} - F_{A}^{L}\right),\ S\left(\tilde{A}\right) \in [-1,1]. \quad (6)$$

Definition 2.6 [17]: Let $\tilde{A}=\left\langle\left[T_{\tilde{A}}^{L},T_{\tilde{A}}^{U}\right],\left[I_{\tilde{A}}^{L},I_{\tilde{A}}^{U}\right],\left[F_{\tilde{A}}^{L},F_{\tilde{A}}^{U}\right]\right\rangle$ be an IVNN, the accuracy function H of $\tilde{A}$ is defined as follows

$$H_{RIDVAN}\left(\tilde{A}\right)=\frac{1}{2}\begin{pmatrix}T_{A}^{L}+T_{A}^{U}-2\\ I_{A}^{U}(1-T_{A}^{U})-I_{A}^{L}(1-T_{A}^{L})-F_{A}^{U}(1-I_{A}^{L})-F_{A}^{L}(1-I_{A}^{U})\end{pmatrix},H\left(\tilde{A}\right)\in[-1,1]. \quad (7)$$

To rank any two IVNNs $\tilde{A}=\left\langle\left[T_{\tilde{A}}^{L},T_{\tilde{A}}^{U}\right],\left[I_{\tilde{A}}^{L},I_{\tilde{A}}^{U}\right],\left[F_{\tilde{A}}^{L},F_{\tilde{A}}^{U}\right]\right\rangle$ and $\tilde{B}=\left\langle\left[T_{\tilde{B}}^{L},T_{\tilde{B}}^{U}\right],\left[I_{\tilde{B}}^{L},I_{\tilde{B}}^{U}\right],\left[F_{\tilde{B}}^{L},F_{\tilde{B}}^{U}\right]\right\rangle$,

Ridvan [17] introduced the following method.

Definition 2.7 [17]: Let $\tilde{A}$ and $\tilde{B}$ be two IVNNs, $S\left(\tilde{A}\right)$ and $S\left(\tilde{B}\right)$ be scores of $\tilde{A}$ and $\tilde{B}$ respectively, and $H\left(\tilde{A}\right)$ and $H\left(\tilde{B}\right)$ be accuracy values of $\tilde{A}$ and $\tilde{B}$ respectively, then

i. If $S(\tilde{A})>S(\tilde{B})$ then $\tilde{A}$ is larger than $\tilde{B}$, denoted $\tilde{A}>\tilde{B}$.
ii. If $S(\tilde{A})=S(\tilde{B})$ then we check their accuracy values and decide as follows:
(a) If $H(\tilde{A})=H(\tilde{B})$, then $\tilde{A}=\tilde{B}$.
(b) However, if $H(\tilde{A})>H(\tilde{B})$, then $\tilde{A}$ is larger than $\tilde{B}$, denoted $\tilde{A}>\tilde{B}$.

Definition 2.8 [12]: Let $\tilde{A}=\left\langle\left[T_{\tilde{A}}^{L},T_{\tilde{A}}^{U}\right],\left[I_{\tilde{A}}^{L},I_{\tilde{A}}^{U}\right],\left[F_{\tilde{A}}^{L},F_{\tilde{A}}^{U}\right]\right\rangle$ be an IVNN, the score function S of $\tilde{A}$ is defined as follows

$$S_{NANCY}\left(\tilde{A}\right)=\frac{4+\left(T_{A}^{L}+T_{A}^{U}-2I_{A}^{L}-2I_{A}^{U}-F_{A}^{L}-F_{A}^{L}\right)\left(4-T_{A}^{L}+T_{A}^{U}-F_{A}^{L}-F_{A}^{L}\right)}{8},\ S\left(\tilde{A}\right)\in[0,1].$$

Remark 2.9: In neutrosophic mathematics, the zero sets are represented by the following form 0_N={<x, [0, 0], [1, 1], [1,1])> :x∈ X}.

3 The proposed algorithm

The following algorithm is a new concept of finding the MST of undirected interval valued neutrosophic graph using the matrix approach.

Algorithm:

Input: the weight matrix M $=\left[W_{ij}\right]_{n\times n}$ for the undirected weighted interval valued neutrosophic graph *G*.

Output: Minimum cost Spanning tree *T* of *G*.

Step 1: Input interval valued neutrosophic adjacency matrix *A*.

Step 2: Convert the interval valued neutrosophic matrix into a score matrix $\left[S_{ij}\right]_{n\times n}$ using the score function.

Step 3: Iterate step 4 and step 5 until all (n-1) entries matrix of S are either marked or set to zero or other words all the nonzero elements are marked.

Step 4: Find the weight matrix M either columns-wise or row-wise to determine the unmarked minimum entries S_{ij} which is the weight of the corresponding edge e_{ij} in M.

Step 5: If the corresponding edge e_{ij} of selected S_{ij} produces a cycle with the previous marked entries of the score matrix S then set $S_{ij} = 0$ else mark S_{ij} .

Step 6: Construct the graph *T* including only the marked entries from the score matrix S which shall be desired minimum cost spanning tree of *G*.

4 Practical example

4.1 Example 1

In this section, a numerical example of IVNMST is used to demonstrate of the proposed algorithm. Consider the following graph G= (V, E) shown Figure 1, with fives nodes and seven edges. Various steps involved in the construction of the minimum cost spanning tree are described as follow –

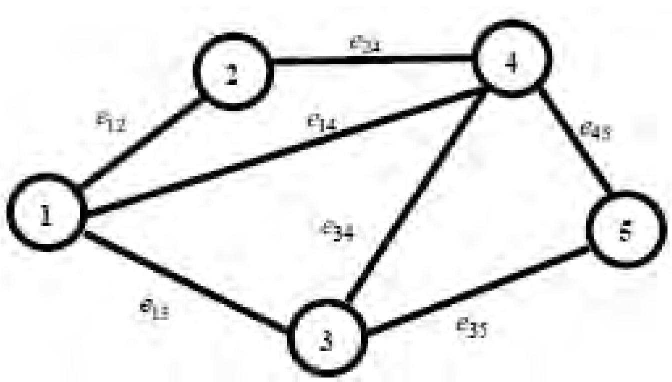


Fig.1. Undirected interval valued neutrosophic graphs

Table 1.

e_{ij}	**Edge length**
e_{12}	<[.3, .4], [.1, .2], <[.2, .4]>
e_{13}	<[.4, .5], [.2, .6], <[.4, .6]>
e_{14}	<[.1, .3], [.6, .8], <[.8, .9]>
e_{24}	<[.4, .5], [.8, .9], <[.3, .4]>
e_{34}	<[.2, .4], [.3, .4], <[.7, .8]>
e_{35}	<[.4, .5], [.6, .7], <[.5, .6]>
e_{45}	<[.5, .6], [.4, .5], <[.3, .4]

The interval valued neutrosophic adjacency matrix A is computed below:

$$A= \begin{bmatrix} 0 & \boldsymbol{e_{12}} & \boldsymbol{e_{13}} & \boldsymbol{e_{14}} & 0 \\ \boldsymbol{e_{12}} & 0 & 0 & \boldsymbol{e_{24}} & 0 \\ \boldsymbol{e_{13}} & 0 & 0 & \boldsymbol{e_{34}} & \boldsymbol{e_{35}} \\ \boldsymbol{e_{14}} & \boldsymbol{e_{24}} & \boldsymbol{e_{34}} & 0 & \boldsymbol{e_{45}} \\ 0 & 0 & \boldsymbol{e_{35}} & \boldsymbol{e_{45}} & 0 \end{bmatrix}$$

Applying the score function proposed by Tan [18], we get the score matrix:

$$S= \begin{bmatrix} 0 & 0.633 & 0.517 & 0.217 & 0 \\ 0.633 & 0 & 0 & 0.5 & 0 \\ 0.517 & 0 & 0 & 0.45 & 0.4 \\ 0.217 & 0.5 & 0.45 & 0 & 0.583 \\ 0 & 0 & 0.4 & 0.583 & 0 \end{bmatrix}$$

In this matrix, the minimum entries 0.217 is selected and the corresponding edge (1, 4) is marked by the green color. Repeat the procedure until termination (Figure 2).

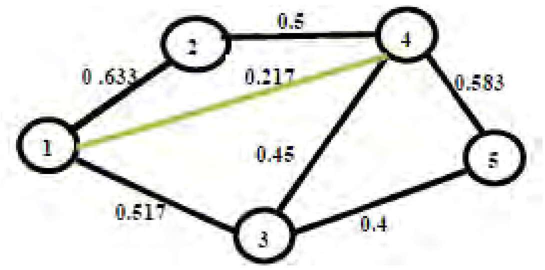


Fig.2. Marked interval valued neutrosophic graphs

The next non-zero minimum entries 0.4 is marked and corresponding edges (3, 5) are also colored (Figure 3).

$$S = \begin{bmatrix} 0 & 0.633 & 0.517 & 0.217 & 0 \\ 0.633 & 0 & 0 & 0.5 & 0 \\ 0.517 & 0 & 0 & 0.45 & 0.4 \\ 0.217 & 0.5 & 0.45 & 0 & 0.583 \\ 0 & 0 & 0.4 & 0.583 & 0 \end{bmatrix}$$

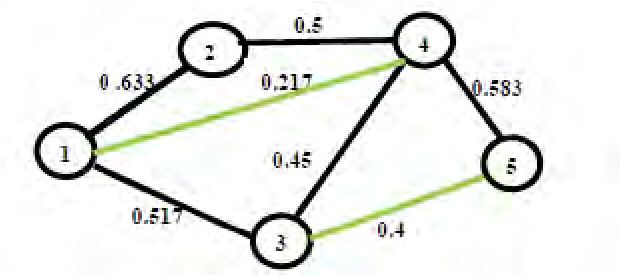


Fig. 3. Marked interval valued neutrosophic graphs in next iteration

$$S = \begin{bmatrix} 0 & 0.633 & 0.517 & 0.217 & 0 \\ 0.633 & 0 & 0 & 0.5 & 0 \\ 0.517 & 0 & 0 & 0.45 & 0.4 \\ 0.217 & 0.5 & 0.45 & 0 & 0.583 \\ 0 & 0 & 0.4 & 0.583 & 0 \end{bmatrix}$$

The next non-zero minimum entries 0.45 is marked. The corresponding marked edges are portrayed in Figure 4.

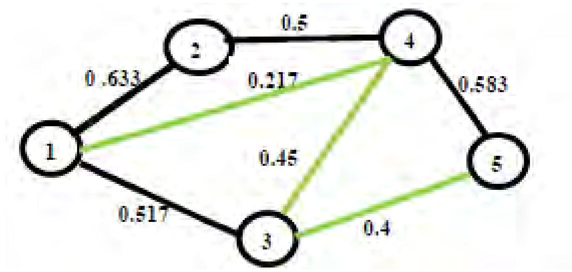


Fig. 4. Marked interval valued neutrosophic graphs in next iteration

	0	0.633	0.517	0.217	0
	0.633	0	0	0.5	0
S=	0.517	0	0	0.45	0.4
	0.217	0.5	0.45	0	0.583
	0	0	0.4	0.583	0

The next non-zero minimum entries 0. 5 is marked. The corresponding marked edges are portrayed in Figure 5.

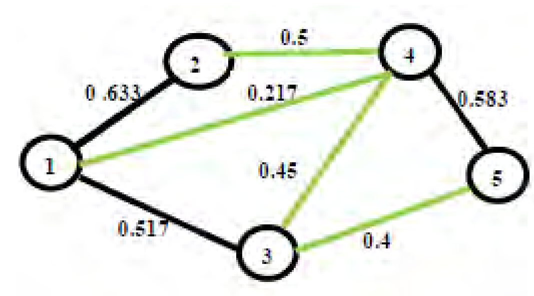


Fig. 5. Marked interval valued neutrosophic graphs in next iteration

	0	0.633	~~0.517~~ 0	0.217	0
	0.633	0	0	0.5	0
S=	0.517	0	0	0.45	0.4
	0.217	0.5	0.45	0	0.583
	0	0	0.4	0.583	0

The next minimum non-zero element 0.517 is marked. However, while drawing the edges, it produces the cycle so we delete and mark it as 0 instead of 0.517 (Figure 6).

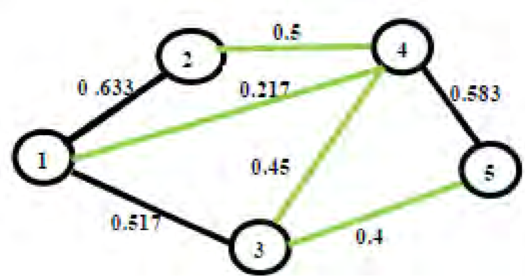


Fig. 6. Cycle {1, 3, 4}

S=

0	0.633	~~0.517~~ 0	0.217	0
0.633	0	0	0.5	0
0.517	0	0	0.45	0.4
0.217	0.5	0.45	0	0.583
0	0	0.4	~~0.583~~ 0	0

The next minimum non-zero element 0.583 is marked. However, while drawing the edges, it produces the cycle so we delete and mark it as 0 instead of 0.583 (Figure 7).

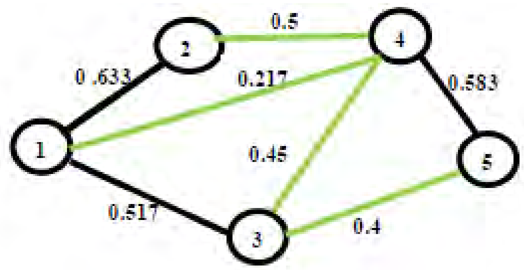


Fig. 7. Cycle {3, 4, 5}

S=

0	0.633	~~0.517~~ 0	0.217	0
~~0.633~~ 0	0	0	0.5	0
0.517	0	0	0.45	0.4
0.217	0.5	0.45	0	0.583
0	0	0.4	~~0.583~~ 0	0

The next minimum non-zero element 0.633 is marked. However, while drawing the edges, it produces the cycle so we delete and mark it as 0 instead of 0.633 (Figure 8).

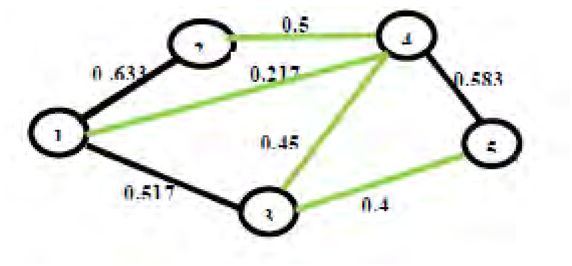


Fig. 8. Marked edges in the next round

Finally, we get the final path of minimum cost of spanning tree of G is portrayed in Figure 9.

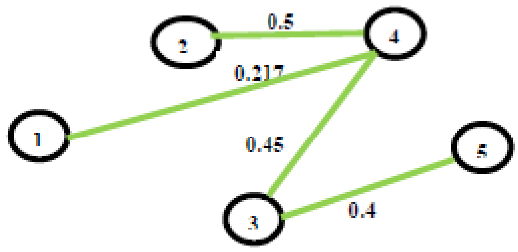


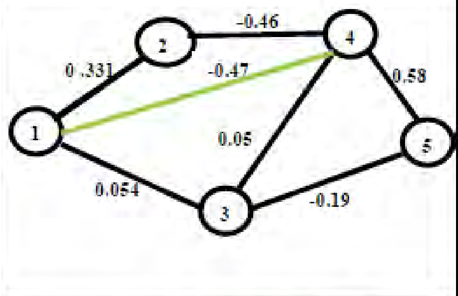
Fig. 9. Final path of minimum cost of spanning tree of the graph

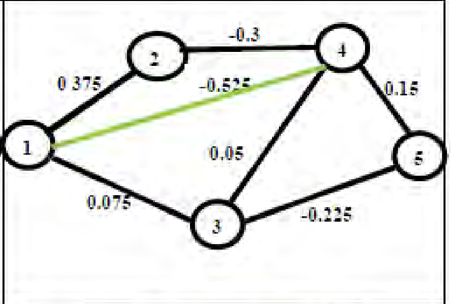
And thus, the crisp minimum cost spanning tree is 1.567 and the final path of minimum cost of spanning tree is{2, 4},{4, 1},{4, 3},{3, 5}. The procedure is termination.

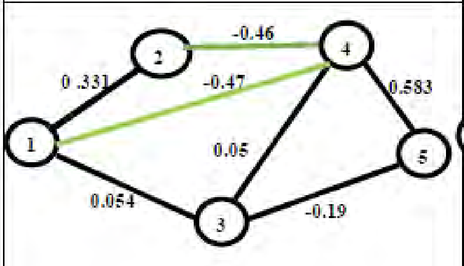
4.2 Example 2

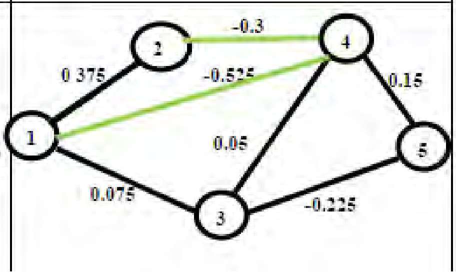
The score function is used in machine learning involved in manipulating probabilities. Here the score functions in the proposed algorithm plays a vital role in identifying the minimum spanning tree of undirected interval valued neutrosophic graphs. Also based on the order of polynomial time computation the score function used are approaching towards different MST for an Neutrosophic graph. We compare our proposed method with these scoring methods used by different researchers and hence compute the MST of undirected interval valued neutrosophic graphs.

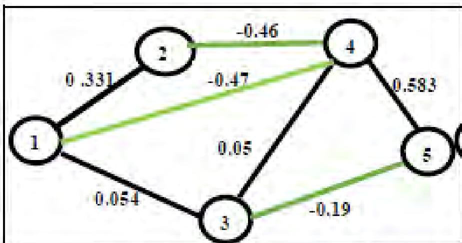
Using the score function proposed by Nancy [12], we get the score matrix	Using the score function proposed by RIDVAN [17], we get the score matrix
$S=\begin{bmatrix} 0 & 0.331 & 0.054 & -0.47 & 0 \\ 0.331 & 0 & 0 & -0.46 & 0 \\ 0.517 & 0 & 0 & 0.05 & -0.19 \\ -0.47 & -0.46 & 0.05 & 0 & 0.115 \\ 0 & 0 & -0.19 & 0.115 & 0 \end{bmatrix}$	$S=\begin{bmatrix} 0 & 0.375 & 0.075 & -0.525 & 0 \\ 0.375 & 0 & 0 & -0.3 & 0 \\ 0.075 & 0 & 0 & 0.05 & -0.225 \\ -0.525 & -0.3 & 0.05 & 0 & 0.15 \\ 0 & 0 & -0.225 & 0.15 & 0 \end{bmatrix}$

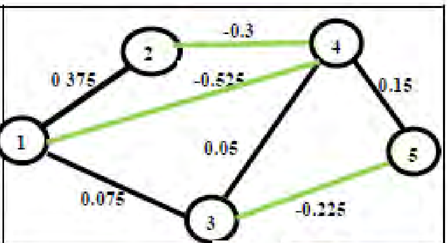


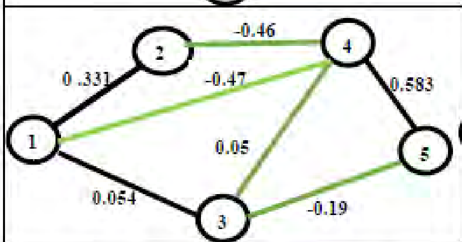


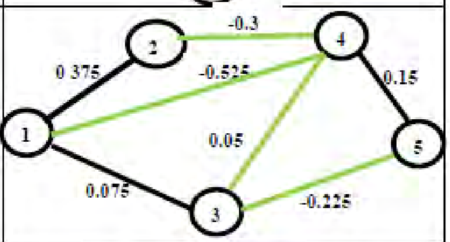


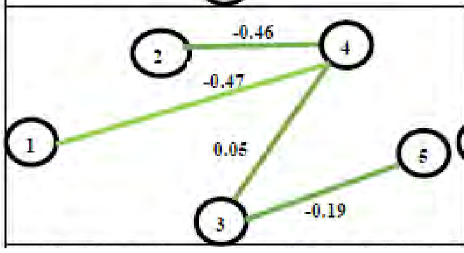


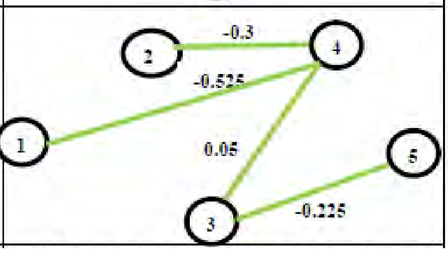












The crisp minimum cost spanning tree is -1.07 and the final path of minimum cost of spanning tree is{2, 4},{4, 1},{4, 3},{3, 5}. The procedure is termination.	The crisp minimum cost spanning tree is -1 and the final path of minimum cost of spanning tree is{2, 4},{4, 1},{4, 3},{3, 5}. The procedure is termination.

5 Comparative study

In what follows we compare the proposed method presented in section 4 with other existing methods including the algorithm proposed by Mullai et al [15] as follow

Iteration 1:

Let $C_1 = \{1\}$ and $\overline{C}_1 = \{2, 3, 4, 5\}$

Iteration 2:

Let $C_2 = \{1, 4\}$ and $\overline{C}_2 = \{2, 3, 5\}$

Iteration 3:

Let $C_3 = \{1, 4, 3\}$ and $\overline{C}_3 = \{2, 5\}$

Iteration 4:

Let $C_4 = \{1, 3, 4, 5\}$ and $\overline{C}_4 = \{2\}$

Finally, the interval valued neutrosophic minimal spanning tree is

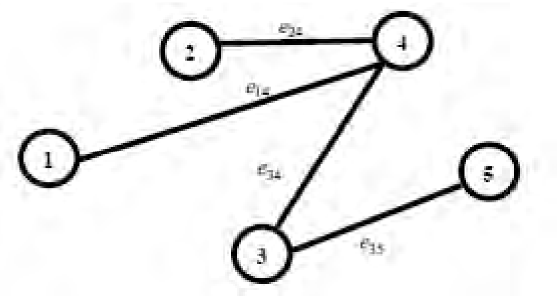


Fig .10. IVN minimal spanning tree obtained by Mullai's algorithm.

So, it can be seen that the interval valued neutrosophic minimal spanning tree {2, 4},{4, 1},{4, 3},{3, 5}.obtained by Mullai's algorithm, After deneutrosophication of edges'weight using the score function, is the same as the path obtained by proposed algorithm. The difference between the proposed algorithm and Mullai's algorithm is that our approach is based on Matrix approch, which can be easily implemented in Matlab, whereas the Mullai's algorithm is based on the comparison of edges in each iteration of the algorithm and this leads to high computation.

7 Conclusions and Future Work

This article analyse about the minimum spanning tree problem where the edges weights are represented by interval valued neutrosophic numbers. In the proposed algorithm, many examples were investigated on MST. The main objective of this study is to focus on algorithmic approach of MST in uncertain environment by using neutrosophic numbers as arc lengths. In addition, the algorithm we use is simple enough and more effective for real time environment. This work could be extended to the case of directed neutrosophic graphs and other kinds of neutrosophic graphs such as bipolar and interval valued bipolar neutrosophic graphs. In future, the proposed algorithm could be implemented to the real time scenarios in transportation and supply chain management in the field of operations research. On the other hand, graph interpretations (decision trees) of syllogistic logics and bezier curves in neutrosophic world could be considered and implemented as the real-life applications of natural logics and geometries of data [31-36].

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A Novel Method for Solving the Fully Neutrosophic Linear Programming Problems

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Abstract

The most widely used technique for solving and optimizing a real-life problem is linear programming (LP), due to its simplicity and efficiency. However, in order to handle the impreciseness in the data, the neutrosophic set theory plays a vital role which makes a simulation of the decision-making process of humans by considering all aspects of decision (i.e., agree, not sure and disagree). By keeping the advantages of it, in the present work, we have introduced the neutrosophic LP models where their parameters are represented with a trapezoidal neutrosophic numbers and presented a technique for solving them. The presented approach has been illustrated with some numerical examples and shows their superiority with the state of the art by comparison. Finally, we conclude that proposed approach is simpler, efficient and capable of solving the LP models as compared to other methods.

1 Introduction

One of the most extremely used OR methods in real-life problems according to empirical surveys is linear programming [1–4]. It is a mathematical programming which contains a linear objective function and a group of linear equalities and inequalities constraints. The petroleum manufacture was the first and most productive application of linear programming. Well-defined data which contain a greater cost of information are required for LP problems. But in real-life problems, the precision of data is overwhelmingly deceitful and this affects optimal solution of LP problems. Probability distributions failed to transact with inaccurate and unclear information. Also fuzzy sets were introduced by Zadeh [5] to handle vague and imprecise information. But also fuzzy set does not represent vague and imprecise information efficiently, because it considers only the truthiness function. After then, Atanassove [6] introduced the concept of intuitionistic fuzzy set to handle vague and imprecise information, by considering both the truth and falsity function. But also intuitionistic fuzzy set does not simulate human decision-making process. Because the proper decision is fundamentally a problem of arranging and explicate facts the concept of neutrosophic set theory was presented by Smarandache, to handle vague, imprecise and inconsistent information [7–10]. Neutrosophic set theory simulates decision-making process of humans, by considering all aspects of decision-making process. Neutrosophic set is a popularization of fuzzy and intuitionistic fuzzy sets; each element of set had a truth, indeterminacy and falsity membership function. So, neutrosophic set can assimilate inaccurate, vague and maladjusted information efficiently and effectively [11, 12]. We now can say that NLP problem is a problem in which at least one coefficient is represented by a neutrosophic number due to vague, inconsistent and uncertain information. The NLP problems are more useful

Keywords Trapezoidal neutrosophic number · Linear programming · Neutrosophic set · Ranking function

than crisp LP problems because decision maker in his/her formulation of the problem is not forced to make a delicate formulation. The use of NLP problems is recommended to avert unrealistic modeling. In this research, it is the first time to present LP problems in a neutrosophic environment with trapezoidal neutrosophic numbers. Two ranking functions are introduced according to the problem type, for converting NLP problem to crisp problem. The proposed model was applied to both maximization and minimization problems.

The remaining part of this research is marshaled as follows: We survey the pertinent fuzzy and intuitionistic FLP problems literature review in Sect. 2. The important concepts of neutrosophic set arithmetic are presented in Sect. 3. The formularization of NLP models is presented in Sect. 4. The proposed method for solving NLP problems is presented in Sect. 5. Numerical examples are disbanded with the suggested method, a comparison of results with different researchers is illustrated and the drawbacks of existing methods are listed in Sect. 6. Finally, conclusions and future trends are clarified in Sect. 7.

2 Literature review

Linear programming problems in the fuzzy environment have classified into two groups which are, symmetric and non-symmetric problems according to Zimmermann [13]. Objectives and constraints weight are equally significant in symmetric FLP problems, but non-symmetric problem weights of objectives and constraints are not equal [14]. Another classification of FLP problems was introduced by Leung [15]: (1) problems with crisp values of objective and fuzzy values of constraints; (2) problems with crisp values of constraints and fuzzy values of objectives; (3) problems with fuzzy objectives and fuzzy constraints; and finally (4) robust programming problems. Three types of fuzzy linear programming models were proposed by Luhandjula [16], which are flexible, mathematical and fuzzy stochastic programming models. Another six models of FLP problems was introduced by Lnuiguchi et al. [17], which are as follows: flexible, possibility programming, possibility LP by using fuzzy max, possibility linear programming with fuzzy preference relations, possibility linear programming with fuzzy objectives and robust programming. An FLP problem with equality and inequality constraints are introduced by Kumar et al. [18]. Various approaches for disbanding FLP with inequality constraints were proposed by several authors [19–21], by firstly converting FLP problems to its equivalent crisp model and then get the optimal fuzzy solution of the original case. A large number of authors have deliberated different properties of FLP problems and suggested various models for finding solutions. The first introduction of fuzzy programming theory was suggested by Tanaka et al. [22]. The first formulation and solving of FLP problems are presented by Zimmerman [23]. Tanaka and Asai [24] suggested an approach for getting the fuzzy optimal solution of FLP problems. Verdegay solved FLP problems by depending on fuzzification principle of objective [25]. The fuzzified version of mathematical problems was examined by Herrera et al. [26]. An FLP problem with fuzzy values of objective function coefficients were proposed by Zhang et al. [27]. They converted FLP problems into multi-objective problems. Another model of FLP problems with fuzzy values of objective function coefficients and constraints was introduced by Stanciulescu et al. [28]. An FLP model with symmetric trapezoidal fuzzy numbers was presented by Ganesan and Veeramani [29]. They obtained the optimal solution of a problem without converting it to the crisp form. A revised version of Ganesan and Veeramani method was proposed by Ebrahimnejad [30]. A ranking function for arranging trapezoidal fuzzy numbers of FLP problems was introduced by Mahdavi and Naasseri [31]. The idealistic stipulation for FLP problems was derived by Wu [32], by presenting the concept of a non-dominated solution of multi-objective programming. By utilizing a defuzzification function, Wu [33] converted the problem into optimization problems. The full FLP problems were introduced by Lotfi et al. [34]. Some researchers have proposed a ranking function for converting FLP problems into its tantamount crisp LP model and then solving it by standard methods. The primal simplex method was extended by Maleki et al. [35], for solving FLP problems. Tavana and Ebrahimnejad introduced a new approach for solving FLP problems with symmetric trapezoidal fuzzy numbers [36]. The fully intuitionistic FLP problems introduced by Bharati and Singh [37] depend on sign distance between triangular intuitionistic fuzzy numbers. A ranking function was used by Sidhu and Kumar [38] for solving intuitionistic FLP problems. Nagoorgani and Ponnalagu [39] introduced an accuracy function to defuzzify triangular intuitionistic fuzzy number. The previous researches motivated us to propose a study for solving NLP problems. There does not exist any researches which solve neutrosophic linear programming problems with trapezoidal neutrosophic numbers [40–45].

3 Preliminaries

A review of important concepts and definitions of neutrosophic set is presented in this section.

Definition 1 [43] A single-valued neutrosophic set N through X taking the form N={□x, T_N(x), $I_N(x)$, $F_N(x)$□:

x□X}, where X be a universe of discourse, $T_N(x)$: X→[0, 1], $I_N(x)$: X→[0, 1] and $F_N(x)$: X→[0, 1] with $0 \le T_N(x) + I_N(x) + F_N(x) \le 3$ for all x□X. $T_N(x)$, $I_N(x)$ and $F_N(x)$ represent truth membership, indeterminacy membership and falsity membership degrees of x to N.

Definition 2 [43] The trapezoidal neutrosophic number $\tilde{A}$ is a neutrosophic set in R with the following truth, indeterminacy and falsity membership functions:

$$T_{\tilde{A}}(x) = \begin{cases} \alpha_{\tilde{A}}\left(\dfrac{x-a_1}{a_2-a_1}\right) & (a_1 \le x \le a_2) \\ \alpha_{\tilde{A}} & (a_2 \le x \le a_3)\,, \\ \alpha_{\tilde{A}} & (a_2 \le x \le a_3) \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

$$I_{\tilde{A}}(x) = \begin{cases} \dfrac{\left(a_2 - x + \theta_{\tilde{A}}(x - a_1')\right)}{(a_2 - a_1')} & (a_1' \le x \le a_2) \\ \theta_{\tilde{A}} & (a_2 \le x \le a_3) \\ \dfrac{\left(x - a_3 + \theta_{\tilde{A}}(a_4' - x)\right)}{(a_4' - a_3)} & (a_3 < x \le a_4') \\ 1 & \text{otherwise} \end{cases} \tag{2}$$

$$F_{\tilde{A}}(x) = \begin{cases} \dfrac{\left(a_2 - x + \beta_{\tilde{A}}(x - a_1'')\right)}{(a_2 - a_1'')} & (a_1'' \le x \le a_2) \\ \beta_{\tilde{A}} & (a_2 \le x \le a_3) \\ \dfrac{\left(x - a_3 + \beta_{\tilde{A}}(a_4'' - x)\right)}{(a_4'' - a_3)} & (a_3 < x \le a_4'') \\ 1 & \text{otherwise} \end{cases} \tag{3}$$

where $\alpha_{\tilde{A}}$, $\theta_{\tilde{A}}$ and $\beta_{\tilde{A}}$ represent the maximum degree of truthiness, minimum degree of indeterminacy, minimum degree of falsity, respectively, $\alpha_{\tilde{A}}$, $\theta_{\tilde{A}}$ and $\beta_{\tilde{A}} \in [0, 1]$.

Also $a_1'' \le a_1 \le a_1' \le a_2 \le a_3 \le a_4' \le a_4 \le a_4''$.

The membership functions of trapezoidal neutrosophic number are presented in Fig. 1.

Definition 3 [43] The mathematical operations on two trapezoidal neutrosophic numbers $\tilde{A} = \langle (a_1, a_2, a_3, a_4); \alpha_{\tilde{A}}, \theta_{\tilde{A}}, \beta_{\tilde{A}} \rangle$ and $\tilde{B} = \langle (b_1, b_2, b_3, b_4); \alpha_{\tilde{B}}, \theta_{\tilde{B}}, \beta_{\tilde{B}} \rangle$ are as follows:

$$\tilde{A} + \tilde{B} = \langle (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4); \alpha_{\tilde{A}} \wedge \alpha_{\tilde{B}}, \theta_{\tilde{A}} \vee \theta_{\tilde{B}}, \beta_{\tilde{A}} \vee \beta_{\tilde{B}} \rangle$$

$$\tilde{A} - \tilde{B} = \langle (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1); \alpha_{\tilde{A}} \wedge \alpha_{\tilde{B}}, \theta_{\tilde{A}} \vee \theta_{\tilde{B}}, \beta_{\tilde{A}} \vee \beta_{\tilde{B}} \rangle$$

$$\tilde{A}^{-1} = \left\langle \left(\frac{1}{a_4}, \frac{1}{a_3}, \frac{1}{a_2}, \frac{1}{a_1} \right); \alpha_{\tilde{A}}, \theta_{\tilde{A}}, \beta_{\tilde{A}} \right\rangle, \quad \text{where } (\tilde{A} \ne 0)$$

$$\gamma\tilde{A} = \begin{cases} \langle (\gamma a_1, \gamma a_2, \gamma a_3, \gamma a_4); \alpha_{\tilde{A}}, \theta_{\tilde{A}}, \beta_{\tilde{A}} \rangle & \text{if } (\gamma > 0) \\ \langle (\gamma a_4, \gamma a_3, \gamma a_2, \gamma a_1); \alpha_{\tilde{A}}, \theta_{\tilde{A}}, \beta_{\tilde{A}} \rangle & \text{if } (\gamma < 0) \end{cases}$$

$$\frac{\tilde{A}}{\tilde{B}} = \begin{cases} \left\langle \left(\frac{a_1}{b_4}, \frac{a_2}{b_3}, \frac{a_3}{b_2}, \frac{a_4}{b_1} \right); \alpha_{\tilde{A}} \wedge \alpha_{\tilde{B}}, \theta_{\tilde{A}} \vee \theta_{\tilde{B}}, \beta_{\tilde{A}} \vee \beta_{\tilde{B}} \right\rangle & \text{if } (a_4 > 0, b_4 > 0) \\ \left\langle \left(\frac{a_4}{b_4}, \frac{a_3}{b_3}, \frac{a_2}{b_2}, \frac{a_1}{b_1} \right); \alpha_{\tilde{A}} \wedge \alpha_{\tilde{B}}, \theta_{\tilde{A}} \vee \theta_{\tilde{B}}, \beta_{\tilde{A}} \vee \beta_{\tilde{B}} \right\rangle & \text{if } (a_4 \langle 0, b_4 \rangle 0) \\ \left\langle \left(\frac{a_4}{b_1}, \frac{a_3}{b_2}, \frac{a_2}{b_3}, \frac{a_1}{b_4} \right); \alpha_{\tilde{A}} \wedge \alpha_{\tilde{B}}, \theta_{\tilde{A}} \vee \theta_{\tilde{B}}, \beta_{\tilde{A}} \vee \beta_{\tilde{B}} \right\rangle & \text{if } (a_4 < 0, b_4 < 0) \end{cases}$$

$$\tilde{A}\tilde{B} = \begin{cases} \langle (a_1 b_1, a_2 b_2, a_3 b_3, a_4 b_4); \alpha_{\tilde{A}} \wedge \alpha_{\tilde{B}}, \theta_{\tilde{A}} \vee \theta_{\tilde{B}}, \beta_{\tilde{A}} \vee \beta_{\tilde{B}} \rangle & \text{if } (a_4 > 0, b_4 > 0) \\ \langle (a_1 b_4, a_2 b_3, a_3 b_2, a_4 b_1); \alpha_{\tilde{A}} \wedge \alpha_{\tilde{B}}, \theta_{\tilde{A}} \vee \theta_{\tilde{B}}, \beta_{\tilde{A}} \vee \beta_{\tilde{B}} \rangle & \text{if } (a_4 \langle 0, b_4 \rangle 0) \\ \langle (a_4 b_4, a_3 b_3, a_2 b_2, a_1 b_1); \alpha_{\tilde{A}} \wedge \alpha_{\tilde{B}}, \theta_{\tilde{A}} \vee \theta_{\tilde{B}}, \beta_{\tilde{A}} \vee \beta_{\tilde{B}} \rangle & \text{if } (a_4 < 0, b_4 < 0) \end{cases}$$

Definition 4 A ranking function of neutrosophic numbers is a function R: $N(R) \to R$, where $N(R)$ is a set of neutrosophic numbers defined on set of real numbers, which convert each neutrosophic number into the real line.

Let $\tilde{A} = \langle (a_1, a_2, a_3, a_4); \alpha_{\tilde{A}}, \theta_{\tilde{A}}, \beta_{\tilde{A}} \rangle$ and $\tilde{B} = \langle (b_1, b_2, b_3, b_4); \alpha_{\tilde{B}}, \theta_{\tilde{B}}, \beta_{\tilde{B}} \rangle$ are two trapezoidal neutrosophic numbers, then

1. If $R(\tilde{A}) > R(\tilde{B})$ then $\tilde{A} > \tilde{B}$,
2. If $R(\tilde{A}) < R(\tilde{B})$ then $\tilde{A} < \tilde{B}$,
3. If $R(\tilde{A}) = R(\tilde{B})$ then $\tilde{A} = \tilde{B}$.

4 Neutrosophic linear programming problem (NLP)

In this section, various types of NLP problems are presented.

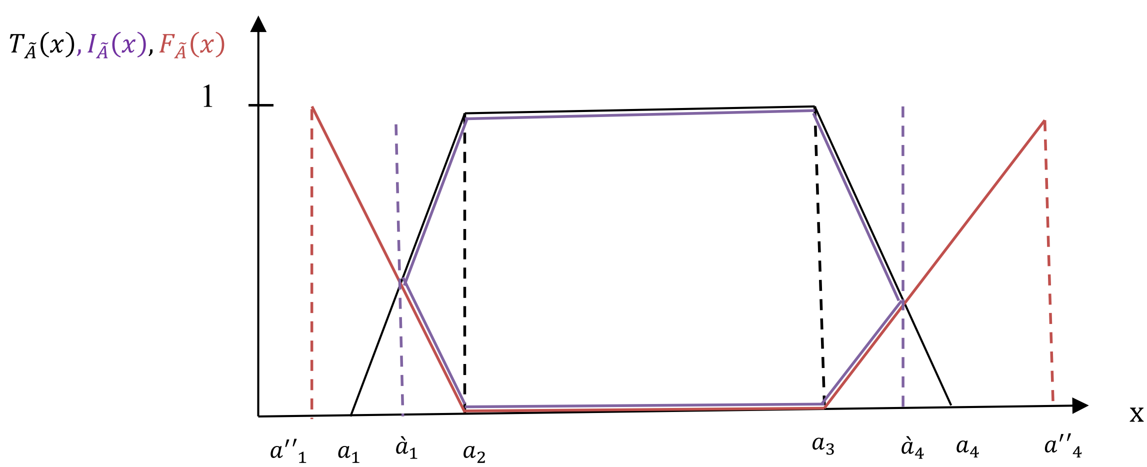

Fig. 1 Truth membership, indeterminacy and falsity membership functions of trapezoidal neutrosophic number

The first type of NLP problem is the problem in which coefficients of objective function variables are represented by trapezoidal neutrosophic numbers, but all other parameters are represented by real numbers.

$$\text{Maximize/minimize } \tilde{Z} \approx \sum_{j=1}^{n} \tilde{c}_j x_j$$
$$\text{Subject to} \tag{4}$$
$$\sum_{j=1}^{n} a_{ij}x_j \leq, =, \geq b_i; \quad i = 1,2,\ldots,m,$$
$$j = 1,2,\ldots,n, \quad x_j \geq 0.$$

In this type of problem, $\tilde{c}_j$ is a trapezoidal neutrosophic number.

The second type of NLP problem is the problem in which objective function variables and coefficients are exemplified by real values but coefficients of constraints variables and right-hand side are represented by trapezoidal neutrosophic numbers.

$$\text{Maximize/minimize } Z = \sum_{j=1}^{n} c_j x_j$$
$$\text{Subject to}$$
$$\sum_{j=1}^{n} \tilde{a}_{ij}x_j \,\tilde{\leq}, \approx, \tilde{\geq}\, \tilde{b}_i; \quad i = 1,2,\ldots,m, \quad j = 1,2,\ldots,n, \quad x_j \geq 0. \tag{5}$$

Here, both $\tilde{a}_{ij}$ and $\tilde{b}_i$ are trapezoidal neutrosophic numbers.

The third type of NLP problem is the problem in which all parameters are represented by trapezoidal neutrosophic numbers, except variables are exemplified only by real values.

$$\text{Maximize / minimize } \tilde{Z} \approx \sum_{j=1}^{n} \tilde{c}_j x_j$$
$$\text{Subject to} \tag{6}$$
$$\sum_{j=1}^{n} \tilde{a}_{ij}x_j \,\tilde{\leq}, \approx, \tilde{\geq}\, \tilde{b}_i; \quad i = 1,2,\ldots,m,$$
$$j = 1,2,\ldots,n, \quad x_j \geq 0.$$

Here, $\tilde{c}_j, \tilde{a}_{ij}$ and $\tilde{b}_i$ are trapezoidal neutrosophic numbers.

The NLP problem may also be a problem with neutrosophic values for variables, coefficients in goal function and right-hand side of constraints.

$$\text{Maximize/minimize } \tilde{Z} \approx \sum_{j=1}^{n} \tilde{c}_j \tilde{x}_j$$
$$\text{Subject to} \tag{7}$$
$$\sum_{j=1}^{n} a_{ij}\tilde{x}_j \,\tilde{\leq}, \approx, \tilde{\geq}\, \tilde{b}_i; \quad i = 1,2,\ldots,m,$$
$$j = 1,2,\ldots,n, \quad x_j \geq 0.$$

Here, $\tilde{c}_j, \tilde{x}_j$ and $\tilde{b}_i$ are trapezoidal neutrosophic numbers. Here, $\tilde{x}_j$ is defined as trapezoidal neutrosophic numbers, if authors want to obtain results in the form of neutrosophic numbers. But in reality, any manager or decision maker want to obtain the crisp optimal solution of problem, through considering vague, imprecise and inconsistent information when defining the problem. So, if we obtain the crisp value of decision variables, this problem can be considered as another formulation of NLP (6).

5 Proposed NLP method

A new approach suggested to find the neutrosophic optimal solution of NLP problems is introduced in this section.

Step 1 Let decision makers insert their NLP problem with trapezoidal neutrosophic numbers. Because we always want to maximize truth degree, minimize indeterminacy and falsity degree of information, and then inform decision makers to apply this concept when entering trapezoidal neutrosophic numbers of NLP model.

Step 2 Regarding to definition 4, we propose a ranking function for trapezoidal neutrosophic numbers.

Step 3 Let ($\tilde{a} = a^l, a^{m1}, a^{m2}, a^u, ; T_{\tilde{a}}, I_{\tilde{a}}, F_{\tilde{a}}$) be a trapezoidal neutrosophic number, where a^l, a^{m1}, a^{m2}, a^u, are lower bound, first, second median value and upper bound for trapezoidal neutrosophic number, respectively. Also $T_{\tilde{a}}, I_{\tilde{a}}, F_{\tilde{a}}$ are the truth, indeterminacy and falsity degree of trapezoidal number. If NLP problem is a maximization problem, then:

Ranking function for this trapezoidal neutrosophic number is as follows:

$$\mathrm{R}(\tilde{a}) = \left(\frac{a^l + a^u + 2\left(a^{m1} + a^{m2}\right)}{2}\right) + \text{confirmation degree.}$$

Mathematically, this function can be written as follows:

$$R(\tilde{a}) = \left(\frac{a^l + a^u + 2(a^{m1} + a^{m2})}{2}\right) + (T_{\tilde{a}} - I_{\tilde{a}} - F_{\tilde{a}}) \tag{8}$$

If NLP problem is a minimization problem, then:

Ranking function for this trapezoidal neutrosophic number is as follows:

$$\mathrm{R}(\tilde{a}) = \left(\frac{a^l + a^u - 3\left(a^{m1} + a^{m2}\right)}{2}\right) + \text{confirmation degree.}$$

Mathematically, this function can be written as follows:

$$R(\tilde{a}) = \left(\frac{a^l + a^u - 3(a^{m1} + a^{m2})}{2}\right) + (T_{\tilde{a}} - I_{\tilde{a}} - F_{\tilde{a}}). \quad (9)$$

Step 4 According to the type of NLP problem, apply the suitable ranking function to convert each trapezoidal neutrosophic number to its equivalent crisp value. This lead to convert NLP problem to its crisp model.

Step 5 Solve the crisp model using the standard method and obtain the optimal solution of problem.

6 Numerical examples

In this section, to prove the applicability and advantages of our proposed model of NLP problems, we solved the same problem which introduced by Ganesan and Veeramani [29] and Ebrahimnejad and Tavana [36].

The difference between fuzzy set and neutrosophic set is that fuzzy set takes into consideration the truth degree only. But neutrosophic set takes into consideration the truth, indeterminacy and falsity degree. The decision makers and problem solver always seek to maximize the truth degree, minimize indeterminacy and falsity degree. Then, in the following example, we consider truth degree (T)=1, indeterminacy (I) and falsity (F) degree=0, as follows $(1, 0, 0)$ for each trapezoidal neutrosophic number and this called the confirmation degree of each trapezoidal neutrosophic number. We should also note that, according to Ganesan, Veeramani and Ebrahimnejad, Tavana each trapezoidal number is symmetric with the following form:

$$\tilde{a} = \left(a^l, a^u, \alpha, \alpha\right),$$

where a^l, a^u, α, α represented the lower, upper bound and first, second median value of trapezoidal number, respectively. The median values of trapezoidal numbers according to Ganesan, Veeramani and by Ebrahimnejad, Tavana are with equal vales (α). Now let us apply our proposed method on the same problem.

6.1 Example 1

$$\begin{aligned}
&\text{Maximize } \tilde{Z} \approx (13, 15, 2, 2)x_1 + (12, 14, 3, 3)x_2 + (15, 17, 2, 2)x_3\\
&\text{Subject to}\\
&12x_1 + 13x_2 + 12x_3 \tilde{\leq} (475, 505, 6, 6),\\
&14x_1 + 13x_3 \tilde{\leq} (460, 480, 8, 8),\\
&12x_1 + 15x_2 \tilde{\leq} (465, 495, 5, 5),\\
&x_1, x_2, x_3 \tilde{\geq} \tilde{0}.
\end{aligned} \quad (10)$$

Because this NLP problem is a maximization problem, then by using Eq. (8) each trapezoidal number will convert to its equivalent crisp number. Remember that confirmation degree of each trapezoidal number is (1, 0, 0) according to decision maker opinion as we illustrated previously at the beginning of example. Then, the crisp model of previous problem will be as follows:

$$\begin{aligned}
&\text{Maximize } Z = 19x_1 + 20x_2 + 21x_3\\
&\text{Subject to}\\
&12x_1 + 13x_2 + 12x_3 \leq 503,\\
&14x_1 + 13x_3 \leq 487,\\
&12x_1 + 15x_2 \leq 491\\
&x_1, x_2, x_3 \geq 0.
\end{aligned} \quad (11)$$

We can structure the standard form of previous problem (11) as follows:

$$\begin{aligned}
&\text{Maximize } Z = 19x_1 + 20x_2 + 21x_3\\
&\text{Subject to}\\
&12x_1 + 13x_2 + 12x_3 + s_4 = 503,\\
&14x_1 + 13x_3 + s_5 = 487,\\
&12x_1 + 15x_2 + s_6 = 491,\\
&x_1, x_2, x_3, s_4, s_5, s_6 \geq 0.
\end{aligned} \quad (12)$$

where s_4, s_5, s_6 are slack variables.

The previous standard form can be solved by the simplex approach. The initial tableau of simplex is presented in Table 1.

The coming variable in Table 2 is x_3 and departing variable is s_5.

The entering variable is x_2 and leaving variable is s_4 as shown in Table 3.

Table 1 Initial simplex form

Basic variables	x_1	x_2	x_3	s_4	s_5	s_6	RHS
s_4	12	13	12	1	0	0	503
s_5	14	0	13	0	1	0	487
s_6	12	15	0	0	0	1	491
Z	19	20	21	0	0	0	0

Table 2 First simplex form

Basic variables	x_1	x_2	x_3	s_4	s_5	s_6	RHS
s_4	− 12/13	13	0	1	− 12/13	0	695/13
x_3	14/13	0	1	0	1/13	0	487/13
s_6	12	15	0	0	0	1	491
Z	− 47/13	20	0	0	− 21/13	0	10,227/13

Table 3 Optimal form

Basic variables	x_1	x_2	x_3	s_4	s_5	s_6	RHS
x_2	− 12/169	1	0	1/13	− 12/169	0	695/169
x_3	14/13	0	1	0	1/13	0	487/13
s_6	2208/169	0	0	− 15/13	180/169	1	429
Z	− 371/169	0	0	− 20/13	− 33/169	0	869

6.2 Comparisons between our proposed model and other existing models

By comparing proposed model results with Ebrahimnejad and Tavana [36] results of the same problem, we noted that:

1. Our proposed model results are better than Ebrahimnejad and Tavana results. Let us look at the optimal tableau of our proposed model as shown in Table 3, it is obvious that the objective function value equal 869 but in Ebrahimnejad and Tavana, the objective function equal 635 by knowing that, the problem is a maximization problem. To make this more obvious, let us introduce the optimal tableau of Ebrahimnejad and Tavana model as presented in Table 4.
2. Ebrahimnejad and Tavana proposed their model to solve only symmetric trapezoidal numbers. But our model can solve symmetric and non-symmetric numbers.
3. When entering symmetric trapezoidal numbers of Ebrahimnejad and Tavana, it take the following form:

$\tilde{a} = \left(a^l, a^u, \alpha, \alpha\right)$, and they did not utilize the value of α in their calculations of ranking function for obtaining the equivalent crisp value, so let us ask ourselves a question "what is the rule of α?". But in our proposed model, we take all values into considerations. Our ranking function has not any missing values of trapezoidal numbers, and then it is very accurate and comprehensive.

4. As we know,a^l, a^u, α, α represented the lower, upper bound, first and second median value of trapezoidal number, respectively. Because two values of α are equals, then the triangular numbers will be more logical than trapezoidal numbers.
5. To solve a problem with not symmetric trapezoidal numbers using Ebrahimnejad and Tavana method, we need to approximate all not symmetric trapezoidal numbers into the closest symmetric numbers. This approximation will make obtained results which are not delicate.
6. The big drawback of Ebrahimnejad and Tavana fuzzy model is the taking of truthiness function only. But in real life, the decision-making process takes the following form "agree, not sure and disagree." We treated this drawback in our model by using neutrosophic. Since, beside the truth function, we take into account the indeterminacy and falsity function.

Table 4 Ebrahimnejad and Tavana optimal tableau

Basis	x_1	x_2	x_3	s_4	s_5	s_6	RHS
x_2	− 12/169	1	0	1/13	− 12/169	0	730/169
x_3	14/13	0	1	0	1/13	0	470/169
s_6	1848/169	0	0	− 15/13	180/169	1	70,170/169
Z	42/13	0	0	1	52/169	0	634.6

Also by comparing our model with Ganesan and Veeramani at the same problem, we also noted that:

1. Our model is more simple and efficient than Ganesan and Veeramani model.
2. Since obtained results of Ebrahimnejad, Tavana and Ganesan and Veeramani are equals then, our results are also better than Ganesan and Veeramani model.
3. Our model represents reality efficiently than Ganesan and Veeramani model, because we consider all aspects of decision-making process in our calculations (i.e., the truthiness, indeterminacy and falsity degree).
4. Ganesan and Veeramani model represented to solve only the symmetric trapezoidal numbers. Our model can solve both the symmetric and non-symmetric.

Also, by comparing our model with Kumar et al. [18] for solving the same problem we founded that:

1. In their model, they convert the FLP problem to its tantamount crisp model. But their model has more variables and constraints.
2. Their models increase the complexity of solving linear programming problem by simplex algorithm.
3. Our model reduces complexity of problem, by reducing the number of constraints and variables.
4. Their model is a time-consuming and complex, but our model is not.
5. Also our model represents reality efficiently and better than their model.

By solving the previous example according to Saati et al. [44] proposed method, then the model will be as follows:

$$\begin{aligned}
&\text{Maximize } Z = 13x_1 + 12x_2 + 15x_3\\
&\text{Subject to}\\
&12x_1^l + 13x_2^l + 12x_3^l \le 475,\\
&12x_1^u + 13x_2^u + 12x_3^u \le 505,\\
&12x_1^{m1} + 13x_2^{m1} + 12x_3^{m1} \le 6,\\
&12x_1^{m2} + 13x_2^{m2} + 12x_3^{m2} \le 6,\\
&14x_1^l + 13x_3^l \le 460,\\
&14x_1^u + 13x_3^u \le 480,\\
&14x_1^{m1} + 13x_3^{m1} \le 8,\\
&14x_1^{m2} + 13x_3^{m2} \le 8,\\
&12x_1^l + 15x_2^l \le 465,\\
&12x_1^u + 15x_2^u \le 495,\\
&12x_1^{m1} + 15x_2^{m1} \le 5,\\
&12x_1^{m2} + 15x_2^{m2} \le 5,\\
&x_1^l + x_1^u \ge 0,\\
&x_2^l + x_2^u \ge 0,\\
&x_3^l + x_3^u \ge 0,\\
&x_1^{m1} + x_1^{m2} \ge 0,\\
&x_2^{m1} + x_2^{m2} \ge 0,\\
&x_3^{m1} + x_3^{m2} \ge 0.
\end{aligned} \tag{13}$$

As an effect, the numbers of constraints and variables are increased, and this lead to increase complexity of problem, increase the space of recording binary bits and also increase computational time when solving it by simplex method. If the numbers of constraints of the original problem are increased, then the solution will become very difficult to apply. But our proposed method solves the same problem with less variables and constraints, and then, with less complexity and also less computational time when solving by simplex method.

6.3 Example 2

In this example, we solve a NLP problem with trapezoidal neutrosophic numbers. The order of element for trapezoidal neutrosophic numbers is as follows: lower, first median value, second median value and finally the upper bound. The decision makers' confirmation degree about each value of trapezoidal neutrosophic number is (0.9, 0.1, 0.1). This example belongs to the second classification of NLP problems as listed in Sect. 4.

Table 5 Initial simplex form

Basic variables	x_1	x_2	s_3	s_4	s_5	RHS
s_3	33	46	1	0	0	90,041
s_4	53	15	0	1	0	48,046
s_5	44	34	0	0	1	56,031
Z	25	48	0	0	0	0

$$\begin{aligned}
&\text{Maximize } Z = 25x_1 + 48x_2\\
&\text{Subject to}\\
&(14, 15, 17, 18)x_1 + (25, 30, 34, 38)x_2\\
&\quad \tilde{\le} (44,980, 45,000, 45,030, 45,070)\\
&(21, 24, 26, 33)x_1 + (4, 6, 8, 11)x_2\\
&\quad \tilde{\le} (23,980, 24,000, 24,050, 24,060)\\
&(17, 21, 22, 26)x_1 + (12, 14, 19, 22)x_2\\
&\quad \tilde{\le} (27,990, 28,000, 28,030, 28,040)\\
&\tilde{x}_1, \tilde{x}_2 \tilde{\ge} \tilde{0}.
\end{aligned} \tag{14}$$

By using Eq. (8), each trapezoidal number will convert to its equivalent crisp number. Then, the crisp model of previous problem will be as follows:

$$\begin{aligned}
&\text{Maximize } Z = 25x_1 + 48x_2\\
&\text{Subject to}\\
&33x_1 + 64x_2 \le 90,041,\\
&53x_1 + 15x_2 \le 48,046,\\
&44x_1 + 34x_2 \le 56,031,\\
&x_1, x_2 \ge 0.
\end{aligned} \tag{15}$$

We can structure the standard form of previous problem (15) as follows:

$$\begin{aligned}
&\text{Maximize } Z = 25x_1 + 48x_2\\
&\text{Subject to}\\
&33x_1 + 64x_2 + s_3 = 90,041\\
&53x_1 + 15x_2 + s_4 = 48,046\\
&44x_1 + 34x_2 + s_5 = 56,031\\
&x_1, x_2, s_3, s_4, s_5 \ge 0.
\end{aligned} \tag{16}$$

where s_3, s_4, s_5 are slack variables.

The previous standard form can be solved by the simplex approach. The initial tableau of simplex is presented in Table 5. The entering variable in Table 6 is x_2 and leaving variable is s_3.

The coming variable is x_1 and departing variable is s_5 as in Table 7.

Table 6 First simplex form

Basic variables	x_1	x_2	s_3	s_4	s_5	RHS
x_2	33/64	1	1/64	0	0	1406.89
s_4	2897/64	0	− 15/64	1	0	26,942.6
s_5	847/32	0	− 17/32	0	1	8196.72
Z	0.25	0	− 0.75	0	0	67,530.8

Table 7 Optimal form

Basic variables	x_1	x_2	s_3	s_4	s_5	RHS
x_2	0	1	2/77	0	− 3/154	1247.21
s_4	0	0	571/847	1	− 1.71015	12,925
x_1	1	0	− 17/847	0	32/847	23,845/77
Z	0	0	− 631/847	0	− 8/847	67,608.2

Table 8 Optimal simplex form

Basis	x_1	x_2	s_3	s_4	RHS
s_3	0	− 7/3	1	− 2/3	10
x_1	1	4/3	0	− 1/3	2
Z	0	2	0	2	12

Table 9 Departments

Products	Wiring	Drilling	Assembly	Inspection	Unit profit
P1	0.5	3	2	0.5	$\tilde{9}\$$
P2	1.5	1	4	1	$\widetilde{12}\$$
P3	1.5	2	1	0.5	$\widetilde{15}\$$
P4	1	3	2	0.5	$\widetilde{11}\$$

6.4 Example 3

Let us introduce another type of problems in this example and making a comparison with other research at the same example.

By solving the same problem which introduced by Saati et al. [44]:

$$\begin{aligned}&\text{Minimize } Z = 6x_1 + 10x_2\\&\text{Subject to}\\&2x_1 + 5x_2 \tilde{\geq} (5, 8, 3, 13),\\&3x_1 + 4x_2 \tilde{\geq} (6, 0, 4, 16),\\&x_1, x_2 \tilde{\geq} \tilde{0}.\end{aligned} \tag{17}$$

Let confirmation degree is (1, 0, 0) according to our assumptions and note that, here the order of trapezoidal neutrosophic number is as follows: lower bound, first, second median value and finally the upper bound, respectively. Let us use Eq. (9) for transforming the previous model to its crisp model as follows:

$$\begin{aligned}&\text{Minimize } Z = 6x_1 + 10x_2\\&\text{Subject to}\\&2x_1 + 5x_2 \geq -6, \quad .\\&3x_1 + 4x_2 \geq 6,\\&x_1, x_2 \geq 0.\end{aligned} \tag{18}$$

The previous problem can be solved by the simplex approach. The optimal tableau of simplex method is presented in Table 8.

From the previous table, the value of objective function =12, $x_1 = 2$ and $x_2 = 0$.

When Saati et al. [35] solved the previous example, the results are nearly equal with our result. Since the value of Z according to their model is equal to 12.857, the value of $x_1 = 1.429$ and $x_2 = 0.429$. It is obvious that two approach results are nearly equal, but our proposed method has several advantages over their method:

1. We obtain the results which also obtained by Saati et al. [44] but with easy and simple method.
2. Number of constraints in our model is the same of the original model, but when Saati solved their model, the number of variables and constraints is significantly increased. Since in Saati et al. [44] model, number of constraints of the previous problem becomes 20 constraints when they trying to solve the previous problem.
3. Due to the big increase in number of variables and constraints of Saati model, the complexity of solving the problem by simplex will increase and computational time will increase sure.
4. Their proposed approach is difficult to apply in large scale of problems.
5. Also their approach does not represent vague, inconsistent information efficiently.

6.5 Case study

A company for electronic industries manufactures four technical products for aerospace companies that conclude NASA contracts. The outputs must get through four parts before they are shipped. These departments are: Wiring, Drilling, Assembly and finally Inspection. The required time for each unit manufactured and its profit is presented in Table 9. The minimum production quantity for fulfilling contracts monthly is presented in Table 10. The objective of company is to produce products in such quantities for maximizing the total profits.

Table 10 Time capacity and minimum production level

Departments	Capacity (in hours)	Products	Minimum production level
Wiring	$\widetilde{1500}$	P1	$\widetilde{150}$
Drilling	$\widetilde{2350}$	P2	$\widetilde{100}$
Assembly	$\widetilde{2600}$	P3	$\widetilde{300}$
Inspection	$\widetilde{1200}$	P4	$\widetilde{400}$

The confirmation degree of previous information according to decision makers' opinions is (0.9, 0.1, 0.1).

Let number of units of p1 produced=x_1,

Let number of units of p2 produced=x_2,

Let number of units of p3 produced=x_3,

Let number of units of p4 produced=x_4.

The formulation of previous problem is as follows:

$$\begin{aligned}
&\text{Maximize } \tilde{Z} \approx \tilde{9}x_1 + \widetilde{12}x_2 + \widetilde{15}x_3 + \widetilde{11}x_4\\
&\text{Subject to}\\
&0.5x_1 + 1.5x_2 + 1.5x_3 + x_4 \le \widetilde{1500},\\
&3x_1 + x_2 + 2x_3 + 3x_4 \le \widetilde{2350},\\
&2x_1 + 4x_2 + x_3 + 2x_4 \le \widetilde{2600},\\
&0.5x_1 + x_2 + 0.5x_3 + 0.5x_4 \le \widetilde{1200},\\
&x_1 \ge \widetilde{150},\\
&x_2 \ge \widetilde{100},\\
&x_3 \ge \widetilde{300},\\
&x_4 \ge \widetilde{400}.\\
&x_1, x_2, x_3, x_4 \ge \tilde{0}.
\end{aligned} \tag{19}$$

Note that the values of each neutrosophic number represented by a trapezoidal neutrosophic number as follows:

$$\begin{aligned}
&\tilde{9} = (6, 8, 9, 12), \widetilde{12} = (9, 10, 12, 14),\\
&\widetilde{15} = (12, 13, 15, 17), \widetilde{11} = (8, 9, 11, 13),\\
&\widetilde{150} = (120, 130, 150, 170),\\
&\widetilde{100} = (70, 80, 100, 120),\\
&\widetilde{300} = (270, 280, 300, 320), \widetilde{400} = (370, 380, 400, 420),\\
&\widetilde{1500} = (1200, 1300, 1500, 1700),\\
&\widetilde{2350} = (2200, 2250, 2350, 2400)\\
&\widetilde{2600} = (2200, 2400, 2600, 2800),\\
&\widetilde{1200} = (1000, 1100, 1200, 1300).
\end{aligned}$$

By using Eq. (8), the previous problem transform to the following crisp model as follows:

$$\begin{aligned}
&\text{Maximize } Z = 27x_1 + 34x_2 + 43x_3 + 31x_4\\
&\text{Subject to}\\
&0.5x_1 + 1.5x_2 + 1.5x_3 + x_4 \le 4251,\\
&3x_1 + x_2 + 2x_3 + 3x_4 \le 6901,\\
&2x_1 + 4x_2 + x_3 + 2x_4 \le 7501,\\
&0.5x_1 + x_2 + 0.5x_3 + 0.5x_4 \le 3451,\\
&x_1 \ge 426,\\
&x_2 \ge 276,\\
&x_3 \ge 876,\\
&x_4 \ge 1176.\\
&x_1, x_2, x_3, x_4 \ge 0.
\end{aligned} \tag{20}$$

By solving the previous model using simplex approach, the results are as follows:

$$\begin{aligned}
&x_1 = 426,\\
&x_2 = 343,\\
&x_3 = 876,\\
&x_4 = 1176,\\
&Z = 97,288.
\end{aligned}$$

7 Conclusions and research directions

By applying the neutrosophic set concept to the linear programming problems, we treated imprecise, vague and inconsistent information efficiently. We also have a better representation of reality through considering all aspects of the decision-making process. We proposed two ranking functions for converting trapezoidal neutrosophic numbers to its equivalent crisp values. The first ranking function is for maximization problems and the second-ranking function is for minimization problems. After using the suitable ranking function and transforming the problem to its equivalent crisp model, then we solve the problem using the standard methods. By comparing our proposed model with other existing fuzzy models, we concluded that our proposed model is simpler, efficient and achieve better results than other researchers. It is also revealed that proposed method is equivalently applied for solving with the symmetric and non-symmetric trapezoidal numbers.

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A Hybrid Approach of Neutrosophic Sets and DEMATEL Method for Developing Supplier Selection Criteria

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Abstract For any organization, the selection of suppliers is a very important step to increase productivity and profitability. Any organization or company seeks to use the best methodology and the appropriate technology to achieve its strategies and objectives. The present study employs the neutrosophic set for decision making and evaluation method (DEMATEL) to analyze and determine the factors influencing the selection of SCM suppliers. DEMATEL is considered a proactive approach to improve performance and achieve competitive advantages. This study applies the neutrosophic set Theory to adjust general judgment, using a new scale to present each value. A case study implementing the proposed methodology is presented (i.e. selecting the best supplier for a distribution company). This research was designed by neutrosophic DEMATEL data collection survey of experts, interviewing professionals in management, procurement and production. The results analyzed in our research prove that quality is the most influential criterion in the selection of suppliers.

Keywords Supply chain management (SCM) · Supplier selection · Neutrosophic set · DEMATEL

1 Introduction

It cannot be denied that the success or the failure of any organization depend on how it chooses the appropriate supply chain management system and suppliers. Many organizations are currently seeking to contract many suppliers from around the world to create a collaborative commerce, and to increase trade, profitability and productivity. Experts are interested in purchasing and holding contracts with major suppliers, since the supplier selection is one of the most important functions of saving raw materials cost, of procurement management, and of increasing competitive advantage. The supply chain is an integral part of the new business management in the design of services from suppliers to customers. Supply chain management enables business participants to effectively combine products and services for a long-term relationship [1]. The effective coordination on information flows between enterprises, material, delivery, product, payment and trading partners can be defined extensively as supply chain management [2]. The economic environment forces organizations and collective institutions to seek competitive alternatives to meet the needs of customers and market. Organizations must have better production technology for internal and external competitiveness. Companies are an important part of the process of increasing the supply chain. Projects seeking to increase the production and compete in the international market must manage the supply chain in a highly effective way, and the suppliers selection is considered a key point of the process [3]. The process of integrating all activities in order to create satisfied customers is called supply chain management, and it is applied by the best companies around the world to control the flow of information, services and materials [4]. Supply chain management improves the competitive position of a company. Companies are always striving to maintain their competitiveness by developing issues such as improved model analysis, road planning, pregnancy planning, or supply chain management. Usually, the managers focus on organizing processes within the company to maximize profits, but the supply chain management seeks to link internal processes and decisions with external enterprise partners to improve and create competitiveness [5]. In recent years, supply chain management has attracted increasing awareness in academic publications. Supply chain management has been used to promote efficiency of the value chain on a wide range of products, services and other manufactured materials. Disagreement may occur in the process of selecting criteria. Many studies have tried to help managers and decision makers in any organization to take a relevant decision in selecting the best criterion suiting their organizations.

The process of supplier evaluation, appraisal, evaluation and contracting is called supplier selection [6]. There are some distinguishing features among suppliers, such as manufacturing procedures, technology, geographical location and larger processes that adopt better suppliers in pursuit of competitiveness [7]. Many researchers are tempted in displaying performance to make the supply chain more and more efficient [8], and consequently an intuitionistic fuzzy sets DEMATEL method was proposed to analyze the influential criteria practices, suggesting that empirical studies should be conducted as future research [9]. DEMATEL, an extended technique for formulating and analyzing influential relationships among difficult criteria, has been extensively used to extract the texture of a complex problem. The current literature review indicates that most papers used traditional intuitionistic fuzzy set to level the ambiguity of experts judgments and opinions (Fig. 1).

Fuzzy set focuses only on membership function and it does not take into account the non-membership and indeterminacy, so it fails to deal with uncertainty and indeterminacy existing in the real world. To overcome the drawbacks of fuzzy set, we integrated DEMATEL method in neutrosophic environment. The neutrosophic set is an extent or generalization of

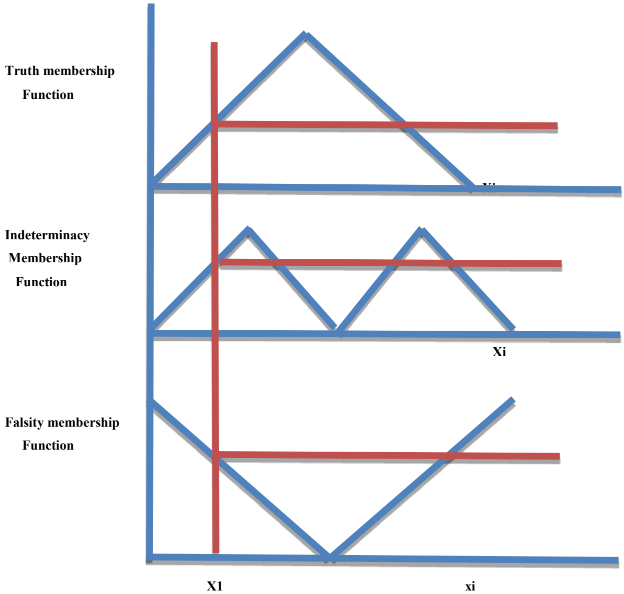

Fig. 1 Neutrosophication process [16]

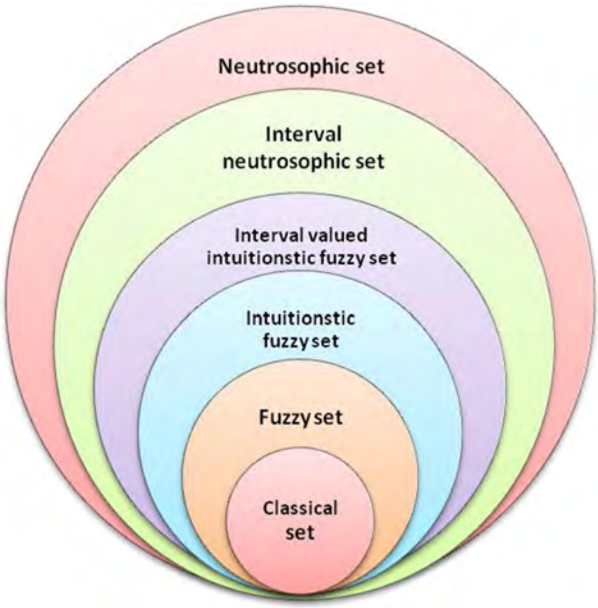

Fig. 2 From classical sets to neutrosophic sets

the intuitionistic fuzzy set. It represents real world problems effectively and efficiently by considering all aspects of a decision situations (i.e. truthiness, indeterminacy and falsity) [10, 11], as shown in Fig. 2.

Neutrosophy was introduced by Smarandache [12, 13] as a branch of philosophy that studies the origin and scope of neutralities. Neutrosophy has been used in various applications to solve various problems as a critical path problem [14], obtaining PERT three times in project management [15]. Normally, the criteria have a degree of interactivity and related relationships. In such cases, it is very difficult for decision makers and experts to avoid interference between criteria and to obtain a specific goal. The main contributions of this research are:

- It introduces a new methodology by aggregating the neutrosophic set and DEMATEL method.
- It presents a case study showing how an organization increases its practices and activities according to specific criteria.

In this research, the DEMATEL method is used to develop mutual relationships and interdependencies. We present a causal diagram to describe relationships and their influence degrees on criteria.

It is important to evaluate the weakness and the strength of each criterion against another. One advantage of this method is showing the relationships and interdependence between features. Neutrosophic set theory is used in this research to express decision maker's preferences [17]. Neutrosophic sets (NSs) are an extension of the intuitionistic fuzzy sets (IFs), presenting more accurately the opinions and better interpreting the ambiguity, where the membership of a value or an element is defined as a number between 0 and 1, by resorting on a hesitation degree in IFs, whilst in NSs on an indeterminacy degree. Neutrosophic Set moves one step further by examining the membership of truth, the membership of indeterminacy and the non-membership of a member of a given set. Also, it is necessary to acquire experts opinions to evaluate influences. Neutrosophic Set has the following benefits:

- It introduces the indeterminacy degree that helps experts to express their opinions more accurately.
- It represents the extent of decision makers disagreements.

The proposed model also combines different interests of decision makers in one opinion in order to eliminate inconsistencies or to address the inconsistencies of expert judgments and improve consistency. A case study is solved to explain the model's suitability.

This research is organized as follows: Sect. 2 is a literature review that presents papers about DEMATEL for supplier selection. Section 3 illustrates the basic definitions of neutrosophic sets. Section 4 presents a methodology of the proposed model. Section 5 validates the model by solving a case study. Section 6 concludes the research and determines the future directions of the work.

2 The related work

In this section, we present some supplier selection related work. The two important stages in supply chain management, which considers all the activities from the purchasing of raw material to the final delivery of the product, are the supplier's selection and the evaluation. The supplier selection problem requires high accuracy methods of multiple criteria decision making for solving it. According to the literature reviews, many researchers proposed methods based on DEMATEL. Chang et al. [18] applied DEMATEL with fuzzy to evaluate and select the best supplier and to improve performance with respect to organizational factors and strategic performances, which included ten evaluating criteria. Dey et al. [19] applied

DEMATEL to establish a long-term relationship with a company and its suppliers, with respect to their criteria. Hsu et al. [20] explored and used DEMATEL for decision making within the green supply chain, and focused on the components of green supply chain management and how they serve as a foundation for the decision framework and for recognizing the influential criterion of carbon footprint in environment. Lin [21] used DEMATEL to enhance environment performance, which is shaped by criteria as green purchasing, green design and product recovery practices. Dalalah et al. [22] employed DEMATEL for a supplier selection problem, implementing and applying it on an industrial case for the selection of cans suppliers at a factory in Amman, with respect to various supplier evaluation criteria. Govindan et al. [23] developed and used intuitionistic fuzzy with DEMATEL for decision making within the green supply chain, and focused on the components of green supply chain management to handle the causal relationships between GSCM practices and performances.

In this study, we aim to select the best supplier with respect to the various criteria using DEMATEL in neutrosophic environment. The selection of supplier problem is still challenging, and selecting the right supplier becomes a critical activity within a company, consequently affecting its efficiency and profitability. Due to its strategic importance, important research is being done to cope with the supplier evaluation and selection problem.

3 Neutrosophic sets

In this section, we give definitions involving neutrosophic sets, single valued neutrosophic sets, trapezoidal neutrosophic numbers, and operations on trapezoidal neutrosophic numbers.

Definition 1 [24] Let X be a space of points and $x \in X$. A neutrosophic set in X is defined by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$ and a falsity-membership function $F_A(x)$, $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or real nonstandard subsets of $]-0, 1+[$. That is $T_A(x){:}X \rightarrow\ ]-0, 1+[$, $I_A(x){:}X \rightarrow\ ]-0, 1+[$ and $F_A(x){:}X \rightarrow\ ]-0, 1+[$. There is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$, so $0- \leq \sup\ (x) + \sup\ x + \sup\ x \leq 3+$.

Definition 2 [16, 24–26] Let X be an universe of discourse. A single valued neutrosophic set over X is an object taking the form $=\{\langle x, T_A(x), I_A(x), F_A(x),\rangle{:}x \in X\}$, where $T_A(x){:}X \rightarrow [0, 1]$, $I_A(x){:}X \rightarrow [0, 1]$ and $F_A(x){:}X \rightarrow [0, 1]$ with $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$ for all $x \in X$. The intervals $T_A(x)$, $I_A(x)$ and $F_A(x)$ represent the truth-membership degree, the indeterminacy-membership degree and the falsity membership degree of x to , respectively. For convenience, a SVN number is represented by $= (a, \mathrm{b}, \mathrm{c})$, where $a, b, c \in [0, 1]$ and $a + b + c \leq 3$.

Definition 3 [27, 28] Suppose $\alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}} \in [0,1]$ and $a_1, a_2, a_3, a_4 \in \mathrm{R}$, where $a_1 \leq a_2 \leq a_3 \leq a_4$. Then, a single valued trapezoidal neutrosophic number $\tilde{a} = \langle (a_1, a_2, a_3, a_4); \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}} \rangle$ is a special neutrosophic set on the real line set R, whose truth-membership, indeterminacy-membership and falsity-membership functions are defined as:

$$T_{\tilde{a}} = \begin{cases} \alpha_{\tilde{a}} \left(\frac{x-a_1}{a_2-a_1}\right) & (a_1 \leq x \leq a_2) \\ \alpha_{\tilde{a}} & (a_2 \leq x \leq a_3) \\ \alpha_{\tilde{a}} \left(\frac{a_4-x}{a_4-a_3}\right) & (a_3 \leq x \leq a_4) \\ 0 & otherwise \end{cases} \tag{1}$$

$$I_{\widetilde{a}}(x) = \begin{cases} \frac{(a_2 - x + \theta_{\widetilde{a}}(x - a_1))}{(a_2 - a_1)} & (a_1 \le x \le a_2) \\ \alpha_{\widetilde{a}} & (a_2 \le x \le a_3) \\ \frac{(x - a_3 + \theta_{\widetilde{a}}(a4 - x))}{(a_4 - a_3)} & (a_3 \le x \le a_4) \\ 1 & otherwise \end{cases} \quad (2)$$

$$F_{\widetilde{a}}(x) = \begin{cases} \frac{(a_2 - x + \beta_{\widetilde{a}}(x - a_1))}{(a_2 - a_1)} & (a_1 \le x \le a_2) \\ \alpha_{\widetilde{a}} & (a_2 \le x \le a_3) \\ \frac{(x - a_3 + \beta_{\widetilde{a}}(a4 - x))}{(a_4 - a_3)} & (a_3 \le x \le a_4) \\ 1 & otherwise \end{cases}, \quad (3)$$

where $\alpha_{\widetilde{a}}$, $\theta_{\widetilde{a}}$ and $\beta_{\widetilde{a}}$ typify the maximum truth-membership degree, the minimum indeterminacy-membership degree and the minimum falsity-membership degree, respectively. A single valued trapezoidal neutrosophic number $\widetilde{a} = \langle (a_1, a_2, a_3, a_4); \alpha_{\widetilde{a}}, \theta_{\widetilde{a}}, \beta_{\widetilde{a}} \rangle$ may express an ill-defined quantity of the range, which is approximately equal to the interval $[a_2, a_3]$.

Definition 4 [16, 28] Let $\widetilde{a} = \langle (a_1, a_2, a_3, a_4); \alpha_{\widetilde{a}}, \theta_{\widetilde{a}}, \beta_{\widetilde{a}} \rangle$ and $\widetilde{b} = \langle (b_1, b_2, b_3, b_4); \alpha_{\widetilde{a}}, \theta_{\widetilde{a}}, \beta_{\widetilde{a}} \rangle$ be two single valued trapezoidal neutrosophic numbers, and $\Upsilon \neq 0$ be any real number. Then:

1. Addition of two trapezoidal neutrosophic numbers:

$$\widetilde{a} + \widetilde{b} = \left\langle (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4);\ \alpha_{\widetilde{a}} \wedge \alpha_{\widetilde{b}}, \theta_{\widetilde{a}} \vee \theta_{\widetilde{b}}, \beta_{\widetilde{a}} \vee \beta_{\widetilde{b}} \right\rangle$$

2. Subtraction of two trapezoidal neutrosophic numbers:

$$\widetilde{a} - \widetilde{b} = \left\langle (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1);\ \alpha_{\widetilde{a}} \wedge \alpha_{\widetilde{b}}, \theta_{\widetilde{a}} \vee \theta_{\widetilde{b}}, \beta_{\widetilde{a}} \vee \beta_{\widetilde{b}} \right\rangle$$

3. Inverse of trapezoidal neutrosophic numbers:

$$\widetilde{a}^{-1} = \left\langle \left(\frac{1}{a_4}, \frac{1}{a_3}, \frac{1}{a_2}, \frac{1}{a_1} \right);\ \alpha_{\widetilde{a}}, \theta_{\widetilde{a}}, \beta_{\widetilde{a}} \right\rangle \quad \text{where } (\widetilde{a} \neq 0)$$

4. Multiplication of trapezoidal neutrosophic numbers by constant value:

$$\Upsilon \widetilde{a} = \begin{cases} \langle (\Upsilon a_1, \Upsilon a_2, \Upsilon a_3, \Upsilon a_4); \alpha_{\widetilde{a}}, \theta_{\widetilde{a}}, \beta_{\widetilde{a}} \rangle & \text{if } (\Upsilon > 0) \\ \langle (\Upsilon a_4, \Upsilon a_3, \Upsilon a_2, \Upsilon a_1); \alpha_{\widetilde{a}}, \theta_{\widetilde{a}}, \beta_{\widetilde{a}} \rangle & \text{if } (\Upsilon < 0) \end{cases}$$

5. Division of two trapezoidal neutrosophic numbers:

$$\frac{\widetilde{a}}{\widetilde{b}} = \begin{cases} \left\langle \left(\frac{a_1}{b_4}, \frac{a_2}{b_3}, \frac{a_3}{b_2}, \frac{a_4}{b_1} \right); \alpha_{\widetilde{a}} \wedge \alpha_{\widetilde{b}}, \theta_{\widetilde{a}} \vee \theta_{\widetilde{b}}, \beta_{\widetilde{a}} \vee \beta_{\widetilde{b}} \right\rangle & \text{if } (a_4 > 0, b_4 > 0) \\ \left\langle \left(\frac{a_4}{b_4}, \frac{a_3}{b_3}, \frac{a_2}{b_2}, \frac{a_1}{b_1} \right); \alpha_{\widetilde{a}} \wedge \alpha_{\widetilde{b}}, \theta_{\widetilde{a}} \vee \theta_{\widetilde{b}}, \beta_{\widetilde{a}} \vee \beta_{\widetilde{b}} \right\rangle & \text{if } (a_4 < 0, b_4 > 0) \\ \left\langle \left(\frac{a_4}{b_1}, \frac{a_3}{b_2}, \frac{a_2}{b_3}, \frac{a_1}{b_4} \right); \alpha_{\widetilde{a}} \wedge \alpha_{\widetilde{b}}, \theta_{\widetilde{a}} \vee \theta_{\widetilde{b}}, \beta_{\widetilde{a}} \vee \beta_{\widetilde{b}} \right\rangle & \text{if } (a_4 < 0, b_4 < 0) \end{cases}$$

6. Multiplication of trapezoidal neutrosophic numbers:

$$\widetilde{a}\widetilde{b} = \begin{cases} \left\langle (a_1 b_1, a_2 b_2, a_3 b_3, a_4 b_4); \alpha_{\widetilde{a}} \wedge \alpha_{\widetilde{b}}, \theta_{\widetilde{a}} \vee \theta_{\widetilde{b}}, \beta_{\widetilde{a}} \vee \beta_{\widetilde{b}} \right\rangle & \text{if } (a_4 > 0, b_4 > 0) \\ \left\langle (a_1 b_4, a_2 b_3, a_3 b_2, a_4 b_1); \alpha_{\widetilde{a}} \wedge \alpha_{\widetilde{b}}, \theta_{\widetilde{a}} \vee \theta_{\widetilde{b}}, \beta_{\widetilde{a}} \vee \beta_{\widetilde{b}} \right\rangle & \text{if } (a_4 < 0, b_4 > 0) \\ \left\langle (a_4 b_4, a_3 b_3, a_2 b_2, a_1 b_1); \alpha_{\widetilde{a}} \wedge \alpha_{\widetilde{b}}, \theta_{\widetilde{a}} \vee \theta_{\widetilde{b}}, \beta_{\widetilde{a}} \vee \beta_{\widetilde{b}} \right\rangle & \text{if } (a_4 < 0, b_4 < 0) \end{cases}$$

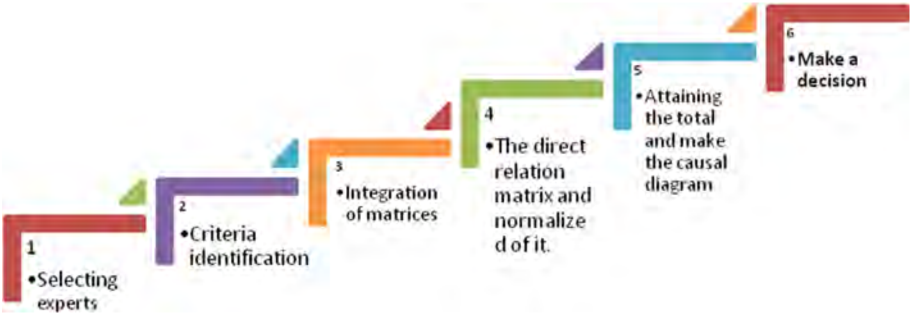

Fig. 3 The general neutrosophic DEMATEL framework

4 Neutrosophic DEMATEL approach

Atanassov [29] developed the intuitionistic fuzzy set theory. To overcome some of its limits, Smarandache [17] proposed the neutrosophic set theory. Neutrosophy handles vagueness and uncertainty, and attend the indeterminacy of values. Neutrosophy has some of advantage with DEMATEL:

- Neutrosophy provides the ability to present unknown information in our model using the indeterminacy degree, so the experts can present opinions about the unsure preferences.
- Neutrosophy depicts the disagreement of decision makers and experts.
- Neutrosophy heeds all aspects of decision making situations by considering truthiness, indeterminacy and falsity altogether.

DEMATEL is used to solve some complex and interrelated problems. In DEMATEL all criteria or factors fall into two categories: cause and effect.

In this section, we present the steps of the proposed model based on the neutrosophic DEMATEL analysis as shown in Fig. 3.

The procedures are explained as follows:

Step 1. Identifying decision goals: collecting relevant information presenting the problem.

1. Selection of experts and decision makers that have experience in the field.
2. Identifying the relevant criteria to the problem.

Step 2. Pairwise comparison matrices between relevant criteria.

1. Identify the criteria, Criteria = (F1, F2, F3… Fm).
2. Experts make pairwise comparisons matrices between criteria.
   a. Interpret each value for each criterion compared to other in a trapezoidal neutrosophic number $(l_{nm}, m_{nml}, m_{nm}, u_{nm})$.
   b. Make comparisons between criteria by each expert as shown in Table 1.
   c. Focuses only on (n – 1) consensus judgments using a scale from 0 to 1 [30, 31].

3. Experts should determine the maximum truth-membership degree (α), the minimum indeterminacy-membership degree (β) and the minimum falsity membership degree (θ) of single valued neutrosophic numbers as shown in Table 2.
4. Determine the crisp value of each opinion as shown in Table 3, using the following equations:

Table 1 The pairwise comparison matrix between criteria

Criteria	F_1	F_2	…	F_n
F_1	$(l_{11}, m_{11l}, m_{11u}, u_{11})$	$(l_{11}, m_{11l}, m_{11u}, u_{11})$	…	$(l_{1n}, m_{1nl}, m_{1nu}, u_{1n})$
F_2	$(l_{21}, m_{21l}, m_{21u}, u_{21})$	$(l_{22}, m_{22l}, m_{22u}, u_{22})$	…	$(l_{2n}, m_{2nl}, m_{2nu}, u_{2n})$
…	…	…	…	…
F_n	$(l_{n1}, m_{n1l}, m_{n1u}, u_{n1})$	$(l_{n2}, m_{n2l}, m_{n2u}, u_{n2})$	…	$(l_{nn}, m_{nnl}, m_{nnu}, u_{nn})$

Table 2 The pairwise comparison matrix between criteria with the α, β and θ degree

C	F_1	F_2	…	F_n
F_1	$(l_{11}, m_{11l}, m_{11u}, u_{11}; \alpha, \beta, \theta)$	$(l_{11}, m_{11l}, m_{11u}, u_{11}; \alpha, \beta, \theta)$	…	$(l_{1n}, m_{1nl}, m_{1nu}, u_{1n}; \alpha, \beta, \theta)$
F_2	$(l_{21}, m_{21l}, m_{21u}, u_{21}; \alpha, \beta, \theta)$	$(l_{22}, m_{22l}, m_{22u}, u_{22}; \alpha, \beta, \theta)$	…	$(l_{2n}, m_{2nl}, m_{2nu}, u_{2n}; \alpha, \beta, \theta)$
…	…	…	…	…
F_n	$(l_{n1}, m_{n1l}, m_{n1u}, u_{n1}; \alpha, \beta, \theta)$	$(l_{n2}, m_{n2l}, m_{n2u}, u_{n2}; \alpha, \beta, \theta)$	…	$(l_{nn}, m_{nnl}, m_{nnu}, u_{nn}; \alpha, \beta, \theta)$

Table 3 The crisp values of comparison matrix

C	F_1	F_2	…	F_n
F_1	CV_{11}	CV_{21}	…	CV_{m1}
F_2	CV_{12}	CV_{22}	…	CV_{m2}
…	…	…	…	…
F_n	CV_{1n}	CV_{2n}	…	CV_{mn}

Table 4 Integration of the average opinions of all experts

C	F_1	F_2	…	F_n
F_1	CV_{11}	CV_{21}	…	CV_{m1}
F_2	CV_{12}	CV_{22}	…	CV_{m2}
…	…	…	…	…
F_n	CV_{1n}	CV_{2n}	…	CV_{mn}

$$\mathrm{S}\left(\widetilde{a}_{ij}\right) = \frac{1}{16}\left[a_1 + b_1 + c_1 + d_1\right] \times \left(2 + \alpha_{\widetilde{a}} - \theta_{\widetilde{a}} - \beta_{\widetilde{a}}\right) \tag{4}$$

$$\mathrm{A}\left(\widetilde{\mathrm{a}}_{ij}\right) = \frac{1}{16}\left[a_1 + b_1 + c_1 + d_1\right] \times \left(2 + \alpha_{\widetilde{a}} - \theta_{\widetilde{a}} - \beta_{\widetilde{a}}\right) \tag{5}$$

Step 3. Integration of matrices.
All opinions of experts need to be integrated into one matrix presenting the average opinions of all experts about each criterion, as shown in Table 4.

$$CV_{11} = \frac{CV_{11n1} + CV_{11}n2 + CV_{11}nm}{n} \tag{6}$$

where n, number of experts. We obtain the average for all values as in the following matrix.

Step 4. Generating the direct relation matrix.

This matrix is obtained from previous step (3), i.e. the integrating of all averaged opinions of experts. An initial direct relation matrix A is a n × n matrix obtained by pairwise comparisons, $S = [s_{ij}]_{n\times n}$. S_{ij} denotes the degree to which the criterion i affects the criterion j.

Step 5. Normalizing the direct relation matrix.
The normalized direct relation matrix can be obtained using the equation:

$$K = \frac{1}{Max_{1\le i\le n}\sum_{j=1}^{n} a_{ij}} \quad (7)$$

$$S = K \times A \quad (8)$$

Step 6. Attaining the total relation matrix.
This step requires use of the Matlab software. The total relation matrix is acquired using the formula (9) from the generalized direct relation matrix S. A total relation matrix (T), in which (I) denotes the identity matrix, is shown as follows:

$$T = S \times (I - S)^{-1} \quad (9)$$

Step 7. Obtaining the sum of rows and columns.
The sum of rows is denoted by (D), and the sum of columns is denoted by (R). Calculate R + D and R − D.
Calculate T, where $T = [a_{ij}]_{n\times n}$, i, j = 1, 2… n

$$D = \left[\sum_{i=1}^{n} a_{ij}\right]_{1\times n} = [a_j]_{n\times 1} \quad (10)$$

$$R = \left[\sum_{j=1}^{n} a_{ij}\right]_{1\times n} = [a_j]_{n\times 1} \quad (11)$$

Step 8. Drawing cause and effect diagram
The causal diagram is obtained by the horizontal axes is presented by (D + R) and the vertical axes (D − R) which is a degree of relation and it depicts the steps of proposed model in Fig. 4.

5 The proposed methodology in a case study

In this section, we describe the details of the proposed methodology of a hybrid approach of neutrosophic sets and DEMATEL method for developing supplier selection criteria. This section is divided into five subsections: (1) the case study, (2) the Neutrosophic DEMATEL questionnaire design, (3) the calculation process of the Neutrosophic DEMATEL Method, (4) the analysis of the evaluation criteria shown in Fig. 5.

5.1 Case study

Flopater Trading Company was established in 2003. The company specializes in supplying plastic pipe fittings, soon becoming one of the largest distributors for large companies in the production of PVC pipes and joints. The company started importing from abroad and took large contracts of polypropylene pipes and fittings produced by Cosmo Plast UAE. Cosmo Plast in the United Arab Emirates is one of the largest factories in the Gulf region, and the company has started to support this plant in projects and accreditation with consultants.

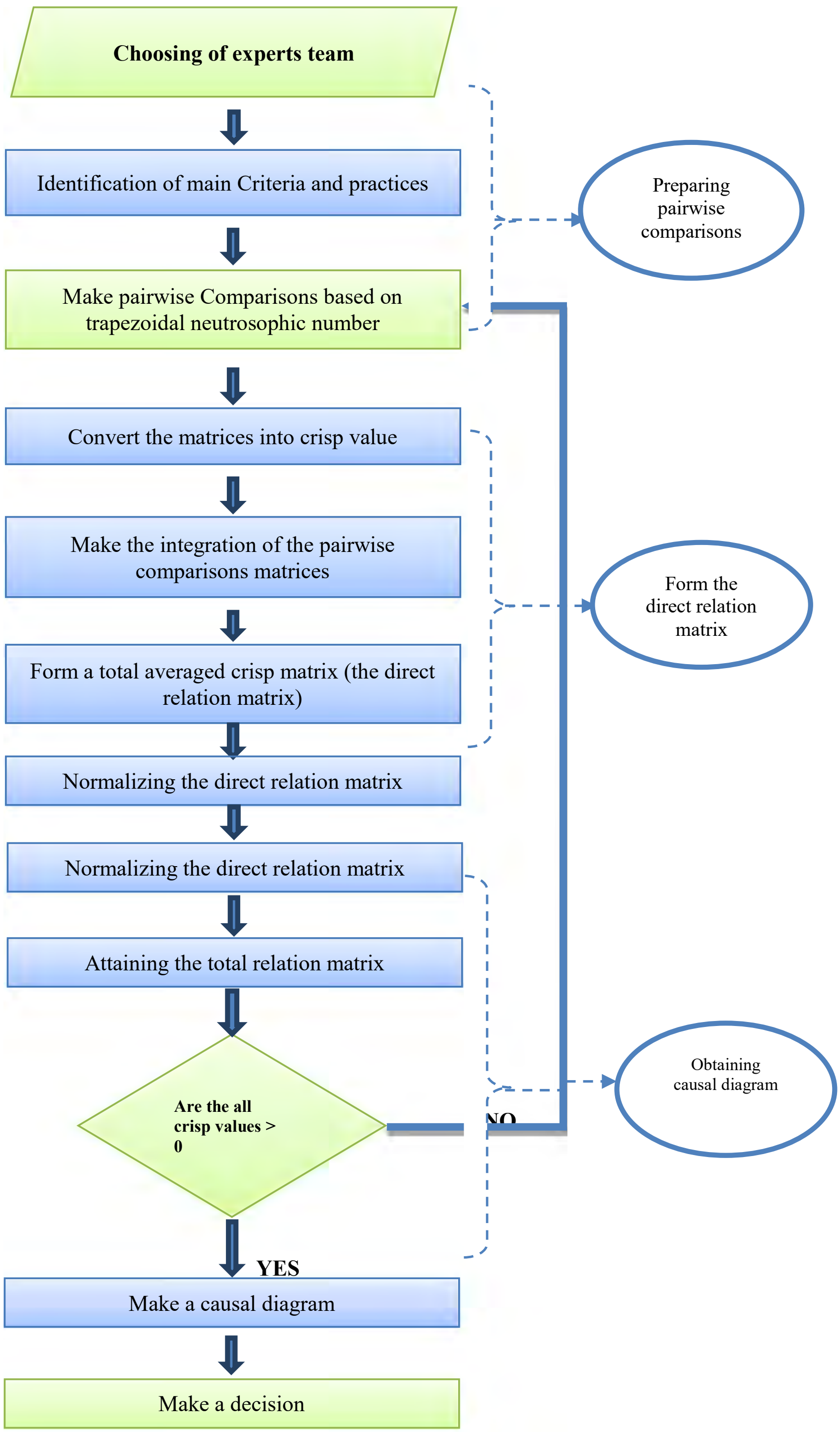

Fig. 4 Schematic diagram of DEMATEL in neutrosophic environment

Fig. 5 Main criteria selected for evaluation

The company also imports chips produced by a major company in Turkey, Zir Kelipas, also providing full procurement to all contracting companies or contractors in projects, helping them to deliver all supplies to the place of the project or their stores. The company offers appropriate ways for payment, committing to always deliver on time. Since it intends to expand trade and increase the number of contracts with customers, one of the most important problems facing the company is the selection criteria of suppliers.

5.2 Neutrosophic DEMATEL questionnaire design

In this research, the design of questionnaire is structured as following: In the first part, we determine the selection criteria. Then, we need to understand each criterion, its definition and its importance in the evaluation of selecting supplier. We employed seven (7) evaluation criteria: (1) cost, (2) time delivery, (3) quality, (4) innovation, (5) reputation, (6) response to customers, (7) location. The influence of every criterion on selecting the best supplier gets evaluated by Neutrosophic DEMATEL method. In the second part, we perform pairwise comparisons matrices to evaluate each criterion based on points of views from experts, using the neutrosophic scale of 0, 1.

5.3 The calculation process of Neutrosophic DEMATEL method

For collecting data, we interviewed three professionals in the management of purchasing and setup of contracts. The three experts determined the most important evaluation criterion to be used. The criteria symbols in this research are as follows: Cost (F1), Time delivery (F2), Quality (F3), Innovation (F4), Reputation (F5), Response to customers (F6), Location (F7). The data collected from the three experts were analyzed by the Neutrosophic DEMATEL method. The steps that were conducted are the following:

Step 1. Choosing the experts team
The first step of Neutrosophic DEMATEL method is selection of the best experts in the field of management purchasing and setup contracts. We selected three expert, to which we further refer as the first expert, the second expert, and the third expert.

Step 2. Identification of main criteria and practices
We sorted seven evaluation criteria as selected by the team of experts, namely: Cost (F1), Time delivery (F2), Quality (F3), Innovation (F4), Reputation (F5), Response to customers (F6), Location (F7).

Step 3. Performing pairwise comparisons matrices based on trapezoidal neutrosophic numbers.

1. Pairwise comparisons matrices to evaluate each feature or criterion against each other, as shown in Tables 5, 6, and 7.
2. Experts should determine the maximum truth membership degree (α), the minimum indeterminacy membership degree (θ) and the minimum falsity membership degree (β) of single valued neutrosophic numbers, as shown in Tables 8, 9, 10.
3. Convert the matrices into crisp values, as shown in Tables 11, 12, 13.

Step 4. Integrating the matrices
We process the integration of the three matrices according to formula (6), where a diagonal is 0.5. The initial direct-relation matrix (S) is shown in Table 14.

Step 5. Normalizing the initial direct relation matrix
We apply the Eq. (7) to obtain the value of K and then the formula (8) to obtain the generalized direct relation matrix X.
Calculation of each row:

$$\begin{bmatrix} Row\ 1 & 1.86 \\ Row\ 2 & 1.76 \\ Row\ 3 & 1.58 \\ Row\ 4 & 1.82 \\ Row\ 5 & 1.78 \\ Row\ 6 & 1.77 \\ Row\ 7 & 1.83 \end{bmatrix} \quad \text{Max} = 1.86 \ \ \text{k} = \frac{1}{1.86}$$

The generalized direct relation matrix X is presented in Table 15.

Step 6. Attaining the total relation matrix
This step is performed using the Matlab software. The total relation matrix is acquired using the formula (9) from the generalized direct relation matrix X. A total relation matrix (T) is obtained, where (I) denotes the identity matrix. The total relation matrix is presented in Table 16.

Step 7. Obtaining the sum of rows and columns
The sum of rows is denoted by (D), and the sum of columns is denoted by (R), using the formulas (10, 11).

Sum of rows and columns

Col 1	8.2182	Row 1	8.8362
Col 2	8.3654	Row 2	8.2445
Col 3	8.2309	Row 3	7.2893
Col 4	7.3264	Row 4	7.5221
Col 5	6.6465	Row 5	7.1926
Col 6	7.5996	Row 6	7.1994
Col 7	7.3456	Row 7	7.4485

Table 5 The pairwise comparison matrix of criteria

Criteria	F_1	F_2	F_3	F_4	F_5	F_6	F_7
F_1	(0.5, 0.5, 0.5, 0.5)	(0.6, 0.7, 0.9, 0.1)	(0.7, 0.9, 1.0, 1.0)	(0.4, 0.7, 1.0, 1.0)	(0.4, 0.7, 0.3, 0.4)	(0.4, 0.3, 0.9, 0.8)	(0.8, 0.4, 0.7, 0.3)
F_2	(0.0, 0.1, 0.3, 0.4)	(0.5, 0.5, 0.5, 0.5)	(0.6, 0.7, 0.8, 0.9)	(0.3, 0.5, 0.9, 1.0)	(0.3, 0.4, 0.6, 1.0)	(0.2, 0.3, 0.5, 0.7)	(0.8, 0.7, 0.5, 0.1)
F_3	(0.4, 0.2, 0.1, 0.3)	(0.3, 0.0, 0.5, 0.8)	(0.5, 0.5, 0.5, 0.5)	(0.2, 0.5, 0.6, 0.8)	(0.3, 0.5, 0.7, 0.6)	(0.1, 0.5, 1.0, 1.0)	(0.1, 0.5, 0.3, 0.7)
F_4	(0.7, 0.3, 0.3, 0.6)	(0.6, 0.1, 0.7, 1.0)	(0.2, 0.4, 0.5, 0.8)	(0.5, 0.5, 0.5, 0.5)	(0.4, 0.3, 0.2, 0.5)	(0.4, 0.2, 0.3, 0.6)	(0.4, 0.3, 0.4, 0.2)
F_5	(0.3, 0.4, 0.5, 0.8)	(0.3, 0.2, 0.4, 0.9)	(0.2, 0.4, 0.5, 0.6)	(0.3, 0.2, 0.6, 0.7)	(0.5, 0.5, 0.5, 0.5)	(0.8, 0.4, 0.8, 0.2)	(0.4, 0.7, 0.5, 0.0)
F_6	(0.7, 0.7, 0.3, 0.5)	(0.3, 0.2, 0.4, 0.9)	(0.2, 0.4, 0.5, 0.6)	(0.8, 0.4, 0.4, 0.1)	(0.7, 0.6, 0.0, 0.5)	(0.5, 0.5, 0.5, 0.5)	(0.5, 0.2, 0.8, 1.0)
F_7	(0.9, 0.7, 0.6, 0.3)	(0.8, 0.9, 0.4, 0.0)	(0.8, 0.4, 0.5, 1.0)	(0.8, 0.4, 0.4, 0.1)	(0.4, 0.7, 0.6, 0.5)	(0.9, 0.6, 0.6, 0.5)	(0.5, 0.5, 0.5, 0.5)

Table 6 The pairwise comparison matrix of criteria

Criteria	F_1	F_2	F_3	F_4	F_5	F_6	F_7
F_1	(0.5, 0.5, 0.5, 0.5)	(0.4, 0.5, 0.6, 0.1)	(0.7, 0.9, 0.4, 0.5)	(0.4, 0.8, 0.1, 0.5)	(0.5, 0.6, 0.3, 0.4)	(0.6, 0.3, 0.5, 0.1)	(0.2, 0.4, 0.3, 0.5)
F_2	(0.5, 0.6, 0.4, 0.3)	(0.5, 0.5, 0.5, 0.5)	(0.5, 0.7, 0.8, 0.2)	(0.3, 0.4, 0.9, 0.7)	(0.3, 0.2, 0.6, 1.0)	(0.5, 0.3, 0.4, 0.7)	(0.5, 0.3, 0.5, 0.1)
F_3	(0.1, 0.5, 0.4, 0.6)	(0.3, 0.6, 0.8, 0.8)	(0.5, 0.5, 0.5, 0.5)	(0.2, 0.3, 0.4, 0.9)	(0.3, 0.4, 0.7, 0.5)	(0.1, 0.5, 0.0, 1.0)	(0.6, 0.4, 0.3, 0.7)
F_4	(1.0, 0.8, 0.9, 0.7)	(0.6, 0.4, 0.7, 1.0)	(0.2, 0.3, 0.4, 0.8)	(0.5, 0.5, 0.5, 0.5)	(0.4, 0.5, 0.2, 0.8)	(0.4, 0.2, 0.5, 0.6)	(0.8, 0.6, 0.4, 0.2)
F_5	(0.9, 0.6, 0.3, 0.6)	(0.4, 0.2, 0.5, 0.6)	(0.2, 0.7, 0.3, 0.6)	(0.3, 0.6, 0.5, 0.7)	(0.5, 0.5, 0.5, 0.5)	(0.8, 0.5, 0.8, 0.2)	(0.9, 0.3, 0.5, 0.5)
F_6	(0.7, 0.5, 0.3, 0.4)	(0.3, 0.5, 0.4, 0.7)	(0.2, 0.4, 0.1, 0.4)	(0.7, 0.4, 0.5, 0.2)	(0.7, 0.1, 0.6, 0.2)	(0.5, 0.5, 0.5, 0.5)	(0.5, 0.7, 0.8, 1.0)
F_7	(0.5, 0.4, 0.6, 0.3)	(0.8, 0.5, 0.4, 0.4)	(0.3, 0.4, 0.6, 1.0)	(0.6, 0.1, 0.4, 0.3)	(0.4, 0.3, 0.6, 0.4)	(0.7, 0.5, 0.4, 0.5)	(0.5, 0.5, 0.5, 0.5)

Table 7 The pairwise comparison matrix of criteria

Criteria	F_1	F_2	F_3	F_4	F_5	F_6	F_7
F_1	(0.5, 0.5, 0.5, 0.5)	(0.4, 0.4, 0.5, 0.1)	(0.5, 0.3, 0.7, 0.5)	(0.4, 0.8, 0.1, 0.5)	(0.1, 0.5, 0.6, 0.4)	(0.4, 0.6, 0.9, 0.8)	(0.8, 0.4,0.7,0.5)
F_2	(0.3, 0.6, 0.5, 0.3)	(0.5, 0.5, 0.5, 0.5)	(0.3, 0.7, 0.8, 0.1)	(0.4, 0.5, 0.9, 0.7)	(0.7, 0.2, 0.6, 1.0)	(0.4, 0.6, 0.5, 0.7)	(0.8, 0.6, 0.5, 0.4)
F_3	(0.3, 0.6, 0.4, 0.5)	(0.3, 0.4, 0.2, 0.8)	(0.5, 0.5, 0.5, 0.5)	(0.9, 0.7, 0.5, 0.3)	(0.4, 0.1, 0.7, 0.6)	(0.7, 0.5, 0.6, 1.0)	(0.1, 0.6, 0.3, 0.7)
F_4	(1.0, 0.4, 0.9, 0.5)	(0.2, 0.4, 0.6, 1.0)	(0.8, 0.2, 0.4, 0.7)	(0.5, 0.5, 0.5, 0.5)	(0.4, 0.5, 0.2, 0.4)	(0.4, 0.2, 0.5, 0.6)	(0.4, 0.3, 0.7, 0.5)
F_5	(0.5, 0.6, 0.2, 0.3)	(0.6, 0.3, 0.5, 0.9)	(0.2, 0.8, 0.3, 0.5)	(0.8, 0.3, 0.5, 0.2)	(0.5, 0.5, 0.5, 0.5)	(0.5, 0.4, 0.8, 0.6)	(0.9, 0.4, 0.5, 0.3)
F_6	(0.7, 0.7, 0.6, 0.4)	(0.3, 0.4, 0.6, 0.9)	(0.5, 0.4, 0.5, 0.6)	(0.9, 0.6, 0.4, 0.5)	(0.8, 0.3, 0.1, 0.5)	(0.5, 0.5, 0.5, 0.5)	(0.5, 0.4, 0.8, 1.0)
F_7	(0.9, 0.7, 0.6, 0.7)	(0.5, 0.9, 0.4, 0.0)	(0.3, 0.7, 0.5, 1.0)	(0.4, 0.6, 0.4, 0.8)	(0.4, 0.7, 0.3, 0.5)	(0.9, 0.7, 0.4, 0.1)	(0.5, 0.5, 0.5, 0.5)

Table 8 The pairwise comparison matrix between criteria with the α, β and θ degree

Criteria	F_1	F_2	F_3	F_4	F_5	F_6	F_7
F_1	(0.5, 0.5, 0.5, 0.5)	(0.6, 0.7, 0.9, 0.1; 0.7, 0.2, 0.5)	(0.7, 0.9, 1.0, 1.0; 0.4, 0.2, 0.3)	(0.4, 0.7, 1.0, 1.0; 0.5, 0.2, 0.1)	(0.4, 0.7, 0.3, 0.4; 0.5, 0.4, 0.2)	(0.4, 0.3, 0.9, 0.8; 0.9, 0.7, 0.4)	(0.8, 0.4, 0.7, 0.3; 0.9, 0.6, 0.7)
F_2	(0.0, 0.1, 0.3, 0.4; 0.8, 0.2, 0.6)	(0.5, 0.5, 0.5, 0.5)	(0.6, 0.7, 0.8, 0.9; 0.4, 0.5, 0.6)	(0.3, 0.5, 0.9, 1.0; 0.5, 0.1, 0.2)	(0.3, 0.4, 0.6, 1.0; 0.6, 0.4, 0.3)	(0.2, 0.3, 0.5, 0.7; 0.7, 0.4, 0.6)	(0.8, 0.7, 0.5, 0.1; 0.8, 0.3, 0.5)
F_3	(0.4, 0.2, 0.1, 0.3; 0.5, 0.3, 0.4)	(0.3, 0.0, 0.5, 0.8; 0.8, 0.5, 0.3)	(0.5, 0.5, 0.5, 0.5)	(0.2, 0.5, 0.6, 0.8; 0.7, 0.3, 0.4)	(0.3, 0.5, 0.7, 0.6; 0.3, 0.6, 0.5)	(0.1, 0.5, 1.0, 1.0; 0.3, 0.5, 0.4)	(0.1, 0.5, 0.3, 0.7; 0.3, 0.4, 0.6)
F_4	(0.7, 0.3, 0.3, 0.6; 0.5, 0.2, 0.1)	(0.6, 0.1, 0.7, 1.0; 0.3, 0.1, 0.5)	(0.2, 0.4, 0.5, 0.8; 0.9, 0.4, 0.6)	(0.5, 0.5, 0.5, 0.5)	(0.4, 0.3, 0.2, 0.5; 0.9, 0.7, 0.4)	(0.4, 0.2, 0.3, 0.6; 0.6, 0.3, 0.5)	(0.4, 0.3, 0.4, 0.2; 0.3, 0.6, 0.5)
F_5	(0.3, 0.4, 0.5, 0.8; 0.2, 0.4, 0.3)	(0.3, 0.2, 0.4, 0.9; 0.4, 0.3, 0.6)	(0.2, 0.4, 0.5, 0.6; 0.7, 0.4, 0.5)	(0.3, 0.2, 0.6, 0.7; 0.7, 0.2, 0.6)	(0.5, 0.5, 0.5, 0.5)	(0.8, 0.4, 0.8, 0.2; 0.8, 0.2, 0.4)	(0.4, 0.7, 0.5, 0.0; 0.3, 0.4, 0.1)
F_6	(0.7, 0.7, 0.3, 0.5; 0.7, 0.2, 0.4)	(0.3, 0.2, 0.4, 0.9; 0.8, 0.5, 0.5)	(0.2, 0.4, 0.5, 0.6; 0.7, 0.6, 0.3)	(0.8, 0.4, 0.4, 0.1; 0.9, 0.1, 0.6)	(0.7, 0.6, 0.0, 0.5; 0.4, 0.3, 0.5)	(0.5, 0.5, 0.5, 0.5)	(0.5, 0.2, 0.8, 1.0; 0.7, 0.6,0.5)
F_7	(0.9, 0.7, 0.6, 0.3; 0.8, 0.4, 0.5)	(0.8, 0.9, 0.4, 0.0; 0.9, 0.5, 0.4)	(0.8, 0.4, 0.5, 1.0; 0.5, 0.7, 0.2)	(0.8, 0.4, 0.4, 0.1; 0.5, 0.2, 0.9)	(0.4, 0.7, 0.6, 0.5; 0.9, 0.6, 0.7)	(0.9, 0.6, 0.6, 0.5; 0.5, 0.8, 0.5)	(0.5, 0.5, 0.5, 0.5)

Table 9 The pairwise comparison matrix between criteria with the α, β and θ degree

Criteria	F_1	F_2	F_3	F_4	F_5	F_6	F_7
F_1	(0.5, 0.5, 0.5, 0.5)	(0.4, 0.5, 0.6, 0.1; 0.5, 0.4, 0.3)	(0.7, 0.9, 0.4, 0.5; 0.2, 0.6, 0.3)	(0.4, 0.8, 0.1, 0.5; 0.8, 0.7, 0.5)	(0.5, 0.6, 0.3, 0.4; 0.2, 0.5, 0.4)	(0.6, 0.3, 0.5, 0.1; 0.4, 0.3, 0.6)	(0.4, 0.4, 0.3, 0.5; 0.3, 0.5, 0.7)
F_2	(0.5, 0.6, 0.4, 0.3; 0.4, 0.3, 0.5)	(0.5, 0.5, 0.5, 0.5)	(0.5, 0.7, 0.8, 0.2; 0.3, 0.6, 0.8)	(0.3, 0.4, 0.9, 0.7; 0.4, 0.8, 0.1)	(0.3, 0.2, 0.6, 1.0; 0.5, 0.4, 0.7)	(0.5, 0.3, 0.4, 0.7; 0.3, 0.8, 0.4)	(0.5, 0.3, 0.5, 0.1; 0.9, 0.4, 0.3)
F_3	(0.1, 0.5, 0.4, 0.6; 0.7, 0.4, 0.6)	(0.3, 0.6, 0.8, 0.8; 0.2, 0.1, 0.7)	(0.5, 0.5, 0.5, 0.5)	(0.2, 0.3, 0.4, 0.9; 0.5, 0.4, 0.6)	(0.3, 0.4, 0.7, 0.5; 0.5, 0.3, 0.3)	(0.1, 0.5, 0.0, 1.0; 0.4, 0.3, 0.8)	(0.6, 0.4, 0.3, 0.7; 0.3, 0.6, 0.5)
F_4	(1.0, 0.8, 0.9, 0.7; 0.5, 0.6, 0.2)	(0.6, 0.4, 0.7, 1.0; 0.6, 0.3, 0.8)	(0.2, 0.3, 0.4, 0.8; 0.5, 0.4, 0.4)	(0.5, 0.5, 0.5, 0.5)	(0.4, 0.5, 0.2, 0.8; 0.8, 0.3, 0.5)	(0.4, 0.2, 0.5, 0.6; 0.2, 0.5, 0.4)	(0.8, 0.6, 0.4, 0.2; 0.4, 0.7, 0.2)
F_5	(0.9, 0.6, 0.3, 0.6; 0.5, 0.6, 0.7)	(0.4, 0.2, 0.5, 0.6; 0.5, 0.2, 0.3)	(0.2, 0.7, 0.3, 0.6; 0.2, 0.3, 0.9)	(0.3, 0.6, 0.5, 0.7; 0.5, 0.3, 0.7)	(0.5, 0.5, 0.5, 0.5)	(0.8, 0.5, 0.8, 0.2; 0.8, 0.3, 0.5)	(0.9, 0.3, 0.5, 0.5; 0.4, 0.1, 0.1)
F_6	(0.7, 0.5, 0.3, 0.4; 0.5, 0.4, 0.6)	(0.3, 0.5, 0.4, 0.7; 0.5, 0.4, 0.3)	(0.2, 0.4, 0.1, 0.4; 0.4, 0.3, 0.6)	(0.7, 0.4, 0.5, 0.2; 0.6, 0.7, 0.5)	(0.7, 0.1, 0.6, 0.2; 0.7, 0.3, 0.6)	(0.5, 0.5, 0.5, 0.5)	(0.5, 0.7, 0.8, 1.0; 0.8, 0.7, 0.5)
F_7	(0.5, 0.4, 0.6, 0.3; 0.8, 0.6, 0.3)	(0.8, 0.5, 0.4, 0.4; 0.5, 0.2, 0.6)	(0.3, 0.4, 0.6, 1.0; 0.2, 0.5, 0.7)	(0.6, 0.1, 0.4, 0.3; 0.5, 0.7, 0.2)	(0.4, 0.3, 0.6, 0.4; 0.8, 0.6, 0.3)	(0.7, 0.5, 0.4, 0.5; 0.7, 0.3, 0.4)	(0.5, 0.5, 0.5, 0.5)

Table 10 The pairwise comparison matrix between criteria with the α, β and θ degrees

Criteria	F_1	F_2	F_3	F_4	F_5	F_6	F_7
F_1	(0.5, 0.5, 0.5, 0.5)	(0.4, 0.4, 0.5, 0.1; 0.8, 0.7, 0.9)	(0.5, 0.3, 0.7, 0.5; 0.7, 0.4, 0.1)	(0.4, 0.8, 0.1, 0.5; 0.5, 0.2, 0.4)	(0.1, 0.5, 0.6, 0.4; 0.4, 0.1, 0.7)	(0.4, 0.6, 0.9, 0.8; 0.3, 0.5, 0.4)	(0.8, 0.4, 0.7, 0.5; 0.8, 0.5, 0.4)
F_2	(0.3, 0.6, 0.5, 0.3; 0.5, 0.4, 0.6)	(0.5, 0.5, 0.5, 0.5)	(0.3, 0.7, 0.8, 0.1; 0.2, 0.3, 0.6)	(0.4, 0.5, 0.9, 0.7; 0.3, 0.1, 0.7)	(0.7, 0.2, 0.6, 1.0; 0.6, 0.3, 0.5)	(0.4, 0.6, 0.5, 0.7; 0.8, 0.4, 0.6)	(0.8, 0.6, 0.5, 0.4; 0.7, 0.8, 0.3)
F_3	(0.3, 0.6, 0.4, 0.5; 0.3, 0.2, 0.4)	(0.3, 0.4, 0.2, 0.8; 0.4, 0.5, 0.8)	(0.5, 0.5, 0.5, 0.5)	(0.9, 0.7, 0.5, 0.3; 0.2, 0.3, 0.6)	(0.4, 0.1, 0.7, 0.6; 0.4, 0.1, 0.9)	(0.7, 0.5, 0.6, 1.0; 0.7, 0.5, 0.8)	(0.1, 0.6, 0.3, 0.7; 0.7, 0.4, 0.6)
F_4	(1.0, 0.4, 0.9, 0.5; 0.2, 0.1, 0.3)	(0.2, 0.4, 0.6, 1.0; 0.9, 0.3, 0.1)	(0.8, 0.2, 0.4, 0.7; 0.8, 0.4, 0.8)	(0.5, 0.5, 0.5, 0.5)	(0.4, 0.5, 0.8, 0.4; 0.8, 0.8, 0.6)	(0.4, 0.2, 0.5, 0.6; 0.6, 0.5, 0.2)	(0.4, 0.3, 0.7, 0.5; 0.5, 0.7, 0.3)
F_5	(0.5, 0.6, 0.2, 0.3; 0.6, 0.5, 0.7)	(0.6, 0.3, 0.5, 0.9; 0.4, 0.3, 0.5)	(0.2, 0.8, 0.3, 0.5; 0.4, 0.3, 0.5)	(0.8, 0.3, 0.5, 0.2; 0.1, 0.5, 0.4)	(0.5, 0.5, 0.5, 0.5)	(0.5, 0.4, 0.8, 0.6; 0.9, 0.3, 0.4)	(0.9, 0.4, 0.5, 0.3; 0.7, 0.2, 0.1)
F_6	(0.7, 0.7, 0.6, 0.4; 0.8, 0.4, 0.7)	(0.3, 0.4, 0.6, 0.9; 0.4, 0.7, 0.6)	(0.5, 0.4, 0.5, 0.6; 0.8, 0.6, 0.4)	(0.9, 0.6, 0.4, 0.5; 0.4, 0.7, 0.4)	(0.8, 0.3, 0.1, 0.5; 0.5, 0.6, 0.5)	(0.5, 0.5, 0.5, 0.5)	(0.5, 0.4, 0.8, 1.0; 0.9, 0.3, 0.4)
F_7	(0.9, 0.7, 0.6, 0.7; 0.4, 0.6, 0.9)	(0.5, 0.9, 0.4, 0.0; 0.8, 0.6, 0.5)	(0.6, 0.7, 0.5, 1.0; 0.7, 0.5, 0.8)	(0.4, 0.6, 0.4, 0.8; 0.4, 0.5, 0.6)	(0.4, 0.7, 0.3, 0.5; 0.7, 0.4, 0.6)	(0.9, 0.7, 0.4, 0.1; 0.3, 0.5, 0.8)	(0.5, 0.5, 0.5, 0.5)

Table 11 The crisp values of comparison matrix

Criteria	F_1	F_2	F_3	F_4	F_5	F_6	F_7
F_1	0.5	0.288	0.428	0.426	0.214	0.270	0.220
F_2	0.100	0.5	0.244	0.371	0.273	0.181	0.263
F_3	0.113	0.200	0.5	0.263	0.158	0.228	0.130
F_4	0.261	0.255	0.226	0.5	0.158	0.169	0.098
F_5	0.188	0.169	0.191	0.214	0.5	0.303	0.181
F_6	0.289	0.203	0.191	0.234	0.180	0.5	0.250
F_7	0.296	0.263	0.270	0.149	0.220	0.195	0.5

Table 12 The crisp values of comparison matrix

Criteria	F_1	F_2	F_3	F_4	F_5	F_6	F_7
F_1	0.5	0.180	0.203	0.180	0.146	0.141	0.110
F_2	0.180	0.5	0.124	0.216	0.184	0.132	0.193
F_3	0.170	0.219	0.5	0.169	0.226	0.130	0.150
F_4	0.361	0.253	0.181	0.5	0.238	0.138	0.118
F_5	0.180	0.213	0.113	0.197	0.5	0.288	0.303
F_6	0.178	0.214	0.103	0.158	0.158	0.5	0.300
F_7	0.214	0.223	0.144	0.140	0.202	0.263	0.5

Table 13 The crisp values of comparison matrix

Criteria	F_1	F_2	F_3	F_4	F_5	F_6	F_7
F_1	0.5	0.105	0.275	0.214	0.160	0.236	0.285
F_2	0.159	0.5	0.154	0.234	0.281	0.248	0.230
F_3	0.191	0.117	0.5	0.195	0.158	0.245	0.181
F_4	0.315	0.344	0.210	0.5	0.184	0.202	0.178
F_5	0.140	0.230	0.180	0.135	0.5	0.316	0.315
F_6	0.255	0.151	0.225	0.195	0.149	0.5	0.317
F_7	0.163	0.191	0.245	0.179	0.202	0.131	0.5

Table 14 The integration matrix

Criteria	F_1	F_2	F_3	F_4	F_5	F_6	F_7
F_1	0.5	0.191	0.302	0.273	0.173	0.216	0.205
F_2	0.146	0.5	0.174	0.274	0.246	0.187	0.229
F_3	0.158	0.179	0.5	0.209	0.181	0.201	0.154
F_4	0.312	0.284	0.206	0.5	0.193	0.170	0.155
F_5	0.169	0.204	0.161	0.182	0.5	0.302	0.266
F_6	0.241	0.189	0.173	0.196	0.162	0.5	0.307
F_7	0.224	0.226	0.220	0.156	0.208	0.208	0.5

Table 15 Normalized matrix

Criteria	F_1	F_2	F_3	F_4	F_5	F_6	F_7
F_1	0.269	0.103	0.162	0.147	0.093	0.116	0.110
F_2	0.079	0.269	0.094	0.147	0.132	0.101	0.123
F_3	0.085	0.096	0.269	0.112	0.097	0.108	0.083
F_4	0.168	0.153	0.111	0.145	0.104	0.091	0.083
F_5	0.091	0.110	0.087	0.098	0.145	0.162	0.143
F_6	0.130	0.102	0.093	0.105	0.087	0.147	0.165
F_7	0.120	0.122	0.118	0.084	0.112	0.159	0.145

Table 16 The total relation matrix

Criteria	F_1	F_2	F_3	F_4	F_5	F_6	F_7
F_1	1.4895	1.3104	1.3747	1.2188	1.0546	1.2165	1.1717
F_2	1.1796	1.4323	1.1966	1.1458	1.0406	1.1287	1.1209
F_3	1.0597	1.0920	1.2834	0.9900	0.8920	1.0158	0.9564
F_4	1.1941	1.1955	1.1315	1.0582	0.9272	1.0271	0.9885
F_5	1.0599	1.0988	1.0526	0.9618	0.9330	1.0672	1.0193
F_6	1.1090	1.0905	1.0660	0.9717	0.8729	1.0503	1.0390
F_7	1.1264	1.1459	1.1261	0.9801	0.9262	1.0940	1.0498

Row + Column and Row – Column

0	Row + Col	Row – Col
1	17.0544	– 0.618
2	16.6099	0.1209
3	15.5202	0.9416
4	14.8485	– 0.1957
5	13.8391	– 0.5461
6	14.799	0.4002
7	14.7941	– 0.1029

Step 8. Drawing cause and effect diagram
The causal diagram is obtained by the horizontal axes, presented by (D + R), and the vertical axes (D – R), which is a degree of relation, as depicted in Fig. 6.

5.4 Analyzing the evaluation criteria

The final step is the analysis of collected data according to the causal diagram. This article integrates several questionnaires from expert interviews to find out the evaluation criteria and to calculate the average of each criterion. The research results determine the most important criterion. From this causal chart, according to the Neutrosophic DEMATEL Method, the importance of all criteria was established. According to experts' opinions, Quality (F3) had the greatest impact and Cost (F1) had the lesser impact on the selection of the company supplier.

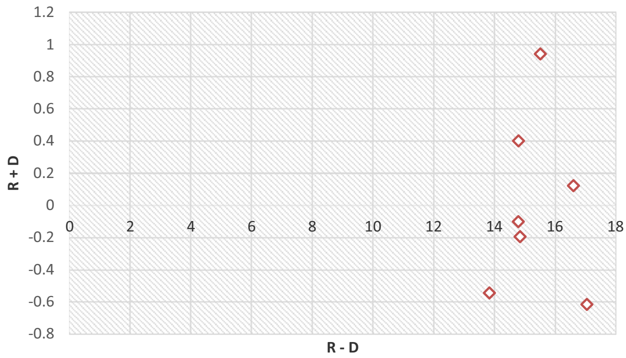

Fig. 6 The causal diagram for evaluation criteria

6 Conclusions and recommendations

This study presents the criteria selected by experts in the field of production and procurement in collective organizations, affecting the productivity and profitability of any organization. Potential supply chain management practices have been developed and performed using the Neutrosophic DEMATEL Method to select the best standards that have a greater impact on other criteria. The proposed approach succeeded in developing the DEMATEL Method by applying to it the Neutrosophic Set Theory, using a new scale from 0 to 1 and employing the maximum truth membership degree (α), the minimum indeterminacy membership degree (θ) and the minimum falsity membership degree (β) of a single valued neutrosophic number. The opinions were collected from experts by interviews, and consequently analyzed using the Neutrosophic DEMATEL Method, by comparisons of each criterion, according to each individual expert, and their formulation of each value according to a single valued neutrosophic number. Finally, we extracted the most important criterion or feature that proved to be important for any organization in order to effectively choose its suppliers. However, this research contains some limitations and difficulties due to the fact that the multitude of standards and features require a large processing team and complex calculation.

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Neutrosophic Logic Based Quantum Computing

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Abstract: We introduce refined concepts for neutrosophic quantum computing such as neutrosophic quantum states and transformation gates, neutrosophic Hadamard matrix, coherent and decoherent superposition states, entanglement and measurement notions based on neutrosophic quantum states. We also give some observations using these principles. We present a number of quantum computational matrix transformations based on neutrosophic logic and clarify quantum mechanical notions relying on neutrosophic states. The paper is intended to extend the work of Smarandache by introducing a mathematical framework for neutrosophic quantum computing and presenting some results.

Keywords: neutrosophic computation; neutrosophic logic; quantum computation; computation; logic

1. Introduction

1.1. Neutrosophy Theory

Neutrosophic set concept, introduced by Smarandache [1,2], is a more universal structure that extends the concepts of the classic set, fuzzy set [3] and intuitionistic fuzzy set [4]. Unlike intuitionistic fuzzy sets, the indeterminacy is explicitly defined in neutrosophic sets. A neutrosophic set has three basic components defined separately: Truth T, indeterminacy I and falsity F, regarding membership. Neutrosophy was proposed as an ambitious project by Smarandache as a new branch of philosophy as well, concerning "the origin, nature, and scope of neutralities, as well as their mutual effects with different intellectual spectra". The key assumption of neutrosophy is that every idea has not only a certain degree of truth, as is generally taken in many-valued logic contexts, but also degrees of falsity and indeterminacy need to be considered independently from each other. Neutrosophy has settled the baseline for a number of new mathematical theories generalizing both their classical and fuzzy counterparts, such as neutrosophic set theory, geometry, statistics, topology, analysis, probability, and logic. The neutrosophic framework has already been applied to practical applications in many different fields, such as decision-making, semantic web, and data analysis in medicine.

Now, let us look at the concepts of some subfields of neutrosophy. *Neutrosophic set* has a formal definition as follows: Let U be a universe of discourse or space, and M be a set in U. An element x from U is stated related to the set M as $x(T, I, F)$ and belongs to M in the following way: it is t % true in the set, i % indeterminate in the set, and f % false, where t varies in T, i varies in I, f varies in F. Statically T, I, F are subsets, but dynamically T, I, F are functions/operators depending on many known or unknown parameters. *Neutrosophic logic* is a general framework for the unification of many existing logics. The main idea of neutrosophic logic is to characterize each logical statement in a 3-dimensional neutrosophic space, where each dimension of the space represents respectively the truth (T), the falsehood (F), and the indeterminacy (I) of the statement under consideration, where T, I, F are standard or non-standard real subsets of $[0^-, 1^+]$. For instance, a statement can be between $[0.21, 0.55]$ true, 0.23 or between $(0.35, 0.45)$ indeterminate, and either 0.32 or 0.75 false. *Neutrosophic statistics* is the analysis of events characterized by the neutrosophic probability. The function that models the neutrosophic probability of a random variable x is called *neutrosophic distribution*: $NP(x) = (T(x), I(x), F(x))$, where $T(x)$ represents the probability that value x occurs, $F(x)$ represents the probability that value x does not occur, and $I(x)$ represents the indeterminate/unknown probability of value x. *Neutrosophic probability* is an extension of the classical probability and imprecise probability where a case, event or fact A occurs is t % true—where t varies in the subset T, i % indeterminate—where i varies in the subset I, and f % false—where f varies in the subset F. In classical probability $n_s up \leq 1$, while in neutrosophic probability $n_s up \leq 3^+$. In imprecise probability, the probability of an event is a subset T in $[0, 1]$, not a number p in $[0, 1]$, the rest was supposed to be the opposite, subset F (also from the unit interval $[0, 1]$); there is no indeterminate subset I in imprecise probability.

1.2. Quantum Mechanics and Computing

Quantum mechanics was started with Planck [5] and interpreted as real life problem by Einstein [6]. The mechanics was developed by Bohr, Heisenberg, Broglie, Schrödinger, Born, Dirac, Hilbert, Sommerfeld, Dyson, Wien, Pauli, Von Neumann and others [7–12] in the first 30 years of the 20th century. Computers are mechanisms that support transaction information by executing algorithms. An algorithm is a well-defined process to perform an information processing task. The task can always be translated into a realization. When creating complicated algorithms for a variety of tasks, working with some improved computational models is very useful, probably very important. However, when examining the actual limitations of a computation mechanism, it is key to remember the connection between computation and realization. Quantum computation explores how efficiently nature allows us to compute. The standard computational model is based on classical mechanics; the mechanics of the Turing machine relies on classical mechanics. Quantum information processing changes not only the physical paradigm used for computing and communication but also the concepts of knowledge and computation. Quantum computation is not synonymous with quantum effects to make calculations. Actual computing mechanisms of the quantum are based on a larger physical reality than is represented by the idealized computational model. Quantum information processing is the result of the use of the physical reality that quantum theory states to perform tasks that were previously thought to be infeasible or impossible. The mechanisms that perform quantum information processing are known as quantum computers. In the last few decades of the twentieth century, researchers tried to follow two of the most influential and revolutionary theories: information science and quantum mechanics. Their success provided an unfamiliar computation and information range of vision. This new insight has significantly changed how the relationship between quantum information theory, computation, knowledge, and physics is considered and has given rise to new applications and epoch-making algorithms. The theory of information, which contains the foundations of computer science and communication, made possible to address the important issues in computer science and communication. The Turing machine is a classical model that behaves entirely according to classical mechanical principles. Quantum mechanics has become an increasingly significant line in the progress of developing more efficient computing mechanisms. Until recently, the effect of quantum mechanics had been limited to low-level applications and it had no effect on how computation or communication was carried or worked. At the beginning of the 1980s, a number of scientists found that quantum mechanics had eye-opening effects that could be used in information processing. Richard Feynman [13], Yuri Manin [14], and other influential scientists realized that some quantum mechanical phenomena could not be efficiently simulated by a standard Turing machine. This observation has led to speculation that perhaps these quantum phenomena could be used to make computations more efficient in general.

Such programme required re-thinking the underlying theoretical model of informatics and completely removed it from the classical circle. Quantum computing, a field that includes quantum information, quantum algorithms, quantum cryptography, quantum communication, and quantum games, explores the effects of using quantum mechanical phenomena for information modeling and processing instead of using the rules of classical mechanics in computations.

In the following sections, we will introduce a mathematical framework of the unification of neutrosophic theory and quantum theory, in a fully computational approach. In this context, we will reveal how one can have a computational approach to the solution of mathematical and algorithmic problems of a model that can be encountered in both the neutrosophic and quantum universes. In this sense, this paper presents a more computational approach to the neutrosophic quantum concept, i.e., neutrosophic quantum computation, whose groundwork was laid by the work of Smarandache [15].

2. Neutrosophic Quantum Computing

In this part, we define some fundamental notions of neutrosophic quantum computing. Some concepts will involve new interpretations and others will be straightforward generalizations. As also mentioned in Smarandache [15], we should note in the beginning of our paper that the reversibility condition of quantum computing has some challenging issues in the neutrosophic counterpart of this ambitious field. It is mainly due to the fact that neutrosophic states involve indeterminacy, so the inverse function of such states might not always be definable, hence the domain may not be uniquely recovered from the image. We propose an interesting open problem regarding a special case of this issue at the end of the paper.

We assume some basic familiarity with linear algebra and complex numbers including their basic properties like the norm of a complex vector, complex conjugation, complex number multiplication, etc. The reader may refer to Yanofsky and Mannucci's [16] or Nielsen and Chuang's [17] book for a detailed account on quantum computing and quantum information.

Definition 1. *A neutrosophic quantum bit (neutrobit) is a three-dimensional complex vector*

$$|\psi\rangle = \begin{bmatrix} \alpha \\ \beta \\ \gamma \end{bmatrix} = \alpha|0\rangle + \beta|1\rangle + \gamma|I\rangle$$

such that $\alpha, \beta, \gamma \in \mathbb{C}$ *are called coefficients (or amplitudes) and* $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$, *where we define the basis vectors* $|0\rangle$, $|1\rangle$, $|I\rangle$ *in the canonical basis as*

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad |I\rangle = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

In comparison to classical quantum computation, the reader may have noticed a new basis vector $|I\rangle$ introduced above. We call this vector the *indeterminacy basis*.

A *coherent neutrosophic quantum state* $|\psi\rangle$ is a linear combination (superposition) of the basis vectors $|0\rangle$, $|1\rangle$ and $|I\rangle$ which is in the form

$$|\psi\rangle = \alpha|0\rangle + \beta|1\rangle + \gamma|I\rangle$$

such that $\alpha, \beta, \gamma \in \mathbb{C}$ and that $|\alpha|^2 + |\beta|^2 + |\gamma|^2 = 1$.

Thus, a coherent neutrosophic quantum state is three-dimensional complex vector, which is of unit length.

Quantum systems evolve via special kind of matrix transformations. We define *neutrosophic Pauli gates* as given below:

$$X = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Y = \begin{bmatrix} 1 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad W = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 1 \end{bmatrix}.$$

The matrix X is actually the *NOT* gate which the reader might be familiar from classical quantum computation. That is, if $k \in \{0,1\}$, then $X|k\rangle = |1-k\rangle$. Notice that $X|I\rangle = |I\rangle$. Thus, we define the negation of the indeterminacy basis as itself. The next two gates are Y-rotation and Z-rotation (phase change). The new gate here is the W-transformation which can be simply thought of as a rotation around the $|I\rangle$ basis with an equal coefficient distribution of the bases between $|I\rangle$ and the basis on which the rotation is applied. The intuition behind these rotation gates will be understood better once we give the unit ball representation of neutrobits later on.

An important quantum gate in classical quantum computing is the *Hadamard transform*, which is defined as the matrix

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix}.$$

Standard Hadamard transform is defined on a single qubit since it is a 2×2 matrix. Hadamard matrix used in classical quantum computing is a unitary matrix. Thus, it is reversible, and is actually its own inverse. To introduce the neutrosophic counterpart of this transformation, we first need to define the notion of indeterminate (decoherent) superpositions to make sense of the use of the Hadamard transform in neutrosophic quantum computing. The terms *coherent* and *decoherent* superpositions of neutrobits were first introduced by Smarandache [15] for denoting quantum states with some indeterminacy. We modify these notions to make the Hadamard transform work on neutrobits.

Definition 2. *The reserved three-dimensional vector*

$$|0_I\rangle = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_I$$

is called the decoherent state of the $|0\rangle$ *basis vector. We define* $|1_I\rangle$ *similarly. That is,*

$$|1_I\rangle = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_I$$

is defined to be the decoherent state of $|1\rangle$*. Any linear combination that includes either of these vectors is called a decoherent superposition.*

The motivation behind this definition is to mix the *coherent* (stable) basis state $|0\rangle$ with the intrinsic property of neutrosophic logic, which is indeterminacy. A quantum system may still have a degree of indeterminancy even if the system appears to be in a pure basis state. A scalar α for any of these decoherent vectors is denoted by α_I. Thus, when we write α_I, for some number α, the reader should understand that we are refering to the coefficient of a decoherent state. For example, the vector

$$|\psi\rangle = \begin{bmatrix} \left(\frac{1}{\sqrt{2}}\right)_I \\ \left(\frac{1}{\sqrt{2}}\right)_I \\ 0 \end{bmatrix}$$

denotes the decoherent superposition state

$$\frac{1}{\sqrt{2}}|0_I\rangle + \frac{1}{\sqrt{2}}|1_I\rangle.$$

We could also define a decoherent state for $|I\rangle$, but, since the state $|I\rangle$ naturally involves an indeterminacy regarding which classical bit the state refers to, there is no need to repeat this decoherence. Thus, we adopt $|0_I\rangle$ and $|1_I\rangle$ as reserved basis vectors that will be used in decoherent superposition states. We should once again emphasize that $|0_I\rangle$ is different than the coherent basis state $|0\rangle$. It is also different than the coherent superposition state $|\psi\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{1}{\sqrt{2}}|I\rangle$. The latter says that the system is in a superposition of basis states $|0\rangle$ and $|I\rangle$, the former says that the system is in a possibly *indetermined* state $|0\rangle$. If $|\psi\rangle = |0\rangle$, this tells us that $|\psi\rangle$ is for certain in the basis state $|0\rangle$. The state $|0_I\rangle + |I\rangle$ says that the system is in a decoherent superposition of $|I\rangle$ and a possibly indetermined state $|0\rangle$. The distinction between coherent and decoherent states should now be clear. However, another way to imagine $|0_I\rangle$ as the state $|0\rangle$ with a bounded error $\epsilon > 0$.

Given the information above, we define the *neutrosophic Hadamard transform* as

$$H_N = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \left(\frac{1}{\sqrt{2}}\right)_I \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \left(\frac{1}{\sqrt{2}}\right)_I \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 \end{bmatrix}.$$

Then, it is easy to verify that

$$H_N|0\rangle = \tfrac{1}{\sqrt{3}}|0\rangle + \tfrac{1}{\sqrt{3}}|1\rangle + \tfrac{1}{\sqrt{3}}|I\rangle,$$

$$H_N|1\rangle = \tfrac{1}{\sqrt{3}}|0\rangle - \tfrac{1}{\sqrt{3}}|1\rangle - \tfrac{1}{\sqrt{3}}|I\rangle,$$

$$H_N|I\rangle = \tfrac{1}{\sqrt{2}}|0_I\rangle + \tfrac{1}{\sqrt{2}}|1_I\rangle.$$

3. Observables and Measurement

In classical mechanics, it is intuitively understood what is meant by an observable. An observable in classical mechanics is a quantity like velocity, momentum, position, temperature, etc. It is intuitively clear what these quantities are. In quantum mechanics, one needs to be more specific when talking about observables.

Definition 3. *Let A be an $n \times n$ matrix. We say that A is Hermitian if $A^\dagger A = AA^\dagger$, where $A^\dagger$ is called the Hermitian conjugate of A and is defined as the transpose of the complex conjugate matrix of A. An $n \times n$ matrix A is called unitary if $A^\dagger A = AA^\dagger = Id$, where Id is the identity matrix.*

We note that, in classical quantum computing, state evolution is obtained by applying unitary operators. There are two reasons for this. The first reason is that classical quantum computations are reversible. The second reason is that unitary transformations preserve inner products, hence they preserve the norm of the vectors. As we shall discuss later, this requirement is questionable in neutrosophic quantum computing.

In classical quantum computing, it is assumed that, for every observable, there corresponds a Hermitian operator. We use the same postulate for the neutrosophic case.

Measurement postulate. Observables in neutrosophic quantum computing are Hermitian operators.

Measurements are the outcomes of observables applied on the physical system in consideration. Classical quantum computing usually takes *projective measurements* in the sense that when we measure a state, the new state of the system becomes one of the basis states of the system. Thus, after the measurement, a general superposition state gets *projected* onto one of the basis vectors. We shall not adopt this requirement in neutrosophic quantum computing. The reason is the following. If the outcome were to be projected onto one of the basis states, the logic used here would be no different than the classical interpretation. Even if the state of the quantum system is projected onto a single basis state, we would still require a degree of probability of the same basis state being on other basis states. This is one reason why we should decoherent superposition states into account in neutrosophic quantum computing. It relies on the very nature of neutrosophic logic. For that matter, observables we take into consideration are non-projective.

Measuring an observable on a neutrosophic quantum bit yields not a single classical state, but a probability distribution of the basis states $|0\rangle, |1\rangle, |I\rangle$. This is perhaps one of the most important difference between classical quantum computation and neutrosophic quantum computation. Given a neutrobit $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle + \gamma|I\rangle$, making a measurement on state $|\psi\rangle$ yields a triplet

$$\langle p_{|0\rangle}, p_{|1\rangle}, p_{|I\rangle}\rangle,$$

where $p_{|0\rangle}$ denoting the probability of $|\psi\rangle$ being in state $|0\rangle$, $p_{|1\rangle}$ denoting the probability of $|\psi\rangle$ being in state $|1\rangle$, and $p_{|I\rangle}$ denoting the probability of $|\psi\rangle$ being in the indetermined basis state $|I\rangle$. Thus, the outcome of observing a neutrobit gives a probability distribution of basis states. In classical quantum computing, the outcome of measurement on a qubit is a classical bit information.

Let us illustrate this idea. For example, given the neutrobit

$$|\psi\rangle = \frac{1}{\sqrt{3}}|0\rangle + \frac{1}{\sqrt{3}}|1\rangle + \frac{1}{\sqrt{3}}|I\rangle,$$

in a coherent superposition, measuring some observable Ω on the state $|\psi\rangle$ should yield a neutrosophic quantum state $|\psi'\rangle = \alpha|0\rangle + \beta|1\rangle + \gamma|I\rangle$, of decoherent superposition.

It should be noted that the neutrosophic quantum state should not be confused with an ordinary superposition state of a classical quantum system. Thus, a pure state in a neutrosophic quantum system always looks like a superposition. A neutrosophic quantum state is in a coherent superposition of three basis states $|0\rangle, |1\rangle, |I\rangle$. However, as soon as we make a measurement on state $|\psi\rangle$, it yields a decoherent superposition, which is merely a triplet containing the information of probability distributions for each basis states. We state this as a theorem.

Theorem 1. *Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle + \gamma|I\rangle$ be a coherent neutrosophic quantum state. The outcome of a measurement on $|\psi\rangle$ is a three-dimensional real vector, particularly a decoherent neutrosophc quantum superposition.*

Proof. Suppose that we are given a coherent state $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle + \gamma|I\rangle$. Without loss of generality, we may assume that the state is in a superposition rather than in a single coherent basis. Assume that we are given a Hermitian operator Ω which is not necessarily unitary and projective. Applying Ω on $|\psi\rangle$, since we assumed that Ω is non-projective, will still yield a linear combination of vectors, particularly a three-dimensional vector. Since the probability of seeing a single coherent basis state is a magnitude square of the coefficient corresponding to that basis vector, the probability of observing

$|0\rangle$ is some $p_{|0\rangle}$. Similarly, the probability of observing $|1\rangle$ is $p_{|1\rangle}$ and the probability of seeing $|I\rangle$ is some $p_{|I\rangle}$. Since Ω is non-projective, we observe a vector containing these probabilites as elements. However, since the outcome is decoherent, it should be that each probability value can be taken to be indetermined. That is, the outcome of the observation will be a vector

$$\begin{bmatrix} (p_{|0\rangle})_I \\ (p_{|1\rangle})_I \\ (p_{|I\rangle})_I \end{bmatrix}.$$

Since $|I\rangle_I = |I\rangle$, we have

$$\begin{bmatrix} (p_{|0\rangle})_I \\ (p_{|1\rangle})_I \\ (p_{|I\rangle}) \end{bmatrix}.$$

The vector above is a decoherent superposition state with numbers $p_{|0\rangle}$, $p_{|1\rangle}$, and $p_{|I\rangle}$. Since each number is the magnitude square of the coefficients of the state vector being measured, they cannot be complex valued. Thus, each of these numbers are real valued. □

4. Tensor Products and Entanglement

The usual tensor product of classical qubits generalizes to the neutrosophic case. Given two neutrobits

$$|\psi\rangle = \begin{bmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{bmatrix}, \quad |\phi\rangle = \begin{bmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{bmatrix},$$

the tensor product is defined as

$$|\psi\rangle \otimes |\phi\rangle = \begin{bmatrix} \alpha_1 \\ \beta_1 \\ \gamma_1 \end{bmatrix} \otimes \begin{bmatrix} \alpha_2 \\ \beta_2 \\ \gamma_2 \end{bmatrix} = \begin{bmatrix} \alpha_1\alpha_2 \\ \alpha_1\beta_2 \\ \alpha_1\gamma_2 \\ \beta_1\alpha_2 \\ \beta_1\beta_2 \\ \beta_1\gamma_2 \\ \gamma_1\alpha_2 \\ \gamma_1\beta_2 \\ \gamma_1\gamma_2 \end{bmatrix}.$$

The tensor product of measurement outcomes can also be defined. Assume that $|\psi'\rangle = \langle p^1_{|0\rangle}, p^1_{|1\rangle}, p^1_{|I\rangle}\rangle$ and $|\phi'\rangle = \langle p^2_{|0\rangle}, p^2_{|1\rangle}, p^2_{|I\rangle}\rangle$ are two probability distributions of two *decoherent* quantum states. Then, we define

$$p^{1\otimes 2}_{|0\rangle} = p^1_{|0\rangle} \cdot p^2_{|0\rangle},$$

$$p^{1\otimes 2}_{|1\rangle} = p^1_{|1\rangle} \cdot p^2_{|1\rangle},$$

$$p^{1\otimes 2}_{|I\rangle} = p^1_{|I\rangle} \cdot p^2_{|I\rangle}.$$

Then, we write the tensor product as $|\psi'\rangle \otimes |\phi'\rangle = \langle p^{1\otimes 2}_{|0\rangle}, p^{1\otimes 2}_{|1\rangle}, p^{1\otimes 2}_{|I\rangle}\rangle$.

The tensor product of measurement outcomes provides us with the ability to use compound outcome information of multiple neutrobit systems. We shall now look at the neutrosophic entanglement property. In classical quantum computation, a two qubit system is entangled if it is not the tensor product of two single-qubit systems. We adopt the same definition for neutrosophic coherent superposition states. However, entanglement is not defined on decoherent states. Suppose that we are given two neutrobits $|\psi\rangle = \alpha_1|0\rangle + \beta_1|1\rangle + \gamma_1|I\rangle$ and $|\phi\rangle = \alpha_2|0\rangle + \beta_2|1\rangle + \gamma_2|I\rangle$, the tensor product is defined exactly the same as in the classical case. That is,

$$|\psi\rangle \otimes |\phi\rangle = \alpha_1\alpha_2|00\rangle + \alpha_1\beta_2|01\rangle + \alpha_1\gamma_2|0I\rangle + \cdots + \gamma_2\gamma_2|II\rangle.$$

This is completely a coherent superposition. If we measure this two-neutrobit system, though, we get a 9-tuple containing probability distributions where each element of the 9-tuple denotes the probability of the compound system $|\psi\rangle \otimes |\phi\rangle$ being in the ith basis state for a two-neutrobit system. The reader should easily be able to verify that, for an n-neutrobit system, there are 3^n basis states.

5. More on Quantum Operators

As noted earlier, most quantum transformations are defined similarly as in the classical case. For a better understanding though, we shall discuss more about the action of the neutrosophic Hadamard transform. The neutrosophic Hadamard transform is defined as

$$H_N = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \left(\frac{1}{\sqrt{2}}\right)_I \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \left(\frac{1}{\sqrt{2}}\right)_I \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & 0 \end{bmatrix}.$$

The indeterminate values $\frac{1}{\sqrt{2}}$ in the neutrosophic Hadamard transform denote the indeterminate decoherent counterpart of the basis states $|0\rangle$ and $|1\rangle$. Any state which involves any of these decoherent vectors is also decoherent. Despite that we leave $H_N|0_I\rangle$ and $H_N|1_I\rangle$ undefined, we define the logical NOT operator over the decoherent states as

$$NOT|0_I\rangle = |1_I\rangle,$$

$$NOT|1_I\rangle = |0_I\rangle.$$

We leave the action of H_N on two reserved decoherent vectors $|0_I\rangle$ and $|1_I\rangle$ undefined for the reason that creating a superposition from an already decoherent neutrosophic quantum state might prevent us to obtain the original input decoherence from the output decoherence. Thus, due to this reversibility problem, it is better if we leave the mentioned transformations undefined. Since $|I\rangle$ is a legitimate coherent state in neutrosophic quantum computation, we defined

$$H_N|I\rangle = \frac{1}{\sqrt{2}}|0_I\rangle + \frac{1}{\sqrt{2}}|1_I\rangle.$$

We may imagine a coherent neutrobit as a vector on a three-dimensional unit ball as given in Figure 1.

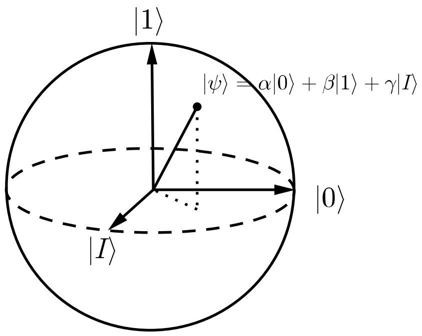

Figure 1. Representation of a neutrobit vector on a unit ball with real coefficients.

Of course, we assume in this image, for simplicity, that the amplitudes are real values. Allowing complex coefficients would require us to represent a neturobit on a four-dimensional geometry since an additional imaginary axis would need to be introduced. The basis vectors here are all mutually orthogonal. That is, the inner product of any of the two basis vectors is 0.

When we make a measurement on the state $|\psi\rangle$, we get a triplet $\langle p_{|0\rangle}, p_{|1\rangle}, p_{|I\rangle} \rangle$ which was defined earlier, where $p_{|0\rangle} = |\alpha|^2$, $p_{|1\rangle} = |\beta|^2$, $p_{|I\rangle} = |\gamma|^2$. The new state of the system in this case is a decoherent superposition of $|0\rangle$, $|1\rangle$ and $|I\rangle$ each with a degree of probability $p_{|0\rangle}$, $p_{|1\rangle}$, $p_{|I\rangle}$, respectively.

6. Results

We introduced a refined mathematical framework for neturosophic quantum computing based on the original work of Smarandache [15] and we gave a few standard transformations and notions that are to be used in neutrosophic quantum computations. Perhaps the most important difference from the classical quantum computation is the involvement of the indeterminacy basis and the separation between coherent and decoherent states. Treating the Hadamard transform as a function creating a superposition from a coherent state, we introduced the reserved decoherent vectors for this purpose. The measurement process is also slightly different in this case. The outcome of any measurement on a neutrobit gives a probability distribution, a decoherent state, of all possible basis states each with a certain degree of probability determined by the corresponding coefficients.

The computational complexity of the neutrosophic quantum gates, when applied to a quantum state, would be the same as their classical counterparts since the size of the transformation matrices in the neutrosophic counterpart does not change asymtotically. That is, for the neutrosophic Hadamard transform for instance, multiplying a 3×3 matrix with a three-dimensional vector does not give any difference in terms of computational complexity compared to its classical counterpart. The same observation can be easily seen with the other gates. The only complexity difference is with the tensor product that, since we are not working on a three-dimensional vector space, the size of the vector space grows by factors of 3 instead of 2 when taking tensor products of n many neutrobits. It should be noted that this is still a constant difference.

A practical application of neutrosophic quantum computing in the future would be used to solve hard problems involving indeterminate cases of multiple states when taken as a whole system. For example, it may not be known which one of the many possible channels that a quantum information is transferred through quantum communication channels. If we were to study the behavior of the transferred superposition quantum state, we would have to use neutrosophic quantum computing notions to describe the state of the transfer process that will involve the probability of the information being transferred on one particular channel, probability of the information not being transferred on the same channel, and a degree of indeterminacy of the information being transferred on that channel. This is required for a single channel. Thus, we would have a superposition of all possible probability distributions if we consider every channel taken together. The entire distribution will naturally define a decoherent quantum superposition state.

As stated in Smarandache [15], satisfying the reversibility condition of quantum computing is more problematic in the neutrosophic case due to the inclusion of indeterminate states. The first attempt to settle this problem is to try to make the neutrosophic Hadamard transform unitary, and hence reversible. We shall give the following open problem, for which we hope to encourage researchers in neutrosophic computation or quantum computing for finding a possible solution.

Open problem. Define a "reasonable" neutrosophic Hadamard transformation matrix, which is unitary.

By "reasonable", we mean preserving the original properties of the standard Hadamard transform such as creating a superposition of basis states, etc.

Another future work is to find a legitimate protocol for the teleportation of the state of a neutrobit from one location to another. This particularly has many applications in networks and communication. A typical quantum teleportation of a standard qubit is performed through classical bit channels. In order to send the state of a qubit, the first party sends two classical bits and the second part recovers the state of a qubit from the received classical bits. What kind of channels do we need to transport the state of a neutrobit? A classical channel may be a solution. A quantum channel, on the other hand, may not be sufficient to teleport a neutrobit due to the fact that the preservation of indeterminate states through the teleportation process becomes questionable. One idea is to separate the indeterminate state from the superposition and treat it as a classical quantum superposition state of all coherent basis states and then use the classical quantum teleportation protocol on this system.

Neutrosophic quantum computing is at its very early stage of development. We believe that this new field will attract many researchers in computer science, physics and mathematics for further advancement along with discovering many useful future applications.

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Validation of the Pedagogical Strategy for the formation of the competence entrepreneurship in high education through the use of Neutrosophic Logic and IADOV technique

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Abstract.The objective of this work is to validate the implementation of the pedagogical strategy for the development of the competence to undertake as a contribution to the comprehensive education in senior high students in "10 de Octubre" borough in Havana, Cuba.The research seeks to increase scientific knowledge; so it is necessary to objectify the requirements of validity and reliability on which it is based. Validity is understood as the consistency and stability shown by the results of research when applying different demonstration methods, based on the assumption that these are conceived and structured with the capacity to determine and measure.Reliability allows establishing that the conclusive results of the research are balanced and refers to the degree to which the same action submitted to a measurement by the same investigative or different subject produces similar results. For the authors, investigative reliability constitutes the degree of stability that when applying the validation they tend not to vary.In order to sustain the derivations of the development of this strategy, a survey instrument was applied to training and recipients whose results were evaluated through a complex methodology, which integrates the Iadov technique and the neutrosophic logic, determining the transcendences and strategic training repercussions and its consequences on the performers of senior high education.

Keywords:Iadov, Neutrosophic, Indeterminacy, Group Satisfaction Index

1 Introduction

The systematization of several definitions led the authors to define the entrepreneurship competence as the complex and systemic set of knowledge, abilities, skills, attitudes and values that interact synergistically and make viable the autonomous and effective performance of the individual, by providing it with tools to create, manage, interpret, understand and transform the social environment with a critical, proactive and innovative vision, sustaining a life model, personal development in present and in future[1].

The formation of competence to undertake makes the learner a protagonist of the community context by enabling the application of knowledge through the selection of methods, procedures and disjunctive proposals, which mobilize the cognitive and attitudinal structures developed during the process, complex arrangements that pay tribute to the integral development of student[2].

In order to contribute to the development of the comprehensive education of the student of senior high education, this pedagogical strategy for the development of entrepreneurship competition was conceived and applied. This moment is concretized in the strategic performance of modeling of ventures, based on proposals resulting from the student's identification of possible solutions to problems that arise from the social demands of the surrounding environment[3].

In the provisions of stages and phases, students are encouraged in the search of alternatives, of conformation of factical schemes, which allows the application, in the social context, of knowledge, skills, attitudes,and values.

The determination of the actors' assessment of the strategy's impact makes up a significant indicator of the validity of the strategy. This action needs to validate the results by the investigation and with this purpose, the Iadov technique is applied. Iadov constitutes an indirect way to study of satisfaction, in this case, the actors' developers and evaluators of the process and the addressees[4].
Iadov's technique uses, as suggested by the original method[5], the related criteria of answers to intercalated questions whose relationship the subject does not know, at the same time the unrelated or complementary questions serve as an introduction and support of objectivity to the respondent who uses them to locate and contrast the answers.The results of these questions interact through what is called the "Iadov Logical Table"[6, 7]. In this paper, the satisfaction of emitting actors (teaching staff and training activity) and those who are beneficiaries of the development strategy, the receiving actors are combined.User criterion techniques should be used as a way to assess results in those cases in which the evaluators are users of what is proposed, that is, in addition to having control over the problem being studied, they are "contextualized", immersed in the context in which is the application of the result[7].
The degree of satisfaction-dissatisfaction is a psychological state that manifests itself in people as an expression of the interaction of a set of affective experiences that move between the positive and negative poles insofar as in the activity that the subject develops, the object, responds to their needs and corresponds to their motives and interests[8]. The relationship between indeterminacy and user importance has not yet been clarified and include in Iadov.
Recently a new theory has been introduced in decision making which is known as neutrosophiclogic and set developed by Florentin Smarandache in 1995[9]. The term neutrosophy means knowledge of neutral thought and thisneutral represents the principal distinction between fuzzy and intuitionistic fuzzy logicand sets [10]. With neutrosophy theory, a new logic is introduced in which each proposition is estimated to have a degree of truth (T), adegree of indeterminacy (I) and a degree of falsity (F)[11]. Many extensions of classical decision-making methods have been proposed for dealing with indeterminacy based on neutrosophy theory like DEMATEL [12] AHP [13], VIKOR[14]and TOPSIS [15].
The original proposal of the Iadov method does not allow an adequate management of the indetermination nor the management of the importance of the users[11].The introduction of the neutrosophic estimation seeks to solve the problems of indeterminacy that appear universally in the evaluations of the surveys and other instruments, by taking advantage of not only the opposing positions but also the neutral or ambiguous ones[16]. Under the principle that every idea <A> tends to be neutralized, diminished, balanced by other ideas, in clear rupture with binary doctrines in the explanation and understanding of phenomena[17].
This work continues as follows: Section 2 is about some important concepts about neutrosophy and Iadov . A case study is presented and discussed in section 3. The paper ends with conclusions and some recommendation for future work.

2 Materials and methods

In Iadov technique the questionnaire used to determine the degree of user satisfaction with the proposed system of indicators to predict, design and measure the impact of the researcher's strategy has a total of seven questions, three of which are closed and four open, whose relationship is ignored by the subject[18].These three closed questions are related through the "Iadov logical table", which is presented adapted to the present investigation. The resulting number of the interrelation of the three questions indicates the position of each subject in the satisfaction scale, that is, your individual satisfaction This satisfaction scale is expressed by SVN numbers[19]. The original definition of true value in the neutrosophic logic is shown below [20]:
Be $N = \{(T, I, F) : T, I, F \subseteq [0, 1]\}$a neutrosophical valuation is a mapping of a group of proportional formulas to$N$, and for each p sentence we have:

$$v\,(p) = (T, I, F) \tag{1}$$

In order to ease the practical application to a decision making and engineering problems, it was carried out the proposal of single valued neutrosophic sets (SVNS) this allows the use of linguistic variables[21, 22], this increase the interpretation of models of recommendation and the usage of the indetermination.

Be X an universe of discourse. A SVNS Aon $X$is an object of the form.

$$A = \{\langle x, u_A(x), r_A(x), v_A(x)\rangle : x \in X\} \tag{2}$$

where,$u_A(x): X \to [0,1]$, $r_A(x), : X \to [0,1]$and $v_A(x): X \to [0,1]$ with $0 \le u_A(x) + r_A(x) + v_A(x) :\le 3$ for all $x \in X$. The intervals $u_A(x), r_A(x)$and $v_A(x)$ denote the memberships to true, indeterminate and false of x in A, respectively. For convenience reasons, an SVN number will be expressed as $A = (a, b, c)$, where $a, b, c \in [0, 1]$, y $a + b + c \leq 3$.

In order to analyze the results, it is established a scoring function. To order the alternatives it is used a score function[23]adapted :

$$s(\mathrm{V}) = \mathrm{T} - \mathrm{F} - \mathrm{I} \qquad (3)$$

In the event that the assessment corresponds to indeterminacy(not defined) (I) a process of de-neutrosophication developed as proposed by Salmerón and Smarandache[24]. In this case, I ∈ [-1,1].Finally, we work with the average of the extreme values $I \in [0,1]$ to obtain a single one.

$$\lambda([a_1, a_2]) = \frac{a_1 + a_2}{2} \qquad (4)$$

Subsequently, the results are aggregated and the weighted average aggregation operator is used to calculate the group satisfaction index (GSI). The weighted average (WA) is one of the most mentioned aggregation operators in the literature[25, 26]. A WA operator has associated a vector of weights, V, with $v_i \in [0,1]$ and $\sum_1^n v_i = 1$, having the following form:

$$WA(a_1, .., a_n) = \sum_1^n v_i a_i \qquad (5)$$

Where v_i represented the importance of the source. This proposal allow to fill a gap in the literature of the Iadov techniques extending it to deal with indeterminacy and importance of user due to expertise or any other reason [27].

3 Survey of teachers and methodologists of senior high education:

The case study was developed for the validation of a pedagogical strategy for the development of the competence to undertake as in "10 de Octubre" borough in Havana, CubaA scale with individual satisfaction and its corresponding score value was used (Table 1).

Expression	Number SVN	Scoring
Clearly pleased	(1, 0, 0)	1
More pleased than unpleased	(1, 0.25, 0.25)	0.5
Not defined	I	0
More unpleased than pleased	(0.25, 0.25, 1)	-0.5
Clearly unpleased	(0,0,1)	-1
Contradictory	(1,0,1)	0

Table 1. Individual satisfaction scale.

A sample of 21 teachers and methodologists from senior high education were surveyed. The survey was elaborated with 7 questions, three closed questions interspersed in four open questions; of which 1 fulfilled the introductory function and three functioned as reaffirmation and sustenance of objectivity to the respondent.

<table>
<tr><td rowspan="2"></td><td colspan="9">Would you consider postponing the development of the competence to undertake as a contribution to the comprehensive education of the student of senior high education?</td></tr>
<tr><td colspan="3">No</td><td colspan="3">I don´t know</td><td colspan="3">yes</td></tr>
<tr><td rowspan="2">Do your expectations meet the application of the strategy for the development of the competence to undertake as a contribution to the comprehensive education of the student of senior high education?</td><td colspan="9">If you could choose freely, a strategy for the formation of competencies in students of senior high education would you choose one with similar characteristics to the one used for the development of the competence to undertake?</td></tr>
<tr><td>yes</td><td>I don´t know</td><td>No</td><td>yes</td><td>I don´t know</td><td>No</td><td>yes</td><td>I don´t know</td><td>No</td></tr>
</table>

Very pleased.	1 (14)	2 (2)	6	2	2	6	6	6	6
Parcially pleased.	2 (2)	2 (2)	3	2 (1)	3	3	6	3	6
It's all the same to me	3	3	3	3	3	3	3	3	3
More unpleased than pleased.	6	3	6	3	4	4	3	4	4
Not pleased	6	6	6	6	4	4	6	4	5
I don´t know what to say	2	3	6	3	3	3	6	3	4

Table 2. The logical picture of the Iadov technique for teachers and methodologists from senior high education.

In this case,the following results are as follows:

Expression	Total	%
Clearly pleased	14	66
More pleased than unpleased	7	33
Not defined	0	0
More unpleased than pleased	0	0
Clearly unpleased	0	0
Contradictory	0	0

Table 3.Results of the application to teachers and methodologists.

The calculation of the score is carried out and it is determined by I. in this case, it was given the same value to each user. The final result of the index of group satisfaction (GSI) that the method portrays, in this case, is: GSI =0.82

This shows a high level of satisfaction according to the satisfaction scale.

For the students, a survey similar to that of the teachers was prepared, and a total of 101 senior high students were interviewed who received the training program with the following results:

	Can you do without undertaking and achieve your professional realization?								
	No			I don´t know			yes		
Are you satisfied with the way in which the program was applied to develop your skills and knowledge to learn to undertake?	Would you like to be an entrepreneur and take on the challenges in your future personal performance?								
	yes	I don´t know	No	yes	I don´t know	No	yes	I don´t know	No
Very pleased.	1 (71)	2 (2)	6	2	2	6	6	6	6
Parcially pleased.	2 (23)	2 (1)	3	2 (1)	3	3	6	3 (1)	6
Its all the same to me.	3 (1)	3	3	3	3 (1)	3	3	3	3
more unpleased than pleased.	6	3	6	3 (1)	4	4	3	4	4
Not pleased	6	6	6	6	4	4	6	4	5
I don´t know what to say	2	3	6	3	3	3	6	3	4

Table 4. Logical picture of Iadov students of senior high education.

In this case, the following results are obtained:

Expression	Total	%
Clearly pleased	71	70.3
More pleased than unpleased	27	26.7
Not defined	3	2.97
More unpleased than pleased	0	0
Clearly unpleased	0	0
Contradictory	0	0

Table 5.Results of the application of the students in senior high.

The calculation of the score is carried out. In this case, it was given the same value of importance to each user. The final result of the index of group satisfaction (GSI) that the method portrays, in this case, is: GSI=0.837 this shows a high level of satisfaction according to the satisfaction scale.

By locating the values reached in the satisfaction scale

- Actors developers: 0.809
- Recipients – actors: 0.837

In both cases, the results are positive, which certifies the effectiveness of the implementation of the strategy, as shown in the graph.

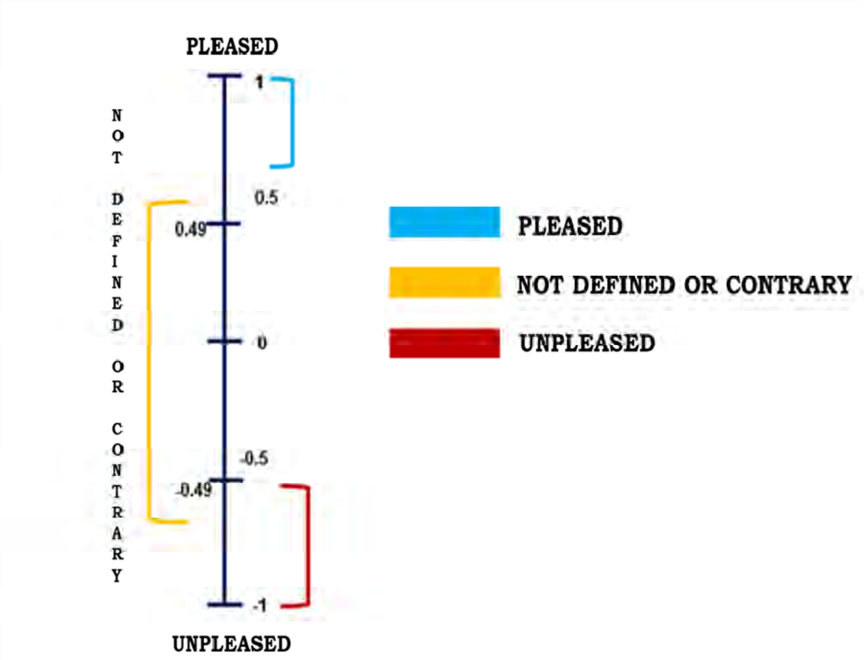

Figure 1. Scale with group satisfaction index

The proposal of extending Iadov method with neutrosophyresults to be easy to use and practical in real-world application. The inclusions of indeterminacy allow a more powerful way to represent information compared with the classical application of the technique. The inclusion of the aggregation operator extends the traditionalIadov method including the importanceof information sources[28]. The application in the real world of the proposal validates the pedagogical strategy for the formation of the competence entrepreneurship in high education.

Conclusions

In this paper, the Iadov method was extended allowing an adequate management of the indeterminacy and the management of the importance of users. Iadov's method with the inclusion of neutrosophic analysis showed its applicability and ease of use in a case study. Among the advantages with respect to the original approach is that it can incorporate the indetermination and contradiction more naturally. Another advantage is that it allows the use of aggregation operators which makes it possible to express, in this case, the importance or expertise of users according to experience or some other criteria.

The validation process using the neutrosophic Iadov technique in users of the implementation of the pedagogical strategy for the formation of the competence entrepreneurship in high education in "10 de Octubre" borough in Havana, Cuba confirmed its feasibility of use. The results were expressed quantitatively in a high Group Satisfaction Index in the two applications presented in the case study.

Future works will concentrate on the uses of the 2-tuplelinguistic model for giving a linguistic output and the use of different aggregation operator. The development of a software tool supporting the proposal is another area of future research.

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About Nonstandard Neutrosophic Logic (Answers to Imamura's "Note on the Definition of Neutrosophic Logic")

Florentin Smarandache

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Abstract.

In order to more accurately situate and fit the neutrosophic logic into the framework of nonstandard analysis, we present the neutrosophic inequalities, neutrosophic equality, neutrosophic infimum and supremum, neutrosophic standard intervals, including the cases when the neutrosophic logic standard and nonstandard components *T, I, F* get values outside of the classical unit interval *[0, 1],* and a brief evolution of neutrosophic operators.

The paper intends to answer Imamura's criticism that we found benefic in better understanding the nonstandard neutrosophic logic – although the nonstandard neutrosophic logic was never used in practical applications.

1. Uselessness of Nonstandard Analysis in Neutrosophic Logic, Set, Probability, et al.

Imamura's discussion [1] on the definition of neutrsphic logic is welcome, but it is useless, since from all neutrosophic papers and books published, from all conference presentations, and from all MSc and PhD theses defended around the world, etc. (more than one thousand) in the last two decades since the first neutrosophic research started *(1998-2018)*, and from hundreds of neutrosophic researchers, not even a single one ever used the nonstandard form of neutrosophic logic, set, or probability and statistics in no occasion (extended researches or applications).

All researchers, with no exception, have used the *Standard Neutrosophic Set and Logic* [so no stance whatsoever of *Nonstandard Neutrosophic Set and Logic*], where the neutrosophic components *T, I, F* are subsets of the standard unit interval *[0, 1]*.

Even more, for simplifying the calculations, the majority of researchers have utilized the *Single-Valued Neutrosophic Set and Logic* {when *T, I, F* are single numbers from *[0, 1]*}, on the second place was *Interval-Valued Neutrosophic Set and Logic* {when *T, I, F* are intervals included in *[0, 1]},* and on the third one the *Hesitant Neutrosophic Set and Logic* {when *T, I, F* were discrete finite sets included in *[0, 1]*}.

In this direction, there have been published papers on single-valued "neutrosophic standard sets" [12, 13, 14], where the neutrosophic components are just *standard real numbers*, considering the particular case when $0 \leq T + I + F \leq 1$ (in the most general case $0 \leq T + I + F \leq 3$).

Actually, Imamura himself acknowledges on his paper [1], page 4, that:

> "neutrosophic logic does not depend on transfer, so the use of non-standard analysis is not essential for this logic, and can be eliminated from its definition".

Entire neutrosophic community has found out about this result and has ignored the non-standard analysis in the studies and applications of neutrosophic logic for two decades.

2. Applicability of Neutrosophic Logic et al. vs. Theoretical Nonstandard Analysis

Neutrosophic logic, set, measure, probability, statistics and so on were designed with the primordial goal of being applied in practical fields, such as:

Artificial Intelligence, Information Systems, Computer Science, Cybernetics, Theory Methods, Mathematical Algebraic Structures, Applied Mathematics, Automation, Control Systems, Big Data, Engineering, Electrical, Electronic, Philosophy, Social Science, Psychology, Biology, Biomedical, Engineering, Medical Informatics, Operational Research, Management Science, Imaging Science, Photographic Technology, Instruments, Instrumentation, Physics, Optics, Economics, Mechanics, Neurosciences, Radiology Nuclear, Medicine, Medical Imaging, Interdisciplinary Applications, Multidisciplinary Sciences etc. [2],

while nonstandard analysis is mostly a pure mathematics.

Since 1990, when I emigrated from a political refugee camp in Turkey to America, working as a software engineer for Honeywell Inc., in Phoenix, Arizona State, I was advised by American co-workers to do theories that have *practical applications*, not pure-theories and abstractizations as "*art pour art*".

3. Theoretical Reason for the Nonstandard Form of Neutrosophic Logic

The only reason I have added the nonstandard form to neutrosophic logic (and similarly to neutrosophic set and probability) was in order to make a distinction between *Relative Truth* (which is truth in some Worlds, according to Leibniz) and *Absolute Truth* (which is truth in all possible Words, according to Leibniz as well) that occur in philosophy.

Another possible reason may be when the neutrosophic degrees of truth, indeterminacy, or falsehood are infinitesimally determined, for example a value infinitesimally bigger than 0.8 (or 0.8^{+}), or infinitesimally smaller than 0.8 (or ${}^{-}0.8$). But these can easily be overcome by roughly using interval neutrosophic values, for example (0.80, 0.81) and (0.79, 0.80) respectively.

I wanted to get the neutrosophic logic as general as possible [6], extending all previous logics (Boolean, fuzzy, intuitionistic fuzzy logic, intuitionistic logic, paraconsistent logic, dialethism), and to have it able to deal with all kind of logical propositions (including paradoxes, nonsensical propositions, etc.).

That's why in 2013 I extended the Neutrosophic Logic to *Refined Neutrosophic Logic* [from generalizations of *2-valued Boolean logic* to fuzzy logic, also from the *Kleene's and*

Lukasiewicz's and *Bochvar's 3-symbol valued logics* or *Belnap's 4-symbol valued logic* to the most general *n-symbol* or *n-numerical* valued refined neutrosophic logic, for any integer $n \geq 1$], the largest ever so far, when some or all neutrosophic components *T, I, F* were respectively split/refined into neutrosophic subcomponents: $T_1, T_2, \ldots; I_1, I_2, \ldots; F_1, F_2, \ldots$ which were deduced from our everyday life [3].

4. From Paradoxism movement to Neutrosophy branch of philosophy and then to Neutrosophic Logic

I started first from *Paradoxism* (that I founded in 1980's as a movement based on antitheses, antinomies, paradoxes, contradictions in literature, arts, and sciences), then I introduced the *Neutrosophy* (as generalization of Dialectics, neutrosophy is a branch of philosophy studying the dynamics of triads, inspired from our everyday life, triads that have the form:

<*A*>, its opposite <*antiA*>, and their neutrals <*neutA*>, (1)

where <*A*> is any item or entity [4].

(Of course, we take into consideration only those triads that make sense in our real and scientific world.)

The Relative Truth neutrosophic value was marked as *1*, while the Absolute Truth neutrosophic value was marked as 1^+ (a tinny bigger than the Relative Truth's value): $1^+ >_N 1$, where $>_N$ is a neutrosophic inequality, meaning 1^+ is neutrosophically bigger than *1*.

Similarly for Relative Falsehood / Indeterminacy (which falsehood / indeterminacy in some Worlds), and Absolute Falsehood / Indeterminacy (which is falsehood / indeterminacy in all possible worlds).

5. Introduction to Nonstandard Analysis [15, 16]

An *infinitesimal number* (ε) is a number ε such that $|\varepsilon| < 1/n$, for any non-null positive integer *n*. An infinitesimal is close to zero, and so small that it cannot be measured.

The infinitesimal is a number smaller, in absolute value, than anything positive nonzero.

Infinitesimals are used in calculus.

An *infinite number (ω)* is a number greater than anything:

$1 + 1 + 1 + \ldots + 1$ (for any finite number terms) (2)

The infinites are reciprocals of infinitesimals.

The set of *hyperreals* (*non-standard reals*), denoted as R^*, is the extension of set of the real numbers, denoted as *R*, and it comprises the infinitesimals and the infinites, that may be represented on the *hyperreal number line*

$1/\varepsilon = \omega/1.$ (3)

The set of hyperreals satisfies the *transfer principle*, which states that the statements of first order in R are valid in R^* as well.

A *monad* (*halo*) of an element $a \in R^*$, denoted by $\mu(a)$, is a subset of numbers infinitesimally close to a.

Let's denote by R_+^* the set of positive nonzero hyperreal numbers.

We consider the left monad and right monad, and we have introduced the *binad* [5]:

Left Monad { that we denote, for simplicity, by (^-a) or only ^-a } is defined as:

$$\mu(^-a) = (^-a) = {}^-a = \{a - x,\ x \in R_+^* \mid x \text{ is infinitesimal}\}. \quad (4)$$

Right Monad { that we denote, for simplicity, by (a^+) or only by a^+ } is defined as:

$$\mu(a^+) = (a^+) = a^+ = \{a + x,\ x \in R_+^* \mid x \text{ is infinitesimal}\}. \quad (5)$$

Bimonad { that we denote, for simplicity, by $(^-a^+)$ or only $^-a^+$ } is defined as:

$$\mu(^-a^+) = (^-a^+) = {}^-a^+$$

$$= \{a - x,\ x \in R_+^* \mid x \text{ is infinitesimal}\} \cup \{a + x,\ x \in R_+^* \mid x \text{ is infinitesimal}\}$$

$$= \{a \pm x,\ x \in R_+^* \mid x \text{ is infinitesimal}\}. \quad (6)$$

The left monad, right monad, and the bimonad are subsets of R^*.

6. Neutrosophic Strict Inequalities

We recall the neutrosophic inequality which is needed for the inequalities of nonstandard numbers.

Let α, β be elements in a partially ordered set M.

We have defined the *neutrosophic strict inequality*

$$\alpha >_N \beta \quad (7)$$

and read as

"α is neutrosophically greater than β" *(8)*

if

α in general is greater than β,

or *α is approximately greater than β,*

or *subject to some indeterminacy* (unknown or unclear ordering relationship between α and β) *or subject to some contradiction (*situation when α is smaller than or equal to β*) α is greater than β.*

It means that in most of the cases, on the set M, *α is greater than β.*

And similarly for the opposite neutrosophic strict inequality $\alpha <_N \beta$.

7. Neutrosophic Equality

We have defined the *neutrosophic inequality*

$$\alpha =_N \beta \quad (9)$$

and read as

"α is neutrosophically equal to β" *(10)*

if

α in general is equal to β,

or *α is approximately equal to β,*

or *subject to some indeterminacy* (unknown or unclear ordering relationship between α and β) *or subject to some contradiction (*situation when α is not equal to β*) α is equal to β.*

It means that in most of the cases, on the set M, *α is equal to β.*

8. Neutrosophic (Non-Strict) Inequalities

Combining the neutrosophic strict inequalities with neutrosophic equality, we get the $\geq_N$ and $\leq_N$ neutrosophic inequalities.

Let α, β be elements in a partially ordered set M.

The *neutrosophic (non-strict) inequality*

$$\alpha \geq_N \beta \quad (11)$$

and read as

"α is neutrosophically greater than or equal to β" *(12)*

if

α in general is greater than or equal to β,

or *α is approximately greater than or equal to β,*

or *subject to some indeterminacy* (unknown or unclear ordering relationship between α and β) *or subject to some contradiction (*situation when α is smaller than β*) α is greater than or equal to β.*

It means that in most of the cases, on the set M, *α is greater than or equal to β.*

And similarly for the opposite neutrosophic (non-strict) inequality $\alpha \leq_N \beta$.

9. Neutrosophically Ordered Set

Let M be a set. $(M, <_N)$ is called a neutrosophically ordered set if:

$$\forall \ \alpha, \beta \in M, \text{ one has: either } \alpha <_N \beta, \text{ or } \alpha =_N \beta, \text{ or } \alpha >_N \beta. \quad (13)$$

10. Neutrosophic Nonstandard Inequalities

Let $\mathscr{P}(R^*)$ be the power-set of R^*. Let's endow $(\mathscr{P}(R^*), <_N)$ with a neutrosophic inequality

Let $a, b \in R$, where R is the set of (standard) real numbers.

And let $({}^-a)$, (a^+), $({}^-a^+) \in \mathscr{P}(R^*)$, and $({}^-b)$, (b^+), $({}^-b^+) \in \mathscr{P}(R^*)$, be the left monads, right monads, and the bimonads of the elements (standard real numbers) a and b respectively. Since all monads are subsets, we may treat the single real numbers $a = [a, a]$ and $b = [b, b]$ as subsets too.

$\mathscr{P}(R^*)$ is a set of subsets, and thus we deal with neutrosophic inequalities between subsets.

i) If the subset α has many of its elements above all elements of the subset β, then $\alpha >_N \beta$ (partially).

ii) If the subset α has many of its elements below all elements of the subset β, then $\alpha <_N \beta$ (partially).

iii) If the subset α has many of its elements equal with elements of the subset β, then $\alpha =_N \beta$ (partially).

If the subset α verifies *i)* and *iii)* with respect to subset β, then $\alpha \geq_N \beta$.

If the subset α verifies *ii)* and *iii)* with respect to subset β, then $\alpha \leq_N \beta$.

If the subset α verifies *i)* and *ii)* with respect to subset β, then there is no neutrosophic order (inequality) between α and β.

{ For example, between $({}^-a^+)$ and a there is no neutrosophic order. }

Similarly, if the subset α verifies *i), ii)* and *iii)* with respect to subset β, then there is no neutrosophic order (inequality) between α and β.

11. Open Neutrosophic Research

The quantity or measure of "many of its elements" of the above *i), ii),* and *iii)* conditions depends on each *neutrosophic application* and on its *neutrosophic experts*.

For the *neutrosophic nonstandard inequalities*, we propose based on the above three conditions the following:

$$(^{-}a) <_N a <_N (a^{+}) \tag{14}$$

because $\forall x \in R_+^*, a - x < a < a + x$, where x is of course a (nonzero) positive infinitesimal (the above double neutrosophic inequality actually becomes a double classical standard real inequality for each fixed positive infinitesimal).

$$(^{-}a) \leq_N (^{-}a^{+}) \leq_N (a^{+}) \tag{15}$$

This double neutrosophic inequality may be justified due to $(^{-}a^{+}) = (^{-}a) \cup (a^{+})$, so:

$$(^{-}a) \leq_N (^{-}a) \cup (a^{+}) \leq_N (a^{+}) \tag{16}$$

whence the left side of the inequality middle term coincides with the inequality first term, while the right side of the inequality middle term coincides with the third inequality term.

If $a > b$, which is a (standard) classical real inequality, then we have the following neutrosophic nonstandard inequalities:

$$a >_N (^{-}b), \quad a >_N (b^{+}), \quad a >_N (^{-}b^{+}); \tag{17}$$

$$(^{-}a) >_N b, (^{-}a) >_N (^{-}b), (^{-}a) >_N (b^{+}), \quad (^{-}a) >_N (^{-}b^{+}); \tag{18}$$

$$(a^{+}) >_N b, \quad (a^{+}) >_N (^{-}b), \quad (a^{+}) >_N (b^{+}), \quad (a^{+}) >_N (^{-}b^{+}); \tag{19}$$

$$(^{-}a^{+}) >_N b, \quad (^{-}a^{+}) >_N (^{-}b), \quad (^{-}a^{+}) >_N (b^{+}), \quad (^{-}a^{+}) >_N (^{-}b^{+}). \tag{20}$$

If $a \geq b$, which is a (standard) classical real inequality, then we have the following neutrosophic nonstandard inequalities:

$$a \geq_N (^{-}b); \tag{21}$$

$$(^{-}a) \geq_N (^{-}b); \tag{22}$$

$$(a^{+}) \geq_N (^{-}b), \quad (a^{+}) \geq_N b, \quad (a^{+}) \geq_N (b^{+}), \quad (a^{+}) \geq_N (^{-}b^{+}); \tag{23}$$

$$(^{-}a^{+}) \geq_N (^{-}b), \quad (^{-}a^{+}) \geq_N (^{-}b^{+}). \tag{24}$$

And similarly for $<_N$ and $\leq_N$ neutrosophic nonstandard inequalities.

12. Neutrosophic Nonstandard Equalities

Let a, b be standard real numbers; if $a = b$ that is a (classical) standard equality, then:

$$(^{-}a) =_N (^{-}b), \quad (a^{+}) =_N (b^{+}), \quad (^{-}a^{+}) =_N (^{-}b^{+}). \tag{25}$$

13. Neutrosophic Infimum and Neutrosophic Supremum

As an extension of the classical infimum and classical supremum, and using the neutrosophic inequalities and neutrosophic equalities, we define the neutrosophic infimum (denoted as inf_N) and the neutrosophic supremum (denoted as sup_N).

Neutrosophic Infimum.

Let $(S, <_N)$ be a set that is neutrosophically partially ordered, and M a subset of S. The neutrosophic infimum of M, denoted as $inf_N(M)$ is the neutrosophically greatest element in S that is neutrosophically less than or equal to all elements of M:

Neutrosophic Supremum.

Let $(S, <_N)$ be a set that is neutrosophically partially ordered, and M a subset of S. The neutrosophic supremum of M, denoted as $sup_N(M)$ is the neutrosophically smallest element in S that is neutrosophically greater than or equal to all elements of M.

14. Classical Infimum and Supremum vs. Neutrosophic Infimum and Supremum.

Giving the definitions of neutrosophic components from my book [5]:

> "Let *T, I, F* be standard or non-standard real subsets of $]^-0, 1^+[$,
> with sup T = t_sup, inf T = t_inf,
> sup I = i_sup, inf I = i_inf,
> sup F = f_sup, inf F = f_inf,
> and n_sup = t_sup+i_sup+f_sup,
> n_inf = t_inf+i_inf+f_inf."

Imamura argues (page 3) that:

> "Subsets of R*, even bounded, may have neither infima nor suprema, because the transfer principle ensures the existences of infima and suprema only for internal sets."

This is true from a classical point of view, yet according to the definitions of the neutrosophic inequalities, the neutrosophic infimum and supremum do exist for the nonstandard intervals, for example:

$$inf_N(\,]^-a, b^+[\,) = {}^-a, \text{ and } sup_N(\,]^-a, b^+[\,) = b^+. \quad (26)$$

Indeed, into my definition above I had to clearly mention that we talk neutrosophically [*mea culpa*] by inserting an "N" standing for neutrosophic (inf_N and sup_N):

> Let *T, I, F* be standard or non-standard real subsets of $]^-0, 1^+[$,
> with sup_N T = t_sup, inf_N T = t_inf,
> sup_N I = i_sup, inf_N I = i_inf,
> sup_N F = f_sup, inf_N F = f_inf,
> and n_sup = t_sup+i_sup+f_sup,
> n_inf = t_inf+i_inf+f_inf.

I was more prudent when I presented the sum of single valued standard neutrosophic components, saying:

Let T, I, F be single valued numbers, T, I, F $\in [0, 1]$*, such that* $0 \le T + I + F \le 3$.

A friend alerted me: *"If T, I, F are numbers in [0, 1], of course their sum is between 0 and 3."* "Yes, I responded, I afford this tautology, because if I did not mention that the sum is up to *3*, readers would take for granted that the sum $T + I + F$ is bounded by *1*, since that is in all logics and in probability!"

15. Notations

Imamura is right when criticizing my confusion of notations between hyperreals (numbers) and monads (subsets). I was rather informal than formal at the beginning.
By ${}^{-}a$ and b^{+} most of times I wanted to mean the subsets of left monad and right monad respectively. Taking an arbitrary positive infinitesimal ε, and writing ${}^{-}a = a\text{-}\varepsilon$ and $b^{+} = b+\varepsilon$ was actually picking up a representative from each class (monad).
Similarly, representations of the monads by intervals were not quite accurate from a classical point of view:

$$({}^{-}a) = (a\text{-}\varepsilon, a), \quad (27)$$

$$(b^{+}) = (b, b+\varepsilon), \quad (28)$$

$$({}^{-}a^{+}) = (a\text{-}\varepsilon, a) \cup (b, b+\varepsilon), \quad (29)$$

but they were rather neutrosophic equalities (approximations):

$$({}^{-}a) =_N (a\text{-}\varepsilon, a), \quad (30)$$

$$(b^{+}) =_N (b, b+\varepsilon), \quad (31)$$

$$({}^{-}a^{+}) =_N (a\text{-}\varepsilon, a) \cup (b, b+\varepsilon). \quad (32)$$

16. Nonarchimedean Ordered Field.
At pages 5-6 of note [1], Imamura proposed the following Nonarchimedean Ordered Field K:

> "Let $x, y \in K$. x and y are said to be *infinitely close* (denoted by $a \approx b$) if $a - b$ is infinitesimal. We say that x is *roughly smaller than y* (and write $x \lesssim y$) if $x < y$ or $x \approx y$."

An ordered field is called nonarchimedian field, if it has non-null infinitesimals.
While it is a beautiful definition to consider that x and y are *infinitely close* (denoted by $a \approx b$) if $a - b$ is infinitesimal, it produces confusions into the nonstandard neutrosophic logic. Why? Because one cannot distinguish any-longer between ${}^{-}a$, a, and a^{+} (which is essential in and the flavor of nonstandard neutrosophic logic, in order to differentiate the *relative truth/indeterminacy/falsehood* from *absolute truth/indeterminacy/falsehood* respectively), since one gets that:

$$({}^{-}a) \approx a \approx (a^{+}) \quad (33)$$

or with the simplest notations:

$$ {}^{-}a \approx a \approx a^{+}. \quad (34)$$

Proof:

$\forall x \in R_{+}^{*}, a-(a-x)=x=$ infinitesimal, whence $a \approx (^{-}a)$ *(35)*

and $\forall x \in R_{+}^{*}, (a+x)-a=x=$ infinitesimal, whence $a^{+} \approx a$. *(36)*

For the definition of nonstandard interval $]^{-}a, b^{+}[$, Imamura proposes at page 6:

"For $a, b \in K$ the set $]^{-}a, b^{+}[_K$ is defined as follows:

$]^{-}a, b^{+}[_K = \{x \in K \mid a \lesssim x \lesssim b\}$."

In nonstandard neutrosophic logic and set, we may have not only $]^{-}a, b^{+}[$, but various forms of nonstandard intervals:

$$]\overset{m_1}{a}, \overset{m_2}{b}[\quad (37)$$

where m_1 and m_2 stand for: left monads ($^{-}$), right monads ($^{+}$), or bimonads ($^{-+}$), in all possible combinations (in total $3 \times 3 = 9$ possibilities).

Yet, Imamura's definition cannot be adjusted for all above nonstandard intervals, for example the nonstandard intervals of the form $]a^{+}, {}^{-}b[$, because if one writes:

$$]a^{+}, b^{-}[_K = \{x \in K \mid a \lesssim x \lesssim b\} \quad (38)$$

one arrives at proving that

$$]^{-}a, b^{+}[_K \subseteq]a^{+}, b^{-}[_K \quad (39)$$

which is obviously false, since: ^{-}a is below a and hence below a^{+}, and in the same way b^{+} is above b and hence above ^{-}b {one gets a bigger nonstandard interval included in or equal to a smaller nonstandard interval}. This occurs because $^{-}a \approx a^{+}$ and $b^{+} \approx b^{-}$ (in Imamura's notation).

17. Nonstandard Unit Interval.

Imamura cites my work:

> "by "–a" one signifies a monad, i.e., a set of hyper-real numbers in non-standard analysis:
> $(–a) = \{ a - x \in R_{*} \mid x$ is infinitesimal $\}$,
> and similarly "b_{+}" is a hyper monad:
> $(b_{+}) = \{ b + x \in R_{*} \mid x$ is infinitesimal $\}$.
> ([5] p. 141; [6] p. 9)"

But these are inaccurate, because my exact definitions of monads, since my 1998 first world neutrosophic publication {see [5], page 9; and [6], pages 385 - 386}, were:

> "$(^{-}a) = \{ a - x: x \in R_{+}^{*} \mid x$ is infinitesimal $\}$,
> and similarly "b+" is a hyper monad:
> $(b^{+}) = \{ b + x: x \in R_{+}^{*} \mid x$ is infinitesimal $\}$"

Imamura says that:

> "The correct definitions are the following:
> $(^{-}a) = \{ a - x \in R^{*} \mid x$ is *positive* infinitesimal $\}$,
> $(b^{+}) = \{ b + x \in R^{*} \mid x$ is *positive* infinitesimal $\}$."

I did not have a chance to see how my article was printed in *Proceedings of the 3rd Conference of the European Society for Fuzzy Logic and Technology* [7], that Imamura talks about, maybe there were some typos, but Imamura can check the *Multiple Valued Logic / An International Journal* [6], published in England in 2002 (ahead of the European Conference from 2003, that Imamura cites) by the prestigious Taylor & Francis Group Publishers, and clearly one sees that it is: $\boldsymbol{R}_{+}^{*}$ (so, x is a *positive* infinitesimal into the above formulas), therefore there is no error.

Then Imamura continues:

> "Ambiguity of the definition of the nonstandard unit interval. Smarandache did not give any explicit definition of the notation $]^{-}0, 1^{+}[$ in [5] (or the notation $\Vdash^{-}0, 1^{+}\dashv\!|$ in [6]). He only said:
> Then, we call $]^{-}0, 1^{+}[$ a non-standard unit interval. Obviously, 0 and 1, and analogously non-standard numbers infinitely small but less than 0 or infinitely small but greater than 1, belong to the non-standard unit interval. ([5] p. 141; [6] p. 9)."

Concerning the notations I used for the nonstandard intervals as $\Vdash \dashv\!|$ or] [, it was imperative to employ notations different from the classical *[ ]* or *( )* intervals, since the extremes of the nonstandard unit interval were unclear, vague.
I thought it was easily understood that:

$$]^{-}0, 1^{+}[\; = (^{-}0) \cup [0, 1] \cup (1^{+}). \tag{40}$$

Or, using the previous neutrosophic inequalities, we may write:

$$]^{-}0, 1^{+}[\; = \{x \in R^{*}, \; ^{-}0 \leq_N x \leq_N 1^{+}\}. \tag{41}$$

Imamura says that:

> "Here $^{-}0$ and 1^{+} are particular real numbers defined in the previous paragraph:
> $^{-}0 = 0-\varepsilon$ and $1^{+} = 1+\varepsilon$, where ε is a fixed non-negative infinitesimal."

This is untrue, I never said that "ε is a *fixed* non-negative infinitesimal", ε was not fixed, I said that for any real numbers *a* and *b* {see again [5], page 9; and [6], pages 385 - 386}:

> "$(^{-}a) = \{ a - x: x \in R_{+}^{*} \mid x \text{ is infinitesimal} \}$,
> $(b^{+}) = \{ b + x: x \in R_{+}^{*} \mid x \text{ is infinitesimal} \}$".

Therefore, once we replace a = 0 and b = 1 we get:

> $(^{-}0) = \{ 0 - x: x \in R_{+}^{*} \mid x \text{ is infinitesimal} \}$,
> $(1^{+}) = \{ 1 + x: x \in R_{+}^{*} \mid x \text{ is infinitesimal} \}$.

Thinking out of box, inspired from the real world, was the first intent, i.e. allowing neutrosophic components (truth / indeterminacy / falsehood) values be outside of the classical (standard) unit real interval *[0, 1]* used in all previous (Boolean, multi-valued etc.) logics if needed in applications, so neutrosophic component values < 0 and > 1 had to occurs due to the Relative / Absolute stuff, with:

$$^{-}0 <_N 0 \quad \text{and} \quad 1^{+} >_N 1. \tag{42}$$

Later on, in 2007, I found plenty of cases and real applications in Standard Neutrosophic Logic and Set (therefore, not using the Nonstandard Neutrosophic Logic and Set), and it was thus possible the extension of the neutrosophic set *to Neutrosophic Overset (when some neutrosophic component is > 1), and to Neutrosophic Underset (when some neutrosophic component is < 0), and to Neutrosophic Offset (when some neutrosophic components are off the interval [0, 1], i.e. some neutrosophic component > 1 and some neutrosophic component < 0). Then, similar extensions to respectively Neutrosophic Over/Under/Off Logic, Measure, Probability, Statistics etc.* [8, 17, 18, 19], extending the unit interval [0, 1] to

$$[\Psi, \Omega], \text{ with } \Psi \leq 0 < 1 \leq \Omega, \tag{43}$$

where Ψ, Ω are standard real numbers.

Imamura says, regarding the definition of neutrosophic logic that:

> "In this logic, each proposition takes a value of the form (T, I, F), where T, I, F are subsets of the nonstandard unit interval $]^{-}0, 1^{+}[$ and represent all possible values of Truthness, Indeterminacy and Falsity of the proposition, respectively."

Unfortunately, this is not exactly how I defined it.

In my first book {see [5], p. 12; or [6] pp. 386 – 387} it is stated:

> "Let T, I, F be real standard or non-standard subsets of]-0, 1+["

meaning that T, I, F may also be "real standard" not only real non-standard.

In *The Free Online Dictionary of Computing*, 1999-07-29, edited by Denis Howe from England, it is written:

> Neutrosophic Logic:
> <*logic*> (Or "Smarandache logic") A generalization of fuzzy logic based on Neutrosophy. A proposition is t true, i indeterminate, and f false, where t, i, and f are real values from the ranges T, I, F, with no restriction on T, I, F, or the sum n=t+i+f. Neutrosophic logic thus generalizes:
> - intuitionistic logic, which supports incomplete theories (for 0<n<100, 0<=t,i,f<=100);

- fuzzy logic (for n=100 and i=0, and 0<=t,i,f<=100);
- Boolean logic (for n=100 and i=0, with t,f either 0 or 100);
- multi-valued logic (for 0<=t,i,f<=100);
- paraconsistent logic (for n>100, with both t,f<100);
- dialetheism, which says that some contradictions are true (for t=f=100 and i=0; some paradoxes can be denoted this way).

Compared with all other logics, neutrosophic logic introduces a percentage of "indeterminacy" - due to unexpected parameters hidden in some propositions. It also allows each component t,i,f to "boil over" 100 or "freeze" under 0. For example, in some tautologies t>100, called "overtrue".
Home.
["Neutrosophy / Neutrosophic probability, set, and logic", F. Smarandache, American Research Press,
1998].

As Denis Howe said in 1999, the neutrosophic components *t, i, f* are "*real values* from the ranges *T, I, F*", not nonstandard values or nonstandard intervals. And this was because nonstandard ones were not important for the neutrosophic logic (the Relative/Absolute plaid no role in technological and scientific applications and future theories).

18. The Logical Connectives ∧, ∨,

Imamura's critics of my first definition of the neutrosophic operators is history for long ago.

All fuzzy, intuitionistic fuzzy, and neutrosophic logic operators are *inferential approximations*, not written in stone. They are improved from application to application.

Let's denote:

$\wedge_F, \wedge_N, \wedge_P$ representing respectively the fuzzy conjunction, neutrosophic conjunction, and plithogenic conjunction;

similarly

$\vee_F, \vee_N, \vee_P$ representing respectively the fuzzy disjunction, neutrosophic disjunction, and plithogenic disjunction,

and

$\rightarrow_F$, $\rightarrow_N$, $\rightarrow_P$ representing respectively the fuzzy implication, neutrosophic implication, and plithogenic implication.

I agree that my beginning neutrosophic operators (when I applied the same *fuzzy t-norm*, or the same *fuzzy t-conorm*, to all neutrosophic components *T, I, F*) were less accurate than others developed later by the neutrosophic community researchers. This was pointed out since 2002 by Ashbacher [9] and confirmed in 2008 by Rivieccio [10]. They observed that if on T_1 and T_2 one applies a *fuzzy t-norm*, on their opposites F_1 and F_2 one needs to apply the *fuzzy t-conorm* (the opposite of fuzzy t-norm), and reciprocally.

About inferring I_1 and I_2, some researchers combined them in the same directions as T_1 and T_2.

Then:

$$(T_1, I_1, F_1) \wedge_N (T_2, I_2, F_2) = (T_1 \wedge_F T_2, I_1 \wedge_F I_2, F_1 \vee_F F_2), \quad (44)$$
$$(T_1, I_1, F_1) \vee_N (T_2, I_2, F_2) = (T_1 \vee_F T_2, I_1 \vee_F I_2, F_1 \wedge_F F_2), \quad (45)$$
$$(T_1, I_1, F_1) \rightarrow_N (T_2, I_2, F_2) = (F_1, I_1, T_1) \vee_N (T_2, I_2, F_2) = (F_1 \vee_F T_2, I_1 \vee_F I_2, T_1 \wedge_F F_2); \quad (46)$$

others combined I_1 and I_2 in the same direction as F_1 and F_2 (since both I and F are negatively qualitative neutrosophic components), the most used one:

$$(T_1, I_1, F_1) \wedge_N (T_2, I_2, F_2) = (T_1 \wedge_F T_2, I_1 \vee_F I_2, F_1 \vee_F F_2), \quad (47)$$
$$(T_1, I_1, F_1) \vee_N (T_2, I_2, F_2) = (T_1 \vee_F T_2, I_1 \wedge_F I_2, F_1 \wedge_F F_2), \quad (48)$$
$$(T_1, I_1, F_1) \rightarrow_N (T_2, I_2, F_2) = (F_1, I_1, T_1) \vee_N (T_2, I_2, F_2) = (F_1 \vee_F T_2, I_1 \wedge_F I_2, T_1 \wedge_F F_2). \quad (49)$$

Now, applying the neutrosophic conjunction suggested by Imamura:

> "This causes some counterintuitive phenomena. Let A be a (true) proposition with value ({ 1 } , { 0 } , { 0 }) and let B be a (false) proposition with value ({ 0 } , { 0 } , { 1 }).
> Usually we expect that the falsity of the conjunction A ∧ B is { 1 }. However, its actual falsity is { 0 }."

we get:

$$(1, 0, 0) \wedge_N (0, 0, 1) = (0, 0, 1), \quad (50)$$

which is correct (so the falsity is *1*).

Even more, recently, in an extension of neutrosophic set to *plithogenic set* [11] (which is a set whose each element is characterized by many attribute values), the *degrees of contradiction c(,)* between the neutrosophic components *T, I, F* have been defined (in order to facilitate the design of the aggregation operators), as follows:
$c(T, F) = 1$ (or *100%,* because they are totally opposite), $c(T, I) = c(F, I) = 0.5$ (or *50%,* because they are only half opposite), then:

$$(T_1, I_1, F_1) \wedge_P (T_2, I_2, F_2) = (T_1 \wedge_F T_2, 0.5(I_1 \wedge_F I_2) + 0.5(I_1 \vee_F I_2), F_1 \vee_F F_2), \quad (51)$$
$$(T_1, I_1, F_1) \vee_P (T_2, I_2, F_2) = (T_1 \vee_F T_2, 0.5(I_1 \vee_F I_2) + 0.5(I_1 \wedge_F I_2), F_1 \wedge_F F_2). \quad (52)$$
$$(T_1, I_1, F_1) \rightarrow_N (T_2, I_2, F_2) = (F_1, I_1, T_1) \vee_N (T_2, I_2, F_2)$$
$$= (F_1 \vee_F T_2, 0.5(I_1 \vee_F I_2) + 0.5(I_1 \wedge_F I_2), T_1 \wedge_F F_2). \quad (53)$$

Conclusion.

We thank very much Dr. Takura Imamura for his interest and critics of *Nonstandard Neutrosophic Logic*, which eventually helped in improving it. {In the history of mathematics, critics on nonstandard analysis, in general, have been made by Paul Halmos, Errett Bishop, Alain Connes and others.} We hope we'll have more dialogues on the subject in the future.

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Soft Rough Neutrosophic Influence Graphs with Application

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Abstract: In this paper, we apply the notion of soft rough neutrosophic sets to graph theory. We develop certain new concepts, including soft rough neutrosophic graphs, soft rough neutrosophic influence graphs, soft rough neutrosophic influence cycles and soft rough neutrosophic influence trees. We illustrate these concepts with examples, and investigate some of their properties. We solve the decision-making problem by using our proposed algorithm.

Keywords: soft rough neutrosophic graphs; soft rough neutrosophic influence graphs; soft rough neutrosophic influence cycles; soft rough neutrosophic influence trees

1. Introduction

Smarandache [1] introduced neutrosophic sets as a generalization of fuzzy sets and intuitionistic fuzzy sets. A neutrosophic set has three constituents: truth-membership, indeterminacy-membership and falsity-membership, in which each membership value is a real standard or non-standard subset of the unit interval

$]0^-, 1^+[$. In real-life problems, neutrosophic sets can be applied more appropriately by using the single-valued neutrosophic sets defined by Smarandache [1] and Wang et al. [2]. Ye [3,4] and Peng et al. [5] further extended the study of neutrosophic sets. Soft set theory [6] was proposed by Molodtsov in 1999 to deal with uncertainty in a parametric manner. Babitha and Sunil discussed the concept of soft set relation [7]. On the other hand, Pawlak [8] proposed the notion of rough sets. It is a rigid appearance of modeling and processing partial information. It has been effectively connected to decision analysis, machine learning, inductive reasoning, intelligent systems, pattern recognition, signal analysis, expert systems, knowledge discovery, image processing, and many other fields [9–12]. In literature, rough theory has been applied in different field of mathematics [13–16]. Dubois and Prade [17] developed two concepts called rough fuzzy sets and fuzzy rough sets and concluded that these two theories are different approaches to handle vagueness. Feng et al. [18] combined soft sets with fuzzy sets and rough sets. Meng et al. [19] dealt with soft rough fuzzy sets and soft fuzzy rough sets. Broumi et al. [20] studied rough neutrosophic sets. Yang et al. [21] proposed single-valued neutrosophic rough sets, and established an algorithm for decision-making problem based on single-valued neutrosophic rough sets on two universes.

A graph is a convenient way of representing information involving relationship between objects. The objects are represented by vertices and relations by edges. When there is vagueness in the description of the objects or in its relationships or in both, it is natural that we need to design a fuzzy graph model. Fuzzy models has vital role as their aspiration in decreasing the irregularity between the traditional numerical models used in engineering and sciences and the symbolic models used in expert systems. The fuzzy graph theory as a generalization of Euler's graph theory was first introduced by Kaufmann [22]. Later, Rosenfeld [23] considered fuzzy graphs and obtained analogs of several graph theoretical concepts. Mordeson and Peng [24] defined some operations on fuzzy graphs. Mathew and Sunitha [25,26] presented some new concepts on fuzzy graphs. Gani et al. [27–30] discussed several concepts, including size, order, degree, regularity and edge regularity in fuzzy graphs and intuitionistic fuzzy graphs. Parvathi and Karunambigai [31] described some operation on intuitionistic fuzzy graph. Recently, Akram et al. [32–36] has introduced several extensions of fuzzy graphs with applications. Denish [37] considered the idea of fuzzy incidence graph. Fuzzy incidence graphs were further studied in [38,39]. Due to the limitation of humans knowledge to understand the complex problems, it is very difficult to apply only a single type of uncertainty method to deal with such problems. Therefore, it is necessary to develop hybrid models by incorporating the advantages of many other different mathematical models dealing uncertainty. Recently, new hybrid models, including rough fuzzy graphs [40,41], fuzzy rough graphs [42], intuitionistic fuzzy rough graphs [43,44], rough neutrosophic graphs [45] and neutrosophic soft rough graphs [46] are constructed. For other notations and definitions, the readers are refereed to [47–51]. In this paper, we apply the notion of soft rough neutrosophic sets to graph theory. We develop certain new concepts, including soft rough neutrosophic graphs, soft rough neutrosophic influence graphs, soft rough neutrosophic influence cycles and soft rough neutrosophic influence trees. We illustrate these concepts with examples, and investigate some of their properties. We solve decision-making problem by using our proposed algorithm.

This paper is organized as follows. In Section 2, some definitions and some properties of soft rough neutrosophic graphs are given. In Section 3, soft rough neutrosophic influence graphs, soft rough neutrosophic influence cycles and soft rough neutrosophic influence trees are discussed. In Section 4, an application is presented. Finally, we conclude our contribution with a summary in Section 5 and an outlook for the further research.

2. Soft Rough Neutrosophic Graphs

Definition 1. *Let B be Boolean set and A a set of attributes. For an arbitrary full soft set S over B such that $S_s(a)\subset B$, for some $a\in A$, where $S_s:A\to\mathcal{P}(B)$ is a set-valued function defined as $S_s(a)=\{b\in B|(a,b)\in S\}$, for all $a\in A$. Let (B,S) be a full soft approximation space. For any neutrosophic set $N=\{(b,T_N(b),I_N(b),F_N(b))|b\in B\}\in\mathcal{N}(B)$, where $\mathcal{N}(B)$ is neutrosophic power set of set B. The upper and lower soft rough neutrosophic approximations of N* w.r.t *(B,S), denoted by $\overline{S}(N)$ and $\underline{S}(N)$, respectively, are defined as follows:*

$$\overline{S}(N)=\{(b,T_{\overline{S}(N)}(b),I_{\overline{S}(N)}(b),F_{\overline{S}(N)}(b))\mid b\in B\},$$
$$\underline{S}(N)=\{(b,T_{\underline{S}(N)}(b),I_{\underline{S}(N)}(b),F_{\underline{S}(N)}(b))\mid b\in B\},$$

where

$$\begin{aligned}
T_{\overline{S}(N)}(b)&=\bigwedge_{b\in S_s(a)}\ \bigvee_{t\in S_s(a)} T_N(t),\ T_{\underline{S}(N)}(b)=\bigvee_{b\in S_s(a)}\ \bigwedge_{t\in S_s(a)} T_N(t),\\
I_{\overline{S}(N)}(b)&=\bigvee_{b\in S_s(a)}\ \bigwedge_{t\in S_s(a)} I_N(t),\ I_{\underline{S}(N)}(b)=\bigwedge_{b\in S_s(a)}\ \bigvee_{t\in S_s(a)} I_N(t),\\
F_{\overline{S}(N)}(b)&=\bigvee_{b\in S_s(a)}\ \bigwedge_{t\in S_s(a)} F_N(t),\ F_{\underline{S}(N)}(b)=\bigwedge_{b\in S_s(a)}\ \bigvee_{t\in S_s(a)} F_N(t).
\end{aligned}\tag{1}$$

In other words,

$$T_{\overline{S}(N)}(b)=\bigwedge_{a\in A}\Big((1-S(a,b))\vee\Big(\bigvee_{t\in B}(S(a,t)\wedge T_N(t))\Big)\Big),$$
$$T_{\underline{S}(N)}(b)=\bigvee_{a\in A}\Big(S(a,b)\wedge\Big(\bigwedge_{t\in B}((1-S(a,t))\vee T_N(t))\Big)\Big),$$

$$I_{\overline{S}(N)}(b) = \bigvee_{a\in A}\Big(S(a,b)\wedge\Big(\bigwedge_{t\in B}((1-S(a,t))\vee I_N(t))\Big)\Big),$$
$$I_{\underline{S}(N)}(b) = \bigwedge_{a\in A}\Big((1-S(a,b))\vee\Big(\bigvee_{t\in B}(S(a,t)\wedge I_N(t))\Big)\Big),$$
$$F_{\overline{S}(N)}(b) = \bigvee_{a\in A}\Big(S(a,b)\wedge\Big(\bigwedge_{t\in B}((1-S(a,t))\vee F_N(t))\Big)\Big),$$
$$F_{\underline{S}(N)}(b) = \bigwedge_{a\in A}\Big((1-S(a,b))\vee\Big(\bigvee_{t\in B}(S(a,t)\wedge F_N(t))\Big)\Big).$$

The pair $(\underline{S}(N),\overline{S}(N))$ *is called soft rough neutrosophic set (SRNS) of* N w.r.t (B,S).

Example 1. *Suppose* $N=\{(b_1,0.8,0.3,0.16),(b_2,0.85,0.24,0.2),(b_3,0.79,0.2,0.2),(b_4,0.85,0.36,0.25),(b_5,0.82,0.25,0.25)\}$ *is a neutrosophic set on the universal set* $B=\{b_1,b_2,b_3,b_4,b_5\}$ *under consideration. Let* $A=\{a_1,a_2,a_3\}$ *be a set of parameter on* B*. A full soft set over* B*, denoted by* S*, is defined in Table 1.*

Table 1. Full soft set S.

S	b_1	b_2	b_3	b_4	b_5
a_1	0	0	1	0	1
a_2	1	0	1	0	0
a_3	0	1	1	1	1

A set-valued function $S_s{:}A\to\mathcal{P}(B)$ *is defined as* $S_s(a_1)=\{b_3,b_5\},S_s(a_2)=\{b_1,b_3\},S_s(a_3)=\{b_2,b_3,b_4,b_5\}$*. From Equation (1) of Definition 1, we have*

$$T_{\overline{S}(A)}(b_1) = \bigvee_{y\in S_s(a_2)} N(y)=\vee\{0.8,0.79\}=0.80,$$
$$I_{\overline{S}(N)}(b_1) = \bigwedge_{y\in S_s(a_2)} N(y)=\wedge\{0.3,0.2\}\ =0.20,$$
$$F_{\overline{S}(N)}(b_1) = \bigwedge_{y\in S_s(a_2)} N(y)=\wedge\{0.16,0.2\}=0.16;$$
$$T_{\underline{S}(N)}(b_1) = \bigwedge_{y\in S_s(a_2)} N(y)=\wedge\{0.8,0.79\}=0.79,$$
$$I_{\underline{S}(N)}(b_1) = \bigvee_{y\in S_s(a_2)} N(y)=\vee\{0.3,0.2\}\ =0.30,$$
$$F_{\underline{S}(N)}(b_1) = \bigvee_{y\in S_s(a_2)} N(y)=\vee\{0.16,0.2\}=0.20.$$

Similarly,

$$T_{\overline{S}(N)}(b_2) = 0.85,\quad I_{\overline{S}(N)}(b_2) = 0.20,\quad F_{\overline{S}(N)}(b_2) = 0.20,$$
$$T_{\overline{S}(N)}(b_3) = 0.80,\quad I_{\overline{S}(N)}(b_3) = 0.20,\quad F_{\overline{S}(N)}(b_3) = 0.20,$$
$$T_{\overline{S}(N)}(b_4) = 0.85,\quad I_{\overline{S}(N)}(b_4) = 0.20,\quad F_{\overline{S}(N)}(b_4) = 0.20,$$
$$T_{\overline{S}(N)}(b_5) = 0.82,\quad I_{\overline{S}(N)}(b_5) = 0.20,\quad F_{\overline{S}(N)}(b_5) = 0.20;$$
$$T_{\underline{S}(N)}(b_2) = 0.79,\quad I_{\underline{S}(N)}(b_2) = 0.36,\quad F_{\underline{S}(N)}(b_2) = 0.25,$$
$$T_{\underline{S}(N)}(b_3) = 0.79,\quad I_{\underline{S}(N)}(b_3) = 0.25,\quad F_{\underline{S}(N)}(b_3) = 0.20,$$
$$T_{\underline{S}(N)}(b_4) = 0.79,\quad I_{\underline{S}(N)}(b_4) = 0.36,\quad F_{\underline{S}(N)}(b_4) = 0.25,$$

$$T_{\underline{S}(N)}(b_5) = 0.79, \quad I_{\underline{S}(N)}(b_5) = 0.25, \quad F_{\underline{S}(N)}(b_5) = 0.25.$$

Thus, we obtain

$$\begin{aligned}\overline{S}(N)=&\{(b_1, 0.80, 0.20, 0.16), (b_2, 0.85, 0.20, 0.20), (b_3, 0.80, 0.20, 0.20),\\ &(b_4, 0.85, 0.20, 0.20), (b_5, 0.82, 0.20, 0.20)\},\\ \underline{S}(N)=&\{(b_1, 0.79, 0.30, 0.20), (b_2, 0.79, 0.36, 0.25), (b_3, 0.79, 0.25, 0.20),\\ &(b_4, 0.79, 0.36, 0.25), (b_5, 0.79, 0.25, 0.25)\}.\end{aligned}$$

Definition 2. *A soft rough neutrosophic relation(SRNR) ($\underline{R}(M)$,$\overline{R}(M)$) on $\tilde{B}$=$B\times B$ is a soft rough neutrosophic set, R:$\tilde{A}(A\times A)\to\mathcal{P}(\tilde{B})$ is a full soft set on $\tilde{B}$ and defined by*

$$R(a_{kl}, b_{ij}) \leq \min\{S(a_k, b_i), S(a_l, b_j)\},$$

for all (a_{kl},b_{ij})$\in R$, such that $R_s(a_{kl})\subset\tilde{B}$ for some $a_{kl}\in\tilde{A}$, where R_s:$\tilde{A}\to\mathcal{P}(\tilde{B})$ is a set-valued function, for all $a_{kl}\in\tilde{A}$, defined by

$$R_s(a_{kl}) = \{b_{ij} \in \tilde{B} \mid (a_{kl}, b_{ij}) \in R\}, \;\; b_{ij} \in \tilde{B}.$$

For any neutrosophic set $M\in\mathcal{N}(\tilde{B})$, the upper and lower soft rough neutrosophic approximation of M w.r.t ($\tilde{B}$,R) *are defined as follows:*

$$\begin{aligned}\overline{R}(M)=&\{(b_{ij}, T_{\overline{R}(M)}(b_{ij}), I_{\overline{R}(M)}(b_{ij}), F_{\overline{R}(M)}(b_{ij})) \mid b_{ij} \in\tilde{B}\}\;,\\ \underline{R}(M)=&\{(b_{ij}, T_{\underline{R}(M)}(b_{ij}), I_{\underline{R}(M)}(b_{ij}), F_{\underline{R}(M)}(b_{ij})) \mid b_{ij} \in\tilde{B}\},\end{aligned}$$

where

$$\begin{aligned}T_{\overline{R}(M)}(b_{ij})=&\bigwedge_{b_{ij}\in R_s(a_{kl})}\;\bigvee_{t_{ij}\in R_s(a_{kl})} T_M(t_{ij}),\;\; T_{\underline{R}(M)}(b_{ij}) = \bigvee_{b_{ij}\in R_s(a_{kl})}\;\bigwedge_{t_{ij}\in R_s(a_{kl})} T_M(t_{ij}),\\ I_{\overline{R}(M)}(b_{ij})=&\bigvee_{b_{ij}\in R_s(a_{kl})}\;\bigwedge_{t_{ij}\in R_s(a_{kl})} I_M(t_{ij})\;,\;\; I_{\underline{R}(M)}(b_{ij}) = \bigwedge_{b_{ij}\in R_s(a_{kl})}\;\bigvee_{t_{ij}\in R_s(a_{kl})} I_M(t_{ij}),\\ F_{\overline{R}(M)}(b_{ij})=&\bigvee_{b_{ij}\in R_s(a_{kl})}\;\bigwedge_{t_{ij}\in R_s(a_{kl})} F_M(t_{ij}),\;\; F_{\underline{R}(M)}(b_{ij}) = \bigwedge_{b_{ij}\in R_s(a_{kl})}\;\bigvee_{t_{ij}\in R_s(a_{kl})} F_M(t_{ij}).\end{aligned} \tag{2}$$

In other words,

$$\begin{aligned}T_{\overline{R}(M)}(b_{ij}) &= \bigwedge_{a_{kl}\in A}\Big((1 - R(a_{kl}, b_{ij})) \vee \Big(\bigvee_{t_{ij}\in B}\big(R(a_{kl}, t_{ij}) \wedge T_M(t_{ij})\big)\Big)\Big),\\ T_{\underline{R}(M)}(b_{ij}) &= \bigvee_{a_{kl}\in A}\Big(R(a_{kl}, b_{ij}) \wedge \Big(\bigwedge_{t_{ij}\in B}\big((1 - R(a_{kl}, t_{ij})) \vee T_M(t_{ij})\big)\Big)\Big),\\ I_{\overline{R}(M)}(b_{ij}) &= \bigvee_{a_{kl}\in A}\Big(R(a_{kl}, b_{ij}) \wedge \Big(\bigwedge_{t_{ij}\in B}\big((1 - R(a_{kl}, t_{ij})) \vee I_M(t_{ij})\big)\Big)\Big),\\ I_{\underline{R}(M)}(b_{ij}) &= \bigwedge_{a_{kl}\in A}\Big((1 - R(a_{kl}, b_{ij})) \vee \Big(\bigvee_{t_{ij}\in B}\big(R(a_{kl}, t_{ij}) \wedge I_M(t_{ij})\big)\Big)\Big),\\ F_{\overline{R}(M)}(b_{ij}) &= \bigvee_{a_{kl}\in A}\Big(R(a_{kl}, b_{ij}) \wedge \Big(\bigwedge_{t_{ij}\in B}\big((1 - R(a_{kl}, t_{ij})) \vee F_M(t_{ij})\big)\Big)\Big),\\ F_{\underline{R}(M)}(b_{ij}) &= \bigwedge_{a_{kl}\in A}\Big((1 - R(a_{kl}, b_{ij})) \vee \Big(\bigvee_{t_{ij}\in B}\big(R(a_{kl}, t_{ij}) \wedge F_M(t_{ij})\big)\Big)\Big).\end{aligned}$$

If $\overline{R}(M)=\underline{R}(M)$, *then it is called induced soft rough neutrosophic relation on soft rough neutrosophic set, otherwise, soft rough neutrosophic relation.*

Remark 1. *For a neutrosophic set M on* $\tilde{B}$ *and a neutrosophic set N on B, we have neutrosophic relation as follow*

$$T_M(b_{ij}) \leq \min_i\{T_N(b_i)\}, \quad I_M(b_{ij}) \leq \min_i\{I_N(b_i)\}, \quad F_M(b_{ij}) \leq \min_i\{F_N(b_i)\}.$$

From Definition 2, it follows that:

$$\begin{aligned}
&T_{\overline{R}(M)}(b_{ij}) \leq \min\{T_{\overline{S}(N)}(b_i), T_{\overline{S}(N)}(b_j)\}, \quad T_{\underline{R}(M)}(b_{ij}) \leq \min\{T_{\underline{S}(N)}(b_i), T_{\underline{S}(N)}(b_j)\},\\
&I_{\overline{R}(M)}(b_{ij}) \leq \max\{I_{\overline{S}(N)}(b_i), I_{\overline{S}(N)}(b_j)\}, \quad I_{\underline{R}(M)}(b_{ij}) \leq \max\{I_{\underline{S}(N)}(b_i), I_{\underline{S}(N)}(b_j)\},\\
&F_{\overline{R}(M)}(b_{ij}) \leq \max\{F_{\overline{S}(N)}(b_i), F_{\overline{S}(N)}(b_j)\}, \quad F_{\underline{R}(M)}(b_{ij}) \leq \max\{F_{\underline{S}(N)}(b_i), F_{\underline{S}(N)}(b_j)\}.
\end{aligned}$$

Definition 3. *In Definition 2* b_{ij} *is called effective, if*

$$\begin{aligned}
&T_{\underline{R}(M)}(b_{ij})=T_{\underline{S}(N)}(b_i) \wedge T_{\underline{S}N}(b_j), \; T_{\overline{R}(M)}(b_{ij})=T_{\overline{S}(N)}(b_i) \wedge T_{\overline{S}N}(b_j),\\
&I_{\underline{R}(M)}(b_{ij})=I_{\underline{S}(N)}(b_i) \vee I_{\underline{S}N}(b_j), \; I_{\overline{R}(M)}(b_{ij})=I_{\overline{S}(N)}(b_i) \vee I_{\overline{S}N}(b_j),\\
&F_{\underline{R}(M)}(b_{ij})=F_{\underline{S}(N)}(b_i) \vee F_{\underline{S}N}(b_j), \; F_{\overline{R}(M)}(b_{ij})=F_{\overline{S}(N)}(b_i) \vee F_{\overline{S}N}(b_j).
\end{aligned}$$

Definition 4. *A soft rough neutrosophic influence (SRNI) is a relation from soft rough neutrosophic set to soft rough neutrosophic relation, denoted by* $(\underline{X}(Q),\overline{X}(Q))$ *on* $\hat{B}=B\times\tilde{B}$, *where* $X:\hat{A}(A\times\tilde{A})\rightarrow\mathcal{P}(\hat{B})$ *is a full soft set on* $\hat{B}$ *defined by*

$$X(a_l a_{mn}, b_i b_{jk}) \leq S(a_l, b_i) \wedge R(a_{mn}, b_{jk}),$$

for all $(a_l a_{mn}, b_i b_{jk})\in X$ *and for some* $i\neq j\neq k$ *and* $l\neq m\neq n$. *Let* $X_s:\hat{A}\rightarrow\mathcal{P}(\hat{B})$ *be a set-valued function defined by*

$$X_s(a_l a_{mn}) = \{b_i b_{jk} \in \hat{B} | (a_l a_{mn}, (b_i, , b_{jk})) \in X\}, \; \forall (a_, l a_{mn}) \in \hat{A},$$

For any $Q\in\mathcal{N}(\hat{B})$, *the upper and lower soft rough neutrosophic approximation of Q* w.r.t $(\hat{B},X)$, *for all* $b_i b_{jk}\in\hat{B}$, *are defined as follows:*

$$\begin{aligned}
&\overline{X}(Q)=\{(b_i b_{jk}, T_{\overline{X}(Q)}(b_i b_{jk}), I_{\overline{X}(Q)}(b_i b_{jk}), F_{\overline{X}(Q)}(b_i b_{jk}))\},\\
&\underline{X}(Q)=\{(b_i b_{jk}, T_{\underline{X}(Q)}(b_i b_{jk}), I_{\underline{X}(Q)}(b_i b_{jk}), F_{\underline{X}(Q)}(b_i b_{jk}))\},
\end{aligned}$$

where

$$\begin{aligned}
&T_{\overline{X}(Q)}(b_i b_{jk})=\bigwedge_{b_i b_{jk}\in X_s(a_l a_{mn})} \; \bigvee_{t_i t_{jk}\in X_s(a_l a_{mn})} T_Q(t_i t_{jk}),\\
&T_{\underline{X}(Q)}(b_i b_{jk})=\bigvee_{b_i b_{jk}\in X_s(a_l a_{mn})} \; \bigwedge_{t_i t_{jk}\in X_s(a_l a_{mn})} T_Q(t_i t_{jk}),\\
&I_{\overline{X}(Q)}(b_i b_{jk})=\bigvee_{b_i b_{jk}\in X_s(a_l a_{mn})} \; \bigwedge_{t_i t_{jk}\in X_s(a_l a_{mn})} I_Q(t_i t_{jk}),\\
&I_{\underline{X}(Q)}(b_i b_{jk})=\bigwedge_{b_i b_{jk}\in X_s(a_l a_{mn})} \; \bigvee_{t_i t_{jk}\in X_s(a_l a_{mn})} I_Q(t_i t_{jk}),\\
&F_{\overline{X}(Q)}(b_i b_{jk})=\bigvee_{b_i b_{jk}\in X_s(a_l a_{mn})} \; \bigwedge_{t_i t_{jk}\in X_s(a_l a_{mn})} F_Q(t_i t_{jk}),
\end{aligned} \tag{3}$$

$$F_{\underline{X}(Q)}(b_ib_{jk})=\bigwedge_{b_ib_{jk}\in X_s(a_la_{mn})}\ \bigvee_{t_it_{jk}\in X_s(a_la_{mn})} F_Q(t_it_{jk})\ .$$

In other words,

$$T_{\overline{X}(Q)}(b_ib_{jk}) = \bigwedge_{a_la_{mn}\in A}\Big((1-X(a_la_{mn},b_ib_{jk}))\vee\Big(\bigvee_{t_it_{jk}\in B}(X(a_la_{mn},t_it_{jk})\wedge T_Q(t_it_{jk}))\Big)\Big),$$
$$T_{\underline{X}(Q)}(b_ib_{jk}) = \bigvee_{a_la_{mn}\in A}\Big(X(a_la_{mn},b_ib_{jk})\wedge\Big(\bigwedge_{t_it_{jk}\in B}((1-X(a_la_{mn},t_it_{jk}))\vee T_Q(t_it_{jk}))\Big)\Big),$$
$$I_{\overline{X}(Q)}(b_ib_{jk}) = \bigvee_{a_la_{mn}\in A}\Big(X(a_la_{mn},b_ib_{jk})\wedge\Big(\bigwedge_{t_it_{jk}\in B}((1-X(a_la_{mn},t_it_{jk}))\vee I_Q(t_it_{jk}))\Big)\Big),$$
$$I_{\underline{X}(Q)}(b_ib_{jk}) = \bigwedge_{a_la_{mn}\in A}\Big((1-X(a_la_{mn},b_ib_{jk}))\vee\Big(\bigvee_{t_it_{jk}\in B}(X(a_la_{mn},t_it_{jk})\wedge I_Q(t_it_{jk}))\Big)\Big),$$
$$F_{\overline{X}(Q)}(b_ib_{jk}) = \bigvee_{a_la_{mn}\in A}\Big(X(a_la_{mn},b_ib_{jk})\wedge\Big(\bigwedge_{t_it_{jk}\in B}((1-X(a_la_{mn},t_it_{jk}))\vee F_Q(t_it_{jk}))\Big)\Big),$$
$$F_{\underline{X}(Q)}(b_ib_{jk}) = \bigwedge_{a_la_{mn}\in A}\Big((1-X(a_la_{mn},b_ib_{jk}))\vee\Big(\bigvee_{t_it_{jk}\in B}(X(a_la_{mn},t_it_{jk})\wedge F_Q(t_it_{jk}))\Big)\Big).$$

Remark 2. *For a neutrosophic set Q on $\hat{B}$ and a neutrosophic set N and M on B and $\tilde{B}$, respectively, we have neutrosophic relation as follow*

$$T_Q(b_ib_{jk}) \le \min_{jk}\{T_M(b_{jk})\},\quad I_Q(b_ib_{jk}) \le \min_{jk}\{I_M(b_{jk})\},\quad F_Q(b_ib_{jk}) \le \min_{jk}\{F_M(b_{jk})\}.$$

From Definition 4, we have

$$T_{\overline{X}(Q)}(b_ib_{jk}) \le \min\{T_{\overline{S}(N)}(b_i),T_{\overline{R}(M)}(b_{jk})\},\quad T_{\underline{X}(Q)}(b_ib_{jk}) \le \min\{T_{\underline{S}(N)}(b_i),T_{\underline{R}(M)}(b_{jk})\},$$
$$I_{\overline{X}(Q)}(b_ib_{jk}) \le \max\{I_{\overline{S}(N)}(b_i),I_{\overline{R}(M)}(b_{jk})\},\quad I_{\underline{X}(Q)}(b_ib_{jk}) \le \max\{I_{\underline{S}(N)}(b_i),I_{\underline{R}(M)}(b_{jk})\},$$
$$F_{\overline{X}(Q)}(b_ib_{jk}) \le \max\{F_{\overline{S}(N)}(b_i),F_{\overline{R}(M)}(b_{jk})\},\quad F_{\underline{X}(Q)}(b_ib_{jk}) \le \max\{F_{\underline{S}(N)}(b_i),F_{\underline{R}(M)}(b_{jk})\}.$$

Definition 5. *In Definition 4 b_ib_{jk} is called influence effective, if*

$$T_{\underline{X}(Q)}(b_ib_{jk})=T_{\underline{S}(N)}(b_i)\wedge T_{\underline{R}M}(b_{ij}),\ T_{\overline{X}(Q)}(b_ib_{jk})=T_{\overline{S}(N)}(b_i)\wedge T_{\overline{R}M}(b_{ij}),$$
$$I_{\underline{X}(Q)}(b_ib_{jk})=I_{\underline{S}(N)}(b_i)\vee I_{\underline{R}M}(b_{ij}),\ I_{\overline{X}(Q)}(b_ib_{jk})=I_{\overline{S}(N)}(b_i)\vee I_{\overline{R}M}(b_{ij}),$$
$$F_{\underline{X}(Q)}(b_ib_{jk})=F_{\underline{S}(N)}(b_i)\vee F_{\underline{R}M}(b_{ij}),\ F_{\overline{X}(Q)}(b_ib_{jk})=F_{\overline{S}(N)}(b_i)\vee F_{\overline{R}M}(b_{ij}).$$

Example 2. *Let a full soft set S on an universal set $B=\{b_1,b_2,b_3,b_4\}$ with $A=\{a_1,a_2,a_3\}$ a set of parameters can be defined in tabular form as Table 2 as follows:*

Table 2. Full soft set S.

S	b_1	b_2	b_3	b_4
a_1	1	1	0	1
a_2	0	0	1	1
a_3	1	1	1	1

Now, we can define set-valued function S_s such that

$$S_s(a_1) = \{b_1,b_2,b_4\}, S_s(a_2) = \{b_3,b_4\}, S_s(a_3) = \{b_1,b_2,b_3,b_4\}.$$

Let $N = \{(b_1,1.0,0.0,0.0),(b_2,0.8,0.0,0.1),(b_3,0.5,0.5,0.5),(b_4,0.4,0.7,0.3)\}$ *be a neutrosophic set on B, then by using Equation (1) of Definition 1, we have*

$$\overline{S}(N) = \{(b_1, 1.0, 0.0, 0.0), (b_2, 1.0, 0.0, 0.0), (b_3, 0.5, 0.5, 0.3), (b_4, 0.5, 0.5, 0.3)\},$$

$$\underline{S}(N) = \{(b_1, 0.4, 0.7, 0.3), (b_2, 0.4, 0.7, 0.3), (b_3, 0.4, 0.7, 0.5), (b_4, 0.4, 0.7, 0.3)\}.$$

Hence $(\underline{S}(N), \overline{S}(N))$ *is soft rough neutrosophic set. Let a full soft set R on* $C=\{b_{12},b_{22},b_{23}, b_{32},b_{42}\}\subseteq\tilde{B}$ *with* $L=\{a_{13},a_{21},a_{32}\}\subseteq\tilde{A}$ *can be defined in Table 3 (from L to C) as follows:*

Table 3. Full soft set R.

R	b_{12}	b_{22}	b_{23}	b_{32}	b_{42}
a_{13}	1	1	1	0	1
a_{21}	0	0	0	1	0
a_{32}	0	0	1	0	0

Now, we can define set-valued function R_s *such that*

$$R_s(a_{13}) = \{b_{12}, b_{22}, b_{23}, b_{42}\}, R_s(a_{21}) = \{b_{32}\}, R_s(a_{32}) = \{b_{23}\}.$$

and $M = \{(b_{12},0.4,0.0,0.0),(b_{22},0.4,0.0,0.0),(b_{23},0.4,0.0,0.0),(b_{32},0.4,0.0,0.0),(b_{42},0.4,0.0,0.0)\}$ *a neutrosophic relation on B, then by using Equation (2) of Definition 2, we get*

$$\overline{R}(M)=\{(b_{12}, 0.4, 0.0, 0.0),(b_{22}, 0.4, 0.0, 0.0),(b_{23}, 0.4, 0.0, 0.0),(b_{32}, 0.4, 0.0, 0.0),(b_{42}, 0.4, 0.0, 0.0)\},$$
$$\underline{R}(M)=\{(b_{12}, 0.4, 0.0, 0.0),(b_{22}, 0.4, 0.0, 0.0),(b_{23}, 0.4, 0.0, 0.0),(b_{32}, 0.4, 0.0, 0.0),(b_{42}, 0.4, 0.0, 0.0)\}.$$

Hence $(\underline{R}(M), \overline{R}(M))$ *is an induced soft rough neutrosophic relation. Let a full soft set X on* $D=\{b_1b_{22},b_1b_{23},b_1b_{32},b_1b_{42},b_3b_{12},b_3b_{22},b_3b_{42},b_4b_{12},b_4b_{22},b_4b_{23},b_4b_{32}\}\subseteq\hat{B}$ *with* $P=\{a_{13},a_{21},a_{32}\}\subseteq\hat{A}$ *can be defined in Table 4 (from P to D) as follows:*

Table 4. Full soft set X.

X	b_1b_{22}	b_1b_{23}	b_1b_{32}	b_1b_{42}	b_3b_{12}	b_3b_{22}	b_3b_{42}	b_4b_{12}	b_4b_{22}	b_4b_{23}	b_4b_{32}
a_1a_{32}	0	1	0	0	0	0	0	0	0	1	0
a_2a_{13}	0	0	0	0	0	1	1	1	1	1	1
a_3a_{21}	0	1	0	0	0	0	0	0	0	0	1

Since X is not full soft set on D, therefore, soft rough neutrosophic influence cannot be defined on D.

Definition 6. *A soft rough neutrosophic graph on a nonempty V is a 5-ordered tuple* $G=(A,S,SN,R,RM)$ *such that*

(i) *A is a set of attributes,*
(ii) *S is an arbitrary full soft set over V,*
(iii) *R is an arbitrary full soft set over* $E \subseteq \tilde{V}$,
(vi) $SN=(\underline{S}(N),\overline{S}(N))$ *is a soft rough neutrosophic set of V,*
(v) $RM=(\underline{R}(M),\overline{R}(M))$ *is a soft rough neutrosophic set on* $E \subset \tilde{V}$,

In other words $G=(\underline{G},\overline{G})=(SN,RM)$ *is a soft rough neutrosophic graph(SRNG), where* $\underline{G}=(\underline{S}(N),\underline{R}(M))$ *and* $\overline{G}=(\overline{S}(N),\overline{R}(M))$ *are lower soft rough neutrosophic approximate graphs (LSRNAGs) and upper soft rough neutrosophic approximate graphs (USRNAGs), respectively, of* $G=(SN,RM)$.

Example 3. *Let $V=\{v_1,v_2,v_3,v_4,v_5,v_6\}$ be a vertex set and $A=\{a_1,a_2,a_3\}$ a set of parameters. A full soft set S from A on V can be defined in tabular form in Table 5 as follows:*

Table 5. Full soft set S.

S	v_1	v_2	v_3	v_4	v_5	v_6
a_1	1	1	1	1	1	0
a_2	0	0	1	1	1	1
a_3	1	1	0	0	1	1

Let $N=\{(v_1,0.8,0.6,0.4),(v_2,0.9,0.4,0.45),(v_3,0.7,0.4,0.35),(v_4,0.6,0.3,0.5),(v_5,0.4,0.7,0.6),(v_6,0.5,0.5,0.5)\}$ be a neutrosophic set on V. Then from Equation (1) of Definition 1, we have

$$\overline{S}(N)=\{(v_1,0.9,0.4,0.4),(v_2,0.9,0.4,0.4),(v_3,0.7,0.3,0.5),(v_4,0.7,0.3,0.5),(v_5,0.7,0.4,0.5),(v_6,0.7,0.4,0.5)\},$$
$$\underline{S}(N)=\{(v_1,0.4,0.7,0.6),(v_2,0.4,0.7,0.6),(v_3,0.4,0.7,1.0),(v_4,0.4,0.7,1.0),(v_5,0.4,0.7,0.6),(v_6,0.4,0.7,0.6)\}.$$

Hence, $SN=(\underline{S}(N),\overline{S}(N))$ is a soft rough neutrosophic set on V. Let $E=\{v_{11},v_{15},v_{16},v_{23},v_{25},v_{34},v_{41},v_{43},v_{56},v_{62},v_{63}\}\subseteq\tilde{V}$ and $L=\{a_{12},a_{13},a_{21},a_{23},a_{31}\}\subseteq\tilde{A}$. Then a full soft set R on E (from L to E) can be defined in Table 6 as follows:

Table 6. Full soft set R.

R	v_{11}	v_{15}	v_{16}	v_{23}	v_{25}	v_{34}	v_{41}	v_{43}	v_{56}	v_{62}	v_{63}
a_{12}	0	1	1	1	1	1	0	1	1	0	0
a_{13}	1	1	1	0	1	0	1	0	1	0	0
a_{21}	0	0	0	0	0	1	1	1	0	1	1
a_{23}	0	0	0	0	0	0	1	0	1	1	0
a_{31}	1	1	0	1	1	0	0	0	0	1	0

Let $M=\{(v_{11},0.4,0.3,0.35),(v_{15},0.3,0.3,0.2),(v_{16},0.3,0.2,0.25),(v_{23},0.4,0.1,0.1),(v_{25},0.4,0.2,0.0),(v_{34},0.3,0.1,0.3),(v_{41},0.2,0.1,0.2),(v_{43},0.4,0.28,0.2),(v_{56},0.4,0.3,0.3),(v_{62},0.35,0.25,0.32),(v_{63},0.4,0.12,0.34)\}$ be a neutrosophic set on E. Then from Equation (2) of Definition 2, we have

$$\overline{R}(M)=\{(v_{11},0.4,0.1,0.00),(v_{15},0.4,0.10,0.00),(v_{16},0.4,0.10,0.00),(v_{23},0.4,0.10,0.00),(v_{25},0.4,0.1,0.00),(v_{34},0.4,0.10,0.20),(v_{41},0.4,0.10,0.30),(v_{43},0.4,0.10,0.20),(v_{56},0.4,0.1,0.30),(v_{62},0.4,0.10,0.30),(v_{63},0.4,0.10,0.20)\},$$
$$\underline{R}(M)=\{(v_{11},0.3,0.3,0.35),(v_{15},0.3,0.30,0.35),(v_{16},0.3,0.30,1.00),(v_{23},0.3,0.30,0.35),(v_{25},0.3,0.3,0.35),(v_{34},0.3,0.28,0.34),(v_{41},0.2,0.28,0.32),(v_{43},0.3,0.28,0.34),(v_{56},0.3,0.3,0.32),(v_{62},0.3,0.28,0.32),(v_{63},0.2,0.28,0.34)\}.$$

Hence, $RM=(\underline{R}(M),\overline{R}(M))$ is soft rough neutrosophic set on E. Thus, $\underline{G}=(\underline{S}(N),\underline{R}(M))$ and $\overline{G}=(\overline{S}(N),\overline{R}(M))$ are LSRNAG and USRNAG, respectively, as shown in Figure 1.

Definition 9. *A strength of soft rough neutrosophic graph, denoted by stren, is defined as*

$$stren=\Big((\bigwedge_{v_{jk}\in \underline{E}^*} T_{\underline{R}(M)}(v_{jk})) \wedge (\bigwedge_{v_{jk}\in \overline{E}^*} T_{\overline{R}(M)}(v_{jk})), (\bigvee_{v_{jk}\in \underline{E}^*} I_{\underline{R}(M)}(v_{jk}))\vee$$
$$(\bigvee_{v_{jk}\in \overline{E}^*} I_{\overline{R}(M)}(v_{jk})), (\bigvee_{v_{jk}\in \underline{E}^*} F_{\underline{R}(M)}(v_{jk})) \vee (\bigvee_{v_{jk}\in \overline{E}^*} F_{\overline{R}(M)}(v_{jk}))\Big).$$

Definition 10. *A strongest path joining any two vertices v_i and v_k is the soft rough neutrosophic path which has maximum strength from v_i and v_k, denoted by $CONN_G(v_i,v_k)$ or $E^{\infty}(v_i,v_k)$, is called strength of connectedness from v_i and v_k.*

Definition 11. *A soft rough neutrosophic graph is a cycle if and only if the underlying graphs of each approximation is a cycle. A soft rough neutrosophic cycle is a soft rough neutrosophic graph if and only if the supporting graph of each approximation graph is a cycle and there exist $v_{lm},v_{ij}\in\underline{E}^*$,$v_{lm},v_{ij}\in\overline{E}^*$ and $v_{lm}\neq v_{ij}$ such that*

$$\underline{R}(M)(v_{ij}) = \bigwedge_{v_{lm}\in \underline{E}^*-v_{ij}} \underline{R}(M)(v_{lm}),\ \overline{R}(M)(v_{ij}) = \bigwedge_{v_{lm}\in \overline{E}^*-v_{ij}} \overline{R}(M)(v_{lm}).$$

Equivalently, each approximation graph is a cycle such that

$$\underline{R}(M)(v_{ij})=\Big(\bigwedge_{v_{lm}\in \underline{E}^*-v_{ij}} T_{\underline{R}(M)}(v_{lm}), \bigvee_{v_{lm}\in \underline{E}^*-v_{ij}} I_{\underline{R}(M)}(v_{lm}), \bigvee_{v_{lm}\in \underline{E}^*-v_{ij}} F_{\underline{R}(M)}(v_{lm})\Big),$$
$$\overline{R}(M)(v_{ij})=\Big(\bigwedge_{v_{lm}\in \overline{E}^*-v_{ij}} T_{\overline{R}(M)}(v_{lm}), \bigvee_{v_{lm}\in \overline{E}^*-v_{ij}} I_{\overline{R}(M)}(v_{lm}), \bigvee_{v_{lm}\in \overline{E}^*-v_{ij}} F_{\overline{R}(M)}(v_{lm})\Big).$$

Example 4. *Let $V=\{v_1,v_2,v_3,v_4\}$ be a vertex set and $A=\{a_1,a_2,a_3,a_4\}$ a set of parameters. A relation S over $A\times V$ can be defined in tabular form in Table 7 as follows:*

Table 7. Full soft set S.

S	v_1	v_2	v_3	v_4
a_1	1	1	1	1
a_2	0	1	0	1
a_3	1	0	1	1
a_4	1	0	1	0

Let $N=\{(v_1,0.3,0.4,0.6),(v_2,0.4,0.5,0.1),(v_3,0.9,0.6,0.4),(v_4,1.0,0.2,0.1)\}$ be a neutrosophic set on V. Then from Equation (1) of Definition 1, we have

$$\overline{S}(N) = \{(v_1,0.9,0.4,0.4),(v_2,1.0,0.2,0.1),(v_3,0.9,0.4,0.4),(v_4,1.0,0.2,0.1)\},$$
$$\underline{S}(N) = \{(v_1,0.3,0.6,0.6),(v_2,0.4,0.5,0.1),(v_3,0.3,0.6,0.6),(v_4,0.4,0.5,0.1)\}.$$

Hence, $SN = (\underline{S}(N),\overline{S}(N))$ is soft rough neutrosophic set on V. Let $E=\{v_{13},v_{32},v_{24},v_{41}\}\subseteq\tilde{V}$ and $L=\{a_{13},a_{32},a_{43}\}\subseteq\tilde{A}$. Then a full soft set R on E (from L to E) can be defined in Table 8 as follows:

Definition 13. *A $H=(SN_2,RM_2)$ is called soft rough neutrosophic spanning subgraph of a soft rough neutrosophic graph $G=(SN_1,RM_1)$, if H is a soft rough neutrosophic subgraph such that*

$$T_{\underline{S}(N_2)}(v)=T_{\underline{S}(N_1)}(v),\ I_{\underline{S}(N_2)}(v)=I_{\underline{S}(N_1)}(v),\ F_{\underline{S}(N_2)}(v)=F_{\underline{S}(N_1)}(v),$$
$$T_{\overline{S}(N_2)}(v)=T_{\overline{S}(N_1)}(v),\ I_{\overline{S}(N_2)}(v)=I_{\overline{S}(N_1)}(v),\ F_{\overline{S}(N_2)}(v)=F_{\overline{S}(N_1)}(v).$$

Definition 14. *A soft rough neutrosophic graph is a forest if and only if each supporting approximation graph is a forest. A soft rough neutrosophic graph $G=(SN_1,RM_1)$ is a soft rough neutrosophic forest if and only if there exist a soft rough neutrosophic spanning subgraph $H=(SN_1,RM_2)$ is a forest such that $v_{ij}\in G-H$*

$$T_{\underline{R}(M_1)}(v_{ij})<T_{CONN_{\underline{H}}}(v_i,v_j),\ I_{\underline{R}(M_1)}(v_{ij})>I_{CONN_{\underline{H}}}(v_i,v_j),\ F_{\underline{R}(M_1)}(v_{ij})>F_{CONN_{\underline{H}}}(v_i,v_j),$$
$$T_{\overline{R}(M_1)}(v_{ij})<T_{CONN_{\overline{H}}}(v_i,v_j),\ I_{\overline{R}(M_1)}(v_{ij})>I_{CONN_{\overline{H}}}(v_i,v_j),\ F_{\overline{R}(M_1)}(v_{ij})>F_{CONN_{\overline{H}}}(v_i,v_j).$$

A soft rough neutrosophic graph is a tree if and only if each supporting approximation graph is a tree. A soft rough neutrosophic graph $G=(SN_1,RM_1)$ is a soft rough neutrosophic tree if and only if there exist a soft rough neutrosophic spanning subgraph $H=(SN_1,RM_2)$ is a tree such that $v_{ij}\in G-H$

$$T_{\underline{R}(M_1)}(v_{ij})<T_{CONN_{\underline{H}}}(v_i,v_j),\ I_{\underline{R}(M_1)}(v_{ij})>I_{CONN_{\underline{H}}}(v_i,v_j),\ F_{\underline{R}(M_1)}(v_{ij})>F_{CONN_{\underline{H}}}(v_i,v_j),$$
$$T_{\overline{R}(M_1)}(v_{ij})<T_{CONN_{\overline{H}}}(v_i,v_j),\ I_{\overline{R}(M_1)}(v_{ij})>I_{CONN_{\overline{H}}}(v_i,v_j),\ F_{\overline{R}(M_1)}(v_{ij})>F_{CONN_{\overline{H}}}(v_i,v_j).$$

Definition 15. *Let $G=(SN,RM)$ be a soft rough neutrosophic graph, an edge v_{ij} is a bridge if the edge v_{ij} is a bridge in both supporting graph of $\underline{G}$ and $\overline{G}$, that is the removal of v_{ij} disconnects both the $\underline{G}$ and $\overline{G}$. An edge v_{ij} is a soft rough neutrosophic bridge in a soft rough neutrosophic graph $G=(SN,RM)$, if $v_{lm}\in G$*

$$T_{CONN_{\underline{G}-v_{ij}}}(v_l,v_m)<T_{CONN_{\underline{G}}}(v_l,v_m),\ T_{CONN_{\overline{G}-v_{ij}}}(v_l,v_m)<T_{CONN_{\overline{G}}}(v_l,v_m),$$
$$I_{CONN_{\underline{G}-v_{ij}}}(v_l,v_m)>I_{CONN\underline{G}}(v_l,v_m),\ I_{CONN_{\overline{G}-v_{ij}}}(v_l,v_m)>I_{CONN\overline{G}}(v_l,v_m),$$
$$F_{CONN\underline{G}-v_{ij}}(v_l,v_m)>F_{CONN_{\underline{G}}}(v_l,v_m),\ F_{CONN\overline{G}-v_{ij}}(v_l,v_m)>F_{CONN_{\overline{G}}}(v_l,v_m).$$

Definition 16. *Let $G=(SN_1,RM_1)$ be a soft rough neutrosophic graph then a vertex v_i in G is a cutnode(cutvertex) if it is a cutnode in each supporting graph of $\underline{G}$ and $\overline{G}$. That is, the deletion of v_i from the supporting graphs of $\underline{G}$ and $\overline{G}$ increase the components in the supporting graphs. A vertex v_i is called soft rough neutrosophic cutnode(cutvertex) in a soft rough neutrosophic graph if the removal of v_i reduces the strength of the connectedness from nodes $v_j to v_k\in\underline{V}^*,\overline{V}^*$*

$$T_{CONN_{\underline{G}-v_i}}(v_j,v_k)<T_{CONN_{\underline{G}}}(v_j,v_k),\ T_{CONN_{\overline{G}-v_i}}(v_j,v_k)<T_{CONN_{\overline{G}}}(v_j,v_k),$$
$$I_{CONN_{\underline{G}-v_i}}(v_j,v_k)>I_{CONN_{\underline{G}}}(v_j,v_k),\ I_{CONN_{\overline{G}-v_i}}(v_j,v_k)>I_{CONN_{\overline{G}}}(v_j,v_k),$$
$$F_{CONN_{\underline{G}-v_i}}(v_j,v_k)>F_{CONN_{\underline{G}}}(v_j,v_k),\ F_{CONN_{\overline{G}-v_i}}(v_j,v_k)>F_{CONN_{\overline{G}}}(v_j,v_k).$$

Definition 17. *An edge v_{ij} in soft rough neutrosophic graph G is called strong soft rough neutrosophic edge if*

$$T_{\underline{R}(M)}(v_{ij})\geq T_{CONN_{\underline{G}-v_{ij}}}(v_i,v_j),\ T_{\overline{R}(M)}(v_{ij})\geq T_{CONN_{\overline{G}-v_{ij}}}(v_i,v_j),$$
$$I_{\underline{R}(M)}(v_{ij})\leq I_{CONN_{\underline{G}-v_{ij}}}(v_i,v_j),\ I_{\overline{R}(M)}(v_{ij})\leq I_{CONN_{\overline{G}-v_{ij}}}(v_i,v_j),$$
$$F_{\underline{R}(M)}(v_{ij})\leq F_{CONN_{\underline{G}-v_{ij}}}(v_i,v_j),\ F_{\overline{R}(M)}(v_{ij})\leq F_{CONN_{\overline{G}-v_{ij}}}(v_i,v_j).$$

Definition 18. *An edge v_{ij} in soft rough neutrosophic graph G is called $\alpha-$strong soft rough neutrosophic edge if*

$$T_{\underline{R}(M)}(v_{ij}) > T_{CONN_{\underline{G}-v_{ij}}}(v_i, v_j),\ T_{\overline{R}(M)}(v_{ij}) > T_{CONN_{\overline{G}-v_{ij}}}(v_i, v_j),$$
$$I_{\underline{R}(M)}(v_{ij}) < I_{CONN_{\underline{G}-v_{ij}}}(v_i, v_j),\ I_{\overline{R}(M)}(v_{ij}) < I_{CONN_{\overline{G}-v_{ij}}}(v_i, v_j),$$
$$F_{\underline{R}(M)}(v_{ij}) < F_{CONN_{\underline{G}-v_{ij}}}(v_i, v_j),\ F_{\overline{R}(M)}(v_{ij}) < F_{CONN_{\overline{G}-v_{ij}}}(v_i, v_j).$$

Definition 19. *An edge v_{ij} in soft rough neutrosophic graph G is called $\beta-$strong soft rough neutrosophic edge if*

$$T_{\underline{R}(M)}(v_{ij}) = T_{CONN_{\underline{G}-v_{ij}}}(v_i, v_j),\ T_{\overline{R}(M)}(v_{ij}) = T_{CONN_{\overline{G}-v_{ij}}}(v_i, v_j),$$
$$I_{\underline{R}(M)}(v_{ij}) = I_{CONN_{\underline{G}-v_{ij}}}(v_i, v_j),\ I_{\overline{R}(M)}(v_{ij}) = I_{CONN_{\overline{G}-v_{ij}}}(v_i, v_j),$$
$$F_{\underline{R}(M)}(v_{ij}) = F_{CONN_{\underline{G}-v_{ij}}}(v_i, v_j),\ F_{\overline{R}(M)}(v_{ij}) = F_{CONN_{\overline{G}-v_{ij}}}(v_i, v_j).$$

Definition 20. *An edge v_{ij} in soft rough neutrosophic graph G is called $\delta-$strong soft rough neutrosophic edge if*

$$T_{\underline{R}(M)}(v_{ij}) < T_{CONN_{\underline{G}-v_{ij}}}(v_i, v_j),\ T_{\overline{R}(M)}(v_{ij}) < T_{CONN_{\overline{G}-v_{ij}}}(v_i, v_j),$$
$$I_{\underline{R}(M)}(v_{ij}) > I_{CONN_{\underline{G}-v_{ij}}}(v_i, v_j),\ I_{\overline{R}(M)}(v_{ij}) > I_{CONN_{\overline{G}-v_{ij}}}(v_i, v_j),$$
$$F_{\underline{R}(M)}(v_{ij}) > F_{CONN_{\underline{G}-v_{ij}}}(v_i, v_j),\ F_{\overline{R}(M)}(v_{ij}) > F_{CONN_{\overline{G}-v_{ij}}}(v_i, v_j).$$

Example 5. *Let $V=\{v_1,v_2,v_3,v_4\}$ be a vertex set and $A=\{a_1,a_2,a_3,a_4\}$ a set of parameters. A relation S over $A \times V$ can be defined in tabular form in Table 9 as follows:*

Table 9. Full soft set S.

S	v_1	v_2	v_3	v_4
a_1	1	1	1	1
a_2	0	1	0	1
a_3	1	0	1	1
a_4	1	0	1	0

Let $N=\{(v_1,0.3,0.4,0.6),(v_2,0.4,0.5,0.1),(v_3,0.9,0.6,0.4),(v_4,1.0,0.2,0.1)\}$ be a neutrosophic set on V. Then from Equation (1) of Definition 1, we have

$$\overline{S}(N) = \{(v_1, 0.9, 0.4, 0.4), (v_2, 1.0, 0.2, 0.1), (v_3, 0.9, 0.4, 0.4), (v_4, 1.0, 0.2, 0.1)\},$$
$$\underline{S}(N) = \{(v_1, 0.3, 0.6, 0.6), (v_2, 0.4, 0.5, 0.1), (v_3, 0.3, 0.6, 0.6), (v_4, 0.4, 0.5, 0.1)\}.$$

Hence, $SN=(\underline{S}(N),\overline{S}(N))$ is soft rough neutrosophic set on V. Let $E=\{v_{13},v_{32},v_{43}\}\subseteq\tilde{V}$ and $L=\{a_{12},a_{24},a_{34}\}\subseteq\tilde{A}$. Then a full soft set R on E (from L to E) can be defined in Table 10 as follows:

Table 10. Full soft set R.

R	v_{13}	v_{32}	v_{43}
a_{12}	0	1	0
a_{24}	1	0	1
a_{34}	0	0	1

Example 6. *Let $V=\{v_1,v_2,v_3,v_4,v_5,v_6\}$ be a vertex set and $A=\{a_1,a_2,a_3,a_4\}$ a set of parameters. A full soft set S over $A \times V$ can be defined in tabular form in Table 11 as follows:*

Table 11. Full soft set S.

S	v_1	v_2	v_3	v_4	v_5	v_6
a_1	1	1	1	0	0	1
a_2	0	1	0	0	1	1
a_3	1	0	1	1	1	1
a_4	1	1	1	1	1	1

Let $N=\{(v_1,1.0,0.4,0.7),(v_2,0.9,0.6,0.55),(v_3,0.7,0.9,0.5),(v_4,0.6,0.5,0.6), (v_5,0.65,0.8,0.65),(v_6,0.8,0.7,0.8)\}$ be a neutrosophic set on V. Then from Equation (1) of Definition 1, we have

$$\overline{S}(N)=\{(v_1,1.0,0.4,0.50),(v_2,0.9,0.6,0.55),(v_3,1.0,0.4,0.5),(v_4,1.0,0.4,0.5),(v_5,0.9,0.6,0.55),(v_6,0.9,0.6,0.55)\},$$
$$\underline{S}(N)=\{(v_1,0.7,0.9,0.80),(v_2,0.7,0.8,0.80),(v_3,0.7,0.9,0.8),(v_4,0.6,0.9,0.8),(v_5,0.65,0.8,0.8)(v_6,0.7,0.8,0.8)\}.$$

Hence, $SN=(\underline{S}(N),\overline{S}(N))$ is soft rough neutrosophic set on V. Let $E=\{v_{12},v_{24},v_{32},v_{42},v_{52},v_{62},\}\subseteq\tilde{V}$ and $L=\{a_{13},a_{24},a_{34},a_{41}\}\subseteq\tilde{A}$. Then a full soft set R on E (from L to E) can be defined in Table 12 as follows:

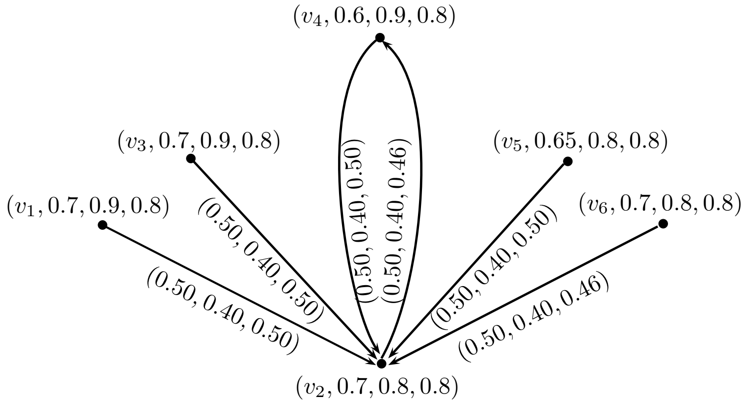
Table 12. Full soft set R.

R	v_{12}	v_{24}	v_{32}	v_{42}	v_{52}	v_{62}
a_{13}	0	1	0	0	0	1
a_{24}	0	1	0	0	1	1
a_{34}	1	0	1	1	1	1
a_{41}	1	1	1	1	1	1

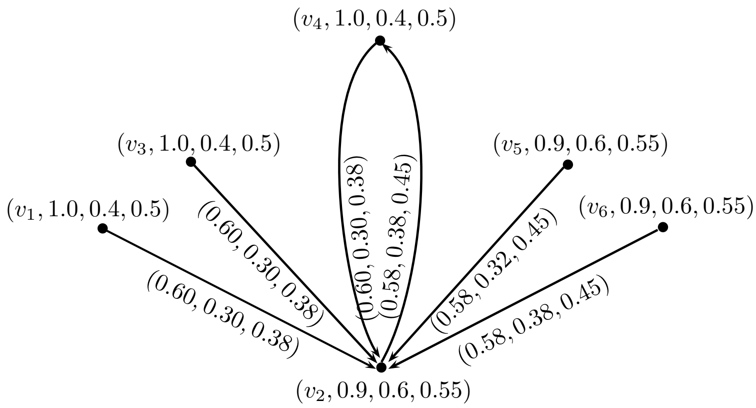
Let $M=\{(v_{12},0.6,0.3,0.4),(v_{24},0.58,0.38,0.46),(v_{32},0.56,0.37,0.47),(v_{42},0.54,0.34,0.38), (v_{52},0.52,0.32,0.5),(v_{62},0.5,0.4,0.45)\}$ be a neutrosophic set on E. Then from Equation (2) of Definition 2, we have

$$\overline{R}(M)=\{(v_{12},0.60,0.30,0.38),(v_{24},0.58,0.38,0.45),(v_{32},0.60,0.30,0.38),(v_{42},0.60,0.30,0.38),(v_{52},0.58,0.32,0.45),(v_{62},0.58,0.38,0.45)\},$$
$$\underline{R}(M)=\{(v_{12},0.50,0.40,0.50),(v_{24},0.50,0.40,0.46),(v_{32},0.50,0.40,0.50),(v_{42},0.50,0.40,0.50),(v_{52},0.50,0.40,0.50),(v_{62},0.50,0.40,0.46)\}.$$

Hence, $RM=(\underline{R}(M),\overline{R}(M))$ is soft rough neutrosophic set on E. Thus, $\underline{G}=(\underline{S}(N),\underline{R}(M))$ and $\overline{G}=(\overline{S}(N),\overline{R}(M))$ are LSRNAG and USRNAG, respectively, as shown in Figure 4. Hence, $G=(\underline{G},\overline{G})$ is SRNG. Let $I=\{v_1v_{24},v_1v_{32},v_1v_{42},v_1v_{52},v_1v_{62},v_3v_{12},v_3v_{24}, v_3v_{42},v_3v_{52},v_3v_{62},v_4v_{12},v_4v_{32},v_4v_{52},v_4v_{62},v_5v_{12}, v_5v_{24},v_5v_{32},v_5v_{42},v_5v_{62},v_6v_{12},v_6v_{24},v_6v_{32},v_6v_{42},v_6v_{52}\}$ $\subseteq V \times E$ and $P=\{a_1a_{24},a_1a_{34},a_2a_{13},a_2a_{34},a_2a_{41},a_3a_{24},a_3a_{41},a_4a_{13}\}\subseteq\hat{A}$. Then and Q a neutrosophic set on I and a full soft set X on I (from P to I) can be defined in Table 13, respectively as follows:



$\underline{G} = (\underline{S}(N), \underline{R}(M))$



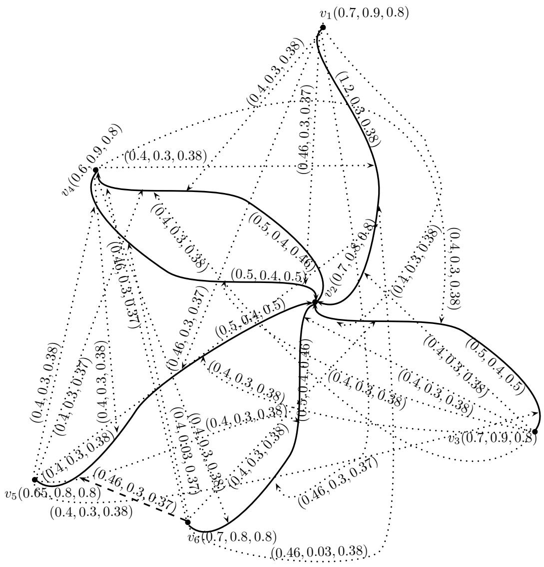
$\overline{G} = (\overline{S}(N), \overline{R}(M))$

$$
\begin{aligned}
Q=\{&(v_1v_{24}, 0.42, 0.3, 0.38), (v_1v_{32}, 0.43, 0.28, 0.37), (v_1v_{42}, 0.49, 0.26, 0.33), (v_1v_{52}, 0.47, 0.29, 0.32),\\
&(v_1v_{62}, 0.46, 0.28, 0.36), (v_3v_{12}, 0.4, 0.29, 0.37), (v_3v_{24}, 0.45, 0.24, 0.36), (v_3v_{42}, 0.48, 0.29, 0.35),\\
&(v_3v_{52}, 0.41, 0.24, 0.36), (v_3v_{62}, 0.42, 0.26, 0.34), (v_4v_{12}, 0.5, 0.25, 0.3), (v_4v_{32}, 0.44, 0.27, 0.32),\\
&(v_4v_{52}, 0.45, 0.23, 0.31), (v_4v_{62}, 0.48, 0.23, 0.38), (v_5v_{12}, 0.46, 0.24, 0.3), (v_5v_{24}, 0.47, 0.26, 0.34),\\
&(v_5v_{32}, 0.4, 0.3, 0.36), (v_5v_{42}, 0.48, 0.29, 0.38), (v_5v_{62}, 0.49, 0.3, 0.37), (v_6v_{12}, 0.49, 0.3, 0.37),\\
&(v_6v_{24}, 0.4, 0.28, 0.35), (v_6v_{32}, 0.47, 0.27, 0.34), (v_6v_{42}, 0.46, 0.29, 0.33), (v_6v_{52}, 0.49, 0.3, 0.32)\}
\end{aligned}
$$

Then the lower and upper soft rough neutrosophic approximation is directly calculated using Equation (3) of Definition 4, we have

$$
\begin{aligned}
\overline{X}(Q)=\{&(v_1v_{24}, 0.49, 0.26, 0.32), (v_1v_{32}, 0.49, 0.26, 0.32), (v_1v_{42}, 0.49, 0.26, 0.32), (v_1v_{52}, 0.49, 0.26,\\
&0.32), (v_1v_{62}, 0.49, 0.26, 0.34), (v_3v_{12}, 0.5, 0.23, 0.3), (v_3v_{24}, 0.49, 0.23, 0.34), (v_3v_{42}, 0.5,\\
&0.23, 0.3), (v_3v_{52}, 0.5, 0.23, 0.3), (v_3v_{62}, 0.49, 0.23, 0.3), (v_4v_{12}, 0.5, 0.23, 0.38), (v_4v_{32},\\
&0.5, 0.23, 0.3), (v_4v_{52}, 0.49, 0.23, 0.31), (v_4v_{62}, 0.49, 0.23, 0.34), (v_5v_{12}, 0.49, 0.24, 0.3),\\
&(v_5v_{24}, 0.49, 0.26, 0.34), (v_5v_{32}, 0.49, 0.24, 0.3), (v_5v_{42}, 0.49, 0.24, 0.3), (v_5v_{62}, 0.49, 0.26,\\
&0.34), (v_6v_{12}, 0.49, 0.26, 0.32), (v_6v_{24}, 0.49, 0.26, 0.34), (v_6v_{32}, 0.49, 0.24, 0.3), (v_6v_{42}, 0.49,\\
&0.26, 0.33), (v_6v_{52}, 0.49, 0.26, 0.32)\};\\
\underline{X}(Q)=\{&(v_1v_{24}, 0.4, 0.3, 0.38), (v_1v_{32}, 0.4, 0.3, 0.38), (v_1v_{42}, 0.46, 0.3, 0.37), (v_1v_{52}, 0.46, 0.3, 0.37),\\
&(v_1v_{62}, 0.46, 0.3, 0.37), (v_3v_{12}, 0.4, 0.3, 0.38), (v_3v_{24}, 0.4, 0.3, 0.38), (v_3v_{42}, 0.4, 0.3, 0.38),\\
&(v_3v_{52}, 0.4, 0.3, 0.38), (v_3v_{62}, 0.4, 0.3, 0.38), (v_4v_{12}, 0.4, 0.3, 0.38), (v_4v_{32}, 0.4, 0.3, 0.38),\\
&(v_4v_{52}, 0.4, 0.3, 0.38), (v_4v_{62}, 0.4, 0.3, 0.38), (v_5v_{12}, 0.4, 0.3, 0.38), (v_5v_{24}, 0.4, 0.3, 0.37),\\
&(v_5v_{32}, 0.4, 0.3, 0.38), (v_5v_{42}, 0.4, 0.3, 0.38), (v_5v_{62}, 0.4, 0.3, 0.37), (v_6v_{12}, 0.46, 0.3, 0.37),\\
&(v_6v_{24}, 0.4, 0.3, 0.37), (v_6v_{32}, 0.4, 0.3, 0.38), (v_6v_{42}, 0.46, 0.3, 0.37), (v_6v_{52}, 0.46, 0.3, 0.37)\}.
\end{aligned}
$$

Thus, $\underline{G}=(\underline{S}(N),\underline{R}(M),\underline{X}(Q))$ *and* $\overline{G}=(\overline{S}(N),\overline{R}(M),X(Q))$ *are LSRNIAG and USRNIAG, respectively, as shown in Figures 5 and 6. Hence,* $G=(\overline{G},\underline{G})$ *is SRNIG.*



where i_k=$(v_k uv)$, e_k=uv, i'_k=(vwv_{k+1}) and $\forall k$=0,1,2,$\cdots$,$n-1$. If $v_0 = v_n$, then it is called closed. If the pairs are distinct, then it is called a soft rough neutrosophic influence trail (SRNI trail). If the edges are distinct, then it is called a soft rough neutrosophic trail (SRN trail). If the vertices are distinct in SRN trail, then it is called a soft rough neutrosophic path (SRN path). If the vertices, edge and pairs are distinct in a walk of SRNIG, then it is called a soft rough neutrosophic influence path (SRNI path). A path is a trail and an influence trail. If a path in a soft rough neutrosophic influence graph is closed, then it is called a cycle.

Definition 25. *A strength of soft rough neutrosophic influence graph, denoted by stren, is defined as*

$$stren=\Big(\big(\bigwedge_{v_{jk}\in\underline{E}^*} T_{\underline{R}(M)}(v_{jk})\big)\wedge\big(\bigwedge_{v_{jk}\in\overline{E}^*} T_{\overline{R}(M)}(v_{jk})\big),\big(\bigvee_{v_{jk}\in\underline{E}^*} I_{\underline{R}(M)}(v_{jk})\big)\vee \big(\bigvee_{v_{jk}\in\overline{E}^*} I_{\overline{R}(M)}(v_{jk})\big),\big(\bigvee_{v_{jk}\in\underline{E}^*} F_{\underline{R}(M)}(v_{jk})\big)\vee\big(\bigvee_{v_{jk}\in\overline{E}^*} F_{\overline{R}(M)}(v_{jk})\big)\Big).$$

An influence strength of soft rough neutrosophic influence graph, denoted by In stren, is defined as

$$In\ stren=\Big(\big(\bigwedge_{v_iv_{jk}\in\underline{I}^*} T_{\underline{X}(Q)}(v_iv_{jk})\wedge\bigwedge_{v_iv_{jk}\in\overline{I}^*} T_{\overline{X}(Q)}(v_iv_{jk})\big),\big(\bigvee_{(v_iv_{jk})\in\underline{I}^*} I_{\underline{R}(M)}(v_iv_{jk})\vee \bigvee_{v_iv_{jk}\in\overline{I}^*} I_{\overline{R}(M)}(v_iv_{jk})\big),\big(\bigvee_{v_iv_{jk}\in\underline{I}^*} F_{\underline{R}(M)}(v_iv_{jk})\vee\bigvee_{v_iv_{jk}\in\overline{E}^*} F_{\overline{R}(M)}(v_iv_{jk})\big)\Big).$$

Definition 26. *In a soft rough neutrosophic influence graph* G*, if in each approximation graph*

$$CONN_{\underline{G}}(v_i,v_k)=\underline{E}^{\infty}(v_i,v_k)=\vee_\alpha\{\underline{E}^{\alpha}(v_i,v_k)\},\ CONN_{\overline{G}}(v_i,v_k)=\overline{E}^{\infty}(v_i,v_k)=\vee_\alpha\{\overline{E}^{\alpha}(v_i,v_k)\}.$$

where

$$\underline{E}^{\alpha}(v_i,v_k)=(\underline{E}^{\alpha-1}\circ\underline{E})(v_i,v_k),\ \overline{E}^{\alpha}(v_i,v_k)=(\overline{E}^{\alpha-1}\circ\overline{E})(v_i,v_k),$$

$$(\underline{E}\circ\underline{E})(v_i,v_k)=\Big(\bigvee_{v_j\in\underline{V}^*}(T_{\underline{R}(M)}(v_{ij})\wedge T_{\underline{R}(M)}(v_{jk})),\ \bigwedge_{v_j\in\underline{V}^*}(I_{\underline{R}(M)}(v_{ij})\vee I_{\underline{R}(M)}(v_{jk})),\ \bigwedge_{v_j\in\underline{V}^*}(F_{\underline{R}(M)}(v_{ij})\vee F_{\underline{R}(M)}(v_{jk}))\Big),$$

$$(\overline{E}\circ\overline{E})(v_i,v_k)=\Big(\bigvee_{v_j\in\overline{V}^*}(T_{\overline{R}(M)}(v_{ij})\wedge T_{\overline{R}(M)}(v_{jk})),\ \bigwedge_{v_j\in\overline{V}^*}(I_{\overline{R}(M)}(v_{ij})\vee I_{\overline{R}(M)}(v_{jk})),\ \bigwedge_{v_j\in\overline{V}^*}(F_{\overline{R}(M)}(v_{ij})\vee F_{\overline{R}(M)}(v_{jk}))\Big).$$

Thus it is the strength of strongest path from v_i to v_k in G*.*

In a soft rough neutrosophic influence graph G*, if in each approximation graph*

$$ICONN_{\underline{G}}(v_i,v_k)=\underline{I}^{\infty}(v_i,v_k)=\vee_\alpha\{\underline{I}^{\alpha}(v_i,v_k)\},\ ICONN_{\overline{G}}(v_i,v_k)=\overline{I}^{\infty}(v_i,v_k)=\vee_\alpha\{\overline{I}^{\alpha}(v_i,v_k)\}.$$

where

$$\underline{I}^{\alpha}(v_i,v_k)=(\underline{I}^{\alpha-1}\circ\underline{I})(v_i,v_k),\ \overline{I}^{\alpha}(v_i,v_k)=(\overline{I}^{\alpha-1}\circ\overline{I})(v_i,v_k),$$

and

$$(\underline{\mathrm{I}}\circ\underline{\mathrm{I}})(v_i,v_k)=\Big(\bigvee_{v_m\in\underline{\mathrm{V}}^*}(T_{\underline{X}(Q)}(v_iv_{lm})\wedge T_{\underline{X}(Q)}(v_mv_{pk})),\bigwedge_{v_m\in\underline{\mathrm{V}}^*}(I_{\underline{X}(Q)}(v_iv_{lm})\vee I_{\underline{X}(Q)}(v_mv_{pk})),\bigwedge_{v_m\in\underline{\mathrm{V}}^*}(F_{\underline{X}(Q)}(v_iv_{lm})\vee F_{\underline{X}(Q)}(v_mv_{pk}))\Big),$$

$$(\overline{\mathrm{I}}\circ\overline{\mathrm{I}})(v_i,v_k)=\Big(\bigvee_{v_m\in\overline{\mathrm{V}}^*}(T_{\overline{X}(Q)}(v_iv_{lm})\wedge T_{\overline{X}(Q)}(v_mv_{pk})),\bigwedge_{v_m\in\overline{\mathrm{V}}^*}(I_{\overline{X}(Q)}(v_iv_{lm})\vee I_{\overline{X}(Q)}(v_mv_{pk})),\bigwedge_{v_m\in\overline{\mathrm{V}}^*}(F_{\overline{X}(Q)}(v_iv_{lm})\vee F_{\overline{X}(Q)}(v_mv_{pk}))\Big).$$

Thus it is the strength of strongest path from v_i to v_k in G.

Definition 27. *A SRNIG is called connected if each two vertex v_j and v_k are joined by a SRN (SRNI) path. Maximal connected partial subgraphs in each approximation subgraph are called component.*

Definition 28. *A soft rough neutrosophic influence graph is a cycle if and only if the underlying graphs of each approximation is a cycle. A soft rough neutrosophic influence graph is a soft rough neutrosophic cycle if and only if the underlying graphs of each approximations is a cycle and there exist $v_{lm},v_{ij}\in\underline{E}^*,v_{lm},v_{ij}\in\overline{E}^*$ and $v_{lm}\neq v_{ij}$, such that*

$$\underline{R}(M)(v_{ij})=\Big(\bigwedge_{v_{lm}\in\underline{\mathrm{E}}^*-v_{ij}}T_{\underline{R}(M)}(v_{lm}),\bigvee_{v_{lm}\in\underline{\mathrm{E}}^*-v_{ij}}I_{\underline{R}(M)}(v_{lm}),\bigvee_{v_{lm}\in\underline{\mathrm{E}}^*-v_{ij}}F_{\underline{R}(M)}(v_{lm})\Big),$$

$$\overline{R}(M)(v_{ij})=\Big(\bigwedge_{v_{lm}\in\overline{\mathrm{E}}^*-v_{ij}}T_{\overline{R}(M)}(v_{lm}),\bigvee_{v_{lm}\in\overline{\mathrm{E}}^*-v_{ij}}I_{\overline{R}(M)}(v_{lm}),\bigvee_{v_{lm}\in\overline{\mathrm{E}}^*-v_{ij}}F_{\overline{R}(M)}(v_{lm})\Big).$$

A soft rough neutrosophic influence graph is a soft rough neutrosophic influence cycle if and only if the graphs is soft rough neutrosophic cycle and there exist $v_lv_{mn},v_iv_{jk}\in\underline{I}^*,v_lv_{mn},v_iv_{jk}\in\overline{I}^*$ and $v_lv_{mn}\neq v_iv_{jk}$, such that

$$\underline{X}(Q)(v_iv_{jk})=\Big(\bigwedge_{v_lv_{mn}\in\underline{\mathrm{I}}^*-v_iv_{jk}}T_{\underline{X}(Q)}(v_lv_{mn}),\bigvee_{v_lv_{mn}\in\underline{\mathrm{I}}^*-v_iv_{jk}}I_{\underline{X}(Q)}(v_lv_{mn}),\bigvee_{v_lv_{mn}\in\underline{\mathrm{I}}^*-v_iv_{jk}}F_{\underline{X}(Q)}(v_lv_{mn})\Big),$$

$$\overline{X}(Q)(v_iv_{jk})=\Big(\bigwedge_{v_lv_{mn}\in\overline{\mathrm{I}}^*-v_iv_{jk}}T_{\overline{X}(Q)}(v_lv_{mn}),\bigvee_{v_lv_{mn}\in\overline{\mathrm{I}}^*-v_iv_{jk}}I_{\overline{X}(Q)}(v_lv_{mn}),\bigvee_{v_lv_{mn}\in\overline{\mathrm{I}}^*-v_iv_{jk}}F_{\overline{X}(Q)}(v_lv_{mn})\Big).$$

Example 7. *Considering Example 4. Let $I=\{v_1v_{32},v_1v_{24},v_2v_{13},v_3v_{24},v_3v_{41},v_4v_{13},v_4v_{32}\}\subseteq\hat{V}$ and $P=\{a_1a_{32},a_2a_{43},a_4a_{13}\}\subseteq\hat{A}$. Then a full soft set X on I (from P to I) can be defined in Table 14 as follows:*

Table 14. Full soft set X.

X	v_1v_{32}	v_1v_{24}	v_2v_{13}	v_3v_{24}	v_3v_{41}	v_4v_{13}	v_4v_{32}
a_1a_{32}	1	0	0	0	0	0	1
a_2a_{43}	0	0	1	0	0	1	0
a_4a_{13}	0	1	0	1	1	0	0

$$T_{\overline{X}(Q_2)}(v_iv_{jk}) \leq T_{\overline{X}(Q_1)}(v_iv_{jk}),\ I_{\overline{X}(Q_2)}(v_iv_{jk}) \geq I_{\overline{X}(Q_1)}(v_iv_{jk}),\ F_{\overline{X}(Q_2)}(v_{ij}) \geq F_{\overline{X}(Q_1)}(v_iv_{jk}).$$

Definition 30. *A* $\mathrm{H}=(SN_2,RM_2,XQ_2)$ *is called soft rough neutrosophic influence spanning subgraph of a soft rough neutrosophic influence graph* $\mathrm{G}=(SN_1,RM_1,XQ_1)$*, if H is a soft rough neutrosophic influence subgraph such that*

$$T_{\underline{S}(N_2)}(v)=T_{\underline{S}(N_1)}(v),\ I_{\underline{S}(N_2)}(v)=I_{\underline{S}(N_1)}(v),\ F_{\underline{S}(N_2)}(v)=F_{\underline{S}(N_1)}(v),$$
$$T_{\overline{S}(N_2)}(v)=T_{\overline{S}(N_1)}(v),\ I_{\overline{S}(N_2)}(v)=I_{\overline{S}(N_1)}(v),\ F_{\overline{S}(N_2)}(v)=F_{\overline{S}(N_1)}(v).$$

Definition 31. *A soft rough neutrosophic influence graph is a forest if and only if each supporting approximation graph is a forest. A soft rough neutrosophic influence graph* $G=(SN_1,RM_1,XQ_1)$ *is a soft rough neutrosophic forest if and only if there exist a soft rough neutrosophic spanning subgraph* $\mathrm{H}=(SN_1,RM_2,XQ_2)$ *is a forest such that* $v_{ij} \in \mathrm{G}-\mathrm{H}$

$$T_{\underline{R}(M_1)}(v_{ij})<T_{CONN_{\underline{\mathrm{H}}}}(v_i,v_j),\ I_{\underline{R}(M_1)}(v_{ij})>I_{CONN_{\underline{\mathrm{H}}}}(v_i,v_j),\ F_{\underline{R}(M_1)}(v_{ij})>F_{CONN_{\underline{\mathrm{H}}}}(v_i,v_j),$$
$$T_{\overline{R}(M_1)}(v_{ij})<T_{CONN_{\overline{\mathrm{H}}}}(v_i,v_j),\ I_{\overline{R}(M_1)}(v_{ij})>I_{CONN_{\overline{\mathrm{H}}}}(v_i,v_j),\ F_{\overline{R}(M_1)}(v_{ij})>F_{CONN_{\overline{\mathrm{H}}}}(v_i,v_j).$$

A soft rough neutrosophic influence graph $\mathrm{G}=(SN_1,RM_1,XQ_1)$ *is a soft rough neutrosophic influence forest if and only if there exist a soft rough neutrosophic spanning subgraph* $\mathrm{H}=(SN_1,RM_1,XQ_2)$ *is a forest such that* $v_iv_{jk} \in \mathrm{G}-\mathrm{H}$

$$T_{\underline{X}(Q_1)}(v_iv_{jk})<T_{ICONN_{\underline{\mathrm{H}}}}(v_i,v_k),\ T_{\overline{X}(Q_1)}(v_iv_{jk})<T_{ICONN_{\overline{\mathrm{H}}}}(v_i,v_k),$$
$$I_{\underline{X}(Q_1)}(v_iv_{jk})>I_{ICONN_{\underline{\mathrm{H}}}}(v_i,v_k),\ I_{\overline{X}(Q_1)}(v_iv_{jk})>I_{ICONN_{\overline{\mathrm{H}}}}(v_i,v_k),$$
$$F_{\underline{X}(Q_1)}(v_iv_{jk})>F_{ICONN_{\underline{\mathrm{H}}}}(v_i,v_k),\ F_{\overline{X}(Q_1)}(v_iv_{jk})>F_{ICONN_{\overline{\mathrm{H}}}}(v_i,v_k).$$

Definition 32. *A soft rough neutrosophic influence graph is a tree if and only if each supporting approximation graph is a tree. A soft rough neutrosophic influence graph* $G=(SN_1,RM_1,XQ_1)$ *is a soft rough neutrosophic tree if and only if there exist a soft rough neutrosophic spanning subgraph* $\mathrm{H}=(SN_1,RM_2,XQ_2)$ *is a tree such that* $v_{ij} \in \mathrm{G}-\mathrm{H}$

$$T_{\underline{R}(M_1)}(v_{ij})<T_{CONN_{\underline{\mathrm{H}}}}(v_i,v_j),\ I_{\underline{R}(M_1)}(v_{ij})>I_{CONN_{\underline{\mathrm{H}}}}(v_i,v_j),\ F_{\underline{R}(M_1)}(v_{ij})>F_{CONN_{\underline{\mathrm{H}}}}(v_i,v_j),$$
$$T_{\overline{R}(M_1)}(v_{ij})<T_{CONN_{\overline{\mathrm{H}}}}(v_i,v_j),\ I_{\overline{R}(M_1)}(v_{ij})>I_{CONN_{\overline{\mathrm{H}}}}(v_i,v_j),\ F_{\overline{R}(M_1)}(v_{ij})>F_{CONN_{\overline{\mathrm{H}}}}(v_i,v_j).$$

A soft rough neutrosophic influence graph $\mathrm{G}=(SN_1,RM_1,XQ_1)$ *is a soft rough neutrosophic influence tree if and only if there exist a soft rough neutrosophic spanning subgraph* $\mathrm{H}=(SN_1,RM_1,XQ_2)$ *is a tree such that* $v_iv_{jk} \in \mathrm{G}-\mathrm{H}$

$$T_{\underline{X}(Q_1)}(v_iv_{jk})<T_{ICONN_{\underline{\mathrm{H}}}}(v_i,v_k),\ T_{\overline{X}(Q_1)}(v_iv_{jk})<T_{ICONN_{\overline{\mathrm{H}}}}(v_i,v_k),$$
$$I_{\underline{X}(Q_1)}(v_iv_{jk})>I_{ICONN_{\underline{\mathrm{H}}}}(v_i,v_k),\ I_{\overline{X}(Q_1)}(v_iv_{jk})>I_{ICONN_{\overline{\mathrm{H}}}}(v_i,v_k),$$
$$F_{\underline{X}(Q_1)}(v_iv_{jk})>F_{ICONN_{\underline{\mathrm{H}}}}(v_i,v_k),\ F_{\overline{X}(Q_1)}(v_iv_{jk})>F_{ICONN_{\overline{\mathrm{H}}}}(v_i,v_k).$$

Definition 33. *Let* $\mathrm{G}=(SN,RM,XQ)$ *be a soft rough neutrosophic influence graph, an edge* v_{ij} *is a bridge if edge* v_{ij} *is a bridge in both underlying graphs of* $\underline{\mathrm{G}}$ *and* $\overline{\mathrm{G}}$*. Let* $\mathrm{G}=(SN,RM,XQ)$ *be a soft rough neutrosophic influence graph, an edge* v_{ij} *is a soft rough neutrosophic bridge if* $v_{lm} \in \mathrm{G}$

$$T_{CONN_{\underline{G}-v_{ij}}}(v_l,v_m)<T_{CONN_{\underline{G}}}(v_l,v_m),\ T_{CONN_{\overline{G}-v_{ij}}}(v_l,v_m)<T_{CONN_{\overline{G}}}(v_l,v_m),$$
$$I_{CONN_{\underline{G}-v_{ij}}}(v_l,v_m)>I_{CONN_{\underline{G}}}(v_l,v_m),\ I_{CONN_{\overline{G}-v_{ij}}}(v_l,v_m)>I_{CONN_{\overline{G}}}(v_l,v_m),$$
$$F_{CONN_{\underline{G}-v_{ij}}}(v_l,v_m)>F_{CONN_{\underline{G}}}(v_l,v_m),\ F_{CONN_{\overline{G}-v_{ij}}}(v_l,v_m)>F_{CONN_{\overline{G}}}(v_l,v_m),$$

Let G=(SN,RM,XQ) *be a soft rough neutrosophic influence graph, an edge* v_{ij} *is an soft rough neutrosophic influence bridge if* $v_{lm}\in$G

$$T_{ICONN_{\underline{G}-v_{ij}}}(v_l,v_m)<T_{ICONN_{\underline{G}}}(v_l,v_m),\ T_{ICONN_{\overline{G}-v_{ij}}}(v_l,v_m)<T_{ICONN_{\overline{G}}}(v_l,v_m),$$
$$I_{ICONN_{\underline{G}-v_{ij}}}(v_l,v_m)>I_{ICONN_{\underline{G}}}(v_l,v_m),\ ;I_{ICONN_{\overline{G}-v_{ij}}}(v_l,v_m)>I_{ICONN_{\overline{G}}}(v_l,v_m),$$
$$F_{ICONN_{\underline{G}-v_{ij}}}(v_l,v_m)>F_{ICONN_{\underline{G}}}(v_l,v_m),\ F_{ICONN_{\overline{G}-v_{ij}}}(v_l,v_m)>F_{ICONN_{\overline{G}}}(v_l,v_m),$$

Definition 34. *Let* G=(SN,RM,XQ) *be a soft rough neutrosophic influence graph, a vertex is a cutnode if a vertex* v_i *is a cutnode in underlying graphs of* $\underline{G}$ *and* $\overline{G}$. *Let* G=(SN,RM,XQ) *be a soft rough neutrosophic influence graph then a vertex* v_i *in* G *is a soft rough neutrosophic cutnode if the deletion of* v_i *from* G *reduces the strength of the connectedness from nodes* $v_j\rightarrow v_k\in\underline{V}^*,\overline{V}^*$

$$T_{CONN_{\underline{G}-v_i}}(v_j,v_k)<T_{CONN_{\underline{G}}}(v_j,v_k),\ T_{CONN_{\overline{G}-v_i}}(v_j,v_k)<T_{CONN_{\overline{G}}}(v_j,v_k),$$
$$I_{CONN_{\underline{G}-v_i}}(v_j,v_k)>I_{CONN_{\underline{G}}}(v_j,v_k),\ I_{CONN_{\overline{G}-v_i}}(v_j,v_k)>I_{CONN_{\overline{G}}}(v_j,v_k),$$
$$F_{CONN_{\underline{G}-v_i}}(v_j,v_k)>F_{CONN_{\underline{G}}}(v_j,v_k),\ F_{CONN_{\overline{G}-v_i}}(v_j,v_k)>F_{CONN_{\overline{G}}}(v_j,v_k).$$

Let G=(SN,RM,XQ) *be a soft rough neutrosophic influence graph then a vertex* v_i *in* G *is an neutrosophic influence cutnode if the deletion of* v_i *from* G *reduces the influence strength of the connectedness from* $v_j\rightarrow v_k\in\underline{V}^*,\overline{V}^*$

$$T_{ICONN_{\underline{G}-v_i}}(v_j,v_k)<T_{ICONN_{\underline{G}}}(v_j,v_k),\ T_{ICONN_{\overline{G}-v_i}}(v_j,v_k)<T_{ICONN_{\overline{G}}}(v_j,v_k),$$
$$I_{ICONN_{\underline{G}-v_i}}(v_j,v_k)>I_{ICONN_{\underline{G}}}(v_j,v_k),\ I_{ICONN_{\overline{G}-v_i}}(v_j,v_k)>I_{ICONN_{\overline{G}}}(v_j,v_k),$$
$$F_{ICONN_{\underline{G}-v_i}}(v_j,v_k)>F_{ICONN_{\underline{G}}}(v_j,v_k),\ F_{ICONN_{\overline{G}-v_i}}(v_j,v_k)>F_{ICONN_{\overline{G}}}(v_j,v_k),$$

Definition 35. *Let* G=(SN,RM,XQ) *be a soft rough neutrosophic influence graph. A pair* v_iv_{jk} *is called a cutpair if and only if* v_iv_{jk} *is a cutpair in both supporting influence graph of* $\underline{G}$ *and* $\overline{G}$. *That is after removing the pair* v_iv_{jk} *there is no path from* v_i *to* v_k *in both supporting influence graph of* $\underline{G}$ *and* $\overline{G}$. *Let* G=(SN,RM,XQ) *be a soft rough neutrosophic influence graph. A pair* v_iv_{jk} *is called a soft rough neutrosophic cutpair if and only if if deleting the pair* v_iv_{jk} *reduces the connectedness from* v_i *to* v_k *in both graph* $\underline{G}$ *and* $\overline{G}$. *That is,*

$$T_{CONN_{\underline{G}-v_iv_{jk}}}(v_i,v_k)<T_{CONN_{\underline{G}}}(v_i,v_k),\ T_{ICONN_{\overline{G}-v_iv_{jk}}}(v_i,v_k)<T_{ICONN_{\overline{G}}}(v_i,v_k),$$
$$I_{CONN_{\underline{G}-v_iv_{jk}}}(v_i,v_k)>I_{CONN_{\underline{G}}}(v_i,v_k),\ I_{ICONN_{\overline{G}-v_iv_{jk}}}(v_i,v_k)>I_{ICONN_{\overline{G}}}(v_i,v_k),$$
$$F_{CONN_{\underline{G}-v_iv_{jk}}}(v_i,v_k)>F_{CONN_{\underline{G}}}(v_i,v_k),\ F_{ICONN_{\overline{G}-v_iv_{jk}}}(v_i,v_k)>F_{ICONN_{\overline{G}}}(v_i,v_k),$$

A soft rough neutrosophic influence cutpair v_iv_{jk} is a pair in a soft rough neutrosophic influence graph G=(SN,RM,XQ) *if there is spanning influence subgraph* $\mathrm{H} = G - v_iv_{jk}$ *reduces the strength of the influence connectedness from v_i to v_k. That is,*

$$T_{ICONN_{\underline{G}-v_iv_{jk}}}(v_i,v_k)<T_{ICONN_{\underline{G}}}(v_i,v_k),\ T_{ICONN_{\overline{G}-v_iv_{jk}}}(v_i,v_k)<T_{ICONN_{\overline{G}}}(v_i,v_k),$$
$$I_{ICONN_{\underline{G}-v_iv_{jk}}}(v_i,v_k)>I_{ICONN_{\underline{G}}}(v_i,v_k),\ I_{ICONN_{\overline{G}-v_iv_{jk}}}(v_i,v_k)>I_{ICONN_{\overline{G}}}(v_i,v_k),$$
$$F_{ICONN_{\underline{G}-v_iv_{jk}}}(v_i,v_k)>F_{ICONN_{\underline{G}}}(v_i,v_k),\ F_{ICONN_{\overline{G}-v_iv_{jk}}}(v_i,v_k)>F_{ICONN_{\overline{G}}}(v_i,v_k),$$

Definition 36. *An edge v_{ij} in soft rough neutrosophic influence graph* G *is called strong soft rough neutrosophic edge if*

$$T_{\underline{R}(M)}(v_{ij})\geq T_{CONN_{\underline{G}-v_{ij}}}(v_i,v_j),\ T_{\overline{R}(M)}(v_{ij})\geq T_{CONN_{\overline{G}-v_{ij}}}(v_i,v_j),$$
$$I_{\underline{R}(M)}(v_{ij})\leq I_{CONN_{\underline{G}-v_{ij}}}(v_i,v_j),\ I_{\overline{R}(M)}(v_{ij})\leq I_{CONN_{\overline{G}-v_{ij}}}(v_i,v_j),$$
$$F_{\underline{R}(M)}(v_{ij})\leq F_{CONN_{\underline{G}-v_{ij}}}(v_i,v_j),\ F_{\overline{R}(M)}(v_{ij})\leq F_{CONN_{\overline{G}-v_{ij}}}(v_i,v_j).$$

A pair v_iv_{jk} in soft rough neutrosophic influence graph G *is called strong pair if*

$$T_{\underline{X}(Q)}(v_iv_{jk})\geq T_{ICONN_{\underline{G}-v_iv_{jk}}}(v_i,v_k),\ T_{\overline{X}(Q)}(v_iv_{jk})\geq T_{ICONN_{\overline{G}-v_iv_{jk}}}(v_i,v_k),$$
$$I_{\underline{X}(Q)}(v_iv_{jk})\leq I_{ICONN_{\underline{G}-v_iv_{jk}}}(v_i,v_k),\ I_{\overline{X}(Q)}(v_iv_{jk})\leq I_{ICONN_{\overline{G}-v_iv_{jk}}}(v_i,v_k),$$
$$F_{\underline{X}(Q)}(v_iv_{jk})\leq F_{ICONN_{\underline{G}-v_iv_{jk}}}(v_i,v_k),\ F_{\overline{X}(Q)}(v_iv_{jk})\leq F_{ICONN_{\overline{G}-v_iv_{jk}}}(v_i,v_k).$$

Definition 37. *An edge v_{ij} in soft rough neutrosophic influence graph* G *is called $\alpha-$strong soft rough neutrosophic edge if*

$$T_{\underline{R}(M)}(v_{ij})>T_{CONN_{\underline{G}-v_{ij}}}(v_i,v_j),\ T_{\overline{R}(M)}(v_{ij})>T_{CONN_{\overline{G}-v_{ij}}}(v_i,v_j),$$
$$I_{\underline{R}(M)}(v_{ij})<I_{CONN_{\underline{G}-v_{ij}}}(v_i,v_j),\ I_{\overline{R}(M)}(v_{ij})<I_{CONN_{\overline{G}-v_{ij}}}(v_i,v_j),$$
$$F_{\underline{R}(M)}(v_{ij})<F_{CONN_{\underline{G}-v_{ij}}}(v_i,v_j),\ F_{\overline{R}(M)}(v_{ij})<F_{CONN_{\overline{G}-v_{ij}}}(v_i,v_j).$$

A pair v_iv_{jk} in soft rough neutrosophic influence graph G *is called $\alpha-$strong pair if*

$$T_{\underline{X}(Q)}(v_iv_{jk})>T_{ICONN_{\underline{G}-v_iv_{jk}}}(v_i,v_k),\ T_{\overline{X}(Q)}(v_iv_{jk})>T_{ICONN_{\overline{G}-v_iv_{jk}}}(v_i,v_k),$$
$$I_{\underline{X}(Q)}(v_iv_{jk})<I_{ICONN_{\underline{G}-v_iv_{jk}}}(v_i,v_k),\ I_{\overline{X}(Q)}(v_iv_{jk})<I_{ICONN_{\overline{G}-v_iv_{jk}}}(v_i,v_k),$$
$$F_{\underline{X}(Q)}(v_iv_{jk})<F_{ICONN_{\underline{G}-v_iv_{jk}}}(v_i,v_k),\ F_{\overline{X}(Q)}(v_iv_{jk})<F_{ICONN_{\overline{G}-v_iv_{jk}}}(v_i,v_k).$$

Definition 38. *An edge v_{ij} in soft rough neutrosophic influence graph* G *is called $\beta-$strong soft rough neutrosophic edge if*

$$T_{\underline{R}(M)}(v_{ij})=T_{CONN_{\underline{G}-v_{ij}}}(v_i,v_j),\ T_{\overline{R}(M)}(v_{ij})=T_{CONN_{\overline{G}-v_{ij}}}(v_i,v_j),$$
$$I_{\underline{R}(M)}(v_{ij})=I_{CONN_{\underline{G}-v_{ij}}}(v_i,v_j),\ I_{\overline{R}(M)}(v_{ij})=I_{CONN_{\overline{G}-v_{ij}}}(v_i,v_j),$$
$$F_{\underline{R}(M)}(v_{ij})=F_{CONN_{\underline{G}-v_{ij}}}(v_i,v_j),\ F_{\overline{R}(M)}(v_{ij})=F_{CONN_{\overline{G}-v_{ij}}}(v_i,v_j).$$

A pair v_iv_{jk} in soft rough neutrosophic influence graph G *is called $\beta-$strong pair if*

$$T_{\underline{X}(Q)}(v_iv_{jk})=T_{ICONN_{\underline{G}-v_iv_{jk}}}(v_i,v_k),\ T_{\overline{X}(Q)}(v_iv_{jk})=T_{ICONN_{\overline{G}-v_iv_{jk}}}(v_i,v_k),$$
$$I_{\underline{X}(Q)}(v_iv_{jk})=I_{ICONN_{\underline{G}-v_iv_{jk}}}(v_i,v_k),\ I_{\overline{X}(Q)}(v_iv_{jk})=I_{ICONN_{\overline{G}-v_iv_{jk}}}(v_i,v_k),$$
$$F_{\underline{X}(Q)}(v_iv_{jk})=F_{ICONN_{\underline{G}-v_iv_{jk}}}(v_i,v_k),\ F_{\overline{X}(Q)}(v_iv_{jk})=F_{ICONN_{\overline{G}-v_iv_{jk}}}(v_i,v_k).$$

Definition 39. *An edge v_{ij} in soft rough neutrosophic influence graph* G *is called $\delta-$strong soft rough neutrosophic edge if*

$$T_{\underline{R}(M)}(v_{ij})<T_{CONN_{\underline{G}-v_{ij}}}(v_i,v_j),\ T_{\overline{R}(M)}(v_{ij})<T_{CONN_{\overline{G}-v_{ij}}}(v_i,v_j),$$
$$I_{\underline{R}(M)}(v_{ij})>I_{CONN_{\underline{G}-v_{ij}}}(v_i,v_j),\ I_{\overline{R}(M)}(v_{ij})>I_{CONN_{\overline{G}-v_{ij}}}(v_i,v_j),$$
$$F_{\underline{R}(M)}(v_{ij})>F_{CONN_{\underline{G}-v_{ij}}}(v_i,v_j),\ F_{\overline{R}(M)}(v_{ij})>F_{CONN_{\overline{G}-v_{ij}}}(v_i,v_j).$$

A pair v_iv_{jk} in soft rough neutrosophic influence graph G *is called $\delta-$strong pair if*

$$T_{\underline{X}(Q)}(v_iv_{jk})<T_{ICONN_{\underline{G}-v_iv_{jk}}}(v_i,v_k),\ T_{\overline{X}(Q)}(v_iv_{jk})<T_{ICONN_{\overline{G}-v_iv_{jk}}}(v_i,v_k),$$
$$I_{\underline{X}(Q)}(v_iv_{jk})>I_{ICONN_{\underline{G}-v_iv_{jk}}}(v_i,v_k),\ I_{\overline{X}(Q)}(v_iv_{jk})>I_{ICONN_{\overline{G}-v_iv_{jk}}}(v_i,v_k),$$
$$F_{\underline{X}(Q)}(v_iv_{jk})>F_{ICONN_{\underline{G}-v_iv_{jk}}}(v_i,v_k),\ F_{\overline{X}(Q)}(v_iv_{jk})>F_{ICONN_{\overline{G}-v_iv_{jk}}}(v_i,v_k).$$

Definition 40. *A $\delta-$strong soft rough neutrosophic edge v_{ij} is called a δ^*-strong soft rough neutrosophic edge if*

$$T_{\underline{R}(M)}(v_{ij})>\bigwedge_{v_{lm}\in\underline{E}^*}T_{\underline{R}(M)}(v_{lm}),\ T_{\overline{R}(M)}(v_{ij})>\bigwedge_{v_{lm}\in\overline{E}^*}T_{\overline{R}(M)}(v_{lm}),$$
$$I_{\underline{R}(M)}(v_{ij})<\bigwedge_{v_{lm}\in\underline{E}^*}I_{\underline{R}(M)}(v_{lm}),\ I_{\overline{R}(M)}(v_{ij})<\bigwedge_{v_{lm}\in\overline{E}^*}I_{\overline{R}(M)}(v_{lm}),$$
$$F_{\underline{R}(M)}(v_{ij})<\bigwedge_{v_{lm}\in\underline{E}^*}F_{\underline{R}(M)}(v_{lm}),\ F_{\overline{R}(M)}(v_{ij})<\bigwedge_{v_{lm}\in\overline{E}^*}F_{\overline{R}(M)}(v_{lm}).$$

A $\delta-$strong pair v_iv_{jk} is called a δ^*-strong pair if

$$T_{\underline{R}(M)}(v_{ij})>\bigwedge_{v_{lm}\in\underline{\mathrm{E}}^*}T_{\underline{R}(M)}(v_{lm}),\ T_{\overline{R}(M)}(v_{ij})>\bigwedge_{v_{lm}\in\overline{\mathrm{E}}^*}T_{\overline{R}(M)}(v_{lm}),$$
$$I_{\underline{R}(M)}(v_{ij})<\bigwedge_{v_{lm}\in\underline{\mathrm{E}}^*}I_{\underline{R}(M)}(v_{lm}),\ I_{\overline{R}(M)}(v_{ij})<\bigwedge_{v_{lm}\in\overline{\mathrm{E}}^*}I_{\overline{R}(M)}(v_{lm}),$$
$$F_{\underline{R}(M)}(v_{ij})<\bigwedge_{v_{lm}\in\underline{\mathrm{E}}^*}F_{\underline{R}(M)}(v_{lm}),\ F_{\overline{R}(M)}(v_{ij})<\bigwedge_{v_{lm}\in\overline{\mathrm{E}}^*}F_{\overline{R}(M)}(v_{lm}).$$

A $\delta-$strong pair v_iv_{jk} is called a δ^*-strong pair if $v_iv_{jk}\neq v_lv_{mn}$

$$T_{\underline{X}(Q)}(v_iv_{jk})>\bigwedge_{v_lv_{mn}\in\underline{\mathrm{I}}^*}T_{\underline{X}(Q)}(v_lv_{mn}),\ T_{\overline{X}(Q)}(v_iv_{jk})>\bigwedge_{v_lv_{mn}\in\overline{\mathrm{I}}^*}T_{\overline{X}(Q)}(v_lv_{mn}),$$
$$I_{\underline{X}(Q)}(v_iv_{jk})<\bigwedge_{v_lv_{mn}\in\underline{\mathrm{I}}^*}I_{\underline{X}(Q)}(v_lv_{mn}),\ I_{\overline{X}(Q)}(v_iv_{jk})<\bigwedge_{v_lv_{mn}\in\overline{\mathrm{I}}^*}I_{\overline{X}(Q)}(v_lv_{mn}),$$
$$F_{\underline{X}(Q)}(v_iv_{jk})<\bigwedge_{v_lv_{mn}\in\underline{\mathrm{I}}^*}F_{\underline{X}(Q)}(v_lv_{mn}),\ F_{\overline{X}(Q)}(v_iv_{jk})<\bigwedge_{v_lv_{mn}\in\underline{\mathrm{I}}^*}F_{\overline{X}(Q)}(v_lv_{mn}).$$

Theorem 4. *G is a soft rough neutrosophic influence forest if and only if in any cycle of G, there is a pair v_iv_{jk} such that*

$$T_{\underline{X}(Q)}(v_iv_{jk})<T_{ICONN_{\underline{G}-v_iv_{jk}}}(v_i,v_k),\ T_{\overline{X}(Q)}(v_iv_{jk})<T_{ICONN_{\overline{G}-v_iv_{jk}}}(v_i,v_k),$$
$$I_{\underline{X}(Q)}(v_iv_{jk})>I_{ICONN_{\underline{G}-v_iv_{jk}}}(v_i,v_k),\ I_{\overline{X}(Q)}(v_iv_{jk})>I_{ICONN_{\overline{G}-v_iv_{jk}}}(v_i,v_k),$$
$$F_{\underline{X}(Q)}(v_iv_{jk})>F_{ICONN_{\underline{G}-v_iv_{jk}}}(v_i,v_k),\ F_{\overline{X}(Q)}(v_iv_{jk})>F_{ICONN_{\overline{G}-v_iv_{jk}}}(v_i,v_k).$$

Proof. The proof is obvious. □

Theorem 5. *A soft rough neutrosophic graph G is a soft rough neutrosophic influence forest if there is at most one path with the most influence strength.*

Proof. Let G be not a soft rough neutrosophic influence forest. Then by Theorem 4, there exist a cycle C in G such that

$$T_{\underline{X}(Q)}(v_iv_{jk})\geq T_{ICONN_{\underline{G}-v_iv_{jk}}}(v_i,v_k),\ T_{\overline{X}(Q)}(v_iv_{jk})\geq T_{ICONN_{\overline{G}-v_iv_{jk}}}(v_i,v_k),$$
$$I_{\underline{X}(Q)}(v_iv_{jk})\leq I_{ICONN_{\underline{G}-v_iv_{jk}}}(v_i,v_k),\ I_{\overline{X}(Q)}(v_iv_{jk})\leq I_{ICONN_{\overline{G}-v_iv_{jk}}}(v_i,v_k),$$
$$F_{\underline{X}(Q)}(v_iv_{jk})\leq F_{ICONN_{\underline{G}-v_iv_{jk}}}(v_i,v_k),\ F_{\overline{X}(Q)}(v_iv_{jk})\leq F_{ICONN_{\overline{G}-v_iv_{jk}}}(v_i,v_k),$$

for every pair v_iv_{jk} of C.

Therefore, v_iv_{jk} is the path within the most influence strength from v_i to v_k. Let v_iv_{jk} be a pair such that

$$T_{\underline{X}(Q)}(v_iv_{jk})>\bigwedge_{v_lv_{mn}\in\underline{I}^*}T_{\underline{X}(Q)}(v_lv_{mn}),\ T_{\overline{X}(Q)}(v_iv_{jk})>\bigwedge_{v_lv_{mn}\in\overline{I}^*}T_{\overline{X}(Q)}(v_lv_{mn}),$$
$$I_{\underline{X}(Q)}(v_iv_{jk})<\bigwedge_{v_lv_{mn}\in\underline{I}^*}I_{\underline{X}(Q)}(v_lv_{mn}),\ I_{\overline{X}(Q)}(v_iv_{jk})<\bigwedge_{v_lv_{mn}\in\overline{I}^*}I_{\overline{X}(Q)}(v_lv_{mn}),$$
$$F_{\underline{X}(Q)}(v_iv_{jk})<\bigwedge_{v_lv_{mn}\in\underline{I}^*}F_{\underline{X}(Q)}(v_lv_{mn}),\ F_{\overline{X}(Q)}(v_iv_{jk})<\bigwedge_{v_lv_{mn}\in\overline{I}^*}F_{\overline{X}(Q)}(v_lv_{mn}),$$

in C. Then remaining part of C is a path with the most influence strength from v_i to v_{jk}. This is a contradiction to the the fact there is at most one path with the most influence strength. Hence, G is a soft rough neutrosophic influence forest. □

Theorem 6. *Assume that G is a cycle. Then G is not a soft rough neutrosophic influence tree if and only if G is a soft rough neutrosophic influence cycle.*

Proof. Let G=(SN,RM,XQ_1) be a soft rough neutrosophic influence cycle. Then there exist at least two distinct edge and two distinct pair such that

$$\underline{R}(M)(v_{ij}) =\Big(\bigwedge_{v_{lm}\in\underline{E}^*-v_{ij}} T_{\underline{R}(M)}(v_{lm}), \bigvee_{v_{lm}\in\underline{E}^*-v_{ij}} I_{\underline{R}(M)}(v_{lm}), \bigvee_{v_{lm}\in\underline{E}^*-v_{ij}} F_{\underline{R}(M)}(v_{lm})\Big),$$
$$\overline{R}(M)(v_{ij}) =\Big(\bigwedge_{v_{lm}\in\overline{E}^*-v_{ij}} T_{\overline{R}(M)}(v_{lm}), \bigvee_{v_{lm}\in\overline{E}^*-v_{ij}} I_{\overline{R}(M)}(v_{lm}), \bigvee_{v_{lm}\in\overline{E}^*-v_{ij}} F_{\overline{R}(M)}(v_{lm})\Big),$$
$$\underline{X}(Q)(v_iv_{jk})=\Big(\bigwedge_{v_lv_{mn}\in\underline{I}^*-v_iv_{jk}} T_{\underline{X}(Q)}(v_lv_{mn}), \bigvee_{v_lv_{mn}\in\underline{I}^*-v_iv_{jk}} I_{\underline{X}(Q)}(v_lv_{mn}), \bigvee_{v_lv_{mn}\in\underline{I}^*-v_iv_{jk}} F_{\underline{X}(Q)}(v_lv_{mn})\Big),$$
$$\overline{X}(Q)(v_iv_{jk})=\Big(\bigwedge_{v_lv_{mn}\in\overline{I}^*-v_iv_{jk}} T_{\overline{X}(Q)}(v_lv_{mn}), \bigvee_{v_lv_{mn}\in\overline{I}^*-v_iv_{jk}} I_{\overline{X}(Q)}(v_lv_{mn}), \bigvee_{v_lv_{mn}\in\overline{I}^*-v_iv_{jk}} F_{\overline{X}(Q)}(v_lv_{mn})\Big).$$

Let H=(SN,RM,XQ_2) be a spanning soft rough neutrosophic influence tree in G. Then there exists a path from v_i to v_k not involving v_iv_{jk} such that $E_1^*-E_2^*=\{(v_iv_{jk})\}$. Hence there does not exist a path in H from v_i to v_k such that

$$T_{\underline{X}(Q_2)}(v_iv_{jk})\leq T_{ICONN_{\underline{G}}}(v_i,v_k),\ T_{\overline{X}(Q_2)}(v_iv_{jk}) \leq T_{ICONN_{\overline{G}}}(v_i,v_k),$$
$$I_{\underline{X}(Q_2)}(v_iv_{jk}) \geq I_{ICONN_{\underline{G}}}(v_i,v_k),\ I_{\overline{X}(Q_2)}(v_iv_{jk}) \geq I_{ICONN_{\overline{G}}}(v_i,v_k),$$
$$F_{\underline{X}(Q_2)}(v_iv_{jk})\geq F_{ICONN_{\underline{G}}}(v_i,v_k),\ F_{\overline{X}(Q_2)}(v_iv_{jk}) \geq F_{ICONN_{\overline{G}}}(v_i,v_k).$$

Thus G is not a soft rough neutrosophic influence tree.

Conversely, suppose that G is not a soft rough neutrosophic influence tree. Since, G is a soft rough neutrosophic influence cycle. So for all $v_iv_{jk}\in\underline{I}^*$ and $v_iv_{jk}\in\overline{I}^*$, we have a soft rough neutrosophic spanning influence subrgraph H=(SN,RM,XQ_2) which is tree and $\underline{X}(Q_2)(v_iv_{jk})$=0, $\overline{X}(Q_2)(v_iv_{jk})$=0 such that $\forall v_iv_{lm} \neq v_lv_{mn}$

$$T_{\underline{X}(Q_2)}(v_iv_{jk})\leq T_{ICONN_{\underline{H}}}(v_i,v_k),\ T_{\overline{X}(Q_2)}(v_iv_{jk}) \leq T_{ICONN_{\overline{G}}}(v_i,v_k),$$
$$I_{\underline{X}(Q_2)}(v_iv_{jk}) \geq I_{ICONN_{\underline{G}}}(v_i,v_k),\ I_{\overline{X}(Q_2)}(v_iv_{jk}) \geq I_{ICONN_{\overline{G}}}(v_i,v_k),$$
$$F_{\underline{X}(Q_2)}(v_iv_{jk})\geq F_{ICONN_{\underline{G}}}(v_i,v_k),\ F_{\overline{X}(Q_2)}(v_iv_{jk}) \geq F_{ICONN_{\overline{G}}}(v_i,v_k),$$

$\forall v_lv_{mn} \in \underline{I}^* - v_iv_{jk}$ and $v_lv_{mn} \in \overline{I}^* - v_iv_{jk}$

$$T_{\underline{X}(Q_2)}(v_iv_{jk})=\bigwedge_{v_lv_{mn}\in\underline{I}^*} T_{\underline{X}(Q_1)}(v_lv_{mn}),\ T_{\overline{X}(Q_2)}(v_iv_{jk})=\bigwedge_{v_lv_{mn}\in\overline{I}^*} T_{\overline{X}(Q_1)}(v_lv_{mn}),$$
$$I_{\underline{X}(Q_2)}(v_iv_{jk})=\bigwedge_{v_lv_{mn}\in\underline{I}^*} I_{\underline{X}(Q_1)}(v_lv_{mn}),\ I_{\overline{X}(Q_2)}(v_iv_{jk})=\bigwedge_{v_lv_{mn}\in\underline{I}^*} I_{\overline{X}(Q_1)}(v_lv_{mn}),$$
$$F_{\underline{X}(Q_2)}(v_iv_{jk})=\bigwedge_{v_lv_{mn}\in\underline{I}^*} F_{\underline{X}(Q_1)}(v_lv_{mn}),\ F_{\overline{X}(Q_2)}(v_iv_{jk})=\bigwedge_{v_lv_{mn}\in\underline{I}^*} F_{\overline{X}(Q_1)}(v_lv_{mn}).$$

Therefore,

$$\underline{X}(Q)(v_iv_{jk})=\Big(\bigwedge_{v_lv_{mn}\in\underline{I}^*-v_iv_{jk}} T_{\underline{X}(Q)}(v_lv_{mn}), \bigvee_{v_lv_{mn}\in\underline{I}^*-v_iv_{jk}} I_{\underline{X}(Q)}(v_lv_{mn}), \bigvee_{v_lv_{mn}\in\underline{I}^*-v_iv_{jk}} F_{\underline{X}(Q)}(v_lv_{mn})\Big),$$
$$\overline{X}(Q)(v_iv_{jk})=\Big(\bigwedge_{v_lv_{mn}\in\overline{I}^*-v_iv_{jk}} T_{\overline{X}(Q)}(v_lv_{mn}), \bigvee_{v_lv_{mn}\in\overline{I}^*-v_iv_{jk}} I_{\overline{X}(Q)}(v_lv_{mn}), \bigvee_{v_lv_{mn}\in\overline{I}^*-v_iv_{jk}} F_{\overline{X}(Q)}(v_lv_{mn})\Big).$$

where $v_iv_{jk} \neq v_lv_{mn}$ not uniquely. Therefore G is a soft rough neutrosophic influence cycle. □

Theorem 7. *If*

$$T_{\underline{X}(Q)}(v_iv_{jk}) > T_{ICONN_{\underline{G}-v_iv_{jk}}}(v_i,v_k),\ \ T_{\overline{X}(Q)}(v_iv_{jk}) > T_{ICONN_{\overline{G}-v_iv_{jk}}}(v_i,v_k),$$
$$I_{\underline{X}(Q)}(v_iv_{jk}) < I_{ICONN_{\underline{G}-v_iv_{jk}}}(v_i,v_k),\ \ I_{\overline{X}(Q)}(v_iv_{jk}) < I_{ICONN_{\overline{G}-v_iv_{jk}}}(v_i,v_k),$$
$$F_{\underline{X}(Q)}(v_iv_{jk}) < F_{ICONN_{\underline{G}-v_iv_{jk}}}(v_i,v_k),\ \ F_{\overline{X}(Q)}(v_iv_{jk}) < F_{ICONN_{\overline{G}-v_iv_{jk}}}(v_i,v_k),$$

in a soft rough neutrosophic graph. Then v_iv_{jk} is a cutpair in soft rough neutrosophic influence graph G.

Proof. Suppose v_iv_{jk} is not a cutapir in soft rough neutrosophic influence graph, then

$$T_{ICONN_{\underline{G}-v_i,v_k}}(v_i,v_k) = T_{ICONN_{\underline{G}}}(v_i,v_k),\ T_{ICONN_{\overline{G}-v_i,v_k}}(v_i,v_k) = T_{ICONN_{\overline{G}}}(v_lv_{mn}),$$
$$I_{ICONN_{\underline{G}-v_i,v_k}}(v_i,v_k) = I_{ICONN_{\underline{G}}}(v_i,v_k),\ I_{ICONN_{\overline{G}-v_i,v_k}}(v_i,v_k) = I_{ICONN_{\overline{G}}}(v_lv_{mn}),$$
$$F_{ICONN_{\underline{G}-v_i,v_k}}(v_i,v_k) = F_{ICONN_{\underline{G}}}(v_i,v_k),\ F_{ICONN_{\overline{G}-v_i,v_k}}(v_i,v_k) = F_{ICONN_{\overline{G}}}(v_lv_{mn}).$$

Since,

$$T_{\underline{X}(Q)}(v_i,v_k) \leq T_{ICONN_{\underline{G}}}(v_i,v_k),\ T_{\overline{X}(Q)}(v_i,v_k) \leq T_{ICONN_{\overline{G}}}(v_lv_{mn}),$$
$$I_{\underline{X}(Q)}(v_i,v_k) \geq I_{ICONN_{\underline{G}}}(v_i,v_k),\ I_{\overline{X}(Q)}(v_i,v_k) \geq I_{ICONN_{\overline{G}}}(v_lv_{mn}),$$
$$F_{\underline{X}(Q)}(v_i,v_k) \geq F_{ICONN_{\underline{G}}}(v_i,v_k),\ F_{\overline{X}(Q)}(v_i,v_k) \geq F_{ICONN_{\overline{G}}}(v_lv_{mn}).$$

Therefore,

$$T_{ICONN_{\underline{G}-v_i,v_k}}(v_i,v_k) \geq T_{\underline{X}(Q)}(v_i,v_k),\ T_{ICONN_{\overline{G}-v_i,v_k}}(v_i,v_k) \geq T_{\overline{X}(Q)}((v_i,v_k)),$$
$$I_{ICONN_{\underline{G}-v_i,v_k}}(v_i,v_k) \leq I_{\underline{X}(Q)}(v_i,v_k),\ I_{ICONN_{\overline{G}-v_i,v_k}}(v_i,v_k) \leq I_{\overline{X}(Q)}((v_i,v_k)),$$
$$F_{ICONN_{\underline{G}-v_i,v_k}}(v_i,v_k) \leq F_{\underline{X}(Q)}(v_i,v_k),\ F_{ICONN_{\overline{G}-v_i,v_k}}(v_i,v_k) \leq F_{\overline{X}(Q)}((v_i,v_k)),$$

which is a contradiction. Hence, it is proved. □

Theorem 8. *If*

$$T_{\underline{X}(Q)}(v_iv_{jk}) > T_{\underline{X}(Q)}(v_lv_{mn}),\ \ T_{\underline{X}(Q)}(v_iv_{jk}) > T_{\underline{X}(Q)}(v_lv_{mn}),$$
$$I_{\underline{X}(Q)}(v_iv_{jk}) < I_{\underline{X}(Q)}(v_lv_{mn}),\ \ I_{\underline{X}(Q)}(v_iv_{jk}) < I_{\underline{X}(Q)}(v_lv_{mn}),$$
$$F_{\underline{X}(Q)}(v_iv_{jk}) < F_{\underline{X}(Q)}(v_lv_{mn}),\ \ F_{\underline{X}(Q)}(v_iv_{jk}) < F_{\underline{X}(Q)}(v_lv_{mn}),$$

for some v_iv_{jk} among all cycles in soft rough neutrosophic influence graph G. *Then*

$$T_{\underline{X}(Q)}(v_iv_{jk}) > T_{ICONN_{\underline{G}-v_iv_{jk}}}(v_i,v_k),\ \ T_{\overline{X}(Q)}(v_iv_{jk}) > T_{ICONN_{\overline{G}-v_iv_{jk}}}(v_i,v_k),$$
$$I_{\underline{X}(Q)}(v_iv_{jk}) < I_{ICONN_{\underline{G}-v_iv_{jk}}}(v_i,v_k),\ \ I_{\overline{X}(Q)}(v_iv_{jk}) < I_{ICONN_{\overline{G}-v_iv_{jk}}}(v_i,v_k),$$
$$F_{\underline{X}(Q)}(v_iv_{jk}) < F_{ICONN_{\underline{G}-v_iv_{jk}}}(v_i,v_k),\ \ F_{\overline{X}(Q)}(v_iv_{jk}) < F_{ICONN_{\overline{G}-v_iv_{jk}}}(v_i,v_k).$$

Proof. Since

$$T_{ICONN_{\underline{G}-v_iv_{jk}}}(v_iv_{jk}) \geq T_{ICONN_{\underline{G}}}(v_iv_{jk}),\ T_{ICONN_{\overline{G}-v_iv_{jk}}}(v_iv_{jk}) \geq T_{ICONN_{\overline{G}}}((v_iv_{jk})),$$
$$I_{ICONN_{\underline{G}-v_iv_{jk}}}(v_iv_{jk}) \leq I_{ICONN_{\underline{G}}}(v_iv_{jk}),\ I_{ICONN_{\overline{G}-v_iv_{jk}}}(v_iv_{jk}) \leq I_{ICONN_{\overline{G}}}((v_iv_{jk})),$$
$$F_{ICONN_{\underline{G}-v_iv_{jk}}}(v_iv_{jk}) \leq F_{ICONN_{\underline{G}}}(v_iv_{jk}),\ F_{ICONN_{\overline{G}-v_iv_{jk}}}(v_iv_{jk}) \leq F_{ICONN_{\overline{G}}}((v_iv_{jk})).$$

Therefore, there exists a path from v_i to v_k not involving (v_iv_{jk}) such that

$$T_{ICONN_{\underline{G}-v_iv_{jk}}}(v_iv_{jk}) \geq T_{\underline{X}(Q)}(v_iv_{jk}),\ T_{ICONN_{\overline{G}-v_iv_{jk}}}(v_iv_{jk}) \geq T_{\overline{X}(Q)}((v_iv_{jk})),$$
$$I_{ICONN_{\underline{G}-v_iv_{jk}}}(v_iv_{jk}) \leq I_{\underline{X}(Q)}(v_iv_{jk}),\ I_{ICONN_{\overline{G}-v_iv_{jk}}}(v_iv_{jk}) \leq I_{\overline{X}(Q)}((v_iv_{jk})),$$
$$F_{ICONN_{\underline{G}-v_iv_{jk}}}(v_iv_{jk}) \leq F_{\underline{X}(Q)}(v_iv_{jk}),\ F_{ICONN_{\overline{G}-v_iv_{jk}}}(v_iv_{jk}) \leq F_{\overline{X}(Q)}((v_iv_{jk})),$$

This along with v_iv_{jk} is a cycle and v_iv_{jk} is least value. □

Theorem 9. *If v_iv_{jk} is a soft rough neutrosophic influence cutpair in soft rough neutrosophic influence graph* G. *Then*

$$T_{\underline{X}(Q)}(v_iv_{jk}) > T_{\underline{X}(Q)}(v_lv_{mn}),\ T_{\underline{X}(Q)}(v_iv_{jk}) > T_{\underline{X}(Q)}(v_lv_{mn}),$$
$$I_{\underline{X}(Q)}(v_iv_{jk}) < I_{\underline{X}(Q)}(v_lv_{mn}),\ I_{\underline{X}(Q)}(v_iv_{jk}) < I_{\underline{X}(Q)}(v_lv_{mn}),$$
$$F_{\underline{X}(Q)}(v_iv_{jk}) < F_{\underline{X}(Q)}(v_lv_{mn}),\ F_{\underline{X}(Q)}(v_iv_{jk}) < F_{\underline{X}(Q)}(v_lv_{mn}),$$

for some v_iv_{jk} among all cycles of G.

Proof. Suppose on contrary in a cycle, we

$$T_{\underline{X}(Q)}(v_iv_{jk}) > T_{\underline{X}(Q)}(v_lv_{mn}),\ T_{\underline{X}(Q)}(v_iv_{jk}) > T_{\underline{X}(Q)}(v_lv_{mn}),$$
$$I_{\underline{X}(Q)}(v_iv_{jk}) < I_{\underline{X}(Q)}(v_lv_{mn}),\ I_{\underline{X}(Q)}(v_iv_{jk}) < I_{\underline{X}(Q)}(v_lv_{mn}),$$
$$F_{\underline{X}(Q)}(v_iv_{jk}) < F_{\underline{X}(Q)}(v_lv_{mn}),\ F_{\underline{X}(Q)}(v_iv_{jk}) < F_{\underline{X}(Q)}(v_lv_{mn}).$$

Then any path involving it can be converted into a path not involving it with influence strength greater than and equal to the value of XQ for previously deleted pairs. So v_iv_{jk} is not a cutpair. This is a contradiction to our assumption. Hence v_iv_{jk} is not a pair with the least value among all cycle. □

Theorem 10. *If* G$=(SN_1,RM_1,XQ_1)$ *is a soft rough neutrosophic forest, then the pairs of neutrosophic spanning subgraph* $H=(SN_1,RM_1,XQ_2)$ *such that*

$$T_{\underline{X}(Q_1)}(v_iv_{jk}) < T_{ICONN_{\underline{H}}}(v_i,v_k),\ T_{\overline{X}(Q_1)}(v_iv_{jk}) < T_{ICONN_{\overline{H}}}(v_i,v_k),$$
$$I_{\underline{X}(Q_1)}(v_iv_{jk}) > I_{ICONN_{\underline{H}}}(v_i,v_k),\ I_{\overline{X}(Q_1)}(v_iv_{jk}) > I_{ICONN_{\overline{H}}}(v_i,v_k),$$
$$F_{\underline{X}(Q_1)}(v_iv_{jk}) > F_{ICONN_{\underline{H}}}(v_i,v_k),\ F_{\overline{X}(Q_1)}(v_iv_{jk}) > F_{ICONN_{\overline{H}}}(v_i,v_k),$$

are exactly the cutpairs of G.

Theorem 11. *A soft rough neutrosophic influence graph* G *is a cycle. Then an edge v_{jk} is a soft rough neutrosophic influence bridge if and only if it is an edge common to atmost two cutpair.*

Theorem 12. *Let* G *be a soft rough neutrosophic influence graph. Then the following conditions are equivalent.*

1. *For a pair* $v_i v_{jk} \in \underline{\mathrm{I}}^* \cap \overline{\mathrm{I}}^*$

$$T_{\underline{X}(Q)}(v_i v_{jk}) > T_{ICONN_{\underline{G}-v_i v_{jk}}}(v_i, v_k),\ T_{\overline{X}(Q)}(v_i v_{jk}) > T_{ICONN_{\overline{G}-v_i v_{jk}}}(v_i, v_k),$$
$$I_{\underline{X}(Q)}(v_i v_{jk}) < I_{ICONN_{\underline{G}-v_i v_{jk}}}(v_i, v_k),\ I_{\overline{X}(Q)}(v_i v_{jk}) < I_{ICONN_{\overline{G}-v_i v_{jk}}}(v_i, v_k),$$
$$F_{\underline{X}(Q)}(v_i v_{jk}) < F_{ICONN_{\underline{G}-v_i v_{jk}}}(v_i, v_k),\ F_{\overline{X}(Q)}(v_i v_{jk}) < F_{ICONN_{\overline{G}-v_i v_{jk}}}(v_i, v_k).$$

2. $v_i v_{jk}$ *is an influence cutpair*

4. Application to Decision-Making

Decision making is a process that plays an important role in our daily lives. Decision making process can help us make more purposeful, thoughtful decisions by systemizing relevant information step by step. The process of decision making involves making a choice among different alternatives, it starts when we do not know what to do.

The selection of the right path for transferring goods from one state to another states illegally. Every state has different polices within or out side the state, there are several factors to take into consideration for selecting the right path. Whether the economy of a country is good, having job opportunity or a safety.

Suppose a trader wants to extend his business to the countries C_1,C_2,C_3,C_4,C_5 and C_6. Initially, he takes C_1 and extends his business one by one. Assume A is set of the parameters, consisting of element a_1 = job, a_2 = economy above average, a_3 = safety, a_4 = other.

Let S be a full soft set from A to parameter set V, as shown in Table 16.

Table 16. Soft Neutrosophic Set S.

S	C_1	C_2	C_3	C_4	C_5	C_6
a_1	1	1	1	0	1	1
a_2	0	0	1	1	1	1
a_3	1	1	1	0	0	1
a_3	1	1	1	1	1	1

Suppose N={(C_1,0.8,0.6, 0.7),(C_2,0.9,0.5,0.65),(C_3,0.75,0.6,0.65),(C_4,1.0,0.55,0.85),(C_5,0.95,0.63,0.8), (C_6,0.85,0.65,0.95)} is most favorable object describes membership of suitable countries foreign polices corresponding to the boolean set V, which is a neutrosophic set on the set V under consideration.

$SN = (\underline{S}(N), \overline{S}(N))$ is a full soft rough set in full soft approximation space (V, S) where

$$\overline{S}(N)=\{(C_1, 0.90, 0.50, 0.65), (C_2, 0.90, 0.50, 0.65), (C_3, 0.90, 0.55, 0.65), (C_4, 1.00, 0.55, 0.65), (C_5, 0.95, 0.55, 0.65), (C_6, 0.9, 0.55, 0.65)\},$$
$$\underline{S}(N)=\{(C_1, 0.75, 0.65, 0.95), (C_2, 0.75, 0.65, 0.95), (C_3, 0.75, 0.65, 0.95), (C_4, 0.75, 0.65, 0.95), (C_5, 0.75, 0.65, 0.95), (C_6, 0.75, 0.65, 0.95)\}.$$

Let E={C_{12},C_{14},C_{15},C_{23},C_{26},C_{34},C_{35},C_{45},C_{46},C_{56}}$\subseteq \tilde{V} = V \times V$ and L={$a_{14}, a_{21}, a_{34}, a_{42}$}$\subseteq \tilde{A} = A \times A$. A full soft relation R on E (from L to E) can be defined as shown in Table 17.

Table 17. Full soft set R.

R	C_{12}	C_{14}	C_{15}	C_{23}	C_{26}	C_{34}	C_{35}	C_{45}	C_{46}	C_{56}
a_{14}	1	1	1	1	1	1	1	0	0	1
a_{21}	0	0	0	0	0	0	1	1	1	1
a_{34}	1	1	1	1	1	1	1	0	0	0
a_{42}	0	1	1	1	1	1	1	1	1	1

Let M={(C_{12},0.74,0.5,0.62),(C_{14},0.75,0.45,0.63),(C_{15},0.74,0.54,0.61),(C_{23},0.72,0.48,0.65),(C_{26},0.71, 0.49,0.64),(C_{34},0.72,0.53,0.64),(C_{35},0.73,0.52,0.63),(C_{45},0.7,0.51,0.61),(C_{46},0.74,0.55,0.6),(C_{56},0.73,0.47, 0.64)} be most favorable object describes membership of countries foreign polices toward others countries corresponding to the boolean set E, which is a neutrosophic set on the set V under consideration.

$RM = (\underline{R}M, \overline{R}M)$ is a soft neutrosophic rough relation, where

$$\overline{R}M=\{(C_{12}, 0.75, 0.45, 0.61), (C_{14}, 0.75, 0.45, 0.61), (C_{15}, 0.75, 0.45, 0.61), (C_{23}, 0.75, 0.45, 0.61), (C_{26}, 0.75, 0.45, 0.61), (C_{34}, 0.75, 0.45, 0.61), (C_{35}, 0.74, 0.47, 0.61), (C_{45}, 0.74, 0.47, 0.6), (C_{46}, 0.74, 0.47, 0.6), (C_{56}, 0.74, 0.47, 0.61)\},$$

$$\underline{R}M=\{(C_{12}, 0.71, 0.54, 0.65), (C_{14}, 0.71, 0.54, 0.65), (C_{15}, 0.71, 0.54, 0.65), (C_{23}, 0.71, 0.54, 0.65), (C_{26}, 0.71, 0.54, 0.65), (C_{34}, 0.71, 0.54, 0.65), (C_{35}, 0.71, 0.54, 0.64), (C_{45}, 0.70, 0.55, 0.64), (C_{46}, 0.70, 0.55, 0.64), (C_{56}, 0.71, 0.54, 0.64)\}.$$

Let $I=\{C_1C_{15}, C_1C_{23}, C_1C_{35}, C_2C_{34}, C_3C_{14}, C_3C_{26}, C_3C_{45}, C_4C_{23}, C_4C_{45}, C_4C_{46}, C_5C_{23}, C_5C_{34}, C_5C_{46}, C_6C_{12}, C_6C_{15}\} \subseteq \hat{V} = V \times E$ and $F=\{a_1a_{42}, a_2a_{14}, a_3a_{34}, a_4a_{21}, a_4a_{42}\} \subseteq \hat{A} = A \times L$.

A full soft relation X on I (from F to I) can be defined in Table 18 as follows:

Table 18. Full soft set X.

X	C_1C_{15} C_4C_{45}	C_1C_{23} C_4C_{46}	C_1C_{35} C_5C_{23}	C_2C_{34} C_5C_{34}	C_3C_{14} C_5C_{46}	C_3C_{26} C_6C_{12}	C_3C_{45}, C_6C_{15}	C_4C_{23}
e_1e_{42}	1 0	1 0	1 1	1 1	1 1	1 0	1 1	0
e_2e_{14}	0 0	0 0	0 1	0 1	1 0	1 1	0 1	0
e_2e_{34}	0 0	0 0	0 0	0 0	0 0	0 0	0 0	1
e_3e_{34}	1 0	1 0	1 0	1 0	1 0	1 1	0 1	0
e_4e_{21}	0 1	0 1	1 0	0 0	0 1	0 0	1 0	0
e_4e_{42}	1 1	1 1	1 1	1 1	1 1	1 0	1 1	0

Let $Q = \{(C_1C_{15}, 0.7, 0.43, 0.58), (C_1C_{23}, 0.65, 0.39, 0.54), (C_1C_{35}, 0.66, 0.37, 0.56), (C_2C_{34}, 0.68, 0.38, 0.59), (C_3C_{14}, 0.6, 0.4, 0.6), (C_3C_{26}, 0.62, 0.42, 0.58), (C_3C_{45}, 0.64, 0.45, 0.54), (C_4C_{23}, 0.7, 0.45, 0.60), (C_4C_{45}, 0.7, 0.36, 0.48), (C_4C_{46}, 0.68, 0.35, 0.5), (C_5C_{23}, 0.69, 0.45, 0.54), (C_5C_{34}, 0.65, 0.42, 0.58), (C_5C_{46}, 0.64, 0.41, 0.59), (C_6C_{12}, 0.63, 0.4, 0.6), (C_6C_{15}, 0.62, 0.39, 0.5)\}$ be most favorable object describes membership of countries impact toward others countries regarding trade corresponding to the boolean set I, which is a neutrosophic set on the set I under consideration.

$XQ = (\underline{X}Q, \overline{X}Q)$ is a soft neutrosophic rough influence, where

$$\overline{X}Q=\{(C_1C_{15},0.70,0.37,0.50),(C_1C_{23},0.70,0.37,0.50),(C_1C_{35},0.70,0.37,0.50),(C_2C_{34},0.70,0.37,0.50),\\ (C_3C_{14},0.69,0.39,0.50),(C_3C_{26},0.69,0.39,0.50),(C_3C_{45},0.70,0.37,0.50),(C_4C_{23},0.7,0.45,0.60),\\ (C_4C_{45},0.70,0.35,0.48),(C_4C_{46},0.70,0.35,0.48),(C_5C_{23},0.69,0.39,0.50),(C_5C_{34},0.69,0.39,0.50),\\ (C_5C_{46},0.70,0.37,0.50),(C_6C_{12},0.69,0.39,0.50),(C_6C_{15},0.69,0.39,0.50)\},$$

$$\underline{X}Q=\{(C_1C_{15},0.60,0.43,0.60),(C_1C_{23},0.60,0.43,0.60),(C_1C_{35},0.64,0.43,0.59),(C_2C_{34},0.60,0.43,0.60),\\ (C_3C_{14},0.60,0.43,0.60),(C_3C_{26},0.60,0.43,0.60),(C_3C_{45},0.64,0.45,0.59),(C_4C_{23},0.7,0.45,0.60),\\ (C_4C_{45},0.64,0.45,0.59),(C_4C_{46},0.64,0.45,0.59),(C_5C_{23},0.60,0.45,0.60),(C_5C_{34},0.60,0.45,0.60),\\ (C_5C_{46},0.64,0.45,0.59),(C_6C_{12},0.60,0.43,0.60),(C_6C_{15},0.60,0.43,0.60)\}.$$

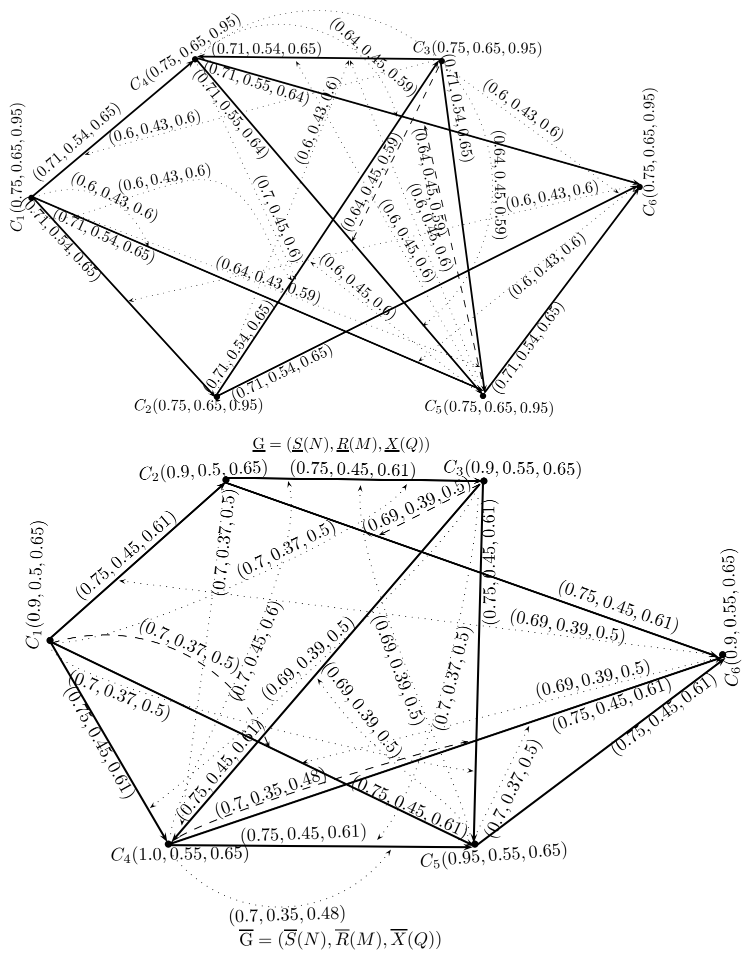
Thus, $G = (\underline{G},\overline{G})$ is a soft neutrosophic rough influence graph as shown in Figure 9. He finds the strength of each path from C_1 to C_6. The paths are

$$P_1 : C_1, C_5, C_2, C_3, C_6,$$

$$P_2 : C_1, C_4, C_5, C_6,$$

$$P_3 : C_1, C_3, C_5, C_2, C_6$$

with their influence strength as (0.6,0.45,0.5), respectively.



Since, there is more than one path, therefore, the trader calculates the score function which is formulated in Equation (4):

$$\text{Score Function}(C_i)=\Big(T_{\underline{S}(N)}(C_i)+T_{\overline{S}(N)}(C_i)+T_{\underline{R}(M)}(C_{ij})+T_{\overline{R}(M)}(C_{ij})+T_{\underline{X}(Q)}(C_iC_{jk})+ T_{\overline{X}(Q)}(C_iC_{jk}), I_{\underline{S}(N)}(C_i)I_{\overline{S}(N)}(C_i)+I_{\underline{R}(M)}(C_{ij})I_{\overline{R}(M)}(C_{ij})+ I_{\underline{X}(Q)}(C_iC_{jk})I_{\overline{X}(Q)}(C_iC_{jk}), F_{\underline{S}(N)}(C_i)F_{\overline{S}(N)}(C_i)+ F_{\underline{R}(M)}(C_{ij})F_{\overline{R}(M)}(C_{ij})+F_{\underline{X}(Q)}(C_iC_{jk})F_{\overline{X}(Q)}(C_iC_{jk})\Big). \quad (4)$$

For each C_i, the score values of C_i is calculated directly and as shown in Table 19.

Table 19. Score Function.

V	Score Values
C_1	(9.97,1.054,2.702)
C_2	(5.87,1.2979,1.7105)
C_3	(8.48,1.3562,2.2994)
C_4	(6.73,1.392,2.3119)
C_5	(7.07,1.3673,1.9029)
C_6	(4.23,0.6929,1.2175)

So, he chooses the path P_3:C_1,C_3,C_5,C_2,C_6. The Algorithm 1 of the application is also be given in Algorithm 1. The flow chart is given in Figure 10.

Algorithm 1: Influence strength of each path in rough neutrosophic influence graph

1. Input the universal sets C and P.
2. Input the full soft set S and neutrosophic set N on V.
3. Calculate the Soft rough neutrosophic sets on V.
4. Input the universal sets E and L.
5. Input the full soft set R and neutrosophic set M on E.
6. Calculate the Soft rough neutrosophic sets on E.
7. Input the universal sets I and F.
8. Input the full soft set X and neutrosophic set Q on I.
9. Calculate the Soft rough neutrosophic sets on I.
10. Find the number of path and calculate their influence strength of each path from C_1 to C_n.
11. Choose that path which has maximum membership, minimum indeterminacy and falsity value. If $i > 1$, than calculate the score values of each C_i, choose that C_i which has maximum membership and come immediately after C_1 in one of the paths.

5. Conclusions

Graph theory has been applied widely in various areas of engineering, computer science, database theory, expert systems, neural networks, artificial intelligence, signal processing, pattern recognition, robotics, computer networks, and medical diagnosis. Present research has shown that two or more theories can be combined into a more flexible and expressive framework for modeling and processing incomplete information in information systems. Various mathematical models that combine rough sets, soft sets and neutrosophic sets have been introduced. A soft rough neutrosophic set is a hybrid tool for handling indeterminate, inconsistent and uncertain information that exist in real life. We have applied this concept to graph theory. We have presented certain concepts, including soft rough neutrosophic graphs, soft rough neutrosophic influence graphs, soft rough neutrosophic influence cycles, soft rough neutrosophic influence trees. We also have considered an application of soft rough neutrosophic influence graph in decision-making to illustrate the best path in the business. In the future, we will study, (1) Neutrosophic rough hypergraphs, (2) Bipolar neutrosophic rough hypergraphs, (3) Neutrosophic soft rough hypergraphs, (4) Decision support systems based on soft rough neutrosophic information.

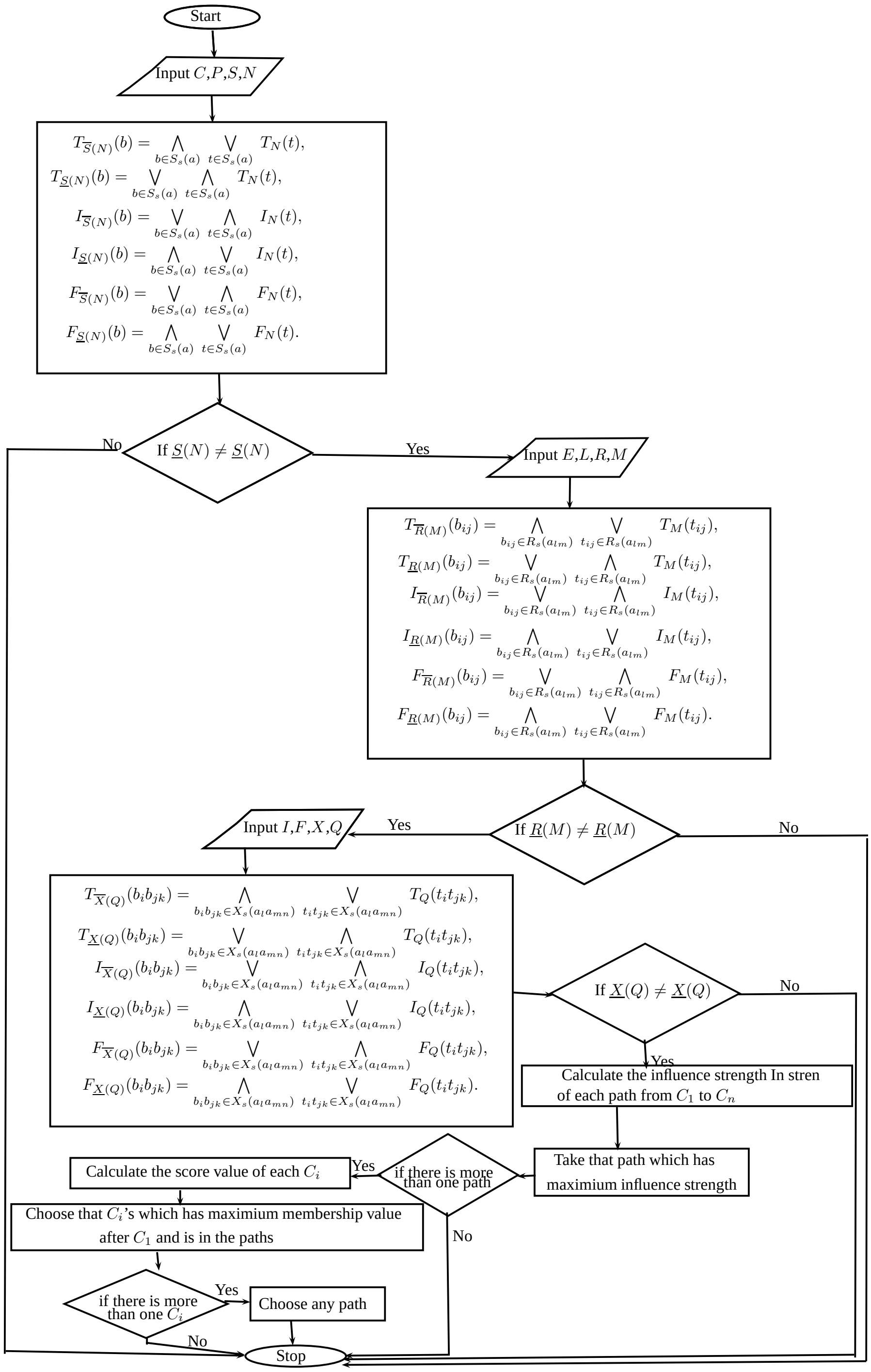


Figure 10: The flow chart of the application

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Some Results on Neutrosophic Triplet Group and Their Applications

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Abstract: This article is based on new developments on a neutrosophic triplet group (NTG) and applications earlier introduced in 2016 by Smarandache and Ali. NTG sprang up from neutrosophic triplet set X: a collection of triplets $(b, neut(b), anti(b))$ for an $b \in X$ that obeys certain axioms (existence of neutral(s) and opposite(s)). Some results that are true in classical groups were investigated in NTG and were shown to be either universally true in NTG or true in some peculiar types of NTG. Distinguishing features between an NTG and some other algebraic structures such as: generalized group (GG), quasigroup, loop and group were investigated. Some neutrosophic triplet subgroups (NTSGs) of a neutrosophic triplet group were studied. In particular, for any arbitrarily fixed $a \in X$, the subsets $X_a = \{b \in X : neut(b) = neut(a)\}$ and $\ker f_a = \{b \in X | f(b) = neut(f(a))\}$ of X, where $f : X \to Y$ is a neutrosophic triplet group homomorphism, were shown to be NTSG and normal NTSG, respectively. Both X_a and $\ker f_a$ were shown to be a-normal NTSGs and found to partition X. Consequently, a Lagrange-like formula was found for a finite NTG X; $|X| = \sum_{a \in X} [X_a : \ker f_a]|\ker f_a|$ based on the fact that $|\ker f_a| \big| |X_a|$. The first isomorphism theorem $X/\ker f \cong \mathrm{Im}\, f$ was established for NTGs. Using an arbitrary non-abelian NTG X and its NTSG X_a, a Bol structure was constructed. Applications of the neutrosophic triplet set, and our results on NTG in relation to management and sports, are highlighted and discussed.

Keywords: generalized group; neutrosophic triplet set; neutrosophic triplet group; group

1. Introduction

1.1. Generalized Group

Unified gauge theory has the algebraic structure of a generalized group abstrusely, in its physical background. It has been a challenge for physicists and mathematicians to find a desirable unified theory for twistor theory, isotopies theory, and so on. Generalized groups are instruments for constructions in unified geometric theory and electroweak theory. Completely simple semigroups are precisely generalized groups (Araujo et al. [1]). As recorded in Adeniran et al. [2], studies on the properties and structures of generalized groups have been carried out in the past, and these have been extended to smooth generalized groups and smooth generalized subgroups by Agboola [3,4], topological generalized groups by Molaei [5], Molaei and Tahmoresi [6], and quotient space of generalized groups by Maleki and Molaei [7].

Definition 1 (Generalized Group(GG)). *A generalized group X is a non-void set with a binary operation called multiplication obeying the set of rules given below.*

(i) $(ab)c = a(bc)$ *for all* $a, b, c \in X$.

(ii) *For each* $a \in X$ *there is a unique* $e(a) \in X$ *such that* $ae(a) = e(a)a = a$ *(existence and uniqueness of identity element).*

(iii) *For each* $a \in X$, *there is* $a^{-1} \in X$ *such that* $aa^{-1} = a^{-1}a = e(a)$ *(existence of inverse element).*

Definition 2. *Let X be a non-void set. Let $(\cdot)$ be a binary operation on X. Whenever $a \cdot b \in X$ for all $a, b \in X$, then $(X, \cdot)$ is called a groupoid.*

Whenever the equation $c \cdot x = d$ (or $y \cdot c = d$) have unique solution with respect to x (or y) i.e., satisfies the left (or right) cancellation law, then $(X, \cdot)$ is called a left (or right) quasigroup. If a groupoid $(X, \cdot)$ is both a left quasigroup and right quasigroup, then it is called a quasigroup. If there is an element $e \in X$ called the identity element such that for all $a \in X$, $a \cdot e = e \cdot a = a$, then a quasigroup $(X, \cdot)$ is called a loop.

Definition 3. *A loop is called a Bol loop whenever it satisfies the identity*

$$((ab)c)b = a((bc)b).$$

Remark 1. *One of the most studied classes of loops is the Bol loop.*

For more on quasigroups and loops, interested readers can check [8–15].

A generalized group X has the following properties:

(i) For each $a \in X$, there is a unique $a^{-1} \in X$.

(ii) $e(e(a)) = e(a)$ and $e(a^{-1}) = e(a)$ if $a \in X$.

(iii) If X is commutative, then X is a group.

1.2. Neutrosophic Triplet Group

Neutrosophy is a novel subdivision of philosophy that studies the nature, origination, and ambit of neutralities, including their interaction with ideational spectra. Florentin Smarandache [16] introduced the notion of neutrosophic logic and neutrosophic sets for the first time in 1995. As a matter of fact, the neutrosophic set is the generalization of classical sets [17], fuzzy sets [18], intuitionistic fuzzy sets [17,19], and interval valued fuzzy sets [17], to cite a few. The growth process of neutrosophic sets, fuzzy sets, and intuitionistic fuzzy sets are still evolving, with diverse applications. Some recent research findings in these directions are [20–27].

Smarandache and Ali [28] were the first to introduce the notion of the neutrosophic triplet, which they had earlier talked about at a conference. These neutrosophic triplets were used by them to introduce the neutrosophic triplet group, which differs from the classical group both in fundamental and structural properties. The distinction and comparison of the neutrosophic triplet group with the classical generalized group were given. They also drew a brief outline of the potential applications of the neutrosophic triplet group in other research fields. For discussions of results on neutrosophic triplet groups, neutrosophic quadruples, and neutrosophic duplets of algebraic structures, as well as new applications of neutrosophy, see Jaiyéọlá and Smarandache [29]. Jaiyéọlá and Smarandache [29] were the first to introduce and study inverse property neutrosophic triplet loops with applications to cryptography for the first time.

Definition 4 (Neutrosophic Triplet Set-NTS). *Let X be a non-void set together with a binary operation $\star$ defined on it. Then X is called a neutrosophic triplet set if, for any $a \in X$, there is a neutral of 'a' denoted by $neut(a)$ (not necessarily the identity element) and an opposite of 'a' denoted by $anti(a)$, with $neut(a), anti(a) \in X$ such that*

$$a \star neut(a) = neut(a) \star a = a \quad \text{and} \quad a \star anti(a) = anti(a) \star a = neut(a).$$

The elements $a, neut(a)$ and $anti(a)$ are together called neutrosophic triplet, and represented by $(a, neut(a), anti(a))$.

Remark 2. *For an $a \in X$, each of $neut(a)$ and $anti(a)$ may not be unique. In a neutrosophic triplet set $(X, \star)$, an element b (or c) is the second (or third) component of a neutrosophic triplet if $a, c \in X$ ($a, b \in X$) such that $a \star b = b \star a = a$ and $a \star c = c \star a = b$. Thus, (a, b, c) is a neutrosophic triplet.*

Example 1 (Smarandache and Ali [28]). *Consider $(\mathbb{Z}_6, \times_6)$ such that $\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ and $\times_6$ is multiplication in modulo 6. $(2, 4, 2)$, $(4, 4, 4)$, and $(0, 0, 0)$ are neutrosophic triplets, but 3 will not give rise to a neutrosophic triplet.*

Definition 5 (Neutrosophic Triplet Group—NTG). *Let $(X, \star)$ be a neutrosophic triplet set. Then $(X, \star)$ is referred to as a neutrosophic triplet group if $(X, \star)$ is a semigroup. Furthermore, if $(X, \star)$ obeys the commutativity law, then $(X, \star)$ is referred to as a commutative neutrosophic triplet group.*

Let $(X, \star)$ be a neutrosophic triplet group. Whenever $neut(ab) = neut(a)neut(b)$ for all $a, b \in X$, then X is referred to as a normal neutrosophic triplet group.

Let $(X, \star)$ be a neutrosophic triplet group and let $H \subseteq X$. H is referred to as a neutrosophic triplet subgroup (NTSG) of X if $(H, \star)$ is a neutrosophic triplet group. Whence, for any fixed $a \in X$, H is called a-normal NTSG of X, written $H \overset{a}{\lhd} X$ if $ay\, anti(a) \in H$ for all $y \in H$.

Remark 3. *An NTG is not necessarily a group. However, a group is an NTG where $neut(a) = e$, the general identity element for all $a \in X$, and $anti(a)$ is unique for each $a \in X$.*

Example 2 (Smarandache and Ali [28]). *Consider $(\mathbb{Z}_{10}, \otimes)$ such that $c \otimes d = 3cd \bmod 10$. $(\mathbb{Z}_{10}, \otimes)$ is a commutative NTG but neither a GG nor a classical group.*

Example 3 (Smarandache and Ali [28]). *Consider $(\mathbb{Z}_{10}, \star)$ such that $c \star d = 5c + d \bmod 10$. $(\mathbb{Z}_{10}, \star)$ is a non-commutative NTG but not a classical group.*

Definition 6 (Neutrosophic Triplet Group Homomorphism). *Let $f : X \to Y$ be a mapping such that X and Y are two neutrosophic triplet groups. Then f is referred to as a neutrosophic triplet group homomorphism if $f(cd) = f(c)f(d)$ for all $c, d \in X$. The kernel of f at $a \in X$ is defined by*

$$\ker f_a = \{x \in X : f(x) = neut(f(a))\}.$$

The Kernel of f is defined by

$$\ker f = \bigcup_{a \in X} \ker f_a$$

such that $f_a = f|_{X_a}$, where $X_a = \{x \in X : neut(x) = neut(a)\}$.

Remark 4. *The definition of neutrosophic triplet group homomorphism above is more general than that in Smarandache and Ali [28]. In Theorem 5, it is shown that, for an NTG homomorphism $f : X \to Y$, $f(neut(a)) = neut(f(a))$ and $f(anti(a)) = anti(f(a))$ for all $a \in X$.*

The present work is a continuation of the study of a neutrosophic triplet group (NTG) and its applications, which was introduced by Smarandache and Ali [28]. Some results that are true in classical groups were investigated in NTG and will be proved to be either generally true in NTG or true in some classes of NTG. Some applications of the neutrosophic triplet set, and our results on NTG in relation to management and sports will be discussed.

The first section introduces GG and NTG and highlights existing results that are relevant to the present study. Section 2 establishes new results on algebraic properties of NTGs and NTG homomorphisms, among which are Lagrange's Theorem and the first isomorphism theorem, and presents a method of the construction of Bol algebraic structures using an NTG. The third section describes applications of NTGs to human management and sports.

2. Main Results

We shall first establish the relationship among generalized groups, quasigroups, and loops with a neutrosophic triplet group assumed.

Lemma 1. *Let X be a neutrosophic triplet group.*

1. *X is a generalized group if it satisfies the left (or right) cancellation law or X is a left (or right) quasigroup.*
2. *X is a generalized group if and only if each element $x \in X$ has a unique $neut(x) \in X$.*
3. *Whenever X has the cancellation laws (or is a quasigroup), then X is a loop and group.*

Proof. 1. Let x have at least two neutral elements, say $neut(x), neut(x)' \in X$. Then $xx = xx \Rightarrow xx\,anti(x) = xx\,anti(x) \Rightarrow x\,neut(x) = x\,neut(x)' \underset{\text{left cancellation law}}{\overset{\text{left quasigroup}}{\Longrightarrow}} neut(x) = neut(x)'$. Therefore, X is a generalized group. Similarly, X is a generalized group if it is has the right cancellation law or if it is a right quasigroup.
2. This follows by definition.
3. This is straightforward because every associative quasigroup is a loop and group. □

2.1. Algebraic Properties of Neutrosophic Triplet Group

We now establish some new algebraic properties of NTGs.

Theorem 1. *Let X be a neutrosophic triplet group. For any $a \in X$, $anti\big(anti(a)\big) = a$.*

Proof. $anti\big(anti(a)\big)anti(a) = neut\big(anti(a)\big) = neut(a)$ by Theorem 1 ([29]). After multiplying by a, we obtain

$$\big[anti\big(anti(a)\big)anti(a)\big]a = neut(a)a = a. \tag{1}$$

$$\begin{aligned} LHS &= anti\big(anti(a)\big)\big(anti(a)a\big) = anti\big(anti(a)\big)neut(a) \\ &= anti\big(anti(a)\big)neut\big(anti(a)\big) = anti\big(anti(a)\big)neut\Big(anti\big(anti(a)\big)\Big) = anti\big(anti(a)\big). \end{aligned} \tag{2}$$

Hence, based on Equations (1) and (2), $anti\big(anti(a)\big) = a$. □

Theorem 2. *Let X be a neutrosophic triplet group such that the left cancellation law is satisfied, and $neut(a) = neut\big(a\,anti(b)\big)$ if and only if $a\,anti(b) = a$. Then X is an idempotent neutrosophic triplet group if and only if $neut(a)anti(b) = anti(b)neut(a)\ \forall\ a, b \in X$.*

Proof. $neut(a)anti(b) = anti(b)neut(a) \Leftrightarrow (a\,neut(a))anti(b) = a\,anti(b)neut(a) \Leftrightarrow a\,anti(b) = a\,anti(b)neut(a) \Leftrightarrow neut(a) = neut\big(a\,anti(b)\big) \Leftrightarrow a\,anti(b) = a \Leftrightarrow a\,anti(b)b = ab \Leftrightarrow a\,neut(b) = ab \Leftrightarrow anti(a)a\,neut(b) = anti(a)ab \Leftrightarrow neut(a)neut(b) = neut(a)b \Leftrightarrow neut(b) = b \Leftrightarrow b = bb$. □

Theorem 3. *Let X be a normal neutrosophic triplet group in which $neut(a)anti(b) = anti(b)neut(a)\ \forall\, a, b \in X$. Then, $anti(ab) = anti(b)anti(a)\ \forall\, a, b \in X$.*

Proof. Since $anti(ab)(ab) = neut(ab)$, then by multiplying both sides of the equation on the right by $anti(b)anti(a)$, we obtain

$$\left[anti(ab)ab\right]anti(b)anti(a) = neut(ab)anti(b)anti(a). \tag{3}$$

Going by Theorem 1([29]),

$$\begin{aligned}\left[anti(ab)ab\right]anti(b)anti(a) &= anti(ab)a\big(b\ anti(b)\big)anti(a) = anti(ab)a(neut(b)anti(a))\\ &= anti(ab)(a\ anti(a))neut(b) = anti(ab)\big(neut(a)neut(b)\big)\\ &= anti(ab)neut(ab) = anti(ab)neut\big(anti(ab)\big) = anti(ab).\end{aligned} \tag{4}$$

Using Equations (3) and (4), we obtain

$$\begin{gathered}\left[anti(ab)ab\right]anti(b)anti(a) = anti(ab) \Rightarrow\\ neut(ab)\big(anti(b)anti(a)\big) = anti(ab) \Rightarrow anti(ab) = anti(b)anti(a).\end{gathered}$$

□

It is worth characterizing the neutrosophic triplet subgroup of a given neutrosophic triplet group to see how a new NTG can be obtained from existing NTGs.

Lemma 2. *Let H be a non-void subset of a neutrosophic triplet group X. The following are equivalent.*

(i) *H is a neutrosophic triplet subgroup of X.*
(ii) *For all $a, b \in H$, $a\ anti(b) \in H$.*
(iii) *For all $a, b \in H$, $ab \in H$, and $anti(a) \in H$.*

Proof. **(i)⇒ (ii)** If H is an NTSG of X and $a, b \in H$, then $anti(b) \in H$. Therefore, by closure property, $a\ anti(b) \in H\ \forall\, a, b \in H$.
(ii)⇒ (iii) If $H \neq \emptyset$, and $a, b \in H$, then we have $b\ anti(b) = neut(b) \in H$, $neut(b)anti(b) = anti(b) \in H$, and $ab = a\ anti(anti(b)) \in H$, i.e., $ab \in H$.
(iii)⇒ (i) $H \subseteq X$, so H is associative since X is associative. Obviously, for any $a \in H$, $anti(a) \in H$. Let $a \in H$, then $anti(a) \in H$. Therefore, $a\ anti(a) = anti(a)a = neut(a) \in H$. Thus, H is an NTSG of X.
□

Theorem 4. *Let G and H be neutrosophic triplet groups. The direct product of G and H defined by*

$$G \times H = \{(g, h) : g \in G \text{ and } h \in H\}$$

is a neutrosophic triplet group under the binary operation $\circ$ defined by

$$(g_1, h_1) \circ (g_2, h_2) = (g_1 g_2, h_1 h_2).$$

Proof. This is simply done by checking the axioms of neutrosophic triplet group for the pair $(G \times H, \circ)$, in which case $neut(g, h) = \big(neut(g), neut(h)\big)$ and $anti(g, h) = \big(anti(g), anti(h)\big)$. □

Lemma 3. *Let $\mathcal{H} = \{H_i\}_{i\in\Omega}$ be a family of neutrosophic triplet subgroups of a neutrosophic triplet group X such that $\bigcap_{i\in\Omega} H_i \neq \emptyset$. Then $\bigcap_{i\in\Omega} H_i$ is a neutrosophic triplet subgroup of X.*

Proof. This is a routine verification using Lemma 2. □

2.2. Neutrosophic Triplet Group Homomorphism

Let us now establish results on NTG homomorphisms, its kernels, and images, as well as a Lagrange-like formula and the First Isomorphism Theorem for NTGs.

Theorem 5. *Let $f : X \to Y$ be a homomorphism where X and Y are two neutrosophic triplet groups.*

1. $f(neut(a)) = neut(f(a))$ *for all* $a \in X$.
2. $f(anti(a)) = anti(f(a))$ *for all* $a \in X$.
3. *If H is a neutrosophic triplet subgroup of X, then $f(H)$ is a neutrosophic triplet subgroup of Y.*
4. *If K is a neutrosophic triplet subgroup of Y, then $\emptyset \neq f^{-1}(K)$ is a neutrosophic triplet subgroup of X.*
5. *If X is a normal neutrosophic triplet group and the set $X_f = \{(neut(a), f(a)) : a \in X\}$ with the product*

$$(neut(a), f(a))(neut(b), f(b)) := (neut(ab), f(ab)),\ \textit{then}$$

X_f *is a neutrosophic triplet group.*

Proof. Since f is an homomorphism, $f(ab) = f(a)f(b)$ for all $a, b \in X$.

1. Place $b = neut(a)$ in $f(ab) = f(a)f(b)$ to obtain $f\big(a\ neut(a)\big) = f(a)f(neut(a)) \Rightarrow f(a) = f(a)f(neut(a))$. Additionally, place $b = neut(a)$ in $f(ba) = f(b)f(a)$ to obtain $f\big(neut(a)a\big) = f(neut(a))f(a) \Rightarrow f(a) = f(neut(a))f(a)$. Thus, $f(neut(a)) = neut(f(a))$ for all $a \in X$.
2. Place $b = anti(a)$ in $f(ab) = f(a)f(b)$ to obtain $f\big(a\ anti(a)\big) = f(a)f(anti(a)) \Rightarrow f(neut(a)) = f(a)f(anti(a)) \Rightarrow neut(f(a)) = f(a)f(anti(a))$. Additionally, place $b = anti(a)$ in $f(ba) = f(b)f(a)$ to obtain $f\big(anti(a)a\big) = f(anti(a))f(a) \Rightarrow f(neut(a)) = f(a)f(anti(a)) \Rightarrow neut(f(a)) = f(anti(a))f(a)$. Thus, $f(anti(a)) = anti(f(a))$ for all $a \in X$.
3. If H is an NTSG of G, then $f(H) = \{f(h) \in Y : h \in H\}$. We shall prove that $f(H)$ is an NTSG of Y by Lemma 2.
   Since $f(neut(a)) = neut(f(a)) \in f(H)$ for $a \in H$, $f(H) \neq \emptyset$. Let $a', b' \in f(H)$. Then $a' = f(a)$ and $b' = f(b)$. Thus, $a'\ anti(b') = f(a)anti(f(b)) = f(a)f(anti(b)) = f(a\ anti(b)) \in f(H)$. Therefore, $f(H)$ is an NTSG of Y.
4. If K is a neutrosophic triplet subgroup of Y, then $\emptyset \neq f^{-1}(K) = \{a \in X : f(a) \in K\}$. We shall prove that $f(H)$ is an NTSG of Y by Lemma 2.
   Let $a, b \in f^{-1}(K)$. Then $a', b' \in K$ such that $a' = f(a)$ and $b' = f(b)$. Thus, $a'\ anti(b') = f(a)anti(f(b)) = f(a)f(anti(b)) = f(a\ anti(b)) \in K \Rightarrow a\ anti(b) \in f^{-1}(K)$. Therefore, $f^{-1}(K)$ is an NTSG of X.
5. Given the neutrosophic triplet group X and the set $X_f = \{(neut(a), f(a)) : a \in X\}$ with the product $(neut(a), f(a))(neut(b), f(b)) := (neut(ab), f(ab))$. X_f is a groupoid.
   $(neut(a), f(a))(neut(b), f(b)) \cdot (neut(z), f(z)) = (neut(ab), f(ab))(neut(z), f(z)) = (neut(abz), f(abz))$
   $= (neut(a), f(a))(neut(bz), f(bz)) = (neut(a), f(a)) \cdot (neut(b), f(b))(neut(z), f(z))$.
   Therefore, X_f is a semigroup.
   For $(neut(a), f(a)) \in X_f$, let $neut(neut(a), f(a)) = \big(neut(neut(a)), neut(f(a))\big)$. Then $neut(neut(a), f(a)) = \big(neut(a), (f(neut(a))\big) \in X_f$. Additionally, let $anti(neut(a), f(a)) = \big(anti(neut(a)), anti(f(a))\big)$. Then $anti(neut(a), f(a)) = \big(neut(a), f(anti(a))\big) \in X_f$.
   Thus, $(neut(a), f(a))neut(neut(a), f(a)) = (neut(a), f(a))\big(neut(a), (f(neut(a))\big) = (neut(a), f(a))\big(neut(anti(a)), (f(neut(a))\big) = \big(neut(a\ \ anti(a)), f(a\ \ neut(a))\big) = \big(neut(neut(a)), f(a\ \ neut(a))\big) = (neut(a), f(a)) \Rightarrow (neut(a), f(a))neut(neut(a), f(a)) = (neut(a), f(a))$ and similarly, $neut(neut(a), f(a))(neut(a), f(a)) = (neut(a), f(a))$.

On the other hand, $(neut(a), f(a))anti(neut(a), f(a)) = (neut(a), f(a)) \cdot (neut(a), f(anti(a))) = (neut(a), f(a))(neut(anti(a)), (f(anti(a))) = (neut(a \ \ anti(a)), f(a \ \ anti(a))) = (neut(neut(a)), f(neut(a))) = (neut(a), (f(neut(a))) = neut(neut(a), f(a)) \Rightarrow (neut(a), f(a)) \cdot anti(neut(a), f(a)) = neut(neut(a), f(a))$ and similarly, $anti(neut(a), f(a)) \cdot (neut(a), f(a)) = neut(neut(a), f(a))$.

Therefore, X_f is a neutrosophic triplet group.

□

Theorem 6. *Let $f : X \to Y$ be a neutrosophic triplet group homomorphism.*

1. $\ker f_a \overset{a}{\triangleleft} X$.
2. $X_a \overset{a}{\triangleleft} X$.
3. *X_a is a normal neutrosophic triplet group.*
4. $anti(cd) = anti(d)anti(c) \ \forall \ c, d \in X_a$.
5. $X_a = \bigcup_{c \in X_a} c \ker f_a$ *for all* $a \in X$.
6. *If X is finite,* $|X_a| = \sum_{c \in X_a} |c \ker f_a| = [X_a : \ker f_a]|\ker f_a|$ *for all* $a \in X$ *where* $[X_a : \ker f_a]$ *is the index of* $\ker f_a$ *in* X_a*, i.e., the number of distinct left cosets of* $\ker f_a$ *in* X_a.
7. $X = \bigcup_{a \in X} X_a$.
8. *If X is finite,* $|X| = \sum_{a \in X} [X_a : \ker f_a]|\ker f_a|$.

Proof. 1. $f(neut(a)) = neut(f(a)) = neut(neut(f(a))) = neut(f(neut(a))) \Rightarrow neut(a) \in \ker f_a \Rightarrow \ker f_a \neq \emptyset$. Let $c, d \in \ker f_a$, then $f(c) = f(d) = neut(f(a))$. We shall use Lemma 2.
$f(c \ \ anti(d)) = f(c)f(anti(d)) = f(c)anti(f(d)) = neut(f(a))anti(neut(f(a))) = neut(f(a))neut(f(a)) = neut(f(a)) \Rightarrow c \ anti(d) \in \ker f_a$.
Thus, $\ker f_a$ is a neutrosophic triplet subgroup of X. For the a-normality, let $d \in \ker f_a$, then $f(d) = neut(f(a))$. Therefore, $f(ad \ \ anti(a)) = f(a)f(d)f(anti(a)) = f(a)neut(f(a))anti(f(a)) = f(a)anti(f(a)) = neut(f(a)) \Rightarrow ad \ anti(a) \in \ker f_a$ for all $d \in \ker f_a$. Therefore, $\ker f_a \overset{a}{\triangleleft} X$.

2. $X_a = \{c \in X : neut(c) = neut(a)\}$. $neut(neut(a)) = neut(a) \Rightarrow neut(a) \in X_a$. Therefore, $X_a \neq \emptyset$. Let $c, d \in X_a$. Then $neut(c) = neut(a) = neut(d)$. $(cd)neut(a) = c(d \ neut(a)) = c(d \ neut(d)) = cd$, and $neut(a)(cd) = (neut(a)c)d = (neut(c)c)d = cd$. Therefore, $neut(cd) = neut(a)$.
$neut(anti(c)) = anti(neut(c)) = anti(neut(a)) = neut(a) \Rightarrow anti(c) \in X_a$. Thus, X_a is a neutrosophic triplet subgroup of X.
$neut(anti(a)) = neut(a) \Rightarrow anti(a) \in X_a$. Therefore, $(ac \ \ anti(a))neut(a) = (ac)(anti(a)neut(a)) = ac \ anti(a)$, and $neut(a)(ac \ anti(a)) = neut(a)a(c \ anti(a)) = ac \ anti(a)$. Thus, $neut(ac \ anti(a)) = neut(a) \Rightarrow ac \ anti(a) \in X_a$. Therefore, $X_a \overset{a}{\triangleleft} X$.

3. Let $c, d \in X_a$. Then $neut(c) = neut(a) = neut(d)$. Therefore, $neut(cd) = neut(a) = neut(a)neut(a) = neut(c)neut(d)$. Thus, X_a is a normal NTG.

4. For all $c, d \in X_a$, $neut(c)anti(d) = neut(a)anti(d) = neut(d)anti(d) = anti(d) = anti(d)neut(d) = anti(d)neut(a) = anti(d)$. Therefore, based on Point 3 and Theorem 3, $anti(cd) = anti(d)anti(c) \ \forall \ c, d \in X_a$.

5. Define a relation $\asymp$ on X_a as follows: $c \asymp d$ if $anti(c)d \in \ker f_a$ for all $c, d \in X_a$. $anti(c)c = neut(c) = neut(a) \Rightarrow anti(c)c \in \ker f_a \Rightarrow c \asymp c$. Therefore, $\asymp$ is reflexive.
$c \asymp d \Rightarrow anti(c)d \in \ker f_a \overset{\text{by 4.}}{\Rightarrow} anti(anti(c)d) \in \ker f_a \Rightarrow anti(d)c \in \ker f_a \Rightarrow d \asymp c$. Therefore, $\asymp$ is symmetric.
$c \asymp d, d \asymp z \Rightarrow anti(c)d, anti(d)z \in \ker f_a \Rightarrow anti(c)d \ anti(d)z = anti(c)neut(d)z = anti(c)neut(a)z = anti(c)z \in \ker f_a \Rightarrow c \asymp z$. Therefore, $\asymp$ is transitive and $\asymp$ is an

equivalence relation.

The equivalence class $[c]_{f_a} = \{d : anti(c)d \in \ker f_a\} = \{d : c\ anti(c)d \in c\ker f_a\} = \{d : neut(c)d \in c\ker f_a\} = \{d : neut(a)d \in c\ker f_a\} = \{d : d \in c\ker f_a\} = c\ker f_a$. Therefore, $X_a/ \asymp = \{[c]_{f_a}\}_{c\in X_a} = \{c\ker f_a\}_{c\in X_a}$.

Thus, $X_a = \bigcup_{c\in X_a} c\ker f_a$ for all $a \in X$.

6. If X is finite, then $|\ker f_a| = |c\ker f_a|$ for all $c \in X_a$. Thus, $|X_a| = \sum_{c\in X_a} |c\ker f_a| = [X_a : \ker f_a]|\ker f_a|$ for all $a \in X$ where $[X_a : \ker f_a]$ is the index of $\ker f_a$ in X_a, i.e., the number of distinct left cosets of $\ker f_a$ in X_a.
7. Define a relation $\sim$ on X: $c \sim d$ if $neut(c) = neut(d)$. $\sim$ is an equivalence relation on X, so $X/ \sim = \{X_c\}_{c\in X}$ and, therefore, $X = \bigcup_{a\in X} X_a$.
8. Hence, based on Point 7, if X is finite, then $|X| = \sum_{a\in X} |X_a| = \sum_{a\in X} [X_a : \ker f_a]|\ker f_a|$.

□

Theorem 7. *Let $a \in X$ and $f : X \to Y$ be a neutrosophic triplet group homomorphism. Then*

1. *f is a monomorphism if and only if $\ker f_a = \{neut(a)\}$ for all $a \in X$;*
2. *the factor set $X/\ker f = \bigcup_{a\in X} X_a/\ker f_a$ is a neutrosophic triplet group (neutrosophic triplet factor group) under the operation defined by*

$$c\ker f_a \cdot d\ker f_b = (cd)\ker f_{ab}.$$

Proof. 1. Let $\ker f_a = \{neut(a)\}$ and let $c, d \in X$. If $f(c) = f(d)$, this implies that $f(c\ anti(d)) = f(d)anti(f(d)) = f(d\ anti(f(d))) \Rightarrow f(c\ anti(d)) = neut(f(d)) \Rightarrow c\ anti(d) \in \ker f_d \Rightarrow$

$$c\ anti(d) = neut(d) = neut(anti(d)). \tag{5}$$

Similarly, $f(anti(d)c) = neut(f(d)) \Rightarrow anti(d)c \in \ker f_d \Rightarrow$

$$anti(d)c = neut(anti(d)). \tag{6}$$

Using Equations (5) and (6), $c = anti(anti(d)) = d$. Therefore, f is a monomorphism.

Conversely, if f is mono, then $f(d) = f(c) \Rightarrow d = c$. Let $k \in \ker f_a$, $a \in X$. Then $f(k) = neut(f(a)) = f(neut(a)) \Rightarrow k = neut(a)$. Therefore, $\ker f_a = \{neut(a)\}$ for all $a \in X$.

2. Let $c\ker f_a, d\ker f_b, z\ker f_c \in X/\ker f = \bigcup_{a\in X} X_a/\ker f_a$.

Groupoid: Based on the multiplication $c\ker f_a \cdot d\ker f_b = (cd)\ker f_{ab}$, the factor set $X/\ker f$ is a groupoid.

Semigroup: $(c\ker f_a \cdot d\ker f_b) \cdot z\ker f_c = (cdz)\ker f_{abc} = c\ker f_a(d\ker f_b \cdot z\ker f_c)$.

Neutrality: Let $neut(c\ker f_a) = neut(c)\ker f_{neut(a)}$. Then $c\ker f_a \cdot neut(c\ker f_a) = c\ker f_a \cdot neut(c)\ker f_{neut(a)} = (c\ neut(c))\ker f_{a\ neut(a)} = c\ker f_a$ and similarly, $neut(c\ker f_a) \cdot c\ker f_a = c\ker f_a$.

Opposite: Let $anti(c\ker f_a) = anti(c)\ker f_{anti(a)}$. Then $c\ker f_a \cdot anti(c\ker f_a) = c\ker f_a \cdot anti(c)\ker f_{anti(a)} = (c\ anti(c))\ker f_{a\ anti(a)} = neut(c)\ker f_{neut(a)}$. Similarly, $anti(c\ker f_a)) \cdot c\ker f_a = neut(c)\ker f_{neut(a)}$.

$\therefore$ $(X/\ker f, \cdot)$ is an NTG.

□

Theorem 8. *Let $\phi : X \to Y$ be a neutrosophic triplet group homomorphism. Then $X/\ker\phi \cong Im\ \phi$.*

Proof. Based on Theorem 6(7), $X = \bigcup_{a\in X} X_a$. Similarly, define a relation $\approx$ on $\phi(X) = \text{Im}\ \phi$: $\phi(c) \approx \phi(d)$ if $neut(\phi(c)) = neut(\phi(d))$. $\approx$ is an equivalence relation on $\phi(X)$, so $\phi(X)/\approx = \{\phi(X_c)\}_{c\in X}$ and $\text{Im}\ \phi = \bigcup_{c\in X} \phi(X_c)$. It should be noted that $X_a \overset{a}{\triangleleft} X$ in Theorem 6(2).

Let $\bar{\phi}_a : X_a/\ker\phi_a \to \phi(X_a)$ given by $\bar{\phi}_a(c\ker\phi_a) = \phi(c)$. It should be noted that, by Theorem 6(1), $\ker\phi_a \overset{a}{\triangleleft} X$. Therefore, $c\ker\phi_a = d\ker\phi_a \Rightarrow anti(d)c\ker\phi_a = anti(d)d\ker\phi_a = neut(d)\ker\phi_a = \ker\phi_a \Rightarrow anti(d)c\ker\phi_a = \ker\phi_a \Rightarrow \phi(anti(d)c) = neut(\phi(a)) \Rightarrow anti(\phi(d))\phi(c) = neut(\phi(a)) \Rightarrow \phi(d)anti(\phi(d))\phi(c) = \phi(d)neut(\phi(a)) \Rightarrow neut(\phi(d))\phi(c) = \phi(d)neut(\phi(a)) \Rightarrow \phi(neut(d))\phi(c) = \phi(d)\phi(neut(a)) \Rightarrow \phi(neut(d)\ c)) = \phi(d\ neut(a)) \Rightarrow \phi(neut(a)\ c)) = \phi(d\ neut(a)) \Rightarrow \phi(neut(c)\ c)) = \phi(d\ neut(c)) \Rightarrow \phi(c) = \phi(d) \Rightarrow \bar{\phi}_a(c\ker\phi_a) = \bar{\phi}_a(d\ker\phi_a)$. Thus, $\bar{\phi}_a$ is well defined.

$\bar{\phi}_a(c\ker\phi_a) = \bar{\phi}_a(d\ker\phi_a) \Rightarrow \phi(c) = \phi(d) \Rightarrow anti(\phi(d))\phi(c) = anti(\phi(d))\phi(d) = neut(\phi(d)) \Rightarrow \phi(anti(d))\phi(c) = neut(\phi(d)) = \phi(neut(d)) = \phi(neut(a)) = neut(\phi(a)) \Rightarrow \phi(anti(d)\ c) = neut(\phi(a)) \Rightarrow anti(d)\ c \in \ker\phi_a \Rightarrow d\ anti(d)\ c \in d\ker\phi_a \Rightarrow neut(d)\ c \in d\ker\phi_a \Rightarrow neut(a)\ c \in d\ker\phi_a \Rightarrow c \in d\ker\phi_a \overset{\text{Theorem 6(1)}}{\Longrightarrow} c\ker\phi_a = d\ker\phi_a$. This means that $\bar{\phi}_a$ is 1-1. $\bar{\phi}_a$ is obviously onto. Thus, $\bar{\phi}_a$ is bijective.

Now, based on the above and Theorem 7(2), we have a bijection

$$\Phi = \bigcup_{a\in X} \bar{\phi}_a : X/\ker\phi = \bigcup_{a\in X} X_a/\ker\phi_a \to \text{Im}\ \phi = \phi(X) = \bigcup_{a\in X} \phi(X_a)$$

defined by $\Phi(c\ker\phi_a) = \phi(c)$. Thus, if $c\ker\phi_a, d\ker\phi_b \in X/\ker\phi$, then

$$\Phi\Big(c\ker\phi_a \cdot d\ker\phi_b\Big) = \Phi\big(cd\ker\phi_a b\big) = \phi(cd) = \phi(c)\phi(d) = \Phi\big(c\ker\phi_a\big)\Phi\big(d\ker\phi_b\big).$$

$\therefore\ X/\ker\phi \cong \text{Im}\ \phi.$ □

2.3. Construction of Bol Algebraic Structures

We now present a method of constructing Bol algebraic structures using an NTG.

Theorem 9. *Let X be a non-abelian neutrosophic triplet group and let $A = X_a \times X$ for any fixed $a \in X$. For $(h_1, g_1), (h_2, g_2) \in A$, define $\circ$ on A as follows:*

$$(h_1, g_1) \circ (h_2, g_2) = (h_1h_2, h_2g_1\ anti(h_2)g_2).$$

Then $(A, \circ)$ is a Bol groupoid.

Proof. Let $a, b, c \in A$. By checking, it is true that $a \circ (b \circ c) \neq (a \circ b) \circ c$. Therefore, $(A, \circ)$ is non-associative. X_a is a normal neutrosophic triplet group by Theorem 6(3). A is a groupoid.

Let us now verify the Bol identity:

$$((a \circ b) \circ c) \circ b = a \circ ((b \circ c) \circ b)$$

$$\text{LHS} \quad = ((a \circ b) \circ c) \circ b = \Big(h_1h_2h_3h_2, h_2h_3h_2g_1\ anti(h_2)g_2\ anti(h_3)g_3\ anti(h_2)g_2\Big).$$

Following Theorem 6(4),

$$\begin{aligned}\text{RHS} &= a \circ ((b \circ c) \circ b) = \\ &\Big(h_1h_2h_3h_2, h_2h_3h_2g_1\ anti\big(h_2h_3h_2\big)h_2h_3g_2\ anti(h_3)g_3\ anti(h_2)g_2\Big) = \\ &\Big(h_1h_2h_3h_2, h_2h_3h_2g_1\ anti(h_2)\big(anti(h_3)\ anti(h_2)h_2h_3\big)g_2\ anti(h_3)g_3\ anti(h_2)g_2\Big) = \\ &\Big(h_1h_2h_3h_2, h_2h_3h_2g_1\ anti(h_2)\big(anti(h_3)\ neut(h_2)h_3\big)g_2\ anti(h_3)g_3\ anti(h_2)g_2\Big) = \\ &\Big(h_1h_2h_3h_2, h_2h_3h_2g_1\ anti(h_2)\big(anti(h_3)\ neut(a)h_3\big)g_2\ anti(h_3)g_3\ anti(h_2)g_2\Big) = \\ &\Big(h_1h_2h_3h_2, h_2h_3h_2g_1\ anti(h_2)anti(h_3)h_3g_2\ anti(h_3)g_3\ anti(h_2)g_2\Big) = \\ &\Big(h_1h_2h_3h_2, h_2h_3h_2g_1\ anti(h_2)neut(h_3)g_2\ anti(h_3)g_3\ anti(h_2)g_2\Big) = \\ &\Big(h_1h_2h_3h_2, h_2h_3h_2g_1\ anti(h_2)neut(a)g_2\ anti(h_3)g_3\ anti(h_2)g_2\Big) = \\ &\Big(h_1h_2h_3h_2, h_2h_3h_2g_1\ anti(h_2)g_2\ anti(h_3)g_3\ anti(h_2)g_2\Big).\end{aligned}$$

Therefore, LHS = RHS. Hence, $(A, \circ)$ is a Bol groupoid. □

Corollary 1. *Let H be a subgroup of a non-abelian neutrosophic triplet group X, and let $A = H \times X$. For $(h_1, g_1), (h_2, g_2) \in A$, define $\circ$ on A as follows:*

$$(h_1, g_1) \circ (h_2, g_2) = (h_1h_2, h_2g_1\ anti(h_2)g_2).$$

Then $(A, \circ)$ is a Bol groupoid.

Proof. A subgroup H is a normal neutrosophic triplet group. The rest of the claim follows from Theorem 9. □

Corollary 2. *Let H be a neutrosophic triplet subgroup (which obeys the cancellation law) of a non-abelian neutrosophic triplet group X, and let $A = H \times X$. For $(h_1, g_1), (h_2, g_2) \in A$, define $\circ$ on A as follows:*

$$(h_1, g_1) \circ (h_2, g_2) = (h_1h_2, h_2g_1\ anti(h_2)g_2).$$

Then $(A, \circ)$ is a Bol groupoid.

Proof. By Theorem 1(3), H is a subgroup of X. Hence, following Corollary 1, $(A, \circ)$ is a Bol groupoid. □

Corollary 3. *Let H be a neutrosophic triplet subgroup of a non-abelian neutrosophic triplet group X that has the cancellation law and let $A = H \times X$. For $(h_1, g_1), (h_2, g_2) \in A$, define $\circ$ on A as follows:*

$$(h_1, g_1) \circ (h_2, g_2) = (h_1h_2, h_2g_1\ anti(h_2)g_2).$$

Then $(A, \circ)$ is a Bol loop.

Proof. By Theorem 1(3), X is a non-abelian group and H is a subgroup of X. Hence, $(A, \circ)$ is a loop and a Bol loop by Theorem 9. □

3. Applications in Management and Sports

3.1. One-Way Management and Division of Labor

Consider a company or work place consisting of a set of people X with $|X|$ number of people. A working unit or subgroup with a leader 'a' is denoted by X_a.

$neut(x)$ for any $x \in X$ represents a co-worker (or co-workers) who has (have) a good (non-critical) working relationship with x, while $anti(x)$ represents a co-worker (or co-workers) whom x considers as his/her personal critic(s) at work.

Hence, X_a can be said to include both critics and non-critics of each worker x. It should be noted that in X_a, $neut(a) = neut(x)$ for all $x \in X_a$. This means that every worker in X_a has a good relationship with the leader 'a'.

Thus, by Theorem 6(7)—$X = \bigcup_{a \in X} X_a$ and $|X| = \Sigma_{a\in X}|X_a|$—the company or work place X can be said to have a good division of labor for effective performance and maximum output based on the composition of its various units (X_a). See Figure 1.

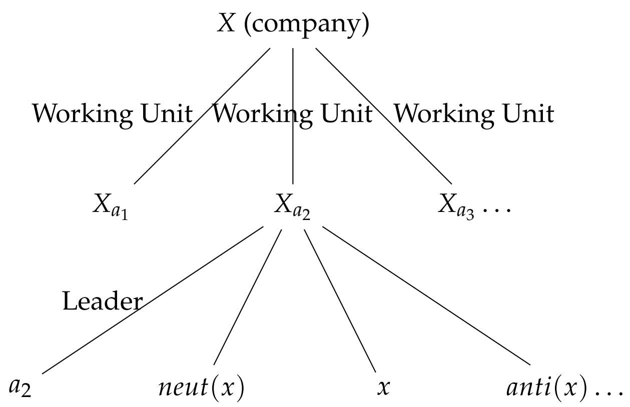

Figure 1. One-way management and division of labor.

3.2. Two-Way Management Division of Labor

Consider a company or work place consisting of a set of people X with $|X|$ number of people at a location A and another company or work place consisting of people Y with $|Y|$ number of people at another location B. Assume that both companies are owned by the same person f. Hence, $f : X \to Y$ can be considered as a movement (transfer) or working interaction between workers at A and at B. The fact that f is a neutrosophic triplet group homomorphism indicates that the working interaction between X and Y is preserved.

Let 'a' be a unit leader at A whose work correlates to another leader $f(a)$ at B. Then $Ker f_a$ represents the set of workers x in a unit at A under the leadership of 'a' such that there are other, corresponding workers $f(x)$ at B under the leadership of $f(a)$. Here, $f(x) = neut(f(a))$ means that workers $f(x)$ at B under the leadership of $f(a)$ are loyal and in a good working relationship. The mapping f_a shows that the operation of a subgroup leader (the operation is denoted by 'a') is subject to the modus operandi of the owner of the two companies, where the owner is denoted by f.

The final formula $|X| = \sum_{x\in X}[X_a : \ker f_a]|ker f_a|$ in Theorem 6(8) shows that the overall performance of the set of people X is determined by how the unit leaders 'a' at A properly harmonize with the unit leaders at B in the effective administration of $\ker f_a$ and X_a (Figure 2).

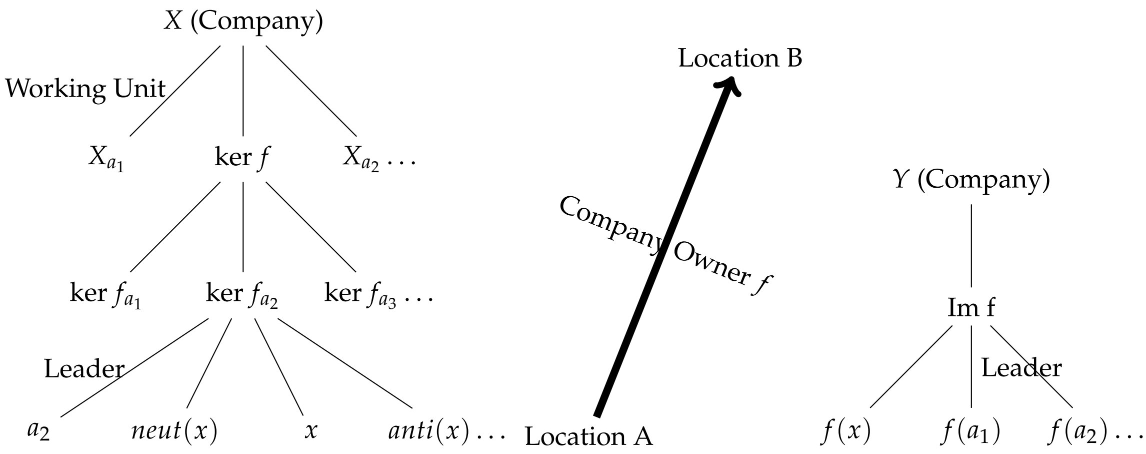

Figure 2. Two-way management division of labor.

3.3. Sports

In the composition of a team, a coach can take X_a as the set of players who play in a particular department (e.g., forward, middle field, or defence), where a is the leader of that department. Let $neut(x)$ represent player(s) whose performance is the same as that of player x, and let $anti(x)$ represent player(s) that can perform better than player x. It should be noted that the condition $neut(x) = neut(a)$ for all $x \in X_a$ means that the department X_a has player(s) who are equal in performance; i.e., those whose performance are equal to that of the departmental leader a. Hence, a neutrosophic triplet $(x, neut(x), anti(x))$ is a triple from which a coach can make a choice of his/her starting player and make a substitution. The neutrosophic triplet can also help a coach to make the best alternative choice when injuries arise. For instance, in the goal keeping department (for soccer/football), three goal keepers often make up the team for any international competition. Imagine an incomplete triplet $(x, neut(x), ?)$, i.e., no player is found to be better than x, which reduces to a duplet.

X_a can also be used for grouping teams in competitions in the preliminaries. If $x = team$, then $anti(x) = teams$ that can beat x and $neut(x) = teams$ that can play draw with x. Therefore, neutrosophic triplet $(x, neut(x), anti(x))$ is a triplet with which competition organizers can draw teams into groups for a balanced competition. The Fédération Internationale de Football Association (FIFA) often uses this template in drawing national teams into groups for its competitions. Club teams from various national leagues, to qualify for continental competitions (e.g., Union of European Football Associations (UEFA) Champions League and Confederation of African Football (CAF) Champions League), have to be among the five. This implies the application of duplets, triplets, quadruples, etc. (Figure 3).

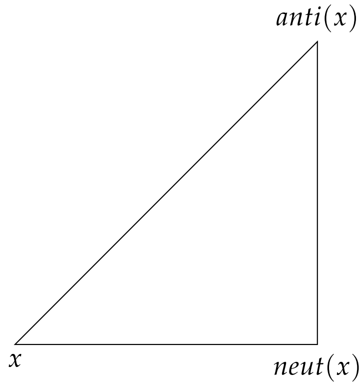

Figure 3. Sports.

Author Contributions: T.G.J. established new properties of neutrosophic triplet groups. He further presented applications of the neutrosophic triplet sets and groups to management and sports. F.S. cointroduced the neutrosophic triplet set and group, as well as their properties. He confirmed the relevance of the neutrosophic duplet and the quadruple in the applications of neutrosophic triplet set.

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Single Valued Neutrosophic Hesitant Fuzzy Computational Algorithm for Multiobjective Nonlinear Optimization Problem

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Abstract: In many real-life situations, it is often observed that the degree of indeterminacy (neutrality) plays an important role along with the satisfaction and dissatisfaction levels of the decision maker(s) (DM(s)) in any decision making process. Due to some doubt or hesitation, it may necessary for DM(s) to take opinions from experts which leads towards a set of conflicting values regarding satisfaction, indeterminacy and dis-satisfaction level of DM(s). In order to highlight the above-mentioned insight, we have developed an effective framework which reflects the reality involved in any decision-making process. In this study, a multiobjective nonlinear programming problem (MO-NLPP) has been formulated in the manufacturing system. A new algorithm, neutrosophic hesitant fuzzy programming approach (NHFPA), based on single-valued neutrosophic hesitant fuzzy decision set has been proposed which contains the concept of indeterminacy hesitant degree along with truth and falsity hesitant degrees of different objectives. In order to show the validity and applicability of the proposed approach, a numerical example has been presented. The superiority of the proposed approach has been shown by comparing with other existing approaches. Based on the present work, conclusions and future scope have been presented.

Keywords: Indeterminacy hesitant membership function, Neutrosophic hesitant fuzzy programming, Multiobjective nonlinear programming problem.

1 Introduction

Many decision-making processes inherently involved different conflicting objectives which are to be optimized (maximize/minimize) under given circumstances. In the present competitive era, it is indispensable for decision maker(s) (DM(s)) to obtain better possible outcomes/results when dealing with multiple objectives at a time. Although, it is quite difficult to have an optimal solution which satisfies all the objectives efficiently a compromise solution is possible which is accepted by DM(s) up to some extent. Literature reveals various approaches for multiobjective optimization problem and continuous effort have been made to obtain the best compromise solution. It is often observed that the modeling and formulation of the problem arising in agriculture production planning, manufacturing system etc. takes the form of nonlinear programming problem with multiple objective which is realistic in nature. Thus, multiobjective nonlinear programming problem (MO-NLPP) is also a challenging problem due to its local and global optimal concept, unlike multiobjective linear programming problem.

Bellman and Zadeh [5] introduced fuzzy set (FS) and based on that set Zimmermann [27] proposed fuzzy programming approach (FPA) for multiobjective optimization problems. The FPA deals only degree of belongingness but sometimes it may necessary to deal with non-membership function (non-belongingness) in order to obtain the results in the more realistic way. To overcome the above fact, Atanassov [4] introduced the intuitionistic fuzzy set (IFS) which is the extension of the FS. The IFS is based on more intuition as compared to FS because it also deals with the non-membership function (non-belongingness) of the element in the set. Based on IFS, intuitionistic fuzzy programming approach (IFPA) gained its own popularity among the existing multiobjective optimization techniques. Angelov [3] first used the optimization technique under intuitionistic fuzzy environment. Mahmoodirad et al. [15] proposed a new approach for the balanced transportation problem by considering all parameters and variables are of triangular intuitionistic fuzzy values and pointed out some shortcomings of existing approaches. Singh and Yadav [19] discussed multiobjective nonlinear programming problem in the manufacturing system and solved by using three approaches namely; *Zimmerman's technique,* γ*- operator* and *Min. bounded sum operator* with intuitionistic fuzzy parameters. Bharati and Singh [6] also proposed a new computational algorithm for multiobjective linear programming problem in the interval-valued intuitionistic fuzzy environment.

In recent years, the extensions or generalizations of FS and IFS have been presented with the fact that indeterminacy degree exists in real life and as a result, a set named neutrosophic set came in existence. Smarandache [20] introduced the concept of the neutrosophic set (NS). The term neutrosophic is the combination of two words, *neutre* from French meaning, neutral, and *sophia* from Greek meaning, skill/wisdom. Thus neutrosophic literally means *knowledge of neutral thoughts* which well enough differentiate it from FS and IFS. The neutrosophic set involves three membership functions, namely; maximization of truth (belongingness), indeterminacy (belongingness to some extent) and minimization of falsity (non-belongingness) in an efficient manner. Based on NS, neutrosophic programming approach (NPA) came into existence and extensively used in real life applications. Abdel-Basset et al. [1] proposed a novel approach to solving fully neutrosophic linear programming problem and applied to production planning problem. Rizk-Allah et al. [16] solved the MO-TPs under neutrosophic environment and compared the obtained results with the existing approach by measuring the ranking degree using TOPSIS approach. Ye et al. [23] formulated neutrosophic number nonlinear programming problem (NN-NPP) and proposed an effective method to solve the problem under neutrosophic number environments. Liu and You [12] extended Muirhead mean to interval neutrosophic set and developed some new operator named as interval neutrosophic Muirhead mean operators which have been further applied to multi-attribute decision making (MADM) problem. Liu et al. [14] have combined the power average operator with Herorian mean operator which results in linguistic neutrosophic power Herorian aggregation operator and extended them for neutrosophic information process. Ahmad and Adhami [2] have also solved the nonlinear transportation problem with fuzzy parameters using neutrosophic programming approach and compared the solution results with other existing approaches. Liu and Shi [10] have introduced the valued neutrosophic uncertain linguistic set and developed some operators which have been further used to multi-attribute group decision making (MAGDM) problem. Liu and Teng [11] have proposed some normal neutrosophic operator based on normal neutrosophic numbers and developed an MADM method based on neutrosophic number generalized weighted power averaging operator. Zhang et al. [25] have proposed some new MAGDM methods in which the attributes are interactive in the form of the interval-valued hesitant uncertain linguistic number. Liu and Zhang [13] have extended the Maclaurian symmetric mean operator to single-valued trapezoidal neutrosophic numbers and developed a method to deal with MAGDM problem based on single-valued trapezoidal neutrosophic weighted Maclaurian symmetric mean operator.

Sometimes, the DM(s) is(are) not sure about the single specific value of the parameters in the set due to doubt or incomplete information but a set of different conflicting values may possible to represent the membership degree for any element to the set. In order to deal with the above fact, Torra and Narukawa [21] introduced the concept of the hesitant fuzzy set (HFS). The HFS is the generalization of fuzzy set and is very useful tools by ensuring the active involvement of different experts' opinions in the decision-making process. Based on HFS, hesitant fuzzy programming approach (HFPA) has been developed which incontinently allows the DM(s) to collaborate with experts in order to collect their incompatible opinions. Bharati [7] developed the hesitant computational algorithm for multiobjective linear programming problem and applied to production planning problem. Zhang et al. [24] developed a hesitant fuzzy programming technique to deal with multi-criteria decision-making problems within the hesitant fuzzy elements environment. Zhou and Xu [26] proposed new portfolio selection and risk investment approaches under hesitant fuzzy environment. All the above-discussed sets have its own limitations regarding the existence of each element in the set. In brief, FS deals only the membership degree of the element in the set whereas IFS considers both membership and non-membership degree of the element in the set simultaneously. NS is the generalization of FS and IFS because it allows the DM(s) to implement the thoughts of neutrality which gives the indeterminacy membership degree for an element to the set. Furthermore, HFS is also an extension of FS as its membership is represented by a set of different conflicting values in the set. Based on the above-mentioned sets, various optimization techniques such as fuzzy optimization techniques, intuitionistic fuzzy optimization techniques, neutrosophic optimization techniques, and hesitant fuzzy optimization techniques have been developed and widely used to solve multiobjective optimization problem which usually exists in real life.

In real life, hesitancy is the most trivial issue in the decision-making process. To deal with it, HFS may be used as an appropriate tool by assigning a set of different membership degree for an element in the set. The limitation of HFS is that it only represents the truth hesitant membership degree and does not deals with indeterminacy hesitant membership degree and falsity hesitant membership degree for an element in the set which arises due to inconsistent, imprecise, inappropriate and incomplete information. On the other hand, a single-valued neutrosophic set (SVNS) is a special case of NS which provides an additional opportunity to the DM(s) by incorporating the thoughts of neutrality. It is only confined to the truth, indeterminacy and a falsity membership degree for an element to the set. It can not ensure the interference of a set of membership values due to doubt and consequently the involvement of different experts' opinions in the decision-making process. The crucial situation arises when the two aspects namely; hesitations and neutral thoughts exist simultaneously in the decision-making process. In this case, HFS and SVNS may not be an appropriate tool to represent the situation in an efficient and effective manner. Thus, this kind of situations are beyond the scope of FS, IFS, SVNS, and HFS and consequently beyond the scope of FPA, IFPA, NPA, and HFPA to decision making process respectively. Therefore, truth, indeterminacy and the falsity situations under hesitant uncertainty is more practical terminology in real life optimization problems.

To get rid of the above limitations, Ye [22] investigated a new set named single-valued neutrosophic hesitant fuzzy set (SVNHFS) which is the combination of HFS and SVNS respectively. The SVNHFS contemplate over truth hesitant fuzzy membership, indeterminacy hesitant fuzzy membership and the falsity hesitant fuzzy membership degrees for an element to the set. Biswas et al. [8] discussed multi-attribute decision-making problems in which the rating values are expressed with single-valued neutrosophic hesitant fuzzy set information and proposed grey relational analysis method for multi-attribute decision making. Şahin and Liu [17] investigated correlation and correlation coefficient of SVNHFSs and discussed its applications in the decision-making process. Biswas et al. [9] proposed a variety of distance measures for single-valued neutrosophic sets and applied these measures to multi-attribute decision-making problems. In this present study, a new computational method, neutrosophic hesitant fuzzy programming approach (NHFPA) has been proposed to obtain the best possible solution of MO-NLPP which is based on SVNHFS. The proposed NHFPA involves the three membership function, namely; maximization of truth hesitant fuzzy (belongingness), indeterminacy hesitant fuzzy (belongingness to some extent) and minimization of falsity hesitant fuzzy (non-belongingness) in an emphatic manner.

To best of our knowledge, no such method has been proposed in the literature to solve the MO-NLPP. The proposed method covers different aspects of impreciseness, vagueness, inaccuracy, the incompleteness that are often encountered in real life optimization problems and provides flexibility in the decision-making process. The remarkable point is that the proposed approach actively seeks opinions from different experts under the neutrosophic environment which is more practical in real life situations and strongly concerned with the involvement of distinguished experts in order to make the fruitful decision. The neutral/indeterminacy hesitant fuzzy concept involved in single-valued neutrosophic hesitant fuzzy set leads towards the future research scope in this domain.

The rest of the paper has been summarized as follows:
In section 2, the preliminaries regarding neutrosophic set, hesitant fuzzy set, and single-valued neutrosophic hesitant fuzzy set have been discussed while section 3 represents the problem formulation and development of the proposed neutrosophic hesitant fuzzy programming approach (NHFPA). In section 4, a numerical study has been presented in order to show the applicability and validity of the proposed approach. A comparative study has also done with other existing approaches. Finally, conclusions and future scope have been discussed based on the present work in section 5.

2 Preliminaries

2.1 Neutrosophic Set (NS)

Definition 2.1.1: [20] Let X be a universe discourse such that x ∈ X, then a neutrosophic set A in X is defined by three membership functions namely, truth $T_A(x)$, indeterminacy $I_A(x)$ and a falsity $F_A(x)$ and is denoted by the following form:

$$A = \{< x, T_A(x), I_A(x), F_A(x) > | x \in X\} \tag{1}$$

where $T_A(x), I_A(x)$ and $F_A(x)$ are real standard or non-standard subsets belong to $]0^-, 1^+[$, also given as, $T_A(x) : X \rightarrow]0^-, 1^+[$, $I_A(x) : X \rightarrow]0^-, 1^+[$, and $F_A(x) : X \rightarrow]0^-, 1^+[$. There is no restriction on the sum of $T_A(x), I_A(x)$ and $F_A(x)$, so we have,

$$0^- \leq \ sup\, T_A(x) + I_A(x) + \ sup\, F_A(x) \leq 3^+ \tag{2}$$

Definition 2.1.2: [20] A single valued neutrosophic set A over universe of discourse X is defined as

$$A = \{< x, T_A(x), I_A(x), F_A(x) > | x \in X\} \tag{3}$$

where $T_A(x), I_A(x)$ and $F_A(x) \in [0, 1]$ and $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$ for each $x \in X$.

2.2 Hesitant Fuzzy Set (HFS)

Definition 2.2.1: [21] Let there be a fixed set X; a hesitant fuzzy set A on X is defined in terms of a function $h_A(x)$ that when applied to X returns a finite subset of [0,1] and mathematically can be represented as follows:

$$A = \{< x, h_A(x) > | x \in X\} \tag{4}$$

where $h_A(x)$ is a set of some different values in [0,1], denoting the possible membership degrees of the element $x \in X$ to A. Also, we call $h_A(x)$ a hesitant fuzzy element.

Definition 2.2.2: [21] For a given hesitant fuzzy element h, its lower and upper bounds are defined as $h^-(x) = min\ h(x)$ and $h^+(x) = max\ h(x)$, respectively.

2.3 Single Valued Neutrosophic Hesitant Fuzzy Set (SVNHFS)

Definition 2.3.1: [22] Let there be a fixed set X; an SVNHFS on X is defined as follows:

$$N_h = \{< x, T_h(x), I_h(x), F_h(x) > | x \in X\} \tag{5}$$

where $T_h(x)$, $I_h(x)$ and $F_h(x)$ are three sets of some values in [0,1], denoting the possible truth hesitant membership degree, indeterminacy hesitant membership degree and the falsity hesitant membership degree of the element $x \in X$ to the set N_h, respectively, with the conditions $0 \leq \alpha,\ \beta,\ \gamma \leq 1$ and $0 \leq \alpha^+,\ \beta^+,\ \gamma^+ \leq 3$, where $\alpha \in T_h(x)$, $\beta \in I_h(x)$, $\gamma \in F_h(x)$ with $\alpha^+ \in T_h^+(x) = \cup_{\alpha \in T_h(x)} max\{\alpha\}$, $\beta^+ \in I_h^+(x) = \cup_{\beta \in I_h(x)} max\{\beta\}$ and $\gamma^+ \in F_h^+(x) = \cup_{\gamma \in F_h(x)} max\{\gamma\}$ for all $x \in X$.
For simplicity, the three-tuple $N_h(x) = \{T_h(x), I_h(x), F_h(x)\}$ is called a single-valued neutrosophic hesitant fuzzy element (SVNHFE) or triple hesitant fuzzy element.

From Definition 2.3.1, it is clear that the SVNHFS comprises three different kinds of membership functions, namely; truth hesitant membership function, indeterminacy hesitant membership function and the falsity hesitant membership function, which consequently results in a more reliable framework and provides pliable access to assign values for each element in the domain, and can deal with three kind of hesitancy in this situation at a time. Thus, classical sets, including fuzzy sets, intuitionistic fuzzy sets, single-valued neutrosophic sets, hesitant fuzzy sets, can be considered as special cases of SVNHFSs (see [22]). Fig. 1 shows the graphical representation of classical sets to SVNHFSs.

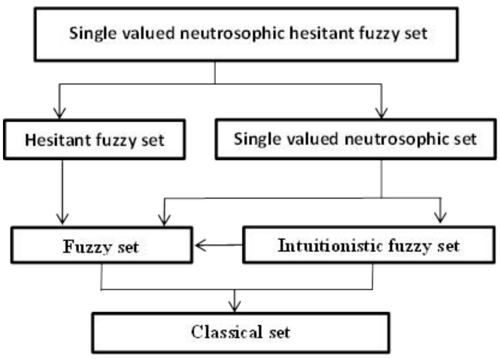


Figure 1: Diagrammatic coverage of classical sets to SVNHFSs.

Definition 2.3.2: [22] Let N_{h_1} and N_{h_2} be two SVNHFSs in a fixed set X; then their union can be defined as follows:

$$\begin{aligned} N_{h_1} \cup N_{h_2} = &\{T_h \in (T_{h_1} \cup T_{h_2}) | T_h \geq max\ (min\ \{T_{h_1} \cup T_{h_2}\}), \\ &I_h \in (I_{h_1} \cup I_{h_2}) | I_h \leq min\ (max\ \{I_{h_1} \cup I_{h_2}\}), \\ &F_h \in (F_{h_1} \cup F_{h_2}) | F_h \leq min\ (max\ \{F_{h_1} \cup F_{h_2}\})\} \end{aligned}$$

Definition 2.3.3: [22] Let N_{h_1} and N_{h_2} be two SVNHFSs in a fixed set X; then their intersection can be defined as follows:

$$\begin{aligned} N_{h_1} \cap N_{h_2} = &\{T_h \in (T_{h_1} \cap T_{h_2}) | T_h \leq min\ (max\ \{T_{h_1} \cap T_{h_2}\}), \\ &I_h \in (I_{h_1} \cap I_{h_2}) | I_h \geq max\ (min\ \{I_{h_1} \cap I_{h_2}\}), \\ &F_h \in (F_{h_1} \cap F_{h_2}) | F_h \geq max\ (min\ \{F_{h_1} \cap F_{h_2}\})\} \end{aligned}$$

3 Problem formulation and solution algorithm

3.1 General mathematical model of multiobjective nonlinear programming problem (MO-NLPP)

Generally, a mathematical programming problem is said to be nonlinear programming problem (NLPP) if either objective function, constraints or both are real-valued nonlinear functions. The objective function(s) is (are) to be optimized (minimize or maximize) under the given constraints. The classical multiobjective nonlinear programming problem (MO-NLPP) is represented in $\mathbf{M}_1$.

$$\begin{aligned}\mathbf{M}_1:\ & Optimize\ Z_k(x),\ \ k=1,2,...,K,\\ & s.t\ \ g_j(x)\le d_j,\ \ j=1,2,...,m_1,\\ & g_j(x)\ge d_j,\qquad j=m_1+1,m_1+2,...,m_2,\\ & g_j(x)=d_j,\qquad j=m_2+1,m_2+2,...,m,\\ & x\ge 0.\end{aligned}$$

where, either $Z_k,\ (k=1,2,...,K), g_j,\ (j=1,2,...,m)$ or both may be real valued nonlinear functions. $x=\ (x_1,x_2,...,x_q)$ is a set of decision variables.

3.2 Development of proposed neutrosophic hesitant fuzzy programming approach (NHFPA)

In this study, a new approach based on single-valued neutrosophic hesitant fuzzy set to solve MO-NLPP has been investigated. The proposed approach is based on the hybrid combination of the two sets, namely; neutrosophic set (Smarandache [20]) and hesitant fuzzy set (Torra and Narukawa [21]) respectively. The proposed neutrosophic hesitant fuzzy programming approach (NHFPA) introduces more realistic aspects in dealing with the indeterminacy hesitation present in the decision-making problem. The interesting point is that the proposed NHFPA also considers the conflicting opinions of different experts regarding some parameters in real life problem which enables the DM(s) to obtain the adequate results under neutrosophic environment.

According to Bellman and Zadeh [5], the fuzzy set includes three concepts, namely; fuzzy decision (D), fuzzy goal (G) and fuzzy constraints (C) and incorporated these concepts in many real-life applications of decision-making under fuzzy environment. So, the fuzzy decision set is defined as follows:

$$D=G\cap C \tag{6}$$

Consequently, the neutrosophic hesitant fuzzy decision set D_h^N, with neutrosophic hesitant objectives and constraints, is defined as follows:

$$\begin{aligned}D_h^N=G\cap C &= (\cap_{k=1}^{K}D_k)(\cap_{i=1}^{m}C_i)\\ &=\{x,\ T_D(x),\ I_D(x),\ F_D(x)\}\\ &=\{T_D\in(T_{G_h}\cap T_{C_h})\mid T_D\le min\ (max\ \{T_{G_h}\cap T_{C_h}\}),\\ &\quad I_D\in(I_{G_h}\cap I_{C_h})\mid I_D\ge max\ (min\ \{I_{G_h}\cap I_{C_h}\}),\\ &\quad F_D\in(F_{G_h}\cap F_{C_h})\mid F_D\ge max\ (min\ \{F_{G_h}\cap F_{C_h}\})\}\end{aligned}$$

Where, $T_D(x),\ I_D(x)$ and $F_D(x)$ are a set of degree of acceptance of neutrosophic hesitant fuzzy decision solution under single-valued neutrosophic hesitant fuzzy decision set. Fig.2 shows the neutrosophic hesitant fuzzy membership degree for the objective function.

On solving each objective function individually, we have k solutions set, $X^1,X^2,...,X^k$, after that the obtained solutions are substituted in each objective function to determine the lower and upper bound for each objective as given below:

$$U_k=max[Z_k(X^k)]\ \ and\ \ L_k=min[Z_k(X^k)]\ \ \forall\, k=1,2,3,...,K. \tag{7}$$

Now, we can define the different hesitant membership function more elaborately under neutrosophic hesitant fuzzy environment as follows:

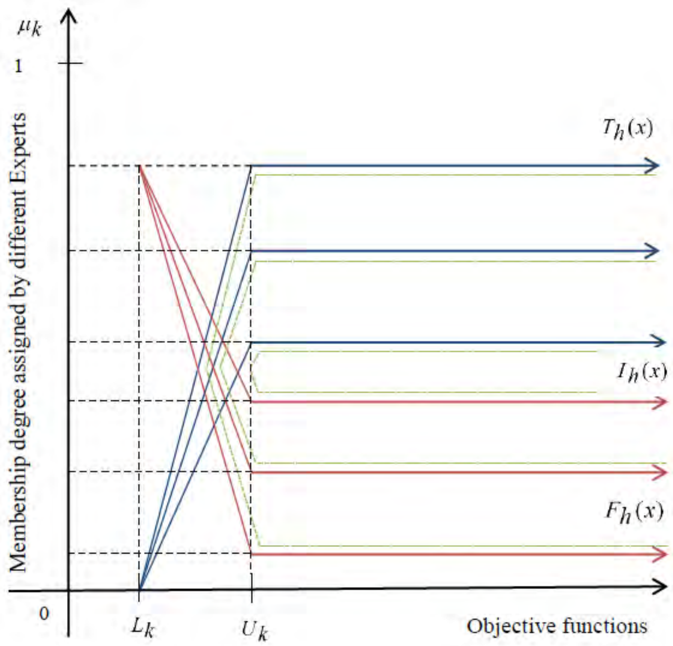


Figure 2: Graphical representation of neutrosophic hesitant fuzzy membership of objective function.

C$ase-I$: For maximization type objective function.

The truth hesitant-membership functions:

$$T_{h^+}^{E_1}(Z_k(x)) = \begin{cases} 0 & if \;\; Z_k(x) < L_k \\ \alpha_1 \frac{(Z_k(x))^t - (L_k)^t}{(U_k)^t - (L_k)^t} & if \;\; L_k \leq Z_k(x) \leq U_k \\ 1 & if \;\; Z_k(x) > U_k \end{cases} \tag{8}$$

$$T_{h^+}^{E_2}(Z_k(x)) = \begin{cases} 0 & if \;\; Z_k(x) < L_k \\ \alpha_2 \frac{(Z_k(x))^t - (L_k)^t}{(U_k)^t - (L_k)^t} & if \;\; L_k \leq Z_k(x) \leq U_k \\ 1 & if \;\; Z_k(x) > U_k \end{cases} \tag{9}$$

$$\begin{matrix} . & ... \\ . & ... \\ . & ... \end{matrix}$$

$$T_{h^+}^{E_n}(Z_k(x)) = \begin{cases} 0 & if \;\; Z_k(x) < L_k \\ \alpha_n \frac{(Z_k(x))^t - (L_k)^t}{(U_k)^t - (L_k)^t} & if \;\; L_k \leq Z_k(x) \leq U_k \\ 1 & if \;\; Z_k(x) > U_k \end{cases} \tag{10}$$

The indeterminacy hesitant-membership functions:

$$I_{h^+}^{E_1}(Z_k(x)) = \begin{cases} 0 & if \;\; Z_k(x) < L_k \\ \beta_1 \frac{(Z_k(x))^t - (L_k)^t}{(s_k)^t} & if \;\; L_k \leq Z_1(x) \leq L_k + s_k \\ 1 & if \;\; Z_k(x) > L_k + s_k \end{cases} \tag{11}$$

$$I_{h^+}^{E_2}(Z_k(x)) = \begin{cases} 0 & if \;\; Z_k(x) < L_k \\ \beta_2 \frac{(Z_k(x))^t - (L_k)^t}{(s_k)^t} & if \;\; L_k \leq Z_k(x) \leq L_k + s_k \\ 1 & if \;\; Z_k(x) > L_k + s_k \end{cases} \tag{12}$$

$$\begin{matrix} . & ... \\ . & ... \\ . & ... \end{matrix}$$

$$I_{h^+}^{E_n}(Z_k(x)) = \begin{cases} 0 & if \;\; Z_k(x) < L_k \\ \beta_n \frac{(Z_k(x))^t - (L_k)^t}{(s_k)^t} & if \;\; L_k \leq Z_k(x) \leq L_k + s_k \\ 1 & if \;\; Z_k(x) > L_k + s_k \end{cases} \tag{13}$$

The falsity hesitant-membership functions:

$$F_{h^+}^{E_1}(Z_k(x)) = \begin{cases} 1 & if \;\; Z_k(x) < L_k + t_k \\ \gamma_1 \frac{(U_k)^t - (Z_k(x))^t - (t_k)^t}{(U_k)^t - (L_k)^t - (t_k)^t} & if \;\; L_k + t_k \leq Z_k(x) \leq U_k \\ 0 & if \;\; Z_k(x) > U_k \end{cases} \tag{14}$$

$$F_{h^+}^{E_2}(Z_k(x)) = \begin{cases} 1 & if \;\; Z_k(x) < L_k + t_k \\ \gamma_2 \frac{(U_k)^t - (Z_k(x))^t - (t_k)^t}{(U_k)^t - (L_k)^t - (t_k)^t} & if \;\; L_k + t_k \leq Z_k(x) \leq U_k \\ 0 & if \;\; Z_k(x) > U_k \end{cases} \tag{15}$$

$$\begin{matrix} . & .. & . \\ . & .. & . \\ . & .. & . \end{matrix}$$

$$F_{h^+}^{E_n}(Z_k(x)) = \begin{cases} 1 & if \;\; Z_k(x) < L_k + t_k \\ \gamma_n \frac{(U_k)^t - (Z_k(x))^t - (t_k)^t}{(U_k)^t - (L_k)^t - (t_k)^t} & if \;\; L_k + t_k \leq Z_k(x) \leq U_k \\ 0 & if \;\; Z_k(x) > U_k \end{cases} \tag{16}$$

where parameter $t > 0$ and $s_k, t_k \in (0, 1) \; \forall k$, are indeterminacy and falsity tolerance values, which is assigned by DM(s) and h^+ represents the maximization type hesitant objective function.
$T_{h^+}^{E_1}(Z_k(x)),\; I_{h^+}^{E_1}(Z_k(x)),\; F_{h^+}^{E_1}(Z_k(x))$ are truth, indeterminacy and the falsity-hesitant-membership degrees assigned by 1^{st} expert.
$T_{h^+}^{E_2}(Z_k(x)),\; I_{h^+}^{E_2}(Z_k(x)),\; F_{h^+}^{E_2}(Z_k(x))$ are truth, indeterminacy and the falsity-hesitant-membership degrees assigned by 2^{nd} expert.
......
......
$T_{h^+}^{E_n}(Z_k(x)),\; I_{h^+}^{E_n}(Z_k(x)),\; F_{h^+}^{E_n}(Z_k(x))$ are truth, indeterminacy and the falsity-hesitant-membership degrees assigned by n^{th} expert.

C$ase - II$: For minimization type objective function.
The truth hesitant-membership functions:

$$T_{h^-}^{E_1}(Z_k(x)) = \begin{cases} 1 & if \;\; Z_k(x) < L_k \\ \alpha_1 \frac{(U_k)^t - (Z_k(x))^t}{(U_k)^t - (L_k)^t} & if \;\; L_k \leq Z_k(x) \leq U_k \\ 0 & if \;\; Z_k(x) > U_k \end{cases} \tag{17}$$

$$T_{h^-}^{E_2}(Z_k(x)) = \begin{cases} 1 & if \;\; Z_k(x) < L_k \\ \alpha_2 \frac{(U_k)^t - (Z_k(x))^t}{(U_k)^t - (L_k)^t} & if \;\; L_k \leq Z_k(x) \leq U_k \\ 0 & if \;\; Z_k(x) > U_k \end{cases} \tag{18}$$

$$\begin{array}{ccc} . & .. & . \\ . & .. & . \\ . & .. & . \end{array}$$

$$T_{h^-}^{E_n}(Z_k(x)) = \begin{cases} 1 & if \;\; Z_k(x) < L_k \\ \alpha_n \; \frac{(U_k)^t - (Z_k(x))^t}{(U_k)^t - (L_k)^t} & if \;\; L_k \leq Z_k(x) \leq U_k \\ 0 & if \;\; Z_k(x) > U_k \end{cases} \tag{19}$$

The indeterminacy hesitant-membership functions:

$$I_{h^-}^{E_1}(Z_k(x)) = \begin{cases} 1 & if \;\; Z_k(x) < U_k - s_k \\ \beta_1 \; \frac{(U_k)^t - (Z_k(x))^t}{(s_k)^t} & if \;\; U_k - s_k \leq Z_k(x) \leq U_k \\ 0 & if \;\; Z_k(x) > U_k \end{cases} \tag{20}$$

$$I_{h^-}^{E_2}(Z_k(x)) = \begin{cases} 1 & if \;\; Z_k(x) < U_k - s_k \\ \beta_2 \; \frac{(U_k)^t - (Z_k(x))^t}{(s_k)^t} & if \;\; U_k - s_k \leq Z_k(x) \leq U_k \\ 0 & if \;\; Z_k(x) > U_k \end{cases} \tag{21}$$

$$\begin{array}{cc} . & ... \\ . & ... \\ . & ... \end{array}$$

$$I_{h^-}^{E_n}(Z_k(x)) = \begin{cases} 1 & if \;\; Z_k(x) < U_k - s_k \\ \beta_n \; \frac{(U_k)^t - (Z_k(x))^t}{(s_k)^t} & if \;\; U_k - s_k \leq Z_k(x) \leq U_k \\ 0 & if \;\; Z_k(x) > U_k \end{cases} \tag{22}$$

The falsity hesitant-membership functions:

$$F_{h^-}^{E_1}(Z_k(x)) = \begin{cases} 0 & if \;\; Z_k(x) < L_k + t_k \\ \gamma_1 \; \frac{(Z_k(x))^t - (L_k)^t - (t_k)^t}{(U_k)^t - (L_k)^t - (t_k)^t} & if \;\; L_k + t_k \leq Z_k(x) \leq U_k \\ 1 & if \;\; Z_k(x) > U_k \end{cases} \tag{23}$$

$$F_{h^-}^{E_2}(Z_k(x)) = \begin{cases} 0 & if \;\; Z_k(x) < L_k + t_k \\ \gamma_2 \; \frac{(Z_k(x))^t - (L_k)^t - (t_k)^t}{(U_k)^t - (L_k)^t - (t_k)^t} & if \;\; L_k + t_k \leq Z_k(x) \leq U_k \\ 1 & if \;\; Z_k(x) > U_k \end{cases} \tag{24}$$

$$\begin{array}{cc} . & ... \\ . & ... \\ . & ... \end{array}$$

$$F_{h^-}^{E_n}(Z_k(x)) = \begin{cases} 0 & if \;\; Z_k(x) < L_k + t_k \\ \gamma_n \; \frac{(Z_k(x))^t - (L_k)^t - (t_k)^t}{(U_k)^t - (L_k)^t - (t_k)^t} & if \;\; L_k + t_k \leq Z_k(x) \leq U_k \\ 1 & if \;\; Z_k(x) > U_k \end{cases} \tag{25}$$

where parameter $t > 0$ and $s_k, t_k \in (0,1)\ \forall k$, are indeterminacy and falsity tolerance values, which is assigned by DM(s) and h^- represents the minimization type hesitant objective function.
$T_{h^-}^{E_1}(Z_k(x)),\ I_{h^-}^{E_1}(Z_k(x)),\ F_{h^-}^{E_1}(Z_k(x))$ are truth, indeterminacy and the falsity-hesitant-membership degrees assigned by 1^{st} expert.
$T_{h^-}^{E_2}(Z_k(x)),\ I_{h^-}^{E_2}(Z_k(x)),\ F_{h^-}^{E_2}(Z_k(x))$ are truth, indeterminacy and the falsity-hesitant-membership degrees assigned by 2^{nd} expert.
......
......
$T_{h^-}^{E_n}(Z_k(x)),\ I_{h^-}^{E_n}(Z_k(x)),\ F_{h^-}^{E_n}(Z_k(x))$ are truth, indeterminacy and the falsity-hesitant-membership degrees assigned by n^{th} expert.

Let $T_h^{E_n} = min\ (T_{h^+}^{E_n},\ T_{h^-}^{E_n})$, $I_h^{E_n} = min\ (I_{h^+}^{E_n},\ I_{h^-}^{E_n})$ and $F_h^{E_n} = max\ (F_{h^+}^{E_n},\ F_{h^-}^{E_n})\ \forall\ k = 1, 2, ..., K$. Now, the motive is to determine the highest degree of satisfaction for DM(s) by establishing a balance between objectives and constraints.
The neutrosophic hesitant fuzzy model for MO-NLPP ($\mathbf{M}_1$) can be represented as follows:

$$\begin{aligned} \mathbf{M}_2 : & Max\ \ min_{k=1,2,3,...,K}\ \ T_h^{E_n}(Z_k(x)) \\ & Max\ \ min_{k=1,2,3,...,K}\ \ I_h^{E_n}(Z_k(x)) \\ & Min\ \ max_{k=1,2,3,...,K}\ \ F_h^{E_n}(Z_k(x)) \\ & s.t\ \ g_j(x) \leq d_j,\ \ j = 1, 2, ..., m_1, \\ & g_j(x) \geq d_j, \qquad j = m_1 + 1, m_1 + 2, ..., m_2, \\ & g_j(x) = d_j, \qquad j = m_2 + 1, m_2 + 2, ..., m, \\ & x \geq 0. \end{aligned}$$

With the help of auxiliary parameters, model $\mathbf{M}_2$ can be transformed into the following form $\mathbf{M}_3$.

$$\begin{aligned}
\mathbf{M}_3 :& Max\ \frac{\sum \alpha_n}{n} \\
& Max\ \frac{\sum \beta_n}{n} \\
& Min\ \frac{\sum \gamma_n}{n} \\
& s.t.\ T_{h^+}^{E_n}(Z_k(x)) \geq \alpha_n,\ \ I_{h^+}^{E_n}(Z_k(x)) \geq \beta_n,\ \ F_{h^+}^{E_n}(Z_k(x)) \leq \gamma_n \\
& T_{h^-}^{E_n}(Z_k(x)) \geq \alpha_n,\ \ I_{h^-}^{E_n}(Z_k(x)) \geq \beta_n,\ \ F_{h^-}^{E_n}(Z_k(x)) \leq \gamma_n \\
& g_j(x) \leq d_j,\ \ j = 1, 2, ..., m_1, \\
& g_j(x) \geq d_j,\ \ j = m_1 + 1, m_1 + 2, ..., m_2, \\
& g_j(x) = d_j,\ \ j = m_2 + 1, m_2 + 2, ..., m, \\
& x \geq 0, \quad \alpha_n, \beta_n, \gamma_n \in (0, 1) \\
& \alpha_n + \beta_n + \gamma_n \leq 3, \quad \alpha_n \geq \beta_n, \quad \alpha_n \geq \gamma_n, \quad \forall\ n.
\end{aligned}$$

Using linear membership function, model $\mathbf{M}_3$ can be written as in $\mathbf{M}_4$.

$$\begin{aligned}
\mathbf{M}_4 :& Max\ \chi = \frac{\alpha_1 + \alpha_2 + ... + \alpha_n}{n} + \frac{\beta_1 + \beta_2 + ... + \beta_n}{n} - \frac{\gamma_1 + \gamma_2 + ... + \gamma_n}{n} \\
& s.t.\ T_{h^+}^{E_1}(Z_k(x)) \geq \alpha_1,\ T_{h^+}^{E_2}(Z_k(x)) \geq \alpha_2,\ ...,\ T_{h^+}^{E_n}(Z_k(x)) \geq \alpha_n \\
& I_{h^+}^{E_1}(Z_k(x)) \geq \beta_1,\ I_{h^+}^{E_2}(Z_k(x)) \geq \beta_2,\ ...,\ I_{h^+}^{E_n}(Z_k(x)) \geq \beta_n \\
& F_{h^+}^{E_1}(Z_k(x)) \leq \gamma_1,\ F_{h^+}^{E_2}(Z_k(x)) \leq \gamma_2,\ ...,\ F_{h^+}^{E_n}(Z_k(x)) \leq \gamma_n \\
& T_{h^-}^{E_1}(Z_k(x)) \geq \alpha_1,\ T_{h^-}^{E_2}(Z_k(x)) \geq \alpha_2,\ ...,\ T_{h^-}^{E_n}(Z_k(x)) \geq \alpha_n \\
& I_{h^-}^{E_1}(Z_k(x)) \geq \beta_1,\ I_{h^-}^{E_2}(Z_k(x)) \geq \beta_2,\ ...,\ I_{h^-}^{E_n}(Z_k(x)) \geq \beta_n \\
& F_{h^-}^{E_1}(Z_k(x)) \leq \gamma_1,\ F_{h^-}^{E_2}(Z_k(x)) \leq \gamma_2,\ ...,\ F_{h^-}^{E_n}(Z_k(x)) \leq \gamma_n \\
& g_j(x) \leq d_j, \quad j = 1, 2, ..., m_1, \\
& g_j(x) \geq d_j, \quad j = m_1 + 1, m_1 + 2, ..., m_2, \\
& g_j(x) = d_j, \quad j = m_2 + 1, m_2 + 2, ..., m, \\
& x \geq 0,\ \ 0 \leq \alpha_1, \alpha_2, ..., \alpha_n \leq 1,\ \ 0 \leq \beta_1, \beta_2, ..., \beta_n \leq 1 \\
& 0 \leq \gamma_1, \gamma_2, ..., \gamma_n \leq 1,\ \ \alpha_n \geq \beta_n,\ \ \alpha_n \geq \gamma_n, \\
& \alpha_n + \beta_n + \gamma_n \leq 3,\ \ \forall\ n.
\end{aligned}$$

Finally, model $\mathbf{M}_4$ gives the compromise solution to MO-NLPP.

3.3 Proposed NHFPA algorithm for MO-NLPP

The whole procedure from problem formulation to final solvable model $\mathbf{M}_4$ discussed in section 3 is summarized as step-wise algorithm.
Step-1. Formulate the multiobjective nonlinear programing problems as in $\mathbf{M}_1$.
Step-2. Determine the bounds U_k and L_k, for each objective by using equation (7).
Step-3. By using U_k and L_k, define the upper and lower bound for truth hesitant, indeterminacy hesitant and falsity hesitant membership functions as given in equation (8)-(25).
Step-4. Ask for the truth hesitant, indeterminacy hesitant and the falsity hesitant membership degrees from different experts or DM(s).
Step-5. Formulate MO-NLPP under neutrosophic hesitant fuzzy environment defined in $\mathbf{M}_4$.
Step-6. Solve the multiobjective nonlinear programing problem in order to obtain the compromise solution using suitable techniques or some optimizing software packages.

4 Experimental study

In order to show the efficiency and validity of the proposed method, we adopted the numerical example of the manufacturing system discussed by Singh and Yadav [19]. The DM(s) of the company intends to maximize the total profit incurred over products and minimize the total time required for each product. Also, assumed that the DM(s) seeks three experts' opinion in the decision-making process. Therefore, the crisp multiobjective non-linear programming problem formulation [19] is given as follows:

$$\begin{aligned}
\mathbf{M}_1 :\ & Max\ Z_1(x) = 99.875x_1^{\frac{1}{2}} - 8x_1 + 119.875x_2^{\frac{1}{2}} - 10.125x_2 + 95.125x_3^{\frac{1}{3}} - 8x_3 \\
& Min\ Z_2(x) = 3.875x_1 + 5.125x_2 + 5.9375x_3 \\
& s.t\ \ 2.0625x_1 + 3.875x_2 + 2.9375x_3 \leq 333.125 \\
& 3.875x_1 + 2.0625x_2 + 2.0625x_3 \leq 365.625 \\
& 2.9375x_1 + 2.0625x_2 + 2.9375x_3 \geq 360 \\
& x_1,\ x_2,\ x_3 \geq 0.
\end{aligned}$$

On solving each objective function individually given in $(\mathbf{M}_1)$, we get the following individual best solution, lower and upper bound for each objective.
$X^1 = (57.82,\ 13.09,\ 55.53)$, $X^2 = (62.26,\ 0,\ 60.28)$ along with $L_1 = 180.72$, $U_1 = 516.70$, $L_2 = 599.23$ and $U_2 = 620.84$.

Since, the first objective $Z_1(x)$ is of maximization type and the satisfaction level of Experts or DMs increases if the values of objective function tends towards its upper bound. Therefore the truth hesitant membership, indeterminacy hesitant membership and falsity hesitant membership functions of upper bound can be represented as follows:

For Z_1: The upper and lower bound for first objective and its membership functions.

$$T_{h^+}^{E_1}(Z_1(x)) = \begin{cases} 0 & if\ \ Z_1(x) < 180.72 \\ 0.98\ \frac{(99.875x_1^{\frac{1}{2}}-8x_1+119.875x_2^{\frac{1}{2}}-10.125x_2+95.125x_3^{\frac{1}{3}}-8x_3)^t-(180.72)^t}{(516.70)^t-(180.72)^t} & if\ \ 180.72 \leq Z_1(x) \leq 516.70 \\ 1 & if\ \ Z_1(x) > 516.70 \end{cases} \tag{26}$$

$$T_{h^+}^{E_2}(Z_1(x)) = \begin{cases} 0 & if\ \ Z_1(x) < 180.72 \\ 0.99\ \frac{(99.875x_1^{\frac{1}{2}}-8x_1+119.875x_2^{\frac{1}{2}}-10.125x_2+95.125x_3^{\frac{1}{3}}-8x_3)^t-(180.72)^t}{(516.70)^t-(180.72)^t} & if\ \ 180.72 \leq Z_1(x) \leq 516.70 \\ 1 & if\ \ Z_1(x) > 516.70 \end{cases} \tag{27}$$

$$T_{h^+}^{E_3}(Z_1(x)) = \begin{cases} 0 & if\ \ Z_1(x) < 180.72 \\ \frac{(99.875x_1^{\frac{1}{2}}-8x_1+119.875x_2^{\frac{1}{2}}-10.125x_2+95.125x_3^{\frac{1}{3}}-8x_3)^t-(180.72)^t}{(516.70)^t-(180.72)^t} & if\ \ 180.72 \leq Z_1(x) \leq 516.70 \\ 1 & if\ \ Z_1(x) > 516.70 \end{cases} \tag{28}$$

$$I_{h^+}^{E_1}(Z_1(x)) = \begin{cases} 0 & if\ \ Z_1(x) < 180.72 \\ 0.98\ \frac{(99.875x_1^{\frac{1}{2}}-8x_1+119.875x_2^{\frac{1}{2}}-10.125x_2+95.125x_3^{\frac{1}{3}}-8x_3)^t-(180.72)^t}{(s_1)^t} & if\ \ 180.72 \leq Z_1(x) \leq 180.72+s_1 \\ 1 & if\ \ Z_1(x) > 180.72+s_1 \end{cases} \tag{29}$$

$$I_{h^+}^{E_2}(Z_1(x)) = \begin{cases} 0 & if\ \ Z_1(x) < 180.72 \\ 0.99\ \frac{(99.875x_1^{\frac{1}{2}}-8x_1+119.875x_2^{\frac{1}{2}}-10.125x_2+95.125x_3^{\frac{1}{3}}-8x_3)^t-(180.72)^t}{(s_1)^t} & if\ \ 180.72 \leq Z_1(x) \leq 180.72+s_1 \\ 1 & if\ \ Z_1(x) > 180.72+s_1 \end{cases} \tag{30}$$

$$I_{h^+}^{E_3}(Z_1(x)) = \begin{cases} 0 & if\ \ Z_1(x) < 180.72 \\ \frac{(99.875x_1^{\frac{1}{2}}-8x_1+119.875x_2^{\frac{1}{2}}-10.125x_2+95.125x_3^{\frac{1}{3}}-8x_3)^t-(180.72)^t}{(s_1)^t} & if\ \ 180.72 \leq Z_1(x) \leq 180.72+s_1 \\ 1 & if\ \ Z_1(x) > 180.72+s_1 \end{cases} \tag{31}$$

$$F_{h^+}^{E_1}(Z_1(x)) = \begin{cases} 1 & if\ \ Z_1(x) < 180.72 \\ 0.98\ \frac{(516.70)^t-(t_1)^t-(99.875x_1^{\frac{1}{2}}-8x_1+119.875x_2^{\frac{1}{2}}-10.125x_2+95.125x_3^{\frac{1}{3}}-8x_3)^t}{(516.70)^t-(180.72)^t-(t_1)^t} & if\ \ 180.72 \leq Z_1(x) \leq 516.70-t_1 \\ 0 & if\ \ Z_1(x) > 516.70-t_1 \end{cases} \tag{32}$$

$$F_{h^+}^{E_2}(Z_1(x)) = \begin{cases} 1 & if\ \ Z_1(x) > 516.70 \\ 0.99\ \frac{(516.70)^t-(t_1)^t-(99.875x_1^{\frac{1}{2}}-8x_1+119.875x_2^{\frac{1}{2}}-10.125x_2+95.125x_3^{\frac{1}{3}}-8x_3)^t}{(516.70)^t-(180.72)^t-(t_1)^t} & if\ \ 180.72+t_1 \leq Z_1(x) \leq 516.70 \\ 0 & if\ \ Z_1(x) < 180.72+t_1 \end{cases} \tag{33}$$

$$F_{h^+}^{E_3}(Z_1(x)) = \begin{cases} 1 & if\ \ Z_1(x) > 516.70 \\ \frac{(516.70)^t-(t_1)^t-(99.875x_1^{\frac{1}{2}}-8x_1+119.875x_2^{\frac{1}{2}}-10.125x_2+95.125x_3^{\frac{1}{3}}-8x_3)^t}{(516.70)^t-(180.72)^t-(t_1)^t} & if\ \ 180.72+t_1 \leq Z_1(x) \leq 516.70 \\ 0 & if\ \ Z_1(x) < 180.72+t_1 \end{cases} \tag{34}$$

Similarly, the second objective $Z_2(x)$ is of minimization type and the satisfaction level of Experts or DMs increases if the values of objective function tends towards its lower bound. Thus the truth hesitant membership, indeterminacy hesitant membership and falsity hesitant membership functions of lower bound can be represented as follows:

For Z_2: The upper and lower bound for second objective and its membership functions.

$$T_{h^-}^{E_1}(Z_2(x)) = \begin{cases} 1 & if\ \ Z_2(x) < 599.23 \\ 0.98\ \frac{(620.84)^t-(3.875x_1+5.125x_2+5.9375x_3)^t}{(620.84)^t-(599.23)^t} & if\ \ 599.23 \leq Z_2(x) \leq 620.84 \\ 0 & if\ \ Z_2(x) > 620.84 \end{cases} \tag{35}$$

$$T_{h^-}^{E_2}(Z_2(x)) = \begin{cases} 1 & if\ \ Z_2(x) < 599.23 \\ 0.99\ \frac{(620.84)^t-(3.875x_1+5.125x_2+5.9375x_3)^t}{(620.84)^t-(599.23)^t} & if\ \ 599.23 \leq Z_2(x) \leq 620.84 \\ 0 & if\ \ Z_2(x) > 620.84 \end{cases} \tag{36}$$

$$T_{h^-}^{E_3}(Z_2(x)) = \begin{cases} 1 & if\ \ Z_2(x) < 599.23 \\ \frac{(620.84)^t-(3.875x_1+5.125x_2+5.9375x_3)^t}{(620.84)^t-(599.23)^t} & if\ \ 599.23 \leq Z_2(x) \leq 620.84 \\ 0 & if\ \ Z_2(x) > 620.84 \end{cases} \tag{37}$$

$$I_{h^-}^{E_1}(Z_2(x)) = \begin{cases} 1 & if\ \ Z_2(x) < 620.84-s_2 \\ 0.98\ \frac{(620.84)^t-(3.875x_1+5.125x_2+5.9375x_3)^t}{(s_2)^t} & if\ \ 620.84-s_2 \leq Z_2(x) \leq 620.84 \\ 0 & if\ \ Z_2(x) > 620.84 \end{cases} \tag{38}$$

$$I_{h^-}^{E_2}(Z_2(x)) = \begin{cases} 1 & if\ \ Z_2(x) < 620.84-s_2 \\ 0.99\ \frac{(620.84)^t-(3.875x_1+5.125x_2+5.9375x_3)^t}{(s_2)^t} & if\ \ 620.84-s_2 \leq Z_2(x) \leq 620.84 \\ 0 & if\ \ Z_2(x) > 620.84 \end{cases} \tag{39}$$

$$I_{h^-}^{E_3}(Z_2(x)) = \begin{cases} 1 & if \;\; Z_2(x) < 620.84 - s_2 \\ \frac{(620.84)^t - (3.875x_1+5.125x_2+5.9375x_3)^t}{(s_2)^t} & if \;\; 620.84 - s_2 \leq Z_2(x) \leq 620.84 \\ 0 & if \;\; Z_2(x) > 620.84 \end{cases} \tag{40}$$

$$F_{h^-}^{E_1}(Z_2(x)) = \begin{cases} 0 & if \;\; Z_2(x) < 599.23 + t_2 \\ 0.98 \, \frac{(3.875x_1+5.125x_2+5.9375x_3)^t - (599.23)^t - (t_2)^t}{(620.84)^t - (599.23)^t - (t_2)^t} & if \;\; 599.23 + t_2 \leq Z_2(x) \leq 620.84 \\ 1 & if \;\; Z_2(x) > 620.84 \end{cases} \tag{41}$$

$$F_{h^-}^{E_2}(Z_2(x)) = \begin{cases} 0 & if \;\; Z_2(x) < 599.23 + t_2 \\ 0.99 \, \frac{(3.875x_1+5.125x_2+5.9375x_3)^t - (599.23)^t - (t_2)^t}{(620.84)^t - (599.23)^t - (t_2)^t} & if \;\; 599.23 + t_2 \leq Z_2(x) \leq 620.84 \\ 1 & if \;\; Z_2(x) > 620.84 \end{cases} \tag{42}$$

$$F_{h^-}^{E_3}(Z_2(x)) = \begin{cases} 0 & if \;\; Z_2(x) < 599.23 + t_2 \\ \frac{(3.875x_1+5.125x_2+5.9375x_3)^t - (599.23)^t - (t_2)^t}{(620.84)^t - (599.23)^t - (t_2)^t} & if \;\; 599.23 + t_2 \leq Z_2(x) \leq 620.84 \\ 1 & if \;\; Z_2(x) > 620.84 \end{cases} \tag{43}$$

The final solution model is given as follows:

$$\mathbf{M}_4 : Max \;\; \chi = \frac{\alpha_1 + \alpha_2 + \alpha_3}{3} + \frac{\beta_1 + \beta_2 + \beta_3}{3} - \frac{\gamma_1 + \gamma_2 + \gamma_3}{3}$$

$$s.t. \; 0.98 \, \frac{(99.875x_1^{\frac{1}{2}} - 8x_1 + 119.875x_2^{\frac{1}{2}} - 10.125x_2 + 95.125x_3^{\frac{1}{3}} - 8x_3)^t - (180.72)^t}{(516.70)^t - (180.72)^t} \geq \alpha_1$$

$$0.99 \, \frac{(99.875x_1^{\frac{1}{2}} - 8x_1 + 119.875x_2^{\frac{1}{2}} - 10.125x_2 + 95.125x_3^{\frac{1}{3}} - 8x_3)^t - (180.72)^t}{(516.70)^t - (180.72)^t} \geq \alpha_2$$

$$\frac{(99.875x_1^{\frac{1}{2}} - 8x_1 + 119.875x_2^{\frac{1}{2}} - 10.125x_2 + 95.125x_3^{\frac{1}{3}} - 8x_3)^t - (180.72)^t}{(516.70)^t - (180.72)^t} \geq \alpha_3$$

$$0.98 \, \frac{(99.875x_1^{\frac{1}{2}} - 8x_1 + 119.875x_2^{\frac{1}{2}} - 10.125x_2 + 95.125x_3^{\frac{1}{3}} - 8x_3)^t - (180.72)^t}{(s_1)^t} \geq \beta_1$$

$$0.99 \, \frac{(99.875x_1^{\frac{1}{2}} - 8x_1 + 119.875x_2^{\frac{1}{2}} - 10.125x_2 + 95.125x_3^{\frac{1}{3}} - 8x_3)^t - (180.72)^t}{(s_1)^t} \geq \beta_2$$

$$\frac{(99.875x_1^{\frac{1}{2}} - 8x_1 + 119.875x_2^{\frac{1}{2}} - 10.125x_2 + 95.125x_3^{\frac{1}{3}} - 8x_3)^t - (180.72)^t}{(s_1)^t} \geq \beta_3$$

$$0.98 \, \frac{(516.70)^t - (t_1)^t - (99.875x_1^{\frac{1}{2}} - 8x_1 + 119.875x_2^{\frac{1}{2}} - 10.125x_2 + 95.125x_3^{\frac{1}{3}} - 8x_3)^t}{(516.70)^t - (180.72)^t - (t_1)^t} \leq \gamma_1$$

$$0.99 \, \frac{(516.70)^t - (t_1)^t - (99.875x_1^{\frac{1}{2}} - 8x_1 + 119.875x_2^{\frac{1}{2}} - 10.125x_2 + 95.125x_3^{\frac{1}{3}} - 8x_3)^t}{(516.70)^t - (180.72)^t - (t_1)^t} \leq \gamma_2$$

$$\frac{(516.70)^t - (t_1)^t - (99.875x_1^{\frac{1}{2}} - 8x_1 + 119.875x_2^{\frac{1}{2}} - 10.125x_2 + 95.125x_3^{\frac{1}{3}} - 8x_3)^t}{(516.70)^t - (180.72)^t - (t_1)^t} \leq \gamma_3$$

$$0.98 \, \frac{(620.84)^t - (3.875x_1 + 5.125x_2 + 5.9375x_3)^t}{(620.84)^t - (599.23)^t} \geq \alpha_1$$

$$0.99 \, \frac{(620.84)^t - (3.875x_1 + 5.125x_2 + 5.9375x_3)^t}{(620.84)^t - (599.23)^t} \geq \alpha_2$$

$$\frac{(620.84)^t - (3.875x_1 + 5.125x_2 + 5.9375x_3)^t}{(620.84)^t - (599.23)^t} \geq \alpha_3$$

$$0.98 \, \frac{(620.84)^t - (3.875x_1 + 5.125x_2 + 5.9375x_3)^t}{(s_2)^t} \geq \beta_1$$

$$0.99 \, \frac{(620.84)^t - (3.875x_1 + 5.125x_2 + 5.9375x_3)^t}{(s_2)^t} \geq \beta_2$$

$$\frac{(620.84)^t - (3.875x_1 + 5.125x_2 + 5.9375x_3)^t}{(s_2)^t} \geq \beta_3$$

$$0.98 \, \frac{(3.875x_1 + 5.125x_2 + 5.9375x_3)^t - (599.23)^t - (t_2)^t}{(620.84)^t - (599.23)^t - (t_2)^t} \leq \gamma_1$$

$$0.99 \, \frac{(3.875x_1 + 5.125x_2 + 5.9375x_3)^t - (599.23)^t - (t_2)^t}{(620.84)^t - (599.23)^t - (t_2)^t} \leq \gamma_2$$

$$\frac{(3.875x_1 + 5.125x_2 + 5.9375x_3)^t - (599.23)^t - (t_2)^t}{(620.84)^t - (599.23)^t - (t_2)^t} \leq \gamma_3$$

$$2.0625x_1 + 3.875x_2 + 2.9375x_3 \leq 333.125$$

$$3.875x_1 + 2.0625x_2 + 2.0625x_3 \leq 365.625$$

$$2.9375x_1 + 2.0625x_2 + 2.9375x_3 \geq 360$$

$$x_1,\ x_2,\ x_3 \geq 0,\quad 0 \leq \alpha_1, \alpha_2, \alpha_3 \leq 1,\ \ 0 \leq \beta_1, \beta_2, \beta_3 \leq 1,$$
$$0 \leq \gamma_1, \gamma_2, \gamma_3 \leq 1,\ \ 0 \leq s_1, t_1 \leq 1,\ \ 0 \leq s_2, t_2 \leq 1,$$
$$\alpha_n \geq \beta_n,\ \ \alpha_n \geq \gamma_n,\ \ \alpha_n + \beta_n + \gamma_n \leq 3,\quad \forall\ n = 1, 2, 3.$$

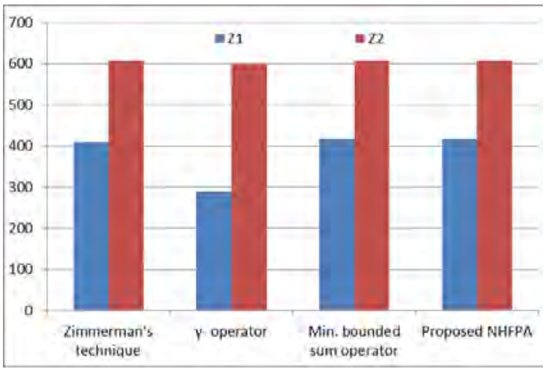
The multiobjective nonlinear programming problem $\mathbf{M}_4$ has been written in AMPL language and solved using solvers available on NEOS server online facility provided by Wisconsin Institutes for Discovery at the University of Wisconsin in Madison for solving Optimization problems, see (Server [18]).
At $t = 2$, the optimal solution of the multiobjective nonlinear programming problem by using the proposed neutrosophic hesitant fuzzy programming approach (NHFPA) is $x = (60.48,\ 5.26,\ 58.37)$, $Z_1 = 416.58,\ Z_2 = 607.88$ with the degree of satisfaction $\chi = 1.20$ respectively.

4.1 Comparative study

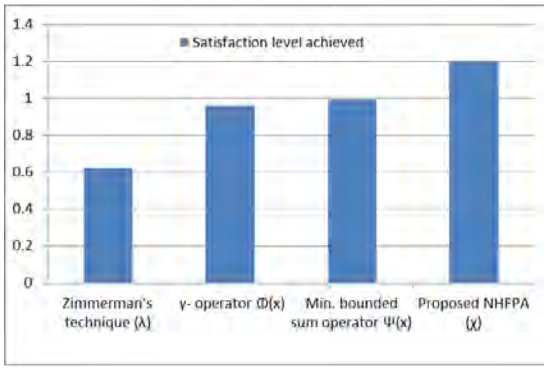
The multiobjective nonlinear programming problem of manufacturing system with conflicting objectives have been solved by using proposed neutrosophic hesitant fuzzy programming approach (NHFPA). The solution results obtained by proposed method and with other existing approaches discussed in [19] have been summarized in Table-1. From the table, it is clear that the minimum deviation from ideal solution of each objective function is 100.12 and 0.41 by using proposed NHFPA and γ- operator respectively. Furthermore, the highest satisfaction level has been attained by proposed approach i.e; χ=1.20, which reveals the superiority of proposed NHFPA over other existing approaches in terms of satisfactory degree of DM(s). Fig-3 shows the graphical representation of the objective functions and satisfaction level obtained by different approaches.

Table 1: Comparison of results with existing methods.

Solution method	Objective values		Deviations from ideal solutions		Satisfaction level
	Max. Z_1	Min. Z_2	$(U_1 - Z_1)$	$(Z_2 - L_2)$	
Zimmerman's technique [19]	409.70	607.28	107	8.05	λ= 0.62
γ- operator [19]	288.86	599.64	227.84	0.41(min.)	$\phi(x)$= 0.96
Min. bounded sum operator [19]	416.58	607.88	100.12	8.65	$\psi(x)$= 0.99
Proposed NHFPA	416.58	607.88	100.12(min.)	8.65	χ= 1.20 (max.)



(a) Objective functions obtained by different approaches.



(b) Satisfaction level achieved by different approaches.

Figure 3: Comparison of results with proposed NHFPA and different existing approaches.

5 Conclusions

In this study, a new approach has been suggested to solve the multiobjective nonlinear programming problem in the neutrosophic hesitant fuzzy environment. The proposed neutrosophic hesitant fuzzy programming approach (NHFPA) comprises three different membership functions, namely; truth hesitant, indeterminacy hesitant and a falsity hesitant membership function which contains a set of different values between 0 and 1. The proposed approach provides the more realistic framework and considers various aspects of the DM's neutral thoughts with hesitations in the decision-making process. The main contribution by introducing the proposed approach is that it allows the DM(s) to express his/her(their) degree of hesitation and neutral thoughts according to the need of adverse situations in a convenient manner. In order to show the superiority of proposed NHFPA, it is applied to solve multiobjective nonlinear programming problem in the manufacturing system. To best of our knowledge, no such approach is suggested in the literature to solve MO-NLPP in such an efficient and effective manner.

Therefore, the proposed NHFPA will be very helpful in such a typical situation when the DM(s) have some neutral thoughts and also with a set some hesitation values in the decision-making process. In future, the proposed approach may be applied to the multiobjective fractional programming problem, bi-level nonlinear programming problem, multilevel fractional programming problem etc.

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Single Valued (2n+1) Sided Polygonal Neutrosophic Numbers and Single Valued (2n) Sided Polygonal Neutrosophic Numbers

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Abstract. This paper introduces a single valued (2n as well as 2n+1) sided polygonal neutrosophic numbers in continuation with other defined single valued neutrosophic numbers. The paper provides basic algebra like addi-tion, subtraction and multiplication of a single valued (2n as well as 2n+1) sided polygonal neutrosophic numbers with examples. In addition, the paper introduces matrix for single valued (2n as well as 2n+1) sided polygonal neutrosophic matrix and its properties.

Keywords: Fuzzy numbers, Intuitionistic fuzzy numbers, Single valued trapezoidal neutrosophic numbers, Single valued triangular neutrosophic numbers, Neutrosophic matrix.

1 Introduction

In the real world problems, uncertainty occurs in many situations which cannot be handled precisely via crisp set theory. To approximate those uncertainties exists in the given linguistics words the fuzzy set theory is introduced by Zadeh [10]. After that, Dubois and Prade [2] defined the fuzzy number as a generalization of real number. In continuation, many authors [5-8, 11-23] introduced various types of fuzzy numbers such as triangular, trapezoidal, pentagonal, hexagonal fuzzy numbers etc. with their membership functions. Atanassov [1] introduced the concept of intuitionistic fuzzy sets that provides precise solutions to the problems in uncertain situations than fuzzy sets with membership and non-membership functions. After developing intuitionistic fuzzy sets, authors in [4, 6, 10, 19] defined various types of intuitionistic fuzzy numbers and different types of operations on intuitionistic fuzzy sets are also established by suitable examples. Smarandache [9] introduced the generalization of both fuzzy and intuitionistic fuzzy sets and named it as neutrosophic set. The Single valued neutrosophic number and its applications are described in [3]. The results of the problems using neutrosophic sets are more accurate than the results given by fuzzy and intuitionistic fuzzy sets [11-20]. Due to which it is applied in various fields for multi-decision tasks [20-32]. The applications of n-valued neutrosophic set [24-26] in data analytics research fields given a thrust to study the neutrosophic numbers. This paper focuses on introducing mathematical operation of 2n and 2n+1 sided polygonal neutrosophic numbers and its matrices with examples.

The rest of the paper is organized as follows: The section 2 contains preliminaries. Section 3 explains single valued 2n+1 polygonal neutrosophic numbers whereas the Section 4 demonstrates Single valued 2n side polygonal neutrosophic numbers. Section 5 provides conclusions followed by acknowledgements and references.

2. Preliminaries

Definition 1 (Fuzzy Number)[4]: A fuzzy number is nothing but an extension of a regular number in the sense that it does not refer to one single value but rather to a connected set of possible values, where each of the possible value has its own weight between 0 and 1. This weight is called the membership function. The complex fuzzy set for a given fuzzy number $\widetilde{A}$ can be defined as $\mu_{\widetilde{A}}(x)$ is non-decreasing for $x \le x_0$ and non-increasing for $\ge x_0$. Similarly other properties can be defined.

Definition 2 (Triangular fuzzy number [4]): A fuzzy number $\widetilde{A}$= {a, b, c } is said to be a triangular fuzzy number if its membership function is given by, where $a \le b \le c$

$$\mu_{\widetilde{A}}(x)= \begin{cases} \frac{(x-a)}{(b-a)} & for\ a \le x \le b \\ \frac{(c-x)}{(c-b)} & for\ b \le x \le c \\ 0 & otherwise \end{cases}$$

Definition 3 (Trapezoidal fuzzy number [4])
A Trapezoidal fuzzy number (TrFN) denoted by $\widetilde{A}_P$ is defined as (a, b, c, d), where the membership function

$$\mu_{\widetilde{A}_P}(x)= \begin{cases} 0 & for\ x \le a \\ \frac{(x-a)}{(b-a)} & for\ a \le x \le b \\ 1 & for\ b \le x \le c \\ \frac{(d-x)}{(d-c)} & for\ c \le x \le d \\ 0 & for\ x \ge d \end{cases}$$

Or, $\mu_{\widetilde{A}_P}(x)$**= max (min (**$\frac{(x-a)}{(b-a)}$**, 1,** $\frac{(d-x)}{(d-c)}$ **),0)**

Definition 4 (Generalized Trapezoidal Fuzzy Number) (GTrFNs)
A Generalized Fuzzy Number (a, b, c, d, w), is called a Generalized Trapezoidal Fuzzy Number "x" if its membership function is given by

$$(x)= \begin{cases} 0 & for\ x \le a \\ \frac{(x-a)}{(b-a)}w & for\ a \le x \le b \\ w & for\ b \le x \le c \\ \frac{(d-x)}{(d-c)}w & for\ c \le x \le d \\ 0 & for\ x \ge d \end{cases}$$

Or, $\mu_{\widetilde{A}_P}(x)$**= max (min (w** $\frac{(x-a)}{(b-a)}$**, w,**$w\ \frac{(d-x)}{(d-c)}$**) ,0)**

Definition 5 (Pentagonal fuzzy number [4])
A pentagonal fuzzy number (PFN) of a fuzzy set $\widetilde{A}_P$= {a, b, c, d, e} and its membership function is given by,

$$\mu_{\widetilde{A}_P}(x)= \begin{cases} 0 & for\ x < a \\ \frac{(x-a)}{(b-a)} & for\ a \le x \le b \\ \frac{(x-b)}{(c-b)} & for\ b \le x \le c \\ 1 & x = c \\ \frac{(d-x)}{(d-c)} & for\ c \le x \le d \\ \frac{(e-x)}{(e-d)} & for\ d \le x \le e \\ 0 & for\ x > d \end{cases}$$

Definition 6 (Hexagonal fuzzy number [4])

A Hexagonal fuzzy number (HFN) of a fuzzy set $\tilde{A}_P$= {a, b, c, d, e, f} and its membership function is given by,

$$\mu_{\tilde{A}_P}(x)= \begin{cases} 0 & for\ x < a \\ \frac{1}{2}(\frac{x-a}{b-a}) & for\ a \le x \le b \\ \frac{1}{2}+\frac{1}{2}(\frac{x-b}{c-b}) & for\ b \le x \le c \\ 1 & c \le x \le d \\ 1-\frac{1}{2}(\frac{x-d}{e-d}) & for\ c \le x \le d \\ \frac{1}{2}(\frac{f-x}{f-e}) & for\ d \le x \le e \\ 0 & for\ x > d \end{cases}$$

Definition 7 (Octagonal fuzzy number [4])

A Octagonal fuzzy number (OFN) of a fuzzy set $\tilde{A}_P$= $\{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}$ and its membership function is given by,

$$\mu_{\tilde{A}_P} = \begin{cases} k\frac{x-a_1}{a_2-a_1}, & a_1 \le x \le a_2 \\ k, & a_2 \le x \le a_3 \\ k+(1-k)\frac{x-a_3}{a_4-a_3}, & a_3 \le x \le a_4 \\ 1, & a_4 \le x \le a_5 \\ k+(1-k)\frac{a_6-x}{a_6-a_5}, & a_5 \le x \le a_6 \\ k, & a_6 \le x \le a_7 \\ k\frac{a_8-x}{a_8-a_7}, & a_7 \le x \le a_8 \\ 0, & \text{Otherwise} \end{cases}$$

Where k= max$\{a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8\}$

Definition 8 (A triangular intuitionistic fuzzy number)[4]

A triangular intuitionistic fuzzy number $\tilde{a}$ is denoted as $\tilde{a} = ((a,b,c),(a',b',c'))$, where $a' \le a \le b \le b' \le c \le c'$ with the following membership function $\mu_{\tilde{a}}(x)$ and non-membership function $\nu_{\tilde{a}}(x)$

$$\mu_{\tilde{a}}(x) = \begin{cases} \frac{x-a}{b-a}, & a \le b \\ \frac{c-x}{c-b}, & b \le c \\ 0, & \text{otherwise} \end{cases}$$

$$\nu_{\tilde{a}}(x) = \begin{cases} \frac{b-x}{b-a'}, & a' \le b \\ \frac{x-b}{c'-b}, & b \le c' \\ 1, & \text{otherwise} \end{cases}$$

Definition 9 (Trapezoidal Intuitionistic fuzzy number)

$$\mu_{\tilde a}(x)=\begin{cases} 0 & x\le 0 \\ \frac{(x-a)}{(b-a)} & \text{for } a<x<b \\ w & \text{for } b\le x\le c \\ \frac{(d-x)}{(d-c)} & \text{for } c<x<d \\ 0 & \text{otherwise}\end{cases},\quad \nu_{\tilde a}(x)=\begin{cases} 1 & x\le 0 \\ \frac{(b-x+u_{\tilde a}(x-a))}{(b-a)} & \text{for } a<x<b \\ u_{\tilde a} & \text{for } b\le x\le c \\ \frac{(x-c+u_{\tilde a}(d-x))}{(d-c)} & \text{for } c<x<d \\ 1 & \text{otherwise}\end{cases}$$

Definition 10 (Single valued triangular neutrosophic number [3]):
A triangular neutrosophic number $\tilde a$=< (a, b,c) ;$w_{\tilde a}$, $u_{\tilde a}$,$y_{\tilde a}$> is a special neutrosophic set on the real number set R, whose truth-membership, indeterminacy– membership and falsity-membership functions are defined as follows:

$$\mu_{\tilde a}(x)=\begin{cases} \frac{(x-a)}{(b-a)}w_{\tilde a} & \text{for } a\le x\le b \\ w_{\tilde a} & \text{for } x=b \\ \frac{(c-x)}{(c-b)}w_{\tilde a} & \text{for } b\le x\le c \\ 0 & \text{otherwise}\end{cases},\quad \nu_{\tilde a}(x)=\begin{cases} \frac{(b-x+u_{\tilde a}(x-a))}{(b-a)} & \text{for } a\le x\le b \\ u_{\tilde a} & \text{for } x=b \\ \frac{(x-b+u_{\tilde a}(c-x))}{(c-b)} & \text{for } b\le x\le c \\ 1 & \text{otherwise}\end{cases}$$

$$\lambda_{\tilde a}(x)=\begin{cases} \frac{(b-x+y_{\tilde a}(x-a))}{(b-a)} & \text{for } a\le x\le b \\ y_{\tilde a} & \text{for } x=b \\ \frac{(x-b+y_{\tilde a}(c-x))}{(c-b)} & \text{for } b\le x\le c \\ 1 & \text{otherwise}\end{cases}$$

A triangular neutrosophic number $\tilde a$ =< (a, b,c) ;$w_{\tilde a}$, $u_{\tilde a}$,$y_{\tilde a}$> may express an ill-known quantity about b which is approximately equal to b.

Definition 11 (Single valued trapezoidal neutrosophic number [3]):
A triangular neutrosophic number $\tilde a$=< $(a, b,c,$ d) ;$w_{\tilde a}$, $u_{\tilde a}$,$y_{\tilde a}$> is a special neutrosophic set on the real number set R, whose truth-membership, indeterminacy– membership and falsity-membership function are defined as follows:

$$\mu_{\tilde a}(x)=\begin{cases} \frac{(x-a)}{(b-a)}w_{\tilde a} & \text{for } a\le x\le b \\ w_{\tilde a} & \text{for } b\le x\le c \\ \frac{(d-x)}{(d-c)}w_{\tilde a} & \text{for } c\le x\le d \\ 0 & \text{otherwise}\end{cases},\quad \nu_{\tilde a}(x)=\begin{cases} \frac{(b-x+u_{\tilde a}(x-a))}{(b-a)} & \text{for } a\le x\le b \\ u_{\tilde a} & \text{for } b\le x\le c \\ \frac{(x-c+u_{\tilde a}(d-x))}{(d-c)} & \text{for } c\le x\le d \\ 1 & \text{otherwise}\end{cases}$$

$$\lambda_{\tilde a}(x)=\begin{cases} \frac{(b-x+y_{\tilde a}(x-a))}{(b-a)} & \text{for } a\le x\le b \\ y_{\tilde a} & \text{for } b\le x\le c \\ \frac{(x-c+y_{\tilde a}(d-x))}{(d-c)} & \text{for } c\le x\le d \\ 1 & \text{otherwise}\end{cases}$$

The single valued trapezoidal neutrosophic numbers are a generalization of the intuitionistic trapezoidal fuzzy numbers, Thus, the neutrosophic number may express more uncertainty than the intuitionstic fuzzy number.

3. Single valued 2n+1 polygonal neutrosophic numbers

Definition 12 (Single valued 2n+1 polygonal neutrosophic number)**:**
A single valued 2n+1 sided polygonal neutrosophic number $\tilde a$=< $(a_1,a_2 ,\ldots,a_n,\ldots,a_{2n},a_{2n+1})$;$w_{\tilde a}$, $u_{\tilde a}$,$y_{\tilde a}$> is a special neutrosophic set on the real number set R, whose truth-membership, indeterminacy– membership and falsity-membership functions are defined as follows:

$$T_A(x) = \begin{cases} \dfrac{x-a_1}{a_2-a_1} w_{\tilde{a}}, & a_1 \le a_2 \\ \dfrac{x-a_2}{a_3-a_2} w_{\tilde{a}}, & a_2 \le a_3 \\ \ldots \\ \ldots \\ \ldots \\ \dfrac{x-a_n}{a_{n+1}-a_n} w_{\tilde{a}}, & a_n \le a_{n+1} \\ w_{\tilde{a}} & x = a_{n+1} \\ \dfrac{a_{n+2}-x}{a_{n+2}-a_{n+1}} w_{\tilde{a}}, & a_{n+1} \le a_{n+2} \\ \dfrac{a_{n+3}-x}{a_{n+3}-a_{n+2}} w_{\tilde{a}}, & a_{n+2} \le a_{n+3} \\ \ldots \\ \ldots \\ \ldots \\ \dfrac{a_{2n+1}-x}{a_{2n+1}-a_{2n}} w_{\tilde{a}}, & a_{2n} \le a_{2n+1} \\ 0, & \text{Otherwise} \end{cases}$$

$$I_A(x) = \begin{cases} \dfrac{a_2 - x + u_{\tilde{a}}(x-a_1)}{a_2-a_1}, & a_1 \le a_2 \\ \dfrac{a_3 - x + u_{\tilde{a}}(x-a_2)}{a_3-a_2}, & a_2 \le a_3 \\ \ldots \\ \ldots \\ \ldots \\ \dfrac{a_{n+1} - x + u_{\tilde{a}}(x-a_n)}{a_{n+1}-a_n}, & a_n \le a_{n+1} \\ u_{\tilde{a}}, & x = a_{n+1} \\ \dfrac{x-a_{n+1} + u_{\tilde{a}}(a_{n+2}-x)}{a_{n+2}-a_{n+1}}, & a_{n+1} \le a_{n+2} \\ \dfrac{x-a_{n+2} + u_{\tilde{a}}(a_{n+3}-x)}{a_{n+3}-a_{n+2}}, & a_{n+2} \le a_{n+3} \\ \ldots \\ \ldots \\ \ldots \\ \dfrac{x-a_{2n} + u_{\tilde{a}}(a_{2n+1}-x)}{a_{2n+1}-a_{2n}}, & a_{2n} \le a_{2n+1} \\ 1, & \text{Otherwise} \end{cases}$$

$$F_A(x) = \begin{cases} \dfrac{a_2 - x + y_{\tilde{a}}(x-a_1)}{a_2-a_1}, & a_1 \le a_2 \\ \dfrac{a_3 - x + y_{\tilde{a}}(x-a_2)}{a_3-a_2}, & a_2 \le a_3 \\ \ldots \\ \ldots \\ \ldots \\ \dfrac{a_{n+1} - x + y_{\tilde{a}}(x-a_n)}{a_{n+1}-a_n}, & a_n \le a_{n+1} \\ y_{\tilde{a}}, & x = a_{n+1} \\ \dfrac{x-a_{n+1} + y_{\tilde{a}}(a_{n+2}-x)}{a_{n+2}-a_{n+1}}, & a_{n+1} \le a_{n+2} \\ \dfrac{x-a_{n+2} + y_{\tilde{a}}(a_{n+3}-x)}{a_{n+3}-a_{n+2}}, & a_{n+2} \le a_{n+3} \\ \ldots \\ \ldots \\ \dfrac{x-a_{2n} + y_{\tilde{a}}(a_{2n+1}-x)}{a_{2n+1}-a_{2n}}, & a_{2n} \le a_{2n+1} \\ 1, & \text{Otherwise} \end{cases}$$

Example:1 If $w_{\tilde{a}}$ = 0.2 ,$u_{\tilde{a}}$ = 0.4 $y_{\tilde{a}}$ = 0.3 and n= 4 , then we have an nanogonal neutrosophic number $\tilde{a}$ and it is taken as $\tilde{a} =< (3,6,8,10,11,21,43,44,56) >$. Figure 1 demonstrates the Example 1.

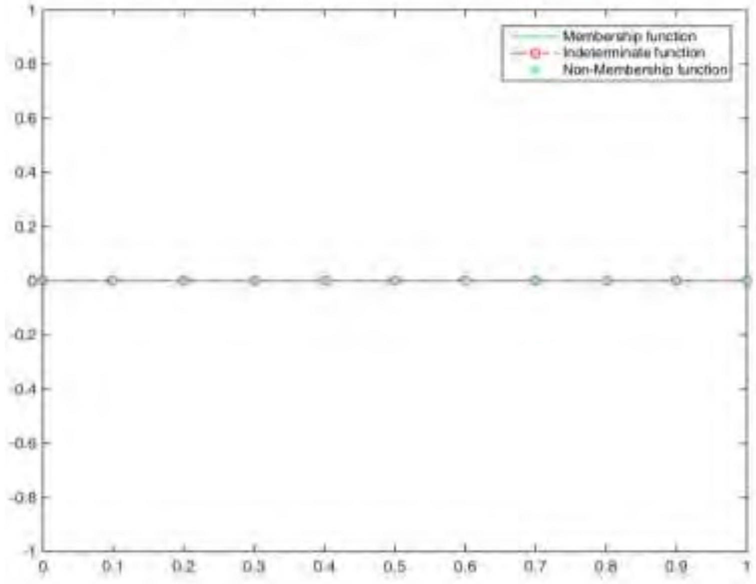

Figure: 1

Example: 2

If $w_{\tilde{a}}$ = 0.2 ,$u_{\tilde{a}}$ = 0.4 $y_{\tilde{a}}$ = 0.3 and n= 4 , then we have an nanogonal neutrosophic number $\tilde{a}$ and it is taken as $\tilde{a} =< (3,6,8,10,1,2,4,7,5) >$. Figure 2 demonstrates the Example 2 and its neutrosophic membership.

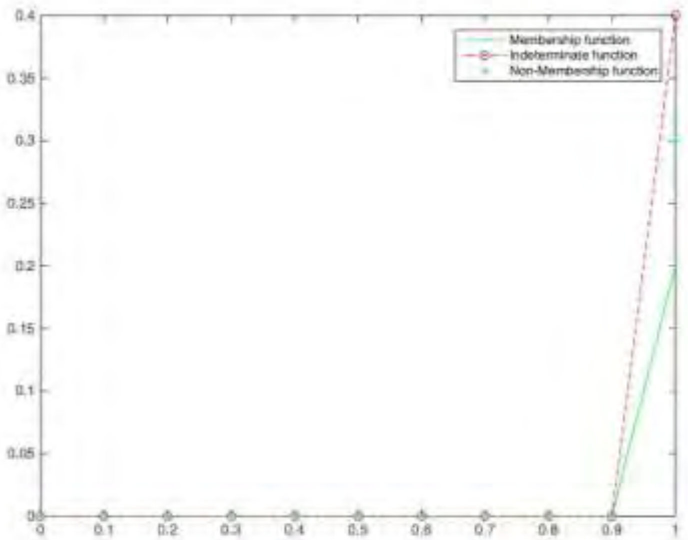

Figure: 2

Note

The single valued triangular neutrosophic number can be generalized to a single valued 2n+1 polygonal neutrosophic number, where n=1,2,3,…,n

$\tilde{a}$=< (a_1,a_2 ,….,a_n,…,a_{2n},a_{2n+1}) ;$w_{\tilde{a}}$, $u_{\tilde{a}}$,$y_{\tilde{a}}$>, where$\tilde{a}$ may express an ill –known quantity about a_n which is gradually equal to a_n.

We mean that $a_2$approximatesa_n, $a_3$approximatesa_n a littel better thana_2,………………. a_{n-1}approximatesa_n a litte better than all previous a_1, a_2,…a_n,

Remark
If $0\leq w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \leq 1$, $0\leq w_{\tilde{a}}+u_{\tilde{a}}+y_{\tilde{a}} \leq 1$, $y_{\tilde{a}}=0$ and the single valued 2n+1 sided polygonal neutrosophic number reduced to the case single valued 2n+1 sided polygonal fuzzy number.

3.1. Operations of single valued 2n+1 sided polygonal neutrosophic numbers
Following are the three operations that can be performed on single valued 2n+1 polygonal neutrosophic numbers suppose $A_{PNN}=<(a_1,a_2,\ldots,a_n,\ldots,a_{2n},a_{2n+1}); w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}}>$ and $B_{PNN}=<(b_1,b_2,\ldots,b_n,\ldots,b_{2n},b_{2n+1}); w_{\tilde{b}}, u_{\tilde{b}}, y_{\tilde{b}}>$ are two single valued 2n+1 polygonal neutrosophic numbers then

(i) Addition:
$A_{PNN}+B_{PNN}=<(a_1+b_1, a_2+b_2, \ldots, a_n+b_n, \ldots, a_{2n}+b_{2n}, a_{2n+1}+b_{2n+1}); w_{\tilde{a}}+w_{\tilde{b}}-w_{\tilde{a}}\cdot w_{\tilde{b}}, u_{\tilde{b}}\cdot u_{\tilde{b}}, y_{\tilde{a}}\cdot y_{\tilde{b}}>$

(ii) Subtraction:
$A_{PNN}-B_{PNN}=<(a_1-b_1, a_2-b_2, \ldots, a_n-b_n, \ldots, a_{2n}-b_{2n}, a_{2n+1}-b_{2n+1}); w_{\tilde{a}}+w_{\tilde{b}}-w_{\tilde{a}}\cdot w_{\tilde{b}}, u_{\tilde{b}}\cdot u_{\tilde{b}}, y_{\tilde{a}}\cdot y_{\tilde{b}}>$

Multiplication:
$A_{PNN}*B_{PNN}=<(a_1\cdot b_1, a_2\cdot b_2, \ldots, a_n\cdot b_n, \ldots, a_{2n}\cdot b_{2n}, a_{2n+1}\cdot b_{2n+1}); w_{\tilde{a}}\cdot w_{\tilde{b}}, u_{\tilde{a}}+u_{\tilde{b}}-u_{\tilde{a}}\cdot u_{\tilde{b}}, y_{\tilde{a}}+y_{\tilde{b}}-y_{\tilde{a}}\cdot y_{\tilde{b}}>$

Remark
If $w_{\tilde{a}}=1$, $u_{\tilde{a}}=0$ $y_{\tilde{a}}=0$ then **single valued 2n+1 sided polygonal neutrosophic number** $A_{PNN}=<(a_1,a_2,\ldots,a_n,\ldots,a_{2n},a_{2n+1}); w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}}>$ **reduced to the case of single valued 2n+1 sided polygonal fuzzy number** $A_{PFN}=<(a_1,a_2,\ldots,a_n,\ldots,a_{2n},a_{2n+1})>$, n=1,2,3,…,n.

Remark
If $0\leq w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \leq 1$, $0\leq w_{\tilde{a}}+u_{\tilde{a}}+y_{\tilde{a}} \leq 3$, and n=1, the single valued 2n+1 -sided polygonal neutrosophic number reduced to the case of the single valued triangular neutrosophic number $A_{PNN}=<(a_1,a_2,a_3); w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}}>$ **[3].**

Example 3: Let $w_{\tilde{a}}=1$, $u_{\tilde{a}}=0$, $y_{\tilde{a}}=0$ and n= 1

If $w_{\tilde{a}}=1$, $u_{\tilde{a}}=0$, $y_{\tilde{a}}=0$ and n= 2, then we have an Pentagonal fuzzy number [5]:

Let A=(1, 2, 3 , 4, 5) and B=(2, 3,4,5,6) be two Pentagonal fuzzy numbers, then

i. A + B = (3, 5, 7, 9,11)

ii. A – B = (-1,-1, -1,-1,-1)

iii. 2A = (2, 4, 6, 8, 10)

iv. A.B = (2, 6, 12, 20, 30)

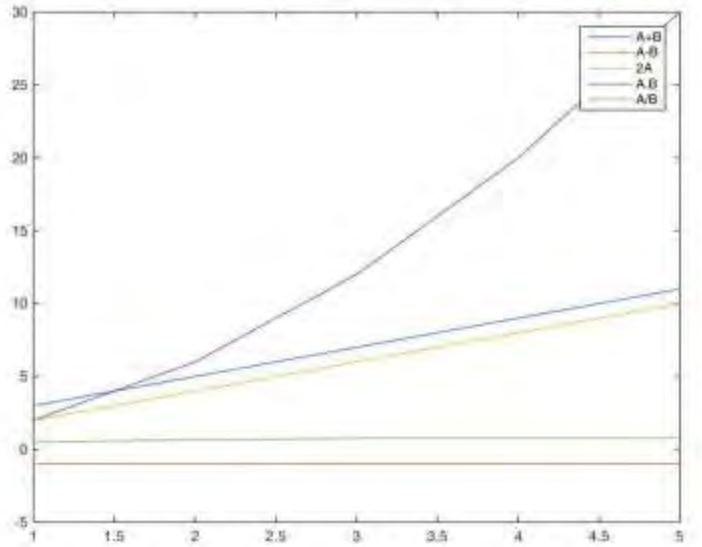

Figure: 3

Figure 3 demonstrates operation given in Example 3. The single valued 2n+1 polygonal neutrosophic number are generalization of the Pentagonal fuzzy number numbers [5] , and single valued triangular neutrosophic number [3]

4. Single valued 2n-sided polygonal neutrosophic numbers

Definition 13: The single valued trapezoidal neutrosophic number can be extended to a single valued 2n sided polygonal neutrosophic number $\tilde{a}$=< $(a_1, a_2, \ldots, a_n, a_{n+1}, \ldots, a_{2n-1}, a_{2n})$; $\boldsymbol{w_{\tilde{a}}}, \boldsymbol{u_{\tilde{a}}}, \boldsymbol{y_{\tilde{a}}}$> where n=1,2,3,…,n, whose truth-membership, indeterminacy– membership and falsity-membership functions are defined as follows:

$$T_A(x) = \begin{cases} k\dfrac{x-a_1}{a_2-a_1}w_{\tilde{a}}, & a_1 \le x \le a_2 \\ k+(1-k)\dfrac{x-a_2}{a_3-a_2}w_{\tilde{a}}, & a_2 \le x \le a_3 \\ \ldots \\ \ldots \\ \ldots \\ k+(1-mk)\dfrac{x-a_{n-1}}{a_n-a_{n-1}}w_{\tilde{a}}, & a_{n-1} \le x \le a_n \\ w_{\tilde{a}}, & a_n \le x \le a_{n+1} \\ k+(1-mk)\dfrac{a_{n+2}-x}{a_{n+2}-a_{n+1}}w_{\tilde{a}}, & a_{n+1} \le x \le a_{n+2} \\ \ldots \\ \ldots \\ \ldots \\ k+(1-k)\dfrac{a_{2n-1}-x}{a_{2n-1}-a_{2n-2}}w_{\tilde{a}}, & a_{2n-2} \le x \le a_{2n-1} \\ k\dfrac{a_{2n}-x}{a_{2n}-a_{2n-1}}w_{\tilde{a}}, & a_{2n-1} \le x \le a_{2n} \\ 0, & \text{Otherwise} \end{cases}$$

$$
I_A(x) = \begin{cases}
k + (1-mk)\dfrac{a_2 - x}{a_2 - a_1}u_{\tilde{a}}, & a_1 \le x \le a_2 \\
k + (1-(m-1)k)\dfrac{a_3 - x}{a_3 - a_2}u_{\tilde{a}}, & a_2 \le x \le a_3 \\
\ldots & \\
\ldots & \\
\ldots & \\
k + (1-k)\dfrac{a_{n-1} - x}{a_{n-1} - a_{n-2}}u_{\tilde{a}}, & a_{n-2} \le x \le a_{n-1} \\
k\dfrac{a_n - x}{a_n - a_{n-1}}u_{\tilde{a}}, & a_{n-1} \le x \le a_n \\
0, & a_n \le x \le a_{n+1} \\
k\dfrac{x - a_{n+1}}{a_{n+2} - a_{n+1}}u_{\tilde{a}}, & a_{n+1} \le x \le a_{n+2} \\
k + (1-k)\dfrac{x - a_{n+2}}{a_{n+3} - a_{n+2}}u_{\tilde{a}}, & a_{n+2} \le x \le a_{n+3} \\
\ldots & \\
\ldots & \\
\ldots & \\
k + (1-(m-1)k)\dfrac{x - a_{2n-2}}{a_{2n-1} - a_{2n-2}}u_{\tilde{a}}, & a_{2n-2} \le x \le a_{2n-1} \\
k + (1-mk)\dfrac{x - a_{2n-1}}{a_{2n} - a_{2n-1}}u_{\tilde{a}}, & a_{2n-1} \le x \le a_{2n} \\
1, & \text{Otherwise}
\end{cases}
$$

$$F_A(x) = \begin{cases} k+(1-mk)\dfrac{a_2-x}{a_2-a_1}y_{\tilde{a}}, & a_1 \le x \le a_2 \\ k+(1-(m-1)k)\dfrac{a_3-x}{a_3-a_2}y_{\tilde{a}}, & a_2 \le x \le a_3 \\ \dots \\ \dots \\ \dots \\ k+(1-k)\dfrac{a_{n-1}-x}{a_{n-1}-a_{n-2}}y_{\tilde{a}}, & a_{n-2} \le x \le a_{n-1} \\ k\dfrac{a_n-x}{a_n-a_{n-1}}y_{\tilde{a}}, & a_{n-1} \le x \le a_n \\ 0, & a_n \le x \le a_{n+1} \\ k\dfrac{x-a_{n+1}}{a_{n+2}-a_{n+1}}y_{\tilde{a}}, & a_{n+1} \le x \le a_{n+2} \\ k+(1-k)\dfrac{x-a_{n+2}}{a_{n+3}-a_{n+2}}y_{\tilde{a}}, & a_{n+2} \le x \le a_{n+3} \\ \dots \\ \dots \\ \dots \\ k+(1-(m-1)k)\dfrac{x-a_{2n-2}}{a_{2n-1}-a_{2n-2}}y_{\tilde{a}}, & a_{2n-2} \le x \le a_{2n-1} \\ k+(1-mk)\dfrac{x-a_{2n-1}}{a_{2n}-a_{2n-1}}y_{\tilde{a}}, & a_{2n-1} \le x \le a_{2n} \\ 1, & \text{Otherwise} \end{cases}$$

where $\tilde{a}$ may represent an ill–known quantity of range, which is gradually approximately equal to the interval $[a_n, a_{n+1}]$.

We mean that (a_2, a_{2n-1}) approximates $[a_n, a_{n+1}]$,

(a, a_{2n-2}) approximates $[a_n, a_{n+1}]$ a little better than (a_2, a_{2n-1}), …………………(a_n, a_{n+1}) approximates $[a_n, a_{n+1}]$ a little better than all previous intervals.

Remark

If $0 \le w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \le 1$, $0 \le w_{\tilde{a}} + u_{\tilde{a}} + y_{\tilde{a}} \le 1$, $y_{\tilde{a}} = 0$ and the single valued 2n -sided polygonal neutrosophic number reduced to the case of single valued 2n-sided polygonal fuzzy number.

4.1 Single valued 2n-sided polygonal neutrosophic number

Following are the three operations that can be performed on single valued 2n-sided polygonal neutrosophic numbers suppose $\boldsymbol{A_{PNN}}$=< $(a_1, a_2, \dots, a_n, a_{n+1}, \dots, a_{2n-1}, a_{2n})$;$\boldsymbol{w_{\tilde{a}}}$, $\boldsymbol{u_{\tilde{a}}}$,$\boldsymbol{y_{\tilde{a}}}$> and $\boldsymbol{B_{PNN}}$=<$(b_1, b_2, \dots, b_n, b_{n+1}, \dots, b_{2n-1}, b_{2n})$;$\boldsymbol{w_{\tilde{b}}}$, $\boldsymbol{u_{\tilde{b}}}$,$\boldsymbol{y_{\tilde{b}}}$>are two**2n-sided polygonal neutrosophic number.**

(i) **Addition:** $\boldsymbol{A_{PNN}}$+$\boldsymbol{B_{PNN}}$=$(a_1+b_1, a_2+b_2, \dots, a_n+b_n, a_{n+1}+b_{n+1}, \dots, a_{2n-1}+b_{2n-1}, a_{2n}+b_{2n})$;$\boldsymbol{w_{\tilde{a}} + w_{\tilde{b}} - w_{\tilde{a}} \cdot w_{\tilde{b}}}$, $\boldsymbol{u_{\tilde{b}} \cdot u_{\tilde{b}}}$,$\boldsymbol{y_{\tilde{a}} \cdot y_{\tilde{b}}}$>

(ii) **Subtraction:**$\boldsymbol{A_{PNN}}$-$\boldsymbol{B_{PNN}}$=<$(a_1-b_{2n}, a_2-b_{2n-1}, \dots, a_n-b_n, a_{n+1}-b_{n-1}, \dots, a_{2n-1}-b_2, a_{2n}-b_1)$;$\boldsymbol{w_{\tilde{a}} + w_{\tilde{b}} - w_{\tilde{a}} \cdot w_{\tilde{b}}}$, $\boldsymbol{u_{\tilde{b}} \cdot u_{\tilde{b}}}$,$\boldsymbol{y_{\tilde{a}} \cdot y_{\tilde{b}}}$>

(iii) **Multiplication:**$\boldsymbol{A_{PNN}}$*$\boldsymbol{B_{PNN}}$ =<$(a_1 \cdot b_1, a_2 \cdot b_2, \dots, a_n \cdot b_n, a_{n+1} \cdot b_{n+1}, \dots, a_{2n-1} \cdot b_{2n-1}, a_{2n} \cdot b_{2n})$;$\boldsymbol{w_{\tilde{a}} \cdot w_{\tilde{b}}}$,$\boldsymbol{u_{\tilde{a}} + u_{\tilde{b}} - u_{\tilde{a}} \cdot u_{\tilde{b}}}$,$\boldsymbol{y_{\tilde{a}} + y_{\tilde{b}} - y_{\tilde{a}} \cdot y_{\tilde{b}}}$>

Remark

If $w_{\tilde{a}} = 1$,$u_{\tilde{a}} = 0$ $y_{\tilde{a}} = 0$ then **single valued 2n**sided**polygonal neutrosophic number** $\boldsymbol{A_{PNN}}$=<$(a_1, a_2, \dots, a_n, a_{n+1}, \dots, a_{2n-1}, a_{2n})$;$\boldsymbol{w_{\tilde{a}}}$, $\boldsymbol{u_{\tilde{a}}}$,$\boldsymbol{y_{\tilde{a}}}$> **reduced to the case of single valued 2n-** sided **polygonal fuzzy number**$\boldsymbol{A_{PFN}}$=<$(a_1, a_2, \dots, a_n, a_{n+1}, \dots, a_{2n-1}, a_{2n})$ **for** n=1,2,3,…,n.

Remark

If $0\leq w_{\tilde{a}}, u_{\tilde{a}}, y_{\tilde{a}} \leq 1$, $0\leq w_{\tilde{a}}+ u_{\tilde{a}}+y_{\tilde{a}} \leq 3$, and n=2, the single valued 2n-sided polygonal neutrosophic number reduced to the case of single valued trapezoidal neutrosophic number $\boldsymbol{A_{PNN}}$=< $(\boldsymbol{a_1},\boldsymbol{a_2},\boldsymbol{a_3}, \boldsymbol{a_4})$;$\boldsymbol{w_{\tilde{a}}}, \boldsymbol{u_{\tilde{a}}},\boldsymbol{y_{\tilde{a}}}$>**[x].**

Example 4: if $w_{\tilde{a}} = 1$,$u_{\tilde{a}} = 0$ $y_{\tilde{a}} = 0$ and n= 3 then we have an Hexagonal fuzzy number [7-8]:
Let A=(1, 2, 3 ,5 ,6) and B=(2, 4,6,8,10,12) be two Hexagonal fuzzy numbers then
A+ B= (3, 6,9, 13,16,19)

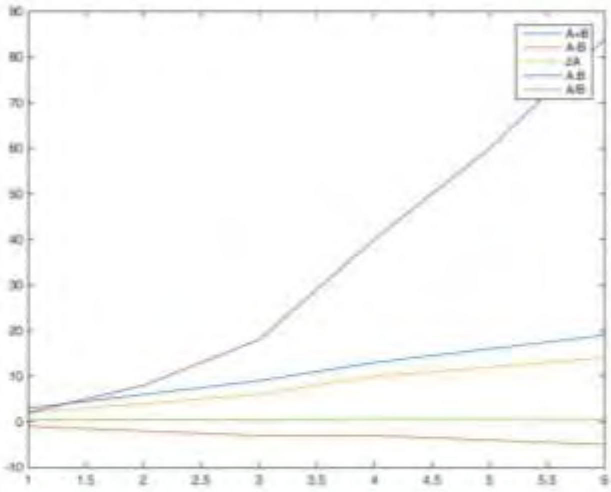

Figure: 4

Figure 4 demonstrates operation given in Example 4.

The single valued 2n-sided polygonal neutrosophic number are generalization of the hexagonal fuzzy numbers [8] ,intuitionistic trapezoidal fuzzy numbers[x] and single valued trapezoidal neutrosophic number [3] with its application [12-23] for multi-decision process [24-26].

5. Conclusion:

This paper introduces single valued (2n and 2n+1) sided polygonal neutrosophic numbers its addition, subtraction, multiplication as well as polygonal neutrosophic matrix with an illustrative example. In near future our focus will be on applications of single-valued 2n sided polygonal neutrosophic numbers and its other mathematical algebra.

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A Novel Methodology Developing an Integrated ANP: A Neutrosophic Model for Supplier Selection

Abduallah Gamal, Mahmoud Ismail, Florentin Smarandache

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Abstract

In this research, the main objectives are to study the Analytic Network Process (ANP) technique in neutrosophic environment, to develop a new method for formulating the problem of Multi-Criteria Decision-Making (MCDM) in network structure, and to present a way of checking and calculating consistency consensus degree of decision makers. We have used neutrosophic set theory in ANP to overcome the situation when the decision makers might have restricted knowledge or different opinions, and to specify deterministic valuation values to comparison judgments. We formulated each pairwise comparison judgment as a trapezoidal neutrosophic number. The decision makers specify the weight criteria in the problem and compare between each criteria the effect of each criteria against other criteria. In decision-making process, each decision maker should make $\frac{n \times (n-1)}{2}$ relations for n alternatives to obtain a consistent trapezoidal neutrosophic preference relation. In this research, decision makers use judgments to enhance the performance of ANP. We introduced a real life example: how to select personal cars according to opinions of decision makers. Through solution of a numerical example, we formulate an ANP problem in neutrosophic environment.

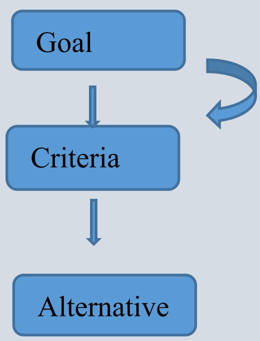
Keywords

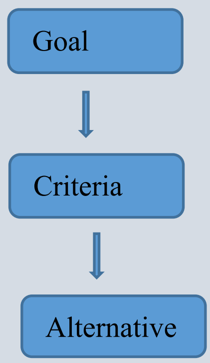
Analytic Network Process, Neutrosophic Set, Multi-Criteria Decision Analysis (MCDM).

1 Introduction

The Analytic Network Process (ANP) is a new theory that extends the Analytic Hierarchy Process (AHP) to cases of dependency and feedback, and generalizes the supermatrix approach introduced by Saaty (1980) for the AHP [1]. This research focuses on ANP method, which is a generalization of AHP. Analytical Hierarchy Process (AHP) [2] is a multi-criteria decision making method where, given the criteria and alternative solutions of a specific model, a graph structure is created, and the decision maker is asked to pair-wisely compare the components, in order to determine their priorities. On the other hand, ANP supports feedback and interaction by having inner and outer dependencies among the models' components [2]. We deal with the problem, analyze it, and specify alternatives and the critical factors that change the decision. ANP is considered one of the most adequate technique for dealing with multi criteria decision-making using network hierarchy [19]. We present a comparison of ANP vs. AHP in *Table 1*: how each technique deals with a problem, the results of each technique, advantages and disadvantages.

Table1. Comparison of ANP vs. AHP.

<table>
<tr><th>Property</th><th>ANP
(Analytic Network Process)</th><th>AHP
(Analytic Hierarch Process)</th></tr>
<tr><td>Structure</td><td>

Network</td><td>

Hierarchy</td></tr>
</table>

Why are the results different	The user learns through feedback comparisons that his/her priority for cost is not nearly as high as originally thought when asked the question abstractly, while prestige gets more weight.	The user going top down makes comparisons, when asked, without referring to actual alternatives, and overestimates the importance of cost.
Advantages	a) Using feedback and interdependence between criteria. b) Deal with complex problem without structure.	a) Straightforward and convenient. b) Simplicity by using pairwise comparisons.
Disadvantages	a) Conflict between decision makers. b) Inconsistencies. c) Hole of large scale 1 to 9. d) Large comparisons matrix.	a) Decision maker's capacity. b) Inconsistencies. c) Hole of large scale 1 to 9. d) Large comparisons matrix.

Analytic network process (ANP) consists of criteria and alternatives by decomposing them into sub-problems, specifying the weight of each criterion and comparing each criterion against other criterion, in a range between 0 and 1. We employ ANP in decision problems, and we make pairwise comparison matrices between alternatives and criteria. In any traditional methods, decision makers face a difficult problem to make $\frac{n \times (n-1)}{2}$ consistent judgments for each alternative.

In this article, we deal with this problem by making decision maker using (n-1) judgments. The analysis of ANP requires applying a scale system for pairwise comparisons matrix, and this scale plays an important role in transforming qualitative analysis to quantitative analysis [4].

Most of previous researchers use the scale 1-9 of analytic network process and hierarchy. In this research, we introduced a new scale from 0 to 1, instead of the scale 1-9. This scale 1-9 creates large hole between ranking results, and we overcome this drawback by using the scale [0, 1] [5], determined by some serious mathematical shortages of Saaty's scale, such as:

- Large hole between ranking results and human judgments;
- Conflicting between ruling matrix and human intellect.

The neutrosophic set is a generalization of the intuitionistic fuzzy set. While fuzzy sets use true and false for express relationship, neutrosophic sets use true membership, false membership and indeterminacy membership [6]. ANP employs network structure, dependence and feedback [7]. MCDM is a formal and structured decision making methodology for dealing with complex problems [8]. ANP was also integrated as a SWOT method [9]. An overview of integrated ANP with intuitionistic fuzzy can be found in Rouyendegh, [10].

Our research is organized as it follows: Section 2 gives an insight towards some basic definitions of neutrosophic sets and ANP. Section 3 explains the proposed methodology of neutrosophic ANP group decision making model. Section 4 introduces a numerical example.

2 Preliminaries

In this section, we give definitions involving neutrosophic set, single valued neutrosophic sets, trapezoidal neutrosophic numbers, and operations on trapezoidal neutrosophic numbers.

2.1 Definition [26-27]

Let X be a space of points and $x \in X$. A neutrosophic set A in X is defined by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$ and a falsity-membership function $F_A(x)$, $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard or real nonstandard subsets of $]^-0, 1^+[$. That is $T_A(x): X \rightarrow]^-0, 1^+[$, $I_A(x): X \rightarrow]^-0, 1^+[$ and $F_A(x): X \rightarrow]^-0, 1^+[$. There is no restriction on the sum of $T_A(x)$, $I_A(x)$ and $F_A(x)$, so $0^- \leq \sup(x) + \sup x + \sup x \leq 3^+$.

2.2 Definition [13, 14, 26]

Let X be a universe of discourse. A single valued neutrosophic set A over X is an object taking the form $A = \{\langle x, T_A(x), I_A(x), F_A(x), \rangle : x \in X\}$, where $T_A(x): X \rightarrow [0,1]$, $I_A(x): X \rightarrow [0,1]$ and $F_A(x): X \rightarrow [0,1]$ with $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$ for all $x \in X$. The intervals $T_A(x)$, $I_A(x)$ and $F_A(x)$ represent the truth-membership degree, the indeterminacy-membership degree and the falsity membership degree of x to A, respectively. For convenience, a SVN number is represented by $A = (a, b, c)$, where $a, b, c \in [0, 1]$ and $a+b+c \leq 3$.

2.3 Definition [14, 15, 16]

Suppose $\alpha_{\tilde{a}}$, $\theta_{\tilde{a}}$, $\beta_{\tilde{a}}$ ϵ [0,1] and a_1 , a_2 , a_3 , a_4 ϵ R, where $a_1 \leq a_2 \leq a_3 \leq a_4$. Then, a single valued trapezoidal neutrosophic number $\tilde{a}$=⟨(a_1 , a_2 , a_3 , a_4); $\alpha_{\tilde{a}}$, $\theta_{\tilde{a}}$, $\beta_{\tilde{a}}$⟩ is a special neutrosophic set on the real line set R, whose truth-membership, indeterminacy-membership and falsity-membership functions are defined as:

$$T_{\tilde{a}}(x) = \begin{cases} \alpha_{\tilde{a}}\left(\frac{x-a_1}{a_2-a_1}\right) & (a_1 \leq x \leq a_2) \\ \alpha_{\tilde{a}} & (a_2 \leq x \leq a_3) \\ \alpha_{\tilde{a}}\left(\frac{a_4-x}{a_4-a_3}\right) & (a_3 \leq x \leq a_4) \\ 0 & otherwise \end{cases} \quad (1)$$

$$I_{\tilde{a}}(x) = \begin{cases} \frac{(a_2-x+\theta_{\tilde{a}}(x-a_1))}{(a_2-a_1)} & (a_1 \leq x \leq a_2) \\ \alpha_{\tilde{a}} & (a_2 \leq x \leq a_3) \\ \frac{(x-a_3+\theta_{\tilde{a}}(a4-x))}{(a_4-a_3)} & (a_3 \leq x \leq a_4) \\ 1 & otherwise \end{cases}, \quad (2)$$

$$F_{\tilde{a}}(x) = \begin{cases} \frac{(a_2-x+\beta_{\tilde{a}}(x-a_1))}{(a_2-a_1)} & (a_1 \leq x \leq a_2) \\ \alpha_{\tilde{a}} & (a_2 \leq x \leq a_3) \\ \frac{(x-a_3+\beta_{\tilde{a}}(a4-x))}{(a_4-a_3)} & (a_3 \leq x \leq a_4) \\ 1 & otherwise \end{cases}, \quad (3)$$

where $\alpha_{\tilde{a}}$, $\theta_{\tilde{a}}$ and $\beta_{\tilde{a}}$ represent the maximum truth-membership degree, the minimum indeterminacy-membership degree and the minimum falsity-membership degree, respectively. A single valued trapezoidal neutrosophic number $\tilde{a}$=⟨(a_1 , a_2 , a_3 , a_4); $\alpha_{\tilde{a}}$, $\theta_{\tilde{a}}$, $\beta_{\tilde{a}}$⟩ may express an ill-defined quantity of the range, which is approximately equal to the interval [a_2 , a_3] .

2.4 Definition [15, 14]

Let $\tilde{a}$=⟨(a_1 , a_2 , a_3 , a_4); $\alpha_{\tilde{a}}$, $\theta_{\tilde{a}}$, $\beta_{\tilde{a}}$⟩ and $\tilde{b}$=⟨(b_1 , b_2 , b_3 , b_4); $\alpha_{\tilde{b}}$, $\theta_{\tilde{b}}$, $\beta_{\tilde{b}}$⟩ be two single valued trapezoidal neutrosophic numbers, and ϒ≠ 0 be any real number. Then:

- Addition of two trapezoidal neutrosophic numbers:

$$\tilde{a} + \tilde{b} = \langle (a_1 + b_1, a_2 + b_2, a_3 + b_3, a_4 + b_4); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}} \rangle$$

- Subtraction of two trapezoidal neutrosophic numbers:

$$\tilde{a} - \tilde{b} = \langle (a_1 - b_4, a_2 - b_3, a_3 - b_2, a_4 - b_1); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}} \rangle$$

- Inverse of trapezoidal neutrosophic number:

$$\tilde{a}^{-1} = \langle (\frac{1}{a_4}, \frac{1}{a_3}, \frac{1}{a_2}, \frac{1}{a_1}) ; \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}} \rangle \qquad \text{where } (\tilde{a} \neq 0)$$

- Multiplication of trapezoidal neutrosophic number by constant value:

$$\Upsilon\tilde{a} = \begin{cases} \langle (\Upsilon a_1, \Upsilon a_2, \Upsilon a_3, \Upsilon a_4); \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}} \rangle & \text{if } (\Upsilon > 0) \\ \langle (\Upsilon a_4, \Upsilon a_3, \Upsilon a_2, \Upsilon a_1); \alpha_{\tilde{a}}, \theta_{\tilde{a}}, \beta_{\tilde{a}} \rangle & \text{if } (\Upsilon < 0) \end{cases}$$

- Division of two trapezoidal neutrosophic numbers:

$$\frac{\tilde{a}}{\tilde{b}} = \begin{cases} \langle (\frac{a_1}{b_4}, \frac{a_2}{b_3}, \frac{a_3}{b_2}, \frac{a_4}{b_1}); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}} \rangle & \text{if } (a_4 > 0, b_4 > 0) \\ \langle (\frac{a_4}{b_4}, \frac{a_3}{b_3}, \frac{a_2}{b_2}, \frac{a_1}{b_1}); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}} \rangle & \text{if } (a_4 < 0, b_4 > 0) \\ \langle (\frac{a_4}{b_1}, \frac{a_3}{b_2}, \frac{a_2}{b_3}, \frac{a_1}{b_4}); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}} \rangle & \text{if } (a_4 < 0, b_4 < 0) \end{cases}$$

- Multiplication of trapezoidal neutrosophic numbers:

$$\tilde{a}\tilde{b} = \begin{cases} \langle (a_1 b_1, a_2 b_2, a_3 b_3, a_4 b_4); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}} \rangle & \text{if } (a_4 > 0, b_4 > 0) \\ \langle (a_1 b_4, a_2 b_3, a_3 b_2, a_4 b_1); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}} \rangle & \text{if } (a_4 < 0, b_4 > 0) \\ \langle (a_4 b_4, a_3 b_3, a_2 b_2, a_1 b_1); \alpha_{\tilde{a}} \wedge \alpha_{\tilde{b}}, \theta_{\tilde{a}} \vee \theta_{\tilde{b}}, \beta_{\tilde{a}} \vee \beta_{\tilde{b}} \rangle & \text{if } (a_4 < 0, b_4 < 0) \end{cases}$$

3 Methodology

In this study, we present the steps of the proposed model, we identify criteria, evaluate them, and decision makers also evaluate their judgments using neutrosophic trapezoidal numbers.

In previous articles, we noticed that the scale (1-9) has many drawbacks illustrated by [5]. We present a new scale from 0 to 1 to avoid this drawbacks. We use (n-1) judgments to obtain consistent trapezoidal neutrosophic preference relations instead of $\frac{n \times (n-1)}{2}$, in order to decrease the workload. ANP is used for ranking and selecting the alternatives.

The model of ANP in neutrosophic environment quantifies four criteria to combine them for decision making into one global variable. To do this, we first present the concept of ANP and determine the weight of each criterion based on opinions of decision makers.

Then, each alternative is evaluated with other criteria, considering the effects of relationships among criteria. The ANP technique is composed of four steps in the traditional way [17].

The steps of our ANP neutrosophic model can be introduced as:

Step - 1 constructing the model and problem structuring:

1. Selection of decision makers (DMs).

Form the problem in a network; the first level represents the goal and the second level represents criteria and sub-criteria and interdependence and feedback between criteria, and the third level represents the alternatives. An example of a network structure:

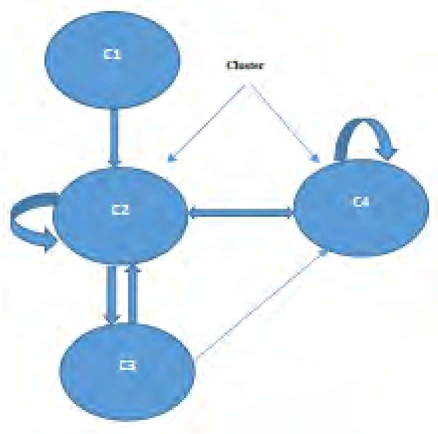


Figure 1. ANP model.

Another example of a network ANP structure [17]:

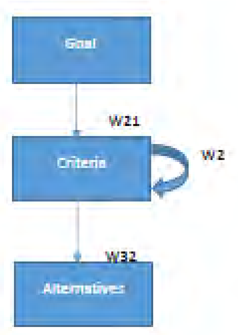


Fig. 2. A Network Structure.

2. Prepare the consensus degree as it follows:

   CD = $\frac{NE}{N} \times 100\%$, where NE is the number of decision makers that have the same opinion and N is the total numbers of experts. Consensus degree should be greater than 50% [16].

Step - 2 Pairwise comparison matrices to determine weighting

1. Identify the alternatives of a problem A = {A1, A2, A3, …, Am}.
2. Identify the criteria and sub-criteria, and the interdependency between them:
   C = {C1, C2, C3, …, Cm}.
3. Determine the weighting matrix of criteria that is defined by decision makers (DMs) for each criterion (W_1).
4. Determine the relationship interdependencies among the criteria and the weights, the effect of each criterion against another in the range from 0 to 1.
5. Determine the interdependency matrix from multiplication of weighting matrix in step 3 and interdependency matrix in step 4.
6. Decision makers make pairwise comparisons matrix between alternatives compared to each criterion, and focus only on (n-1) consensus judgments instead of using $\frac{n\times(n-1)}{2}$ [16].

$$\tilde{R}=\begin{bmatrix}(l_{11},m_{11l},m_{11u},u_{11}) & (l_{11},m_{11l},m_{11u},u_{11}) & \cdots & (l_{1n},m_{1nl},m_{1nu},u_{1n})\\ (l_{21},m_{21l},m_{21u},u_{21}) & (l_{22},m_{22l},m_{22u},u_{22}) & \cdots & (l_{2n},m_{2nl},m_{2nu},u_{2n})\\ \cdots & \cdots & \cdots & \cdots\\ (l_{n1},m_{n1l},m_{n1u},u_{n1}) & (l_{n2},m_{n2l},m_{n2u},u_{n2}) & \cdots & (l_{nn},m_{nnl},m_{nnu},u_{nn})\end{bmatrix}$$

To make the comparisons matrix accepted, we should check the consistency of the matrix.

Definition 5 The consistency of a trapezoidal neutrosophic reciprocal preference relations $\tilde{R}$ = ($\check{r}_{ij}$) n × n can be expressed as:

$\check{r}_{ij}$ = $\check{r}_{ik}$ + $\check{r}_{kj}$ – (0.5, 0.5, 0.5, 0.5) where i, j, k = 1, 2 … n. can also be written as l_{ij} = l_{ik} + l_{kj} – (0.5, 0.5, 0.5, 0.5), m_{ijl} = m_{ikl} + m_{mjl} – (0.5, 0.5, 0.5, 0.5), m_{iju} = m_{iku} + m_{kju} - (0.5, 0.5, 0.5, 0.5), u_{ij} = m_{ik} + m_{kj} – (0.5, 0.5, 0.5, 0.5) , where i, j, k = 1, 2 … n and for $\check{r}_{ik}$ = 1- $\check{r}_{kj}$ {Abdel-Basset, 2017 [16]}.

Definition 6 In order to check whether a trapezoidal neutrosophic reciprocal preference relation $\tilde{R}$ is additive approximation - consistency or not [16].

$$\check{r}_{ij} = \frac{\check{r}_{ij}+c_x}{1+2c_x} \quad (5)$$

$$\check{r}_{ij} = \frac{-\check{r}_{ij}+c_x}{1+2c_x} \quad (6)$$

$$u_{ij} - m_{ij} = \Delta \quad (7)$$

We transform the neutrosophic matrix to pairwise comparison deterministic matrix by adding (α, θ, β), and we use the following equation to calculate the accuracy and score

$$S\,(\tilde{a}_{ij}) = \frac{1}{16}\,[a_1 + b_1 + c_1 + d_1] \times (2 + \alpha_{\tilde{a}} - \theta_{\tilde{a}} - \beta_{\tilde{a}}) \quad (8)$$

and

$$A\,(\tilde{a}_{ij}) = \frac{1}{16}\,[a_1 + b_1 + c_1 + d_1] \times (2 + \alpha_{\tilde{a}} - \theta_{\tilde{a}} + \beta_{\tilde{a}}) \quad (9)$$

We obtain the deterministic matrix by using S $(\tilde{a}_{ij})$.

From the deterministic matrix, we obtain the weighting matrix by dividing each entry by the sum of the column.

Step - 3 Formulation of supermatrix

The supermatrix concept is similar to the Markov chain process [18].

1. Determine scale and weighting data for the n alternatives against n criteria $w_{21}, w_{22}, w_{23}, \dots, w_{2n}$.
2. Determine the interdependence weighting matrix of criteria comparing it against another criteria in range from 0 to 1, defined as:

$$W_3 = \begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ C_n \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_n \end{array} \\ \begin{bmatrix} (0-1) & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & (0-1) \end{bmatrix} \end{array} \quad (10)$$

3. We obtain the weighting criteria $W_c = W_3 \times W_1$.
4. Determine the interdependence matrix $\tilde{A}_{criteria}$ among the alternatives with respect to each criterion.

$$\tilde{A}_{criteria} = \begin{bmatrix} (0.5, 0.5, 0.5, 0.5) & (l_{11}, m_{11l}, m_{11u}, u_{11}) & \dots & (l_{1n}, m_{1nl}, m_{1nu}, u_{1n}) \\ (l_{21}, m_{21l}, m_{21u}, u_{21}) & (0.5, 0.5, 0.5, 0.5) & \dots & (l_{2n}, m_{2nl}, m_{2nu}, u_{2n}) \\ \dots & \dots & (0.5, 0.5, 0.5, 0.5) & \dots \\ (l_{n1}, m_{n1l}, m_{n1u}, u_{n1}) & (l_{n2}, m_{n2l}, m_{n2u}, u_{n2}) & \dots & (0.5, 0.5, 0.5, 0.5) \end{bmatrix}$$

Step - 4 Selection of the best alternatives

1. Determine the priorities matrix of the alternatives with respect to each of the n criteria W_{An} where n is the number of criteria.

   Then, $W_{A1} = W_{\tilde{A}_{C1}} \times W_{21}$

   $W_{A2} = W_{\tilde{A}_{C1}} \times W_{22}$

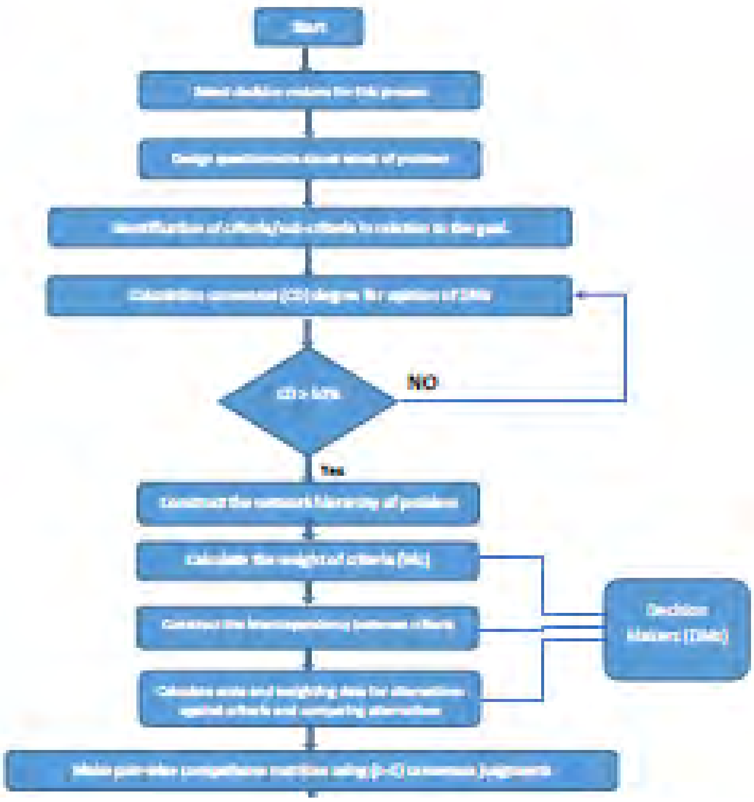
   $W_{A3} = W_{\tilde{A}_{C1}} \times W_{23}$

   $W_{An} = W_{\tilde{A}_{Cn}} \times W_{2n}$

   Then, $W_A = [W_{A1}, W_{A2}, W_{A3}, \ldots, W_{An}]$.

2. In the last we rank the priorities of criteria and obtain the best alternatives by multiplication of the W_A matrix by the Weighting criteria matrix W_c, i.e.

$W_A \times W_c$



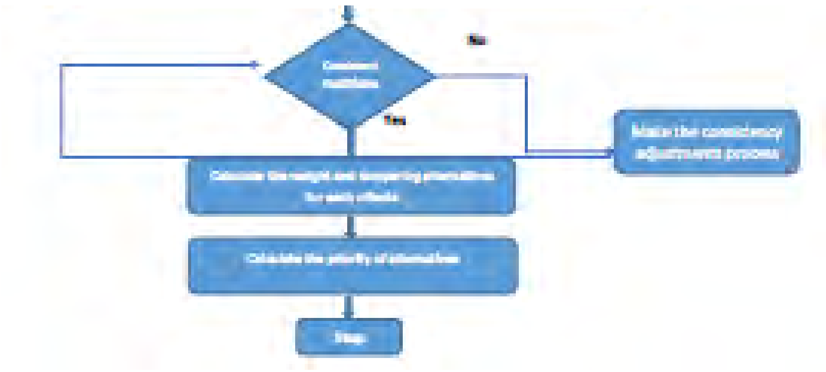


Figure 3. Schematic diagram of ANP with neutrosophic.

4 Numerical Example

In this section, we present an example to illustrate the ANP in neutrosophic environment - selecting the best personal car from four alternatives: Crossover is alternative A1, Sedan is alternative A2, Diesel is alternative A3, Nissan is alternative A4. We have four criteria C_j (j = 1, 2, 3, and 4), as follows: C_1 for price, C_2 for speed, C_3 for color, C_4 for model. The criteria to be considered is the supplier selections, which are determined by the DMs from a decision group. The team is split into four groups, namely DM_1, DM_2, DM_3 and DM_4, formed to select the most suitable alternatives. The criteria to be considered in the supplier's selection are determined by the DMs team from the expert's procurement office.

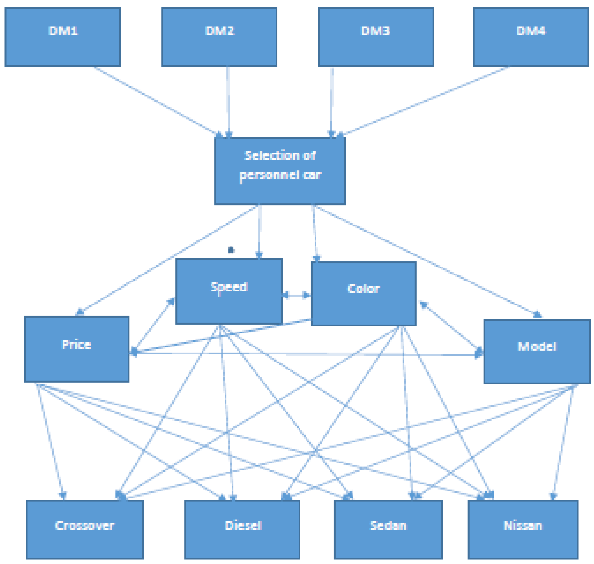


Figure 4. Network structure of the illustrative example.

In this example, we seek to illustrate the improvement and efficiency of ANP, the interdependency among criteria and feedback, and how a new scale from 0 to 1 improves and facilitates the solution and the ranking of the alternatives.

Step - 1: In order to compare the criteria, the decision makers assume that there is no interdependency among criteria. This data reflects relative weighting without considering interdependency among criteria. The weighting matrix of criteria that is defined by decision makers is W_1= (P, S, C, M) = (0.33, 0.40, 0.22, 0.05).

Step - 2: Assuming that there is no interdependency among the four alternatives, (A_1 A_2, A_3, A_4), they are compared against each criterion. Decision makers determine the relationships between each criterion and alternative, establishing the neutrosophic decision matrix between four alternatives (A_1, A_2,A_3, A_4) and four criteria (C_1, C_2 , C_3 , C_4):

$$R = \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} \begin{bmatrix} & C_1 & C_2 & C_3 & C_4 \\ & (0.3 , 0.5,0.2,0.5) & (0.6, 0.7,0.9,0.1) & (0.7 , 0.2,0.4,0.6) & (0.3 , 0.6,0.4,0.7) \\ & (0.6 , 0.3,0.4,0.7) & (0.2 ,0.3,0.6,0.9) & (0.6 , 0.7,0.8,0.9) & (0.3 , 0.5,0.2,0.5) \\ & (0.3 , 0.5,0.2,0.5) & (0.3 , 0.7,0.4,0.3) & (0.8 , 0.2,0.4,0.6) & (0.2 , 0.5,0.6,0.8) \\ & (0.4 , 0.3,0.1,0.6) & (0.1 , 0.4,0.2,0.8) & (0.5 , 0.3,0.2,0.4) & (0.6 , 0.2,0.3,0.4) \end{bmatrix}$$

The last matrix appears consistent to definition 6 (5, 6, 7). Then, by ensuring consistency of trapezoidal neutrosophic additive reciprocal preference relations, decision makers (DMs) should determine the maximum truth-membership degree (α), minimum indeterminacy-membership degree (θ), and minimum falsity-membership degree (β) of single valued neutrosophic numbers, as in definition 6 (c). Therefore:

$$R = \begin{matrix} A1 \\ A2 \\ A3 \\ A3 \end{matrix} \begin{bmatrix} C_1 & C_2 & C_3 & C_4 \\ (0.3,0.5,0.2,0.5; 0.3,0.4,0.6) & (0.6,0.7,0.9,0.1; 0.4,0.3,0.5) & (0.7,0.2,0.4,0.6; 0.8,0.4,0.2) & (0.3,0.6,0.4,0.7; 0.4,0.5,0.6) \\ (0.6,0.3,0.4,0.7; 0.2,0.5,0.8) & (0.2,0.3,0.6,0.9; 0.6,0.2,0.5) & (0.6,0.7,0.8,0.9; 0.2,0.5,0.7) & (0.3,0.5,0.2,0.5; 0.5,0.7,0.8) \\ (0.3,0.5,0.2,0.5; 0.4,0.5,0.7) & (0.3,0.7,0.4,0.3; 0.2,0.5,0.9) & (0.8,0.2,0.4,0.6; 0.4,0.6,0.5) & (0.2,0.5,0.6,0.8; 0.4,0.3,0.8) \\ (0.4,0.3,0.1,0.6; 0.2,0.3,0.5) & (0.1,0.4,0.2,0.8; 0.7,0.3,0.6) & (0.5,0.3,0.2,0.4; 0.3,0.4,0.7) & (0.6,0.2,0.3,0.4; 0.6,0.3,0.4) \end{bmatrix}$$

$$S(\tilde{a}_{ij}) = \frac{1}{16}[a_1 + b_1 + c_1 + d_1] \times (2 + \alpha_{\tilde{a}} - \theta_{\tilde{a}} - \beta_{\tilde{a}})$$

And

$$A(\tilde{a}_{ij}) = \frac{1}{16}[a_1 + b_1 + c_1 + d_1] \times (2 + \alpha_{\tilde{a}} - \theta_{\tilde{a}} + \beta_{\tilde{a}})$$

The deterministic matrix can be obtained by S ($\tilde{a}_{ij}$) equation in the following step:

$$\begin{array}{cc} & \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ R = \begin{array}{c} A1 \\ A2 \\ A3 \\ A3 \end{array} & \begin{bmatrix} 0.122 & 0.23 & 0.261 & 0.163 \\ 0.113 & 0.238 & 0.188 & 0.10 \\ 0.113 & 0.085 & 0.163 & 0.17 \\ 0.123 & 0.169 & 0.105 & 0.178 \end{bmatrix} \end{array}$$

Scale and weighting data for four alternatives against four criteria is derived by dividing each element by the sum of each column. The comparison matrix of four alternatives and four criteria is the following:

$$\begin{array}{cc} & \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{array}{c} A1 \\ A2 \\ A3 \\ A3 \end{array} & \begin{bmatrix} 0.259 & 0.319 & 0.364 & 0.268 \\ 0.240 & 0.329 & 0.262 & 0.164 \\ 0.240 & 0.118 & 0.227 & 0.278 \\ 0.261 & 0.234 & 0.146 & 0.291 \end{bmatrix} \\ & \begin{array}{cccc} w_{21} & w_{22} & w_{23} & w_{24} \end{array} \end{array}$$

Step - 3: Decision makers take into consideration the interdependency among criteria. When one alternative is selected, more than one criterion should be considered. Therefore, the impact of all the criteria needs to be examined by using pairwise comparisons. By decision makers' group interviews, four sets of weightings have been obtained. The data that the decision makers prepare for the relationships between criteria reflect the relative impact degree of the four criteria with respect to each of four criteria. We make a graph to show the relationship between the interdependency among four criteria, and the mutual effect.

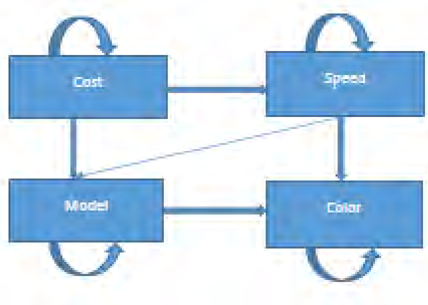


Figure 5. Interdependence among the criteria.

The interdependency weighting matrix of criteria is defined as:

$$\mathrm{w}_3 = \begin{matrix} \\ C_1 \\ C_2 \\ C_3 \\ C_4 \end{matrix} \begin{matrix} C_1 & C_2 & C_3 & C_4 \\ \end{matrix}$$

$$\mathrm{w}_3 = \begin{bmatrix} 1 & 0.8 & 0.4 & 0 \\ 0 & 0.2 & 0.5 & 0.6 \\ 0 & 0 & 0.1 & 0.3 \\ 0 & 0 & 0 & 0.1 \end{bmatrix}$$

$$\mathrm{w}_c = \mathrm{w}_3 \times \mathrm{w}_1 = \begin{bmatrix} 1 & 0.8 & 0.4 & 0 \\ 0 & 0.2 & 0.5 & 0.6 \\ 0 & 0 & 0.1 & 0.3 \\ 0 & 0 & 0 & 0.1 \end{bmatrix} \times \begin{bmatrix} 0.33 \\ 0.40 \\ 0.22 \\ 0.05 \end{bmatrix} = \begin{bmatrix} 0.738 \\ 0.220 \\ 0.037 \\ 0.005 \end{bmatrix}$$

Thus, it is derived that $\mathrm{w_c}$= (C_1,C_2 ,C_3 ,C_4) = (0.738, 0.220, 0.037, 0.005).

Step - 4: The interdependency among alternatives with respect to each criterion is calculated by respect of consistency ratio that the decision makers determined. In order to satisfy the criteria 1 (C_1), which alternative contributes more to the action of alternative 1 against criteria 1 and how much more? We defined the project interdependency weighting matrix for criteria C_1 as:

a. First criteria (C_1)

DMs compare criteria with other criteria, and determine the weighting of every criteria:

$$\tilde{A}_{C1} = \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} \begin{bmatrix} (0.5,0.5,0.5,0.5) & (0.3,0.2,0.4,0.5) & y & y \\ y & (0.5,0.5,0.5,0.5) & (0.1,0.2,0.4,0.8) & y \\ y & y & (0.5,0.5,0.5,0.5) & (0.2,0.3,0.4,0.7) \\ y & y & y & (0.5,0.5,0.5,0.5) \end{bmatrix}$$

(columns A_1, A_2, A_3, A_4)

where y indicates preference values that are not determined by decision makers. Then, we can calculate these values and make them consistent with their judgments. Let us complete the previous matrix according to definition 5 as follows:

$\widetilde{R}_{13} = \tilde{r}_{12} + \tilde{r}_{23}$ - (0.5 , 0.5,0.5,0.5) = (−0.1, −0.1,0.3,0.8)

$\widetilde{R}_{31}$ = 1 - $\widetilde{R}_{13}$ = 1 - (−0.1, −0.1,0.3,0.8) = (0.2, 0.7,1.1,1.1)

$\widetilde{R}_{32} = \tilde{r}_{31} + \tilde{r}_{12}$ - (0.5 , 0.5,0.5,0.5) = (0.0,0.4 ,1.0,1.1)

$\widetilde{R}_{21}$ = 1 - $\widetilde{R}_{12}$ = 1 – (0.3, 0.2,0.4,0.5) = (0.5, 0.6, 0.8, 0.7)

$\widetilde{R}_{14} = \tilde{r}_{13} + \tilde{r}_{34}$ - (0.5 , 0.5,0.5,0.5) = (−0.1, −0.3,0.2,1.1)

$\widetilde{R}_{24} = \tilde{r}_{21} + \tilde{r}_{14}$ - (0.5 , 0.5,0.5,0.5) = (−0.1, −0.2,0.5,1.2)

$\widetilde{R}_{41}$ = 1 - $\widetilde{R}_{14}$ = 1 – (−0.1, −0.3,0.2,1.0) = (1.0, 0.8, 1.3, 1.1)

$\widetilde{R}_{42}$ = 1 - $\widetilde{R}_{24}$ = 1 – (−0.1, −0.2,0.5,1.2) = (0.2, 0.5, 1.2, 1.1)

$\widetilde{R}_{43}$ = 1 - $\widetilde{R}_{34}$ = 1 – (0.2, 0.3,0.4,0.7) = (0.3, 0.6, 0.7, 0.8)

The comparison matrix will be as follows:

$$\tilde{A}_{C1} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} (0.5, 0.5,0.5,0.5) & (0.3, 0.2,0.4,0.5) & (-0.1, -0.1,0.3,0.8) & (-0.1, -0.3,0.2,1.1) \\ (0.5, 0.6, 0.8, 0.7) & (0.5, 0.5,0.5,0.5) & (0.1, 0.2,0.4,0.8) & (-0.1, -0.2,0.5,1.2) \\ (0.2, 0.7,1.1,1.1) & (0.0,0.4, 1.0,1.1) & (0.5, 0.5,0.5,0.5) & (0.2, 0.3,0.4,0.7) \\ (1.0, 0.8, 1.3, 1.1) & (0.2, 0.5, 1.2, 1.1) & (0.3, 0.6, 0.7, 0.8) & (0.5, 0.5,0.5,0.5) \end{bmatrix} \end{array}$$

According to definition 6, one can see that this relation is not a trapezoidal neutrosophic additive reciprocal preference relation. By using Eq. 5, Eq. 6 and Eq. 7 in definition 6, we obtain the following:

$$\tilde{A}_{C1} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} (0.5, 0.5,0.5,0.5) & (0.3, 0.2,0.4,0.5) & (0.1, 0.1,0.3,0.8) & (0.1, 0.3,0.2,1.0) \\ (0.5, 0.6, 0.8, 0.7) & (0.5, 0.5,0.5,0.5) & (0.1, 0.2,0.4,0.8) & (0.1, 0.2,0.5,1.0) \\ (0.2, 0.7,1.0,1.0) & (0.0,0.4, 1.0,1.0) & (0.5, 0.5,0.5,0.5) & (0.2, 0.3,0.4,0.7) \\ (1.0, 0.8, 1.0, 1.0) & (0.2, 0.5, 1.0, 1.0) & (0.3, 0.6, 0.7, 0.8) & (0.5, 0.5,0.5,0.5) \end{bmatrix} \end{array}$$

We check if the matrix is consistent according to definition 6. By ensuring consistency of trapezoidal neutrosophic additive reciprocal preference relations, decision makers (DMs) should determine the maximum truth-membership degree (α), the minimum indeterminacy-membership degree (θ) and the minimum falsity-membership degree (β) of single valued neutrosophic numbers as in definition 6.

$$\tilde{A}_{C1} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{bmatrix} (0.5, 0.5,0.5,0.5) & (0.3, 0.2,0.4,0.5; 0.7, 0.2, 0.5) & (0.1, 0.1,0.3,0.8; 0.5, 0.2,0.1) & (0.1, 0.3,0.2,1.0; 0.5,0.2,0.1) \\ (0.5, 0.6, 0.8, 0.7; 0.7, 0.2, 0.5) & (0.5, 0.5,0.5,0.5) & (0.1, 0.2,0.4,0.8; 0.4, 0.5, 0.6) & (0.1, 0.2,0.5,1.0; 0.5,0.1,0.2) \\ (0.2, 0.7,1.0,1.0; 0.8, 0.2, 0.1) & (0.0,0.4, 1.0,1.0; 0.3, 0.1, 0.5) & (0.5, 0.5,0.5,0.5) & (0.2, 0.3,0.4,0.7; 0.7, 0.2, 0.5) \\ (1.0, 0.8, 1.0, 1.0; 0.6,0.2,0.3) & (0.2, 0.5, 1.0, 1.0; 0.6,0.2,0.3) & (0.3, 0.6, 0.7, 0.8; 0.9,0.4,0.6) & (0.5, 0.5,0.5,0.5) \end{bmatrix} \end{array}$$

We make sure the matrix is deterministic, or we transform the previous matrix to be a deterministic pairwise comparison matrix, to calculate the weight of each criterion using equation (8, 9) in definition 6.

The deterministic matrix can be obtained by S ($\tilde{a}_{ij}$) equation in the following step:

$$\tilde{A}_{C1} = \begin{bmatrix} 0.5 & 0.175 & 0.179 & 0.22 \\ 0.325 & 0.5 & 0.122 & 0.25 \\ 0.453 & 0.265 & 0.5 & 0.2 \\ 0.38 & 0.354 & 0.285 & 0.5 \end{bmatrix}$$

We present the weight of each alternatives according to each criteria from the deterministic matrix easily by dividing each entry by the sum of the column; we obtain the following matrix as:

$$\tilde{A}_{C1} = \begin{bmatrix} 0.30 & 0.135 & 0.165 & 0.188 \\ 0.196 & 0.386 & 0.112 & 0.214 \\ 0.273 & 0.198 & 0.460 & 0.171 \\ 0.229 & 0.274 & 0.262 & 0.427 \end{bmatrix}$$

b. Second criteria (C_2)

DMs compare criteria with other criteria, and determine the weighting of every criteria:

$$\tilde{A}_{C2} = \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} \overset{\begin{matrix} A_1 & A_2 & A_3 & A_4 \end{matrix}}{\begin{bmatrix} (0.5\,,0.5,0.5,0.5) & (0.3,0.6,0.4,0.5) & y & y \\ y & (0.5\,,0.5,0.5,0.5) & (0.5\,,0.2,0.4,0.9) & y \\ y & y & (0.5\,,0.5,0.5,0.5) & (0.5\,,0.3,0.4,0.7) \\ y & y & y & (0.5\,,0.5,0.5,0.5) \end{bmatrix}}$$

where y indicates preference values that are not determined by decision makers, then we can calculate these values and make them consistent with their judgments.

We complete the previous matrix according to definition 5 as follows:

The comparison matrix will be as follows:

$$\tilde{A}_{C2} = \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} \overset{\begin{matrix} A_1 & A_2 & A_3 & A_4 \end{matrix}}{\begin{bmatrix} (0.5\,,0.5,0.5,0.5) & (0.3,0.6,0.4,0.5) & (0.3,0.3,0.3,0.9) & (0.3,0.1,0.2,1.1) \\ (0.5,0.6,0.4,0.7) & (0.5\,,0.5,0.5,0.5) & (0.5\,,0.2,0.4,0.9) & (0.3,0.2,0.1,1.3) \\ (0.1,0.7,0.7,0.7) & (-0.1,0.8\,,0.3,0.5) & (0.5\,,0.5,0.5,0.5) & (0.5\,,0.3,0.4,0.7) \\ (1.0,0.8,0.9,0.7) & (0.3,0.9,0.8,0.7) & (0.3,0.6,0.7,0.5) & (0.5\,,0.5,0.5,0.5) \end{bmatrix}}$$

According to definition 6, one can see that this relation is not a trapezoidal neutrosophic additive reciprocal preference relation. By using Eq. 5, Eq. 6 and Eq. 7 in definition 6, we obtain the following:

$$\tilde{A}_{C2} = \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} \overset{\begin{matrix} A_1 & A_2 & A_3 & A_4 \end{matrix}}{\begin{bmatrix} (0.5\,,0.5,0.5,0.5) & (0.3,0.6,0.4,0.5) & (0.3,0.3,0.3,0.9) & (0.3,0.1,0.2,1.0) \\ (0.5,0.6,0.4,0.7) & (0.5\,,0.5,0.5,0.5) & (0.5\,,0.2,0.4,0.9) & (0.3,0.2,0.1,1.0) \\ (0.1,0.7,0.7,0.7) & (0.1,0.8\,,0.3,0.5) & (0.5\,,0.5,0.5,0.5) & (0.5\,,0.3,0.4,0.7) \\ (1.0,0.8,0.9,0.7) & (0.3,0.9,0.8,0.7) & (0.3,0.6,0.7,0.5) & (0.5\,,0.5,0.5,0.5) \end{bmatrix}}$$

Let us check that the matrix is consistent according to definition 6. Then, by ensuring consistency of trapezoidal neutrosophic additive reciprocal preference relations, decision makers (DMs) should determine the maximum truth-membership degree (α), the minimum indeterminacy-membership degree (θ) and the minimum falsity-membership degree (β) of single valued neutrosophic numbers, as in definition 6. Then:

$$\tilde{A}_{C2} = \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} \overset{\begin{matrix} A_1 & & & A_2 & & & A_3 & & & A_4 \end{matrix}}{\begin{bmatrix} (0.5\,,0.5,0.5,0.5) & (0.3,0.6,0.4,0.5\,;0.7\,,0.3,0.5) & (0.3,0.3,0.3,0.9\,;0.5,0.2,0.1) & (0.3,0.1,0.2,1.0;0.5,0.2,0.1) \\ (0.5,0.6,0.4,0.7;0.7\,,0.3,0.5) & (0.5\,,0.5,0.5,0.5) & (0.5\,,0.2,0.4,0.9;0.4,0.5,0.6) & (0.3,0.2,0.1,1.0;0.5,0.1,0.4) \\ (0.1,0.7,0.7,0.7;0.8,0.2\,,0.3) & (0.1,0.8\,,0.3,0.5;0.4,0.2,0.5) & (0.5\,,0.5,0.5,0.5) & (0.5\,,0.3,0.4,0.7;0.6,0.2,0.5) \\ (1.0,0.8,0.9,0.7;0.6,0.4,0.3) & (0.3,0.9,0.8,0.7;0.6,0.2,0.3) & (0.3,0.6,0.7,0.5;0.9,0.4,0.5) & (0.5\,,0.5,0.5,0.5) \end{bmatrix}}$$

Let us be sure the matrix is deterministic, or transform the previous matrix to be deterministic pairwise comparison matrix, to calculate the weight of each criteria using equation (8, 9) in definition 6.

The deterministic matrix can be obtained by S ($\tilde{a}_{ij}$) equation in the following step:

$$\tilde{A}_{C2} = \begin{bmatrix} 0.5 & 0.214 & 0.247 & 0.22 \\ 0.216 & 0.5 & 0.163 & 0.20 \\ 0.316 & 0.181 & 0.5 & 0.226 \\ 0.404 & 0.354 & 0.3 & 0.5 \end{bmatrix}$$

We present the weight of each alternatives according to each criteria from the deterministic matrix by dividing each entry by the sum of the column; we obtain the following matrix:

$$\tilde{A}_{C2} = \begin{bmatrix} 0.50 & 0.215 & 0.244 & 0.192 \\ 0.216 & 0.503 & 0.161 & 0.175 \\ 0.273 & 0.182 & 0.495 & 0.197 \\ 0.229 & 0.356 & 0.259 & 0.436 \end{bmatrix}$$

c. Third criteria (C_3)

DMs compare criteria with other criteria, and determine the weight of every criteria.

$$\tilde{A}_{C3} = \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} \overset{\begin{matrix} A_1 & & & A_2 & & & A_3 & & & A_4 \end{matrix}}{\begin{bmatrix} (0.5\,,0.5,0.5,0.5) & (0.6,0.7,0.9,0.1) & y & y \\ y & (0.5\,,0.5,0.5,0.5) & (0.6\,,0.7,0.8,0.9) & y \\ y & y & (0.5\,,0.5,0.5,0.5) & (0.2\,,0.5,0.6,0.8) \\ y & y & y & (0.5\,,0.5,0.5,0.5) \end{bmatrix}}$$

where y indicates preference values that are not determined by decision makers; then, we can calculate these values and make them consistent with their judgments.

We complete the previous matrix according to definition 5 as follows:

$$\tilde{A}_{C3} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \\ \end{array}$$

$$\tilde{A}_{C3} = \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} \begin{bmatrix} (0.5, 0.5, 0.5, 0.5) & (0.6, 0.7, 0.9, 0.1) & (0.7, 0.9, 1.2, 1.4) & (0.4, 0.7, 1.3, 1.7) \\ (0.0, 0.1, 0.3, 0.4) & (0.5, 0.5, 0.5, 0.5) & (0.6, 0.7, 0.8, 0.9) & (0.3, 0.5, 0.9, 1.2) \\ (-0.4, -0.2, 0.1, 0.3) & (-0.3, 0.0, 0.5, 0.8) & (0.5, 0.5, 0.5, 0.5) & (0.2, 0.5, 0.6, 0.8) \\ (-0.7, -0.3, 0.3, 0.6) & (-0.6, -0.1, 0.7, 1.1) & (0.2, 0.4, 0.5, 0.8) & (0.5, 0.5, 0.5, 0.5) \end{bmatrix}$$

According to definition 6, one can see that the relation is not a trapezoidal neutrosophic additive reciprocal preference relation. By using Eq. 5, Eq. 6 and Eq. 7 in definition 6, we obtain the following:

$$\begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array}$$

$$\tilde{A}_{C3} = \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} \begin{bmatrix} (0.5, 0.5, 0.5, 0.5) & (0.6, 0.7, 0.9, 0.1) & (0.7, 0.9, 1.0, 1.0) & (0.4, 0.7, 1.0, 1.0) \\ (0.0, 0.1, 0.3, 0.4) & (0.5, 0.5, 0.5, 0.5) & (0.6, 0.7, 0.8, 0.9) & (0.3, 0.5, 0.9, 1.0) \\ (0.4, 0.2, 0.1, 0.3) & (0.3, 0.0, 0.5, 0.8) & (0.5, 0.5, 0.5, 0.5) & (0.2, 0.5, 0.6, 0.8) \\ (0.7, 0.3, 0.3, 0.6) & (0.6, 0.1, 0.7, 1.0) & (0.2, 0.4, 0.5, 0.8) & (0.5, 0.5, 0.5, 0.5) \end{bmatrix}$$

Then, let us check that the matrix is consistent according to definition 6. Then, by ensuring consistency of trapezoidal neutrosophic additive reciprocal preference relations, decision makers (DMs) should determine the maximum truth-membership degree (α), the minimum indeterminacy-membership degree (θ) and the minimum falsity-membership degree (β) of the single valued neutrosophic numbers as in definition 6. Then:

$$\begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array}$$

$$\tilde{A}_{C3} = \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix}$$

$$\begin{bmatrix} (0.5, 0.5, 0.5, 0.5) & (0.6, 0.7, 0.9, 0.1; 0.7, 0.2, 0.5) & (0.7, 0.9, 1.0, 1.0; 0.5, 0.2, 0.1) & (0.4, 0.7, 1.0, 1.0; 0.5, 0.2, 0.3) \\ (0.0, 0.1, 0.3, 0.4; 0.8, 0.2, 0.6) & (0.5, 0.5, 0.5, 0.5) & (0.6, 0.7, 0.8, 0.9; 0.5, 0.2, 0.1) & (0.3, 0.5, 0.9, 1.0; 0.5, 0.1, 0.2) \\ (0.4, 0.2, 0.1, 0.3; 0.5, 0.3, 0.4) & (0.3, 0.0, 0.5, 0.8; 0.8, 0.5, 0.3) & (0.5, 0.5, 0.5, 0.5) & (0.2, 0.5, 0.6, 0.8; 0.6, 0.4, 0.2) \\ (0.7, 0.3, 0.3, 0.6; 0.5, 0.2, 0.1) & (0.6, 0.1, 0.7, 1.0; 0.3, 0.1, 0.5) & (0.2, 0.4, 0.5, 0.8; 0.3, 0.1, 0.5) & (0.5, 0.5, 0.5, 0.5) \end{bmatrix}$$

Let us be sure the matrix is deterministic, or transform the previous matrix to be deterministic pairwise comparison matrix, to calculate the weight of each criteria using equation (8, 9) in definition 6.

The deterministic matrix can be obtained by S ($\tilde{a}_{ij}$) equation in the following step:

$$\tilde{A}_{C3} = \begin{bmatrix} 0.5 & 0.4 & 0.49 & 0.41 \\ 0.1 & 0.5 & 0.41 & 0.37 \\ 0.18 & 0.24 & 0.5 & 0.56 \\ 0.38 & 0.30 & 0.20 & 0.5 \end{bmatrix}$$

We present the weight of each alternatives according to each criteria from the deterministic matrix by dividing each entry by the sum of the column; we obtain the following matrix:

$$\tilde{A}_{C3} = \begin{bmatrix} 0.43 & 0.27 & 0.30 & 0.22 \\ 0.08 & 0.35 & 0.26 & 0.20 \\ 0.15 & 0.16 & 0.31 & 0.30 \\ 0.33 & 0.21 & 0.12 & 0.27 \end{bmatrix}$$

d. Four criteria (C_4)

DMs compare criteria with other criteria, and determine the weighting of every:

$$\tilde{A}_{C4} = \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} \overset{\begin{matrix} A_1 & A_2 & A_3 & A_4 \end{matrix}}{\begin{bmatrix} (0.5\ ,0.5,0.5,0.5) & (0.4,0.5,0.3,0.7) & y & y \\ y & (0.5\ ,0.5,0.5,0.5) & (0.4\ ,0.2,0.7,0.5) & y \\ y & y & (0.5\ ,0.5,0.5,0.5) & (0.4,0.6,0.5,0.8) \\ y & y & y & (0.5\ ,0.5,0.5,0.5) \end{bmatrix}}$$

Where y indicates the preference values that are not determined by decision makers; then, we can calculate these values and make them consistent with their judgments.

We complete the previous matrix according to definition 5 as follows:

$$\tilde{A}_{C4} = \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} \overset{\begin{matrix} A_1 & A_2 & A_3 & A_4 \end{matrix}}{\begin{bmatrix} (0.5\ ,0.5,0.5,0.5) & (0.4,0.5,0.3,0.7) & (0.3,0.2,0.5,0.7) & (0.2,0.3,0.5,1.0) \\ (0.3,0.7,0.5,0.6) & (0.5\ ,0.5,0.5,0.5) & (0.4\ ,0.2,0.7,0.5) & (0.0,0.5,0.5,1.1) \\ (0.3,0.7,0.5,0.6) & (0.2,0.5\ ,0.6,0.9) & (0.5\ ,0.5,0.5,0.5) & (0.4,0.6,0.5,0.8) \\ (0.3,0.7,0.5,0.6) & (-0.1,0.5,0.5,1.0) & (0.2,0.5,0.4,0.6) & (0.5\ ,0.5,0.5,0.5) \end{bmatrix}}$$

According to definition 6, one can see that this relation is not a trapezoidal neutrosophic additive reciprocal preference relation. By using Eq. 5, Eq. 6 and Eq. 7 in definition 6, we obtain the following:

$$\tilde{A}_{C4} = \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} \overset{\begin{matrix} A_1 & A_2 & A_3 & A_4 \end{matrix}}{\begin{bmatrix} (0.5\ ,0.5,0.5,0.5) & (0.4,0.5,0.3,0.7) & (0.3,0.2,0.5,0.7) & (0.2,0.3,0.5,1.0) \\ (0.3,0.7,0.5,0.6) & (0.5\ ,0.5,0.5,0.5) & (0.4\ ,0.2,0.7,0.5) & (0.0,0.5,0.5,1.0) \\ (0.3,0.7,0.5,0.6) & (0.2,0.5\ ,0.6,0.9) & (0.5\ ,0.5,0.5,0.5) & (0.4,0.6,0.5,0.8) \\ (0.3,0.7,0.5,0.6) & (0.1,0.5,0.5,1.0) & (0.2,0.5,0.4,0.6) & (0.5\ ,0.5,0.5,0.5) \end{bmatrix}}$$

Then, we check that the matrix is consistent according to definition 6. By ensuring consistency of trapezoidal neutrosophic additive reciprocal preference relations, decision makers (DMs) should determine the maximum truth-membership degree (α), the minimum indeterminacy-membership degree (θ) and the minimum falsity-membership degree (β) of the single valued neutrosophic numbers, as in definition 6.

$$\tilde{A}_{C4} = \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} \begin{bmatrix} & A_1 & A_2 & A_3 & A_4 \\ & (0.5, 0.5, 0.5, 0.5) & (0.4, 0.5, 0.3, 0.7 ; 0.4, 0.3, 0.6) & (0.3, 0.2, 0.5, 0.7 ; 0.2, 0.3, 0.5) & (0.2, 0.3, 0.5, 1.0; 0.3, 0.1, 0.8) \\ & (0.3, 0.7, 0.5, 0.6; 0.7, 0.4, 0.5) & (0.5, 0.5, 0.5, 0.5) & (0.4, 0.2, 0.7, 0.5; 0.3, 0.5, 0.6) & (0.0, 0.5, 0.5, 1.0; 0.4, 0.3, 0.2) \\ & (0.3, 0.5, 0.8, 0.7; 0.7, 0.4, 0.5) & (0.2, 0.5, 0.6, 0.9; 0.7, 0.4, 0.3) & (0.5, 0.5, 0.5, 0.5) & (0.4, 0.6, 0.5, 0.8; 0.7, 0.3, 0.5) \\ & (0.0, 0.5, 0.7, 0.8; 0.5, 0.2, 0.4) & (0.1, 0.5, 0.5, 1.0; 0.5, 0.3, 0.6) & (0.2, 0.5, 0.4, 0.6; 0.4, 0.6, 0.2) & (0.5, 0.5, 0.5, 0.5) \end{bmatrix}$$

Let us be sure the matrix is deterministic, or transform the previous matrix to be deterministic pairwise comparison matrix, to calculate the weight of each criteria using equation (8, 9) in definition 6.

The deterministic matrix can be obtained by S ($\tilde{a}_{ij}$) equation in the following step:

$$\tilde{A}_{C4} = \begin{bmatrix} 0.5 & 0.18 & 0.15 & 0.17 \\ 0.24 & 0.5 & 0.13 & 0.23 \\ 0.29 & 0.27 & 0.5 & 0.27 \\ 0.23 & 0.21 & 0.17 & 0.5 \end{bmatrix}$$

We present the weight of each alternative according to each criteria from the deterministic matrix by dividing each entry by the sum of the column; we obtain the following matrix:

$$\tilde{A}_{C4} = \begin{bmatrix} 0.40 & 0.16 & 0.16 & 0.15 \\ 0.19 & 0.43 & 0.14 & 0.19 \\ 0.23 & 0.23 & 0.5 & 0.23 \\ 0.18 & 0.18 & 0.18 & 0.42 \end{bmatrix}$$

Step 4: The priorities of the alternative W_A with respect to each of the four criteria are given by synthesizing the results from Steps 2 and 4 as follows:

$$W_{A1} = W_{\tilde{A}_{C_1}} \times W_{21} = \begin{bmatrix} 0.199 \\ 0.172 \\ 0.273 \\ 0.299 \end{bmatrix}$$

$$W_{A2} = W_{\tilde{A}_{C_2}} \times W_{22} = \begin{bmatrix} 0.303 \\ 0.294 \\ 0.251 \\ 0.347 \end{bmatrix}$$

$$W_{A3} = W_{\tilde{A}_{C_3}} \times W_{23} = \begin{bmatrix} 0.327 \\ 0.209 \\ 0.210 \\ 0.241 \end{bmatrix}$$

$$W_{A4} = W_{\tilde{A}_{C_4}} \times W_{24} = \begin{bmatrix} 0.222 \\ 0.216 \\ 0.305 \\ 0.250 \end{bmatrix}$$

The matrix W_A is defined by grouping together the above four columns:

W_A =[$W_{A1}, W_{A2}, W_{A3}, W_{A4}$]

Step 5: The overall priorities for the candidate alternatives are finally calculated by multiplying W_A and W_C:

$$= W_A \times W_c = \begin{matrix} W_{A1} & W_{A2} & W_{A3} & W_{A4} \end{matrix} \begin{bmatrix} 0.199 & 0.303 & 0.327 & 0.222 \\ 0.172 & 0.294 & 0.209 & 0.216 \\ 0.273 & 0.251 & 0.210 & 0.305 \\ 0.299 & 0.347 & 0.241 & 0.250 \end{bmatrix} \times \begin{bmatrix} 0.738 \\ 0.220 \\ 0.037 \\ 0.005 \end{bmatrix} = \begin{bmatrix} 0.226 \\ 0.200 \\ 0.265 \\ 0.307 \end{bmatrix}$$

The final results in the ANP Neutrosophic Phase are (A1, A2, A3, A4) = (0.226, 0.200, 0.265, 0.307). These ANP Neutrosophic results are interpreted as follows. The highest weighting of criteria in this problem selection example is A4. Next is A1. These weightings are used as priorities in selecting the best personnel car.

Then, it is obvious that the four alternative has the highest rank, meaning that Nissan is the best car according to this criteria, followed by Crossover, Diesel and, finally, Sedan.

Table 2. Ranking of alternatives.

Car Name	Priority
Crossover	0.22
Diesel	0.20
Nissan	0.26
Sedan	0.30

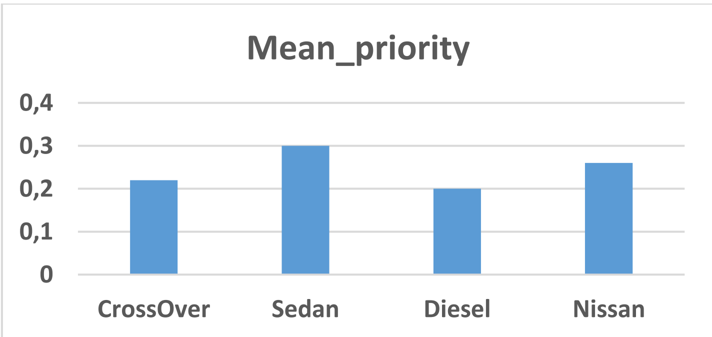


Figure 6. ANP ranking of alternatives.

5 Conclusion

This research employed the ANP technique in neutrosophic environment for solving complex problems, showing the interdependence among criteria, the feedback and the relative weight of decision makers (DMs). We analyzed how to determine the weight for each criterion, and the interdependence among criteria, calculating the weighting of each criterion to each alternative. The proposed model of ANP in neutrosophic environment is based on using of $(n-1)$ consensus judgments instead of $\frac{n\times(n-1)}{2}$ ones, in order to decrease the workload. We used a new scale from 0 to 1 instead of that from 1 to 9. We also presented a real life example as a case study. In the future, we plan to apply ANP in neutrosophic environment by integrating it with other techniques, such as TOPSIS.

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A Neutrosophic Technique Based Efficient Routing Protocol for MANET Based on Its Energy and Distance

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Abstract. In the last decade, characterizing the energy in MANET based on its acceptation, rejection and uncer-tain part is addressed as one of the major issues by the researchers. An efficient energy routing protocol for MA-NET is another issue. To resolve these issues current paper focuses on utilizing the properties of neutrosophic technique. The essential idea of the protocol is to choose an energy efficient route with respect to neutrosophic technique. In this neutrosophic set, we have three components such as (T, I, F). Each parameter such as energy and distance is taken from these neutrosophic sets to determine the efficient energy route in MANET. After taking a brief survey about energy efficient routing for MANET using various methods, we are trying to implement the neutrosophic set technique to find the efficient energy route for MANET which provides the better energy route in uncertain situations. The comparative analysis between vague set MANET and neutrosophic MANET for the val-ues of energy functions and distance functions is done by using Matlab and the result is discussed graphically

1 Introduction

Wireless networking technologies play a vital role for giving rise to many new applications in internet world. Mobile ad-hoc network (MANET) is one of the most leading fields for research and development of wireless network. Now a days, wireless ad-hoc network has become one of the most vibrant and active field of communi-cation and networks due to the popularity of mobile devices. Also, mobile or wireless network has become one of the indeed requirement for the users around the world. In this network, there are no groundwork stations or mobile switching centres and other structures of these types. The topology of Mobile ad-hoc network (MANET) changes dynamically. Each node is within others node's radio range via wireless networks. In the present era, nearly everyone has a mobile phone and most of it are smart phones. These devices are very cheaper and more powerful which make Mobile ad-hoc network (MANET) as the speed-growing network [1, 26, 36, 37]. Because of frequent braking of communication links, the nodes in mobile ad-hoc networks are free to move to anywhere. Also, a node in Mobile ad-hoc network (MANET) performs complete access to send data from one node to the other very fast and provides accurate services. Mobile ad-hoc network (MANET) is user friendly network which is easy to add or remove from the network. In this, each node contains some energy with limited battery capacity. The energy has been lost very speed in ad-hoc networks by transforming the data from one node to another node and also over all network's lifetime. Therefore the energy efficient routing indicates that the selecting route re-quires high energy and shortest distance. In this regard recently one of the authors has utilized implications of weighted concept lattice [31] and its implications using three-way neutrosophic environment [32-33] at different threshold [25] beyond the fuzzy logic [40]. It is shown that the computing paradigm of neutrosophic logic pro-vides an authorization to deal with indeterminacy in the given network when compared to any other approaches available in fuzzy logic. Hence the current paper focused on introducing the concept of neutrosophic logic for analyzing the energy efficient routing protocol in Mobile ad-hoc network (MANET).

Neutrosophic set was in-troduced by Florentin Smarandache [34] in 1995. Neutrosophic set is the generalization of fuzzy set, intuition-istic set fuzzy set, classical set and paraconsistent set etc. In intuitionistic fuzzy sets[2], the uncertainty is de-pendent on the degree of belongingness and degree of non-belongingness. In case of neutrosophy theory, the in-determinacy factor is independent of truth and falsity membership-values. Also neutrosophic sets are more gen-eral than IFS, because there are no conditions between the degree of truth, degree of indeterminacy and degree of falsity. In 2005, Wang et.al [38] introduced single valued neutrosophic sets which can be used in real world ap-plications. In this case, a problem is addressed while dealing with efficient route in routing protocol based on its distance or energy. To shoot this problem, the current paper introduces a method to characterize the energy effi-cient route in Mobile ad-hoc network (MANET) based on its acceptation, rejection and uncertain part. In the same time the analysis of the proposed method is compared with one of the existing methods to validate the re-sults. The motivation is to discover the precise and efficient path based on its maximal acceptance, minimum re-jection, and minimal indeterminacy. The objective is to provide an optimal routing in Mobile ad-hoc network (MANET) in minimal energy utilization when compared to vague set [18]. One of the significant outputs of the proposed method is that it deals with uncertainty independent from truth and false membership-values.

The remaining part of the paper is organized as follows: Section 2 provides preliminaries about each of the set theories. Section 3 provides proposed method with its comparative analysis in Section 4. Section 5 provides conclusions and future research.

2 Overview of Mobile ad-hoc networks[28]

Mobile Ad Hoc networking (MANET) can be classified into first, second and third generations. The first generation of mobile ad-hoc network came up with "packet radio" networks (PRNET) in 1970s and it has evolved to be a robust, reliable, operational experimental network. The PRNET used a combination of ALOHA and channel access approaches CSMA for medium access, and a distance-vector routing to give packet-switched networking to mobile field elements in an infrastructure less, remote environment. The second generation evolved in early 1980's when SURAN (Survivable Adaptive Radio Networks) significantly improved upon the radios, scalability of algorithms, and resilience to electronic attacks. During this period include GloMo (Global Mobile Information System) and NTDR (Near Term Digital Radio)were developed. The aim of GloMo was to give office-environment Ethernet-type multimedia connectivity anytime, anywhere, in handheld devices. Channel access approaches were in the CSMA/CA and TDMA molds, and several novel routing and topology control schemes were developed. The NTDR used clustering and link- state routing, and self-organized into a two-tier ad hoc network. Now used by the US Army, NTDR is the only "real" (non-prototypical) ad hoc network in use today. The third generation evolved in 1990's also termed as commercial network with the advent of Notebooks computers, open source software and equipments based on RF and infrared. IEEE 802.11 subcommittee adopted the term "ad hoc networks." The development of routing within the Mobile ad-hoc networking (MANET) working group and the larger community forked into reactive (routes on- demand) and proactive (routes ready-to-use) routing protocols 141. The 802.1 1 subcommittee standardized a medium access protocol that was based on collision avoidance and tolerated hidden terminals, making it usable, if not optimal, for building mobile ad hoc network prototypes out of notebooks and 802.11 PCMCIA cards. HIPERLAN and Bluetooth were some other standards that addressed and benefited ad hoc networking. With the increase of portable devices with wireless communication, ad-hoc networking plays an important role in many applications such as commercial, military and sensor networks, data networks etc., Mobile ad-hoc networks allow users to access and exchange information regardless of their geographic position or proximity to infrastructure. Since Mobile ad-hoc networking (MANET) has no static infrastructure, it offers an advantageous decentralized character to the network. Decentralization makes the networks more flexible and more robust.

3 Preliminaries

Definition of Fuzzy Set:

Fuzzy set was introduced by Zadeh in 1965 [40] and it gives new trend in application of mathematics. Every value of the fuzzy set consisting of order pair one is true membership and another one is false membership which lies between 0 and 1. Several authors [30, 39, 21-23, 27, 29] used fuzzy set theory in ad-hoc network and wireless sensor network to solve routing problems. The logic in fuzzy set theory is vastly used in all fields of mathematics like networks, graphs, topological space …etc.

Definition:[9]Intuitionistic Fuzzy Set:

Intuitionistic Fuzzy Sets are the extension of usual fuzzy sets. All outcomes which are applicable for fuzzy sets can be derived here also. Almost all the research works for fuzzy sets can be used to draw information of IFSs.

Further, there have been defined over IFSs not only operations similar to those of ordinary fuzzy sets, but also operators that cannot be defined in the case of ordinary fuzzy sets.

Definition:[17,24] Adroit System:

Adroit system [17, 24] is a computer program that efforts to act like a human effect in a particular subject area to give the solution to the particular unpredictable problem. Sometimes, adroit systems are used instead of human minds. Its main parts are knowledge based system and inference engine. In that the software is the knowledge based system which can be solved by artificial intelligence technique to find efficient route. The second part is inference engine which processes data by using rule based knowledge.

Definition:[34] Neutrosophic Set:

A neutrosophic set is a triplet which contains a truth membership function, a false membership function and indeterminacy function. Many authors extended this neutrosophic theory in different fields of mathematics such as decision making, optimization, graph theory etc.,[3-16, 42-52]. In particular, with the best knowledge, this is the first time to calculate efficient energy protocol for MANET based on the neutrosophic technique.

Let U be the universe. The neutrosophic set A in U is characterized by a truth-membership function T_A, a indeterminacy-membership function I_A and a falsity-membership function F_A. $T_A(x)$, $I_A(x)$ and $F_A(x)$ are real standard elements of [0,1]. It can be written as

$$A_{NS} = \{< T_A(x), I_A(x), F_A(x) >, x \in U, T_A(x), I_A(x), F_A(x) \in]^-0,1^+[,\} \quad (1)$$

There is no restriction on the sum of $T_A(x)$, $I_A(x)$ and F (x). So $0^- \leq T_A(x)+I_A(x)+F_A(x) \leq 3^+$.

Definition:[35] Let U be a universe of discourse and A the neutrosophic set A $\subset$ U. Let $T_A(x), I_A(x), F_A(x)$ be the functions that describe the degree of membership, indeterminate membership and non-membership respectively of a generic element x $\in$ U with respect to the neutrosophic set A. A single valued neutrosophic overset (SVNOV) A on the universe of discourse U is defined as:

$$A_{SVNOV} = \{< T_A(x), I_A(x), F_A(x) >, x \in U, T_A(x), I_A(x), F_A(x) \in [0,\Omega]\} \quad (2)$$

where $T_A(x), I_A(x), F_A(x)$: U$\rightarrow [0,\Omega]$, 0<1<Ω and Ω is called overlimit. Then there exists at least one element in A such that it has at least one neutrosophic component >1and no element has neutrosophic component <0

Definition:[35]Let U be a universe of discourse and the neutrosophic set A$\subset$ U. Let $T_A(x), I_A(x), F_A(x)$ be the functions that describe the degree of membership, indeterminate membership and non-membership respectively of a generic element x $\in$ U with respect to the neutrosophic set A. A single valued neutrosophic underset (SVNU) A on the universe of discourse U is defined as:

$$A_{SVNU} = \{< T_A(x), I_A(x), F_A(x) >, x \in U, T_A(x), I_A(x), F_A(x) \in [\Psi,1]\} \quad (3)$$

where $T_A(x), I_A(x), F_A(x)$: U$\rightarrow [\Psi,1]$, Ψ <0<1 and Ψ is called lowerlimit.Then there exists at least one element in A such that it has at least one neutrosophic component<0 and no element has neutrosophic component >1

Definition:[35] Let U be a universe of discourse and the neutrosophic set A$\subset$ U. Let $T_A(x), I_A(x), F_A(x)$ be the functions that describe the degree of membership, indeterminate membership and non-membership respectively of a generic element x $\in$ U with respect to the neutrosophic set A. A single valued neutrosophic offset (SVNOF) A on the universe of discourse U is defined as:

$$A_{SVNOF} = \{< T_A(x), I_A(x), F_A(x) >, x \in U, T_A(x), I_A(x), F_A(x) \in [\Psi,\Omega]\} \quad (4)$$

where $T_A(x), I_A(x), F_A(x)$: U$\rightarrow [\Psi,1]$, Ψ <0<1<Ω and Ψ is called underlimit while Ω is called overlimit. Then there exist some elements in A such that at least one neutrosophic component >1, and at least another neutrosophic component < 0

Example 1: Let A={(x_1,<1.2, 0.4,0.1>),(x_2 ,<0.2, 0.3,-0.7>)},since $T(x_1)$=1.2 >1 , $F(x_2)$= - 0.7 <0

Definition:[35]The complement of a single valued neutrosophic overset/ underset/offset A is denoted by C(A) and is defined by

$$C(A) = \{(x, < F_A(x), \Psi + \Omega - I_A(x), T_A(x) >, x \in U \} \quad (5)$$

Definition:[35]The intersection of two single valued neutrosophic overset/ underset/offset A and B is a single valued neutrosophic overset/ underset/offset denoted C and is denoted by C= A$\cap$ B and is defined by

$$C = A \cap B = \{(x, < \min(T_A(x), T_B(x)), \max(I_A(x), I_B(x)), \max(F_A(x), F_B(x))), x \in U\} \quad (6)$$

Definition:[35]The union of two single valued neutrosophic overset/ underset/offset A and B is a single valued neutrosophic overset/ underset/offset denoted C and is denoted by C= A$\cup$ B and is defined by

$$C = A \cup B = \{(x, < \max(T_A(x), T_B(x)), \min(I_A(x), I_B(x)), \min(F_A(x), F_B(x))), x \in U\} \quad (7)$$

The following table 1, describe the neutrosophic oversets, neutrosophic undersets, neutrosophic offsets and Single valued neutrosophic sets

Types of neutrosophic sets	Ψ **(under limit)**	Ω **(overlimit)**
neutrosophic oversets	0	1<Ω
neutrosophic undersets	Ψ <0	1
neutrosophic offsets	Ψ <0	1<Ω
Single valued neutrosophic sets	0	1

Table 1. Some type of neutrosophic sets

It can be observed that, the algebra of neutrosophic set provides an independent way to deal with indeterminacy beyond the truth and false membership-values of a vague set. However characterizing the distance of routing protocol in MANET based on its truth, falsity and indeterminacy membership-values is complex problem. To deal with this problem, one of the algorithms is proposed in the next section with an illustrative example.

4 PROPOSED PROTOCOL

In this section, a method is proposed to characterize the efficient routing path in MANET based on the neutrosophic technique using energy and distance. In this proposed protocol, energy function may be low, medium and high and also in a similar way distance may be short, medium and long. To represent these levels a neutrosophic set based membership function μ , Indeterminacy σ and non-membership γ is defined in this paper.

All these energy membership functions $E_L, E_M and E_H$ and distance membership functions $D_S, D_M and D_L$ are given in Table 2 and Table3.

Linguistic value	**Notation**	**Neutrosophic range**	**Basic value**
Low	E_L	(E_L^+,E_L^0,E_L^-)	(0,0.9,1.8)
Medium	E_M	(E_M^+,E_M^0,E_M^-)	(1.8,2.7,3.5)
High	E_H	(E_H^+,E_H^0,E_H^-)	(3.5,4.4,5)

Table 2. A neutrosophic set based representation of energy function

Linguistic value	**Notation**	**Neutrosophic range**	**Basic value**
Short	Ds	(DL+,DL0,DL-)	(0,9,17)
Medium	DM	(DM+,DM0,DM-)	(17,26,34)
Large	DL	(DH+,DH0,DH-)	(34,42,50)

Table 3. A neutrosophic set based representation of distance function

The a single valued neutrosophic overset/ underset/offset are characterized by three memberships, the truth-membership, indeterminacy-membership and false membership functions as described in definitions above.

It gives an interpretation of membership grades. Low, medium and high of the energy and distance functions are written as follows:

Neutrosophic Energy function values:

$\mu(E_L)$ = (0.3, 0.7, 1.2), $\mu(E_M)$ = (1.4, 2.3, 3), $\mu(E_H)$ = (3.2, 4, 4.8)

$\sigma(E_L)$ = (0.4, 0.9, 1.4), $\sigma(E_M)$ = (1.3, 2.6, 3.2), $\sigma(E_H)$ = (3.4, 3.8, 4.6)

$\gamma(E_L)$ = (0.2, 0.6, 1.4), $\gamma(E_M)$ = (1.2, 2.5, 3.4), $\gamma(E_H)$ = (3, 4.2, 4.5)

Neutrosophic Distance function values:

$\mu(D_s)$ = (0.2, 4, 10), $\mu(D_M)$ = (12, 20, 32), = (30, 38, 44)

$\sigma(D_s)$ = (0.5, 5, 12), $\sigma(D_M)$ = (10, 23, 30), $\sigma(D_L)$ = (29, 41, 49)

$\gamma(D_s)$ = (0.3, 6, 8), $\gamma(D_M)$ = (14, 21, 28), $\gamma(D_L)$ = (32, 40, 47)

Recently, several authors tried to deduce the neutrosophic values in various fields [40]. The current paper tried most suitable and ideal solution deduced by considering true members function μ for the better solution. These neutrosophic values are used for efficient route selection in MANET which is given below in table 3. By comparing different routes of the MANET, rating of the route is calculated by the Eq. 8 as given below:

$$NR_{i,j} = {meanof \quad \mu(E_i)} \Big/ {meanof \quad \mu(D_i)} \qquad (8)$$

From the rating of different route given in Table 4, each value of $NR_{i,j}$ is a neutrosophic route having different values which determine the nature of the route in MANET.

S.No	Neutrosophic possible route
1	If Energy is $\mu(E_L)$ and (Distance is $\mu(D_s)$,then the route is R1.
2	If Energy is $\mu(E_L)$ and (Distance is $\mu(D_M)$, then the route is R2.
3	If Energy is $\mu(E_L)$ and (Distance is $\mu(D_L)$,then the route is R3.
4	If Energy is $\mu(E_M)$ and (Distance is $\mu(D_s)$,then the route is R4.
5	If Energy is $\mu(E_M)$ and (Distance is $\mu(D_M)$,then the route is R5.
6	If Energy is $\mu(E_M)$ and (Distance is $\mu(D_L)$,then the route is R6.
7	If Energy is $\mu(E_H)$ and (Distance is $\mu(D_s)$,then the route is R7.
8	If Energy is $\mu(E_H)$ and (Distance is $\mu(D_M)$,then the route is R8.
9	If Energy is $\mu(E_H)$ and (Distance is $\mu(D_L)$,then the route is R9.

Table 4. A neutrosophic technique based efficient route selection

Route number	**Neutrosophic Rating of route**	**Enlightenment of Rating**
R1	0.154929	Good
R2	0.034375	Bad
R3	0.019642	Very Bad
R4	0.471830	Excellent
R5	0.104687	Poor
R6	0.059821	Very poor
R7	0.873239	Very excellent
R8	0.19375	Very good
R9	0.110714	Medium

Table 5. Enlightenment of rating of different routes in Neutrosophic Technique

Hence, each neutrosophic route has a specific rating in MANET. Table 4 provides a way to defined different neutrosophic routes by considering various energy functions and as well as distance functions. Following that the sequences of different routes based on their rating is given in Table 5. The decreasing order according to rating on the routes is R3 < R2< R5 < R9 < R1 <R8 < R4 < R7.Table 5 represents that based on neutrosophic ordering defined by the proposed method route R7 is one of the best energy efficient route among them for the given MANET.

5 Comparative Analysis

While comparing vague set and neutrosophic set, vague set is equivalent to intuitionistic fuzzy set because both of them having only truth and false membership functions. Also neutrosophic set is the generalization of fuzzy and intuitionistic fuzzy sets. Hence the results obtained by using neutrosophic set is better than the results obtained by using vague set. In this section, the comparative analysis among neutrosophic and vague set based routing protocol is discussed. The membership values of energy and distance functions of vague set Manet and neutrosophic set Manet are given in Table 6. Comparison between Vague set rating of route (VSR) and Neutrosophic rating of route (NRR) are given in Table 7.

Table 6. Membership values of energy and distance function

Notation	Base Value of Energy function		Notation	Base Value of Distance function	
	VM	NM		VM	NM
EL	(0,1.8)	(0,0.9,1.8)	Ds	(0, 17)	(0,9,17)
EM	(1.8, 3.5)	(1.8,2.7,3.5)	DM	(17, 34)	(17,26,34)
EH	(3.5, 5)	(3.5,4.4,5)	DL	(34, 50)	(34,42,50)

Table 7. Comparison between Vague Set Rating of route(VSR) and Neutrosophic Rating of route (NRR):

Route number	Vague Set Rating of route(VSR)	Neutrosophic Rating of route (NRR)	Enlightenment of Rating	
			VSR	NRR
R1	0.011842	0.154929	Very Bad	Good
R2	0.021176	0.034375	Bad	Bad
R3	0.105882	0.019642	Satisfactory	Very Bad
R4	0.059211	0.471830	Medium	Excellent
R5	0.105882	0.104687	Less Good	Poor
R6	0.529412	0.059821	Good	Very poor

R7	0.1	0.873239	Very good	Very excellent
R8	0.178824	0.19375	Excellent	Very good
R9	0.894118	0.110714	Very excellent	Medium

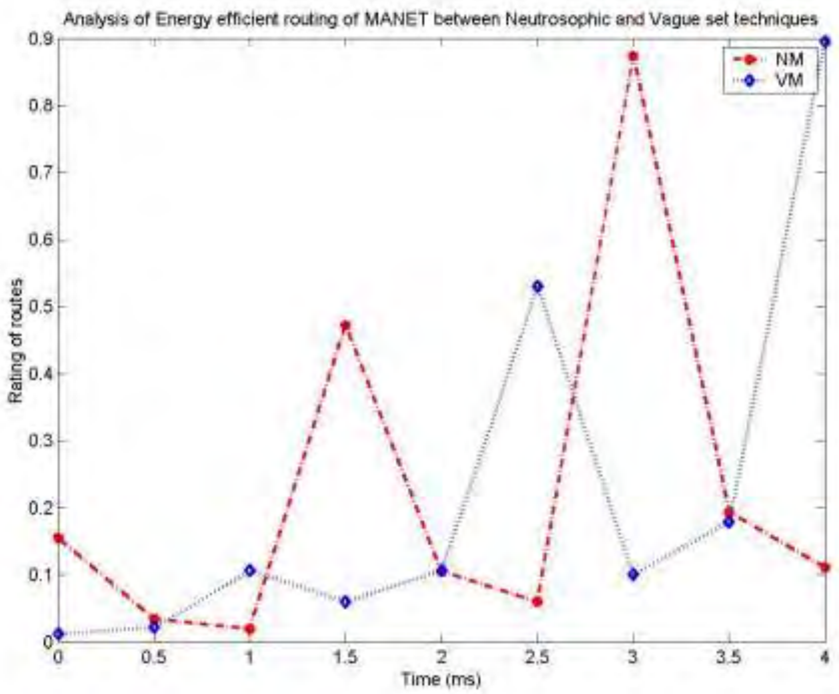

Figure. 1: The comparison of energy efficient MANET using neutrosophic and vague set

The graph for rating of routes of MANET using neutrosophic set and vague set techniques are plotted in Figure 1 for the values of energy functions and distance functions by using Table 6 and Table 7 with the help of Matlab software. It provides an information that, the efficient energy routing of mobile ad-hoc network using neutrosophic set technique(NM) is much better than the efficient energy routing of MANET using vague set technique(VM) in uncertain environment. However, the proposed method is focused on static environment in case the node and data set changes at each interval of time then the proposed unable to represent the case precisely. To deal with dynamic environment author will focus in near future to introduce the extensive properties of neutrosophic set and its applications to wireless ad-hoc network(WANET), flying ad-hoc network(FANET) and vehicular ad-hoc network(VANET).

Conclusion and future work

This paper utilizes properties of single valued neutrosophic for finding an efficient routing protocol for MANET based on distance and energy. In this regard, several algorithms are proposed to characterize it based on truth and falsity membership-values of a defined vague set. However the current paper aimed at dealing with uncertainty in routing protocol of MANET based on its truth, falsity and indeterminacy membership-values, indeterminacy. It is shown that the proposed method provides a precise representation and selection of energy efficient routing protocol when compared to vague sets as shown in Table 6 and 7. In future, the authors will focuses on investigating the energy efficient routes for WANET, FANET, VANET for dynamic environment

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A Review of Seven Applications of Neutrosophic Logic: In Cultural Psychology, Economics Theorizing, Conflict Resolution, Philosophy of Science, etc.

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Abstract: In this short communication, we review seven applications of NFL that we have explored in a number of papers: (1) Background: the purpose of this study is to review how neutrosophic logic can be found useful in a number of diverse areas of interest; (2) Methods: we use logical analysis based on NL; (3) Results: some fields of study may be found elevated after analyzed by NL theory; and (4) Conclusions: we can expect NL theory to be applied in many areas of research too, in applied mathematics, economics, and physics. Hopefully the readers will find a continuing line of thoughts in our research from the last few years.

Keywords: neutrosophic logic; cultural psychology; economics; conflict resolution; philosophy of science; cosmology

1. Introduction

First, let us discuss a commonly asked question: what is Neutrosophic Logic? Here, we offer a short answer.

Vern Poythress argues that sometimes we need a modification of the basic philosophy of mathematics, in order to re-define and redeem mathematics [1]. In this context, allow us to argue in favor of Neutrosophic logic as a starting point, in lieu of the Aristotelian logic that creates so many problems in real world.

In Neutrosophy, we can connect an idea with its opposite and with its neutral and get common parts, i.e. <A> ∧ <non-A> = nonempty set. This constitutes the common part of the uncommon things! It is true/real—paradox. From neutrosophy, it all began: neutrosophic logic, neutrosophic set, neutrosophic probability, neutrosophic statistics, neutrosophic measures, neutrosophic physics, and neutrosophic algebraic structures [2].

It is true in a restricted case, i.e. Hegelian dialectics considers only the dynamics of opposites (<A> and <anti-A>), but in our everyday life, not only the opposites interact, but the neutrals < neut-A > between them too. For example, if you fight with a man (so you both are the opposites to each other), but neutral people around both of you (especially the police) interfere to reconcile both of you. Neutrosophy considers the dynamics of opposites and their neutrals.

So, neutrosophy means that: <A>, <anti-A> (the opposite of <A>), and < neut-A > (the neutrals between <A> and <anti-A>) interact among themselves. A neutrosophic set is characterized by a truth-membership function (T), an indeterminacy-membership function (I), and a falsity-membership function (F), where T, I, F are subsets of the unit interval [0, 1].

As particular cases we have a single-valued neutrosophic set {when T, I, F are crisp numbers in [0, 1]}, and an interval-valued neutrosophic set {when T, I, F are intervals included in [0, 1]}.

From a different perspective, we can also say that neutrosophic logic is (or "Smarandache logic") a generalization of fuzzy logic based on Neutrosophy (http://fs.unm.edu/NeutLog.txt). A proposition is t true, i indeterminate, and f false, where t, i, and f are real values from the ranges T, I, F, with no restriction on T, I, F, or the sum n = t + i + f. Neutrosophic logic thus generalizes:

- Intuitionistic logic, which supports incomplete theories (for 0 < n < 100 and i = 0, 0 < = t, i, f < = 100);
- Fuzzy logic (for n = 100 and i = 0, and 0 < = t, i, f < = 100);
- Boolean logic (for n = 100 and i = 0, with t, f either 0 or 100);
- Multi-valued logic (for 0 < = t, i, f < = 100);
- Paraconsistent logic (for n > 100 and i = 0, with both t, f < 100);
- Dialetheism, which says that some contradictions are true (for t = f = 100 and i = 0; some paradoxes can be denoted this way).

Compared with all other logics, neutrosophic logic introduces a percentage of "indeterminacy"—due to unexpected parameters hidden in some propositions. It also allows each component t, i, f to "boil over" 100 or "freeze" under 0. For example, in some tautologies t > 100, called "overtrue." Neutrosophic Set is a powerful structure in expressing indeterminate, vague, incomplete and inconsistent information.

In this short review article, we will review seven applications of NL theory in diverse fields of science.

We introduce a number of key terms here. For example, from a NL perspective, we can find a reconciliation between "push" and "pull" type of gravitation, by considering both forces are in place. To speak more plainly, pull force takes place on an astronomical scale, while push force takes place at geological scale, and this effect can be found for instance: a. the fact that the Moon is receding from Earth (around 4 cm/yr), b. the fact that the Earth is expanding caused by dissipative geodynamics process, and c. the Pangea hypothesis.

In the context of cosmology, we argue that neutrosophic logic is in agreement with Kant and Vaas's position, it offers a resolution to the long standing disputes between beginning and eternity of the Universe. In other words, in this respect we agree with Vaas: "how a conceptual and perhaps physical solution of the temporal aspect of Immanuel Kant's "*first antinomy of pure reason*" is possible, i.e. how our universe in some respect could have both a beginning and an eternal existence. Therefore, paradoxically, *there might have been a time before time or a beginning of time in time*."

2. Seven Applications of Neutrosophic Logic in Diverse Fields of Science

2.1. Cultural Psychology

Culture is a shared meaning system, found among those who speak a particular language dialect, during a specific historic period, and in a definable geographic region. Collectivism is a cultural pattern found in most traditional societies, especially in Asia, Latin America, and Africa. It contrasts with individualism, which is a cultural pattern found mostly in America and Europe.

This theme was explored by Prof. Harry Triandis (https://www.researchgate.net/profile/Harry_Triandis). Triandis was born in Greece in 1926. During the Second World War, he learned four foreign languages and developed his curiosity about the differences that exist between cultures. His time getting to know people across various European nations inspired him to research cultural disparities in the way people think. This issue can be reconciled with the help of NL theory, which may be appropriate for socio-economics theorizing, as we will discuss in the next subsection.

2.2. Socio-Economics Theorizing [3]

In a series of papers, we outlined a more general approach to reconcile classical tensions between individualism and collectivism, between cooperation and competition, and so on. In our opinion, our tendency to cooperate or compete is partly influenced by the culture that we inherit from our ancestors. One of us (VC) once lived for a while in Russia, and he found that many people there are rather cold and distant (of course *not all* of them, some are warm and friendly). He learned that such a trait may be found as quite common in many countries in Europe. They tend to be individual and keep certain distance from each other. In physics term, they are like fermions. Our proposed simplistic analogy of human behaviour, i.e. individualism and collectivism, is not uncommon. (Indeed such cultural psychology research has been reported since Harry C. Triandis et al. See, for example, (a) The Self and Social Behaviour in Differing Cultural Contexts, *Psychological Review*, vol. 96, no. 3; (b) Harry C. Triandis and Eunkook M. Suh, Cultural Influences on Personality, *Annu. Rev. Psychol*. 2002. 53:133–60; (c) J. Allik and A. Realo, Individualism-collectivism and social capital, *J. Cross-Cultural Psychology*, Vol. 35 No. 1, January 2004, 29–49. This last mentioned paper includes a quote from Emile Durkheim: "The question that has been the starting point for our study has been that of the connection between the individual personality and social solidarity. How does it come about that the individual, whilst becoming more autonomous, depends ever more closely upon society? How can he become at the same time more of an individual and yet more linked to society?")

There is a developmental psychology hypothesis suggesting that perhaps such a trait co-relates to the fact that many children in Europe lack nurturing and human touch from their parents in their childhood, which possibly make them rather cold and individual. Of course, whether this is true, is yet to be verified.

On the contrary, most people in Asia and Africa are *gregariously groupie* (except perhaps in large metropolitan areas). They tend to spend much time with family and friends, just like many Italians do. They attend religious rituals regularly or watch music festival together, and so on. In physics term, they are bosons. Of course, such a sweeping generalization may be oversimplifying. (After writing up this article, we found that Sergey Rashkovskiy also wrote on a quite similar theme, albeit with statistical mechanics in mind. The title of his recent paper is: "*'Bosons' and 'fermions' in social and economic systems*." Here is abstract from his paper: "We analyze social and economic systems with a hierarchical structure and show that for such systems, it is possible to construct thermostatistics, based on the intermediate Gentile statistics. We show that in social and economic hierarchical systems there are elements that obey the Fermi-Dirac statistics and can be called fermions, as well as elements that are approximately subject to Bose-Einstein statistics and can be called bosons. We derive the first and second laws of thermodynamics for the considered economic system and show that such concepts as temperature, pressure and financial potential (which is an analogue of the chemical potential in thermodynamics) that characterize the state of the economic system as a whole, can be introduced for economic systems." Url: https://arxiv.org/ftp/arxiv/papers/1805/1805.05327.pdf)

Therefore, it seems quite natural to us, that Adam Smith wrote a book on philosophy suggesting that individual achievement is the key to national welfare (because he was British and thus emphasized individualism). If only Adam Smith had been born in Bangkok or Manila, he would have probably written this book in a different way.

It was more than a hundred years before mathematicians like John F. Nash, Jr. figured out that individual pursuit towards one's own goals does not lead to achieve a common goal as a society. (For example, let us imagine 10 players of a football team try simultaneously to score a goal against the opposite team, will they succeed? Of course no, they should arrange according to their coach's instruction: 1-4-4-2, or some other type of arrangement.)

At this point, some readers may ask: which is better, to be like fermions or bosons? Our opinion is as follows: just like in particle physics, both fermions and bosons are required. In the same way, fermion behavior and boson behavior are both needed to advance quality of life. Fermion people tend to strive toward human progress, while boson people are those who enrich our life.

This issue again can be reconciled with the help of NL theory, i.e. such a human tension is always there, but there does not need to be conflicts. Similarly, from such a fermion-boson perspective (which we propose a new term: *ferson*), a classic tension between capitalism (emphasizing individual achievements) and socialism can be reconciled, for example by considering a range of possibilities, including a new term (possibly): *capicialism*. (This is reminiscent of a term introduced by Alvin Toffler in 70 s, in which he predicted as culture shock, that describes the combined behavior of consumerism and producers: *prosumerism*.)

2.3. Conflict Resolution [4]

Binary choices are another source of problems. As a one-liner joke says:

There are two kinds of people in the world: Those who think there are two kinds of people in the world and those who don't. (Plus some others who aren't sure.)

—(http://philippe.ameline.free.fr/humor/TwoKindOfPeople.htm)

A funnier joke on binary logic:

There are 10 kinds of people in the world: Those who understand binary and those who don't.

—(http://philippe.ameline.free.fr/humor/TwoKindOfPeople.htm)

As Phillipe Schweizer remarked:

"These two possibilities, these alternatives, are the basis of cognition, and allow choice and therefore action through the fact that a preference becomes possible: either I prefer there is X, or I prefer there is no X. Then autonomy appears. And indeed the valuation or affect too: "I like" or "I don't like", and it goes with it together. The stages described here are not as distinct as those of Piaget, they overlap, include and extend. The "there is no" is opposed to the "there is" forming the opposite. Thus the binary appears and the logic of the same name also: either "there is", or "there is not": X or non-X, one and the other being mutually exclusive.

... There is this and that and that again: a perception of the environment, a representation of a situation as a collection of objects. Our other most frequent and fundamental conception is opposition: there is or there is not. What also gives one thing and its opposite: day and night, hot and cold, big and small ... The importance of this simplifying binary conception of two situations sliced diametrically away in opposite is the most prominent form of mental life. It is the emblematic *form of a choice."*

(Quote from Phillipe Schweizer. Thinking on Thinking: The Elementary forms of Mental Life Neutrosophical representation as enabling cognitive heuristics. Submitted for review.)

In this regards, one of us (FS) recently published a new book, with the following title: Neutropsychic Personality [5]. In this book, FS described possible extension of Freudian mental model: id-ego-superego, using his neutrosophic logic theory. His definition of Neutropsychic is as follows:

"Neutropsyche is the psychological theory that studies the soul or spirit using the neutrosophy and neutrosophic theories. It is based on triadic neutrosophic psychological concepts, procedures, ideas, and theories of the form (<A>,< neut-A >,<anti-A>), such as (positive, neutral, negative), (good behavior, ignorant behavior, bad behavior), (taking the decision to act, pending, taking the decision not to act), (sensitive, moderate, insensitive), (under-reacting, normally reacting, over-reacting), (under-thinking, normal thinking, over-thinking), and so on, and their refinements as (< Aj >,< neut-Aj >,< anti-Aj >)." ([5], p.29)

Perhaps it is necessary to develop an improved model of the neutropsychic basis of decision making.

Another possible way to resolve this fundamental problem of human societies, is to accept otherness (cf. Milad Hanna, [6]), without being absorbed by the otherness.

Such a logical analysis derived from Kolmogorov's principle of contradiction eventually remind us of the following:

(a) To keep humble mind before Nature (God's creation), and perhaps we should not rely too much on our logic system and mathematical prowess;

(b) In developing a theory one should keep complications and abstractions to a minimum;

(c) To build theory in the nearest correspondence to the facts; it is the best if each parameter can be mapped to a measurable quantity.

We hope the above three criteria can be a useful set of practical guidelines for building mathematical models in theoretical physics or cosmology.

2.5. Cosmology [11]

Questions regarding the formation of the Universe and what was there before the existence of the Early Universe have been of great interest to mankind at all times. In recent decades, the Big Bang as described by the Lambda CDM-Standard Model Cosmology has become widely accepted by majority of physics and cosmology communities. Among other things, we can cite A.A. Grib and Pavlov who pointed out possible heavy particles creation out of vacuum and also other proposals such as *Creatio Ex-Nihilo theory* (CET).

However, philosophical problems remain, as Vaas pointed out: Did the universe have a beginning or does it exist forever, i.e. is it eternal at least in relation to the past? This fundamental question was the main topic in ancient philosophy of nature and the Middle Ages. Philosophically it was more or less banished then by Immanuel Kant's *Critique of Pure Reason*. However, it has been revived in modern physical cosmology both in the controversy between the big bang and steady state models some decades ago and in the contemporary attempts to explain the big bang within a quantum cosmological framework.

Interestingly, Vaas also noted that Immanuel Kant, in his *Critique of Pure Reason* (1781/1787), argued that it is possible to prove both that the world has a beginning and that it is eternal (first antinomy of pure reason, A426f/B454f). As Kant believed he could overcome this "self-contradiction of reason" ("*Widerspruch der Vernunft mit ihr selbst*", A740) by what he called "*transcendental idealism*", the question whether the cosmos exists forever or not has almost vanished in philosophical discussions.

In a paper accepted recently by *Asia Mathematika J.*, we take a closer look at Genesis 1:2 to see whether the widely-accepted notion of *creatio ex-nihilo* is supported by Hebrew Bible or not [11].

It turns out that neutrosophic logic is in agreement with Kant and Vaas's position, it offers a resolution to the long standing disputes between beginning and eternity of the Universe. In other words, in this respect we agree with Vaas: "how a conceptual and perhaps physical solution of the temporal aspect of Immanuel Kant's "*first antinomy of pure reason*" is possible, i.e. how our universe in some respect could have both a beginning and an eternal existence. Therefore, paradoxically, *there might have been a time before time or a beginning of time in time*."

Summarizing, neutrosophic logic studies the dynamics of opposites and neutralities; from this viewpoint, we can understand that it is indeed a real possibility that the Universe had both initial start but with eternal background (that may be called "primordial fluid"). This is exactly the picture we got after our closer look at Gen. 1:1-2.

2.6. American Football Game

(This section is after discussion with Robert Neil Boyd.)

Let us look at a situation in a football game (American style football).

The offense and the defense are lined up. The offense is in range to try kick a field goal to score 3 points. When the ball is passed from the center to the holder, so that the kicker may try to kick it through the upright poles that are the goal posts, many different things may happen. This is not a simple situation of the ball going between the uprights or not. The defense may be able to get a man in position to block the kick.

If the kick is blocked, according to the rules, the defense may pick up the ball and carry it towards their side of the field. If the man who picked up the ball and ran with it, is not tackled to the ground before he crosses the goal line, the play results in a touchdown, a 6 point score for the defending players.

Or the player who picked up the ball after the kick attempt was blocked runs several yards towards his goal line, where he is tackled by one of the members of the kicking team, which causes him to lose the ball he was carrying. The kicking team recovers the fumble and the play is over.

Or the holder fails to catch the pass from the center, or the holder may drop the pass from center and either pick it up and run with it, or drops it to the ground before he can do anything, or the pass may sail over the head of everyone (whereupon, many things are possible), or the holder may fail to place the ball properly for the kicker, resulting in a failed attempt.

Or the defense may commit one of several possible rule infractions before, or during the kick, so that the result of the play is a penalty against the defending team. If the penalty is large enough, it can result in a new set of downs for the kicking team, so the place-kicker leaves the field so that the normal offense players can take 4 more tries to gain 10 yards.

Or there can be a penalty against the kicking team that may result in the kicking team being forced out of range to try the kick. So the kicker leaves the field without attempting to kick a field goal.

Or the offensive team has the ball lined up to try and score. When the ball is passed to the holder, it is a fake kick and the holder runs for a first down or a touch down or passes the ball to an offensive player for a first down or passes the ball and it is not caught, which means the defense obtains the ball at the spot where the ball was placed before the kick attempt.

Or the kicker attempts to kick the ball through the uprights and succeeds, scoring 3 points for his team.

The kicker can get the snap directly from the center and try to make a pass completion, or he can run while carrying the ball, which can result in interception or fumbling or touchdown or first down, or the kicker being tackled before he reaches the line. Or he completes a pass and the receiver makes a first down or a touch down or get tackled to the ground before the line to gain, or the receiver fumbles the ball as he is tackled, leading to a potential touchdown for the other team. Many additional possibilities exist, but most of them are very rare.

During any play in a football game, it is possible for any player on either team to score a touchdown for and gain 6 points for their team. This is possible because human beings are interacting in a game played with goals and goal lines and an oddly shaped biconvex bi-conical ball inflated with high pressure air that is surrounded by a rubber sack that is surrounded by a leather case which is held in place with stitches and laces. The shape of the ball causes it to bounce in unpredictable ways when it is dropped or kicked or thrown. In addition, hot temperatures make the ball softer and cold temperatures make the ball harder. Both of the factors cause the ball to behave in different ways. When the ball is harder, it is like kicking a rock. When the ball is harder, it becomes more slippery so it is harder to throw and harder to catch, and it hits you harder when you catch it.

So a field goal attempt does not merely involve two possibilities, but an almost infinite variety of events may happen, before the attempt, during the attempt, or after the attempt.

Neutrosophic logic may be expanded to more than three possible states, since in an infinite universe, an infinite number of things may happen. I understand the tri-state basis of it as being valuable in many circumstances. There should be ways to extend the logic into larger numbers of choices, so that there is a range of yesses, to 1000 kinds of maybes or almosts, or something elses, or something unexpected that was outside the starting point of the data set, and so on, to the No of the equation. The null-A of non-Aristotelian logic, which is what Neutrosophic logic is, can involve much more than just the simplistic null set.

Question: How to extend the center, null-A state, to provide for abnormalities or exigencies?

Right now, the easiest thing to do seems to be to widen the null state to include all the possibilities that are additional to, or contingent on one or more rules, internal to the null state. So now the null state becomes much broader, and able to handle much more complicated situations, such as a field goal attempt during an American football game.

It seems that the "expanded middle" would be a good option for problem structure in Neutrosophy.

2.7. Gravitation

Despite majority of physical theories of gravitation assuming it is a pull force, a number of researchers have begun to work out a push gravity, which is known as Le Sage/Laplace gravitation theory. An interesting remark on impetus to Le Sage gravitation theory can be found in article by the late Prof. Halton Arp on his work with Narlikar:

> *"Nevertheless the ball had started rolling down hill so to speak and in 1991, with Narlikar's help, I outlined in Apeiron the way in which particle masses growing with time would account for the array of accumulated extragalactic paradoxes. Later Narlikar and Arp (1993) published in the Astrophysical Journal Narlikar's original, 1977 solution of the basic dynamical equations along with the Apeiron applications to the quasar/galaxy observations.*
>
> ...
>
> *The first insight came when I realized that the Friedmann solution of 1922 was based on the assumption that the masses of elementary particles were always and forever constant, m = const. He had made an approximation in a differential equation and then solved it. This is an error in mathematical procedure. What Narlikar had done was solve the equations for m = f(x, t). This a more general solution, what Tom Phipps calls a covering theory.*
>
> ...
>
> *But Narlikar had overwhelmed me with the beauty of the variable mass solution by showing how the local dynamics could be recovered by the simple conformal transformation from t time (universal) to what we called τ time (our galaxy) time. The advertisement here was that our solution inherited all the physics triumphs much heralded in general relativity but also accounted for the non-local phenomena like quasar and extragalactic redshifts."* [12]

Therefore, there are many reasons to support Le Sage gravity, despite majority of physicists preferring Einsteinian view. Summarizing, there should be a hidden dynamical matter creation process, suggesting that Newton second law was actually not just F = ma, but it should be written in complete form: F = d[mv]/dt = m[dv/dt] + v[dm/dt], therefore there is matter creation term. (In fact, it is known that Newton's second law was written originally as the momentum change over time, that is F_g = dp/dt.) All physics of Earth etc. assumes the Earth is static, but actually it is increasing in size and mass. This approach has been explored by both of us and also Robert Neil Boyd in a number of papers, see for instance [13,14].

Moreover, from a NL perspective, we can find a reconciliation between "push" and "pull" type of gravitation, by considering both forces are in place. To speak more plainly, pull force takes place at astronomical scale, while push force takes place at geological scale, and this effect can be found for instance: a. the fact that the Moon is receding from Earth (at a constant rate of around 4 cm/yr), b. the fact that the Earth is expanding, caused by dissipative geodynamics process, c. Pangea hypothesis. We will present our result in a paper to be presented in forthcoming 5th *EuroSciCon* 2019.

Allow us to introduce another new term in order to reconcile push and pull gravitational force, pullsh force. Such an idea is presently under investigation.

3. Results

Some fields of science are improved by being analyzed by NL theory; therefore we can expect NL theory will be applied in many areas of research too, in applied mathematics, economics, and also physics. For example, we also explored on how NL theory may be used to reconcile the "push" and "pull" gravitation theories. This is still a preliminary exploration, so we include this topic in discussion section.

In the context of cosmology, we argued that neutrosophic logic is in agreement with Kant and Vaas's position, it offers a resolution to the long standing disputes between beginning and eternity of the Universe. In other words, in this respect we agree with Vaas: "how a conceptual and perhaps physical solution of the temporal aspect of Immanuel Kant's "*first antinomy of pure reason*" is possible, i.e. how our universe in some respect could have both a beginning and an eternal existence. Therefore, paradoxically, *there might have been a time before time or a beginning of time in time*."

4. Discussion

We have discussed among other things, a few applications of NL theory in a number of fields, such as cultural psychology and economics theory. The essence of our discussion is that NL allows one to study the dynamics of opposites and neutralities. It is a generalization of dialectics.

Moreover, from a NL perspective, we can find a reconciliation between "push" and "pull" type of gravitation, by considering both forces are in place. To speak more plainly, pull force takes place at astronomical scale, while push force takes place at geological scale, and this effect can be found for instance: a. the fact that the Moon is receding from Earth (at a constant rate of around 4 cm/yr), b. the fact that the Earth is expanding, caused by dissipative geodynamics process, c. Pangea hypothesis. We will present our result in a paper to be presented in forthcoming 5th *EuroSciCon* 2019. Such an idea will be investigated later on.

We hope these discussions will be found useful in other areas as well; for instance in international relations and peace keeping efforts.

5. Conclusions

In this short article, we review seven applications of NFL that we have explored in a number of papers. Hopefully the readers will find a continuing line of thoughts in our research in the last few years, emphasizing our improved understanding of various branches of human knowledge. All of these branches have been enhanced and elevated to a higher level through applications of NL theory.

To summarize our results: we introduced a number of key terms here. For example, from a NL perspective, we can find a reconciliation between "push" and "pull" types of gravitation, by considering both forces. To speak more plainly, pull force takes place at astronomical scale, while push force takes place at geological scale, and this effect can be found for instance: a. the fact that the Moon is receding from Earth (at a constant rate of around 4 cm/yr), b. the fact that the Earth is expanding, caused by dissipative geodynamics process, c. Pangea hypothesis.

In the context of cosmology, we argue that neutrosophic logic is in agreement with Kant and Vaas's position; it offers a resolution to long-standing dispute between the beginning and the eternity of the Universe. In other words, in this respect we agree with Vaas: "how a conceptual and perhaps physical solution of the temporal aspect of Immanuel Kant's "*first antinomy of pure reason*" is possible, i.e. how our universe in some respect could have both a beginning and an eternal existence. Therefore, paradoxically, *there might have been a time before time or a beginning of time in time*."

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Multiple Attribute Group Decision Making Based on 2-Tuple Linguistic Neutrosophic Dombi Power Heronian Mean Operators

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Peide Liu, Qaisar Khan, Tahir Mahmood, Florentin Smarandache, Ying Li (2019). Multiple Attribute Group Decision Making Based on 2-Tuple Linguistic Neutrosophic Dombi Power Heronian Mean Operators. *IEEE Access* 7: 100205-100230; DOI: 10.1109/ACCESS.2019.2925344

ABSTRACT As an expansion of 2-tuple linguistic intuitionistic fuzzy set, the newly developed 2-tuple linguistic neutrosophic set (2-TLNS) is more satisfactory to define decision maker's assessment information in decision making problems. 2-TLN aggregation operators are of great significance in multiple attribute group decision making (MAGDM) problems with a 2-tuple linguistic environment. Therefore, in this article our main contribution is to develop novel 2-TLN power Heronian aggregation (2-TLNPHM) operators. Firstly, we develop new operational laws established on Dombi T-norm (DTN) and Dombi T-conorm (DTCN). Secondly, Taking full advantages of the power average (PA) operator and Heronian mean (HM) operator, we develop some new novel power Heronian mean operator and discuss its related properties and special cases. The main advantages of developed aggregation operators are that not only remove the effect of awkward data which may be too high or too low, but also have a good capacity to model the extensive correlation between attributes, making them more worthy for successfully solving more and more complicated MAGDM problems. Thus, we develop a new algorithm to handle MAGDM based on the developed aggregation operators. Lastly, we apply the proposed method and algorithm to risk assessment for construction of engineering projects to show the efficiency of the developed method and algorithm. The dominant novelties of this contribution are triplex. Firstly, new operational laws are proposed for 2-TLNNs. Secondly, novel 2-TLNPHM operators are developed. Thirdly, a new approach for 2-tuple linguistic neutrosophic MAGDM is developed.

INDEX TERMS 2-TLNS, Dombi T-norm, Dombi T-conorm, PA operator, Heronian mean, MAGDM.

I. INTRODUCTION

In actual life, multiple attribute group decision making (MAGDM) problems are the vital part of decision theory in which we select the optimal one from the group of finite alternatives based on the overall information. Conventionally, it has been accepted that the information concerning acquiring the alternatives is taken in the form of real number. But in our daily life, it is hard for a decision maker to give his evaluations regarding the object in crisp values due to vagueness and insufficient information. Rather, it has been enhance acceptable that these evaluations are given by fuzzy set (FS) or its extended form. Intuitionistic fuzzy set (IFS) [1] is the vigorous augmentation of FS [2] to deal with vagueness by including an identical falsity-membership into the analysis. A lot of studies by different researchers were conducted on IFS in different fields. IFSs have good capability to explain and articulate decision maker's (DMs) fuzzy decision information in MAGDM problems. However, IFS still have shortcomings and there exist relatively a few situations in which it is inappropriate to employ IFS to articulate DMs preference information. The key motive is that the hesitancy/indeterminacy degree is dependent of membership degree and non-membership degree in IFSs, for example when a DM utilizes an IFN (0.6, 0.2) to represent his/her assessment on a certain attribute. Then, the indeterminacy/hesitancy degree of the DM is 1 − 0.6 − 0.2 = 0.2. In simple words, once the truth-membership and falsity-membership degrees are determined, the degree of indeterminacy is determined automatically. Some other generalizations FS are proposed by some scholars such as Pythagorean fuzzy sets [3], hesitant Pythagorean fuzzy sets [4]. However, these are rather different from real MAGDM problems. In real MAGDM, the indeterminacy/hesitancy degrees should not be determined automatically and should be provided by DMs. For example, if a DM thinks the membership degree is 0.6, the membership degree is 0.4, and the degree that he/she is not sure about the result is 0.2, then the DMs evaluation value can be denoted as (0.6, 0.4, 0.2), which cannot be represented by IFSs. In order to deal with this case, Smarandache [5, 6] initially developed the concept of neutrosophic set (NS), which has the capacity of dealing inconsistent and indeterminate information. In the NS, its degree of membership $\overline{\overline{TR}}_A(a)$, degree of indeterminacy $\overline{\overline{IN}}_A(a)$ and degree of falsity $\overline{\overline{FL}}_A(a)$ are expressed independently, which lie real standard or non-standard subsets of $]0^-,1^+[$, that is $\overline{\overline{TR}}_A(a):U \to]0^-,1^+[$, $\overline{\overline{IN}}_A(a):U \to]0^-,1^+[$ and $\overline{\overline{FL}}_A(a):U \to]0^-,1^+[$, such that $0^- \le \overline{\overline{TR}}_A(a)+\overline{\overline{IN}}_A(a)+\overline{\overline{FL}}_A(a) \le 3^+$. Thus, the use of nonstandard interval $]0^-,1^+[$ may verdict some difficulty in real applications. To utilize NS easily in real application Wang et al. [7] proposed the concept of single valued neutrosophic set (SVNS) by changing the non-standard unit interval into the standard unit interval $[0,1]$. Further, Wang et al. [8] proposed the concept of interval neutrosophic set (INS). Ye [9] developed simplified neutrosophic set (SNS), which consist of both concepts of SVNS and INS. Some researcher developed improved operational laws for these sets [10,11].

In recent time, information aggregation operators [12-15] have enticed comprehensive recognitions of researchers and have become a vital part of MAAGDM. Generally, for aggregating a group of data, it is mandatory to assess the functions and the operations of aggregation operators. For the functions, the conventional aggregation operator developed Xu, Xu and Yager [16, 17] only can aggregate a group of real values into a single real value. In the past few years, some expanded aggregation operators have been developed by different researchers. For example, Sun et al. [18] developed some Choquet integral operator for INS. Liu and Tang, Peng et al. [19, 20] extended the power average (PA) operator developed by Yager [21] to interval neutrosophic and multi-valued neutrosophic environment, which has the capacity of removing the bad impact of awkward data. Wu et al [22] developed cross entropy and prioritized aggregation operators for SNNs, which take the priorities of criterion by priority weights. Besides, some aggregation operators can consider interrelationship among aggregated arguments. That is Bonferroni mean (BM) operator developed by Bonferroni [23], Heronian mean (HM) operator developed by Sykora [24].

All the above aggregation operators are capable to deal with information available in the form of real numbers. However, in various actual situations, mostly for various actual MAGDM problems, the assessment information associated with every alternatives are normally unpredictable or vague, due to the increasing complexity such as lack of time, lack of knowledge and various other limitations. Therefore, it is often hard for DMs to represent the assessment information about alternatives in the form of numeric values. Hence, to deal with such type of situations, Zadeh [25] initially proposed the concept of linguistic variable. It has also been generalized to various linguistic environments such as 2-tuple linguistic representation model [26-30], intuitionistic 2-tuple linguistic model [31] and so on [32, 33]. These developed concepts have also the same limitations to that of FS and IFS have. To overcome these limitations, Wang et al. [34] developed the concept of 2-tuple linguistic neutrosophic set (2-TLNS) based on the SVNS and 2-tuple linguistic information model , which is the generalization of several concepts such as 2-tuple linguistic set, 2-tuple linguistic fuzzy set and 2-tuple linguistic intuitionistic fuzzy set [35]. They described some operational laws for 2-tuple linguistic neutrosophic number (2-TLNN) , proposed some aggregation operators and apply these aggregation operators to solve MADM problems. Wang et al. [36, 37] further developed MAGDM method based TODIM and Muirhead mean operators to deal with 2-tuple linguistic environment. Wu et al. [38] proposed some 2-tuple linguistic neutrosophic Hamy mean (2-TLNHM) operators. Wu et al. [39] proposed the idea of SVN 2-tuple linguistic set (SVN2TLS), SVN 2 tuple linguistic number (SVN2TN), basic operational laws based on Hamacher triangular norm and conorm. Then based on these operational laws propose some aggregation operators and apply these aggregation operator to deal with MAGDM problem under SVN2TL information.

The Dombi t-norm (DTN) and Dombi t-conorm (DTCN) proposed by Dombi [40] have general parameter, which makes the information aggregation process more flexible. In the past few years, some researchers proposed Dombi operational laws for various sets and based on these Dombi operational laws they developed different aggregation operators [41-56].

Due to the increasing complexity in real decision making problems day by day, we have to look at the following questions, when selecting the best alternative. (1) In various situations, the assessment values of the attributes presented by the DMs may be too high or too low, have a negative effect on the final ranking results. The PA operator is a useful aggregation operator that authorizes the assessed values to equally supported and improved. Therefore, we may utilize the PA operator to vanish such bad effect by choosing different weights constructed by the support measure. (2) In various practical decision making problems the assessment values of attribute are dependent. Therefore, the interrelationship among the values of the attributes should be scrutinized. The HM operator can gain this function. However, HM operator has some advantages over BM. From the existing literature, we can notice that there is a need to combine PA operator with HM operator to deal with 2-TLN environment and achieved the above advantages.

Therefore, the main aim of this article is to propose some Dombi operational laws for 2-TLNNs, combine PA operator with HM operator, and extend the idea to 2-TLN environment, and develop some new aggregation operators such as 2-TLN power HM (2-TLNHM) operator, its weighted form, 2-LN power geometric HM (2-TLNHM) operator, its weighted form and discussed some special cases of the developed aggregation operator and apply them to MAGDM to achieve the two requirements discussed above.

To do so, the rest of the article is organized as follows.

In section 1, some basic definitions about SVNS, 2-TLNS, PA operator, HM operator and related properties are discussed. In section 3, we developed some operational laws for 2-TLNNs. In section 4, based on these operational laws we developed some 2-tuple linguistic Dombi power Heronian mean operators, related properties and special cases are discussed. In section 5, MAGDM method is developed based on these newly developed aggregation operators and a numerical example is given to show the effectiveness of the proposed MAGDM approach. In section 6, comparison of the developed approach and some existing approaches are given. At the end Conclusion, future work and references are given.

II. Preliminaries

In this part, we gave some basic definitions and results about 2-TLNSs, PA operator and HM operator.

A. 2-TLNS and their operations

Definition 1[7]. Let Θ be a space of points (objects), with a common component in Θ denoted by η. A SVNS $\widetilde{SN}$ in Θ is expressed by,

$$\widetilde{SN}=\left\{\left\langle \eta,\xi_{\widetilde{SN}}(\eta),\psi_{\widetilde{SN}}(\eta),\zeta_{\widetilde{SN}}(\eta)\right\rangle \mid \eta\in\Theta\right\} \tag{1}$$

Where $\xi_{\widetilde{SN}}(\eta),\psi_{\widetilde{SN}}(\eta)$ and $\zeta_{\widetilde{SN}}(\eta)$ respectively denote the TMD, IMD and FMD of the element $\eta\in\Theta$ to the set $\widetilde{SN}$. For each point $\eta\in\Theta$, we have $\xi_{\widetilde{SN}}(\eta),\psi_{\widetilde{SN}}(\eta),\zeta_{\widetilde{SN}}(\eta)\in[0,1]$, and $0\le\xi_{\widetilde{SN}}(\eta)+\psi_{\widetilde{SN}}(\eta)+\zeta_{\widetilde{SN}}(\eta)\le 3$.

Definition 2 [34]. Suppose that $\Gamma=\{\Gamma_1,\Gamma_2,....,\Gamma_p\}$ is a 2-TLSs with $p+1$ cardinality. That is the order of 2-TLSs is odd. If $\Gamma=\left\langle (s_t,\Xi),(s_i,\Psi),(s_f,\Upsilon)\right\rangle$ is described for $(s_t,\Xi),(s_i,\Psi),(s_f,\Upsilon)\in\Gamma$ and $\Xi,\Psi,\Upsilon\in[o,p]$, where $(s_t,\Xi),(s_i,\Psi)$ and (s_f,Υ) respectively, represents the truth-membership degree, indeterminacy-membership degree and falsity-membership degree by 2-TLNSs, then the 2-TLNSs is described as follows:

$$\Gamma_g=\left\{\left\langle \left(s_{t_g},\Xi_g\right),\left(s_{i_g},\Psi_g\right),\left(s_{f_g},\Upsilon_g\right)\right\rangle\right\} \tag{2}$$

where, $0\le\Delta^{-1}\left(s_{t_g},\Xi\right)\le p, 0\le\Delta^{-1}\left(s_{i_g},\Psi\right)\le p, 0\le\Delta^{-1}\left(s_{f_g},\Upsilon\right)\le p$ such that $0\le\Delta^{-1}\left(s_{t_g},\Xi\right)+\Delta^{-1}\left(s_{i_g},\Psi\right)+\Delta^{-1}\left(s_{f_g},\Upsilon\right)\le 3p.$

Definition 3 [34]. Let $\Gamma=\left\langle (s_t,\Xi),(s_i,\Psi),(s_f,\Upsilon)\right\rangle$ be a 2-TLNN. Then, the score and accuracy functions are described as follows:

$$\overline{\overline{SR}}(\Gamma)=\Delta\left\{\frac{2p+\Delta^{-1}(s_t,\Xi)-\Delta^{-1}(s_i,\Psi)-\Delta^{-1}(s_f,\Upsilon)}{3p}\right\},\overline{\overline{SR}}(\Gamma)\in[0,1]; \tag{3}$$

$$\overline{\overline{AC}}(\Gamma)=\Delta\left\{\Delta^{-1}(s_t,\Xi)-\Delta^{-1}(s_f,\Upsilon)\right\},\overline{\overline{AC}}(\Gamma)\in[-p,p]. \tag{4}$$

Definition 4 [34]. Let $\Gamma_1=\left\langle (s_{t_1},\Xi_1),(s_{i_1},\Psi_1),(s_{f_1},\Upsilon_1)\right\rangle$ and $\Gamma_2=\left\langle (s_{t_2},\Xi_2),(s_{i_2},\Psi_2),(s_{f_2},\Upsilon_2)\right\rangle$ be any two arbitrary 2-TLNNs. Then, the comparison rules are described as follows:

(1) If $\overline{\overline{SR}}(\Gamma_1)>\overline{\overline{SR}}(\Gamma_2)$, then $\Gamma_1>\Gamma_2$;

(2) If $\overline{\overline{SR}}(\Gamma_1)=\overline{\overline{SR}}(\Gamma_2)$, then

i. If $\overline{\overline{AC}}(\Gamma_1)>\overline{\overline{AC}}(\Gamma_2)$, then $\Gamma_1>\Gamma_2$;

ii. If $\overline{\overline{AC}}(\Gamma_1)=\overline{\overline{AC}}(\Gamma_2)$, then $\Gamma_1=\Gamma_2$.

Definition 5 [36]. Let $\Gamma_1=\left\langle (s_{t_1},\Xi_1),(s_{i_1},\Psi_1),(s_{f_1},\Upsilon_1)\right\rangle$ and $\Gamma_2=\left\langle (s_{t_2},\Xi_2),(s_{i_2},\Psi_2),(s_{f_2},\Upsilon_2)\right\rangle$ be any two arbitrary 2-TLNNs. Then, the normalized Hamming distance is described as follows:

$$\overline{\overline{D}}_H(\Gamma_1,\Gamma_2)=\frac{1}{3p}\left\{\begin{aligned}&\left|\Delta^{-1}\left(s_{t_1},\Xi_1\right)-\Delta^{-1}\left(s_{t_2},\Xi_2\right)\right|+\left|\Delta^{-1}\left(s_{i_1},\Psi_1\right)-\Delta^{-1}\left(s_{i_2},\Psi_2\right)\right|+\\&\left|\Delta^{-1}\left(s_{f_1},\Upsilon_1\right)-\Delta^{-1}\left(s_{f_2},\Upsilon_2\right)\right|\end{aligned}\right\} \tag{5}$$

B. The PA operator

Yager [21] was the first one who presented the concept of the PA which is one of the important aggregation operators. The PA operator diminishes some negative effects of unnecessarily high or unnecessarily low arguments given by experts. The conventional PA operator can only deal with crisp numbers, and is defined as follows.

Definition 6 [21]. Let $b_i (i=1,2,...,m)$ be a group of non-negative crisp numbers, the PA is a function defined by

$$PA(b_1,b_2,....,b_m)=\frac{\sum_{i=1}^{m}\left(1+T(b_i)\right)b_i}{\sum_{i=1}^{m}\left(1+T(b_i)\right)} \tag{6}$$

Where $T(b_i)=\sum_{\substack{j=1\\ j\ne i}}^{m}Sup(b_i,b_j)$ and $Sup(b,c)$ is the support degree for b from c, which satisfies some axioms. 1) $Sup(b,c)\in[0,1]$; 2) $Sup(b,c)=Sup(c,b)$; 3) $Sup(b,c)\ge Sup(d,e)$, if $|b-c|<|d-e|$.

C. HM operator

HM [24] is also an important tool, which can represent the interrelationships of the input values, and it is defined as follows:

Definition 7 [24]. Let $I=[0,1], x,y\geq 0, H^{x,y}: I^m \to I$, if $H^{x,y}$ satisfies;

$$H^{x,y}(b_1,b_2,...,b_m)=\left(\frac{2}{m^2+m}\sum_{i=1}^{m}\sum_{j=i}^{m} b_i^x b_j^y\right)^{\frac{1}{x+y}} \tag{7}$$

Then the mapping $H^{x,y}$ is said to be HM operator with parameters. The HM satisfies the properties of idem potency, boundedness and monotonicity.

III. Dombi operational laws for 2-TLNNs

A. Dombi TN and TCN

Dombi operations consist of the Dombi sum and Dombi product.

Definition 8 [40]. Let α and β be any two real number. Then, the DTN and DTCN among α and β are explain as follows:

$$T_D(\alpha,\beta)=\frac{1}{1+\left\{\left(\frac{1-\alpha}{\alpha}\right)^{\Im}+\left(\frac{1-\beta}{\beta}\right)^{\Im}\right\}^{\frac{1}{\Im}}}; \tag{8}$$

$$T_D^*(\alpha,\beta)=1-\frac{1}{1+\left\{\left(\frac{\alpha}{1-\alpha}\right)^{\Im}+\left(\frac{\beta}{1-\beta}\right)^{\Im}\right\}^{\frac{1}{\Im}}}. \tag{9}$$

Where $\Im\geq 1$, and $(\alpha,\beta)\in[0,1]\times[0,1]$.

According to the DTN and DTCN, we develop few operational rules for 2-TLNNs.

Definition 9. Let $\Gamma_1=\left\langle (s_{t_1},\Xi_1),(s_{i_1},\Psi_1),(s_{f_1},\Upsilon_1)\right\rangle$ and $\Gamma_2=\left\langle (s_{t_2},\Xi_2),(s_{i_2},\Psi_2),(s_{f_2},\Upsilon_2)\right\rangle$ be an arbitrary 2-TLNNs and $\Im>0$, for simplicity, we assume that $\frac{\Delta^{-1}(s_{t_g},\Xi_g)}{h}=\bar{\bar{t}}_g, \frac{\Delta^{-1}(s_{i_g},\Psi_g)}{h}=\bar{\bar{i}}_g, \frac{\Delta^{-1}(s_{f_g},\Upsilon_g)}{h}=\bar{\bar{f}}_g$ for $g=1,2$. Then, the operational laws can be described as follows:

$$(1)\Gamma_1\otimes\Gamma_2=\left\langle \begin{array}{l}\Delta\left(h\left(\frac{1}{1+\left(\left(\frac{1-\bar{\bar{t}}_1}{\bar{\bar{t}}_1}\right)^{\Im}+\left(\frac{1-\bar{\bar{t}}_2}{\bar{\bar{t}}_2}\right)^{\Im}\right)^{1/\Im}}\right)\right),\\ \Delta\left(h\left(1-\frac{1}{1+\left(\left(\frac{\bar{\bar{i}}_1}{1-\bar{\bar{i}}_1}\right)^{\Im}+\left(\frac{\bar{\bar{i}}_2}{1-\bar{\bar{i}}_2}\right)^{\Im}\right)^{1/\Im}}\right)\right),\Delta\left(h\left(1-\frac{1}{1+\left(\left(\frac{\bar{\bar{f}}_1}{1-\bar{\bar{f}}_1}\right)^{\Im}+\left(\frac{\bar{\bar{f}}_2}{1-\bar{\bar{f}}_2}\right)^{\Im}\right)^{1/\Im}}\right)\right)\end{array}\right\rangle \tag{10}$$

$$(2)\Gamma_1\otimes\Gamma_2=\left\langle \begin{array}{l}\Delta\left(h\left(1-\frac{1}{1+\left(\left(\frac{\bar{\bar{t}}_1}{1-\bar{\bar{t}}_1}\right)^{\Im}+\left(\frac{\bar{\bar{t}}_2}{1-\bar{\bar{t}}_2}\right)^{\Im}\right)^{1/\Im}}\right)\right),\\ \Delta\left(h\left(\frac{1}{1+\left(\left(\frac{1-\bar{\bar{i}}_1}{\bar{\bar{i}}_1}\right)^{\Im}+\left(\frac{1-\bar{\bar{i}}_2}{\bar{\bar{i}}_2}\right)^{\Im}\right)^{1/\Im}}\right)\right),\Delta\left(h\left(\frac{1}{1+\left(\left(\frac{1-\bar{\bar{f}}_1}{\bar{\bar{f}}_1}\right)^{\Im}+\left(\frac{1-\bar{\bar{f}}_2}{\bar{\bar{f}}_2}\right)^{\Im}\right)^{1/\Im}}\right)\right)\end{array}\right\rangle; \tag{11}$$

$$(3)\,\xi\Gamma_1=\left\langle \begin{array}{l}\Delta\left(h\left(1-\frac{1}{1+\left(\xi\left(\frac{\bar{\bar{t}}_1}{1-\bar{\bar{t}}_1}\right)^{\Im}\right)^{1/\Im}}\right)\right),\Delta\left(h\left(\frac{1}{1+\left(\xi\left(\frac{1-\bar{\bar{i}}_1}{\bar{\bar{i}}_1}\right)^{\Im}\right)^{1/\Im}}\right)\right),\\ \Delta\left(h\left(\frac{1}{1+\left(\xi\left(\frac{1-\bar{\bar{f}}_1}{\bar{\bar{f}}_1}\right)^{\Im}\right)^{1/\Im}}\right)\right)\end{array}\right\rangle,\xi>0; \tag{12}$$

$$(4)\,\Gamma_1^{\xi}=\left\langle \begin{array}{l}\Delta\left(h\left(\frac{1}{1+\left(\xi\left(\frac{1-\bar{\bar{t}}_1}{\bar{\bar{t}}_1}\right)^{\Im}\right)^{1/\Im}}\right)\right),\Delta\left(h\left(1-\frac{1}{1+\left(\xi\left(\frac{\bar{\bar{i}}_1}{1-\bar{\bar{i}}_1}\right)^{\Im}\right)^{1/\Im}}\right)\right),\\ \Delta\left(h\left(1-\frac{1}{1+\left(\xi\left(\frac{\bar{\bar{f}}_1}{1-\bar{\bar{f}}_1}\right)^{\Im}\right)^{1/\Im}}\right)\right)\end{array}\right\rangle,\xi>0. \tag{13}$$

IV. The 2-tuple linguistic neutrosophic Dombi Heronian aggregation operators

In this part, based on the Dombi operational laws for 2-TLNNs, we combine PA operator and HM operator to propose 2-TLNDPHM operator, 2-TLNDWPHM operator, 2-TLNDPGHM operator, 2-TLNDWPGHM operator and discuss some related properties.

A. The 2-LNDPHM and 2-LNDWPHM operators

Definition 10. Let $\Gamma_g\,(g=1,2,...,p)$ be a group of 2-TLNNs, $x,y\geq 0$. Then, the 2-TLNNDPHM operator is described as follows:

$$2-TLNDPHM^{x,y}(\Gamma_1,\Gamma_2,...,\Gamma_p)$$

$$=\left(\frac{2}{p^2+p}\sum_{g=1}^{p}\sum_{q=g}^{p}\left(\frac{p(1+T(\Gamma_g))}{\sum_{r=1}^{p}(1+T(\Gamma_r))}\Gamma_g\right)^{x}\otimes_D\left(\frac{p(1+T(\Gamma_q))}{\sum_{r=1}^{p}(1+T(\Gamma_r))}\Gamma_q\right)^{y}\right)^{\frac{1}{x+y}}. \tag{14}$$

Where $T(\Gamma_g)=\sum_{q=1,g\neq q}^{p} Sup(\Gamma_g,\Gamma_q), Sup(\Gamma_g,\Gamma_q)=1-\bar{\bar{D}}(\Gamma_g,\Gamma_q)$ is the support degree for Γ_g from Γ_q, which satisfy the following conditions: (1) $Sup(\Gamma_g,\Gamma_q)\in[0,1]$; (2) $Sup(\Gamma_g,\Gamma_q)=Supp(\Gamma_g,\Gamma_q)$; (3) $Sup(\Gamma_g,\Gamma_q)\geq Sup(\Gamma_r,\Gamma_s)$, if $\bar{\bar{D}}(\Gamma_g,\Gamma_q)<\bar{\bar{D}}(\Gamma_r,\Gamma_s)$, in which $\bar{\bar{D}}(\Gamma_g,\Gamma_q)$ is the distance measure between 2-TLNNs Γ_g and Γ_q defined in Definition (5).

In order, to represent Equation (14) in a simple form, we assume that

$$\aleph_g=\frac{(1+T(\Gamma_g))}{\sum_{r=1}^{p}(1+T(\Gamma_r))} \tag{15}$$

Therefore, Equation (14) takes the form

$$2-TLNDPHM^{x,y}(\Gamma_1,\Gamma_2,\dots,\Gamma_p)=\left(\frac{2}{p^2+p}\sum_{g=1}^{p}\sum_{q=g}^{p}(p\aleph_g\Gamma_g)^x\otimes_D(\aleph_q\Gamma_q)^y\right)^{\frac{1}{x+y}}. \tag{16}$$

Theorem 1. Let $x,y\geq 0,$ and x,y do not take the value 0 at the same time, $\Gamma_g(g=1,2,\dots,p)$ be a group of 2-TLNNs and let $\frac{\Delta^{-1}(s_{t_g},\Xi_g)}{h}=\bar{\bar{t}}_g,\frac{\Delta^{-1}(s_{i_g},\Psi_g)}{h}=\bar{\bar{i}}_g,\frac{\Delta^{-1}(s_{f_g},\Upsilon_g)}{h}=\bar{\bar{f}}_g$. Then, the aggregated value utilizing Equation (14), is still a 2-TLNN, and

$$2-TLNPHM(\Gamma_1,\Gamma_2,\dots,\Gamma_p)=$$

$$\left\langle \begin{array}{l}\Delta\left(h\left(1\Big/\left(1+\left(\frac{p^2+p}{2(x+y)}\times 1\Big/\left(\sum_{g=1,q=q}^{p}1\Big/\left(\frac{x}{p\aleph_g\left(\frac{\bar{\bar{t}}_g}{1-\bar{\bar{t}}_g}\right)^3}+\frac{y}{p\aleph_q\left(\frac{\bar{\bar{t}}_q}{1-\bar{\bar{t}}_q}\right)^3}\right)\right)\right)\right)^{\frac{1}{3}}\right)\right),\\ \Delta\left(h\left(1-1\Big/\left(1+\left(\frac{p^2+p}{2(x+y)}\times 1\Big/\left(\sum_{g=1,q=g}^{p}1\Big/\left(\frac{x}{p\aleph_g\left(\frac{1-\bar{\bar{i}}_g}{\bar{\bar{i}}_g}\right)^3}+\frac{y}{p\aleph_q\left(\frac{1-\bar{\bar{i}}_q}{\bar{\bar{i}}_q}\right)^3}\right)\right)\right)\right)^{\frac{1}{3}}\right)\right),\\ \Delta\left(h\left(1-1\Big/\left(1+\left(\frac{p^2+p}{2(x+y)}\times 1\Big/\left(\sum_{g=1,q=g}^{p}1\Big/\left(\frac{x}{p\aleph_g\left(\frac{1-\bar{\bar{f}}_g}{\bar{\bar{f}}_g}\right)^3}+\frac{y}{p\aleph_q\left(\frac{1-\bar{\bar{f}}_q}{\bar{\bar{f}}_q}\right)^3}\right)\right)\right)\right)^{\frac{1}{3}}\right)\right)\end{array}\right\rangle. \tag{17}$$

Proof. According to operational laws, we have

$$p\aleph_g\Gamma_g=\left\langle\Delta\left(h\left(1-1\Big/\left(1+\left(p\aleph_g\left(\frac{\bar{\bar{t}}_g}{1-\bar{\bar{t}}_g}\right)^3\right)\right)^{\frac{1}{3}}\right)\right),\right.$$

$$\left.\Delta\left(h\left(1\Big/\left(1+\left(p\aleph_g\left(\frac{1-\bar{\bar{i}}_g}{\bar{\bar{i}}_g}\right)^3\right)\right)^{\frac{1}{3}}\right)\right),\Delta\left(h\left(1\Big/\left(1+\left(p\aleph_g\left(\frac{1-\bar{\bar{f}}_g}{\bar{\bar{f}}_g}\right)^3\right)\right)^{\frac{1}{3}}\right)\right)\right\rangle,$$

and

$$p\aleph_q\Gamma_q=\left\langle\Delta\left(h\left(1-1\Big/\left(1+\left(p\aleph_q\left(\frac{\bar{\bar{t}}_q}{1-\bar{\bar{t}}_q}\right)^3\right)\right)^{\frac{1}{3}}\right)\right),\right.$$

$$\left.\Delta\left(h\left(1\Big/\left(1+\left(p\aleph_q\left(\frac{1-\bar{\bar{i}}_q}{\bar{\bar{i}}_q}\right)^3\right)\right)^{\frac{1}{3}}\right)\right),\Delta\left(h\left(1\Big/\left(1+\left(p\aleph_q\left(\frac{1-\bar{\bar{f}}_q}{\bar{\bar{f}}_q}\right)^3\right)\right)^{\frac{1}{3}}\right)\right)\right\rangle.$$

Let $\bar{\bar{a}}_g=\frac{\bar{\bar{t}}_g}{1-\bar{\bar{t}}_g},\bar{\bar{a}}_q=\frac{\bar{\bar{t}}_q}{1-\bar{\bar{t}}_q},\bar{\bar{b}}_g=\frac{1-\bar{\bar{i}}_g}{\bar{\bar{i}}_g},\bar{\bar{b}}_q=\frac{1-\bar{\bar{i}}_q}{\bar{\bar{i}}_q},\bar{\bar{c}}_g=\frac{1-\bar{\bar{f}}_g}{\bar{\bar{f}}_g},\bar{\bar{c}}_q=\frac{1-\bar{\bar{f}}_q}{\bar{\bar{f}}_q}.$

Then, we can obtain

$$p\aleph_g\Gamma_g=\left\langle\Delta\left(h\left(1-1\Big/1+(p\aleph_g)^{\frac{1}{3}}\bar{\bar{a}}_g\right)\right),\right.$$

$$\left.\Delta\left(h\left(1\Big/1+(p\aleph_g)^{\frac{1}{3}}\bar{\bar{b}}_g\right)\right),\Delta\left(h\left(1\Big/1+(p\aleph_g)^{\frac{1}{3}}\bar{\bar{c}}_g\right)\right)\right\rangle,$$

$$p\aleph_q\Gamma_q=\left\langle\Delta\left(h\left(1-1\Big/1+(p\aleph_q)^{\frac{1}{3}}\bar{\bar{a}}_q\right)\right),\right.$$

$$\left.\Delta\left(h\left(1\Big/1+(p\aleph_q)^{\frac{1}{3}}\bar{\bar{b}}_q\right)\right),\Delta\left(h\left(1\Big/1+(p\aleph_q)^{\frac{1}{3}}\bar{\bar{c}}_q\right)\right)\right\rangle,$$

and

$$(p\aleph_g\Gamma_g)^x=\left\langle\Delta\left(h\left(1\Big/1+x^{\frac{1}{3}}\Big/(p\aleph_g)^{\frac{1}{3}}\bar{\bar{a}}_g\right)\right),\right.$$

$$\left.\Delta\left(h\left(1-1\Big/1+x^{\frac{1}{3}}\Big/(p\aleph_g)^{\frac{1}{3}}\bar{\bar{b}}_g\right)\right),\Delta\left(h\left(1-1\Big/1+x^{\frac{1}{3}}\Big/(p\aleph_g)^{\frac{1}{3}}\bar{\bar{c}}_g\right)\right)\right\rangle,$$

$$(p\aleph_q\Gamma_q)^y=\left\langle\Delta\left(h\left(1\Big/1+y^{\frac{1}{3}}\Big/(p\aleph_q)^{\frac{1}{3}}\bar{\bar{a}}_q\right)\right),\right.$$

$$\left.\Delta\left(h\left(1-1\Big/1+y^{\frac{1}{3}}\Big/(p\aleph_q)^{\frac{1}{3}}\bar{\bar{b}}_q\right)\right),\Delta\left(h\left(1-1\Big/1+y^{\frac{1}{3}}\Big/(p\aleph_q)^{\frac{1}{3}}\bar{\bar{c}}_q\right)\right)\right\rangle.$$

Furthermore, we can have

$$(p\aleph_g\Gamma_g)^x\otimes_D(p\aleph_q\Gamma_q)^y=\left\langle\Delta\left(h\left(1\Big/1+\left(x\Big/p\aleph_g\bar{\bar{a}}_g^3+y\Big/p\aleph_q\bar{\bar{a}}_q^3\right)^{\frac{1}{3}}\right)\right),\right.$$

$$\left.\Delta\left(h\left(1-1\Big/1+\left(x\Big/p\aleph_g\bar{\bar{b}}_g^3+y\Big/p\aleph_q\bar{\bar{b}}_q^3\right)^{\frac{1}{3}}\right)\right),\Delta\left(h\left(1-1\Big/1+\left(x\Big/p\aleph_g\bar{\bar{c}}_g^3+y\Big/p\aleph_q\bar{\bar{c}}_q^3\right)^{\frac{1}{3}}\right)\right)\right\rangle,$$

and

$$\sum_{g=1,\,q=g}^{p}\left(p\aleph_g\Gamma_g\right)^x\otimes_D\left(p\aleph_q\Gamma_q\right)^y$$

$$=\left\langle \Delta\left(h\left(1-1\Big/\left(1+\sum_{g=1,\,q=g}^{p}\left(\left(\frac{1}{1+\left(\frac{x}{p\aleph_g\bar{\bar{a}}_g^{\Im}}+\frac{y}{p\aleph_q\bar{\bar{a}}_q^{\Im}}\right)^{\frac{1}{\Im}}}\right)\Big/\left(1-\frac{1}{1+\left(\frac{x}{p\aleph_g\bar{\bar{a}}_g^{\Im}}+\frac{y}{p\aleph_q\bar{\bar{a}}_q^{\Im}}\right)^{\frac{1}{\Im}}}\right)\right)^{\Im}\right)^{\frac{1}{\Im}}\right)\right),\right.$$

$$\Delta\left(h\left(1\Big/\left(1+\sum_{g=1,\,q=g}^{p}\left(\left(1-1+\frac{1}{1+\left(\frac{x}{p\aleph_g\bar{\bar{b}}_g^{\Im}}+\frac{y}{p\aleph_q\bar{\bar{b}}_q^{\Im}}\right)^{\frac{1}{\Im}}}\right)\Big/\left(1-\frac{1}{1+\left(\frac{x}{p\aleph_g\bar{\bar{b}}_g^{\Im}}+\frac{y}{p\aleph_q\bar{\bar{b}}_q^{\Im}}\right)^{\frac{1}{\Im}}}\right)\right)^{\Im}\right)^{\frac{1}{\Im}}\right)\right),$$

$$\left.\Delta\left(h\left(1\Big/\left(1+\sum_{g=1,\,q=g}^{p}\left(\left(1-1+\frac{1}{1+\left(\frac{x}{p\aleph_g\bar{\bar{c}}_g^{\Im}}+\frac{y}{p\aleph_q\bar{\bar{c}}_q^{\Im}}\right)^{\frac{1}{\Im}}}\right)\Big/\left(1-\frac{1}{1+\left(\frac{x}{p\aleph_g\bar{\bar{c}}_g^{\Im}}+\frac{y}{p\aleph_q\bar{\bar{c}}_q^{\Im}}\right)^{\frac{1}{\Im}}}\right)\right)^{\Im}\right)^{\frac{1}{\Im}}\right)\right)\right\rangle,$$

$$=\left\langle \Delta\left(h\left(1-1\Big/\left(1+\left(\sum_{g=1,\,q=g}^{p}1\Big/\left(\frac{x}{p\aleph_g\bar{\bar{a}}_g^{\Im}}+\frac{y}{p\aleph_q\bar{\bar{a}}_q^{\Im}}\right)\right)^{\frac{1}{\Im}}\right)\right)\right),\right.$$

$$\Delta\left(h\left(1\Big/\left(1+\left(\sum_{g=1,\,q=g}^{p}1\Big/\left(\frac{x}{p\aleph_g\bar{\bar{b}}_g^{\Im}}+\frac{y}{p\aleph_q\bar{\bar{b}}_q^{\Im}}\right)\right)^{\frac{1}{\Im}}\right)\right)\right),$$

$$\left.\Delta\left(h\left(1\Big/\left(1+\left(\sum_{g=1,\,q=g}^{p}1\Big/\left(\frac{x}{p\aleph_g\bar{\bar{c}}_g^{\Im}}+\frac{y}{p\aleph_q\bar{\bar{c}}_q^{\Im}}\right)\right)^{\frac{1}{\Im}}\right)\right)\right)\right\rangle.$$

So, we can have

$$\frac{2}{p^2+p}\sum_{g=1,\,q=g}^{p}\left(p\aleph_g\Gamma_g\right)^x\otimes_D\left(p\aleph_q\Gamma_q\right)^y$$

$$=\left\langle \Delta\left(h\left(1-1\Big/\left(1+\left(\frac{2}{p^2+p}\left(\left(1-1\Big/\left(1+\left(\sum_{g=1,\,q=g}^{p}1\Big/\left(\frac{x}{p\aleph_g\bar{\bar{a}}_g^{\Im}}+\frac{y}{p\aleph_q\bar{\bar{a}}_q^{\Im}}\right)\right)^{\frac{1}{\Im}}\right)\right)\Big/\left(1-1+1\Big/\left(1+\left(\sum_{g=1,\,q=g}^{p}1\Big/\left(\frac{x}{p\aleph_g\bar{\bar{a}}_g^{\Im}}+\frac{y}{p\aleph_q\bar{\bar{a}}_q^{\Im}}\right)\right)^{\frac{1}{\Im}}\right)\right)\right)^{\Im}\right)^{\frac{1}{\Im}}\right)\right)\right),\right.$$

$$\Delta\left(h\left(1\Big/\left(1+\left(\frac{2}{p^2+p}\left(\left(1-1\Big/\left(1+\left(\sum_{g=1,\,q=g}^{p}1\Big/\left(\frac{x}{p\aleph_g\bar{\bar{b}}_g^{\Im}}+\frac{y}{p\aleph_q\bar{\bar{b}}_q^{\Im}}\right)\right)^{\frac{1}{\Im}}\right)\right)\Big/\left(1\Big/\left(1+\left(\sum_{g=1,\,q=g}^{p}1\Big/\left(\frac{x}{p\aleph_g\bar{\bar{b}}_g^{\Im}}+\frac{y}{p\aleph_q\bar{\bar{b}}_q^{\Im}}\right)\right)^{\frac{1}{\Im}}\right)\right)\right)^{\Im}\right)^{\frac{1}{\Im}}\right)\right)\right),$$

$$\left.\Delta\left(h\left(1\Big/\left(1+\left(\frac{2}{p^2+p}\left(\left(1-1\Big/\left(1+\left(\sum_{g=1,\,q=g}^{p}1\Big/\left(\frac{x}{p\aleph_g\bar{\bar{c}}_g^{\Im}}+\frac{y}{p\aleph_q\bar{\bar{c}}_q^{\Im}}\right)\right)^{\frac{1}{\Im}}\right)\right)\Big/\left(1\Big/\left(1+\left(\sum_{g=1,\,q=g}^{p}1\Big/\left(\frac{x}{p\aleph_g\bar{\bar{c}}_g^{\Im}}+\frac{y}{p\aleph_q\bar{\bar{c}}_q^{\Im}}\right)\right)^{\frac{1}{\Im}}\right)\right)\right)^{\Im}\right)^{\frac{1}{\Im}}\right)\right)\right)\right\rangle,$$

$$=\left\langle \Delta\left(h\left(1-1\Big/\left(1+\left(\frac{2}{p^2+p}\sum_{g=1,\,q=g}^{p}1\Big/\left(\frac{x}{p\aleph_g\bar{\bar{a}}_g^{\Im}}+\frac{y}{p\aleph_q\bar{\bar{a}}_q^{\Im}}\right)\right)^{\frac{1}{\Im}}\right)\right)\right),\right.$$

$$\Delta\left(h\left(1\Big/\left(1+\left(\frac{2}{p^2+p}\sum_{g=1,\,q=g}^{p}1\Big/\left(\frac{x}{p\aleph_g\bar{\bar{b}}_g^{\Im}}+\frac{y}{p\aleph_q\bar{\bar{b}}_q^{\Im}}\right)\right)^{\frac{1}{\Im}}\right)\right)\right),$$

$$\left.\Delta\left(h\left(1\Big/\left(1+\left(\frac{2}{p^2+p}\sum_{g=1,\,q=g}^{p}1\Big/\left(\frac{x}{p\aleph_g\bar{\bar{c}}_g^{\Im}}+\frac{y}{p\aleph_q\bar{\bar{c}}_q^{\Im}}\right)\right)^{\frac{1}{\Im}}\right)\right)\right)\right\rangle.$$

Then

$$\left(\frac{2}{p^2+p}\sum_{g=1,\,q=g}^{p}\left(p\aleph_g\Gamma_g\right)^x\otimes_D\left(p\aleph_q\Gamma_q\right)^y\right)^{\frac{1}{x+y}}$$

$$=\left\langle \Delta\left(h\left(1\Big/\left(1+\left(\frac{p^2+p}{2(x+y)}\sum_{g=1,\,q=g}^{p}1\Big/\left(\frac{x}{p\aleph_g\bar{\bar{a}}_g^{\Im}}+\frac{y}{p\aleph_q\bar{\bar{a}}_q^{\Im}}\right)\right)^{\frac{1}{\Im}}\right)\right)\right),\right.$$

$$\Delta\left(h\left(1-1\Big/\left(1+\left(\frac{p^2+p}{2(x+y)}\sum_{g=1,\,q=g}^{p}1\Big/\left(\frac{x}{p\aleph_g\bar{\bar{b}}_g^{\Im}}+\frac{y}{p\aleph_q\bar{\bar{b}}_q^{\Im}}\right)\right)^{\frac{1}{\Im}}\right)\right)\right),\quad(18)$$

$$\left.\Delta\left(h\left(1-1\Big/\left(1+\left(\frac{p^2+p}{2(x+y)}\sum_{g=1,\,q=g}^{p}1\Big/\left(\frac{x}{p\aleph_g\bar{\bar{c}}_g^{\Im}}+\frac{y}{p\aleph_q\bar{\bar{c}}_q^{\Im}}\right)\right)^{\frac{1}{\Im}}\right)\right)\right)\right\rangle.$$

Now put $\bar{\bar{a}}_g=\frac{\bar{\bar{t}}_g}{1-\bar{\bar{t}}_g},\bar{\bar{a}}_q=\frac{\bar{\bar{t}}_q}{1-\bar{\bar{t}}_q},\bar{\bar{b}}_g=\frac{1-\bar{\bar{i}}_g}{\bar{\bar{i}}_g},\bar{\bar{b}}_q=\frac{1-\bar{\bar{i}}_q}{\bar{\bar{i}}_q},\bar{\bar{c}}_g=\frac{1-\bar{\bar{f}}_g}{\bar{\bar{f}}_g},\bar{\bar{c}}_q=\frac{1-\bar{\bar{f}}_q}{\bar{\bar{f}}_q}$ in Equation (18), we can have

$$=\left\langle \Delta\left(h\left(1\Big/\left(1+\left(\frac{p^2+p}{2(x+y)}\sum_{g=1,\,q=g}^{p}1\Big/\left(\frac{x}{p\aleph_g\left(\frac{\bar{\bar{t}}_g}{1-\bar{\bar{t}}_g}\right)^{\Im}}+\frac{y}{p\aleph_q\left(\frac{\bar{\bar{t}}_q}{1-\bar{\bar{t}}_q}\right)^{\Im}}\right)\right)^{\frac{1}{\Im}}\right)\right)\right),\right.$$

$$\Delta\left(h\left(1-1\Big/\left(1+\left(\frac{p^2+p}{2(x+y)}\sum_{g=1,\,q=g}^{p}1\Big/\left(\frac{x}{p\aleph_g\left(\frac{1-\bar{\bar{i}}_g}{\bar{\bar{i}}_g}\right)^{\Im}}+\frac{y}{p\aleph_q\left(\frac{1-\bar{\bar{i}}_q}{\bar{\bar{i}}_q}\right)^{\Im}}\right)\right)^{\frac{1}{\Im}}\right)\right)\right),$$

$$\left.\Delta\left(h\left(1-1\Big/\left(1+\left(\frac{p^2+p}{2(x+y)}\sum_{g=1,\,q=g}^{p}1\Big/\left(\frac{x}{p\aleph_g\left(\frac{1-\bar{\bar{f}}_g}{\bar{\bar{f}}_g}\right)^{\Im}}+\frac{y}{p\aleph_q\left(\frac{1-\bar{\bar{f}}_g}{\bar{\bar{f}}_g}\right)^{\Im}}\right)\right)^{\frac{1}{\Im}}\right)\right)\right)\right\rangle.$$

This completes the proof of Theorem (1).

Theorem 2 (Idempotency). Let $\Gamma_g(g=1,2,...,p)$ be a group of 2-TLNNs, if all $\Gamma_g(g=1,2,...,p)$ are same, that is

$\Gamma_g = \Gamma = \langle (s_t, \Xi), (s_i, \Psi), (s_f, \Upsilon) \rangle (g = 1,2,...,p)$. Assume that $\frac{\Delta^{-1}(s_{t_g}, \Xi_g)}{h} = \overset{=}{t}_g, \frac{\Delta^{-1}(s_{i_g}, \Psi_g)}{h} = \overset{=}{i}_g, \frac{\Delta^{-1}(s_{f_g}, \Upsilon_g)}{h} = \overset{=}{f}_g$, then

$$2-TLNPHM(\Gamma_1, \Gamma_2, ..., \Gamma_p) = \Gamma. \quad (19)$$

Proof. Since all $\Gamma_g = \Gamma = \langle (s_t, \Xi), (s_i, \Psi), (s_f, \Upsilon) \rangle (g = 1,2,...,p)$, so we can have $Sup(\Gamma_g, \Gamma_q) = 1$, for all $g, q = 1,2,...,p$, so $\aleph_g = \frac{1}{p}$, for all $g = 1,2,...,p$. Then

$$2-TLNPHM^{x,y}(\Gamma_1, \Gamma_2, ..., \Gamma_p) = 2-TLNPHM^{x,y}(\Gamma, \Gamma, ..., \Gamma)$$

$$= \left\langle \Delta\left(h\left(1 \Big/ \left(1 + \left(\frac{p^2+p}{2(x+y)} \sum_{\substack{g=1,\\ q=g}}^{p} 1 \Big/ \left(\frac{x}{p\frac{1}{p}\left(\frac{\overset{=}{t}}{1-\overset{=}{t}}\right)^3} + \frac{y}{p\frac{1}{p}\left(\frac{\overset{=}{t}}{1-\overset{=}{t}}\right)^3}\right)\right)^{\frac{1}{3}}\right)\right)\right),\right.$$

$$\Delta\left(h\left(1 - 1 \Big/ \left(1 + \left(\frac{p^2+p}{2(x+y)} \sum_{\substack{g=1,\\ q=g}}^{p} 1 \Big/ \left(\frac{x}{p\frac{1}{p}\left(\frac{1-\overset{=}{i}}{\overset{=}{i}}\right)^3} + \frac{y}{p\frac{1}{p}\left(\frac{1-\overset{=}{i}}{\overset{=}{i}}\right)^3}\right)\right)^{\frac{1}{3}}\right)\right)\right),$$

$$\left.\Delta\left(h\left(1 - 1 \Big/ \left(1 + \left(\frac{p^2+p}{2(x+y)} \sum_{\substack{g=1,\\ q=g}}^{p} 1 \Big/ \left(\frac{x}{p\frac{1}{p}\left(\frac{1-\overset{=}{f}}{\overset{=}{f}}\right)^3} + \frac{y}{p\frac{1}{p}\left(\frac{1-\overset{=}{f}}{\overset{=}{f}}\right)^3}\right)\right)^{\frac{1}{3}}\right)\right)\right)\right\rangle,$$

$$= \left\langle \Delta\left(h\left(1 \Big/ \left(1 + \left(\frac{p^2+p}{2(x+y)} \sum_{\substack{g=1,\\ q=g}}^{p} 1 \Big/ \left(\frac{x+y}{\left(\frac{\overset{=}{t}}{1-\overset{=}{t}}\right)^3}\right)\right)^{\frac{1}{3}}\right)\right)\right),\right.$$

$$\Delta\left(h\left(1 - 1 \Big/ \left(1 + \left(\frac{p^2+p}{2(x+y)} \sum_{\substack{g=1,\\ q=g}}^{p} 1 \Big/ \left(\frac{x+y}{\left(\frac{1-\overset{=}{i}}{\overset{=}{i}}\right)^3}\right)\right)^{\frac{1}{3}}\right)\right)\right),$$

$$\left.\Delta\left(h\left(1 - 1 \Big/ \left(1 + \left(\frac{p^2+p}{2(x+y)} \sum_{\substack{g=1,\\ q=g}}^{p} 1 \Big/ \left(\frac{x+y}{\left(\frac{1-\overset{=}{f}}{\overset{=}{f}}\right)^3}\right)\right)^{\frac{1}{3}}\right)\right)\right)\right\rangle,$$

$$= \left\langle \Delta\left(h\left(1 \Big/ \left(1 + \left(\frac{1}{(x+y)} \times 1 \Big/ \left(\frac{x+y}{\left(\frac{\overset{=}{t}}{1-\overset{=}{t}}\right)^3}\right)\right)^{\frac{1}{3}}\right)\right)\right),\right.$$

$$\Delta\left(h\left(1 - 1 \Big/ \left(1 + \left(\frac{1}{(x+y)} \times 1 \Big/ \left(\frac{x+y}{\left(\frac{1-\overset{=}{i}}{\overset{=}{i}}\right)^3}\right)\right)^{\frac{1}{3}}\right)\right)\right),$$

$$\left.\Delta\left(h\left(1 - 1 \Big/ \left(1 + \left(\frac{1}{(x+y)} \times 1 \Big/ \left(\frac{x+y}{\left(\frac{1-\overset{=}{f}}{\overset{=}{f}}\right)^3}\right)\right)^{\frac{1}{3}}\right)\right)\right)\right\rangle,$$

$$= \langle (s_t, \Xi), (s_i, \Psi), (s_f, \Upsilon) \rangle = \Gamma$$

Theorem 3 (Boundedness). Let $\Gamma_g (g = 1,2,...,p)$ be a group of 2-TLNNs. If $\overset{=}{m}^- = \langle \min_g(s_{t_g}, \Xi_g), \max_g(s_{i_g}, \Psi_g), \max_g(s_{f_g}, \Upsilon_g) \rangle$ and $\overset{=}{m}^+ = \langle \max_g(s_{t_g}, \Xi_g), \min_g(s_{i_g}, \Psi_g), \min_g(s_{f_g}, \Upsilon_g) \rangle$, then

$$\overset{=}{m}^- \le 2-TLNPHM^{x,y}(\Gamma_1, \Gamma_2, ..., \Gamma_p) \le \overset{=}{m}^+ \quad (20)$$

Proof. To prove this let us assume that, $\frac{\Delta^{-1}(s_{t_g}, \Xi_g)}{h} = \overset{=}{t}_g, \frac{\Delta^{-1}(s_{i_g}, \Psi_g)}{h} = \overset{=}{i}_g, \frac{\Delta^{-1}(s_{f_g}, \Upsilon_g)}{h} = \overset{=}{f}_g$,

$\frac{\Delta^{-1}(s_{t_g^-}, \Xi_g^-)}{h} = \overset{=}{t}_g^-, \frac{\Delta^{-1}(s_{i_g^-}, \Psi_g^-)}{h} = \overset{=}{i}_g^-, \frac{\Delta^{-1}(s_{f_g^-}, \Upsilon_g^-)}{h} = \overset{=}{f}_g^-$,

$\frac{\Delta^{-1}(s_{t_g^+}, \Xi_g^+)}{h} = \overset{=}{t}_g^+, \frac{\Delta^{-1}(s_{i_g^+}, \Psi_g^+)}{h} = \overset{=}{i}_g^+, \frac{\Delta^{-1}(s_{f_g^+}, \Upsilon_g^+)}{h} = \overset{=}{f}_g^+$.

Since $\overset{=}{m}^+ = \langle \max_g(s_{t_g}, \Xi_g), \min_g(s_{i_g}, \Psi_g), \min_g(s_{f_g}, \Upsilon_g) \rangle$, $\overset{=}{m}^- = \langle \min_g(s_{t_g}, \Xi_g), \max_g(s_{i_g}, \Psi_g), \max_g(s_{f_g}, \Upsilon_g) \rangle$. Then, there are $\overset{=}{t}^- \le \overset{=}{t}(\Gamma_g) \le \overset{=}{t}^+, \overset{=}{i}^- \le \overset{=}{i}(\Gamma_g) \le \overset{=}{i}^+, \overset{=}{f}^- \le \overset{=}{f}(\Gamma_g) \le \overset{=}{f}^+$ for all $g = 1,2,....,p$. So, we have

$$\bar{\bar{t}}(\Gamma_g)=\Delta\left(h\left(1\Big/\left(1+\left(\frac{p^2+p}{2(x+y)}\sum_{g=1,\,q=g}^{p}1\Big/\left(\frac{x}{p\aleph_g\left(\frac{\bar{\bar{t}}_g}{1-\bar{\bar{t}}_g}\right)^{\Im}}+\frac{y}{p\aleph_q\left(\frac{\bar{\bar{t}}_q}{1-\bar{\bar{t}}_q}\right)^{\Im}}\right)\right)^{\frac{1}{\Im}}\right)\right)\right)$$

$$\geq\Delta\left(h\left(1\Big/\left(1+\left(\frac{p^2+p}{2(x+y)}\sum_{g=1,\,q=g}^{p}1\Big/\left(\frac{x}{p\aleph_g\left(\frac{\bar{\bar{t}}_g^{-}}{1-\bar{\bar{t}}_g^{-}}\right)^{\Im}}+\frac{y}{p\aleph_q\left(\frac{\bar{\bar{t}}_q^{-}}{1-\bar{\bar{t}}_q^{-}}\right)^{\Im}}\right)\right)^{\frac{1}{\Im}}\right)\right)\right)=\bar{\bar{t}}^{-},$$

$$\bar{\bar{i}}(\Gamma_g)=\Delta\left(h\left(1-1\Big/\left(1+\left(\frac{p^2+p}{2(x+y)}\sum_{g=1,\,q=g}^{p}1\Big/\left(\frac{x}{p\aleph_g\left(\frac{1-\bar{\bar{i}}_g}{\bar{\bar{i}}_g}\right)^{\Im}}+\frac{y}{p\aleph_q\left(\frac{1-\bar{\bar{i}}_q}{\bar{\bar{i}}_q}\right)^{\Im}}\right)\right)^{\frac{1}{\Im}}\right)\right)\right)$$

$$\leq\Delta\left(h\left(1-1\Big/\left(1+\left(\frac{p^2+p}{2(x+y)}\sum_{g=1,\,q=g}^{p}1\Big/\left(\frac{x}{p\aleph_g\left(\frac{1-\bar{\bar{i}}_g}{\bar{\bar{i}}_g^{+}}\right)^{\Im}}+\frac{y}{p\aleph_q\left(\frac{1-\bar{\bar{i}}_q}{\bar{\bar{i}}_q^{+}}\right)^{\Im}}\right)\right)^{\frac{1}{\Im}}\right)\right)\right)=\bar{\bar{i}}^{+},$$

$$\bar{\bar{f}}(\Gamma_g)=\Delta\left(h\left(1-1\Big/\left(1+\left(\frac{p^2+p}{2(x+y)}\sum_{g=1,\,q=g}^{p}1\Big/\left(\frac{x}{p\aleph_g\left(\frac{1-\bar{\bar{f}}_g}{\bar{\bar{f}}_g}\right)^{\Im}}+\frac{y}{p\aleph_q\left(\frac{1-\bar{\bar{f}}_q}{\bar{\bar{f}}_q}\right)^{\Im}}\right)\right)^{\frac{1}{\Im}}\right)\right)\right)$$

$$\leq\Delta\left(h\left(1-1\Big/\left(1+\left(\frac{p^2+p}{2(x+y)}\sum_{g=1,\,q=g}^{p}1\Big/\left(\frac{x}{p\aleph_g\left(\frac{1-\bar{\bar{f}}_g^{+}}{\bar{\bar{f}}_g^{+}}\right)^{\Im}}+\frac{y}{p\aleph_q\left(\frac{1-\bar{\bar{f}}_q^{+}}{\bar{\bar{f}}_q^{+}}\right)^{\Im}}\right)\right)^{\frac{1}{\Im}}\right)\right)\right)=\bar{\bar{f}}^{+}.$$

Then, there is the following comparison:

(1) For the expected value:

$$\overline{\overline{SR}}(\Gamma)=\Delta\left\{\frac{2h+\Delta^{-1}(s_t,\Xi)-\Delta^{-1}(s_i,\Psi)-\Delta^{-1}(s_f,\Upsilon)}{3h}\right\}$$

$$\geq\Delta\left\{\frac{2h+\Delta^{-1}(s_{t^-},\Xi^-)-\Delta^{-1}(s_{i^+},\Psi^+)-\Delta^{-1}(s_{f^+},\Upsilon^+)}{3h}\right\}$$

$$=\overline{\overline{SR}}\left(\left\langle(s_{t^-},\Xi)-\Delta^{-1}(s_{i^-},\Psi)-\Delta^{-1}(s_{f^-},\Upsilon)\right\rangle\right).$$

If $\overline{\overline{SR}}(\Gamma)>\overline{\overline{SR}}\left(\left\langle(s_{t^-},\Xi)-\Delta^{-1}(s_{i^-},\Psi)-\Delta^{-1}(s_{f^-},\Upsilon)\right\rangle\right)$,

then

$\left\langle(s_{t^-},\Xi)-\Delta^{-1}(s_{i^-},\Psi)-\Delta^{-1}(s_{f^-},\Upsilon)\right\rangle<2-TLNPHM^{x,y}(\Gamma_1,\Gamma_2,...,\Gamma_p)$,

Else $\overline{\overline{SR}}(\Gamma)=\overline{\overline{SR}}\left(\left\langle(s_{t^-},\Xi)-\Delta^{-1}(s_{i^-},\Psi)-\Delta^{-1}(s_{f^-},\Upsilon)\right\rangle\right)$,

then we have the score function

(2)

$$\overline{\overline{AC}}(\Gamma)=\Delta\left\{\Delta^{-1}(s_t,\Xi)-\Delta^{-1}(s_f,\Upsilon)\right\}\geq\Delta\left\{\Delta^{-1}(s_{t^-},\Xi^-)-\Delta^{-1}(s_{f^+},\Upsilon^+)\right\}$$

$$=\overline{\overline{AC}}\left(\left\langle(s_{t^-},\Xi)-\Delta^{-1}(s_{i^-},\Psi)-\Delta^{-1}(s_{f^-},\Upsilon)\right\rangle\right).$$

If $\overline{\overline{AC}}(\Gamma)>\overline{\overline{AC}}\left(\left\langle(s_{t^-},\Xi)-\Delta^{-1}(s_{i^-},\Psi)-\Delta^{-1}(s_{f^-},\Upsilon)\right\rangle\right)$,

then

$\left\langle(s_{t^-},\Xi)-\Delta^{-1}(s_{i^-},\Psi)-\Delta^{-1}(s_{f^-},\Upsilon)\right\rangle<2-TLNPHM^{x,y}(\Gamma_1,\Gamma_2,...,\Gamma_p)$,

Else

$$\overline{\overline{AC}}(\Gamma)=\overline{\overline{AC}}\left(\left\langle(s_{t^-},\Xi)-\Delta^{-1}(s_{i^-},\Psi)-\Delta^{-1}(s_{f^-},\Upsilon)\right\rangle\right),$$

$$=2-TLNPHM^{x,y}(\Gamma_1,\Gamma_2,...,\Gamma_p)$$

So, we have

$\bar{\bar{m}}^{-}\leq 2-TLNPHM^{x,y}(\Gamma_1,\Gamma_2,...,\Gamma_p)$.

In a similar way, we can show that $2-TLNPHM^{x,y}(\Gamma_1,\Gamma_2,...,\Gamma_p)\leq\bar{\bar{m}}^{+}$.

Hence we have

$\bar{\bar{m}}^{-}\leq 2-TLNPHM^{x,y}(\Gamma_1,\Gamma_2,...,\Gamma_p)\leq\bar{\bar{m}}^{+}$.

In the following, we shall discuss some special cases with respect to the parameter parameters x and y.

(1) When $y\to 0,\Im>0,$ we can have

$$2-TLNDPHM^{x,0}(\Gamma_1,\Gamma_2,...,\Gamma_p)$$

$$=\left(\frac{2}{p^2+p}\sum_{g=1}^{p}\sum_{q=g}^{p}\left(\frac{p(1+T(\Gamma_g))}{\sum_{r=1}^{p}(1+T(\Gamma_r))}\Gamma_g\right)^{x}\otimes\left(\frac{p(1+T(\Gamma_q))}{\sum_{r=1}^{p}(1+T(\Gamma_r))}\Gamma_q\right)^{y}\right)^{\frac{1}{x+y}}$$

$$=\left(\frac{2}{p^2+p}\sum_{g=1}^{p}\left((p+1-g)\left(\frac{p(1+T(\Gamma_g))}{\sum_{r=1}^{p}(1+T(\Gamma_r))}\Gamma_g\right)^{x}\right)\right)^{\frac{1}{x}}.$$

That is, the 2-TLDPHM operator degenerates into the 2-tuple linguistic neutrosophic descending Dombi power average operator.

(2) When $x\to 0,\Im>0,$ we can have

$$2-TLNDPHM^{o,y}(\Gamma_1,\Gamma_2,...,\Gamma_p)$$

$$=\left(\frac{2}{p^2+p}\sum_{g=1}^{p}\sum_{q=g}^{p}\left(\frac{p(1+T(\Gamma_g))}{\sum_{r=1}^{p}(1+T(\Gamma_r))}\Gamma_g\right)^{x}\otimes\left(\frac{p(1+T(\Gamma_q))}{\sum_{r=1}^{p}(1+T(\Gamma_r))}\Gamma_q\right)^{y}\right)^{\frac{1}{x+y}}$$

$$=\left(\frac{2}{p^2+p}\sum_{g=1}^{p}\left((g)\left(\frac{p(1+T(\Gamma_g))}{\sum_{r=1}^{p}(1+T(\Gamma_r))}\Gamma_g\right)^{y}\right)\right)^{\frac{1}{y}}.$$

That is, the 2-TLDPHM operator degenerates into the 2-tuple linguistic neutrosophic ascending Dombi power average operator.

(3) When $y \to 0, \Im > 0,$ and $Sup(\Gamma_g, \Gamma_q) = \beta(\beta \in [0,1])$ for all $g \neq q,$ then, we can have

$$2-TLNDPHM^{x,0}(\Gamma_1, \Gamma_2, ..., \Gamma_p) = \left(\frac{2}{p^2+p} \sum_{g=1}^{p} \sum_{q=g}^{p} \left(\frac{p(1+T(\Gamma_g))}{\sum_{r=1}^{p}(1+T(\Gamma_r))} \Gamma_g \right)^x \otimes \left(\frac{p(1+T(\Gamma_q))}{\sum_{r=1}^{p}(1+T(\Gamma_r))} \Gamma_q \right)^y \right)^{\frac{1}{x+y}} = \left(\frac{2}{p^2+p} \sum_{g=1}^{p} \left((p+1-g)(\Gamma_g)^x \right) \right)^{\frac{1}{x}}.$$

That is, the 2-TLDPHM operator degenerates into the 2-tuple linguistic neutrosophic linear descending Dombi weighted average operator. Certainly, the weight vector of Γ_g^x is $(p, p-1, ..., 1)$.

(4) When $x \to 0, \Im > 0,$ and $Sup(\Gamma_g, \Gamma_q) = \beta(\beta \in [0,1])$ for all $g \neq q,$ then, we can have

$$2-TLNDPHM^{0,y}(\Gamma_1, \Gamma_2, ..., \Gamma_p) = \left(\frac{2}{p^2+p} \sum_{g=1}^{p} \sum_{q=g}^{p} \left(\frac{p(1+T(\Gamma_g))}{\sum_{r=1}^{p}(1+T(\Gamma_r))} \Gamma_g \right)^x \otimes \left(\frac{p(1+T(\Gamma_q))}{\sum_{r=1}^{p}(1+T(\Gamma_r))} \Gamma_q \right)^y \right)^{\frac{1}{x+y}} = \left(\frac{2}{p^2+p} \sum_{g=1}^{p} \left((g)(\Gamma_g)^y \right) \right)^{\frac{1}{y}}.$$

That is, the 2-TLDPHM operator degenerates into the 2-tuple linguistic neutrosophic linear descending Dombi weighted average operator.

(5) When $x = y = 1, \Im > 0,$ and $Sup(\Gamma_g, \Gamma_q) = \beta(\beta \in [0,1])$ for all $g \neq q,$ then, we can have

$$2-TLNDPHM^{0,y}(\Gamma_1, \Gamma_2, ..., \Gamma_p) = \left(\frac{2}{p^2+p} \sum_{g=1}^{p} \sum_{q=g}^{p} \left((\Gamma_g) \otimes (\Gamma_q) \right) \right)^{\frac{1}{2}}.$$

That is, the 2-TLDPHM operator degenerates into the 2-tuple linguistic neutrosophic linear Dombi Heronian mean operator.

In the above developed 2-TLNDPHM operator, only power weight vector and the correlation between input arguments are taken under consideration and are not to consider the weight vector of the input arguments. Therefore, to remove this deficiency, we will propose it weighted form, that is 2-TPLNDWPHM operator.

Definition 11. Let $\Gamma_g (g = 1, 2, ..., p)$ be a group of 2-TLNNs, $x, y \geq 0, \bar{\bar{W}} = \left(\bar{\bar{w}}_1, \bar{\bar{w}}_2, ..., \bar{\bar{w}}_p \right)^T$ be the weight vector such that $\bar{\bar{w}}_g \in [0,1]$ and $\sum_{g=1}^{p} \bar{\bar{w}}_g = 1.$ Then, the 2-TLNNDWPHM operator is described as follows:

$$2-TLNDWPHM^{x,y}(\Gamma_1, \Gamma_2, ..., \Gamma_p) = \left(\frac{2}{p^2+p} \sum_{g=1}^{p} \sum_{q=g}^{p} \left(\frac{p\bar{\bar{w}}_g(1+T(\Gamma_g))}{\sum_{t=1}^{p}\bar{\bar{w}}_t(1+T(\Gamma_r))} \Gamma_g \right)^x \otimes \left(\frac{p\bar{\bar{w}}_q(1+T(\Gamma_q))}{\sum_{t=1}^{p}\bar{\bar{w}}_t(1+T(\Gamma_r))} \Gamma_q \right)^y \right)^{\frac{1}{x+y}}. \quad (21)$$

Where $T(\Gamma_g) = \sum_{q=1, g\neq q}^{p} Sup(\Gamma_g, \Gamma_q), Sup(\Gamma_g, \Gamma_q) = 1 - \bar{\bar{D}}(\Gamma_g, \Gamma_q)$ is the support degree for Γ_g from Γ_q, which satisfy the following conditions: (1) $Sup(\Gamma_g, \Gamma_q) \in [0,1]$; (2) $Sup(\Gamma_g, \Gamma_q) = Supp(\Gamma_g, \Gamma_q)$; (3) $Sup(\Gamma_g, \Gamma_q) \geq Sup(\Gamma_r, \Gamma_s)$, if $\bar{\bar{D}}(\Gamma_g, \Gamma_q) < \bar{\bar{D}}(\Gamma_r, \Gamma_s)$, in which $\bar{\bar{D}}(\Gamma_g, \Gamma_q)$ is the distance measure between 2-TLNNs Γ_g and Γ_q defined in Definition (5).

In order, to represent Equation (21) in a simple form, we assume that

$$\Theta_g = \frac{\bar{\bar{w}}_g(1+T(\Gamma_g))}{\sum_{r=1}^{p}\bar{\bar{w}}_r(1+T(\Gamma_r))} \quad (22)$$

Therefore, Equation (21) takes the form

$$2-TLNDWPHM^{x,y}(\Gamma_1, \Gamma_2, ..., \Gamma_p) = \left(\frac{2}{p^2+p} \sum_{g=1}^{p} \sum_{q=g}^{p} \left(p\Theta_g\Gamma_g \right)^x \otimes \left(\Theta_q\Gamma_q \right)^y \right)^{\frac{1}{x+y}}. \quad (23)$$

Theorem 4. Let $x, y \geq 0,$ and x, y do not take the value 0 at the same time, $\Gamma_g (g = 1, 2, ..., p)$ be a group of 2-TLNNs and let $\frac{\Delta^{-1}(s_{t_g}, \Xi_g)}{h} = \bar{\bar{t}}_g, \frac{\Delta^{-1}(s_{i_g}, \Psi_g)}{h} = \bar{\bar{i}}_g, \frac{\Delta^{-1}(s_{f_g}, \Upsilon_g)}{h} = \bar{\bar{f}}_g$. Then, the aggregated value utilizing Equation (21), is still a 2-TLNN, and

$$2-TLNDWPHM(\Gamma_1, \Gamma_2, ..., \Gamma_p) = \left\langle \Delta\left(h \left(1 \middle/ \left(1 + \left(\frac{p^2+p}{2(x+y)} \times 1 \middle/ \left(\sum_{g=1, q=q}^{p} 1 \middle/ \left(\frac{x}{p\Theta_g\left(\frac{\bar{\bar{t}}_g}{1-\bar{\bar{t}}_g}\right)^{\Im}} + \frac{y}{p\Theta_q\left(\frac{\bar{\bar{t}}_q}{1-\bar{\bar{t}}_q}\right)^{\Im}} \right) \right) \right)^{\frac{1}{\Im}} \right) \right) \right), \right.$$
$$\Delta\left(h \left(1 - 1 \middle/ \left(1 + \left(\frac{p^2+p}{2(x+y)} \times 1 \middle/ \left(\sum_{g=1, q=g}^{p} 1 \middle/ \left(\frac{x}{p\Theta_g\left(\frac{1-\bar{\bar{i}}_g}{\bar{\bar{i}}_g}\right)^{\Im}} + \frac{y}{p\Theta_q\left(\frac{1-\bar{\bar{i}}_q}{\bar{\bar{i}}_q}\right)^{\Im}} \right) \right) \right)^{\frac{1}{\Im}} \right) \right) \right),$$
$$\left. \Delta\left(h \left(1 - 1 \middle/ \left(1 + \left(\frac{p^2+p}{2(x+y)} \times 1 \middle/ \left(\sum_{g=1, q=g}^{p} 1 \middle/ \left(\frac{x}{p\Theta_g\left(\frac{1-\bar{\bar{f}}_g}{\bar{\bar{f}}_g}\right)^{\Im}} + \frac{y}{p\Theta_q\left(\frac{1-\bar{\bar{f}}_q}{\bar{\bar{f}}_q}\right)^{\Im}} \right) \right) \right)^{\frac{1}{\Im}} \right) \right) \right) \right\rangle. \quad (24)$$

Proof. Same is Theorem 1.

It is worthy to note that the 2-TLNDWPHM operator has only the property of boundedness and does not have the properties of idempotency and monotonicity.

B. The 2-TLNDPGHM Operator and 2-TLNDWPGHM operator

Definition 12. Let $\Gamma_g\,(g=1,2,...,p)$ be a group of 2-TLNNs, $x, y \geq 0.$ Then, the 2-TLNNDPGHM operator is described as follows:

$$2-TLNDPGHM^{x,y}\left(\Gamma_1,\Gamma_2,...,\Gamma_p\right) = \frac{1}{x+y}\left(\prod_{g=1}^{p}\prod_{q=g}^{p}\left(x\left(\Gamma_g\right)^{\frac{p\left(1+T\left(\Gamma_g\right)\right)}{\sum_{r=1}^{p}\left(1+T\left(\Gamma_r\right)\right)}} \oplus y\left(\Gamma_q\right)^{\frac{p\left(1+T\left(\Gamma_q\right)\right)}{\sum_{r=1}^{p}\left(1+T\left(\Gamma_r\right)\right)}}\right)\right)^{\frac{2}{p^2+p}}. \tag{25}$$

Where $T\left(\Gamma_g\right)=\sum_{q=1,g\neq q}^{p} Sup\left(\Gamma_g,\Gamma_q\right), Sup\left(\Gamma_g,\Gamma_q\right)=1-\overline{\overline{D}}\left(\Gamma_g,\Gamma_q\right)$ is the support degree for Γ_g from Γ_q, which satisfy the following conditions: (1) $Sup\left(\Gamma_g,\Gamma_q\right)\in[0,1]$; (2) $Sup\left(\Gamma_g,\Gamma_q\right)=Supp\left(\Gamma_g,\Gamma_q\right)$; (3) $Sup\left(\Gamma_g,\Gamma_q\right)\geq Sup\left(\Gamma_r,\Gamma_s\right)$, if $\overline{\overline{D}}\left(\Gamma_g,\Gamma_q\right)<\overline{\overline{D}}\left(\Gamma_r,\Gamma_s\right)$, in which $\overline{\overline{D}}\left(\Gamma_g,\Gamma_q\right)$ is the distance measure between 2-TLNNs Γ_g and Γ_q defined in Definition (5).

In order, to represent Equation (25) in a simple form, we assume that

$$\aleph_g=\frac{\left(1+T\left(\Gamma_g\right)\right)}{\sum_{r=1}^{p}\left(1+T\left(\Gamma_r\right)\right)} \tag{26}$$

Therefore, Equation (25) takes the form

$$2-TLNDPGHM^{x,y}\left(\Gamma_1,\Gamma_2,...,\Gamma_p\right) = \frac{1}{x+y}\left(\prod_{g=1}^{p}\prod_{q=g}^{p}\left(x\left(\Gamma_g\right)^{p\aleph_g}\oplus y\left(\Gamma_q\right)^{p\aleph_q}\right)\right)^{\frac{2}{p^2+p}}. \tag{27}$$

Theorem 5. Let $x, y \geq 0,$ and x, y do not take the value 0 at the same time, $\Gamma_g (g=1,2,...,p)$ be a group of 2-TLNNs and let $\frac{\Delta^{-1}\left(s_{t_g},\Xi_g\right)}{h}=\overline{\overline{t}}_g, \frac{\Delta^{-1}\left(s_{i_g},\Psi_g\right)}{h}=\overline{\overline{i}}_g, \frac{\Delta^{-1}\left(s_{f_g},\Upsilon_g\right)}{h}=\overline{\overline{f}}_g$. Then, the aggregated value utilizing Equation (25), is still a 2-TLNN, and

$$2-TLNDWPGHM\left(\Gamma_1,\Gamma_2,...,\Gamma_p\right)$$

$$=\left\langle \Delta\left(h\left(1-1\middle/\left(1+\left(\frac{p^2+p}{2(x+y)}\times 1\middle/\left(\sum_{\substack{g=1\\q=g}}^{p} 1\middle/\left(\frac{x}{p\aleph_g\left(\frac{1-\overline{\overline{t}}_g}{\overline{\overline{t}}_g}\right)^{\Im}}+\frac{y}{p\aleph_q\left(\frac{1-\overline{\overline{t}}_q}{\overline{\overline{t}}_q}\right)^{\Im}}\right)\right)\right)^{\frac{1}{\Im}}\right)\right)\right),\right.$$

$$\Delta\left(h\left(1\middle/\left(1+\left(\frac{p^2+p}{2(x+y)}\times 1\middle/\left(\sum_{\substack{g=1,\\q=q}}^{p} 1\middle/\left(\frac{x}{p\aleph_g\left(\frac{\overline{\overline{i}}_g}{1-\overline{\overline{i}}_g}\right)^{\Im}}+\frac{y}{p\aleph_q\left(\frac{\overline{\overline{i}}_q}{1-\overline{\overline{i}}_q}\right)^{\Im}}\right)\right)\right)^{\frac{1}{\Im}}\right)\right)\right),$$

$$\left.\Delta\left(h\left(1\middle/\left(1+\left(\frac{p^2+p}{2(x+y)}\times 1\middle/\left(\sum_{\substack{g=1,\\q=q}}^{p} 1\middle/\left(\frac{x}{p\aleph_g\left(\frac{\overline{\overline{f}}_g}{1-\overline{\overline{f}}_g}\right)^{\Im}}+\frac{y}{p\aleph_q\left(\frac{\overline{\overline{f}}_q}{1-\overline{\overline{f}}_q}\right)^{\Im}}\right)\right)\right)^{\frac{1}{\Im}}\right)\right)\right)\right\rangle. \tag{28}$$

Proof. According to operational laws, we have

$$\Gamma_g^{p\aleph_g}=\left\langle \Delta\left(h\left(1\middle/1+\left(p\aleph_g\left(\frac{1-\overline{\overline{t}}_g}{\overline{\overline{t}}_g}\right)^{\Im}\right)^{\frac{1}{\Im}}\right)\right),\right.$$

$$\left.\Delta\left(h\left(1-1\middle/1+\left(p\aleph_g\left(\frac{\overline{\overline{i}}_g}{1-\overline{\overline{i}}_g}\right)^{\Im}\right)^{\frac{1}{\Im}}\right)\right),\Delta\left(h\left(1-1\middle/1+\left(p\aleph_g\left(\frac{\overline{\overline{f}}_g}{1-\overline{\overline{f}}_g}\right)^{\Im}\right)^{\frac{1}{\Im}}\right)\right)\right\rangle,$$

and

$$\Gamma_q^{p\aleph_q}=\left\langle \Delta\left(h\left(1\middle/1+\left(p\aleph_q\left(\frac{1-\overline{\overline{t}}_q}{\overline{\overline{t}}_q}\right)^{\Im}\right)^{\frac{1}{\Im}}\right)\right),\right.$$

$$\left.\Delta\left(h\left(1-1\middle/1+\left(p\aleph_q\left(\frac{\overline{\overline{i}}_q}{1-\overline{\overline{i}}_q}\right)^{\Im}\right)^{\frac{1}{\Im}}\right)\right),\Delta\left(h\left(1-1\middle/1+\left(p\aleph_q\left(\frac{\overline{\overline{f}}_q}{1-\overline{\overline{f}}_q}\right)^{\Im}\right)^{\frac{1}{\Im}}\right)\right)\right\rangle,$$

Let

$$\overline{\overline{a}}_g=\frac{1-\overline{\overline{t}}_g}{\overline{\overline{t}}_g},\overline{\overline{a}}_q=\frac{1-\overline{\overline{t}}_q}{\overline{\overline{t}}_q},\overline{\overline{b}}_g=\frac{\overline{\overline{i}}_g}{1-\overline{\overline{i}}_g},\overline{\overline{b}}_q=\frac{\overline{\overline{i}}_q}{1-\overline{\overline{i}}_q},\overline{\overline{c}}_g=\frac{\overline{\overline{f}}_g}{1-\overline{\overline{f}}_g},\overline{\overline{c}}_q=\frac{\overline{\overline{f}}_q}{1-\overline{\overline{f}}_q}.$$

Then, we can obtain

$$\Gamma_g^{p\aleph_g}=\left\langle \Delta\left(h\left(1\middle/1+\left(p\aleph_g\right)^{\frac{1}{\Im}}\overline{\overline{a}}_g\right)\right),\right.$$

$$\left.\Delta\left(h\left(1-1\middle/1+\left(p\aleph_g\right)^{\frac{1}{\Im}}\overline{\overline{b}}_g\right)\right),\Delta\left(h\left(1-1\middle/1+\left(p\aleph_g\right)^{\frac{1}{\Im}}\overline{\overline{c}}_g\right)\right)\right\rangle,$$

$$\Gamma_q^{p\aleph_q}=\left\langle \Delta\left(h\left(1\middle/1+\left(p\aleph_q\right)^{\frac{1}{\Im}}\overline{\overline{a}}_q\right)\right),\right.$$

$$\left.\Delta\left(h\left(1-1\middle/1+\left(p\aleph_q\right)^{\frac{1}{\Im}}\overline{\overline{b}}_q\right)\right),\Delta\left(h\left(1-1\middle/1+\left(p\aleph_q\right)^{\frac{1}{\Im}}\overline{\overline{c}}_q\right)\right)\right\rangle,$$

and

$$x\Gamma_g^{p\aleph_g}=\left\langle \Delta\left(h\left(1-1\middle/1+x^{\frac{1}{\Im}}\middle/\left(p\aleph_g\right)^{\frac{1}{\Im}}\overline{\overline{a}}_g\right)\right),\right.$$

$$\left.\Delta\left(h\left(1\middle/1+x^{\frac{1}{\Im}}\middle/\left(p\aleph_g\right)^{\frac{1}{\Im}}\overline{\overline{b}}_g\right)\right),\Delta\left(h\left(1\middle/1+x^{\frac{1}{\Im}}\middle/\left(p\aleph_g\right)^{\frac{1}{\Im}}\overline{\overline{c}}_g\right)\right)\right\rangle,$$

$$y\Gamma_q^{p\aleph_q} = \left\langle \Delta\left(h\left(1-1\Big/1+y^{\frac{1}{\mathfrak{Z}}}\Big/\left(p\aleph_q\right)^{\frac{1}{\mathfrak{Z}}}\bar{\bar{a}}_q\right)\right),\right.$$
$$\left.\Delta\left(h\left(1\Big/1+y^{\frac{1}{\mathfrak{Z}}}\Big/\left(p\aleph_q\right)^{\frac{1}{\mathfrak{Z}}}\bar{\bar{b}}_q\right)\right), \Delta\left(h\left(1\Big/1+y^{\frac{1}{\mathfrak{Z}}}\Big/\left(p\aleph_q\right)^{\frac{1}{\mathfrak{Z}}}\bar{\bar{c}}_q\right)\right)\right\rangle,$$

Furthermore, we can have

$$x\Gamma_g^{p\aleph_g}\oplus_D x\Gamma_q^{p\aleph_q} = \left\langle \Delta\left(h\left(1-1\Big/1+\left(\frac{x}{p\aleph_g\bar{\bar{a}}_g^{\mathfrak{Z}}}+\frac{y}{p\aleph_q\bar{\bar{a}}_q^{\mathfrak{Z}}}\right)^{\frac{1}{\mathfrak{Z}}}\right)\right),\right.$$
$$\Delta\left(h\left(1\Big/1+\left(\frac{x}{p\aleph_g\bar{\bar{b}}_g^{\mathfrak{Z}}}+\frac{y}{p\aleph_q\bar{\bar{b}}_q^{\mathfrak{Z}}}\right)^{\frac{1}{\mathfrak{Z}}}\right)\right),$$
$$\left.\Delta\left(h\left(1\Big/1+\left(\frac{x}{p\aleph_g\bar{\bar{c}}_g^{\mathfrak{Z}}}+\frac{y}{p\aleph_q\bar{\bar{c}}_q^{\mathfrak{Z}}}\right)^{\frac{1}{\mathfrak{Z}}}\right)\right)\right\rangle,$$

and

$$\prod_{\substack{g=1,\\q=g}}^{p} x\Gamma_g^{p\aleph_g}\oplus_D y\Gamma_q^{p\aleph_q}$$
$$=\left\langle \Delta\left(h\left(1\Big/\left(1+\sum_{\substack{g=1,\\q=g}}^{p}\left(\left(1-1+\frac{1}{1+\left(\frac{x}{p\aleph_g\bar{\bar{a}}_g^{\mathfrak{Z}}}+\frac{y}{p\aleph_q\bar{\bar{a}}_q^{\mathfrak{Z}}}\right)^{\frac{1}{\mathfrak{Z}}}}\right)\Big/\left(1-\frac{1}{1+\left(\frac{x}{p\aleph_g\bar{\bar{a}}_g^{\mathfrak{Z}}}+\frac{y}{p\aleph_q\bar{\bar{a}}_q^{\mathfrak{Z}}}\right)^{\frac{1}{\mathfrak{Z}}}}\right)\right)^{\mathfrak{Z}}\right)^{\frac{1}{\mathfrak{Z}}}\right)\right),\right.$$
$$\Delta\left(h\left(1-1\Big/\left(1+\sum_{\substack{g=1,\\q=g}}^{p}\left(\left(\frac{1}{1+\left(\frac{x}{p\aleph_g\bar{\bar{b}}_g^{\mathfrak{Z}}}+\frac{y}{p\aleph_q\bar{\bar{b}}_q^{\mathfrak{Z}}}\right)^{\frac{1}{\mathfrak{Z}}}}\right)\Big/\left(1-\frac{1}{1+\left(\frac{x}{p\aleph_g\bar{\bar{b}}_g^{\mathfrak{Z}}}+\frac{y}{p\aleph_q\bar{\bar{b}}_q^{\mathfrak{Z}}}\right)^{\frac{1}{\mathfrak{Z}}}}\right)\right)^{\mathfrak{Z}}\right)^{\frac{1}{\mathfrak{Z}}}\right)\right),$$
$$\left.\Delta\left(h\left(1-1\Big/\left(1+\sum_{\substack{g=1,\\q=g}}^{p}\left(\left(\frac{1}{1+\left(\frac{x}{p\aleph_g\bar{\bar{c}}_g^{\mathfrak{Z}}}+\frac{y}{p\aleph_q\bar{\bar{c}}_q^{\mathfrak{Z}}}\right)^{\frac{1}{\mathfrak{Z}}}}\right)\Big/\left(1-\frac{1}{1+\left(\frac{x}{p\aleph_g\bar{\bar{c}}_g^{\mathfrak{Z}}}+\frac{y}{p\aleph_q\bar{\bar{c}}_q^{\mathfrak{Z}}}\right)^{\frac{1}{\mathfrak{Z}}}}\right)\right)^{\mathfrak{Z}}\right)^{\frac{1}{\mathfrak{Z}}}\right)\right)\right\rangle,$$

$$=\left\langle \Delta\left(h\left(1\Big/\left(1+\left(\sum_{\substack{g=1,\\q=g}}^{p}1\Big/\left(\frac{x}{p\aleph_g\bar{\bar{a}}_g^{\mathfrak{Z}}}+\frac{y}{p\aleph_q\bar{\bar{a}}_q^{\mathfrak{Z}}}\right)\right)^{\frac{1}{\mathfrak{Z}}}\right)\right)\right),\right.$$
$$\Delta\left(h\left(1-1\Big/\left(1+\left(\sum_{\substack{g=1,\\q=g}}^{p}1\Big/\left(\frac{x}{p\aleph_g\bar{\bar{b}}_g^{\mathfrak{Z}}}+\frac{y}{p\aleph_q\bar{\bar{b}}_q^{\mathfrak{Z}}}\right)\right)^{\frac{1}{\mathfrak{Z}}}\right)\right)\right),$$
$$\left.\Delta\left(h\left(1-1\Big/\left(1+\left(\sum_{\substack{g=1,\\q=g}}^{p}1\Big/\left(\frac{x}{p\aleph_g\bar{\bar{c}}_g^{\mathfrak{Z}}}+\frac{y}{p\aleph_q\bar{\bar{c}}_q^{\mathfrak{Z}}}\right)\right)^{\frac{1}{\mathfrak{Z}}}\right)\right)\right)\right\rangle.$$

So, we can have

$$\left(\prod_{\substack{g=1,\\q=g}}^{p} x\Gamma_g^{p\aleph_g}\oplus_D y\Gamma_q^{p\aleph_q}\right)^{\frac{2}{p^2+p}}$$
$$=\left\langle \Delta\left(h\left(1\Big/\left(1+\left(\frac{2}{p^2+p}\left(\left(1-1\Big/\left(1+\left(\sum_{\substack{g=1,\\q=g}}^{p}1\Big/\left(\frac{x}{p\aleph_g\bar{\bar{a}}_g^{\mathfrak{Z}}}+\frac{y}{p\aleph_q\bar{\bar{a}}_q^{\mathfrak{Z}}}\right)\right)^{\frac{1}{\mathfrak{Z}}}\right)\right)\Big/\left(1\Big/1+\left(\sum_{\substack{g=1,\\q=g}}^{p}1\Big/\left(\frac{x}{p\aleph_g\bar{\bar{a}}_g^{\mathfrak{Z}}}+\frac{y}{p\aleph_q\bar{\bar{a}}_q^{\mathfrak{Z}}}\right)\right)^{\frac{1}{\mathfrak{Z}}}\right)\right)^{\mathfrak{Z}}\right)^{\frac{1}{\mathfrak{Z}}}\right)\right)\right),\right.$$
$$\Delta\left(h\left(1-1\Big/\left(1+\left(\frac{2}{p^2+p}\left(\left(1-1\Big/\left(1+\left(\sum_{\substack{g=1,\\q=g}}^{p}1\Big/\left(\frac{x}{p\aleph_g\bar{\bar{b}}_g^{\mathfrak{Z}}}+\frac{y}{p\aleph_q\bar{\bar{b}}_q^{\mathfrak{Z}}}\right)\right)^{\frac{1}{\mathfrak{Z}}}\right)\right)\Big/\left(1-1+1\Big/1+\left(\sum_{\substack{g=1,\\q=g}}^{p}1\Big/\left(\frac{x}{p\aleph_g\bar{\bar{b}}_g^{\mathfrak{Z}}}+\frac{y}{p\aleph_q\bar{\bar{b}}_q^{\mathfrak{Z}}}\right)\right)^{\frac{1}{\mathfrak{Z}}}\right)\right)^{\mathfrak{Z}}\right)^{\frac{1}{\mathfrak{Z}}}\right)\right)\right),$$
$$\left.\Delta\left(h\left(1-1\Big/\left(1+\left(\frac{2}{p^2+p}\left(\left(1-1\Big/\left(1+\left(\sum_{\substack{g=1,\\q=g}}^{p}1\Big/\left(\frac{x}{p\aleph_g\bar{\bar{c}}_g^{\mathfrak{Z}}}+\frac{y}{p\aleph_q\bar{\bar{c}}_q^{\mathfrak{Z}}}\right)\right)^{\frac{1}{\mathfrak{Z}}}\right)\right)\Big/\left(1-1+1\Big/1+\left(\sum_{\substack{g=1,\\q=g}}^{p}1\Big/\left(\frac{x}{p\aleph_g\bar{\bar{c}}_g^{\mathfrak{Z}}}+\frac{y}{p\aleph_q\bar{\bar{c}}_q^{\mathfrak{Z}}}\right)\right)^{\frac{1}{\mathfrak{Z}}}\right)\right)^{\mathfrak{Z}}\right)^{\frac{1}{\mathfrak{Z}}}\right)\right)\right)\right\rangle,$$

$$=\left\langle \Delta\left(h\left(1\Big/\left(1+\left(\frac{2}{p^2+p}\sum_{\substack{g=1,\\q=g}}^{p}1\Big/\left(\frac{x}{p\aleph_g\bar{\bar{a}}_g^{\mathfrak{Z}}}+\frac{y}{p\aleph_q\bar{\bar{a}}_q^{\mathfrak{Z}}}\right)\right)^{\frac{1}{\mathfrak{Z}}}\right)\right)\right),\right.$$
$$\Delta\left(h\left(1-1\Big/\left(1+\left(\frac{2}{p^2+p}\sum_{\substack{g=1,\\q=g}}^{p}1\Big/\left(\frac{x}{p\aleph_g\bar{\bar{b}}_g^{\mathfrak{Z}}}+\frac{y}{p\aleph_q\bar{\bar{b}}_q^{\mathfrak{Z}}}\right)\right)^{\frac{1}{\mathfrak{Z}}}\right)\right)\right),$$
$$\left.\Delta\left(h\left(1-1\Big/\left(1+\left(\frac{2}{p^2+p}\sum_{\substack{g=1,\\q=g}}^{p}1\Big/\left(\frac{x}{p\aleph_g\bar{\bar{c}}_g^{\mathfrak{Z}}}+\frac{y}{p\aleph_q\bar{\bar{c}}_q^{\mathfrak{Z}}}\right)\right)^{\frac{1}{\mathfrak{Z}}}\right)\right)\right)\right\rangle.$$

Then

$$\frac{1}{x+y}\left(\prod_{\substack{g=1,\\q=g}}^{p} x\Gamma_g^{p\aleph_g}\oplus_D y\Gamma_q^{p\aleph_q}\right)^{\frac{2}{p^2+p}}$$
$$=\left\langle \Delta\left(h\left(1-1\Big/\left(1+\left(\frac{p^2+p}{2(x+y)}\sum_{\substack{g=1,\\q=g}}^{p}1\Big/\left(\frac{x}{p\aleph_g\bar{\bar{a}}_g^{\mathfrak{Z}}}+\frac{y}{p\aleph_q\bar{\bar{a}}_q^{\mathfrak{Z}}}\right)\right)^{\frac{1}{\mathfrak{Z}}}\right)\right)\right),\right.$$
$$\Delta\left(h\left(1\Big/\left(1+\left(\frac{p^2+p}{2(x+y)}\sum_{\substack{g=1,\\q=g}}^{p}1\Big/\left(\frac{x}{p\aleph_g\bar{\bar{b}}_g^{\mathfrak{Z}}}+\frac{y}{p\aleph_q\bar{\bar{b}}_q^{\mathfrak{Z}}}\right)\right)^{\frac{1}{\mathfrak{Z}}}\right)\right)\right), \quad (29)$$
$$\left.\Delta\left(h\left(1\Big/\left(1+\left(\frac{p^2+p}{2(x+y)}\sum_{\substack{g=1,\\q=g}}^{p}1\Big/\left(\frac{x}{p\aleph_g\bar{\bar{c}}_g^{\mathfrak{Z}}}+\frac{y}{p\aleph_q\bar{\bar{c}}_q^{\mathfrak{Z}}}\right)\right)^{\frac{1}{\mathfrak{Z}}}\right)\right)\right)\right\rangle.$$

Now put $\bar{\bar{a}}_g=\frac{1-\bar{\bar{t}}_g}{\bar{\bar{t}}_g}, \bar{\bar{a}}_q=\frac{1-\bar{\bar{t}}_q}{\bar{\bar{t}}_q}, \bar{\bar{b}}_g=\frac{\bar{\bar{i}}_g}{1-\bar{\bar{i}}_g}, \bar{\bar{b}}_q=\frac{\bar{\bar{i}}_q}{1-\bar{\bar{i}}_q}, \bar{\bar{c}}_g=\frac{\bar{\bar{f}}_g}{1-\bar{\bar{f}}_g}, \bar{\bar{c}}_q=\frac{\bar{\bar{f}}_q}{1-\bar{\bar{f}}_q}$ in Equation (29), we can have

$$=\left\langle \Delta\left(h\left(1-1\Big/\left(1+\left(\frac{p^2+p}{2(x+y)}\sum_{\substack{g=1,\\q=g}}^{p}1\Big/\left(\frac{x}{p\aleph_g\left(\frac{1-\bar{\bar{t}}_g}{\bar{\bar{t}}_g}\right)^{\Im}}+\frac{y}{p\aleph_q\left(\frac{1-\bar{\bar{t}}_q}{\bar{\bar{t}}_q}\right)^{\Im}}\right)\right)^{\frac{1}{\Im}}\right)\right)\right),\right.$$

$$\Delta\left(h\left(1\Big/\left(1+\left(\frac{p^2+p}{2(x+y)}\sum_{\substack{g=1,\\q=g}}^{p}1\Big/\left(\frac{x}{p\aleph_g\left(\frac{\bar{\bar{i}}_g}{1-\bar{\bar{i}}_g}\right)^{\Im}}+\frac{y}{p\aleph_q\left(\frac{\bar{\bar{i}}_q}{1-\bar{\bar{i}}_q}\right)^{\Im}}\right)\right)^{\frac{1}{\Im}}\right)\right)\right),$$

$$\left.\Delta\left(h\left(1\Big/\left(1+\left(\frac{p^2+p}{2(x+y)}\sum_{\substack{g=1,\\q=g}}^{p}1\Big/\left(\frac{x}{p\aleph_g\left(\frac{\bar{\bar{f}}_g}{1-\bar{\bar{f}}_g}\right)^{\Im}}+\frac{y}{p\aleph_q\left(\frac{\bar{\bar{f}}_q}{1-\bar{\bar{f}}_q}\right)^{\Im}}\right)\right)^{\frac{1}{\Im}}\right)\right)\right)\right\rangle.$$

This completes the proof of Theorem.

Theorem 6 (Idempotency). Let $\Gamma_g(g=1,2,...,p)$ be a group of 2-TLNNs, if all $\Gamma_g(g=1,2,...,p)$ are same, that is $\Gamma_g=\Gamma=\langle(s_t,\Xi),(s_i,\Psi),(s_f,\Upsilon)\rangle(g=1,2,...,p)$. Assume that $\frac{\Delta^{-1}(s_{t_g},\Xi_g)}{h}=\bar{\bar{t}}_g,\frac{\Delta^{-1}(s_{i_g},\Psi_g)}{h}=\bar{\bar{i}}_g,\frac{\Delta^{-1}(s_{f_g},\Upsilon_g)}{h}=\bar{\bar{f}}_g$, then

$$2-TLNPGHM(\Gamma_1,\Gamma_2,...,\Gamma_p)=\Gamma. \quad (30)$$

Theorem 7 (Boundedness). Let $\Gamma_g(g=1,2,...,p)$ be a group of 2-TLNNs. If $\bar{\bar{m}}^-=\langle\min_g(s_{t_g},\Xi_g),\max_g(s_{i_g},\Psi_g),\max_g(s_{f_g},\Upsilon_g)\rangle$ and $\bar{\bar{m}}^+=\langle\max_g(s_{t_g},\Xi_g),\min_g(s_{i_g},\Psi_g),\min_g(s_{f_g},\Upsilon_g)\rangle$, then

$$\bar{\bar{m}}^-\le 2-TLNPHM^{x,y}(\Gamma_1,\Gamma_2,...,\Gamma_p)\le\bar{\bar{m}}^+. \quad (31)$$

By specifying different values to the parameters x and y, some particular cases of the 2-TLNDPGHM operator are described below:

(1) If $y\to 0,\Im>0$, then we can have

$$2-TLNDPGHM^{x,0}(\Gamma_1,\Gamma_2,...,\Gamma_p)=\frac{1}{x+y}\left(\prod_{g=1}^{p}\prod_{q=g}^{p}\left(x(\Gamma_g)^{\frac{p(1+T(\Gamma_g))}{\sum_{r=1}^{p}(1+T(\Gamma_r))}}\oplus y(\Gamma_q)^{\frac{p(1+T(\Gamma_q))}{\sum_{r=1}^{p}(1+T(\Gamma_r))}}\right)\right)^{\frac{2}{p^2+p}}$$

$$=\frac{1}{x}\left(\prod_{g=1}^{p}\left(x(\Gamma_g)^{\frac{p(1+T(\Gamma_g))}{\sum_{r=1}^{p}(1+T(\Gamma_r))}}\right)^{(p+1-g)}\right)^{\frac{2}{p^2+p}}.$$

That is, the 2-TLNDPGHM operator degenerates into the 2-tuple linguistic neutrosophic Dombi descending power geometric average operator.

(2) If $x\to 0,\Im>0$, then we can have

$$2-TLNDPGHM^{0,y}(\Gamma_1,\Gamma_2,...,\Gamma_p)=\frac{1}{x+y}\left(\prod_{g=1}^{p}\prod_{q=g}^{p}\left(x(\Gamma_g)^{\frac{p(1+T(\Gamma_g))}{\sum_{r=1}^{p}(1+T(\Gamma_r))}}\oplus y(\Gamma_q)^{\frac{p(1+T(\Gamma_q))}{\sum_{r=1}^{p}(1+T(\Gamma_r))}}\right)\right)^{\frac{2}{p^2+p}}$$

$$=\frac{1}{y}\left(\prod_{g=1}^{p}\left(y(\Gamma_g)^{\frac{p(1+T(\Gamma_g))}{\sum_{r=1}^{p}(1+T(\Gamma_r))}}\right)^{(g)}\right)^{\frac{2}{p^2+p}}.$$

That is, the 2-TLNDPGHM operator degenerates into the 2-tuple linguistic neutrosophic Dombi descending power geometric average operator.

(3) If $y\to 0,\Im>0,$ and $Sup(\Gamma_g,\Gamma_q)=\beta(\beta\in[0,1])$ for all $g\ne q.$ Then, we can have

$$2-TLNDPGHM^{x,0}(\Gamma_1,\Gamma_2,...,\Gamma_p)=\frac{1}{x+y}\left(\prod_{g=1}^{p}\prod_{q=g}^{p}\left(x(\Gamma_g)^{\frac{p(1+T(\Gamma_g))}{\sum_{r=1}^{p}(1+T(\Gamma_r))}}\oplus y(\Gamma_q)^{\frac{p(1+T(\Gamma_q))}{\sum_{r=1}^{p}(1+T(\Gamma_r))}}\right)\right)^{\frac{2}{p^2+p}}$$

$$=\frac{1}{x}\left(\prod_{g=1}^{p}(x\Gamma_g)^{(p+1-g)}\right)^{\frac{2}{p^2+p}}.$$

That is, the 2-TLNDPGHM operator degenerates into 2-tuple linguistic neutrosophic Dombi descending geometric average operator.

(4) If $x\to 0,\Im>0,$ and $Sup(\Gamma_g,\Gamma_q)=\beta(\beta\in[0,1])$ for all $g\ne q.$ Then, we can have

$$2-TLNDPGHM^{x,0}(\Gamma_1,\Gamma_2,...,\Gamma_p)=\frac{1}{x+y}\left(\prod_{g=1}^{p}\prod_{q=g}^{p}\left(x(\Gamma_g)^{\frac{p(1+T(\Gamma_g))}{\sum_{r=1}^{p}(1+T(\Gamma_r))}}\oplus y(\Gamma_q)^{\frac{p(1+T(\Gamma_q))}{\sum_{r=1}^{p}(1+T(\Gamma_r))}}\right)\right)^{\frac{2}{p^2+p}}$$

$$=\frac{1}{y}\left(\prod_{g=1}^{p}(y\Gamma_g)^{(g)}\right)^{\frac{2}{p^2+p}}.$$

That is, the 2-TLNDPGHM operator degenerates into 2-tuple linguistic neutrosophic Dombi ascending geometric average operator.

Similar to 2-TLNDPHM operator, the 2-TLNDPGHM operator have only power weight vector and the correlation between input arguments are taken under consideration and are not to consider the weight vector of the input arguments. Therefore, to remove this deficiency, we will propose its weighted form, that is 2-TPLNDWPGHM operator.

Definition 13. Let $\Gamma_g\,(g=1,2,...,p)$ be a group of 2-TLNNs, $x,y\geq 0$. Then, the 2-TLNNDWPGHM operator is described as follows:

$$2-TLNDWPGHM^{x,y}\left(\Gamma_1,\Gamma_2,...,\Gamma_p\right)=\frac{1}{x+y}\left(\prod_{g=1}^{p}\prod_{q=g}^{p}\left(x\left(\Gamma_g\right)^{\frac{p\overline{\overline{w}}_g\left(1+T\left(\Gamma_g\right)\right)}{\sum_{r=1}^{p}\overline{\overline{w}}_r\left(1+T\left(\Gamma_r\right)\right)}}\oplus y\left(\Gamma_q\right)^{\frac{p\overline{\overline{w}}_g\left(1+T\left(\Gamma_q\right)\right)}{\sum_{r=1}^{p}\overline{\overline{w}}_r\left(1+T\left(\Gamma_r\right)\right)}}\right)\right)^{\frac{2}{p^2+p}}. \quad (32)$$

Where $T\left(\Gamma_g\right)=\sum_{q=1,g\neq q}^{p}Sup\left(\Gamma_g,\Gamma_q\right), Sup\left(\Gamma_g,\Gamma_q\right)=1-\overline{\overline{D}}\left(\Gamma_g,\Gamma_q\right)$ is the support degree for Γ_g from Γ_q, which satisfy the following conditions: (1) $Sup\left(\Gamma_g,\Gamma_q\right)\in[0,1]$; (2) $Sup\left(\Gamma_g,\Gamma_q\right)=Supp\left(\Gamma_g,\Gamma_q\right)$; (3) $Sup\left(\Gamma_g,\Gamma_q\right)\geq Sup\left(\Gamma_r,\Gamma_s\right)$, if $\overline{\overline{D}}\left(\Gamma_g,\Gamma_q\right)<\overline{\overline{D}}\left(\Gamma_r,\Gamma_s\right)$, in which $\overline{\overline{D}}\left(\Gamma_g,\Gamma_q\right)$ is the distance measure between 2-TLNNs Γ_g and Γ_q defined in Definition (5).

In order, to represent Equation (32) in a simple form, we assume that

$$\Theta_g=\frac{\overline{\overline{w}}_g\left(1+T\left(\Gamma_g\right)\right)}{\sum_{r=1}^{p}\overline{\overline{w}}_r\left(1+T\left(\Gamma_r\right)\right)} \quad (33)$$

Therefore, Equation (32) takes the form

$$2-TLNDWPGHM^{x,y}\left(\Gamma_1,\Gamma_2,...,\Gamma_p\right)=\frac{1}{x+y}\left(\prod_{g=1}^{p}\prod_{q=g}^{p}\left(x\left(\Gamma_g\right)^{p\Theta_g}\oplus y\left(\Gamma_q\right)^{p\Theta_q}\right)\right)^{\frac{2}{p^2+p}}. \quad (34)$$

Theorem 8. Let $x,y\geq 0$, and x,y do not take the value 0 at the same time, $\Gamma_g(g=1,2,...,p)$ be a group of 2-TLNNs and let $\frac{\Delta^{-1}\left(s_{t_g},\Xi_g\right)}{h}=\overline{\overline{t}}_g, \frac{\Delta^{-1}\left(s_{i_g},\Psi_g\right)}{h}=\overline{\overline{i}}_g, \frac{\Delta^{-1}\left(s_{f_g},\Upsilon_g\right)}{h}=\overline{\overline{f}}_g$. Then, the aggregated value utilizing Equation (32), is still a 2-TLNN, and

$$2-TLNDWPGHM\left(\Gamma_1,\Gamma_2,...,\Gamma_p\right)=$$

$$\left\langle \Delta\left(h\left(1-1\middle/\left(1+\left(\frac{p^2+p}{2(x+y)}\times 1\middle/\left(\sum_{\substack{g=1\\q=g}}^{p}1\middle/\left(\frac{x}{p\Theta_g\left(\frac{1-\overline{\overline{t}}_g}{\overline{\overline{t}}_g}\right)^{\Im}}+\frac{y}{p\Theta_q\left(\frac{1-\overline{\overline{t}}_q}{\overline{\overline{t}}_q}\right)^{\Im}}\right)\right)\right)^{\frac{1}{\Im}}\right)\right)\right),\right.$$

$$\Delta\left(h\left(1\middle/\left(1+\left(\frac{p^2+p}{2(x+y)}\times 1\middle/\left(\sum_{\substack{g=1,\\q=q}}^{p}1\middle/\left(\frac{x}{p\Theta_g\left(\frac{\overline{\overline{i}}_g}{1-\overline{\overline{i}}_g}\right)^{\Im}}+\frac{y}{p\Theta_q\left(\frac{\overline{\overline{i}}_q}{1-\overline{\overline{i}}_q}\right)^{\Im}}\right)\right)\right)^{\frac{1}{\Im}}\right)\right)\right),$$

$$\left.\Delta\left(h\left(1\middle/\left(1+\left(\frac{p^2+p}{2(x+y)}\times 1\middle/\left(\sum_{\substack{g=1,\\q=q}}^{p}1\middle/\left(\frac{x}{p\Theta_g\left(\frac{\overline{\overline{f}}_g}{1-\overline{\overline{f}}_g}\right)^{\Im}}+\frac{y}{p\Theta_q\left(\frac{\overline{\overline{f}}_q}{1-\overline{\overline{f}}_q}\right)^{\Im}}\right)\right)\right)^{\frac{1}{\Im}}\right)\right)\right)\right\rangle. \quad (35)$$

Similar to 2-TLNDWPHM, the 2-TLNDWPGHM operator has only the property of boundedness and does not have the properties of idempotency and monotonicity.

V. An application of 2-TLNDWPHM and 2-TLNDWPGHM operator to group decision making

In this section, we pertains the afore-presented Dombi power Heronian aggregation operators to establish constructive approach for MAGDM under 2-TLNN environments. Let $\overline{\overline{AT}}=\left\{\overline{\overline{AT}}_1,\overline{\overline{AT}}_2,...,\overline{\overline{AT}}_m\right\}$ be the set of discrete alternatives, the set of attributes is expressed by $\overline{\overline{CT}}=\left\{\overline{\overline{CT}}_1,\overline{\overline{CT}}_2,...,\overline{\overline{CT}}_n\right\}$, the weight vector of the attributes is represented by $\overline{\overline{W}}=\left(\overline{\overline{w}}_1,\overline{\overline{w}}_2,...,\overline{\overline{w}}_n\right)^T$ such that $\overline{\overline{w}}_e\in[0,1],\sum_{e=1}^{n}\overline{\overline{w}}_e=1$, and $\overline{\overline{DE}}=\left(\overline{\overline{de}}_1,\overline{\overline{de}}_2,...,\overline{\overline{de}}_a\right)$ denote the set of a decision makers, with weight vector expressed by $\varpi=\left(\varpi_1,\varpi_2,...,\varpi_a\right)^T$ such that $\varpi_b\in[0,1],\sum_{b=1}^{a}\varpi_b=1$. Assume that $\overline{\overline{DT}}^b=\left(\Gamma_{ce}^b\right)_{m\times n}$ is the decision matrix, where $\Gamma_{ce}^b=\left\langle\left(s_{t_{ce}^b},\Xi_{ce}^b\right),\left(s_{i_{ce}^b},\Psi_{ce}^b\right),\left(s_{f_{ce}^b},\Upsilon_{ce}^b\right)\right\rangle$ takes the form of 2-TLNN, given by decision maker $\overline{\overline{de}}_b$ for alternative $\overline{\overline{AT}}_c$ with respect to the attribute $\overline{\overline{CT}}_e$.

Then, depending on real decision situations where the weight vector of both attributes and decision makers are completely known in advance. Therefore, in the following we present a MAGDM approach based on the developed 2-TLNDWPHM and 2-TLNDWPGHM operators. To do so, just follow the step below:

Step 1. Calculate the support degrees by the following formula:

$$Sup\left(\Gamma_{ce}^b,\Gamma_{ce}^l\right)=1-\overline{\overline{D}}_H\left(\Gamma_{ce}^b,\Gamma_{ce}^l\right),(b,l=1,2,...,a;c=1,2,...,m;e=1,2,...,n). \quad (36)$$

Which satisfy the axioms for support functions, $\overline{\overline{D}}_H\left(\Gamma_{ce}^b,\Gamma_{ce}^l\right)$ is the distance measure given in Definition (5).

Step 2. Determine the support degree $T\left(\Gamma_{ce}^b\right)$ that IFN Γ_{ce}^b receives from other 2-TLNNs $\Gamma_{ce}^l\left(l=1,2,...,a;l\neq b\right)$, where

$$T\left(\Gamma_{ce}^b\right)=\sum_{l=1,l\neq b}^{a}\sup\left(\Gamma_{ce}^b,\Gamma_{ce}^l\right). \quad (37)$$

Step 3. Utilize weights $\varpi_b\left(b=1,2,....,a\right)$ for decision makers $\overline{\overline{de}}_b$ to determine weights $\aleph_{ce}^b$ associated with the 2-TLNN Γ_{ce}^b,

$$\aleph_{ce}^b=\frac{\varpi_b\left(1+T\left(\Gamma_{ce}^b\right)\right)}{\sum_{b=1}^{a}\varpi_b\left(1+T\left(\Gamma_{ce}^b\right)\right)},(b=1,2,...,a). \quad (38)$$

Where $\aleph_{ce}^{b} \geq 0$ and $\sum_{b=1}^{a} \aleph_{ce}^{b} = 1$.

Step 4. Aggregate all the individual decision matrices $\overline{\overline{DT}}^{b} = \left(\Gamma_{ce}^{b}\right)_{m \times n} (b = 1,2,...,a)$ into group decision matrix $\widetilde{DT} = \left(\Gamma_{ce}\right)_{m \times n}$ by utilizing 2-TLNDWPHM or 2-TLNDWPGHM operators, where

$$\Gamma_{ce}^{b} = 2 - TLNDWPHM\left(\Gamma_{ce}^{1}, \Gamma_{ce}^{2}, ..., \Gamma_{ce}^{a}\right) \tag{39}$$

Or

$$\Gamma_{ce}^{b} = 2 - TLNDWPGHM\left(\Gamma_{ce}^{1}, \Gamma_{ce}^{2}, ..., \Gamma_{ce}^{a}\right) \tag{40}$$

Step 5. Determine support degrees $Sup\left(\Gamma_{ce}, \Gamma_{cx}\right)$ by the following formula;

$$Sup\left(\Gamma_{ce}, \Gamma_{cx}\right) = 1 - \overline{\overline{D}}_{H}\left(\Gamma_{ce}, \Gamma_{cx}\right); \quad (c = 1,2,...,m, e = 1,2,...,n, e \neq x) \tag{41}$$

where $\overline{\overline{D}}_{H}\left(\Gamma_{ce}, \Gamma_{cx}\right)$ is distance measure given in Definition(5).

Step 6. Determine the support degree $T\left(\Gamma_{ce}\right)$ that 2-TLNNs Γ_{ce} collects from other 2-TLNNs $\Gamma_{cx} (x = 1,2,...,n; e \neq x)$, where

$$T\left(\Gamma_{ce}\right) = \sum_{x=1, x \neq e}^{n} \overline{\overline{w}}_{x} Sup\left(\Gamma_{ce}, \Gamma_{cx}\right). \tag{42}$$

Step 7. Determine weighting vector $\Phi_{ce} (c = 1,2,....,m, e = 1,2,...,n)$ associated with Γ_{ce},

$$\Phi_{ce} = \frac{\overline{\overline{w}}_{e}\left(1 + T\left(\Gamma_{ce}\right)\right)}{\sum_{e=1}^{n} \overline{\overline{w}}_{e}\left(1 + T\left(\Gamma_{ce}\right)\right)}. \tag{43}$$

Step 8. Utilize 2-TLNDWPHM or 2-TLNNDWPGHM operators to aggregate all assessment values $\Gamma_{ce} (c = 1,2,....,m, e = 1,2,...,n)$ into overall assessment value $\Gamma_{c} (c = 1,2,....,m)$ corresponding to the alternatives $\overline{\overline{AL}}_{c} \, (c = 1,2,...,m)$:

$$\Gamma_{c} = 2 - TLNDWPHM\left(\Gamma_{c1}, \Gamma_{c2}, ..., \Gamma_{cn}\right) \tag{44}$$

Or

$$\Gamma_{c} = 2 - TLNDWPGHM\left(\Gamma_{c1}, \Gamma_{c2}, ..., \Gamma_{cn}\right) \tag{45}$$

Step 9. Determine the scores $\overline{\overline{SC}}\left(\overline{\overline{if}}_{d}\right)$ for the overall IFN of the alternatives $\overline{\overline{AL}}_{d} \, (d = 1,2,...,g)$ by utilizing Definition (3).

Step 10. Rank all alternatives $\overline{\overline{AL}}_{d} \, (d = 1,2,...,g)$ and select the optimal one (s) with the ranking order $\Gamma_{d} (d = 1,2,...,g)$.

A. Numerical Examples and Comparative analysis

The following example is adapted from [38], to show the validity and practicality of the developed aggregation operators.

Example 1. Let us assume that there are five potential construction engineering projects (alternatives) $\overline{\overline{AL}}_{b}\left(b = 1,2,...,5\right)$ to be assess. These five potential alternatives are assessed by decision makers with respect to the following four attributes (1) the construction work environment denoted by $\overline{\overline{CT}}_{1}$;
(2) the construction site safety protection measure denoted by $\overline{\overline{CT}}_{2}$; (3) The safety management ability of the engineering projects management denoted by $\overline{\overline{CT}}_{3}$ and (4) the safety production responsibility system denoted by $\overline{\overline{CT}}_{4}$, with weight vector $\left(0.5, 0.3, 0.1, 0.1\right)^{T}$ and expert weight vector is $\left(0.2, 0.5, 0.3\right)^{T}$. The experts provide information in the form of 2-TLNNs, which are listed in Tables 1-3.

Table.1 The 2-TLN decision matrix $\overline{\overline{DT}}^{1}$

	$\overline{\overline{CT}}_{1}$	$\overline{\overline{CT}}_{2}$	$\overline{\overline{CT}}_{3}$	$\overline{\overline{CT}}_{4}$
$\overline{\overline{AL}}_{1}$	$\left\langle (s_4,0),(s_3,0),(s_2,0) \right\rangle$	$\left\langle (s_5,0),(s_3,0),(s_1,0) \right\rangle$	$\left\langle (s_4,0),(s_1,0),(s_2,0) \right\rangle$	$\left\langle (s_2,0),(s_3,0),(s_2,0) \right\rangle$
$\overline{\overline{AL}}_{2}$	$\left\langle (s_3,0),(s_2,0),(s_4,0) \right\rangle$	$\left\langle (s_4,0),(s_2,0),(s_2,0) \right\rangle$	$\left\langle (s_3,0),(s_2,0),(s_2,0) \right\rangle$	$\left\langle (s_4,0),(s_3,0),(s_3,0) \right\rangle$
$\overline{\overline{AL}}_{3}$	$\left\langle (s_5,0),(s_4,0),(s_3,0) \right\rangle$	$\left\langle (s_4,0),(s_4,0),(s_3,0) \right\rangle$	$\left\langle (s_2,0),(s_1,0),(s_2,0) \right\rangle$	$\left\langle (s_4,0),(s_2,0),(s_2,0) \right\rangle$
$\overline{\overline{AL}}_{4}$	$\left\langle (s_2,0),(s_1,0),(s_2,0) \right\rangle$	$\left\langle (s_5,0),(s_1,0),(s_2,0) \right\rangle$	$\left\langle (s_4,0),(s_3,0),(s_5,0) \right\rangle$	$\left\langle (s_3,0),(s_1,0),(s_1,0) \right\rangle$
$\overline{\overline{AL}}_{5}$	$\left\langle (s_4,0),(s_3,0),(s_1,0) \right\rangle$	$\left\langle (s_5,0),(s_2,0),(s_2,0) \right\rangle$	$\left\langle (s_3,0),(s_2,0),(s_1,0) \right\rangle$	$\left\langle (s_3,0),(s_2,0),(s_2,0) \right\rangle$

Table.2 The 2-TLN decision matrix $\overline{\overline{DT}}^{2}$

	$\overline{\overline{CT}}_{1}$	$\overline{\overline{CT}}_{2}$	$\overline{\overline{CT}}_{3}$	$\overline{\overline{CT}}_{4}$
$\overline{\overline{AL}}_{1}$	$\left\langle (s_3,0),(s_2,0),(s_3,0) \right\rangle$	$\left\langle (s_3,0),(s_3,0),(s_2,0) \right\rangle$	$\left\langle (s_3,0),(s_1,0),(s_2,0) \right\rangle$	$\left\langle (s_4,0),(s_1,0),(s_3,0) \right\rangle$
$\overline{\overline{AL}}_{2}$	$\left\langle (s_2,0),(s_3,0),(s_4,0) \right\rangle$	$\left\langle (s_3,0),(s_3,0),(s_3,0) \right\rangle$	$\left\langle (s_3,0),(s_4,0),(s_3,0) \right\rangle$	$\left\langle (s_2,0),(s_4,0),(s_4,0) \right\rangle$
$\overline{\overline{AL}}_{3}$	$\left\langle (s_2,0),(s_3,0),(s_3,0) \right\rangle$	$\left\langle (s_3,0),(s_2,0),(s_2,0) \right\rangle$	$\left\langle (s_2,0),(s_3,0),(s_1,0) \right\rangle$	$\left\langle (s_3,0),(s_2,0),(s_4,0) \right\rangle$
$\overline{\overline{AL}}_{4}$	$\left\langle (s_3,0),(s_2,0),(s_2,0) \right\rangle$	$\left\langle (s_2,0),(s_2,0),(s_3,0) \right\rangle$	$\left\langle (s_3,0),(s_4,0),(s_2,0) \right\rangle$	$\left\langle (s_3,0),(s_1,0),(s_2,0) \right\rangle$
$\overline{\overline{AL}}_{5}$	$\left\langle (s_3,0),(s_2,0),(s_1,0) \right\rangle$	$\left\langle (s_3,0),(s_4,0),(s_3,0) \right\rangle$	$\left\langle (s_4,0),(s_1,0),(s_1,0) \right\rangle$	$\left\langle (s_2,0),(s_3,0),(s_2,0) \right\rangle$

Table.3 The 2-TLN decision matrix $\overline{\overline{DT}}^{3}$

	$\overline{\overline{CT}}_{1}$	$\overline{\overline{CT}}_{2}$	$\overline{\overline{CT}}_{3}$	$\overline{\overline{CT}}_{4}$

$\overline{\overline{AL}}_1$	$\langle(s_3,0),(s_3,0),(s_1,0)\rangle$	$\langle(s_4,0),(s_2,0),(s_1,0)\rangle$	$\langle(s_4,0),(s_4,0),(s_3,0)\rangle$	$\langle(s_4,0),(s_1,0),(s_3,0)\rangle$
$\overline{\overline{AL}}_2$	$\langle(s_2,0),(s_3,0),(s_3,0)\rangle$	$\langle(s_4,0),(s_4,0),(s_4,0)\rangle$	$\langle(s_2,0),(s_4,0),(s_3,0)\rangle$	$\langle(s_2,0),(s_3,0),(s_4,0)\rangle$
$\overline{\overline{AL}}_3$	$\langle(s_2,0),(s_1,0),(s_2,0)\rangle$	$\langle(s_3,0),(s_2,0),(s_2,0)\rangle$	$\langle(s_3,0),(s_4,0),(s_5,0)\rangle$	$\langle(s_2,0),(s_4,0),(s_4,0)\rangle$
$\overline{\overline{AL}}_4$	$\langle(s_3,0),(s_1,0),(s_2,0)\rangle$	$\langle(s_2,0),(s_3,0),(s_2,0)\rangle$	$\langle(s_3,0),(s_4,0),(s_5,0)\rangle$	$\langle(s_5,0),(s_3,0),(s_2,0)\rangle$
$\overline{\overline{AL}}_5$	$\langle(s_3,0),(s_3,0),(s_2,0)\rangle$	$\langle(s_3,0),(s_2,0),(s_2,0)\rangle$	$\langle(s_3,0),(s_2,0),(s_3,0)\rangle$	$\langle(s_5,0),(s_3,0),(s_4,0)\rangle$

Step 1. Calculate the support degrees by utilizing formula (36). For simplicity we shall denote
$Sup\left(\Gamma_{ce}^{b},\Gamma_{ce}^{l}\right)=S_{ce}^{bl},(b,l=1,2,3;c=1,...,5;e=1,...,4)$.

$S_{11}^{12}=S_{11}^{21}=0.8333,S_{11}^{13}=S_{11}^{31}=0.8889,S_{11}^{23}=S_{11}^{23}=0.8333,$
$S_{12}^{12}=S_{12}^{21}=0.8333,S_{12}^{13}=S_{12}^{31}=0.8889,S_{12}^{23}=S_{12}^{32}=0.8333;$
$S_{13}^{12}=S_{13}^{21}=0.9444,S_{13}^{13}=S_{13}^{31}=0.7778,S_{13}^{23}=S_{13}^{32}=0.7222,$
$S_{14}^{12}=S_{14}^{21}=0.7222,S_{14}^{13}=S_{14}^{31}=0.7222,S_{14}^{23}=S_{14}^{32}=1.000;$
$S_{21}^{12}=S_{21}^{21}=0.8889,S_{21}^{13}=S_{21}^{31}=0.8333,S_{21}^{23}=S_{21}^{32}=0.9444,$
$S_{22}^{12}=S_{22}^{21}=0.8333,S_{22}^{13}=S_{22}^{31}=0.7778,S_{22}^{23}=S_{22}^{32}=0.8333;$
$S_{23}^{12}=S_{23}^{21}=0.8333,S_{23}^{13}=S_{23}^{31}=0.7778,S_{23}^{23}=S_{23}^{32}=0.9444,$
$S_{24}^{12}=S_{24}^{21}=0.7778,S_{24}^{13}=S_{24}^{31}=0.8333,S_{24}^{23}=S_{24}^{32}=0.9444;$
$S_{31}^{12}=S_{31}^{21}=0.777778,S_{31}^{13}=S_{31}^{31}=0.6111,S_{31}^{23}=S_{31}^{23}=0.8333,$
$S_{32}^{12}=S_{32}^{21}=0.7778,S_{32}^{13}=S_{32}^{31}=0.7778,S_{32}^{23}=S_{32}^{32}=1.000;$
$S_{33}^{12}=S_{33}^{21}=0.833333,S_{33}^{13}=S_{33}^{31}=0.6111,S_{33}^{23}=S_{33}^{32}=0.6667,$
$S_{34}^{12}=S_{34}^{21}=0.8333,S_{34}^{13}=S_{34}^{31}=0.6667,S_{34}^{23}=S_{34}^{32}=0.8333;$
$S_{41}^{12}=S_{41}^{21}=0.8889,S_{41}^{13}=S_{41}^{31}=0.9444,S_{41}^{23}=S_{41}^{32}=0.9444,$
$S_{42}^{12}=S_{42}^{21}=0.722222,S_{42}^{13}=S_{42}^{31}=0.7222,S_{42}^{23}=S_{42}^{32}=0.8889;$
$S_{43}^{12}=S_{43}^{21}=0.7222,S_{43}^{13}=S_{43}^{31}=0.8889,S_{43}^{23}=S_{43}^{32}=0.8333,$
$S_{44}^{12}=S_{44}^{21}=0.9444,S_{44}^{13}=S_{44}^{31}=0.7222,S_{44}^{23}=S_{44}^{32}=0.7778;$
$S_{51}^{12}=S_{51}^{21}=0.8889,S_{51}^{13}=S_{51}^{31}=0.8889,S_{51}^{23}=S_{51}^{32}=0.8889,$
$S_{52}^{12}=S_{52}^{21}=0.722222,S_{52}^{13}=S_{52}^{31}=0.8889,S_{52}^{23}=S_{52}^{32}=0.8333;$
$S_{53}^{12}=S_{53}^{21}=0.8889,S_{53}^{13}=S_{53}^{31}=0.8889,S_{53}^{23}=S_{53}^{32}=0.7778,$
$S_{54}^{12}=S_{54}^{21}=0.8889,S_{54}^{13}=S_{54}^{31}=0.7222,S_{54}^{23}=S_{54}^{32}=0.7222;$

Step 2. Determine the support degree $T\left(\Gamma_{ce}^{b}\right)$ by utilizing formula (37). For simplicity, we shall denote $T\left(\Gamma_{ce}^{b}\right)$ by $T_{ce}^{b}\left(b=1,2,3;c=1,...,5;e=1,...,4\right)$.

$T_{11}^{1}=1.7222,T_{11}^{2}=1.6667,T_{11}^{3}=1.7222,T_{12}^{1}=1.7222,T_{12}^{2}=1.6667,T_{12}^{3}=1.7222;$
$T_{13}^{1}=1.7222,T_{13}^{2}=1.6667,T_{13}^{3}=1.5000,T_{14}^{1}=1.4444,T_{14}^{2}=1.7222,T_{14}^{3}=1.7222;$
$T_{21}^{1}=1.7222,T_{21}^{2}=1.8333,T_{21}^{3}=1.7778,T_{22}^{1}=1.6111,T_{22}^{2}=1.6667,T_{22}^{3}=1.6111;$
$T_{23}^{1}=1.6111,T_{23}^{2}=1.7778,T_{23}^{3}=1.7222,T_{24}^{1}=1.6111,T_{24}^{2}=1.7222,T_{24}^{3}=1.7222;$
$T_{31}^{1}=1.3889,T_{31}^{2}=1.6111,T_{31}^{3}=1.4444,T_{32}^{1}=1.5556,T_{32}^{2}=1.7778,T_{32}^{3}=1.7778;$
$T_{33}^{1}=1.444444,T_{33}^{2}=1.5000,T_{33}^{3}=1.2778,T_{34}^{1}=1.5000,T_{34}^{2}=1.6667,T_{34}^{3}=1.6667;$
$T_{41}^{1}=1.8333,T_{41}^{2}=1.8333,T_{41}^{3}=1.8889,T_{42}^{1}=1.4444,T_{42}^{2}=1.6111,T_{42}^{3}=1.6111;$
$T_{43}^{1}=1.6111,T_{43}^{2}=1.5556,T_{43}^{3}=1.7222,T_{44}^{1}=1.6667,T_{44}^{2}=1.7222,T_{44}^{3}=1.7222;$
$T_{51}^{1}=1.7778,T_{51}^{2}=1.7778,T_{51}^{3}=1.7778,T_{52}^{1}=1.6111,T_{52}^{2}=1.5556,T_{52}^{3}=1.7222;$
$T_{53}^{1}=1.7778,T_{53}^{2}=1.6667,T_{53}^{3}=1.6667,T_{54}^{1}=1.6111,T_{54}^{2}=1.6111,T_{54}^{3}=1.6111;$

Step 3. Utilize weights $\varpi_b\left(b=1,2,....,a\right)$ for decision makers $\overline{\overline{de}}_b$ to determine weights $\aleph_{ce}^{b}$ utilizing formula (38), we have

$\aleph_{11}^{1}=0.2021,\aleph_{11}^{2}=0.4949,\aleph_{11}^{3}=0.3031,\aleph_{12}^{1}=0.2021,$
$\aleph_{12}^{2}=0.4949,\aleph_{12}^{3}=0.3031;\aleph_{13}^{1}=0.2072,\aleph_{13}^{2}=0.5074,$
$\aleph_{13}^{3}=0.2854,\aleph_{14}^{1}=0.1833,\aleph_{14}^{2}=0.5104,\aleph_{14}^{3}=0.3063;$
$\aleph_{21}^{1}=0.1948,\aleph_{21}^{2}=0.5070,\aleph_{21}^{3}=0.2982,\aleph_{22}^{1}=0.1979,$
$\aleph_{22}^{2}=0.5053,\aleph_{22}^{3}=0.2968;\aleph_{23}^{1}=0.1915,\aleph_{23}^{2}=0.5092,$
$\aleph_{23}^{3}=0.2994,\aleph_{24}^{1}=0.1934,\aleph_{24}^{2}=0.5041,\aleph_{24}^{3}=0.3025;$
$\aleph_{31}^{1}=0.1898,\aleph_{31}^{2}=0.5188,\aleph_{31}^{3}=0.2914,\aleph_{32}^{1}=0.1870,$
$\aleph_{32}^{2}=0.5081,\aleph_{32}^{3}=0.3049;\aleph_{33}^{1}=0.2018,\aleph_{33}^{2}=0.5161,$
$\aleph_{33}^{3}=0.2821,\aleph_{34}^{1}=0.1899,\aleph_{34}^{2}=0.5063,\aleph_{34}^{3}=0.3038;$
$\aleph_{41}^{1}=0.1988,\aleph_{41}^{2}=0.4971,\aleph_{41}^{3}=0.3041,\aleph_{42}^{1}=0.1897,$
$\aleph_{42}^{2}=0.5065,\aleph_{42}^{3}=0.3039;\aleph_{43}^{1}=0.1996,\aleph_{43}^{2}=0.4883,$
$\aleph_{43}^{3}=0.3121,\aleph_{44}^{1}=0.1967,\aleph_{44}^{2}=0.5020,\aleph_{44}^{3}=0.3012;$
$\aleph_{51}^{1}=0.2000,\aleph_{51}^{2}=0.5000,\aleph_{51}^{3}=0.3000,\aleph_{52}^{1}=0.1996,$
$\aleph_{52}^{2}=0.4883,\aleph_{52}^{3}=0.3121;\aleph_{53}^{1}=0.2066,\aleph_{53}^{2}=0.4959,$
$\aleph_{53}^{3}=0.2975,\aleph_{54}^{1}=0.2000,\aleph_{54}^{2}=0.5000,\aleph_{54}^{3}=0.3000.$

Step 4. Aggregate all the individual decision matrices $\overline{\overline{DT}}^{b}=\left(\Gamma_{ce}^{b}\right)_{m\times n}(b=1,2,3;c=1,...,5;e=1,2....,4)$ into group decision matrix $\widetilde{DT}=\left(\Gamma_{ce}\right)_{m\times n}$ by utilizing formula (39) or (40), we have (assume $x=y=1,\Im=2$)

Table 4. Overall decision matrix utilizing 2-TLNDWPHM operator

	$\overline{\overline{CT}}_1$	$\overline{\overline{CT}}_2$
$\overline{\overline{AL}}_1$	$\langle(s_3,0.3121),(s_3,-0.4416),(s_2,-0.2546)\rangle$	$\langle(s_4,0.1659),(s_3,-0.3613),(s_1,0.2902)\rangle$
$\overline{\overline{AL}}_2$	$\langle(s_2,0.2801),(s_3,-0.3045),(s_4,-0.3296)\rangle$	$\langle(s_4,-0.3597),(s_3,-0.0893),(s_3,-0.0893)\rangle$
$\overline{\overline{AL}}_3$	$\langle(s_4,-0.2651),(s_2,-0.0778),(s_3,-0.3501)\rangle$	$\langle(s_3,0.2982),(s_2,0.3239),(s_2,0.2406)\rangle$
$\overline{\overline{AL}}_4$	$\langle(s_2,-0.2647),(s_1,0.3901),(s_2,0.0337)\rangle$	$\langle(s_4,-0.2654),(s_2,-0.2069),(s_2,0.3631)\rangle$
$\overline{\overline{AL}}_5$	$\langle(s_3,0.3096),(s_3,-0.4428),(s_1,0.2318)\rangle$	$\langle(s_4,-0.0301),(s_3,-0.4330),(s_2,0.3512)\rangle$

Table 4. Overall decision matrix utilizing 2-TLNDWPHM operator

	$\overline{\overline{CT}}_3$	$\overline{\overline{CT}}_4$
$\overline{\overline{AL}}_1$	$\langle (s_4,-0.3581),(s_1,0.3112),(s_2,0.2905) \rangle$	$\langle (s_4,-0.3215),(s_1,0.2170),(s_3,-0.2948) \rangle$
$\overline{\overline{AL}}_2$	$\langle (s_3,-0.3265),(s_2,0.2579),(s_3,-0.3015) \rangle$	$\langle (s_3,-0.2540),(s_3,0.3221),(s_4,-0.3122) \rangle$
$\overline{\overline{AL}}_3$	$\langle (s_2,0.3217),(s_2,0.1066),(s_2,-0.4061) \rangle$	$\langle (s_3,0.0781),(s_2,0.4413),(s_3,0.2604) \rangle$
$\overline{\overline{AL}}_4$	$\langle (s_3,0.3115),(s_4,-0.2868),(s_3,-0.0090) \rangle$	$\langle (s_4,0.1354),(s_1,0.3029),(s_2,-0.3720) \rangle$
$\overline{\overline{AL}}_5$	$\langle (s_3,0.4424),(s_1,0.4666),(s_3,0.2970) \rangle$	$\langle (s_4,0.0490),(s_3,-0.3096),(s_2,0.4329) \rangle$

Table 5. Overall decision matrix utilizing 2-TLNDWPGHM operator

	$\overline{\overline{CT}}_1$	$\overline{\overline{CT}}_2$
$\overline{\overline{AL}}_1$	$\langle (s_3,0.2654),(s_3,-0.3807),(s_2,0.2226) \rangle$	$\langle (s_3,0.3613),(s_3,-0.3211),(s_1,0.4161) \rangle$
$\overline{\overline{AL}}_2$	$\langle (s_2,0.2434),(s_3,-0.2657),(s_4,-0.2991) \rangle$	$\langle (s_4,-0.4152),(s_3,0.2068),(s_3,0.2068) \rangle$
$\overline{\overline{AL}}_3$	$\langle (s_2,0.3580),(s_3,-0.0182),(s_3,-0.3261) \rangle$	$\langle (s_3,0.2628),(s_3,-0.2657),(s_2,0.2741) \rangle$
$\overline{\overline{AL}}_4$	$\langle (s_3,-0.2587),(s_1,0.4168),(s_2,-0.0334) \rangle$	$\langle (s_2,0.3539),(s_2,0.1742),(s_2,0.4146) \rangle$
$\overline{\overline{AL}}_5$	$\langle (s_3,0.2659),(s_3,-0.3839),(s_1,0.3244) \rangle$	$\langle (s_3,0.3685),(s_3,0.1333),(s_2,0.4107) \rangle$

Table 5. Overall decision matrix utilizing 2-TLNDWPHM operator

	$\overline{\overline{CT}}_3$	$\overline{\overline{CT}}_4$
$\overline{\overline{AL}}_1$	$\langle (s_4,-0.4167),(s_3,-0.3578),(s_2,0.3254) \rangle$	$\langle (s_3,0.2725),(s_1,0.6568),(s_3,-0.2619) \rangle$
$\overline{\overline{AL}}_2$	$\langle (s_3,-0.3557),(s_4,-0.3280),(s_3,-0.2649) \rangle$	$\langle (s_2,-0.3282),(s_3,0.4432),(s_4,-0.2421) \rangle$
$\overline{\overline{AL}}_3$	$\langle (s_2,0.2928),(s_3,0.1344),(s_4,-0.0620) \rangle$	$\langle (s_3,-0.2141),(s_3,0.0877),(s_4,-0.3258) \rangle$
$\overline{\overline{AL}}_4$	$\langle (s_3,0.2627),(s_4,-0.2405),(s_5,-0.4560) \rangle$	$\langle (s_3,0.4863),(s_2,0.1564),(s_2,-0.2633) \rangle$
$\overline{\overline{AL}}_5$	$\langle (s_3,0.3799),(s_2,-0.3948),(s_2,-0.1612) \rangle$	$\langle (s_3,-0.2028),(s_3,-0.2659),(s_3,0.0921) \rangle$

Step 5. Calculate the support degrees of Table4, by utilizing formula (41). For simplicity we shall denote $Sup(\Gamma_{ce},\Gamma_{ce}) = S_c^{ce}, (c=1,...,5; e=1,...,4)$.

$S_1^{12}=S_1^{21}=0.9228, S_1^{13}=S_1^{31}=0.8821, S_1^{14}=S_1^{41}=0.8518,$
$S_1^{23}=S_1^{32}=0.8416, S_1^{24}=S_1^{42}=0.8153, S_1^{34}=S_1^{43}=0.9697;$
$S_2^{12}=S_2^{21}=0.8703, S_2^{13}=S_2^{31}=0.8929, S_2^{14}=S_2^{41}=0.9383,$
$S_2^{23}=S_2^{32}=0.9152, S_2^{24}=S_2^{42}=0.8843, S_2^{34}=S_2^{43}=0.9374;$
$S_3^{12}=S_3^{21}=0.9307, S_3^{13}=S_3^{31}=0.8526, S_3^{14}=S_3^{41}=0.9008,$
$S_3^{23}=S_3^{32}=0.8977, S_3^{24}=S_3^{42}=0.9246, S_3^{34}=S_3^{43}=0.8468;$
$S_4^{12}=S_4^{21}=0.9038, S_4^{13}=S_4^{31}=0.7857, S_4^{14}=S_4^{41}=0.8948,$
$S_4^{23}=S_4^{32}=0.8349, S_4^{24}=S_4^{42}=0.9097, S_4^{34}=S_4^{43}=0.7446;$
$S_5^{12}=S_5^{21}=0.9006, S_5^{13}=S_5^{31}=0.9284, S_5^{14}=S_5^{41}=0.8848,$
$S_5^{23}=S_5^{32}=0.8510, S_5^{24}=S_5^{42}=0.9842, S_5^{34}=S_5^{43}=0.8352;$

or

Calculate the support degrees of Table 5, by utilizing formula (41). For simplicity we shall denote

$S_1^{12}=S_1^{21}=0.9261, S_1^{13}=S_1^{31}=0.9754, S_1^{14}=S_1^{41}=0.9175,$
$S_1^{23}=S_1^{32}=0.9393, S_1^{24}=S_1^{42}=0.8444, S_1^{34}=S_1^{43}=0.9051;$
$S_2^{12}=S_2^{21}=0.8718, S_2^{13}=S_2^{31}=0.8720, S_2^{14}=S_2^{41}=0.9527,$
$S_2^{23}=S_2^{32}=0.8957, S_2^{24}=S_2^{42}=0.8864, S_2^{34}=S_2^{43}=0.9129;$
$S_3^{12}=S_3^{21}=0.9138, S_3^{13}=S_3^{31}=0.9177, S_3^{14}=S_3^{41}=0.9168,$
$S_3^{23}=S_3^{32}=0.8314, S_3^{24}=S_3^{42}=0.8858, S_3^{34}=S_3^{43}=0.9456;$
$S_4^{12}=S_4^{21}=0.9115, S_4^{13}=S_4^{31}=0.6977, S_4^{14}=S_4^{41}=0.9221,$
$S_4^{23}=S_4^{32}=0.7431, S_4^{24}=S_4^{42}=0.8811, S_4^{34}=S_4^{43}=0.7252;$
$S_5^{12}=S_5^{21}=0.9052, S_5^{13}=S_5^{31}=0.9089, S_5^{14}=S_5^{41}=0.8794,$
$S_5^{23}=S_5^{32}=0.8827, S_5^{24}=S_5^{42}=0.9185, S_5^{34}=S_5^{43}=0.8455;$

Step 6. Determine the support degree $T(\Gamma_{ce})$ by utilizing formula (42)

$T_{11}=2.6567, T_{12}=2.5797, T_{13}=2.6934, T_{14}=2.6368, T_{21}=2.7015,$
$T_{22}=2.6698, T_{23}=2.7456, T_{24}=2.7601; T_{31}=2.6840, T_{32}=2.7530,$
$T_{33}=2.5971, T_{34}=2.6721, T_{41}=2.5844, T_{42}=2.6484, T_{43}=2.3653,$
$T_{44}=2.5491; T_{51}=2.7138, T_{52}=2.7358, T_{53}=2.6146, T_{54}=2.7042.$

Or

Determine the support degree $T(\Gamma_{ce})$ by utilizing formula (42)

$T_{11}=2.8189, T_{12}=2.7097, T_{13}=2.8197, T_{14}=2.6669, T_{21}=2.6965,$
$T_{22}=2.6539, T_{23}=2.6806, T_{24}=2.7521; T_{31}=2.7482, T_{32}=2.6311,$
$T_{33}=2.6947, T_{34}=2.7482, T_{41}=2.5313, T_{42}=2.5357, T_{43}=2.1660,$
$T_{44}=2.5284; T_{51}=2.6936, T_{52}=2.7064, T_{53}=2.6372, T_{54}=2.6434.$

Step 7. Determine weighting vector Φ_{ce} by utilizing formula (43),

$\Phi_{11}=0.5029, \Phi_{12}=0.2954, \Phi_{13}=0.1016, \Phi_{14}=0.1000, \Phi_{21}=0.4999,$
$\Phi_{22}=0.2974, \Phi_{23}=0.1012, \Phi_{24}=0.1016; \Phi_{31}=0.4985, \Phi_{32}=0.3047,$
$\Phi_{33}=0.0974, \Phi_{34}=0.0994, \Phi_{41}=0.5009, \Phi_{42}=0.3059, \Phi_{43}=0.0941,$
$\Phi_{44}=0.0992; \Phi_{51}=0.5006, \Phi_{52}=0.3021, \Phi_{53}=0.0974, \Phi_{54}=0.0999.$

Or

Determine weighting vector Φ_{ce} by utilizing formula (43),

$\Phi_{11}=0.5063, \Phi_{12}=0.2951, \Phi_{13}=0.1013, \Phi_{14}=0.0972, \Phi_{21}=0.5012,$
$\Phi_{22}=0.2973, \Phi_{23}=0.0998, \Phi_{24}=0.1017; \Phi_{31}=0.5055, \Phi_{32}=0.2938,$
$\Phi_{33}=0.0996, \Phi_{34}=0.1011, \Phi_{41}=0.1430, \Phi_{42}=0.3034, \Phi_{43}=0.0906,$
$\Phi_{44}=0.1009; \Phi_{51}=0.5009, \Phi_{52}=0.3016, \Phi_{53}=0.0987, \Phi_{54}=0.0988.$

Step 8. Utilize 2-TLNDWPHM or 2-TLNNDWPGHM operators given in formula (44) or formula (45) to aggregate all assessment values (assume $x=y=1, \Im=2$)

$\overline{\overline{AL}}_1 = \langle (s_4,-0.3978),(s_2,0.0053),(s_2,-0.1452) \rangle;$

$\overline{\overline{AL}}_2 = \langle (s_3,-0.2250),(s_3,0.0835),(s_3,0.3512) \rangle;$

$\overline{\overline{AL}}_3 = \langle (s_3, 0.1907), (s_2, 0.2747), (s_2, 0.4501) \rangle;$

$\overline{\overline{AL}}_4 = \langle (s_3, 0.3506), (s_2, -0.2546), (s_2, 0.2681) \rangle;$

$\overline{\overline{AL}}_5 = \langle (s_4, -0.4224), (s_2, 0.4077), (s_2, -0.2722) \rangle.$

or

$\overline{\overline{AL}}_1 = \langle (s_4, -0.4206), (s_2, 0.3839), (s_2, 0.0258) \rangle;$

$\overline{\overline{AL}}_2 = \langle (s_3, -0.2382), (s_3, 0.0972), (s_3, 0.3152) \rangle;$

$\overline{\overline{AL}}_3 = \langle (s_3, -0.2307), (s_3, -0.1970), (s_3, -0.0111) \rangle;$

$\overline{\overline{AL}}_4 = \langle (s_3, 0.2394), (s_2, 0.0406), (s_3, -0.3743) \rangle;$

$\overline{\overline{AL}}_5 = \langle (s_3, 0.3430), (s_3, -0.4474), (s_2, -0.0295) \rangle.$

Step 9. Calculate the score values utilizing Definition (3), we have

$\overline{\overline{SR}}(\overline{\overline{AL}}_1) = 0.6523, \overline{\overline{SR}}(\overline{\overline{AL}}_2) = 0.4634, \overline{\overline{SR}}(\overline{\overline{AL}}_3) = 0.5814,$

$\overline{\overline{SR}}(\overline{\overline{AL}}_4) = 0.6298, \overline{\overline{SR}}(\overline{\overline{AL}}_5) = 0.6357.$

Calculate the score values utilizing Definition (3), we have

$\overline{\overline{SR}}(\overline{\overline{AL}}_1) = 0.6205, \overline{\overline{SR}}(\overline{\overline{AL}}_2) = 0.4639, \overline{\overline{SR}}(\overline{\overline{AL}}_3) = 0.4987,$

$\overline{\overline{SR}}(\overline{\overline{AL}}_4) = 0.5874, \overline{\overline{SR}}(\overline{\overline{AL}}_5) = 0.6011.$

Step 10. Rank all the alternatives and select the best one according to their score values.

$\overline{\overline{AL}}_1 > \overline{\overline{AL}}_5 > \overline{\overline{AL}}_4 > \overline{\overline{AL}}_3 > \overline{\overline{AL}}_3.$

or

$\overline{\overline{AL}}_1 > \overline{\overline{AL}}_5 > \overline{\overline{AL}}_4 > \overline{\overline{AL}}_3 > \overline{\overline{AL}}_3.$

$\overline{\overline{AL}}_1$ is the best one while the worst one is $\overline{\overline{AL}}_3$.

VI. Discussion

In the following, we will further analyze the effect of the parameters x, y and $\Im$ on the final ranking result of Example 1. Then we can adopt the different values of x and y in step 4 and step 8, while the value $\Im$ is fix. The results are given in Table 6 and Table 7. Moreover, the effect of general parameter $\Im$, is shown in Table 8 and Table 9, while the parameters x, y are fix.

From Table 6 and Table 7, we can notice that the ranking orders are different for different values of the parameters x, y. However, the best alternative $\overline{\overline{AL}}_1$ or $\overline{\overline{AL}}_5$. From Table 6 and Table 7, we can also notice that, when the values of the parameter x or y increases, the score values increases utilizing 2-TLNDWPHM operator, while the score values decreases utilizing2-TLNDWPGHM operator. Generally, for computational simplicity one may select $x = y = \frac{1}{2}$ according to the actual need of decision making problems.

Table 6. Effect of parameter x and y on ranking result utilizing 2-TLNDWPHM operator

Parameter values	Score values	Ranking orders
$x=1, y=2, \Im=2$	$\overline{\overline{SR}}(\overline{\overline{AL}}_1)=0.6537, \overline{\overline{SR}}(\overline{\overline{AL}}_2)=0.4574, \overline{\overline{SR}}(\overline{\overline{AL}}_3)=0.5764, \overline{\overline{SR}}(\overline{\overline{AL}}_4)=0.6273, \overline{\overline{SR}}(\overline{\overline{AL}}_5)=0.6325.$	$\overline{\overline{AL}}_1 > \overline{\overline{AL}}_5 > \overline{\overline{AL}}_4 > \overline{\overline{AL}}_3 > \overline{\overline{AL}}_2.$
$x=3, y=5, \Im=2$	$\overline{\overline{SR}}(\overline{\overline{AL}}_1)=0.6530, \overline{\overline{SR}}(\overline{\overline{AL}}_2)=0.4585, \overline{\overline{SR}}(\overline{\overline{AL}}_3)=0.5772, \overline{\overline{SR}}(\overline{\overline{AL}}_4)=0.6275, \overline{\overline{SR}}(\overline{\overline{AL}}_5)=0.6328.$	$\overline{\overline{AL}}_1 > \overline{\overline{AL}}_5 > \overline{\overline{AL}}_4 > \overline{\overline{AL}}_3 > \overline{\overline{AL}}_2.$
$x=2, y=7, \Im=2$	$\overline{\overline{SR}}(\overline{\overline{AL}}_1)=0.6570, \overline{\overline{SR}}(\overline{\overline{AL}}_2)=0.4560, \overline{\overline{SR}}(\overline{\overline{AL}}_3)=0.5757, \overline{\overline{SR}}(\overline{\overline{AL}}_4)=0.6284, \overline{\overline{SR}}(\overline{\overline{AL}}_5)=0.6332.$	$\overline{\overline{AL}}_1 > \overline{\overline{AL}}_5 > \overline{\overline{AL}}_4 > \overline{\overline{AL}}_3 > \overline{\overline{AL}}_2.$
$x=6, y=19, \Im=2$	$\overline{\overline{SR}}(\overline{\overline{AL}}_1)=0.6563, \overline{\overline{SR}}(\overline{\overline{AL}}_2)=0.4561, \overline{\overline{SR}}(\overline{\overline{AL}}_3)=0.5756, \overline{\overline{SR}}(\overline{\overline{AL}}_4)=0.6280, \overline{\overline{SR}}(\overline{\overline{AL}}_5)=0.6329.$	$\overline{\overline{AL}}_1 > \overline{\overline{AL}}_5 > \overline{\overline{AL}}_4 > \overline{\overline{AL}}_3 > \overline{\overline{AL}}_2.$
$x=14, y=30, \Im=2$	$\overline{\overline{SR}}(\overline{\overline{AL}}_1)=0.6540, \overline{\overline{SR}}(\overline{\overline{AL}}_2)=0.4571, \overline{\overline{SR}}(\overline{\overline{AL}}_3)=0.5761, \overline{\overline{SR}}(\overline{\overline{AL}}_4)=0.6273, \overline{\overline{SR}}(\overline{\overline{AL}}_5)=0.6324.$	$\overline{\overline{AL}}_1 > \overline{\overline{AL}}_5 > \overline{\overline{AL}}_4 > \overline{\overline{AL}}_3 > \overline{\overline{AL}}_2.$
$x=2, y=100, \Im=2$	$\overline{\overline{SR}}(\overline{\overline{AL}}_1)=0.6754, \overline{\overline{SR}}(\overline{\overline{AL}}_2)=0.4631, \overline{\overline{SR}}(\overline{\overline{AL}}_3)=0.5926, \overline{\overline{SR}}(\overline{\overline{AL}}_4)=0.6476, \overline{\overline{SR}}(\overline{\overline{AL}}_5)=0.6526.$	$\overline{\overline{AL}}_1 > \overline{\overline{AL}}_5 > \overline{\overline{AL}}_4 > \overline{\overline{AL}}_3 > \overline{\overline{AL}}_2.$
$x=50, y=2, \Im=2$	$\overline{\overline{SR}}(\overline{\overline{AL}}_1)=0.6897, \overline{\overline{SR}}(\overline{\overline{AL}}_2)=0.5290, \overline{\overline{SR}}(\overline{\overline{AL}}_3)=0.6451, \overline{\overline{SR}}(\overline{\overline{AL}}_4)=0.6745, \overline{\overline{SR}}(\overline{\overline{AL}}_5)=0.6976.$	$\overline{\overline{AL}}_5 > \overline{\overline{AL}}_1 > \overline{\overline{AL}}_4 > \overline{\overline{AL}}_3 > \overline{\overline{AL}}_2.$
$x=35, y=6, \Im=2$	$\overline{\overline{SR}}(\overline{\overline{AL}}_1)=0.6685, \overline{\overline{SR}}(\overline{\overline{AL}}_2)=0.4998, \overline{\overline{SR}}(\overline{\overline{AL}}_3)=0.6142, \overline{\overline{SR}}(\overline{\overline{AL}}_4)=0.6526, \overline{\overline{SR}}(\overline{\overline{AL}}_5)=0.6669.$	$\overline{\overline{AL}}_5 > \overline{\overline{AL}}_1 > \overline{\overline{AL}}_4 > \overline{\overline{AL}}_3 > \overline{\overline{AL}}_2.$
$x=80, y=4, \Im=2$	$\overline{\overline{SR}}(\overline{\overline{AL}}_1)=0.6869, \overline{\overline{SR}}(\overline{\overline{AL}}_2)=0.5256, \overline{\overline{SR}}(\overline{\overline{AL}}_3)=0.6409, \overline{\overline{SR}}(\overline{\overline{AL}}_4)=0.6713, \overline{\overline{SR}}(\overline{\overline{AL}}_5)=0.6937.$	$\overline{\overline{AL}}_5 > \overline{\overline{AL}}_1 > \overline{\overline{AL}}_4 > \overline{\overline{AL}}_3 > \overline{\overline{AL}}_2.$

Table 7. Effect of parameter x and y on decision result utilizing 2-TLNDWPGHM operator

Parameter values	Score values	Ranking orders
$x=1, y=2, \Im=2$	$\overline{\overline{SR}}(\overline{\overline{AL}}_1)=0.6252, \overline{\overline{SR}}(\overline{\overline{AL}}_2)=0.4663, \overline{\overline{SR}}(\overline{\overline{AL}}_3)=0.5037, \overline{\overline{SR}}(\overline{\overline{AL}}_4)=0.5810, \overline{\overline{SR}}(\overline{\overline{AL}}_5)=0.5998.$	$\overline{\overline{AL}}_1 > \overline{\overline{AL}}_5 > \overline{\overline{AL}}_4 > \overline{\overline{AL}}_3 > \overline{\overline{AL}}_2.$

$x=3, y=5,$ $\Im=2$	$\overline{\overline{SR}}(\overline{\overline{AL}}_1)=0.6245, \overline{\overline{SR}}(\overline{\overline{AL}}_2)=0.4662,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_3)=0.5029, \overline{\overline{SR}}(\overline{\overline{AL}}_4)=0.5832,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_5)=0.6006.$	$\overline{\overline{AL}}_1>\overline{\overline{AL}}_5>\overline{\overline{AL}}_4>\overline{\overline{AL}}_3>\overline{\overline{AL}}_2.$
$x=2, y=7,$ $\Im=2$	$\overline{\overline{SR}}(\overline{\overline{AL}}_1)=0.6251, \overline{\overline{SR}}(\overline{\overline{AL}}_2)=0.4643,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_3)=0.5042, \overline{\overline{SR}}(\overline{\overline{AL}}_4)=0.5718,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_5)=0.5951.$	$\overline{\overline{AL}}_1>\overline{\overline{AL}}_5>\overline{\overline{AL}}_4>\overline{\overline{AL}}_3>\overline{\overline{AL}}_2.$
$x=6, y=19,$ $\Im=2$	$\overline{\overline{SR}}(\overline{\overline{AL}}_1)=0.6254, \overline{\overline{SR}}(\overline{\overline{AL}}_2)=0.4649,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_3)=0.5043, \overline{\overline{SR}}(\overline{\overline{AL}}_4)=0.5737,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_5)=0.5961.$	$\overline{\overline{AL}}_1>\overline{\overline{AL}}_5>\overline{\overline{AL}}_4>\overline{\overline{AL}}_3>\overline{\overline{AL}}_2.$
$x=14, y=30,$ $\Im=2$	$\overline{\overline{SR}}(\overline{\overline{AL}}_1)=0.6253, \overline{\overline{SR}}(\overline{\overline{AL}}_2)=0.4662,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_3)=0.5039, \overline{\overline{SR}}(\overline{\overline{AL}}_4)=0.5800,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_5)=0.5994.$	$\overline{\overline{AL}}_1>\overline{\overline{AL}}_5>\overline{\overline{AL}}_4>\overline{\overline{AL}}_3>\overline{\overline{AL}}_2.$
$x=2, y=100,$ $\Im=2$	$\overline{\overline{SR}}(\overline{\overline{AL}}_1)=0.6082, \overline{\overline{SR}}(\overline{\overline{AL}}_2)=0.4388,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_3)=0.4828, \overline{\overline{SR}}(\overline{\overline{AL}}_4)=0.5177,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_5)=0.5575.$	$\overline{\overline{AL}}_1>\overline{\overline{AL}}_5>\overline{\overline{AL}}_4>\overline{\overline{AL}}_3>\overline{\overline{AL}}_2.$
$x=50, y=2,$ $\Im=2$	$\overline{\overline{SR}}(\overline{\overline{AL}}_1)=0.5583, \overline{\overline{SR}}(\overline{\overline{AL}}_2)=0.4136,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_3)=0.4440, \overline{\overline{SR}}(\overline{\overline{AL}}_4)=0.5533,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_5)=0.5620.$	$\overline{\overline{AL}}_5>\overline{\overline{AL}}_1>\overline{\overline{AL}}_4>\overline{\overline{AL}}_3>\overline{\overline{AL}}_2.$
$x=35, y=6,$ $\Im=2$	$\overline{\overline{SR}}(\overline{\overline{AL}}_1)=0.5868, \overline{\overline{SR}}(\overline{\overline{AL}}_2)=0.4367,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_3)=0.4683, \overline{\overline{SR}}(\overline{\overline{AL}}_4)=0.5771,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_5)=0.5835.$	$\overline{\overline{AL}}_1>\overline{\overline{AL}}_5>\overline{\overline{AL}}_4>\overline{\overline{AL}}_3>\overline{\overline{AL}}_2.$
$x=80, y=4,$ $\Im=2$	$\overline{\overline{SR}}(\overline{\overline{AL}}_1)=0.5618, \overline{\overline{SR}}(\overline{\overline{AL}}_2)=0.4163,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_3)=0.4470, \overline{\overline{SR}}(\overline{\overline{AL}}_4)=0.5569,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_5)=0.5650.$	$\overline{\overline{AL}}_5>\overline{\overline{AL}}_1>\overline{\overline{AL}}_4>\overline{\overline{AL}}_3>\overline{\overline{AL}}_2.$

Table 8. Effect of parameter $\Im$ on decision result 2-TLNDWPHM operator

Parameter values	Score values	Ranking orders
$x=1, y=2,$ $\Im=3$	$\overline{\overline{SR}}(\overline{\overline{AL}}_1)=0.6855, \overline{\overline{SR}}(\overline{\overline{AL}}_2)=0.4851,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_3)=0.6316, \overline{\overline{SR}}(\overline{\overline{AL}}_4)=0.6667,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_5)=0.6713.$	$\overline{\overline{AL}}_1>\overline{\overline{AL}}_5>\overline{\overline{AL}}_4>\overline{\overline{AL}}_3>\overline{\overline{AL}}_2.$
$x=1, y=2,$ $\Im=5$	$\overline{\overline{SR}}(\overline{\overline{AL}}_1)=0.7326, \overline{\overline{SR}}(\overline{\overline{AL}}_2)=0.5291,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_3)=0.7017, \overline{\overline{SR}}(\overline{\overline{AL}}_4)=0.7181,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_5)=0.7226.$	$\overline{\overline{AL}}_1>\overline{\overline{AL}}_5>\overline{\overline{AL}}_4>\overline{\overline{AL}}_3>\overline{\overline{AL}}_2.$
$x=1, y=2,$ $\Im=9$	$\overline{\overline{SR}}(\overline{\overline{AL}}_1)=0.7795, \overline{\overline{SR}}(\overline{\overline{AL}}_2)=0.5840,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_3)=0.7632, \overline{\overline{SR}}(\overline{\overline{AL}}_4)=0.7668,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_5)=0.7734.$	$\overline{\overline{AL}}_1>\overline{\overline{AL}}_5>\overline{\overline{AL}}_4>\overline{\overline{AL}}_3>\overline{\overline{AL}}_2.$
$x=1, y=2,$ $\Im=15$	$\overline{\overline{SR}}(\overline{\overline{AL}}_1)=0.8025, \overline{\overline{SR}}(\overline{\overline{AL}}_2)=0.6176,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_3)=0.7934, \overline{\overline{SR}}(\overline{\overline{AL}}_4)=0.7954,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_5)=0.7991.$	$\overline{\overline{AL}}_1>\overline{\overline{AL}}_5>\overline{\overline{AL}}_4>\overline{\overline{AL}}_3>\overline{\overline{AL}}_2.$
$x=1, y=2,$ $\Im=30$	$\overline{\overline{SR}}(\overline{\overline{AL}}_1)=0.8184, \overline{\overline{SR}}(\overline{\overline{AL}}_2)=0.6425,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_3)=0.8142, \overline{\overline{SR}}(\overline{\overline{AL}}_4)=0.8152,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_5)=0.8169.$	$\overline{\overline{AL}}_1>\overline{\overline{AL}}_5>\overline{\overline{AL}}_4>\overline{\overline{AL}}_3>\overline{\overline{AL}}_2.$
$x=1, y=2,$ $\Im=100$	$\overline{\overline{SR}}(\overline{\overline{AL}}_1)=0.8290, \overline{\overline{SR}}(\overline{\overline{AL}}_2)=0.6595,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_3)=0.8278, \overline{\overline{SR}}(\overline{\overline{AL}}_4)=0.8281,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_5)=0.8285.$	$\overline{\overline{AL}}_1>\overline{\overline{AL}}_5>\overline{\overline{AL}}_4>\overline{\overline{AL}}_3>\overline{\overline{AL}}_2.$
$x=1, y=2,$ $\Im=200$	$\overline{\overline{SR}}(\overline{\overline{AL}}_1)=0.8312, \overline{\overline{SR}}(\overline{\overline{AL}}_2)=0.6631,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_3)=0.8306, \overline{\overline{SR}}(\overline{\overline{AL}}_4)=0.8307,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_5)=0.8310.$	$\overline{\overline{AL}}_5>\overline{\overline{AL}}_1>\overline{\overline{AL}}_4>\overline{\overline{AL}}_3>\overline{\overline{AL}}_2.$

Table 9. Effect of parameter $\Im$ on decision result 2-TLNDWPGHM operator

Parameter values	Score values	Ranking orders
$x=1, y=2,$ $\Im=3$	$\overline{\overline{SR}}(\overline{\overline{AL}}_1)=0.5940, \overline{\overline{SR}}(\overline{\overline{AL}}_2)=0.4411,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_3)=0.4610, \overline{\overline{SR}}(\overline{\overline{AL}}_4)=0.5084,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_5)=0.5644.$	$\overline{\overline{AL}}_1>\overline{\overline{AL}}_5>\overline{\overline{AL}}_4>\overline{\overline{AL}}_3>\overline{\overline{AL}}_2.$
$x=1, y=2,$ $\Im=5$	$\overline{\overline{SR}}(\overline{\overline{AL}}_1)=0.5531, \overline{\overline{SR}}(\overline{\overline{AL}}_2)=0.4105,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_3)=0.4008,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_4)=0.4233, \overline{\overline{SR}}(\overline{\overline{AL}}_5)=0.5123.$	$\overline{\overline{AL}}_1>\overline{\overline{AL}}_5>\overline{\overline{AL}}_4>\overline{\overline{AL}}_2>\overline{\overline{AL}}_3.$
$x=1, y=2,$ $\Im=9$	$\overline{\overline{SR}}(\overline{\overline{AL}}_1)=0.5146, \overline{\overline{SR}}(\overline{\overline{AL}}_2)=0.3794,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_3)=0.3443, \overline{\overline{SR}}(\overline{\overline{AL}}_4)=0.3570,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_5)=0.4478.$	$\overline{\overline{AL}}_1>\overline{\overline{AL}}_5>\overline{\overline{AL}}_2>\overline{\overline{AL}}_4>\overline{\overline{AL}}_3.$
$x=1, y=2,$ $\Im=15$	$\overline{\overline{SR}}(\overline{\overline{AL}}_1)=0.4875, \overline{\overline{SR}}(\overline{\overline{AL}}_2)=0.3609,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_3)=0.3163, \overline{\overline{SR}}(\overline{\overline{AL}}_4)=0.3240,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_5)=0.4019.$	$\overline{\overline{AL}}_1>\overline{\overline{AL}}_5>\overline{\overline{AL}}_2>\overline{\overline{AL}}_4>\overline{\overline{AL}}_3.$
$x=1, y=2,$ $\Im=30$	$\overline{\overline{SR}}(\overline{\overline{AL}}_1)=0.4656, \overline{\overline{SR}}(\overline{\overline{AL}}_2)=0.3470,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_3)=0.2964, \overline{\overline{SR}}(\overline{\overline{AL}}_4)=0.3003,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_5)=0.3670.$	$\overline{\overline{AL}}_1>\overline{\overline{AL}}_5>\overline{\overline{AL}}_2>\overline{\overline{AL}}_4>\overline{\overline{AL}}_3.$
$x=1, y=2,$ $\Im=100$	$\overline{\overline{SR}}(\overline{\overline{AL}}_1)=0.4507, \overline{\overline{SR}}(\overline{\overline{AL}}_2)=0.3374,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_3)=0.2833, \overline{\overline{SR}}(\overline{\overline{AL}}_4)=0.2844,$ $\overline{\overline{SR}}(\overline{\overline{AL}}_5)=0.3433.$	$\overline{\overline{AL}}_1>\overline{\overline{AL}}_5>\overline{\overline{AL}}_2>\overline{\overline{AL}}_4>\overline{\overline{AL}}_3.$

$x=1, y=2,$ $\Im=200$	$\overline{\overline{SR}}\left(\overline{\overline{AL_1}}\right)=0.4476, \overline{\overline{SR}}\left(\overline{\overline{AL_2}}\right)=0.3354,$ $\overline{\overline{SR}}\left(\overline{\overline{AL_3}}\right)=0.2805, \overline{\overline{SR}}\left(\overline{\overline{AL_4}}\right)=0.2811,$ $\overline{\overline{SR}}\left(\overline{\overline{AL_5}}\right)=0.3383.$	$\overline{\overline{AL_1}} > \overline{\overline{AL_5}} > \overline{\overline{AL_2}} > \overline{\overline{AL_4}} > \overline{\overline{AL_3}}.$

From Table 8 and Table 9, we can notice that the ranking orders are different for different values of the parameters $\Im$. However, the best alternative $\overline{\overline{AL_1}}$ or $\overline{\overline{AL_5}}$. From Table 8 and Table 9, we can also notice that, when the values of the parameter $\Im$ increases, the score values increases utilizing 2-TLNDWPHM operator, while the score values decreases utilizing2-TLNDWPGHM operator. So, one may select the parameter value according to the actual need of decision making problem.

A. Compare with existing methods

In order to confirm the efficacy of the developed approach and describe its advantages, we can compare our developed method with some existing methods.

B. Validity of the developed method

In order to confirm the validity of the developed approach, we can utilize some existing methods to solve the same example. Since the developed approach is based on the combination of PA, HM operators and Dombi operations. So, we can utilize the methods in which the interrelationships between two input arguments are considered. Therefore, the reference methods of comparison are 2-TLNNWBM, 2-TLNNWGBM operators and 2-TLNHM, 2-TLNDHM operators. The score values and ranking orders of the above example by solving these two methods and the developed method as given in Table 10. From Table 10, we can notice that the ranking order obtained by the existing methods is the same as that obtained from the proposed approach. This shows the developed approach is valid.

Table 10. The score values and ranking orders obtained from different methods

Approach	Score values	Ranking order
2-TLNNWBM [34] ($p=q=1$)	$\overline{\overline{SR}}\left(\overline{\overline{AL_1}}\right)=0.6298, \overline{\overline{SR}}\left(\overline{\overline{AL_2}}\right)=0.4648,$ $\overline{\overline{SR}}\left(\overline{\overline{AL_3}}\right)=0.5642, \overline{\overline{SR}}\left(\overline{\overline{AL_4}}\right)=0.6145,$ $\overline{\overline{SR}}\left(\overline{\overline{AL_5}}\right)=0.6243.$	$\overline{\overline{AL_1}} > \overline{\overline{AL_5}} > \overline{\overline{AL_4}} > \overline{\overline{AL_3}} > \overline{\overline{AL_2}}.$
2-TLNNWGBM [34] ($p=q=1$)	$\overline{\overline{SR}}\left(\overline{\overline{AL_1}}\right)=0.6259, \overline{\overline{SR}}\left(\overline{\overline{AL_2}}\right)=0.4606,$ $\overline{\overline{SR}}\left(\overline{\overline{AL_3}}\right)=0.5622, \overline{\overline{SR}}\left(\overline{\overline{AL_4}}\right)=0.6080,$ $\overline{\overline{SR}}\left(\overline{\overline{AL_5}}\right)=0.6198.$	$\overline{\overline{AL_1}} > \overline{\overline{AL_5}} > \overline{\overline{AL_4}} > \overline{\overline{AL_3}} > \overline{\overline{AL_2}}.$
2-TLNWHM[42] ($k=2$)	$\overline{\overline{SR}}\left(\overline{\overline{AL_1}}\right)=0.9013, \overline{\overline{SR}}\left(\overline{\overline{AL_2}}\right)=0.8395,$ $\overline{\overline{SR}}\left(\overline{\overline{AL_3}}\right)=0.8751, \overline{\overline{SR}}\left(\overline{\overline{AL_4}}\right)=0.8895,$ $\overline{\overline{SR}}\left(\overline{\overline{AL_5}}\right)=0.8962.$	$\overline{\overline{AL_1}} > \overline{\overline{AL_5}} > \overline{\overline{AL_4}} > \overline{\overline{AL_3}} > \overline{\overline{AL_2}}.$
2-TLNWDHM[42] ($k=2$)	$\overline{\overline{SR}}\left(\overline{\overline{AL_1}}\right)=0.2062, \overline{\overline{SR}}\left(\overline{\overline{AL_2}}\right)=0.1327,$ $\overline{\overline{SR}}\left(\overline{\overline{AL_3}}\right)=0.1718, \overline{\overline{SR}}\left(\overline{\overline{AL_4}}\right)=0.1946,$ $\overline{\overline{SR}}\left(\overline{\overline{AL_5}}\right)=0.2005.$	$\overline{\overline{AL_1}} > \overline{\overline{AL_5}} > \overline{\overline{AL_4}} > \overline{\overline{AL_3}} > \overline{\overline{AL_2}}.$
Proposed 2-TLNDWPHM operator	$\overline{\overline{SR}}\left(\overline{\overline{AL_1}}\right)=0.6523, \overline{\overline{SR}}\left(\overline{\overline{AL_2}}\right)=0.4634,$ $\overline{\overline{SR}}\left(\overline{\overline{AL_3}}\right)=0.5814, \overline{\overline{SR}}\left(\overline{\overline{AL_4}}\right)=0.6298,$ $\overline{\overline{SR}}\left(\overline{\overline{AL_5}}\right)=0.6357.$	$\overline{\overline{AL_1}} > \overline{\overline{AL_5}} > \overline{\overline{AL_4}} > \overline{\overline{AL_3}} > \overline{\overline{AL_2}}.$
Proposed 2-TLNDW PGHM	$\overline{\overline{SR}}\left(\overline{\overline{AL_1}}\right)=0.6205, \overline{\overline{SR}}\left(\overline{\overline{AL_2}}\right)=0.4639,$ $\overline{\overline{SR}}\left(\overline{\overline{AL_3}}\right)=0.4987, \overline{\overline{SR}}\left(\overline{\overline{AL_4}}\right)=0.5874,$ $\overline{\overline{SR}}\left(\overline{\overline{AL_5}}\right)=0.6011.$	$\overline{\overline{AL_1}} > \overline{\overline{AL_5}} > \overline{\overline{AL_4}} > \overline{\overline{AL_3}} > \overline{\overline{AL_2}}.$

From Table 10, we can see that the ranking order obtained from the proposed method based on developed aggregation operator and the methods developed Wang et al. [34], Wu et al. [38] are same. This shows the validity of the proposed method. Yet, it cannot manifest the advantages of the developed method due to same ranking results.

Further, in the following we will show the advantages of the developed method.

C. The advantages of the developed method

(1) The developed method is based on the 2-TLNDWPHM operator and the method presented by Wei [34] is based on 2-TLNNWBM operator. Both the methods have the characteristics of considering interrelationship among two input arguments and the only difference between them is that the developed aggregation operators also remove the effect of awkward data which may be too low or too high. In order to show this advantage, we give the following example.

Example 2. We can only change some data in the Example 1. We slightly change the value of alternative $\overline{\overline{AL_1}}$ with respect to the attribute $\overline{\overline{CT_4}}$. That is the value $\langle (s_4,0),(s_1,0),(s_3,0)\rangle$ is changed to $\langle (s_3,0),(s_2,0),(s_4,0)\rangle$ and the score values and ranking order are given in Table 11.

Table 11. The score values and ranking orders obtained from different methods

Approach	Score values	Ranking order
2-LNNWBM [34]	$\overline{\overline{SR}}\left(\overline{\overline{AL_1}}\right)=0.6215, \overline{\overline{SR}}\left(\overline{\overline{AL_2}}\right)=0.4648,$ $\overline{\overline{SR}}\left(\overline{\overline{AL_3}}\right)=0.5642, \overline{\overline{SR}}\left(\overline{\overline{AL_4}}\right)=0.6145,$ $\overline{\overline{SR}}\left(\overline{\overline{AL_5}}\right)=0.6243.$	$\overline{\overline{AL_5}} > \overline{\overline{AL_4}} > \overline{\overline{AL_4}} > \overline{\overline{AL_3}} > \overline{\overline{AL_2}}.$
2-TLNNWGBM [34]	$\overline{\overline{SR}}\left(\overline{\overline{AL_1}}\right)=0.6178, \overline{\overline{SR}}\left(\overline{\overline{AL_2}}\right)=0.4606,$ $\overline{\overline{SR}}\left(\overline{\overline{AL_3}}\right)=0.5622, \overline{\overline{SR}}\left(\overline{\overline{AL_4}}\right)=0.6080,$ $\overline{\overline{SR}}\left(\overline{\overline{AL_5}}\right)=0.6198.$	$\overline{\overline{AL_5}} > \overline{\overline{AL_1}} > \overline{\overline{AL_4}} > \overline{\overline{AL_3}} > \overline{\overline{AL_2}}.$
Proposed Method 2-TLNDWPHM operator	$\overline{\overline{SR}}\left(\overline{\overline{AL_1}}\right)=0.6392, \overline{\overline{SR}}\left(\overline{\overline{AL_2}}\right)=0.4629,$ $\overline{\overline{SR}}\left(\overline{\overline{AL_3}}\right)=0.5814, \overline{\overline{SR}}\left(\overline{\overline{AL_4}}\right)=0.6298,$ $\overline{\overline{SR}}\left(\overline{\overline{AL_5}}\right)=0.6357.$	$\overline{\overline{AL_1}} > \overline{\overline{AL_5}} > \overline{\overline{AL_4}} > \overline{\overline{AL_3}} > \overline{\overline{AL_2}}.$

Proposed Method 2-TLNDWPGHM	$\overline{\overline{SR}}\left(\overline{\overline{AL}}_1\right)=0.6056, \overline{\overline{SR}}\left(\overline{\overline{AL}}_2\right)=0.4639,$ $\overline{\overline{SR}}\left(\overline{\overline{AL}}_3\right)=0.4987, \overline{\overline{SR}}\left(\overline{\overline{AL}}_4\right)=0.5874,$ $\overline{\overline{SR}}\left(\overline{\overline{AL}}_5\right)=0.6011.$	$\overline{\overline{AL}}_1 > \overline{\overline{AL}}_5 > \overline{\overline{AL}}_4 > \overline{\overline{AL}}_3 > \overline{\overline{AL}}_2.$

From Table 11, we can notice that when we slightly change the value of the alternative $\overline{\overline{AL}}_1$ with respect to the attribute $\overline{\overline{CT}}_4$ in Table 2, then the ranking order obtained from the proposed method remain the same, while that acquired from the method developed by Wang et al.[34] is totally different. The best alternative remains the same in the proposed approach while utilizing the Wang et el. [34] approach based on 2-TLNNWBM and 2-TLNNWGBM, the best alternative is $\overline{\overline{AL}}_5$. The main reason behind these different ranking orders is that, the aggregation operators developed by Wang et al. [34] just only consider the interrelationship among input arguments and does not have the capacity of removing the bad impact of awkward data on final ranking result. While, the proposed approach is based on the proposed aggregation operators have the property of removing the effect of awkward data and consider the interrelationship among input arguments. The proposed aggregation operators are based on Dombi operational laws which have a general parameter, that makes the decision process more flexible. So the developed aggregation operator in this article is more general and practical to be used in solving MAGDM problems.

(2) Compare with the approach based on Hamy mean operator

To compare the developed approach with that of Hamy mean operator proposed by Wu et al. [38], we take another Example adapted from [12]. The Hamy mean operator proposed by Wu et al. [38] can also consider the interrelationship among input arguments.

Example 3. Let there is an investment company who wants to invest some money in the available four companies as a group of alternatives $\overline{\overline{AL}}_b\,(b=1,2,...,4)$. These four companies are respectively, a car company denoted by $\overline{\overline{AL}}_1$, a food company denoted by $\overline{\overline{AL}}_2$, , a computer company denoted by $\overline{\overline{AL}}_3$ and an arm company denoted by $\overline{\overline{AL}}_4$. These four potential alternatives are assessed by decision makers with respect to the following three attributes (1) the risk denoted by $\overline{\overline{CT}}_1$; (2) the growth denoted by $\overline{\overline{CT}}_2$; and (3) The environmental impact denoted by $\overline{\overline{CT}}_3$ with weight vector $(0.4,0.2,0.4)^T$. The assessment information is provided in the form of 2-TLNNs and is given in Table 12.

Table.12. The 2-TLN decision matrix

	$\overline{\overline{CT}}_1$	$\overline{\overline{CT}}_2$	$\overline{\overline{CT}}_3$
$\overline{\overline{AL}}_1$	$\langle(s_2,0),(s_4,0),(s_4,0)\rangle$	$\langle(s_4,0),(s_1,0),(s_3,0)\rangle$	$\langle(s_2,0),(s_4,0),(s_4,0)\rangle$
$\overline{\overline{AL}}_2$	$\langle(s_4,0),(s_3,0),(s_3,0)\rangle$	$\langle(s_2,0),(s_3,0),(s_2,0)\rangle$	$\langle(s_4,0),(s_1,0),(s_2,0)\rangle$
$\overline{\overline{AL}}_3$	$\langle(s_5,0),(s_1,0),(s_3,0)\rangle$	$\langle(s_3,0),(s_2,0),(s_2,0)\rangle$	$\langle(s_2,0),(s_4,0),(s_2,0)\rangle$
$\overline{\overline{AL}}_4$	$\langle(s_3,0),(s_5,0),(s_1,0)\rangle$	$\langle(s_3,0),(s_1,0),(s_2,0)\rangle$	$\langle(s_3,0),(s_1,0),(s_2,0)\rangle$

The score values and ranking results obtained by the proposed aggregation operators and the 2-TLNWHM operator, 2-TLNWDHM operator are given in Table 13.

From Table 13, one can notice that the ranking order obtained from the developed aggregation operators and that of obtained by 2-TLNWHM operator, and 2-TLNWDHM operator are totally different. From the proposed aggregation operator the best alternative is $\overline{\overline{AL}}_3$, while the worst one is $\overline{\overline{AL}}_1$, and from the 2-TLNWHM operator or 2-TLNWDHM operator proposed in Wu et al. [38], the best alternative is $\overline{\overline{AL}}_4$, while the worst one remain the same. The main reason behind different ranking order is that the both the aggregation operators can consider the interrelationship between input arguments, but the developed aggregation operator have two more characteristics. It can remove the effect of awkward data and proposed aggregation operators are based on Dombi operational laws, which have a general parameter that makes the information aggregation process more flexible. Therefore the developed aggregation operators are more flexible and general to be used in solving MAGDM problems.

Table 13. The score values and ranking orders obtained from different methods

Approach	Score values	Ranking order
2-LNNWHM [38]	$\overline{\overline{SR}}\left(\overline{\overline{AL}}_1\right)=0.7337, \overline{\overline{SR}}\left(\overline{\overline{AL}}_2\right)=0.7917,$ $\overline{\overline{SR}}\left(\overline{\overline{AL}}_3\right)=0.8367, \overline{\overline{SR}}\left(\overline{\overline{AL}}_4\right)=0.8406.$	$\overline{\overline{AL}}_4 > \overline{\overline{AL}}_3 > \overline{\overline{AL}}_2 > \overline{\overline{AL}}_1.$
2-TLNNWDHM [38]	$\overline{\overline{SR}}\left(\overline{\overline{AL}}_1\right)=0.1691, \overline{\overline{SR}}\left(\overline{\overline{AL}}_2\right)=0.2082,$ $\overline{\overline{SR}}\left(\overline{\overline{AL}}_3\right)=0.2695, \overline{\overline{SR}}\left(\overline{\overline{AL}}_4\right)=0.2868.$	$\overline{\overline{AL}}_4 > \overline{\overline{AL}}_3 > \overline{\overline{AL}}_2 > \overline{\overline{AL}}_1.$
Proposed Method 2-TLNDWPHM operator	$\overline{\overline{SR}}\left(\overline{\overline{AL}}_1\right)=0.5228, \overline{\overline{SR}}\left(\overline{\overline{AL}}_2\right)=0.5246,$ $\overline{\overline{SR}}\left(\overline{\overline{AL}}_3\right)=0.7052, \overline{\overline{SR}}\left(\overline{\overline{AL}}_4\right)=0.6902.$	$\overline{\overline{AL}}_3 > \overline{\overline{AL}}_4 > \overline{\overline{AL}}_2 > \overline{\overline{AL}}_1.$
Proposed Method 2-TLNDWPGHM	$\overline{\overline{SR}}\left(\overline{\overline{AL}}_1\right)=0.3678, \overline{\overline{SR}}\left(\overline{\overline{AL}}_2\right)=0.4755,$ $\overline{\overline{SR}}\left(\overline{\overline{AL}}_3\right)=0.4998, \overline{\overline{SR}}\left(\overline{\overline{AL}}_4\right)=0.4921.$	$\overline{\overline{AL}}_3 > \overline{\overline{AL}}_4 > \overline{\overline{AL}}_2 > \overline{\overline{AL}}_1.$

VII CONCLUSION

In this article firstly, we proposed some new operational laws for 2-TLNNs based on Dombi T-norm and Dombi T-conorm. Secondly, we proposed some new aggregation operators on these operational laws such as 2-tuple linguistic neutrosophic Dombi power Heronian mean operator, 2-tuple linguistic neutrosophic Dombi weighted power Heronian mean operator, 2-tuple linguistic neutrosophic Dombi power geometric Heronian mean operator and 2-tuple linguistic neutrosophic Dombi weighted power geometric Heronian mean operator. We also discussed it properties and few special cases with respect to parameters. Furthermore, we developed an algorithm for solving MAGDM problems under 2-tuple linguistic neutrosophic environment. We also show the advantages of the developed MAGDM approaches by comparing with some existing MAGDM approaches. The main advantages of the developed aggregation operators are The developed aggregation operators are based on Dombi operational laws, which consists of general parameter, that makes the information aggregation process more flexible. The developed aggregation operators have two characteristics at a time, firstly, it can vanish the effect of awkward data by taking the advantage of PA operator, Secondly, it can consider the interrelationship among the input arguments by taking the advantages of HM operator. For these reasons the developed MAGDM method based on these developed aggregation operator is more general and reasonable.

In future research, we will extend power Heronian mean operators to some new extension such as 2-tuple linguistic cubic neutrosophic, 2-tuple linguistic Double valued neutrosophic and so on. At the same time, we also research on some applications in energy and supply chain management.

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Introduction on Some New Results on Interval-Valued Neutrosophic Graphs

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Abstract

In this paper, inspired by the concept of generalized single-valued neutrosophic graphs(GSVNG) of the first type, we define yet another generalization of neutrosophic graph called the generalized interval- valued neutrosophic graph of 1 type (GIVNG1) in addition to our previous work on complex neutrosophic graph (CNG1) in [47]. We will also show a matrix representation for this new generalization. Many of the fundamental properties and characteristics of this new concept is also studied. Like the concept CNG1 in [47], the concept of GIVNG1 is another extension of generalized fuzzy graphs 1 (GFG1) and GSVNG1.

1. Introduction

In order to efficiently handle real life scenarios that conatins uncertain information,neutrosophic set(NS) theory, established by Smarandache [32], is put forward from the perspective of philosophical standpoints through regarding the degree of indeterminacy or neutrality as an independent element. As a result, many extended forms of fuzzy sets such as classical fuzzy sets [45], intuitionistic fuzzy sets [3-4], interval-valued fuzzy sets [40] and interval-valued intuitionistic fuzzy sets [5] could be seen as reduced forms of NS theory. In a NS, a true membership degree T, an indeterminacy membership degree I and a falsity membership degree F constitute the whole independent membership degrees owned by each element. However, it is noticed that the range of T , I and F falls within a real standard or nonstandard unit interval$]^-0, 1^+[$, hence it is difficult in applying NSs to many kinds of real world situations due to the limitation of T , I and F . Therefore, an updated form called single valued neutrosophic sets (SVNSs) was designed by Smarandache firstly [32]. Then, several properties in terms of SVNSs were further explored by Wang et al. [43]. In addition, it is relatively tough for experts to provide the three membership degrees with exact values, sometimes the form of interval numbers outperform the exact values in many practical situations. Inspired by this issue, Wang et al. [43] constructed interval neutrosophic sets concept (INSs) that performs better in precision and flexibility. Thus, INSs could be regarded as an extension of SVNSs. Moreover, some recent works about NSs, INSs and SVNSs along with their applications could be found in [13-15, 22,35, 53-59].

To studying the relationship between objects or events, the concept of Graph is thus created. In classical crisp graph theory, each of the two vertices (representing object or event) can assign two crisp value, 0 (not related/connected) or 1 (related/connected). The approach of fuzzy graph is a generalization the classical graph by allowing the degree of relationship (i.e. the membership value) to be anywhere in [0,1] for the edges, and it also assign membership values for the vertices. In the context of fuzzy graph, there is a rule that must be satisfied by all the edges and vertices, as follows:

the membership value of an edge must always be less than or equal to both the membership values of its two adjacent vertices. (*)

In over one hundred research papers, the further generalization of fuzzy graphs were studied, such as intuitionistic graphs, interval valued fuzzy graphs [7, 25, 28, 29]and interval-valued intuitionistic fuzzy graphs [24].However, such generalization still preserve (*) that was established since the period of fuzzy graphs.

As a result, Samanta et al. [39]analysed the concept of generalized fuzzy graphs (GFG), which was derived from the concept of fuzzy graph while removing the confinement of (*) . He had also studied some major advantages of GFG, such as completeness and regularity, by some proven facts. These authors had further developed GFG into two types, namely: generalized fuzzy graphs of first type (GFG1), generalized fuzzy graphs based on second type (GFG2). Each type of GFG can likewise be created by matrices just as in the case of some fuzzy graphs. The authors had also justified that the concept of fuzzy graphs on previous literatures are limited to representing some very particular systems such as social network, and therefore GFG is claimed to be capable to put to use on a much wider range of different scenario.

On the other hand, when the description of an object or a relation is both indeterminate and uncertain, it may be handled by fuzzy[23], intuitionistic fuzzy, interval-valued fuzzy, interval-valued intuitionistic fuzzy graphs and Set-valued graphs [2]. So, for this purpose, another new concept: neutrosophic graphs based on literal indeterminacy (I),were proposed by Smarandache [34]to deal with such situations. Such concept was published in a book by the same author collaborating with Vasantha et al.[42]. Later on, Smarandache[30-31] further introduced yet a new concept for neutrosophic graph theory, this time using the neutrosophic truth-values (*T,I,F*). He also gave various characterization on neutrosophicgraph, such as theneutrosophic edge graphs, neutrosophic vertex graphs and neutrosophic vertex-edge graphs. Later on[33], Smarandache himself further generalized the concept of neutrosophic graphs, and yield even more new structures such as neutrosophic offgraph, neutrosophic bipolar graphs, neutrosophic tripolar graphs and neutrosophic multipolar graphs. After which, the study of neutrosophic vertex-edge graphs has captured the attention of most researchers, and thus having more generalizations derived from it.

In 2016, using the concepts of SVNSs, Broumi et al.[8] investigated on the concept of single-valued neutrosophic graphs, and formulated certain types of single-valued neutrosophic graphs (SVNGs). After that, Broumi et al.introduced in [9,10,16,17,36]: the necessity of neighbourhood degree of a vertices and closed neighborhood degree of vertices in single-valued neutrosophic graph, isolated-SVNGs, Bipolar-SVNGs, complete bipolar-SVNGs, regular bipolar-SVNGs, uniform-SVNGs. In[11-12,18], also they studied the concept of interval-valued neutrosophic graphs and the importance of strong interval-valued neutrosophic graph, where different methods such as union, join, intersection and complement have been further investigated. In [35], Broumi et al. proposed some computing procedure in Matlab for neutrosophic operational matrices. Broumi et al.[37] developed a Matlab toolbox for interval valued neutrosophic matrices for computer applications. Akram and Shahzadi [6] introduced a new version of SVNGs that are different from those proposed in [8,36],and studied some of their properties. Ridvan[20] presented a new approach to neutrosophic graph theory with applications. Malarvizhi and Divya[38] presented the the ideas of antipodal single valued neutrosophic graph. Karaaslan and Davvaz[21] explore some interesting properties of single-valued neutrosophic graphs.Krishnarajet al.[1] introduced the concept of perfect and status in single valued neutrosophic graphs and investigated some of their properties.

Krishnaraj et al. [26] also analysed the concepts self-centered single valued neutrosophic graphs and discussed the properties of this concept with various examples, while Mohmed Ali et al.[41]extended it further to interval valued neutrosophic graphs[11].Kalyan and Majumdar [27] introduce the concept of single valued neutrosophic digraphs and implemented it in solving a multicriterion decision making problems.

The interval-valued neutrosophic graphs studied in the literature [11, 12], like the concept of fuzzy graph, is nonetheless bounded with the following condition familiar to (*):

The edge membership value is lessser than the minimum of its end vertex values, whereas the edge indeterminacy-membership value is lesser than the maximum of its end vertex values or greater than the maximum of its end vertex values. Also the edge non-membership value is lesser than the minimum of its end vertex values or is greater than the maximum of its end vertex values. (**)

Broumi et al.[19]had thus followed the approach of Samanta et al. [39], by suggesting the removal of (**) and presented the logic of generalized single-valued neutrosophic graph of type1 (GSVNG1). This is also a generalization from generalized fuzzy graph of type1 [39].

The main goal of this work is to further generalize the method of GSVNG1 to interval-valued neutrosophic graphs of first type (GIVNG1), for which all the true, indeterminacy, and false membership values, are inconsistent. Similarly, the appropriate matrix representation of GIVNG1 will also be given.

The results in this article is further derived from a conference paper [46] that we have published one year ago in IEEE. On the other hand, we have just published a paper on complex neutrosophic graph (CNG1), which is another extension of GFG1 and GSVNG1 in [47]. The approach ofGIVNG1 and CNG1, however, are distint from one another. This is becausethe concept of CNG1 extends the existing theory by generalizing real numbers into complex numbers, while all the entries remain single valued; whereas in this paper,the concept of GIVNG1 extends the existing theory by generalizing the single valued entriesinto inter-valued entries,while all those inter-valued entries remains as real numbers

Thus, following the format of our recent conference paper [46], this paper has been aligned likewise: In Section 2, the concept on neutrosophic sets, single- valued neutrosophic sets, interval valued neutrosophic graph and generalized single-valued neutrosophic graphs of type 1are described in detail, which serves as cornerstones for all the contents in later parts of the article. In Section 3, we present the ideas of GIVNG1 illustrated with an example. Section 4 gives the appropriate way to represent the matrix of GIVNG1.

2. Some preliminary results

In this part, we briefly include some basic definitions in [19, 32, 43,47] related to NS, SVNSs, interval- valued neutrosophic graphs(IVNG) and generalized single-valued neutrosophic graphs of type 1(GSNG1).

Definition 2.1 [32]. Let X be a series of points with basic elements in X presented by x; then the neutrosophic set(NS) A (is an object in the form A = {< x: $T_A(x)$, $I_A(x)$, $F_A(x)$ >, $x \in X$}, defines the functions $T, I, F: X \to]^-0,1^+[$denoted by the truth-membership, indeterminacy-membership, and falsity-membership of the element x X to the set A showing the condition:

$$^-0 \leq T_A(x) + I_A(x) + F_A(x) 3^+. \qquad (1)$$

The functions $T_A(x)$, $I_A(x)$ and $F_A(x)$ are absolute standard or non-standard subsets of $]^-0,1^+[$.

As it is very complex i applying NSs to real issues, Smarandache [32] developed the notion of a SVNS, which is an occurrence of a NS and can be employed in practical scientific and engineering applications.

Definition 2.2 [43]**.** Let X be a series of points (objects) with basic elements in X presented by x. A single valued neutrosophic set A ($SVNS$ A) is characterized by truth-membership $T_A(x)$, an indeterminacy-membership $I_A(x)$, and a falsity-membership $F_A(x)$.$\forall x \in X$, $T_A(x)$, $I_A(x)$, $F_A(x) \in [0, 1]$. A SVNS A can be rewritten as

$$A = \{< x: T_A(x), I_A(x), F_A(x) >, x \in X\} \qquad (2)$$

Definition 2.3 [19] Suppose the following conditions are expected:

a) V is a null-void set.

b) $\rho_T, \rho_I, \rho_F : \mathrm{V} \to [0,1]$

c) E = $\{(\rho_T(u),\rho_T(v)) \mid u, v \in V\}$,
F = $\{(\rho_I(u),\rho_I(v)) \mid u, v \in V\}$,
G = $\{(\rho_F(u),\rho_F(v)) \mid u, v \in V\}$.

d) α: E $\to$ [0, 1] , β: F $\to$ [0, 1], δ: G $\to$ [0, 1] are three functions.

e) $\rho = (\rho_T, \rho_I, \rho_F)$; and
$\omega = (\omega_T, \omega_I, \omega_F)$ with
$\omega_T(u, v) = \alpha((\rho_T(x), \rho_T(v)))$,
$\omega_I(u, v) = \beta((\rho_I(x), \rho_I(v)))$,
$\omega_F(u, v) = \delta((\rho_F(x), \rho_F(v)))$, $\forall u, v \in V$.

Then:

i) The structure $\xi = < V, \rho, \omega >$ is considered to a GSVNG1.

Remark: ρ which depends on ρ_T, ρ_I, ρ_F. And ω which depends on α, β. Hence there are 7 mutually alone parameters in total which make up a CNG1: V, ρ_T, ρ_I, ρ_F, α, β, δ.

ii) $\forall \in V$, x is considered to bea *vertex* of ξ. The whole set V is termed as the *vertex set* of ξ.

iii) $\forall\, u, v \in V$, (u, v) is considered to be a *directed edge* of ξ .
In special, (u, v) is considered to be a *loop* of ξ.

iv) For all vertex : $\rho_T(v)$, $\rho_I(v)$, $\rho_F(v)$ are considered to be the T, I, and F *membership value*, respectively of that vertex v . Moreover, if $\rho_T(v) = \rho_I(v) = \rho_F(v) = 0$, then v is supposed to be a *null* vertex.

v) Correspondingly, for all edge (u, v) : $\omega_T(u, v)$, $\omega_I(u, v)$, $\omega_F(u, v)$ considered to have T, I, and F respectively membership value, of that directed edge (u, v). In addition, if $\omega_T(u, v) = \omega_I(u, v) = \omega_F(u, v) = 0$,then (u, v) is considered to be a *null* directed edge.

Remark : It obeys that: $V\times V\rightarrow[0,1]$.

3. Concepts related to Generalized Interval Valued Neutrosophic Graph of First Type

In the modelling of real life scenarios with neutrosophic system (i.e. neutrosophic sets, neutrosophic graphs, etc), the truth-membership value, indeterminate-membership value, and false-membership value are often taken to mean the *ratio out of a population* who find reasons to “agree”, “be neutral” and “disagree”. It can also by any 3 analogous descriptions, such as “seek excitement” “loft around” and “relax”. However, there are real life situations where even such ratio out of the population are subject to conditions. One typical example will be having the highest and the lowest value. For example “It is expected that 20% to 30% of the population of country X will disagree with the Prime Minister’s decision”.

To model such an event, therefore, we generalize Definition 2.3 so that the truth-membership value, indeterminate-membership value, and false-membership value can be any closed subinterval of [0,1], instead of a single number from [0,1].Such generalization is further derived from [46], which is a conference paper that we have just published on this topic.

Note: For all the other parts of this work, we will define:

$$\Delta_1 = \{[x, y]: 0 \le x \le y \le 1\}$$

Definition 3.1 [46]. Let the statements below holds good:

a) V is considered as a non-empty set.

b) $\tilde{\rho}_T, \tilde{\rho}_I, \tilde{\rho}_F$ are three functions, eachfrom V to Δ_1.

c) $E = \{(\tilde{\rho}_T(u), \tilde{\rho}_T(v)) \mid u, v \in V\}$,
$F = \{(\tilde{\rho}_I(u), \tilde{\rho}_I(v)) \mid u, v \in V\}$,
$G = \{(\tilde{\rho}_F(u), \tilde{\rho}_F(v)) \mid u, v \in V\}$.

d) $\alpha: E \to \Delta_1$, $\beta: F \to \Delta_1$, $\delta: G \to \Delta_1$ are three functions.

e) $\tilde{\rho} = (\tilde{\rho}_T, \tilde{\rho}_I, \tilde{\rho}_F)$; and
$\tilde{\omega} = (\tilde{\omega}_T, \tilde{\omega}_I, \tilde{\omega}_F)$ with
$\tilde{\omega}_T(u, v) = \alpha((\tilde{\rho}_T(x), \tilde{\rho}_T(v)))$,
$\tilde{\omega}_I(u, v) = \alpha((\tilde{\rho}_I(x), \tilde{\rho}_I(v)))$,,
$\tilde{\omega}_F(u, v) = \alpha((\tilde{\rho}_F(x), \tilde{\rho}_F(v)))$,,
for every $u, v \in V$.

Then:

i) The structure $\xi = <V, \tilde{\rho}, \tilde{\omega}>$is said to be a *generalized interval-valued neutrosophic graph of type* 1 (GIVNG1) .

ii) For each $\in V$, x is termed to be a *vertex* of ξ. The spanned set V is named the *vertex set* of ξ.

iii) $\forall\, u, v \in V$, (u, v) is termed to be a *directed edge* of ξ In particular, (u, v) is said to be a *loop* of ξ .

iv) $\forall$ vertex : $\tilde{\rho}_T(v)$, $\tilde{\rho}_I(v)$, $\tilde{\rho}_F(v)$ are said to be the *truth-membership value*, *indeterminate-membership value*, and *false-membership value*, respectively, of that vertex v. Moreover, if $\tilde{\rho}_T(v) = \tilde{\rho}_I(v) = \tilde{\rho}_F(v) = [0,0]$, then v is deemed as *void* vertex.

v) Similarly, for each edge (u, v) : $\tilde{\omega}_T(u, v)$, $\tilde{\omega}_I(u, v)$, $\tilde{\omega}_F(u, v)$ are said to be the *T*, *I*, and *F membership value* respectively of that directed edge (u, v). Moreover, if $\tilde{\omega}_T(u, v) = \tilde{\omega}_I(u, v) = \tilde{\omega}_F(u, v) = [0,0]$,then (u, v) is said to be a *void* directed edge.

Remark : It follows that : $V \times V \to \Delta_1$.

Note that every vertex v in a GIVNG1 have a single, undirected loop, whether void or not. Also each of the distinct vertices u, v in a GIVNG1possses *two* directed edges, resulting from (u, v) and (v, u), whether void or not.

We study that in classical graph theory, we handle ordinary (or undirected) graphs, and also some simple graphs. Further we relate our GIVNG1 with it, we now give the below definition.

Definition 3.2. [46] Given ξ = <V, $\tilde{\rho}$, $\tilde{\omega}$> be a GIVNG1.

a) If $\tilde{\omega}_T(a,b) = \tilde{\omega}_T(b,a)$, $\tilde{\omega}_I(a,b) = \tilde{\omega}_I(b,a)$ and $\tilde{\omega}_F(a,b) = \tilde{\omega}_F(b,a)$, then $\{a, b\} = \{(a, b), (b, a)\}$ is said to be an (*ordinary*) *edge* of ξ. Moreover, $\{a, b\}$ is said to be a *void* (ordinary) edge if both (a, b) and (b, a) are void.

b) If $\tilde{\omega}_T(u,v) = \tilde{\omega}_T(v,u), \tilde{\omega}_I(u,v) = \tilde{\omega}_I(v,u)$ and $\tilde{\omega}_F(u,v) = \tilde{\omega}_F(v,u)$ holds good for all $v \in V$, then ξ is considered to be *ordinary* (or *undirected*), else it is considered to be *directed*.

c) When all the loops of ξ are becoming void, then ξ is considered to be *simple*.

In the following section, we discuss a real life scenario, for which GSVNG1 is insufficient to model it - it can only be done by using GIVNG1.

Example 3.3. Part 3.3.1 The scenario

Country X has 4 cities $\{a, b, c, d\}$. The cities are connected with each other by some roads, there are villages along the four roads (all of them are two way) $\{a, b\}$, $\{c, b\}$, $\{a, c\}$ and $\{d, b\}$. As for the other roads, such as $\{c, b\}$, they are either non-exitsant, or there are no population living along them (e.g. industrial area, national park, or simply forest).The legal driving age of Country X is 18.The prime minister of Country X would like to suggest an amendment of the legal driving age from 18 to 16. Before conducting a countrywide survey involving all the citizens, the prime minister discuss with all members of the parliament about the expected outcomes.

The culture and living standard of all the cities and villages differ from one another. In particular:

The public transport in c is so developed that few will have to drive. The people are rich enough to buy even air tickets. People in d tend to be more open minded in culture. Moreover, sports car exhibitions and shows are commonly held there. A fatal road accident just happened along {c,b}, claiming the lives of five unlicensed teenagers racing at 200km/h. {a, c} is governed by an opposition leader who is notorious for being very uncooperative in all parliament affairs.

Eventually the parliament meeting was concluded with the following predictions:

		Expected percentage of citizens that will -					
		support		be neutral		Against	
		at least	at most	at least	at most	at least	at most
Cities	*a*	0.1	0.4	0.2	0.6	0.3	0.7
	b	0.3	0.5	0.2	0.5	0.2	0.5
	c	0.1	0.2	0.0	0.3	0.1	0.2
	d	0.5	0.7	0.2	0.4	0.1	0.2
Villages along the roads	{*a*,*b*}	0.2	0.3	0.1	0.4	0.4	0.7
	{*c*,*b*}	0.1	0.2	0.1	0.2	0.5	0.8
	{*a*,*c*}	0.1	0.7	0.1	0.8	0.1	0.7
	{*d*,*b*}	0.2	0.3	0.3	0.6	0.2	0.5

Without loss of generality: It is either $\{c, d\}$ does not exist, or there are no people living there, so all the six values – support (least, most), neutral (least, most), against(least, most), are all zero.

Part 3.3.2 Representing with GIVNG1

When we start from step a to e in def. 3.1 , to illustrate the schema with a special GIVNG1

a) Take$V_0 = \{a, b, c, d\}$

b) In line with the scenario, present the three functions $\tilde{\rho}_T, \tilde{\rho}_I, \tilde{\rho}_F$, as illustrated in the following table.

	a	*b*	*c*	*d*
$\tilde{\rho}_T$	[0.1,0.4]	[0.3,0.5]	[0.1,0.2]	[0.5,0.7]
$\tilde{\rho}_I$	[0.2,0.6]	[0.2,0.5]	[0.0,0.3]	[0.2,0.4]
$\tilde{\rho}_F$	[0.3,0.7]	[0.2,0.5]	[0.1,0.2]	[0.1,0.2]

c) By statement c) from Definition 3.1: Let

$E_0 = \{(\tilde{\rho}_T(u), \tilde{\rho}_T(v)) \mid u, v \in \{a, b, c, d\}\}$

$F_0 = \{(\tilde{\rho}_I(u), \tilde{\rho}_I(v)) \mid u, v \in \{a, b, c, d\}\}$

$G_0 = \{(\tilde{\rho}_F(u), \tilde{\rho}_F(v)) \mid u, v \in \{a, b, c, d\}\}$

d) In accordance with the scenario, define

$\alpha : \mathrm{E}_0 \to \Delta_1, \beta : \mathrm{F}_0 \to \Delta_1, \delta : \mathrm{G}_0 \to \Delta_1,$

as illustrated in the following tables.

$\alpha((\tilde{\rho}_T(u), \tilde{\rho}_T(v)))$:

v u	a	b	c	d
a	0	[0.2,0.3]	[0.1,0.7]	0
b	[0.2,0.3]	0	[0.1,0.2]	[0.2,0.3]
c	[0.1,0.7]	[0.1,0.2]	0	0
d	0	[0.2,0.3]	0	0

$\alpha((\tilde{\rho}_I(u), \tilde{\rho}_I(v)))$:

v u	a	b	c	d
a	0	[0.1,0.4]	[0.1,0.8]	0
b	[0.1,0.4]	0	[0.1,0.2]	[0.3,0.6]
c	[0.1,0.8]	[0.1,0.2]	0	0
d	0	[0.3,0.6]	0	0

$\alpha((\tilde{\rho}_F(u), \tilde{\rho}_F(v)))$:

v u	a	b	c	d
a	0	[0.4,0.7]	[0.1,0.7]	0
b	[0.4,0.7]	0	[0.5,0.8]	[0.2,0.5]
c	[0.1,0.7]	[0.5,0.8]	0	0
d	0	[0.2,0.5]	0	0

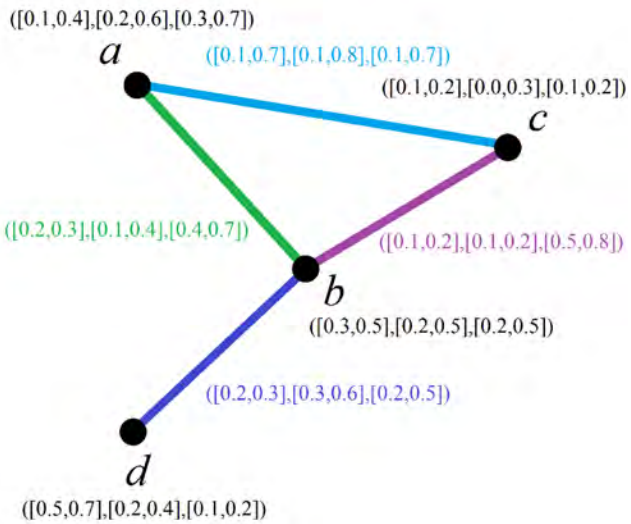

Figure 1

e) By statement e) from Definition 3.1, let

$\tilde{\rho}_0 = (\tilde{\rho}_T, \tilde{\rho}_I, \tilde{\rho}_F)$; and

$\tilde{\omega}_0=(\tilde{\omega}_T, \tilde{\omega}_I, \tilde{\omega}_F)$ with

$\tilde{\omega}_T(u, v) = \alpha\,((\tilde{\rho}_T(u), \tilde{\rho}_T(v))),$

$\tilde{\omega}_T(u, v) = \beta\,((\tilde{\rho}_I(u), \tilde{\rho}_I(v))),$

$\tilde{\omega}_T(u, v) = \delta\,((\tilde{\rho}_F(u), \tilde{\rho}_F(v))),$

for all $u, v \in V_0$. We now have formed $<V_0, \tilde{\rho}_0, \tilde{\omega}_0>$, which is a GIVNG1.

The way of showing the concepts of $<V_0, \tilde{\rho}_0, \tilde{\omega}_0>$ is by exerting a diagram that is similar with graphs as in classical graph theory, as given in the figure 1 below

That is to say, only the non-void edges (whether directed or ordinary) and vertices been drawn in the picture shown above.

Also, understanding the fact that, in classical graph theory GT, a graph isdenoted by adjacency matrix, for which the entries are either a positive integer (connected) or 0 (which is not connected).

This motivates us to present a GIVNG1, by a matrix as well. However, instead of a single value which defines the value that is either 0 or 1, there are *three* values to handle: $\tilde{\omega}_T, \tilde{\omega}_I, \tilde{\omega}_F$, with each of them being elements of Δ_1. Moreover, each of the vertices themselves also contains $\tilde{\rho}_T, \tilde{\rho}_I, \tilde{\rho}_F$, which should be taken into account as well.

4. Illustration of GIVNG1by virtue adjacency matrix

Section 4.1 Algorithms representing GIVNG1

In light of two ways that are similar to other counterparts, the focal point of interest in the following part is to express the notion of GIVNG1.

Suppose $\xi = <V, \tilde{\rho}, \tilde{\omega}>$ is a GIVNG1 where $V=\{v_1, v_2,\ldots, v_n\}$ denotes the vertex set (i.e. GIVNG1 has finite vertices). Remember that GIVNG1has its edge membership values (T,I,F) depending on the membership values (T,I,F) of adjacent vertices, in accordance with the functions α, β, δ.

Furthermore:

$\tilde{\omega}_T(u, v) = \alpha\,((\tilde{\rho}_T(u), \tilde{\rho}_T(v)))$ for all $v \in V$, where

$\alpha : \mathrm{E} \rightarrow \Delta_1$, and $E = \{(\tilde{\rho}_T(u), \tilde{\rho}_T(v)) \mid u, v \in V\}$,

$\tilde{\omega}_I(u, v) = \beta\,((\tilde{\rho}_I(u), \tilde{\rho}_I(v)))$ for all $u, v \in V$, where

$\beta : \mathrm{F} \rightarrow \Delta_1$, and $\mathrm{F} = \{(\tilde{\rho}_I(u), \tilde{\rho}_I(v)) \mid u, v \in V\}$,

$\tilde{\omega}_F(u, v) = \delta\,((\tilde{\rho}_F(u), \tilde{\rho}_F(v)))$ for all $u, v \in V$, where

$\delta : \mathrm{G} \rightarrow \Delta_1$, and $\mathrm{G} = \{(\tilde{\rho}_F(u), \tilde{\rho}_F(v)) \mid u, v \in V\}$.

First we will form an $n \times n$ matrix as presented

$$\tilde{\mathbf{S}} = \left[\tilde{\mathbf{a}}_{i,j}\right]_n = \begin{pmatrix} \tilde{\mathbf{a}}_{1,1} & \tilde{\mathbf{a}}_{1,2} & \cdots & \tilde{\mathbf{a}}_{1,n} \\ \tilde{\mathbf{a}}_{2,1} & \tilde{\mathbf{a}}_{2,2} & & \tilde{\mathbf{a}}_{2,n} \\ \vdots & & \ddots & \vdots \\ \tilde{\mathbf{a}}_{n,1} & \tilde{\mathbf{a}}_{n,2} & \cdots & \tilde{\mathbf{a}}_{n,n} \end{pmatrix},$$

For each i, j, $\tilde{\mathbf{a}}_{i,j} = (\tilde{\omega}_T(v_i, v_j), \tilde{\omega}_I(v_i, v_j), \tilde{\omega}_F(v_i, v_j))$

That is to say, for an element of the matrix $\tilde{S}$, different from taking numbers 0 or 1 according to classical literatures, we usually take the element as an ordered set involving 3 closed subintervals of [0,1].

Remark : Due to the fact that ξ could only have undirected loops, the dominating diagonal elements of $\tilde{S}$ is not multiplied by 2, which is shown as adjacency matrices from classical literatures. It is noted that 0 represents void, 1 for directed ones and 2 for undirected ones.

At the same time, considering $\tilde{\rho}_T$, $\tilde{\rho}_I$, $\tilde{\rho}_F$ is included in ξ, which also deserves to be considered.

Therefore another matrix $\tilde{\boldsymbol{R}}$ is given in the following part.

$$\tilde{\mathbf{R}} = \left[\tilde{\mathbf{R}}_i\right]_{n,1} = \begin{pmatrix} \tilde{\mathbf{r}}_1 \\ \tilde{\mathbf{r}}_2 \\ \vdots \\ \tilde{\mathbf{r}}_n \end{pmatrix},$$

Where

$$\begin{aligned} \tilde{\mathbf{r}}_i &= (\tilde{\rho}_T(v_i), \tilde{\rho}_I(v_i), \tilde{\rho}_F(v_i)) \\ &= \left([\rho_T^L(v_i), \rho_T^U(v_i)], [\rho_I^L(v_i), \rho_I^U(v_i)], [\rho_F^L(v_i), \rho_F^U(v_i)]\right) \forall. \end{aligned}$$

In order to complete the task of describing the whole ξ in our way, the matrix $\tilde{\boldsymbol{R}}$ is augmented with $\tilde{S}$. Then $[\tilde{\boldsymbol{R}} \,|\, \tilde{S}]$ is represented as an *adjacency matrix of* GIVNG, which is presented below.

$$[\mathbf{R} \,|\, \mathbf{S}] = \begin{pmatrix} \tilde{\mathbf{r}}_1 & \tilde{\mathbf{a}}_{1,1} & \tilde{\mathbf{a}}_{1,2} & \cdots & \tilde{\mathbf{a}}_{1,n} \\ \tilde{\mathbf{r}}_2 & \tilde{\mathbf{a}}_{2,1} & \tilde{\mathbf{a}}_{2,2} & & \tilde{\mathbf{a}}_{2,n} \\ & \vdots & & \ddots & \vdots \\ \tilde{\mathbf{r}}_n & \tilde{\mathbf{a}}_{n,1} & \tilde{\mathbf{a}}_{n,2} & \cdots & \tilde{\mathbf{a}}_{n,n} \end{pmatrix},$$

where $\tilde{\mathbf{a}}_{i,j} = (\tilde{\omega}_T(v_i, v_j), \tilde{\omega}_I(v_i, v_j), \tilde{\omega}_F(v_i, v_j))$,

and $\tilde{\mathbf{r}}_i = ([\rho_T^L(v_i), \rho_T^U(v_i)], [\rho_I^L(v_i), \rho_I^U(v_i)], [\rho_F^L(v_i), \rho_F^U(v_i)])$, $\forall i$ and j.

It is worth noticing $[\tilde{\boldsymbol{R}}\,|\,\tilde{S}]$ is not a square matrix ($n \times (n+1)$matrix), thus this kind of representation will aid us to save another *divided* ordered set to denote the values of vertices as $\tilde{\rho}_T$, $\tilde{\rho}_I$, $\tilde{\rho}_F$.

For both edges and vertices, it is imperative to separately handle each of three kinds of membership values in several situations. Consequently, by means of three $n\times(n+1)$ matrices, we aim to give a brand-new way for expressing the whole ξ, denoted as $[\tilde{\boldsymbol{R}}\,|\,\tilde{S}]_T$, $[\tilde{\boldsymbol{R}}\,|\,\tilde{S}]_I$ and $[\tilde{\boldsymbol{R}}\,|\,\tilde{S}]_F$,each of them is resulted from $[\tilde{\boldsymbol{R}}\,|\,\tilde{S}]$ through taking a single kind of membership values from the corresponding elements.

$$[\tilde{\mathbf{R}}\,|\,\tilde{\mathbf{S}}]_T = [\tilde{\mathbf{R}}_T\,|\,\tilde{\mathbf{S}}_T] = \begin{pmatrix} \tilde{\rho}_T(v_1) & \tilde{\omega}_T(v_1,v_1) & \tilde{\omega}_T(v_1,v_2) & & \tilde{\omega}_T(v_1,v_n) \\ \tilde{\rho}_T(v_2) & \tilde{\omega}_T(v_2,v_1) & \tilde{\omega}_T(v_2,v_2) & \cdots & \tilde{\omega}_T(v_2,v_n) \\ & \vdots & & \ddots & \vdots \\ \tilde{\rho}_T(v_n) & \tilde{\omega}_T(v_n,v_1) & \tilde{\omega}_T(v_n,v_2) & \cdots & \tilde{\omega}_T(v_n,v_n) \end{pmatrix},$$

$$[\tilde{\mathbf{R}}\,|\,\tilde{\mathbf{S}}]_I = [\tilde{\mathbf{R}}_I\,|\,\tilde{\mathbf{S}}_I] = \begin{pmatrix} \tilde{\rho}_I(v_1) & \tilde{\omega}_I(v_1,v_1) & \tilde{\omega}_I(v_1,v_2) & & \tilde{\omega}_i(v_1,v_n) \\ \tilde{\rho}_I(v_2) & \tilde{\omega}_I(v_2,v_1) & \tilde{\omega}_I(v_2,v_2) & \cdots & \tilde{\omega}_I(v_2,v_n) \\ & \vdots & & \ddots & \vdots \\ \tilde{\rho}_I(v_n) & \tilde{\omega}_I(v_n,v_1) & \tilde{\omega}_I(v_n,v_2) & \cdots & \tilde{\omega}_I(v_n,v_n) \end{pmatrix},$$

$$[\tilde{\mathbf{R}}\,|\,\tilde{\mathbf{S}}]_F = [\tilde{\mathbf{R}}_F\,|\,\tilde{\mathbf{S}}_F] = \begin{pmatrix} \tilde{\rho}_F(v_1) & \tilde{\omega}_F(v_1,v_1) & \tilde{\omega}_F(v_1,v_2) & & \tilde{\omega}_F(v_1,v_n) \\ \tilde{\rho}_F(v_2) & \tilde{\omega}_F(v_2,v_1) & \tilde{\omega}_F(v_2,v_2) & \cdots & \tilde{\omega}_F(v_2,v_n) \\ & \vdots & & \ddots & \vdots \\ \tilde{\rho}_F(v_n) & \tilde{\omega}_F(v_n,v_1) & \tilde{\omega}_F(v_n,v_2) & \cdots & \tilde{\omega}_F(v_n,v_n) \end{pmatrix}.$$

$[\tilde{\boldsymbol{R}}\,|\,\tilde{S}]_T$, $[\tilde{\boldsymbol{R}}\,|\,\tilde{S}]_I$ and $[\tilde{\boldsymbol{R}}\,|\,\tilde{S}]_F$ should be stated respectively with the *true adjacency matrix*, the *indeterminate adjacency matrix*, and *false adjacency matrix* of ξ .

Remark 1 : If $[\tilde{\boldsymbol{R}}\,|\,\tilde{S}]_I = [\tilde{\boldsymbol{R}}\,|\,\tilde{S}]_F = [[0,0]]_{n,\,n+1}$, $\tilde{\boldsymbol{R}}_T = [[1,1]]_{n,1}$, all the entries of $\tilde{\boldsymbol{S}}_T$ are either [1, 1] or [0, 0] , then ξ is reduced to a graph in classical literature. Moreover, if that $\tilde{\boldsymbol{S}}_T$ is symmetric and the main diagonal elements are being 0, we have ξ is further condensed to a simple ordinary graph in literature.

Remark 2 : If $[\tilde{\boldsymbol{R}}\,|\,\tilde{S}]_I = [\tilde{\boldsymbol{R}}\,|\,\tilde{S}]_F = [[0,0]]_{n,\,n+1}$, and all the entries of $[\tilde{\boldsymbol{R}}\,|\,\tilde{S}]_T = [[a_{i,j}, a_{i,j}]]_{n,\,n+1}$, then ξ is reduced to a generalized fuzzy graph type 1 (GFG1).

Remark 3 : If $[\tilde{\boldsymbol{R}}\,|\,\tilde{S}]_T = [[a_{i,j}, a_{i,j}]]_{n,\,n+1}$, $[\tilde{\boldsymbol{R}}\,|\,\tilde{S}]_T = [[b_{i,j}, b_{i,j}]]_{n,\,n+1}$, $[\tilde{\boldsymbol{R}}\,|\,\tilde{S}]_T = [[c_{i,j}, c_{i,j}]]_{n,\,n+1}$, then ξ is thus reduced to GSVNG1.

Section 4.2 : Case study to illustrate our example in this paper

For our example in the set-up by the last way i.e. with three matrices: $[\tilde{\boldsymbol{R}}\,|\,\tilde{\boldsymbol{S}}]_T$, $[\tilde{\boldsymbol{R}}\,|\,\tilde{\boldsymbol{S}}]_I$ and $[\tilde{\boldsymbol{R}}\,|\,\tilde{\boldsymbol{S}}]_F$:

$$[R\,|\,S]_T = \begin{pmatrix} [0.1,0.4] & [0,0] & [0.2,0.3] & [0.1,0.7] & [0,0] \\ [0.3,0.5] & [0.2,0.3] & [0,0] & [0.1,0.2] & [0.2,0.3] \\ [0.1,0.2] & [0.1,0.7] & [0.1,0.2] & [0,0] & [0,0] \\ [0.1,0.7] & [0,0] & [0.2,0.3] & [0,0] & [0,0] \end{pmatrix}$$

$$[R\,|\,S]_I = \begin{pmatrix} [0.2,0.6] & [0,0] & [0.1,0.4] & [0.1,0.8] & [0,0] \\ [0.2,0.5] & [0.1,0.4] & [0,0] & [0.1,0.2] & [0.3,0.6] \\ [0.0,0.3] & [0.1,0.8] & [0.1,0.2] & [0,0] & [0,0] \\ [0.2,0.4] & [0,0] & [0.3,0.6] & [0,0] & [0,0] \end{pmatrix}$$

$$[R\,|\,S]_F = \begin{pmatrix} [0.3,0.7] & [0,0] & [0.4,0.7] & [0.1,0.7] & [0,0] \\ [0.2,0.5] & [0.4,0.7] & [0,0] & [0.5,0.8] & [0.2,0.5] \\ [0.1,0.2] & [0.1,0.7] & [0.5,0.8] & [0,0] & [0,0] \\ [0.1,0.2] & [0,0] & [0.2,0.5] & [0,0] & [0,0] \end{pmatrix}$$

5. Postulated results on ordinary GIVNG1

We now illustrate some theoretical results that are derived from the definition of ordinary GIVNG1, as well as its indication with adjacency matrix. Since we focus on the basic GIVNG1, all the edges which we will be referring to are termed as ordinary edges.

Definition 5.1 The addition operation + is defined on Δ_1 as follows: $[x\,,\,y] + [z\,,\,t] = [\,x + y\,,\,z + t]$ for all $x, y, z, t \in [0,1]$.

Definition 5.2 Let $\xi = <V, \tilde{\rho}, \tilde{\omega}>$ be an ordinary GIVNG1. Let $V = \{v_1, v_2, \ldots, v_n\}$ to be the vertex set of ξ. Then, $\forall$ *i*, the *degree* of v_i, symbolised as $\tilde{D}(v_i)$, is well-defined to be the ordered set

$$(\tilde{D}_T(v_i), \tilde{D}_I(v_i), \tilde{D}_F(v_i)),$$

for which, $\tilde{D}_T(v_i)$ represents the *degree* of v_i and

$$a) \quad \tilde{D}_T(v_i) = \left[\sum_{r=1}^{n} \omega_T^L(v_i, v_r) + \omega_T^L(v_i, v_i), \sum_{r=1}^{n} \omega_T^U(v_i, v_r) + \omega_T^U(v_i, v_i)\right]$$

$$b) \quad \tilde{D}_I(v_i) = \left[\sum_{r=1}^{n} \omega_I^L(v_i, v_r) + \omega_I^L(v_i, v_i), \sum_{r=1}^{n} \omega_I^U(v_i, v_r) + \omega_I^U(v_i, v_i)\right]$$

$$c) \quad \tilde{D}_F(v_i) = \left[\sum_{r=1}^{n} \omega_F^L(v_i, v_r) + \omega_F^L(v_i, v_i), \sum_{r=1}^{n} \omega_F^U(v_i, v_r) + \omega_F^U(v_i, v_i)\right]$$

Remark 1 : In resemblance to classical graph theory, each undirected loop has both its ends connected to the similar vertex and so is counted twice.

Remark 2 : Every value of $\tilde{D}_T(v_i)$, $\tilde{D}_I(v_i)$ and $\tilde{D}_F(v_i)$ are elements of Δ_1 instead of a single number.

Definition 5.3 : Given $\xi = <V, \tilde{\rho}, \tilde{\omega}>$ and $V = \{v_1, v_2, \ldots, v_n\}$ are respectively an ordinary GIVNG1 and the vertex set of ξ . Then, the *quantity of edges in* ξ, represented as E_ξ and we describe the ordered set $(\tilde{E}_T, \tilde{E}_I, \tilde{E}_F)$ for which

$$a) \quad \tilde{E}_T = \left[\sum_{\{r,s\}\subseteq\{1,2,\ldots,n\}} \omega_T^L(v_r, v_s) \; , \sum_{\{r,s\}\subseteq\{1,2,\ldots,n\}} \omega_T^U(v_r, v_s) \right]$$

$$b) \quad \tilde{E}_I = \left[\sum_{\{r,s\}\subseteq\{1,2,\ldots,n\}} \omega_I^L(v_r, v_s) \; , \sum_{\{r,s\}\subseteq\{1,2,\ldots,n\}} \omega_I^U(v_r, v_s) \right]$$

$$c) \quad \tilde{E}_F = \left[\sum_{\{r,s\}\subseteq\{1,2,\ldots,n\}} \omega_F^L(v_r, v_s) \; , \sum_{\{r,s\}\subseteq\{1,2,\ldots,n\}} \omega_F^U(v_r, v_s) \right]$$

Remark 1: We count each edge only once in classical graph theory, as given by $\{r,s\} \subseteq \{1,2,\ldots,n\}$.

For instance, if $\tilde{\omega}_T(v_a, v_b)$ is added, we will not add $\tilde{\omega}_T(v_b, v_a)$ again since $\{a, b\} = \{b, a\}$.

Remark 2 : Each values of $\tilde{E}_T$, $\tilde{E}_I$ and $\tilde{E}_F$ are elements of Δ_1 instead of a single number, and need not be 0 or 1 as in classical graph literature. Consequently, it is called "amount" of edges, instead of the "number" of edges as in the classical reference.

$\tilde{E}_T$, $\tilde{E}_I$, $\tilde{E}_F$ are closed subintervals of [0,1], and $\tilde{D}_T(v_i)$, $\tilde{D}_I(v_i)$, $\tilde{D}_F(v_i)$ are also closed subintervals of [0,1] for each vertex v_i. These give rise to the following lemmas

Lemma 5.4 : *Let* $\xi = <V, \tilde{\rho}, \tilde{\omega}>$ *be an ordinary GIVNG1. Let* $V = \{v_1, v_2, \ldots, v_n\}$ *to be the vertex set of* ξ *. Denote*

$a)$ $\tilde{\omega}_T(v_i, v_j) = [\phi_{T,(i,j)}, \psi_{T,(i,j)}]$

$b)$ $\tilde{\omega}_I(v_i, v_j) = [\phi_{I,(i,j)}, \psi_{I,(i,j)}]$

$c)$ $\tilde{\omega}_F(v_i, v_j) = [\phi_{F,(i,j)}, \psi_{F,(i,j)}], \forall\, i, j$

For each i we have:

i) $\tilde{D}_T(v_i) = \left[\sum_{r=1}^{n}\phi_{T,(i,r)} + \phi_{T,(i,i)}, \sum_{r=1}^{n}\psi_{T,(i,r)} + \psi_{T,(i,i)}\right]$,

ii) $\tilde{D}_I(v_i) = \left[\sum_{r=1}^{n}\phi_{I,(i,r)} + \phi_{I,(i,i)}, \sum_{r=1}^{n}\psi_{I,(i,r)} + \psi_{I,(i,i)}\right]$,

iii) $\tilde{D}_F(v_i) = \left[\sum_{r=1}^{n}\phi_{F,(i,r)} + \phi_{T,(i,i)}, \sum_{r=1}^{n}\psi_{F,(i,r)} + \psi_{F,(i,i)}\right]$.

Furthermore:

iv) $\tilde{E}_T = \left[\sum_{\{r,s\}\subseteq\{1,2,\ldots,n\}}\phi_{T,(r,s)}, \sum_{\{r,s\}\subseteq\{1,2,\ldots,n\}}\psi_{T,(r,s)}\right]$,

v) $\tilde{E}_I = \left[\sum_{\{r,s\}\subseteq\{1,2,\ldots,n\}}\phi_{I,(r,s)}, \sum_{\{r,s\}\subseteq\{1,2,\ldots,n\}}\psi_{I,(r,s)}\right]$,

vi) $\tilde{E}_F = \left[\sum_{\{r,s\}\subseteq\{1,2,\ldots,n\}}\phi_{F,(r,s)}, \sum_{\{r,s\}\subseteq\{1,2,\ldots,n\}}\psi_{F,(r,s)}\right]$.

Proof : We can proof it directly by applying Def.5.1, Def. 5.2 and Def. 5.3. In the following two theorems, we introduce two theorems which both as a modified version of the well-known theorem in classical graph theory.

"We know that the sum of the degree of invariably its vertices is twice the number of its edges for any classical graph."

Theorem 5.5 : Let $\xi = <V, \tilde{\rho}, \tilde{\omega}>$ bean ordinary GIVNG1. Then

$$\sum_{r=1}^{n}\tilde{D}(v_r) = 2\tilde{E}_\xi$$

Proof : As $\tilde{D}(v_i) = (\tilde{D}_T(v_i), \tilde{D}_I(v_i), \tilde{D}_F(v_i))$ for all i , and $\tilde{E}_\xi = (\tilde{E}_T, \tilde{E}_I, \tilde{E}_F)$. It is enough to show that $2\tilde{E}_T = \sum_{r=1}^{n}\tilde{D}_T(v_r)$:

$$\tilde{E}_T = \left[\sum_{\{r,s\}\subseteq\{1,2,\ldots,n\}}\omega_T^L(v_r,v_s), \sum_{\{r,s\}\subseteq\{1,2,\ldots,n\}}\omega_T^U(v_r,v_s)\right]$$

$$= \left[\sum_{\substack{\{r,s\}\subseteq\{1,2,\ldots,n\}\\ r\neq s}}\omega_T^L(v_r,v_s) + \sum_{r=1}^{n}\omega_T^L(v_r,v_r),\right.$$

$$\left.\sum_{\substack{\{r,s\}\subseteq\{1,2,\ldots,n\}\\ r\neq s}}\omega_T^U(v_r,v_s) + \sum_{r=1}^{n}\omega_T^U(v_r,v_r)\right]$$

Since $\{r, s\} = \{s, r\}$ for all s and r ,

$$2\tilde{E}_T = \begin{bmatrix} 2\sum_{\substack{\{r,s\}\subseteq\{1,2,\ldots,n\}\\ r\neq s}} \omega_T^L(v_r, v_s) + 2\sum_{r=1}^{n}\omega_T^L(v_r, v_r), \\ 2\sum_{\substack{\{r,s\}\subseteq\{1,2,\ldots,n\}\\ r\neq s}} \omega_T^U(v_r, v_s) + 2\sum_{r=1}^{n}\omega_T^U(v_r, v_r) \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{\substack{r\in\{1,2,\ldots,n\}\\ s\in\{1,2,\ldots,n\}\\ r\neq s}} \omega_T^L(v_r, v_s) + 2\sum_{r=1}^{n}\omega_T^L(v_r, v_r), \\ \sum_{\substack{r\in\{1,2,\ldots,n\}\\ s\in\{1,2,\ldots,n\}\\ r\neq s}} \omega_T^U(v_r, v_s) + 2\sum_{r=1}^{n}\omega_T^U(v_r, v_r) \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{\substack{r\in\{1,2,\ldots,n\}\\ s\in\{1,2,\ldots,n\}}} \omega_T^L(v_r, v_s) + \sum_{r=1}^{n}\omega_T^L(v_r, v_r), \\ \sum_{\substack{r\in\{1,2,\ldots,n\}\\ s\in\{1,2,\ldots,n\}}} \omega_T^U(v_r, v_s) + \sum_{r=1}^{n}\omega_T^U(v_r, v_r) \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{r=1}^{n}\sum_{s=1}^{n}\omega_T^L(v_r, v_s) + \sum_{r=1}^{n}\omega_T^L(v_r, v_r), \\ \sum_{r=1}^{n}\sum_{s=1}^{n}\omega_T^U(v_r, v_s) + \sum_{r=1}^{n}\omega_T^U(v_r, v_r) \end{bmatrix}$$

$$= \begin{bmatrix} \sum_{r=1}^{n}\left(\sum_{s=1}^{n}\omega_T^L(v_r, v_s) + \omega_T^L(v_r, v_r)\right), \\ \sum_{r=1}^{n}\left(\sum_{s=1}^{n}\omega_T^U(v_r, v_s) + \omega_T^U(v_r, v_r)\right), \end{bmatrix}$$

$$= \sum_{r=1}^{n}\tilde{D}_T(v_r).$$

This finishes the proof. ∎

6. Conclusion

The idea of GSVNG1 was extended to introduce the concept of generalized interval-valued neutrosophic graph of type 1(GIVNG1). The matrix representation of GIVNG1 was also introduced. The future direction of this research includes the study of completeness, regularity of GIVNG1, and also denote the notion of generalized interval-valued neutrosophic graphs of type 2.As GIVNG1 (in this paper) and CNG1 (from [47]) are both extensions of the existing concepts of CFG1 and GSVNG1, but in two entirely different directions, the future direction of this research also includes further extensions from GIVNG1 and CNG1, that incorporates *both* the inter-valued entries (as in GIVNG1) and complexity of numbers (as in CNG1), and the study of scenarios that necessitate such extensions [48-52].

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Intuitionistic Bipolar Neutrosophic Set and Its Application to Intuitionistic Bipolar Neutrosophic Graphs

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ABSTRACT. This manuscript is devoted to study a new concept of intuitionistic bipolar neutrosophic set with the operations like union, intersection and complement. Also, an application to intuitionistic bipolar neutrosophic graph with examples are developed. Further, we presented the Cartesian product, cross product, lexicographic product and strong product with suitable examples.

1. INTRODUCTION

The neutrosophic set has three independent parts, namely truth-membership degree, indeterminacy-membership degree and falsity-membership degree provided the sum of these values lies between 0 and 3; therefore, it is applied to many different areas, such as algebra [21, 22] and decision-making problems (see [26] and references therein). Au-thor Smarandache [25] remarks the difference between neutrosophic set and logic, and intuitionistic fuzzy set and logic. Interval neutrosophic sets with applications in BCK/BCI-algebra and KU-algebras are developed in [1, 2, 18, 22, 24]. Single valued neutrosophic graphs with their degree, order and size are established in [12, 13]. Intuitionistic fuzzy set is initiated by Atanassov as a significant generalization of fuzzy set. Intuitionistic fuzzy sets are very useful while representing a problem by a linguistic variable, given in terms of a membership function only, seems too complicated. Recently intuitionistic fuzzy sets have been applied to many fields such as logic programming, medical diagnosis, decision mak-ing problems etc. On the other hand, bipolar fuzzy sets are extension of fuzzy sets whose membership degree ranges from $[-1, 1]$. The membership degree $(0, 1]$ represents that an object satisfies a certain property whereas the membership degree $[-1, 0)$ represents that the element satisfies the implicit counter-property. The positive information indicates that the consideration to be possible and negative information indicates that the consideration is granted to be impossible. Application to decision making of bipolar neutrosophic sets and bipolar neutrosophic graph structures are studied in [3, 4], respectively. Neutrosophic bipolar vague sets and its application to graph theory are analysed in [19, 20]. Similarity measures of bipolar neutrosophic sets and its application to decision making are established in [26]. In [10, 15], intuitionistic neutrosophic sets and its relations are discussed. Furthermore, intuitionistic neutrosophic graph structures are extensively studied in [6, 7]. Motivated by these works, we established intuitionistic bipolar neutrosophic set and its application to intuitionistic bipolar neutrosophic graphs.

The major contribution of this work as follows:

- Newly introduced intuitionistic bipolar neutrosophic set with the operations like union, intersection and complement.
- Its application to Intuitionistic Bipolar Neutrosophic Graph (IBNG) with example are developed. Also neutrosophic bipolar vague subgraph, induced subgraph, strong and complete IBNG are established.
- Further we presented the Cartesian product, cross product, lexicographic product and strong product with suitable examples. The obtained results give the generalization of above mentioned works.

2. Preliminaries

Definition 2.1. [12] Let X be a space of points (objects), with a generic element in X denoted by x. A Single Valued Neutrosophic Set (SVNS) A in X is characterized by truth-membership function $T_A(x)$, indeterminacy-membership function $I_A(x)$ and falsity-membership-function $F_A(x)$. For each point x in X, $T_A(x), F_A(x), I_A(x) \in [0,1]$,

$$A = \{\langle x, T_A(x), F_A(x), I_A(x)\rangle,\ x \in X\} \text{ and } 0 \le T_A(x) + I_A(x) + F_A(x) \le 3.$$

Definition 2.2. [13] A neutrosophic graph is defined as a pair $G^* = (V, E)$ where (i) $V = \{v_1, v_2, .., v_n\}$ such that $T_1 : V \to [0,1]$, $I_1 : V \to [0,1]$ and $F_1 : V \to [0,1]$ denote the degree of truth-membership function, indeterminacy function and falsity-membership function, respectively and

$$0 \le T_A(u) + I_A(u) + F_A(u) \le 3, \text{ for } u \in V.$$

(ii) $E \subseteq V \times V$ where $T_2 : E \to [0,1]$, $I_2 : E \to [0,1]$ and $F_2 : E \to [0,1]$ are such that

$$T_2(uv) \le \min\{T_1(u), T_1(v)\},\ I_2(uv) \le \min\{I_1(u), I_1(v)\},$$
$$F_2(uv) \le \max\{F_1(u), F_1(v)\} \quad \text{and } 0 \le T_2(uv) + I_2(uv) + F_2(uv) \le 3,\ \forall uv \in E.$$

Definition 2.3. [16] A bipolar neutrosophic set A in X is defined as an object of the form

$$A = \{\langle x, T^P(x), I^P(x), F^P(x), T^N(x), I^N(x), F^N(x)\rangle,\ x \in X\},$$

where $T^P, I^P, F^P : X \to [0,1]$ and $T^N, I^N, F^N : X \to [-1,0]$. The Positive membership degree $T^P(x), I^P(x), F^P(x)$ denote the truth membership, indeterminate membership and false membership of an element $x \in X$ corresponding to a bipolar neutrosophic set A and the negative membership degree $T^N(x), I^N(x), F^N(x)$ denotes the truth membership, indeterminate membership and false membership of an element $x \in X$ to some implicit counter-property corresponding to a bipolar neutrosophic set A.

Definition 2.4. [16] Let X be a non-empty set. Then we call

$$A = \{\langle x, T^P(x), I^P(x), F^P(x), T^N(x), I^N(x), F^N(x)\rangle, x \in X\}$$

a bipolar single valued neutrosophic relation on X such that $T_A^P(x,y) \in [0,1], I_A^P(x,y) \in [0,1], F_A^P(x,y) \in [0,1]$ and $T_A^N(x,y) \in [-1,0], I_A^N(x,y) \in [-1,0], F_A^N(x,y) \in [-1,0]$.

Definition 2.5. [3, 4] Let $A = (T_A^P, I_A^P, F_A^P, T_A^N, I_A^N, F_A^N)$ and $B = (T_B^P, I_B^P, F_B^P, T_B^N, I_B^N, F_B^N)$ be bipolar single valued neutrosophic set on X. If $B = (T_B^P, I_B^P, F_B^P, T_B^N, I_B^N, F_B^N)$ is a bipolar single valued neutrosophic relation on $A = (T_A^P, I_A^P, F_A^P, T_A^N, I_A^N, F_A^N)$ then

$$T_B^P(xy) \leq \min(T_A^P(x), T_A^P(y)), \qquad T_B^N(xy) \geq \max(T_A^N(x), T_A^N(y)),$$
$$I_B^P(xy) \geq \max(I_A^P(x), I_A^P(y)), \qquad I_B^N(xy) \leq \min(I_A^N(x), I_A^N(y)),$$
$$F_B^P(xy) \geq \max(F_A^P(x), F_A^P(y)), \qquad F_B^N(xy) \leq \min(F_A^N(x), F_A^N(y)).$$

A bipolar single valued neutrosophic relation B on X is called symmetric if $T_B^P(xy) = T_B^P(yx), I_B^P(xy) = I_B^P(yx), F_B^P(xy) = F_B^P(yx)$ and $T_B^N(xy) = T_B^N(yx), I_B^N(xy) = I_B^N(yx), F_B^N(xy) = F_B^N(yx)$ for all $xy \in X$.

Definition 2.6. [3, 4] A bipolar single-valued neutrosophic graph on a nonempty set X is a pair $G = (C, D)$, where C is a bipolar single-valued neutrosophic set on X and D is a bipolar single-valued neutrosophic relation in X such that

$$\text{(i) } T_D^P(xy) \leq \min(T_C^P(x), T_C^P(y)), I_D^P(xy) \leq \min(I_C^P(x), I_C^P(y)),$$
$$F_D^P(xy) \leq \max(F_C^P(x), F_C^P(y)),$$
$$\text{(ii) } T_D^N(xy) \geq \max(T_C^N(x), T_C^P(y)), I_D^N(xy) \geq \max(I_C^N(x), I_C^P(y)),$$
$$F_D^N(xy) \geq \min(F_C^N(x), F_C^P(y)),$$

for all $x, y \in X$.

Definition 2.7. [10, 15] An element x of X is called significant with respect to neutrosophic set A of X if the degree of truth-membership or indeterminacy-membership or falsity membership value, i.e $T_A(x)$ or $I_A(x)$ or $F_A(x) \geq 0.5$. Otherwise, we call it insignificant. Also, for neutrosophic set the truth-membership, indeterminacy-membership and falsity- membership all can not be significant.
we define an intuitionistic neutrosophic set by $A = \langle x, T_A(x), I_A(x), F_A(x)\rangle$, where $\min\{T_A(x), F_A(x)\} \leq 0.5$, $\min\{T_A(x), I_A(x)\} \leq 0.5$, & $\min\{I_A(x), F_A(x)\} \leq 0.5$, for all $x \in X$ with the condition $0 \leq \{T_A(x) + I_A(x) + F_A(x)\} \leq 2$

Definition 2.8. [10, 15] A INS Relation (INSR) is defined as a intuitionistic subset of $X \times Y$, having the form

$$R = \{< (x, y), T_R(x, y), I_R(x, y), F_R(x, y) >: x \in X, y \in Y\}$$

where,

$$T_R : X \times Y \to [0, 1], I_R : X \times Y \to [0, 1], F_R : X \times Y \to [0, 1]$$

satisfies the conditions
(i) at least one of this $T_R(x, y), I_R(x, y)$ and $F_R(x, y)$ is ≥ 0.5 and
(ii) $0 \leq \{T_A(x) + I_A(x) + F_A(x)\} \leq 2$. The colllection of all INSR on $X \times Y$ is denoted as $GR(X \times Y.)$

Definition 2.9. [6, 7] An intuitionistic neutrosophic graph is a pair $G = (A, B)$ with underlying set V, where $T_A, F_A, I_A : V \to [0, 1]$ denote the truth, falsity and indeterminacy membership values of the vertices in V and $T_B, F_B, I_B : E \subseteq V \times V \to [0, 1]$ denote the truth, falsity and indeterminacy membership values of the edges $kl \in E$ such that

$$(i) T_B(kl) \leq T_A(k) \wedge T_A(l), I_B(kl) \leq I_A(k) \wedge I_A(l), F_B(kl) \geq F_A(k) \wedge F_A(l)$$
$$(ii) T_B(kl) \wedge I_B(kl) \leq 0.5, T_B(kl) \wedge F_B(kl) \leq 0.5, I_B(kl) \wedge F_B(kl) \leq 0.5,$$
$$(iii) 0 \leq T_B(kl) + I_B(kl) + F_B(kl) \leq 2 \ \forall k, l \in V.$$

3. INTUITIONISTIC BIPOLAR NEUTROSOPHIC SET

Definition 3.1. An element x of X is called significant with respect to neutrosophic set A of X if the degree of truth-membership or indeterminacy-membership or falsity membership value, i.e $T_A(x)$ or $I_A(x)$ or $F_A(x) \geq 0.5$. Otherwise, we call it insignificant. Also, for neutrosophic set the truth-membership, indeterminacy-membership and falsity-membership all can not be significant. we define an intuitionistic bipolar neutrosophic set by

$$A = \langle x, T_A^P(x), I_A^P(x), F_A^P(x), T_A^N(x), I_A^N(x), F_A^N(x)\rangle$$

where

$$\min\{T_A^P, F_A^P\} \leq 0.5, \max\{T_A^N, F_A^N\} \geq -0.5, \min\{T_A^P, I_A^P\} \leq 0.5,$$
$$\max\{T_A^N, I_A^N\} \geq -0.5, \min\{F_A^P, I_A^P\} \leq 0.5, \max\{F_A^N, I_A^N\} \geq -0.5$$

$T_A^P : X \to [0,1], T_A^N : X \to [-1,0], I_A^P : X \to [0,1]$,
$I_A^N : X \to [-1,0], F_A^P : X \to [0,1], F_A^N : X \to [-1,0]$, with the conditions

$$0 \leq T_A^P(x) + I_A^P(x) + F_A^P(x) \leq 2, -2 \geq T_A^P(x) + I_A^P(x) + F_A^P(x) \geq 0.$$

Definition 3.2. A IBNS relation (IBNSR) is defined as a intuitionistic bipolar subset of $X \times Y$, having the form

$$R = \{< (x,y), T_R^P(x,y), I_R^P(x,y), F_R^P(x,y), T_R^N(x,y), I_R^N(x,y), F_R^N(x,y) >: x \in X, y \in Y\}$$

where,

$$T_R^P : X \times Y \to [0,1], I_R^P : X \times Y \to [0,1], F_R^P : X \times Y \to [0,1]$$
$$T_R^N : X \times Y \to [-1,0], I_R^N : X \times Y \to [-1,0], F_R^N : X \times Y \to [-1,0]$$

satisfy the conditions (i) at least one of this $T_R^P(x,y), I_R^P(x,y)$ and $F_R^P(x,y)$ is ≥ 0.5 at least one of this $T_R^N(x,y), I_R^N(x,y)$ and $F_R^N(x,y)$ is ≤ -0.5 and
(ii) $0 \leq T_A^P(x) + I_A^P(x) + F_A^P(x) \leq 2$, $-2 \geq T_A^P(x) + I_A^P(x) + F_A^P(x) \geq 0$.

Definition 3.3. Let $A_1 =< x, T_{A_1}^P(x), I_{A_1}^P(x), F_{A_1}^P(x), T_{A_1}^N(x), I_{A_1}^N(x), F_{A_1}^N(x) >$ and $A_2 =< x, T_{A_2}^P(x), I_{A_2}^P(x), F_{A_2}^P(x), T_{A_2}^N(x), I_{A_2}^N(x), F_{A_2}^N(x) >$ be two IBNSs. then $A_1 \subset A_2$ if any only if

$$T_{A_1}^P(x) \leq T_{A_2}^P(x), T_{A_1}^N(x) \geq T_{A_2}^N(x).$$
$$I_{A_1}^P(x) \leq I_{A_2}^P(x), I_{A_1}^N(x) \geq I_{A_2}^N(x).$$
$$F_{A_1}^P(x) \geq F_{A_2}^P(x), F_{A_1}^N(x) \leq F_{A_2}^N(x). \forall x \in X.$$

Definition 3.4. The union of two IBNSs A and B is also IBNS, whose truth membership, intermediate membership and false membership functions are,

$$T_{(A\cup B)}^P(x) = \max\{T_A^P(x), T_B^P(x)\}$$
$$I_{(A\cup B)}^P(x) = \min\{I_A^P(x), I_B^P(x)\}$$
$$F_{(A\cup B)}^P(x) = \min\{F_A^P(x), F_B^P(x)\},$$

and

$$T_{(A\cup B)}^N(x) = \min\{T_A^N(x), T_B^N(x)\}$$
$$T_{(A\cup B)}^N(x) = \max\{T_A^N(x), T_B^N(x)\}$$
$$T_{(A\cup B)}^N(x) = \max\{T_A^N(x), T_B^N(x)\},$$

for all $x \in X$.

Example 3.5. Let $A = \{((x_1, 0.7, 0.3, 0.4)^P(-0.6, -0.4, -0.3)^N), ((x_2, 0.5, 0.5, 0.8)^P (-0.6, -0.5, -0.4)^N)\}$ and $B = \{((x_1, 0.4, 0.7, 0.4)^P(-0.4, -0.7, -0.3)^N), ((x_2, 0.4, 0.3, 0.9)^P (-0.5, -0.6, -0.2)^N)\}$ be two IBNSs of X. Then by definition of union we get,

$A \cup B = \{((x_1, 0.7, 0.3, 0.3)^P(-0.6, -0.7, -0.3)^N), ((x_2, 0.5, 0.3, 0.8)^P(-0.6, -0.5, -0.2)^N)\}$

Definition 3.6. The intersection of two IBNSs A and B is also IBNS, whose truth-membership, indeterminacy-membership and falsity-membership functions are,

$$T^P_{(A\cap B)}(x) = \min\{T^P_A(x), T^P_B(x)\}$$
$$I^P_{(A\cap B)}(x) = \max\{I^P_A(x), I^P_B(x)\}$$
$$F^P_{(A\cap B)}(x) = \max\{F^P_A(x), F^P_B(x)\},$$

and

$$T^N_{(A\cap B)}(x) = \max\{T^N_A(x), T^N_B(x)\}$$
$$T^N_{(A\cap B)}(x) = \min\{T^N_A(x), T^N_B(x)\}$$
$$T^N_{(A\cap B)}(x) = \min\{T^N_A(x), T^N_B(x)\},$$

for all $x \in X$.

Example 3.7. For above example, then by definition of intersection, we obtain

$A \cap B = \{((x_1, 0.4, 0.3, 0.4)^P(-0.4, -0.4, -0.3)^N), ((x_2, 0.4, 0.3, 0.9)^P(-0.5, -0.5, -0.4)^N)\}$

Definition 3.8. The complement of IBNSs
$A =< x, T^P_A(x), I^P_A(x), F^P_A(x), T^N_A(x), I^N_A(x), F^N_A(x) >$ for all $x \in X$, is defined as

$$(T^P(x))^C = F^P(x), (I^P(x))^C = 1 - I^P(x), (F^P(x))^C = T^P(x),$$

and

$$(T^N(x))^C = F^N(x), (I^N(x))^C = -1 - I^N(x), (F^N(x))^C = T^N(x),$$

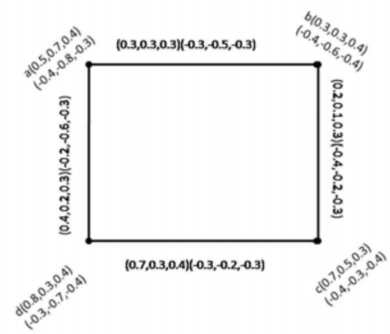
for all $x \in X$.

4. Intuitionistic Bipolar Neutrosophic Graphs

Definition 4.1. An Intuitionistic Bipolar Neutrosophic Graph (IBNG) is defined as a pair $G = (R, S)$, $R = (A^P, A^N)$ and $S = (B^P, B^N)$ where

(i) $R = \{r_1, r_2, ..., r_n\}$ such that, $T^P_A : R \to [0, 1], I^P_A : R \to [0, 1], F^P_A : R \to [0, 1], T^N_A : R \to [-1, 0], I^N_A : R \to [-1, 0]$, and $F^N_A : R \to [-1, 0]$ denote the degree of truth-membership, indeterminacy-membership and falsity-membership functions, respectively,
(ii) $S \subseteq R \times R$ where $T^P_B : R \times R \to [0, 1], I^P_B : R \times R \to [0, 1], F^P_B : R \times R \to [0, 1], T^N_B : R \times R \to [-1, 0], I^N_B : R \times R \to [-1, 0]$, and $F^N_B : R \times R \to [-1, 0]$
(iii) $T^P_B(rs) \le \min(T^P_A(r), T^P_A(s)), I^P_B(rs) \le \min(I^P_A(r), I^P_A(s)), F^P_B(rs) \le \max(F^P_A(r), F^P_A(s)),$
(iv) $T^P_B(rs) \wedge I^P_B(rs) \le 0.5, T^P_B(rs) \wedge F^P_B(rs) \le 0.5, I^P_B(rs) \wedge F^P_B(rs) \le 0.5.$
(v) $0 \le T^P_B(rs) + I^P_B(rs) + F^P_B(rs) \le 2.$
(vi) $T^N_B(rs) \ge \max(T^N_A(r), T^N_A(s)), I^N_B(rs) \ge \max(I^N_A(r), I^N_A(s)), F^N_B(rs) \ge \min(F^N_A(r), F^N_A(s)),$
(vii) $T^N_B(rs) \vee I^N_B(rs) \ge -0.5, T^N_B(rs) \vee F^N_B(rs) \ge -0.5, I^N_B(rs) \vee F^N_B(rs) \ge -0.5$
(viii) $0 \ge T^N_B(rs) + I^N_B(rs) + F^N_B(rs) \ge -2.$

Example 4.2. Consider a IBNGs such that $A = \{a, b, c, d\}$, $B = \{ab, bc, cd\}$ by routine condition we have,



$Figure\,1$: INTUITIONISTIC BIPOLAR NEUTROSOPHIC GRAPH

Definition 4.3. [3] A graph $G^{'} = (R^{'}, S^{'})$ is said to be subgraph of $G = (R, S)$ if

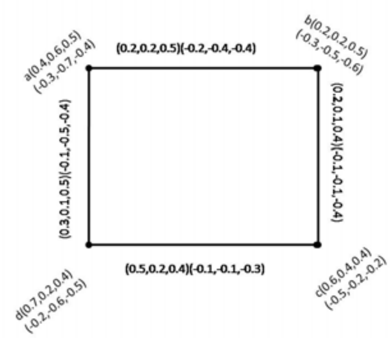
$$(T_A^{'})^P(r) \leq T_A^P(r), (I_A^{'})^P(r) \leq I_A^P(r), (F_A^{'})^P(r) \geq F_A^P(r)$$
$$(T_A^{'})^N(r) \geq T_A^N(r), (I_A^{'})^N(r) \geq T_A^N(r), (F_A^{'})^N(r) \leq F_A^N(r),$$

for all $r \in R$ and

$$(T_B^{'})^P(rs) \leq T_B^P(rs), (I_A^{'})^P(rs) \leq I_B^P(rs), (F_B^{'})^P(rs) \geq F_B^P(rs)$$
$$(T_B^{'})^N(rs) \geq T_B^N(rs), (I_B^{'})^N(rs) \geq T_B^N(rs), (F_B^{'})^N(rs) \leq F_B^N(rs),$$

for all $rs \in S$

Example 4.4. An IBNG subgraph is represented as Figure 2



$Figure\,2$: INTUITIONISTIC BIPOLAR NEUTROSOPHIC SUBGRAPH

Definition 4.5. A graph $G^{'} = (R', S')$ is said to be induced subgraph of $G = (R, S)$ if

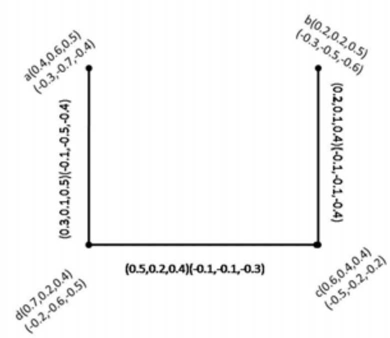
$$(T_A^{'})^P(r) = T_A^P(r), (I_A^{'})^P(r) = I_A^P(r), (F_A^{'})^P(r) = F_A^P(r)$$
$$(T_A^{'})^N(r) = T_A^N(r), (I_A^{'})^N(r) = T_A^N(r), (F_A^{'})^N(r) = F_A^N(r),$$

for all $r \in R$ and

$$(T_B^{'})^P(rs) = T_B^P(rs), (I_A^{'})^P(rs) = I_B^P(rs), (F_B^{'})^P(rs) = F_B^P(rs)$$
$$(T_B^{'})^N(rs) = T_B^N(rs), (I_B^{'})^N(rs) = T_B^N(rs), (F_B^{'})^N(rs) = F_B^N(rs),$$

for $rs \in S$

Example 4.6. An IBNG induced subgraph is represented as Figure 3.



$Figure\, 3$: INTUITIONISTIC BIPOLAR NEUTROSOPHIC INDUCED SUBGRAPH

Definition 4.7. A graph $G^{'} = (R', S')$ is said to be spanning subgraph of $G = (R, S)$ if

$$(T_B^{'})^P(rs) \leq T_B^P(rs), (I_A^{'})^P(rs) \leq I_B^P(rs), (F_B^{'})^P(rs) \geq F_B^P(rs)$$
$$(T_B^{'})^N(rs) \geq T_B^N(rs), (I_B^{'})^N(rs) \geq T_B^N(rs), (F_B^{'})^N(rs) \leq F_B^N(rs),$$

for all $rs \in S$

Definition 4.8. An IBNG $G = (R, S)$ is called strong IBNG if

$$(T_B^{'})^P(rs) = T_B^P(rs), (I_A^{'})^P(rs) = I_B^P(rs), (F_B^{'})^P(rs) = F_B^P(rs)$$
$$(T_B^{'})^N(rs) = T_B^N(rs), (I_B^{'})^N(rs) = T_B^N(rs), (F_B^{'})^N(rs) = F_B^N(rs),$$

for all $rs \in S$. S is the set of edges.

Definition 4.9. An IBNG $G = (R, S)$ is called complete IBNG if

$$(T_B^{'})^P(rs) = T_B^P(rs), (I_A^{'})^P(rs) = I_B^P(rs), (F_B^{'})^P(rs) = F_B^P(rs)$$
$$(T_B^{'})^N(rs) = T_B^N(rs), (I_B^{'})^N(rs) = T_B^N(rs), (F_B^{'})^N(rs) = F_B^N(rs),$$

for all $rs \in S$. R is the set of nodes.

Definition 4.10. The Cartesian product of two IBNGs G_1 and G_2 is denoted by the pair $G_1 \times G_2 = (R_1 \times R_2, S_1 \times S_2)$ and defined as

$$\begin{aligned} T^P_{A_1\times A_2}(kl) &= T^P_{A_1}(k) \wedge T^P_{A_2}(l)\\ I^P_{A_1\times A_2}(kl) &= I^P_{A_1}(k) \wedge I^P_{A_2}(l)\\ F^P_{A_1\times A_2}(kl) &= F^P_{A_1}(k) \vee F^P_{A_2}(l)\\ T^N_{A_1\times A_2}(kl) &= T^N_{A_1}(k) \vee T^N_{A_2}(l)\\ I^N_{A_1\times A_2}(kl) &= I^N_{A_1}(k) \vee I^N_{A_2}(l)\\ F^N_{A_1\times A_2}(kl) &= F^N_{A_1}(k) \wedge F^N_{A_2}(l), \end{aligned}$$

for all $kl \in R_1 \times R_2$. The membership value of the edges in $G_1 \times G_2$ can be calculated as,

$$\begin{aligned} (1)T^P_{B_1\times B_2}(k,l_1)(k,l_2) &= T^P_{A_1}(k) \wedge T^P_{B_2}(l_1l_2)\\ T^N_{B_1\times B_2}(k,l_1)(k,l_2) &= T^N_{A_1}(k) \vee T^N_{B_2}(l_1l_2), \end{aligned}$$

$$\begin{aligned} (2)I^P_{B_1\times B_2}(k,l_1)(k,l_2) &= I^P_{A_1}(k) \wedge I^P_{B_2}(l_1l_2)\\ I^N_{B_1\times B_2}(k,l_1)(k,l_2) &= I^N_{A_1}(k) \vee I^N_{B_2}(l_1l_2), \end{aligned}$$

$$\begin{aligned} (3)F^P_{B_1\times B_2}(k,l_1)(k,l_2) &= F^P_{A_1}(k) \vee F^P_{B_2}(l_1l_2)\\ F^N_{B_1\times B_2}(k,l_1)(k,l_2) &= F^N_{A_1}(k) \wedge F^N_{B_2}(l_1l_2), \end{aligned}$$

for all $k \in R_1, l_1l_2 \in S_2$.

$$\begin{aligned} (4)T^P_{B_1\times B_2}(k_1,l)(k_2,l) &= T^P_{A_2}(l) \wedge T^P_{B_2}(k_1k_2)\\ T^N_{B_1\times B_2}(k_1,l)(k_2,l) &= T^N_{A_2}(l) \vee T^N_{B_2}(k_1k_2), \end{aligned}$$

$$\begin{aligned} (5)I^P_{B_1\times B_2}(k_1,l)(k_2,l) &= I^P_{A_2}(l) \wedge I^P_{B_2}(k_1k_2)\\ I^N_{B_1\times B_2}(k_1,l)(k_2,l) &= I^N_{A_2}(l) \vee I^N_{B_2}(k_1k_2), \end{aligned}$$

$$\begin{aligned} (6)F^P_{B_1\times B_2}(k_1,l)(k_2,l) &= F^P_{A_2}(l) \vee F^P_{B_2}(k_1k_2)\\ F^N_{B_1\times B_2}(k_1,l)(k_2,l) &= F^N_{A_2}(l) \wedge F^N_{B_2}(k_1k_2), \end{aligned}$$

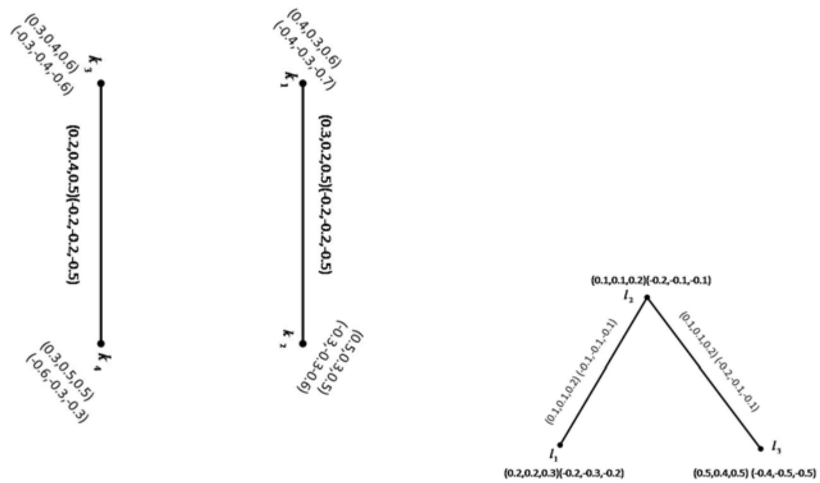
for all $k_1k_2 \in S_1, l \in R_2$.

Example 4.11. Consider $G_1 = (R_1, S_1)$ and $G_2 = (R_2, S_2)$ are two IBNG of $G = (R, S)$ respectively, as represented in Figure 4, now we get $G_1 \times G_2$ as follows Figure 5

Theorem 4.1. *The Cartesian product $G_1 \times G_2 = (R_1 \times R_2, S_1 \times S_2)$ of IBNG of IBNG G_1 and G_2 is an IBNG of $G_1 \times G_2$.*

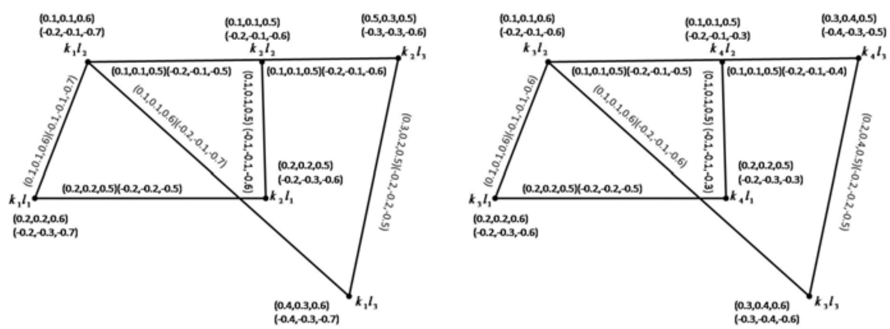
Proof. We consider:
Case 1: for $k \in R_1, l_1l_2 \in S_2$

$$\begin{aligned} T^P_{(B_1\times B_2)}((kl_1)(kl_2)) &= T^P_{A_1}(k) \wedge T^P_{B_2}(l_1l_2)\\ &\le T^P_{A_1}(k) \wedge [T^P_{A_2}(l_1) \wedge T^P_{A_2}(l_2)]\\ &= [T^P_{A_1}(k) \wedge T^P_{A_2}(l_1)] \wedge [T^P_{A_1}(k) \wedge T^P_{A_2}(l_2)]\\ &= T^P_{(A_1\times A_2)}(k,l_1) \wedge T^P_{(A_1\times A_2)}(k,l_2) \end{aligned}$$



G_1 G_2

$Figure\,4$



$G_1 \times G_2$

$Figure\,5$: Cartesian product of IBNG

$$\begin{aligned}
I^P_{(B_1\times B_2)}((kl_1)(kl_2)) &= I^P_{A_1}(k) \wedge I^P_{B_2}(l_1l_2)\\
&\le I^P_{A_1}(k) \wedge [I^P_{A_2}(l_1) \wedge I^P_{A_2}(l_2)]\\
&= [I^P_{A_1}(k) \wedge I^P_{A_2}(l_1)] \wedge [I^P_{A_1}(k) \wedge I^P_{A_2}(l_2)]\\
&= I^P_{(A_1\times A_2)}(k,l_1) \wedge I^P_{(A_1\times A_2)}(k,l_2)
\end{aligned}$$

$$\begin{aligned}
F^P_{(B_1\times B_2)}((kl_1)(kl_2)) &= F^P_{A_1}(k) \vee F^P_{B_2}(l_1l_2)\\
&\le F^P_{A_1}(k) \vee [F^P_{A_2}(l_1) \vee F^P_{A_2}(l_2)]\\
&= [F^P_{A_1}(k) \vee F^P_{A_2}(l_1)] \vee [F^P_{A_1}(k) \vee F^P_{A_2}(l_2)]\\
&= F^P_{(A_1\times A_2)}(k,l_1) \vee F^P_{(A_1\times A_2)}(k,l_2)
\end{aligned}$$

for all $kl_1, kl_2 \in G_1 \times G_2$.
Case 2: for $k \in R_2, l_1 l_2 \in S_1$

$$\begin{aligned} T^P_{(B_1 \times B_2)}((l_1 k)(l_2 k)) &= T^P_{A_2}(k) \wedge T^P_{B_1}(l_1 l_2) \\ &\leq T^P_{A_2}(k) \wedge [T^P_{A_1}(l_1) \wedge T^P_{A_1}(l_2)] \\ &= [T^P_{A_2}(k) \wedge T^P_{A_1}(l_1)] \wedge [T^P_{A_2}(k) \wedge T^P_{A_1}(l_2)] \\ &= T^P_{(A_1 \times A_2)}(l_1, k) \wedge T^P_{(A_1 \times A_2)}(l_2, k) \end{aligned}$$

$$\begin{aligned} I^P_{(B_1 \times B_2)}((l_1 k)(l_2 k)) &= I^P_{A_2}(k) \wedge I^P_{B_1}(l_1 l_2) \\ &\leq I^P_{A_2}(k) \wedge [I^P_{A_1}(l_1) \wedge I^P_{A_1}(l_2)] \\ &= [I^P_{A_2}(k) \wedge I^P_{A_1}(l_1)] \wedge [I^P_{A_2}(k) \wedge I^P_{A_1}(l_2)] \\ &= I^P_{(A_1 \times A_2)}(l_1, k) \wedge I^P_{(A_1 \times A_2)}(l_2, k) \end{aligned}$$

$$\begin{aligned} F^P_{(B_1 \times B_2)}((l_1 k)(l_2 k)) &= F^P_{A_2}(k) \vee F^P_{B_1}(l_1 l_2) \\ &\leq F^P_{A_2}(k) \vee [F^P_{A_1}(l_1) \vee F^P_{A_1}(l_2)] \\ &= [F^P_{A_2}(k) \vee F^P_{A_1}(l_1)] \vee [F^P_{A_2}(k) \vee F^P_{A_1}(l_2)] \\ &= F^P_{(A_1 \times A_2)}(l_1, k) \vee F^P_{(A_1 \times A_2)}(l_2, k), \end{aligned}$$

for all $l_1 k, l_2 k \in G_1 \times G_2$.
Similarly, one can prove the result for negative part also. □

Definition 4.12. The Cross product of two IBNGs G_1 and G_2 is denoted by the pair $G_1 \times G_2 = (R_1 \times R_2, S_1 \times S_2)$ and defined as

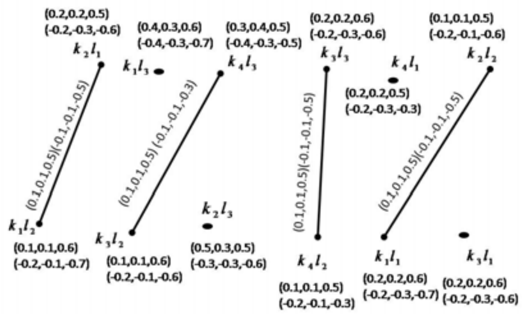
$$\begin{aligned} (i) T^P_{A_1 \times A_2}(kl) &= T^P_{A_1}(k) \wedge T^P_{A_2}(l) \\ I^P_{A_1 \times A_2}(kl) &= I^P_{A_1}(k) \wedge I^P_{A_2}(l) \\ F^P_{A_1 \times A_2}(kl) &= F^P_{A_1}(k) \vee F^P_{A_2}(l) \\ T^N_{A_1 \times A_2}(kl) &= T^N_{A_1}(k) \vee T^N_{A_2}(l) \\ I^N_{A_1 \times A_2}(kl) &= I^N_{A_1}(k) \vee I^N_{A_2}(l) \\ F^N_{A_1 \times A_2}(kl) &= F^N_{A_1}(k) \wedge F^N_{A_2}(l), \end{aligned}$$

for all $k, l \in R_1 \times R_2$.

$$\begin{aligned} (ii) T^P_{(B_1 \times B_2)}(k_1 l_1)(k_2 l_2) &= T^P_{B_1}(k_1 k_2) \wedge T^P_{B_2}(l_1 l_2) \\ I^P_{(B_1 \times B_2)}(k_1 l_1)(k_2 l_2) &= I^P_{B_1}(k_1 k_2) \wedge I^P_{B_2}(l_1 l_2) \\ F^P_{(B_1 \times B_2)}(k_1 l_1)(k_2 l_2) &= F^P_{B_1}(k_1 k_2) \vee F^P_{B_2}(l_1 l_2) \\ (iii) T^N_{(B_1 \times B_2)}(k_1 l_1)(k_2 l_2) &= T^N_{B_1}(k_1 k_2) \vee T^N_{B_2}(l_1 l_2) \\ I^N_{(B_1 \times B_2)}(k_1 l_1)(k_2 l_2) &= I^N_{B_1}(k_1 k_2) \vee I^N_{B_2}(l_1 l_2) \\ F^N_{(B_1 \times B_2)}(k_1 l_1)(k_2 l_2) &= F^N_{B_1}(k_1 k_2) \wedge F^N_{B_2}(l_1 l_2), \end{aligned}$$

for all $k_1 k_2 \in S_1, l_1 l_2 \in S_2$.

Example 4.13. Consider $G_1 = (R_1, S_1)$ and $G_2 = (R_2, S_2)$ are two IBNG of $G = (R, S)$ respectively, as represented in Figure 4. Now, we get cross product $G_1 \times G_2$ as follows Figure 6.



$G_1 \times G_2$

$Figure\,6$: CROSS PRODUCT OF INTUITIONISTIC BIPOLAR NEUTROSOPHIC GRAPH

Theorem 4.2. *Cross product $G_1 \times G_2 = (R_1 \times R_2, S_1 \times S_2)$ of two IBNG of G_1 and G_2 is an IBNG of $G_1 \times G_2$.*

Proof. For all $k_1l_1, k_2l_2 \in G_1 \times G_2$

$$\begin{aligned} T^P_{(B_1\times B_2)}((k_1l_1)(k_2l_2)) &= T^P_{B_1}(k_1k_2) \wedge T^P_{B_2}(l_1l_2) \\ &\leq [T^P_{A_1}(k_1) \wedge T^P_{A_1}(k_2)] \wedge [T^P_{A_2}(l_1) \wedge T^P_{A_2}(l_2)] \\ &= [T^P_{A_1}(k_1) \wedge T^P_{A_2}(l_1)] \wedge [T^P_{A_1}(k_2) \wedge T^P_{A_2}(l_2)] \\ &= T^P_{(A_1\times A_2)}(k_1l_1) \wedge T^P_{(A_1\times A_2)}(k_2, l_2), \end{aligned}$$

$$\begin{aligned} I^P_{(B_1\times B_2)}((k_1l_1)(k_2l_2)) &= I^P_{B_1}(k_1k_2) \wedge I^P_{B_2}(l_1l_2) \\ &\leq [I^P_{A_1}(k_1) \wedge I^P_{A_1}(k_2)] \wedge [I^P_{A_2}(l_1) \wedge I^P_{A_2}(l_2)] \\ &= [I^P_{A_1}(k_1) \wedge I^P_{A_2}(l_1)] \wedge [I^P_{A_1}(k_2) \wedge I^P_{A_2}(l_2)] \\ &= I^P_{(A_1\times A_2)}(k_1l_1) \wedge I^P_{(A_1\times A_2)}(k_2, l_2), \end{aligned}$$

$$\begin{aligned} F^P_{(B_1\times B_2)}((k_1l_1)(k_2l_2)) &= F^P_{B_1}(k_1k_2) \vee F^P_{B_2}(l_1l_2) \\ &\leq [F^P_{A_1}(k_1) \vee F^P_{A_1}(k_2)] \vee [F^P_{A_2}(l_1) \vee F^P_{A_2}(l_2)] \\ &= [F^P_{A_1}(k_1) \vee F^P_{A_2}(l_1)] \vee [F^P_{A_1}(k_2) \vee F^P_{A_2}(l_2)] \\ &= F^P_{(A_1\times A_2)}(k_1l_1) \vee F^P_{(A_1\times A_2)}(k_2, l_2) \end{aligned}$$

Similarly, we can prove the result for negative part also. □

Definition 4.14. The lexicographic product of two IBNGs G_1 and G_2 is denoted by the pair $G_1 \bullet G_2 = (R_1 \bullet R_2, S_1 \bullet S_2)$ and defined as

$$\begin{aligned}(i)T^P_{(A_1\bullet A_2)}(kl) &= T^P_{A_1}(k) \wedge T^P_{A_2}(l)\\ I^P_{(A_1\bullet A_2)}(kl) &= I^P_{A_1}(k) \wedge I^P_{A_2}(l)\\ F^P_{(A_1\bullet A_2)}(kl) &= F^P_{A_1}(k) \vee F^P_{A_2}(l)\\ T^N_{(A_1\bullet A_2)}(kl) &= T^N_{A_1}(k) \vee T^N_{A_2}(l)\\ I^N_{(A_1\bullet A_2)}(kl) &= I^N_{A_1}(k) \vee I^N_{A_2}(l)\\ F^N_{(A_1\bullet A_2)}(kl) &= F^N_{A_1}(k) \wedge F^N_{A_2}(l),\end{aligned}$$

for all $k, l \in R_1 \times R_2$

$$\begin{aligned}(ii)T^P_{(B_1\bullet B_2)}(kl_1)(kl_2) &= T^P_{A_1}(k) \wedge T^P_{B_2}(l_1l_2)\\ I^P_{(B_1\bullet B_2)}(kl_1)(kl_2) &= I^P_{A_1}(k) \wedge I^P_{B_2}(l_1l_2)\\ F^P_{(B_1\bullet B_2)}(kl_1)(kl_2) &= F^P_{A_1}(k) \vee F^P_{B_2}(l_1l_2)\\ T^N_{(B_1\bullet B_2)}(kl_1)(kl_2) &= T^N_{A_1}(k) \vee T^N_{B_2}(l_1l_2)\\ I^N_{(B_1\bullet B_2)}(kl_1)(kl_2) &= I^N_{A_1}(k) \vee I^N_{B_2}(l_1l_2)\\ F^N_{(B_1\bullet B_2)}(kl_1)(kl_2) &= F^N_{A_1}(k) \wedge F^N_{B_2}(l_1l_2),\end{aligned}$$

for all $k \in R_1, l_1l_2 \in S_2$.

$$\begin{aligned}(iii)T^P_{(B_1\bullet B_2)}(k_1l_1)(k_2l_2) &= T^P_{B_1}(k_1k_2) \wedge T^P_{B_2}(l_1l_2)\\ I^P_{(B_1\bullet B_2)}(k_1l_1)(k_2l_2) &= I^P_{B_1}(k_1k_2) \wedge I^P_{B_2}(l_1l_2)\\ F^P_{(B_1\bullet B_2)}(k_1l_1)(k_2l_2) &= F^P_{B_1}(k_1k_2) \vee F^P_{B_2}(l_1l_2)\\ T^N_{(B_1\bullet B_2)}(k_1l_1)(k_2l_2) &= T^N_{B_1}(k_1k_2) \vee T^N_{B_2}(l_1l_2)\\ I^N_{(B_1\bullet B_2)}(k_1l_1)(k_2l_2) &= I^N_{B_1}(k_1k_2) \vee I^N_{B_2}(l_1l_2)\\ F^N_{(B_1\bullet B_2)}(k_1l_1)(k_2l_2) &= F^N_{B_1}(k_1k_2) \wedge F^N_{B_2}(l_1l_2),\end{aligned}$$

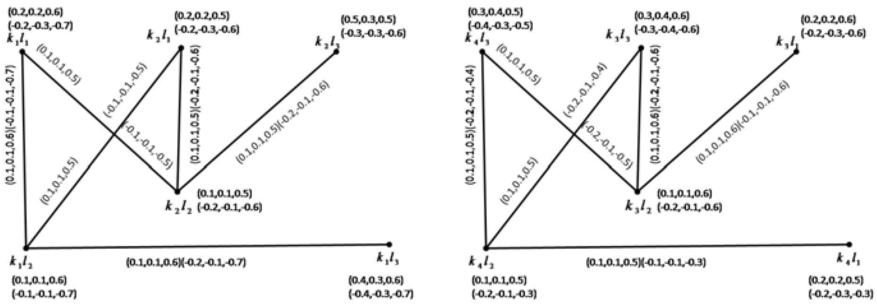
for all $k_1k_2 \in S_1, l_1l_2 \in S_2$.

Example 4.15. Lexicographic product of IBNG $G_1 = (R_1, S_1)$ and $G_2 = (R_2, S_2)$ shown in Figure 2 are defined as $G_1 \bullet G_2 = (R_1 \bullet R_2, S_1 \bullet S_2)$ and is represented in Figure 7.

Theorem 4.3. *Lexicographic product $G_1 \bullet G_2 = (R_1 \bullet R_2, S_1 \bullet S_2)$ of two IBNG of G_1 and G_2 is an IBNG of $G_1 \bullet G_2$.*

Proof. We consider two cases:
Case 1: for $k \in R_1, l_1l_2 \in S_2$

$$\begin{aligned}T^P_{(B_1\bullet B_2)}((kl_1)(kl_2)) &= T^P_{A_1}(k) \wedge T^P_{B_2}(l_1l_2)\\ &\leq T^P_{A_1}(k) \wedge [T^P_{A_2}(l_1) \wedge T^P_{A_2}(l_2)]\\ &= [T^P_{A_1}(k) \wedge T^P_{A_2}(l_1)] \wedge [T^P_{A_1}(k) \wedge T^P_{A_2}(l_2)]\\ &= T^P_{(A_1\bullet A_2)}(k, l_1) \wedge T^P_{(A_1\bullet A_2)}(k, l_2)\end{aligned}$$



$G_1 \bullet G_2$

$Figure\,7$: LEXICOGRAPHIC PRODUCT INTUITIONISTIC BIPOLAR NEUTROSOPHIC GRAPH

$$\begin{aligned}
I^P_{(B_1 \bullet B_2)}((kl_1)(kl_2)) &= I^P_{A_1}(k) \wedge I^P_{B_2}(l_1 l_2) \\
&\leq I^P_{A_1}(k) \wedge [I^P_{A_2}(l_1) \wedge I^P_{A_2}(l_2)] \\
&= [I^P_{A_1}(k) \wedge I^P_{A_2}(l_1)] \wedge [I^P_{A_1}(k) \wedge I^P_{A_2}(l_2)] \\
&= I^P_{(A_1 \bullet A_2)}(k, l_1) \wedge I^P_{(A_1 \bullet A_2)}(k, l_2)
\end{aligned}$$

$$\begin{aligned}
F^P_{(B_1 \bullet B_2)}((kl_1)(kl_2)) &= F^P_{A_1}(k) \vee F^P_{B_2}(l_1 l_2) \\
&\leq F^P_{A_1}(k) \vee [F^P_{A_2}(l_1) \vee F^P_{A_2}(l_2)] \\
&= [F^P_{A_1}(k) \vee F^P_{A_2}(l_1)] \vee [F^P_{A_1}(k) \vee F^P_{A_2}(l_2)] \\
&= F^P_{(A_1 \bullet A_2)}(k, l_1) \vee F^P_{(A_1 \bullet A_2)}(k, l_2)
\end{aligned}$$

for all $kl_1, kl_2 \in S_1 \times S_2$.
Case 2: For all $k_1k_2 \in S_1$, $l_1l_2 \in S_2$

$$\begin{aligned}
T^P_{(B_1 \bullet B_2)}((k_1l_1)(k_2l_2)) &= T^P_{B_1}(k_1k_2) \wedge T^P_{B_2}(l_1 l_2) \\
&\leq [T^P_{A_1}(k_1) \wedge T^P_{A_1}(k_2)] \wedge [T^P_{A_2}(l_1) \wedge T^P_{A_2}(l_2)] \\
&= [T^P_{A_1}(k_1) \wedge T^P_{A_2}(l_1)] \wedge [T^P_{A_1}(k_2) \wedge T^P_{A_2}(l_2)] \\
&= T^P_{(A_1 \bullet A_2)}(k_1, l_1) \wedge T^P_{(A_1 \bullet A_2)}(k_2, l_2)
\end{aligned}$$

$$\begin{aligned}
I^P_{(B_1 \bullet B_2)}((k_1l_1)(k_2l_2)) &= I^P_{B_1}(k_1k_2) \wedge I^P_{B_2}(l_1 l_2) \\
&\leq [I^P_{A_1}(k_1) \wedge I^P_{A_1}(k_2)] \wedge [I^P_{A_2}(l_1) \wedge I^P_{A_2}(l_2)] \\
&= [I^P_{A_1}(k_1) \wedge I^P_{A_2}(l_1)] \wedge [I^P_{A_1}(k_2) \wedge I^P_{A_2}(l_2)] \\
&= I^P_{(A_1 \bullet A_2)}(k_1, l_1) \wedge I^P_{(A_1 \bullet A_2)}(k_2, l_2)
\end{aligned}$$

$$\begin{aligned}
F^P_{(B_1 \times B_2)}((k_1l_1)(k_2l_2)) &= F^P_{B_1}(k_1k_2) \vee F^P_{B_2}(l_1 l_2) \\
&\leq [F^P_{A_1}(k_1) \vee F^P_{A_1}(k_2)] \vee [F^P_{A_2}(l_1) \vee F^P_{A_2}(l_2)] \\
&= [F^P_{A_1}(k_1) \vee F^P_{A_2}(l_1)] \vee [F^P_{A_1}(k_2) \vee F^P_{A_2}(l_2)] \\
&= F^P_{(A_1 \bullet A_2)}(k_1, l_1) \vee F^P_{(A_1 \bullet A_2)}(k_2, l_2)
\end{aligned}$$

for all $k_1l_1, k_2l_2 \in R_1 \bullet R_2$. Similarly, we can prove the result for negative part also. □

Definition 4.16. The strong product of two IBNGs G_1 and G_2 is denoted by the pair $G_1 \boxtimes G_2 = (R_1 \boxtimes R_2, S_1 \boxtimes S_2)$ and defined as

$$(i)T^P_{(A_1 \boxtimes A_2)}(kl) = T^P_{A_1}(k) \wedge T^P_{A_2}(l),$$
$$I^P_{(A_1 \boxtimes A_2)}(kl) = I^P_{A_1}(k) \wedge I^P_{A_2}(l),$$
$$F^P_{(A_1 \boxtimes A_2)}(kl) = F^P_{A_1}(k) \vee F^P_{A_2}(l),$$
$$T^N_{(A_1 \boxtimes A_2)}(kl) = T^N_{A_1}(k) \vee T^N_{A_2}(l),$$
$$I^N_{(A_1 \boxtimes A_2)}(kl) = I^N_{A_1}(k) \vee I^N_{A_2}(l),$$
$$F^N_{(A_1 \boxtimes A_2)}(kl) = F^N_{A_1}(k) \wedge F^N_{A_2}(l),$$

for all $k, l \in R_1 \boxtimes R_2$

$$(ii)T^P_{(B_1 \boxtimes B_2)}(kl_1)(kl_2) = T^P_{A_1}(k) \wedge T^P_{B_2}(l_1l_2),$$
$$I^P_{(B_1 \boxtimes B_2)}(kl_1)(kl_2) = I^P_{A_1}(k) \wedge I^P_{B_2}(l_1l_2),$$
$$F^P_{(B_1 \boxtimes B_2)}(kl_1)(kl_2) = F^P_{A_1}(k) \vee F^P_{B_2}(l_1l_2),$$
$$T^N_{(B_1 \boxtimes B_2)}(kl_1)(kl_2) = T^N_{A_1}(k) \vee T^N_{B_2}(l_1l_2),$$
$$I^N_{(B_1 \boxtimes B_2)}(kl_1)(kl_2) = I^N_{A_1}(k) \vee I^N_{B_2}(l_1l_2),$$
$$F^N_{(B_1 \boxtimes B_2)}(kl_1)(kl_2) = F^N_{A_1}(k) \wedge F^N_{B_2}(l_1l_2),$$

for all $k \in R_1, l_1l_2 \in S_2$.

$$(iii)T^P_{B_1 \boxtimes B_2}(k_1, l)(k_2, l) = T^P_{A_2}(l) \wedge T^P_{B_2}(k_1k_2),$$
$$I^P_{B_1 \boxtimes B_2}(k_1, l)(k_2, l) = I^P_{A_2}(l) \wedge I^P_{B_2}(k_1k_2),$$
$$F^P_{B_1 \boxtimes B_2}(k_1, l)(k_2, l) = F^P_{A_2}(l) \vee F^P_{B_2}(k_1k_2),$$
$$T^N_{B_1 \boxtimes B_2}(k_1, l)(k_2, l) = T^N_{A_2}(l) \vee T^N_{B_2}(k_1k_2),$$
$$I^N_{B_1 \boxtimes B_2}(k_1, l)(k_2, l) = I^N_{A_2}(l) \vee I^N_{B_2}(k_1k_2),$$
$$F^N_{B_1 \boxtimes B_2}(k_1, l)(k_2, l) = F^N_{A_2}(l) \wedge F^N_{B_2}(k_1k_2),$$

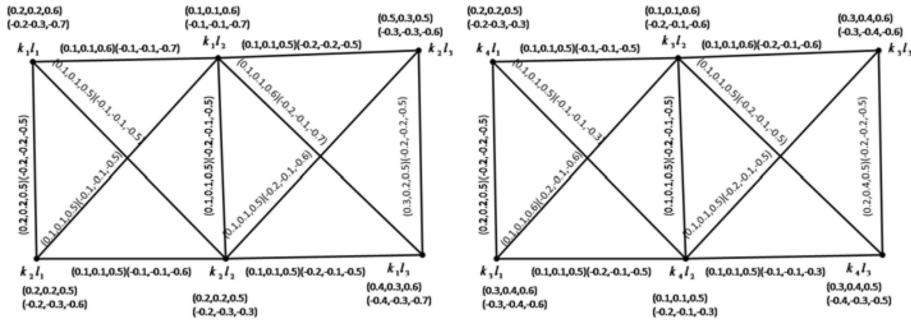
for all $k_1k_2 \in S_1, l \in R_2$.

$$(iv)T^P_{(B_1 \boxtimes B_2)}(k_1l_1)(k_2l_2) = T^P_{B_1}(k_1k_2) \wedge T^P_{B_2}(l_1l_2)$$
$$I^P_{(B_1 \boxtimes B_2)}(k_1l_1)(k_2l_2) = I^P_{B_1}(k_1k_2) \wedge I^P_{B_2}(l_1l_2)$$
$$F^P_{(B_1 \boxtimes B_2)}(k_1l_1)(k_2l_2) = F^P_{B_1}(k_1k_2) \vee F^P_{B_2}(l_1l_2)$$
$$T^N_{(B_1 \boxtimes B_2)}(k_1l_1)(k_2l_2) = T^N_{B_1}(k_1k_2) \vee T^N_{B_2}(l_1l_2)$$
$$I^N_{(B_1 \boxtimes B_2)}(k_1l_1)(k_2l_2) = I^N_{B_1}(k_1k_2) \vee I^N_{B_2}(l_1l_2)$$
$$F^N_{(B_1 \boxtimes B_2)}(k_1l_1)(k_2l_2) = F^N_{B_1}(k_1k_2) \wedge F^N_{B_2}(l_1l_2)$$

for all $k_1k_2 \in S_1, l_1l_2 \in S_2$.

Example 4.17. Strong product of IBNG $G_1 = (R_1, S_1)$ and $G_2 = (R_2, S_2)$ shown in Figure 2 is defined as $G_1 \boxtimes G_2 = (R_1 \boxtimes R_2, S_1 \boxtimes S_2)$ and is represented in Figure 8.

Theorem 4.4. *Strong product $G_1 \boxtimes G_2 = (R_1 \boxtimes R_2, S_1 \boxtimes S_2)$ of two IBNG of G_1 and G_2 is an IBNG of $G_1 \boxtimes G_2$.*



$G_1 \boxtimes G_2$

$Figure\,8$: STRONG PRODUCT INTUITIONISTIC BIPOLAR NEUTROSOPHIC GRAPH

Proof. There are three cases:

Case 1: for $k \in R_1, l_1 l_2 \in S_2$

$$\begin{aligned}
T^P_{(B_1 \boxtimes B_2)}((kl_1)(kl_2)) &= T^P_{A_1}(k) \wedge T^P_{B_2}(l_1 l_2) \\
&\leq T^P_{A_1}(k) \wedge [T^P_{A_2}(l_1) \wedge T^P_{A_2}(l_2)] \\
&= [T^P_{A_1}(k) \wedge T^P_{A_2}(l_1)] \wedge [T^P_{A_1}(k) \wedge T^P_{A_2}(l_2)] \\
&= T^P_{(A_1 \boxtimes A_2)}(k, l_1) \wedge T^P_{(A_1 \boxtimes A_2)}(k, l_2)
\end{aligned}$$

$$\begin{aligned}
I^P_{(B_1 \boxtimes B_2)}((kl_1)(kl_2)) &= I^P_{A_1}(k) \wedge I^P_{B_2}(l_1 l_2) \\
&\leq I^P_{A_1}(k) \wedge [I^P_{A_2}(l_1) \wedge I^P_{A_2}(l_2)] \\
&= [I^P_{A_1}(k) \wedge I^P_{A_2}(l_1)] \wedge [I^P_{A_1}(k) \wedge I^P_{A_2}(l_2)] \\
&= I^P_{(A_1 \boxtimes A_2)}(k, l_1) \wedge I^P_{(A_1 \boxtimes A_2)}(k, l_2)
\end{aligned}$$

$$\begin{aligned}
F^P_{(B_1 \boxtimes B_2)}((kl_1)(kl_2)) &= F^P_{A_1}(k) \vee F^P_{B_2}(l_1 l_2) \\
&\leq F^P_{A_1}(k) \vee [F^P_{A_2}(l_1) \vee F^P_{A_2}(l_2)] \\
&= [F^P_{A_1}(k) \vee F^P_{A_2}(l_1)] \vee [F^P_{A_1}(k) \vee F^P_{A_2}(l_2)] \\
&= F^P_{(A_1 \boxtimes A_2)}(k, l_1) \vee F^P_{(A_1 \boxtimes A_2)}(k, l_2)
\end{aligned}$$

for all $kl_1, kl_2 \in R_1 \boxtimes R_2$.

Case 2: for $k \in R_2, l_1 l_2 \in S_1$

$$\begin{aligned}
T^P_{(B_1 \boxtimes B_2)}((l_1 k)(l_2 k)) &= T^P_{A_2}(k) \wedge T^P_{B_1}(l_1 l_2) \\
&\leq T^P_{A_2}(k) \wedge [T^P_{A_1}(l_1) \wedge T^P_{A_1}(l_2)] \\
&= [T^P_{A_2}(k) \wedge T^P_{A_1}(l_1)] \wedge [T^P_{A_2}(k) \wedge T^P_{A_1}(l_2)] \\
&= T^P_{(A_1 \boxtimes A_2)}(l_1, k) \wedge T^P_{(A_1 \boxtimes A_2)}(l_2, k)
\end{aligned}$$

$$
\begin{aligned}
I^{P}_{(B_1\boxtimes B_2)}((l_1k)(l_2k)) &= I^{P}_{A_2}(k)\wedge I^{P}_{B_1}(l_1l_2)\\
&\leq I^{P}_{A_2}(k)\wedge [I^{P}_{A_1}(l_1)\wedge I^{P}_{A_1}(l_2)]\\
&= [I^{P}_{A_2}(k)\wedge I^{P}_{A_1}(l_1)]\wedge [I^{P}_{A_2}(k)\wedge I^{P}_{A_1}(l_2)]\\
&= I^{P}_{(A_1\boxtimes A_2)}(l_1,k)\wedge I^{P}_{(A_1\boxtimes A_2)}(l_2,k)
\end{aligned}
$$

$$
\begin{aligned}
F^{P}_{(B_1\boxtimes B_2)}((l_1k)(l_2k)) &= F^{P}_{A_2}(k)\vee F^{P}_{B_1}(l_1l_2)\\
&\leq F^{P}_{A_2}(k)\vee [F^{P}_{A_1}(l_1)\vee F^{P}_{A_1}(l_2)]\\
&= [F^{P}_{A_2}(k)\vee F^{P}_{A_1}(l_1)]\vee [F^{P}_{A_2}(k)\vee F^{P}_{A_1}(l_2)]\\
&= F^{P}_{(A_1\boxtimes A_2)}(l_1,k)\vee F^{P}_{(A_1\boxtimes A_2)}(l_2,k)
\end{aligned}
$$

for all $l_1k, l_2k \in R_1 \boxtimes R_2$. □

Case 3: For all $k_1k_2 \in S_1$,$l_1l_2 \in S_2$,

$$
\begin{aligned}
T^{P}_{(B_1\boxtimes B_2)}((k_1l_1)(k_2l_2)) &= T^{P}_{B_1}(k_1k_2)\wedge T^{P}_{B_2}(l_1l_2)\\
&\leq [T^{P}_{A_1}(k_1)\wedge T^{P}_{A_1}(k_2)]\wedge [T^{P}_{A_2}(l_1)\wedge T^{P}_{A_2}(l_2)]\\
&= [T^{P}_{A_1}(k_1)\wedge T^{P}_{A_2}(l_1)]\wedge [T^{P}_{A_1}(k_2)\wedge T^{P}_{A_2}(l_2)]\\
&= T^{P}_{(A_1\boxtimes A_2)}(k_1,l_1)\wedge T^{P}_{(A_1\boxtimes A_2)}(k_2,l_2)
\end{aligned}
$$

$$
\begin{aligned}
I^{P}_{(B_1\boxtimes B_2)}((k_1l_1)(k_2l_2)) &= I^{P}_{B_1}(k_1k_2)\wedge I^{P}_{B_2}(l_1l_2)\\
&\leq [I^{P}_{A_1}(k_1)\wedge I^{P}_{A_1}(k_2)]\wedge [I^{P}_{A_2}(l_1)\wedge I^{P}_{A_2}(l_2)]\\
&= [I^{P}_{A_1}(k_1)\wedge I^{P}_{A_2}(l_1)]\wedge [I^{P}_{A_1}(k_2)\wedge I^{P}_{A_2}(l_2)]\\
&= I^{P}_{(A_1\boxtimes A_2)}(k_1,l_1)\wedge I^{P}_{(A_1\boxtimes A_2)}(k_2,l_2)
\end{aligned}
$$

$$
\begin{aligned}
F^{P}_{(B_1\boxtimes B_2)}((k_1l_1)(k_2l_2)) &= F^{P}_{B_1}(k_1k_2)\vee F^{P}_{B_2}(l_1l_2)\\
&\leq [F^{P}_{A_1}(k_1)\vee F^{P}_{A_1}(k_2)]\vee [F^{P}_{A_2}(l_1)\vee F^{P}_{A_2}(l_2)]\\
&= [F^{P}_{A_1}(k_1)\vee F^{P}_{A_2}(l_1)]\vee [F^{P}_{A_1}(k_2)\vee F^{P}_{A_2}(l_2)]\\
&= F^{P}_{(A_1\boxtimes A_2)}(k_1,l_1)\vee F^{P}_{(A_1\boxtimes A_2)}(k_2,l_2)
\end{aligned}
$$

for all $k_1l_1, k_2l_2 \in R_1 \boxtimes R_2$. Similarly, we can prove the result for negative part also.

Definition 4.18. The composition of two IBNGs G_1 and G_2 is denoted by the pair $G_1 \circ G_2 = (R_1 \circ R_2, S_1 \circ S_2)$ and defined as

$$
\begin{aligned}
(i)T^{P}_{(A_1\circ A_2)}(kl) &= T^{P}_{A_1}(k)\wedge T^{P}_{A_2}(l)\\
I^{P}_{(A_1\circ A_2)}(kl) &= I^{P}_{A_1}(k)\wedge I^{P}_{A_2}(l)\\
F^{P}_{(A_1\circ A_2)}(kl) &= F^{P}_{A_1}(k)\vee F^{P}_{A_2}(l)\\
T^{N}_{(A_1\circ A_2)}(kl) &= T^{N}_{A_1}(k)\vee T^{N}_{A_2}(l)\\
I^{N}_{(A_1\circ A_2)}(kl) &= I^{N}_{A_1}(k)\vee I^{N}_{A_2}(l)\\
F^{N}_{(A_1\circ A_2)}(kl) &= F^{N}_{A_1}(k)\wedge F^{N}_{A_2}(l),
\end{aligned}
$$

for all $k, l \in R_1 \circ R_2$

$$(ii) T^P_{(B_1 \circ B_2)}(kl_1)(kl_2) = T^P_{A_1}(k) \wedge T^P_{B_2}(l_1 l_2)$$
$$I^P_{(B_1 \circ B_2)}(kl_1)(kl_2) = I^P_{A_1}(k) \wedge I^P_{B_2}(l_1 l_2)$$
$$F^P_{(B_1 \circ B_2)}(kl_1)(kl_2) = F^P_{A_1}(k) \vee F^P_{B_2}(l_1 l_2)$$
$$T^N_{(B_1 \circ B_2)}(kl_1)(kl_2) = T^N_{A_1}(k) \vee T^N_{B_2}(l_1 l_2)$$
$$I^N_{(B_1 \circ B_2)}(kl_1)(kl_2) = I^N_{A_1}(k) \vee I^N_{B_2}(l_1 l_2)$$
$$F^N_{(B_1 \circ B_2)}(kl_1)(kl_2) = F^N_{A_1}(k) \wedge F^N_{B_2}(l_1 l_2),$$

for all $k \in R_1, l_1 l_2 \in S_2$.

$$(iii) T^P_{B_1 \circ B_2}(k_1 l)(k_2 l) = T^P_{A_2}(l) \wedge T^P_{B_2}(k_1 k_2)$$
$$I^P_{B_1 \circ B_2}(k_1 l)(k_2 l) = I^P_{A_2}(l) \wedge I^P_{B_2}(k_1 k_2)$$
$$F^P_{B_1 \circ B_2}(k_1 l)(k_2 l) = F^P_{A_2}(l) \vee F^P_{B_2}(k_1 k_2)$$
$$T^N_{B_1 \circ B_2}(k_1 l)(k_2 l) = T^N_{A_2}(l) \vee T^N_{B_2}(k_1 k_2)$$
$$I^N_{B_1 \circ B_2}(k_1 l)(k_2 l) = I^N_{A_2}(l) \vee I^N_{B_2}(k_1 k_2)$$
$$F^N_{B_1 \circ B_2}(k_1 l)(k_2 l) = F^N_{A_2}(l) \wedge F^N_{B_2}(k_1 k_2),$$

for all $k_1 k_2 \in S_1, l \in R_2$.

$$(iv) T^P_{(B_1 \circ B_2)}(k_1 l_1)(k_2 l_2) = T^P_{B_1}(k_1 k_2) \wedge T^P_{A_2}(l_1) \wedge T^P_{A_2}(l_2)$$
$$I^P_{(B_1 \circ B_2)}(k_1 l_1)(k_2 l_2) = I^P_{B_1}(k_1 k_2) \wedge I^P_{A_2}(l_1) \wedge I^P_{A_2}(l_2)$$
$$F^P_{(B_1 \circ B_2)}(k_1 l_1)(k_2 l_2) = F^P_{B_1}(k_1 k_2) \vee F^P_{A_2}(l_1) \vee F^P_{A_2}(l_2)$$

$$T^N_{(B_1 \circ B_2)}(k_1 l_1)(k_2 l_2) = T^N_{B_1}(k_1 k_2) \vee T^N_{A_2}(l_1) \vee T^N_{A_2}(l_2)$$
$$I^N_{(B_1 \circ B_2)}(k_1 l_1)(k_2 l_2) = I^N_{B_1}(k_1 k_2) \vee I^N_{A_2}(l_1) \vee I^N_{A_2}(l_2)$$
$$F^N_{(B_1 \circ B_2)}(k_1 l_1)(k_2 l_2) = F^N_{B_1}(k_1 k_2) \wedge F^N_{A_2}(l_1) \wedge F^N_{A_2}(l_2),$$

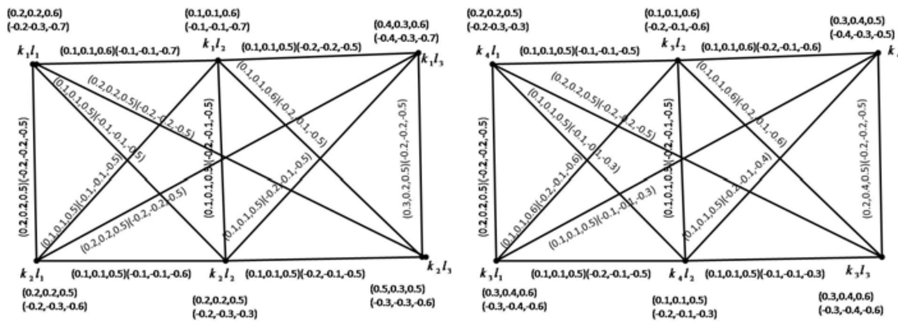
for all $k_1 k_2 \in S_1, l_1 l_2 \in S_2$ such that $l_1 \neq l_2$

Example 4.19. Composition of IBNG $G_1 = (R_1, S_1)$ and $G_2 = (R_2, S_2)$ shown in Figure 2 is defined as $G_1 \circ G_2 = (R_1 \circ R_2, S_1 \circ S_2)$ and is represented in Figure 9.

Theorem 4.5. *Composition $G_1 \circ G_2 = (R_1 \circ R_2, S_1 \circ S_2)$ of two IBNG of G_1 and G_2 is an IBNG of $G_1 \circ G_2$.*

Proof. There are three cases:
Case 1: for $k \in R_1, l_1 l_2 \in S_2$

$$\begin{aligned} T^P_{(B_1 \circ B_2)}((kl_1)(kl_2)) &= T^P_{A_1}(k) \wedge T^P_{B_2}(l_1 l_2) \\ &\leq T^P_{A_1}(k) \wedge [T^P_{A_2}(l_1) \wedge T^P_{A_2}(l_2)] \\ &= [T^P_{A_1}(k) \wedge T^P_{A_2}(l_1)] \wedge [T^P_{A_1}(k) \wedge T^P_{A_2}(l_2)] \\ &= T^P_{(A_1 \circ A_2)}(k, l_1) \wedge T^P_{(A_1 \circ A_2)}(k, l_2) \end{aligned}$$



$G_1 \circ G_2$

$Figure\,9$: COMPOSITION INTUITIONISTIC BIPOLAR NEUTROSOPHIC GRAPH

$$\begin{aligned}
I^P_{(B_1\circ B_2)}((kl_1)(kl_2)) &= I^P_{A_1}(k) \wedge I^P_{B_2}(l_1l_2) \\
&\leq I^P_{A_1}(k) \wedge [I^P_{A_2}(l_1) \wedge I^P_{A_2}(l_2)] \\
&= [I^P_{A_1}(k) \wedge I^P_{A_2}(l_1)] \wedge [I^P_{A_1}(k) \wedge I^P_{A_2}(l_2)] \\
&= I^P_{(A_1\circ A_2)}(k, l_1) \wedge I^P_{(A_1\circ A_2)}(k, l_2)
\end{aligned}$$

$$\begin{aligned}
F^P_{(B_1\circ B_2)}((kl_1)(kl_2)) &= F^P_{A_1}(k) \vee F^P_{B_2}(l_1l_2) \\
&\leq F^P_{A_1}(k) \vee [F^P_{A_2}(l_1) \vee F^P_{A_2}(l_2)] \\
&= [F^P_{A_1}(k) \vee F^P_{A_2}(l_1)] \vee [F^P_{A_1}(k) \vee F^P_{A_2}(l_2)] \\
&= F^P_{(A_1\circ A_2)}(k, l_1) \vee F^P_{(A_1\circ A_2)}(k, l_2)
\end{aligned}$$

for all $kl_1, kl_2 \in R_1 \circ R_2$.
Case 2: for $k \in R_2, l_1l_2 \in S_1$

$$\begin{aligned}
T^P_{(B_1\circ B_2)}((l_1k)(l_2k)) &= T^P_{A_2}(k) \wedge T^P_{B_1}(l_1l_2) \\
&\leq T^P_{A_2}(k) \wedge [T^P_{A_1}(l_1) \wedge T^P_{A_1}(l_2)] \\
&= [T^P_{A_2}(k) \wedge T^P_{A_1}(l_1)] \wedge [T^P_{A_2}(k) \wedge T^P_{A_1}(l_2)] \\
&= T^P_{(A_1\circ A_2)}(l_1, k) \wedge T^P_{(A_1\circ A_2)}(l_2, k)
\end{aligned}$$

$$\begin{aligned}
I^P_{(B_1\circ B_2)}((l_1k)(l_2k)) &= I^P_{A_2}(k) \wedge I^P_{B_1}(l_1l_2) \\
&\leq I^P_{A_2}(k) \wedge [I^P_{A_1}(l_1) \wedge I^P_{A_1}(l_2)] \\
&= [I^P_{A_2}(k) \wedge I^P_{A_1}(l_1)] \wedge [I^P_{A_2}(k) \wedge I^P_{A_1}(l_2)] \\
&= I^P_{(A_1\circ A_2)}(l_1, k) \wedge I^P_{(A_1\circ A_2)}(l_2, k)
\end{aligned}$$

$$\begin{aligned}
F^P_{(B_1\circ B_2)}((l_1k)(l_2k)) &= F^P_{A_2}(k) \vee F^P_{B_1}(l_1l_2) \\
&\leq F^P_{A_2}(k) \vee [F^P_{A_1}(l_1) \vee F^P_{A_1}(l_2)] \\
&= [F^P_{A_2}(k) \vee F^P_{A_1}(l_1)] \vee [F^P_{A_2}(k) \vee F^P_{A_1}(l_2)] \\
&= F^P_{(A_1\circ A_2)}(l_1, k) \vee F^P_{(A_1\circ A_2)}(l_2, k)
\end{aligned}$$

for all $l_1k, l_2k \in R_1 \circ R_2$.
Case 3: For $k_1k_2 \in S_1, l_1, l_2 \in R_2$ such that $l_1 \neq l_2$

$$\begin{aligned} T^P_{(B_1\circ B_2)}((k_1l_1)(k_2l_2)) &= T^P_{B_1}(k_1, k_2) \wedge T^P_{A_2}(l_1) \wedge T^P_{A_2}(l_2) \\ &\leq [T^P_{A_1}(k_1) \wedge T^P_{A_1}(k_2)] \wedge [T^P_{A_2}(l_1) \wedge T^P_{A_2}(l_2)] \\ &= [T^P_{A_1}(k_1) \wedge T^P_{A_2}(l_1)] \wedge [T^P_{A_1}(k_2) \wedge T^P_{A_2}(l_2)] \\ &= T^P_{(A_1\circ A_2)}(k_1l_1) \wedge T^P_{(A_1\circ A_2)}(k_2l_2) \end{aligned}$$

$$\begin{aligned} I^P_{(B_1\circ B_2)}((k_1, l_1)(k_2, l_2)) &= I^P_{B_1}(k_1, k_2) \wedge I^P_{A_2}(l_1) \wedge I^P_{A_2}(l_2) \\ &\leq [I^P_{A_1}(k_1) \wedge I^P_{A_1}(k_2)] \wedge [I^P_{A_2}(l_1) \wedge I^P_{A_2}(l_2)] \\ &= [I^P_{A_1}(k_1) \wedge I^P_{A_2}(l_1)] \wedge [I^P_{A_1}(k_2) \wedge I^P_{A_2}(l_2)] \\ &= I^P_{(A_1\circ A_2)}(k_1l_1) \wedge I^P_{(A_1\circ A_2)}(k_2l_2) \end{aligned}$$

$$\begin{aligned} F^P_{(B_1\circ B_2)}((k_1, l_1)(k_2, l_2)) &= F^P_{B_1}(k_1, k_2) \vee F^P_{A_2}(l_1) \vee F^P_{A_2}(l_2) \\ &\leq [F^P_{A_1}(k_1) \vee F^P_{A_1}(k_2)] \vee [F^P_{A_2}(l_1) \vee F^P_{A_2}(l_2)] \\ &= [F^P_{A_1}(k_1) \vee F^P_{A_2}(l_1)] \vee [F^P_{A_1}(k_2) \vee F^P_{A_2}(l_2)] \\ &= F^P_{(A_1\circ A_2)}(k_1l_1) \vee F^P_{(A_1\circ A_2)}(k_2l_2) \end{aligned}$$

for all $k_1l_1, k_2l_2 \in R_1 \circ R_2$. Similarly, we can prove the result for negative part also. □

5. Conclusions

In this work, a new concept of intuitionistic bipolar neutrosophic set with the operations like union, intersection and complement have been developed. Also, an application to intuitionistic bipolar neutrosophic graph with examples have established. Further, we presented the Cartesian product, cross product, lexicographic product and strong product with suitable examples. In future, isomorphic properties will be investigated.

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n-Refined Neutrosophic Vector Spaces

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Abstract

This paper introduces the concept of n-refined neutrosophic vector spaces as a generalization of neutrosophic vector spaces, and it studies elementary properties of them. Also, this work discusses some corresponding concepts such as weak/strong n-refined neutrosophic vector spaces, and n-refined neutrosophic homomorphisms.

Keywords: n-Refined weak neutrosophic vector space, n-Refined strong neutrosophic vector space, n-Refined neutrosophic homomorphism.

1.Introduction

Neutrosophy as a part of philosophy founded by F. Smarandache to study origin, nature, and indeterminacies became a strong tool in studying algebraic concepts. Neutrosophic algebraic structures were defined and studied such as neutrosophic modules,and neutrosopohic vector spaces, etc.See [1,2,3,4,5,6,7,8,9]. In 2013 Smarandacheintroduceda perfect idea, when he extended the neutrosophic set to refined [n-valued] neutrosophic set, i.e. the truth value T is refined/split into types of sub-truths such as (T_1, T_2, …,) similarly indeterminacy I is refined/split into types of sub-indeterminacies (I_1, I_2, …,) and the falsehood F is refined/split into sub-falsehood (F_1, F_2,..,) [10,11]. Refined neutrosophic algebraic structures were studied such as refined neutrosophic rings, refined neutrosophic modules, and n-refined neutrosophic rings [4,12].

In this article authors try to define n-refined neutrosophic vector spaces, subspaces, and homomorphisms and to present some of their elementary properties.

For our purpose we use multiplication operation (defined in [12]) between indeterminacies $I_1, I_2, \dots, I_n$ as follows:

$$I_m I_s = I_{\min(m,s)}.$$

This work is a continuation of the study on the n-refined neutrosophic structures that began in [12].

2. Preliminaries

Definition 2.1: [12]

Let (R,+,×) be a ring and $I_k; 1 \leq k \leq n$ be n indeterminacies. We define R_n(I)={$a_0 + a_1 I + \cdots + a_n I_n; a_i \in R$} to be an n-refined neutrosophic ring.

Definition 4.3: [12]

(a) Let R_n(I) be an n-refined neutrosophic ring and P = $\sum_{i=0}^{n} P_i I_i$ = {$a_0 + a_1 I + \cdots + a_n I_n; a_i \in P_i$}, where P_i is a subset of R, we define P to be an AH-subring if P_i is a subring of R for all . AHS-subring is defined by the condition $P_i = P_j$ for all i, j .

(b) P is an AH-ideal if P_i are two-side ideals of R for all i, the AHS-ideal is defined by the condition $P_i = P_j$ for all i, j .

(c) The AH-ideal P is said to be null if $P_i = R$ or $P_i = \{0\}$ for all i.

Definition 2.3 :[5]

Let (V , + , ·) be a vector space over the field K; then (V(I) , + , ·) is called a weak neutrosophic vector space over the field K, and it is called a strong neutrosophic vector space if it is a vector space over the neutrosophic field K(I).

Definition 2.4 : [5]

Let V(I) be a strong neutrosophic vector space over the neutrosophic field K(I) and W(I) be a non empty set of V(I) then W(I) is called a strong neutrosophic subspace if W(I) is itself a strong neutrosophic vector space.

Definition 2.6 :[5]

Let U(I) , W(I) be two strong neutrosophic subspaces of V(I) and let $f: V(I) \rightarrow W(I)$, we say that f is a neutrosophic vector space homomorphism if

(a) f(I)=I,

(b) f is a vector space homomorphism.

We define the kernel of f by Ker(f) = { x∈ $V(I)$; f(x) = $0_{W(I)}$}.

Definition 2.7 :[5]

Let $v_1, v_2, \ldots, v_s \in V(I)$ and $x \in V(I)$; we say that x is a linear combination of { $v_i; i = 1, \ldots, s$} if

x = $a_1 v_1 + \cdots + a_s v_s$ such that $a_i \in K(I)$.

The set { $\boldsymbol{v_i; i = 1, \ldots, s}$} is called linearly independent if $\boldsymbol{a_1 v_1} + \cdots + \boldsymbol{a_s v_s} = \boldsymbol{0}$ implies $\boldsymbol{a_i} = \boldsymbol{0}$ for all i.

3. Main concepts and results

Definition 3.1:

Let (K,+,·) be a field, we say that $K_n(I) = K + KI_1 + \cdots + KI_n = \{a_0 + a_1 I_1 + \cdots + a_n I_n; a_i \in K\}$ is an n-refined neutrosophic field.

It is clear that $K_n(I)$ is an n-refined neutrosophic field, but not a field in the classical meaning.

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Example 3.2 :

Let $K = Q$ be the field of rationals. The corresponding 3-refined neutrosophic field is

$$Q_3(I) = \{a + bI_1 + cI_2 + dI_3; a, b, c, d \in Q\}.$$

Definition 3.3 :

Let (V,+,·) be a vector space over the field K. Then we say that $V_n(I) = V + VI_1 + \cdots + VI_n = \{x_0 + x_1 I_1 + \cdots + x_n I_n; x_i \in V\}$ is a weak n-refined neutrosophic vector space over the field K. Elements of $V_n(I)$ are called n-refined neutrosophic vectors, elements of K are called scalars.

If we take scalars from the n-refined neutrosophicfield $K_n(I)$, we say that $V_n(I)$ is a strong n-refined neutrosophic vector space over the n-refined neutrosophic field $K_n(I)$. Elements of $K_n(I)$ are called n-refined neutrosophic scalars.

Remark 3.4:

If we take n=1 we get the classical neutrosophic vector space.

Addition on $V_n(I)$ is defined as:

$$\sum_{i=0}^{n} a_i I_i + \sum_{i=0}^{n} b_i I_i = \sum_{i=0}^{n} (a_i + b_i) I_i.$$

Multiplication by a scalar $m \in K$ is defined as:

$$m \cdot \sum_{i=0}^{n} a_i I_i = \sum_{i=0}^{n} (m a_i) I_i.$$

Multiplication by an n-refined neutrosophic scalar $m = \sum_{i=0}^{n} m_i I_i \in K_n(I)$ is defined as:

$$\left(\sum_{i=0}^{n} m_i I_i\right) \cdot \left(\sum_{i=0}^{n} a_i I_i\right) = \sum_{i,j=0}^{n} (m_i a_j) I_i I_j,$$

where $a_i \in V, m_i \in K, I_i I_j = I_{\min(i,j)}$.

Theorem 3.5 :

Let (V,+,·) be a vector space over the field K. Then a weak n-refined neutrosophic vector space $V_n(I)$ is a vector space over the field K. A strong n-refined neutrosophic vector space is not a vector space but a module over the n-refinedneutrosophic field K_n(I).

Proof:

It is similar to that of Theorem 2.3 in [5].

Example 3.6:

Let $V = Z_2$ be the finite vector space of integers modulo 2 over itself:

(a) The corresponding weak 2-refined neutrosophic vector space over the field Z_2 is

$V_n(I) = \{0,1, I_1, I_2, I_1 + I_2, 1 + I_1 + I_2, 1 + I_1, 1 + I_2\}$.

Definition 3.7:

Let $V_n(I)$ be a weak n-refined neutrosophic vector space over the field K; a nonempty subset $W_n(I)$ is called a weak n-refined neutrosophic subspace of $V_n(I)$ if $W_n(I)$ is a subspace of $V_n(I)$ itself.

Definition 3.8:

Let $V_n(I)$ be a strong n-refined neutrosophic vector space over the n-refined neutrosophic field $K_n(I)$; a nonempty subset $W_n(I)$ is called a strong n-refined neutrosophic subspace of $V_n(I)$ if $W_n(I)$ is a submodule of $V_n(I)$ itself.

Theorem 3.9:

Let $V_n(I)$ be a weak n-refined neutrosophic vector space over the field K, $W_n(I)$ be a nonempty subset of $V_n(I)$. Then $W_n(I)$ is a weak n-refined neutrosophic subspace if and only if:

$x+y \in W_n(I), m \cdot x \in W_n(I)$ for all $x, y \in W_n(I), m \in K$.

Proof:

It holds directly from the condition of subspace.

Theorem 3.10:

Let $V_n(I)$ be a strong n-refined neutrosophic vector space over an n-refined neutrosophic field $K_n(I)$, $W_n(I)$ be a nonempty subset of $V_n(I)$. Then $W_n(I)$ is a strong n-refined neutrosophic subspace if and only if:

$x+y \in W_n(I), m \cdot x \in W_n(I)$ for all $x, y \in W_n(I), m \in K_n(I)$.

Proof:

It holds directly from the condition of submodule.

Example 3.11:

Let $V = R^2$ be a vector space over the field R, $W =< (0,1) >$ is a subspace of V, $R_2^2(I) = \{(a,b) + (m,s)I_1 + (k,t)I_2; a,b,m,s,k,t \in R\}$ is the corresponding weak/strong 2-refined neutrosophic vector space.

$W_2(I) = \{a_0 + a_1 I_1 + a_2 I_2\} = \{(0,x) + (0,y)I_1 + (0,z)I_2; x,y,z \in R\}$ is a weak 2-refined neutrosophic subspace of the weak 2-refined neutrosophic vector space $R_2^2(I)$ over the field R.

$W_2(I) = \{a_0 + a_1 I_1 + a_2 I_2\} = \{(0,x) + (0,y)I_1 + (0,z)I_2; x,y,z \in R\}$ is a strong 2-refined neutrosophic subspace of the strong 2-refined neutrosophic vector space $R_2^2(I)$ over the n-refined neutrosophic field $R_2(I)$.

Definition 3.12:

Let $V_n(I)$ be a weak n-refined neutrosophic vector space over the field K, x be an arbitrary element of $V_n(I)$, we say that x is a linear combination of $\{x_1, x_2, \dots, x_m\} \subseteq V_n(I)$, or $x = a_1 x_1 + a_2 x_2 + \cdots + a_m x_m: a_i \in K, x_i \in V_n(I)$.

Example 3.13:

Consider the weak 2-refined neutrosophic vector space in Example 3.11,

$x = (0,2) + (1,3)I \in R_2^2(I)$, $x = 2(0,1) + 1(1,0)I_1 + 3(0,1)I_2$, i.e x is a linear combination of the set $\{(0,1), (1,0)I_1, (0,1)I_2\}$ over the field R.

Definition 3.14:

Let $V_n(I)$ be a strong n-refined neutrosophic vector space over an n-refined neutrosophic field $K_n(I)$, x be an arbitrary element of $V_n(I)$, we say that x is a linear combination of $\{x_1, x_2, \dots, x_m\} \subseteq V_n(I)$ is $x = a_1x_1 + a_2x_2 + \cdots + a_mx_m$: $a_i \in K_n(I), x_i \in V_n(I)$.

Example 3.15:

Consider the strong 2-refined neutrosophic vector space $R_2^2(I) = \{(a,b) + (m,s)I_1 + (k,t)I_2; a,b,m,s,k,t \in R\}$ over the 2-refined neutrosophic field $R_2(I)$,

$x = (0,2) + (3,3)I_1 + (-1,0)I_2 = (2 + I_1) \cdot (0,1) + (1 + I_2) \cdot (1,1)I_1 + (I_1 - I_2) \cdot (1,0)I_2$, hence x is a linear combination of the set $\{(0,1), (1,1)I_1, (1,0)I_2\}$ over the 2-refined neutrosophic field $R_2(I)$.

Definition 3.16:

Let $X = \{x_1, \dots, x_m\}$ be a subset of a weak n-refined neutrosophic vector space $V_n(I)$ over the field K, X is a weak linearly independent set if $\sum_{i=0}^{m} a_ix_i = 0$ implies $a_i = 0, a_i \in K$.

Definition 3.17:

Let $X = \{x_1, \dots, x_m\}$ be a subset of a strong n-refined neutrosophic vector space $V_n(I)$ over the n-refined neutrosophic field $K_n(I)$, X is a weak linearly independent set if $\sum_{i=0}^{m} a_ix_i = 0$ implies $a_i = 0, a_i \in K_n(I)$.

Definition 3.18:

Let $V_n(I), W_n(I)$ be two strong n-refined neutrosophic vector space over the n-refined neutrosophic field $K_n(I)$, let $f: V_n(I) \to U_n(I)$ be a well defined map. It is called a strong n-refined neutrosophic homomorphism if:

$f(ax + by) = af(x) + bf(y)$ for all $x, y \in V_n(I), a, b \in K_n(I)$.

A weak n-refined neutrosophic homomorphism can be defined as the same.

We can understand the strong n-refined homomorphism as a module homomorphism, weak n-refined neutrosophic homomorphism can be understood as a vector space homomorphism.

Remark:

The previous definition of n-refined homomorphism between two strong/weak n-refined vector spaces is a classical homomorphism between two modules/spaces. We can not add a similar condition to the concept of neutrosophic homomorphism ($f(I_i) = I_i$), since I_i is not supposed to be an element of $V_n(I)$if V has more than one dimension for example. According to our definition,Ker(f) will be a subspace (which is different from classical neutrosophic vector space case) sicne$f$ was defined as a classical homomorphism without any additional condition.

Definition 3.19:

Let $f: V_n(I) \to U_n(I)$ be a weak/strong n-refined neutrosophic homomorphism, we define:

(a) $Ker(f) = \{x \in V_n(I); f(x) = 0\}$.

(b) $Im(f) = \{y \in U_n(I) \exists x \in V_n(I) \text{ and } y = f(x)\}$.

Theorem 3.20:

Let $f: V_n(I) \to U_n(I)$ be a weak n-refined neutrosophic homomorphism. Then

(a) $Ker(f)$ is a weak n-refined neutrosophic subspace of V_n(I).

(b) $Im(f)$ is a weak n-refined neutrosophic subspace of $U_n(I)$.

Proof:

(a) f is a vector space homomorphism since $V_n(I), U_n(I)$ are vector spaces, hence $Ker(f)$ is a subspace of the vector space $V_n(I)$, thus $Ker(f)$ is a weak n-refined neutrosophic subspace of $V_n(I)$.

(b) It holds by similar argument.

Theorem 3.21:

Let $f: V_n(I) \to U_n(I)$ be a strong n-refined neutrosophic homomorphism. Then

(a) $Ker(f)$ is a strong n-refined neutrosophic subspace of V_n(I).

(b) $Im(f)$ is a strong n-refined neutrosophic subspace of $U_n(I)$.

Proof:

(a) f is a module homomorphism since $V_n(I), U_n(I)$ are modules over the n-refined neutrosophic field $K_n(I)$, hence $Ker(f)$ is a submodule of the vector space $V_n(I)$, thus $Ker(f)$ is a strong n-refined neutrosophic subspace of $V_n(I)$.

(b) Holds by similar argument.

Example 3.22:

Let $R_2^2(I) = \{x_0 + x_1 I_1 + x_2 I_2 ; x_0, x_1, x_2 \in R^2\}$, $R_2^3(I) = \{y_0 + y_1 I_1 + y_2 I_2 ; y_0, y_1, y_2 \in R^3\}$ be two weak 2-refined neutrosophic vector space over the field R. Consider $f: R_2^2(I) \to R_2^3(I)$, where

$f[(a,b) + (m,n)I_1 + (k,s)I_2] = (a,0,0) + (m,0,0)I_1 + (k,0,0)I_2$, f is a weak 2-refined neutrosophic homomorphism over the field R.

$Ker(f) = \{(0,b) + (0,n)I_1 + (0,s)I_2 ; b,n,s \in R\}$.

$Im(f) = \{(a,0,0) + (m,0,0)I_1 + (k,0,0)I_2 ; a,m,k \in R\}$.

Example 3.23:

Let $W_2(I) =< (0,0,1)I_1 >= \{q.\ (0,0,a)I_1 ; a \in R, q \in R_2(I)\}$, $U_2(I) =< (0,1,0)I_1 >= \{q.\ (0,a,0)I_1 ; a \in R ; q \in R_2(I)\}$ be two strong 2-refined neutrosophic subspaces of the strong 2-refined neutrosophic vector space $R_2^3(I)$ over 2-refined neutrosophic field R_2(I). Define $f: W_2(I) \to U_2(I); f\ [q(0,0,a)I_1] = q(0,a,0)I_1 ; q \in R_2(I)$.

f is a strong 2-refinedneutrosophic homomorphism:

Let $A = q_1(0,0,a)I_1, B = q_2(0,0,b)I_1 \in W_2(I) ; q_1, q_2 \in R_2(I)$, we have

$$A + B = (q_1 + q_2)(0,0,a+b)I_1, f(A+B) = (q_1 + q_2).(0,a+b,0)I_1 = f(A) + f(B).$$

Let $m = c + dI_1 + eI_2 \in R_2(I)$ be a 2-refined neutrosophic scalar, we have

$m \cdot A = c \cdot q_1(0,0,a)I_1 + d \cdot q_1(0,0,a)I_1I_1 + e \cdot q_1(0,0,a)I_2I_1 = q_1(0,0,ca+da+ea)I_1$,

$f(mA) = q_1(0,ca+da+ea,0)I_1 = m \cdot f(A)$, hence f is a strong 2-refined neutrosophic homomorphism.

$Ker(f) = (0,0,0) + (0,0,0)I_1 + (0,0,0)I_2$.

$Im(f) = U_2(I)$.

Remark 3.24:

A union of two n-refined neutrosophic vector spaces $V_n(I)$and$W_n(I)$ is not supposed to be an n-refined neutrosophic vector space, since the addition operation can not be defined. For example consider $V = R^3, W = R^2, n = 2$.

5. Conclusion

In this paper we have introducedthe concept of weak/strong n-refined neutrosophic vector space. Also, some related concepts such as weak/strong n-refined neutrosophic subspace, weak/strong n-refined neutrosophic homomorphism have been presented and studied.

Future research

Authors hope that some corresponding notions will be studied in future such as weak/strong n-refined neutrosophic basis, and AH-subspaces.

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Modified Neutrosophic Fuzzy Optimization Model for Optimal Closed-Loop Supply Chain Management under Uncertainty

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1. Introduction

Needs for commodities or products for human life explored and enhanced a revolution in industrial sectors. An initially integrated mechanism for production to consumption of commodities was only the goals of any firm. Decision policies concerning the production and use of products were the prime concern. Therefore, systematic and business-oriented managerial practices were designed for the flow of products, termed as supply chain management (SCM). SCM is the procedure of procurement, processing, distribution, and consumption of finished products in a clear planning timescale. The general structural domain of SCM includes a raw material supplier point, a manufacturing plant, a distribution center, and the end-users or customers. These echelons are interconnected or interlocked to each other for the movement of different materials and products. The organizational and managerial perspective of SCM terminates at the end-users of finished products and terminates from ultimately the next stages related to the three R's (reduce, reuse, and recycle). End-of-use and end-of-life products create various environmental issues due to improper management of used products. Consequently, harmful impacts due to landfills, contamination of freshwater resources, and toxic air pollution generated on a large scale influenced human life drastically. These issues could not be compensated at any cost. To ensure that environmental questions and social concerns arise during supply chain design, a government has taken the initiative and established laws that include wholesome supply chain practices, termed as the closed loop supply chain (CLSC) network. The CLSC design helps in strengthening the ecofriendly practices with end-of-use products and reduces environmental impacts. Therefore, to reveal pervasiveness in SCM, extension of echelons has been located. Hence the concept of the reverse chain has been identified to execute backward processes for used products. Generally, the reverse chain consists of different echelons, such as the collection center, recycling point, and disposal sites. The CLSC design contemplates the flow of different materials, products, and parts of the used commodity in a well-defined interconnected path. Various facility centers in the reverse chain reduce the environmental impacts and substantiates ecofriendly production-consumption planning scenarios. For successful completion of sustainable trade practices, the significant role of the CLSC design network would be crucial or at least prominent. Ultimate destinations for end-of-use and end-of-life products would be more rigorously depicted in CLSC design. Refurbishing and recycling centers inevitably provide services to the used products and parts to transform into their useful life. The marginal reduction in different kinds of costs and a significant increase in revenues are the counterpart for enhancement in net profit throughout the CLSC planning network.

Consumerism has been a considerable part of the sustainability problem for years by imposing a burden with harmful waste through flooding and landfill issues. The CLSC business model implements highly efficient management of materials and waste minimization strategies that lead to zero-waste generation. The CLSC management network includes either putting all outputs back into the system or incineration. A combination of forwarding and reverse material flows to reuse and recycle all metals and transform waste into energy. The CLSC can enable manufacturers to take a proactive stance toward and ensure easy compliance with electronic waste regulations. Environmental value is the ease of agreement to be more conscious about the environment. A CLSC can allow the business to respond to ecological concerns by saving energy and decreasing the input of new materials. Consumer value can be achieved by a well-organized customer product returns system that can help ensure hassle-free warranties and improve customer loyalty. Improved parts management helps the business deliver extended warranties and service agreements that can boost customer satisfaction. The acquisition process in CLSC management provides valuable data on common production issues, supply defects, failure rates, product lifecycle, consumer complaints, and consumer usage patterns. This information can be used to improve product design and development. Minimize wastewater and industrial sludge production by reducing the amount of water needed for the manufacturing process. Procure raw material in bulk (where possible) to reduce the amount of packaging material that enters the waste stream. Assure precautions to avoid the process that causes hazardous waste to be mixed with nonhazardous waste, minimizing the amount of dangerous waste that must be stored, treated, and disposed of. Practice quality control strategies like ISI 14001 and Six Sigma to help minimize product defects.

The implicitness of uncertainty is trivial in real-life scenarios. Inconsistent, incomplete, inappropriate, inexact, and improper information about various input parameters such as costs, capacity, and demand in the CLSC design network lead to the existence of uncertainty theory. Several aspects inherently affect the modeling and optimizing procedure of real-life optimization problem. Abrupt changes in the prices of raw materials, hike in fuel rates, increases in required facility locations, behavior of fluctuating markets, competition among different companies' policies for customer satisfaction, environmental conditions, failing in timely shipment of ordered products, political and governmental policies regarding various taxes over procurement, production, distribution, and management of end-of-use products are the most dominating factors for causing uncertainty in modeling approaches. Impreciseness may be represented in different forms. The difficulty involved in parameters due to vague information can be dealt with by different fuzzy techniques. Fuzziness among parameters most frequently encounters and results in uncertainty modeling. To reflect the most common aspect of uncertainty, we have assumed that all the input parameters are a triangular and trapezoidal fuzzy number rather than stochastic random variables. Defuzzification or the ranking function executes the process of obtaining the crisp or deterministic version of a fuzzy number. A robust technique has been used, which covers an extensive range of feasibility degrees. Most of the conventional methods are limited to fuzzy-based solution schemes by defining the marginal evaluation of each objective using the membership function. Apart from metaheuristic techniques, a tremendous number of research papers have investigated and implemented the different fuzzy optimization techniques to obtain the global compromise solution of the CLSC planning problems. A detailed list of such fuzzy approaches can be found in Govindan et al. [1] and Govindan and Bouzon [2]. Here in this study, a neutrosophic fuzzy programming approach (NFPA) based on the neutrosophic decision has been suggested to solve the proposed CLSC design problem. Intuitionistic fuzzy imprecise preference relations among different objectives have also been investigated and successfully incorporated with an NFPA which is together termed as modified NFPA with intuitionistic fuzzy importance relations.

The rest part of this chapter is as follows: In Section 15.2, a literature review related to the CLSC network is presented whereas Section 15.3 highlights the significant research contribution. Section 15.4 discusses the modeling CLSC design network under uncertainty while Section 15.5 represents the solution methodology to solve the final model. A real-life case study based on a laptop manufacturing firm is examined in Section 15.6, which shows the applicability and validity of the proposed approach efficiently. Finally, conclusions are highlighted based on the present work in Section 15.7.

2. Literature review

The CLSC planning problem has rapidly gained popularity among many researchers. The complex and challenging situation during the flow of goods and products from different sources to destination points has immensely attracted attention toward emerging research scope for the optimal policy implementation or decision-making processes to CLSC planning problems. Consequently, different approaches to solve the CLSC planning model have been introduced, along with their promising features in the context of optimality and applicability under different environments. Thus, here we review some existing CLSC models under different uncertainty and discuss the approaches adopted to solve them.

A well-defined set of the interconnected network for the flow of multiple products has also created a very complex configuration of multiechelon CLSC design. Most of the existing studies have been presented on multiproduct and multiechelon CLSC planning problems. Gupta and Evans [3] have addressed multiple-echelon CLSC frameworks for electrical and electronic gadget scrap products. They designed a weighted nonprimitive goal programming model for the CLSC model and solved the proposed model with the aid of a discrete weighting scheme to the corresponding goal preference. Pishvaee et al. [4] designed a robust optimization model for CLSC configuration under randomly distributed parameters. The developed modeling approach then turned into the deterministic mixed-integer linear programming model and they solved this using a robust optimization technique. Özceylan and Paksoy [5] also presented a mixed-integer fuzzy mathematical model for CLSC under uncertainty with multiparts and multiperiods. The fuzzy solution approach has been applied for both fuzzy objectives and parameters with the help of a linear membership function. Özkır and Başlıgil [6] developed a multiobjective CLSC model with particular emphasis on the satisfaction level of trade, customer, and net profit incurred over the current product's lifetime in the supply chain network. They adopted a fuzzy set (FS) theory-based solution method to deal with the proposed CLSC model. Yin and Nishi [7] also discussed an SCM problem with a quantity discount and uncertain demand at each echelon. The constructed SCM model resulted in the form of a mixed-integer nonlinear programming problem (MINLPP) with integral functions. An outer-approximation method has been suggested to solve the MINLPP. An improvement in efficiency performance has been achieved by reconstructing the MINLPP model into a stochastic programming model with the replacement of integral functions by incorporating the normalization method. Özceylan and Paksoy [8] addressed the CLSC planning model under tactical and strategic decision scenarios. The developed CLSC planning model has emerged as an MINLPP. They applied a fuzzy interactive solution approach to solve the propounded CLSC network design. Garg et al. [9] also designed a sustainable CLSC network with the core emphasis on environmental issues raised after the end of use and end of life of the used products. They formulated a bio-bjective integer nonlinear programming problem for the proposed CLSC network. The solution scheme has been adopted and applied by balancing the trade-off between socioeconomic and environmental aspects. The interactive multiobjective programming approach has been used to obtain the optimal allocation of different products. Alshamsi and Diabat [10] presented the reverse logistic (RL) system in the CLSC design network. The proposed RL texture initiates at the customer level and terminates at remanufacturing facilities in the reverse supply chain. The presented study was found to be limited to the RLs system. They modeled the deterministic mixed-integer linear programming problem with a single objective. A sustainable supply chain network has been designed by Arampantzi and Minis [11] and incorporates various factors, such as social, capital investment, environmental, political, etc., that affect the supply chain network directly and indirectly. They formulated a multiobjective mixed-integer linear programming problem and solved it by using two different conventional techniques: the goal programming method and the ϵ-constrained method.

Ma and Li [12] discussed a CLSC model for hazardous products under different uncertain parameters. The motive was to determine the optimal quantity of a shipment under a probabilistic environment. To address the scenario efficiently, the proposed model has been reformulated as a two-stage stochastic programming model along with risk and reward constraints. The two solution approaches, the Parallel Enumeration Method and the Genetic Algorithm (GA), have been applied to solve the designed CLSC model.

Fard and Hajaghaei-Keshteli [13] also addressed a tri-level location-allocation planning problem for a CLSC network. The modeling study undertaken comprises three echelons: distribution center, customer zone, and recovery facility. The propounded tri-level CLSC planning model has been solved by using a Variable Neighborhood Search, Tabu Search (TS), and Particle Swarm Optimization in addition to these approaches; Fard and Hajaghaei-Keshteli further applied two recent metaheuristic algorithms, the Keshtel Algorithm and Water Wave Optimization, to obtain a feasible solution to the location-allocation problem. Zhen et al. [14] also designed a CLSC model with the capacitated allocation of products under uncertain demand for new and returned merchandise. The proposed decision-making model turned into a two-stage stochastic mixed-integer nonlinear programming problem (SMINLPP). Thus, the transformed model resulted in the deterministic demand and capacities parameters involved in the designed CLSC model. They also implemented the TS algorithm to solve the SMINLPP. Tsao et al. [15] formulated a sustainable supply chain design under economic and environmental objectives. The proposed supply chain model has taken the form of a multiobjective mathematical programming problem under stochastic demand and fuzzy costs. An interactive two-phase fuzzy probabilistic multiobjective programming problem has been introduced to deal with both sorts of uncertainty. Hasanov et al. [16] addressed the optimal quantity of products under four-level CLSC with a hybrid remanufacturing facility. The reverse chain includes the recovered process, which ensures the reuse of used products at a different level. The mathematical modeling framework has been carried out with a particular emphasis on remanufactured or returned products, or both. The developed modeling approach is aiming to minimize the overall cost incurred over the policies implemented during a single time horizon. Fakhrzad et al. [17] presented multiple products, periods, levels, and indices in the green CLSC planning model under uncertainty. The propounded model was then transformed into the multiobjective mixed-integer linear programming problem. Since the proposed model was NP-hard, to deal with it Nondominated Sorting Genetic Algorithm-II (NSGA-II) has been adopted to solve the proposed green CLSC network. Singh and Goh [18] also discussed the multiobjective mixed-integer linear programming problem under intuitionistic fuzzy parameters. Further, they transformed the multiobjective optimization problem into a single objective to solve the model. To achieve an acceptable satisfaction degree, different scalarization techniques such as the γ-connective approach and minimum sum bounded operator have been used. The proposed solution scheme has also been implemented to solve the pharmaceutical SCM model. Fathollahi-Fard et al. [19] designed a multiobjective stochastic CLSC model with an exclusive focus on the social issues associated with individual requirement and responsibility (such as job opportunity). The addressed stochastic CLSC model has been solved by using a couple of different nature-inspired algorithms and hybridized into the benefits of both, that is, social and environmental domains. Liao [20] presented a reverse logistics network design (RLND) for product recovery and remanufacturing processes. The proposed model emerged into a conventional mixed-integer nonlinear programming model for RLND under multiple echelons. The GA has been adopted as the solution method of the proposed RLND model. The formulated modeling structure has been validated and implemented with the help of the recycling bulk waste example in Taiwan. Zarbakhshnia et al. [21] have also discussed the green closed loop logistics network model as the mixed-integer linear programming problem. The undertaken study has mainly been concerned with the multiple stages, products, and objectives in the proposed model. A solution scheme, the ϵ-constraint method, has been chosen to solve numerous targets. Dominguez et al. [22] also investigated the role of manufactured and remanufactured products in the CLSC with capacitated constraints. The research background explicitly reveals the four relevant uncertain factors to determine the efficiency of executed policies in the system. A managerial insight has been propounded that could contribute to understanding decision-making processes. Eskandarpour et al. [23] presented a study on the literature review of approximately 80 research papers in the field of CLSC planning problems. The chosen study area has been classified based on four questions: (i) What kind of socioeconomic and environmental issues have been included? (ii) How the problems related to the matters discussed have been unified or integrated in the supply chain model? (iii) What sort of solution schemes have been applied to solve the modeling problem? and (iv) Which numerical illustrations or computational studies have been taken from real-life applications? Furthermore, the shortcomings and drawbacks of different models have been pointed out, and consequently, the scope for future research has also been intimated. The interested reader may refer to the recent publications by Govindan et al. [1] and Govindan and Bouzon [2], based on reviewed work in the reverse logistic barriers and drivers.

3. Research contribution

A tremendous amount of work has been developed and applied successfully on the CLSC network in the last few decades. Only a few research works are available that have included the testing center as a facility location for the dissembled parts/components [3, 9]. Therefore, this chapter has put more emphasis on the reverse chain and is mainly concerned with end-of-use products and end-of-life. The modified neutrosophic fuzzy optimization techniques have been used for the first time in the field of SCM.

The following are the significant and remarkable contributions to this presented research work.

- The proposed CLSC planning model has been designed for multidimensional echelons, in which five multiple echelons have been included in the forwarding chain, whereas six various echelons have been integrated into the reverse chain which shows the great concern or influence regarding the end-of-use and end-of-life products. The different facility centers in the reverse chain ensure that the CLSC planning model is socioeconomic and environmentally friendly.

- The different objective functions have been presented to analyze the shares in total capital investment over the raw materials and products in the forward and reverse chain individually. A new preference scheme has been investigated to achieve better outcomes for the preferred objective functions.
- The uncertainty among parameters has been represented with fuzzy numbers and dealt with the expected interval and expected values of the involved parameters. Three constraints have been depicted with fuzzy equality in restrictions, which reveals the reality more closely. The fuzzy equality constraints are then efficiently transformed into two subconstraints.
- The NFPA has been developed to solve the proposed CLSC designed model. The proposed solution approach has been inspired by the indeterminacy degree that emerged in decision-making processes. Indeterminacy/neutral thoughts are the region of negligence for propositions' values, between the degree of acceptance and rejection. It is the first time that the NFPA has been applied to solve the CLSC planning model.
- A novel intuitionistic fuzzy linguistic preference scheme has been investigated to assign weight/preference to the most preferred objective functions. The intuitionistic fuzzy linguistic preference relations have been efficiently integrated with an NFPA and termed as a modified NFPA.
- The proposed CLSC designed model has been implemented on real case study data to show the validity and applicability of the proposed solution methods. A variety of different solutions sets has been generated and summarized under the optimal choices of quantity allocation.
- The sensitivity analysis has also been performed on the obtained solution results based on the feasibility degree β and crisp weight parameter α by tuning them at different values between 0 and 1.
- The significance of the obtained results has been analyzed along with the remarkable findings. Conclusions and future research scope have been set out based on the present study.

4. Description of CLSC network

A well-organized systematic and interconnected network for the flow of materials, products, and parts is much needed to survive in the competitive market. Production processes explicitly adhere to the different perspectives of the finished products. The conventional supply chain design initiates with the availability of raw resources to finished goods and terminates at the consumption points. The globalization of markets, governmental legislation, and environmental practices creates many concerns for the used products and leads to the existence of a CLSC that inherently ensures the best management of end-of-use and end-of-life products. The efficiently expanded texture of the supply chain network designed has been widely adopted by the decision maker(s) with the inclusion of the reverse chain. Therefore, the CLSC network consists of two phases: forward chain and reverse chain for the flow of material, products, and used parts. In this study, a CLSC design is presented, which consists of five echelons in the forward chain and six echelons in the reverse chain, which is shown in Fig. 15.1.

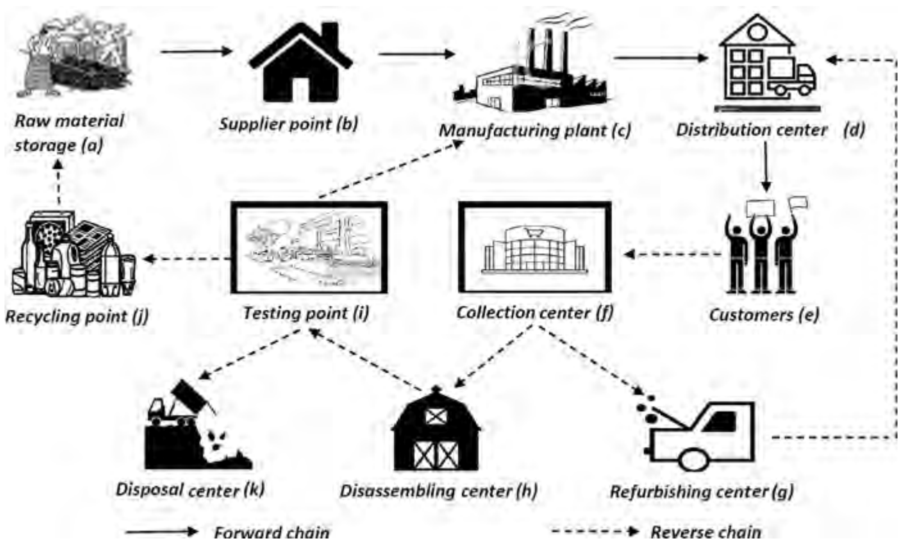

Fig. 15.1 Proposed closed loop supply chain network design.

The initialization of production processes starts with the procurement of raw materials from the storage center to the supplier point, which in turn supplies the relevant raw materials to the manufacturing plant for the production of new products. Afterward, the finished products are delivered to the distribution center to fulfill the demand of customers or markets. Unlike the forward logistics flow, the reverse logistics flow consists of a few more steps. The first step involves collecting defective products from customers at the collection center. The end-of-use products are outsourced from different customers, either directly or via markets. Collection of used products initiates the sustainable reverse chain, and collecting the used products maintains the flow cycle of products into a different facility phase. The collection center is responsible for the optimal distribution of used products for further required services. In most cases, returns processors collect fewer defective products, undertake repairs at refurbishing centers, and return them to the buyers. At this point, it is worth noting that returns processors may remanufacture defective products and ship them back to retailers and distributors, who in turn sell them to end users. Alternatively, returns processors may recycle defective products to extract materials and parts that can be reused in the production process by sending them to a disassembling center. Further, the parts and materials are taken to a testing point for the inspection of their further utility, and from there the elements that can be used to make new goods are sent back to the hybrid manufacturing plant. On the other hand, only the parts that are recyclable are shipped to the recycling point, and move forward through the supply chain until they reach the end users. The final step involves any materials or parts that are not utilized throughout the steps discussed earlier, which reach the disposal center for incineration or dumping purposes.

To and fro movement of materials, products, and parts throughout the CLSC network contemplates over multifarious objectives associated with the entire phenomenon. Procurement, processing, distribution, and transportation processes turn into a significant investment in costs which should be optimized under optimal allocation of the commodity. The cost of purchasing raw materials and used products is also a measure of great concern. Delivery time of the finished products to the customers must be reduced to overcome cancelations of ordered products. Revenues from sales of new products and recyclable parts encourage the enhancement of shares in the net profit. Hence the proposed CLSC model comprises multiple conflicting objectives such as minimization of processing, purchasing, and transportation costs, minimization of expected product delivery time, and maximization revenue from the selling of the products.

The propounded CLSC planning network configuration is based on the following postulated assumptions:

- The propounded CLSC network has been designed for multiple raw materials/parts multi-products, and multiechelons along with single time horizons. Each facility location is well established and functional for the associated services over the stipulated period.
- Movement of new products initiates from manufacturing plants to customers, and the flow of used products starts from customers to the disassembling center. Meanwhile, the recovered products are also shipped after renovation to the distribution center. Therefore, the demand for new and refurbished products is met through the distribution center only.
- Set-up costs associated with different echelons are assumed to be included in the processing costs. Revenues are only derived from the selling prices of new products and recyclable products, which turn into a contribution to the net profit.
- The disposal facility is the only route to remove the scrap parts/components from the proposed CLSC planning model. The rest of the quantity is assumed to remain in its useful life.
- Uncertainty among different parameters has been considered as fuzzy numbers. The fuzzy linguistic term has assigned the preference among different objective functions.

Indices	Descriptions
a	The number of raw materials/parts/components storage points ($a = 1, 2, \ldots, A$)
b	The number of supplier points ($b = 1, 2, \ldots, B$)
c	The number of manufacturing/remanufacturing plants ($c = 1, 2, \ldots, C$)
d	The number of distribution center ($d = 1, 2, \ldots, D$)
e	The number of customers/markets ($e = 1, 2, \ldots, E$)
f	The number of collection center ($f = 1, 2, \ldots, F$)
g	The number of refurbishing/repairing center ($g = 1, 2, \ldots, G$)
h	The number of disassembling center ($h = 1, 2, \ldots, H$)
i	The number of raw materials/parts/components testing points ($i = 1, 2, \ldots, I$)
j	The number of recycling points ($j = 1, 2, \ldots, J$)
k	The number of disposal centers ($k = 1, 2, \ldots, K$)
l	The number of different products ($l = 1, 2, \ldots, L$)

m	The number of raw materials/parts/components storage points ($m = 1, 2, \ldots, M$)
Decision variables	Descriptions
$X1_{m,a,b}$	The quantity of raw material m shipped from raw material storage point a to supplier point b
$X2_{m,b,c}$	The quantity of raw material m shipped from supplier point b to manufacturing plant c
$X3_{l,c,d}$	The quantity of different products l shipped from manufacturing plant c to different distribution center d
$X4_{l,d,e}$	The quantity of different products l shipped from different distribution center d to different customers/markets e
$X5_{l,e,f}$	The quantity of different used products l shipped from different customers/markets e to collection center f
$X6_{l,f,g}$	The quantity of different repairable products l shipped from collection center f to refurbishing center g
$X7_{l,g,d}$	The quantity of different recovered products l shipped from refurbishing center g to different distribution center d
$X8_{l,f,h}$	The quantity of different unrepairable products l shipped from collection center f to disassembling center h
$X9_{m,h,i}$	The quantity of parts/components m shipped from disassembling center h to testing point i
$X10_{m,i,c}$	The quantity of raw material m shipped from testing point i to manufacturing plant c
$X11_{m,i,j}$	The quantity of recyclable parts/components m shipped from testing point i to recycling point j
$X12_{m,i,k}$	The quantity of scrap parts/components m shipped from testing point i to disposal center k
$X13_{m,j,a}$	The quantity of recovered parts/components m shipped from recycling point j to raw materials storage point a
Parameters	Descriptions
rf_l	Recovery rate of used products l at refurbishing center
$rc_{l,e}$	Collection rate of used products l from customer or market e
rt_m	Testing rate of different parts/components m at testing center
rm_m	Reuse rate of different tested parts/components m at manufacturing plant
rr_m	Recycling rate of different recyclable parts/components m at recycling center
rd_m	Disposal rate of raw materials/parts/components m at disposal center

Parameters	**Descriptions**
$PC1_{m,a}$	Unit storage cost incurred over raw material m at raw material storage center a
$PC2_{m,b}$	Unit safety cost incurred over raw material m at supplier point b
$PC3_{l,c}$	Unit production cost levied over product l at manufacturing plant c
$PC4_{l,d}$	Unit inventory holding cost levied over product l at distribution center d
$PC5_{l,f}$	Unit collection facility cost levied over product l at collection center f
$PC6_{l,g}$	Unit refurbishing cost levied over product l at refurbishing center g
$PC7_{l,h}$	Unit disassembling cost levied over product l at disassembling center h
$PC8_{m,i}$	Unit testing cost levied over each component m at testing center i

$PC9_{m,j}$	Unit recycling cost levied over raw material *m* at recycling point *j*
$PC10_{m,k}$	Unit disposal cost levied over each component *m* at disposal center *k*
$TC1_{m,a,b}$	Unit transportation cost of raw material *m* shipped from raw material storage point *a* to supplier point *b*
$TC2_{m,b,c}$	Unit transportation cost of raw material *m* shipped from supplier point *b* to manufacturing plant *c*
$TC3_{l,c,d}$	Unit transportation cost of different products *l* shipped from manufacturing plant *c* to different distribution center *d*
$TC4_{l,d,e}$	Unit transportation cost of different products *l* shipped from different distribution center *d* to different customer/market *e*
$TC5_{l,e,f}$	Unit transportation cost of different used products *l* shipped from different customers/markets *e* to collection center *f*
$TC6_{l,f,g}$	Unit transportation cost of different repairable products *l* shipped from collection center *f* to refurbishing center *g*
$TC7_{l,g,d}$	Unit transportation cost of different recovered products *l* shipped from refurbishing center *g* to different distribution center *d*
$TC8_{l,f,h}$	Unit transportation cost of different unrepairable products *l* shipped from collection center *f* to disassembling center *h*
$TC9_{m,h,i}$	Unit transportation cost of parts/components *m* shipped from disassembling center *h* to testing point *i*
$TC10_{m,i,c}$	Unit transportation cost of parts/components *m* shipped from testing point *i* to manufacturing plant *c*
$TC11_{m,i,j}$	Unit transportation cost of different recyclable parts/components *m* shipped from testing point *i* to recycling point *j*
$TC12_{m,i,k}$	Unit transportation cost of disposable parts/components *m* shipped from testing point *i* to disposal center *k*
$TC13_{m,j,a}$	Unit transportation cost of recovered parts/components *m* shipped from recycling point *j* to raw materials storage point *a*
$T_{l,d,e}$	Unit transportation time required to ship different products *l* from distribution center *d* to different customers/markets *e*
$PU1_m$	Unit purchasing cost of raw materials/parts/components *m*
$PU2_l$	Unit purchasing cost of different used products *l*
$SP1_m$	Unit selling price of different recyclable parts/components *m*
$SP2_l$	Unit selling price of different new products *l*
$MC1_{m,a}$	Maximum available quantity of raw material *m* at raw material storage center *a*
$MC2_{m,b}$	Maximum available quantity of raw material *m* at supplier *b*
$MC3_{m,c}$	Minimum required quantity of raw material *m* at manufacturing plant *c*
$MC4_{l,d}$	Maximum available quantity of new products *l* at distribution center *d*
$MC5_{l,e}$	Minimum demand quantity of different new products *l* by customers or at markets *e*
$MC6_{l,f}$	Maximum collection capacity of different used products *l* at collection center *f*
$MC7_{l,g}$	Maximum refurbishing capacity of different repairable products *l* at refurbishing center *g*
$MC8_{m,h}$	Maximum disassembling capacity of different parts/components *m* at disassembling center *h*

$MC9_{m,i}$	Maximum testing capacity different scrap parts/components m at testing point i
$MC10_{m,j}$	Maximum capacity of recyclable parts/components m at recycling point j
$MC11_{m,k}$	Maximum disposal capacity of disposable parts/components m at disposal center k

4.1 Multiple objective function

The typical and efficient CLSC model always comprises multiple conflicting objectives for both forward and reverse chains, which are to be attained simultaneously. Here, we highlight the different costs associated with ahead and change strings separately to analyze the echelon-wise effects in terms of expenditure on the overall CLSC planning problem.

Objective 1: Total processing costs. Initially, the raw materials have been stored at a raw material storage center to ensure the smooth running of the CLSC design. The processing cost indicates the different sort of value at each echelon such as storage cost at the raw material storage center, safety cost at the supplier point, production cost at the manufacturing center and inventory or distribution cost at the distribution center, levied on the unit raw material or new products. The significant reduction in these processing costs automatically results in the maximum margin of profit. The reverse chain also contains multiple echelons with different processing costs associated with them. Here, the processing cost refers to the value of the collection at the collection center, the cost of disassembly at the disassembling center, the refurbishing cost at the refurbishing center, the cost of testing at the testing center, the cost of recycling at the recycling center, and the disposal cost at the disposal point, respectively. The designed network facility executed by each echelon ensures that the commonly used products in the reverse supply chain survive at their end-of-life use or disposable condition. Thus the first objective function ensures the minimization of the processing costs at different echelon in the forward chain under the optimal quantity allocation.

$$
\begin{aligned}
\text{Minimize } Z_1 = & \sum_{m=1}^{M}\sum_{a=1}^{A} PC1_{m,a}X1_{m,a,b} + \sum_{m=1}^{M}\sum_{b=1}^{B} PC2_{m,b}X2_{m,b,c} \\
& + \sum_{l=1}^{L}\sum_{c=1}^{C} PC3_{l,c}X3_{l,c,d} + \sum_{l=1}^{L}\sum_{d=1}^{D} PC4_{l,d}X4_{l,d,e} \\
& + \sum_{l=1}^{L}\sum_{f=1}^{F} PC5_{l,f}X5_{l,e,f} + \sum_{l=1}^{L}\sum_{g=1}^{G} PC6_{l,g}X6_{l,f,g} \\
& + \sum_{l=1}^{L}\sum_{h=1}^{H} PC7_{l,h}X8_{l,f,h} + \sum_{i=1}^{M}\sum_{i=1}^{I} PC8_{m,i}X10_{m,i,c} \\
& + \sum_{m=1}^{M}\sum_{j=1}^{J} PC9_{m,j}X11_{m,i,j} + \sum_{m=1}^{M}\sum_{k=1}^{K} PC10_{m,k}X12_{m,i,k} \quad \forall\ c,f,g,h,a,b,c,d,e,i.
\end{aligned}
$$

Objective 2: Total transportation costs.

The transportation cost is one of the well-known objective functions under CLSC design. Typical and interconnected transportation networks within each echelon in CLSC design yield high transportation costs. In the forward chain, the shipment of raw material from the raw material storage point to the supplier point and from the supplier point to the manufacturing plant integrates the marginal shares in the total transportation cost. The delivery of new products from the manufacturing plant to the distribution center and from the distribution center to customers also has a significant role in attaining the gross profit in the proposed CLSC network. The propounded CLSC network has put more emphasis on the reverse chain by including more facility locations compared to the forward chain. The to and fro shipment of used products and raw parts/components results in high transportation costs. The reverse chain network allows the recovered products and tested parts/components to enter into the forward chain directly from the refurbishing center to the distribution center and from the testing point to the manufacturing plant without touching the recycling facility. Hence to and fro shipment of products and parts/components from multiple different echelons is turned into high transportation costs. Therefore, the second objective function results in the minimization of to and fro transportation costs to varying echelons in the forward chain for the maximum shipment quantity of products under the optimal allocation policy.

$$
\begin{aligned}
\text{Minimize } Z_2 = & \sum_{m=1}^{M}\sum_{a=1}^{A}\sum_{b=1}^{B} TC1_{m,a,b}X1_{m,a,b} + \sum_{m=1}^{M}\sum_{b=1}^{B}\sum_{c=1}^{C} TC2_{m,b,c}X2_{m,b,c} \\
& + \sum_{l=1}^{L}\sum_{c=1}^{C}\sum_{d=1}^{D} TC3_{l,c,d}X3_{l,c,d} + \sum_{l=1}^{L}\sum_{d=1}^{D}\sum_{e=1}^{E} TC4_{l,d,e}X4_{l,d,e} \\
& + \sum_{l=1}^{L}\sum_{e=1}^{E}\sum_{f=1}^{F} TC5_{l,e,f}X5_{l,e,f} + \sum_{l=1}^{L}\sum_{f=1}^{F}\sum_{g=1}^{G} TC6_{l,f,g}X6_{l,f,g} \\
& + \sum_{l=1}^{L}\sum_{g=1}^{G}\sum_{d=1}^{D} TC7_{l,g,d}X7_{l,g,d} \\
& + \sum_{l=1}^{L}\sum_{f=1}^{F}\sum_{h=1}^{H} TC8_{l,f,h}X8_{l,f,h} + \sum_{m=1}^{M}\sum_{i=1}^{I}\sum_{h=1}^{H} TC9_{m,h,i}X9_{m,h,i} \\
& + \sum_{m=1}^{M}\sum_{i=1}^{I}\sum_{c=1}^{C} TC10_{m,i,c}X10_{m,i,c} \\
& + \sum_{m=1}^{M}\sum_{i=1}^{I}\sum_{j=1}^{J} TC11_{m,i,j}X11_{m,i,j} + \sum_{m=1}^{M}\sum_{i=1}^{I}\sum_{k=1}^{K} TC12_{m,i,k}X12_{m,i,k} \\
& + \sum_{m=1}^{M}\sum_{j=1}^{J}\sum_{a=1}^{A} TC13_{m,j,a}X13_{m,j,a}.
\end{aligned}
$$

Objective 3: Total purchasing cost of used products and raw materials.
In this proposed CLSC design, the purchasing of raw material and used products at two echelons has been allowed. The purchasing cost of raw materials from the supplier point and the purchasing cost of used products from customers yields the total purchasing cost. However, these costs leave a significant margin among the new outsourced products by contributing less operational costs on the recovered products. Therefore, the third objective function ensures the minimization of the total purchasing cost of raw materials and different used products from suppliers and customers to maintain the efficiency of the manufacturing plant.

$$\text{Minimize } Z_3 = \sum_{m=1}^{M} PU1_m X2_{m,b,c} + \sum_{l=1}^{L} PU2_l X5_{l,e,f} \quad \forall\ b,c,e,f.$$

Objective 4: Products delivery time.
The most critical issue in CLSC design is to determine the optimal time policy during the whole process. Notably, the shipment time of new products from the distribution center to different customers must be attained under the stipulated delivery period at the time of the ordered quantity. The goodwill and reputation of the company are strongly connected with delivery time. The latter also reduces the loss of any perishable products that happens due to delay. Moreover, cancelation from the customers' side would be almost negligible with the timely transshipment of the products. Henceforth, the fourth objective dynamically ensures the minimization of the total shipment time of different new products from the distributor to customers to maintain the reputation and reliability of the company.

$$\text{Minimize } Z_4 = \sum_{l=1}^{L} \sum_{d=1}^{D} \sum_{e=1}^{E} T_{l,d,e} X4_{l,d,e} \quad \forall\ d,e.$$

Objective 5: Revenues from the sale of new products and recyclable parts/ components.
By the significant increase in the sales ratio of new products and recyclable parts/ components, a marginal profit could be extracted. Selling of new products at higher quantities covers the maximum part of the capital investment during the production and distribution processes—recyclable parts/components are also a reliable source of profit from its sales. The selling price of the new products has a significant contribution toward the net profit and simultaneously yields in the contribution to gross profit. Thus the fifth or last objective function ensures the maximization of new products selling to survive in the competitive market with the maximum turnover of the new products under the optimal production policy.

$$\text{Maximize } Z_5 = \sum_{m=1}^{M} SP1_m X11_{m,i,j} + \sum_{l=1}^{L} SP2_l X4_{l,d,e} \quad \forall\ i,j,d,e.$$

4.2 Constraints

The following are the relevant constraints or restrictions under which the objective functions are to be optimized by yielding the most promising and systematic strategies for allocating different raw materials or parts/components and various products among multiple echelons in the proposed CLSC designed model. For the sake of convenience, we have categorized all the constraints under six different groups, and these can be summarized as follows.

4.2.1 Constraints related to the capacity of different echelons in the CLSC network

The procurement of raw materials initiates from the raw material storage center where the abundance or stock of raw materials has been kept to fulfill the demand from suppliers. Therefore, the total shipment quantity of different raw materials from the raw materials storage center to the supplier must not exceed its capacities and can be represented by Eq. (15.1). Supplier points also have a limited ability for the flow of different raw materials to maintain the intake and outsourced ratio. It is essential for the supplier to hold some raw materials for distribution at times of scarcity, when raw material storage functioning is interrupted over a stipulated time. Hence the constraints imposed over the number of raw materials shipped from the supplier point to a different manufacturing plant must less than or equal to the capacity of suppliers and can be presented by Eq. (15.2). The collection of used products from different customers starts the key functioning role of the reverse chain. It is the very first stage at which the end-of-use products are collected by the collection center. It must be assured that the accumulation quantity of used products from different customers must be less than or equal to the capacity of various collection centers and can be represented by Eq. (15.3). A well-organized system of collection centers provides frequent services to the used products so that all the end-of-use products are refurbished and can be used further without significantly affecting the demand. After ensuring the required services for used products, it has been allowed to ship the used merchandise from the collection center to the refurbishing center for renovating processes. Hence the total quantity of used products must not exceed the capacity of the refurbishing center and can be given in Eq. (15.4). The number of used products that need testing services for their further utilization has been shipped to the disassembling facility to disassemble the used products into different components or parts. To ensure that the number of used products which have been sent for dismantling purpose must be less than or equal to its capacity and can be represented by Eq. (15.5). After completing the required test for the parts/components, the recyclable quantity of parts/components has been sent to the recycling facility, which denotes the last echelon of the CLSC network. To ensure the number of recyclable parts/components received from different testing points must not exceed the maximum capacity of the recycling center and can be stated in Eq. (15.6). After testing procedures, end-of-life parts or components are declared as disposable parts/components and shipped to the disposal center in good time to reduce environmental issues. Hence to avoid the burden on landfills and underground disposal, the number of disposable parts/components must not exceed the maximum disposal capacity at the disposal center and this can be represented by Eq. (15.7). After recycling processes, the parts/components are transformed into new raw materials and ready the shipment to the raw material storage center. To fulfill the stock capacity of a natural material storage center, the number of raw materials must be greater than or equal to its minimum storage capacity for the smooth running of the production system, and this can be represented by Eq. (15.8).

$$\sum_{b=1}^{B} X1_{m,a,b} \leq MC1_{m,a} \quad \forall\ m,a, \tag{15.1}$$

$$\sum_{c=1}^{C} X2_{m,b,c} \leq MC2_{m,b} \quad \forall\ m,b, \tag{15.2}$$

$$\sum_{e=1}^{E} rc_{l,e} X5_{l,e,f} \leq MC6_{l,f} \quad \forall\ l,f, \tag{15.3}$$

$$\sum_{g=1}^{G} X6_{l,f,g} \leq MC7_{l,g} \quad \forall\ l,g, \tag{15.4}$$

$$\sum_{h=1}^{H} X8_{l,f,h} \leq MC8_{l,h} \quad \forall\ l,h, \tag{15.5}$$

$$\sum_{i=1}^{I} rr_m X11_{m,i,j} \leq MC10_{m,j} \quad \forall\ m,j, \tag{15.6}$$

$$\sum_{i=1}^{I} rd_m X12_{m,i,k} \leq MC11_{m,k} \quad \forall\ m,k, \tag{15.7}$$

$$\sum_{j=1}^{J} X13_{m,j,a} \geq MC1_{m,a} \quad \forall\ m,a. \tag{15.8}$$

4.2.2 Constraints related to production requirement

An efficient production system is an integral part of the CLSC network. Hybrid manufacturing/remanufacturing plants play a vital role in the optimal production of new products. Therefore, particular minimum requirements must be met to start the production processes. To ascertain the minimum condition of raw materials from two sources, the supplier point and testing center, the number of raw materials from two references must be greater than or equal to their production capacity at different manufacturing plants, and this can represented by Eq. (15.9).

$$\sum_{b=1}^{B} X2_{m,b,c} + \sum_{i=1}^{I} rm_m X10_{m,i,c} \geq MC3_{m,c} \quad \forall\ m,c. \tag{15.9}$$

4.2.3 Constraints related to maximum inventory at the distribution center

The distribution center is responsible for the shipment of products to different customers/markets. The demand for a new product is uncertain and only can be predicted based on previous information. Thus, to avoid the inventory cost and ascertain the maximum capacity restriction at the distribution center, the incoming products from manufacturing plants as well as refurbishing centers must be less than or equal to the maximum capacity of inventory at the distribution center, and this can be achieved by Eq. (15.10).

$$\sum_{c=1}^{C} X3_{l,c,d} + \sum_{g=1}^{G} rf_l X7_{l,g,d} \le MC4_{l,d} \quad \forall\ l,d. \tag{15.10}$$

4.2.4 Constraints related to demand of new and refurbished products

The most important and critical aspect of integrated CLSC is to fulfill the demand of customers or markets. The need for products is seldom stable. However, it can be predicted through prior information from the demand pattern. The only distribution center is responsible for the delivery of new products to the customers in this proposed CLSC network. To ensure this, the number of shipped products from the distribution center to different markets must be higher than its tentative demand over the stipulated ordered period, and this can be represented by Eq. (15.11).

$$\sum_{e=1}^{E} X4_{l,d,e} \ge MC5_{l,e} \quad \forall\ l,e. \tag{15.11}$$

4.2.5 Constraints related to the testing capacity at testing facility centers

The testing facility has been designed for taking the final decision over the parts or components regarding at which echelon they are to be transported. From a testing point, there are three facility options for the processing of tested parts/components. The manufacturing plant, recycling center, and disposal center have been structured for the final termination of the reverse supply chain. Hence the total sum of the number of parts/components that are transported from the testing plant to different facility locations must be less than or equal to the maximum capacity of the testing point, and this can be represented by Eq. (15.12).

$$\sum_{c=1}^{C} rt_m X10_{m,i,c} + \sum_{j=1}^{J} rt_m X11_{m,i,j} + \sum_{k=1}^{K} rt_m X12_{m,i,k} \le MC9_{m,i} \quad \forall\ m,i. \tag{15.12}$$

4.3 Proposed CLSC model formulation under uncertainty

The formulation of different conflicting objective functions and with some dynamic constraints under the proposed CLSC network has been presented in previous sections. Usually, the modeling texture of the CLSC network has been regarded as deterministic, which means that all the introduced parameters and constraints are known and predetermined well in advance. However, it is often observed that a deterministic modeling approach under CLSC design may not be an appropriate framework in decision-making processes. The typical multiechelon interconnected CLSC design model inherently yields some uncertainty. Impreciseness, vagueness, ambiguousness, randomness, incompleteness, etc., are the most common and frequent issues in the CLSC model. Different factors are responsible for the creation of uncertainty in the modeling of the CLSC network. Random fluctuation in the demand quantity, competitive market scenario, natural tragedy, variation in different kinds of costs, etc., laid down the base of uncertainty. In various adverse circumstances, the complete information about different parameters is not predetermined, but some inconsistent, improper, and incomplete information may be available to determine the deterministic value of the parameters. Uncertainty may exist in different forms, such as fuzzy, stochastic, and other types of risk. Vagueness or ambiguousness is responsible for fuzzy parameters which can be dealt with using the fuzzy techniques, whereas randomness gives birth to the stochastic parameters and can be quickly sorted out by using stochastic programming techniques with known means and variances of the parameters. Therefore, to highlight the most critical insight of the uncertainty, we have incorporated fuzzy parameters and few fuzzy equality constraints in the proposed CLSC designed network. Various cost parameters, such as processing costs, transportation costs, purchasing cost, selling prices, and time, have been taken as fuzzy parameters. The capacities or volumes of different echelons are also considered as fuzzy numbers. Inequality restrictions imposed over different constraints may avoid some aspects of getting better results from the CLSC planning model. Flexibility, among some preferred limitations, has been postulated to reveal reality more clearly. Hence we have developed a couple of fuzzy equality constraints ($\tilde{=}$) which means "essentially equal to" which signifies that the restrictions should more or less be satisfied and are more flexible than inequality constraints (Eqs. 15.22–15.24). The customer demand constraint has been assured with fuzzy equality constraints due to the change in utility or satisfaction behavior of the customers. The disposal facility is a single way for the removal of scrap parts/components out of the CLSC network. The testing facility plays a vital role in inspecting different parts/components. The optimum allocation of used parts/products has been decided at the testing facility point. Three various service destinations have been designed for the parts/components according to their potential utility after inspection. Therefore, more or less shipment quantity of parts/components is justifiable to ensure the optimum allocation to different facility centers. Hence, the proposed model with multiple objective functions and various constraints under uncertainty has been presented in model M_1.

$$M_1: \text{Minimize } Z_1 = \sum_{m=1}^{M}\sum_{a=1}^{A}\widetilde{PC1}_{m,a}X1_{m,a,b} + \sum_{m=1}^{M}\sum_{b=1}^{B}\widetilde{PC2}_{m,b}X2_{m,b,c}$$
$$+ \sum_{l=1}^{L}\sum_{c=1}^{C}\widetilde{PC3}_{l,c}X3_{l,c,d} + \sum_{l=1}^{L}\sum_{d=1}^{D}\widetilde{PC4}_{l,d}X4_{l,d,e}$$
$$+ \sum_{l=1}^{L}\sum_{f=1}^{F}\widetilde{PC5}_{l,f}X5_{l,e,f} + \sum_{l=1}^{L}\sum_{g=1}^{G}\widetilde{PC6}_{l,g}X6_{l,f,g}$$
$$+ \sum_{l=1}^{L}\sum_{h=1}^{H}\widetilde{PC7}_{l,h}X8_{l,f,h} + \sum_{i=1}^{M}\sum_{i=1}^{I}\widetilde{PC8}_{m,i}X10_{m,i,c}$$
$$+ \sum_{m=1}^{M}\sum_{j=1}^{J}\widetilde{PC9}_{m,j}X11_{m,i,j} + \sum_{m=1}^{M}\sum_{k=1}^{K}\widetilde{PC10}_{m,k}X12_{m,i,k}$$

$$\text{Minimize } Z_2 = \sum_{m=1}^{M}\sum_{a=1}^{A}\sum_{b=1}^{B}\widetilde{TC1}_{m,a,b}X1_{m,a,b} + \sum_{m=1}^{M}\sum_{b=1}^{B}\sum_{c=1}^{C}\widetilde{TC2}_{m,b,c}X2_{m,b,c}$$
$$+ \sum_{l=1}^{L}\sum_{c=1}^{C}\sum_{d=1}^{D}\widetilde{TC3}_{l,c,d}X3_{l,c,d} + \sum_{l=1}^{L}\sum_{d=1}^{D}\sum_{e=1}^{E}\widetilde{TC4}_{l,d,e}X4_{l,d,e}$$
$$+ \sum_{l=1}^{L}\sum_{e=1}^{E}\sum_{f=1}^{F}\widetilde{TC5}_{l,e,f}X5_{l,e,f} + \sum_{l=1}^{L}\sum_{f=1}^{F}\sum_{g=1}^{G}\widetilde{TC6}_{l,f,g}X6_{l,f,g}$$
$$+ \sum_{l=1}^{L}\sum_{g=1}^{G}\sum_{d=1}^{D}\widetilde{TC7}_{l,g,d}X7_{l,g,d}$$
$$+ \sum_{l=1}^{L}\sum_{f=1}^{F}\sum_{h=1}^{H}\widetilde{TC8}_{l,f,h}X8_{l,f,h} + \sum_{m=1}^{M}\sum_{i=1}^{I}\sum_{h=1}^{H}\widetilde{TC9}_{m,h,i}X9_{m,h,i}$$
$$+ \sum_{m=1}^{M}\sum_{i=1}^{I}\sum_{c=1}^{C}\widetilde{TC10}_{m,i,c}X10_{m,i,c} + \sum_{m=1}^{M}\sum_{i=1}^{I}\sum_{j=1}^{J}\widetilde{TC11}_{m,i,j}X11_{m,i,j}$$
$$+ \sum_{m=1}^{M}\sum_{i=1}^{I}\sum_{k=1}^{K}\widetilde{TC12}_{m,i,k}X12_{m,i,k} + \sum_{m=1}^{M}\sum_{j=1}^{J}\sum_{a=1}^{A}\widetilde{TC13}_{m,j,a}X13_{m,j,a}$$

$$\text{Minimize } Z_3 = \sum_{m=1}^{M}\widetilde{PU1}_{m}X2_{m,b,c} + \sum_{l=1}^{L}\widetilde{PU2}_{l}X5_{l,e,f}$$

$$\text{Minimize } Z_4 = \sum_{l=1}^{L}\sum_{d=1}^{D}\sum_{e=1}^{E}\widetilde{T}_{l,d,e}X4_{l,d,e}$$

$$\text{Maximize } Z_5 = \sum_{m=1}^{M}\widetilde{SP1}_{m}X2_{m,b,c} + \sum_{l=1}^{L}\widetilde{SP2}_{l}X5_{l,e,f}$$

subject to

$$\sum_{b=1}^{B} X1_{m,a,b} \leq \widetilde{MC1}_{m,a}, \tag{15.13}$$

$$\sum_{c=1}^{C} X2_{m,b,c} \leq \widetilde{MC2}_{m,b}, \tag{15.14}$$

$$\sum_{b=1}^{B} X2_{m,b,c} + \sum_{i=1}^{I} rm_m X10_{m,i,c} \geq \widetilde{MC3}_{m,c}, \tag{15.15}$$

$$\sum_{c=1}^{C} X3_{l,c,d} + \sum_{g=1}^{G} rf_l X7_{l,g,d} \leq \widetilde{MC4}_{l,d}, \tag{15.16}$$

$$\sum_{e=1}^{E} rc_{l,e} X5_{l,e,f} \leq \widetilde{MC6}_{l,f}, \tag{15.17}$$

$$\sum_{g=1}^{G} X6_{l,f,g} \leq \widetilde{MC7}_{l,g}, \tag{15.18}$$

$$\sum_{h=1}^{H} X8_{l,f,h} \leq \widetilde{MC8}_{l,h}, \tag{15.19}$$

$$\sum_{i=1}^{I} rr_m X11_{m,i,j} \leq \widetilde{MC10}_{m,j}, \tag{15.20}$$

$$\sum_{j=1}^{J} X13_{m,j,a} \geq \widetilde{MC1}_{m,a}, \tag{15.21}$$

$$\sum_{e=1}^{E} X4_{l,d,e} \widetilde{=} \widetilde{MC5}_{l,e}, \tag{15.22}$$

$$\sum_{i=1}^{I} rd_m X12_{m,i,k} \widetilde{=} \widetilde{MC11}_{m,k}, \tag{15.23}$$

$$\sum_{c=1}^{C} rt_m X10_{m,i,c} + \sum_{j=1}^{J} rt_m X11_{m,i,j} + \sum_{k=1}^{K} rt_m X12_{m,i,k} \widetilde{=} \widetilde{MC9}_{m,i}. \tag{15.24}$$

Where notations $(\tilde{\cdot})$ over different parameters represent the triangular/trapezoidal fuzzy number for all indices' sets, the fuzzy crisp inequality constraint has been described by $(\leq, \geq)$. The fuzzy equality constraints indicate that more or less attainment has been represented by $(\widetilde{=})$ for the given indices' sets.

5. Solution methodology

5.1 Treating fuzzy parameters and constraints

The addressed CLSC mathematical model inherently involves some vagueness and ambiguousness in the value of different parameters such as costs, capacity, revenues, etc. Defuzzification and the ranking function are the processes to obtain crisp versions of the fuzzified parameters based on the upper and lower magnitude of the vague parameters. On the other hand, the vagueness or uncertainty present in the equality or inequality constraints also needs to be defuzzified, and then converted into the strict crisp equality or inequality form of the constraints. To deal with vague or fuzzy parameters and constraints, different defuzzification techniques have been used in the literature. Among all the defuzzification approaches for uncertain parameters and constraints, Jiménez [24] and Jiménez et al. [25] discussed the combo defuzzification or ranking approach, which deals efficiently with the vague parameters as well as vague constraints. They also elaborately discussed the strong justification for ranking approaches with the help of different properties such as robustness, distinguishability, fuzzy or linguistic notations, and rationality. Later on, it has been extensively used by many researchers (see [25–27]). Without more justification on the ranking function, this chapter has adopted the defuzzification or ranking function for both vague parameters and constraints based on the Jiménez [24] approaches.

Definition 15.1. Jiménez et al. [25]
An FS defined over any universe of discourse is said to be a fuzzy number if the membership function is increasing semicontinuously in the upper interval and decreasing semicontinuously in the lower range, respectively. Therefore, the membership function of a fuzzy number along with $f_\phi(x)$ and $g_\phi(x)$, which are the left- and right-hand sides of the membership function, can be given as follows:

$$\mu_\phi(x) = \begin{cases} 0 & \text{if } x \leq \phi_1 \; or \; x \geq \phi_4 \\ f_\phi(x) & \text{if } \phi_1 \leq x \leq \phi_2 \\ g_\phi(x) & \text{if } \phi_3 \leq x \leq \phi_4 \, , \\ 1 & \text{if } \phi_2 \leq x \leq \phi_3 \end{cases} \tag{15.25}$$

where $\widetilde{\phi} = (\phi_1, \phi_2, \phi_3, \phi_4; 1)$ represents a fuzzy number. A fuzzy number $\widetilde{\phi} = (\phi_1, \phi_2, \phi_3, \phi_4)$ is said to be trapezoidal if $f_\phi(x)$ and $g_\phi(x)$ exist. Also, if $\phi_2 = \phi_3$, then one can obtain a triangular fuzzy number.

Definition 15.2. Jiménez et al. [25]
The representation of an expected interval for the fuzzy number $\widetilde{\phi}$ can be provided as follows:

$$EI(\widetilde{\phi}) = [E_1^\phi, E_2^\phi] = \left[\int_0^1 f_\phi^{-1}(x)dx, \int_0^1 f_\phi^{-1}(x)dx \right]. \tag{15.26}$$

The half point of the expected interval of the fuzzy number $\widetilde{\phi}$ is termed as its expected value and can be shown as follows:

$$EV(\widetilde{\phi}) = \left[\frac{E_1^{\phi} + E_2^{\phi}}{2}\right]. \tag{15.27}$$

Hence the expected interval and expected value for a trapezoidal fuzzy number $\widetilde{\phi} = (\phi_1, \phi_2, \phi_3, \phi_4)$ can be obtained as follows:

$$EI(\phi) \ = \left[\frac{\phi_1 + \phi_2}{2}, \frac{\phi_3 + \phi_4}{2}\right], \tag{15.28}$$

$$EV(\phi) \ = \left[\frac{\phi_1 + \phi_2 + \phi_3 + \phi_4}{4}\right]. \tag{15.29}$$

For any trapezoidal fuzzy number $\widetilde{\phi} = (\phi_1, \phi_2, \phi_3, \phi_4)$, if $\phi_2 = \phi_3$ (say ϕ) then it reduces into a triangular fuzzy number $\widetilde{\phi} = (\phi_1, \phi, \phi_4)$ and; its expected interval and expected value can be derived as follows:

$$EI(\phi) \ = \left[\frac{\phi_1 + \phi}{2}, \frac{\phi + \phi_4}{2}\right], \tag{15.30}$$

$$EV(\phi) \ = \left[\frac{\phi_1 + 2\phi + \phi_4}{4}\right]. \tag{15.31}$$

Definition 15.3. Jiménez et al. [25]
Suppose that there are two fuzzy $\widetilde{\phi}$ and $\widetilde{\psi}$ such that both have semicontinuous increasing and decreasing membership functions for upper and lower intervals, then the degree in which $\widetilde{\phi}$ is greater than $\widetilde{\psi}$ can be easily pointed out by constructing the following membership function:

$$\delta_V(\widetilde{\phi}, \widetilde{\psi}) = \begin{cases} 0 & \text{if } E_2^{\phi} - E_1^{\psi} < 0 \\ \dfrac{E_2^{\phi} - E_1^{\psi}}{E_2^{\phi} - E_1^{\psi} - (E_1^{\phi} - E_2^{\psi})} & \text{if } 0 \in [E_1^{\phi} - E_2^{\psi}, E_2^{\phi} - E_1^{\psi}], \\ 1 & \text{if } E_2^{\phi} - E_1^{\psi} > 0 \end{cases} \tag{15.32}$$

where $[E_1^{\phi}, E_2^{\phi}]$ and $[E_1^{\psi}, E_2^{\psi}]$ represent the expected intervals of $\widetilde{\phi}$ and $\widetilde{\psi}$. If $\delta_V(\widetilde{\phi}, \widetilde{\psi}) = 0.5$, then one can say that both $\widetilde{\phi}$ and $\widetilde{\psi}$ are indifferent.

Consequently, if $\delta_V(\widetilde{\phi}, \widetilde{\psi}) \geq \beta$, then one can say that $\widetilde{\phi}$ is greater than or equal to $\widetilde{\psi}$, at least in a degree β, and can be mathematically represented as $\widetilde{\phi}_i \geq_{\beta} \widetilde{\psi}_i$.

Definition 15.4. Jiménez et al. [25]
Introducing a decision vector X such that $x \in R^n$, then we can assign a feasibility degree β if for at least

$$\min_{i \in V} [\delta_V(\widetilde{\phi}_i X, \widetilde{\psi}_i)] = \beta, \tag{15.33}$$

where $\widetilde{\phi}_i = (\widetilde{\phi}_{i1}, \widetilde{\phi}_{i2}, \ldots, \widetilde{\phi}_{iv})$.
Intuitionally, in another sense, it can be written as

$$\widetilde{\phi}_i X \geq_\beta \widetilde{\psi}_i \quad \forall\ i = 1, 2, \ldots, v. \tag{15.34}$$

Incorporating the concept of (Jiménez et al. [25]) in the above inequality, equivalently we have

$$\frac{E_2^{\phi_i X} - E_1^{\psi_i}}{E_2^{\phi_i X} - E_1^{\psi_i} - (E_1^{\phi_i X} - E_2^{\psi_i})} \geq \beta \quad \forall\ i = 1, 2, \ldots, v. \tag{15.35}$$

On simplifying the above inequality equation, the equivalent inequality relations with feasibility degree β have been derived as follows:

$$((1-\beta)E_2^{\phi_i} + \beta E_1^{\phi_i})\, X \geq (\beta E_2^{\psi_i} + (1-\beta)E_1^{\psi_i}). \tag{15.36}$$

Furthermore, it can be concluded that the β-feasible fuzzy equalities, such as

$$\widetilde{\phi}_i X \cong_\beta \widetilde{\psi}_i \quad \forall\ i = v+1, v+2, \ldots, V, \tag{15.37}$$

can also be defuzzified in a similar fashion to the ranking function approach for fuzzy inequalities and can be given as follows:

$$\left(\left(1 - \frac{\beta}{2}\right) E_2^{\phi_i} + \frac{\beta}{2} E_1^{\phi_i}\right) X \geq \left(\frac{\beta}{2} E_2^{\psi_i} + \left(1 - \frac{\beta}{2}\right) E_1^{\psi_i}\right), \tag{15.38}$$

$$\left(\frac{\beta}{2} E_2^{\phi_i} + \left(1 - \frac{\beta}{2}\right) E_1^{\phi_i}\right) X \leq \left(\left(1 - \frac{\beta}{2}\right) E_2^{\psi_i} + \frac{\beta}{2} E_1^{\psi_i}\right). \tag{15.39}$$

Therefore, the fuzzy equality constraints result in the doubly crisp auxiliary inequality constraints for representing the restrictions with half of the β-feasibility degree by balancing an equilibrium state for the fuzzy equality constraints.

In order to obtain the crisp version of the proposed CLSC model, we have used the expected values [25] of the triangular fuzzy parameters present in the objective functions such as transportation cost, processing cost, purchasing cost, time, and revenues, whereas the trapezoidal fuzzy parameters such as different capacities involved in the constraints have been defuzzified by using the concept of the expected interval [25] of the parameters. Based on the above-discussed defuzzification approaches, the fuzzy parameters and constraints have been converted into their crisp versions, which has been also shown in Table 15.1.

$$M_2: \text{Minimize } Z_1 = \sum_{m=1}^{M}\sum_{a=1}^{A} EV(\widetilde{PC1})_{m,a} X1_{m,a,b} + \sum_{m=1}^{M}\sum_{b=1}^{B} EV(\widetilde{PC2})_{m,b} X2_{m,b,c}$$
$$+ \sum_{l=1}^{L}\sum_{c=1}^{C} EV(\widetilde{PC3})_{l,c} X3_{l,c,d} + \sum_{l=1}^{L}\sum_{d=1}^{D} EV(\widetilde{PC4})_{l,d} X4_{l,d,e}$$
$$+ \sum_{l=1}^{L}\sum_{f=1}^{F} EV(\widetilde{PC5})_{l,f} X5_{l,e,f} + \sum_{l=1}^{L}\sum_{g=1}^{G} EV(\widetilde{PC6})_{l,g} X6_{l,f,g}$$
$$+ \sum_{l=1}^{L}\sum_{h=1}^{H} EV(\widetilde{PC7})_{l,h} X8_{l,f,h} + \sum_{i=1}^{M}\sum_{i=1}^{I} EV(\widetilde{PC8})_{m,i} X10_{m,i,c}$$
$$+ \sum_{m=1}^{M}\sum_{j=1}^{J} EV(\widetilde{PC9})_{m,j} X11_{m,i,j} + \sum_{m=1}^{M}\sum_{k=1}^{K} EV(\widetilde{PC10})_{m,k} X12_{m,i,k}$$

$$\text{Minimize } Z_2 = \sum_{m=1}^{M}\sum_{a=1}^{A}\sum_{b=1}^{B} EV(\widetilde{TC1})_{m,a,b} X1_{m,a,b} + \sum_{m=1}^{M}\sum_{b=1}^{B}\sum_{c=1}^{C} EV(\widetilde{TC2})_{m,b,c} X2_{m,b,c}$$
$$+ \sum_{l=1}^{L}\sum_{c=1}^{C}\sum_{d=1}^{D} EV(\widetilde{TC3})_{l,c,d} X3_{l,c,d} + \sum_{l=1}^{L}\sum_{d=1}^{D}\sum_{e=1}^{E} EV(\widetilde{TC4})_{l,d,e} X4_{l,d,e}$$
$$+ \sum_{l=1}^{L}\sum_{e=1}^{E}\sum_{f=1}^{F} EV(\widetilde{TC5})_{l,e,f} X5_{l,e,f} + \sum_{l=1}^{L}\sum_{f=1}^{F}\sum_{g=1}^{G} EV(\widetilde{TC6})_{l,f,g} X6_{l,f,g}$$
$$+ \sum_{l=1}^{L}\sum_{g=1}^{G}\sum_{d=1}^{D} EV(\widetilde{TC7})_{l,g,d} X7_{l,g,d} + \sum_{l=1}^{L}\sum_{f=1}^{F}\sum_{h=1}^{H} EV(\widetilde{TC8})_{l,f,h} X8_{l,f,h}$$
$$+ \sum_{m=1}^{M}\sum_{i=1}^{I}\sum_{h=1}^{H} EV(\widetilde{TC9})_{m,h,i} X9_{m,h,i} + \sum_{m=1}^{M}\sum_{i=1}^{I}\sum_{c=1}^{C} EV(\widetilde{TC10})_{m,i,c} X10_{m,i,c}$$
$$+ \sum_{m=1}^{M}\sum_{i=1}^{I}\sum_{j=1}^{J} EV(\widetilde{TC11})_{m,i,j} X11_{m,i,j} + \sum_{m=1}^{M}\sum_{i=1}^{I}\sum_{k=1}^{K} EV(\widetilde{TC12})_{m,i,k} X12_{m,i,k}$$
$$+ \sum_{m=1}^{M}\sum_{j=1}^{J}\sum_{a=1}^{A} EV(\widetilde{TC13})_{m,j,a} X13_{m,j,a}$$

$$\text{Minimize } Z_3 = \sum_{m=1}^{M} EV(\widetilde{PU1})_m X2_{m,b,c} + \sum_{l=1}^{L} EV(\widetilde{PU2})_l X5_{l,e,f}$$

$$\text{Minimize } Z_4 = \sum_{l=1}^{L}\sum_{d=1}^{D}\sum_{e=1}^{E} EV(\widetilde{T})_{l,d,e} X4_{l,d,e}$$

$$\text{Maximize } Z_5 = \sum_{m=1}^{M} EV(\widetilde{SP1})_m X2_{m,b,c} + \sum_{l=1}^{L} EV(\widetilde{SP2})_l X5_{l,e,f}$$

Table 15.1 Information regarding triangular/trapezoidal fuzzy parameters.

Fuzzy parameter	Triangular/trapezoidal fuzzy number	$EI(.) = [E_1^{(.)}, E_2^{(.)}]$	EV(.)
$\widetilde{PC*}_{*,*}$	$(PC*_{*,*}^{(1)}, PC*_{*,*}^{(2)}, PC*_{*,*}^{(3)})$	$\left[\frac{PC*_{*,*}^{(1)}+PC*_{*,*}^{(2)}}{2}, \frac{PC*_{*,*}^{(2)}+PC*_{*,*}^{(3)}}{2}\right]$	$\frac{PC*_{*,*}^{(1)}+2PC*_{*,*}^{(2)}+PC*_{*,*}^{(3)}}{4}$
$\widetilde{TC*}_{*,*,*}$	$(TC*_{*,*,*}^{(1)}, TC*_{*,*,*}^{(2)}, TC*_{*,*,*}^{(3)})$	$\left[\frac{TC*_{*,*,*}^{(1)}+TC*_{*,*,*}^{(2)}}{2}, \frac{TC*_{*,*,*}^{(2)}+TC*_{*,*,*}^{(3)}}{2}\right]$	$\frac{TC*_{*,*,*}^{(1)}+2TC*_{*,*,*}^{(2)}+TC*_{*,*,*}^{(3)}}{4}$
$\widetilde{T}_{*,*,*}$	$(T_{*,*,*}^{(1)}, T_{*,*,*}^{(2)}, T_{*,*,*}^{(3)})$	$\left[\frac{T_{*,*,*}^{(1)}+T_{*,*,*}^{(2)}}{2}, \frac{T_{*,*,*}^{(2)}+T_{*,*,*}^{(3)}}{2}\right]$	$\frac{T_{*,*,*}^{(1)}+2T_{*,*,*}^{(2)}+T_{*,*,*}^{(3)}}{4}$
$\widetilde{PU*}_{*,*}$	$(PU*_{*,*}^{(1)}, PU*_{*,*}^{(2)}, PU*_{*,*}^{(3)})$	$\left[\frac{PU*_{*,*}^{(1)}+PU*_{*,*}^{(2)}}{2}, \frac{PU*_{*,*}^{(2)}+PU*_{*,*}^{(3)}}{2}\right]$	$\frac{PU*_{*,*}^{(1)}+2PU*_{*,*}^{(2)}+PU*_{*,*}^{(3)}}{4}$
$\widetilde{SP*}_{*,*}$	$(SP*_{*,*}^{(1)}, SP*_{*,*}^{(2)}, SP*_{*,*}^{(3)})$	$\left[\frac{SP*_{*,*}^{(1)}+SP*_{*,*}^{(2)}}{2}, \frac{SP*_{*,*}^{(2)}+SP*_{*,*}^{(3)}}{2}\right]$	$\frac{SP*_{*,*}^{(1)}+2SP*_{*,*}^{(2)}+SP*_{*,*}^{(3)}}{4}$
$\widetilde{MC*}_{*,*}$	$(MC*_{*,*}^{(1)}, MC*_{*,*}^{(2)}, MC*_{*,*}^{(3)}, MC*_{*,*}^{(4)})$	$\left[\frac{MC*_{*,*}^{(1)}+MC*_{*,*}^{(2)}}{2}, \frac{MC*_{*,*}^{(3)}+MC*_{*,*}^{(4)}}{2}\right]$	$\frac{MC*_{*,*}^{(1)}+MC*_{*,*}^{(2)}+MC*_{*,*}^{(3)}+MC*_{*,*}^{(4)}}{4}$

Notes: $*$ represents the different numbers 1, 2, 3, … used in parameters.
$(*, *)$ and $(*, *, *)$ in suffixes represent the different indices set.

subject to

$$\sum_{b=1}^{B} X1_{m,a,b} \leq (1-\beta)E_2^{\widetilde{MC1_{m,a}}} + \beta E_1^{\widetilde{MC1_{m,a}}}, \tag{15.40}$$

$$\sum_{c=1}^{C} X2_{m,b,c} \leq (1-\beta)E_2^{\widetilde{MC2_{m,b}}} + \beta E_1^{\widetilde{MC2_{m,b}}}, \tag{15.41}$$

$$\sum_{b=1}^{B} X2_{m,b,c} + \sum_{i=1}^{I} rm_m X10_{m,i,c} \geq \beta E_2^{\widetilde{MC3_{m,c}}} + (1-\beta)E_1^{\widetilde{MC3_{m,c}}}, \tag{15.42}$$

$$\sum_{c=1}^{C} X3_{l,c,d} + \sum_{g=1}^{G} rf_l X7_{l,g,d} \leq (1-\beta)E_2^{\widetilde{MC4_{l,d}}} + \beta E_1^{\widetilde{MC4_{l,d}}}, \tag{15.43}$$

$$\sum_{e=1}^{E} rc_{l,e} X5_{l,e,f} \leq (1-\beta)E_2^{\widetilde{MC6_{l,f}}} + \beta E_1^{\widetilde{MC6_{l,f}}}, \tag{15.44}$$

$$\sum_{g=1}^{G} X6_{l,f,g} \leq (1-\beta)E_2^{\widetilde{MC7_{l,g}}} + \beta E_1^{\widetilde{MC7_{l,g}}}, \tag{15.45}$$

$$\sum_{h=1}^{H} X8_{l,f,h} \leq (1-\beta)E_2^{\widetilde{MC8_{l,h}}} + \beta E_1^{\widetilde{MC8_{l,h}}}, \tag{15.46}$$

$$\sum_{i=1}^{I} rr_m X11_{m,i,j} \leq (1-\beta)E_2^{\widetilde{MC10_{m,j}}} + \beta E_1^{\widetilde{MC10_{m,j}}}, \tag{15.47}$$

$$\sum_{j=1}^{J} X13_{m,j,a} \geq \beta E_2^{\widetilde{MC1_{m,a}}} + (1-\beta)E_1^{\widetilde{MC1_{m,a}}}, \tag{15.48}$$

$$\sum_{i=1}^{I} rd_m X12_{m,i,k} \geq \frac{\beta}{2} E_2^{\widetilde{MC11_{m,k}}} + \left(1-\frac{\beta}{2}\right) E_1^{\widetilde{MC11_{m,k}}}, \tag{15.49}$$

$$\sum_{i=1}^{I} rd_m X12_{m,i,k} \leq \left(1-\frac{\beta}{2}\right) E_2^{\widetilde{MC11_{m,k}}} + \frac{\beta}{2} E_1^{\widetilde{MC11_{m,k}}}, \tag{15.50}$$

$$\sum_{e=1}^{E} X4_{l,d,e} \geq \frac{\beta}{2} E_2^{\widetilde{MC5_{l,e}}} + \left(1-\frac{\beta}{2}\right) E_1^{\widetilde{MC5_{l,e}}}, \tag{15.51}$$

$$\sum_{e=1}^{E} X4_{l,d,e} \leq \left(1-\frac{\beta}{2}\right) E_2^{\widetilde{MC5_{l,e}}} + \frac{\beta}{2} E_1^{\widetilde{MC5_{l,e}}}, \tag{15.52}$$

$$\sum_{c=1}^{C} rt_m X10_{m,i,c} + \sum_{j=1}^{J} rt_m X11_{m,i,j} + \sum_{k=1}^{K} rt_m X12_{m,i,k} \geq \frac{\beta}{2} E_2^{\widetilde{MC9_{m,i}}} + \left(1-\frac{\beta}{2}\right) E_1^{\widetilde{MC9_{m,i}}}, \tag{15.53}$$

$$\sum_{c=1}^{C} rt_m X10_{m,i,c} + \sum_{j=1}^{J} rt_m X11_{m,i,j} + \sum_{k=1}^{K} rt_m X12_{m,i,k} \leq \left(1-\frac{\beta}{2}\right) E_2^{\widetilde{MC9_{m,i}}} + \frac{\beta}{2} E_1^{\widetilde{MC9_{m,i}}}. \tag{15.54}$$

5.2 Neutrosophic fuzzy programming approach

The multiobjective optimization problems are prevalent in real-life scenarios. Due to the existence of complex and conflicting multiple goals or objectives, the task of obtaining optimal solutions is a vital issue. The different conventional optimization techniques for obtaining the compromise solution of multiobjective programming problems are based on the marginal evaluation (degree of validity) for each objective (say Z_o) in the feasible solution set. By marginal evaluation, we mean a transformation function (say $\mu(Z_o) \rightarrow [0, 1]|\alpha \in [0, 1]$) that assigned the values between 0 and 1 to each objective function which shows that the decision makers' preferences have been fulfilled up to α level of satisfaction. Therefore, the quantification of marginal evaluation is based on the different decision set theory. Initially, Zadeh [28] proposed the FS theory, which explicitly contains the membership function (degree of belongingness) of the element into the feasible solution set. Later on, Zimmermann [29] introduced the fuzzy programming approach to solve multiobjective optimization problems. In a fuzzy programming approach, the quantification of marginal evaluation is represented by a membership function, which only maximizes the degree of belongingness under the fuzzy decision set. The extended version of the fuzzy optimization technique has been applied in a wide range of real-life applications. Furthermore, the generalizations or extensions of the FS were initially proposed by Atanassov [30] and named the intuitionistic fuzzy set (IFS). The analytical coverage spectrum of IFS is versatile and flexible compared to FS as it deals with the membership (degree of belongingness) as well as nonmembership (degree of nonbelongingness) functions of the element into the feasible set. Based on IFS, first Angelov [31] proposed the intuitionistic fuzzy programming approach to obtain the compromise solution of the multiobjective optimization problems. The quantification of marginal evaluation of each objective function under the IF decision set depends on the membership and nonmembership functions, which are to be achieved by maximizing the membership function and minimizing the nonmembership functions simultaneously. The intuitionistic fuzzy programming approach has been extensively studied with various real-life problems.

In the past few decades, it has been observed that the situation may arise in real-life decision-making problems where the indeterminacy or neutral thoughts about an element into the feasible set exists. Indeterminacy/neutral is the region of the negligence of a proposition's value and lies between a truth and falsity degree. Therefore, the further generalization of FS and IFS has been presented by introducing a new member into the feasible decision set. First, Smarandache [32] investigated the neutrosophic set (NS) which comprises three membership functions: truth (degree of belongingness), indeterminacy (degree of belongingness up to some extent), and falsity (degree of nonbelongingness) functions of the element into the NS. The word *neutrosophic* is the hybrid mixture of two different words, *neutre*, taken from the French, meaning neutral, and *sophia*, derived from the Greek, meaning skill/wisdom, which literally gives the meaning *knowledge of neutral thoughts* (see [32]). The independent indeterminacy degree is sufficient to differentiate itself from FS and IFS. Recent literature on the NS reveals that many researchers have taken an interest in the neutrosophic domain (see [33–36]) and this is likely to be a prominent emerging research area in the future. This study has also taken advantage of the versatile and effective texture of a neutrosophic fuzzy decision set to develop the NFPA. The NFPA has been designed to solve the proposed CLSC model with multiple objectives under the set of constraints. The NFPA quantifies the marginal evaluation of each objective function under three different membership functions: truth, indeterminacy, and falsity membership functions. Thus the NFPA optimization techniques for the multiobjective optimization problem has a significant role in the implementation and execution of the neutral thoughts in decision-making processes.

Definition 15.5. Neutrosophic set [32]

Let there be a universe discourse Y such that $y \in Y$, then an NS W in Y is defined by three membership functions, truth $p_W(y)$, indeterminacy $q_W(y)$, and falsity $r_W(y)$, and denoted by the following form:

$$W = \{\langle y, p_W(y), q_W(y), r_W(y)\rangle | y \in Y\},$$

where $p_W(y)$, $q_W(y)$, and $r_W(y)$ are real standard or nonstandard subsets belonging to $]0^-, 1^+[$, also given as $p_W(y) : y \rightarrow]0^-, q^+[$, $r_W(y) : Y \rightarrow]0^-, 1^+[$, and $r_W(y) : Y \rightarrow]0^-, 1^+[$. There is no restriction on the sum of $p_W(y)$, $q_W(y)$, and $r_W(y)$, so we have

$$0^- \leq \sup p_W(y) + q_W(y) + \sup r_W(y) \leq 3^+.$$

Definition 15.6. Smarandache [32]

Let there be two single-valued NSs A and B, then $C = (A \cup B)$ with truth $p_C(y)$, indeterminacy $q_C(y)$, and falsity $r_C(y)$ membership functions are given by

$p_C(y) = \max\ (p_A(y), p_B(y))$,
$q_C(y) = \min\ (q_A(y), q_B(y))$, and
$r_C(y) = \min\ (r_A(y), r_B(y))$ for each $y \in Y$.

Definition 15.7. Smarandache [32]

Let there be two single-valued NSs A and B, then $C = (A \cap B)$ with truth $p_C(y)$, indeterminacy $q_C(y)$, and falsity $r_C(y)$ membership functions are given by

$p_C(y) = \min\ (p_A(y), p_B(y))$,
$q_C(y) = \max\ (q_A(y), q_B(y))$, and
$r_C(y) = \max\ (r_A(y), r_B(y))$ for each $y \in Y$.

First, Bellman and Zadeh [37] introduced the idea of the fuzzy decision set (D) which contains a set of fuzzy goals (G) and fuzzy constraints (C). Later on, it was widely used in many real-life decision-making problems. Thus, a fuzzy decision set (D) can be stated as follows:

$$D = G \cap C.$$

Equivalently, the neutrosophic decision set $D_{Neutrosophic}$, with a set of neutrosophic goals and constraints, can be given as follows:

$$D_{Neutrosophic} = \left(\cap_{o=1}^{O} G_o\right)\left(\cap_{n=1}^{N} C_n\right) = \left(y, p_D(y), q_D(y), r_D(y)\right),$$

where

$$p_D(y) = \min \left\{ \begin{array}{l} p_{G_1}(y), p_{G_2}(y), \ldots, p_{G_O}(y) \\ p_{C_1}(y), p_{C_2}(y), \ldots, p_{C_N}(y) \\ \end{array} \right\} \quad \forall\, y \in Y,$$

$$q_D(y) = \max \left\{ \begin{array}{l} q_{G_1}(y), q_{G_2}(y), \ldots, q_{G_O}(y) \\ q_{C_1}(y), q_{C_2}(y), \ldots, q_{C_N}(y) \\ \end{array} \right\} \quad \forall\, y \in Y,$$

$$r_D(y) = \max \left\{ \begin{array}{l} r_{G_1}(y), r_{G_2}(y), \ldots, r_{G_O}(y) \\ r_{C_1}(y), r_{C_2}(y), \ldots, r_{C_N}(y) \\ \end{array} \right\} \quad \forall\, y \in Y,$$

where the truth, indeterminacy, and falsity membership functions have been represented by $p_W(y)$, $q_W(y)$, and $r_W(y)$ under neutrosophic decision set $D_{Neutrosophic}$, respectively.

The marginal evaluation for each objective function by using the transformation functions of truth $p_W(y)$, indeterminacy $q_W(y)$, and falsity $r_W(y)$ membership functions can be derived with the help of the upper and lower bounds of each objective function. The solution of each single objective under the given set of constraints provides the upper and lower bounds for each objective function and can be denoted as U_o and L_o with a set of decision variables X^1, X^2, …, X^o, respectively.

Mathematically, it can be shown as follows:

$$U_o = \max\left[Z_o(X^o)\right] \text{ and } L_o = \min\left[Z_o(X^o)\right] \quad \forall\, o = 1, 2, 3, \ldots, O. \tag{15.55}$$

The upper and lower bounds for o objective function under the neutrosophic environment can be obtained as follows:

$$\begin{array}{ll} U_o^p = U_o,\ L_o^p = L_o & \text{for truth membership,} \\ U_o^q = L_o^p + s_o,\ L_o^q = L_o^p & \text{for indeterminacy membership,} \\ U_o^r = U_o^p,\ L_o^r = L_o^p + t_o & \text{for falsity membership,} \end{array}$$

where s_o and $t_o \in (0,\ 1)$ are predetermined real numbers assigned by the decision maker(s). With the help of upper and lower bounds for each of the three membership functions, we have presented the linear membership function under a neutrosophic decision-making framework.

$$p_o(Z_o(x)) = \begin{cases} 1 & \text{if } Z_o(x) < L_o^p \\ \dfrac{U_o^p - Z_o(x)}{U_o^p - L_o^p} & \text{if } L_o^p \leq Z_o(x) \leq U_o^p, \\ 0 & \text{if } Z_o(x) > U_o^p \end{cases} \tag{15.56}$$

$$q_o(Z_o(x)) = \begin{cases} 1 & \text{if } Z_o(x) < L_o^q \\ \dfrac{U_o^q - Z_o(x)}{U_o^q - L_o^q} & \text{if } L_o^q \leq Z_o(x) \leq U_o^q, \\ 0 & \text{if } Z_o(x) > U_o^q \end{cases} \tag{15.57}$$

$$r_o(Z_o(x)) = \begin{cases} 1 & \text{if } Z_o(x) > U_o^r \\ \dfrac{Z_o(x) - L_o^r}{U_o^r - L_o^r} & \text{if } L_o^r \leq Z_o(x) \leq U_o^r. \\ 0 & \text{if } Z_o(x) < L_o^r \end{cases} \tag{15.58}$$

In the above-discussed membership functions, $L_o^{(.)} \neq U_o^{(.)}$ for all o objective functions. The value of these membership will be equal to 1, if for any membership $L_o^{(.)} = U_o^{(.)}$. The diagrammatic representation of the objective function with different components of membership functions under a neutrosophic decision set is shown in Fig. 15.2.

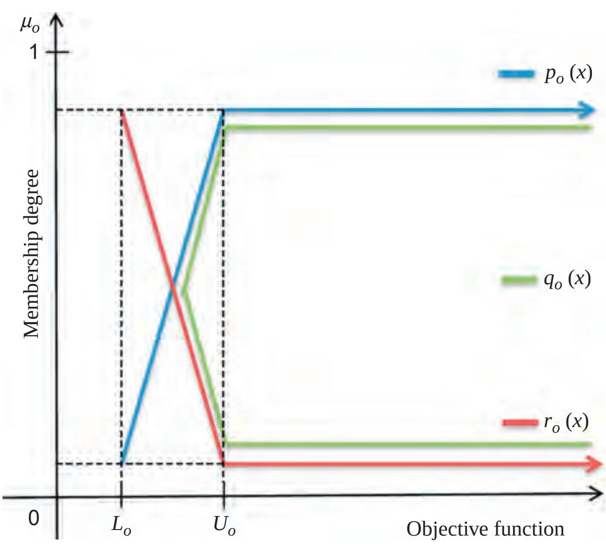

Fig. 15.2 Diagrammatic representation of truth, indeterminacy, and falsity membership degrees for the objective function.

Logically, the aim of developing the different achievement function is to achieve the maximum satisfaction degree or level according to the preference of the decision maker(s). Therefore, here also we have defined the individual achievement variables for each membership function, such as by maximization of truth membership, maximization of indeterminacy degree, and minimization of a falsity degree of each objective function efficiently. With the aid of linear truth, indeterminacy, and falsity membership functions under a neutrosophic environment, the neutrosophic fuzzy mathematical programming model can be presented as follows:

$$
\begin{aligned}
M_3:\ & \text{Max } \min_{o=1,2,3,\ldots,O}\ p_o(Z_o(x)) \\
& \text{Max } \min_{o=1,2,3,\ldots,O}\ q_o(Z_o(x)) \\
& \text{Min } \max_{o=1,2,3,\ldots,O}\ r_o(Z_o(x)) \\
& \text{subject to} \\
& p_o(Z_o(x)) \geq q_o(Z_o(x)),\ p_o(Z_o(x)) \geq r_o(Z_o(x)) \\
& 0 \leq p_o(Z_o(x)) + q_o(Z_o(x)) + r_o(Z_o(x)) \leq 3. \\
& \text{Eqs. } (15.40) - (15.54)
\end{aligned}
$$

With the help of auxiliary parameters, model M_3 can be transformed into the following form M_4.

$$
\begin{aligned}
M_4:\ & \text{Max } \lambda_o \\
& \text{Max } \theta_o \\
& \text{Min } \eta_o \\
& \text{subject to} \\
& p_o(Z_o(x)) \geq \lambda_o \\
& q_o(Z_o(x)) \geq \theta_o \\
& r_o(Z_o(x)) \leq \eta_o \\
& \lambda_o \geq \theta_o,\ \lambda_o \geq \eta_o,\ \ 0 \leq \lambda_o + \theta_o + \eta_o \leq 3 \\
& \lambda_o, \theta_o, \eta_o \in (0,1). \\
& \text{Eqs. } (15.40) - (15.54)
\end{aligned}
$$

Without loss of generality, the model M_4 can be rewritten as in M_5.

$$
\begin{aligned}
M_5:\ & \text{Max } \sum_{o=1}^{O} (\lambda_o + \theta_o - \eta_o) \\
& \text{subject to} \\
& Z_o(x) + (U_o^p - L_o^p)\lambda_o \leq U_o^p \\
& Z_o(x) + (U_o^q - L_o^q)\theta_o \leq U_o^q \\
& Z_o(x) - (U_o^r - L_o^r)\eta_o \leq L_o^r \\
& \lambda_o \geq \theta_o,\ \lambda_o \geq \eta_o,\ \ 0 \leq \lambda_o + \theta_o + \eta_o \leq 3 \\
& \lambda_o, \theta_o, \eta_o \in (0,1), \\
& \text{Eqs. } (15.40) - (15.54)
\end{aligned}
$$

where λ_o, θ_o, and η_o are auxiliary achievement variables for truth, indeterminacy, and falsity membership functions, respectively. Therefore, the proposed NFPA is a convenient conventional optimization technique that is only preferred over others due to the existence of its independent indeterminacy degree.

5.3 Modified neutrosophic fuzzy programming with intuitionistic fuzzy preference relations

The effective modeling and optimization framework of multiobjective optimization problems explicitly results in the best possible compromise solution under adverse circumstances, since, while dealing with multiple objectives or goals, most often, DM(s) intends to provide priorities among the different objectives over each other. Generally, the preferences among the objective function have been defined by assigning the maximum crisp weight parameter (say $w_o = 0.1, 0.2, \ldots, 1 | \sum_o^O w_o = 1$) to the preferred objective function. In the past few decades, Aköz and Petrovic [38] proposed a new methodology to assign the preference among different objectives or goals based on the linguistic importance relation and investigated three different fuzzy linguistic importance relationship such as *slightly more important than*, *moderately more important than*, and *significantly more important than* for different conflicting objectives. These linguistic terms have taken the advantages of membership functions associated with corresponding objectives or goals between which the important relation has been defined. Later on, this linguistic preference scheme was adopted by several researchers (see [27, 39–46]) in various real-life applications and decision-making processes. The appropriate selection of membership functions is always a crucial task for decision makers. Since the quantification of preference, the membership function has been done for the three linguistic fuzzy preference relations, but it would be more convenient and realistic to consider the nonmembership function as well as the similar linguistic fuzzy preference relations.

Therefore, to incorporate the membership and nonmembership function for linguistic preference relations among the objective, we have designed the structure of our proposed linguistic preference relations among different objectives or goals. Again, we have developed the linear membership and nonmembership function for each linguistic preference relation among the different objectives in the intuitionistic fuzzy environment. The transformation function has been defined with the help of truth membership functions of each objective. The information regarding linguistic preference relations under the intuitionistic fuzzy environment is shown in Table 15.2. The membership and nonmembership function for intuitionistic fuzzy linguistic preference relations is shown in Fig. 15.3.

The linear membership function for each linguistic preference relation can be defined as follows and achieved by maximizing it [38].

$$\mu_{\widetilde{R}_{1(o,u)}} = \begin{cases} (p_o - p_u + 1) & \text{if } -1 \le p_o - p_u \le 0 \\ 1 & \text{if } 0 \le p_o - p_u \le 1 \end{cases}, \tag{15.59}$$

Table 15.2 Linguistic relative preferences of objective o over u.

Linguistic term	Intuitionistic fuzzy relation	Membership and nonmembership functions	Transform function
Slightly more important than	$\widetilde{R}_1$	$\mu_{\widetilde{R}_1}$ and $\nu_{\widetilde{R}_1}$	
Moderately more important than	$\widetilde{R}_2$	$\mu_{\widetilde{R}_2}$ and $\nu_{\widetilde{R}_2}$	$p_o(X)-p_u(X)\ \forall\ o, u \in (1\ldots O)$
Significantly more important than	$\widetilde{R}_3$	$\mu_{\widetilde{R}_3}$ and $\nu_{\widetilde{R}_3}$	

$$\mu_{\widetilde{R}_2(o,u)} = \left\{ \left(\frac{p_o - p_u + 1}{2}\right) \quad \text{if} -1 \leq p_o - p_u \leq 1, \right. \tag{15.60}$$

$$\mu_{\widetilde{R}_3(o,u)} = \begin{cases} 0 & \text{if} -1 \leq p_o - p_u \leq 0 \\ (p_o - p_u) & \text{if } 0 \leq p_o - p_u \leq 1 \end{cases}. \tag{15.61}$$

The linear nonmembership function for the linguistic preference relations can be given as follows and achieved by minimizing it.

$$\nu_{\widetilde{R}_1(o,u)} = \begin{cases} -(p_o - p_u) & \text{if} -1 \leq p_o - p_u \leq 0 \\ 0 & \text{if } 0 \leq p_o - p_u \leq 1 \end{cases}. \tag{15.62}$$

$$\nu_{\widetilde{R}_2(o,u)} = \left\{ \frac{1 - (p_o - p_u)}{2} \quad \text{if} -1 \leq p_o - p_u \leq 1, \right. \tag{15.63}$$

$$\nu_{\widetilde{R}_3(o,u)} = \begin{cases} 1 & \text{if} -1 \leq p_o - p_u \leq 0 \\ 1 - (p_o - p_u) & \text{if } 0 \leq p_o - p_u \leq 1 \end{cases}, \tag{15.64}$$

where $\widetilde{R}_1$, $\widetilde{R}_2$, and $\widetilde{R}_3$ are the importance relations defined by the linguistic term *slightly more important than*, *moderately more important than*, and *significantly more important than*, respectively.

The new achievement function for satisfaction degrees of the imprecise linguistic importance relations can be defined with the aid of the membership and nonmembership function for intuitionistic fuzzy linguistic preference relations. We have defined a score function $S_{\widetilde{R}(o,u)} = (\mu_{\widetilde{R}(o,u)} - \nu_{\widetilde{R}(o,u)})$, which has been used to express the satisfactory degree of decision makers' linguistic importance relations. Let us define a binary variable $BI_{(o,\ u)}$; $o, u = 1, 2, \ldots, O$, where $o \neq u$ such that

$$BI_{o,u} = \begin{cases} 1 & \text{if a linguistic preference relation is defined between the objective } Z_o \text{ and } Z_u \\ 0 & \text{otherwise} \end{cases}.$$

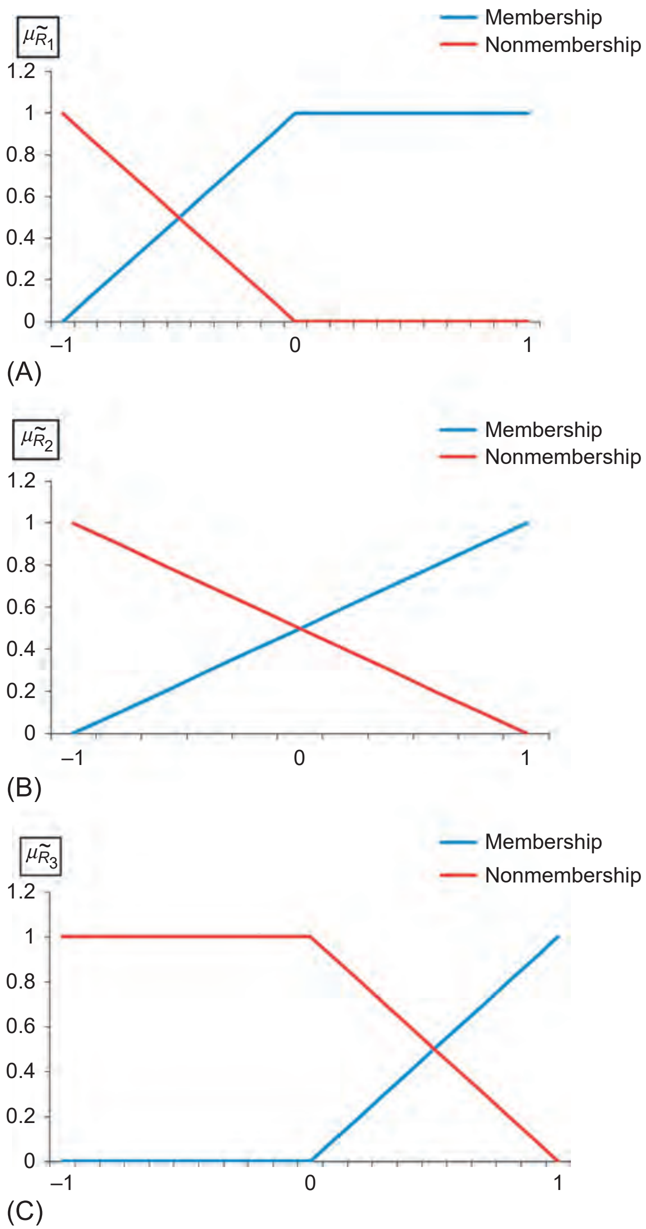

Fig. 15.3 Linear membership and nonmembership functions for intuitionistic fuzzy linguistic preference relations. (A) $R_1(o,u)=\widetilde{R}_1$. (B) $R_2(o,u)=\widetilde{R}_2$. (C) $R_3(o,u)=\widetilde{R}_3$.

The modified NFPA with intuitionistic fuzzy linguistic preference relations has been designed with the hybrid integration of the achievement function under the NFPA model and score functions for the satisfaction degree of decision makers. The achievement function for the modified NFPA can be defined as the convex combination of the sum of individual truth membership, indeterminacy function, and falsity membership function of each objective or goals and the sum of score functions of the imprecise linguistic importance relations. Thus the proposed modified NFPA can be given as follows:

$$
\begin{aligned}
M_6:\ & \text{Max } \alpha \sum_{o=1}^{O}(\lambda_o+\theta_o-\eta_o)+(1-\alpha)\sum_{o=1}^{O}\sum_{u=1}^{O} BI_{o,u} S_{\tilde{R}(o,u)} \\
& \text{subject to} \\
& Z_o(x)+(U_o^p-L_o^p)\lambda_o \le U_o^p \\
& Z_o(x)+(U_o^q-L_o^q)\theta_o \le U_o^q \\
& Z_o(x)-(U_o^r-L_o^r)\eta_o \le L_o^r \\
& (p_o-p_u+1) \ge \mu_{\tilde{R}_1(o,u)} \\
& \left(\frac{p_o-p_u+1}{2}\right) \ge \mu_{\tilde{R}_2(o,u)} \\
& (p_o-p_u) \ge \mu_{\tilde{R}_3(o,u)} \\
& -(p_o-p_u) \le \nu_{\tilde{R}_1(o,u)} \\
& \frac{1-(p_o-p_u)}{2} \le \nu_{\tilde{R}_2(o,u)} \\
& 1-(p_o-p_u) \le \nu_{\tilde{R}_3(o,u)} \\
& S_{\tilde{R}(o,u)} = \left(\mu_{\tilde{R}(o,u)}-\nu_{\tilde{R}(o,u)}\right) \\
& \mu_{\tilde{R}(o,u)} \ge \nu_{\tilde{R}(o,u)} \\
& 0 \le \mu_{\tilde{R}(o,u)}+\nu_{\tilde{R}(o,u)} \le 1 \\
& 0 \le \mu_{\tilde{R}(o,u)},\ \nu_{\tilde{R}(o,u)} \le 1 \quad \forall\ BI_{o,u}=1 \\
& \lambda_o \ge \theta_o,\ \lambda_o \ge \eta_o,\quad 0 \le \lambda_o+\theta_o+\eta_o \le 3 \\
& \lambda_o,\ \theta_o,\ \eta_o \in (0,1), \\
& \text{Eqs. } (15.40) - (15.54)
\end{aligned}
$$

where α is a nonzero parameter taking values between 0 and 1 and can be assigned by tuning it for either the membership function of objectives or linguistic preference relations.

The proposed modified NFPA modeling approach considers the degree of belongingness and nonbelongingness simultaneously, which is a better representation of uncertain importance relations among objectives because it enhances the membership degree as well as efficiently reducing the nonmembership degree. In spite of all this, while dealing with a large number of goals at a time, assigning the different crisp weight to all objectives according to the decision-makers' priority level is not feasible, because it may be time-consuming. To avoid the weight assignment complexity, it would be the best technique to assign linguistic priorities among different objectives.

5.3.1 Stepwise solution algorithm

The stepwise solution procedures for the proposed modified NFPA with intuitionistic fuzzy preference relations can be represented as follows:

Step 1. Design the proposed CLSC planning problem under uncertainty as given in model M_1.
Step 2. Convert each fuzzy parameter involved in model M_1 into its crisp form by using the expected intervals and values method as given in Eqs. (15.28)–(15.31) or presented in Table 15.1. Transform fuzzy constraints into their crisp versions by using Eqs. (15.38)–(15.39).
Step 3. Modify model M_1 into M_2 and solve for each objective function individually in order to obtain the best and worst solution set.
Step 4. Determine the upper and lower bounds for each objective function by using Eq. (15.55). With the aid of U_o and L_o, define the upper and lower bounds for truth, indeterminacy, and falsity memberships as given in Eqs. (15.56)–(15.58).
Step 5. Develop the neutrosophic optimization model M_5 with the aid of auxiliary variables.
Step 6. Assign linguistic importance relations among different objectives under an intuitionistic fuzzy environment (see Eqs. 15.59–15.64). Integrate the preference relation into model M_5 and transform into model M_6, which includes constraints of CLSC given in Eqs. (15.40)–(15.54).
Step 7. Model M_6 represents the modified neutrosophic fuzzy optimization model with intuitionistic fuzzy importance relations. Solve the model in order to obtain the compromise solution using suitable techniques or some optimizing software packages.

6. Computational study

The city of Nizam (Deccan), currently known as Hyderabad, is one of the leading IT hubs of India. It is well known for its IT hub service-oriented firms. A Hyderabad-based ABC (name changed) reputed multinational laptop manufacturing company has intended to model the production, transportation, distribution, and collection problems, due to the existence of a testing center facility in the proposed CLSC designed network. The prominent features of the CLSC design made it possible for the modeling and optimization approach under uncertainty. Regardless, unique, potentially functional components of the proposed CLSC design model have attracted the attention of decision makers. Less opportunity for the disposal of scrap parts/components is also a leading factor to adopt the model which ensures less accountability toward governmental managerial laws. The ecofriendly environmental nature of the modeling approach is a beneficial factor and guarantees freedom from the different governmental legislative traps. The interference of uncertainty among the various parameters reveals the realistic modeling approach. Ample scope for generating different solutions set by tuning the weight parameter and feasibility degree is the crucial promising factor for modeling choice by decision makers. To maintain sustainability in the competitive market, it would be more effective and efficient to develop the proposed CLSC design network.

The company has a fully functional multiechelon facility location and a well-organized decision policy scheme. In the forward chain, five multiechelon facilities are the main constituent part of the forward process. Three raw material storage centers, three supplier points, three hybrid manufacturing/remanufacturing plants, three distribution centers, and six customer/market zones explicitly represent the forward flow chain. In the reverse flow chain, six multiechelon facilities are taken into consideration, which signifies more emphasis on the opposite chain. The reverse flow chain consists of three collection centers of used laptops, three refurbishing or repair centers, three disassembling centers, three testing points, three recycling centers, and three disposal sites at which the end-of-life parts/components are removed from the designed CLSC network.

Every new and refurbished laptop is a hybrid combination of three different types of raw materials and parts/components. Refurbished laptops are also usable and acceptable in the market. Manufacturing plants provide a new laptop whereas the refurbishing center is responsible for renovated or refurbished laptops. The forward chain starts from the shipment of raw parts from the raw materials storage center to three supplier points. All three suppliers are responsible for the delivery of raw materials to hybrid manufacturing plants. Afterward, the newly manufactured laptops are shipped to three distribution centers. The demand quantity of the laptops must be fulfilled by the distribution center only. There is no scope for direct shipment from the manufacturing plant to the hybrid facility center. The collection center is accountable for the accumulation of end-of-use products from customers/market zones. The used products are disassembled into three parts or components. The testing facility carefully inspects the various parts/components and decides to implement a particular service to make it usable. From the testing center, three different destinations—the manufacturing plant, recycling point, and disposal center—have been postulated. Recyclable products are sent to the recycling center, whereas scrap or end-of-life parts/components are dumped at the disposal center. Parts/components that can constitute raw materials are entered into the forward chain through manufacturing plants. The recycling process turns the pieces into new raw materials, which ensures the procurement of raw materials and initiates the forward chain. Hence to implement the proposed CLSC model efficiently, the triangular fuzzy input data for transportation cost, purchasing cost, revenues, and time have been summarized in Table 15.3. Various capacities at each echelon in the CLSC chain network have been represented by trapezoidal fuzzy data, whereas processing cost parameters have been considered as triangular fuzzy input data. Since numerous objective functions have been developed in the proposed CLSC model, the following preference relations have been decided among different objective functions. However, the preference scheme has been randomly assigned, and there are no hard and fast rules. It solely depends upon the decision maker's choices. The type of preference relations between the objectives have been defined as follows:

- Objective Z_2 is moderately more important than objective Z_1 (i.e., $\widetilde{R}_2(2,1)$).
- Objective Z_4 is slightly more important than objective Z_3 (i.e., $\widetilde{R}_1(4,3)$).
- Objective Z_3 is significantly more important than objective Z_5 (i.e., $\widetilde{R}_3(3,5)$).
- Objective Z_4 is slightly more important than objective Z_5 (i.e., $\widetilde{R}_1(4,5)$).

Table 15.3 Input fuzzy data for the parameters.

Transportation cost from sources to destinations (*, *)	**Types of raw materials (*m*) or products (*l*)**		
	1	**2**	**3**
$\widetilde{TC1}_{m,a,b}$	(14, 24, 34)	(22, 32, 44)	(34, 36, 38)
$\widetilde{TC2}_{m,b,c}$	(30, 32, 34)	(34, 36, 38)	(38, 40, 42)
$\widetilde{TC3}_{l,c,d}$	(52, 56, 60)	(60, 63, 66)	(66, 67, 68)
$\widetilde{TC4}_{l,d,e}$	(60, 65, 70)	(66, 69, 72)	(71, 74, 77)
$\widetilde{TC5}_{l,e,f}$	(28, 29, 30)	(35, 37, 39)	(42, 44, 46)
$\widetilde{TC6}_{l,f,g}$	(32, 34, 36)	(35, 39, 43)	(41, 42, 43)
$\widetilde{TC7}_{l,g,d}$	(40, 42, 44)	(45, 48, 51)	(50, 55, 60)
$\widetilde{TC8}_{l,f,h}$	(44, 48, 52)	(50, 53, 56)	(55, 59, 63)
$\widetilde{TC9}_{m,h,i}$	(50, 51, 52)	(50, 55, 60)	(60, 63, 66)
$\widetilde{TC10}_{m,i,c}$	(25, 27, 29)	(30, 32, 34)	(35, 39, 41)
$\widetilde{TC11}_{m,i,j}$	(55, 60, 65)	(65, 67, 69)	(71, 73, 75)
$\widetilde{TC12}_{m,i,k}$	(33, 36, 39)	(40, 43, 46)	(44, 49, 54)
$\widetilde{TC13}_{m,j,a}$	(68, 71, 74)	(73, 75, 77)	(60, 62, 64)
Time			
$\widetilde{T}_{l,d,e}$	(05, 07, 09)	(04, 06, 08)	(01, 03, 06)
Purchasing cost			
$\widetilde{PU1}_{m}$	(36, 38, 40)	(45, 47, 49)	(24, 26, 28)
$\widetilde{PU2}_{l}$	(25, 27, 29)	(15, 17, 19)	(15, 17, 19)
Selling price			
$\widetilde{SP1}_{m}$	(42, 46, 50)	(40, 45, 50)	(21, 23, 26)
$\widetilde{SP2}_{l}$	(36, 38, 40)	(42, 45, 48)	(24, 26, 28)
rf_l	0.71	0.53	0.58
$rc_{l,e}$	0.82	0.76	0.38
rt_m	0.23	0.49	0.73
rm_m	0.81	0.67	0.35
rr_m	0.32	0.43	0.61
rd_m	0.12	0.19	0.23
Processing cost at each echelon			
$\widetilde{PC1}_{m,a}$	(14, 24, 34)	(22, 32, 44)	(34, 36, 38)
$\widetilde{PC2}_{m,b}$	(30, 32, 34)	(34, 36, 38)	(38, 40, 42)
$\widetilde{PC3}_{l,c}$	(52, 56, 60)	(60, 63, 66)	(66, 67, 68)
$\widetilde{PC4}_{l,d}$	(60, 65, 70)	(66, 69, 72)	(71, 74, 77)
$\widetilde{PC5}_{l,f}$	(28, 29, 30)	(35, 37, 39)	(42, 44, 46)
$\widetilde{PC6}_{l,g}$	(32, 34, 36)	(35, 39, 43)	(41, 42, 43)

Table 15.3 Continued

Transportation cost from sources to destinations (*, *)	Types of raw materials (*m*) or products (*l*)		
	1	**2**	**3**
$\widetilde{PC7}_{l,h}$	(40, 42, 44)	(45, 48, 51)	(50, 55, 60)
$\widetilde{PC8}_{m,i}$	(44, 48, 52)	(50, 53, 56)	(55, 59, 63)
$\widetilde{PC9}_{m,j}$	(50, 51, 52)	(50, 55, 60)	(60, 63, 66)
$\widetilde{PC10}_{m,k}$	(25, 27, 29)	(30, 32, 34)	(35, 39, 41)
Capacity/demand at each echelon			
$\widetilde{MC1}_{m,a}$	(512, 514, 516, 518)	(622, 624, 626, 628)	(718, 724, 726, 728)
$\widetilde{MC2}_{m,b}$	(613, 614, 615, 616)	(514, 516, 518, 520)	(512, 514, 516, 518)
$\widetilde{MC3}_{m,c}$	(724, 725, 726, 727)	(812, 813, 814, 815)	(914, 916, 918, 920)
$\widetilde{MC4}_{l,d}$	(212, 214, 216, 218)	(221, 222, 223, 224)	(217, 218, 219, 220)
$\widetilde{MC5}_{l,e}$	(314, 318, 322, 326)	(312, 314, 316, 318)	(329, 339, 349, 359)
$\widetilde{MC6}_{l,f}$	(115, 116, 117, 118)	(119, 120, 121, 122)	(114, 116, 118, 120)
$\widetilde{MC7}_{l,g}$	(124, 125, 126, 127)	(113, 114, 115, 116)	(117, 119, 121, 123)
$\widetilde{MC8}_{m,h}$	(110, 111, 112, 113)	(114, 116, 118, 120)	(119, 120, 121, 122)
$\widetilde{MC9}_{m,i}$	(224, 225, 226, 227)	(212, 214, 216, 218)	(314, 316, 318, 320)
$\widetilde{MC10}_{m,j}$	(324, 325, 326, 327)	(212, 213, 214, 215)	(214, 216, 218, 220)
$\widetilde{MC11}_{m,k}$	(212, 214, 216, 218)	(221, 222, 223, 224)	(317, 318, 319, 320)

6.1 Results and discussions

The modified neutrosophic fuzzy optimization model for the proposed CLSC network has been written in AMPL language and solved using the solver Kintro 10.3.0 through the NEOS server version 5.0 online facility provided by Wisconsin Institutes for Discovery at the University of Wisconsin in Madison for solving optimization problems; see Refs. [47,48]. The characteristic description of the problem is presented as follows: The final multiobjective optimization model along with a set of well-defined multiple objectives comprises 459 variables including 42 binary variables and 417 linear variables, 530 constraints including 498 linear one-sided inequalities constraints and 32 linear equality constraints, respectively. The total computational time for obtaining the final solution was 0.113 seconds (CPU time). Due to space limitations, only the final solution results of all decision variables obtained at a feasibility degree ($\beta = 0.5$) with weight parameter ($\alpha = 0.5$) have been discussed in detail. The optimum allocation of raw materials, new products, and used parts/components among different echelons has been depicted in Tables 15.4 and 15.5. In the forward chain, procurement of raw materials initiates from a raw material storage center (RMS) to a supplier point (SP). The total allocation of raw materials from RMS 1 to all three SPs is found to be 504.17, 592.38, and 681.51, whereas from RMS 2 and 3 to all three SPs have been obtained as 572.14, 553.84, and 703.15, and 497.57, 457.32, and 646.87, respectively. The maximum shipment quantity has been observed from RMS 2 to SP 3 due to the lowest transportation and processing cost incurred over the raw materials. Suppliers are responsible for fulfilling the requirement for starting the manufacturing processes at the hybrid manufacturing plant (MP). The optimum shipment quantity from SP 1 to all three MPs is 706.25, 630.15, and 625.18, respectively. Similarly, from SP 2 and 3 to all three MPs have been obtained as 630.25, 630.21, and 656.51, and 563.70, 498.34, and 533.18, respectively. The highest shipment amount of raw materials has been allocated to MP 1 whereas the least amount of raw materials has been delivered to MP 2 bearing in mind the fact that the outbound capacity of manufacturing plant receives the maximum raw materials and parts from the SPs and testing points (TPs). SP 3 also provides the maximum amount of raw materials to all three MPs and are obtained as 563.7, 488.34, and 533.18 bearing in mind the fact that outbound restrictions on manufacturing plants have been satisfied, and tested and approved parts/components are sent back to the manufacturing plant for further utilization. Newly built products are transferred to the distribution center (DC) so that the demand from customers (Cs) could be met. The optimal distribution scheme among different customers has been obtained. From DC 1 to all six Cs, the total shipment of products is found to be 332.23, 400.85, 350.61, 297.21, 274.95, and 266.61, respectively. However, DC 1 has a negligible contribution to meet the demand of C 2, 3, 4, and 5, with other types of products to avoid the maximum transportation cost and late expected delivery time. Similarly, the total quantities of each product distributed from DC 2 and 3 to all six Cs have been depicted, which ensures the minimum transportation costs along with the timely shipment of products. It has been observed that no product has been shipped from DC 2 to C 1, 2, and 3 due to the maximum chances for late delivery of the products. Hence a minimum transportation cost and shipment time have been achieved without significantly affecting the demand constraint. Overall, DC 2 outsourced the maximum shipment of products to all six Cs and revealed a significant contribution to fulfilling the demand. Since refurbished products are also acceptable in the market, approximately 13.32% of total used products are renovated and shipped to DCs for the fulfillment of further needs.

The significant role of the collection center (CC) starts when end-of-use and end-of-life products come into existence. The potential accumulation framework for used products from the customer zone is much needed. The designed CLSC model inherently involves the CC, which is the first echelon of the reverse supply chain network. The exclusive collection of the end-of-use product from customers is found to be a significant percentage, that is, approximately 91.34% of the total fulfilled demand,

Table 15.4 Optimal quantities of raw materials and products shipped from different sources to various destinations.

Raw material storage center (*a*)	Supplier point (*b*)	Types of raw material (*m*)		
		1	**2**	**3**
Storage center 1	1	127.83	232.78	143.56
Storage center 1	2	248.23	151.62	192.53
Storage center 1	3	201.32	312.28	167.91
Storage center 2	1	164.21	264.67	143.26
Storage center 2	2	321.34	109.23	123.27
Storage center 2	3	221.63	368.83	112.69
Storage center 3	1	116.94	219.43	161.20
Storage center 3	2	213.52	142.20	101.60
Storage center 3	3	329.53	127.64	189.70
Supplier point (*b*)	**Manufacturing plant (*c*)**	**Types of raw materials (*m*)**		
		1	**2**	**3**
Supplier point 1	1	261.24	243.12	201.89
Supplier point 1	2	291.64	124.15	214.36
Supplier point 1	3	236.39	213.56	175.23
Supplier point 2	1	218.95	189.67	221.63
Supplier point 2	2	253.68	128.63	247.90
Supplier point 2	3	287.25	112.46	256.80
Supplier point 3	1	212.54	187.62	163.54
Supplier point 3	2	202.35	142.37	143.62
Supplier point 3	3	298.34	116.52	118.32
Manufacturing plant (*c*)	**Distribution center (*d*)**	**Types of products (*l*)**		
		1	**2**	**3**
Manufacturing plant 1	1	127.83	132.78	143.56
Manufacturing plant 1	2	148.23	151.62	192.53
Manufacturing plant 1	3	121.32	112.28	67.91
Manufacturing plant 2	1	164.21	64.67	163.26
Manufacturing plant 2	2	171.34	119.23	143.27
Manufacturing plant 2	3	181.63	68.83	152.69
Manufacturing plant 3	1	196.94	89.43	61.20
Manufacturing plant 3	2	113.52	42.20	101.60
Manufacturing plant 3	3	129.53	127.64	189.70
Distribution center (*d*)	**Customers (*e*)**	**Types of products (*l*)**		
		1	**2**	**3**
Distribution center 1	1	112.34	145.26	74.63
Distribution center 1	2	85.26	163.23	152.36
Distribution center 1	3	152.36	–	198.35
Distribution center 1	4	163.98	–	115.23

Continued

Table 15.4 Continued

Distribution center (*d*)	Customers (*e*)	Types of products (*l*) 1	2	3
Distribution center 1	5	165.32	–	109.63
Distribution center 1	6	154.23	–	112.38
Distribution center 2	1	198.43	167.23	–
Distribution center 2	2	165.24	144.23	–
Distribution center 2	3	180.50	143.20	–
Distribution center 2	4	155.96	124.27	127.52
Distribution center 2	5	169.58	153.65	65.87
Distribution center 2	6	187.65	84.59	154.23
Distribution center 3	1	169.75	–	159.86
Distribution center 3	2	–	173.89	168.27
Distribution center 3	3	–	196.43	149.26
Distribution center 3	4	–	142.35	149.37
Distribution center 3	5	184.26	73.68	163.87
Distribution center 3	6	179.35	97.36	135.98
Customers (*e*)	**Collection center (*f*)**	**Types of products (*l*) 1**	**2**	**3**
Customer 1	1	61.32	53.68	94.38
Customer 1	2	85.23	78.56	145.80
Customer 1	3	84.32	58.50	145.23
Customer 2	1	52.31	16.78	61.83
Customer 2	2	47.50	51.32	134.62
Customer 2	3	79.68	45.23	84.23
Customer 3	1	52.63	89.45	79.56
Customer 3	2	74.96	98.74	112.34
Customer 3	3	47.89	114.90	78.46
Customer 4	1	89.56	89.45	74.68
Customer 4	2	76.34	94.68	52.60
Customer 4	3	58.35	78.89	53.46
Customer 5	1	61.23	106.83	45.3
Customer 5	2	63.85	117.40	47.6
Customer 5	3	86.34	127.63	76.85
Customer 6	1	74.68	121.69	44.62
Customer 6	2	85.90	153.45	57.67
Customer 6	3	84.32	173.65	79.85
Collection center (*f*)	**Refurbishing center (*g*)**	**Types of products (*l*) 1**	**2**	**3**
Collection center 1	1	36.24	45.32	32.65
Collection center 1	2	41.58	31.25	21.32
Collection center 1	3	16.23	74.32	24.12
Collection center 2	1	14.23	61.32	42.37

Table 15.4 Continued

Collection center (*f*)	**Refurbishing center (*g*)**	**Types of products (*l*)** 1	2	3
Collection center 2	2	71.20	41.23	54.64
Collection center 2	3	24.53	85.93	27.65
Collection center 3	1	34.53	22.38	68.53
Collection center 3	2	74.30	72.30	23.60
Collection center 3	3	25.90	33.56	67.84

Table 15.5 Optimal quantities of used products and parts shipped from different sources to various destinations.

Refurbishing plant (*g*)	**Distribution center (*d*)**	**Types of products (*l*)** 1	2	3
Refurbishing plant 1	1	34.28	24.89	52.37
Refurbishing plant 1	2	25.36	41.98	56.35
Refurbishing plant 1	3	27.85	39.38	49.35
Refurbishing plant 2	1	23.89	54.23	63.45
Refurbishing plant 2	2	31.45	47.68	54.38
Refurbishing plant 2	3	43.56	42.89	47.86
Refurbishing plant 3	1	44.87	57.98	53.78
Refurbishing plant 3	2	38.45	47.56	63.45
Refurbishing plant 3	3	37.84	49.63	57.68
Collection center (*f*)	**Disassembling center (*h*)**	**Types of products (*l*)** 1	2	3
Collection center 1	1	146.23	98.29	154.78
Collection center 1	2	131.26	157.23	74.39
Collection center 1	3	157.89	158.96	84.97
Collection center 2	1	98.46	143.69	87.56
Collection center 2	2	87.60	89.63	178.87
Collection center 2	3	89.68	63.84	187.20
Collection center 3	1	107.35	84.96	172.86
Collection center 3	2	118.35	97.63	166.34
Collection center 3	3	112.57	98.68	136.94
Disassembling center (*h*)	**Testing center (*i*)**	**Types of products (*m*)** 1	2	3
Disassembling center 1	1	98.86	47.52	112.36
Disassembling center 1	2	187.34	145.26	75.40
Disassembling center 1	3	85.32	143.26	146.37

Continued

Table 15.5 Continued

Disassembling center (*h*)	Testing center (*i*)	Types of products (*m*)		
		1	2	3
Disassembling center 2	1	55.85	121.35	141.23
Disassembling center 2	2	65.36	185.98	124.36
Disassembling center 2	3	80.45	178.90	142.58
Disassembling center 3	1	60.85	42.38	173.45
Disassembling center 3	2	75.03	63.57	156.89
Disassembling center 3	3	86.08	53.76	154.36
Recycling center (*j*)	**Raw material storage facility (*a*)**	**Types of parts (*m*)**		
		1	**2**	**3**
Recycling center 1	1	24.23	227.35	16.35
Recycling center 1	2	12.54	14.80	22.35
Recycling center 1	3	28.34	14.25	25.36
Recycling center 2	1	21.50	22.24	16.80
Recycling center 2	2	17.35	12.36	13.52
Recycling center 2	3	18.53	11.98	16.39
Recycling center 3	1	24.37	22.35	14.35
Recycling center 3	2	17.98	27.85	17.68
Recycling center 3	3	19.63	14.32	13.84
Testing center (*i*)	**Manufacturing plant (*c*)**	**Types of tested parts (*m*)**		
		1	**2**	**3**
Testing center 1	1	42.35	17.43	11.75
Testing center 1	2	13.40	16.98	27.06
Testing center 1	3	17.31	12.37	18.08
Testing center 2	1	08.32	29.56	21.07
Testing center 2	2	21.43	39.75	19.01
Testing center 2	3	17.35	18.56	12.89
Testing center 3	1	38.96	21.29	28.34
Testing center 3	2	17.51	10.37	12.34
Testing center 3	3	21.27	03.78	22.48
Testing center (*i*)	**Recycling facility (*j*)**	**Types of recyclable parts (*m*)**		
		1	**2**	**3**
Testing center 1	1	34.53	45.05	48.35
Testing center 1	2	54.27	35.64	53.42
Testing center 1	3	58.34	56.34	24.35
Testing center 2	1	62.78	49.64	31.70
Testing center 2	2	28.34	51.46	41.32
Testing center 2	3	25.32	47.86	105.06

Table 15.5 Continued

Testing center (*i*)	**Recycling facility (*j*)**	**Types of recyclable parts (*m*)**		
		1	**2**	**3**
Testing center 3	1	78.35	48.36	62.37
Testing center 3	2	48.32	42.36	56.28
Testing center 3	3	51.43	118.36	29.23
Testing center (*i*)	**Disposal facility (*k*)**	**Types of scrap parts (*m*)**		
		1	**2**	**3**
Testing center 1	1	14.25	–	16.52
Testing center 1	2	17.24	–	13.25
Testing center 1	3	–	–	11.24
Testing center 2	1	–	21.85	18.54
Testing center 2	2	–	19.65	–
Testing center 2	3	16.35	12.35	19.32
Testing center 3	1	12.89	14.22	–
Testing center 3	2	15.45	–	19.34
Testing center 3	3	17.40	–	21.30

which indicates the vast need for the reverse supply chain to tackle used products. The required service at different echelons in the reverse chain has been designed especially for socioenvironmental concerns. An optimal amount of used products has been collected by all three CCs from all six customer zones. The maximum amount of used products has been received by CC 3 from C 6 which is 337.82, and the least quantity 190.70 by CC 3 from C 4 to ensure the least collection and transportation costs levied over each type of product. At CCs, complete inspection of the collected, used products has been performed and a decision taken to ship either to the disassembling center (DS) or refurbishing center (RC) to initiate the required services. The total amounts of used products transported from CC 1 to all three RCs are obtained as 114.21, 94.15, and 114.67, whereas the total shipment quantities from CC 2 and 3 to all three RCs have been allocated as 117.92, 167.07, and 138.11, and 125.44, 170.20, and 127.30, respectively. The maximum quantity of used products has been transported from CC 3 to RC 2 whereas the minimum shipment quantity is found to be shipped from CC 1 to RC 2 because of the lowest transportation cost and availability of the required service for particular types of products. The quantity of used products is approximately 88.21% of the total capacity of the RC, which ensures the significant need for such a functional echelon in CLSC. The disassembling center (DS) only receives those end-of-use products that require reliability tests of each part/component. At the

DS, used products are disassembled into different parts/components for the testing process where all necessary measures would be taken regarding the useful life of parts. From CC 1 to all three DSs, the total shipment amounts of used products have been obtained as 399.30, 362.88, and 401.82, which shows approximately 31.98% of the entire collection of used products. Likewise, the net amount of used products transported from CC 2 and 3 to all three DSs are 329.71, 356.10, and 340.72, and 365.17, 382.32, and 348.19, respectively. The shipment of end-of-use products at DS 2 and 3 are found to be 14.29% and 39.47% of the net used products collected at all three CCs, which shows that approximately 94% of the total raised used products have been completely dealt with at the CC and signify that the design of the reverse chain is much needed to avoid environmental issues. The total disassembled parts that have been shipped from DS 1 to all three TPs are found to be 258.74, 408, and 374.95, which comprise 31.35% of the disassembled parts/components and ensures that transportation and inspection costs incurred over these parts would be minimal. Similarly, from DS 2 and 3 to all TPs the optimal amounts of pieces have been shipped, which are found to be 33.84%, and 29.27% of the total disassembled parts at DSs to minimize the total cost of inspection by ensuring the capacity restrictions at TPs, respectively.

The testing point (TP) inevitably inspects the reliability or usefulness of parts/ components and provides the best decision to deal with tested parts. TPs are interconnected with three echelons: manufacturing plants, recycling centers, and disposal sites. The TP is also a promising source for the procurement of raw materials to hybrid manufacturing/remanufacturing plants. Approximately 9.84% of the total requirement for raw materials has been met by different TPs with the aid of dissembled parts of used products. However, the recycling point (RP) receives a significant amount of tested parts that ensures green practice with recyclable components. The net quantity of recyclable parts that have been transported from all three TPs to RP 1 is found to be 127.93, 143.33, and 139.03, which is 93.66% of the total recyclable capacity of tested parts at RP 1. The maximum quantity of recyclable parts has been received by RP 3 whereas the least amount of certified parts has been shipped to RP 2 bearing in mind the fact that transportation and recycling costs levied over each component are minimal at these facilities. Finished recycled products have been sent back to raw material storage centers and recycled for the smooth running of the production processes. After the inspection procedure, the declared disposable parts have been shipped for disposal. The optimal quantity of disposable parts has been obtained with the satisfaction of disposal capacity constraints of each disposal facility center (DF). The obtained results showing that at some DFs, there is no amount of tested parts for disposal. However, the total shipped amounts from all three TPs to DF 1 are found to be 30.77, 40.39, and 27.11, which is 47.32% of the full capacity of DF 1. Moreover, from all three TPs to DF 2 and 3, the net amounts of disposable parts that have been transported are found to be 30.49, 19.65, and 34.79, and 11.24, 48.02, and 38.70, respectively. At these DFs, approximately 53.57% and 69.38% shares of the total disposal capacity have been disposed of, which strictly ensures that there is still an abundant opportunity for incineration.

Multiechelon CLSC design networks require potential capital investment to the flow of products throughout the supply chain processes. Processing cost, transportation cost, and purchasing cost have been depicted as different objectives that inherently require capital. The obtained results of these three objectives show a remarkable contribution to the total capital investment. At each feasibility degree β and weight parameter α, the average share of processing cost has been obtained as approximately 83.39%, the total ordinary dividends of the transportation cost is found to be approximately 14.78% and that of the average purchasing cost is approximately 1.83% of the total investment in the proposed CLSC network. The maximum shares have been exhausted by the processing charge with the fact that multiple different echelons have been associated with specific functional services to raw materials, new products, and used parts in the proposed CLSC network. Transportation costs hold a slightly smaller portion of the total investment, which shows the reduced to and fro movement of products among different echelons. Due to the interconnected systematic facility centers, the optimal shipment strategy turns into fewer transportation costs. The purchasing of raw materials in bulk from raw material storage centers and used products from customers has comparatively very low in the total capital investment. The expected whole delivery time and revenues from sales have also been included as conflicting potential objectives in the proposed CLSC model, which sufficiently reflects the effective exogenous solution results. The flow of new products in the forward chain and end-of-use products in reverse chain much depends on keen managerial insight and decision-making strategy. The potential performances of each echelon would be recognized in the context of allocation and required service to the different products and parts. The solution results have been presented only for $\alpha = 0.5$ and $\beta = 0.5$, but more information could be extracted by obtaining the solution results at different values of α and β regarding the optimum allocation of products and parts, respectively.

6.2 Sensitivity analyses

Sensitivity analyses have been performed for all the objective functions by tuning the feasibility degree (β) and weight parameter (α) simultaneously. The feasibility degree (β) referred to the preference or acceptance level of decision makers. The higher value of (β) ensures the maximum satisfaction level of decision makers. The feasibility degree among parameters reflects the satisfaction level by offering different choices. Hence more substantial feasibility degree generally gives the worse solution of objectives. The weight parameter (α) provides the weight to either the membership function of all the objectives or the score functions of the intuitionistic fuzzy preference relations among different objectives. Therefore, a higher value of (α) signifies a higher weight to either the corresponding membership functions or the score function of linguistic preference relations. The priority structure has been designed as the convex combination between the membership function of the objectives and the score function of the linguistic preference relations. The weight parameter (α) is directly assigned to the membership functions of each goal whereas $(1 - \alpha)$ has been assigned to the score function of linguistic preference relations. The solution results of all the objective functions and preference relations are shown in Fig. 15.4.

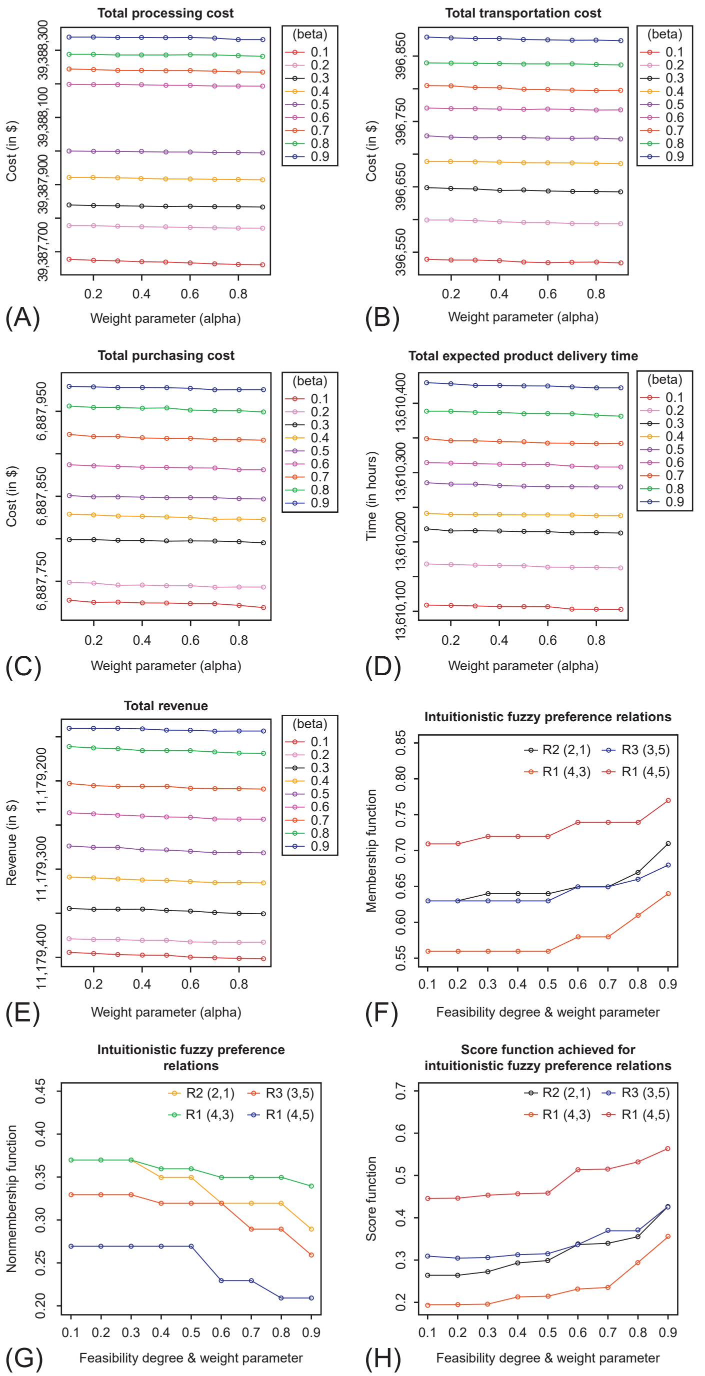

Fig. 15.4 Graphical representation of obtained results. (A) First objective (Z_1). (B) Second objective (Z_2). (C) Third objective (Z_3). (D) Fourth objective (Z_4). (E) Fifth objective (Z_5). (F) Membership degree. (G) Nonmembership degree. (H) Score function.

6.2.1 Sensitivity analyses of objective functions

The first objective (Z_1) is to minimize the total processing cost (TPC) supply chain. At $\beta = 0.1$ and $\alpha = 0.9$, the minimum (best) value of (Z_1) has been attained, which is $39,387,662. As ($\alpha$) decreases, the values of (Z_1) either increase or remain the same for some (α). With an increase in the feasibility degree (β), a significant increment in the objective function (Z_1) has been observed. The maximum (worst) value of objective (Z_1) has been obtained as $39,388,338 at $\beta = 0.9$ and $\alpha = 0.1$. Hence it has been concluded that with the increase in feasibility degree (β) and the decrease in weight parameter (α), the value of objective (Z_1) reaches its worst values. The different solution results of (Z_1) ranging between $39,387,662 and $39,388,338 are summarized in Table 15.6, and Fig. 15.4A shows the trending behavior of (Z_1) at different feasibility degree (β) and weight parameters (α), respectively. Furthermore, the effects of feasibility degrees (β) are severe, as the marginal increment in the value of (Z_1) rapidly approaches the worst solutions, whereas the effect of the weight parameter (α) on the objective (Z_1) is almost negligible. The TPC has been obtained that solely occurred over four echelons in the forwarding chain. Hence, the obtained results for TPC are due to the high processing cost at the raw material storage center, supplier point, and manufacturing plants. Inbound capacity restrictions at these echelons are also a key factor for increment in TPC. The maximum numbers of raw materials and new products require different processing costs, which turn into more capital investment in the material and product processing purposes.

The minimization of total transportation costs in the CLSC has been represented by the second objective (Z_2). At $\beta = 0.1$ and $\alpha = 0.9$, the minimum (best) value of transportation cost is $396,534. As a feasibility degree (β) increases, there is a significant marginal increment in the objective (Z_2) that has been found. The values of (Z_2) either increase or remain stable for different values of (α) with the decrease in the weight parameter (α). The maximum (worst) value of (Z_2) has been attained as $396,879 at $\beta = 0.9$ and $\alpha = 0.1$, respectively. Thus it has emerged that with the increase in the feasibility degree (β) and the decrease in the weight parameter (α), the value of the objective (Z_2) approaches its worst outcomes. The different solution results of (Z_2) have been generated, which lie between $396,534 and $396,879, and are presented in Table 15.7. The fluctuating behavior of (Z_2) has also been shown in Fig. 15.4B at a different feasibility degree (β) and weight parameter (α). The utmost influencing capability of the feasibility degree (β) has been reflected by the significant increase in the objective (Z_2) and which lead (Z_2) toward its worst values. The weight parameter (α) has fewer effects on the objective (Z_2) compared to the feasibility degree (β) among all the solution choices. Due to the low processing charges at each echelon in the reverse chain, the total TPC has been obtained much less compared to TPC in the forwarding chain. Each echelon in the reverse chain dealt with either end-of-use products or end-of-life products. To perform the different required services on such products would not necessarily result in higher costs, because of less complexity in dealing with used and returned products compared to the manufacturing of new parts and products.

Table 15.6 Total processing costs (Z_1) at different feasibility degrees (β) and weight parameters (α).

Feasibility degree (β)	Weight parameter (α)								
	0.9	**0.8**	**0.7**	**0.6**	**0.5**	**0.4**	**0.3**	**0.2**	**0.1**
0.1	39,387,662	39,387,664	39,387,666	39,387,668	39,387,670	39,387,672	39,387,674	39,387,676	39,387,678
0.2	39,387,768	39,387,769	39,387,771	39,387,773	39,387,774	39,387,774	39,387,776	39,387,778	39,387,779
0.3	39,387,834	39,387,834	39,387,835	39,387,836	39,387,836	39,387,836	39,387,838	39,387,838	39,387,839
0.4	39,387,914	39,387,916	39,387,916	39,387,916	39,387,918	39,387,918	39,387,921	39,387,921	39,387,922
0.5	39,387,994	39,387,996	39,387,996	39,387,996	39,387,997	39,387,997	39,387,998	39,387,998	39,387,999
0.6	39,388,194	39,388,194	39,388,194	39,388,196	39,388,196	39,388,197	39,388,197	39,388,197	39,388,198
0.7	39,388,234	39,388,234	39,388,236	39,388,238	39,388,239	39,388,241	39,388,241	39,388,243	39,388,244
0.8	39,388,282	39,388,283	39,388,285	39,388,285	39,388,285	39,388,286	39,388,286	39,388,288	39,388,288
0.9	39,388,331	39,388,331	39,388,334	39,388,336	39,388,336	39,388,336	39,388,337	39,388,338	39,388,338

Table 15.7 Total transportation costs (Z_2) at different feasibility degrees (β) and weight parameters (α).

Feasibility degree (β)	Weight parameter (α)								
	0.9	**0.8**	**0.7**	**0.6**	**0.5**	**0.4**	**0.3**	**0.2**	**0.1**
0.1	396,534	396,535	396,535	396,534	396,535	396,537	396,538	396,538	396,539
0.2	396,593	396,593	396,593	396,595	396,595	396,596	396,598	396,599	396,599
0.3	396,642	396,642	396,642	396,643	396,644	396,644	396,646	396,648	396,649
0.4	396,686	396,686	396,687	396,687	396,687	396,688	396,688	396,689	396,689
0.5	396,723	396,724	396,724	396,724	396,725	396,725	396,725	396,726	396,728
0.6	396,767	396,767	396,768	396,768	396,768	396,769	396,769	396,769	396,770
0.7	396,798	396,798	396,798	396,799	396,799	396,802	396,802	396,805	396,805
0.8	396,837	396,837	396,838	396,838	396,838	396,839	396,839	396,839	396,840
0.9	396,873	396,874	396,874	396,874	396,875	396,876	396,876	396,877	396,879

The minimization of the total purchasing cost has been represented by the third objective (Z_3). The minimum (best) value of the objective (Z_3) has been obtained as \$6,887,719 at $\beta = 0.1$ and $\alpha = 0.9$. At $\beta = 0.9$ and $\alpha = 0.1$, the maximum (worst) value of the total purchasing cost has been obtained, which is \$6,887,980. As (α) decreases, the values of (Z_3) either increase or remain inert for some (α). With an increase in the feasibility degree (β), the significant increment in the objective function (Z_3) has been noticed. Thus it has emerged that with the increase in the feasibility degree (β) and the decrease in the weight parameter (α), the value of the objective (Z_3) approaches its worst outcomes. The different solution results of (Z_3) have been generated, which lie between \$6,887,719 and \$6,887,980 and are represented in Table 15.8. The declining performance of (Z_3) has also been shown in Fig. 15.4C at different feasibility degrees (β) and weight parameters (α). Furthermore, the effect of the feasibility degree (β) is more influential, as the significant increase in the value of (Z_3) rapidly approaches the worst solutions whereas the effect of the weight parameter (α) on the objective (Z_3) is almost negligible.

The fourth objective (Z_4) is the minimization of total product delivery time to different customers/market zones. At $\beta = 0.1$ and $\alpha = 0.9$, the minimum (best) value of the total products delivery time is 13,610,103 hours. As the feasibility degree (β) increases, a significant marginal increment in the objective (Z_4) is observed. The values of (Z_4) either increase or remain inactive for different values of (α) with the decrease in the weight parameter (α). The maximum (worst) value of (Z_4) has been attained as 13,610,429 hours at $\beta = 0.9$ and $\alpha = 0.1$, respectively. Thus it has been concluded that with the increase in the feasibility degree (β) and the decrease in the relative weight parameter (α), the value of the objective (Z_4) approaches its worst results. The various solution outcomes of (Z_4) have been generated, which lie between 13,610,103 and 13,610,429 hours, and are presented in Table 15.9. The trending feature of (Z_4) is also shown in Fig. 15.4D at different feasibility degrees (β) and weight parameters (α). The powerful performance of the feasibility degree (β) has been observed by the significant increase in the objective (Z_4) and which leads (Z_4) toward its worst values. The weight parameter (α) has fewer effects on the objective (Z_4) compared to the feasibility degree (β) among all the solution sets.

The maximization of revenues earned from the selling of new products has been represented by the fifth objective (Z_5). The maximum (best) value of the objective (Z_5) has been obtained as \$11,179,402 at $\beta = 0.1$ and $\alpha = 0.9$. At $\beta = 0.9$ and $\alpha = 0.1$, the minimum (worst) value of revenues has been obtained, which is \$11,179,140. As (α) decreases, the values of (Z_5) either decrease or remain stable for some (α). With an increase in the feasibility degree (β), the significant decrease in the objective function (Z_5) has been found. Thus it has been elicited that with the increase in the feasibility degree (β) and the decrease in the weight parameter (α), the value of the objective (Z_5) approaches its worst outcomes. The different solution results of (Z_5) ranging between \$6,887,719 and \$6,887,980, and are summarized

Table 15.8 Total purchasing costs (Z_3) at different feasibility degrees (β) and weight parameters (α).

Feasibility degree (*β*)	**Weight parameter (*α*)**								
	0.9	**0.8**	**0.7**	**0.6**	**0.5**	**0.4**	**0.3**	**0.2**	**0.1**
0.1	6,887,719	6,887,721	6,887,723	6,887,723	6,887,724	6,887,724	6,887,725	6,887,725	6,887,727
0.2	6,887,743	6,887,743	6,887,743	6,887,744	6,887,744	6,887,745	6,887,745	6,887,747	6,887,748
0.3	6,887,795	6,887,796	6,887,797	6,887,797	6,887,797	6,887,798	6,887,798	6,887,799	6,887,799
0.4	6,887,823	6,887,823	6,887,823	6,887,825	6,887,826	6,887,827	6,887,827	6,887,828	6,887,829
0.5	6,887,847	6,887,847	6,887,848	6,887,848	6,887,848	6,887,848	6,887,849	6,887,849	6,887,851
0.6	6,887,881	6,887,881	6,887,883	6,887,883	6,887,884	6,887,884	6,887,885	6,887,886	6,887,887
0.7	6,887,916	6,887,917	6,887,917	6,887,918	6,887,918	6,887,919	6,887,921	6,887,921	6,887,923
0.8	6,887,949	6,887,951	6,887,951	6,887,952	6,887,954	6,887,954	6,887,955	6,887,955	6,887,956
0.9	6,887,976	6,887,976	6,887,976	6,887,977	6,887,978	6,887,978	6,887,978	6,887,979	6,887,980

Table 15.9 Total expected product delivery times (Z_4) at different feasibility degrees (β) and weight parameters (α).

	Weight parameter (α)								
Feasibility degree (β)	**0.9**	**0.8**	**0.7**	**0.6**	**0.5**	**0.4**	**0.3**	**0.2**	**0.1**
0.1	13,610,103	13,610,103	13,610,103	13,610,106	13,610,106	13,610,106	13,610,107	13,610,108	13,610,108
0.2	13,610,162	13,610,163	13,610,163	13,610,163	13,610,165	13,610,166	13,610,166	13,610,167	13,610,168
0.3	13,610,213	13,610,213	13,610,213	13,610,215	13,610,215	13,610,216	13,610,216	13,610,216	13,610,218
0.4	13,610,237	13,610,237	13,610,238	13,610,238	13,610,238	13,610,239	13,610,239	13,610,239	13,610,241
0.5	13,610,279	13,610,279	13,610,279	13,610,279	13,610,280	13,610,281	13,610,283	13,610,283	13,610,285
0.6	13,610,308	13,610,308	13,610,309	13,610,311	13,610,311	13,610,311	13,610,312	13,610,313	13,610,314
0.7	13,610,342	13,610,342	13,610,342	13,610,342	13,610,345	13,610,345	13,610,346	13,610,346	13,610,348
0.8	13,610,381	13,610,383	13,610,385	13,610,385	13,610,385	13,610,386	13,610,386	13,610,388	13,610,388
0.9	13,610,422	13,610,422	13,610,423	13,610,424	13,610,424	13,610,425	13,610,425	13,610,428	13,610,429

Table 15.10 Total revenues (Z_5) at different feasibility degrees (β) and weight parameters (α).

Feasibility degree (β)	Weight parameter (α)								
	0.9	**0.8**	**0.7**	**0.6**	**0.5**	**0.4**	**0.3**	**0.2**	**0.1**
0.1	11,179,402	11,179,402	11,179,401	11,179,400	11,179,398	11,179,398	11,179,397	11,179,396	11,179,395
0.2	11,179,383	11,179,383	11,179,383	11,179,383	11,179,381	11,179,381	11,179,380	11,179,380	11,179,379
0.3	11,179,351	11,179,351	11,179,349	11,179,348	11,179,347	11,179,346	11,179,346	11,179,346	11,179,345
0.4	11,179,316	11,179,316	11,179,316	11,179,314	11,179,313	11,179,313	11,179,311	11,179,310	11,179,309
0.5	11,179,281	11,179,281	11,179,281	11,179,280	11,179,278	11,179,278	11,179,275	11,179,275	11,179,274
0.6	11,179,243	11,179,243	11,179,243	11,179,241	11,179,241	11,179,240	11,179,239	11,179,237	11,179,236
0.7	11,179,209	11,179,209	11,179,209	11,179,208	11,179,206	11,179,206	11,179,206	11,179,205	11,179,203
0.8	11,179,168	11,179,168	11,179,167	11,179,165	11,179,165	11,179,165	11,179,163	11,179,162	11,179,161
0.9	11,179,143	11,179,143	11,179,143	11,179,142	11,179,142	11,179,141	11,179,140	11,179,140	11,179,140

in Table 15.10, and Fig. 15.4E shows the inclining behavior of (Z_5) at different feasibility degrees (β) and weight parameters (α), respectively. Furthermore, the effect of the feasibility degree (β) is more influential, as the significant increase in the value of (Z_5) rapidly approaches the worst solutions whereas the weight parameter (α) affects the objective (Z_5) almost trivially.

6.2.2 Sensitivity analyses of intuitionistic fuzzy linguistic preference relations

Imprecise importance relations have been represented by an intuitionistic fuzzy preference hierarchy for three different linguistic terms. The membership functions for importance relations $\widetilde{R}_2(2,1)$, $\widetilde{R}_1(4,3)$, $\widetilde{R}_3(3,5)$, and $\widetilde{R}_1(4,5)$ have been obtained and shown in Table 15.11 and Fig. 15.4F. With the increase in the feasibility degree (β) and the weight parameter (α), the preference membership function for $\widetilde{R}_2(2,1)$ also increases and reaches its maximum, that is, 0.71 at $\beta = 0.9$ and $\alpha = 0.9$. Similarly, the preference membership functions for $\widetilde{R}_1(4,3)$, $\widetilde{R}_3(3, 5)$, and $\widetilde{R}_1(4,5)$ also reveal increasing behavior with the increase in the feasibility degree (β) and the weight parameter (α), and reaches their maximum attainment, that is, 0.64, 0.68, and 0.77 at $\beta = 0.9$ and $\alpha = 0.9$, respectively. Moreover, the nonmembership functions for different linguistic preferences are summarized in Table 15.11 and are shown in Fig. 15.4G. The motive is to minimize the nonmembership functions of each linguistic preference relation. Hence the minimum attainment degrees of nonmembership functions for $\widetilde{R}_2(2,1)$, $\widetilde{R}_1(4,3)$, $\widetilde{R}_3(3,5)$, and $\widetilde{R}_1(4,5)$ have been obtained as 0.29, 0.34, 0.26, and 0.21 at $\beta = 0.9$ and $\alpha = 0.9$, respectively. The overall satisfaction degree of linguistic preference relations has been represented by the score function. The maximization of the score function ensures the maximum satisfaction degree for the intended preferences among different objectives and is shown in Fig. 15.4H. In Table 15.11, with the increase in value of β and α, the score function shows the enhancing trend. At $\beta = 0.9$ and $\alpha = 0.9$, it approaches the maximum satisfactory degree, that is, 0.4256, 0.3557, 0.4253, and 0.5637 for $\widetilde{R}_2(2,1)$, $\widetilde{R}_1(4,3)$, $\widetilde{R}_3(3,5)$, and $\widetilde{R}_1(4,5)$, respectively. By tuning the parameters β and α, various sets of score functions for satisfaction level could be obtained effectively. Hence, intuitionistic fuzzy linguistic preference relations would be a good representative of priority structure among objectives according to the interest of decision maker(s). They would also be an effective and promising tool for assigning the preference when large numbers of objectives and goals have been dealt with simultaneously. The assignment of crisp weight (such as $w_o = 0.1, 0.2, \ldots, 1 | \sum_o^O w_o = 1$) to significant number objectives might be time-consuming and would involve more complexity to search for the best combination of crisp weight among different objectives or goals. Hence it would be tricky to assign the linguistic preferences among different objectives, which reduced the time and exempted from the best combination of crisp weight.

Table 15.11 Achievement degree of intuitionistic fuzzy linguistic preference relations at different feasibility degrees (β) and weight parameters (α).

Feasibility degree	Weight parameter	Intuitionistic fuzzy preference relations											
		Membership functions				Nonmembership functions				Score function achieved for intuitionistic fuzzy preference relations			
(β)	($1-\alpha$)	$\mu_{\tilde{R}_2(2,1)}$	$\mu_{\tilde{R}_1(4,3)}$	$\mu_{\tilde{R}_3(3,5)}$	$\mu_{\tilde{R}_1(4,5)}$	$\nu_{\tilde{R}_2(2,1)}$	$\nu_{\tilde{R}_1(4,3)}$	$\nu_{\tilde{R}_3(3,5)}$	$\nu_{\tilde{R}_1(4,5)}$	$S_{\tilde{R}_2(2,1)}$	$S_{\tilde{R}_1(4,3)}$	$S_{\tilde{R}_3(3,5)}$	$S_{\tilde{R}_1(4,5)}$
0.1	0.1	0.63	0.56	0.63	0.71	0.37	0.37	0.33	0.27	0.2634	0.1924	0.3091	0.4453
0.2	0.2	0.63	0.56	0.63	0.71	0.37	0.37	0.33	0.27	0.2637	0.1938	0.3037	0.4467
0.3	0.3	0.64	0.56	0.63	0.72	0.37	0.37	0.33	0.27	0.2721	0.1957	0.3052	0.4531
0.4	0.4	0.64	0.56	0.63	0.72	0.35	0.36	0.32	0.27	0.2932	0.2122	0.3127	0.4567
0.5	0.5	0.64	0.56	0.63	0.72	0.35	0.36	0.32	0.27	0.2981	0.2143	0.3149	0.4579
0.6	0.6	0.65	0.58	0.65	0.74	0.32	0.35	0.32	0.23	0.3381	0.2311	0.3351	0.5133
0.7	0.7	0.65	0.58	0.65	0.74	0.32	0.35	0.29	0.23	0.3393	0.2341	0.3691	0.5148
0.8	0.8	0.67	0.61	0.66	0.74	0.32	0.35	0.29	0.21	0.3547	0.2934	0.3712	0.5321
0.9	0.9	0.71	0.64	0.68	0.77	0.29	0.34	0.26	0.21	0.4256	0.3557	0.4253	0.5637

7. Conclusions

In this study, an effective modeling and optimization framework for the CLSC design has been formulated as a mixed-integer neutrosophic fuzzy programming problem under uncertainty. The proposed CLSC designed model comprises multiproduct, multiechelon, and multiobjective scenarios for the optimum allocation of new and end-of-use products. In the forward chain, five functional echelons have been designed, whereas the reverse chain consists of six potential echelons to deal with end-of-use and end-of-life products. The testing center has been depicted in the CLSC model, which ensures the promising useful life of the product. Multiple-conflicting objectives with a well-defined set of constraints reveal typical complexity under a fuzzy environment. To deal with fuzzy parameters and constraints, a fuzzy robust ranking function technique depending on a feasibility degree has been suggested. Fuzzy inequality constraints have been converted into their crisp forms by using the ranking function, whereas fuzzy equality constraints have been transformed into two equivalent auxiliary crisp inequalities. Then the obtained fresh model has been solved by using a modified NFPA which consists of independent indeterminacy thoughts in decision-making processes. A novel linguistic importance scheme named intuitionistic fuzzy preference relations among different objectives has been investigated. With the aid of the linear preference membership and nonmembership function, the marginal achievement of each linguistic preference has been attained. The overall satisfaction level has been represented by the convex combination of membership functions of each objective and score function of intuitionistic fuzzy preference relations. By tuning the feasibility degree and weight parameter, a different set of optimal solution results has been generated. A sensitivity analysis of the obtained results has been performed. Therefore, the presented CLSC modeling study under uncertainty may be helpful for practitioners and decision makers who are actively dedicated in the decision-making process of procurement, production, distribution, transportation, and management of end-of-use and end-of-life products in the CLSC network.

The propounded CLSC study has some limitations that can be addressed in future research. The CLSC network has been designed for a single period, but modeling with multiple periods is much needed in real-life scenarios. Incorporation of the triple bottom lines concept, which means sustainable development of the CLSC model comprising economic policies, environmental issues, and social concerns, would be a remarkable extension of the proposed model. Uncertainty among parameters due to randomness or other uncertain forms would be a significant enhancement of the discussed CLSC model. Various metaheuristic approaches may be applied to solve the proposed model as a future research scope.

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A Review of Fuzzy Soft Topological Spaces, Intuitionistic Fuzzy Soft Topological Spaces and Neutrosophic Soft Topological Spaces

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Abstract

The notion of fuzzy sets initiated to overcome the uncertainty of an object. Fuzzy topological space, intuitionistic fuzzy sets in topological structure space, vagueness in topological structure space, rough sets in topological space, theory of hesitancy and neutrosophic topological space, etc. are the extension of fuzzy sets. Soft set is a family of parameters which is also a set. Fuzzy soft topological space, intuitionistic fuzzy soft and neutrosophic soft topological space are obtained by incorporating soft sets with various topological structures. This motivates to write a review and study on various soft set concepts. This paper shows the detailed review of soft topological spaces in various sets like fuzzy, Intuitionistic fuzzy set and neutrosophy. Eventually, we compared some of the existing tools in the literature for easy understanding and exhibited their advantages and limitations.
Keywords: Soft sets, fuzzy soft topological space, intuitionistic fuzzy soft topological space, neutrosophic soft topological space.

1 Introduction

In the year 1999, Molodtsov[47] proposed the concept of soft sets (SS). This concept developed to overcome the difficulty to fix membership for each case. SS is a family of parameterization of the universe of discourse. Parameters may be numbers, meaningful words, sentences, etc. Anyone could define the parameterization for their convenient. This technique is very useful to model the uncertainties. Also, Molodtsov defined some basic operations and presented some uses of SS, such as stability and operations research, etc.

The first definition of soft spaces was introduced by the authors Shabir and Naz[70] and it is defined on the universe of discourse with a fixed set of parameters. Also they proved that a soft topological space provides a parameterized family of topological spaces. The researchers[14,21,52,74] are developed the concept of soft set theory.

Fuzzy set (FS) was introduced by Zadeh[81] in the year 1965, every element 'a' in A has a membership value, where A is mapped from the universe of discourse to [0, 1]. Later Chang[24] introduced the concept fuzzy topology in the literature, which satisfies the three axioms of topology and also Chang used the same notation in fuzzy topology as Zadeh used for FS. After few years, Lowen[39] defined fuzzy topology which is different from definition by Chang. Maji et al.[42] proposed the concept of fuzzy soft set (FSS) and defined some basic operations. Later, Tanay et al.[72] introduced fuzzy soft topological space (FSTS)and established the basic definitions of FSTS by incorporating the fuzzy topology and soft set. FSTS was applied in various ways say, game theory, analysis, etc. Fuzzy soft set in topological space further studied by Roy.[61,63] The authors[10,22,30,36,45,50,54] are successfully applied FSTS in real life.

FS failed to address the rejection of an object in the set. So Atanassov[12] proposed the theory where every object in a set has both acceptance and rejection with subject to the constraint that sum of acceptance and rejection should not exceed 1 and non-negative and that theory is called Intuitionistic fuzzy set theory. Intuitionistic fuzzy set A(a) is an generalization of fuzzy set where every element a in A is a subset of universal set have degree of membership and degree of non- membership and each function map from the universe of discourse to the interval [0, 1]. Researchers developed the theory by generalising it and got new result through extension.[6,15,16,57–59] Later, Maji et al.[43] introduced the notion of intuitionistic fuzzy soft set (IFSS). D. Coker[27] initiated the concept of IFSTS and followed by[34,53,76] developed the concept in decision making.

Atanassov failed to address the problem when indeterminacy occurs in the object. To address the difficulty, Smarandache[67,68] originated the concept called neutrosophic set. Every element in the neutrosophic set has truth, indeterminacy and falsity values respectively and which are maps from universe of discourse to [0, 1] with the constraint that truth, indeterminacy and falsity values should not exceed 3 and not less than zero under addition. Many complex problems in statistics, in graph theory when relationship between the object have acceptance, rejection and also indeterminacy, physics, image processing, networking and in decision making which can't be solved by existing classical methods. The generalisation of this notion also exist in the literature, namely neutrosophic soft topology, neutrosophic nano topology, neutrosophic nano ideals topology, neutrosophic support soft set,[56] neutrosophic soft supra topological spaces in various sets, etc. Maji et al.[41] presented the concepts of neutrosophic soft set. Maji[44] successfully applied the concept of neutrosophic soft sets (NSS) in pattern recognition, reasoning etc. Thereafter, Bera[18,19] initiated the concept of neutrosophic soft topological space (NSTS). The following authors[7,8,19,28,66] are developed NSTS. In this work, soft sets in various topological spaces are studied in detail. Advantages and limitations of different soft topological spaces are presented. Eventually comparison table of classical soft topological space, FSTS, IFSTS, NSTS are also presented.

2 Preliminaries

This section is a collection of definitions initiated by[12,18,27,40–43,64,67,72,81] for the development of soft sets in various uncertainty sets.

Definition 2.1. Fuzzy set A(a) of universal set X is defined by $A = \{(a, \mu_A(a)) : a \in X\}$, where μ_A represent the degree of membership and it is mapped from the universal set X to the unit interval [0, 1].

Definition 2.2. The pair (F, A) is called a FSS over X, where the mapping $F : A \to F(U)$, F(U) is the set of all fuzzy subsets of non-empty set X and A is a subset of the set of parameters E.

Definition 2.3. Let the pair (X, τ) be a FSTS and τ be a family of FSS over $X \neq \emptyset$. The pair (X, τ) is said to be a FSTS if it satisfying the following conditions: (i) $0_E, 1_E \in \tau$. (ii) If f_{A_1}, f_{A_2} in τ, then $f_{A_1} \wedge f_{A_2}$ in τ. (ii) If $(f_A)_i$ in τ, for all i in J, then union of $(f_A)_i$ in τ. Then τ is called a topology of fuzzy soft sets on X.

Definition 2.4. Let (X, τ) be FSTS and $f_A \in F(X, E)$. The closure of fuzzy soft set f_A is intersection of all fuzzy soft closed supersets of f_A.

Definition 2.5. Let (X, τ) be FSTS and $f_A \in F(X, E)$. The interior of fuzzy soft set f_A is union of all fuzzy soft open subsets of f_A.

Definition 2.6. Intuitionistic fuzzy set A(a) on the non-empty set X is defined by $A = \{(a, \mu_A(a), \nu_A(a)) : a \in X\}$, where μ_A denotes truth value and ν_A denote the falsehood and the map of truth value and falsehood from the universal set X to the interval [0, 1] and satisfying the constraint that sum of truth and falsehood value is lies between 0 and 1, for each $a \in X$.

Definition 2.7. Let X and E be the initial universe and set of all parameters respectively and A is a subset of the parameter set E. Let IF(U) be the set of all power set of X. If the mapping F from the set A to IF(U), then the pair (F, A) is said to be intuitionistic soft set over X.

Definition 2.8. Neutrosophic set A (a) on the non-empty set X is defined by $A = \{(a, \mu_A(a), \sigma_A(a), \nu_A(a)) : a \in X\}$, where μ_A represent the degree of membership, σ_A represent the degree of indeterminacy and ν_A represent the degree of non-membership and the map of membership , indeterminacy and non-membership from the universal set X to the interval [0, 1] and with the constraint $0 \leq \mu_A(a) + \sigma_A(a) + \nu_A(a) \leq 3$, for each $a \in X$.

Definition 2.9. Let X be a non-empty set and the parameters set be E. The power set of X is denoted by P(X) and is defined as the collection of all neutrosophic set. The pair (F, I) is called the NSS over X, where I is a subset of X and the map F from I to P(X).

Definition 2.10. Let X and E are the non-empty set and set of parameters respectively and NSTS (X, τ) is a subset of NSS (X, E) satisfying the following:

(1) Null and universal soft set are the members of τ. (2) Finite intersection of member of any finite sub collection of τ also in τ. (3) Arbitrary union of member of any sub collection of τ also a member of τ.

3 Soft Topological Spaces in Various Sets

3.1 Fuzzy soft set

In 2008, Yao[78] presented the concept of soft fuzzy set and this concept tested for the significance of existing soft fuzzy set. Lastly, FSS relations and soft fuzzy set relations are compared with some example. Cagman[23] modified the definition of FSS and studied the concept with some of its properties. Finally, fuzzy soft aggregation operator is defined for effective construction of decision process.

Generalized FSS introduced by Majumdar[46] in 2010. Some properties of generalized FSS and its applications are presented by Manjumdar. Tanay[72] first introduced the concept of FSTS to the literature. The authors also defined the notion of neighbourhood, family of neighborhood, interior and closure of FSS, Basis for FSTS and finally subspace of FSTS along with the some properties. Gunduz. C[31] defined interior, closure of FSTS. Further, Gunduz introduced open and closed sets in respect of FSS and continuous mapping in FSS, homeomorphism of FSTS. Characteristics of fuzzy soft topological structure also discussed. In 2011, Zhi Kong et al.[82] discussed FSS to present a real life problem with grey relation analysis theory. The result is verified with some cases. Mahanta[48] introduced and studied fuzzy soft point and its neighbourhood in a FSTS. Closure and interior of FSS are studied and investigated separation axioms and connectedness of FSTS. Abd El – Latif[4] developed and studied the concept of pre-connected, pre-separated, pre-soft subspace of FSTS. Generally, Pre - disconnectedness of FSTS is not traditional property proved by Abd El – Latif.[4] In 2012, Varol[73] brought the notion of fuzzy soft continuity and projection mapping of FSTS. Simsekler[67] defined fuzzy soft open sets and fuzzy soft closed sets in FSTS and also fuzzy Q-neighbourhood of fuzzy soft points are defined. Roy et al.[61,63] defined the concept of accumulation point using Q-neighbourhood and also proved that separation axioms exists using Q-neighbourhood in FSS.

Yang et al.[77] combined the concept of multi-fuzzy set and soft set to produce a new result called multi-fuzzy soft set. Also defined some theoretic operations say, union, intersection and complementary. Yang et al.[77] developed an algorithm using multi-fuzzy soft set. Eventually using the proposed algorithm, decision making problem is analysed. Roy and Maji[62] analysed the decision making problems using fuzzy soft sets. They construct an algorithm for selecting object from universe of discourse by considering maximum value among the score using score function. Cetkin[25] established the concept of continuous mappings in FSTS and presented the idea of anti-chain and isomorphism to FSTS.

In 2015, Kandil[37] introduced the concept of semi connected set, semi s-connected set, semi separated set in FSTS. Sabir Hussain[64] defined the soft pre-open set, soft alpha-open set in FSS and studied soft neighbourhood at fuzzy soft points. Also introduced soft regular open set and studied further. Finally, the relationships between the above proposed concepts are presented. Pre-open, pre-closed set of FSTS introduced by Abd El-Latif[1,3] and studied some properties of pre-regular, pre-normal space of FSTS. Fuzzy α-connected set, fuzzy α-separated set, fuzzy α-S-connected set in FSTS established by A.M. Abd El-Latif.[2]

A. Kandil et al.[36] defined fuzzy soft point based on equivalence classes in the year 2015 and described that Universal fuzzy soft set can be written as the union of disjoint connected component. G. Kalpana et al.[35] introduced fuzzy soft r-open and fuzzy soft r-closed mappings, fuzzy soft r-closure, fuzzy soft r-interior, fuzzy soft r-continuous mapping through fuzzy soft set. Abd El-Latif[3] initiated the notion of β-open soft sets and β-separation axioms in FSTS and established the properties of β-closure and β – regular, β-normal space in FSTS.

3.2 Intuitionistic Fuzzy Soft Set

Here, we present the initialization, extensions and generalization of intuitionistic fuzzy soft set in topological structure. Yang[76] originated the concept of interval - valued IFSS, defined the set theoretic operations and finally decision making problem solved by adopting existing algorithm. Mukherjee[49] proposed and studied a new type of sequence of intuitionistic fuzzy soft multi sets and some of its properties are investigated. Also the increasing, decreasing and convergent sequences of intuitionistic fuzzy soft multi- topological spaces are introduced by Mukherjee.[49] Finally, cluster intuitionistic fuzzy soft multi topological space and their properties are studied. In 2010, Xu[74] presented the concept of IFSS by merging K.Atanassov intuitionistic fuzzy set and soft set. Developed some basic operations and applying this tool to target the type recognition problem. Jiang et al.[33] combined the two classical methods viz. soft set and interval-valued intuitionistic fuzzy set and produced a new result called interval-valued IFSS. Union, intersection and complement of interval-valued IFSS defined and established some basic properties.

In 2012 Yin et al.[79] introduced further the concept of IFSS. In particular, theoretical operations such as union, intersection and complement, etc. are introduced. Mapping on IFSS introduced and their basic properties also presented. Li et al.[40] proposed the novel notion called IFSTS in the year 2013. The author also defined the interior, closure, base, relative complement and absolute IFSS and IFSTS. Some properties of IFSTS also presented.

In 2013, Agarwal et al.[5] developed the concept of generalized IFSS and this developed a generalized parameter to pool the intuitionistic fuzzy numbers. The author has developed three different algorithms mainly for decision making. One is for in medical diagnosis to compare the intuitionistic fuzzy numbers and remaining for measure the similarity, if any in selecting the supplier. Kumud Borgohain[38] studied IFSTS and defined intuitionistic fuzzy soft separation axioms, normal space and finally completely normal space of IFSTS. Osmanoglu et al.[55] introduced intuitionistic fuzzy soft finer and coarser topological space, Intuitionistic fuzzy soft discrete topology and intuitionistic fuzzy soft indiscrete topology. Further, soft points and complement of intuitionistic fuzzy soft points and separation axioms of the same introduced and their properties also studied. Cetkin[26] introduced the definition of closure intuitionistic supra fuzzy soft topological space.

In 2014, Bayramov. S[13] introduced the basic definitions of IFSTS namely, null and absolute IFSTS, interior and closure, associated closure of IFSTS. Some basic properties also investigated. Mukherjee[51] established the notion of intuitionistic fuzzy soft multi topological space for the parameterized family and also established the basic structure of intuitionistic fuzzy soft multi topological structure.

Shuker Mahmood[71] studied and established soft b-closed, soft b-continuous mapping, soft b- closed disconnectedness of IFSTS. In 2017, Yogalakshmi[80] initiated the concept of various compactness of IFSTS, namely almost compact, nearly compact, etc and also studied intuitionistic soft fuzzy filter and intuitionistic soft fuzzy prime filter of IFSTS.

3.3 Neutrosophic soft set

This section contains the overview of various studies on NSS. In 2012, Maji[41] defined the concept of NSS by combining soft set and neutrosophic set. Some basic operations of NSS, such as union, intersection and complement are defined and developed some properties of NSS. In the year 2013, Said Broumi[20] presented the concept of generalized NSS with basic definitions and properties of generalized NSS. Deli[29] defined the notion of relation on NSS. The composition of NSS is used to compose two different NSS. Deli[29] examined the following concepts, namely equivalence relation, equivalence class and quotient of NSS. Deli also analyzed the decision making problem using NSS relation. Arockiarani[9] defined a distance measure and score function to present a decision making problem using the existing tool called NSS.

In 2017, Al-Quran[11] introduced the notion of neutrosophic vague soft set which is an extended concept of classical soft set. Some basic operations and properties are defined and studied and at the end of the work, presented the decision making problem using the proposed concept to show the effectiveness. Parimala et al.[56] introduced an algorithm to analyze the medical diagnosis problem using interval-valued FSTS. In their work, some basic theoretic operations are also investigated.

Bera[18] introduced the concept of NSTS. In the introduction paper, the authors are also defined interior, closure, base for NSTS, subspace of NSTS and regular NSTS. Finally some properties of NSTS and separation axioms with different characteristics are studied and investigated.

In 2018 Bera[17] introduced neutrosophic soft connected and compact topological space along with some properties. Finally the concept of continuous mapping on NSTS introduced and studied. Gunduz Aras et al.[32] established the definition of NSS and introduced the neutrosophic soft point. Finally separation axioms and subspace of NSTS are studied in detail. Parimala et al.[60] proposed a new concept by incorporating NSS with hesitancy degree, which is exclusively for finding the residual of NSS.

4 Advantages and Limitations

Advantages and limitation of classical topological space and other topological spaces, such as FSTS, IFSTS, NSTS are presented here.

Types	Advantages	Limitations
General topology	It's a classical method and it is basic for all other topological space.	We could not apply the classical approach for uncertainties and for many real life fields.
Fuzzy topology	In fuzzy topology, every element has membership grade which lies between [0, 1].	Rejection part of membership does not exist in the fuzzy topology.
Intuitionistic fuzzy topology	Every element in the set has truth and falsehood value.	It's difficult to apply when some element have indeterminacy or indeterminate form.
Neutrosophic topology	Every element in the non-empty set has acceptance, rejection and indeterminacy value. So all variables in the universe of discourse have value between [0,1].	Residual part may lead to some obvious errors in the solution.
Fuzzy soft topology	A non-empty set can be written as disjoint union of parameters set. One can separate the characteristic from the universal set and investigate according to the need of problem.	Non-acceptance of an element in the parameter does not consider.
Intuitionistic fuzzy soft topology	Every element in the parameterization has possibility of acceptance and possibility of non-acceptance value.	Omitting the possibility of neutrality.
Neutrosophic soft topology	Here we consider acceptance rate, non-acceptance rate and neutrality rate of all elements in the parameter.	Accuracy may affect if the residual rate is high.

The following table emphasize the comparison of various tools which we discussed in this overview.

Sets	Image	Pre-Image	Uncertainty	Truth Value of Parameter	False value of Parameter	Indeterminacy of parameter.
Classical sets	Universal set	Integer Set.	-	-	-	-
Soft Topology	Initial Universe	Power set whose ranges from closed interval 0 to 1	-	-	-	-
Fuzzy Soft Topology	Initial Universe	Power set whose ranges from closed interval 0 to 1	Present	Present	-	-
Intuitionistic Soft Topological Space	Initial Universe	Power set whose ranges from closed interval 0 to 1	Present	Present	Present	-
Neutrosophic soft topological space	Initial Universe	Power set whose ranges from closed interval 0 to 1	Present	Present	Present	present

5 Conclusions

Topological space has several applications in mathematics and in other fields like operations research, physics, data science, etc. But sometimes applying the concept of topology for real life application is difficult, because of uncertainties, inconsistent, incomplete information of the element. Fuzzy soft topological space introduced to overcome the difficulty in classical set which deals uncertainty of the object and intuitionistic fuzzy soft topological space established to solve some problem which encounter in fuzzy soft topology. Some cases, object has indeterminacy value, for those cases the previous tools can't be used. So Neutrosophic set has been introduced to deal the uncertainty, incomplete and inconsistent. This paper is thorough study of all these tools. Advantages and limitations of all existing tools are discussed.

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A short remark on Bong Han duct system (PVS) as a Neutrosophic bridge between Eastern and Western Medicine paradigms

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Abstract

In a previous paper in this journal (IJNS), it is mentioned about a possible approach of "curemony" as a middle way in order to reconcile Eastern and Western's paradigms of medicine [1]. Although it is known in literature that there are some attempts to reconcile between Eastern and Western medicine paradigms, known as "integrative medicine," here a new viewpoint is submitted, i.e. Bong Han duct system (PVS), which is a proof of Meridian system, can be a bridge between those two medicine paradigms in neutrosophic sense. This can be considered as a Neutrosophic Logic way to bridge or reconcile the age-old debates over the Western and Eastern approach to medicine. It is also hoped that there will be further research in this direction, especially to clarify the distinction between Pasteur's germ theory and Bechamp's microzyma theory. More research is obviously recommended. Motivation of this paper: to prove that Neutrosophic Logic offers a reconciliation towards better dialogue between Western and Eastern medicine systems. Novelty aspect: it is discussed here how Bong Han Duct system offers a proven and observable way to Meridian system, which in turn it can be a good start to begin meaningful dialogue between Western and Eastern systems.

Keywords : Pasteur, microzyma, Bechamp theory, meridian system, Bong Han Kim, Bong Han duct system, neutrosophic logic

1.Introduction

In the light of recent advancements on the use of Neutrosophic Logic in various branches of science and mathematics, this paper discuss possible application in medicine philosophy. See for instance [13-19].

This paper is inspired partly by the movie, Jewel in the Palace (Dae Jang Geum). One of these authors (VC) has a younger brother who likes to watch that movie. He already completed watching the entire series (more than 70 episodes) more than three times. According to a good documentary on that movie [11]:

A history book courageous woman is reawakened in a hit TV dramatization. In 1392, the Joseon Dynasty appeared. The rulers of Joseon would lead the Korean landmass until the administration fell, to be supplanted by a Japanese provincial system, in 1910. All things considered, Joseon's heritage suffers: It was one of the world's longest-running imperial administrations. In the "Joseon-Wangjo-Sillok" - "The Annals of the Joseon Dynasty;" the official record of the realm - a lady named "Daejanggeum" is referenced. She lived during the rule of King Jungjong (1506~1544), and the archives disclose to us that she had been a low-positioning court woman who picked up the ruler's trust and was elevated to the most noteworthy positioned woman in the kitchen, and furthermore to regal doctor. In one notice in the archives, the ruler states, "I have nearly recuperated from the sickness of a couple of months. So I should offer honors to the individuals who put forth bunches of attempts to fix me. Give the imperial doctors and euinyeo (female associate) Daejanggeum blessings."

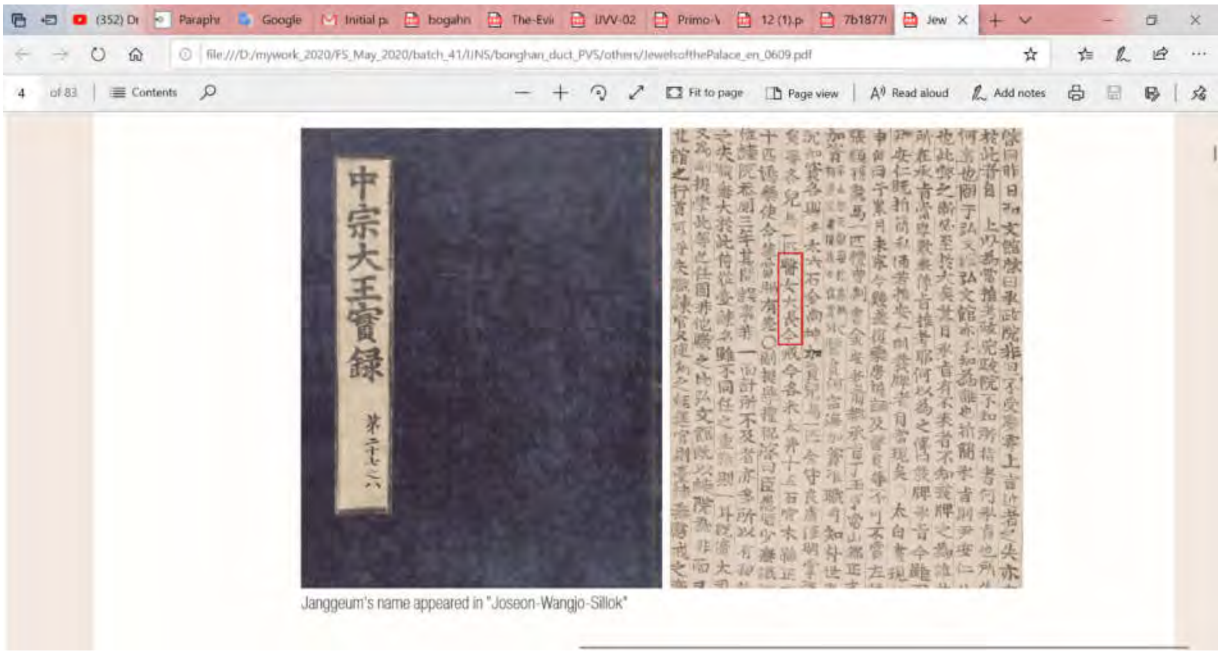

Figure 1. Jang Geum name was recorded in the "Joseon-Wangjo-Sillok" - "The Annals of the Joseon Dynasty." After Kang Min Su [11]

What is more interesting to these authors, is not only the depiction of royal palace at the time, but also the use of royal cuisine as medication, beside the use of acupuncture methods.[11]

Now it seems obvious for Western scholars to pause at this point and ask: "What? Acupuncture? Are you joking?"

This short review paper is discussing that approach: whether it is possible to reconcile both Eastern and Western medicine paradigms from the view point of Bong Han Kim's duct system (PVS) and its relation to Bechamp's microzyma.

As it is brought up in [1], it is notable by most medication experts, that Western way to deal with medication depends on "assaulting" an infection, individually. This is called germ hypothesis: one remedy for one ailment (Pasteur). On the contrary side, Eastern medication is situated specifically on old knowledge of restoring the parity (harmonious functions) of the body, at the end of the day: to blend our body and our live with nature. In spite of the fact that those two methodologies in medication and social insurance have caused such a large number of contentions and false impressions, really it is conceivable to do an exchange between them. From Neutrosophic Logic's point of view, a goal to the above clashing ideal models can be found in creating novel methodologies which acknowledge the two conventions in medication, or it is conceivable to call such a methodology: "curemony," for example by simultaneously relieving an infection and reestablishing harmony and returning concordance in one's body-mind-soul all in all.

Now it is known that one of the objections by Western scholars about the Eastern medicine (based on meridian points) is the unobservability of meridian vascular/duct system. This makes meridian system neglected in almost all textbooks taught in Western medicine schools. Therefore, here a new viewpoint is submitted, i.e. Bong Han duct system (PVS), which is a proof of Meridian system, can be a bridge between those two medicine paradigms in neutrosophic sense. This can be considered as a Neutrosophic Logic way to bridge or reconcile the age old debates over the Western and Eastern approach to medicine.

It would be a lot easier to merge both the eastern (ancient) and the modern western curative system in terms of neutrosophy.These neutrosophic intermediates will help further to boost dialogues between those Western and Eastern system and their useful information. This neutrosophic intermediator is actually dealing with conscious of both non-matter and matter in terms of ancients and modern techniques.

2. Introduction to Bong Han duct system

Nonetheless, in literature it is recorded that Bong Han Kim is a Professor in Biology in Korea. Around 1950-1960 he found the vessel which is a "duct" to known Eastern Meridian system, which is already known in acupuncture medicine system. Therefore it seems like a bridge between Western and Eastern medicine paradigms. As it is mentioned in previous paper [1], this paper will discuss how those paradigms can be reconciled in Neutrosophic Logic, using a degree of Western medicine and a degree of Eastern medicine, as the neutral part of neutrosophy. To us, Bong Han duct system is a good way to start a healthy and meaningful dialogue between those two paradigms in medicine.

As Vitaly Vodanoy wrote, which can be rephrased as follows:

> "In the 1960's Bong Han Kim found and described another vascular framework. He had the option to separate it unmistakably from vascular blood and lymph frameworks by the utilization of an assortment of techniques, which were accessible to him in the mid-twentieth century. He gave nitty gritty portrayal of the framework and made thorough graphs and photos in his distributions. He showed that this framework is made out of hubs and vessels, and it was answerable for tissue recovery. In any case, he didn't reveal in subtleties his

> techniques. Thus, his outcomes are moderately dark from the vantage purpose of contemporary researchers. The stains that Kim utilized had been idealized and being used for over 100 years. In this manner, the names of the stains coordinated to the unequivocal conventions for the use with the specific cells or particles. Generally, it was not typically important to portray the strategy utilized except if it is altogether strayed from the first technique."[9]

Although his method was almost forgotten until recently, it has been recovered again in the past decade. It is clear therefore, that Bonghan Kim's work, who essentially (and without being aware of the work previously done by Bechamp) discovered that what we call the 'Meridian System' (known as the Kyungrak System in the Korean tongue) which exists in the body as an actual third anatomical vascular system, comprised of ducts, ductules, corpuscles, and a unique type of fluid, the contents of which tie directly back to Bechamp's own discoveries (work is still being done today on the mapping out of this anatomical system, as it is far more extensive than the old Oriental texts gave it credit.) See [4].

Remark on terminology:

> "In a matter of seconds before the primary International Symposium on Primo Vascular System, which was held in Jecheon, Korea during September 17–18, 2010, Dr. Kwang-Sup Soh, recommended that it is critical to concur upon a solitary phrasing for the Bonghan framework. It was concurred that following terms would be embraced: Bong-Han System (BHS) - Primo Vascular System (PVS); Bonghan Duct (BHD)- Primo Vessel (PV); Bonghan Corpuscle (BHC)- Primo Node (PN); Bonghan Ductule - P-Subvessel; Bonghan Liquor-Primo Fluid (P-liquid); Sanalp-Microcell"[9].

Now in the next section, it will be discussed virus research, especially at their beginning.

Hidden the introduction of virology is a conviction that infections are monomorphism, they are fixed species, unchangeable; that each neurotic kind produces (typically) just a single explicit illness; that microforms never emerge endogenously, i.e., have supreme source with the host. Thus the worldview prompts conviction called "germ hypothesis" of Pasteur: for example one remedy for one disease.[6-7]

Bechamp recorded standard as the premise of another hypothesis about "infections." Briefly, this guideline holds that in every single living life form are organically indestructible anatomical components, which he called microzymas. They are freely living sorted out matures, equipped for creating compounds and fit for advancement into increasingly complex microforms, for example, microbes. Bechamp's proposition is that infection is a state of one's interior condition (landscape); that ailment (and its indications) are "conceived of us and in us"; and that malady isn't created by an assault of microentities yet considers forward their endogenous cause. [8]

All things considered, it is realized that Pasteur duplicated whatever he discovered Bechamp thoughts would fit in his own hypothesis. Consequently, Bechamp was unmistakably increasingly unique researcher contrasted with Pasteur.

3. A re-interpretation of diseases and virus from Bechamp's theory

This section begin by citing [4], which can be paraphrased as follows:

> "Through a cautious perception of the wonders of the thickening of the blood just as the procedure of maturation; and as a methods for all the more accurately deciphering the basic idea of these marvels; Bechamp straightforwardly saw that there exist a layers of subcellular, miniaturized scale natural living structures known as 'microzymas', a word which when interpreted signifies 'minor ages'. These structures were alluded to without anyone else and by other people (who came later, and mentioned a similar objective facts) as some type of 'atomic granulations' (more on this beneath). These microzyma are littler in size than some other known types of small scale natural life, and fill in as the base establishment for the development of every other type of such life."

Moreover, on a more recent setting, see Andrew Kaufmann's report on WHO's early investigation of the corona virus, before it was declared globally as an epidemic.[1]

According to Dr. Andrew Kaufman's report, a "virus" as observed is actually an exosome. That is not impossible. Even if you want to be more assertive. It's not just the PCR test that is inaccurate. So the so-called virus is indeed questionable. Because it relates to the germ theory of Pasteur, meaning each disease will need one type of medicine [1][2].

That's not right. Pasteur's theory draws a lot from the real expert at the time: Bechamp.[4]

In essence, according to Bechamp, the source of the disease is most likely to be endogenous. Meaning from within the body when adjusting itself to the environment.

What is interesting to ask here is what kind of the changes in the environment that triggers the emergence of symptoms such as excessive thirs? Actually, it is known as one of the symptoms known for electromagnetic radiation. Therefore, it is no surprise that there are some allegations by experts: severe radiation disturbances arise in Wuhan and Italy and also the USA because of they are the locations where the massive 5G network has begun to be installed (see also Firstenberg's report [5]).

But this short paper is not intended to discuss more detailed about relation between 5G and covid-19, so this problem will be left to others to take up this matter and investigate further.

4. Concluding remarks

This paper continued our previous article, where possible approach of "curemony" is discussed as a middle Neutrosophic way in order to reconcile Eastern and Western's paradigms of medicine [1]. Although it is known in literature, that there are some attempts to reconcile between Eastern and Western medicine paradigms, known as "integrative medicine," here it is submitted a viewpoint that Bong Han duct system (PVS) which is a proof of meridian system, can be a neutrosophic bridge between those two medicine paradigms.

Here a new viewpoint is submitted, i.e. Bong Han duct system (PVS), which is a proof of Meridian system, can be a bridge between those two medicine paradigms in neutrosophic sense. This can be considered as a Neutrosophic Logic way to bridge or reconcile the age old debates over the Western and Eastern approach to medicine.

It would be a lot easier to merge both the eastern (ancient) and the modern western curative system in terms of neutrosophy.These neutrosophic intermediates will help further to boost dialogues between those Western and Eastern system and their useful information. This neutrosophic intermediator is actually dealing with conscious of both non-matter and matter in terms of ancients and modern techniques.

As mentioned in our previous paper [1], it is also discussed how those paradigms can be reconciled in Neutrosophic Logic. To us, Bong Han duct system (PVS) is a good way to start a healthy and meaningful dialogue between those two paradigms in medicine.

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A New Group Decision Making Method with Distributed Indeterminacy Form Under Neutrosophic Environment: An Introduction to Neutrosophic Social Choice Theory

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ABSTRACT We present a novel social choice theory based multi-criteria decision making method under neutrosophic environment and a new form of truth representation of neutrosophic theory called Distributed Indeterminacy Form (DIF). Our hybrid method consists of classical methods and an aggregation operator used in social choice theory. In addition to this, we also use DIF function to provide a more sensitive indeterminacy approach towards accuracy functions. We also consider reciprocal property for all individuals. This provides, as in intuitionistic fuzzy decision making theory, a consistent decision making for each individual. The solution approach presented in this paper in group decision making is treated under neutrosophic individual preference relations. These new approaches seem to be more consistent with natural human behaviour, hence should be more plausible and feasible. Moreover, the use of a similar approach to develop some *deeper soft* degrees of consensus is outlined. Finally, we give a Python implementation of our work in the Appendix section.

INDEX TERMS Neutrosophic logic, group decision making, neutrosophic preference relations, distributed indeterminacy form, social choice theory, neutrosophic social choice theory.

I. INTRODUCTION

In most cases, it is intricate for decision-makers to accurately reveal a preference when solving multi-criteria decision-making (MCDM) problems with imprecise, vague or incomplete information. Under these conditions, fuzzy sets (fs) [1], where the membership degree is represented by a real number in [0, 1], are viewed as a strong mechanism method for solving MCDM problems [2], as well as reasoning approximation and pattern recognition problems. However, fs cannot cope with particular situations where it is not easy to define the membership degree using a specific value. In order to obviate the absence of knowledge of non-membership degrees, Atanassov [3] introduced intuitionistic fuzzy sets (IFS), an extension of fs. IFS have been widely used in the solution of some significant MCDM problems [4]–[6], including multigranulation [7]–[12], neural networks [13], [14], and medical diagnosis problems [15]. Smarandache [16] introduced neutrosophic logic and neutrosophic sets (NS) and Rivieccio [17] later raised concern about that an NS is a set where each element of the universe has a degree of truth, indeterminacy and falsity and it lies within $]^{-}0, 1^{+}[$, i.e. the non-standard unit interval. Clearly this is an extension of the standard interval [0, 1]. Furthermore, the uncertainty presented here, i.e. the indeterminacy factor, is dependent on the truth and falsity values, whereas the incorporated uncertainty is dependent on the degrees of belongingness and non-belongingness of IFS [18]. Recent studies show that neutrosophy can in fact be used in many applications. Ye [21]–[34], Lui and Wang [35], Lui *et al.* [36], Liu and Li [37], Liu and Shi [38], [39], Liu and Tang [40], Şahin and Liu [41], Chi and Liu [41], Biswas *et al.* [41], Biswas *et al.* [44]–[49], Monda and Pramanik [50]–[54], Peng *et al.* [55], Zhang *et al.* [56], [57], Peng *et al.* [58], Zhang *et al.* [59], [60], Tian *et al.* [61], [62], Ji *et al.* [63]–[65], Peng and Dai [66], Peng *et al.* [67], Peng and Liu [68], Peng and Dai [69], and Blin and Whinston [70] are some of the significant works on and introduced innovative methods on decision making under fuzzy and neutrosophic environments.

In this study, we propose to distribute the indeterminacy on truth and falsity to be aligned with real life applications and to take into consideration such situations in which uncertainty in social choices have an effective role in truth and falsity. We determine a rational social choice solely by the preferences of individuals in a society. A rational choice is possible only if every individual in the society is rational. Social choice theory investigates solutions to the problem of making a collective decision on a fair and democratic ground. The main purpose and subject area of social choice theory is to study the decision making problem for collectives to make a collective decision in a democratic manner. Of course our main concern will be to devise a method to make a cumulative decision rather than judging how fair the decisions of individuals are. The collective decision will manifest itself in neutrosophic values that the individuals give assignments to the preferences. Every individual is assumed to be able to assign to every preference some neutrosophic comparison value as pairs. We benefit from fuzzy and intuitionistic fuzzy social choice in solving the decision problems concerning neutrosophic social choice. Some well known works in fuzzy social choice and fuzzy decision making can be found in [70]–[75]. As for the intuitionistic fuzzy choice, we refer the reader to [76]–[78]. The advantage of our method is that we take care of Indeterminacy as well into neutrosophic social choice, while the previous methods involving fuzzy and intuitionistic fuzzy into social choice ignored the indeterminacy ? which is not accurate. This paper is about not only a classical decision making paper but also has a paper that considers decision making, truth maker theory and a new accuracy function interpretation (DIF). Addition to these, on the other hand, social choice theory under neutrosophic environment is studied for the first time, so we cannot compare other existing methods to the method in our paper. The comparison method is to cite some papers related to decision making. Many of the computational social choice theories that have been studied are based on rational individuals and their consistent preferences. Knowing the fact that the consistency of these pairwise comparisons forms the main theme, such theories devise appropriate methods based on the winner of the consensus of the group or based on an ordering of the preferences with respect to a priority as a result of voting of each individual. In any social choice, the consensus winner is defined as the choice of the dominant individual or the collective decision of rational individuals. The goal is to determine the best preference picked by the group. For the fuzzy solutions of finding a consensus, we refer the reader to Kacprzyk *et al.* [79]. We introduce a mathematical model for determining a consensus winner as a result of a collective decision, and in case of otherwise, we present a model which orders the preferences with respect to their weights. We also give an example in the last part of the paper to explain the model better. Compared with fuzzy and intuitionistic social choice theories, our model extends the social choice theory to neutrosophic based social choice theory in solving practical decision problems and present a richer language discourse.

II. FUNDAMENTAL DEFINITIONS

In classical set (cs) theory, an element either belongs to a set or not. The membership of elements in a set is interpreted in binary terms according to a divalent case. In fuzzy set theory, introduced by Zadeh [1], a gradual assessment of the membership of elements in a set is permitted by a membership function which takes values in the real unit interval [0, 1]. In fuzzy set theory, classical divalent sets are usually called crisp sets. Fuzzy set theory is a generalization of the classical set theory. IFS are sets whose elements have degrees of membership and non-membership. IFS have been introduced by Atanassov [3] as an extension of the notion of fuzzy set, which itself extends the classical notion of a set. Neutrosophic set theory is a generalization of IFS, CS, FS, paraconsistent set, dialetheist set, paradoxist set, tautological set based on Neutrosophy [16]. An element $x(T, I, F)$ belongs to the set in the following way: it is true in the set with a degree of $t \in [0, 1]$, indeterminate with a degree of $i \in [0, 1]$, and it is false with a degree of $f \in [0, 1]$.

We will now give some definitions of the fundamental concepts related to our study.

Definition 1 [1]: Given a universal set U and a generic element, denoted by x, a *fuzzy set* X in U is a set of ordered pairs defined as

$X = \{(x, \mu_X(x))|x \in U\}$, where $\mu_X : U \longmapsto [0, 1]$ is called the *membership function* of A and $\mu_X(x)$ is the *degree of membership* of the element x in X.

Definition 2 [3]: An *intuitionistic fuzzy set* X over a universe of discourse U is represented as

$X = \{(x, \mu_X(x), \nu_X(x))|x \in U\}$, where $\mu_X : U \longmapsto [0, 1]$ and $\nu_X : U \longmapsto [0, 1]$ are called respectively the *membership function* of A and the *non-membership function* of A for x in X. The *degree of non-membership* of the element x in X is defined as $\mu_X(x) = 1 - \nu_X(x)$.

Definition 3 [16], [19]: Let U be a universe of discourse. A *neutrosophic set* is defined as

$$N = \{(x, T(x), I(x), F(x)) : x \in U\},$$

which is identified by a *truth-membership function* $T_N : U \longmapsto]0^-, 1^+[$, *indeterminacy-membership function* $I_N : U \longmapsto]0^-, 1^+[$ and *falsity-membership function* $F_N : U \longmapsto]0^-, 1^+[$.

Definition 4 [16], [19]: Let U be a universe of discourse. A *single valued neutrosophic set* is defined as

$$N = \{(x, T(x), I(x), F(x)) : x \in U\},$$

which is identified by *a truth-membership function* $T_N : U \longmapsto [0, 1]$, *indeterminacy-membership function* $I_N : U \longmapsto [0, 1]$ and *falsity-membership function* $F_N : U \longmapsto [0, 1]$ with $0 \leq T_N(x) + I_N(x) + F_N(x) \leq 3$. A *single-valued neutrosophic number* (SVNN) is denoted by $a = (T, I, F)$.

Definition 5 [20]: Let a be a single-valued neutrosophic number. An *accuracy function* H of a single-valued neutrosophic number is represented as follows.

$$H(a) = \frac{1 + T_a - I_a(1 - T_a) - F_a(1 - I_a))}{2}, \quad (1)$$

where for all a, $H(a) \in [0, 1]$. H is an order relation which gives an accuracy score of information of a. If $H(a_1) = H(a_2)$, then $a_1 = a_2$, that is, they have the same information. If $H(a_1) < H(a_2)$, then a_2 is larger than a_1.

III. ACCURACY FUNCTION AND DISTRIBUTED INDETERMINACY FORM

For a neutrosophic value, the accuracy function H is calculated by the values T, I and F. However, in the process of making a decision, such independent values may not yield results consistent with the decision-making process on objects. Suppose, one has truth, falsity and indeterminacy values applied on a concept. We cannot speak about truth by ignoring indeterminacy. The reason is that we make a decision on the basis of including indeterminacy and the truth-maker gives the values by taking into account the indeterminacy. Sorensen [80]–[82] who published many papers on truth-maker theory, buries the theory of indeterminacy in the truth-maker theory. By a similar approach, we desire to calculate the the accuracy function dependent on T and F, taking the indeterminacy into consideration. The direct application of this idea to neutrosophic decision making helps us to approximate the outcomes with a better precision by distributing the indeterminacy on neutrosophic values. Let H be an accuracy function. This time we reflect the indeterminacy value on the truth and falsity values in the following way: Let $a = (T_a, I_a, F_a)$ be a single valued neutrosophic number with truth value T_a, indeterminacy value I_a, and falsity value F_a. *Distributed Indeterminacy Form* (DIF) of a is defined as $a_{DIF} = (T_a - T_aI_a, 0, F_a - F_aI_a)$. Here, we distribute indeterminacy effect on truth and falsity. In other words, we decrease the power of truth and falsity in proportion to the magnitude of indeterminacy. Our aim here is to determine how the value of truth and falsity is affected by the degree of growth of indeterminacy. Consider the following case for the accuracy function H. Despite that $H(0.5, 0.5, 0.6) = 0.475$, we have that $H(0.5, 0.6, 0.6) = 0.48$. In other words, even though the precision should have been decreased when the indeterminacy increases, we observe the opposite here. This, at first might, may seem contradictory but the case will become clear in a moment. So DIF gives us a method to keep a neutrosophic number as small as possible in the ordering of the preferences in proportional to the increment of the indeterminacy value, provided that the truth or falsity values are fixed.

A. SELF COMPARISON

All comparisons on the same alternative should be assigned a balanced value by rational individuals. The values 0.5, (0.5, 0.5), and (0.5, 0.5, 0.5) are assigned respectively for self-comparison by individuals in fuzzy set, intuitionistic fuzzy set and neutrosophic set. Assigned self comparison of a neutrosophic value a is (0.5, 0.5, 0.5) and outcome of this number under H function is naturally $H(a) = 0.5$. The DIF of this value is $a_{DIF} = (0.25, 0, 0.25)$ and $H(0.25, 0, 0.25) = 0.375$. This in turn gives us a result quite different from self-comparison. One of the most important reasons that we introduce the distributed indeterminacy concept is the effect of indeterminacy over the other two values, i.e truth and falsity. Moreover, we would like to see this effect as a rational assignment in the self-comparison process, so we would like to use the triplet (0.5, 0, 0.5) instead of (0, 5, 0.5, 0.5). As it can be seen, we pull the indeterminacy factor down to zero. Moreover, the DIF of (0.5, 0, 0.5) is equal to itself, that is (0.5, 0, 0.5). Furthermore, the image of (0.5, 0, 0.5) under the function H takes the value 0.5, which is just the appropriate value for the self-comparison process.

IV. RECIPROCAL PROPERTY AND HESITATION FUNCTION

In this section, we will the define reciprocal property and hesitation function for neutrosophy theory by reviewing the properties and the functions in fuzzy and intuitionistic theories.

A. RECIPROCAL PROPERTY IN FUZZY THEORY

Reference [83] A *fuzzy preference relation* $R = (r_{ij})$ on a finite set of alternatives X is a relation in $X \times X$ which is characterised by the membership function $\mu_R : X \times X \longmapsto [0, 1]$. Pairwise comparisons concentrate on two alternatives at a time which enable individuals when giving their preferences. If an individual prefers an alternative x_i to another alternative x_j, then she/he should not simultaneously prefers x_j to x_i. Then, the numerical representation of the comparison of two alternatives is denoted by a reciprocal preference relation R as follows:

$$\begin{aligned} r_{ij} = 1 &\quad \Leftrightarrow \quad x_i \succ x_j \\ r_{ij} = 0 &\quad \Leftrightarrow \quad x_j \succ x_i \\ r_{ij} = 0.5 &\quad \Leftrightarrow \quad x_j \sim x_i \end{aligned}$$

In fuzzy social choice theory, we also see binary crisp preference relations or [0, 1]-valued (fuzzy) preference relations. $x_{ij} = 1$ shows the absolute degree of preference for x_i over x_j. A definite preference for x_i over x_j is $r_{ij} \in (0.5, 1)$. Indifference between x_i and x_j is $r_{ij} = 0.5$. Reciprocal [0, 1]-valued relations ($R = (r_{ij}; \forall i, j : 0 \leq r_{ij} \leq 1, r_{ij} + r_{ji} = 1$) are widely used in fuzzy set theory for representing preferences.

B. RECIPROCAL PROPERTY AND HESITATION FUNCTION IN INTUITIONISTIC FUZZY THEORY

[76] An *intuitionistic fuzzy preference relation* P on a finite set of alternatives $X = \{x_1, \ldots, x_n\}$ is characterised by a membership function $\mu_P : X \times X \longrightarrow [0, 1]$ and a non-membership function $\nu_P : X \times X \longrightarrow$ such that $0 \leq \mu_P(x_i, x_j) + \nu_P(x_i, x_j) \leq 1$, $\forall (x_i, x_j) \in X \times X$. As in

the case for fuzzy preference relation, an *intuitionistic fuzzy preference relation* is represented by the matrix $P = (p_{ij})$ with $p_{ij} =< \mu_{ij}, \nu_{ij} >, \forall i, j = 1, 2, \ldots, n$. Obviously, when the hesitancy function is the null function we have that $\mu_{ij} + \nu_{ij} = 1$ $(\forall i, j)$, and the intuitionistic fuzzy preference relation $P = (p_{ij})$ is mathematically equivalent to the reciprocal fuzzy preference relation $R = (r_{ij})$, with $r_{ij} = \mu_{ij}$. An intuitionistic fuzzy preference relation is referred to as *reciprocal* when the following additional conditions are imposed:

$$(i)\ \mu_{ii} = \nu_{ii} = 0.5, \quad \forall i \in \{1, \ldots, n\}$$
$$(ii)\ \mu_{ij} = \nu_{ji}, \quad \forall i, j \in \{1, \ldots, n\}.$$

In intuitionistic fuzzy studies, the relations do not need to have reciprocity but must satisfy $r_{ij} \leq 1 - r_{ji}$ due to intuitionistic index. In other words, for an IFS A, $\pi_A(x)$ is determined by the following expression: $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$ is called the *hesitancy degree* of the element $x \in X$ to the set A, and $\pi_A(x) \in [0, 1], \forall x \in X$.

C. RECIPROCAL PROPERTY AND HESITATION FUNCTION IN NEUTROSOPHY THEORY

Let $S = \{s_1, s_2, s_3, \ldots, s_n\}$ be a set of alternatives (or options) and m be a set of individuals. Each individual declares his or her own preferences over S which are represented by an individual neutrosophic preference relation R_k such that

$$N_{R_k} : S \times S \longmapsto [0, 1] \times [0, 1] \times [0, 1]$$

which is traditionally represented by a matrix $R_k = [r_{ij}^k = N_{R_k}(r_i^k, r_j^k)], i, j = 1, 2, 3, .., n; k = 1, 2, 3, \ldots, m$.

$$R_k = \begin{bmatrix} (0.5, 0.5, 0.5) & r_{12}^k & r_{13}^k & r_{14}^k \\ r_{21}^k & (0.5, 0.5, 0.5) & r_{23}^k & r_{24}^k \\ r_{31}^k & r_{32}^k & (0.5, 0.5, 0.5) & r_{33}^k \\ r_{41}^k & r_{42}^k & r_{43}^k & (0.5, 0.5, 0.5) \end{bmatrix}$$

The matrix above shows that neutrosophic preferences of an individual k are among s_1, s_2, s_3, s_4. Also that $N_{R_k}(s_1, s_1) = N_{R_k}(s_2, s_2) = N_{R_k}(s_3, s_3) = N_{R_k}(s_4, s_4) = (0.5, 0.5, 0.5)$, $N_{R_k}(s_1, s_2) = r_{12}^k$, $N_{R_k}(s_3, s_4) = r_{34}^k$, etc. We require that there is no larger outcome when an alternative is compared to itself. Almost all studies in the literature on decision making assign no value or assign zero degree to their underlying discourse for self-comparisons. We follow a entirely computational approach here. On the other hand though, zeros given in other previous studies may lead us have a false perception to compare any s_i. For a neutrosophic preference function *mu*, if $mu(s_i, s_j) = 0$, then s_i is definitely larger than s_j. If we had a rational individual, $mu(s_i, s_i)$ would have been 0.5, since if we do self-comparison, an alternative can not have any advantage over itself. We use the H function in Definition 2.5 for preciseness and to act as a neutrosophic index of SVNNs. If $i = j$, then we take $N_{R_k}(s_i, s_j)$ to be $(0.5, 0.5, 0.5)$ without DIF, and $(0.5, 0, 0.5)$ with DIF. So, we have the following matrix:

$$R_k = \begin{bmatrix} (0.5, 0, 0.5) & r_{12}^k & r_{13}^k & r_{14}^k \\ r_{21}^k & (0.5, 0, 0.5) & r_{23}^k & r_{24}^k \\ r_{31}^k & r_{32}^k & (0.5, 0, 0.5) & r_{33}^k \\ r_{41}^k & r_{42}^k & r_{43}^k & (0.5, 0, 0.5) \end{bmatrix}$$

The function H (called *neutrosophic index* or *neutrosophic hesitation function*) assigns each a_{ij} neutrosophic value to a number in [0, 1].

We have that

$$H(a_{ij}) = \frac{1 + T(a_{ij}) - I(a_{ij})(1 - T(a_{ij})) - F(a_{ij})(1 - I(a_{ij}))}{2} \tag{2}$$

Now, we have a new matrix $R_k^H = [H(r_{ij}^k) = H^k(N_{R_k}(s_i, s_j))]$, where $i, j = 1, 2, 3, \ldots, n$; $k = 1, 2, 3, \ldots, m$. More explicitly,

$$R_k^H = \begin{bmatrix} H((0.5, 0, 0.5)) & H(r_{12}^k) & H(r_{13}^k) & H(r_{14}^k) \\ H(r_{21}^k) & H((0.5, 0, 0.5)) & H(r_{23}^k) & H(r_{24}^k) \\ H(r_{31}^k) & H(r_{32}^k) & H((0.5, 0, 0.5)) & H(r_{34}^k) \\ H(r_{41}^k) & H(r_{42}^k) & H(r_{43}^k) & H((0.5, 0, 0.5)) \end{bmatrix}$$

We find it more appropriate to use the notion of *hesitation* in order to have consistency between the choosers (individuals) and their preference. Here, we benefit from the IFS. In utilizing IFS, we provide a hybrid account of the neutrosophic accuracy function by hesitation. We adopt intuitionistic index in our study since we use the function H as a solid index throughout the paper. Not every $H^k(r_{ij})$ needs to be reciprocal, i.e. $H^k(r_{ij}) \neq 1 - H^k(r_{ji})$ but should be quasi-reciprocal. That is, $H(r_{ij}^k) \leq 1 - H(r_{ji}^k)$, for each $i, j = 1, \ldots, n$. If k is not quasi-reciprocal, we call k an *irrational individual*. If $i = j$, then we just take $N_{Rk}(a_i, a_j) = (0.5, 0.5, 0.5)$ since $H((0.5, 0.5, 0.5)) = 0.5$ irrespective of DIF. Furthermore, when we consider DIF, the neutrosophic value of the assignment made by a rational individual on the same preference is $(0.5, 0, 0.5)$ from now on, and $H((0.5, 0, 0.5)) = 0.5$ as desired.

$$DIF(R_k) = \begin{bmatrix} (0.5, 0, 0.5) & DIF(r_{12}^k) & DIF(r_{13}^k) & DIF(r_{14}^k) \\ DIF(r_{21}^k) & (0.5, 0, 0.5) & DIF(r_{23}^k) & DIF(r_{24}^k) \\ DIF(r_{31}^k) & DIF(r_{32}^k) & (0.5, 0, 0.5) & DIF(r_{33}^k) \\ DIF(r_{41}^k) & DIF(r_{42}^k) & DIF(r_{43}^k) & (0.5, 0, 0.5) \end{bmatrix}$$

R_i : preference matrix of the ith individual,

$DIF(R_i)$: DIF of preference matrix of the ith individual,

R_i^H : range of preference matrix of the ith individual under H function,

$r_k^H(ij)$: represents the element at the row i and column j of R_i^H for individual k,

$h^k(ij)$: distribution of the kth individual's votes for each pairwise comparison of alternative's value is determined through 0.5 derived from R_i^H,

$[[h^k]]$: the matrix obtained by each element of $h^k(ij)$,

$[[H_{ij}]]$: matrix of the group vote,

A_k : the degree for preference k assigned by the group,

a_{ij}^k : majority determination value for preference k of the group (the element at the row i and column j of $[[h^k]]$),

H_{ij}^k : majority determination value for preference k of the group under H function,

$$h^k(ij) = \begin{cases} 1, & r_k^H(ij) > 0.5 \\ 0, & otherwise \end{cases}$$

$H_{\pi_{ij}}$: average majority determination value of the group under H function,

H_π : consensus winner determination matrix,

$C(s_i)$: social aggregation function for the alternative (preference) s_i,

Example 6: Suppose that there are three experts $m_1,\ m_2,\ m_3$ and four facilities $s_1,\ s_2,\ s_3,\ s_4$ in the same business industry. We assume that all experts are rational and so we assume all neutrosophic values satisfy quasi-reciprocal property. We also take the self-comparison value to be (0.5, 0, 0.5). Each expert assigns some neutrosophic opinion value by comparing the facilities in pairs as follows:

R_{m_i} is the set of assigned values (preferences) by m_i to pairs in the facilities where $1 \leq i \leq 3$.

$$\begin{aligned} &R_{m_1} \\ &= \{(s_1, s_1) = (0.5, 0, 0.5), (s_1, s_2) = (0.45, 0.24, 0.27), \\ &\quad (s_1, s_3) = (0.31, 0.14, 0.66), (s_1, s_4) = (0.8, 0.3, 0), \\ &\quad (s_2, s_1) = (0.1, 0.45, 0.52), (s_2, s_2) = (0.5, 0, 0.5), \\ &\quad (s_2, s_3) = (0.48, 0.26, 0.37), (s_2, s_4) = (0.2, 0.7, 0.8), \\ &\quad (s_3, s_1) = (0.61, 0.43, 0.71), (s_3, s_2) = (0.31, 0, 0.71), \\ &\quad (s_3, s_3) = (0.5, 0, 0.5), (s_3, s_4) = (0.76, 0.23, 0.27), \\ &\quad (s_4, s_1) = (0.1, 0.6, 0.9), (s_4, s_2) = (0.81, 0.55, 0.33), \\ &\quad (s_4, s_3) = (0.11, 0.32, 0.59), (s_4, s_4) = (0.5, 0, 0.5)\} \end{aligned}$$

$$\begin{aligned} &R_{m_2} \\ &= \{(s_1, s_1) = (0.5, 0, 0.5), (s_1, s_2) = (0.2, 0.4, 0.7), \\ &\quad (s_1, s_3) = (0.21, 0.55, 0.95), (s_1, s_4) = (0.4, 0.5, 0.3), \\ &\quad (s_2, s_1) = (0.29, 0.53, 0.38), (0.29, 0.53, 0.38), \\ &\quad (s_2, s_2) = (0.5, 0, 0.5), (s_2, s_3) = (0.62, 0.45, 0.16), \\ &\quad (s_2, s_4) = (0.2, 0.7, 0.8), (s_3, s_1) = (0.72, 0.15, 0.18), \\ &\quad (s_3, s_2) = (0.11, 0.13, 0.79), (s_3, s_3) = (0.5, 0, 0.5), \\ &\quad (s_3, s_4) = (0.51, 0.45, 0.53), (s_4, s_1) = (0.15, 0.35, 0.23), \\ &\quad (s_4, s_2) = (0.81, 0.55, 0.33), (s_4, s_3) = (0.17, 0.57, 0.36), \\ &\quad (s_4, s_4) = (0.5, 0, 0.5)\} \end{aligned}$$

$$\begin{aligned} &R_{m_3} \\ &= \{(s_1, s_1) = (0.5, 0, 0.5), (s_1, s_2) = (0.3, 0.45, 0.7), \\ &\quad (s_1, s_3) = (0.1, 0.85, 0.78), (s_1, s_4) = (0.4, 0.5, 0.3), \\ &\quad (s_2, s_1) = (0.36, 0.51, 0.39), (s_2, s_2) = (0.5, 0, 0.5), \\ &\quad (s_2, s_3) = (0.62, 0.45, 0.16), (s_2, s_4) = (0.1, 0.8, 0.21), \\ &\quad (s_3, s_1) = (0.92, 0.1, 0.16), (s_3, s_2) = (0.11, 0.13, 0.79), \\ &\quad (s_3, s_3) = (0.5, 0, 0.5), (s_3, s_4) = (0.23, 0.45, 0.74), \\ &\quad (s_4, s_1) = (0.15, 0.35, 0.23), (s_4, s_2) = (0.6, 0.2, 0.1), \\ &\quad (s_4, s_3) = (0.57, 0.57, 0.36), (s_4, s_4) = (0.5, 0, 0.5)\} \end{aligned}$$

$$\begin{aligned} &R_{m_4} \\ &= \{(s_1, s_1) = (0.5, 0, 0.5), (s_1, s_2) = (0.2, 0.4, 0.7), \\ &\quad (s_1, s_3) = (0.25, 0.87, 0.38), (s_1, s_4) = (0.4, 0.5, 0.3), \\ &\quad (s_2, s_1) = (0.29, 0.53, 0.38), (s_2, s_2) = (0.5, 0, 0.5), \\ &\quad (s_2, s_3) = (0.62, 0.45, 0.16), (s_2, s_4) = (0.34, 0.66, 0.21), \\ &\quad (s_3, s_1) = (0.73, 0.87, 0.56), (s_3, s_2) = (0.14, 0.19, 0.79), \\ &\quad (s_3, s_3) = (0.5, 0, 0.5), (s_3, s_4) = (0.21, 0.45, 0.66), \\ &\quad (s_4, s_1) = (0.16, 0.35, 0.23), (s_4, s_2) = (0.6, 0.4, 0.8), \\ &\quad (s_4, s_3) = (0.68, 0.57, 0.36), (s_4, s_4) = (0.5, 0, 0.5)\} \end{aligned}$$

We now represent each R_{m_i} in matrix form and then calculate their distributed indeterminacy forms $DIF(R_{m_i})$.

$$R_{m_1} = \begin{bmatrix} (0.5, 0, 0.5) & (0.45, 0.24, 0.27) & (0.31, 0.14, 0.66) & (0.8, 0.3, 0) \\ (0.1, 0.45, 0.52) & (0.5, 0, 0.5) & (0.48, 0.26, 0.37) & (0.2, 0.7, 0.8) \\ (0.61, 0.43, 0.71) & (0.31, 0, 0.71) & (0.5, 0, 0.5) & (0.76, 0.23, 0.27) \\ (0.1, 0.6, 0.9) & (0.81, 0.55, 0.33) & (0.11, 0.32, 0.59) & (0.5, 0, 0.5) \end{bmatrix}$$

$$DIF(R_{m_1}) = \begin{bmatrix} (0.5, 0.5, 0.5) & (0.342, 0, 0.2052) & (0.2666, 0, 0.5676) & (0.56, 0, 0) \\ (0.055, 0, 0.286) & (0.5, 0, 0.5) & (0.3552, 0, 0.2738) & (0.06, 0, 0.24) \\ (0.3477, 0, 0.4047) & (0.31, 0, 0.71) & (0.5, 0, 0.5) & (0.5852, 0, 0.2079) \\ (0.04, 0, 0.36) & (0.3645, 0, 0.1485) & (0.0748, 0, 0.4012) & (0.5, 0, 0.5) \end{bmatrix}$$

$$R_{m_2} = \begin{bmatrix} (0.5, 0, 0.5) & (0.2, 0.4, 0.7) & (0.21, 0.55, 0.95) & (0.4, 0.5, 0.3) \\ (0.29, 0.53, 0.38) & (0.5, 0, 0.5) & (0.62, 0.45, 0.16) & (0.2, 0.7, 0.8) \\ (0.72, 0.15, 0.18) & (0.11, 0.13, 0.79) & (0.5, 0, 0.5) & (0.51, 0.45, 0.53) \\ (0.15, 0.35, 0.23) & (0.81, 0.55, 0.33) & (0.17, 0.57, 0.36) & (0.5, 0, 0.5) \end{bmatrix}$$

$$DIF(R_{m_2}) = \begin{bmatrix} (0.5, 0.5, 0.5) & (0.12, 0, 0.42) & (0.0945, 0, 0.4275) & (0.2, 0, 0.15) \\ (0.1363, 0, 0.1786) & (0.5, 0, 0.5) & (0.341, 0, 0.088) & (0.06, 0, 0.24) \\ (0.612, 0, 0.153) & (0.0957, 0, 0.6873) & (0.5, 0, 0.5) & (0.2805, 0, 0.2915) \\ (0.0975, 0, 0.1495) & (0.3645, 0, 0.1485) & (0.0731, 0, 0.1548) & (0.5, 0, 0.5) \end{bmatrix}$$

$$R_{m_3} = \begin{bmatrix} (0.5, 0, 0.5), (0.3, 0.45, 0.7), (0.76, 0.35, 0.38), (0.4, 0.5, 0.3) \\ (0.36, 0.51, 0.39), (0.5, 0, 0.5), (0.62, 0.45, 0.16), (0.46, 0.46, 0.21) \\ (0.92, 0.86, 0.35), (0.11, 0.13, 0.79), (0.5, 0, 0.5), (0.23, 0.45, 0.74) \\ (0.15, 0.35, 0.23), (0.6, 0.4, 0.8), (0.57, 0.57, 0.36), (0.5, 0, 0.5) \end{bmatrix}$$

$$DIF(R_{m_3}) = \begin{bmatrix} (0.5, 0, 0.5) & (0.165, 0, 0.385) & (0.494, 0, 0.247) & (0.2, 0, 0.15) \\ (0.1764, 0, 0.1911) & (0.5, 0, 0.5) & (0.341, 0, 0.088) & (0.2484, 0, 0.1134) \\ (0.1288, 0, 0.049) & (0.0957, 0, 0.6873) & (0.5, 0, 0.5) & (0.1265, 0, 0.407) \\ (0.0975, 0, 0.1495) & (0.36, 0, 0.48) & (0.2451, 0, 0.1548) & (0.5, 0, 0.5) \end{bmatrix}$$

$$R_{m_4} = \begin{bmatrix} (0.5, 0, 0.5), (0.2, 0.4, 0.7), (0.51, 0.35, 0.38), (0.4, 0.5, 0.3) \\ (0.29, 0.53, 0.38), (0.5, 0, 0.5), (0.62, 0.45, 0.16), (0.34, 0.66, 0.21) \\ (0.73, 0.87, 0.56), (0.14, 0.19, 0.79), (0.5, 0, 0.5), (0.21, 0.45, 0.66) \\ (0.16, 0.35, 0.23), (0.6, 0.4, 0.8), (0.68, 0.57, 0.36), (0.5, 0, 0.5) \end{bmatrix}$$

$$DIF(R_{m_4}) = \begin{bmatrix} (0.5, 0, 0.5) & (0.12, 0, 0.42) & (0.3315, 0, 0.247) & (0.2, 0, 0.15) \\ (0.1363, 0, 0.1786) & (0.5, 0, 0.5) & (0.341, 0, 0.088) & (0.1156, 0, 0.0714) \\ (0.0949, 0, 0.0728) & (0.1134, 0, 0.6399) & (0.5, 0, 0.5) & (0.1155, 0, 0.363) \\ (0.104, 0, 0.1495) & (0.36, 0, 0.48) & (0.2924, 0, 0.1548) & (0.5, 0, 0.5) \end{bmatrix}$$

Now we apply the H function to $DIF(R_i)$ and then obtain R_i^H.

$$R_{m_1}^H = \begin{bmatrix} 0.5 & 0.5684 & 0.3495 & 0.78 \\ 0.3844 & 0.5 & 0.5407 & 0.41 \\ 0.4715 & 0.3 & 0.5 & 0.6886 \\ 0.34 & 0.608 & 0.3368 & 0.5 \end{bmatrix}$$

$$h^{m_1}(ij) = \begin{cases} 1, & r_{m_1}^H(ij) > 0.5 \\ 0, & otherwise \end{cases}$$

$$[[h^{m_1}]] = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

$$R_{m_2}^H = \begin{bmatrix} 0.5 & 0.35 & 0.3335 & 0.525 \\ 0.4788 & 0.5 & 0.6265 & 0.41 \\ 0.7295 & 0.2041 & 0.5 & 0.4945 \\ 0.474 & 0.474 & 0.4591 & 0.5 \end{bmatrix}$$

$$h^{m_2}(ij) = \begin{cases} 1, & r^H_{m_2}(ij) > 0.5 \\ 0, & otherwise \end{cases}$$

$$[[h^{m_2}]] = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$R^H_{m_3} = \begin{bmatrix} 0.5 & 0.39 & 0.6234 & 0.525 \\ 0.4926 & 0.5 & 0.6265 & 0.5675 \\ 0.5399 & 0.2041 & 0.5 & 0.35975 \\ 0.474 & 0.4399 & 0.54515 & 0.5 \end{bmatrix}$$

$$h^{m_3}(ij) = \begin{cases} 1, & r^H_{m_3}(ij) > 0.5 \\ 0, & otherwise \end{cases}$$

$$[[h^{m_3}]] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

$$R^H_{m_4} = \begin{bmatrix} 0.5 & 0.35 & 0.5422 & 0.525 \\ 0.4788 & 0.5 & 0.6265 & 0.5221 \\ 0.511 & 0.2367 & 0.5 & 0.3762 \\ 0.477 & 0.439 & 0.5688 & 0.5 \end{bmatrix}$$

$$h^{m_4}(ij) = \begin{cases} 1, & r^H_{m_4}(ij) > 0.5 \\ 0, & otherwise \end{cases}$$

$$[[h^{m_4}]] = \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

The next step is to collect and compare the preferences. To do this, we add the columns of $[[H_{ij}]]$ and divide it to number of the alternatives.

$$A_k = \frac{1}{m} \sum [[H_{ik}]]$$

such that $1 \leq k \leq m$

$$H_{\pi_{ij}} = \begin{cases} \frac{1}{m} \sum_{k=1}^{m} a^k_{ij}, & i \neq j \\ 0, & i = j \end{cases}$$

such that $i, j = 1, 2, \ldots n$ and $k = 1, 2, \ldots, m$.

$$H_{\pi_{12}} = \frac{a^{m_1}_{12} + a^{m_2}_{12} + a^{m_3}_{12} + a^{m_4}_{12}}{4} = \frac{1+0+0+0}{4} = \frac{1}{4},$$

$$H_{\pi_{13}} = \frac{1}{2}, \quad H_{\pi_{14}} = 1, \; H_{\pi_{21}} = 0, H_{\pi_{23}} = 1, H_{\pi_{24}} = \frac{1}{2},$$

$$H_{\pi_{31}} = \frac{3}{4}, \quad H_{\pi_{32}} = 0, \; H_{\pi_{34}} = \frac{1}{4}, H_{\pi_{41}} = 0, H_{\pi_{42}} = \frac{1}{4},$$

$$H_{\pi_{43}} = \frac{1}{2}$$

$$H_\pi = \begin{bmatrix} - & \frac{1}{4} & \frac{1}{2} & 1 \\ 0 & - & 1 & \frac{1}{2} \\ \frac{3}{4} & 0 & - & \frac{1}{2} \\ 0 & \frac{1}{4} & \frac{1}{2} & - \end{bmatrix}$$

We now define the notion of a consensus winner.

Definition 7 [74]: $s_i \in W$ is called a *consensus winner* if and only if $\forall s_j \neq s_i : r_{ij} > 0.5$, where $r_{ij} \in H_\pi$.

In our example above, there is no winner because there are multiple numbers greater than 0.5. If there is a consensus winner, it must be unique and the set W must be a singleton since the reciprocal property must hold. Of course, it is easy to define that α-consensus winner for different α-values. So we define a *social aggregation average function* C to calculate the order of s_i in the group to the extent that individuals are not against option s_i.

$$C(s_i) = \frac{1}{m-1} \sum_{i \neq j} r_{ij}, \tag{3}$$

where $i, j = 1, 2, \ldots, m$.

$$C(s_1) = \frac{7}{12}, \quad C(s_2) = \frac{6}{12}, \; C(s_3) = \frac{5}{12}, C(s_4) = \frac{3}{12}.$$

So, $C(s_1) > C(s_2) > C(s_3) > C(s_4)$.

V. CONCLUSION

The main aim of this paper is to bring into attention the interplay between neutrosophy and social choice theory. Within the framework of this intention, we have taken inheritance from studies on fuzzy and intuitionistic fuzzy social choice theory and developed *the neutrosophic based social choice theory*. First we defined the *DIF*, which was used in Sorensen's truth-maker theory to distribute the indeterminacy on truth and falsity values for certain neutrosophic calculations. We believe that the notion of *DIF* gives a new insight, breath and different perspectives for neutrosophic studies. Through *DIF*, we emphasize hesitation and reciprocal characteristics in self-comparisons and other pairwise comparisons to define a consistent decision maker. We determine a consensus winner if exists. In case of otherwise, we obtain orders of the given alternatives by defining a social aggregation average function. Finally we give in the Appendix, a Python implementation of an algorithm computing the output in the order of $\frac{n}{11}$ seconds, where n is the input size (the number of matrices), when executed in a mid-end computer.

A. FURTHER RESEARCH DIRECTIONS

Some future researches to extend and diversify this work may include the following ideas:

- studying the quantifiers *most, at most,* etc [86],
- considering interval valued neutrosophic sets [87],
- considering bipolar valued neutrosophic sets [88],
- introducing different forms of DIF depending on underlying models,
- presenting several forms of aggregation operators [89],
- applications on plithogenic sets [90].
- applications on Maclaurin symmetric mean, q-rung orthopair 2-tuple linguistic aggregation and continuous interval-valued Pythagorean operators [91]- [93].

APPENDIX

A Python implementation [84], [85] of the group decision making method with distributed indeterminacy form under neutrosophic environment is as follows:

```
from __future__ import division
from collections import defaultdict
import math
import sys


R1=[ [ (0.5,0,0.5),(0.45,0.24,0.27), (0.31,0.14,0.66) , (0.8,0.3,0)],
     [(0.1,0.45,0.52) , (0.5,0,0.5), (0.48,0.26,0.37) , (0.2,0.7,0.8)],
     [(0.61,0.43,0.71) , (0.31,0,0.71) , (0.5,0,0.5)  , (0.76,0.23,0.27)],
     [(0.1,0.6,0.9) , (0.81,0.55,0.33) , (0.11,0.32,0.59) , (0.5,0,0.5)]]


R2=[ [ (0.5,0,0.5),(0.2,0.4,0.7), (0.21,0.55,0.95) , (0.4,0.5,0.3)],
     [(0.29,0.53,0.38) , (0.5,0,0.5), (0.62,0.45,0.16) , (0.2,0.7,0.8)],
     [(0.72,0.15,0.18) , (0.11,0.13,0.79)  , (0.5,0,0.5)  , (0.51,0.45,0.53)],
     [(0.15,0.35,0.23) , (0.81,0.55,0.33) , (0.17,0.57,0.36) , (0.5,0,0.5)] ]


R3=[ [ (0.5,0,0.5),(0.3,0.45,0.7), (0.1,0.85,0.78) , (0.4,0.5,0.3)],
     [(0.36,0.51,0.39) , (0.5,0,0.5), (0.62,0.45,0.16) , (0.1,0.8,0.21)],
     [(0.92,0.1,0.16) , (0.11,0.13,0.79)  , (0.5,0,0.5)  , (0.23,0.45,0.74)],
     [(0.15,0.35,0.23) , (0.6,0.2,0.1) , (0.57,0.57,0.36) , (0.5,0,0.5)] ]


R4=[ [ (0.5,0,0.5),(0.2,0.4,0.7), (0.25,0.87,0.38) , (0.4,0.5,0.3)],
     [(0.29,0.53,0.38) , (0.5,0,0.5), (0.62,0.45,0.16) , (0.34,0.66,0.21)],
     [(0.73,0.87,0.56) , (0.14,0.19,0.79)  , (0.5,0,0.5)  , (0.21,0.45,0.66)],
     [(0.16,0.35,0.23) , (0.6,0.4,0.8) , (0.68,0.57,0.36) , (0.5,0,0.5)] ]


AllTogether= {'R1': [[ (0.5,0,0.5),(0.45,0.24,0.27), (0.31,0.14,0.66) , (0.8,0.3,0)],
                     [(0.1,0.45,0.52) , (0.5,0,0.5), (0.48,0.26,0.37), (0.2,0.7,0.8)],
     [(0.61,0.43,0.71) , (0.31,0,0.71) , (0.5,0,0.5)  , (0.76,0.23,0.27)],
                     [(0.1,0.6,0.9) , (0.81,0.55,0.33) , (0.21,0.32,0.59) , (0.5,0,0.5)]],
    'R2': [[ (0.5,0,0.5),(0.2,0.4,0.7), (0.21,0.55,0.95) , (0.4,0.5,0.3)],
           [(0.29,0.53,0.38) , (0.5,0,0.5), (0.62,0.45,0.16) , (0.83,0.46,0.21)],
    [(0.72,0.15,0.18) , (0.11,0.13,0.79)  , (0.5,0,0.5)  , (0.51,0.45,0.53)],
           [(0.15,0.35,0.23) , (0.6,0.4,0.8) , (0.47,0.57,0.36) , (0.5,0,0.5)]],
    'R3': [[ (0.5,0,0.5),(0.3,0.45,0.7), (0.76,0.35,0.38) , (0.4,0.5,0.3)],
           [(0.36,0.51,0.39) , (0.5,0,0.5), (0.62,0.45,0.16) , (0.46,0.46,0.21)],
           [(0.92,0.86,0.35) , (0.11,0.13,0.79)  , (0.5,0,0.5)  , (0.23,0.45,0.74)],
           [(0.15,0.35,0.23) , (0.6,0.4,0.8) , (0.57,0.57,0.36) , (0.5,0,0.5)]],
    'R4': [[ (0.5,0,0.5),(0.2,0.4,0.7), (0.51,0.35,0.38) , (0.4,0.5,0.3)],
           [(0.29,0.53,0.38) , (0.5,0,0.5), (0.62,0.45,0.16) , (0.34,0.66,0.21)],
           [(0.73,0.87,0.56) , (0.14,0.19,0.79)  , (0.5,0,0.5)  , (0.21,0.45,0.66)],
           [(0.16,0.35,0.23) , (0.6,0.4,0.8) , (0.68,0.57,0.36) , (0.5,0,0.5)]]}


def AccuracyFunction(T,I,F):

    HV= (1+ T - I*(1-T)-F*(1-I))/2
    return HV


def DIF(T,I,F):
    T1=math.fabs(T-I*T)
    F1=math.fabs(F-I*F)

    DIFi='('+str(T1)+','+str(0)+','+str(F1)+')'
    return DIFi


def AccuracyIntedeteminacyDistubition(T,I,F):
    T1=math.fabs(T-I*T)
    F1=math.fabs(F-I*F)

    ID=AccuracyFunction(T1,I,F1)
    return ID


def RationalityChecker(R):
    columnR=len(R)
    idn=0

    rowR=len(R[0])
    for i in range(0,rowR-1):
        if R[i][i] != (0.5, 0, 0.5):
            print '(,',i,i,") is not (0.5, 0, 0.5), so, s ",i, ' is not rational agent'
            idn=1


    for i in range(0,rowR):
        for j in range(0,rowR):
            if i !=j:
                t1=R[i][j][0]
                i1=R[i][j][1]
                f1=R[i][j][2]
                A1=AccuracyIntedeteminacyDistubition(t1,i1,f1)

                t2=R[j][i][0]
                i2=R[j][i][1]
                f2=R[j][i][2]

                A2=AccuracyIntedeteminacyDistubition(t2,i2,f2)

                if A1 > 1-A2 : # A1~must be less than~or~equal to 1-A2
                    idn=1

                    print R[i][j], ' and ', R[j][i], ' does not satisfy hesitation
                    property'

    return idn


def RHcreation(K):

    global RHtogether
    RHtogether= defaultdict()
    for i in K.keys():

            columnAll=len(K[i])

            rowAll1=len(K[i][0])
            rowAll2=len(K[i][0])

            for j in range(0,rowAll1):


                for k in range(0,rowAll2):

                    t1=K[i][j][k][0]
                    i1=K[i][j][k][1]
                    f1=K[i][j][k][2]
                    A=AccuracyIntedeteminacyDistubition(t1,i1,f1)


                    if i not in  RHtogether.keys():
                        RHtogether[i]=[A]

                    else:
                        RHtogether[i].extend([A])

            number= int(math.sqrt(len(RHtogether[i])))
            m=0
            new_list=[]
            while m<len(RHtogether[i]):
                new_list.append( RHtogether[i][m:m + number])
                m+= number

            RHtogether[i]=new_list


    return RHtogether


def OneZero(K):
    global H
    H=defaultdict()
    for i in K.keys():
            columnAllin=len(K[i])

            rowAll=len(K[i][0])
            for j in range(0,columnAllin):
```

```
                for k in range(0,rowAll):
                    if K[i][j][k]>0.5:
                        if i not in H:
                            H[i]=[1]
                        else:
                            H[i].append(1)
                    else:
                        if i not in H:
                            H[i]=[0]
                        else:
                            H[i].append(0)
            number= int(math.sqrt(len(H[i])))
            m=0
            new_list=[]
            while m<len(H[i]):
                new_list.append( H[i][m:m + number])
                m+= number

            H[i]=new_list

    return H

def H_pi_ij(K):
    global Hpij
    Hpij= defaultdict()
    columnAllin112=len(H)


    for i in range(0,columnAllin112):



        Topij=0

        for j in range(0,columnAllin112):
                Topij=0

                for k in H.keys():

                    if i != j:

                        Topij = Topij + H[k][i][j]

                    else:
                        Topij=0

                aij=str(i+1)+str(j+1)
                TopijAvarage= Topij/len(H)

                if aij not in Hpij.keys():
                    TopijAvarage= Topij/len(H)
                    Hpij[aij]=TopijAvarage
                else:
                    Hpij[aij]=TopijAvarage
    return Hpij


def Alternative_Ordinary(Hpij):
    global ORD
    ORD= defaultdict()

    Number_Of_Alternatives=int(math.sqrt(len(Hpij)))
    for i in range(1,Number_Of_Alternatives+1):
        istr=str(i)

        Top=0
        for k in Hpij.keys():

            if istr==k[1]:

                Top=Top+Hpij[k]

        TopJavarage= Top/Number_Of_Alternatives


        if istr not in ORD.keys():
            istA='Alternative '+istr
            ORD[istA]=TopJavarage

        else:
            ORD[istA]=TopJavarage
    return ORD
def GroupDecisionWithID(m):
    for i in AllTogether.keys():
        if RationalityChecker(AllTogether[i])==1:
            print 'inconsistent agent'


    Step1=RHcreation(m)
    Step2=OneZero(Step1)
    Step3=H_pi_ij(Step2)
    Step4=Alternative_Ordinary(Step3)
    return Step4
```

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Interval-Valued Neutrosophic Subgroup Based on Interval-Valued Triple T-Norm

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ABSTRACT

Presently, interval-valued neutrosophic set theory has become an important research topic. It is widely used in various pure as well as applied fields. This chapter will provide some essential scopes to study interval-valued neutrosophic subgroup. Here the notion of interval-valued triple T-norm has been introduced, and based on that, interval-valued neutrosophic subgroup has been defined. Furthermore, some homomorphic characteristics of this notion have been studied. Additionally, based on interval-valued triple T-norm, interval-valued neutrosophic normal subgroup has been introduced and some of its homomorphic characteristics have been analyzed.

1. INTRODUCTION

In our real physical world, many uncertainties are involved. To tackle these ambiguities, crisp set (CS) theory is not always enough. As a result, researchers needed more capable set theories. Consequently, different set theories have emerged, for instance, fuzzy set (FS) (Zadeh, Fuzzy sets, 1965), intuitionistic fuzzy set (IFS) (Atanassov, 1986), neutrosophic set (NS) (Smarandache, 1999), plithogenic set (PS) (Smarandache, 2018), etc. FS theory is capable of handling real-life uncertainties very well. Still, in some ambiguous situations, researchers need sets that are more general i.e. IFSs or sometimes even more general sets like NSs or PSs, etc. Presently, NS theory has grabbed quite lot attentions of different researchers from various fields. Presently, NS theory has become an important and fruitful research field. Furthermore, Smarandache has also developed neutrosophic measure and probability (Smarandache, 2013), calculus (Smarandache & Khalid, 2015), psychology (Smarandache, 2018), etc. At present, NS theory is used in different applied fields, for instance, in pattern recognition problem (Vlachos & Sergiadis, 2007), image segmentation (Guo & Cheng, 2009), decision making problem (Majumdar, 2015;

Table 1. Some applications of IVNS in various fields

Author & Year	Applications of IVNS in various fields
(Broumi et. al., 2015)	Introduced the concept of n-valued IVNS and mentioned how it can be applied in medical diagnosing.
(Broumi et. al., 2014)	Presented the definition of parameterized soft set in IVNS environment and its application in DMPs.
(Ye, 2014)	Defined Hamming and Euclidean distances between two IVNSs and introduced similarity measures in IVNSs with an application in DMP.
(Ye, 2014)	Introduced a correlation coefficient (improved) of single-valued NSs and extended it to a correlation coefficient between IVNSs. Further, applied it in multiple attribute DMPs.
(Zhang et. al., 2014)	Proposed a technique based on IVNS to solve multi-criteria DMPs.
(Aiwu et. al., 2015)	Proposed an aggregation operation rules (improved) for IVNS and extended the generalized weighted aggregation operator.
(Zhang et. al., 2016)	Illustrated a novel outranking method for multi-criteria DMPs with IVNSs.
(Broumi et. al., 2016)	Extended the notion of neutrosophic graph-based multi-criteria decision-making approach in interval-valued neutrosophic graph theory.
(Deli, 2017)	Proposed the concept of the soft IVNS and investigated its application in DMP.
(Yuan et. al., 2019)	Applied IVNSs in image segmentation.
(Thong et. al., 2019)	Proposed dynamic IVNS for dynamic DMP.

Abdel-Basset et. al., 2017; Abdel-Basset et. al., 2019), mobile-edge computing (Abdel-Basset et. al., 2019), neutrosophic forecasting (Abdel-Basset et. al., 2019), supply chain management (Abdel-Basset et. al., 2019; Abdel-Basset et. al., 2019), supplier selection problems (Abdel-Basset et. al., 2018; Abdel-Basset et. al., 2018), goal programming problem (Abdel-Basset et. al., 2016), multi-objective programming problem (Hezam et. al., 2015), medical diagnosis (Kumar et. al., 2015; Deli et. al., 2015), shortest path problem (Kumar, et al., 2019; Kumar et. al., 2018; Kumar et. al., 2020), transportation problem (Kumar et. al., 2019) etc. Again, gradually some other set theories, like, interval-valued FS (IVFS) (Zadeh, 1975), interval-valued IFS (IVIFS) (Atanassov, 1999) and interval-valued NS (IVNS) (Wang et. al., 2005), etc. have evolved. These notions are generalizations of CS, FS, IFS, and NSs. Presently; these set theories are extensively applied in different fields, mainly in decision-making problems (DMP). In the following Table 1 some applications of IVNSs have been discussed.

Based on FS, Rosenfeld introduced the notion of fuzzy subgroup (FSG) (Rosenfeld, 1971). Gradually, various mathematicians have developed intuitionistic fuzzy subgroup (IFSG) (Biswas, 1989), neutrosophic subgroup (NSG) (Çetkin & Aygün, 2015), etc. Furthermore, they have studied effects of homomorphism on them. Some researchers have analyzed their normal forms also. Furthermore, the notions of interval-valued fuzzy subgroup (IVFSG) (Biswas, 1994), interval-valued intuitionistic fuzzy subgroup (IVIFSG) (Aygünoğlu et. al., 2012), etc. have been defined. In addition, different researchers have studied their normal forms, homomorphic image, homomorphic pre-image, etc. Still, the concept of the interval-valued neutrosophic subgroup is undefined. Also, different algebraic aspects of IVNSGs are needed to be studied.

This Chapter has been arranged as follows: In Segment 2, literature surveys of FS, IFS, INS, FSG, IFSG, and NSG are given. In Segment 3, the notions of IVFS, IVIFS, IVFSG, IVIFSG, etc. have been mentioned. In Segment 4, interval-valued triple T-norm (IVTTN), IVNSG, normal form of IVNSG, etc. are introduced and the effects of homomorphism on these notions are mentioned. Finally, in segment 5, the conclusion has been provided and some scopes of future researches are given.

2. LITERATURE SURVEY

In this segment, some essential notions, like, FS, IFS, NS, FSG, IFSG, NSG, level set, level subgroup, etc., are discussed and also, some of their basic fundamental properties are given. All these notions play vital roles in the development of IVNSG.

Definition 2.1 (Zadeh, 1965) Let J=[0,1]. A FS σ of a CS M is a mapping from M to J i.e σ: $M\rightarrow J$.

Definition 2.2 (Atanassov, 1986) A IFS **γ** of a CS M is denoted as $\gamma= \{(k,t_\gamma(k), f_\gamma(k)): k \in M\}$, where both t_γ and f_γ are FSs of R and $\forall k \in M$ t_γ and f_γ satisfy the criteria $1 \geq t_\gamma(k) + f_\gamma(k) \geq 0$.

Definition 2.3 (Smarandache, 1999) A NS η of a CS M is denoted as $\eta= \{(s,t_\eta(s),i_\eta(s),f_\eta(s)); s \in M\}$, where f_η,i_η,t_η: $M\rightarrow]^-0,1^+[$ are the respective degree of falsity, indeterminacy, and truth and of any element $k \in R$. Here $\forall s \in M$, f_η, i_η and t_η satisfy the criteria $3+ \geq f_\eta(s) + i_\eta(s) + t_\eta(s) \geq {}^-0$.

Definition 2.4 (Zadeh, 1965) Let α be a FS of M. Then $\forall t \in J$ the sets $\alpha_t= \{k \in M: \alpha(k) \geq t\}$ are called level subsets of α.

2.1. Fuzzy Subgroup, Intuitionistic Fuzzy Subgroup and Neutrosophic Subgroup

Definition 2.5 (Rosenfeld, 1971) A FS α of a crisp group M is called a FSG of R iff $\forall k,s \in M$, conditions given below are fulfilled:

1. $\alpha(ks) \geq \min\{\alpha(k),\alpha(s)\}$
2. $\alpha(s^{-1}) \geq \alpha(s)$.

Here $\alpha(s^{-1}) = \alpha(s)$ and $\alpha(s) \leq \alpha(e)$ (e is the neutral element of M). Also, in the above definition if only condition (i) is satisfied by α then we call it a fuzzy subgroupoid.

Theorem 2.1 (Rosenfeld, 1971) α is a FSG of M iff $\forall k, s \in R$ $\boldsymbol{\alpha}(ks^{-1}) \geq \min\{\alpha(k),\alpha(s)\}$.

Definition 2.6 (Das, 1981) Suppose α is a FSG of a group M. Then $\forall t \in J$ the level subgroups of α are α_t, where $\alpha(e) \geq t$.

Definition 2.7 (Biswas, 1989)An IFS $\boldsymbol{\gamma}= \{(k,t_\gamma(k),f_\gamma(k)): k \in M\}$ of a crisp set M is called an IFSG of M iff $\forall k, s \in M$

1. $t_\gamma(ks^{-1}) \geq \min\{t_\gamma(k), t_\gamma(s)\}$
2. $f_\gamma(ks^{-1}) \leq \max\{f_\gamma(k), f_\gamma(s)\}$

The collection of all IFSG of M will be denoted as IFSG(M).

Definition 2.8 (Çetkin & Aygün, 2015) Let M be a group and δ be a NS of M. δ is called a NSG of M iff the conditions given below are fulfilled:

1. $\delta(k \bullet s) \geq \min\{\delta(k), \delta(s)\}$, i.e. $t_\delta(k \cdot s) \geq \min\{t_\delta(k), t_\delta(s)\}$, $i_\delta(k \cdot s) \geq \min\{i_\delta(k), i_\delta(s)\}$ and $f_\delta(k \cdot s) \leq \max\{f_\delta(k), f_\delta(s)\}$
2. $\delta(s^{-1}) \geq \delta(s)$ i.e. $t_\delta(s^{-1}) \geq t_\delta(s)$, $i_\delta(s^{-1}) \geq i_\delta(s)$ and $f_\delta(s^{-1}) \leq f_\delta(s)$

The collection of all NSG will be denoted as NSG(R). Here notice that t_δ and i_δ are following Definition 2.5 i.e. both of them are actually FSGs of R.

Example 2.1 (Çetkin & Aygün, 2015) Suppose $M=\{1, -1, i, -i\}$ and δ is a NS of M, where $\delta = \{(1, 0.6, 0.5, 0.4), (-1, 0.7, 0.4, 0.3), (i, 0.8, 0.4, 0.2), (-i, 0.8, 0.4, 0.2)\}$. Notice that $\delta \in$ NSG(M).

Theorem 2.2 (Çetkin & Aygün, 2015) Let M be a group and δ be a NS of M. Then $\delta \in$ NSG(M) iff $\forall k, s \in M$ $\delta(k \cdot s^{-1}) \geq \min\{\delta(k), \delta(s)\}$.

Theorem 2.3 (Çetkin & Aygün, 2015) $\delta \in$ NSG(M) iff $\forall p \in [0,1]$ the p-level sets $(t_\delta)_p$, $(i_\delta)_p$ and p-lower-level set $(\overline{f_\delta})_p$ are CSGs of M.

Definition 2.9 (Çetkin & Aygün, 2015) Let M be a group and δ be a NS of M. Here δ is called a neutrosophic normal subgroup (NNSG) of M iff $\forall k,s \in M$ $\delta(k \cdot s \cdot k^{-1}) \leq \delta(s)$ i.e. $t_\delta(k \cdot s \cdot k^{-1}) \leq t_\delta(s)$, $i_\delta(k \cdot s \cdot k^{-1}) \leq i_\delta(s)$ and $f_\delta(k \cdot s \cdot k^{-1}) \geq f_\delta(s)$.

The collection of all NNSGs of M will be denoted as NNSG(M).

Definition 2.10 (Anthony & Sherwood, 1979) A FS α of M is said to have supremum property if for any $\alpha' \circ \alpha$ $\exists k_0 \circ \alpha'$ such that $\alpha(k_0) \circ \sup_{k \in \alpha'} \alpha(k)$.

Theorem 2.4 (Anthony & Sherwood, 1979) Suppose α is a fuzzy subgroupoid of M based on a continuous TN T and l be a homomorphism on M, then the image (supremum image) of α is a fuzzy subgroupoid on $l(M)$ based on T.

Theorem 2.5 (Rosenfeld, 1971) Homomorphic image or pre-image of any FSG having supremum property is a FSG.

Theorem 2.6 (Sharma, 2011) Let M_1 and M_2 are two crisp groups. Also, suppose l is a homomorphism of M_1 into M_2 then preimage of an IFSG γ of M_2 i.e. $l^{-1}(\gamma)$ is an IFSG of M_1.

Theorem 2.7 (Sharma, 2011) Let l be a surjective homomorphism of a group M_1 to another group M_2, then the image of an IFSG γ of M_1 i.e. $l(\gamma)$ is an IFSG of M_2.

Theorem 2.8 (Çetkin & Aygün, 2015) Homomorphic image or pre-image of any NSG is a NSG.

Theorem 2.9 (Çetkin & Aygün, 2015) Let $\delta \in$ NNSG(M) and l be a homomorphism on M. Then the homomorphic pre-image of δ i.e. $l^{-1}(\delta) \in$ NNSG(M).

Theorem 2.10 (Çetkin & Aygün, 2015) Let $\delta \in$ NNSG(M) and l be a surjective homomorphism on M. Then the homomorphic image of δ i.e. $l(\delta) \in$NNSG(M).

Table 2. Significance and influences of some authors in FSG, IFSG, and NSG

Author and Year	Different contributions in FSG, IFSG, and NSG
(Rosenfeld, 1971)	Introduced FSG.
(Das, 1981)	Introduced level subgroup.
(Anthony & Sherwood, 1979)	Introduced FSG using general T-norm.
(Anthony & Sherwood, 1982)	Introduced subgroup generated and function generated FSG.
(Sherwood, 1983)	Studied product of FSGs.
(Mukherjee & Bhattacharya, 1984)	Introduced fuzzy normal subgroups and cosets.
(Biswas, 1989)	Introduced IFSG.
(Eroğlu, 1989)	Studied homomorphic image of FSG.
(Hur et. al., 2003)	Investigated some properties of IFSGs and inutionistic fuzzy subrings.
(Hur et. al., 2004)	Defined normal version of IFSG and intuitionistic fuzzy cosets.
(Sharma, 2011)	Studied homomorphism of IFSG.
(Çetkin & Aygün, 2015)	Introduced NSG and NNSG and studied some of their fundamental properties by introducing homomorphism.

In the following Table 2, some sources have been mentioned which have some major contributions in the fields of FSG, IFSG, and NSG.

2.2. A List of Abbreviations

CS signifies “crisp set”.
FS signifies “fuzzy set”.
IFS signifies “intuitionistic fuzzy set”.
NS signifies “neutrosophic set”.
FSG signifies “fuzzy subgroup”.
IFSG signifies “intuitionistic fuzzy subgroup”.
NSG signifies “neutrosophic subgroup”.
TN signifies “T-norm”.
TC signifies “T-conorm”.
IVTN signifies “interval-valued T-norm”.
IVTC signifies “interval-valued T-conorm”.
IVDTN signifies “interval-valued double T-norm”.
IVTTN signifies “interval-valued triple T-norm”.
IVFS signifies “interval-valued fuzzy set”.
IVIFS signifies “interval-valued intuitionistic fuzzy set”.
IVNS signifies “interval-valued neutrosophic set”.

IVFSG signifies “interval-valued fuzzy subgroup”.
IVIFSG signifies “interval-valued intuitionistic fuzzy subgroup”.
IVNSG signifies “interval-valued neutrosophic subgroup”.
IVIFNSG signifies “interval-valued intuitionistic fuzzy normal subgroup”.
IVNNSG signifies “interval-valued neutrosophic normal subgroup”.

2.3. Motivation of the Work

So far, IVFSG and IVIFSG have grabbed a lot of attention and hence, as a result, as a result, it has yielded a lot of promising research fields. Some researchers have introduced functions in the environments of IVFSG and IVIFSG. Furthermore, they have introduced homomorphism in IVFSG and IVIFSG environments and studied some of their fundamental algebraic properties. IVNSG is relatively new and may become fruitful research field in near future. Also, the notion of IVNNSG is needed to be introduced. Furthermore, functions are needed to be introduced in the interval-valued neutrosophic environment and some homomorphic characteristics of IVNSG and IVNNSG are needed to be introduced. In this chapter, the subsequent research gaps are discussed:

- Still, the notion of IVNSG is undefined.
- Homomorphic image and preimage of IVNSG are needed to be studied.
- Still, the notion of IVNNSG is undefined.
- Also, some homomorphic characteristics of IVNNSG are needed to be analyzed.

Therefore, this inspires us to introduce and develop these notions of IVNSG and IVNNSG and analyze some of their algebraic characteristics.

2.4. Contribution of the Work

On the basis of the above gaps, the purpose of this chapter is to give some important definitions, examples and, theories in the field of IVNSG. Also, function has been introduced in interval-valued neutrosophic environment and some homomorphic properties of IVNSG and IVNNSG are discussed properly. The following are some goals that are planned and accomplished during this research work:

- To define IVNSG and study its algebraic properties.
- To define IVNNSG and study its algebraic properties.
- To introduce a function in interval-valued neutrosophic environment.
- To study some properties of homomorphic images and preimages of IVNSG and IVNNSG.

3. DESCRIPTION OF THE WORK

3.1. Research Problem

Until now, several researchers have studied different fundamental properties and algebraic structures of FSG, IFSG, as well as NSG. Again, some researchers have introduced IVFSG, IVIFSG and analyzed their fundamental properties. It is known that homomorphism preserves algebraic structures of any entity. Therefore, it is an essential tool to study some fundamental algebraic properties. Hence, several researchers have introduced and studied homomorphism in the environments of FSG, IFSG, NSG, IVFSG, IVIFSG, etc. In addition, some researchers have introduced the normal forms of FSG, IFSG, NSG, IVFSG, IVIFSG and studied their homomorphic properties. Until now, the notion of IVNSG is undefined and unexplored. Also, the normal form of IVNSG is undefined. Hence, these notions are yet to be introduced. Furthermore, the effects of homomorphism on these notions i.e. fundamental properties of homomorphic images and preimages of these notions are needed to be analyzed.

In this chapter, these essential notions of IVNSG and its normal form have been introduced and analyzed with proper examples. In the following subsection, some important notions have been discussed, which were introduced earlier.

3.1.1. Preliminaries

In this segment, the notions of interval number, IVFS, IVIFS, IVFSG, IVDTN, IVIFSG, IVTTN, etc. have been discussed. These notions are essential for introducing IVNSG.

Definition 3.1 Let J=[0,1]. An interval number of J is denoted as $\bar{g}=[g^{\circ},g^{+}]$, where $0 \le g^{\circ} \le g^{+} \le 1$.

The set of all the interval numbers of J will be denoted as ρ(J) where $\rho(J) \circ \{\bar{g} \circ [g^{-},g^{+}] : g^{-} \le g^{+}, g^{-}, g^{+} \in J\}$.

Again, $\forall g \in J$, $g=[g, g] \in \rho(J)$ i.e. interval numbers are more general than ordinary numbers.

Let $\forall i$, $\bar{u}_i = [u_i^{-}, u_i^{+}] \circ J$. Then supremum and infimum of $\bar{u}_i$ are defined as:

$$\sup(\bar{u}_i) = [\bigcirc u_i^{-}, \bigcirc u_i^{+}] \text{ and } \inf(\bar{u}_i) = [\wedge u_i^{-}, \wedge u_i^{+}].$$

Also, let

$\bar{g} = [g^{-}, g^{+}] \circ \ \rho(J)$ and $\bar{u} = [u^{-}, u^{+}] \circ \ \rho(J)$,

then the subsequent are true:

1. $\bar{g} \le \bar{u}$ iff $g^{-} \le u^{-}$ and $g^{+} \le u^{+}$.
2. $\bar{g} = \bar{u}$ iff $g^{-}=u^{-}$and $g^{+}=u^{+}$.
3. $\bar{g} < \bar{u}$ iff $g^{-} \le u^{-}$and $g^{+1}u^{+}$.

Definition 3.2 (Zadeh, 1975) Let M be crisp set, then the mapping $\bar{\mu} : M \to \circ(J)$ is called an IVFS of M.

A set of all IVFS of M is denoted as IVFS(M). For each $\bar{\mu} \circ$ IVFS(M), $\bar{\mu}^{\circ}(k) \le \bar{\mu}^{+}(k)$ for all $k \in M$. Here, $\bar{\mu}^{\circ}(k)$ and $\bar{\mu}^{\circ}(k)$ are fuzzy sets of $\bar{\mu}$. Also, Let $(\bar{g}, \bar{u}_i) \in \rho(J) \circ \ \rho(J)$, where $\bar{g}_i = [g_i^{\circ}, g_i^{+}]$ and $\bar{u}_i = [u_i^{\circ}, u_i^{+}]$ with $g_i^{+} + u_i^{+} \circ 1$, for all i. Then supremum and infimum $(\bar{g}_i, \bar{u}_i)$ are defined as:

(1.) $\bigwedge_{i\in\lambda}(\bar{g}_i, \bar{u}_i) = (\bigwedge_{i\in\lambda} \bar{g}_i, \bigcirc_{i\in\lambda} \bar{u}_i) = ([\bigwedge_{i\in\lambda} g_i^{-}, \bigwedge_{i\in\lambda} g_i^{+}], [\bigcirc_{i\in\lambda} u_i^{+}, \bigcirc_{i\in\lambda} u_i^{+}])$

(2.) $\bigvee_{i\in\lambda}(\bar{g}_i, \bar{u}_i) = (\bigvee_{i\in\lambda} \bar{g}_i, \bigcirc_{i\in\lambda} \bar{u}_i) = ([\bigvee_{i\in\lambda} g_i^{-}, \bigvee_{i\in\lambda} g_i^{+}], [\bigcirc_{i\in\lambda} u_i^{+}, \bigcirc_{i\in\lambda} u_i^{+}])$

Again, for all

$(\bar{g}_1, \bar{u}_1), (\bar{g}_2, \bar{u}_2) \in \rho(J) \circ \ \rho(J)$,

with

$(\bar{g}_1, \bar{u}_1) = ([g_1^{\circ}, g_1^{+}], [u_1^{\circ}, u_1^{+}]), (\bar{g}_2, \bar{u}_2) = ([g_2^{\circ}, g_2^{+}], [u_2^{\circ}, u_2^{+}])$,

the subsequent are true:

1. $(\bar{g}_1, \bar{u}_1) \le (\bar{g}_2, \bar{u}_2)$ iff $\bar{g}_1 \le \bar{g}_2$ and $\bar{u}_1 \circ \bar{u}_2$,
2. $(\bar{g}_1, \bar{u}_1) = (\bar{g}_2, \bar{u}_2)$ iff $\bar{g}_1 = \bar{g}_2$ and $\bar{u}_1 = \bar{u}_2$,
3. $(\bar{g}_1, \bar{u}_1) < (\bar{g}_2, \bar{u}_2)$ iff $\bar{g}_1 \circ \bar{g}_2, \bar{u}_1 \ge \bar{u}_2$ and $\bar{g}_1 \ne \bar{g}_2, \bar{u}_1 \ne \bar{u}_2$.

Definition 3.3 (Atanassov, 1999) Let M be a crisp set, then a mapping $\tilde{\gamma} : M \to \circ(J) \times \circ(J)$ defined as $\tilde{\gamma}(k) \circ (\bar{t}_{\tilde{\gamma}}(k), \bar{f}_{\tilde{\gamma}}(k))$, with $\bar{t}_{\tilde{\gamma}}^{\circ}(k) \circ \bar{f}_{\tilde{\gamma}}^{\circ}(k) \le 1$, for all $k \in M$ is called an IVIFS of M.

In the above definition, both $\bar{t}_{\tilde{\gamma}}(k)$ and $\bar{f}_{\tilde{\gamma}}(k)$ are IVFSs of M. A set of all IVIFSs of M will be denoted as IVFS(M).

Definition 3.4 (Mondal & Samanta, 2001) Suppose M_1 and M_2 are two crisp sets and l: $M_1 \to M_2$ be a function. Let $\tilde{\gamma}_1 \circ$ IVIFS(M_1) and $\tilde{\gamma}_2 \circ$ IVIFS(M_2). Then $\forall k \in M_1$ the image of $\tilde{\gamma}_1$ i.e. $l(\tilde{\gamma}_1)$ is denoted as $l(\tilde{\gamma}_1)(s) \circ (l(\bar{t}_{\tilde{\gamma}_1})(s), l(\bar{f}_{\tilde{\gamma}_1})(s))$ and $\forall s \in M_2$ the preimage of $\tilde{\gamma}_2$ i.e. $l^{-1}(\sigma_2)$ is denoted as $l^{-1}(\tilde{\gamma}_2)(k) \circ \tilde{\gamma}_2(l(k))$. where

$$l(\tilde{\gamma}_1)(s) \circ \left[\bigvee_{k \in l^{-1}(s)} (\overline{t}_{\tilde{\gamma}_1})(k), \bigwedge_{k \in l^{-1}(s)} (\overline{f}_{\tilde{\gamma}_1})(k) \right]$$
$$\circ \left[[l(t^-_{\tilde{\gamma}_1})(s), l(t^+_{\tilde{\gamma}_1})(s)], [l(f^-_{\tilde{\gamma}_1})(s), l(f^+_{\tilde{\gamma}_1})(s)] \right]$$

and

$$l^{-1}(\tilde{\gamma}_2)(k) = \left[[l^{-1}(t^-_{\tilde{\gamma}_2})(k), l^{-1}(t^\circ_{\tilde{\gamma}_2})(k)], [l^{-1}(f^-_{\tilde{\gamma}_2})(k), l^{-1}(f^\circ_{\tilde{\gamma}_2})(k)] \right]$$
$$= \left[[t^-_{\tilde{\gamma}_2}(l(k)), t^\circ_{\tilde{\gamma}_2}(l(k))], [f^-_{\tilde{\gamma}_2}(l(k)), f^\circ_{\tilde{\gamma}_2}(l(k))] \right]$$

Definition 3.5 (Gupta & Qi, 1991) A function T: $J \rightarrow J$ is called a TN iff $\forall k,s,t \in J$, conditions given below are fulfilled:

(1.) $T(k, 1) = k$

(2.) $T(k, s) = T(s, k)$

(3.) $T(k, s) \leq T(t, s)$ if $k \leq t$

(4.) $T(k, T(s, t)) = T(T(k, s), t)$

Notice that, T is an idempotent TN iff T is minimum TN or $T = \wedge$.

Definition 3.6 (Klement et. al., 2013) Suppose T is a TN, then the function $\overline{T}: \rho(J) \circ \rho(J) \rightarrow \rho(J)$ defined as $\overline{T}(\overline{g}, \overline{u}) = [T(g^\circ, u^\circ), T(g^+, u^+)]$ is called an IVTN.

Notice that, $\overline{T}$ is idempotent if T is idempotent.

Definition 3.7 (Gupta & Qi, 1991) A function S: $J \rightarrow J$ is called a TC iff $\forall k,s,t \in J$, subsequent conditions are fulfilled:

(1.) $S(k,0)=k$

(2.) $S(k,s)=S(s,k)$

(3.) $S(k,s) \leq S(t,s)$ if $k \leq t$

(4.) $S(k,S(s,t))=S(S(k,s),t)$

Note that, S is an idempotent TC iff S is maximum TC or $S=\vee$.

Definition 3.8 (Klement et. al., 2013) Let S be a TC, then the mapping $\overline{S}: \rho(J) \circ \rho(J) \rightarrow \rho(J)$ defined as $S(\overline{g}, \overline{u}) = [\overline{S}(g^\circ, u^\circ), S(g^+, u^+)]$ is called an IVTC.

Note that, $\overline{S}$ is idempotent if S is idempotent.

Definition 3.9 (Aygünoğlu et. al., 2012) Suppose $\overline{T}$ is an IVTN and $\overline{S}$ is an IVTC. Then a mapping $\tilde{T}: (\rho(J) \circ \rho(J))^2 \rightarrow \rho(J) \circ \rho(J)$ denoted as $\tilde{T}((\overline{g}_1, \overline{u}_1), (\overline{g}_2, \overline{u}_2)) = (\overline{T}(\overline{g}_1, \overline{g}_2), \overline{S}(\overline{u}_1, \overline{u}_2))$ is called an IVDTN.

Note that, $\tilde{T}$ is idempotent if both $\overline{T}$ and $\overline{S}$ are idempotent.

Definition 3.10 (Aygünoğlu et. al., 2012) Let M be a crisp group. An IVIFS $\tilde{\gamma} \circ \{(s, t_{\tilde{\gamma}}(s), f_{\tilde{\gamma}}(s)) : s \in M\}$ of M is called an IVIFSG of M with respect to IVDTN $\tilde{T}$ if the conditions given below are fullfilled:

1. $\tilde{\gamma}(k \circ s) \geq \tilde{T}(\tilde{\gamma}(k), \tilde{\gamma}(s)),\ \forall k, s \in M,$
2. $\tilde{\gamma}(s^{\circ 1}) \geq \tilde{\gamma}(s),\ \forall s \in M.$

Where condition (1.) implies that,

$$\overline{t}_{\tilde{\gamma}}(k \circ s) \geq \overline{T}(\overline{t}_{\tilde{\gamma}}(k), \overline{t}_{\tilde{\gamma}}(s)), \overline{f}_{\tilde{\gamma}}(k \circ s) \leq \overline{S}(\overline{f}_{\tilde{\gamma}}(k), \overline{f}_{\tilde{\gamma}}(s))$$

and condition (2.) implies that, $\overline{t}_{\tilde{\gamma}}(s^{\circ 1}) \geq \overline{t}_{\tilde{\gamma}}(s),\ \overline{f}_{\tilde{\gamma}}(s^{\circ 1}) \leq \overline{f}_{\tilde{\gamma}}(s)$.

The set of all IVIFSG of a group M based on IVDTN $\tilde{T}$ will be mentioned as $\text{IVIFSG}(M, \tilde{T})$.

Theorem 3.1 (Aygünoğlu et. al., 2012) Suppose M is a group and $\tilde{\gamma} \circ \text{IVIFS}(M)$. Then $\tilde{\gamma} \circ \text{IVIFSG}(M, \tilde{T})$ iff $\forall k, s \in M$. $\tilde{\gamma}(k \circ s^{-1}) \geq \tilde{T}(\tilde{\gamma}(k), \tilde{\gamma}(s))$.

Theorem 3.2 (Aygünoğlu et. al., 2012) Let M_1 and M_2 be two crisp groups with l: $M_1 \rightarrow M_2$ be a homomorphism and $\tilde{T}$ be a continuous IVDTN. If $\tilde{\gamma} \circ \text{IVIFSG}(M_1, \tilde{T})$, then $l(\tilde{\gamma}) \circ \text{IVIFSG}(M_2, \tilde{T})$.

Theorem 3.3 (Aygünoğlu et. al., 2012) Suppose M_1 and M_2 are two crisp groups and l be a homomorphism from M_1 into M_2. If $\tilde{\gamma}' \circ \text{IVIFSG}(M_2, \tilde{T})$, then $l^{-1}(\tilde{\gamma}) \circ \text{IVIFSG}(M_1, \tilde{T})$.

Definition 3.11 (Aygünoğlu et. al., 2012) Let M be a crisp group and $\tilde{\gamma} \circ \text{IVIFSG}(M, \tilde{T})$. Then $\tilde{\gamma}$ is called an IVIFNSG of M with respect to IVDTN $\tilde{T}$ if $\forall k, s \in M$, $\tilde{\gamma}(k \circ s) = \tilde{\gamma}(s \circ k)$.

The set of all IVIFNSG of a crisp group M with respect to $\tilde{T}$ will be denoted as $\text{IVIFNSG}(M, \tilde{T})$.

Theorem 3.4 (Aygünoğlu et. al., 2012) Suppose M_1 and M_2 are two crisp groups and l be a homomorphism from M_1 into M_2. If $\tilde{\gamma}' \circ \text{IVIFNSG}(M_2, \tilde{T})$, then $l^{-1}(\tilde{\gamma}^{\circ}) \in \text{IVIFNSG}(M_1, \tilde{T})$.

Theorem 3.5 (Aygünoğlu et. al., 2012) Let M_1 and M_2 be two crisp groups and l be a surjective homomorphism from M_1 into M_2. If $\tilde{\gamma} \circ \text{IVIFNSG}(M_1, \tilde{T})$, then $l(\tilde{\gamma}) \circ \text{IVIFNSG}(M_1, \tilde{T})$.

In the following Table 3, some sources have been mentioned which have some major contributions in the fields of IVFS, IVIFS, IVFSG and IVIFSG.

Table 3. Some important contributions in the fields of IVFS, IVIFS, IVFSG, and IVIFSG

Author and Year	Different contributions in IVFS, IVIFS, IVFSG and IVIFSG
(Zadeh, 1975)	Introduced IVFS
(Biswas, 1994)	Defined IVFSG which is of Rosenfeld's nature.
(Guijun & Xiaoping, 1996)	Introduced IVSGs induced by triangular norms.
(Atanassov, 1999)	Introduced IVIFS.
(Mondal & Samanta, 1999)	Defined topology of IVFSs is and studied some of its properties.
(Davvaz, Interval-valued fuzzy subhypergroups, 1999)	Introduced the concepts of interval-valued fuzzy subhypergroup of a hypergroup.
(Li & Wang, 2000)	Introduced the notion of S_H-IVFSG.
(Mondal & Samanta, 2001)	Defined topology of IVIFSs is and studied some of its properties.
(Davvaz, 2001)	Extended the notion of fuzzy ideal of a near-ring by introducing interval-valued L-fuzzy ideal of a near-ring.
(Jun & Kim, 2002)	Introduced interval-valued fuzzy R-subgroups in near rings.
(Aygünoğlu et. al., 2012)	Defined IVDTN and using that introduced IVIFSG.

In the following section, the notion of IVNSG has been defined, which is based on IVTTN. Also, some essential homomorphic properties of IVNSG has been investigated. Furthermore, the normal form of IVNSG has been defined and its homomorphic characteristics have been studied.

4. PROPOSED NOTION OF INTERVAL-VALUED NEUTROSOPHIC SUBGROUP

Definition 4.1 Suppose $\bar{T}$ and $\bar{I}$ are two IVTNs and $\bar{F}$ be an IVTC. The function

$$\breve{T}:(\rho(J)^\circ\ \rho(J)^\circ\ \rho(J))^2 \rightarrow \rho(J)^\circ\ \rho(J)^\circ\ \rho(J)$$

denoted as

$$\breve{T}((\bar{g}_1,\bar{u}_1,\bar{t}_1),(\bar{g}_2,\bar{u}_2,\bar{t}_2)=(\bar{T}(\bar{g}_1,\bar{g}_2),\bar{I}(\bar{u}_1,\bar{u}_2),\bar{F}(\bar{t}_1,\bar{t}_2))$$

is called an IVTTN.

Definition 4.2 Suppose M is a crisp group. An IVNS $\delta\ \breve{\circ}\ \{(s,\bar{t}_{\breve{\delta}}(s),\bar{i}_{\breve{\delta}}(s),\bar{f}_{\breve{\delta}}(s)) : s \in M\}$ of M is called an IVNSG of M with respect to IVTTN $\breve{T}$ if the conditions given below are fulfilled:

1. $\breve{\delta}(k \circ s) \geq \breve{T}(\breve{\delta}(k),\breve{\delta}(s)),\ \forall\, k,s \in M,$
2. $\breve{\delta}(s^{\circ 1}) \geq \breve{\delta}(s),\ \forall\, s \in M.$

Now, by condition (1.)

$$\bar{t}_{\breve{\delta}}(k \circ s) \geq \bar{T}(\bar{t}_{\breve{\delta}}(k),\bar{t}_{\breve{\delta}}(s)), \bar{i}_{\breve{\delta}}(k \circ s) \geq \bar{I}(\bar{i}_{\breve{\delta}}(k),\bar{i}_{\breve{\delta}}(s)), \bar{f}_{\breve{\delta}}(k \circ s) \leq \bar{F}(\bar{f}_{\breve{\delta}}(k),\bar{f}_{\breve{\delta}}(s))$$

and by condition (2.) $\bar{t}_{\breve{\delta}}(s^{\circ 1}) \geq \bar{t}_{\breve{\delta}}(s), i_{\breve{\delta}}(s^{\circ 1}) \geq i_{\breve{\delta}}(s)$ and $\bar{f}_{\breve{\delta}}(s^{\circ 1}) \leq \bar{f}_{\breve{\delta}}(s)$.

The set of all IVNSG of a group M with respect to an IVTTN $\breve{T}$ will be denoted as IVNSG$(M,\breve{T})$.

Example 4.1 Let $R=\{e,k,s,ks\}$ be the Klein's four group. Let

$$\breve{\delta} \circ \{(e,[0.1,0.3],[0.2,0.4],[0.1,0.4]),(k,[0.1,0.3],[0.2,0.3],[0.2,0.4]),$$
$$(s,[0.1,0.2],[0.2,0.4],[0.2,0.5]),\ (ks,[0.1,0.2],[0.2,0.3],[0.2,0.5])\}$$

be a IVNS of M. Also, let in IVTTN $\breve{T}$, the corresponding IVTNs $\bar{T}$ and $\bar{I}$ consist of minimum TN and corresponding IVTC $\bar{F}$ consists of maximum TC. In Table 4 all possible compositions of elements in $\breve{\delta}$ and their corresponding interval-valued memberships are mentioned.

Clearly, from Table 4, $\breve{\delta}$ satisfies condition (i) of Definition 4.2. Again, each element belonging to $\breve{\delta}$ is its own inverse. Hence, $\breve{\delta}$ satisfies condition (ii) of Definition 4.2. So, $\breve{\delta} \circ$ IVNSG$(M,\breve{T})$.

Theorem 4.1 Let M be a group and $\breve{\delta} \circ$ IVNSG$(M,\breve{T})$. Then $\forall s \in M$

1. $\breve{\delta}(s^{\circ 1}) = \breve{\delta}(s)$ and
2. $\breve{\delta}(e) \circ \breve{\delta}(s)$, where e is the neutral element of M.

Proof:

1. From Definition 4.2, we have $\breve{\delta}(s^{\circ 1}) \geq \breve{\delta}(s),\ \forall\, s \in M$. Again, for any $s \in M$, $\breve{\delta}(s) \circ \breve{\delta}((s^{-1})^{-1}) \geq \breve{\delta}(s^{-1})$. So, $\breve{\delta}(s^{\circ 1}) = \breve{\delta}(s)$.
2. For any $k \in R$, $\breve{\delta}(e) \circ \breve{\delta}(s \cdot s^{-1}) \circ \breve{T}(\breve{\delta}(s),\breve{\delta}(s^{-1})) = \breve{T}(\breve{\delta}(s),\breve{\delta}(s)) = \breve{\delta}(s)$.

Theorem 4.2 Suppose M is a crisp group and $\breve{\delta} \circ$ IVNS(M). Then $\breve{\delta} \circ$ IVNSG$(M,\breve{T})$ iff $\forall k,s \in M$, $\breve{\delta}(k \circ s^{-1}) \geq \breve{T}(\breve{\delta}(k),\breve{\delta}(s))$.

Theorem 4.3 Suppose M is a crisp group and $\breve{\delta}_1,\breve{\delta}_2 \circ$ IVNSG$(M,\breve{T})$. Then $\breve{\delta}_1 \circ \breve{\delta}_2 \in$ IVNSG$(M,\breve{T})$.

Proof: Let $\breve{\delta}_1,\breve{\delta}_2 \circ$ IVNSG$(M,\breve{T})$. To prove $\breve{\delta}_1 \circ \breve{\delta}_2 \in$ IVNSG$(M,\breve{T})$, it is needed to show that

Table 4. All possible compositions of elements in $\breve{\delta}$ and their interval-valued memberships

e•e	$\bar{t}_{\breve{\delta}}(e\circ e)=\bar{t}_{\breve{\delta}}(e)\geq\bar{T}(\bar{t}_{\breve{\delta}}(e),\bar{t}_{\breve{\delta}}(e))$, $\bar{i}_{\breve{\delta}}(e\circ e)=\bar{i}_{\breve{\delta}}(e)\geq\bar{I}(\bar{i}_{\breve{\delta}}(e),\bar{i}_{\breve{\delta}}(e))$, $\bar{f}_{\breve{\delta}}(e\circ e)=\bar{f}_{\breve{\delta}}(e)\leq\bar{F}(\bar{f}_{\breve{\delta}}(e),\bar{f}_{\breve{\delta}}(e))$
e•k	$\bar{t}_{\breve{\delta}}(e\circ k)=\bar{t}_{\breve{\delta}}(k)=[0.1,0.3]\geq[0.1,0.3]=\bar{T}([0.1,0.3],[0.1,0.3])=\bar{T}(\bar{t}_{\breve{\delta}}(e),\bar{t}_{\breve{\delta}}(k))$, $\bar{i}_{\breve{\delta}}(e\circ k)=\bar{i}_{\breve{\delta}}(k)=[0.2,0.3]\geq[0.2,0.3]=\bar{I}([0.2,0.4],[0.2,0.3])=\bar{I}(\bar{i}_{\breve{\delta}}(e),\bar{i}_{\breve{\delta}}(k))$, $\bar{f}_{\breve{\delta}}(e\circ k)=\bar{f}_{\breve{\delta}}(k)=[0.2,0.4]\leq[0.2,0.4]=\bar{F}([0.1,0.4],[0.2,0.4])=\bar{F}(\bar{f}_{\breve{\delta}}(e),\bar{f}_{\breve{\delta}}(k))$
e•s	$\bar{t}_{\breve{\delta}}(e\circ s)=\bar{t}_{\breve{\delta}}(s)=[0.1,0.2]\geq[0.1,0.2]=\bar{T}([0.1,0.3],[0.1,0.2])=\bar{T}(\bar{t}_{\breve{\delta}}(e),\bar{t}_{\breve{\delta}}(s))$, $\bar{i}_{\breve{\delta}}(e\circ s)=\bar{i}_{\breve{\delta}}(s)=[0.2,0.4]\geq[0.2,0.4]=\bar{I}([0.2,0.4],[0.2,0.4])=\bar{I}(\bar{i}_{\breve{\delta}}(e),\bar{i}_{\breve{\delta}}(s))$, $\bar{f}_{\breve{\delta}}(e\circ s)=\bar{f}_{\breve{\delta}}(s)=[0.2,0.5]\leq[0.2,0.5]=\bar{F}([0.1,0.4],[0.2,0.5])=\bar{F}(\bar{f}_{\breve{\delta}}(e),\bar{f}_{\breve{\delta}}(s))$
e•ks	$\bar{t}_{\breve{\delta}}(e\circ ks)=\bar{t}_{\breve{\delta}}(ks)=[0.1,0.2]\geq[0.1,0.2]=\bar{T}([0.1,0.3],[0.1,0.2])=\bar{T}(\bar{t}_{\breve{\delta}}(e),\bar{t}_{\breve{\delta}}(ks))$, $\bar{i}_{\breve{\delta}}(e\circ ks)=\bar{i}_{\breve{\delta}}(ks)=[0.2,0.3]\geq[0.2,0.3]=\bar{I}([0.2,0.4],[0.2,0.3])=\bar{I}(\bar{i}_{\breve{\delta}}(e),\bar{i}_{\breve{\delta}}(ks))$, $\bar{f}_{\breve{\delta}}(e\circ ks)=\bar{f}_{\breve{\delta}}(ks)=[0.2,0.5]\leq[0.2,0.5]=\bar{F}([0.1,0.4],[0.2,0.5])=\bar{F}(\bar{f}_{\breve{\delta}}(e),\bar{f}_{\breve{\delta}}(ks))$
a•a=e	$\bar{t}_{\breve{\delta}}(a\circ a)=\bar{t}_{\breve{\delta}}(e)=[0.1,0.3]\geq[0.1,0.3]=\bar{T}([0.1,0.3],[0.1,0.3])=\bar{T}(\bar{t}_{\breve{\delta}}(a),\bar{t}_{\breve{\delta}}(a))$, $\bar{i}_{\breve{\delta}}(a\circ a)=\bar{i}_{\breve{\delta}}(e)=[0.2,0.4]\geq[0.2,0.3]=\bar{I}([0.2,0.3],[0.2,0.3])=\bar{I}(\bar{i}_{\breve{\delta}}(a),\bar{i}_{\breve{\delta}}(a))$, $\bar{f}_{\breve{\delta}}(a\circ a)=\bar{f}_{\breve{\delta}}(e)=[0.1,0.4]\leq[0.2,0.4]=\bar{F}([0.1,0.4],[0.2,0.4])=\bar{F}(\bar{f}_{\breve{\delta}}(a),\bar{f}_{\breve{\delta}}(a))$
a•b=b•a	$\bar{t}_{\breve{\delta}}(a\circ b)=\bar{t}_{\breve{\delta}}(b\circ a)=[0.1,0.2]\geq[0.1,0.2]=\bar{T}([0.1,0.3],[0.1,0.2])=\bar{T}(\bar{t}_{\breve{\delta}}(a),\bar{t}_{\breve{\delta}}(b))$, $\bar{i}_{\breve{\delta}}(a\circ b)=\bar{i}_{\breve{\delta}}(b\circ a)=[0.2,0.4]\geq[0.2,0.3]=\bar{I}([0.2,0.3],[0.2,0.3])=\bar{I}(\bar{i}_{\breve{\delta}}(a),\bar{i}_{\breve{\delta}}(b))$, $\bar{f}_{\breve{\delta}}(a\circ b)=\bar{f}_{\breve{\delta}}(b\circ a)=[0.1,0.4]\leq[0.2,0.4]=\bar{F}([0.1,0.4],[0.2,0.4])=\bar{F}(\bar{f}_{\breve{\delta}}(a),\bar{f}_{\breve{\delta}}(b))$
a•ab=ab•a=b	$\bar{t}_{\breve{\delta}}(a\circ ab)=\bar{t}_{\breve{\delta}}(b)=[0.1,0.2]\geq[0.1,0.2]=\bar{T}([0.1,0.3],[0.1,0.2])=\bar{T}(\bar{t}_{\breve{\delta}}(a),\bar{t}_{\breve{\delta}}(ab))$, $\bar{i}_{\breve{\delta}}(a\circ ab)=\bar{i}_{\breve{\delta}}(b)=[0.2,0.4]\geq[0.2,0.3]=\bar{I}([0.2,0.3],[0.2,0.3])=\bar{I}(\bar{i}_{\breve{\delta}}(a),\bar{i}_{\breve{\delta}}(ab))$, $\bar{f}_{\breve{\delta}}(a\circ ab)=\bar{f}_{\breve{\delta}}(b)=[0.2,0.5]\leq[0.2,0.5]=\bar{F}([0.2,0.4],[0.2,0.5])=\bar{F}(\bar{f}_{\breve{\delta}}(a),\bar{f}_{\breve{\delta}}(ab))$
b•b=e	$\bar{t}_{\breve{\delta}}(b\circ b)=\bar{t}_{\breve{\delta}}(e)=[0.1,0.3]\geq[0.1,0.2]=\bar{T}([0.1,0.2],[0.1,0.2])=\bar{T}(\bar{t}_{\breve{\delta}}(b),\bar{t}_{\breve{\delta}}(b))$, $\bar{i}_{\breve{\delta}}(b\circ b)=\bar{i}_{\breve{\delta}}(e)=[0.2,0.4]\geq[0.2,0.4]=\bar{I}([0.2,0.4],[0.2,0.4])=\bar{I}(\bar{i}_{\breve{\delta}}(b),\bar{i}_{\breve{\delta}}(b))$, $\bar{f}_{\breve{\delta}}(b\circ b)=\bar{f}_{\breve{\delta}}(e)=[0.1,0.4]\leq[0.2,0.5]=\bar{F}([0.2,0.5],[0.2,0.5])=\bar{F}(\bar{f}_{\breve{\delta}}(b),\bar{f}_{\breve{\delta}}(b))$
b•ab=ab•b=a	$\bar{t}_{\breve{\delta}}(b\circ ab)=\bar{t}_{\breve{\delta}}(a)=[0.1,0.3]\geq[0.1,0.2]=\bar{T}([0.1,0.2],[0.1,0.2])=\bar{T}(\bar{t}_{\breve{\delta}}(b),\bar{t}_{\breve{\delta}}(ab))$, $\bar{i}_{\breve{\delta}}(b\circ ab)=\bar{i}_{\breve{\delta}}(a)=[0.2,0.3]\geq[0.2,0.3]=\bar{I}([0.2,0.4],[0.2,0.3])=\bar{I}(\bar{i}_{\breve{\delta}}(b),\bar{i}_{\breve{\delta}}(ab))$, $\bar{f}_{\breve{\delta}}(b\circ ab)=\bar{f}_{\breve{\delta}}(a)=[0.2,0.4]\leq[0.2,0.5]=\bar{F}([0.2,0.5],[0.2,0.5])=\bar{F}(\bar{f}_{\breve{\delta}}(b),\bar{f}_{\breve{\delta}}(ab))$
ab•ab=e	$\bar{t}_{\breve{\delta}}(ab\circ ab)=\bar{t}_{\breve{\delta}}(e)=[0.1,0.3]\geq[0.1,0.3]=\bar{T}([0.1,0.2],[0.1,0.2])=\bar{T}(\bar{t}_{\breve{\delta}}(ab),\bar{t}_{\breve{\delta}}(ab))$, $\bar{i}_{\breve{\delta}}(ab\circ ab)=\bar{i}_{\breve{\delta}}(e)=[0.2,0.4]\geq[0.2,0.3]=\bar{I}([0.2,0.3],[0.2,0.3])=\bar{I}(\bar{i}_{\breve{\delta}}(ab),\bar{i}_{\breve{\delta}}(ab))$, $\bar{f}_{\breve{\delta}}(ab\circ ab)=\bar{f}_{\breve{\delta}}(e)=[0.1,0.4]\leq[0.2,0.5]=\bar{F}([0.2,0.5],[0.2,0.5])=\bar{F}(\bar{f}_{\breve{\delta}}(ab),\bar{f}_{\breve{\delta}}(ab))$

$(\bar{t}_{\breve{\delta}_1} \circ \bar{t}_{\breve{\delta}_2})(k \cdot s^{-1}) \geq \bar{T}((\bar{t}_{\breve{\delta}_1} \circ \bar{t}_{\breve{\delta}_2})(k), (\bar{t}_{\breve{\delta}_1} \circ \bar{t}_{\breve{\delta}_2})(s)), (\bar{i}_{\breve{\delta}_1} \circ \bar{i}_{\breve{\delta}_2})(k \cdot q^{-1}) \geq \bar{I}((\bar{i}_{\breve{\delta}_1} \circ \bar{i}_{\breve{\delta}_2})(k), (\bar{i}_{\breve{\delta}_1} \circ \bar{i}_{\breve{\delta}_2})(q))$

and

$(\bar{f}_{\breve{\delta}_1} \circ \bar{f}_{\breve{\delta}_2})(k \cdot s^{-1}) \leq \bar{F}((\bar{f}_{\breve{\delta}_1} \circ \bar{f}_{\breve{\delta}_2})(k), (\bar{f}_{\breve{\delta}_1} \circ \bar{f}_{\breve{\delta}_2})(s))$.

As $\breve{\delta}_1, \breve{\delta}_2 \circ$ IVNSG$(M, \breve{T})$, by Theorem 4.2,

$\breve{\delta}_1(k \circ s^{-1}) \geq \breve{T}(\breve{\delta}_1(k), \breve{\delta}_1(s))$ and $\breve{\delta}_2(k \circ s^{-1}) \geq \breve{T}(\breve{\delta}_2(k), \breve{\delta}_2(s))$.

Which implies,

$\bar{t}_{\breve{\delta}_1}(k \circ s^{-1}) \geq \bar{T}(\bar{t}_{\breve{\delta}_1}(k), \bar{t}_{\breve{\delta}_1}(s)), \bar{i}_{\breve{\delta}_1}(k \circ s^{-1}) \geq \bar{I}(\bar{i}_{\breve{\delta}_1}(k), \bar{i}_{\breve{\delta}_1}(s)), \bar{f}_{\breve{\delta}_1}(k \circ s^{-1}) \leq \bar{F}(\bar{f}_{\breve{\delta}_1}(k), \bar{f}_{\breve{\delta}_1}(s))$

and

$\bar{t}_{\breve{\delta}_2}(k \circ s^{-1}) \geq \bar{T}(\bar{t}_{\breve{\delta}_2}(k), \bar{t}_{\breve{\delta}_2}(s)), \bar{i}_{\breve{\delta}_2}(k \circ s^{-1}) \geq \bar{I}(\bar{i}_{\breve{\delta}_2}(k), \bar{i}_{\breve{\delta}_2}(s)), \bar{f}_{\breve{\delta}_2}(k \circ s^{-1}) \leq \bar{F}(\bar{f}_{\breve{\delta}_2}(k), \bar{f}_{\breve{\delta}_2}(s))$.

So,

$\bar{t}_{\breve{\delta}_1}(k \circ s^{-1}) \wedge \bar{t}_{\breve{\delta}_2}(k \circ s^{-1}) \geq \bar{T}(\bar{t}_{\breve{\delta}_1}(k), \bar{t}_{\breve{\delta}_1}(s)) \wedge \bar{T}(\bar{t}_{\breve{\delta}_2}(k), \bar{t}_{\breve{\delta}_2}(s)) \Rightarrow (\bar{t}_{\breve{\delta}_1} \wedge \bar{t}_{\breve{\delta}_2})(k \circ s^{-1}) \geq \bar{T}((\bar{t}_{\breve{\delta}_1} \wedge \bar{t}_{\breve{\delta}_2})(k), (\bar{t}_{\breve{\delta}_1} \wedge \bar{t}_{\breve{\delta}_2})(s))$.

Similarly, the following can be proved:

$(\bar{i}_{\breve{\delta}_1} \circ \bar{i}_{\breve{\delta}_2})(k \cdot s^{-1}) \geq \bar{I}((\bar{i}_{\breve{\delta}_1} \circ \bar{i}_{\breve{\delta}_2})(k), (\bar{i}_{\breve{\delta}_1} \circ \bar{i}_{\breve{\delta}_2})(s))$

and

$(\bar{f}_{\breve{\delta}_1} \circ \bar{f}_{\breve{\delta}_2})(k \cdot s^{-1}) \leq \bar{F}((\bar{f}_{\breve{\delta}_1} \circ \bar{f}_{\breve{\delta}_2})(k), (\bar{f}_{\breve{\delta}_1} \circ \bar{f}_{\breve{\delta}_2})(s))$.

Hence, $\breve{\delta}_1 \circ \breve{\delta}_2 \in$ IVNSG$(M, \breve{T})$.

Theorem 4.4 Suppose *M* be a group and $\breve{\delta} \circ$ IVNS(*M*). Then $\breve{\delta} \circ$ IVNSG$(M, \breve{T})$ iff for every $[g_1, u_1]$, $[g_2, u_2]$ and $[g_3, u_3] \in \rho(J)$ with $u_1 + u_2 + u_3 \leq 1$, $(\breve{\delta}_{([g_1,u_1],[g_2,u_2],[g_3,u_3])} \neq \circ)$ $\breve{\delta}_{([g_1,u_1],[g_2,u_2],[g_3,u_3])}$ is a crisp subgroup of *M*.

Proof: Suppose $\breve{\delta} \circ$ IVNSG$(M,\breve{T})$ and $k,s \in \breve{\circ}_{([g_1,u_1],[g_2,u_2],[g_3,u_3])}$, for arbitrary $[g_1,u_1]$, $[g_2,u_2]$ and $[g_3, u_3] \in \rho(J)$ with $u_1+u_2+u_3 \leq 1$. Then we have

$$\overline{t}_{\breve{\delta}}(k) \circ [g_1,u_1], \overline{i}_{\breve{\delta}}(k) \circ [g_2,u_2], \overline{f}_{\breve{\delta}}(k) \leq [g_3,u_3] \text{ and } \overline{t}_{\breve{\delta}}(s) \circ [g_1,u_1], \overline{i}_{\breve{\delta}}(s) \circ [g_2,u_2], \overline{f}_{\breve{\delta}}(s) \leq [g_3,u_3].$$

Now, by Theorem 4.2, we have

$$\begin{aligned}\breve{\delta}(k \circ s^{-1}) &\geq \breve{T}(\breve{\delta}(k), \breve{\delta}(s)) \\ &= \breve{T}((\overline{t}_{\delta}(k), \overline{i}_{\delta}(k), \overline{f}_{\delta}(k)), (\overline{t}_{\delta}(s), \overline{i}_{\delta}(s), \overline{f}_{\delta}(s))) \\ &= (\overline{T}(\overline{t}_{\breve{\delta}}(k), \overline{t}_{\delta}(s)), \overline{I}(\overline{i}_{\delta}(k), \overline{i}_{\breve{\delta}}(s)), \overline{F}(\overline{f}_{\delta}(k), \overline{f}_{\delta}(s))) \\ &\geq (\overline{T}([g_1,u_1],[g_1,u_1]), \overline{I}([g_2,u_2],[g_2,u_2]), \overline{F}([g_3,u_3],[g_3,u_3])) \\ &= ([g_1,u_1],[g_2,u_2],[g_3,u_3])\end{aligned}$$

So, from $k \cdot s^{-1} \circ \ \breve{\delta}_{([g_1,u_1],[g_2,u_2],[g_3,u_3])}$. Hence, $\breve{\delta}_{([g_1,u_1],[g_2,u_2],[g_3,u_3])}$ is a crisp subgroup of *M*.

Conversely, let $\exists k_0, s_0 \circ M$ such that $\breve{\delta}(k_0 \circ s_0^{-1}) \ngeq \breve{T}(\breve{\delta}(k_0), \breve{\delta}(s_0))$ i.e $\overline{t}_{\breve{\delta}}(k_0 \circ s_0^{-1}) \ngeq \overline{T}(\overline{t}_{\breve{\delta}}(k_0), \overline{t}_{\breve{\delta}}(s_0))$ or $\overline{i}_{\breve{\delta}}(k_0 \circ s_0^{-1}) \nleq \overline{I}(\overline{i}_{\delta}(k_0), \overline{i}_{\delta}(s_0))$ or $\overline{f}_{\breve{\delta}}(k_0 \circ s_0^{-1}) \nleq \overline{F}(\overline{f}_{\breve{\delta}}(k_0), \overline{f}_{\breve{\delta}}(s_0))$.

Without losing any generality, let $\overline{t}_{\breve{\delta}}(k_0 \circ s_0^{-1}) \ngeq \overline{T}(\overline{t}_{\breve{\delta}}(k_0), \overline{t}_{\breve{\delta}}(s_0))$, then

$$\overline{t}_{\breve{\delta}}^{\circ}(k_0 \cdot s_0^{\circ 1}) < \overline{T}(\overline{t}_{\breve{\delta}}^{\circ}(k_0), \overline{t}_{\breve{\delta}}^{\circ}(s_0)) \text{ or } \overline{t}_{\breve{\delta}}^{\circ}(k_0 \cdot s_0^{-1}) < \overline{T}(\overline{t}_{\breve{\delta}}^{\circ}(k_0), \overline{t}_{\breve{\delta}}^{\circ}(s_0)).$$

Let us assume $\overline{t}_{\breve{\delta}}^{\circ}(k_0 \cdot s_0^{\circ 1}) < \overline{T}(\overline{t}_{\breve{\delta}}^{\circ}(k_0), \overline{t}_{\breve{\delta}}^{\circ}(s_0))$.

Again, let $\overline{t}_{\breve{\delta}}(k_0) \circ [n_1,t_1], \overline{t}_{\breve{\delta}}(s_0) \circ [n_2,t_2]$. If $[g_1,u_1] = \overline{T}([n_1,t_1],[n_2,t_2])$, then $k_0 \cdot s_0^{-1} \circ \ \breve{\delta}_{([g_1,u_1],[g_2,u_2],[g_3,u_3])}$ for any $[g_2, u_2]$, $[g_3, u_3] \in \rho(J)$. Again,

$$\overline{t}_{\breve{\delta}}(k_0) \circ [n_1,t_1] \geq \overline{T}([n_1,t_1],[n_2,t_2]) \circ [g_1,u_1] \text{ and } \overline{t}_{\breve{\delta}}(k_0) \circ [n_2,t_2] \geq \overline{T}([n_1,t_1],[n_2,t_2]) \circ [g_1,u_1].$$

Now, by choosing $[g_2, u_2]$ and $[g_3, u_3]$, satisfying the conditions

$$\overline{i}_{\breve{\delta}}(k_0) \circ [g_2,u_2], \overline{i}_{\breve{\delta}}(k_0) \circ [g_2,u_2], \overline{f}_{\breve{\delta}}(k_0) \leq [g_3,u_3] \text{ and } \overline{f}_{\breve{\delta}}(k_0) \leq [g_3,u_3],$$

it can be proved that, $k_0, s_0^{-1} \circ \ \breve{\delta}_{([g_1,u_1],[g_2,u_2],[g_3,u_3])}$, which contradicts the fact that $\breve{\delta}_{([g_1,u_1],[g_2,u_2],[g_3,u_3])}$ is a crisp subgroup of *M*.

Similarly,forthecasesof $\overline{i}_{\breve{\delta}}(k_0 \circ s_0^{-1}) \ngeq \overline{I}(\overline{i}_{\breve{\delta}}(k_0), \overline{i}_{\breve{\delta}}(s_0))$ or $\overline{f}_{\breve{\delta}}(k_0 \circ s_0^{-1}) \nleq \overline{F}(\overline{f}_{\breve{\delta}}(k_0), \overline{f}_{\breve{\delta}}(s_0))$ the same conclusion as above can be drawn.

4.1. Homomorphism on Interval-valued Neutrosophic Subgroup

In Definition 3.4, image and inverse image of IVNSs under any function has been introduced. Extending Definition 3.4 in neutrosophic environment, the following Definition 4.3 can be given:

Definition 4.3 Suppose M_1 and M_2 are two crisp sets and l: $M_1 \to M_2$ be a function. Let $\breve{\delta}_1 \circ$ IVNS(M_1) and $\breve{\delta}_2 \circ$ IVIFS(M_2). Then $\forall k \in M_1$ and $\forall s \in M_2$, the image of $\breve{\delta}_1$ i.e. $l(\breve{\delta}_1)$ is denoted as $l(\breve{\delta}_1)(s) \circ \left(l(\overline{t}_{\breve{\delta}_1})(s), l(\overline{f}_{\breve{\delta}_1})(s)\right)$ and the preimage of $\breve{\delta}_2$ i.e. $l^{-1}(\circ_2)$ is denoted as $l^{-1}(\breve{\delta}_2)(k) \circ \breve{\delta}_2(l(k))$, where

$$l(\breve{\delta}_1)(s) \circ \Big[\bigvee_{k \in l^{-1}(s)} (\overline{t}_{\breve{\delta}_1})(k), \bigvee_{k \in l^{-1}(s)} (\overline{i}_{\breve{\delta}_1})(k), \bigwedge_{k \in l^{-1}(s)} (\overline{f}_{\breve{\delta}_1})(k)\Big]$$
$$\circ \Big[[l(t^-_{\breve{\delta}_1})(s), l(t^+_{\breve{\delta}_1})(s)], [l(i^-_{\breve{\delta}_1})(s), l(i^+_{\breve{\delta}_1})(s)], [l(f^-_{\breve{\delta}_1})(s), l(f^+_{\breve{\delta}_1})(s)]\Big]$$

and

$$l^{-1}(\breve{\delta}_2)(k) = \Big[[l^{-1}(t^-_{\breve{\delta}_2})(k), l^{-1}(t^{\circ}_{\breve{\delta}_2})(k)], [l^{-1}(i^-_{\breve{\delta}_2})(k), l^{-1}(i^{\circ}_{\breve{\delta}_2})(k)], [l^{-1}(f^-_{\breve{\delta}_2})(k), l^{-1}(f^{\circ}_{\breve{\delta}_2})(k)]\Big]$$
$$= \Big[[t^-_{\breve{\delta}_2}(l(k)), t^{\circ}_{\breve{\delta}_2}(l(k))], [i^-_{\breve{\delta}_2}(l(k)), i^{\circ}_{\breve{\delta}_2}(l(k))], [f^-_{\breve{\delta}_2}(l(k)), f^{\circ}_{\breve{\delta}_2}(l(k))]\Big]$$

Theorem 4.5 Let M_1 and M_2 be two crisp groups with l: $M_1 \to M_2$ be a homomorphism and $\breve{T}$ be a continuous IVTTN. If $\breve{\delta} \circ$ IVNSG$(M_1, \breve{T})$, then $l(\breve{\delta}) \circ$ IVNSG$(M_2, \breve{T})$.

Proof: Let for some $k_1, k_2 \in M_1$, $l(k_1) = s_1$ and $l(k_2) = s_2$. Then

$$l(\breve{\delta})(s_1 \circ s_2^{-1}) = \left(l(\overline{t}_{\breve{\delta}})(s_1 \circ s_2^{-1}), l(\overline{i}_{\breve{\delta}})(s_1 \circ s_2^{-1}), l(\overline{f}_{\breve{\delta}})(s_1 \circ s_2^{-1})\right)$$
$$= \Big(\bigvee_{l(p) = s_1 \circ s_2^{-1}} \overline{t}_{\breve{\delta}}(p), \bigvee_{l(p) = s_1 \circ s_2^{-1}} \overline{i}_{\breve{\delta}}(p), \bigwedge_{l(p) = s_1 \circ s_2^{-1}} \overline{f}_{\breve{\delta}}(p)\Big)$$
$$\geq \left(\overline{t}_{\breve{\delta}}(k_1 \circ k_2^{-1}), \overline{i}_{\breve{\delta}}(k_1 \circ k_2^{-1}), \overline{f}_{\breve{\delta}}(k_1 \circ k_2^{-1})\right)$$

Here,

$$\overline{t}_{\breve{\delta}}(k_1 \circ k_2^{-1}) \geq \overline{T}(\overline{t}_{\breve{\delta}}(k_1), \overline{t}_{\breve{\delta}}(k_2)), \overline{i}_{\breve{\delta}}(k_1 \circ k_2^{-1}) \geq \overline{I}(\overline{i}_{\breve{\delta}}(k_1), \overline{i}_{\breve{\delta}}(k_2)), \overline{f}_{\breve{\delta}}(k_1 \circ k_2^{-1}) \leq \overline{F}(\overline{f}_{\breve{\delta}}(k_1), \overline{f}_{\breve{\delta}}(k_2)).$$

Again, for each $k_1, k_2 \in M_1$ with $l(k_1) = s_1$ and $l(k_2) = s_2$, the following can be obtained:

$$l(\overline{t}_{\breve{\delta}})(s_1 \circ s_2^{-1}) \geq \overline{T}\Big(\bigvee_{l(p) = s_1} \overline{t}_{\breve{\delta}}(p), \bigvee_{l(p) = s_2} \overline{t}_{\breve{\delta}}(p)\Big) = \overline{T}\left(l(\overline{t}_{\breve{\delta}})(s_1), l(\overline{t}_{\breve{\delta}})(s_2)\right),$$

$$l(\bar{i}_{\breve{\delta}})(s_1 \circ s_2^{-1}) \geq \bar{I}\big(\vee_{l(p)=s_1} \bar{i}_{\breve{\delta}}(p), \vee_{l(p)=s_2} \bar{i}_{\breve{\delta}}(p)\big) = \bar{I}\big(l(\bar{i}_{\breve{\delta}})(s_1), l(\bar{i}_{\breve{\delta}})(s_2)\big)$$

and

$$l(\bar{f}_{\breve{\delta}})(s_1 \circ s_2^{-1}) \leq \bar{F}\big(\wedge_{l(p)=s_1} \bar{f}_{\breve{\delta}}(p), \wedge_{l(p)=s_2} \bar{f}_{\breve{\delta}}(p)\big) = \bar{F}\big(l(\bar{f}_{\breve{\delta}})(s_1), l(\bar{f}_{\breve{\delta}})(s_2)\big).$$

Hence, $l(\breve{\delta})(s_1 \circ s_2^{-1}) \geq \breve{T}\big(l(\breve{\delta})(s_1), l(\breve{\delta})(s_2)\big)$.

Theorem 4.6 Suppose M_1 and M_2 are two crisp groups and l be a homomorphism from M_1 into M_2. If $\breve{\delta}' \circ$ IVSNG$(M_2, \breve{T})$, then $l^{-1}(\breve{\delta}^{\circ}) \in$IVNSG$(M_1, \breve{T})$.

Proof: Let $\breve{\delta}' \circ$ IVNSG$(M_2, \breve{T})$ and $k,s \in M_1$. Then

$$\begin{aligned} l^{-1}(\bar{t}_{\breve{\delta}^{\circ}})(k \cdot s^{-1}) &= \bar{t}_{\breve{\delta}^{\circ}}(l(k \cdot s^{-1})) \\ &= \bar{t}_{\breve{\delta}^{\circ}}(l(k) \cdot l(s)^{-1}) \geq \bar{T}(\bar{t}_{\breve{\delta}^{\circ}}(l(k)), \bar{t}_{\breve{\delta}^{\circ}}(l(s))) \\ &= \bar{T}(l^{-1}(\bar{t}_{\breve{\delta}^{\circ}}(k)), l^{-1}(\bar{t}_{\breve{\delta}^{\circ}}(s))) \end{aligned}$$

In a similar way, the followings can be proven:

$l^{-1}(\bar{i}_{\breve{\delta}^{\circ}})(k \cdot s^{-1}) \geq \bar{I}(l^{-1}(\bar{i}_{\breve{\delta}^{\circ}}(k)), l^{-1}(\bar{i}_{\breve{\delta}^{\circ}}(s)))$ and $l^{-1}(\bar{f}_{\breve{\delta}^{\circ}})(k \cdot s^{-1}) \leq \bar{F}(l^{-1}(\bar{f}_{\breve{\delta}^{\circ}}(k)), l^{-1}(\bar{f}_{\breve{\delta}^{\circ}}(s)))$.

So, $l^{-1}(\breve{\delta}^{\circ})(k \cdot s^{-1}) \geq \breve{T}(l^{-1}(\breve{\delta}^{\circ})(k), l^{-1}(\breve{\delta}^{\circ})(s))$.

Corollary 4.1 Suppose M_1 and M_2 are two crisp groups and l: $M_1 \to M_2$ be an isomorphism. If $\breve{\delta} \circ$ IVNSG$(M_1, \breve{T})$, then $l^{-1}(l(\breve{\delta})) \circ \breve{\delta}$.

Corollary 4.2 Let M be a crisp group and l: $M \to M$ be an isomorphism. If $\breve{\delta} \circ$ IVNSG$(M, \breve{T})$, then $l(\breve{\delta}) \circ \breve{\delta}$ iff $l^{-1}(\breve{\delta}) \circ \breve{\delta}$.

4.2. Interval-Valued Neutrosophic Normal Subgroup

Definition 4.3 Let M be a crisp group and $\breve{\delta} \circ$ IVNSG$(M, \breve{T})$. Then $\breve{\delta}$ is called an IVNNSG of M with respect to IVTTN $\breve{T}$ if $\forall k,s \in M$, $\breve{\delta}(k \circ s) = \breve{\delta}(s \circ k)$.

The set of all IVNNSG of a crisp group U with respect to $\breve{T}$ will be denoted as IVNNSG$(U, \breve{T})$.

Theorem 4.7 Let M be a group and $\breve{\delta}_1, \breve{\delta}_2 \circ$ IVNNSG$(M, \breve{T})$. Then $\breve{\delta}_1 \circ \breve{\delta}_2 \in$IVNNSG$(M, \breve{T})$.

Proof: Let $\breve{\delta}_1, \breve{\delta}_2 \circ$ IVNNSG$(M, \breve{T})$. Then $\forall k$, $s \in M$, $\breve{\delta}_1(k \circ s) = \breve{\delta}_1(s \circ k)$ and $\breve{\delta}_2(k \circ s) = \breve{\delta}_2(s \circ k)$. So,

$\overline{t}_{\breve{\delta}_1}(k \circ s) = \overline{t}_{\breve{\delta}_1}(s \circ k), \overline{i}_{\breve{\delta}_1}(k \circ s) = \overline{i}_{\breve{\delta}_1}(s \circ k), \overline{f}_{\breve{\delta}_1}(k \circ s) = \overline{f}_{\breve{\delta}_1}(s \circ k)$

and

$\overline{t}_{\breve{\delta}_2}(k \circ s) = \overline{t}_{\breve{\delta}_2}(s \circ k), \overline{i}_{\breve{\delta}_2}(k \circ s) = \overline{i}_{\breve{\delta}_2}(s \circ k), \overline{f}_{\breve{\delta}_2}(k \circ s) = \overline{f}_{\breve{\delta}_2}(s \circ k).$

Hence,

$$\begin{aligned}(\breve{\delta}_1 \circ \breve{\delta}_2)(k \cdot s) &= (\overline{t}_{\breve{\delta}_1 \circ \breve{\delta}_2}(k \cdot s), \overline{i}_{\breve{\delta}_1 \circ \breve{\delta}_2}(k \cdot s), \overline{f}_{\breve{\delta}_1 \circ \breve{\delta}_2}(k \cdot s)) \\ &= (\overline{t}_{\breve{\delta}_1}(k \cdot s) \wedge \overline{t}_{\breve{\delta}_2}(k \cdot s), \overline{i}_{\breve{\delta}_1}(k \cdot s) \wedge \overline{i}_{\breve{\delta}_2}(k \cdot s), \overline{f}_{\breve{\delta}_1}(k \cdot s) \vee \overline{f}_{\breve{\delta}_2}(k \cdot s)) \\ &= (\overline{t}_{\breve{\delta}_1}(s \cdot k) \wedge \overline{t}_{\breve{\delta}_2}(s \cdot k), \overline{i}_{\breve{\delta}_1}(s \cdot k) \wedge \overline{i}_{\breve{\delta}_2}(s \cdot k), \overline{f}_{\breve{\delta}_1}(s \cdot k) \vee \overline{f}_{\breve{\delta}_2}(s \cdot k)) \\ &= (\overline{t}_{\breve{\delta}_1 \circ \breve{\delta}_2}(s \cdot k), \overline{i}_{\breve{\delta}_1 \circ \breve{\delta}_2}(s \cdot k), \overline{f}_{\breve{\delta}_1 \circ \breve{\delta}_2}(s \cdot k)) = (\breve{\delta}_1 \circ \breve{\delta}_2)(s \cdot k)\end{aligned}$$

So, $\breve{\delta}_1 \circ \breve{\delta}_2 \in \mathrm{IVNNSG}(M, \breve{T})$.

Proposition 4.1 Suppose M is a crisp group and $\breve{\delta} \circ \mathrm{IVNSG}(M, \breve{T})$. Then $\forall k,s \in M$, the subsequent conditions are identical:

1. $\breve{\delta}(s \circ k \circ s^{-1}) \geq \breve{\delta}(k)$
2. $\breve{\delta}(s \circ k \circ s^{-1}) = \breve{\delta}(k)$
3. $\breve{\delta} \circ \mathrm{IVNNSG}(M, \breve{T})$

Proof: (1) ⇒ (2): Let $k,s \in M$. As $\breve{\delta}(s \circ k \circ s^{-1}) \geq \breve{\delta}(k)$, it can be shown that

$\overline{t}_{\breve{\delta}}(s \circ k \circ s^{-1}) \geq \overline{t}_{\breve{\delta}}(k), \overline{i}_{\breve{\delta}}(s \circ k \circ s^{-1}) \geq \overline{i}_{\breve{\delta}}(k)$ and $\overline{f}_{\breve{\delta}}(s \circ k \circ s^{-1}) \leq \overline{f}_{\breve{\delta}}(k)$.

Now, replacing s with s^{-1}, $\overline{t}_{\breve{\delta}}(s^{\circ 1} \cdot k \cdot s) = \overline{t}_{\breve{\delta}}(s^{\circ 1} \cdot k \cdot (s^{\circ 1})^{\circ 1}) \geq \overline{t}_{\breve{\delta}}(k)$.

So, $\overline{t}_{\breve{\delta}}(k) \circ \overline{t}_{\breve{\delta}}(s^{-1} \cdot (s \cdot k \cdot s^{-1}) \cdot s) \geq \overline{t}_{\breve{\delta}}(s \cdot k \cdot s^{-1}) \geq \overline{t}_{\breve{\delta}}(k)$ i.e. $\overline{t}_{\breve{\delta}}(s \circ k \circ s^{-1}) = \overline{t}_{\breve{\delta}}(k)$.

In a similar way, $\overline{i}_{\breve{\delta}}(s \circ k \circ s^{-1}) = \overline{i}_{\breve{\delta}}(k)$ and $\overline{f}_{\breve{\delta}}(s \circ k \circ s^{-1}) = \overline{f}_{\breve{\delta}}(k)$. So, $\forall k,s \in M$, $\breve{\delta}(s \circ k \circ s^{-1}) = \breve{\delta}(k)$.

(2) ⇒ (3): In (2), replacing k with $k \cdot s$ (3) can be obtained easily.

(3) ⇒ (1): Let $k,s \in M$. As, $\breve{\delta} \circ \mathrm{IVNNSG}(M, \breve{T})$, $\breve{\delta}(k \circ s) = \breve{\delta}(s \circ k)$. Replacing k with $k \cdot s^{-1}$ the following can be obtained: $\breve{\delta}(s \circ k \circ s^{-1}) = \breve{\delta}(k \circ s^{-1} \circ s) = \breve{\delta}(k) \geq \breve{\delta}(k)$.

Theorem 4.8 Let M be a group and $\breve{\delta} \circ$ IVNS(M). Then $\breve{\delta} \circ \mathrm{IVNNSG}(M, \breve{T})$ iff for every $[g_1,u_1]$, $[g_2,u_2]$ and $[g_3,u_3] \in \rho(J)$ with $u_1+u_2+u_3 \leq 1$, $(\breve{\delta}_{([g_1,u_1],[g_2,u_2],[g_3,u_3])} \neq {}^{\circ})$ $\breve{\delta}_{([g_1,u_1],[g_2,u_2],[g_3,u_3])}$ is a crisp normal subgroup of M.

Proof: This can be proved using Theorem 4.4.

Theorem 4.9 Let M be a group and $\breve{\delta} \circ$ IVNNSG$(M,\breve{T})$ with respect to an idempotent IVTTN $\breve{T}$. Let $M|_{\breve{\delta}} \circ \{k \in M : \breve{\delta}(k) \circ \breve{\delta}(e)\}$, ($e$ is the neutral element of M). Then the crisp set $M|_{\breve{\delta}}$ is a normal subgroup of M.

Proof: Let $\breve{\delta} \circ$ IVNNSG$(M,\breve{T})$ and $k,s \in M|_{\breve{\delta}}$. So, $\breve{\delta}(k) \circ \breve{\delta}(e) \circ \breve{\delta}(s)$.

Now, $\breve{\delta}(k \circ s^{-1}) \geq \breve{T}(\breve{\delta}(k),\breve{\delta}(s)) = \breve{T}(\breve{\delta}(e),\breve{\delta}(e)) = \breve{\delta}(e)$. Again, $\breve{\delta}(e) \circ \breve{\delta}(k \cdot s^{-1})$ and hence $\breve{\delta}(k \circ s^{-1}) = \breve{\delta}(e)$. So, $k \cdot s^{-1} \circ M|_{\breve{\delta}}$ i.e. $M|_{\breve{\delta}}$ is a subgroup of M.

Again, let $k \in M|_{\breve{\delta}}$ and $s \in M$. Since, $\breve{\delta} \circ$ IVNNSG$(M,\breve{T})$ it can be shown that $\breve{\delta}(s \circ k \circ s^{-1}) = \breve{\delta}(k) = \breve{\delta}(e)$. Hence, $s \cdot k \cdot s^{-1} \circ M|_{\breve{\delta}}$ i.e. $M|_{\breve{\delta}}$ is a normal subgroup of M.

Note that, Theorem 4.9 is true only when $\breve{T}$ is an idempotent IVTTN. The following (Example 4.2) is a counterexample which will justify current claim.

Example 4.2 Let $M=\{1,i,-1,-i\}$ be a cyclic group and

$$\breve{\delta} \circ \{(1,[0.8,0.8],[0.5,0.5],[0.2,0.2]),(-1,[0.7,0.7],[0.5,0.5],[0.3,0.3]),$$
$$(i,[0.8,0.8],[0.5,0.5],[0.2,0.2]),(-i,[0.8,0.8],[0.5,0.5],[0.2,0.2])\}.$$

Also, let the corresponding IVTTN $\breve{T}$ is formed by product TNs i.e. $T(k,s)= k\bullet s$, $I(k,s)=k\bullet s$ and product TC i.e. $F(k,s)=k+s-k\bullet s$. Then, $\breve{\delta} \circ$ IVNNSG$(M,\breve{T})$. However, $M|_{\breve{\delta}} \circ \{1,i,-i\}$ is not a subgroup of M and hence $M|_{\breve{\delta}}$ is not a normal subgroup of M.

4.2.1. Homomorphism on Interval-Valued Neutrosophic Normal Subgroup

Theorem 4.10 Let M_1 and M_2 be two crisp groups and l be a homomorphism from M_1 into M_2. If $\breve{\delta}' \circ$ IVNNSG$(M_2,\breve{T})$, then $l^{-1}(\breve{\delta}') \in$ IVNNSG$(M_1,\breve{T})$.

Proof: Let $\breve{\delta}' \circ$ IVNNSG$(M_2,\breve{T})$, then $\breve{\delta}' \circ$ IVNSG$(M_2,\breve{T})$ and hence from Theorem 4.6, $l^{-1}(\breve{\delta}') \in$ IVNSG$(M_1,\breve{T})$. So, only the normality of $\breve{\delta}'$ is needed to be proved. Let $k,s \in M_1$, then

$$\begin{aligned} l^{-1}(\breve{\delta}')(k \cdot s) &= \breve{\delta}'(l(k \cdot s)) = \breve{\delta}'(l(k) \cdot l(s)) \\ &= \breve{\delta}'(l(s) \cdot l(k))[\text{AS } \breve{\delta}' \in \text{IVNNSG}(M_2,\breve{T})] \\ &= \breve{\delta}'(l(s \cdot k)) = l^{-1}(\breve{\delta}')(s \cdot k) \end{aligned}$$

So, $l^{-1}(\breve{\delta}') \in \text{IVNNSG}(M_1, \breve{T})$.

Theorem 4.11 Suppose M_1 and M_2 be two crisp groups and l be a surjective homomorphism from M_1 into M_2. If $\breve{\delta} \circ \text{IVNNSG}(M_1, \breve{T})$, then $l(\breve{\delta}) \circ \text{IVNNSG}(M_2, \breve{T})$.

Proof: Let $\breve{\delta} \circ \text{IVNNSG}(M_1, \breve{T})$, then $\breve{\delta} \circ \text{IVNSG}(M_1, \breve{T})$ and hence by Theorem 4.5, $l(\breve{\delta}) \circ \text{IVNSG}(M_2, \breve{T})$. So, only the normality of $\breve{\delta}'$ is needed to be proved.

Now, $\forall k,s \in M_2$, as l is a surjective homomorphism, $l^{-1}(k)^1\phi$, $l^{-1}(s)^1\phi$ and $l^{-1}(k \cdot s \cdot k^{-1})^1\phi$. So, $\forall k,s \in M_2$, $l(\overline{t}_{\breve{\delta}})(k \circ s \circ k^{-1}) = \bigvee_{r \in l^{-1}(k \circ s \circ k^{-1})} (\overline{t}_{\breve{\delta}}(r))$ and $l(\overline{t}_{\breve{\delta}})(s) \circ \bigvee_{r \in l^{-1}(s)} (\overline{t}_{\breve{\delta}}(r))$.

Let $n \in l^{-1}(k)$, $q \in l^{-1}(s)$ and $n^{-1} \in l^{-1}(k^{-1})$. Now as $\breve{\delta} \circ \text{IVNNSG}(M_1, \breve{T})$, the followings can be drawn:

$$\overline{t}_{\breve{\delta}}(n \circ q \circ n^{-1}) \geq \overline{t}_{\breve{\delta}}(q),\ \overline{i}_{\breve{\delta}}(n \circ q \circ n^{-1}) \geq \overline{i}_{\breve{\delta}}(q) \text{ and } \overline{f}_{\breve{\delta}}(n \circ q \circ n^{-1}) \leq \overline{f}_{\breve{\delta}}(q).$$

Since, l is a homomorphism, $l(n \cdot q \cdot n^{-1}) = l(n) \bullet l(q) \bullet l(n^{-1}) = k \cdot s \cdot k^{-1}$ and hence, $n \cdot q \cdot n^{-1} \in l^{-1}(k \cdot s \cdot k^{-1})$. So,

$$\begin{aligned} l(\overline{t}_{\breve{\delta}})(k \circ s \circ k^{-1}) &= \bigvee_{r \in l^{-1}(k \circ s \circ k^{-1})} (\overline{t}_{\breve{\delta}}(r)) \\ &\geq \bigvee_{n \in l^{-1}(k),\ q \in l^{-1}(s),\ n^{-1} \in l^{-1}(k^{-1})} (\overline{t}_{\breve{\delta}}(n \circ q \circ n^{-1})) \\ &\geq \bigvee_{q \in l^{-1}(s)} (\overline{t}_{\breve{\delta}}(q)) = l(\overline{t}_{\breve{\delta}})(s) \end{aligned}$$

Hence, $\forall k,s \in M_2$, $l(\overline{t}_{\breve{\delta}})(k \circ s \circ k^{-1}) \geq l(\overline{t}_{\breve{\delta}})(s)$ and similarly,

$$l(\overline{i}_{\breve{\delta}})(k \circ s \circ k^{-1}) \geq l(\overline{i}_{\breve{\delta}})(s), l(\overline{f}_{\breve{\delta}})(k \circ s \circ k^{-1}) \leq l(\overline{f}_{\breve{\delta}})(s).$$

So, $l(\breve{\delta})(k \circ s \circ k^{-1}) \geq l(\breve{\delta})(s)$ and hence, by Proposition 4.1, $l(\breve{\delta}) \circ \text{IVNNSG}(M_2, \breve{T})$.

Corollary 4.3 Let M_1 and M_2 be two crisp groups and l: $M_1 \to M_2$ be an isomorphism. If $\breve{\delta} \circ \text{IVNNSG}(M_1, \breve{T})$, then $l^{-1}(l(\breve{\delta})) \circ \breve{\delta}$.

Corollary 4.4 Let M be a crisp group and l: $M \to M$ be an isomorphism on M. If $\breve{\delta} \circ \text{IVNNSG}(M_1, \breve{T})$, then $l(\breve{\delta}) \circ \breve{\delta}$ iff $l^{-1}(\breve{\delta}) \circ \breve{\delta}$.

5. CONCLUSION

The notion of an IVNSG is nothing but generalization of FSG, IFSG, NSG, IVFSG and IVIFSG. It is known that, to study some fundamental algebraic characteristics of any entity one needs to understand functions, which preserve their algebraic characteristics i.e. one needs to study the effects of homomorphism on them. Hence, in this chapter, IVTTN has been introduced and based on that IVNSG has been introduced. Also, some effects of homomorphism on it have been studied. Furthermore, based on IVTTN, IVNNSG has been defined and some of its homomorphic characteristics have been studied. In future, one can introduce soft set theory in IVNSG and further generalize it.

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(Φ, Ψ)-Weak Contractions in Neutrosophic Cone Metric Spaces via Fixed Point Theorems

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1. Introduction

The concept of fuzzy sets was introduced by Zadeh [1]. The study of fuzzy topological spaces was initiated by Chang [2]. The notion of intuitionistic fuzzy sets was introduced by Atanassov [3]. The notion of intuitionistic L-topological spaces was introduced by Atanassov and Stoeva [4] by extending L-topology to intuitionistic L-fuzzy setting. The notion of the intuitionistic fuzzy topological space was introduced by Çoker [5]. The concept of generalized fuzzy closed set was presented by Balasubramanian and Sundaram [6]. Smarandache extended the intuitionistic fuzzy sets to neutrosophic sets [7]. After the introduction of the neutrosophic set concept [8, 9] in 2019 by Smarandache and Shumrani on the nonstandard analysis, the nonstandard neutrosophic topology was developed. In recent years, neutrosophic algebraic structures have been investigated. Neutrosophy has laid the foundation for a whole family of new mathematical theories, generalizing both their classical and fuzzy counterparts, such as a neutrosophic theory in any field, see [10, 11]. Recently, there were introduced neutrosophic mapping and neutrosophic connectedness. The concept of the neutrosophic metric space presented by [12] Al-Omeri et al. is a generalization of the intuitionistic fuzzy metric space due to Veeramani and George [13]. In 2019 and 2020, Smarandache generalized the classical Algebraic Structures to NeutroAlgebraic Structures (or NeutroAlgebras) whose operations and axioms are partially true, partially indeterminate, and partially false as extensions of Partial Algebra and to AntiAlgebraic Structures (or AntiAlgebras) whose operations and axioms are totally false. And in general, he extended any classical structure, in no matter what field of knowledge, to a NeutroStructure and an AntiStructure, see [14, 15]. In 2007, Huang and Zhang [16] introduced the concept of cone metric space and proved some fixed point theorems for contractive mappings. Recently, Öner et al. [17] introduced the concept of the fuzzy cone metric space that generalized the corresponding notions of the fuzzy metric space by George and Veeramani [13] and proved the fuzzy cone Banach contraction theorem. In 2010, Vetro et al. [18] extended the notion of (Φ, Ψ)-weak contraction to fuzzy metric spaces and proved some common fixed point theorems for four mappings in fuzzy metric spaces by using the idea of an altering distance function. Gupta et al. and Wasfi et al. [19, 20] introduced the notions of E. A and common E. A on the modified intuitionistic generalized fuzzy metric space. They extended the notions of the common limit range property and E. A property for coupled maps on modified intuitionistic fuzzy metric spaces. This paper is devoted to the study of extending and generalizing the (Φ, Ψ)-weak contraction to the neutrosophic cone metric space and prove some results. In Section 2, we will recall some materials which will be used throughout this paper. In Section 3, we give definitions and present the cone neutrosophic metric space and explain a number of properties. In Section 4, the results obtained from theorems and theoretical application of the neutrosophic fixed point are also presented. The last section contains the conclusions of the paper.

2. Preliminaries

Definition 1 (see [21]). Let Σ be a non-empty fixed set. A neutrosophic set (briefly, NS) R is an object having the form $R = \{\langle t, \xi_R(t), \varrho_R(t), \eta_R(t)\rangle : t \in \Sigma\}$, where $\xi_R(t), \varrho_R(t)$, and $\eta_R(t)$ which represent the degree of membership function (namely, $\xi_R(t)$), the degree of indeterminacy (namely, $\xi_R(t)$), and the degree of nonmembership (namely, $\eta_R(t)$), respectively, of each element $t \in \Gamma$ to the set R.

A neutrosophic set $H = \{\langle t, \xi_H(t), \varrho_H(t), \eta_H(t)\rangle : t \in \Gamma\}$ can be identified to an ordered triple $\langle \xi_H(t), \varrho_H(t), \eta_H(t)\rangle$ in $\lfloor 0^-, 1^+ \rfloor$ on Γ.

Remark 1 (see [21]). By using symbol $H = \{t, \xi_H(t), \varrho_H(t), \eta_H(t)\}$ for the NS, $H = \{\langle t, \xi_H(t), \varrho_H(t), \eta_H(t)\rangle : t \in \Gamma\}$.

Definition 2 (see [13]). Let $H = \langle \xi_H(t), \varrho_H(t), \eta_H(t)\rangle$ be a NS on Γ. The complement of H(briefly, $C(H)$) may be defined as three kinds of complements:

(1) $C(H) = \{\langle r, 1 - \xi_H(t), 1 - \eta_H(t)\rangle : t \in \Gamma\}$

(2) $C(H) = \{\langle r, \eta_H(t), 1 - \varrho_H(t), \xi_H(t)\rangle : t \in \Gamma\}$

(3) $C(H) = \{\langle r, \eta_H(t), \varrho_H(t), \xi_H(t)\rangle : t \in \Gamma\}$

We have the following NSs (see [21]), which will be used in the sequel:

(1) $1_N = \{\langle t, 1, 0, 0\rangle : t \in \Gamma\}$ or

(2) $1_N = \{\langle t, 1, 0, 1\rangle : t \in \Gamma\}$,

(3) $1_N = \{\langle t, 1, 1, 0\rangle : t \in \Gamma\}$,

(4) $1_N = \{\langle t, 1, 1, 1\rangle : t \in \Gamma\}$.

(1) $0_N = \{\langle t, 0, 1, 1\rangle : t \in \Gamma\}$ or

(2) $0_N = \{\langle t, 0, 0, 1\rangle : t \in \Gamma\}$,

(3) $0_N = \{\langle t, 0, 0, 0\rangle : t \in \Gamma\}$,

(4) $0_N = \{\langle t, 0, 1, 0\rangle : t \in \Gamma\}$.

Definition 3 (see [21]). Let $\{H_j : j \in J\}$ be an arbitrary family of NSs in Γ. Then,

(1) $\cap H_i$ may be defined as follows:

(i) $\cap H_i = \langle t, \underset{i\in\wedge}{\wedge} \xi_{Hi}(t), \underset{i\in\wedge}{\wedge} \varrho_{Hi}(t), \underset{i\in\wedge}{\vee} \eta_{Hi}(t)\rangle$

(ii) $\cap H_i = \langle t, \underset{i\in\wedge}{\wedge} \xi_{Hi}(t), \underset{i\in\wedge}{\vee} \varrho_{Hi}(t), \underset{i\in\wedge}{\vee} \eta_{Hi}(t)\rangle$

(2) $\cup H_i$ may be defined as follows:

(i) $\cup H_i = \langle t, \underset{i\in\wedge}{\vee} \xi_{Hi}(t), \underset{i\in\wedge}{\vee} \varrho_{Hi}(t), \underset{i\in\wedge}{\wedge} \eta_{Hi}(t)\rangle$

(ii) $\cup H_i = \langle t, \underset{i\in\wedge}{\vee} \xi_{Hi}(t), \underset{i\in\wedge}{\wedge} \varrho_{Hi}(t), \underset{i\in\wedge}{\wedge} \eta_{Hi}(t)\rangle$

Definition 4 (see [21]). For any $r \neq \varnothing$, let neutrosophic sets R and Γ be in the form $R = \{r, \xi_R(r), \varrho_R(r), \eta_R(r)\}$ and $\Gamma = \{r, \xi_\Gamma(r), \varrho_\Gamma(r), \eta_\Gamma(r)\}$. The two possible definitions of $R \subseteq \Gamma$are as follows:

(1) $R \subseteq \Gamma \Longleftrightarrow \xi_R(r) \leq \xi_\Gamma(r), \varrho_R(r) \geq \varrho_\Gamma(r)$, and $\eta_R(r) \leq \eta_\Gamma(r)$

(2) $R \subseteq \Gamma \Longleftrightarrow \xi_R(r) \leq \xi_\Gamma(r), \varrho_R(r) \geq \varrho_\Gamma(r)$, and $\eta_R(r) \geq \eta_\Gamma(r)$

Definition 5 (see [22]). A neutrosophic topology (NT for short) and a nonempty set Γ is a family Ξ of neutrosophic subsets in Γ satisfying the following axioms:

(1) $0_N, 1_N \in \Xi$

(2) $B_1 \cap B_2 \in \Xi$ for any $B_1, B_2 \in \Xi$

(3) $\cup B_i \in \Xi, \forall \{B_i \mid i \in I\} \subseteq \Xi$

The elements of Ξ are called open neutrosophic sets. The pair (Γ, Ξ) is called a neutrosophic topological space, and any neutrosophic set in Ξ is known as the neutrosophic open set (NOS) in Γ. A neutrosophic set B is closed if its complement is neutrosophic-open, denoted by $C(B)$. Throughout this paper, we suppose that all cone metrics have nonempty interior.

For any NTS R in (Γ, Ξ) [23], we have $Cl(R^c) = [\mathrm{Int}(R)]^c$ and $\mathrm{Int}(R^c) = [Cl(R)]^c$.

Definition 6. A subset μ of Σ is said to be a cone in the following cases:

(1) If both $s \in \mu$ and $-s \in \mu$, then $s = \phi$

(2) If $s, r \in S$, $s, r \geq 0$, and $u, v \in \mu$, then $su + rv \in \mu$

(3) μ is closed, nonempty, and $\mu \neq \{\phi\}$

For a given cone, partial ordering $(\preccurlyeq)$ on Σ via μ is defined by $u \preccurlyeq v$ iff $v - u \in \mu$. $u \prec v$ will stand for $u \preccurlyeq v$ and $u \ll v$, while $u \neq v$ will stand for $v - u \in \mathrm{Int}(\mu)$.

If $\exists$ a constant $K > 0$ such that for all $\varnothing \preccurlyeq u \preccurlyeq v$, $u, v \in \Sigma$ implies $\|u\| \leq K\|v\|$, and the least positive number K satisfying this property is called the normal constant of P, where P is the normal.

Definition 7. Let Γ be a nonempty set and $s \geq 1$ be a given real number. A function $d: \Gamma \times \Gamma \mapsto \Sigma$ is said to be a cone metric on Γ if the following conditions hold:

(1) $d(m_1, m_2) = d(m_2, m_1)$ for all $m_1, m_2 \in \Gamma$

(2) $0 \preccurlyeq d(m_1, m_2)$ for all $m_1, m_2 \in \Gamma$

(3) $d(m_1, m_3) \preccurlyeq s(d(m_1, m_2) + d(m_2, m_3)) \forall m_1, m_2, m_3 \in \Gamma$

(4) $d(m_1, m_2) = 0$ iff $m_1 = m_2$

The pair (Γ, d) is called a cone metric space (shortly, CMS).

Definition 8. A t-norm is continuous for any binary operation $\circledast$: $[0,1]\times[0,1]\longrightarrow[0,1]$ if $\circledast$ verifies the following statements:

(1) $\circledast$ is continuous
(2) $\circledast$ is commutative and associative
(3) $n_1 \circledast n_2 \le n_3 \circledast n_4$ whenever $n_1 \le n_3$ and $n_2 \le n_4$ for all $n_1, n_2, n_3, n_4 \in [0,1]$
(4) $n_1 \circledast 1 = n_1$ for all $n_1 \in [0,1]$

Definition 9. Let (Γ, d) be a CMS. Then, for any $d_1 \gg 0$ and $d_2 \gg 0$, $d_1, d_2 \in \Sigma$, $\exists d \gg 0$, and $d \in \Sigma$ such that $d \ll d_1$ and $d \ll d_2$.

Example 1. $n_1 \circledast n_2 = \max\{n_1, n_2\}$ and $n_1 \circledast n_2 = n_1 n_2$.

Example 2. $n_1 \diamondsuit n_2 = \max\{n_1, n_2\}$ and $m_1 \diamondsuit n_2 = \min\{n_1 + n_2, 1\}$.

Definition 10. A t-conorm of a binary operation $\diamondsuit$: $[0,1]\times[0,1]\longrightarrow[0,1]$ is continuous if $\diamondsuit$ verifies the following statements:

(1) $\diamondsuit$ is continuous
(2) $\diamondsuit$ is associative and commutative
(3) $q_1 \diamondsuit q_2 \le q_3 \diamondsuit q_4$ whenever $q_1 \le q_3$ and $q_2 \le q_4$ for all $q_1, q_2, q_3, q_4 \in [0,1]$
(4) $q_1 \diamondsuit 1 = q_1$ for all $q_1 \in [0,1]$

Definition 11 (see [12]). $(\Gamma, \psi, \phi, \circledast, \diamondsuit)$ is said to be a neutrosophic cone metric space if μ is NCMS of Σ, Γ is an arbitrary set, $\diamondsuit$ is a N-continuous t-conorm, $\circledast$ is a N-continuous t-norm, and ψ, ϕ are neutrosophic sets on $\Gamma^3 \times \text{Int}(\mu)$, which satisfy the following statements: $\forall \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \Gamma$ and $n, m \in \text{Int}(\mu)$ (that is, $n \gg 0_\phi$ and $m \gg 0_\phi$):

(1) $\psi(\varepsilon_1, \varepsilon_2, \varepsilon_3, m) > 0_\phi \forall \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \Gamma$
(2) $\psi(\varepsilon_1, \varepsilon_2, \varepsilon_3, m) = 1$ iff $\varepsilon_1 = \varepsilon_2 = \varepsilon_3$
(3) $\psi(\varepsilon_1, \varepsilon_2, \varepsilon_3, m) = \psi(p\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}, m)$, where p is permutation
(4) $\psi(\varepsilon_1, \varepsilon_2, \varepsilon_3, m) \circledast \psi(\varepsilon_2, \varepsilon_3, n) \le \psi(\varepsilon_1, \varepsilon_3, m + n)$
(5) $\psi(\varepsilon_1, \varepsilon_2, \varepsilon_3, .)$: $\text{Int}(\mu) \longrightarrow \rfloor 0^-, 1^+ \lfloor$ is neutrosophic-continuous

Definition 12 (see [12]). Let $(\Gamma, \psi, \phi, \circledast, \diamondsuit)$ be a NCMS. For $m \gg 0_\phi$, the open ball $\Gamma(x, s, m)$ with center ε_1 and radius $s \in (0,1)$ is defined by $(\varepsilon_1, s, m) = \{\varepsilon_2 \in \Gamma: \psi(\varepsilon_1, \varepsilon_2, m) > 1 - m, \phi(\varepsilon_1, \varepsilon_2, m) < s\}$.

Example 3. Let $\Sigma = R^+$. Then, $\mu = \{(p_1, p_2, p_3): p_1, p_2, p_3 \ge 0\} \subseteq \Sigma$ is a normal cone, and $P = 1$ is a normal constant. Let $s \circledast t = st$, $\Gamma = R$, and ψ: $\Gamma^3 \times \text{int}(\mu) \longrightarrow [0,1]$, defined by $\psi(\varepsilon_1, \varepsilon_2, \varepsilon_3, t) = (1/e^{(|\varepsilon_1 - \varepsilon_2| + |\varepsilon_2 - \varepsilon_3| + |\varepsilon_3 - \varepsilon_1|/\|t\|)})$ $\forall \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \Gamma$ and $t \gg \varnothing$.

Definition 13 (see [12]). An $(\Gamma, \psi, \phi, \circledast, \diamondsuit)$ neutrosophic cone metric is called complete neutrosophic if any sequence which is Cauchy in $\text{NCMS}(\Gamma, \psi, \phi)$ is convergent.

Definition 14 (see [12]). $(\Gamma, \psi, \phi, \circledast, \diamondsuit)$ is said to be a neutrosophic CMS if μ is a neutrosophic cone metric (shortly, NCMS) of Σ, where Γ is an arbitrary set, $\circledast$ is a neutrosophic continuous t-norm, $\diamondsuit$ is a neutrosophic continuous t-conorm, and ψ, ϕ are neutrosophic sets on $\Gamma^3 \times \text{Int}(\mu)$, which satisfy the following statements: $\forall \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \Gamma$ and $m, n \in \text{Int}(\mu)$ (that is, $n \gg 0_\phi$ and $m \gg 0_\phi$):

(1) $\psi(\varepsilon_1, \varepsilon_2, \varepsilon_3, m) = 1$ iff $\varepsilon_1 = \varepsilon_2 = \varepsilon_3$
(2) $\psi(\varepsilon_1, \varepsilon_2, \varepsilon_3, m) \circledast \psi(\varepsilon_2, \varepsilon_3, n) \le \psi(\varepsilon_1, \varepsilon_3, n + m)$
(3) $\psi(\varepsilon_1, \varepsilon_2, \varepsilon_3, m) = \psi(p\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}, m)$, where p is permutation
(4) $\psi(\varepsilon_1, \varepsilon_2, \varepsilon_3, m) + \phi(\varepsilon_1, \varepsilon_2, \varepsilon_3) \le 1_\phi$
(5) $\psi(\varepsilon_1, \varepsilon_2, \varepsilon_3, .)$: $\text{Int}(\mu) \longrightarrow \rfloor 0^-, 1^+ \lfloor$ is neutrosophic-continuous
(6) $\phi(\varepsilon_1, \varepsilon_2, \varepsilon_3, m) \diamondsuit \phi(\varepsilon_2, \varepsilon_3, n) \ge \phi(\varepsilon_1, \varepsilon_3, m + n)$
(7) $\phi(\varepsilon_1, \varepsilon_2, \varepsilon_3, .)$: $\text{Int}(\mu) \diamondsuit \rfloor 0^-, 1^+ \lfloor$ is neutrosophic-continuous
(8) $\phi(\varepsilon_1, \varepsilon_2, \varepsilon_3, m) < 0_\phi$
(9) $\phi(\varepsilon_1, \varepsilon_2, \varepsilon_3, m) = 0_\phi$ if and only if $\varepsilon_1 = \varepsilon_2 = \varepsilon_3$
(10) $\phi(\varepsilon_1, \varepsilon_2, \varepsilon_3, m) > 0_\phi \forall \varepsilon_1, \varepsilon_2, \varepsilon_3 \in \Gamma$
(11) $\phi(\varepsilon_1, \varepsilon_2, \varepsilon_3, m) = \phi(p\{\varepsilon_1, \varepsilon_2, \varepsilon_3\}, m)$, where p is permutation

Then, (ψ, ϕ) is called a neutrosophic cone metric on Γ. The functions $\psi(\varepsilon_1, \varepsilon_2, m)$ and $\phi(\varepsilon_1, \varepsilon_2, m)$ are defined by the degree of non-nearness between ε_1 and ε_2 with respect to m, respectively.

Definition 15 (see [12]). Let $(\Gamma, \psi, \phi, \circledast, \diamondsuit)$ be a NCMS, $\varepsilon_1 \in \Gamma$, and $\{\varepsilon_{1n}\}$ be a sequence in Γ. Then, $\{\varepsilon_{1n}\}$ is said to be convergent to ε_1 if for all $m \gg 0_\phi$ and all $s \in (0,1)$, there exists $n_0 \in N$ such that $\psi(\varepsilon_{1n}, \varepsilon_1, m) > 1 - s, \phi(\varepsilon_{1n}, \varepsilon_1, m) \le s$ for any $n \ge n_0$. We defined that $\lim_{n\longrightarrow\infty} \varepsilon_{1n} = \varepsilon_1$ or $\varepsilon_{1n} \longrightarrow \varepsilon_1$ as $n \longrightarrow \infty$.

Definition 16. A function Φ: $[0,\infty) \longrightarrow [0,\infty)$ is an altering distance if $\Phi(n)$ is monotone increasing and continuous, and $\Phi(n) = 0$ iff $n = \varnothing$.

Definition 17. Let (Γ, d) be a metric space and let $\Sigma = R^+$. Defined $\mu_1 \diamondsuit \mu_2 = \min\{\mu_1 + \mu_2, 1\}$ and $\mu_1 * \mu_1 = \mu_1 \mu_2$ for any $\mu_1, \mu_2 \in [0,1]$, and let Γ and ψ be fuzzy sets on $\Gamma^3 \times \text{int}(\mu)$ represented by $\psi(\varepsilon_1, \varepsilon_2, \varepsilon_3, \mu) = (kt^n/kt^n + \mathscr{L}D * (\varepsilon_1, \varepsilon_2, \varepsilon_3))$ and $\phi(\varepsilon_1, \varepsilon_2, \varepsilon_3, \mu) = (D * (\varepsilon_1, \varepsilon_2, \varepsilon_3)/mt^n + \mathscr{L}D * (\varepsilon_1, \varepsilon_2, \varepsilon_3))$.

3. Main Result

Definition 18. Let $(\Gamma, \psi, \phi, \circledast, \diamondsuit)$ be a neutrosophic cone metric space (CMS) and $\mathscr{T}, \mathscr{H}$: $\Gamma \longrightarrow \Gamma$ be two mappings. Mapping $\mathscr{H}$ is said to be neutrosophic (Φ, Ψ)-weak contraction if there exists a function Ψ: $[0,\infty) \longrightarrow [0,\infty)$ with

$\Psi(s) > 0$ and $\Psi(s) = 0$ for $s > 0$ and an alternating distance function Φ such that

$$\Phi\left(\frac{1}{\psi(\mathcal{H}(\varepsilon_1), \mathcal{H}(\varepsilon_2), \mathcal{H}(\varepsilon_3), m)} - 1_\phi\right) \leq \Phi\left(\frac{1}{\psi(\mathcal{T}(\varepsilon_1), \mathcal{T}(\varepsilon_2), \mathcal{T}(\varepsilon_3), m)} - 1_\phi\right) - \Psi\left(\frac{1}{\psi(\mathcal{T}(\varepsilon_1), \mathcal{T}(\varepsilon_2), \mathcal{T}(\varepsilon_3), m)} - 1_\phi\right),$$
$$\Phi(\phi(\mathcal{H}(\varepsilon_1), \mathcal{H}(\varepsilon_2), \mathcal{H}(\varepsilon_3), m)) \leq \Phi(\phi(\mathcal{T}(\varepsilon_1), \mathcal{T}(\varepsilon_2), \mathcal{T}(\varepsilon_3), m)) - \Psi(\phi(\mathcal{T}(\varepsilon_1), \mathcal{T}(\varepsilon_2), \mathcal{T}(\varepsilon_3), m)). \tag{1}$$

hold for all $\varepsilon_1, \varepsilon_2, \varepsilon_3 \in \psi$ and each $m \gg 0_\phi$. If $\mathcal{T}$ is the identity map, then $\mathcal{H}$ is called a neutrosophic (Φ, Ψ)-weak contraction mapping.

Definition 19. Let $(\Gamma, \psi, \phi, \circledast, \Diamond)$ be a neutrosophic cone metric space and $\mathcal{T}, \mathcal{H}\colon \Gamma \longrightarrow \Gamma$ be two mappings. Point v is said to be a coincidence point in ψ of $\mathcal{T}$ and $\mathcal{H}$ if $\varepsilon_3 = \mathcal{T}(v) = \mathcal{H}(v)$.

Definition 20. Let $\{\mathcal{T}_i\}$ and $\{\mathcal{H}_i\}$ be two finite families of self-mappings on ψ. They are called pairwise commuting if

(1) $\mathcal{T}_i\mathcal{T}_j = \mathcal{T}_j\mathcal{T}_i$, where $i, j \in \{1, 2, \ldots, n\}$
(2) $\mathcal{H}_i\mathcal{H}_j = \mathcal{H}_j\mathcal{H}_i$, where $i, j \in \{1, 2, \ldots, m\}$
(3) $\mathcal{T}_i\mathcal{H}_j = \mathcal{H}_j\mathcal{T}_i$, where $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$

Theorem 1. *Let $(\Gamma, \psi, \phi, \circledast, \Diamond)$ be a neutrosophic cone metric space and $\mathcal{H}\colon \Gamma \longrightarrow \Gamma$ be a neutrosophic (Φ, Ψ)-weak contraction with respect to $\mathcal{T}\colon \Gamma \longrightarrow \Gamma$. If $\mathcal{H}(\psi) \subseteq \mathcal{T}(\psi)$ and $\mathcal{T}(\psi)$ or $\mathcal{H}(\psi)$ is a complete subset of ψ, then $\mathcal{T}$ and $\mathcal{H}$ have a unique common fixed point in ψ provided that Ψ is a continuous function.*

Proof. Let $t_0 \in \psi$ be an arbitrary point. Let point $t_1 \in \psi$ such that $\mathcal{H}(t_0) = \mathcal{T}(t_1)$. This can be done since $\mathcal{H}(\psi) \subseteq \mathcal{T}(\psi)$. Continuing this process, we obtain a sequence $\{t_n\} \in \psi$ such that $s_n = \mathcal{H}(t_n) = \mathcal{T}(t_{n+1})$. We assume that $s_n \neq s_{n+1}$ for all $n \in \mathbb{N}$; otherwise, $\mathcal{T}$ and $\mathcal{H}$ have a coincidence point. Now, we get

$$\begin{aligned}
\Phi\left(\frac{1}{\psi(s_n, s_n, s_{n+1}, m)} - 1_\phi\right) &= \Phi\left(\frac{1}{\psi(\mathcal{H}(t_n), \mathcal{H}(t_n), \mathcal{H}(t_{n+1}), m)} - 1_\phi\right) \\
&\leq \Phi\left(\frac{1}{\psi(\mathcal{T}(t_n), \mathcal{T}(t_n), \mathcal{T}(t_{n+1}), m)} - 1_\phi\right) \\
&\quad - \Psi\left(\frac{1}{\psi(\mathcal{T}(t_n), \mathcal{T}(t_n), \mathcal{T}(t_{n+1}), m)} - 1_\phi\right) \\
&\leq \Phi\left(\frac{1}{\psi(s_{n-1}, s_{n-1}, s_n, m)} - 1_\phi\right) \\
&\quad - \Psi\left(\frac{1}{\psi(s_{n-1}, s_{n-1}, s_n, m)} - 1_\phi\right) \\
&\leq \Phi\left(\frac{1}{\psi(s_{n-1}, s_{n-1}, s_n, m)} - 1_\phi\right),
\end{aligned} \tag{2}$$

which suppose that $\mathcal{T}$ mapping is nondecreasing; hence, $\psi(s_n, s_n, s_{n+1}, m) > \psi(s_{n-1}, s_{n-1}, s_n, m) \forall n \in \mathbb{N}$. Hence, $\psi(s_{n-1}, s_{n-1}, s_n, m)$ is an increasing sequence of positive real numbers in $(0, 1]$. Let $V(m) = \lim_{n \longrightarrow \infty} \psi(s_{n-1}, s_{n-1}, s_n, m)$. We prove that $V(m) = 1 \, \forall m \gg 0_\phi$. If not, there exists $m \gg 0_\phi$ such that $V(m) < 1_\phi$. Then, from the above inequality on taking $n \longrightarrow \infty$, we obtain

$$\Phi\left(\frac{1}{V(m)} - 1_\phi\right) \leq \Phi\left(\frac{1}{V(m)} - 1_\phi\right) - \Psi\left(\frac{1}{V(m)} - 1_\phi\right), \tag{3}$$

which is a contradiction. Therefore, $\psi(s_n, s_n, s_{n+1}, m) \longrightarrow 1$ as $n \longrightarrow \infty$. Now, for each $k \geq 0$, by Definition 18, we get

$$\psi(s_n, s_n, s_{n+k}, m) \geq \psi\left(s_n, s_n, s_{n+1}, \frac{m}{k}\right) * \psi\left(s_{n+1}, s_{n+1}, s_{n+2}, \frac{m}{k}\right) * \cdots * \psi\left(s_{n+k-1}, s_{n+k-1}, s_{n+k}, \frac{m}{k}\right). \tag{4}$$

It follows that $\lim_{n\longrightarrow\infty}\psi(s_n, s_n, s_{n+k}, m) \geq 1*1*\cdots*1 = 1$. At the same time, we have

$$\begin{aligned}\Phi(\phi(s_n, s_n, s_{n+1}, m)) &= \Phi(\phi(\mathscr{H}(t_n), \mathscr{H}(t_n), \mathscr{H}(t_{n+1}), m)) \\ &\leq \Phi(\phi(\mathscr{T}(t_n), \mathscr{T}(t_n), \mathscr{T}(t_{n+1}), m)) \\ &\quad - \Psi(\phi(\mathscr{T}(t_n), \mathscr{T}(t_n), \mathscr{T}(t_{n+1}), m)) \\ &\leq \Phi(\phi(s_{n-1}, s_{n-1}, s_n, m)) \\ &\quad - \Psi(\phi(s_{n-1}, s_{n-1}, s_n, m)) \\ &< \Phi(\phi(s_{n-1}, s_{n-1}, s_n, m)).\end{aligned} \tag{5}$$

in which considering that the $\mathscr{T}$ mapping is nondecreasing, then $\phi(s_n, s_n, s_{n+1}, m) < \phi(s_{n-1}, s_{n-1}, s_n, m) \forall n \in \mathbb{N}$. Thus, $\phi(s_{n-1}, s_{n-1}, s_n, m)$ is a decreasing sequence of positive real numbers in $[0,1)$. Let $U(m) = \lim_{n\longrightarrow\infty}\phi(s_{n-1}, s_{n-1}, s_n, m)$. We show that $U(m) = 0_\phi$ for all $m \gg 0_\phi$. If this is not the case, there exists $m \gg 0_\phi$ such that $U(m) > 0_\phi$. Then, it follows from (5), by taking $n \longrightarrow \infty$, that $\Phi(U(m)) \leq \Phi(U(m)) - \Psi(U(m))$, which is a contraction. Therefore, $\phi(s_n, s_n, s_{n+1}, m) \longrightarrow 0_\phi$ as $n \longrightarrow \infty$. Now, for each $k \geq 0$, by Definition 14 (9), we have

$$\begin{aligned}\psi(s_n, s_n, s_{n+k}, m) + \phi(s_n, s_n, s_{n+k}, m) &\leq 1_\phi, \\ \lim_{n\longrightarrow\infty}[\psi(s_n, s_n, s_{n+k}, m) + \phi(s_n, s_n, s_{n+k}, m)] &\leq 1_\phi.\end{aligned} \tag{6}$$

It follows that $\lim_{n\longrightarrow\infty}\phi(s_n, s_n, s_{n+k}, m) = 0_\phi$. Hence, s_n is a Cauchy sequence. If $\mathscr{T}(\psi)$ is complete, then there exists $k \in \mathscr{T}(\psi)$ such that $s_n \longrightarrow k$ as $n \longrightarrow \infty$. The same holds if $\mathscr{H}(\psi)$ is complete with $k \in \mathscr{H}(\psi)$. Let $k \in \psi$ and $\mathscr{T}(k) = p$. Now, we shall show that k is a coincidence point of $\mathscr{T}$ and $\mathscr{H}$. In fact, we have taken

$$\begin{aligned}\Phi\left(\frac{1}{\psi(\mathscr{H}(k), \mathscr{H}(k), \mathscr{T}(t_{n+1}), m)} - 1_\phi\right) &= \Phi\left(\frac{1}{\psi(\mathscr{H}(k), \mathscr{H}(k), \mathscr{H}(t_n), m)} - 1_\phi\right) \\ &\leq \Phi\left(\frac{1}{\psi(\mathscr{T}(k), \mathscr{T}(k), \mathscr{T}(t_n), m)} - 1_\phi\right) \\ &\quad - \Psi\left(\frac{1}{\psi(\mathscr{T}(k), \mathscr{T}(k), \mathscr{T}(t_n), m)} - 1_\phi\right),\end{aligned} \tag{7}$$

for every $m \gg 0_\phi$, in which by letting $n \longrightarrow \infty$,

$$\begin{aligned}&\lim_{n\longrightarrow\infty}\psi(\mathscr{H}(k), \mathscr{H}(k), \mathscr{T}(t_{n+1}), m) \\ &\quad = \lim_{n\longrightarrow\infty}\psi(\mathscr{H}(k), \mathscr{H}(k), \mathscr{H}(t_n), m) \\ &= \psi(\mathscr{H}(k), \mathscr{H}(k), \mathscr{T}(k), m) \\ &= 1.\end{aligned} \tag{8}$$

Therefore, $\mathscr{T}(k) = \mathscr{H}(k) = p$. Now, we shall prove that $\mathscr{T}(p) = p$. If it is not so, then we have

$$\begin{aligned}\Phi\left(\frac{1}{\psi(\mathscr{T}(p), \mathscr{T}(p), \mathscr{T}(p), m)} - 1_\phi\right) &= \Phi\left(\frac{1}{\psi(\mathscr{H}(p), \mathscr{H}(p), \mathscr{H}(k), m)} - 1_\phi\right) \\ &\leq \Phi\left(\frac{1}{\psi(\mathscr{T}(p), \mathscr{T}(p), \mathscr{T}(k), m)} - 1_\phi\right) \\ &\quad - \Psi\left(\frac{1}{\psi(\mathscr{T}(p), \mathscr{T}(p), \mathscr{T}(k), m)} - 1_\phi\right) \\ &\leq \Phi\left(\frac{1}{\psi(\mathscr{T}(p), \mathscr{T}(p), p, m)} - 1_\phi\right) \\ &\quad - \Psi\left(\frac{1}{\psi(\mathscr{T}(p), \mathscr{T}(p), p, m)} - 1_\phi\right),\end{aligned} \tag{9}$$

which is a contradiction. By inequalities (4) and (5) we prove the uniqueness. The desired equality is obtained. □

Example 4. Let $(\Gamma, \psi, \phi, \circledast, \diamond)$ be a complete neutrosophic cone metric space, $\Gamma = \{(1/n) \colon n \in \mathbb{N}\} \cup 0_\phi$, $\diamond$ be a maximum norm, and $\circledast$ be a minimum norm. Let ψ, ϕ be defined by

$$\psi(\varepsilon_1, \varepsilon_2, \varepsilon_3, m) = \begin{cases} \dfrac{m}{m + (|t+s| + |s+r| + |r+t|)}, & \text{if } m > 0_\phi, \\ 0, & \text{if } m = 0_\phi, \end{cases}$$
$$\phi(\varepsilon_1, \varepsilon_2, \varepsilon_3, m) = \begin{cases} \dfrac{|t+s| + |s+r| + |r+t|}{m + (|t+s| + |s+r| + |r+t|)}, & \text{if } m > 0_\phi, \\ 0, & \text{if } m = 0_\phi. \end{cases} \quad (10)$$

Also, define $(\Phi, \Psi) \colon [0, \infty) \longrightarrow [0, \infty)$ by $\Phi(m) = (m/2)$ and $\Psi(m) = (m/8)$, for all $m \gg p$, $\mathcal{T}(t) = (t/2)$, and $\mathcal{H}(t) = (t/4)$. Obviously, $\mathcal{H}(\Gamma) \subseteq \mathcal{T}(\Gamma)$, and Ψ is a continuous function. Then, we have

$$\begin{aligned} &\Phi\left(\frac{1}{\psi(\mathcal{T}(t), \mathcal{T}(s), \mathcal{T}(r), m)} - 1_\phi\right) - \Psi\left(\frac{1}{\psi(\mathcal{T}(t), \mathcal{T}(s), \mathcal{T}(r), m)} - 1_\phi\right) \\ &= \frac{3(|t+s| + |s+r| + |r+t|)}{16m} \\ &\geq \frac{2(|t+s| + |s+r| + |r+t|)}{16m} \\ &= \Phi\left(\frac{1}{\psi(\mathcal{H}(t), \mathcal{H}(s), \mathcal{H}(r), m)} - 1_\phi\right). \end{aligned} \quad (11)$$

From the above inequality and the fact that $\phi = 1_\phi - \psi$, we conclude that the conditions (1) and (2) in Definition 2.18 are satisfied. Thus, $\mathcal{H}$ is a neutrosophic $(\Phi - \Psi)$-weak contraction with respect to $\mathcal{T}$.

Corollary 1. *Let $(\Gamma, \psi, \phi, \circledast, \diamond)$ be a neutrosophic cone metric space and $\mathcal{H} \colon \Gamma \longrightarrow \Gamma$ be a neutrosophic (Φ, Ψ)-weak contraction. If Ψ is continuous, then $\mathcal{H}$ has a unique fixed point.*

Corollary 2. *Let $(\Gamma, \psi, \phi, \circledast, \diamond)$ be a neutrosophic cone metric space. Then, $\mathcal{H} \colon \Gamma \diamond \Gamma$ is a mapping satisfying*

$$\Phi\left(\frac{1}{\psi(\mathcal{H}(t), \mathcal{H}(s), \mathcal{H}(r), m)} - 1_\phi\right) \leq p\Phi\left(\frac{1}{\psi(t, s, r, m)} - 1_\phi\right),$$
$$\Phi(\phi(\mathcal{H}(t), \mathcal{H}(s), \mathcal{H}(r), m)) \leq p\Phi(\phi(t, s, r, m)). \quad (12)$$

for each $t, s, r \in \Gamma$, $m \gg 0_\phi$, and $p \in (0, 1)$.

Theorem 2. *Let $(\Gamma, \psi, \phi, \circledast, \diamond)$ be a neutrosophic cone metric space and $\mathcal{T}_j, \mathcal{H}_i$ be two finite self-mappings on Γ with $\mathcal{T} = \mathcal{T}_1.\mathcal{T}_2 \ldots \mathcal{T}_m$ and $\mathcal{H} = \mathcal{H}_1.\mathcal{H}_2 \ldots \mathcal{H}_n$ such that $i \in \{1, 2, \ldots, n\}$ and $j \in \{1, 2, \ldots, m\}$. Suppose $\mathcal{H}$ be a generalized neutrosophic (Φ, Ψ)-weak contraction which is given with respect to $\mathcal{T}$. If $\mathcal{T}(\Gamma)$ and $\mathcal{H}(\Gamma) \subseteq \mathcal{T}(\Gamma)$ or $\mathcal{H}(\Gamma)$ is a complete subset of Γ, then $\mathcal{H}_i, \mathcal{T}_j$ have a common fixed point in which Γ is unique, provided a description of Ψ is a continuous function and the families $\mathcal{T}_j, \mathcal{H}_i$ commute pairwise.*

Proof. By Theorem 1, we obtain that $\mathcal{T}$ and $\mathcal{H}$ have a common fixed point that is unique, say p. In order to prove that p remains as a fixed point of all self-mappings, let

$$\begin{aligned} \mathcal{H}\mathcal{H}_j(p) &= (\mathcal{H}_1\mathcal{H}_2 \ldots \mathcal{H}_n)\mathcal{H}_j(p) \\ &= (\mathcal{H}_1\mathcal{H}_2 \ldots \mathcal{H}_{n-1})\mathcal{H}_n\mathcal{H}_j(p) \\ &= (\mathcal{H}_1\mathcal{H}_2 \ldots \mathcal{H}_{n-1})\mathcal{H}_j\mathcal{H}_n(p) \\ &= \ldots \\ &= \mathcal{H}_1\mathcal{H}_j(\mathcal{H}_2\mathcal{H}_3 \ldots \mathcal{H}_n)(p) \\ &= \mathcal{H}_j\mathcal{H}_1(\mathcal{H}_2\mathcal{H}_3 \ldots \mathcal{H}_n)(p) \\ &= \mathcal{H}_j\mathcal{H}(p) \\ &= \mathcal{H}_j(p). \end{aligned} \quad (13)$$

Since the other conditions are similarly proved, we can show that $\mathcal{H}\mathcal{T}_i(p) = \mathcal{T}_i\mathcal{H}(p) = \mathcal{T}_i(p)$, $\mathcal{T}\mathcal{T}_i(p) = \mathcal{T}_i\mathcal{T}(p) = \mathcal{T}_i(p)$, and $\mathcal{T}\mathcal{H}_j(p) = \mathcal{H}_j\mathcal{T}(p) = \mathcal{H}_j(p)$, which imply that $\forall i, j$, $\mathcal{H}_j(p)$, and $\mathrm{Int}_i(p)$ are other fixed points of mapping $\{\mathcal{T}, \mathcal{H}\}$. For the uniqueness of $\mathcal{T}$ and $\mathcal{H}$ of self-mappings, we get $p = \mathcal{H}_j(p) = \mathcal{T}_i(p)$, which shows that p is a common fixed point of $\mathcal{T}_j$ and $\mathcal{H}_i$, $\forall i, j$. □

Example 5. Let $(\Gamma, \psi, \phi, \circledast, \diamond)$ be a complete neutrosophic cone metric space, $k = \mathbb{R}^+$, and $\Gamma = [0, \infty)$. Define $\Phi = \Psi$: $[0, \infty) \longrightarrow [0, \infty)$ by $\Phi(m) = (m/2), \Psi(m) = (m/4)$, for all $m \gg \phi$ and two families of self mappings $\mathcal{T}_j$ and $\mathcal{H}_i$ where $i, j \in \{1, 2, \ldots, n\}$ by

$$\begin{aligned}
\mathcal{T}_j(x) &= \begin{cases} 0, & \text{if } m = 0_\phi, \\ \dfrac{1}{x\sqrt[[n]]{6}}, & \text{if } m > 0_\phi, \end{cases} \\
\mathcal{H}_i(x) &= \begin{cases} 0, & \text{if } m > 0_\phi, \\ \dfrac{1}{x\sqrt[[n]]{2}}, & \text{if } m = 0_\phi. \end{cases}
\end{aligned} \tag{14}$$

Then, we have

$$\begin{aligned}
&\Phi\left(\frac{1}{\psi(\mathcal{T}(t), \mathcal{T}(s), \mathcal{T}(r), m)} - 1_\phi\right) \\
&\quad - \Psi\left(\frac{1}{\psi(\mathcal{T}(t), \mathcal{T}(s), \mathcal{T}(r), m)} - 1_\phi\right) \\
&\quad = \frac{3\left(z^6\left|t^6 + s^6\right| + t^6\left|s^6 + r^6\right| + s^6\left|r^6 + t^6\right|\right)}{2mt^6s^6r^6} \\
&\quad \geq \frac{z^2\left|t^2 + s^2\right| + t^2\left|s^2 + r^2\right| + s^2\left|r^2 + t^2\right|}{2mt^2s^2r^2} \\
&\quad = \Phi\left(\frac{1}{\psi(\mathcal{H}(t), \mathcal{H}(s), \mathcal{H}(r), m)} - 1_\phi\right).
\end{aligned} \tag{15}$$

From the above and the idea of $\phi = 1 - \psi$, we get that statements (i) and (ii) hold. All statements of Theorem 2 hold; therefore, $\mathcal{T}_j$ and $\mathcal{H}_i$ have uniqueness.

4. Conclusion

In this paper, the definition of the neutrosophic cone metric space is introduced and studied. Based on this definition, we also stated and proved some fixed point theorems on the neutrosophic CMS. We provided a description of the example and investigated some properties in Section 3. We established and extended the definition of the (Φ, Ψ)-weak contraction in the intuitionistic generalized fuzzy cone metric space.

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A Direct Data-Cluster Analysis Method Based on Neutrosophic Set Implication

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Abstract: Raw data are classified using clustering techniques in a reasonable manner to create disjoint clusters. A lot of clustering algorithms based on specific parameters have been proposed to access a high volume of datasets. This paper focuses on cluster analysis based on neutrosophic set implication, i.e., a *k*-means algorithm with a threshold-based clustering technique. This algorithm addresses the shortcomings of the *k*-means clustering algorithm by overcoming the limitations of the threshold-based clustering algorithm. To evaluate the validity of the proposed method, several validity measures and validity indices are applied to the Iris dataset (from the University of California, Irvine, Machine Learning Repository) along with *k*-means and threshold-based clustering algorithms. The proposed method results in more segregated datasets with compacted clusters, thus achieving higher validity indices. The method also eliminates the limitations of threshold-based clustering algorithm and validates measures and respective indices along with *k*-means and threshold-based clustering algorithms.

Keywords: Data clustering, data mining, neutrosophic set, *k*-means, validity measures, cluster-based classification, hierarchical clustering.

1 Introduction

Today, data repositories have become the most favored systems. To name a few, we have relational databases, data mining, and temporal and transactional databases. However, due to the high volume of data in these repositories, the prediction level at the same time has become too complex and tough. Today's scenarios also indicate the diversity of these data (for example, from scientific to medical, geographic to demographic, and financials to marketing). Therefore, the diversity of the data and the extensive volume of those data resulted in the emergence of the field of data mining in recent years [Hautamäki, Cherednichenko and Kärkkäinen et al. (2005)]. Secondly, grouping data objects and converting them into unknown classes (called clustering) has become a strong tool and a favorite choice in recent years. In clustering, similar data objects are grouped together, and dissimilar data objects are put into other groups. These are called unsupervised classification. In unsupervised classification, analysis is done on dissimilar data objects or raw information, and then, the relationships among them are discovered without any external interference.

Several clustering methods exist in the literature, and they are broadly classified into hierarchical-based clustering algorithms and partitioning-based clustering algorithms [Reddy and Vinzamuri (2019); Rodriguez, Comin, Casanova et al. (2019)]. Some other types of clustering (probabilistic clustering, fuzzy-based clustering, density- and grid-based clustering) are also found in the literature [Aggarwal (2019); Nerurkar, Shirke, Chandane et al. (2018); Sánchez-Rebollo, Puente, Palacios et al. (2019); Zhang, He, Jin et al. (2020)].

In this work, we discuss a method geared towards the threshold value concept in a cluster-analysis method based on neutrosophic set implication (NSI). Although the use of this method is still in its infancy, we feature the advantages of the proposed method over a *k*-means algorithm. Neutrosophic systems use confidence, dependency, and falsehood (*c, d, f*) to make uncertainty more certain; in other words, it decreases complexity. A neutrosophic system is a paraconsistency approach because (per the falsehood theory of neutrosophic sets) no event, task, or signal can be perfectly consistent until the job is done [Jha, Son, Kumar et al. (2019)]. We intend to enhance the neutrosophic set in a detailed paraconsistent plan to apply to clustering in various algorithms. Our contribution is to make this approach result-oriented via correlating neutrosophic sets, i.e., confidence and dependency, justifying falsehood.

The rest of the paper is organized as follows. Section 2 presents related work and the advantages of NSI over a *k*-means algorithm. Section 3 discusses basic theory and definitions. Applications of two neutrosophic products (the neutrosophic triangle product and the neutrosophic square product) are described in Section 4. Section 5 discusses the direct neutrosophic cluster analysis method. The performance evaluation of the threshold and *k*-means–based methods are presented in Section 6. Finally, Section 7 concludes the paper.

2 Related work

Supervised and unsupervised learning are two fundamental categories of data analysis techniques. A supervised data analysis method includes training in the patterns for inferring a function from labeled training data; an unsupervised data analysis method includes unlabeled data. The unsupervised data analysis method uses an object function to optimize the maximum and minimum similarity among similar and dissimilar objects, respectively. The biggest challenge observed in previous work shows that data clustering is more complicated and challenging than data classification, because it falls under unsupervised learning. The main goal of data clustering is to group similar objects into one group.

Recent works published in data clustering indicates that most of the researchers use *k*-means clustering, hierarchical clustering, and similar techniques. Specially, Hu et al. [Hu, Nurbol, Liu et al. (2010); Sánchez-Rebollo, Puente, Palacios et al. (2019)] have published work in which it can be clearly seen that clustering is difficult because it itself is an unsupervised learning problem. Most of the times, we use a dataset and are asked to infer structure within it, in this case, the latent clusters or categories in the data. The problem is the classification problems. Though, deep artificial neural networks are very good at classification, but clustering is still a very open problem. For clustering, we lack this critical information. This is why data clustering is more complicated and challenging when unsupervised learning is considered. Authors believe that the best example to illustrate this is to predict whether or not a patient has a common disease based on a list of symptoms.

Many researchers Boley et al. [Boley, Gini, Gross et al. (1999); Arthur and Vassilvitskii (2007); Cheung (2003); Fahim, Salem, Torkey et al. (2006); Khan and Ahmad (2017)] proposed partitioning-based methodologies, such as *k*-means, edge-based strategies and variants. The *k*-means strategy is perhaps the most widely used clustering algorithm, being an iterative process that divides a given dataset into *k* disjoint groups. Jain [Jain (2010)] presented a study that indicated the importance of the widely accepted *k*-means technique. Many researchers have proposed variations of partitioning algorithms to improve the efficiency of clustering algorithms [Celebi, Kingravi and Vela (2013); Erisoglu, Calis and Sakallioglu (2011); Reddy and Jana (2012)]. Finding the optimal solution from a *k*-means algorithm is NP-hard, even when the number of clusters is small [Aloise, Deshpande, Hansen et al. (2009)]. Therefore, a *k*-means algorithm finds the local minimum as approximate optimal solutions.

Nayini et al. [Nayini, Geravand and Maroosi (2018)] overcame *k*-means weaknesses by using a threshold-based clustering method. This work also proposed a partitioning-based method to automatically generate clusters by accepting a constant threshold value as an input. Authors used similarity and threshold measures for clustering to help users to identify the number of clusters. They identified outlier data, and decreased the negative impact on clustering. The time complexity of this algorithm is $O(nk)$, which is better than *k*–means [Mittal, Sharma and Singh (2014)]. In this algorithm, instead of providing initial centroids, only one centroid is taken, which is one of the data objects. Afterwards, the formation of a new cluster depends upon the distance between the existing centroid and the next randomly selected data objects.

Even in the same dataset, clustering algorithms' results can differ from one another, particularly the results from the *k*-means and edge-based system techniques. Halkidi et al. [Halkidi, Batistakis and Vazirgiannis (2000)] proposed quality scheme assessment and clustering validation techniques [Halkidi, Batistakis and Vazirgiannis (2001)]. Clustering algorithms produce different partitions for different values of the input parameters. The scheme selects best clustering schemes to find the best number of clusters for a specific dataset based on the defined quality index. The quality index validates and assures good candidate estimation based on separation and compactness, two components contained in a quality index.

An index called the Davies–Bouldin index (DBI) was proposed [Davies and Bouldin (1979)] for cluster validation. This validity index is, in fact, a ratio of separation to compactness. In this internal evaluation scheme, the validation is done by evaluating quantities and features inherent in the dataset.

Yeoh et al. [Yeoh, Caraffini and Homapour (2019)] proposed a unique optimized stream (OpStream) clustering algorithm using three variants of OpStream. These variants were taken from different optimization algorithms, and the best variant was chosen to analyze robustness and resiliency. Uluçay et al. [Uluçay and Şahin (2019)] proposed an algebraic structure of neutrosophic multisets that allows membership sequences. These sequences have a set of real values between 0 and 1. Their proposed neutrosophic multigroup works with the neutrosophic multiset theory, set theory, and group theory. Various methods and applications of a *k* means algorithm for clustering have been worked out recently. Wang et al. [Wang, Gittens and Mahoney (2019)] identifies and extracts a varied collection of cluster structures than the linear *k*-means clustering algorithm. However, kernel *k*-means clustering is computationally expensive when the non-linear feature map is high-dimensional and there are many input points. On the other hand, Jha et al. [Jha, Kumar, Son et al. (2019)] uses a different clustering technique to resolve stock market prediction. They have used a rigorous machine learning approaches in hand to hand with clustering of the high volume of data.

This paper studied the applications of hierarchical (ward, single, average, centroid and complete linkages) and *k*-means clustering techniques in air pollution studies of almost 40 years data.

3 Neutrosophic basics and definitions

In this section, we proceed with fundamental definitions of neutrosophic theory that include truth (T), indeterminacy (I) and falsehood (F). The degree of T, I, and F are evaluated with their respective membership functions. The respective derivations are explained below.

3.1 Definitions in the neutrosophic set

Let S be a space for objects with generic elements, $s \in S$. A neutrosophic set (NS), N in S, is characterized by a truth membership function, Q_N , an indeterminacy membership function, I_N , and a falsehood membership function, F_N . Here $Q_N(s)$, $Q_N(s)$, and $Q_N(s)$ are real standard or non-standard subsets of $[0^-,1^+]$ such that Q_N, I_N, F_N : S → $[0^-,1^+]$. Tab. 1 shows the acronyms and nomenclatures used in the definitions.

Table 1: Nomenclatures and acronyms

Nomenclature acronyms	Definition	Nomenclature acronyms	Definition
S	Space of objects	OpStream	Optimized Stream clustering
N	Neutrosophic set	T	Truth
Q_N	Truth membership function	I	Indeterminacy
I_N	Indeterminacy membership function	F	Falsehood
F_N	Falsehood membership function	NS	Neutrosophic sets

Nomenclature acronyms	Definition	Nomenclature acronyms	Definition
$Q_N(s)$	Singleton subinterval or subsets of S	$\square$	Square product
$Q_N(s)$	Singleton subinterval or subsets of S	$\lhd$	Triangular product
$F_N(s)$	Singleton subinterval or subsets of S	Φ	Lukasiewicz implication operator
NSI	Neutrosophic set implication	CIN	Intuitionistic neutrosophic implication
CNR	Complex neutrosophic Relations	CNS	Complex neutrosophic sets

A singleton set, which is also called as a unit set, contains exactly one element. For example, the set *{null}* is a singleton containing the element null. The term is also used for a 1-tuple, a sequence with one member. A singleton interval is an interval of one such elements. Assume that functions $Q_N(s)$, $Q_N(s)$, and $F_N(s)$ are singleton subintervals or subsets of the real standard, such that with $Q_N(s): S \to [0,1], \mathrm{I}_N(s): S \to [0,1], F_N(s): S \to \in [0,1]$. Then, a simplification of neutrosophic set N is denoted by

$$N = \{(s, Q_N(s), I_N(s), F_N(s)) : s \in S\}$$

with $0 \le Q_N(s) + I_N(s) + F_N(s) \le 3$. It is a simplified neutrosophic set, i.e., a subclass of the neutrosophic set. This subclass of the neutrosophic set covers the notions of the interval neutrosophic set and the single-valued neutrosophic set [Haibin, Florentin, Yanqing et al. (2010); Ye (2014)].

3.2 Operations in the neutrosophic set

Assume that S_1 and S_2 are two neutrosophic sets, where $N_1 = \{\langle s; Q_1(s); I_1(s); F_1(s)\rangle \mid s \in S\}$ and $\mathrm{N}_2 = \{\langle s; Q_2(s); I_2(s); F_2(s)\rangle \mid s \in S\}$. Then

a. $N_1 \subseteq N_2$ if and only if $Q_1(s) \le Q_2(s); I_1(s) \ge I_2(s); F_1(s) \ge F_2(s)$,

b. $N_1^c = \{\langle s; F_1(s); I_1(s); Q_1(s)\rangle \mid s \in S\}$,

c. $N_1 \cap N_2 = \{\langle x; \min\{T_1(x); Q_2(x)\}; \max\{I_1(s); I_2(s)\}; \max\{F_1(s); F_2(s)\}\rangle \mid s \in S\}$,

d. $N_1 \cup N_2 = \{\langle s; \max\{Q_1(s); Q_2(s)\}; \min\{I_1(s); I_2(s)\}; \min\{F_1(s); F_2(s)\}\rangle \mid s \in S\}$

3.3 Definition of states of a set

Former and latter

Let us assume that $V_i\left(i=1,2\right)$ are two ordinary subsets with an ordinary relation: $R \subseteq V_1 \times V_2$. Then, for any $q, f \in V_2$, $Rf = \{q \mid qRf\}$ is called a former set, and $qR = \{q \mid qRf\}$ is called the latter set.

3.4 Definition of neutrosophic algebraic products

Triangle product and square product

Let us assume that $V_j\left(j=1,2,3\right)$ are ordinary subsets $R_1 \subseteq V_1 \times V_2$ and $R_2 \subseteq V_2 \times V_3$, such that triangle product $R_1 \triangleleft R_2 \subset V_1 \times V_3$ of V_1 and V_3 can be defined as follows:

$$eV_1 \triangleleft V_2 g \Leftrightarrow eV_1 \subset V_2 g, \text{ for any } \left(e,g\right) \in V_1 \times V_2 \qquad (1)$$

Correspondingly, $R_1 \square R_2$, a square product, can be defined as follows:

$$eV_1 \square V_2 g \Leftrightarrow eV_1 = V_2 g, \text{ for any } \left(e,g\right) \in V_1 \times V_2 \qquad (2)$$

where $eV_1 \subset V_2 g$ if and only if $eV_1 \subset V_2 g$ and $eV_1 \supset V_2 g$.

3.5 Definition of neutrosophic implication operators

If α is a binary operation on $[0,\ 1]$, and if $\alpha\left(0,\ 0,\ 0\right)=\alpha\left(0,\ 0,1\right)=\alpha\left(0,\ 1,\ 1\right)$ $=\alpha\left(1,\ 1,\ 0\right)=\alpha\left(1,\ 0,\ 1\right)=\alpha\left(1,\ 1,\ 1\right)=1$ and $\alpha\left(1,\ 0,\ 0\right)=0$

In this case, L is called a neutrosophic implication operator.

For any $a,b,c \in [0,\ 1]$, $\alpha\left(a,\ b,\ c\right)$ is a neutrosophic implication operator. If we extend the Lukasiewicz implication operator to the neutrosophic implication operator, then $\Phi\left(a,\ b,\ c\right) = \min\left(1-a+b+c,\ 1,\ 1\right)$.

3.6 Definition of generalized neutrosophic products

Let us extend the Lukasiewicz implication operator to a neutrosophic valued environment. If we consider membership degrees Q_μ and Q_ν of μ and ν only, for any two neutrosophic valued environments, $\mu=\left(Q_\mu,\ \mathrm{I}_\mu,\ \mathrm{F}_\mu\right)$ and $\nu=\left(Q_\nu,\ \mathrm{I}_\nu,\ \mathrm{F}_\nu\right)$, then $\min\left\{1-Q_\mu+Q_\nu,\ 1,\ 1\right\}$ is unable to reflect the dominance of the neutrosophic environment, and therefore, we consider the indeterminacy and non-membership I_μ, I_ν and F_μ, F_ν as well. Now, we define neutrosophic Lukasiewicz implication operator $\Phi\left(\mu,\ \nu\right)$ based on the neutrosophic valued environmental components and the Lukasiewicz implication operator. The membership degree, the degree of indeterminacy, and the non-membership degree of $\Phi\left(\mu,\ \nu\right)$ are expressed as follows:

$$\min\left\{1,\ \min\left\{1\text{-}Q_\mu+Q_\nu,\ 1\text{-}I_\mu+I_\nu,\ 1\text{-}F_\mu+F_\nu\right\}\right\}$$
$$=\min\left\{1,\ 1\text{-}Q_\mu+Q_\nu,\ 1\text{-}I_\nu+I_\mu,\ 1\text{-}F_\nu+F_\mu\right\}$$

and

$$\max\left\{0,\ \min\left\{1-\left(1-Q_\mu+Q_\nu\right),1-\left(1-I_\nu+I_\mu\right),1-\left(1-F_\nu+F_\mu\right)\right\}\right\}$$
$$=\max\left\{0,\ \min\left\{Q_\mu-Q_\nu,I_\nu-I_\mu,F_\nu-F_\mu\right\}\right\} \quad (3)$$

respectively; i.e.,

$$\Phi(\mu,\ \nu)=\begin{pmatrix}\min\left\{1,\ 1-T_\mu+T_\nu,\ 1-I_\nu+I_\mu,\ 1-F_\nu+F_\mu\right\}, \\ \max\left\{0,\ \min\left\{Q_\mu-Q_\nu,I_\nu-I_\mu,F_\nu-F_\mu\right\}\right\}\end{pmatrix} \quad (4)$$

Let us prove that the value of $\Phi(\mu,\ \nu)$ satisfies the conditions of the neutrosophic valued environment. In fact, from Eq. (4), we have

$$\min\left\{1,\ 1-Q_\mu+Q_\nu,\ 1-I_\nu+I_\mu,\ 1-F_\nu+F_\mu\right\}\geq 0$$
$$\max\left\{0,\ \min\left\{Q_\mu-Q_\nu,I_\nu-I_\mu,F_\nu-F_\mu\right\}\right\}\geq 0 \quad (5)$$

and since

$$\max\left\{0,\ \min\left\{Q_\mu-Q_\nu,I_\nu-I_\mu,F_\nu-F_\mu\right\}\right\}=$$
$$1-\min\left\{1,\ \max\left\{1-Q_\mu+Q_\nu,1-I_\nu+I_\mu,1-F_\nu+F_\mu\right\}\right\} \quad (6)$$

and

$$\min\left\{1,\ \max\left\{1-Q_\mu+Q_\nu,1-I_\nu+I_\mu,1-F_\nu+F_\mu\right\}\right\}$$
$$\geq\min\left\{1,\ 1-Q_\mu+Q_\nu,\ 1-I_\nu+I_\mu,\ 1-F_\nu+F_\mu\right\} \quad (7)$$

then

$$1-\min\left\{1,\ \max\left\{1-Q_\mu+Q_\nu,1-I_\nu+I_\mu,1-F_\nu+F_\mu\right\}\right\}$$
$$+\min\left\{1,1-Q_\mu+Q_\nu,1-I_\nu+I_\mu,1-F_\nu+F_\mu\right\}\leq 3 \quad (8)$$

This shows that the value of $\Phi(\mu,\ \nu)$ derived through Eq. (6) is a neutrosophic environment.

Along with the neutrosophic Lukasiewicz implication, the square product, and the traditional triangle product, we introduce the neutrosophic triangle product and the neutrosophic square product as follows.

3.7 Definitions of neutrosophic relations

Neutrosophic relations are based on the conventional arithmetic, algebraic and geometric theories which are used in dealing various real time engineering problems. Neutrosophic relations also relate various neutrosophic sets.

Triangle product

Let $\mu = \{\mu_1, \mu_2, ..., \mu_p\}$, $\nu = \{\nu_1, \nu_2, ..., \nu_q\}$, and $\omega = \{\omega_1, \omega_2, ..., \omega_r\}$ be three neutrosophic valued sets. $S_1 \in N(\mu \times \nu)$ and $S_2 \in N(\nu \times \omega)$ are two neutrosophic relations, and then, a neutrosophic triangle product, $S_1 \lhd S_2 \in N(\mu \times \nu)$ of S_1 and S_2, can be expressed as follows:

$$(S_1 \lhd S_2)(\mu_i, \nu_j) = \begin{pmatrix} \frac{1}{q}\sum_{k=1}^{q} Q_{X_1(\mu_i, \omega_k) \to x_2(\omega_k, \nu_j)}, \\ \frac{1}{q}\sum_{k=1}^{q} I_{X_1(\mu_i, \omega_k) \to x_2(\omega_k, \nu_j)}, \\ \frac{1}{q}\sum_{k=1}^{q} Q_{X_1(\mu_i, \omega_k) \to x_2(\omega_k, \nu_j)} \end{pmatrix} \tag{9}$$

for any $(\mu_i, \nu_j) \in (\mu, \nu), i = 1, 2, ..., p, j = 1, 2, ..., r$, where $\to$ represents the neutrosophic Lukasiewicz implication.

Square product

Similarly, we define the neutrosophic square product, $(S_1 \square S_2) \in N(\mu \times \nu)$ of S_1 and X_2, as follows:

$$(S_1 \square S_2)(\mu_i, \nu_j) = \min_{1 \le k \le q} \begin{pmatrix} Q_{\min\left(S_1(\mu_i, \omega_k) \to S_2(\omega_k, \nu_j), S_2(\omega_k, \nu_j) \to S_1(\mu_i, \omega_k)\right)}, \\ \mathrm{I}_{\min\left(S_1(\mu_i, \omega_k) \to S_2(\omega_k, \nu_j), S_2(\omega_k, \nu_j) \to S_1(\mu_i, \omega_k)\right)}, \\ F_{\min\left(S_1(\mu_i, \omega_k) \to S_2(\omega_k, \nu_j), S_2(\omega_k, \nu_j) \to S_1(\mu_i, \omega_k)\right)}, \end{pmatrix} \tag{10}$$

for any $(\mu_i, \nu_j) \in (\mu, \nu), i = 1, 2, ..., p, j = 1, 2, ..., r$.

Denote X_{ik} as $S(\mu_i, \omega_k)$ for short, similar to the others, for convenience. Subsequently, we can simplify Eq. (9) and Eq. (10) as follows:

$$(S_1 \lhd S_2)(\mu_i, \nu_j) = \begin{pmatrix} \frac{1}{q}\sum_{j=1}^{q} Q_{S_{ik} \to S_{kj}}, \\ \frac{1}{q}\sum_{k=1}^{q} I_{S_{ik} \to S_{kj}}, \\ \frac{1}{q}\sum_{k=1}^{q} F_{S_{ik} \to S_{kj}} \end{pmatrix} \tag{11}$$

$$\left(S_1 \square S_2\right)\left(\mu_i,\ \nu_j\right) = \min_{1\le k\le q}\begin{pmatrix} Q_{\min\left(S_{ik}\to S_{kj},S_{kj}\to S_{ik}\right)}, \\ \mathrm{I}_{\min\left(S_{ik}\to S_{kj},S_{kj}\to S_{ik}\right)}, \\ F_{\min\left(S_{ik}\to S_{kj},S_{kj}\to S_{ik}\right)} \end{pmatrix} \tag{12}$$

Indeed, the neutrosophic triangle product and the neutrosophic square product are firmly related to each other. That is, the neutrosophic triangle product is the basis of the neutrosophic square product, and because of that, $\left(S_1 \square S_2\right)\left(\mu_i,\ \nu_j\right)$ is directly derived from $\left(S_1 \lhd S_2\right)\left(\mu_i,\ \nu_j\right)$ and $\left(S_2 \lhd S_1\right)\left(\mu_i,\ \nu_j\right)$.

4 Applications of the two neutrosophic products

In this subsection, we use the neutrosophic triangle product to compare multi-attribute decision making with neutrosophic information. Subsequently, we use the neutrosophic square product for constructing an anneutrosophic similarity matrix. This anneutrosophic similarity matrix is used for analyzing the neutrosophic clustering method.

Assume a multiple attribute decision making issue. Let $W = \{w_1, w_2, ..., w_p\}$ and $N = \{n_1, n_2, ..., n_q\}$ define sets of p alternatives and q attributes, respectively. The attribute values (also called a characteristic) of each alternative w_i under all the attributes $N_j\left(j = 1,2,...,m\right)$ represent the neutrosophic set. We make a decision based on the multiple attributes:

$$w_i = \left\{\left\langle N_j,\ Q_{w_i}\left(N_j\right),\ I_{w_i}\left(N_j\right), F_{w_i}\left(N_j\right)\right\rangle \middle| N_j \in N\right\},\ i = 1,2,..,n \ and\ \ j = 1,2,...,m \tag{13}$$

where $Q_{w_i}\left(N_j\right)$ denotes the degree of membership, $I_{w_i}\left(N_j\right)$ denotes the degree of indeterminacy, and $F_{w_i}\left(N_j\right)$ denotes the degree of non-membership of w_i to N_j. Apparently, the degree of uncertainty of w_i to N_j is $\Psi_{w_i}\left(N_j\right) = 3 - Q_{w_i}\left(N_j\right) - I_{w_i}\left(N_j\right) - F_{w_i}\left(N_j\right)$.

Let $s_{ij} = \left(Q_{ij},\ I_{ij},\ F_{ij}\right) = \left(Q_{wi}\left(N_j\right),\ I_{wi}\left(N_j\right),\ F_{wi}\left(N_j\right)\right)$ be a neutrosophic value. An $n \times m$ neutrosophic decision matrix, $S = \left(s_{ij}\right)_{n\times m}$, can be constructed based on the neutrosophic valued set $s_{ij}\left(i = 1,2,...,n;\ j = 1,2,...,\mathrm{m}\right)$.

4.1 Neutrosophic triangle product's application

The characteristic vectors of two alternatives for the issues described above, say S_i and S_j, can be expressed as $S_i = (s_{i1}, s_{i2}, ..., s_{im})$ and $S_j = (s_{j1}, s_{j2}, ..., s_{jm})$, respectively. The neutrosophic triangle product can be calculated as follows:

$$\left(S_i \lhd S_j^{-1}\right)_{ij} = \begin{pmatrix} \frac{1}{m}\sum_{k=1}^{m} Q_{s_{ik} \to s_{jk}}, \\ \frac{1}{m}\sum_{k=1}^{m} I_{s_{ik} \to s_{jk}}, \\ \frac{1}{m}\sum_{k=1}^{m} F_{x_{ik} \to x_{jk}} \end{pmatrix} \tag{14}$$

This shows the degree of the alternative, w_j, for preferred alternative w_i, where S_j^{-1} is the inverse of S_j and can be defined as $\left(S_j^{-1}\right)_{kj} = \left(S_j\right)_{kj} = s_{jk}$, $Q_{s_{ik} \to s_{jk}}$, $I_{s_{ik} \to s_{jk}}$ and $F_{s_{ik} \to s_{jk}}$.

Similarly, we can calculate

$$\left(S_j \lhd S_i^{-1}\right)_{ji} = \begin{pmatrix} \frac{1}{m}\sum_{k=1}^{m} Q_{s_{jk} \to s_{ik}}, \\ \frac{1}{m}\sum_{k=1}^{m} I_{s_{jk} \to s_{ik}}, \\ \frac{1}{m}\sum_{k=1}^{m} F_{s_{jk} \to s_{ik}} \end{pmatrix} \tag{15}$$

This shows that degree alternative w_i is preferred to alternative w_j. The alternatives ordering w_i and w_j can be obtained from Eqs. (14) and (15). In fact,

a. if $\left(S_i \lhd S_j^{-1}\right)_{ij} > \left(S_j \lhd S_i^{-1}\right)_{ji}$, alternative w_j is preferred to w_i;

b. if $\left(S_i \lhd S_j^{-1}\right)_{ij} = \left(S_j \lhd S_i^{-1}\right)_{ji}$, there is similarity between w_i and w_j;

c. if $\left(S_i \lhd S_j^{-1}\right)_{ij} < \left(S_j \lhd S_i^{-1}\right)_{ji}$, then w_i is preferred to w_j.

4.2 Neutrosophic square product's application

As we know from Eq. (10), mathematically, neutrosophic square product $(S_1 \times S_2)_{ij}$ can be deciphered as follows: $(S_1 \times S_2)_{ij}$ measures the degree of similarities of the i^{th} row of neutrosophic matrix S_1 and the j^{th} row of neutrosophic matrix S_2. Therefore, considering the issue expressed at the start of Section 4, $(S_i \times S_j^{-1})_{ij}$ expresses the similarity of alternatives w_i and w_j. The following formula can be used for constructing a neutrosophic similarity matrix for $w_i = (i = 1, 2, ..., n)$.

$$sim(w_i,\ w_j) = (S_i \square S_j^{-1})_{ij} = \min_{1\le k\le n}\begin{pmatrix} Q_{\min(s_{ik\to}s_{jk},\, s_{jk}\to s_{ik})}, \\ I_{\min(s_{ik\to}s_{jk},\, s_{jk}\to s_{ik})}, \\ F_{\min(s_{ik\to}s_{jk},\, s_{jk}\to s_{ik})} \end{pmatrix} \tag{16}$$

Eq. (16) has the following desirable properties:

1. $sim(w_i,\ w_j)$ is the neutrosophic value.
2. $sim(w_i,\ w_i) = (1,\ 0)\ (i = 1, 2, ..., n)$.
3. $sim(w_i,\ w_j) = sim(w_j,\ w_i)\ (i = 1, 2, ..., n)$.

Proof for property 1

We can prove that $sim(w_i,\ w_j)$ is the neutrosophic value.

Since the results $s_{ik} \longrightarrow s_{jk}$ and $s_{jk} \longrightarrow s_{ik}$ are all neutrosophic valued sets as proven previously, then $\begin{pmatrix} Q_{\min(s_{ik\to}s_{jk},\, s_{jk}\to s_{ik})}, \\ I_{\min(s_{ik\to}s_{jk},\, s_{jk}\to s_{ik})}, \\ F_{\min(s_{ik\to}s_{jk},\, s_{jk}\to s_{ik})} \end{pmatrix}$ is the neutrosophic value for any k.

Proof for property 2

Since

$$sim(w_i,\ w_i) = (S_i \square S_i^{-1})_{ii} = \min_{1\le k\le n}\begin{pmatrix} Q_{\min(s_{ik\to}s_{ik},\, s_{ik}\to s_{ik})}, \\ I_{\min(s_{ik\to}s_{ik},\, s_{ik}\to s_{ik})}, \\ F_{\min(s_{ik\to}s_{ik},\, s_{ik}\to s_{ik})} \end{pmatrix}$$

we know from definition (10) that $sim(w_i,\ w_i) = (1,\ 0)$.

Proof for property 3

Since

$$sim\left(w_i,\ w_j\right)=\left(S_i\Box S_j^{-1}\right)_{ij}=\min_{1\le k\le n}\begin{pmatrix} Q_{\min\left(s_{ik\to}s_{jk},s_{jk}\to s_{ik}\right)}, \\ I_{\min\left(s_{ik\to}s_{jk},s_{jk}\to s_{ik}\right)}, \\ F_{\min\left(s_{ik\to}s_{jk},s_{jk}\to s_{ik}\right)} \end{pmatrix}$$

$$=\min_{1\le k\le m}\begin{pmatrix} Q_{\min\left(s_{jk}\to s_{ik},s_{ik\to}s_{jk}\right)}, \\ I_{\min\left(s_{jk}\to s_{ik},s_{ik\to}s_{jk}\right)}, \\ F_{\min\left(s_{jk}\to s_{ik},s_{ik\to}s_{jk}\right)} \end{pmatrix}$$

$$=\left(X_j\Box X_i\right)=sim\left(w_j,\ w_i\right)$$

then, $sim\left(w_i,\ w_j\right)=sim\left(w_j,\ w_i\right)\ \left(i,j=1,2,...,n\right)$.

At that point, from the above analyses, we can determine that Eq. (16) satisfies the neutrosophic similarity relation conditions. Thus, this can be used to construct a neutrosophic similarity matrix.

5 Direct neutrosophic cluster analysis method

After constructing a neutrosophic similarity matrix with the abovementioned method, the equivalent matrix is not required before cluster analysis. The required cluster analysis results can be obtained with the neutrosophic equivalent matrix, starting with the neutrosophic similarity matrix. In fact, Luo [Luo (1989)] proposed a direct method for clustering fuzzy sets. This method considers only membership degrees of fuzzy sets. Our proposed direct neutrosophic cluster analysis technique considers the enrollment degrees, indeterminacy degrees, and non-participation degrees of the neutrosophic esteemed set under the neutrosophic conditions presented below. The proposed method is based on Luo's method, which includes following stages.

Stage A. Let $S=\left(s_{ij}\right)_{n\times n}$ be the neutrosophic similarity matrix, where $s_{ij}=\left(Q_{ij},\ I_{ij},\ F_{ij}\right)\ \left(i,j=1,2,...,n\right)$ is a neutrosophic valued set for determining the confidence level, λ_1. Select one of the elements, S, which obeys the following principles.

a. Rank the degrees of membership of $s_{ij}\ \left(i,j=1,2,...,n\right)$ in descending order. Take $\lambda_1=\left(Q_{\lambda_1},\ I_{\lambda_1},\ F_{\lambda_1}\right)=\left(Q_{i_1j_1},\ I_{i_1j_1},\ F_{i_1j_1}\right)$, where $Q_{i_1j_1}=\max_{i,j}\left\{Q_{ij}\right\}$.

b. If there exist two neutrosophic valued sets, $\left(Q_{i_1j_1},\ I_{i_1j_1},\ F_{i_1j_1}\right)$ and $\left(\overline{Q}_{i_1j_1},\ \overline{I}_{i_1j_1},\ \overline{F}_{i_1j_1}\right)$ in (1), such that $I_{i_1j_1} \neq \overline{I}_{i_1j_1}$ and $F_{i_1j_1} \neq \overline{F}_{i_1j_1}$ (without loss of generality, let $I_{i_1j_1} < \overline{I}_{i_1j_1}$ and $F_{i_1j_1} < \overline{F}_{i_1j_1}$), then we choose the first one as λ_1 , i.e., $\lambda_1 = \left(Q_{i_1j_1},\ I_{i_1j_1},\ F_{i_1j_1}\right)$.

Then, for each alternative w_i , let

$$\left[w_i\right]_S^{(1)} = \left\{w_j \left| s_{ij} = \lambda_1\right.\right\} \tag{17}$$

Here, w_i and all alternatives in $\left[w_i\right]_S^{(1)}$ are clustered into one category, and other alternatives are clustered into another category.

Stage B. Select the confidence level, $\lambda_2 = \left(Q_{\lambda_2},\ I_{\lambda_2},\ F_{\lambda_2}\right) = \left(Q_{i_2j_2},\ I_{i_2j_2},\ F_{i_2j_2}\right)$, with $Q_{i_2j_2} = \max_{(i,j)\neq(i_1,j_1)}\left\{Q_{ij}\right\}$, specifically if there exist at least two neutrosophic esteemed sets where the membership degrees have the same value as $Q_{i_2j_2}$. At that point, we can follow strategy (b) in Stage A. Now, let alternatives $\left[w_i\right]_S^{(2)}$ be $\left\{w_j \left| s_{ij} = \lambda_2\right.\right\}$, and then, w_i and all the alternatives are clustered into one type. Let the merger of $\left[w_i\right]_S^{(1)}$ and $\left[w_i\right]_S^{(2)}$ be $\left[w_i\right]_S^{(1,2)}$. Then, the merged alternatives $\left[w_i\right]_S^{(1,2)} = \left\{w_j \left| s_{ij} \in \left\{\lambda_1, \lambda_2\right\}\right.\right\}$, and therefore, w_i and all alternatives in $\left[w_i\right]_S^{(1,2)}$ are clustered into one set. The other alternatives remain unaltered.

Stage C. In this stage, we take other confidence levels and analyze clusters according to the procedure in Stage B. The procedure is carried out until all alternatives are clustered into one category. One of the significant advantages of the proposed direct neutrosophic cluster analysis method is that cluster analysis can be acknowledged by simply depending on the subscripts of the alternatives. We observed from the process described above that, in this method, getting even an λ-cutting matrix is not necessary.

In real-world application scenarios, we simply need to affirm their areas in the neutrosophic similarity matrix after choosing some appropriate confidence levels, and afterward, we can get the kinds of considered objects on the basis of their area subscripts.

6 Performance evaluation

For the performance evaluation, a *k*-means algorithm and a threshold-based algorithm were used on the Iris dataset from the University of California, Irvine (UCI) Machine Learning Repository. A variable number of clusters (from 2 to 10) were generated for the experiments. For the k-means and threshold-based algorithms, cluster number is the input parameter. The k data objects were selected randomly in the k-means algorithm (k was also taken as an initial centroid of the clusters). On the other hand, only one object was selected randomly in the threshold-based method. The selected object was assigned as the initial centroid of the cluster, and was a member of the first cluster. We observed that this method generates more segregated and compact clusters. Finally, we observed that there was significant enhancement in the indices of validity. The following mathematical analysis proves the above statements.

For any cluster-based intuitionistic neutrosophic implication, let $X(T_i, F_a) \rightarrow Y(T_j, F_b)$, where T and F depict truthfulness and falsehood.

Then, we can define various classes of cluster-based neutrosophic set (CNSS) implications, as expressed below:

$$CNSS=([(1-T_i) f \vee T_j] F \wedge [(1 - f_b)f \vee fX], fY_f \wedge (1-T_i)) \quad (18)$$

The proposed new cluster-based intuitionistic neutrosophic (CIN) implication is now extended with $X(T_i, ia, fX) N \rightarrow Y(T_j, iY, fY)$, as follows:

$CIN1\ (T_i f/If \rightarrow T_j, iXf\ iY \wedge, fXf\ fY \wedge)$

where $T_i f/If \rightarrow T_j$ is any cluster of intuitionistic neutrosophic implications, while f is any $\wedge$ neutrosophic conjunction:

$CIN2\ (T_i f/If \rightarrow T_j, iXf\ iY \vee, fXf\ fY \wedge)$, where f is any ∨ fuzzy disjunction:

$CIN3\ (T_i f/If \rightarrow T_j, iX+iY\ 2, fXf\ fY \wedge)$

$CIN4\ (T_i f/If \rightarrow T_j, iX+iY\ 2, fX+fY\ 2)$

Referring to the definition proposed by Broumi et al. [Broumi, Smarandache and Dhar (2014)], the classical logical equivalence and predicate relationship now becomes

$(X \rightarrow Y) \leftrightarrow (\neg X \vee Y)$, where, $(X\ N \rightarrow Y)\ N \leftrightarrow (NX \neg N\ Y \vee)$

The above class of neutrosophic implications can now be depicted with the operators $(NX \neg N\ Y \vee)$. Let us have two cluster-based neutrosophic propositions: $X\langle 0.3, 0.4, 0.2\rangle$ and $Y\langle 0.7, 0.1, 0.4\rangle$.

Then, $X\ N \rightarrow Y$ has the neutrosophic truth value of $X\ YN \vee N \neg$, i.e., $\langle 0.2, 0.4, 0.3\rangle \langle N\ 0.7, 0.1, 0.4\rangle \vee$, or $\langle \max\{0.2, 0.7\}, \min\{0.4, 0.1\}, \min\{0.3, 0.4\}\rangle$, or $\langle 0.7, 0.1, 0.3\rangle$.

Therefore,

$N\langle t, i, f\rangle = \langle f, i, t\rangle \neg$ for neutrosophic negation

and

$\langle t_1, i_1, f_1\rangle \langle t_2, i_2, f_2\ N\rangle \vee = \langle \max\{t_1, t_2\}, \min\{i_1, i_2\}, \min\{f_1, f_2\}\rangle$ for the neutrosophic disjunction.

The dataset that we referred to from Stappers et al. [Stappers, Cooper, Brooke et al. (2016)] and [Systems (2020)] contains 16,259 spurious examples caused by radio frequency interference (RFI)/noise, and 1,639 real pulsar examples with each candidate having eight continuous variables. The first four variables are obtained from the integrated pulse profile. This is an array of continuous variables that describe a longitude-resolved version of the signal. The remaining four variables were similarly obtained from the dispersion measure (DM)-SNR curve. These are summarized in Tab. 2.

Tab. 2 shows a dataset describing a sample of pulsar candidates collected during the high time–resolution universe survey. The first column is the mean of the integrated profile. Mean1 is the mean of the DM-SNR curve, and SD1 is the standard deviation of the DM-SNR curve. Finally, ET1 is the excess kurtosis of the DM-SNR curve, and Skewness1 is the skewness of the DM-SNR curve.

Table 2: Pulsar candidate samples collected during the high time–resolution universe survey

Mean	SD	ET	Skewness	Mean1	SD1	ET1	Skewness1	T F
140.5625	55.68378	-0.23457	-0.69965	3.199833	19.11043	7.975532	74.24222	0
102.5078	58.88243	0.465318	-0.51509	1.677258	14.86015	10.57649	127.3936	0
103.0156	39.34165	0.323328	1.051164	3.121237	21.74467	7.735822	63.17191	0
136.7500	57.17845	-0.06841	-0.63624	3.642977	20.95928	6.896499	53.59366	0
88.72656	40.67223	0.600866	1.123492	1.17893	11.46872	14.26957	252.5673	0
93.57031	46.69811	0.531905	0.416721	1.636288	14.54507	10.62175	131.3940	0
119.4844	48.76506	0.03146	-0.11217	0.999164	9.279612	19.20623	479.7566	0
130.3828	39.84406	-0.15832	0.38954	1.220736	14.37894	13.53946	198.2365	0
107.2500	52.62708	0.452688	0.170347	2.33194	14.48685	9.001004	107.9725	0
107.2578	39.49649	0.465882	1.162877	4.079431	24.98042	7.397080	57.78474	0
142.0781	45.28807	-0.32033	0.283953	5.376254	29.0099	6.076266	37.83139	0
133.2578	44.05824	-0.08106	0.115362	1.632107	12.00781	11.97207	195.5434	0
134.9609	49.55433	-0.1353	-0.08047	10.69649	41.34204	3.893934	14.13121	0
117.9453	45.50658	0.325438	0.661459	2.83612	23.11835	8.943212	82.47559	0
138.1797	51.52448	-0.03185	0.046797	6.330268	31.57635	5.155940	26.14331	0
114.3672	51.94572	-0.0945	-0.28798	2.738294	17.19189	9.050612	96.6119	0
109.6406	49.01765	0.137636	-0.25670	1.508361	12.0729	13.36793	223.4384	0
100.8516	51.74352	0.393837	-0.01124	2.841137	21.63578	8.302242	71.58437	0
136.0938	51.691	-0.04591	-0.27182	9.342809	38.0964	4.345438	18.67365	0
99.36719	41.5722	1.547197	4.154106	27.55518	61.71902	2.208808	3.66268	1
100.8906	51.89039	0.627487	-0.02650	3.883779	23.04527	6.953168	52.27944	0
105.4453	41.13997	0.142654	0.320420	3.551839	20.75502	7.739552	68.51977	0
95.86719	42.05992	0.326387	0.803502	1.832776	12.24897	11.24933	177.2308	0

In Tab. 2, the mean of the integrated profile is compared with pulsar candidates that vary significantly with Mean1. Here, Mean1 is the mean of popular candidates at high time resolution. The dataset that we have referred from Stappers et al. [Stappers, Cooper, Brooke et al. (2016)] and [Systems (2020)] that contains 16,259 spurious examples caused by radio-frequency interference (RFI) or noise, and 1,639 real pulsar examples with each candidate having 8 continuous variables. The first four variables are obtained from the integrated pulse profile. This is an array of continuous variables that describe a longitude-resolved version of the signal. The remaining 4 variables are similarly obtained from the dispersion measure (DM)-SNR curve. These are summarized in Tab. 2.

7 Conclusions and future work

One of the major issues in data clustering is the selection of the right candidates. In addition, the appropriate algorithm to choose the right candidates has been a challenging issue in cluster analysis, especially for an efficient approach that best fits the right sets of data. In this paper, a cluster analysis method based on neutrosophic set implication generates the clusters automatically and overcomes the limitation of the *k*-means algorithm. Our proposed method generates more segregated and compact clusters and achieves higher validity indices, in comparison to the mentioned algorithms. The experimentation carried out in this work focused on cluster analysis based on NSI through a *k*-means algorithm along with a threshold-based clustering technique. We found that the proposed algorithm eliminates the limitations of the threshold-based clustering algorithm. The validity measures and respective indices applied to the Iris dataset along with *k*-means and threshold-based clustering algorithms prove the effectiveness of our method.

Future work will handle data clustering in various dynamic domains using neutrosophic theory. We also intend to apply a periodic search routine by using propagations between datasets of various domains. The data clustering used by our proposed algorithms was found to be workable in a low computational configuration. In the future, we will also use more datasets.

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Indeterminate Likert Scale: Feedback Based on Neutrosophy, Its Distance Measures and Clustering Algorithm

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Abstract
Likert scale is the most widely used psychometric scale for obtaining feedback. The major disadvantage of Likert scale is information distortion and information loss problem that arise due to its ordinal nature and closed format. Real-world responses are mostly inconsistent, imprecise and indeterminate depending on the customers' emotions. To capture the responses realistically, the concept of neutrosophy (study of neutralities and indeterminacy) is used. Indeterminate Likert scale based on neutrosophy is introduced in this paper. Clustering according to customer feedback is an effective way of classifying customers and targeting them accordingly. Clustering algorithm for feedback obtained using indeterminate Likert scaling is proposed in this paper. While dealing real-world scenarios, indeterminate Likert scaling is better in capturing the responses accurately.

1 Introduction

Likert scaling introduced by Likert (1932) is the most commonly used psychometric scale for collecting responses from the user/customer in terms of level of agreement. It has been used in several surveys like organizational behaviour in learning institutes (Kiedrowski 2006; Rus et al. 2014), music education (Orr and Ohlsson 2005), prioritization of routine in dental care (Postma 2007), sports for athlete characteristics and outcome (Brown et al. 2007), etc. Likert scaling suffers from several drawbacks like information distortion and information lost problem due to its ordinal nature and closed format (Li 2013).

Zadeh's (1965) fuzzy set theory functions as an important constructive tool that enables soft division of sets. It gives an extension to fuzzy set as intuitionistic fuzzy set (A-IFS) by Atanassov (1986) where each element is given a membership and a non-membership degree in Atanassov (1986). A fuzzy Likert scale was introduced by Li in Li (2013).

To represent inconsistent, imprecise and uncertain information from the real world, indeterminacy membership is represented independently along with truth and falsity membership in neutrosophic set (Smarandache 2000). It generalizes the concept of several sets like classic set, fuzzy set and paradoxist set, and $T_A(x)$, $I_A(x)$ and $F_A(x)$ are membership function which can be real standard or nonstandard subsets.

In this form, it was not possible to apply it in real-world problems of the scientific and engineering areas. Wang et al. (2010) proposed a single-valued neutrosophic set (SVNS), to overcome this. Neutrosophy has found applications in many real-world practical problems like decision-making problems (Liu and Wang 2014; Liu and Shi 2015; Liu and Teng 2017; Liu and Li 2017; Ye 2013, 2014a, b, c), image processing (Sengur and Guo 2011; Cheng and Guo 2008; Zhang et al. 2010), analysis of social network (Salama et al. 2014) and socio-economic and political problems (Vasantha and Smarandache 2003, 2004), etc.

To offer better accuracy and give expression to imprecision in the indeterminacy, the indeterminacy membership existing in the neutrosophic set is categorized as indeterminacy leaning towards truth and towards false memberships. This makes the indeterminacy in the scenario to be more accurate and less imprecise. This was defined as double-valued neutrosophic set (DVNS) by Kandasamy (2018a, 2016a). Distance measure, cross-entropy measure and clustering algorithm of DVNS were introduced in Kandasamy (2018a). Dice measures on DVNS were proposed in Khan et al. (2018).

To improve the precision and accuracy of the data analysis and to fit in the Likert's scale that is most frequently used psychometric scale, the indeterminacy concept was subdivided into three: indeterminacy leaning towards truth, indeterminacy and false memberships. This refined neutrosophic set is known as the triple refined indeterminate neutrosophic set (TRINS). TRINS was used recently for personality test and classification based on personality (Kandasamy and Smarandache 2016b). TRINS is redefined here as positive membership, positive indeterminate membership, indeterminate membership, negative indeterminate membership and negative membership, to give the best possible mapping of Likert Scaling.

To conceptualize a real-world example of TRINS, consider the scenario where a customer orders four different food items from the restaurant's menu. He might have immensely enjoyed two of the dishes, regretted ordering a particular dish and be unsure about the other dish thinking it might have been better if it was prepared in a different way. If he is asked to provide feedback using Likert scale, he will obviously give an average/neutral score.

Let TRINS A under consideration be represented by $P_A(x), I_{PA}(x), I_A(x), I_{NA}(x), N_A(x)$, where $P_A(x)$ denotes positive membership, $I_{PA}(x)$ is positive indeterminate membership, $I_A(x)$ is indeterminate, $I_{NA}(x)$ is negative indeterminate and $N_A(x)$ is negative. The scenario is represented as $\langle 0.5, 0.25, 0, 0, 0.25\rangle$ that is giving a value of 0.5 for the two dishes that he enjoyed immensely, 0.25 to the dish he regretted and 0.25 to the dish he was unsure about. The scenario can be captured accurately with needed precision which is vital to the result obtained. All the various choices are captured, thereby evading the preferential choice that is selected in the conventional method of Likert scaling.

Clustering analysis basically exploits the notion of distance measures between any two entities, and based on this clusters are formed. This plays a significant role in research fields in the form of data mining, social networking, pattern recognition and machine learning. Traditionally, clustering analysis has been a hard one, which assigns an item to a particular cluster. Since elements in the given scenario do not have rigid restrictions, it is essential to fragment them softly. In this paper, a clustering algorithm is introduced to handle feedback obtained using indeterminate Likert scaling.

Section one is introductory in nature, and section two recalls some basic concepts about Likert scaling/star rating and neutrosophy. Section three discusses the limitation and problems with Likert scaling and provides justification for using indeterminacy. Indeterminate Likert scaling which maps every degree of agreement individually is introduced in section four. Indeterminacy-based minimum spanning tree (MST) clustering algorithm is proposed in the next section, and an illustrative real-world example is provided. Comparison of indeterminate Likert scaling with existing rating scheme and Likert scaling is carried out in section six. Section seven provides the conclusions and future study.

2 Preliminaries

2.1 Likert scaling

Likert scale is the most often used psychometric scale to collect responses from people in a survey. A typical Likert scale survey does not let its respondents simply select from "yes/no"; it provides specific choices that are degrees of "agreeing" or "disagreeing". The most basic Likert scaling format is a 5-column answer, with choices like: strongly disagree, disagree, neither agree nor disagree (do not know), agree and strongly agree. The neutral option is generally opted by the person who is unsure. A study in Armstrong (1987) found negligible differences between the use of "undecided" and "neutral" as a middle option in a 5-point Likert scale.

A sample of Likert scaling for a simple question "How satisfied are you with our services?" is given in Fig. 1.

The star rating scheme is almost similar to Likert scale. 1 star is taken to be equivalent to the lowest rating while the 5 star is considered as the maximum rating. Stars are used as a common experimental or heuristic element for evaluating quality. A sample questionnaire used to elicit responses from the customers of a restaurant using 5 star rating is given in Table 1. Similarly, a questionnaire that uses Likert scale is given in Table 2. The analysis of Likert scale responses is generally carried out using bar charts to show results, mode in the case of the most common response and range and interquartile ranges in the case of analysing variability.

2.2 Neutrosophy and refined neutrosophic set

Neutrosophy, familiarized by Smarandache (2000), studies a perception or event or entity, "A" in relation to its opposite, "Anti-A" and not A, "Non-A", and as neither "A" nor "Anti-A", denoted by "Neut-A".

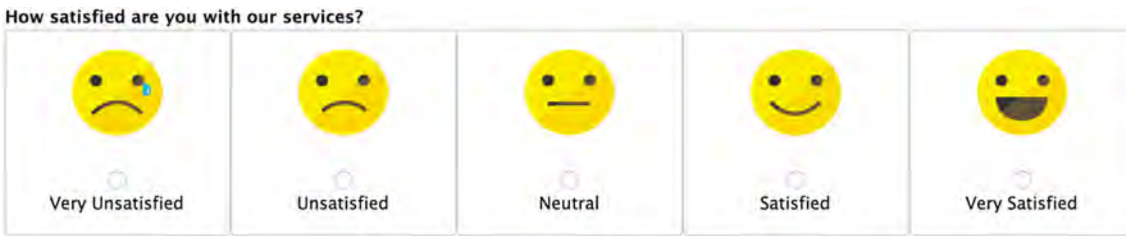

Fig. 1 Sample Likert scale

Table 1 Sample questionnaire using five-star rating scheme for restaurant

Question	Scale
Quality of Food	★★★★☆
Service	★★★★☆
Hygiene	★★★☆☆
Value for money	★★☆☆☆
Ambience	★★★☆☆
Overall Experience	★★★☆☆

Let X be a space of points (objects) with elementary elements in X represented by x. A single-valued neutrosophic set (SVNS) A in X is characterized by truth $T_A(x)$, indeterminacy $I_A(x)$ and falsity $F_A(x)$ membership functions. For each point x in X, there are $T_A(x), I_A(x), F_A(x) \in [0, 1]$ and $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$. A is denoted by $A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle \mid x \in X\}$. The refined neutrosophic logic defined by Smarandache (2013) is as follows:

Definition 1 The truth T is divided into several types of truths: $T_1, T_2, \ldots, T_p$, and I into various indeterminacies: $I_1, I_2, \ldots, I_r$, and F into various falsities: $F_1, F_2, \ldots, F_s$, where all $p, r, s \geq 1$ are integers, and $p + r + s = n$.

Triple refined indeterminate neutrosophic sets have the indeterminacy concept divided into three memberships: indeterminacy favouring positive opinion, indeterminacy favouring negative opinion and indeterminacy. This division helps in improving the accuracy and precision and fits the Likert scale. TRINS (Kandasamy and Smarandache 2016b) has been used to classify personality. In double-valued neutrosophic set (DVNS), the indeterminacy concept is divided into two.

Definition 2 A triple refined indeterminate neutrosophic set (TRINS) A in X as given above is characterized by positive $P_A(x)$, indeterminacy $I_A(x)$, negative $N_A(x)$, positive indeterminacy $I_{PA}(x)$ and negative indeterminacy $I_{NA}(x)$ membership functions. Each has a weight $w_m \in [0, 5]$ associated with it. For each $x \in X$, there are

$$P_A(x), I_{PA}(x), I_A(x), I_{NA}(x), N_A(x) \in [0, 1],$$
$$w_P^m(P_A(x)), w_{I_P}^m(I_{PA}(x)), w_I^m(I_A(x)),$$
$$w_{I_N}^m(I_{NA}(x)), w_N^m(N_A(x)) \in [0, 5]$$

and $0 \leq P_A(x) + I_{PA}(x) + I_A(x) + I_{NA}(x) + N_A(x) \leq 5$. Therefore, a TRINS A can be represented by

$$A = \{\langle x, P_A(x), I_{PA}(x), I_A(x), I_{NA}(x), N_A(x)\rangle \mid x \in X\}.$$

Consider $Q = [q_1, q_2]$ where q_1 is question 1 (quality) and q_2 is question 2 (service) from Table 2. The values of q_1 and q_2 are in [0, 1], and the weight of the membership is applied the values are in [0, 5]. Take the same scenario where a customer orders 4 different items from the menu. He might have immensely enjoyed two of the dishes, regretted ordering a particular dish and may be undecided about the other dish being good. If he is asked to provide feedback using Likert

Table 2 Sample questionnaire using Likert scaling for restaurant

Question	Terrible	Bad	Average	Good	Excellent
Quality of food	☐	☐	☐	☑	☐
Service	☐	☐	☐	☑	☐
Hygiene	☐	☐	☑	☐	☐
Value for money	☐	☑	☐	☐	☐
Ambience	☐	☐	☑	☐	☐
Overall experience	☐	☐	☑	☐	☐

scale, he will obviously give a average/neutral score. This is mapped to TRINS as follows:

Option for "quality" would be a degree of excellent food that is the dishes he enjoyed immensely, a degree of indeterminacy choice towards "good food" will be the dish he was undecided about but thought it was a little good but is unsure, a degree of uncertain and indeterminate combination of good food and not so good food, a degree of indeterminate choice bordering close to bad food and a degree of poor quality food which will be the food that he regretted ordering, instead of a forced single choice. Similarly, the service will vary and can be marked accordingly to different degrees.

This can be represented by TRINS A of X as

$$A = \langle 0.5, 0.25, 0, 0, 0.25\rangle / x_1 + \langle 0.5, 0.1, 0.1, 0.1, 0.2\rangle / x_2.$$

Operators related to set theory like associativity, distributivity, commutativity, idempotency, absorption and the DeMorgan's laws were defined over TRINS (Kandasamy and Smarandache 2016b).

3 Justification for applying indeterminacy-based scaling

Generally in Likert scaling, the user is forced to select the most dominant choice. For example, the normal five-level Likert item would be

– Strongly disagree
– Disagree
– Neither agree or disagree
– Agree
– Strongly agree

Any user will have feelings/options which actually vary from strongly agree to strongly disagree and which are not definite; they are always a mixture of feelings. A small amount of disagreement might bring down the option from "strongly agree" to "agree", whereas a different person might choose to go with the dominant choice of "strongly agree" ignoring the small/meagre amount of disagreement. Some other person might mark the option "neither agree nor disagree" due to the same negative experience. However, it is very obvious and clear that people react differently to the same experience while answering the same question in the questionnaire. The questionnaire using a Likert scale will fail to capture the feelings/exact degree of strong agreement, degrees of weak agreement, degrees of neither agreement nor disagreement, degrees of weak disagreement and the degree of strong disagreement. The respondent/person is generally forced to go with the dominant choice or the choice which he feels at that time or the choice which may be only a shade dominant than the other choice; thereby, the degree of the memberships with other choices is completely lost.

Only a measure of coarse ordinal scale with closed format is used by Likert method. It fails in approximating interval data, and a substantial amount of information is gone and distorted due to the built-in limitations of Likert scaling as said by Russell and Bobko (1992).

A person who opts for "strongly agree" option might not be 100% agree with the statement. There might have been some amount of disagreement which the user was forced to override or only a small difference in mind between any two of the 5 attributes. To exactly capture the various degrees of membership TRINS is used to represent the choices. Using of TRINS and creating a Likert type scale for questionnaire will result in capturing the uncertainty, incomplete and indeterminate nature of the persons opinion in the collected data.

Every option will be given a degree of membership, and the person need not be forced to go with the dominant choice. The various degrees and choices will be captured more accurately with good precision, in fact in a sensitive, accurate and realistic way and not in an approximate way. This will eventually aid in better understanding of the customers and their needs; thereby, better marketing can be carried out.

Generally, the Likert scale is a bipolar scaling technique, determining either positive or negative response to a statement, whereas the TRINS- or DVNS-based Likert type scaling will be measuring both/all responses to a statement, thereby collecting the indeterminate/incomplete details about the options of the persons. This will provide a clear and more detailed view of the various degrees of membership. In the Likert scale, sometimes even point scale is used, where the middle option of "neither agree nor disagree" is removed. This is known as the forced choice method. This can be appropriately represented by DVNS. However, the neutral option is generally opted by the person who is unsure. A study in Armstrong (1987) found insignificant differences between the usage of "undecided" and "neutral" as a central option in a 5-point Likert scale.

There is actually a lot of difference between someone who is undecided and someone who is neutral; in a TRINS-based Likert scale, there can be a separate option for undecided, since equal amount of agreement and disagreement can be represented in degree of weak agreement and degree of weak disagreement, individually.

Indeterminate Likert scaling will remove the necessity to go with the dominant choice or a forced option which cannot always be true if it is varying from the other option only be a small or a shade of difference. The users exact feelings/thinking/option cannot be captured very realistically by Likert scaling, but certainly indeterminate Likert scaling based on TRINS can do this very accurately.

4 Indeterminate Likert scale

The normal five-level Likert scale items are

- Strongly disagree
- Disagree
- Neither agree or disagree
- Agree
- Strongly agree

They will get mapped in indeterminate Likert scale as follows:

- Negative membership
- Indeterminacy leaning towards negative membership
- Indeterminate membership
- Indeterminacy leaning towards positive membership
- Positive membership

While using a five-star rating, this will be mapped from one star to five stars. An indeterminacy-based Likert scale will have a negative membership which will capture the degree of strongly disagree of usual Likert scaling and degree of one-star rating in the star rating scheme. Similarly, the membership of indeterminacy leaning towards negative will capture the degree of disagree or two-star rating. Neural/degree of neither agree not disagree/do not know of the usual Likert scale or the three-star rating will be captured by the indeterminacy membership. Similarly, for degree of agree and degree of strongly agree will be mapped to indeterminacy or neutrality leaning towards positive membership and positive membership, respectively.

An indeterminate Likert scale will be given a representation as shown in Fig. 2: very unsatisfied, unsatisfied, neutral, satisfied, very satisfied with individual scales for grading.

A five-star rating will be like the one represented in Fig. 3. If a user is asked to rate the service provided in the restaurant, the user might have several different types of emotions about the service. The service of the waiters might have been excellent; he will give a 0.5 to "very satisfied". He might have waited for a long time for the food he ordered to arrive, hence a 0.25 to very unsatisfied. Regarding the politeness of the waiters/staff he might not be in a position to make up his mind, he might nevertheless be unable to map it as good or bad, hence a 0.25 for the indeterminate/neural options.

Such a case is given as example in Fig. 2. The user basically has option to slide using the slider provided in each level of agreement. Similarly, in a five-star rating scheme the user can fill the star to provide the degree of membership for each level as shown in Fig. 3. This can be easily implemented in mobile applications. As soon as a negative feedback is obtained, the user can be asked to provide more details by asking particular questions and making the feedback interactive. Due to the nature of indeterminate Likert scale, identifying and isolating a negative experience of the customer become easy. Table 3 gives the input received from the user using a indeterminate Likert scaling-based questionnaire.

This indeterminate Likert scale can be extended to 7-point Likert scale, or any multipoint Likert scale. In fact, it can be altered to the needs of researchers. Truth, indeterminate and Falsity memberships can be divided according to the researchers. These are known as multipoint indeterminate Likert scale. Studies in this direction is left open.

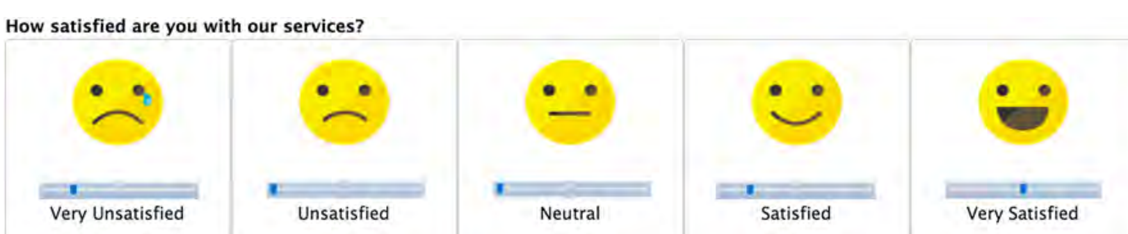

Fig. 2 Indeterminate Likert scale

Fig. 3 Indeterminate rating scale

Table 3 Sample questionnaire using indeterminate Likert scaling for restaurant

Question	P(A)	IP(A)	I(A)	IN(A)	N(A)
Quality of food	0.9	0.03	0.05	0.02	0
Service	0.8	0.05	0.05	0.1	0
Hygiene	0.7	0	0.1	0.1	0.1
Value for money	0.8	0.1	0	0	0.1
Ambience	0.7	0.1	0.1	0.05	0.05
Overall experience	0.75	0	0.05	0.1	0

5 Indeterminate MST clustering algorithm using distance measures

5.1 Distance measures of TRINS

The distance measures and its related algorithm of TRINS are defined in the following:

Consider two TRINS A and B in a universe of discourse, $X = x_l, x_2, \ldots, x_n$, which are denoted by

$$A = \{\langle x_i, P_A(x_i), I_{PA}(x_i), I_A(x_i), I_{NA}(x_i), N_A(x_i)\rangle \mid x_i \in X\}, and B = \{\langle x_i, P_B(x_i), I_{PB}(x_i), I_B(x_i), I_{NB}(x_i), N_B(x_i)\rangle \mid x_i \in X\},$$

where $P_A(x_i)$, $I_{PA}(x_i)$, $I_A(x_i)$, $I_{NA}(x_i)$, $N_A(x_i)$, $P_B(x_i)$, $I_{PB}(x_i)$, $I_B(x_i)$, $I_{NB}(x_i)$, $N_B(x_i) \in [0, 5]$ for every $x_i \in X$.

Let $w_i (i = 1, 2, \ldots, n)$ be the weight of an element $x_i (i = 1, 2, \ldots, n)$, with $w_i \geq 0$; $(i = 1, 2, \ldots, n)$ and $\sum_{i=1}^{n} w_i = 1$.

Then, the generalized TRINS weighted distance is as follows:

$$d_\lambda(A, B) = \left\{ \frac{1}{5} \sum_{i=1}^{n} w_i [\mid P_A(x_i) - P_B(x_i) \mid^\lambda + \mid I_{PA}(x_i) - I_{PB}(x_i) \mid^\lambda + \mid I_A(x_i) - I_B(x_i) \mid^\lambda + \mid I_{NA}(x_i) - I_{NB}(x_i) \mid^\lambda + \mid N_A(x_i) - N_B(x_i) \mid^\lambda] \right\}^{1/\lambda} \tag{1}$$

where $\lambda > 0$.

When $\lambda = 1$, Eq. 1 reduces to TRINS weighted Hamming distance; when $\lambda = 2$, it reduces to TRINS weighted Euclidean distance and is given as

```
Input: TVNS A_1, ......, A_m,
Output: Distance matrix D with elements d_ij
begin
  for i ← 1, m do
    for j ← 1, m do
      if i = j then
        d_ij ← 0
      else
        d_ij ← {d_λ (A_i, A_j)}
      // Using Equation 3
```

Algorithm 1: TRINS weighted distance matrix D

$$d_\lambda(A, B) = \left\{ \frac{1}{5} \sum_{i=1}^{n} w_i [\mid P_A(x_i) - P_B(x_i) \mid^2 + \mid I_{PA}(x_i) - I_{PB}(x_i) \mid^2 + \mid I_A(x_i) - I_B(x_i) \mid^2 + \mid I_{NA}(x_i) - I_{NB}(x_i) \mid^2 + \mid N_A(x_i) - N_B(x_i) \mid^2] \right\}^{1/2} \tag{2}$$

where $\lambda = 2$ in Eq. 1.

The TRINS distance matrix D is as follows:

Definition 3 Let $A_j (j = 1, 2, \ldots, m)$ be a collection of m TRINS, then we define the TRINS distance matrix $D = (d_{ij})_{m \times m}$, where $d_{ij} = d_\lambda(A_i, A_j)$ is the generalized TRINS distance between A_i and A_j and satisfies the following:

1. $d_{ij} \in [0, 5], \forall i, j = 1, 2, \ldots, m$;
2. $d_{ij} = 0$ if and only if $A_i = A_j$;
3. $d_{ij} = d_{ji}$ for all $i, j = 1, 2, \ldots, m$.

The algorithm to calculate the TRINS weighted distance matrix D is given in Algorithm 1.

5.2 Indeterminate MST clustering algorithm

Indeterminate minimum spanning tree (MST) clustering algorithm is proposed as a generalization of the IFMST, SVN-MST and DVN-MST clustering algorithms here.

Consider $X = \{x_1, x_2, \ldots, x_n\}$ to be an attribution space and the weight vector of an element $x_i (i = 1, 2, \ldots, n)$ be $w = \{w_1, w_2, \ldots, w_n\}$, where $w_i \geq 0 (i = 1, 2, \ldots, n)$ and $\sum_{i=1}^{n} w_i = 1$. Let the m samples that need to be clustered be represented as $F_j (j = 1, 2, \ldots, m)$, a collection of m TRINSs. It is $F_j = \{\langle x_j, P_{F_j}(x_j), I_{PF_j}(x_j), I_{F_j}(x_j), I_{NF_j}(x_j), N_{F_j}(x_j)\rangle \mid x_j \in X\}$.

The triple refined indeterminate neutrosophic minimum spanning tree (TRIN-MST) clustering algorithm is provided in Algorithm 2. The description of the algorithm is done along with an example.

```
Input: D = (d_ij)_{m×m}
Output: Minimum Spanning Tree S and Clusters
begin
  Step 1: Calculate D distance matrix (F_1, ..., F_m)
  // Using Algo 2
  Step 2: Create TRINS graph G(V, E)
  for i ← 1, m do
    for j ← 1, m do
      if i != j then
        Insert edge from F_i to F_j with d_ij
  Step 3: Compute MST of G: // by use of
     Kruskal's algorithm
  Sort the edges in order (increasing) of weight in E.
  while No. of edges in subgraph S of G < (V − 1) do
    Select edge (v_i, v_j) with minimum weight.
    Delete (v_i, v_j) from E
    if (v_i, v_j) creates a cycle in S then
      Discard edge v_i, v_j
    else
      Include the edge v_i, v_j in S
  S is the MST of G(V, E).
  Step 4: Clustering S with threshold r
  for i ← 1, m do
    for j ← 1, m do
      if d_ij ≥ r then
        Remove edge
  Clusters are created automatically
```

Algorithm 2: Indeterminate Minimum Spanning Tree (MST) Clustering algorithm

5.3 Illustrative examples

To demonstrate the effectiveness of the proposed TRIN-MST clustering algorithm in the real-world applications, a descriptive example is presented. The results of the indeterminate feedback of ten different people which are represented by TRINS are clustered using the indeterminate MST clustering algorithm.

Example 1 The real-world problem of feedback given by customers of a restaurant (restaurant name is kept anonymous) was taken. The six evaluation questions based on Table 2 were considered and transformed to indeterminate questionnaire as given in Table 3. The answers of the indeterminate feedback of ten different people $F_j(j = 1, 2, \ldots, 10)$ are taken for clustering. The questionnaire has been changed accordingly so as to ensure the use of distance measures. The responses collected from 10 people are given in Kandasamy (2018b).

The weight vector $w_i = 0.167$ is taken uniformly for the attribute $x_i(i = 1, 2, \ldots, 6)$. The TRIN-MST clustering algorithm provided in Algorithm 2 is used to group the ten people of $F_j(j = 1, 2, \ldots, 10)$ into clusters.

The stepwise working of the TRINS-MST clustering algorithm is as follows:

Step 1 The distance matrix $D = d_{ij} = d_\lambda(F_i, F_j)$ is calculated by using Algorithm 1 (taking $\lambda = 2$). $D = (d_{ij})_{m\times m}$ is obtained as given in Fig. 4:

Step 2 Based on D the TRINS graph $G(V, E)$ is constructed where every edge between F_i and $F_j(i, j = 1, 2, \ldots, 10)$ is assigned the TRINS weighted distance d_{ij} that represents the degree of dissimilarity between the elements F_i and F_j.

Step 3 Construction of the MST of the TRINS graph $G(V, E)$ is done as follows:

1. The distances of edges of G sorted in increasing order by weights.
2. A subgraph (empty) S of G is taken and the edge e with minimum weight is added to S, if the end points of e are not connected in S. Here the smallest edge is between F_1 and F_4; $d_{14} = 0.08456$ is added to S and removed from the sorted list.
3. The next minimum weight edge is selected from G; if no cycle is created in S, it is deleted from the list and added to S.
4. Process (3) is repeated until the obtained subgraph S spans all the ten nodes.

The MST S of the TRINS graph so obtained is illustrated in Fig. 5.

Step 4 A threshold r is selected, and all the edges with weights more than r are disconnected to get the subtrees (clusters), as listed in Table 4.

The clusters that are formed when the threshold value r is taken as 0.2928 are given in Fig. 6. It can be clearly seen that there are three different clusters grouped based on their feedback as satisfied customers (F_2, F_5, F_7), unsatisfied customers $(F_1, F_4, F_3, F_8, F_9)$ and indeterminate customers (F_6, F_{10}). Based on the clusters, targeted and interactive marketing can be carried out. This type of clustering is not possible with Likert scaling.

Clustering of customer feedback can be carried only on the basis of particular questions, and from these clusters and other information several insights can be gained.

6 Comparison and discussions

6.1 Comparison with Likert scale

It is known that Likert scaling has drawbacks like information distortion and information loss. These problems are over-

0	0.456	0.1087	0.08456	0.4006	0.3568	0.3865	0.1645	0.2271	0.3834
0.456	0	0.4278	0.507	0.1263	0.3938	0.139	0.4213	0.4649	0.4599
0.1087	0.4278	0	0.1554	0.3684	0.3267	0.3531	0.1686	0.2435	0.3776
0.08456	0.507	0.1554	0	0.453	0.3944	0.4394	0.186	0.2404	0.4157
0.4006	0.1263	0.3684	0.453	0	0.3155	0.134	0.3558	0.4007	0.3928
0.3568	0.3938	0.3267	0.3944	0.3155	0	0.3258	0.3238	0.3618	0.258
0.3865	0.139	0.3531	0.4394	0.134	0.3258	0	0.3375	0.3837	0.3753
0.1645	0.4213	0.1686	0.186	0.3558	0.3238	0.3375	0	0.1412	0.3059
0.2271	0.4649	0.2435	0.2404	0.4007	0.3618	0.3837	0.1412	0	0.2928
0.3834	0.4599	0.3776	0.4157	0.3928	0.258	0.3753	0.3059	0.2928	0

Fig. 4 Distance matrix

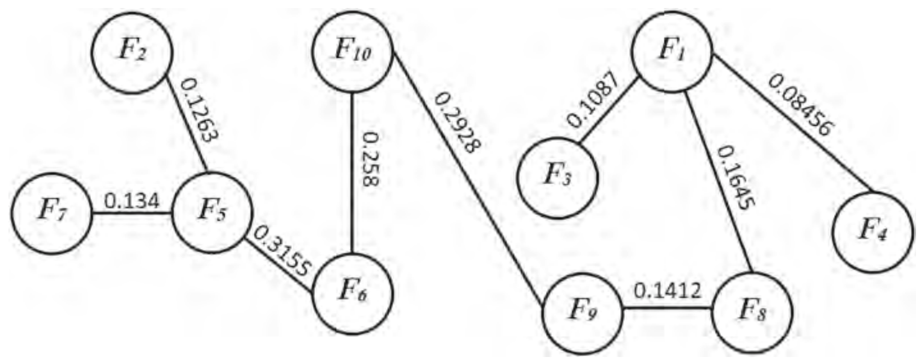

Fig. 5 MST of the TRIN

Table 4 Clustering results using TRIN-MST clustering algorithm

Threshold r	Corresponding clustering result
$r = d_{68} = 0.3155$	$\{F_1, F_3, F_4, F_6, F_8, F_9, F_{10}\}, \{F_2, F_5, F_7\}$
$r = d_{56} = 0.2928$	$\{F_1, F_3, F_4, F_8, F_9\}, \{F_6, F_{10}\}, \{F_2, F_5, F_7\}$

come when TRINS is used for collecting feedback from the user. It captures the feedback in a sensitive, accurate and realistic way as it deals with incomplete, imprecision, uncertain and indeterminate information. It is clearly seen that indeterminate Likert scale when compared to Likert scale gives more option to the customer to express themselves. In Likert scale, only the dominant choice is selected and vital information is lost.

6.2 Comparison with fuzzy Likert scale

Neutrosophic set is generalized as TRINS, intuitionistic fuzzy information is generalized as neutrosophic information/SVNS sets, and fuzzy information is generalized as intuitionistic fuzzy information. Thus, TRINS has the capacity to provide better precision and accuracy to represent the

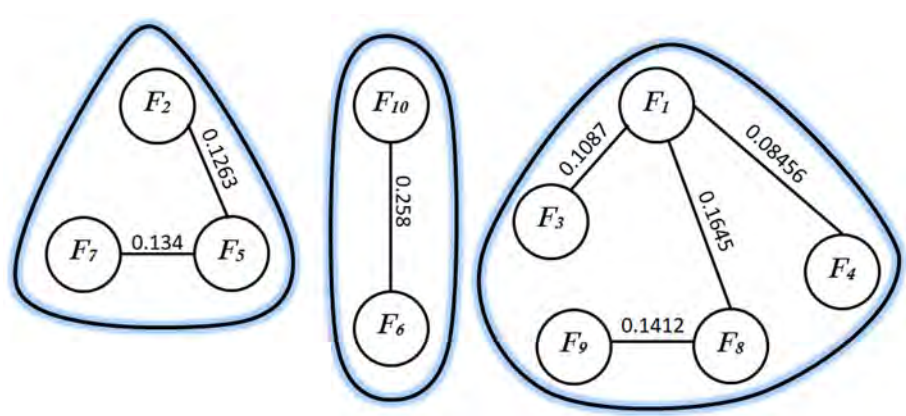

Fig. 6 Clusters of customers

existing uncertain, indeterminate, vague, imperfect and unreliable information.

It has the supplementary ability to designate with more sensitivity the indeterminate and unreliable information. Whereas the SVNS can deal indeterminate and unreliable information, it cannot designate with accuracy the existing indeterminacy. It is acknowledged that neither fuzzy theory nor IFS can deal with information that is indeterminate and inconsistent in nature; however, IFS has provisions to deal and describe with incomplete information. In SVNS, truth, indeterminacy and falsity membership are characterized individualistically, and they can also be defined with respect to any of them (no restriction). This enables SVNS to be prepared to deal with indeterminate information better than IFS, whereas in TRINS, more scope is given to deal with the prevailing indeterminate and unreliable information because the indeterminacy concept is sub-classified as three distinct values. This provides more accurateness and exactness to the indeterminacy present in the data in TRINS than in SVNS.

TRINS deals particularly with the indeterminacy leaning towards (favouring) positive (truth), the indeterminacy leaning towards negative (false) and indeterminacy itself which other methods are incapable of doing it. It is acknowledged that when fuzzy set membership is defined with respect to truth *T*, the information related to indeterminacy and non-membership is missing. In IFS, memberships are defined in terms of truth and false only; here the indeterminacy is taken as what is left after the truth and false membership. The IFS cannot represent the indeterminate and inconsistent information, but it has provisions to describe and work with incomplete information. In SVNS, truth, indeterminacy and falsity membership are represented individualistically, and they can also be defined with respect to any of them (no restriction). This makes SVNS better at dealing information than IFS. TRINS when compared to SVNS/DVNS has better scope to describe and deal with the existing indeterminacy and inconsistent information because the indeterminacy concept is classified as three different values. This provides more accuracy and precision to indeterminacy in TRINS, than in SVNS. However, TRINS is better equipped to deal with indeterminacy than Fuzzy theory. Fuzzy Likert scale cannot capture indeterminate, imprecise and incomplete data. TRINS-based indeterminate Likert scale captures data in a more precise, accurate and realistic way than fuzzy Likert scale.

6.3 Further study

Multipoint indeterminate Likert scale which functions on 7 points or 10 points will be taken up for further studies. These multipoint Likert scales can be used to study a variety of sociological, economical and psychological problems. As future research, we also propose to map the middle 3 terms of TRINS to neutrosophic triplets (Vasantha et al. 2018b) and then they can be automatically mapped to neutrosophic duplets (Vasantha et al. 2018a, c) in the case that indeterminacy leaning towards false is zero.

7 Conclusions

In this paper, indeterminate Likert scaling based on TRINS was introduced; it is equipped to deal with inconsistent, uncertain, imprecise and indeterminate information which Likert scale is incapable of. Generally feedback from the customer depends on the human emotions which are mostly uncertain, inconsistent, imprecise or indeterminate in nature. Hence, indeterminate Likert scale is more apt to use for feedback than Likert scale. Indeterminate Likert scale can be easily implemented and used in mobile apps for collecting feedback. Indeterminate MST clustering algorithm was introduced to cluster the feedback obtained using distance matrices as a main measurement. Results from the clustering can be used for targeted and interactive marketing separately for each cluster.

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Structure, NeutroStructure, and AntiStructure in Science

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Abstract

In any science, a classical Theorem, defined on a given space, is a statement that is 100% true (i.e. true for all elements of the space). To prove that a classical theorem is false, it is sufficient to get a single counter-example where the statement is false.

Therefore, the classical sciences do not leave room for partial truth of a theorem (or a statement). But, in our world and in our everyday life, we have many more examples of statements that are only partially true, than statements that are totally true.

The NeutroTheorem and AntiTheorem are generalizations and alternatives of the classical Theorem in any science.

More general, by the process of NeutroSophication, we have extended any classical Structure, in no matter what field of knowledge, to some NeutroStructure, and by the process of AntiSophication to some AntiStructure.

Keywords : Structure, NeutroStructure, and AntiStructure

1. The Neutrosophic Triplet (<A>, <neutA>, <antiA>)

Let S be a given non-empty space (or set) from a universe of discourse U.

In neutrosophy, the general neutrosophic triplet (<A>,<neutA>,<antiA>), sometimes using the notation <neutroA> for the middle term, can be written as:

$(\langle A(1,0,0)\rangle, \langle A(T,I,F)\rangle, \langle A(0,0,1)\rangle)$, where $(T,I,F) \notin \{(1,0,0),(0,0,1)\}$;

i.e. $A(1,0,0)$ means that <A> is 100% true (T = 1), 0% indeterminate (I = 0), and 0% false (F = 0);

$A(T,I,F)$ means that <A> is T% true, I% indeterminate, and F% false, where (T, I, F) $\notin \{(1,0,0),(0,0,1)\}$;

and $A(0,0,1)$ means that <A> is 0% true (T = 0), 0% indeterminate (I = 0), and 100% false (F = 1), respectively.

2. Example of Neutrosophic Triplet when <A> = Operation

In the case when <A> is an Operation (or Operator, Function, Law), on the given space S, then <A(T, I, F)> means:

Operation <A> is T% well-defined (or inner-defined, inside of S),
I% indeterminate-defined (undefined, unknown),
and F% outer-defined (outside of S).

3. Neutrosophic Triplet Concepts

We have the following particular neutrosophic triplets of notions defined on S:

(<Theorem>, <NeutroTheorem>, <AntiTheorem>),
(<Lemma>, <NeutroLemma>, <AntiLemma>),
(<Consequence>, <NeutroConsequence>, <AntiConsequence>),
(<Proposition>, <NeutroProposition>, <AntiProposition>),
(<Definition>, <NeutroDefinition>, <AntiDefinition>),
(<Property>, <NeutroProperty>, <AntiProperty>),
(<Function>, <NeutroFunction>, <AntiFunction>),
(<Operation>, <NeutroOperation>, <AntiOperation>),
(<Axiom>, <NeutroAxiom>, <AntiAxiom>), etc.

These neutrosophic triplets are referred to any field of knowledge, not only to mathematics.

4. Theorem, Neutro Theorem, Anti Theorem

Let's take the first neutrosophic triplet:
(<Theorem>, <NeutroTheorem>, <AntiTheorem>).
For the other neutrosophic triplets, it will be similar.
Let $T, I, F \in [0,1]$ be single-valued numbers representing respectively the degree of truth (T), degree of indeterminacy (I), and degree of falsehood (F).

(i) A classical Theorem is a statement that is true (T) for all elements of the space S. Therefore, (T, I, F) = (1, 0, 0).
(ii) A NeutroTheorem is a statement that is partially (T), partially indeterminate (I), and partially false (F), where (T, I, F) $\notin \{(1,0,0),(0,0,1)\}$;
(iii) An AntiTheorem is a statement that is false (F) for all elements of the space S. Therefore, (T, I, F) = (0, 0, 1).

We can rewrite this neutrosophic triplet as:
$(\langle Theorem(1,0,0)\rangle, \langle Theorem(T,I,F)\rangle, \langle Theorem(0,0,1)\rangle)$,
where (T, I, F) $\notin \{(1,0,0),(0,0,1)\}$.

Let T, I, F be intervals (and in general any subsets) from [0, 1].
If a Theorem is, let's say, between 90%-100% true, i.e. Theorem([0.9, 1], 0, 0), it does not satisfy the classical Theorem(1, 0, 0), since there is some uncertainty (unclearness) with respect to its degree of truth [0.9, 1] $\neq$ 1. So, this case goes under NeutroTheorem.

If the Theorem is, let's say again, between 99%-100% false, i.e. Theorem(0, 0, [0.99, 1]), it does not satisfy the AntiTheorem, since similarly there is some uncertainty (unclearness) with respect to its degree of falsehood [0.99, 1] $\neq$ 1. This case goes also under NeutroTheorem.

In conclusion, no matter if T, I, F are single-valued numbers, intervals, and in general any subsets of [0, 1], the neutrosophic triplet of each concept is the same.

Classical Theorem = <Theorem(1, 0, 0)>
NeutroTheorem = <Theorem(T, I, F)>, with (T, I, F) $\notin \{(1,0,0),(0,0,1)\}$;
AntiTheorem = < Theorem(0, 0, 1)>.

5. Remark

Let $T, I, F \in [0,1]$ and an axiom $\langle A(T, I, F)\rangle$, which means that the axiom has the neutrosophic degree of truth T, the neutrosophic degree of indeterminacy I, and the neutrosophic degree of falsehood F.
If $I > 0$ or $0 < F < 1$, the $\langle A\rangle$ is a NeutroAxiom.

Proof:
If $I > 0$, then $(T, I, F) \notin \{(1,0,0),(0,0,1)\}$, because the last two neutrosophic triplets have both $I = 0$.

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If $0 < F < 1$, then again $(T, I, F) \notin \{(1,0,0), (0,0,1)\}$ because the last two neutrosophic triplets have $F = 0$ and respectively $F = 1$.

6. Elementary Examples of Neutro Concepts and Anti Concepts

6.1. NeutroOperation

Let Z be the set of integers, and Q the set of rational numbers that is considered the universal set, with $Z \subset Q$.
Let's define the operation of division:

$$\div : Z \times Z \longrightarrow Z.$$

This is a NeutroDivision because:
(i) There exist the integers $15, 5 \in \mathbb{Z}$ such that $15 \div 5 = 3 \in Z$; this is degree of well-defined (inner-defined);
(ii) There exist the integers $7, 0 \in \mathbb{Z}$ such that $7 \div 0 =$ undefined; this is degree of indeterminacy;
(iii) There exist the integers $11, 2 \in \mathbb{Z}$ such that $11 \div 2 = 5.5 \notin Z$ or $5.5 \in Q \setminus \mathbb{Z}$; this is degree of outer-defined.

6.2. AntiOperation
Let $Z^- = \{-1, -2, -3, \dots, -\infty\}$ the set of negative integers,
$\mathbb{C} = \{a + bi; a, b \in \mathbb{R}, i = \sqrt{-1}\}$ the set of complex numbers that acts as the universal set of Z^-.
Let's define the operation of square root (√):
$\surd: Z^- \longrightarrow Z^-$.
For any negative integer $-a$, where $a > 0$ is a positive integer,
$\sqrt{-a} = i\sqrt{a} \notin \mathbb{Z}^-$, or $\sqrt{-a} \in C - Z^-$.
Therefore the operation √ is totally outer-defined.

6.3. NeutroFunction
On the set of integers $\mathbb{Z}$ and its universe of discourse Q, which is the set of rational numbers, we define the function:
$f: \mathbb{Z} \longrightarrow \mathbb{Z}, f(x) = \frac{12}{x}$.

(i) There exists the integer, for example $x = 6$, such that:
$f(6) = \frac{12}{6} = 2 \in \mathbb{Z}$;
this is degree of well-defined.

(ii) There exists the integer $0 \in \mathbb{Z}$, such that:
$f(0) = \frac{12}{0} = undefined$;
this is degree of indeterminacy.

(iii) There exists the integer, for example $5 \in \mathbb{Z}$, such that:
$f(5) = \frac{12}{5} = 2.4 \notin \mathbb{Z}$; or $2.4 \in Q \setminus \mathbb{Z}$;
this is degree of outer-defined.

6.4. NeutroTheorem
Let $Z^+ = \{1, 2, 3, \dots, +\infty\}$ be the set of positive integers.
We consider the following:
Statement
If x and $y \in \mathbb{Z}^+$, then x^y is a perfect square.

In classical algebraic structures this statement is considered false, because it is not 100% true, and to prove it, it is sufficient to get a single particular counter-example. For example, if $x = 2$ and $y = 3$, then $2^3 = 8$ which is not a perfect square.

The classical algebraic structures do not leave room for partial truth of a theorem. But, in our world and in our everyday life, we have many more examples of statements that are only partially true, than statements that are totally true.

The NeutroTheorem and AntiTheorem are generalizations and alternatives of the classical Theorem.

In the above statement, we have:

(i) Degree of truth, when $x = a^2$ or $y = 2b$, where $a, b \in \mathbb{Z}^+$. Since we get $x^y = (a^2)^y = (a^y)^2$ and respectively $x^y = x^{2b} = (x^b)^2$.

Therefore we have two double-infinity many cases when the statement is true.

$x = a^2$ means that x can be written as a perfect square. For example, if $x = 3^8$ we can re-write it as $x = (3^4)^2 = 81^2$.

And $y = 2b$ means y is an even number.

(ii) Degree of indeterminacy is zero, since x^y is always well-defined for non-zero $x, y \in \mathbb{Z}^+$.

(iii) Degree of falsehood, as shown above, for example when $x = 2$ and $y = 3$.

Herein we also have infinitely many cases when the statement is false (for example when $x \neq a^2$ and $y \neq 2b$).

7. Structure, NeutroStructure, AntiStructure in any field of knowledge

In general, by NeutroSophication [1, 2, 3, 4], we have extended any classical Structure, in no matter what field of knowledge, to NeutroStructure, and by AntiSophication to AntiStructure.

A classical Structure, in any field of knowledge, is composed of: a non-empty space, populated by some elements, and both (the space and all elements) are characterized by some relations among themselves (such as: laws, operations, operators, axioms, properties, functions, theorems, lemmas, consequences, algorithms, charts, hierarchies, equations, inequalities, etc.), and their attributes (size, weight, color, shape, location, etc.).

8. Relation, NeutroRelation, AntiRelation

(i) A classical Relation on a given set is a relation that is true for all elements of the set (degree of truth T = 1). Neutrosophically we write Relation(1,0,0).

(ii) A NeutroRelation is a relation that is true for some of the elements (degree of truth T), indeterminate for other elements (degree of indeterminacy I), and false for the other elements (degree of falsehood F). Neutrosophically we write Relation(T,I,F), where (T,I,F) is different from (1,0,0) and from (0,0,1).

(iii) An AntiRelation is a relation that is false for all elements (degree of falsehood F = 1). Neutrosophically we write Relation(0,0,1).

9. Attribute, NeutroAttribute, AntiAttribute

(i) A classical Attribute of the elements of a given set is an attribute that is true for all elements of the set (degree of truth T = 1). Neutrosophically we write Attribute(1,0,0).

(ii) A NeutroAttribute is an attribute that is true for some of the elements (degree of truth T), indeterminate for other elements (degree of indeterminacy I), and false for the other elements (degree of falsehood F). Neutrosophically we write Attribute(T,I,F), where (T,I,F) is different from (1,0,0) and (0,0,1).

(iii) An AntiAttribute is an attribute that is false for all elements (degree of falsehood F = 1). Neutrosophically we write Attribute(0,0,1).

10. Definitions of Structure, NeutroStructure, AntiStructure

(i) A classical Structure is a structure whose all elements are characterized by the same given Relationships and Attributes.

(ii) A NeutroStructure is a structure that has at least one NeutroRelation or one NeutroAttribute, and no AntiRelation nor AntiAttribute.

(iii) An AntiStructure is a structure that has at least one AntiRelation or one AntiAttribute.

11. Example of Neutro Structure

In the Christian society the marriage is defined as the union between a male and a female (degree of truth).

But, in the last decades, this law has become less than 100% true, since persons of the same sex were allowed to marry as well (degree of falsehood).

On the other hand, there are transgender people (whose sex is not well-determined, or whose sex is undetermined), and people who have changed the sex by surgical procedures, and these people (and their marriage) cannot be included in the first two categories (degree of indeterminacy).

Therefore, since we have a NeutroLaw (with respect to the Law of Marriage) we have a Christian NeutroStructure.

Conclusion

A classical Structure, in any field of knowledge, is composed of: a non-empty space, populated by some elements, and both (the space and all elements) are characterized by some relations among themselves, and by some attributes. Classical Structures are mostly in theoretical, abstract, imaginary spaces.

Of course, when analysing a structure, it counts with respect to what relations and attributes we analyse it.

In our everyday life almost all structures are NeutroStructures, since they are neither perfect nor uniform, and not all elements of the structure's space have the same relations and same attributes in the same degree (not all elements behave in the same way).

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Multi-Strategy Decision-Making on Enhancing Customer Acquisition Using Neutrosophic Soft Relational Maps

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Abstract. Decision making by the business managerial on framing strategies to foster customer acquisition is a challenging task. The aim of this paper is to introduce a new method of Multi-Strategy Decision-Making (MSDM) integrated with neutrosophic soft relational maps to determine the significant and feasible strategies of customer acquisition and their inter impacts. The proposed method comprises of two-stage processes and it is validated with twenty strategies, five factors associated with customer acquisition and expert 's opinion based on multivalued neutrosophic soft sets.

Keywords: Multi-Strategy, Decision-Making, Neutrosophic soft sets, Relational maps.

1. Introduction

Decision theory is characterized by various Multi-Criteria Decision making (MCDM) (otherwise called as Multi-Objective or Multi-Attribute or Multi-Dimension Decision-Making) methods such as Analytical Hierarchy Process, ELECTRE, COPRAS, PROMTHEE, TOPSIS, SAW. MCDM methods are used in selection of alternatives subjected to criteria satisfaction. MCDM methods are extended to Fuzzy MCDM to handle uncertainty in decision making. The criterion – alternative association is represented as fuzzy values in fuzzy MCDM. Wang et al. developed Fuzzy MCDM method for sustainable supplier selection and evaluation. Peng et al. [10], Saini et al. [12] developed intuitionistic MCDM (IFMCDM) approaches with intuitionistic representation comprising of membership and non-membership values. Neutrosophic sets introduced by Smarandache [13] comprises of truth , indeterminacy and falsity values and it has been extensively used in MCDM. Athar [5], Abdel-Basset [1, 2], Nada et al. [9], Garg et al. [6] developed neutrosophic MCDM models with neutrosophic representations of criterion alternative association. Another kind of sets that also play a key role in decision making is Soft sets introduced by Molodtsov [8], which was later extended to fuzzy soft sets by Maji [7]. Dey et al. [3] presented the applications of multi-fuzzy soft sets in decision-making. Tripathy et al. [14] described the key role of intuitionistic fuzzy soft sets in group decision making. Faruk Karaaslan [4] elicited the implications of neutrosophic soft sets in decision making. Abu and Omar [11] extended neutrosophic soft sets to Q-neutrosophic soft sets and these sets are applied in comprehensive decision-making. In these neutrosophic soft MCDM models, the optimal ranking of the alternatives are determined. But these model do not cater to determine the impact of exercising the alternatives.

In this paper the new decision making approach based on MCDM is developed with the replacement of alternatives by strategies to make decisions and the criteria by the objectives to be fulfilled. The proposed method comprises of two-stage processes. The first stage ranks the proposed alternatives based on criteria satisfaction rate with the representation of neutrosophic soft sets and in the second stage the chosen alternatives are associated with the principles of decision making using neutrosophic soft relational maps. The integration of soft sets in relational maps is an innovative initiative of this research work. The proposed two-stage decision making process is a ground-breaking endeavor and it is validated by applying to decision making on customer acquisition strategies. Though researchers have explored strategically decision- making in various perspectives, the mathematical approach of strategy selection has not been explored so far to the best of our knowledge and this research work is an opening to it. The content of the paper is organized as follows: the methodology is presented in Section 2, the application of the proposed approach is validated in Section 3, the results are discussed in Section 4, the last section concludes the work.

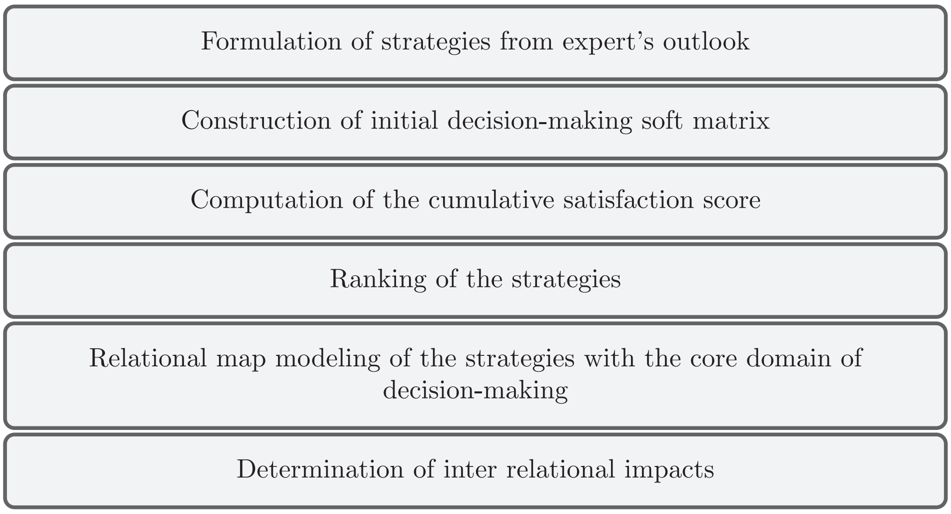
2. Materials and Methods

This section presents the significance and need of MSDM and the algorithmic approach of determining optimal solution.

2.1. Multi-Strategy Decision-Making. In the approach of MSDM, the primary aim is to rank the strategies. In general, all the productions sectors construct their goals and work towards accomplishing the same. The managerial formulate strategies to achieve the goals, but the major challenge is selection and implementation of feasible strategies to yield optimum benefits. The decision-making environment does not involve only selection of alternatives with respect to criteria satisfaction, rather it involves the other dimension of choosing the right optimizing strategies. Strategical decision making is another dominating phenomenon and it has to be focused and this is how the approach of MSDM has evolved. In this new approach the method of finding the optimal strategy is a two-step process. The first step ranks the strategies and the second step associates their inter relationship with the principles of decision making. The steps are as follows:

Characterization of decision-making problem

Selection of objectives of the firm



3. Application of the Proposed MSDM Approach

This section applies the proposed two stage processes of MSDM to the decision making on customer acquisition strategies based on expert's opinion presented as below.

S_1 Selection of Advertising medium to propagate the product,
S_2 Designing user friendly products,
S_3 Customizing the product's utility to the needs of the buyers,
S_4 Attending to the diverse needs of the customers,
S_5 Developing multi-faceted products reflecting the ethos of the customers,
S_6 Scaling the cost of the product to customer's budget,
S_7 Periodic Propagation of the attributes of the product,
S_8 Product outlook modification,
S_9 Creating smart products,
S_{10} Developing innovative kind of products suiting the dynamic needs of the consumers,
S_{11} Create an ambiance to purchase product by providing offers,
S_{12} Communicating the attributes of the product to the customers,
S_{13} On line engagement with the customers,
S_{14} Establishing Trade mark of the product,
S_{15} Provision of various kinds of payment portals,
S_{16} Enrichment of the quality of the product using modern technology,
S_{17} Strengthening the consistency and reliability of the product,
S_{18} Designing products with values adding to consumer's image,
S_{19} Periodical review of product sales and marketing,
S_{20} Integrating eco-friendly characteristics with the products.

In the perspective of soft sets, let $U = \{S_1, S_2, \ldots, S_{20}\}$ and $A = \{A_1, A_2, \ldots, A_5\}$ be the set of purchasing behavior influencing factors, where $A_1 =$ Psychological, $A_2 =$ Personal, $A_3 =$ Product, $A_4 =$ Social and $A_5 =$ Cultural.

A multivalued neutrosophic soft mapping $G : A \longrightarrow P(U)$ is represented as follows:

$$
\begin{aligned}
G(A_1) \quad = \quad & \{\frac{\langle(0.9,0.1,0.2),(0.8,0.3,0.2),(0.9,0.1,0.2)\rangle}{S_1}, \frac{\langle(0.6,0.3,0.3),(0.6,0.1,0.3),(0.6,0.3,0.3)\rangle}{S_2}, \\
& \frac{\langle(0.8,0.3,0.5),(0.9,0.3,0.5),(0.8,0.3,0.5)\rangle}{S_3}, \frac{\langle(0.6,0.2,0.3),(0.6,0.2,0.3),(0.6,0.2,0.3)\rangle}{S_4}, \\
& \frac{\langle(0.7,0.5,0.2),(0.6,0.4,0.2),(.07,0.5,0.2)\rangle}{S_5}, \frac{\langle(0.9,0.1,0.1),(0.9,0.1,0.1),(0.9,0.1,0.1)\rangle}{S_6}, \\
& \frac{\langle(0.8,0.3,0.5),(0.8,0.2,0.5),(0.8,0.3,0.5)\rangle}{S7}, \frac{\langle(0.6,0.4,0.4),(0.6,0.4,0.3),(0.6,0.4,0.4)\rangle}{S_8}, \\
& \frac{\langle(0.7,0.5,0.2),(0.6,0.1,0.2),(0.7,0.5,0.2)\rangle}{S_9}, \frac{\langle(0.6,0.4,0.3),(0.6,0.4,0.3),(0.6,0.4,0.3)\rangle}{S_{10}}, \\
& \frac{\langle(0.9,0.1,0.1),(0.9,0.1,0.1),(0.9,0.1,0.1)\rangle}{S_{11}}, \frac{\langle(0.9,0.1,0.2),(0.9,0.1,0.1),(0.9,0.1,0.2)\rangle}{S_{12}}, \\
& \frac{\langle(0.8,0.3,0.5),(0.8,0.3,0.5),(0.8,0.3,0.5)\rangle}{S_{13}}, \frac{\langle(0.9,0.1,0.2),(0.9,0.1,0.2),(0.9,0.1,0.2)\rangle}{S_{14}}, \\
& \frac{\langle(0.8,0.3,0.5),(0.8,0.2,0.4),(0.8,0.3,0.5)\rangle}{S_{15}}, \frac{\langle(0.6,0.4,0.3),(0.6,0.5,0.3),(0.6,0.4,0.3)\rangle}{S_{16}}, \\
& \frac{\langle(0.8,0.3,0.5),(0.8,0.2,0.5),(0.8,0.3,0.5)\rangle}{S_{17}}, \frac{\langle(0.8,0.3,0.5),(0.8,0.2,0.5),(0.8,0.3,0.5)\rangle}{S_{18}}, \\
& \frac{\langle(0.6,0.4,0.3),(0.6,0.4,0.4),(0.6,0.4,.3)\rangle}{S_{19}}, \frac{\langle(0.7,0.5,0.2),(0.7,0.5,0.1),(0.7,0.5,0.2)\rangle}{S_{20}}\},
\end{aligned}
$$

$$
\begin{aligned}
G(A_2) \quad = \quad & \{\frac{\langle(0.7,0.5,0.2),(0.6,0.4,0.2)),(0.7,0.5,0.2)\rangle}{S_1}, \frac{\langle(0.7,0.5,0.2),(0.9,0.1,0.3),(0.9,0.1,0.2)\rangle}{S_2}, \\
& \frac{\langle(0.8,0.2,0.4),(0.8,0.2,0.3),(0.8,0.2,0.4)\rangle}{S_3}, \frac{\langle(0.9,0.1,0.2),(0.9,0.3,0.2),(0.9,0.1,0.2)\rangle}{S_4}, \\
& \frac{\langle(0.8,0.2,0.4),(0.7,0.2,0.4),(0.8,0.2,0.4)\rangle}{S_5}, \frac{\langle(0.6,0.4,0.3),(0.6,0.4,0.4),(0.6,0.4,0.3)\rangle}{S_6}, \\
& \frac{\langle(0.6,0.4,0.3),(0.6,0.3,0.3),(0.6,0.4,0.3)\rangle}{S7}, \frac{\langle(0.6,0.4,0.3),(0.6,0.2,0.3),(0.6,0.4,0.3)\rangle}{S_8}, \\
& \frac{\langle(0.8,0.2,0.4),(0.7,0.2,0.4),(0.8,0.2,0.4)\rangle}{S_9}, \frac{\langle(0.9,0.2,0.3),(0.9,0.2,0.3),(0.9,0.2,0.3)\rangle}{S_{10}}, \\
& \frac{\langle(0.8,0.2,0.4),(0.8,0.2,0.4),(0.8,0.2,0.4)\rangle}{S_{11}}, \frac{\langle(0.6,0.4,0.3),(0.6,0.4,0.3),(0.6,0.4,0.3)\rangle}{S_{12}}, \\
& \frac{\langle(0.9,0.1,0.1),(0.9,0.2,0.1),(0.9,0.1,0.1)\rangle}{S_{13}}, \frac{\langle(08,0.2,0.4),(0.7,0.2,0.4),(0.8,0.2,0.4)\rangle}{S_{14}}, \\
& \frac{\langle(0.9,0.1,0.2),(0.8,0.1,0.2)(0.9,0.1,0.2)\rangle}{S_{15}}, \frac{\langle(0.9,0.1,0.1),(0.9,0.1,0.2),(0.9,0.1,0.1)\rangle}{S_{16}}, \\
& \frac{\langle(0.6,0.4,0.3),(0.6,0.4,0.2),(0.6,0.4,0.3)\rangle}{S_{17}}, \frac{\langle(0.9,0.1,0.1),(0.9,0.1,0.3),(0.9,0.1,0.1)\rangle}{S_{18}}, \\
& \frac{\langle(0.9,0.1,0.1),(0.9,0.1,0.1)\rangle,(0.9,0.1,0.1)\rangle}{S_{19}}, \frac{\langle(0.8,0.2,0.4),(0.8,0.1,0.3),(0.8,0.2,0.4)\rangle}{S_{20}}\},
\end{aligned}
$$

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\begin{aligned}
G(A_3) \quad = \quad & \{\frac{\langle(0.8,0.3,0.5),(0.8,0.1,0.3),(0.8,0.3,0.5)\rangle}{S_1}, \frac{\langle(0.8,0.2,0.4),(0.7,0.2,0.4),(0.8,0.2,0.4)\rangle}{S_2}, \\
& \frac{\langle(0.5,0.4,0.6),(0.6,0.4,0.3),(0.5,0.4,0.6)\rangle}{S_3}, \frac{\langle(0.8,0.2,0.4),(0.7,0.5,0.3),(0.8,0.2,0.4)\rangle}{S_4}, \\
& \frac{\langle(0.9,0.2,0.3),(0.7,0.5,0.3),(0.9,0.2,0.3)\rangle}{S_5}, \frac{\langle(0.7,0.5,0.2),(0.9,0.3,0.2),(0.7,0.5,0.2)\rangle}{S_6}, \\
& \frac{\langle(0.6,0.4,0.3),(0.6,0.3,0.3),(0.6,0.4,0.3)\rangle}{S7}, \frac{\langle(0.7,0.5,0.2),(0.8,0.5,0.2),(0.7,0.5,0.2)\rangle}{S_8}, \\
& \frac{\langle(0.9,0.2,0.3),(0.8,0.1,0.4),(0.9,0.2,0.3)\rangle}{S_9}, \frac{\langle(0.8,0.2,0.4),(0.7,0.3,0.2),(0.8,0.2,0.4)\rangle}{S_{10}}, \\
& \frac{\langle(0.8,0.2,0.4),(0.8,0.5,0.2),(0.8,0.2,0.4)\rangle}{S_{11}}, \frac{\langle(0.6,0.4,0.3),(0.4,0.5,0.2),(0.7,0.5,0.2)\rangle}{S_{12}},
\end{aligned}
$$

$$
\begin{aligned}
& \frac{\langle(0.9,0.1,0.2),(0.9,0.1,0.3),(0.7,0.5,0.2)\rangle}{S_{13}}, \frac{\langle(0.9,0.1,0.2),(0.9,0.1,0.3),(0.7,0.5,0.2)\rangle}{S_{14}}, \\
& \frac{\langle(0.6,0.4,0.3),(0.6,0.4,0.3),(0.7,0.5,0.2)\rangle}{S_{15}}, \frac{\langle(0.9,0.2,0.3),(0.7,0.5,0.1),(0.7,0.5,0.2)\rangle}{S_{16}}, \\
& \frac{\langle(0.9,0.1,0.2),(0.7,0.5,0.1),(0.7,0.5,0.2)\rangle}{S_{17}}, \frac{\langle(0.6,0.4,0.3),(0.7,0.5,0.1),(0.7,0.5,0.2)\rangle}{S_{18}}, \\
& \frac{\langle(0.7,0.5,0.2),(0.9,0.2,0.2),(0.9,0.2,0.3)\rangle}{S_{19}}, \frac{\langle(0.9,0.1,0.1),(0.6,0.4,0.4),(0.6,0.4,0.3)\rangle}{S_{20}}\},
\end{aligned}
$$

$$
\begin{aligned}
G(A_4) \quad = \quad & \{\frac{\langle(0.6,0.4,0.3),(0.5,0.2,0.3),(0.6,0.4,0.3)\rangle}{S_1}, \frac{\langle(0.9,0.1,0.1),(0.9,0.1,0.1),(0.9,0.1,0.1)\rangle}{S_2}, \\
& \frac{\langle(0.7,0.5,0.2),(0.7,0.4,0.2),(0.7,0.5,0.2)\rangle}{S_3}, \frac{\langle(0.7,0.5,0.2),(0.7,0.5,0.3),(0.7,0.5,0.2)\rangle}{S_4}, \\
& \frac{\langle(0.7,0.5,0.2),(0.7,0.5,0.3),(0.7,0.5,0.2)\rangle}{S_5}, \frac{\langle(0.9,0.1,0.2),(0.9,0.3,0.2),(0.9,0.1,0.2)\rangle}{S_6}, \\
& \frac{\langle(0.5,0.4,0.6),(0.5,0.4,0.7),(0.5,0.4,0.6)\rangle}{S_7}, \frac{\langle(0.7,0.5,0.2),(0.8,0.5,0.2),(0.7,0.5,0.2)\rangle}{S_8}, \\
& \frac{\langle(0.8,0.2,0.4),(0.8,0.1,0.4),(0.8,0.2,0.4)\rangle}{S_9}, \frac{\langle(0.7,0.5,0.2),(0.7,0.3,0.2),(0.7,0.5,0.2)\rangle}{S_{10}}, \\
& \frac{\langle(0.7,0.5,0.2),(0.8,0.5,0.2),(0.7,0.5,0.2)\rangle}{S_{11}}, \frac{\langle(0.7,0.5,0.2),(0.4,0.5,0.2),(0.7,0.5,0.2)\rangle}{S_{12}}, \\
& \frac{\langle(0.9,0.1,0.2),(0.9,0.1,0.3),(0.9,0.1,0.2)\rangle}{S_{13}}, \frac{\langle(0.9,0.4,0.3),(0.7,0.2,0.3),(0.6,0.4,0.3)\rangle}{S_{14}}, \\
& \frac{\langle(0.7,0.5,0.2),(0.6,0.4,0.3),(0.7,0.5,0.2)\rangle}{S_{15}}, \frac{\langle(0.7,0.5,0.2),(0.7,0.5,0.1),(0.9,0.2,0.3)\rangle}{S_{16}}, \\
& \frac{\langle(0.7,0.5,0.2),(0.7,0.5,0.1),(0.9,0.1,0.2)\rangle}{S_{17}}, \frac{\langle(0.7,0.5,0.2),(0.7,0.5,0.4),(0.6,0.4,0.3)\rangle}{S_{18}}, \\
& \frac{\langle(0.9,0.2,0.3),(0.9,0.2,0.2),(0.9,0.2,0.3)\rangle}{S_{19}}, \frac{\langle(0.6,0.2,0.3),(0.9,0.2,0.1),(0.9,0.2,0.3)\rangle}{S_{20}}\}
\end{aligned}
$$

and

$$
\begin{aligned}
G(A_5) \quad = \quad & \{\frac{\langle(0.9,0.2,0.3),(0.9,0.1,0.2),(0.9,0.2,0.3)\rangle}{S_1}, \frac{\langle(0.7,0.5,0.2),(0.8,0.5,0.2),(0.7,0.5,0.2)\rangle}{S_2}, \\
& \frac{\langle(0.8,0.2,0.4),(0.7,0.2,0.4),(0.8,0.2,0.4)\rangle}{S_3}, \frac{\langle(0.9,0.2,0.3),(0.9,0.2,0.4),(0.9,0.2,0.3)\rangle}{S_4}, \\
& \frac{\langle(0.8,0.2,0.4),(0.8,0.2,0.5),(0.8,0.2,0.4)\rangle}{S_5}, \frac{\langle(0.7,0.5,0.2),(0.8,0.5,0.2),(0.7,0.5,0.2)\rangle}{S_6}, \\
& \frac{\langle(0.9,0.2,0.3),(0.9,0.2,0.1),(0.9,0.2,0.3)\rangle}{S_7}, \frac{\langle(0.8,0.2,0.4),(0.8,0.2,0.4),(0.8,0.2,0.4)\rangle}{S_8}, \\
& \frac{\langle(0.9,0.1,0.1),(0.9,0.2,0.1),(0.9,0.1,0.1)\rangle}{S_9}, \frac{\langle(0.9,0.2,0.3),(0.8,0.2,0.3),(0.9,0.2,0.3)\rangle}{S_{10}}, \\
& \frac{\langle(0.8,0.2,0.4),(0.8,02,0.4),(0.8,0.2,0.4)\rangle}{S_{11}}, \frac{\langle(0.8,0.2,0.4),(0.7,0.2,0.4),(0.8,0.2,0.4)\rangle}{S_{12}}, \\
& \frac{\langle(0.7,0.5,0.2),(0.8,0.3,0.2),(0.7,0.5,0.2)\rangle}{S_{13}}, \frac{\langle(0.9,0.1,0.1),(0.9,0.2,0.1),(0.9,0.1,0.1)\rangle}{S_{14}}, \\
& \frac{\langle(0.7,0.5,0.2),(0.7,0.4,0.2),(0.7,0.5,0.2)\rangle}{S_{15}}, \frac{\langle(0.8,0.2,0.4),(0.8,0.2,0.3),(0.8,0.2,0.4)\rangle}{S_{16}}, \\
& \frac{\langle(0.9,0.2,0.3),(0.8,0.2,0.3),(0.9,0.2,0.3)\rangle}{S_{17}}, \frac{\langle(0.9,0.2,0.3),(0.9,0.2,0.2),(0.9,0.2,0.3)\rangle}{S_{18}}, \\
& \frac{\langle(0.8,0.2,0.4),(0.8,0.2,0.3),(0.8,0.2,0.4)\rangle}{S_{19}}, \frac{\langle(0.9,0.2,0.3),(0.8,0.1,0.3),(0.9,0.2,0.3)\rangle}{S_{20}}\}.
\end{aligned}
$$

The score values of each of the strategies with respect to the respective association with the factors are determined by using the algorithm was discussed in [5] (See Figure 1). The following factors are considered as the core factors for the

FIGURE 1. Ranking of the Factors.

next step.

CS_1 Developing multi-faceted products reflecting the ethos of the customers,
CS_2 Scaling the cost of the product to customer's budget,
CS_3 Enrichment of the quality of the product using modern technology,
CS_4 Strengthening the consistency and reliability of the product,
CS_5 Designing products with values adding to consumer's image,
CS_6 Periodical review of product sales and marketing.

These factors are related to the various management systems of the business. The relational impacts are represented linguistic neutrosophic sets and are quantified using neutrosophic triangular fuzzy number as presented in Table 1.

TABLE 1. Quantification of Linguistic Variable.

Linguistic Variable	**Neutrosophic Triangular Number**	**Crisp Value**
Very Low (VL)	((0,0.10,0.15,0.20),0.6,0.2,0.3)	0.06
Low (L)	((0.15,0.2,0.25,0.3),0.6,0.1,0.1)	0.14
Medium (M)	((0.3,0.35,0.4,0.5),0.7,0.1,0.2)	0.23
High (H)	((0.5,0.6,0.7,0.8),0.8,0.2,0.1)	0.41
Very High (VH)	((0.8,0.9,0.95,1),0.9,0.1,0.1)	0.62

Let $U = \{CS_1, CS_2, \ldots, CS_6\}$ and $M = \{M_1, M_2, M_3, M_4\}$ be the set of management systems of business, where

$$\begin{aligned} M_1 &= \text{Product Quality Management,} \\ M_2 &= \text{Customer Loyalty Management,} \\ M_3 &= \text{Customer Relationship Management,} \\ M_4 &= \text{Marketing Management.} \end{aligned}$$

A single valued neutrosophic soft mapping $H : M \longrightarrow P(U)$ is represented as follows:

$$H(M_1) = \{\frac{VH}{CS_1}, \frac{L}{CS_2}, \frac{VH}{CS_3}, \frac{H}{CS_4}, \frac{M}{CS_5}, \frac{H}{CS_6}\},$$

Table 2. Fixed points of the vectors.

Initial Vector	Fixed Point
$X = (100000)$	$X^*M = (0.620.410.410.14)(1110) := X1$ $X1^*MT = (1.440.691.441.651.470.87) = (100110) := Y$ $Y^*M = (1.261.651.650.78)(1110) := X2$ $X2^*MT = (1.440.691.441.651.470.87) = (100110) := Y1$ $(1110)(100110)$
$X = (010000)$	$X^*M = (0.140.140.410.23)(0011) := X_1$ $X1^*MT = (0.550.640.641.030.850.85) = (010111) := Y$ $Y^*M = (1.191.611.881.49)(0110) := X_2$ $X_2^*MT = (0.820.550.821.241.240.46) = (111110) := Y_1$ $Y_1^*M = (2.022.22.471.24)(0110) := X3$ $X_3^*MT = (0.820.550.821.241.240.46) = (111110) := Y_2$ $(0110)(111110)$
$X = (001000)$	$X^*M = (0.620.410.410.23)(1110) := X_1$ $X_1^*MT = (1.440.691.441.651.470.87) = (100110) := Y$ $Y^*M = (1.261.651.650.78)(1110) := X_2$ $X_2^*MT = (1.440.691.441.651.470.87) = (100110) := Y1$ $(1110)(100110)$
$X = (000100)$	$X^*M = (0.410.620.620.41)(1111) := X_1$ $X_1^*MT = (1.580.921.672.061.71.49) = (000110) := Y$ $Y^*M = (0.641.241.240.64)(1111) := X_2$ $X_2^*MT = (1.580.921.672.061.71.49) = (000110) := Y_1$ $(1111)(000110)$
$X = (000010)$	$X^*M = (0.230.620.620.23)(1111) := X_1$ $X_1^*MT = (1.580.921.672.061.71.49) = (000110) := Y$ $Y^*M = (0.641.241.240.64)(1111) := X_2$ $X_2^*MT = (1.580.921.672.061.71.49) = (000110) := Y_1$ $(1111)(000110)$
$X = (000001)$	$X^*M = (0.410.230.230.62)(1001) := X_1$ $X_1^*MT = (0.760.370.850.820.461.03) = (001001) := Y$ $Y^*M = (1.030.640.640.85)(1001) := X_2$ $X_2^*MT = (0.760.370.850.820.461.03) = (001001) := Y_1$ $(1001)(001001)$

$$
\begin{aligned}
H(M_2) &= \{\frac{H}{CS_1}, \frac{L}{CS_2}, \frac{H}{CS_3}, \frac{VH}{CS_4}, \frac{VH}{CS_5}, \frac{M}{CS_6}\},\\
H(M_3) &= \{\frac{H}{CS_1}, \frac{H}{CS_2}, \frac{H}{CS_3}, \frac{VH}{CS_4}, \frac{VH}{CS_5}, \frac{M}{CS_6}\},\\
H(M_4) &= \{\frac{L}{CS_1}, \frac{M}{CS_2}, \frac{M}{CS_3}, \frac{H}{CS_4}, \frac{M}{CS_5}, \frac{VH}{CS_6}\}.
\end{aligned}
$$

The relational impacts are determined by using the procedure discussed in [15] (See Table 2).

4. Results and Discussions

The multivalued neutrosophic soft representation takes in the opinion of three experts into consideration. The twenty strategies taken for study are confined to six strategies based on the final scores of the association rate with the factors.

The six core factors are related with the principles of business management in various dimensions. Each of the core factors is kept in on position. The associational impacts are analyzed and the fixed points are determined. If the core factor CS_1 is kept in on position, the limit point (1110)(100110) is obtained. The factor CS_1 is highly associated with CS_4, CS_5 and M_1, M_2, M_3. By repeating the same mechanism, the associational impacts between the other core factors are determined. This approach of Multi-Strategy Decision-Making with neutrosophic soft sets representations facilitate the decision-making process and it eases the procedure of minimizing the number of strategies. The decision makers evolve many strategies, but implementing all the strategies is not possible, it is quite mandatory to explore the core strategies and to detect its relation with other decision-making principles. To make the process much comprehensive, MSDM approach is constructed in this research work.

5. Conclusion

This paper introduces the approach of Multi-Strategy Decision-Making with two stage process of decision-making. The proposed approach is validated with the decision-making environment of enhancing the customer acquisition strategies. The multivalued neutrosophic soft set representations in the first stage results in confining the number of strategies and the neutrosophic soft relational maps in the second stage is used to determine the relational impacts. This approach can be extended with other kinds of representation. This MSDM approach can be applied to any kind of decision-making environment.

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Neutrosophic Soft Bitopological Spaces

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Abstract: In this paper, we built bitopological space on the concept of neutrosophic soft set, we defined the basic topological concepts of this spaces which are N_3-(bi)*-open set, N_3-(bi)*-closed set, (bi)*-neutrosophic soft interior, (bi)*-neutrosophic soft closure, (bi)*-neutrosophic soft boundary, (bi)*-neutrosophic soft exterior and we introduced their properties. In addition, we investigated the relations of these basic topological concepts with their counterparts in neutrosophic soft topological spaces and we introduced many examples.

Keywords : Neutrosophic soft bitopological spaces, Star bineutrosophic soft open set, star bi neutrosophic soft closed set, fuzzy set.

1. Introduction

The concept of soft set is defined by Molodtsov [1] as follows: Let M be an initial universe set and E be a set of parameters. Let P(M) denotes the set of all the subsets of M. Consider $B \neq \emptyset$, $B \subseteq E$. The collection (β,B) is termed to be the soft set, where β is a mapping by $\beta : B \rightarrow P(M)$, and later this concept has been redefined by Naim Cagman [20]. Smarandache [2] introduced neutrosophic set as a generalization of fuzzy set [3] and intuitionistic fuzzy set [4]. P. K. Maji [5] defended the concept of neutrosophic soft set by combining the concept of neutrosophic set and soft set. This the concept is defined as follows: let M be an initial universe set and E be a set of parameters. Let P(M) denote the set of all the neutrosohpic sets of M. Consider $B \neq \emptyset$, $B \subseteq E$. The collection (β,B) is termed to be the soft neutrosophic set, where β is a mapping by $\beta : B \rightarrow P(M)$. This concept has been modified by [6,7].The concept of neutrosohpic soft topological space was introduced by Bera [8]. Taha et al.[9] redefined the neutrosop-hic soft topological spaces differently from the study [8]. Other theoretical studies on these concepts were presented by a number of researchers, for example, Narmada, Georgiou, Cageman, Al-Nafee, Evanzalin and Salama, (see [10, 11, 22, 13, 14, 15, 16, 17, 18, 19, 20]).

Kelly, [21] introduced the concept of bitopological space. This concept is introduced as an extension of topological space. This concept has been introduced with interest in fuzzy set, soft set and neutrosophic set (see [22, 23, 24, 25]). Therefore, we find it important and necessary to build a bitopological spaces on the concept of neutrosophic soft set. In this paper, bitopological space on the concept of neutrosophic soft set is built, the basic topological concepts of this spaces which are N_3-(bi)*-open set, N_3-(bi)*-closed set, (bi)*-neutrosophic soft interior, (bi)*-neutrosophic soft closure, (bi)*-neutrosophic soft boundary, (bi)*-neutrosophic soft exterior are defined, the relations of these basic topological concepts with their counterparts in neutrosophic soft topological spaces are investigated and many examples on this concepts are given.

2. Preliminary

In this section, we will refer to the basic definitions required in our work.

2.1. Definition [26]

The neutrosohpic set N over M is defined as follows:

$$N=\{< m, H_N(m), G_N(m), J_N(m) >) : m \in M\}.$$

where, the functions H,G,J : M→] − 0,+1[and $-0 \leq H_N(m) + G_N(m) + J_N(m) \leq +3$.

From philosophical point of view the neutrosophic set takes the value from real standard or non-standard subsets of] − 0,+1[. But in real life application in scientific and engineering problems it is difficult to use a neutrosophic set with value from real standard or non-standard subset of] − 0,+1[. Hence we consider the neutrosophic set which takes the value from the subset of [0, 1] .

Firstly, neutrosophic soft set defined by Maji [5] and later this concept and its operations have been redefined by [7]. Our work in this research is based on the definition below:

2.2. Definintion [7]

Let M be an initial universe set and B be a set of parameters. Let P(M) denote the set of all the neutronsohpic sets of M. Then, a neutrosophic soft set β_B over M is a set defined by a set valued function β representing a mapping from B to P(M), where e β is called approximate function of the neutrosophic soft set β_B.

In other words, β_B is a parameterized family of some elements of the set P(M) and therefore it can be written as a set of ordered pairs,

$$\beta_B = \{\big(r, \{< m^{(H_{\beta(r)}(m), G_{\beta(r)}(m), J_{\beta(r)}(m))} > : m \in M\}\big), r \in B\}.$$

Where,

$H_{\beta(r)}(m)$, $G_{\beta(r)}(m)$, $J_{\beta(r)}(m) \in [0,1]$, respectively called the truth-membership, indeterminacymembership,falsity-membership function of β(r). Since supremum of each H, G, J is 1 so the inequality,

$0 \leq H_{\beta(r)}(m) + G_{\beta(r)}(m) + J_{\beta(r)}(m) \leq 3$ is obvious.

From now on, the set of all neutrosohpic sets over M is denoted by $N_3(M)$.

2.3. Definition [9,5]

Let β_B and $\mu_B \in N_3(M)$ such that;

$\beta_B = \{\big(r, \{< m^{(H_{\beta(r)}(m), G_{\beta(r)}(m), J_{\beta(r)}(m))} > : m \in M\}\big), r \in B\}.$

$\mu_B = \{\big(r, \{< m^{(H_{\mu(r)}(m), G_{\mu(r)}(m), J_{\mu(r)}(m))} > : m \in M\}\big), r \in B\}.$Then:

- $\widetilde{M}_B = \{\big(r, \{< m^{(1,1,0)} > : m \in M\}\big), r \in B\}$ [Absolute neutrosohpic soft set].
- $\widetilde{\emptyset}_B = \{\big(r, \{< m^{(0,0,1)} > : m \in M\}\big), r \in B\}$ [Null neutrosohpic soft set].
- $\beta_B \sqsubseteq \mu_B \leftrightarrow \{\big(r, \{< m^{(H_{\beta(r)}(m) \leq H_{\mu(r)}(m), G_{\beta(r)}(m) \leq G_{\mu(r)}(m), J_{\beta(r)}(m) \geq J_{\mu(r)}(m))} > : m \in M\}\big), r \in B\}.$
- $\beta_B \sqcup \mu_B = \{\big(r, \{< m^{(H_{\beta(r)}(m) \vee H_{\mu(r)}(m), G_{\beta(r)}(m) \vee G_{\mu(r)}(m), J_{\beta(r)}(m) \wedge J_{\mu(r)}(m))} >\big) : m \in M\}, r \in B\}.$
- $\beta_B \sqcap \mu_B = \{\big(r, \{< m^{(H_{\beta(r)}(m) \wedge H_{\mu(r)}(m), G_{\beta(r)}(m) \wedge G_{\mu(r)}(m), J_{\beta(r)}(m) \vee J_{\mu(r)}(m))} > : m \in M\}\big), r \in B\}.$

2.4. Definition

Let $\beta_B \in N_3(M)$, The complement of β_B is denoted by $(\beta_B)^C$ and is definded as:

$$(\beta_B)^C = \big(r, \{< m^{(1-H_{\beta(r)}(m), 1-G_{\beta(r)}(m), 1-J_{\beta(r)}(m))} > : m \in M\}\big), r \in B\}.$$

2.5. Definition [9]

Let T ⊆ (N3(M)). The collection T is called a neutrosophic soft topology on M, if the following conditions are true:

1) $\widetilde{M}_B$, $\widetilde{\emptyset}_B$ belong to T.
2) If $\beta_{j_B} \in T$; $j \in J$, then $\sqcup_{j\in J} \beta_{j_B} \in T \ \forall j \in J$.
3) If $\beta_B, \mu_B \in T$, then $\beta_B \sqcap \mu_B \in T$.

Then the triplet (M,B,T) is a neutrosophic soft topological space or (N3-Top for short).
Members of T are called a neutrosohpic soft open sets (N3-T-open for short) and their complements are a neutrosohpic soft open sets (N3-T-closed for short).

The neutrosophic soft interior of $\beta_B \in$ N3(M) ($(\beta_B)^0$ for short) is defended as:

$$(\beta_B)^0 = \sqcup\{(\omega_B): \omega_B \text{ is a N3-T-open set}, \omega_B \sqsubseteq \beta_B\}.$$

The neutrosophic soft closure of $\beta_B \in$ N3(M) ($\overline{(\beta_B)}$ for short) is defended as:

$$\overline{(\beta_B)} = \sqcap\{(\omega_B): \omega_B \text{ is a N3-T-closed set}, \beta_B \sqsubseteq \omega_B\}.$$

2.6. Example

Let M = $\{m_1, m_2, m_3\}$, B = {r} and $\beta_B, \mu_B, \gamma_B \in$ N3(M).

Such that

$$\beta_B = \{(r, \{< m_1^{(1,1,0)} >, < m_2^{(0,0,1)} >, < m_3^{(0,0,1)} >\})\},$$

$$\mu_B = \{(r, \{< m_1^{(1,1,0)} >, < m_2^{(0,0,1)} >, < m_3^{(1,1,0)} >\})\},$$

$$\gamma_B = \{(r, \{< m_1^{(0,0,1)} >, < m_2^{(0,0,1)} >, < m_3^{(1,1,0)} >\})\}.$$

Then, T2 = $\{\widetilde{\emptyset}_B, \widetilde{M}_B, \beta_B, \mu_B\}$ is a neutrosohpic soft topology on M.

2.7. Example

Let M = $\{m_1, m_2, m_3\}$, B = {r} and $\beta_B, \mu_B, \gamma_B, \delta_B \in$ N3(M). Such that

$$\beta_B = \{(r, \{< m_1^{(1,1,0)} >, < m_2^{(0,0,1)} >, < m_3^{(0,0,1)} >\})\}.$$

$$\mu_B = \{(r, \{< m_1^{(1,1,0)} >, < m_2^{(0,0,1)} >, < m_3^{(1,1,0)} >\})\}.$$

$$\gamma_B = \{(r, \{< m_1^{(1,1,0)} >, < m_2^{(1,1,0)} >, < m_3^{(0,0,1)} >\})\}.$$

$$\delta_B = \{(r, \{< m_1^{(0,0,1)} >, < m_2^{(1,1,0)} >, < m_3^{(1,1,0)} >\})\}.$$

Then, T2 = $\{\widetilde{\emptyset}_B, \widetilde{M}_B, \beta_B, \mu_B, \gamma_B\}$ is a neutrosohpic soft topology on M.

3. Neutrosophic soft bitopological space

In this section, we defined the neutrosophic soft bitopological space or (N3-Bi-Top for short) on the concept of neutrosophic soft set and the basic topological concepts of this spaces which are N3-biopen and N3-biclosed.

3.1. Definition

Let (M,B,T1) and (M,B,T2) be two N3-Top spaces defined on M. Then (M,B,T1,T2) is called a neutrosophic soft bitopological space or (N3-Bi-Top for short).

3.2. Example

Let M = $\{m_1,m_2\}$, B = {r} and $\beta_B, \mu_B \in$ N3(M) such that

β_B = {(r, {< m1$^{(0.6, 0.2, 0.5)}$ >, < m2$^{(0.5, 0.4, 0.9)}$>})}, μ_B = {(r, {< m1$^{(0.6, 0.2, 0.4)}$ >, < m2$^{(0.6, 0.4, 0.7)}$ >})}.

Then, T_1 = {$\widetilde{\emptyset}_B$, $\widetilde{M}_B$, β_B} is an N_3-Top on M and T_2 = {$\widetilde{\emptyset}_B$, $\widetilde{M}_B$, μ_B} is an N_3-Top on M.

Therefore, (M,B,T_1,T_2) is an N_3-Bi-Top space.

3.3. Definition

Let (M,B,T_1,T_2) be an N_3-Bi-Top space. The members of (M,B,T_1,T_2) are called bineutrosohpic soft open sets (N_3-biopen for short) and their complements are bineutrosohpic soft closed sets (N_3-biclosed for short).

3.4. Remark

a) Every neutrosophic soft open (closed) set in (M,B,T_1) or (M,B,T_2) is an N_3-biopen (N_3-biclosed) set.

b) Every N_3-Bi-Top space (M,B,T_1,T_2) induces two N_3-Top spaces as (M,B,T_1) and (M,B,T_2).

c) If (M,B,T_1) is an N_3-Top space then (M,B,T_1,T_1) is an N_3-Bi-Top space.

3.5. Theorem

If (M,B,T_1,T_2) is an N_3-Bi-Top space, then (M,B,$T_1 \cap T_2$) is an N_3-Top space.

Proof

Let (M,B,T_1,T_2) be an N_3-Bi-Top space.

(1) Clearly that $\widetilde{\emptyset}_B$, $\widetilde{M}_B \in (T_1 \cap T_2)$.

(2) Let $\beta_B, \mu_B \in (T_1 \cap T_2)$, then $\beta_B, \mu_B \in T_1$ and $\beta_B, \mu_B \in T_2$. This implies that, $\beta_B \sqcap \mu_B \in T_1$ and $\beta_B \sqcap \mu_B \in T_1$. Therefore, $\beta_B \sqcap \mu_B \in (T_1 \cap T_2)$.

(3) Let $\beta_{j_B} \in (T_1 \cap T_2)$; $j \in J$. Then $\beta_{j_B} \in T_1$ and $\beta_{j_B} \in T_2$; $j \in J$. Therefore $\sqcup_{j\in J} \beta_{j_B} \in T_1$ and $\sqcup_{j\in J} \beta_{j_B} \in T_2$ $\forall j \in J$. Thus, we have $\sqcup_{j\in J} \beta_{j_B} \in (T_1 \cap T_2)$.

Hence, (M,B,$T_1 \cap T_2$) is an N_3-Top space.

3.6. Remark

If we take the operation of union instead of the operation of intersection, then the above theorem is not generally correct.

3.7. Example

Let M = {m_1,m_2}, B = {r} and $\beta_B, \mu_B \in N_3(M)$ such that

β_B = {(r, {< m1$^{(0.3, 0.5, 0.7)}$ >, < m2$^{(0.2, 0.4, 0.6)}$ >})}, μ_B = {(r, {< m1$^{(0.5, 0.7, 0.8)}$ >, < m2$^{(0.3, 0.6, 0.8)}$ >})}.

Then, T_1 = {$\widetilde{\emptyset}_B$, $\widetilde{M}_B$, μ_B} is an N_3-Top on M and T_2 = {$\widetilde{\emptyset}_B$, $\widetilde{M}_B$, β_B} is an N_3-Top on M. Thus, (M,B,T_1,T_2) is an N_3-Bi-Top space. But, (M,B,$T_1 \cup T_2$) is not an N_3-Top space. Because, $\beta_B \sqcup \mu_B$ does not belong to ($T_1 \cup T_2$).

4. N_3-(bi)*-open set in neutrosophic soft bitopological space

In this section, N_3-(bi)*-open set, N_3-(bi)*-closed set, (bi)*-neutrosophic soft interior, (bi)*-neutrosophic soft closure, (bi)*-neutrosophic soft boundary, (bi)*-neutrosophic soft exterior are defined based on the idea of δ-open set which was defined in [27].

4.1. Definition

A subset $\beta_B \in N_3(M)$ of an N_3-Bi-Top space (M,B,T_1,T_2) is called star bineutrosophic soft open (N_3-(bi)*-open, for short) in (M,B,T_1,T_2) if and only if $\beta_B \sqsubseteq {\overline{(\beta_B)^{oT}}^{(T1)}}^{oT2}$ and their complement is an N_3-(bi)*-closed set. The set of all N_3-(bi)*-open [N_3-(bi)*-closed] sets in (M,B,T_1,T_2) is denoted by $M^{(Bi)*-N}$ [$M^{(Bi)*-NSC}$] respectively.

4.2. Example

Let $M = \{m_1,m_2,m_3\}$, $B = \{r\}$ and $\beta_B, \mu_B \in N_3(M)$ such that

$$\beta_B = \{(r, \{< m_1^{(1,1,0)} >, < m_2^{(0,0,1)} >, < m_3^{(0,0,1)} >\})\},$$

$$\mu_B = \{(r, \{< m_1^{(1,1,0)} >, < m_2^{(1,1,0)} >, < m_3^{(0,0,1)} >\})\}.$$

$T_1 = \{\widetilde{\emptyset}_B, \widetilde{M}_B\}$ is an N_3-Top on M and $T_2 = \{\widetilde{\emptyset}_B, \widetilde{M}_B, \beta_B, \mu_B\}$ is an N_3-Top on M. Thus, (M, B,T_1,T_2) is an N_3-Bi-Top space.

Note that:

$\beta_B = \{(r, \{< m_1^{(1,1,0)} >, < m_2^{(0,0,1)} >, < m_3^{(0,0,1)} >\})\} \sqsubseteq \overline{(\beta_B)^{oT2}}^{(T1)^{oT2}} = \{(r, \{< m_1^{(1,1,0)} >, < m_2^{(1,1,0)} >, < m_3^{(1,1,0)} >\})\}$.

$\therefore\ \beta_B \sqsubseteq \overline{(\beta_B)^{oT}}^{(T1)^{oT2}}$.

$\mu_B = \{(r, \{< m_1^{(1,1,0)} >, < m_2^{(1,1,0)} >, < m_3^{(0,0,1)} >\})\} \sqsubseteq \overline{(\mu_B)^{oT2}}^{(T1)^{oT2}} = \{(r, \{< m_1^{(1,1,0)} >, < m_2^{(1,1,0)} >, < m_3^{(1,1,0)} >\})\}$.

$\therefore\ \mu_B \sqsubseteq \overline{(\mu_B)^{oT2}}^{(T1)^{oT2}}$.

$\gamma_B = \{(r, \{< m_1^{(1,1,0)} >, < m_2^{(0,0,1)} >, < m_3^{(1,1,0)} >\})\}$.

$\{(r, \{< m_1^{(1,1,0)} >, < m_2^{(0,0,1)} >, < m_3^{(1,1,0)} >\})\} \sqsubseteq \overline{(\gamma_B)^{oT2}}^{(T1)^{oT2}} = \{(r, \{< m_1^{(1,1,0)} >, < m_2^{(1,1,0)} >, < m_3^{(1,1,0)} >\})\}$.

$\therefore\ \gamma_B \sqsubseteq \overline{(\gamma_B)^{oT2}}^{(T1)^{oT2}}$.

$\delta_B = \{(r, \{< m_1^{(0,0,1)} >, < m_2^{(1,1,0)} >, < m_3^{(1,1,0)} >\})\}$.

$\{(r, \{< m_1^{(0,0,1)} >, < m_2^{(1,1,0)} >, < m_3^{(1,1,0)} >\})\} \not\sqsubseteq \overline{(\delta_B)^{oT2}}^{(T1)^{oT2}} = \{(r, \{< m_1^{(0,0,1)} >, < m_2^{(0,0,1)} >, < m_3^{(0,0,1)} >\})\}$.

$\therefore\ \delta_B \not\sqsubseteq \overline{(\delta_B)^{oT2}}^{(T1)^{oT2}}$.

$\varepsilon_B = \{(r, \{< m_1^{(0,0,1)} >, < m_2^{(1,1,0)} >, < m_3^{(1,1,0)} >\})\}$.

$\{(r, \{< m_1^{(0,0,1)} >, < m_2^{(1,1,0)} >, < m_3^{(1,0,0)} >\})\} \not\sqsubseteq \overline{(\varepsilon_B)^{oT2}}^{(T1)^{oT2}} = \{(r, \{< m_1^{(0,0,1)} >, < m_2^{(0,0,1)} >, < m_3^{(0,0,1)} >\})\}$.

$\therefore\ \varepsilon_B \not\sqsubseteq \overline{(\varepsilon_B)^{oT2}}^{(T1)^{oT2}}$.

$\vartheta_B = \{(r, \{< m_1^{(0,0,1)} >, < m_2^{(0,0,1)} >, < m_3^{(1,1,0)} >\})\}$.

$\{(r, \{< m_1^{(0,0,1)} >, < m_2^{(0,0,1)} >, < m_3^{(1,1,0)} >\})\} \not\sqsubseteq \overline{(\vartheta_B)^{oT}}^{(T1)^{oT2}} = \{(r, \{< m_1^{(0,0,1)} >, < m_2^{(0,0,1)} >, < m_3^{(0,0,1)} >\})\}$.

$\therefore\ \vartheta_B \not\sqsubseteq \overline{(\vartheta_B)^{oT}}^{(T1)^{oT2}}$.

In general in any N_3-Bi-Top space, $\widetilde{\emptyset}_B$, $\widetilde{M}_B$ are clearly N_3-(bi)*-open sets.

Hence:

$$M^{(Bi)*-NSO} = \{\widetilde{\emptyset}_B, \widetilde{M}_B, \beta_B, \mu_B, \gamma_B\}.$$

$$M^{(Bi)*-NSC} = \{\ \{(r, \{< m_1^{(0,0,1)} >, < m_2^{(1,1,0)} >, < m_3^{(1,1,0)} >\})\},$$
$$\{(r, \{< m_1^{(0,0,1)} >, < m_2^{(0,0,1)} >, < m_3^{(1,1,0)} >\})\},$$
$$\{(r, \{< m_1^{(0,0,1)} >, < m_2^{(1,1,0)} >, < m_3^{(0,0,1)} >\})\},$$
$$\widetilde{\emptyset}_B,$$
$$\widetilde{M}_B\ \}.$$

4.3. Remark

Let β_B and μ_B be an N_3-(bi)*-open sets, then $\beta_B \sqcap \mu_B$ is not necessary an N_3-(bi)*-open set.

4.4. Example

Let $M = \{m_1, m_2, m_3, m_4, m_5\}$, $B = \{r\}$ and $\beta_B, \mu_B, \gamma_B, \varepsilon_B, \vartheta_B, \alpha_B \in N_3(M)$.

Such that

$$\beta_B = \{(r, \{< m_1^{(1,1,0)} >, < m_2^{(0,0,1)} >, < m_3^{(0,0,1)} >, < m_4^{(0,0,1)} >\})\}.$$

$$\mu_B = \{(r, \{< m_1^{(0,0,1)} >, < m_2^{(0,0,1)} >, < m_3^{(0,0,1)} >, < m_4^{(1,1,0)} >\})\}.$$

$$\gamma_B = \{(r, \{< m_1^{(1,1,0)} >, < m_2^{(0,0,1)} >, < m_3^{(0,0,1)} >, < m_4^{(1,1,0)} >\})\}.$$

$$\varepsilon_B = \{(r, \{< m_1^{(0,0,1)} >, < m_2^{(1,1,0)} >, < m_3^{(1,1,0)} >, < m_4^{(0,0,1)} >\})\}.$$

$$\vartheta_B = \{(r, \{< m_1^{(1,1,0)} >, < m_2^{(1,1,0)} >, < m_3^{(1,1,0)} >, < m_4^{(0,0,1)} >\})\}.$$

$$\alpha_B = \{(r, \{< m_1^{(0,0,1)} >, < m_2^{(1,1,0)} >, < m_3^{(1,1,0)} >, < m_4^{(1,1,0)} >\})\}.$$

$T_1 = \{\widetilde{\emptyset}_B, \widetilde{M}_B, \beta_B, \mu_B, \gamma_B\}$ is an N_3-Top on M and $T_2 = \{\widetilde{\emptyset}_B, \widetilde{M}_B, \beta_B, \mu_B, \gamma_B, \varepsilon_B, \vartheta_B, \alpha_B\}$ is an N_3-Top on M. Thus, (M,B,T_1,T_2) is an N_3-Bi-Top space. Then

ε_B and $\{(r, \{< m_1^{(1,1,0)} >, < m_2^{(0,0,1)} >, < m_3^{(1,1,0)} >, < m_4^{(0,0,1)} >\})\}$ are an N_3-(bi)*-open sets, but the intersection of them $\{(r, \{< m_1^{(0,0,1)} >, < m_2^{(0,0,1)} >, < m_3^{(1,1,0)} >, < m_4^{(0,0,1)} >\})\}$ is not an N_3-(bi)*-open set.

4.5. Theorem

Let (M,B,T_1,T_2) be an N_3-Bi-Top space, then every neutrosophic soft open set in (M,B,T_2) is an N_3-(bi)*-open set in (M,B,T_1,T_2).

Proof

Let β_B be a neutrosophic soft open set in (M,B,T_2). Then $(\beta_B)^{oT2} = \beta_B$. Since $\beta_B \sqsubseteq \overline{(\beta_B)}^{T1}$,

$\beta_B \sqsubseteq \overline{(\beta_B)^{oT2}}^{T1}$, $(\beta_B)^{oT} \sqsubseteq \overline{(\beta_B)^{oT}}^{T1\,oT2}$. Thererfor $\beta_B \sqsubseteq \overline{(\beta_B)^{oT}}^{T1\,oT2}$ and thus β_B is an N_3-(bi)*-open set in (M,B,T_1,T_2).

4.6. Remark

The converse of above remark is not true in general. In Example 3.4 note that, $\{(r, \{< m_1^{(1,1,0)} >, < m_2^{(0,0,1)} >, < m_3^{(1,1,0)} >, < m_4^{(0,0,1)} >\})\}$ is an N_3-(bi)*-open set in (M,B,T_1,T_2), but not a neutrosophic soft open set in (M,B,T_2).

4.7. Definition

If (M,B,T_1,T_2) is an N_3-(Bi)*-Top space and $\beta_B \in N_3(M)$, then the largest N_3-(bi)*-open set contained in β_B is called (bi)*-neutrosophic soft interior of β_B, ($(\beta_B)^{0(bi)*}$ for short). i.e.

$$(\beta_B)^{0(bi)*} = \sqcup\{(\omega_B) : \omega_B \text{ is a } N_3\text{-(bi)*-open set}, \omega_B \sqsubseteq \beta_B\}.$$

4.8. Theorem

Let (M,B,T_1,T_2) be an N_3-(Bi)*-Top space and $\beta_B \in N_3(M)$. Then β_B is an N_3-(bi)*-open set if and only if $\beta_B = (\beta_B)^{0(bi)*}$.

Proof

Let β_B be an N_3-(bi)*-open set. Then β_B is itself an N_3-(bi)*-open set which contains β_B. Therefore, β_B is the largest N_3-(bi)*-open set contained in β_B and $\beta_B = (\beta_B)^{0(bi)*}$. Conversely, suppose that $\beta_B = (\beta_B)^{0(bi)*}$,then β_B is the largest N_3-(bi)*-open set contained in β_B. Thus, β_B is an N_3-(bi)*-open set.

4.9. Theorem

Let $\beta_B, \mu_B \in N_3(M)$.

a) $(\beta_B)^{0(bi)*} \sqsubseteq \beta_B$.
b) $((\beta_B)^{0(bi)*})^{0(bi)*} = (\beta_B)^{0(bi)*}$.
c) $(\beta_B)^{0(bi)*} \sqsubseteq (\mu_B)^{0(bi)*}$; whenever $\beta_B \sqsubseteq \mu_B$.
d) $(\beta_B \sqcap \mu_B)^{0(bi)*} = (\beta_B)^{0(bi)*} \sqcap (\mu_B)^{0(bi)*}$.
e) $(\beta_B \sqcup \mu_B)^{0(bi)*} \sqsupseteq (\beta_B)^{0(bi)*} \sqcup (\mu_B)^{0(bi)*}$.
f) $(\widetilde{M}_B)^{0(bi)*} = \widetilde{M}_B$.
g) $(\widetilde{\emptyset}_B)^{0(bi)*} = \widetilde{\emptyset}_B$.

Proof

(a), (f), (g), (c) (Straightforward).

(b) Let $\mu_B = (\beta_B)^{0(bi)*}$. Then $\mu_B = (\mu_B)^{0(bi)*}$(from Theorem 4.8). Thus$((\beta_B)^{0(bi)*})^{0(bi)*} = (\beta_B)^{0(bi)*}$.

(d) Since, $(\beta_B \sqcap \mu_B)^{0(bi)*} \sqsubseteq (\beta_B)^{0(bi)*}$ and $(\beta_B \sqcap \mu_B)^{0(bi)*} \sqsubseteq (\mu_B)^{0(bi)*}$. Then, $(\beta_B \sqcap \mu_B)^{0(bi)*} \sqsubseteq (\beta_B)^{0(bi)*} \sqcap (\mu_B)^{0(bi)*}$...(1.

Since, $(\beta_B)^{0(bi)*} \sqsubseteq \beta_B$ and $(\mu_B)^{0(bi)*} \sqsubseteq \mu_B$, then $(\beta_B)^{0(bi)*} \sqcap (\mu_B)^{0(bi)*} \sqsubseteq \beta_B \sqcap \mu_B$. But $(\beta_B)^{0(bi)*} \sqcap (\mu_B)^{0(bi)*}$ is a N3-(bi)*-open subset of $\beta_B \sqcap \mu_B$. Therefore, from the detention, we have that $(\beta_B \sqcap \mu_B)^{0(bi)*} \sqsupseteq (\beta_B)^{0(bi)*} \sqcap (\mu_B)^{0(bi)*}$…(2.

Hence, $(\beta_B \sqcap \mu_B)^{0(bi)*} = (\beta_B)^{0(bi)*} \sqcap (\mu_B)^{0(bi)*}$.

(e) Since, $\beta_B \sqsubseteq (\beta_B \sqcup \mu_B)$ and $\mu_B \sqsubseteq (\beta_B \sqcup \mu_B)$, therefore $(\beta_B \sqcup \mu_B)^{0(bi)*} \sqsupseteq (\beta_B)^{0(bi)*}$and $(\beta_B \sqcup \mu_B)^{0(bi)*} \sqsupseteq (\mu_B)^{0(bi)*}$. So, $(\beta_B \sqcup \mu_B)^{0(bi)*} \sqsupseteq (\beta_B)^{0(bi)*} \sqcup (\mu_B)^{0(bi)*}$.

4.10. Example

Let us consider $\beta_B, \mu_B, \gamma_B \in N_3(M)$ in Example 2.6. Such that, $T_2 = \{\widetilde{\emptyset}_B, \widetilde{M}_B, \beta_B, \mu_B\}$ is an N3-Top on M and $T_1 = \{\widetilde{\emptyset}_B, \widetilde{M}_B, \beta_B\}$ is an N3-Top on M. Thus, (M,B,T_1,T_2) is an N3-Bi-Top space.

Note that: 1) $(\beta_B \sqcup \gamma_B)^{0(bi)*} \not\sqsubseteq (\beta_B)^{0(bi)*} \sqcup (\gamma_B)^{0(bi)*}$. 2) $\gamma_B \not\sqsubseteq (\gamma_B)^{0(bi)*}$.

4.11. Definition

If (M,B,T_1,T_2) is an N3-(Bi)*-Top space and $\beta_B \in N_3(M)$, then the intersection of all N3-(bi)*-closed sets containing β_B is called a (bi)*-neutrosophic soft closure of β_B, ($\overline{(\beta_B)}^{(bi)*}$ for short). i.e.

$$\overline{(\beta_B)}^{(bi)*} = \sqcap\{(\omega_B): \omega_B \text{ is an } N_3\text{-(bi)*-closed set}, \beta_B \sqsubseteq \omega_B\}.$$

4.12. Theorem

Let $\beta_B, \mu_B \in N_3(M)$.

a) $\beta_B \sqsubseteq \overline{(\beta_B)}^{(bi)*}$.
b) $\overline{\left(\overline{(\beta_B)}^{(bi)*}\right)}^{(bi)*} = \overline{(\beta_B)}^{(bi)*}$.
c) $\overline{(\beta_B)}^{(bi)*} \sqsubseteq \overline{(\mu_B)}^{(bi)*}$; whenever $\beta_B \sqsubseteq \mu_B$.
d) $\overline{(\beta_B \sqcap \mu_B)}^{(bi)*} \sqsubseteq \overline{(\beta_B)}^{(bi)*} \sqcap \overline{(\mu_B)}^{(bi)*}$.
e) $\overline{(\beta_B \sqcup \mu_B)}^{(bi)*} = \overline{(\beta_B)}^{(bi)*} \sqcup \overline{(\mu_B)}^{(bi)*}$.

f) $\overline{(\widetilde{M}_B)}^{(bi)*} = \widetilde{M}_B$.

g) $\overline{(\widetilde{\emptyset}_B)}^{(bi)*} = \widetilde{\emptyset}_B$.

Proof Straightforward.

4.13. Remark

In above theorem, it is not necessary the converse of (a) and (d) be true.

4.14. Example

Let us take, $\beta_B, \mu_B, \gamma_B, \delta_B \in N_3(M)$ in Example 2.7.

$T_2 = \{\widetilde{\emptyset}_B, \widetilde{M}_B, \beta_B, \mu_B, \gamma_B\}$ is an N_3-Top on M and $T_1 = \{\widetilde{\emptyset}_B, \widetilde{M}_B\}$ is an N_3-Top on M. Thus, (M,B,T_1,T_2) is an N_3-Bi-Top space.

Note that:

1) $(\beta_B \sqcup \gamma_B)^{0(bi)*} \not\sqsubseteq (\beta_B)^{0(bi)*} \sqcup (\gamma_B)^{0(bi)*}$. 2) $\gamma_B \not\sqsubseteq (\gamma_B)^{0(bi)*}$.

4.15. Theorem

Let (M,B,T_1,T_2) be an N_3-(Bi)*-Top space and $\beta_B \in N_3(M)$.

a) $((\beta_B)^C)^{0(bi)*} = \left(\overline{(\beta_B)}^{(bi)*}\right)^C$.

b) $\overline{(\beta_B)^C}^{(bi)*} = \left((\beta_B)^{0(bi)*}\right)^C$.

Proof

(a) We know that, $\overline{(\beta_B)}^{(bi)*} = \sqcap\{\omega_B : (\omega_B)^C$ is a N_3-(bi)*-open set, $\beta_B \sqsubseteq \omega_B\}$. So, we have that, $\left(\overline{(\beta_B)}^{(bi)*}\right)^C = \sqcup\{(\omega_B)^C : (\omega_B)^C$ is an N_3-(bi)*- open set, $(\omega_B)^C \sqsubseteq (\beta_B)^C\} = ((\beta_B)^C)^{0(bi)*}$. Thus, $\left(\left(\beta_B\right)^C\right)^{0(bi)*} = \left(\overline{(\beta_B)}^{(bi)*}\right)^C$.

(b) If we take, $(\beta_B)^C$ instead of β_B in (a), we get that,

$\left(\overline{((\beta_B)^c)}^{(bi)*}\right)^C = (((\beta_B)^C)^C)^{0(bi)*} = ((\beta_B))^{0(bi)*}$. So, $\overline{(\beta_B)^C}^{(bi)*} = \left((\beta_B)^{0(bi)*}\right)^C$.

4.16. Theorem

If (M,T_1,T_2) is an N_3-(Bi)*-Top space and $\beta_B \in N_3(M)$, then β_B is an N_3-(bi)*-closed set if and only if $\beta_B = \overline{(\beta_B)}^{(bi)*}$.

Proof

Let β_B be an N_3-(bi)*-closed set, then β_B is itself an N_3-(bi)*-closed set which contains β_B. Therefore, β_B is the intersection of all N_3-(bi)*-closed sets containing β_B and $\beta_B = \overline{(\beta_B)}^{(bi)*}$.

Conversely, suppose that $\beta_B = \overline{(\beta_B)}^{(bi)*}$, then β_B is the intersection of all N_3-(bi)*-closed sets containing β_B. Thus, β_B is an N_3-(bi)*-closed set.

4.17. Definition

If (M,T_1,T_2) is an N_3-(Bi)*-Top space and $\beta_B \in N_3(M)$, then the (bi)*-neutrosophic soft exterior of β_B, (bi)*-ext(β_B) for short) is defined as, (bi)*-ext(β_B) = $((\beta_B)^C)^{0(bi)*}$.

4.18. Definition

If (M,B,T_1,T_2) is an N_3-(Bi)*-Top space and $\beta_B \in N_3(M)$, then the (bi)*-neutrosophic soft boundary of β_B, ((bi)*-br(β_B) for short) is defined as, (bi)*-br(β_B) = $\overline{(\beta_B)^C}^{(bi)*} \sqcap \overline{(\beta_B)}^{(bi)*}$.

4.19. Theorem

Assume that (M,B,T_1,T_2) is an N_3-(Bi)*-Top space and $\beta_B \in N_3(M)$.

- (bi)*-br $((\beta_B)^C)$ = (bi)*-ext$(\beta_B) \sqcup (\beta_B)^{0(bi)*}$.
- $\overline{(\beta_B)}^{(bi)*}$ = (bi)*-br $(\beta_B) \sqcup (\beta_B)^{0(bi)*}$.
- (bi)*-br $(\beta_B) \sqcap (\beta_B)^{0(bi)*} = \widetilde{\emptyset}_B$.
- (bi)*-br $(\beta_B)^{0(bi)*} \sqsubseteq$ (bi)*-br (β_B).

Proof Straightforward.

4.20. Theorem

Assume that (M,B,T_1,T_2) is an N_3-(Bi)*-Top space and $\beta_B \in N_3(M)$.

- $\beta_B \in M^{(Bi)*-NSO}$ if and only if (bi)*-br $(\beta_B) \sqcap \beta_B = \widetilde{\emptyset}_B$.
- $\beta_B \in M^{(Bi)*-NSC}$ if and only if (bi)*-br $(\beta_B) \sqsubseteq \beta_B$.

Proof Straightforward.

Conclusion

In this research, bitopological space on the concept of neutrosophic soft set is built , the basic topological concepts of this spaces which are N_3-(bi)*-open set, N_3-(bi)*-closed set, (bi)*-neutrosophic soft interior, (bi)*-neutrosophic soft closure, (bi)*-neutrosophic soft boundary, (bi)*-neutrosophic soft exterior are defined and many examples on this concepts are given.

This paper is just a beginning of a new structure and we have studied a few ideas only , it will be necessary to carry out more theoretical research to establish a general framework for the practical application .

we hope that the findings in this paper will help researchers enhance and promote the further study on neutrosophic soft bitopological space .

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Introduction to Neutrosophic Reliability Theory

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Abstract

Reliability is one of the most important indicators of the quality of any system or product, ranging from the simplest machines as a product in any factory to the most complex system, such as phone or aircraft or missile engines, etc. The accuracy of these products indicates their high reliability, and therefore the customer or business owner will have confidence in products and will request more quantity. To reach the highest level of accuracy for the reliability of any system, the corresponding data must be very accurate. For this purpose, we proposed to add accuracy to reliability by adding data that contains more pieces of information about a specific product or problem. We introduced a new logic (Neutrosophic logic) of data instead of classical logic which gives us more accuracy of data that contain indeterminacy such as extremist, vague, and unclear data. We defined the neutrosophic reliability according to the modern neutrosophic logic by constructing a neutrosophic reliability function. We have used the type of series as an application of neutrosophic reliability and introduced some examples. Neutrosophic reliability theory can be applied in computer science and decision support systems.

Keywords: Neutrosophic probability, Neutrosophic Set, Neutrosophic reliability, Neutrosophic random variable, classical reliability, neutrosophic series reliability.

1.Introduction

In a world full of indeterminacy and therefore the traditional set with its boundaries of truth and false has not infused itself with the ability to reflect reality. For this reason, neutrosophic found its place in contemporary research as an alternative representation of the real world. Established by Florentin Smarandache [1, 2, 10, 11, 12], neutrosophy was presented as the study of "the origin, nature, and scope of neutralities, as well as their interactions with different identical spectra". Salama et al. introduced the neutrosophic crisp set theory and many applications in computer science and information system in [3-6] and [18-26]. The theory of reliability is considered as a collection of measures, mathematical systems, improving methods used to obtain solutions to some problems of prediction, estimation, optimal survival probabilities, expected life, or the life distributions of elements of the system. In (2020)

Smarandache et al. introduced an approach for the reliability of data contained in a single-valued neutrosophic number and its application [32,33].

Reliability theory also considers some of the problems related to calculating the actual probability of providing some systems at (a certain time or at an optional time, or through a portion of the time during) which some systems are operating efficiently and accurately. that is, reliability of the system is a measure of a system's ability to operate successfully under conditions and for a specific period with the recent development in production systems, products have become more complex in their manufacture (a collection of components that work as an integrated system), which increases the probability that they will collapse if one component fails in them. One of the most important things in maintaining the system's reliability is the use of highly reliable components. In the classic procedure, we have encountered many problems in determining the reliability of any electrical system, device, product, etc. For example, some data are lost or the value of one of the vehicles is unclear or the basic component on which the system works are not identified (indeterminacy), or one of the paths of an electrical or electronic loop may be unclear or not specified, however, we need the reliability to be more exact and clear. In this case, we use the modern procedure to redefine the reliability according to neutrosophic logic introduced by Smarandache in 1995 [9], as neutrosophic logic allows dealing with all previous cases and others with high flexibility. Neutrosophic logic is considered as a generalization for the fuzzy logic and intuitionistic fuzzy logic [9, 10], and the fundamental concepts of neutrosophic set and Neutrosophic set introduced by Smarandache in [8, 9, 10]. Smarandache extended the fuzzy set to the neutrosophic set [10, 11, 12], introducing the neutrosophic components T, I, F, which represent the membership, indeterminacy, and non-membership values respectively, where] -0, 1+ [is the non-standard unit interval. In this paper, we presented the concept of reliability according to neutrosophic logic and called it neutrosophic reliability. Neutrosophic reliability is a new tool and one of the most important indicators in measuring the quality and reliability of systems in all fields.

2. Fundament als

Neutrosophy theory is applied in different aspects of life to solve problems related to indeterminacy, such as mathematical, engineering, geography, medicine, psychology [9].

Definition 1 [10]

Neutrosophy is a generalization of dialectics (that depended on <A> and <anti- A> only), however in neutrosophic theory considered every entity <A> tends to be neutralized and balanced by < anti-A> and < non-A> entities - as a state of equilibrium. In a classical way <A>, <neut- A>, < anti-A> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, <A>, <neut- A>, <anti- A> and <non- A> may have common parts two by two, or even all three of them as well.

Definition 2 [8,9]

Let U be a universe of discourse; then the neutrosophic set A is an object having the form

$$A = \{< x: T_A(x), I_A(x), F_A(x) >, x \in U\}. \quad (1)$$

Where the functions T, I, F: U $\rightarrow$ $]^-0,1^+[$ define respectively the degree of membership, the degree of indeterminacy, and the degree of non-membership of the element x $\in$ U to the set A with the condition: $^-0 \leq T_A(x)+I_A(x)+F_A(x) \leq 3^+$.

Definition 3 [7,8]

Let X be a space of points (objects) with generic elements in X denoted by x. An interval neutrosophic set A in X is characterized by truth-membership function, indeterminacy-membership function, and falsity-membership function. For each point x in X, we have that T_A (x), I_A (x), F_A (x) subset [0 ,1].

Defination 4 [12]

Neutrosophic random variable is a random variable with some indeterminate if suppose Ω is a sample space of neutrosophic random experiment such as X is function define on Ω, such that the domain or codomain or the relationship between them may contain some indeterminacy,

$$X: \Omega \rightarrow \mathcal{R} \cup I \qquad (2)$$

Definition 5 [12,14]

Neutrosophic probability (or likelihood) is a particular case of the neutrosophic measure. It is an estimation of an event (different from indeterminacy) to occur, together with an estimation that some indeterminacy may occur, and the estimation that the event does not occur.

NP(E) = (chance that event E occurs, indeterminate chance that E occurs or not, a chance that event does not occur)

NP(E) =(ch(E), ch(neut A) ,ch(anti A)) = (T ,I ,F) (3)

3. Classical Reliability

Reliability function

Let $\mathcal{T}$ denote the lifetime of a system, the reliability of that system at the point in time t, that

$R(t) = P(\mathcal{T} > t)$, it is called the reliability at the time t, and we can define it as the probability that the time at which the system could fail is greater than t.

We can find the reliability by cumulative distribution function for a random variable $\mathcal{T}$ as:

$$R(t) = \int_t^\infty f(t)\ dt = 1 - P(\mathcal{T} > t) = 1 - F(t). \text{ [15,30]} \qquad (4)$$

Example (1) Suppose that the company offers a two-year guarantee of its product. So the probability of this product operates as expected during the guarantee should be large. As a measure of reliability, probability can be used to indicate the life of that product (not failed). Let $\mathcal{T}$ is denoted the time of life product will not fail during this period, for example, if R= P($\mathcal{T}$ > 720 days), that is: This standard is a useful indicator for measuring how this product does its intended function.

If R= 0.999, this means that one in a thousand units can fail for two years.

Example (2) What is the probability of mission success, if seven helicopters are sent on a mission and five must succeed for a mission to be successful? Bearing in mind that the probability of a certain type of helicopter surviving a mission is 0.9 [15].

Solution:-

If the number of successes is 5 or more, this indicates to the mission will be a success. Hence, the probability of mission success or mission reliability is:

$$R_s(t) = \sum_{i=k}^{m} \binom{m}{i} R^i\, R^{m-i} = \sum_{i=5}^{7} \binom{7}{i} R^i\, R^{7-i}$$

$$= \binom{7}{5} 0.91^{\,5}(0.09)^2 + \binom{7}{6} 0.91^{\,6}(0.09)^1 + \binom{7}{7} 0.91^{\,7}(0.09)^0 = 0.9806.$$

4. Modern Reliability (Neutrosophic Reliability)

Let $\mathcal{T}$ be a Neutrosophic random variable representing the time of the system failure, and t be the interval of the operation of this system. We defined the reliability according to Neutrosophic probability as follow:

$NR(t) = \int_t^\infty f_N(t)dt = NP(\mathcal{T} > t)$, where t can be an interval or set or neutrosophic number maybe contain some indeterminacy, and $f_N(t)$ is the neutrosophic probability distribution.

Then, $NR(t)$ is the neutrosophic reliability with respect to a neutrosophic probability distribution.

Example (3) Neutrosophic Weibull Distribution [7]

Alhasan, Florentin (2019), define Weibull distribution according to neutrosophic logic as:

$$f_N(X) = \frac{\beta_N}{\alpha_N^{\beta_N}} X^{\beta_N - 1} e^{-(X/\alpha_N)^{\beta_N}}\ X > 0$$

And the reliability of neutrosophic Weibull as:

$$NR(t) = \int_t^\infty f_N(t)dt = NP(\mathcal{T} > t) = e^{-(X/\alpha_N)^{\beta_N}}$$

Such that, the parameters of Weibull distribution as a number neutrosophic. that is, it may be a set or interval.

To find the neutrosophic reliability, we take the following example:

Suppose the product be an electric generator produced with a high capacity of the trademark that has a Weibull distribution with parameter α=1, β=[1.5,2].
Estimate the reliability of the electric generator after the expiration of a five years warranty operation.
The neutrosophic reliability is:

$$NR(t) = e^{-(X/\alpha_N)^{\beta_N}}$$

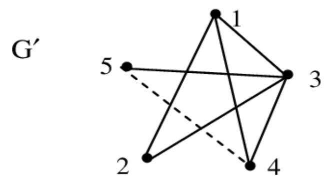
Since the shape parameter is determined β=[1.5,2].

When β= 1.5 and α=1
Then, $NR(5) = e^{-(5/1)^{1.5}} = 14 \times 10^{-4}$

And,
when β= 2 and α=1
$NR(5) = e^{-(5/1)^{2}} = 10^{-5}$
Thus, the reliability of the electric machine operation after 5 years has the range between $[14 \times 10^{-4}, 10^{-5}]$.

5. System Neutrosophic Reliability Modeling

The reliability of the system is depended on the reliability of its components and therefore all its subsystems and components have to be studied to design and analyze the reliability of a system. This can be done through the formulation based on a logical and mathematical model of the system that shows the structure function.
If we take any system that contains vertices and edges, some of them work, others fail, and others are indeterminacy. Each component of the system is identified as passing from one vertice to another.



Figure(1): the system contains indeterminacy

In figure(1), the graph of the system is a neutrosophic graph since it contains the edge < 5,4> is indeterminacy edge.

The device is considered successful if there is a successful path from the source to the sink. The device will be considered as indeterminacy if at least one path is indeterminacy (unclear) from the source to sink in this case the device is unsuccessful. We define the neutrosophic reliability according to modern logic (neutrosophic logic) as follow:

5.1 Structure-Function of Neutrosophic Reliability

The reliability for any system at the time t is denoted by R(t), where t < T can be defined as the probability of the operation of the system within the interval [0, t].

Let's define the structure-function of neutrosophy reliability based on the classical reliability after adding the new component which is the indeterminacy component to truth and falsity components.

In classical reliability, the structure-function of reliability of a device is: [35]

Let $X_1, X_2, \dots, X_i, \dots X_n$ are components of a system (device), and

$\varphi(x_1, x_2, \dots \dots x_n) = \varphi(X)$,

$\varphi : \{0,1\}^n \rightarrow \{0,1\}$ is structure- function defined as:

$$\varphi(X) = \begin{cases} 1 & device\ is\ working\ at\ [0,t] \\ 0 & device\ is\ fails\ at\ [0,t] \end{cases}$$

The neutrosophic reliability is a triple function (truth, indeterminacy, falsehood) that indicates the status of the device (works, indeterminacy, not work) given the status of each component as in the following:

$\varphi_N(x_1, x_2, \dots \dots x_n) = \varphi_N(X)$, is the neutrosophic structure-function of reliability device.

$\varphi_N : \{0,1,I\}^n \rightarrow \{0,1,I\}$

Such that,

$$x_i = \begin{cases} 1 & if\ component\ i\ working\ during\ time\ [0,t] \\ I & if\ component\ i\ indetermincy\ during\ [0,t] \\ 0 & if\ component\ i\ fails\ during\ time\ [0,t] \end{cases} \quad (5)$$

The performance of the device is measured by the triple random variables, that is

$$\varphi_N(X) = \begin{cases} 1 & device\ is\ working\ at\ [0,t] \\ I & device\ is\ idetermincy\ at\ [0,t] \\ 0 & device\ is\ fails\ at\ [0,t] \end{cases} \quad (6)$$

The reliability of a component p_i is the probability that the component i is working correctly. The component i of the device is indeterminacy probability denoted by I_i, and the component failure probability, q_i, is the probability that the component has failed (not working). And we can denote the triple components as follow:

$p_i = NP\{X_i = 1\}$,

$q_i = NP\{X_i = 0\}$ and

$d_i = NP\{X_i = I\}$ (7)

Such that, NP the Neutrosophic Probability [14,12] that an event A occurs is

$NP(X) = \{ch(X), ch(neutX), ch(antiX)\} = (T, I, F)$,

where T, I, F are standard or nonstandard subsets of the nonstandard unitary interval]-0, 1+[, and T is the chance that X occurs, denoted ch(X); I is the indeterminate chance related to X, ch(neut X); and F is the chance that X does not occur, ant(X).

Using the reliability neutrosophic to improve system reliability, such as series, parallel, composed series-parallel, or mixed.

5-2 Neutrosophic Reliability Of Series System

When we configured the reliability of the system, for example, type series: that is in a series system, a failure of any component in the series system, implies failure for the whole system.

If we have N of the components, which contains some indeterminacy components implies the system is a failure. That is if at least one of the components that are indeterminacy is a failure.

Let $X_1, X_2, \ldots, X_i, \ldots X_n$ are components of the system (device), if consider X_i that component is indeterminacy maybe more one, and N=1,2,…, i,..,n.

The neutrosophic structure-function of a series system with N components is

$\varphi_N(X) = X_1 X_2 \ldots X_i \ldots X_n$

X_i Indictor to the indeterminacy component

Such that $\varphi_N(X) = X_1 X_2 \ldots X_i \ldots X_n = (T_i, I_i, F_i), i = 1,2,\ldots n$ (8)

If the series system is successful, the structure-function must be equal to (1,1,1), otherwise, it's a failure. To find the reliability neutrosophic series its equal to the neutrosophic probability that all the components in the series system are true.

If the components N are independent,

Then $NR = Np_1 \dots \dots . Np_n = NR_1 \dots \dots . NR_n$. (9)

Exampl e (4)

Suppose a device is series types contains 4 components and the components have exponential lifetimes and give constant failure rate of each component, 0.3, 0.2, 0.1, 0.4 respectively per 20 days,

In classical reliability, $R(t) = e^{-\lambda t}$

If we suppose the 1st component is A, then $R_A(t) = e^{-0.3t} = 0.00247$

, the 2nd component is B, then $R_B(t) = e^{-0.2t} = 0.0183$

, the third component is C, then $R_c(t) = e^{-0.1t} = 0.1353$

And the fourth component is D, then $R_D(t) = e^{-0.4t}$

Hence $R(t = 20) = e^{-0.3t}. e^{-0.2t}. e^{-0.1t}. e^{-0.4t} = e^{-1.0t} = e^{(-1)(20)} = 2.06115 \times 10^{-9}$

Now, if these components have the neutrosophic exponential distribution and neutrosophic time series [13, 14], in this case, we can consider the constant failure rate is an undetermined number or set or internal which forms the number Neutrosophic, if the constant failure rate in each component as [0.28,0.32], [0.17, 0.28],[0.09, 0.17], [0.32, 0.42] respectively per 20 days.

Therefore, to find the Neutrosophic reliability of the above components, A, B, C, and D as follow:

$NR_A(t) = e^{-[0.28,0.32]t}$, if $\lambda_N = 0.28$ implies that $NR_A(t) = e^{-5.6} = 0.00369$

if $\lambda_N = 0.32$ implies that $NR_A(t) = e^{-6.4} = 0.00166$

$NR_B(t) = e^{-[0.17,0.28]t}$, if $\lambda_N = 0.17$ implies that $NR_B(t) = e^{-3.4} = 0.03337$

if $\lambda_N = 0.28$ implies that $NR_B(t) = e^{-5.6} = 0.00369$

$NR_c(t) = e^{-[0.09,0.17]t}$, if $\lambda_N = 0.09$ implies that $NR_c(t) = e^{-1.8} = 0.16529$

if $\lambda_N = 0.17$ implies that $NR_c(t) = e^{-3.4} = 0.03337$

$NR_D(t) = e^{-[0.32,0.42]t}$, if $\lambda_N = 0.32$ implies that $NR_D(t) = e^{-6.4} = 0.00166$

if $\lambda_N = 0.41$ implies that $NR_D(t) = e^{-8.2} = 2.746 \times 10^{-4}$

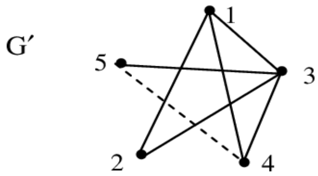
hence, $NR(t = 20) = [3.38949 \times 10^{-8}, 5.6318 \times 10^{-11}]$.

Similarly, give an example of the neutrosophic reliability of neutrosophic Weibull, see[7].

6. Some Applications

1- In the design of an electric cable Extension, the Direct Current which contains three, L, N, and E, is the earth, when the Loop Electric Circuit for work, we need to only L and N without E, in this case, we consider E is indeterminacy line.

2- In any neutrosophic graph, see [16,27,28,29,31,33,34] of the network of any system which contains some indeterminacy edge or indeterminacy vertices. Then in any graph system which contains some indeterminacy edge and indeterminacy vertices, this implies reliability Neutrosophic.



3- In an electrical circulation, we notice some time in the power of the electric current that the electric current does not reach 220 volts, or very invalid that maybe half the power or the power is excessive, and this means that nothing works in electronic systems, such as TV, freeze, Air conditioner,.., etc.

4- To ensure that the vehicle engine works at full capacity and gives the required services, there are three essential factors that we need to consider which are the production of sparks, fuel circulation, and flow of air. in any one of them does not good work, there is indeterminacy hence reliability Neutrosophic.

5- In the field of medicine, to know to measure the quality of a drug for any disease, we need here very high and accurate reliability of a drug to ensure people's lives.

6- In psychology,[2] we need high reliability to measure the balance in a personality(Neutrosophic personality), or in measuring the level of intelligence of children, whenever all the data, including extreme or abnormal ones, are taken into account, the more accurate the data will be.

7- Reliability In Neutrosophic correlation, whenever reliability is a measure of data quality and then give a good Neutrosophic correlation, see [3,4,5]

8- S., H., A.Salama (2016) [6] defined in a neutrosophic graph, every path from a node to other nodes (vertices) contains three functions (every component has weight), maybe this weight is value or area or time and distance.

9- The use of Neutrosophic reliability in knowing the reliability of the devices used for early detection of the Corona COVID 19 virus, as well as the devices for examination (such as a swab, oximeter, laser, or thermal devices for measuring temperature), as well as the reliability of data in modeling Scientific mathematical to study the type of virus or study the virus series.

10- In communications, we need high reliability for quality image compression or message encryption.

Example (5)

To ensure that the vehicle engine works at full capacity and gives the required services, there are three essential factors that we need to consider which are the production of sparks, fuel circulation, and flow of air.

Production of spark. The spark plugs generate the sparks inside the engine. The engine needs to have an efficient quantity of sparks to ensure that the engine works efficiently reliability. If any of the plugs work inefficiently, it will negatively impact the overall performance of the engine.

Fuel circulation. If there are any issues with fuel intake as a result of a blockage in the fuel injections or malfunctioning of any of the injections or shortage in fuel pumping as a result of issues with the fuel pump, it would impact the overall performance of the engine as this will impact the quantity of fuel inside the engine.

6. Conclusions

To obtain a high level of reliability of any product such as the system of machine or medicine, engineering, psychology, a measure of statistic or mathematic, etc. we need accurate and whole data. In this paper, we proposed a new concept whith neutrosophic reliability that depends on classical data and indeterminacy data together. This means that we need to study all data including vague, unclear data. We defined the structure-function for reliability according to the modern logic duch as "neutrosophic logic" that depends on triplet functions (truth, falsehood, indeterminacy) and it's using neutrosophic probability. The series of neutrosophic reliability was also discussed in this paper and some examples were illustrated.

For future work, we'll apply the neutrosophic reliability to improving many methods of reliability in networks, (series or parallel or compound) systems, and many other fields that require high accuracy (high reliability) in its systems. And we can find the neutrosophic reliability for any distribution (Expositional, Gamma, Normal, etc.) and any probability function.

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An Application of Neutrosophic Logic in the Confirmatory Data Analysis of the Satisfaction with Life Scale

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Abstract

The main concept of neutrosophy is that any idea has not only a certain degree of truth but also a degree of falsity and indeterminacy in its own right. Although there are many applications of neutrosophy in different disciplines, the incorporation of its logic in education and psychology is rather scarce compared to other fields. In this study, the Satisfaction with Life Scale was converted into the neutrosophic form and the results were compared in terms of confirmatory analysis by convolutional neural networks. To sum up, two different formulas are proposed at the end of the study to determine the validity of any scale in terms of neutrosophy. While the Lawshe methodology concentrates on the dominating opinions of experts limited by a one-dimensional data space analysis, it should be advocated that the options can be placed in three-dimensional data space in the neutrosophic analysis. The effect may be negligible for a small number of items and participants, but it may create enormous changes for a large number of items and participants. Secondly, the degree of freedom of Lawshe technique is only 1 in 3D space, whereas the degree of freedom of neutrosophical scale is 3, so researchers have to employ three separate parameters of 3D space in neutrosophical scale while a resarcher is restricted in a 1D space in Lawshe technique in 3D space. The third distinction relates to the analysis of statistics. The Lawhe technical approach focuses on the experts' ratio of choices, whereas the importance and correlation level of each item for the analysis in neutrosophical logic are analysed. The fourth relates to the opinion of experts. The Lawshe technique is focused on expert opinions, yet in many ways the word expert is not defined. In a neutrosophical scale, however, researchers primarily address actual participants in order to understand whether the item is comprehended or opposed to or is imprecise. In this research, an alternative technique is presented to construct a valid scale in which the scale first is transformed into a neutrosophical one before being compared using neural networks. It may be concluded that each measuring scale is used for the desired aim to evaluate how suitable and representative the measurements obtained are so that it's content validity can be evaluated.

1 | Introduction

Scale development is an important part of computational social science research, especially for quantitative research. Therefore, this research mostly relies on psychometric research. Usually, psychometricians assess human differences by administering test batteries that have been found to have accurate measuring properties. Effects from these tests are then evaluated by factor analysis and multidimensional scaling to classify latent variables or factors responsible for similar trends of correlations. Specific differences for aimed cognitive skills are generally represented in terms of factors in those studies [1]. The main objective of those who support the psychometric strategy is to allow for the assessment to be made objective. From this standpoint, assessment should be based on objective determinations. For this reason, the psychometric approach emphasizes scales based on statistical methods such as factor analysis, item analysis, and test analysis, and tests its validity and reliability with scientific methods [2].

Neutrosophical set is a potent field of study that has shown its efficiency and strength in various applications. In the meantime, most contributions were theoretical and only validated using mathematical examples or limited data sets and did not use other applications in general [37]. When the literature is reviewed, although it has many applications in natural sciences, recent works focus on the applications of the neutrosophic logic in social sciences [38]. Neutrosophic sets are even more suitable than fuzzy sets to represent the possible responses to questionnaires. The former enables the individual polled to communicate their genuine ideas and emotions even more precisely, thanks to the indeterminacy function of their membership. The benefit of the neutrosophical method is that responders may describe their ideas and emotions more correctly, since both indeterminacy and an independent membership function of falsehood are taken into account [39], [40]. In this respect, this research aims to use the application of the neutrosophic philosophy in social sciences especially in education and assessment and evaluation methods of scale development.

2| Preliminaries

The numerical properties obtained depending on the group to which a test is applied are generally called test statistics. Some of the test statistics can be calculated based on item statistics. In general, the test statistics like the average of the test, the average difficulty of the test, the variance of the test, and other test statistics are highly useful [3]. Researchers want to show whether there is harmony in an instrument's responses. Factor analysis is one of the multivariate approaches that social scientists use to validate psychological aspects. When several independent variables are grouped in a single study, statistical analysis can become rather challenging. It is often advantageous to group together those variables that are correlated with one another. Factor analysis is a technique that allows researchers to see whether many variables can be portrayed as a few factors [4]. Factor analysis seeks to identify some new specific factors by putting together a small number of factors that aren't connected (a p-dimensional space) [5]. It is recommended that the scale of the explanatory factor analysis process should be tested through confirmatory factor analysis [6]. Confirmatory factor analysis could be considered as a way to verify the validity of factor structures. Using this method, it is attempted to prove that the observed variables are connected with the hidden variables and hidden variables are connected. To investigate these relationships, measurement models were built [7].

There are three types of factors for developing a more grounded scale: (i) reliability; (ii) validity; and (iii) sensitivity. Reliability refers to the extent to which a measurement of a phenomenon produces consistent results as given in *Fig. 1* [8]. Therefore, reliability means consistency or stability. Consistency of any measurement scale is important for objective scientific research and this concept is related to 'agreement', 'reproducibility', and 'repeatability' of any measurement. The agreement is the closeness of two measurements made on the same subject as opposed to one another. Reliability includes repeatability. Repeatability means measuring accurately the same variable again and again for the same circumstances [9]. A test or measure is said to be reliable if there are always identical results using the same testing procedure [10]. This means that regardless of how many times the measurement has been taken or by whom it has been performed, you will always obtain the same value. This means two things: first of all, you should get the same result each time you use the measure, and secondly, you should use the measure as many times as possible. This can be an issue in data collection when several people are involved [11]. Reproducibility referred to variations in test results while tests were performed on subjects on different occasions. The changed circumstances may be due to the use of various methods of measurement or instruments, measurement by several observers or raters, or measurements during a period in which the variable's error-free level may undergo a non-negligible change [9].

Reliability is, therefore, the level of error-free. As the amount of error decreases as a result of measurement, reliability increases, and as the number of errors increases, reliability decreases.

Reliability levels of measurement tools are determined by reliability analysis. Reliability is best expressed with the reliability coefficient (r) ranging from 0.00 to +1.00. The closer the reliability coefficient of the measurement tool is to 1, the higher the reliability, the closer to 0, the lower the reliability [12].

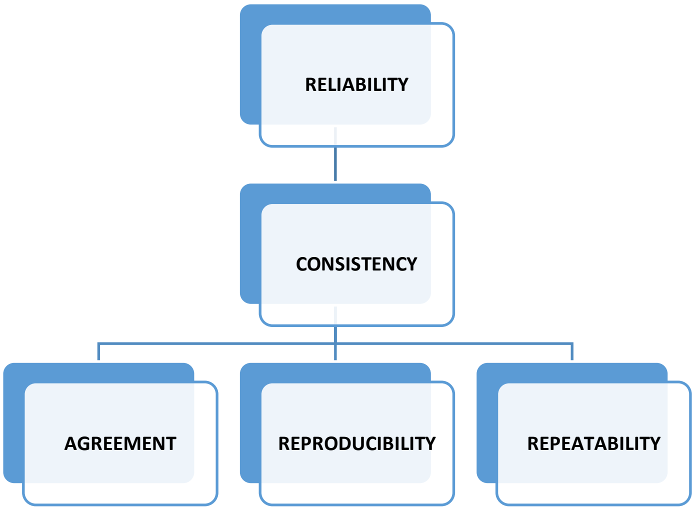


Fig. 1. Reliability and its components.

Validity simply means "measure what is intended to be measured" [13]. There are different types of validity in social sciences (*Fig. 2*). Face validity is a subjective judgment on the operationalization of a construct whether it is appropriate, unambiguous, simple, and proper [14]. Content validity refers to how appropriate and representative the measurements collected are for the desired assessment purpose. The representativeness criterion may have two definitions. Quantifying the extent of sampling is one of them. The second is the extent to which items reflect the structures of the whole scale [15]. Construct is a pattern formed by certain elements that are thought to be related to each other or by the relationships between them. The construct validity measurement tool shows to what extent it can accurately measure the structure and concept that it claims to measure [12]. Construct validity refers to how well you translated or transformed a concept, idea, or behavior that is a construct into a functioning and operating reality, the operationalization [14]. Construct validity is used when trying to quantify a hypothetical construct, like fear. Convergent and discriminant validity should be used to determine the validity of a construct by suggesting that the new measurements are correlated with other measurements of that construct and that the dimensions proposed are inappropriate to the construct unrelated, respectively [16]. Discriminant validity is the extent to which latent variable a discriminates from other latent variables. The Convergent Validity is the degree to which two measurements of a construct are connected theoretically [14]. The validity of the criterion is also divided into concurrent and predictive validity, where the validity of the criterion deals with the correlation between the current measurement and the criterion measurement (such as the gold standard) [16]. Content and construct validity in social sciences are defined as credibility/internal validity. Internal validity is related to the question of whether the research findings fit with reality in the external world. Internal validity is determined by experimenting with specific characteristics and no specific biases. For example, the question of "can we recognize people by looking at their faces?" can be examined. This question is answered by asking two more questions. First, is the independent variable the cause of the dependent variable? Second, can other possible explanations for the relationship between independent variable and dependent be logically eliminated? If the answer to these questions is yes, the researchers can claim that the experiment has internal validity [17]. Criterion validity is the degree to which it is empirically relevant to the outcome. This is something that calculates how well one measure predicts another measure. There are three types of criterion validity namely; concurrent validity, predictive and postdictive validity [14].

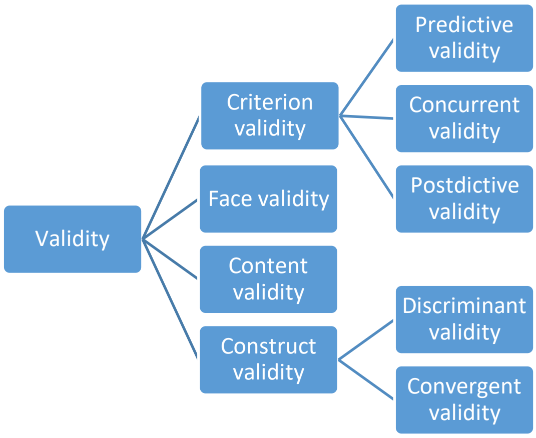


Fig. 2. Subtypes of various forms of validity tests.

Fig. 3. illustrates how reliability and validity are related. In the first target, the shots reached the same spot, but none were effective in reaching the same point. The second target can be regarded as valid but not reliable since the points are expanding over the entire place. The third target did not present reliability or validity, since they hit spread points. The fourth target stands as an indicator of reliability and validity; the shots landed right in the target center and were consistent, right in the target center [18].

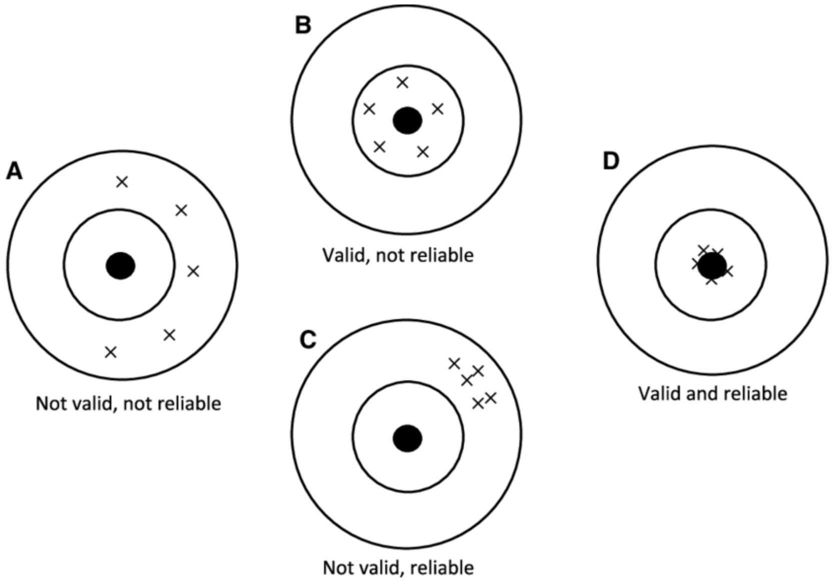


Fig. 3. Possible combinations of validity and reliability of measurement instruments [18].

Sensitivity is defined as the consensus closeness between randomly selected individual measurements or results. It is therefore concerned with the variance of repeated measurements. A measurement tool with low variance is more sensitive than those with a higher variance. For example, as a researcher, one wants to know what is the smallest sample you can use that will take into consideration the variability in the dependent measure and yet be sensitive enough to notice a statistically meaningful difference, whether there is one. Our capacity to distinguish significant differences between groups is defined in part by the variability of individuals in our sample and how much variability occurs among them. Therefore, less variability may contribute to greater sensitivity, and more variability results in less sensitivity [19].

As mentioned above, the key aim of developing questionnaires or scales is to collect correct and appropriate data. The reliability and validity of scale or questionnaire formats is an important feature of testing methodology [14]. The reliable and accurate measurement may, in the simplest intuitive terms, indicate that the current measurement is equal with, or follows, the truth. However, it is often impractical to require the new measurement to be identical to the truth, either because 1) we accept the measurement of a tolerable (or acceptable) error or 2) the truth is simply impossible for us (either because it is not measurable or because it is only measurable with some degree of error) [16]. In this regard, data space and data range are the important dimensions of developing scales because it also changes the data type, the logical space of the analysis, methodology, and validity and reliability of the results (*Fig. 4*).

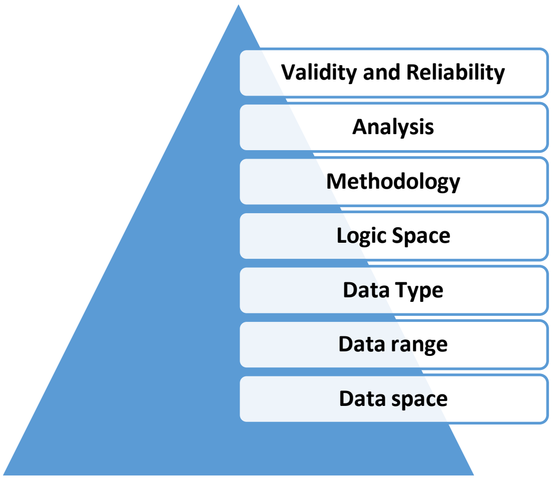


Fig. 4. Data space and data range determines the validity and reliability of any scale.

Data space in measurement tools like scale refers to the set of independent options regarding the particular item of the scale. For example, on any Likert-type scale, the participant can express only one option, so the data space is 1d, whereas on the neutrosophic scale, there are three independent dimensions regarding any item as undecided, agree, and disagree.As it can be seen, data space is 1d in any Likert-type scale and 3d in neutrosophic space and if our measurement tools become more qualitative, like having items requiring free opinions in a paragraph like choices, it has more dimensions, even in ideal cases it has infinite dimensions. However, although n-dimensional space is more appropriate for better valid and reliable results, less dimensional spaces have less vagueness in terms of the interpretation of the data and they can be more easily statistically handled. Additionally, as the dimension of space increases, the objectivity of the measurement tool in terms of measuring common characteristics decreases. The advantage of the 3-dimensional neutrosophic scale is that it both seeks the agreement, disagreement, and confusion levels of the participants. In daily life, many items are encountered to give an opinion about them and we are not restricted within a 1-dimensional space where we can only choose one answer regarding whether we agree, disagree or express uncertainly about a particular case. However, in the three-dimensional neutrosophic space, participants express both their agreement and disagreement level as well as the uncertainty in the items or dimensions of the scale. People sometimes think that they understand a statement, but one word in the statement makes us uncertain whether it is the "right meaning" intended by the source. Similarly, people sometimes agree on some propositions, but just because of the source of the message itself, they also disagree with the item. Therefore, the neutrosophic scale is different from the classical Likert-type scales in terms of data space (*Fig. 5*).

The second important point that distinguishes any measurement tool from each other is the data range. The range of a set of data is the difference between the highest and lowest values in the set. Likert-type scales are commonly arranged in terms of data, ranging from 3 point Likert-type scales to 10 point Likert-type scales. However, the range of the neutrosophic scale is broader than the Likert-type scales. It includes any rational number in a range between 0 and 100. As a result, neutrosophic scales have continuous variable types, whereas Likert-type scales have discrete value types in terms of rational numbers, so data analysis may differ as a result. This can contribute to increasing the sensitivity of the measurement tool in this respect. This is actually what is called as neutrosophic data in some recent researches is the piece of information that contains some indeterminacy. Similar to the classical statistics, it can be classified as [39]:

– *Discrete neutrosophic data, if the values are isolated points.*
– *Continuous neutrosophic data, if the values form one or more intervals.*
– *Quantitative (numerical) neutrosophic data; for example: a number in the interval [2, 5] (we do not know exactly), 47, 52, 67 or 69 (we do not know exactly).*
– *Qualitative (categorical) neutrosophic data; for example: blue or red (we do not know exactly), white, black or green or yellow (not knowing exactly).*
– *The univariate neutrosophic data is a neutrosophic data that consists of observations on a neutrosophic single attribute.*

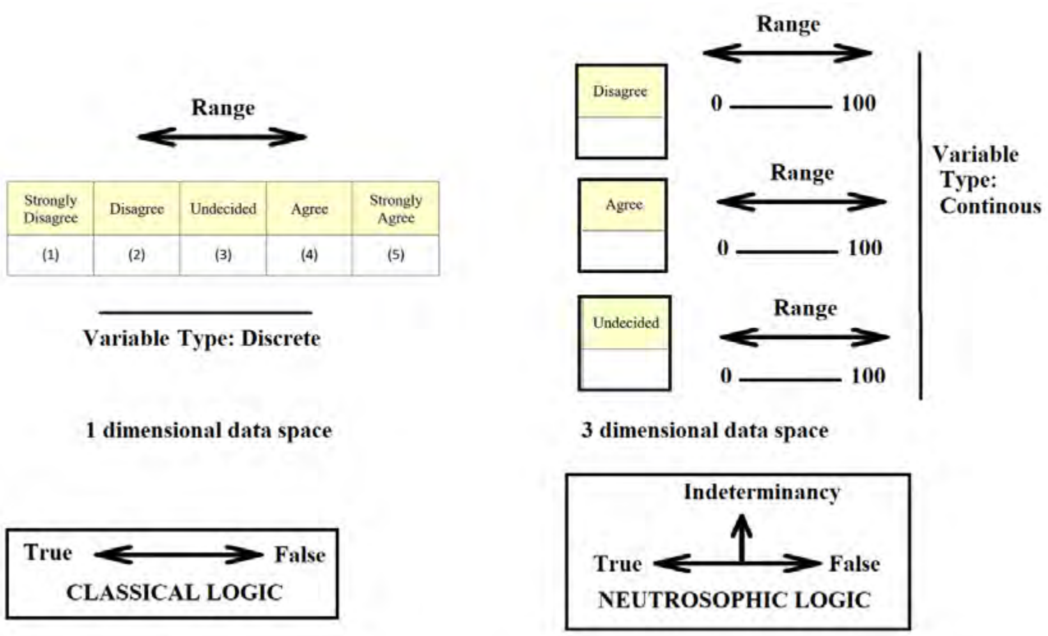


Fig. 5. Data space of classical Likert-type scale, neutrosophic scale.

The third important point of any measurement tool is its logic space. Logic space is important because "in any field of knowledge, each structure is composed from two parts: a space, and a set of axioms (or laws) acting (governing) on it. If the space, or at least one of its axioms (laws), has some indeterminacy of the form (t, i, f) ≠ (1, 0, 0), that structure is a (t, i, f)-Neutrosophic Structure" [41]. Therefore the logic which is in our focus, Neutrosophic Logic, is an emerging field where each proposition is reckoned to have the proportion (percentage) of truth in a subset T, the proportion of indeterminacy in a subset I, and the proportion of falsity in a subset F. A subset of truth (or falsity or indeterminacy) here is considered, rather than just a number, since in many situations can not be precisely determined the proportions of truth and falsity but we can only approach them. For example, suppose that a statement (or proposition) is between 32% and 48% true and 59% to 73% false; worse: 32% to 39% or 41 to 52% true (according to various observers) and 57% or 62% to 71% false. Subsets are not basic intervals but are any set (open or closed or semi-open/semi-closed intervals, discrete, continuous, intersections or unions of previous sets, etc.) following the given proposition. The adventure of gaining meaning and mathematical results from situations of uncertainty was initiated by Zadeh [20]. Fuzzy sets added a new wrinkle to the concept of classical set theory. Elements of the sets have degrees of belongingness (in other words, membership) according to the underlying sets. Atanassov defined intuitionistic fuzzy sets including belongingness and non-belongingness degrees [21], [32]-[34]. Smarandache suggested neutrosophy as a computational solution to the idea of neutrality [22]. Neutrosophic sets consider belongingness, non-belongingness, and indeterminacy degrees. Intuitionistic fuzzy sets are defined by the degree of belongingness and non-belongingness and uncertainty degrees by the 1-(membership degree plus non-membership degree), while the degree of uncertainty is assessed independently of the degree of belongingness and non-belongingness in neutrosophic sets. Here, belongingness, non-belongingness, and degree of uncertainty (uncertainty), like degrees of truth and falsity, can be assessed according to the interpretation of the places to be utilized.

This indicates a difference between the neutrosophic set and the intuitionistic fuzzy set. The definition of neutrosophy is, in this sense, a potential solution and representation of problems in different fields. Twodetailed and mathematical fundamental differences between relative truth (IFL) and absolute truth (NL) are as follows:

I. NL can distinguish absolute truth (truth in all possible worlds, according to Leibniz) from the relative truth (truth in at least one world) because NL (absolute truth) = 1^+ while IFL (relative truth) = 1. This has been practiced in philosophy and linguistics (see the Neutrosophy). The standard interval [0, 1] used in IFL has been extended to the unitary non-standard interval $]^-0, 1^+[$ in NL. Parallel distinctiveness for absolute or relative falsehood and absolute or relative indeterminacy are allowed to consider in NL.

II. There do not exist any limits on T, I, F apart from they are subsets of $]^-0, 1^+[$, thus: $^-0 \leq \inf T + \inf I + \inf F \leq \sup T + \sup I + \sup F \leq 3^+$ in NL. This permission allows dialetheist, paraconsistent, and incomplete information to be identified in NL, while these situations impossible to be identified in IFL since F (falsehood), T (truth), I (indeterminacy) are restricted either to $t + i + f = 1$ or to $t2 + f2 \leq 1$, if T, I, F are all reduced to the points t, i, f respectively, or to sup $T + \sup I + \sup F = 1$ if T, I, F are subsets of [0, 1] in IFL.

Although there are usually three options in Likert-type scales: agreement, disagreement, and vagueness, its logic is based on one valued option located on the opposite sides of true and false values. However, the neutrosophic set has three independent components, giving more freedom for analysis so that it brings different logical operations as well. Therefore, the methodology of the analysis of the data should be changed based on the logical structure of the scale. For instance, while factor analysis is used for classical Likert-type scales, neural networks are more appropriate for the analysis of the data of the neutrosophic scales. Nevertheless, it should be noted that classical analysis and methods can indeed be used for neutrosophic scales based on different analysis procedures. To sum up, "a space with an item, it means an opinion, another element induces another opinion, another element in turn induces another opinion, and so on. The opinion of each element of the structure must be respected. In this way it builds a neutrosophic social structure. The result is a very large socio-neutrosophic structure that is intended to be filtered, evaluated, analyzed by scientific algorithms" [42]. Hence, we can conclude that the validity and reliability of the measurement tools can change based on the logical structure of the scale. As a result, in this study, we take The Satisfaction with Life Scale developed by Diener et al. [23] and adapted in Turkish by Dağlı and ve Baysal [24] and convert it into neutrosophic form, compare the results, and use this analysis to propose new type confirmatory analysis procedures and develop neutrosophic scales. There are many ways to evaluate and interpret data. Some recent studies reveal important developments based on the interpretation and effective use of data [42]-[44].

2.1| The Difference between Lawshe Technique and Neutrosophic Scale

Some argue that the well-known Lawshe technique is very similar to neutrosophic analysis and propose what is the reason behind the logic of neutrosophic forms. Initially suggested in a seminal 1975 paper in Lawshe [25], the method of Lawshe was common in various areas including health care, education, organizational development, personnel psychology, and market research for determining and quantifying content validity [26], [27].

Lawshe [25] has proposed a quantitative measure to evaluate validity of the content termed as the Content Validity Ratio (CVR). The validity ratio of content provides information about validity of items. The approach includes the use of an expert panel to evaluate items based on their relevance to the scale domain. Each item on a scale is classified as a three-point rating system (1) point is irrelevant, 2) item is important, but not essential, and 3) item is essential). The percent of experts considering items significant or essential for the substantive content of the scale is calculated for every element of a CVR. Also a overall measurement of the validity of the content of the scale may be created. The index is calculated as a mean of the CVR scores for items [36].

A quantitative criteria is necessary in the Lawshe approach for determining the validity of content. The Content Validity Index and CVR are the criterion for validity used by experts. In order for each item to be included in the Scale, the content validity ratio is an internationally accepted standard. For all finished items, the Content Validity Index is the average CVR. The CVR should assess whether or not each item is essential, and the Content Validity Index should identify the relationships between the scale items and scale . The Content Validity Index is calculated by using the degree of agreement of the experts on the relevance and clarity of the items. According to CVR values,

– *If all the experts in the panel answered "not necessary" for any item, that item is completely unnecessary.*
– *If all of the experts on the panel gave the answer "useful but not necessary" for any item, that item is significantly necessary.*
– *If the number of experts who give the answer "required" for any item is more than half, it can be commented that the item has a certain validity value, and the validity value of the item will increase as the number of experts who give the answer "required" increases [35].*

First of all, the main difference between those two techniques is in their data space. Although there are three choices in the Lawshe technique for each item as an a-Essential? b-Useful but not essential? Why? c-Not necessary? Why, while membership in neutrosophic logic is very similar to Truth T, indeterminacy I, and falsity F, their dimensions are different from each other because there is only one option regarding each item, which corresponds to one-dimensional data space, but there are three independent data spaces in the neutrosophic form where each data represents a different. According to this, whether all participants agree that the information or ability that has been tested is necessary, or whether none says it is relevant, we are sure that the component has been added or omitted. If there is no majority, the dilemma emerges. There are two hypotheses, both compatible with existing psychophysical principles [28].

– *Every item for which more than half of the experts consider any item to be "essential" has content validity.*
– *The wider the extent or degree of its validity is the more experts (above 50 percent) who view an item as "essential."*

Therefore, the Lawshe technique focuses on the dominant opinions of the experts which are restricted by one-dimensional data space so that it might hide their indeterminacy or disagreement because they are weak compared to the other options. It should be pointed out that altough Lawshe technique is not strictly restricted by the one dimensional options for experts because it also take their suggestions, in the statistical analysis process it focuses on only one options. For a small number of items, the effect of this can be negligible, but for a huge number of items, it can make huge differences.

There is one parameter in the Lawshe technique. Researcher can only choose one option among agreement, disagreement, and indeterminacy based on his/her dominant view. Hence it is actually a 1d dimensional function in a 3-dimensional space. There are three parameters in the Neutrosophic scale. A researcher must choose three options among agreement, disagreement, and indeterminacy. Hence it is actually a 3d dimensional function in a 3-dimensional space. Therefore, the degree of freedom of the Lawshe technique is 1 in 3-d space whereas the degree of freedom of the neutrosophic scale is 3, that is, a researcher is restricted in 1-d space in 3d space of possibilities in Lawshe technique whereas researchers must use three independent parameters of 3d space in neutrosophic scale (*Fig. 6*).

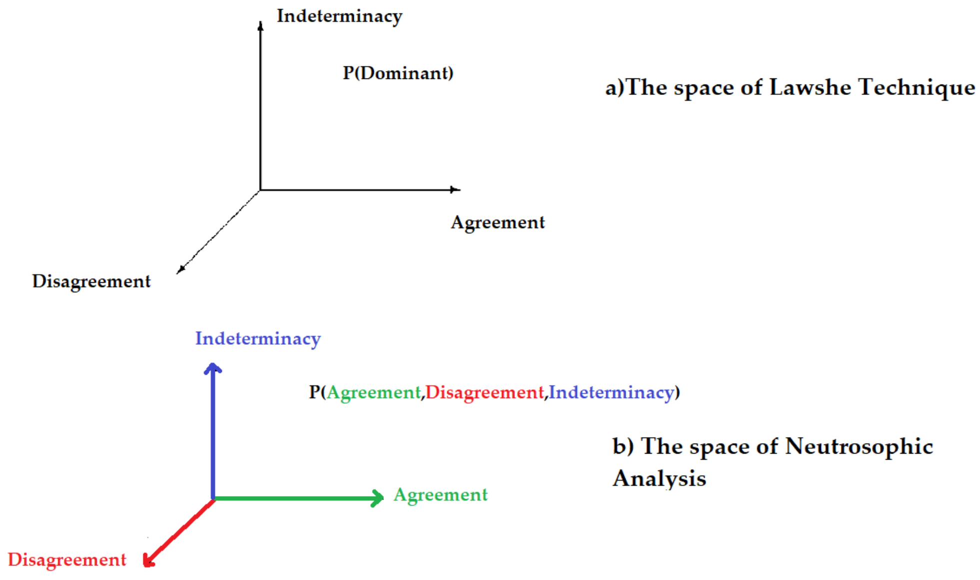


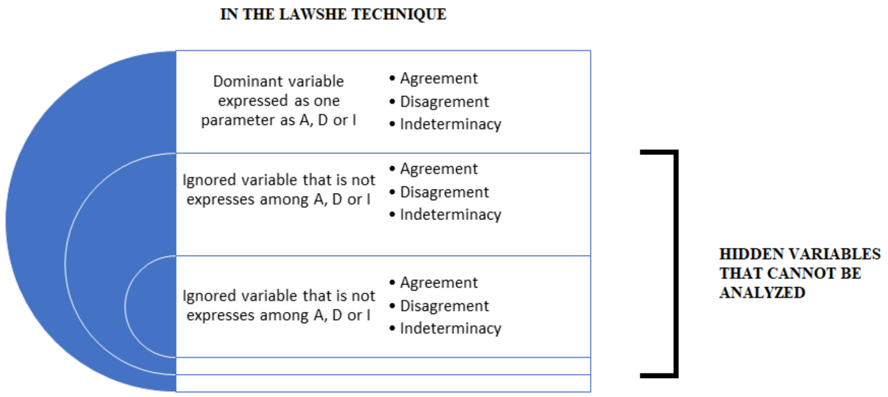
Fig. 6. The difference between the space and parameters of the Lawshe technique with neutrosophic scale.

a) There is one parameter in the Lawshe technique. The analysis focus on one option among agreement, disagreement and indeterminacy based on the dominant view. Hence it is actually a 1d function in 3d space b) there are three parameters in the Neutrosophic scale. The analysis focuses on three options among agreement, disagreement and indeterminacy. Therefore, it is a 3D function in the 3D space.

Therefore, for the participation of a huge number of researchers, the dominant view of the researcher restricted within 1d space in the Lawshe technique may dismiss the other two parameters that cannot be ignored in the actual case. These hidden variables can lead to huge differences especially in the case of the analysis of the options of a huge number of participants and even this cannot be realized. However, in neutrosophic logic, it is impossible to dismiss three parameters since the researchers must give their opinions on them (*Fig. 7*).

The second difference is related to the data range. The Lawshe technique is limited by discrete data that can be manipulated with qualitative comments. Although qualitative comments make the item better, in terms of generalizability we may not be confident that the item is suitable for its content. Opinions of the experts may indicate different content, but the understanding of common participants may indicate different content in this respect.

The third difference is related to statistical analysis. In the Lawshe technique, it is focused on the ratio of decisions of the experts, whereas in the neutrosophic logic we focus on the importance and correlation level of each item for the analysis. In the Lawshe technique, there is no distinction between the importance level and correlation, so it means that the item that is seen as important by experts might not be correlated with the content in the actual applications (*Fig. 8*). In daily life, we wonder about particular features and we seek them in particular sets, but the items of the set can be seen as important but are not relevant to what we want to seek. For example, we may meet a close relative whom we have not seen in a long time and look for him/her in a specific location, and the individuals resembling our relative are important to us, but the importance is diminished when we discover that there is no correlation between the actual close relative and the similar person resembling him/her.



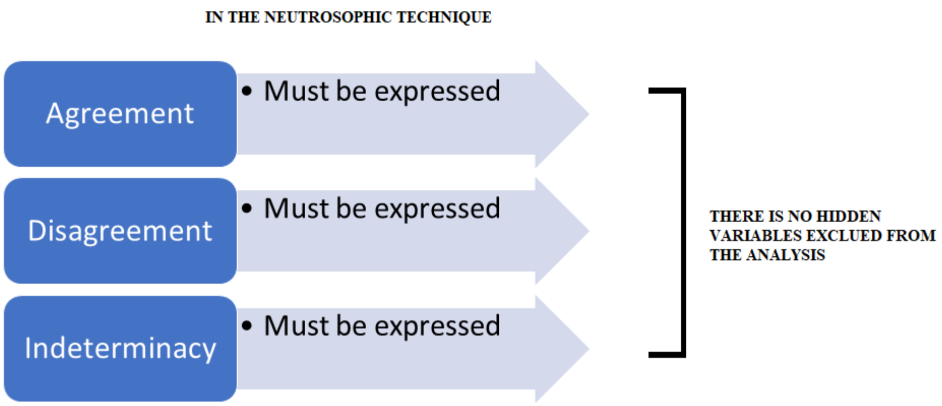


Fig. 7. There is no hidden variable in the neutrosophic technique but there are hidden variables in the Lawshe technique.

Actually Sartre's vivid description [29] regarding his hypothetical appointment with Pierre can be given as a more explicit example for the importance and correlation as follows:

I have an appointment with Pierre at four o'clock. I arrive at the cafe a quarter of an hour late. Pierre is always punctual. Will he have waited for me? I look at the room, the patrons, and I say, "he is not here." Is there an intuition of Pierre's absence, or does negation indeed enter in only with judgment? At first sight it seems absurd to speak here of intuition since to be exact there could not be an intuition of nothing and since the absence of Pierre is this nothing…..

Similarly Pierre's actual presence in a place which I do not know is also a plenitude of being. We seem to have found fullness everywhere. But we must observe that in perception there is always the construction of a figure on a ground. No one object, no group of objects is especially designed to be organized as specifically either ground or figure; all depends on the direction of my attention. When i enter this cafe to search for PIerre, there is formed a synthetic organization of all the objects in the cafe, on the ground of which Pierre is given as about to appear. This organization of the cafe as the ground is an original nihilation. Each element of the setting, a person, a table, a chair, attempts to isolate itself, to lift itself upon the ground constituted by the totality of the other objects, only to fall back once more into the undifferentiation of this ground; it melts into the ground. For the ground is that which is seen only in addition, that which is the object of a purely marginal attention. Thus the original nihilation of all the figures which appear and are swallowed up in the total neutrality of a ground is the necessary condition for the appearance of the principle figure, which is here the person of Pierre.

This nihilation is given to my intuition; i am witness to the successive disappearance of all the objects which i look at-in particular of the faces, which detain me for an instant (could this be Pierre?) and which as quickly decompose precisely because they "are not" theface of Pierre. Nevertheless, if i should finally discover Pierre, my intuition would be filled by a solid element, i should be suddenly arrested by his face and the whole cafe would organize itself around him as a discrete presence.

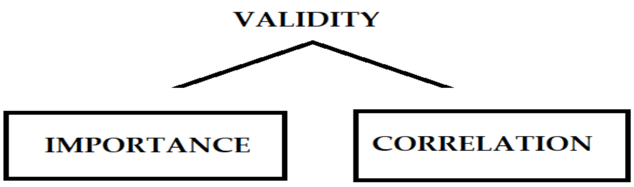


Fig. 8. There is a distinction between the concept of importance and correlation in neutrosophic logic.

Therefore, when experts make a decision, there is no clear distinction between their decision-making process in terms of importance or correlation.

The fourth one is related to expert opinion. Lawshe technique focuses on expert opinion, but the term expert is not clear in many respects. For example, if somebody studies a novel concept that has not been studied previously, how an expert decides whether the item is suitable or not besides deciding on its grammar or meaning. Furthermore, we need different experts for decision-making about the suitability of the item, but the ratio of those experts shouldn't be equal in the proportion of the decision-making process. For example, on some scales, the opinion of a psychologist might be more important than the other experts and their contribution should vary by this. However, in the neutrosophic scales, we mainly aim at the real participants so that we can understand to the extent whether the item is understood or objected or vague.

3| Methodology

In the methodology, first, the items of the Satisfaction with Life Scale were converted into the neutrosophic form where each item has three independent components referring to the agreement, disagreement, and indeterminacy. However, to compare the neutrosophic scale, the classical scale were also used as well. Secondly, each item of neutrosophic scale were analyzed in terms of classical scale in terms of neural networks and Spearman correlation constant. In the second part of the study, the Neutrosophic Life Satisfaction Scale were analyzed in terms of whole structure for confirmatory factor analysis. Finally, the decision-making formula were created to decide to remove or keep the items on the neutrosophic scale (*Fig. 9*).

In this analysis var1 refers to the variable number and a (such as var1a) stands for agreement b stands for indeterminacy and c refers to disagreement. In the neural network analysis for the study, for the level of the analysis of each item, the input variables are three sub-items of each item on the neutrosophic scale and the output variable is each classical scale. Similarly, for the whole structure for confirmatory factor analysis, the input variables are all the items on the neutrosophic scale and output variables are the classical items of the classical scale. The activation function both for the hidden and output layer was chosen as the sigmoid function. The number of hidden layers in each analysis was chosen to be two (*Fig. 10*). Criteria training=batch optimization=gradientdescent was chosen as the criterion. In the analysis of the data, independent variable importance analysis was used.

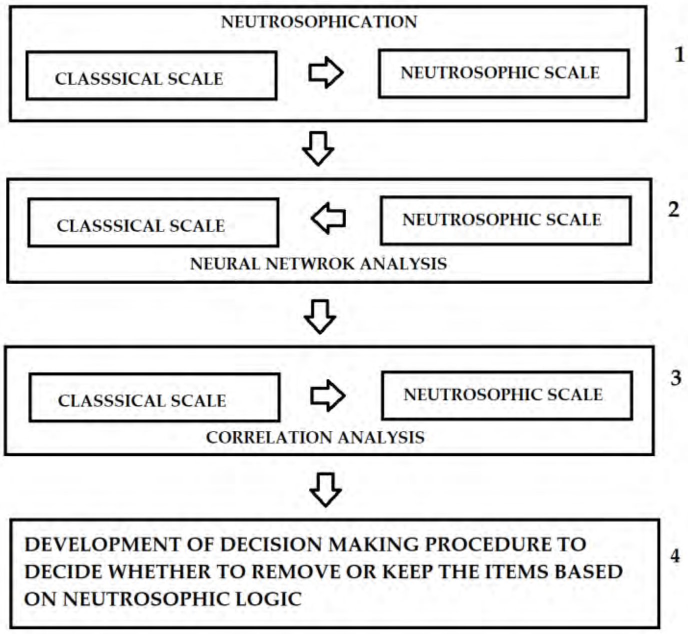


Fig. 9. The procedure for the development of neutrosophic scale.

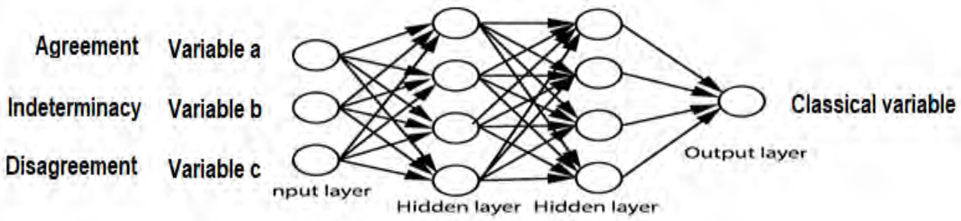


Fig. 10. The general structure of the Convolutional Neural Network (CNN) we used in this study is a three-layer neural network with three input neurons, two hidden layers of four neurons each, and one output layer [30].

Independent variable importance analysis performs a sensitivity analysis, which computes the importance of each predictor in determining the neural network. The importance of an independent variable is a measure of how much the network's model-predicted value varies with different values of the independent variable. Normalized importance is just the importance values that are grouped by and represented as percentages of importance values. In another words, the importance of an independent variable is a measure of how much the network's model-predicted value changes for different values of the independent variable. Normalized importance is simply the importance values divided by the largest importance values and expressed as percentages. However, it should be underlined that you cannot tell is the "direction" of the relationship between these variables and the predicted probability of default" [31], [41]. The importance chart is simply a bar chart of the values in the importance table, sorted in descending value of importance. It allows to guess that a larger amount of debt indicates a greater likelihood of default, but to be sure, you would need to use a model with more easily interpretable parameters [41]. Therefore, the spearman correlations between the variables are examined to see the direction and relationship of the items to decide whether they are suitable or not.

3.1| Measurement Tools

In this study, the satisfaction with Life Scale adapted into Turkish by Dağlı and ve Baysal [24] which was developed by Diener et al. [23] was converted into the neutrosophic form and the results were compared in terms of confirmatory analysis by convolutional neural networks. One might ask why an adapted version of a scale was chosen rather than adapting or developing a new scale in the neutrosophic form. The first reason for this is that the method based on neutrosophic logic is a very new one so that in more grounded levels it must be tested rather than directly using it to assess and develop scales. Secondly, the neutrosophic form could be compared with the classical one and infer the advantageous and disadvantageous sides of the neutrosophic scale in terms of its different aspects. Thirdly, this study is aimed at conducting confirmatory analysis so that a particular measurement tool must be used to assess whether the neutrosophic form can be used for the analysis. In classical confirmatory analysis, similar measurement tools can be used to analyze this, but in this article, the main aim is to use the neutrosophic form to conduct confirmatory analysis.

4| Findings

In this section, we give our findings.

4.1| Analysis of Neutrosophic Life Satisfaction Scale in terms of Reliability

Before using the neutrosophic scale it can be wondered about its reliability before comparing it with the classical one. Cronbach's Alpha constant can be used for the neutrosophic scale. However, it should be noted that Cronbach's Alpha constant should be used three times for three independent factors as given in *Table 1* below.

Table 1. Cronbach's Alpha constant for three dimensions.

Cronbach's Alpha Constant	Variables
0.863	VAR1a VAR2a VAR3a VAR4a VAR5a
0.777	VAR1b VAR2b VAR3b VAR4b VAR5b
0.792	VAR1c VAR2c VAR3c VAR4c VAR5c

Results show that our neutrosophic scale is also reliable which also supports the reliability of the classical scale because Cronbach's Alpha constant is an acceptable level for three dimensions.

4.2| Analysis of Neutrosophic Life Satisfaction Scale in terms of Items of Validity

According to Spearman's rho correlation coefficient, classical variable 1 has a high positive significant correlation with var1a which is related to the agreeing level of the participants and it has an average level negative significant level of correlation variable 1c which is related to the disagreeing level of the participants. Both correlations can be related to the points of a participant who has either a high level of life satisfaction or not. Besides, no correlation between vagueness and classical items shows that there is no indeterminacy about this item.

Table 2. Correlation among neutrosophic item 1 and classical item 1.

	VAR1a	VAR1b	VAR1c
VAR1 Correlation Coefficient	0.678**	-0.022	-0.417**
Sig. (2-tailed)	0.000	0.768	0.000
N	189	189	189

Neural network analysis of the items reveals that participants with positive life satisfaction for item 1a contribute 100% to classical variable 1 and participants with negative life satisfaction for item 1c contribute 26.4% to classical variable 1. This might be related to the differentiation of the number of participants having high-level life satisfaction and a low level of life satisfaction. However, it should be noted that the vagueness of this item is 57.5% implies that there is a moderate level of confusion about this article either because of meaning or the usage of the words or some unknown parameters, although there is no correlation between var1b and classical variable.

Table 3. Independent variable ımportance for classical item 1 in terms of neutrosophic items.

Independent Variable Importance		
	Importance	Normalized Importance
VAR1a	0.544	100,0%
VAR1b	0.313	57,5%
VAR1c	0.143	26,4%

According to Spearman's rho correlation coefficient, classical variable 2a has a significant positive correlation with var2a, which is related to the participants' agreeing level, and variable 2c has a negative significant low level of correlation, which is related to the participants' disagreeing level. Both correlations can be related to the points of participants who have either a high level of life satisfaction or not. Besides, no correlation between vagueness and classical items shows that there is no indeterminacy about this item.

Table 4. Correlation among neutrosophic item 2 and classical item 2.

		VAR2a	VAR2b	VAR2c
VAR2	Correlation Coefficient	0.732**	0.120	-0.277**
	Sig. (2-tailed)	0.000	0.099	0.000
	N	189	189	189

Neural network analysis of the items reveals that participants with positive life satisfaction for item 2a contribute 100% to classical variable 2 and participants with negative life satisfaction for item 2c contribute 26.6% to classical variable 2. This might be related to the differentiation of the number of participants having high-level life satisfaction and a low level of life satisfaction. However, it should be noted that the vagueness of this item is 31.7% implies that there is a weak level of confusion about this article either because of meaning or the usage of the words or some unknown parameters, although there is no correlation between var1b and classical variable.

Table 5. Independent variable importance for classical item 2 in terms of neutrosophic items.

Independent Variable Importance		
	Importance	Normalized Importance
VAR2a	0.632	100,0%
VAR2b	0.200	31,7%
VAR2c	0.168	26,6%

According to Spearman's rho correlation coefficient classical variable 3 has a moderate positive significant correlation with var3a which is related to the agreeing level of the participants and it has a negative significant moderate level of correlation which is related to the disagreeing level of the participants. Both correlations can be related to the points of participants who have either a high level of life satisfaction or not. However, the weak level of significant correlation between vagueness and classical item shows that there is an indeterminacy about this item.

Table 6. Correlation among neutrosophic item 3 and classical item 3.

		VAR3a	VAR3b	VAR3c
VAR3	Correlation Coefficient	0.474**	-0.178*	-0.430**
	Sig. (2-tailed)	0.000	0.014	0.000
	N	189	189	189

According to the results of the neural network analysis for the items, participants with positive life satisfaction for item 3a have a 100% contribution to classical variable 3, while participants with negative life satisfaction for item 3c have a 38, 0% contribution to classical variable 3. This might be related to the differentiation of the number of participants having high-level life satisfaction and a low level of life satisfaction. However, it should be noted that the vagueness of this item 3c, which is 21,7%, implies that there is a weak level of confusion about this article either because of meaning or the usage of the words or some unknown parameters. It should be noted that there is also a weak level significant correlation between item 3b and item 3.

Table 7. Independent variable importance for classical item 3 in terms of neutrosophic items.

Independent Variable Importance		
	Importance	**Normalized Importance**
VAR3a	0.626	100,0%
VAR3b	0.136	21,7%
VAR3c	0.238	38,0%

According to Spearman's rho correlation coefficient classical variable 4 has a high-level significant correlation with var4a which is related to agreeing on the level of the participants and it has a negative moderate level significant correlation which is related to the disagreeing level of the participants. Both correlations can be related to the points of participants who have either a high level of life satisfaction or not. Besides, no correlation between vagueness and classical items shows that there is no indeterminacy about this item (*Table 8*).

Table 8. Correlation among neutrosophic item 4 and classical item 4.

		VAR4a	VAR4b	VAR4c
VAR4	Correlation Coefficient	0.715**	-0.115	-0.475**
	Sig. (2-tailed)	0.000	0.115	0.000
	N	189	189	189

Neural network analysis of the items reveals that participants with positive life satisfaction for item 4a contribute 95.8% to classical variable 4 and participants with negative life satisfaction for item 4c contribute 100.0% to classical variable 4. This might be related to the differentiation of the number of participants having high-level life satisfaction and a low level of life satisfaction. However, it should be noted that the vagueness of this item 4c is 27,0%, implies that there is a weak level of confusion about this article either because of meaning or the usage of the words or some unknown parameters, although there is no correlation between variable 4b and classical variable (*Table 9*).

Table 9. Independent variable importance for classical item 4 in terms of neutrosophic items.

Independent Variable Importance		
	Importance	Normalized Importance
VAR4a	0.430	95,8%
VAR4b	0.121	27,0%
VAR4c	0.449	100,0%

According to Spearman's rho correlation coefficient classical variable 5 has a high level of positive significant correlation with var5a which is related to the agreeing level of the participants and it has a weak level of negative significant correlation which is related to the disagreeing level of the participants. Both correlations can be related to the points of participants who have either a high level of life satisfaction or not. Besides, there is a weak level significant correlation between variable 5 and variable 5b. Therefore, the weak level significant correlation between vagueness and classical item shows that there is an indeterminacy about this item (*Table 10*).

Table 10. Correlation among neutrosophic item 5 and classical item 5.

		VAR5a	VAR5b	VAR5c
VAR5	Correlation Coefficient	0.706**	0.149*	-0.347**
	Sig. (2-tailed)	0.000	0.040	0.000
	N	189	189	189

The results of the neural network analysis for the items show that participants with positive life satisfaction for item 5a have a 100% contribution to the classical variable 4 and participants with negative life satisfaction for item 5c have an 84.2% contribution to the classical variable 4. This might be related to the differentiation of the number of participants having high-level life satisfaction and a low level of life satisfaction. However, it should be noted that the vagueness of this item 4c is 39,6%, implies that there is a weak level of confusion about this article either because of the meaning of the usage of the words or some unknown parameters (*Table 11*).

Table 11. Correlation among neutrosophic item 5 and classical item 5.

Independent Variable Importance		
	Importance	Normalized Importance
VAR5a	0.447	100,0%
VAR5b	0.177	39,6%
VAR5c	0.376	84,2%

4.3| Analysis of Neutrosophic Life Satisfaction Scale in terms of whole Structure for Confirmatory Factor Analysis

Neural network analysis results for two scales can be given as follows. It seems that variable 2 and variable 5 might be problematic when considering the overall contribution of the items for the whole scale since variable …b items are related to the vagueness of the participants. (*Table 12*).

Table 12. Independent variable importance for the whole scales.

Independent Variable Importance		
	Importance	Normalized Importance
VAR5c	0.162	100.00%
VAR2a	0.133	82.30%
VAR5a	0.121	74.70%
VAR3a	0.1	61.50%
VAR1c	0.096	59.30%
VAR2b	0.09	55.70%
VAR5b	0.083	51.10%
VAR4a	0.075	46.60%
VAR3c	0.035	21.50%
VAR1a	0.032	20.00%
VAR2c	0.022	13.30%
VAR4b	0.018	11.20%
VAR4c	0.015	9.00%
VAR1b	0.013	7.80%
VAR3b	0.005	2.90%

5 | Discussion and Conclusion

Content validity refers to how appropriate and representative the measurements collected are for the desired assessment purpose. Content validity refers to how appropriate and representative the measurements obtained are for the desired assessment purpose. The representativeness criterion may have two definitions. Quantifying the extent of sampling is one of them. The second is the extent to which items reflect the structures of the whole scale [15]. In this regard, the most obviating factor in determining whether an item should be removed or not is to use the participants' vagueness choices for each item. In this respect, we have two kinds of variables to formalize our decision-making as correlation constant and importance level. If the decision function is labelled as d where r stands for correlation constant and I stands for importance level, the function for decision making can be written as like this:

$$D=R*I. \quad (1)$$

The interpretation of this formula can be given in *Table 1*. It should be noted that the correlation constant is the absolute value of r as |*R*|.

Table 13. The interpretation of the formula D=R*I.

The Interpretation of The Correlation Coefficient (r)	The Interpretation of The Importance Level	Decision Criteria for Accepting or Rejecting The Item where 0<cc<1 Decision=[correlation coefficient for vagueness (r)]*[Importance level for vagueness]
Very weak correlation or no correlation if r <0.2	Very weak importance level if <20%	if 0≤cc≤20, item acceptable
Weak correlation between 0.2-0.4	Weak importance level 20%-40%	if 20<cc≤40, item acceptable
A moderate correlation between 0.4-0.6	Moderate importance level 40%-60%	if 40<cc≤60, the item should be modified or removed
The high correlation between 0.6-0.8	High importance level 60%-80%	if 60<cc≤80, the item should be modified or removed
If r>0.8, it is interpreted that there is a very high correlation	If 80%>, it is interpreted that there is a very high importance level	if 80<cc≤100, the item should be removed

The formula 5.1 can be applied for the findings of the items of the neutrosophic Life Satisfaction Scale for confirmatory analysis. Let's look at our findings based on item levels with the *Eq. (1)* as given in *Table 14*. The results show that this scale is valid because all the items are at an acceptable level.

Table 14. Application of the Eq. (1) for each item.

	Importance Level (i)	Correlation Constant (r)	Decision Result (d=i*r)	
Var1	57.5	0.22	12.65	Acceptable
Var2	31.7	0.12	3.804	Acceptable
Var3	21.7	0.178	3.8626	Acceptable
Var4	27	0.115	3.105	Acceptable
Var5	39	0.149	5.811	Acceptable

In *Table 11*, independent variable importance for the whole scale shows that variable 2 and variable 5 might be problematic when considering the overall contribution of the items for the whole scale since variable …b items are related to the vagueness of the participants. However, formula 4.1 shows that although the importance level is high, it is not significant, so that all the items on the scale are valid. Finally, one might ask that if the item related to vagueness is only focused on, why do we need the other two items regarding agreement and disagreement ? Although on this scale such a conflict is not seen, this data can be used to evaluate the validity and reliability of the scale. For instance, if both agreement and disagreement items have a similar sign to the target item, it can be concluded that this item is also problematic because it reflects both agreement and disagreement at the same time, implying that there is confusion about it for determining the aimed question. Let label that the correlation of agreement item is α and the correlation of disagreement item is β since these items are opposite to each other their correlation should naturally be opposite to each other so that $\alpha*\beta=-1$. If $\alpha*\beta=+1$ it can be concluded that there is a contradiction in this item. If the *Eq. (1)* is modified for these values where i_1 is the importance level of the first item and i_2 is the importance level of the second item as follows:

$$i_1 *\alpha* i_2* \beta)/100= d. \quad (2)$$

Because we don't want to deal with huge numbers in all the importance levels 100 and correlations 1 or-1, the multiplication is divided by 100 simply by scaling the value into a more simple form.Let apply the rule of our correlation constants in the finding section for each item in *Table 3*. An opposite sign indicates that our data is consistent. Otherwise, the effect of the correlations can be examined and evaluated to be whether the item should be removed or not just as in the classification given in *Table 13*.

Table 15. Decision matrix evaluating the consistency of the items in terms of agreement and disagreement items of the neutrosophic scale.

	i_1	A	i_2	β	$i_1 *\alpha* i_2* \beta$	Decision
Variable 1	100	0.678	26.4	-0.417	-7.4639664	Acceptable
Variable 2	100	0.732	26.6	-0.277	-5.3935224	Acceptable
Variable 3	100	0.474	38	-0.430	-7.74516	Acceptable
Variable 4	95.8	0.715	100	-0.475	-32.536075	Acceptable
Variable 5	100	0.706	84.2	-0.347	-20.6274844	Acceptable

5.1| Future Directions

A neutrosophic scale can be used to confirm the reliability of the classical one because the neutrosophic scale is just an extended form of the classical one. The results show that our neutrosophic scale is also reliable, which also supports the reliability of the classical scale because Cronbach's Alpha constant is an acceptable level for three dimensions. In this respect, it can be understood the Agreement dimension of reliability because the classical scale can be extended into the neutrosophic one and assess the closeness of two measurements made on the same subject as opposed to one another. The repeatability of the scale can be also assessed because the same variable can be measured again and again for the same circumstances [9]. The reproducibility of the scale can be also tested because the variations in test results can also be tested while tests are performed on subjects on different occasions.

Validity simply means "measure what is intended to be measured" [13]. To decide whether a scale is valid or not, its validity can be compared by comparing similar scales or decisions based on expert opinion can be made. In this study, it is offered an alternative method for developing a valid scale where first the scale is converted into a neutrosophic one and then they are compared through neural networks. It can be inferred that any scale to assess how appropriate and representative the measurements collected are for the desired assessment purpose so that its content validity can be evaluated. It can bee can understood how well a concept, idea, or behavior is translated or transformed that is a construct into a functioning and operating reality, the operationalization [14] on any scale so that its construct validity can be understood. This method can also be used for criterion validity because how well one measure predicts another measure can also be calculated.

This research is limited by Three-Valued Logic but it can be extended higher n-valued logics as well. It is limited by classical statistics such as correlation or neural networks but neutrosophic statistics can be also used or the whole data. It is limited by investigating the validity in terms of neutrosophy but this research can be extended into more broader concepts in education. Additionally, more sophisticated formulas can be also developed for subsequent analysis.

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Neutrosophic Dynamic Set

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A.A. Salama, K.F. Alhasan, H. A. Elagamy, Florentin Smarandache (2021). Neutrosophic Dynamic Set. *Neutrosophic Knowledge* 3, 5

Abstract: In this paper, we introduced the concept of the dynamic set according to modern logic, is neutrosophic logic. We study the neutrosophic dynamic set according to time and random variable depended on dynamic set. Neutrosophic dynamic is a dynamic analysis of a sequence of data through of time. It used in many problems in life such as a mathematical statistic, philosophy, medicine, engineering. Some examples and notes are presented.

Keywords: Neutrosophic Dynamic Set, Neutrosophic, Crisp Set, Dynamic Set

1. Introduction

Usually, the neutrosophic set used in available to us information has some indeterminacy [1] and for this, its extensions have become widely applied in almost areas, such as decision-making [6,4], clustering analysis[2], image processing [5], etc. However, in some complex problems in real- life, data may be collected from a different time that needs dynamic decision making for such situations. The term 'dynamic' can be is a series of decisions required to reach a target or the condition that dependent taking of decision and the state of problems. In this paper, we consider dynamic Neutrosophic according to time. The time of the employees 'arrival to their place of work, the follow-up of the students' arrival at their universities Patient care, and record the development of all health changes within a specified time.

Neutrosophic set [1]

The part function (indeterminacy function) that Smarandache (1999) added to intuitionistic fuzzy sets and it is called Neutrosophic Sets. This theory is a robust generalization of the classic set theory, fuzzy set theory by Zadeh, 1965, intuitionistic fuzzy set theory by Atanassov, 1986. Neutrosophic sets present a new part called "indeterminacy" differently from other fuzzy sets, and this part makes meaning more information than other approaches (Wen & Cheng, 2013).[9]

A neutrosophic set contains three parameters (parts), which are: truthiness (T), indeterminacy (I), and falsity (F). Truthiness and falsity correspond to membership (μ) and non-membership (μ^-) in intuitionistic fuzzy sets. Indeterminacy means that decision-makers assess for a decided indicating neutral idea [3].

Concepts of Neutrosophic sets

2. Neutrosophy set

Let A be a set in universal set U, represent A by $\mu_A(x)$, a truth membership function, $\mu_A(x)$: $X \to]0^-, 1^+[$, $I_A(x)$, an indetermanced membership function, $I_A(x)$: $X \to]0^-, 1^+[$ and $\mu_{\tilde{A}}(x)$ a falls membership function, $\mu_{\tilde{A}}{}_A(x) : X \to]0^-, 1^+[$, all these functions are real standard or nonstandard subsets of $]0^-, 1^+[$, where X is non empty set [1, 10].

Let Ω is a neutrosophic sample space that contains some or all of the data that are indeterminacy for the neutrosophic experiment. Then we can define Neutrosophic random variable X is a function defined on Ω.

This function may contain the undetermined in a domain or codomain of function, denoted by $X: \Omega \to$ any values (can be real or indeterminate values), that is, if $u \in \Omega$ then X(u) is equal to me or real number.

3. Dynamic Neutrosophic set

Let $0 \neq T$, A is neutrosophic set , we will define A with respect to time t , such that t belong to T, T > 0 as follows:

$DA_t = \{\mu_A(t), I_A(t), \mu_{\tilde{A}}(t); t \in T\}$, this DA_t, is called dynamic neutrosophic set according to time t.

In the field of technolog

The dynamic neutrosophic class can be defined as

Dynamic neutrosophic data sets are a way of narrowing the number of choices with three degress a user can make on a form field. By narrowing a user's choices, they can enter data faster and more efficiently. You can also use dynamic neutrosophic data sets as a way of eliminating fields that are not necessary for specific situations.

Dynamic neutrosophic data sets are governed by a master element that dictates what some fields in the set will show and how others will behave. Data sets are considered "dynamic" because the values of the elements in the set change, depending on what the user chooses in the master element field. Dynamic data sets work with pull-down lists and radio buttons.

4. Dynamic Neutrosophic random variable

Consider Ω is neutrosophic sample space as T, such that T= (t_1, t_2, .., t_n), where t_i is equal to interval or real number or set or indeterminacy.

Define the Neutrosophic random variable X with respect to t, X(t), such that $X: \Omega \to]0^-, 1^+[$ or I .

Some examples of dynamic Neutrosophic

Example 1:-

If the time to arrival students to university between [7:30 -8:30], we can represent the interval of time according to time dynamically as follow:

Computed numbers of the students who arrive at this time [7:30 - 8:30] surly,

computed numbers of the students who not the arrival at this time [7:30 - 8:30] and

computed numbers of the students whose time arrivals are not determined at this time [7:30 - 8:30].

In other words, we can represent the students which arrival at this time [7:30 - 8:30] surly by X(t);

Represented the students who not the arrival at this time [7:30 - 8:30] by Y(t);

Represent of the students who time arrivals are not determined at [7:30 - 8:30] by Z(t).

Now, if suppose the number of students who came through this time is 50%, the number of students who did not arrive at this time 30%, and the number of students who arrive not determined at [7:30 - 8:30] are 20%. Thus, we can study define dynamical of arrival students according to this time as: $DA_t = \{X_A(t), Z_A(t), Y_A(t), ; t \in T\}$, = {50%, 20%, 30%} Such that A represent the arrival students.

Remark

In the above example, if to need to study according to t more precisely, where t = [7:30 - 8:30], in this case using the exponential distribution for all cases, that is study X(t) by exponential distribution and for Y(t), and Z(t), too .

Example:-2

Assuming we have a set of people, we want to know whether they have had a virus COVID-19 test during a specific time for three months since we can identify people who have an infection or immunity to this virus.

In this case, we consider the set of people as follow:

Let A is the neutrosophic set, some of the peoples are tested denoted by$X_A(t)$, some peoples are not tested, denoted by $Y_A(t)$ and other people are undefined who tested or not tested $Z_A(t)$ (that is: error of test, unknown who test or not, data of their not identified).

Let $DA_t = \{X_A(t), Z_A(t), Y_A(t), ; t \in T\}$ and T= [0 day - 90 day]

Such that, $X_A(t)$ represent the person who tests;

$Y_A(t)$ represent the person who not test;

$Z_A(t)$ represent the person who doesn't know about the test.

If, $X_A(t)$ =24%;

$Y_A(t)$=55%;

$Z_A(t)$ =67%

Then $DA_t = \{ 24\%, 55\%, 67\% \}$ and T= [0 day - 90 day]

In some data, if suppose number the person who tests 30%, if suppose number the person who does not test 70%, if suppose number the person who does not know about test 60%. Then $DA_t = \{30\%, 70\%, 60\%\}$ and T= [0 day - 20 day].

Example:-3

The following represent the neutrosophic dynamic data structure for Security A=ASL (NDS), B=KCR (NDS), C=PKI (NDS) and M=A∨B∨C

No.Nodes	A=ASL(NDS)	B=KCR(NDS)	C=PKI(NDS)	M=A∨B∨C
25	<0.026, 0.034, 0.94>	<0.95, 0.93, 0.07>	<0.15, 0.85, 0.15>	<0.95, 0.034, 0.15>
50	<0.021, 0.036, 0.943>	<0.021, 0.036, 0.943>	<0.2, 0.85, 0.15>	<0.2, 0.036, 0.15>
75	<0.025, 0.038, 0.937>	<0.95, 0.85, 0.15>	<0.23, 0.85, 0.15>	<0.95, 0.038, 0.15>
100	<0.022, 0.038, 0.939>	<0.96, 0.92, 0.08>	<0.26, 0.92, 0.08>	<0.96, 0.038, 0.08>
125	<0.015, 0.004, 0.981>	<0.96, 0.93, 0.07>	<0.3, 0.93, 0.07>	<0.96, 0.004, 0.07>
150	<0.017, 0.004, 0.979>	<0.96, 0.94, 0.06>	<0.32, 0.94, 0.06>	<0.96, 0.004, 0.06>
175	<0.014, 0.004, 0.982>	<0.96, 0.94, 0.06>	<0.36, 0.94, 0.06>	<0.95, 0.004, 0.06>
200	<0.023, 0.004, 0.973>	<0.96, 0.94, 0.06>	<0.4, 0.94, 0.06>	<0.96, 0.004, 0.06>
225	<0.02, 0.004, 0.976>	<0.02, 0.004, 0.976>	<0.44, 0.94, 0.06>	<0.44, 0.004, 0.06>
250	<0.015, 0.004, 0.981>	<0.96, 0.94, 0.06>	<0.45, 0.94, 0.06>	<0.96, 0.004, 0.06>

5. Discussion

Neutrosophic dynamic is an important technique of study the problem according to time in topology, choice, dynamical for some functions, and particularly in mathematical statistics. Neutrosophic dynamic is a dynamic analysis of a sequence of data according to time. Its employment in many problems in life such as a mathematical statistic, philosophy, medicine, and engineering.

In this paper, we defined this technique and it can use in the analysis of many problems by exponential distribution and distribution with the prior conjugate.

6. Conclusion and results

1. In this paper we were able to introduce a new concept of the neutrosophic technique is called a dynamic neutrosophic set, this concept is very important to applied in many phenomena in life .
2. The dynamic neutrosophic set is used in analysis dynamic according to time
3. Application to explain some problems in statistics, choice, topology.

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Decision-Making Problems under the Environment of m-Polar Diophantine Neutrosophic N-Soft Set

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ABSTRACT

Fuzzy models are present everywhere from natural to artificial structures, embodying the dynamic processes in physical, biological, and social systems. As real-life problems are often uncertain on account of inconsistent and indeterminate information, it seems very demanding for an expert to solve those problems using a fuzzy model. In this regard, we develop a hybrid new model m-polar Diophantine neutrosophic N-soft set which is based on neutrosophic set and soft set. Additionally, we define several different sorts of compliments on the proposed set. A proposed set is a generalized form of fuzzy, soft, Pythagorean fuzzy, Pythagorean fuzzy soft, and Pythagorean fuzzy N-soft sets. In this manner, m-polar Diophantine neutrosophic N-soft set is more proficient, a versatile model to oversee vulnerabilities as it likewise survives the downsides of existing models which are to be summed up. Furthermore, we give the application of the proposed set in multi-attribute decision-making problems by defining a new choice-value function.

KEYWORDS

Neutrosophic set; soft set; N-soft set; m-polar diophantine neutrosophic N-soft set; decision making

1 Introduction

The idea of a set and set theory are incredible assets in arithmetic. Shockingly, a non-condition basic set theory for example that the component can either have a place in a set or not, is frequently not appropriate in genuine a daily existence where numerous unclear terms as “enormous benefit,” “high pressing factor,” “moderate temperature,” “dependable instruments,” “safe conditions,” and so forth are broadly utilized. Tragically, such loose depictions cannot be sufficiently taken care of by ordinary mathematical tools.

In fuzzy theory, a recently characterized model by and large beats the downsides of recently characterized models. Because of uncertainty and weaknesses issues in numerous days by day life issues, routine math is not continuously accessible. To manage such issues, different methods such as the theory of possibility, rough set assumption, and fuzzy set theory has been considered as elective models and to keep away from weaknesses too. Inopportunely, the greater part of the options such as science have their own disadvantages and downsides. For example, a large portion of the words like expert, amazing, best, significant is most certainly not quantifiable and uncertain. The rules for words like superb, best, famous, and so forth, hesitate from individual to person.

To deal with such sort of equivocal and unsure data, Zadeh [1] investigated the idea of fuzzy set which is mapping from a universal set X to [0, 1]. Atanassov [2] proposed the fortuitous of intuitionistic fuzzy sets as an expansion of fuzzy sets by presenting the idea of membership and non-membership grades. Molodtsov [3] began the thought of soft set as a mathematical model to oversee vulnerabilities. The chance of soft set has another objective for the researchers due to them utilizes in a wide range of exuberant issues.

Ali et al. [4] presented some new operations on soft set theory. They introduced the ideas of expanded and restricted union and intersections in detail. In [5–7], Yager introduced and investigated several relations on Pythagorean fuzzy set. Peng et al. [8] discussed certain results on Pythagorean fuzzy sets and also defined the Pythagorean fuzzy number in. Peng et al. [9] set up some Pythagorean fuzzy data measures and their applications. Peng et al. [10] proposed some new approaches to manage single-regarded neutrosophic MADM reliant on MABAC, TOPSIS, and new closeness measure with score function. In [11–21], many decision-making problems and algebraic structures are discussed over different fuzzy environments.

Smarandache's introduced neutrosophic set and then proposed many operation on it [22–24]. Neutrosophic set (NS) based on three parameters namely, membership, indeterminacy, and non-membership. Wang et al. [25] propsed the concept of single valued neutrosophic sets. Deli et al. [26] introduced the idea of bipolar neutrosophic set and their applications in multi-criteria decision making problems in. Fatimah et al. [22] introduced the notion of an N-soft set which is an extension of a soft set. Many problems related to decision-making are discussed by using different kind of environments in [27–30].

There are many problems regarding decision-making that need to improveby investigating a new set or model. In this regard, we develop a proposed setthat provides a more batter approximation than existing sets. The proposed work is arranged as follows. In Section 2, some preliminary concepts are given to understand the proposed work. In Section 3, we define the notion of m-polar Diophantine neutrosophic N-soft set and then define some operations on it. In Section 4, we discuss different types of compliments on the proposed set. We give the comparison table and application in multi-attribute decision-making problems in Section 5.

2 Preliminaries

In this section, we give some preliminary concepts related to previous existing sets.

Definition 2.1. [1] A fuzzy set is a mapping μ from universal set X to [0, 1] such that, $\mu: X \rightarrow [0,1]$. The fuzzy set can be written in the form of

$$F_X = \{(x, \mu_F(x)) : x \in X\}.$$

Definition 2.2. [2] An intuitionistic fuzzy set on universal set X is defined as

$$\mathcal{I}_X = \{(x, \mu_I(x), \nu_I(x)) : x \in I\},$$

where $\mu_I: X \to [0,1]$ and $\nu_I: X \to [0,1]$ are the membership and non-membership functions, respectively.

Definition 2.3. [31] The m-polar fuzzy set on a universal set M is a mapping $\mu: M \to [0,1]^m$ and $m(X)$ is the collection of all m-polar fuzzy set on M.

Definition 2.4. [3] The soft set is defined by the set valued mapping $\varphi: \mathcal{I} \to 2^T$, where $\mathcal{I}$ denotes the set of parameters and 2^T is the power set of T. The soft set can be written as,

$$\varphi_{\mathcal{I}} = (\varphi, \mathcal{I}) = \left\{(t, \varphi(t)) : t \in \mathcal{I}, \varphi(t) \in 2^T\right\}.$$

Definition 2.5. [32,33] A fuzzy soft set is defined as

$$\Gamma_S = \{(t, \gamma_S(t)) : t \in T, \gamma_S \in \mathcal{F}(L)\},$$

where $\gamma_S: T \to \mathcal{F}(L)$ and $\mathcal{F}(L)$ is the collection of all fuzzy sets on L and T is the set of parameters with $S \subseteq T$.

Definition 2.6. [34] Let X be the crisp set. Intuitionistic fuzzy soft set (IFSS) is interpreted by multi-valued mapping $\psi: B \to IF^X$, where IF^X represents the collection of all IF-subsets defined over crisp set X (where $B \subseteq X$). Thus the IFSS can be expressed as

$$\mho_B = \left\{(e, \psi_B(e)) : e \in X, \psi_B \in IF^X\right\}.$$

Definition 2.7. [5] Let X be the crisp set. A *Pythagorean fuzzy set* (PFS) can be expressed as

$$P = \left\{< \rho, \mu_P(\rho), \nu_P(\rho) >: 0 \le \mu_P^2(\rho) + \nu_P^2(\rho) \le 1, \rho \in X\right\},$$

where $\mu_P: X \to [0,1]$ and $\nu_P: X \to [0,1]$ with the condition that $0 \le \mu_P^2(\rho) + \nu_P^2(\rho) \le 1$, is known as the degrees of membership and non-membership of $\rho \in X$ to the set P.

Definition 2.8. [35] The score function and accuracy function of Pythagorean fuzzy number $\alpha = (\mu_\alpha, \nu_\alpha)$ over X is defined as, $S(\gamma) = \mu_\gamma^2 - \nu_\gamma^2$ and $Q(\gamma) = \mu_\gamma^2 + \nu_\gamma^2$, with S–$1 \le \gamma \le 1$ and $0 \le Q(\gamma) \le 1$.

Definition 2.9. [36] The ranking function of Pythagorean fuzzy number $\gamma = (\mu_\gamma, \nu_\gamma)$ over X is defined as

$$R(\gamma) = \frac{1}{2} + r_\gamma \left(\frac{1}{2} - \frac{2\theta_\gamma}{\pi}\right),$$

where $r_\gamma = \sqrt{\mu_\gamma^2 + \nu_\gamma^2}$ is called commitment strength and θ_γ is the angle between r_γ and μ_γ. The direction of commitment d_γ is $d_\gamma = 1 - \frac{2\theta_\gamma}{\pi}$, where $\mu_\gamma = r_\gamma cos\theta_\gamma$, $\nu_\gamma = r_\gamma sin\theta_\gamma$.

Definition 2.10. [37] Let X be a non-empty universal set, S be the set of attributes, and $Y \subseteq S$. Let $D = \{0, 1, 2, \ldots, N - 1\}$ be set of grading. The triple (F_p, Y, N) is said to be a Pythagorean fuzzy N-soft set on X, if F_p is a mapping $F_p: Y \to 2^{X \times D} \times PFN$, in which $F: Y \to 2^{X \times D}$, and $P: Y \to PFN$, where PFN is Pythagorean fuzzy number. That is $\mu: Y \to [0,1]$ and $\nu: Y \to [0,1]$ such that

$$0 \le \mu_y^2(x) + \nu_y^2(x) \le 1, \quad \forall y \in Y, \quad \forall x \in X.$$

Hence,

$\left(F_p, Y, N\right) = \left(\left(x, d_y\right), \left(\mu_y(x), \nu_y(x)\right)\right), \quad d_y \in D.$

Definition 2.11. [38] A neutrosophic fuzzy set (NS), S over the universal set X is defined as $S = \{(\psi, \mu_S(\psi), \lambda_S(\psi), \nu_S(\psi))\}$,

where mappings μ_S, λ_S, ν_S stand for degree of truth, degree of indeterminacy, degree of falsity. $\mu_S, \lambda_S, \nu_S \in [0, 1]$, with $0 \leqslant \mu_S + \lambda_S + \nu_S \leqslant 3$.

3 m-Polar Diophantine Neutrosophic N-Soft Set

Definition 3.1 Let $\mathfrak{L} = \{0, 1, 2, \ldots, N - 1\}$ be the set of grades where $N \in \{2, 3, 4, \ldots\}$. If X is a non-emty set and E is the family of attributes. Let A be a non-empty subset of E. A *m-poler Diophantine neutrosophic N-soft* (MPDNNS) set on X is denoted as $(\mho, A, m, N)$ or $\mho_A^{(m,N)}$, where $\mho\colon A \to \mathbf{P}\left(PF^X \times \mathfrak{L}\right)$ is a mapping (where PF^X is the aggregate of all Diophantine neutrosophic subsets over X). That is

$$(\mho, A, m, N) = \left\{\left(e, \left\{\frac{\langle \rho, \iota_e(\rho)\rangle}{(\mu_1(\rho), \mu_2(\rho), \ldots, \mu_m(\rho); \lambda_1(\rho), \lambda_2(\rho), \ldots, \lambda_m(\rho); \nu_1(\rho), \nu_2(\rho), \ldots, \nu_m(\rho)}\right\}\right) \mid e \in A, \rho \in X, \iota_e(\rho) \in \mathfrak{L}\right\},$$

where $\mu_e\colon X \to [0, 1]^m$, $\nu_e\colon X \to [0, 1]^m$, and $\lambda_e\colon X \to [0, 1]^m$ are mappings along with the property,

$$0 \leq \sum_{i=1}^{m} \mu_i^m(\rho) + \sum_{i=1}^{m} \nu_i^m(\rho) + \sum_{i=1}^{m} \lambda_i^m(\rho) \leq 3m.$$

In particular, $\mu_i(\rho)$ represents the truth-membership, $\nu_i(\rho)$ denotes degree of falsity-membership, $\lambda_i(\rho)$ is the degree of indeterminacy and $\iota_e(\rho)$ denotes the grading value of the element $\rho \in X$ corresponding to the attribute $e \in A$ to the set $(\mho, A, m, N)$. If we write $a_{ij} = \mu_{ei}(\rho_j)$, $b_{ij} = \nu_{e_i}\left(\rho_j\right)$, $d_{ij} = \lambda_{ei}\ (\rho_j)$, and $c_{ij} = \iota_{e_j}\left(\rho_j\right)$ where i runs from 1 to m and j runs from 1 to n then the MPDNNS set $\mho_A^{(m,N)}$ may be represented in tabular form as

$\mho_A^{(m,N)}$	e_1	e_2	$\cdots$	e_m
ρ_1	$\langle c_{11}, (a_{11}, a_{21}, \cdots, a_{m1};$	$\langle c_{12}, (a_{11}, a_{21}, \cdots, a_{m1};$		$\langle c_{1m}, (a_{11}, a_{21}, \cdots, a_{m1};$
	$d_{11}, d_{21}, \cdots, d_{m1};$	$d_{11}, d_{21}, \cdots, d_{m1};$	$\cdots$	$d_{11}, d_{21}, \cdots, d_{m1};$
	$b_{11}, b_{21}, \cdots, b_{m1})\rangle$	$b_{11}, b_{21}, \cdots, b_{m1})\rangle$	$\cdots$	$b_{11}, b_{21}, \cdots, b_{m1})\rangle$
ρ_2	$\langle c_{21}, (a_{12}, a_{22}, \cdots, a_{m2};$	$\langle c_{22}, (a_{12}, a_{22}, \cdots, a_{m2};$		$\langle c_{2m}, (a_{12}, a_{22}, \cdots, a_{m2};$
	$d_{12}, d_{22}, \cdots, d_{m2};$	$b_{12}, b_{22}, \cdots, b_{m2};$	$\cdots$	$b_{12}, b_{22}, \cdots, b_{m2};$
	$b_{12}, b_{22}, \cdots, b_{m2})\rangle$	$b_{12}, b_{22}, \cdots, b_{m2})\rangle$	$\cdots$	$b_{12}, b_{22}, \cdots, b_{m2})\rangle$
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
ρ_n	$\langle c_{n1}, (a_{1n}, a_{2n}, \cdots, a_{mn};$	$\langle c_{n2}, (a_{1n}, a_{2n}, \cdots, a_{mn};$		$\langle c_{nm}, (a_{1n}, a_{2n}, \cdots, a_{mn};$
	$d_{1n}, d_{2n}, \cdots, d_{mn};$	$d_{1n}, d_{2n}, \cdots, d_{mm};$	$\cdots$	$d_{1n}, d_{2n}, \cdots, d_{mn};$
	$b_{1n}, b_{2n}, \cdots, b_{mn})\rangle$	$b_{1n}, b_{2n}, \cdots, b_{mm})\rangle$	$\cdots$	$b_{1n}, b_{2n}, \cdots, b_{mn})\rangle$

and in matrix form as

$$(\mho, A, m, N) = \left[\langle c_{ij}, (a_{ij}, d_{ij}, b_{ij})\rangle\right]_{n\times m}$$

$$= \begin{pmatrix} \langle c_{11}, (a_{11}, a_{21}, \cdots, a_{m1}; & \langle c_{12}, (a_{11}, a_{21}, \cdots, a_{m1}; & & \langle c_{1m}, (a_{11}, a_{21}, \cdots, a_{m1}; \\ d_{11}, d_{21}, \cdots, d_{m1}; & d_{11}, d_{21}, \cdots, d_{m1}; & \cdots & d_{11}, d_{21}, \cdots, d_{m1}; \\ b_{11}, b_{21}, \cdots, b_{m1})\rangle & b_{11}, b_{21}, \cdots, b_{m1})\rangle & \cdots & b_{11}, b_{21}, \cdots, b_{m1})\rangle \\ \langle c_{21}, (a_{12}, a_{22}, \cdots, a_{m2}; & \langle c_{22}, (a_{12}, a_{22}, \cdots, a_{m2}; & & \langle c_{2m}, (a_{12}, a_{22}, \cdots, a_{m2}; \\ d_{12}, d_{22}, \cdots, d_{m2}; & d_{12}, d_{22}, \cdots, d_{m2}: & \cdots & d_{12}, d_{22}, \cdots, d_{m2}; \\ b_{12}, b_{22}, \cdots, b_{m2})\rangle & b_{12}, b_{22}, \cdots, b_{m2})\rangle & \cdots & b_{12}, b_{22}, \cdots, b_{m2})\rangle \\ \vdots & \vdots & \ddots & \vdots \\ \langle c_{n1}, (a_{1n}, a_{2n}, \cdots, a_{mn}; & \langle c_{n2}, (a_{1n}, a_{2n}, \cdots, a_{mn}; & & \langle c_{nm}, (a_{1n}, a_{2n}, \cdots, a_{mn}; \\ d_{1n}, d_{2n}, \cdots, d_{mn}; & d_{1n}, d_{2n}, \cdots, d_{mm}; & \cdots & d_{1n}, d_{2n}, \cdots, d_{mn}; \\ b_{1n}, b_{2n}, \cdots, b_{mn})\rangle & b_{1n}, b_{2n}, \cdots, b_{mm})\rangle & \cdots & b_{1n}, b_{2n}, \cdots, b_{mn})\rangle \end{pmatrix}$$

This matrix is called *m-Polar Diophantine neutrosophic N-Soft matrix* or shortly MPDNNS matrix.

Note: We use the expression $x^n + y^n = z^n$ in Definition 3.1, which is similar to Diophantine equation. Thats why we call m-polar Diophantine neutrosophic N-soft set instead of m-polar neutrosophic N-soft set.

Definition 3.2. An MPDNNS set $\mho_A^{(m,N)}$ over X is known as *null MPDNNS set*, symbolized as $\mho_{\phi;}^{(m,0)}$ and defined as

$$\mho_\phi^{(m,0)} = \left\{\left(e, \left\{\frac{\langle \rho, \mathfrak{l}_\phi(\rho)\rangle}{(\mu_{\phi_1}(\rho), \mu_{\phi_2}(\rho), \cdots, \mu_{\phi_m}(\rho); \lambda_{\phi_1}(\rho), \lambda_{\phi_2}(\rho), \cdots, \lambda_{\phi_m}(\rho); \nu_{\phi_1}(\rho), \nu_{\phi_2}(\rho), \cdots, \nu_{\phi_m}(\rho)}\right\}\right) \right.$$

$$\left. | e \in A, \rho \in X, \mathfrak{l}_\phi(\rho) \in \mathfrak{L}\right\},$$

where, $\mu_{\phi_i}(\rho) = 0$, $\lambda_{\phi_i}(\rho) = 1$, $\nu_{\phi_i}(\rho) = 1$, $1 \leq i \leq m$, and $\mathfrak{l}_\phi(\rho) = 0$.

Definition 3.3. An MPDNNS set $\mho_A^{(m,N)}$ over X is known as *absolute MPDNNS set*, symbolized as $\mho_E^{(m,N-1)}$ and defined as

$$\mho_E^{(m,N-1)} = \left\{\left(e, \left\{\frac{\langle \rho, \mathfrak{l}_E(\rho)\rangle}{\left(\mu_{E_1}(\rho), \mu_{E_2}(\rho), \cdots, \mu_{E_m}(\rho); \nu_{E_1}(\rho)\right), \nu_{E_2}(\rho), \cdots, \nu_{E_m}(\rho)}\right\}\right) : \right.$$

$$\left. e \in A, \rho \in X, \mathfrak{l}_E(\rho) \in \mathfrak{L}\right\},$$

where $\mu_{E_i}(\rho) = 1$, $\lambda_{E_i}(\rho) = 0$, $\nu_{E_i}(\rho) = 0$, $1 \leq i \leq m$, and $\mathfrak{l}_\phi(\rho) = N - 1$.

Example 3.1. Let $X = \{\rho_1, \rho_2, \rho_3, \rho_4, \rho_5, \rho_6, \rho_7\}$, $E = \{e_1, e_2, e_3\}$ and $\mathfrak{L} = \{0, 1, 2, \ldots, 5\}$. Suppose that $A = \{e_1, e_3\}$. Then

$$\begin{aligned}\mho_A^{(3,6)} = \{&(e_1, \{\langle\rho_1, 3, (.3, .4, .5; .5, .4, .3; .1, .6, .4)\rangle, \quad \langle\rho_4, 4, (.4, .3, .1; .4, .4, .2; .3, .4, .5)\rangle, \\ &\langle\rho_7, 5, (.6, .2, .55; .3, .1, .2; .4, .3, .5)\rangle\}), \quad (e_3, \{\langle\rho_2, 2, (.2, .1, .7; .6, .6, .5; .8, .4, .6)\rangle, \\ &\langle\rho_3, 1, (.5, .6, .7; .7, .5, .4; .4, .3, .1)\rangle, \quad \langle\rho_5, 2, (.4, .6, .2; .7, .3, .5; .7, .5, .3)\rangle\})\}\end{aligned}$$

is a 3PDN6S set over X. The tabular representation of $\mho_A^{(3,6)}$ is

$\mho_A^{(3,6)}$	e_1	e_2	e_3
ρ_1	$\langle 3, (.3, .4, .5; .5, .4, .3; .1, .6, .4)\rangle$	$\langle 0, (0, 0, 0; 1, 1, 1; 1, 1, 1)\rangle$	$\langle 0, (0, 0, 0; 1, 1, 1; 1, 1, 1)\rangle$
ρ_2	$\langle 0, (0, 0, 0; 1, 1, 1; 1, 1, 1)\rangle$	$\langle 0, (0, 0, 0; 1, 1, 1; 1, 1, 1)\rangle$	$\langle 2, (.2, .1, .7; .6, .6, .5; .8, .4, .6)\rangle$
ρ_3	$\langle 0, (0, 0, 0; 1, 1, 1; 1, 1, 1)\rangle$	$\langle 0, (0, 0, 0; 1, 1, 1; 1, 1, 1)\rangle$	$\langle 1, (.5, .6, .7; .7, .5, .4; .4, .3, .1)\rangle$
ρ_4	$\langle 4, (.4, .3, .1; .4, .4, .2; .3, .4, 0.5)\rangle$	$\langle 0, (0, 0, 0; 1, 1, 1; 1, 1, 1)\rangle$	$\langle 0, (0, 0, 0; 1, 1, 1; 1, 1, 1)\rangle$
ρ_5	$\langle 0, (0, 0, 0; 1, 1, 1)\rangle$	$\langle 0, (0, 0, 0; 1, 1, 1; 1, 1, 1)\rangle$	$\langle 2, (.4, .6, .2; .7, .3, .5; .7, .5, .3)\rangle$
ρ_6	$\langle 0, (0, 0, 0; 1, 1, 1; 1, 1, 1)\rangle$	$\langle 0, (0, 0, 0; 1, 1, 1; 1, 1, 1)\rangle$	$\langle 0, (0, 0, 0; 1, 1, 1; 1, 1, 1)\rangle$
ρ_7	$\langle 5, (.6, .2, .55; .3, .1, .2; .4, .3, 0.5)\rangle$	$\langle 0, (0, 0, 0; 1, 1, 1; 1, 1, 1)\rangle$	$\langle 0, (0, 0, 0; 1, 1, 1; 1, 1, 1)\rangle$

The corresponding 3PDN6S matrix is $(\mho, A, 3, 6) = [a_{ij}, d_{ij}, b_{ij}]_{7\times 3}$

$$= \begin{pmatrix} \langle 3, (.3, .4, .5; .5, .4, .3; .1, .6, .4)\rangle & \langle 0, (0, 0, 0; 1, 1, 1; 1, 1, 1)\rangle & \langle 0, (0, 0, 0; 1, 1, 1; 1, 1, 1)\rangle \\ \langle 0, (0, 0, 0; 1, 1, 1; 1, 1, 1)\rangle & \langle 0, (0, 0, 0; 1, 1, 1; 1, 1, 1)\rangle & \langle 2, (.2, .1, .7; .6, .6, .5; .8, .4, .6)\rangle \\ \langle 0, (0, 0, 0; 1, 1, 1; 1, 1, 1)\rangle & \langle 0, (0, 0, 0; 1, 1, 1; 1, 1, 1)\rangle & \langle 1, (.5, .6, .7; .7, .5, .4; .4, .3, .1)\rangle \\ \langle 4, (.4, .3, .1; .4, .4, .2; .3, .4, 0.5)\rangle & \langle 0, (0, 0, 0; 1, 1, 1; 1, 1, 1)\rangle & \langle 0, (0, 0, 0; 1, 1, 1; 1, 1, 1)\rangle \\ \langle 0, (0, 0, 0; 1, 1, 1; 1, 1, 1)\rangle & \langle 0, (0, 0, 0; 1, 1, 1; 1, 1, 1)\rangle & \langle 2, (.4, .6, .2; .7, .3, .5; .7, .5, .3)\rangle \\ \langle 0, (0, 0, 0; 1, 1, 1; 1, 1, 1)\rangle & \langle 0, (0, 0, 0; 1, 1, 1; 1, 1, 1)\rangle & \langle 0, (0, 0, 0; 1, 1, 1; 1, 1, 1)\rangle \\ \langle 5, (.6, .2, .55; .3, .1, .2; .4, .3, .5)\rangle & \langle 0, (0, 0, 0; 1, 1, 1; 1, 1, 1)\rangle & \langle 0, (0, 0, 0; 1, 1, 1; 1, 1, 1)\rangle \end{pmatrix}$$

Definition 3.4. Let $\mho_A^{(m,N_1)}$ and $\mho_B^{(m,N_2)}$ be MPDNNS sets over X. Then $\mho_A^{(m,N_1)}$ is *MPDNNS subset* of $\mho_B^{(m,N_2)}$, i.e., $\mho_A^{(m,N_1)} \widetilde{\subseteq} \mho_B^{(m,N_2)}$, if

- $A \subseteq B$,
- $N_1 \leq N_2$,
- $\mu_{i,A}(\rho) \leq \mu_{i,B}(\rho)$, $1 \leq i \leq m$,
- $\lambda_{i,A}(\rho) \geq \lambda_{i,B}(\rho)$, $1 \leq i \leq m$,
- $\nu_{i,A}(\rho) \geq \nu_{i,B}(\rho)$, $1 \leq i \leq m$, and
- $\mathfrak{l}_A(\rho) \leq \mathfrak{l}_B(\rho)$.

It is worth mentioning that $\mho_A^{(m,N_1)} \widetilde{\subseteq} \mho_B^{(m,N_2)}$ it is not necessary each element of $\mho_A^{(m,N_1)}$ is also in ${\mho_B}^{(m,N_2)}$.

Definition 3.5. Let $\mho_A^{(m,N_1)}$ and $\mho_B^{(m,N_2)}$ be MPDNNS sets over X. Then $\mho_A^{(m,N_1)}$ is *MPDNNS equal* of $\mho_B^{(m,N_2)}$ i.e., $\mho_A^{(m,N_1)} \cong \mho_B^{(m,N_2)}$, if

- $A = B$,
- $N_1 = N_2$,
- $\mu_{i,A}(\rho) = \mu_{i,B}(\rho),\ 1 \le i \le m$,
- $\lambda_{i,A}(\rho) = \lambda_{i,B}(\rho),\ 1 \le i \le m$,
- $\nu_{i,A}(\rho) = \nu_{i,B}(\rho),\ 1 \le i \le m$, and
- $\mathfrak{l}_A(\rho) = \mathfrak{l}_B(\rho)$.

Proposition 3.1. If $\mho_A^{(m,N)}$ is any MPDNNS set over X, then

(i) $\mho_\phi^{(m,0)} \widetilde{\subseteq} \mho_A^{(m,N)}$,

(ii) $\mho_A^{(m,N)} \widetilde{\subseteq} \mho_E^{(m,N-1)}$.

Proof.

(i) The truth-membership, degree of indeterminacy and degree of falsity always fall in [0, 1] according to Definition 3.1 of MPDNNS set. So, $0 \le \mu_{i,A}(\rho)$, $\nu_{i,A}(\rho) \le 1$, $\lambda_{i,A}(\rho) \le 1$, and grading value $0 \le \mathfrak{l}_A(\rho)$, $\forall \rho \in X$ and $1 \le i \le m$.

Thus, it follows from Definitions 3.4 and 3.2 of null MPDNNS set $\mho_\phi^{(m,0)}$. Hence, $\mho_\phi^{(m,0)} \widetilde{\subseteq} \mho_A^{(m,N)}$.

(ii) Clearly, $\mu_{i,A}(\rho) \le 1$, $\nu_{i,A}(\rho) \ge 0$ and $\nu_{i,A}(\rho) \ge 0$ and grading value $\mathfrak{l}_A(\rho) \le N-1$ for all $\rho \in X$ and $1 \le i \le m$. Thus, it follows from Definitions 3.4 and 3.3 of absolute MPDNNS set. Hence, $\mho_A^{(m,N)} \widetilde{\subseteq} \mho_E^{(m,N-1)}$.

Proposition 3.2. If $\mho_A^{(m,N)}$, $\mho_B^{(m,N)}$ and $\mho_C^{(m,N)}$ are MPDNNS sets over X, then

(i) $\mho_A^{(m,N)} \widetilde{\subseteq} \mho_A^{(m,N)}$.

(ii) $\mho_A^{(m,N)} \widetilde{\subseteq} \mho_B^{(m,N)}$ and $\mho_B^{(m,N)} \widetilde{\subseteq} \mho_C^{(m,N)} \Rightarrow \mho_A^{(m,N)} \widetilde{\subseteq} \mho_C^{(m,N)}$.

Definition 3.6. Let $\mho_A{}^{(m,N_1)}$ and $\mho_B{}^{(m,N_2)}$ be MPDNNS sets over X. Then their *extended union* is defined as $\mho_C^{(m,N)^*} = \mho_A^{(m,N_1)} \widetilde{\cup}_E \mho_B^{(m,N_2)}$,

$$\mho_C^{(m,N)^*} = \left\{ \left(e, \left\{ \frac{\langle \rho, \mathfrak{l}_C(\rho) \rangle}{(\mu_{1,C}(\rho), \mu_{2,C}(\rho), \cdots, \mu_{m,C}(\rho); \lambda_{1,C}(\rho), \lambda_{2,C}(\rho) \cdots, \lambda_{m,C}(\rho); \nu_{1,C}(\rho), \nu_{2,C}(\rho) \cdots, \nu_{m,C}(\rho)} \right\} \right) \middle| e \in C, \rho \in X, \mathfrak{l}_C(\rho) \in \mathfrak{L} \right\}$$

where,

- $C = A \cup B$,
- $(m,\ N)^* = (m,\ \max\{N_1,\ N_2\})$,

- $\mu_C(\rho) = \max\{\mu_{i,A}(\rho),\ \mu_{i,B}(\rho)\},\ 1 \le i \le m,$
- $\lambda_C(\rho) = \min\{\lambda_{i,A}(\rho),\ \lambda_{i,B}(\rho)\}\ 1 \le i \le m,$
- $\nu_C(\rho) = \min\{\nu_{i,A}(\rho),\ \nu_{i,B}(\rho)\}\ 1 \le i \le m,$ and
- $\mathfrak{l}_C(\rho) = \max\{\mathfrak{l}_A(\rho),\ \mathfrak{l}_B(\rho)\}\ \forall\ e \in C.$

Definition 3.7. Let $\mho_A^{(m,N_1)}$ and $\mho_B^{(m,N_2)}$ be MPDNNS sets over X. Then their *restricted union* is defined as $\mho_D^{(m,N)^\diamond} = \mho_A^{(m,N_1)} \widetilde{\cup}_R \mho_B^{(m,N_2)}$,

$$\mho_D^{(m,N)^\diamond} = \left\{ \left(e, \left\{ \frac{\langle \rho, \mathfrak{l}_D(\rho) \rangle}{(\mu_{1,D}(\rho), \mu_{2,D}(\rho), \cdots, \mu_{m,D}(\rho); \lambda_{1,D}(\rho), \lambda_{2,D}(\rho) \cdots, \lambda_{m,D}(\rho); \nu_{1,D}(\rho), \nu_{2,D}(\rho) \cdots, \nu_{m,D}(\rho)} \right\} \right) \mid e \in D, \rho \in X, \mathfrak{l}_D(\rho) \in \mathfrak{L} \right\}$$

where,

- $D = A \cap B,$
- $(m, N)^\diamond = (m,\ \max\{N_1,\ N_2\}),$
- $\mu_D(\rho) = \max\{\mu_{i,A}(\rho),\ \mu_{i,B}(\rho)\},\ 1 \le i \le m,$
- $\lambda_D(\rho) = \min\{\lambda_{i,A}(\rho),\ \lambda_{i,B}(\rho)\}\ 1 \le i \le m,$
- $\nu_D(\rho) = \min\{\nu_{i,A}(\rho),\ \nu_{i,B}(\rho)\}\ 1 \le i \le m,$ and
- $\mathfrak{l}_D(\rho) = \max\{\mathfrak{l}_A(\rho),\ \mathfrak{l}_B(\rho)\}\ \forall\ e \in D.$

Example 3.2. Let $X = \{\rho_i\!: i = 1, 2, \ldots, 8\}$, $E = \{e_i\!: i = 1, 2, 3\}$ and $\mathfrak{L} = \{0, 1, 2, \ldots, 12\}$. Assume that $A = \{e_1, e_3\} \subseteq E$ and $B = \{e_2, e_3\} \subseteq E$. Then

$$\begin{aligned}\mho_A^{(3,13)} = \{&(e_1, \{\langle \rho_2, 11, (.1, .9, .7; .5, .3.2; .4, .6, .8)\rangle, \langle \rho_5, 9, (.2, .1, .2; .3, .4, .5; .6, .7, .8)\rangle,\\ &\langle \rho_7, 8, (.3, .4, .5; .6, .7, .6; .5, .4, .3)\rangle\}), (e_3, \{\langle \rho_3, 3, (.4, .3, .4; .5, .6, .8; .7, .6, .5)\rangle,\\ &\langle \rho_4, 6, (.5, .4, .5; .6, .7, .7; .6, .4, .2)\rangle, \langle \rho_6, 5, (.6, .5, .6; .7, .8, .9; .8, .7, .6)\rangle\})\}\end{aligned}$$

is a 3PDN13S set over X. The corresponding $\mho_A^{(3,13)}$ matrix is

$$\mho_A^{(3,13)} = \left[\langle c_i, \left(a_{ij}; d_{ij}; b_{ij}\right)\rangle\right]_{8\times 3}$$

$$= \begin{pmatrix} \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 11,(.1,.9,.7;.5,.3,0.2;.4,.6,.8)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 3,(.4,.3,.4;.5,.6,.8;.7,.6,.5)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 6,(.5,.4,.5;.6,.7,.7;.6,.4,.2)\rangle \\ \langle 9,(.2,.1,.2;.3,.4.5;.6,.7,.8)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 5,(.6,.5,.6;.7,.8.9;.8,.7,.6)\rangle \\ \langle 8,(.3,.4,.5;.6,.7,.6;.5,.4,.3)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \end{pmatrix} \tag{1}$$

$$\mho_B^{(3,10)} = \{(e_2,\{\langle\rho_1,9,(.7,.6,.4,:.2,.1,.4;.3,.5,.7)\rangle,\langle\rho_4,6,(.8,.6,.5;.6,.4,.7;.5.8,.9)\rangle, \langle\rho_8,8,(.1,.3,.4;.8,.7,.4;.7,.3,.1)\rangle\}),(e_3,\{\langle\rho_2,1,(.2,.1,.3;.6,.7,.9;.7,.3,.6)\rangle, \langle\rho_5,5,(.1,.2,.4;.8,.7,.1;.8,.3,.7)\rangle,\langle\rho_7,3,(.2,.3,.6;.9,.8,.7;.8,.7,.5)\rangle\})\}$$

is a 3PDF10S set over X. The corresponding $\mho_B^{(3,10)}$ matrix is

$$\mho_B^{(3,10)} = [\langle c_i,(a_{ij};b_{ij})\rangle]_{8\times 3}$$

$$\mho_B^{(3,10)} = \begin{pmatrix} \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 9,(.7,.6,.4,:.2,.1,.4;.3,.5,.7)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 1,(.2,.1,.3;.6,.7,.9;.7,.3,.6)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 6,(.8,.6,.5;.6,.4,.7;.5,.8,.9)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 5,(.1,.2,.4;.8,.7,.1;.8,.3,.7)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 3,(.2,.3,.6;.9,.8,.7;.8,.7,.5)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 8,(.1,.3,.4;.8,.7,.4;.7,.3,.1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \end{pmatrix} \tag{2}$$

Extended union of $\mho_A^{(3,13)}$ and $\mho_B^{(3,10)}$ is given below in Eq. (3):

$\mho_C^{(3,13)} = \mho_A^{(3,13)} \widetilde{\cup}_E \mho_B^{(3,10)}$ where $C = \{e_1, e_2, e_3\}$

$$= \begin{pmatrix} \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 9,(.7,.6,.4,:.2,.1,.4;.3,.5,.7)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 11,(.1,.9,.7;.5,.3,.2;.4,.6,.8)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 1,(.2,.1,.3;.6,.7,.9;.7,.3,.6)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 3,(.4,.3,.4;.5,.6,.8;.7,.6,.5)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 6,(.8,.6,.5;.6,.4,.7;.5.8,.9)\rangle & \langle 6,(.5,.4,.5;.6,.7,.7;.6,.4,.2)\rangle \\ \langle 9,(.2,.1,.2;.3,.4.5;.6,.7,.8)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 5,(.1,.2,.4;8,.7,.1;.8,.3,.7)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 5,(.6,.5,.6;.7,.8.9;.8,.7,.6)\rangle \\ \langle 8,(.3,.4,.5;.6,.7,.6;.5,.4,.3)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 3,(.2,.3,.6;.9,.8,.7;.8,.7,.5\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 8,(.1,.3,.4;.8,.7,.4;.7,.3,.1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \end{pmatrix} \tag{3}$$

Restricted union of $\mho_A^{(3,13)}$ and $\mho_B^{(3,10)}$ is given below in Eq. (4)

$\mho_D^{(3,13)} = \mho_A^{(3,13)} \widetilde{\cup}_R \mho_B^{(3,10)}$ where $D = \{e_3\}$

$$= \begin{pmatrix} \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 1,(.2,.1,.3;.6,.7,.9;.7,.3,.6)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 3,(.4,.3,.4;.5,.6,.8;.7,.6,.5)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 6,(.5,.4,.5;.6,.7,.7;.6,.4,.2)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 5,(.1,.2,.4;8,.7,.1;.8,.3,.7)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 5,(.6,.5,.6;.7,.8.9;.8,.7,.6)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 3,(.2,.3,.6;.9,.8,.7;.8,.7,.5\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \end{pmatrix} \tag{4}$$

Definition 3.8. Let $\mho_A^{(m,N_1)}$ and $\mho_B^{(m,N_2)}$ be MPDFNS set over X. Then their *extended intersection* is defined as

$\mho_C^{(m,N)*} = \mho_A^{(m,N_1)} \widetilde{\cap}_E \mho_B^{(m,N_2)}$,

$$\mho_C^{(m,N)*} = \left\{ \left(e, \left\{ \frac{\langle \rho, \mathfrak{l}_C(\rho) \rangle}{(\mu_{1,C}(\rho), \mu_{2,C}(\rho), \cdots, \mu_{m,C}(\rho); \nu_{1,C}(\rho), \nu_{2,C}(\rho) \cdots, \nu_{m,C}(\rho)} \right\} \right) : \right.$$

$$\left. e \in C, \rho \in X, \mathfrak{l}_C(\rho) \in \mathfrak{L} \right\}$$

where,

- $C = A \cup B$,
- $(m,\ N)^* = (m,\ \min\{N_1,\ N_2\})$,
- $\mu_C(\rho) = \min\{\mu_{i,A}(\rho),\ \mu_{i,B}(\rho)\},\ 1 \le i \le m$,
- $\nu_C(\rho) = \max\{\nu_{i,A}(\rho),\ \nu_{i,B}(\rho)\}\ 1 \le i \le m$, and
- $\mathfrak{l}_C(\rho) = \min\{\mathfrak{l}_A(\rho),\ \mathfrak{l}_B(\rho)\}\ \forall\, e \in C$.

Definition 3.9. Let $\mho_A^{(m,N_1)}$ and $\mho_B^{(m,N_2)}$ be MPDFNS set over X. Then their *restricted intersetion* is defined as

$$\mho_D^{(m,N)_\diamond} = \mho_A^{(m,N_1)} \widetilde{\cup}_R \mho_B^{(m,N_2)},$$

$$\mho_D^{(m,N)_\diamond} = \left\{ \left(e, \left\{ \frac{\langle \rho, \mathfrak{l}_D(\rho) \rangle}{(\mu_{1,D}(\rho), \mu_{2,D}(\rho), \cdots, \mu_{m,D}(\rho); \nu_{1,D}(\rho), \nu_{2,D}(\rho) \cdots, \nu_{m,D}(\rho)} \right\} \right) : e \in D, \rho \in X, \mathfrak{l}_D(\rho) \in \mathfrak{L} \right\}$$

where,

- $D = A \cap B$,
- $(m, N)^\diamond = (m, \min\{N_1, N_2\})$,
- $\mu_D(\rho) = \min\{\mu_{i,A}(\rho), \mu_{i,B}(\rho)\}, 1 \le i \le m$,
- $\nu_D(\rho) = \max\{\nu_{i,A}(\rho), \nu_{i,B}(\rho)\}$ $1 \le i \le m$, and
- $\mathfrak{l}_D(\rho) = \min\{\mathfrak{l}_A(\rho), \mathfrak{l}_B(\rho)\}$ $\forall e \in D$.

Example 3.3. Consider $\mho_A^{(3,13)}$ and $\mho_B^{(3,10)}$ as given in Example 3.2 by Eqs. (1) and (2), respectively. Extended intersection of $\mho_A^{13}$ and $\mho_B^{10}$ is given below in Eq. (5):

$\mho_C^{(3,10)} = \mho_A^{(3,13)} \widetilde{\cap}_E \mho_B^{(3,10)}$ where $C = \{e_1, e_2, e_3\}$

$$\mho_C^{(3,10)} = \begin{pmatrix} \langle 0, (0,0,0;1,1,1;1,1,1) \rangle & \langle 0, (0,0,0;1,1,1;1,1,1) \rangle & \langle 0, (0,0,0;1,1,1;1,1,1) \rangle \\ \langle 0, (0,0,0;1,1,1;1,1,1) \rangle & \langle 0, (0,0,0;1,1,1;1,1,1) \rangle & \langle 0, (0,0,0;1,1,1;1,1,1) \rangle \\ \langle 0, (0,0,0;1,1,1;1,1,1) \rangle & \langle 0, (0,0,0;1,1,1;1,1,1) \rangle & \langle 0, (0,0,0;1,1,1;1,1,1) \rangle \\ \langle 0, (0,0,0;1,1,1;1,1,1) \rangle & \langle 0, (0,0,0;1,1,1;1,1,1) \rangle & \langle 0, (0,0,0;1,1,1;1,1,1) \rangle \\ \langle 0, (0,0,0;1,1,1;1,1,1) \rangle & \langle 0, (0,0,0;1,1,1;1,1,1) \rangle & \langle 0, (0,0,0;1,1,1;1,1,1) \rangle \\ \langle 0, (0,0,0;1,1,1;1,1,1) \rangle & \langle 0, (0,0,0;1,1,1;1,1,1) \rangle & \langle 0, (0,0,0;1,1,1;1,1,1) \rangle \\ \langle 0, (0,0,0;1,1,1;1,1,1) \rangle & \langle 0, (0,0,0;1,1,1;1,1,1) \rangle & \langle 0, (0,0,0;1,1,1;1,1,1) \rangle \\ \langle 0, (0,0,0;1,1,1;1,1,1) \rangle & \langle 0, (0,0,0;1,1,1;1,1,1) \rangle & \langle 0, (0,0,0;1,1,1;1,1,1) \rangle \end{pmatrix} \quad (5)$$

Restricted intersection of $\mho_A^{(3,13)}$ and $\mho_B^{(3,10)}$ is given below Eq. (6):

$\mho_D^{(3,10)} = \mho_A^{(3,13)} \widetilde{\cap}_R \mho_B^{(3,10)}$ where $D = \{e_3\}$

$$\mho_D^{(3,10)} = \begin{pmatrix} \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \end{pmatrix} \tag{6}$$

Proposition 3.3. If $\mho_A^{(m,N_1)}$, $\mho_B^{(m,N_2)}$, $\mho_C^{(m,N_3)}$ are MPDNFNS set over X, then

(i) $\mho_A^{(m,N_1)} \widetilde{\cap}_E \mho_A^{(m,N_1)} = \mho_A^{(m,N_1)}$.

(ii) $\mho_A^{(m,N_1)} \widetilde{\cup}_E \mho_A^{(m,N_1)} = \mho_A^{(m,N_1)}$.

(iii) $\mho_A^{(m,N_1)} \widetilde{\cap}_E \mho_B^{(m,N_2)} = \mho_B^{(m,N_2)} \widetilde{\cap}_E \mho_A^{(m,N_1)}$.

(iv) $\mho_A^{(m,N_1)} \widetilde{\cup}_E \mho_B^{(m,N_2)} = \mho_B^{(m,N_2)} \widetilde{\cup}_E \mho_A^{(m,N_1)}$.

(v) $\mho_A^{(m,N_1)} \widetilde{\cap}_E \left(\mho_B^{(m,N_2)} \widetilde{\cap}_E \mho_C^{(m,N_3)}\right) = \left(\mho_A^{(m,N_1)} \widetilde{\cap}_E \mho_B^{(m,N_2)}\right) \widetilde{\cap} \mho_C^{(m,N_3)}$.

(vi) $\mho_A^{(m,N_1)} \widetilde{\cup}_E \left(\mho_B^{(m,N_2)} \widetilde{\cup}_E \mho_C\right) = \left(\mho_A^{(m,N_1)} \widetilde{\cup}_E \mho_B^{(m,N_2)}\right) \widetilde{\cup}_E \mho_C^{(m,N_3)}$.

(vii) $\mho_A^{(m,N_1)} \widetilde{\cap}_R \mho_A^{(m,N_1)} = \mho_A^{(m,N_1)}$.

(viii) $\mho_A^{(m,N_1)} \widetilde{\cup}_R \mho_A^{(m,N_1)} = \mho_A^{(m,N_1)}$.

(ix) $\mho_A^{(m,N_1)} \widetilde{\cap}_R \mho_B^{(m,N_2)} = \mho_B^{(m,N_2)} \widetilde{\cap}_R \mho_A^{(m,N_1)}$.

(x) $\mho_A^{(m,N_1)} \widetilde{\cup}_R \mho_B^{(m,N_2)} = \mho_B^{(m,N_2)} \widetilde{\cup}_R \mho_A^{(m,N_1)}$.

(xi) $\mho_A^{(m,N_1)} \widetilde{\cap}_R \left(\mho_B^{(m,N_2)} \widetilde{\cap}_R \mho_C^{(m,N_3)}\right) = \left(\mho_A^{(m,N_1)} \widetilde{\cap}_R \mho_B^{(m,N_2)}\right) \widetilde{\cap} \mho_C^{(m,N_3)}$.

(xii) $\mho_A^{(m,N_1)} \widetilde{\cup}_R \left(\mho_B^{(m,N_2)} \widetilde{\cup}_R \mho_C\right) = \left(\mho_A^{(m,N_1)} \widetilde{\cup}_R \mho_B^{(m,N_2)}\right) \widetilde{\cup}_R \mho_C^{(m,N_3)}$.

Proof. The proof is obvious from Definitions 3.4, 3.5, 3.6, 3.7, 3.8 and 3.9.

Proposition 3.4. If $\mho_A^{(m,N_1)}$ and $\mho_B^{(m,N_2)}$ are MPDNFNS sets over X, then

(i) $\left(\mho_A^{(m,N_1)} \widetilde{\cap}_E \mho_B^{(m,N_2)}\right) \widetilde{\subseteq} \mho_A^{(m,N_1)} \widetilde{\subseteq} \left(\mho_A^{(m,N_1)} \widetilde{\cup}_E \mho_B^{(m,N_2)}\right)$.

(ii) $\left(\mho_A^{(m,N_1)} \widetilde{\cap}_E \mho_B^{(m,N_2)}\right) \widetilde{\subseteq} \mho_B^{(m,N_2)} \widetilde{\subseteq} \left(\mho_A^{(m,N_1)} \widetilde{\cup}_E \mho_B^{(m,N_2)}\right)$.

(iii) $\left(\mho_A^{(m,N_1)} \widetilde{\cap}_R \mho_B^{(m,N_2)}\right) \widetilde{\subseteq} \mho_A^{(m,N_1)} \widetilde{\subseteq} \mho_A^{(m,N_1)} \widetilde{\cup}_E \mho_B^{(m,N_2)}\Big)$.

(iv) $\left(\mho_A^{(m,N_1)} \widetilde{\cap}_R \mho_B^{(m,N_2)}\right) \widetilde{\subseteq} \mho_B^{(m,N_2)} \widetilde{\subseteq} \left(\mho_A^{(m,N_1)} \widetilde{\cup}_E \mho_B^{(m,N_2)}\right)$.

Proof. Proof of this proposition is obvious from Definitions 3.4, 3.6, 3.8 and 3.9.

4 Complements of m-Polar Neutrosophic Diophantine N-Soft Set

Definition 4.1. Let $\mho_A^{(m,N)}$ be MPDNNS sets over X. Then their *weak complement* is represented as $\left(\mho_A^{(m,N)}\right)^{wc}$ and defined by

$$\left(\mho_A^{(m,N)}\right)^{wc} = \left\{\left(e, \left\{\frac{\left\langle \rho, \mathfrak{l}_A^{wc}(\rho)\right\rangle}{\left(\mu_{1,A}^{wc}(\rho), \mu_{2,A}^{wc}(\rho), \cdots, \mu_{m,A}^{wc}(\rho); \lambda_{1,A}^{wc}(\rho), \lambda_{2,A}^{wc}(\rho), \cdots, \lambda_{m,A}^{wc}(\rho); \nu_{1,A}^{wc}(\rho), \nu_{2,A}^{wc}(\rho) \cdots, \nu_{m,A}^{wc}(\rho)\right)}\right\}\right) | e \in A, \rho \in X, \mathfrak{l}_A^{wc}(\rho) \in \mathfrak{L}\right\}$$

where,

- $\mu_{i,A}^{wc}(\rho) = \nu_{i,A}(\rho),\ 1 \leq i \leq m,$
- $\lambda_{i,A}^{wc}(\rho) = 1 - \lambda_{i,A}(\rho),\ 1 \leq i \leq m,$
- $\nu_{i,A}^{wc}(\rho) = \mu_{i,A}(\rho),\ 1 \leq i \leq m,$ and
- $\mathfrak{l}_A^{wc}(\rho) \cap \mathfrak{l}_A(\rho) = \varnothing,\ \forall\ e \in A.$

Example 4.1. Consider $\mho_A^{(3,13)}$ and $\mho_B^{(3,10)}$ as given in Example 3.2 by Eqs. (1) and (2), respectively. Weak complement of $\mho_A^{(3,13)}$ and $\mho_B^{(3,10)}$ are given below in Eqs. (7) and (8):

$$\mho_A^{(3,13)wc} = \begin{pmatrix} \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 10,(.4,.6,.8;.5,.7,0.8;.1,.9,.7)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 5,(.7,.6,.5;.5,.4,.2;.4,.3,.4)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 8,(.6,.4,.2;.4,.3,.3;.5,.4,.5)\rangle \\ \langle 8,(.6,.7,.8;.7,.6,.5;.2,.1,.2)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 7,(.8,.7,.6;.3,.2.1;.6,.5,.6)\rangle \\ \langle 6,(.5,.4,.3;.4,.3,.4;.3,.4,.5)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \end{pmatrix} \quad (7)$$

$$\mho_B^{(3,13)wc} = \begin{pmatrix} \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 8,(.3,.5,.7;.8,.9,.6;.7,.6,.4)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 7,(.7,.3,.6;.4,.3,.1;.2,.1,.3)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 4,(.5,.8,.9;.4,.6,.3;.8,.6,.5)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 6,(.8,.3,.7;.2,.3,.9;.1,.2,.4)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 2,(.8,.7,.5;.1,.2,.3;.2,.3,.6)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 6,(.7,.3,.1;.8,.7,.4;.1,.3,.4)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \end{pmatrix} \tag{8}$$

Remark 4.1 Again taking week complement of $\left(\mho_A^{(3,13)}\right)^{wc}$ and $\left(\mho_B^{(3,10)}\right)^{wc}$ which given above in Eqs. (7), (8), respectively. There compliments are given below in Eqs. (9) and (10), respectively.

$$\left(\left(\mho_A^{(3,13)wc}\right)^{wc}\right) = \begin{pmatrix} \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 6,(.1,.9,.7;.5,.3,0.2;.4,.6,.8)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 4,(.4,.3,.4;.5,.6,.8;.7,.6,.5)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 5,(.5,.4,.5;.6,.7,.7;.6,.4,.2)\rangle \\ \langle 7,(.2,.1,.2;.3,.4.5;.6,.7,.8)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 7,(.6,.5,.6;.7,.8.9;.8,.7,.6)\rangle \\ \langle 7,(.3,.4,.5;.6,.7,.6;.5,.4,.3)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \end{pmatrix} \tag{9}$$

$$\left(\left(\mho_A^{(3,10)wc}\right)^{wc}\right) = \begin{pmatrix} \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 4,(.7,.6,.4,:.2,.1,.4;.3,.5,.7)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 3,(.2,.1,.3;.6,.7,.9;.7,.3,.6)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 7,(.8,.6,.5;.6,.4,.7;.5,.8,.9)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 6,(.1,.2,.4;.8,.7,.1;.8,.3,.7)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 5,(.2,.3,.6;.9,.8,.7;.8,.7,.5)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 9,(.1,.3,.4;.8,.7,.4;.7,.3,.1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \end{pmatrix} \tag{10}$$

Hence from Eqs. (1), (2), (9) and (10). We conclude that $((\mho_A^{(3,13)})^{wc})^{wc} \neq \mho_A^{(3,13)}$ and $((\mho_B^{(3,10)})^{wc})^{wc} \neq \mho_B^{(3,10)}$.

Since null 3PDN10S set is

$$\mho_{\varnothing}^{(3,0)} = \begin{pmatrix} \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \end{pmatrix} \quad (11)$$

Since absolute 3PDN10S set is

$$\mho_{E}^{(3,10)} = \begin{pmatrix} \langle 9,(1,1,1;0,0,0;0,0,0)\rangle & \langle 9,(1,1,1;0,0,0;0,0,0)\rangle & \langle 9,(1,1,1;0,0,0;0,0,0)\rangle \\ \langle 9,(1,1,1;0,0,0;0,0,0)\rangle & \langle 9,(1,1,1;0,0,0;0,0,0)\rangle & \langle 9,(1,1,1;0,0,0;0,0,0)\rangle \\ \langle 9,(1,1,1;0,0,0;0,0,0)\rangle & \langle 9,(1,1,1;0,0,0;0,0,0)\rangle & \langle 9,(1,1,1;0,0,0;0,0,0)\rangle \\ \langle 9,(1,1,1;0,0,0;0,0,0)\rangle & \langle 9,(1,1,1;0,0,0;0,0,0)\rangle & \langle 9,(1,1,1;0,0,0;0,0,0)\rangle \\ \langle 9,(1,1,1;0,0,0;0,0,0)\rangle & \langle 9,(1,1,1;0,0,0;0,0,0)\rangle & \langle 9,(1,1,1;0,0,0;0,0,0)\rangle \\ \langle 9,(1,1,1;0,0,0;0,0,0)\rangle & \langle 9,(1,1,1;0,0,0;0,0,0)\rangle & \langle 9,(1,1,1;0,0,0;0,0,0)\rangle \\ \langle 9,(1,1,1;0,0,0;0,0,0)\rangle & \langle 9,(1,1,1;0,0,0;0,0,0)\rangle & \langle 9,(1,1,1;0,0,0;0,0,0)\rangle \end{pmatrix} \quad (12)$$

Weak complement of $\mho_{\varnothing}^{(3,0)}$ and $\mho_{E}^{(3,10)}$ are given below in Eqs. (13) and (14), respectively.

$$(\mho_{\Phi}^{(3,0)})^{wc} = \begin{pmatrix} \langle 7,(1,1,1;0,0,0;0,0,0)\rangle & \langle 6,(1,1,1;0,0,0;0,0,0)\rangle & \langle 4,(1,1,1;0,0,0;0,0,0)\rangle \\ \langle 1,(1,1,1;0,0,0;0,0,0)\rangle & \langle 8,(1,1,1;0,0,0;0,0,0)\rangle & \langle 7,(1,1,1;0,0,0;0,0,0)\rangle \\ \langle 6,(1,1,1;0,0,0;0,0,0)\rangle & \langle 7,(1,1,1;0,0,0;0,0,0)\rangle & \langle 3,(1,1,1;0,0,0;0,0,0)\rangle \\ \langle 5,(1,1,1;0,0,0;0,0,0)\rangle & \langle 2,(1,1,1;0,0,0;0,0,0)\rangle & \langle 4,(1,1,1;0,0,0;0,0,0)\rangle \\ \langle 7,(1,1,1;0,0,0;0,0,0)\rangle & \langle 8,(1,1,1;0,0,0;0,0,0)\rangle & \langle 6,(1,1,1;0,0,0;0,0,0)\rangle \\ \langle 8,(1,1,1;0,0,0;0,0,0)\rangle & \langle 7,(1,1,1;0,0,0;0,0,0)\rangle & \langle 2,(1,1,1;0,0,0;0,0,0)\rangle \\ \langle 1,(1,1,1;0,0,0;0,0,0)\rangle & \langle 6,(1,1,1;0,0,0;0,0,0)\rangle & \langle 5,(1,1,1;0,0,0;0,0,0)\rangle \end{pmatrix} \quad (13)$$

$$(\mho_E^{(3,10)})^{wc} = \begin{pmatrix} \langle 9,(0,0,0;1,1,1;1,1,1)\rangle & \langle 5,(0,0,0;1,1,1;1,1,1)\rangle & \langle 5,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 1,(0,0,0;1,1,1;1,1,1)\rangle & \langle 3,(0,0,0;1,1,1;1,1,1)\rangle & \langle 9,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 8,(0,0,0;1,1,1;1,1,1)\rangle & \langle 9,(0,0,0;1,1,1;1,1,1)\rangle & \langle 6,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 2,(0,0,0;1,1,1;1,1,1)\rangle & \langle 5,(0,0,0;1,1,1;1,1,1)\rangle & \langle 5,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 7,(0,0,0;1,1,1;1,1,1)\rangle & \langle 4,(0,0,0;1,1,1;1,1,1)\rangle & \langle 7,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 4,(0,0,0;1,1,1;1,1,1)\rangle & \langle 6,(0,0,0;1,1,1;1,1,1)\rangle & \langle 5,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 1,(0,0,0;1,1,1;1,1,1)\rangle & \langle 8,(0,0,0;1,1,1;1,1,1)\rangle & \langle 9,(0,0,0;1,1,1;1,1,1)\rangle \end{pmatrix} \tag{14}$$

From Eqs. (11)–(14), we have $\left(\mho_\varnothing^{(3,0)}\right)^{wc} \neq \mho_E^{(3,10)}$ and $\left(\mho_E^{(3,10)}\right)^{wc} \neq \mho_\varnothing^{(3,0)}$. Hence, we have following result.

Remark 4.2 Let $\mho_A^{(m,N)}$ be MPDNNS sets, $\mho_\varnothing^{(m,0)}$ be null MPDNNS and $\mho_E^{(m,N-1)}$ absolute MPDNNS over X, then following results that hold in crisp set theory but not hold in MPDNNS set theory

(*i*) $(\mho_\varnothing^{(m,0)})^{wc} \neq \mho_E^{(m,N-1)}$.

(*ii*) $(\mho_E^{(m,N-1)})^{wc} \neq \mho_\varnothing^{(m,0)}$.

(*iii*) $((\mho_A^{(m,N)})^{wc})^{wc} \neq \mho_A^{(m,N)}$.

Definition 4.2. Let $\mho_A^{(m,N)}$ be MPDNNS set over X. Then their *top weak complement* is represented as $(\mho_A^{(m,N)})^{twc}$ and defined by

$$(\mho_A^{(m,N)})^{twc} = \left\{\left(e, \left\{\frac{\langle \rho, \mathfrak{l}_A^{twc}(\rho)\rangle}{(\mu_{1,A}^{twc}(\rho), \mu_{2,A}^{twc}(\rho), \cdots, \mu_{m,A}^{twc}(\rho); \lambda_{1,A}^{twc}(\rho), \lambda_{2,A}^{twc}(\rho), \cdots, \lambda_{m,A}^{twc}(\rho); \nu_{1,A}^{twc}(\rho), \nu_{2,A}^{twc}(\rho)\cdots, \nu_{m,A}^{twc}(\rho)}\right\}\right) | e \in A, \rho \in X, \mathfrak{l}_A^{twc}(\rho) \in \mathfrak{L}\right\}$$

where,

- $\mu_{i,A}^{twc}(\rho) = \nu_{i,A}(\rho),\ 1 \leq i \leq m,$
- $\lambda_{i,A}^{twc}(\rho) = 1 - \lambda_{i,A}(\rho),\ 1 \leq i \leq m,$
- $\nu_{i,A}^{twc}(\rho) = \mu_{i,A}(\rho),\ 1 \leq i \leq m,$ and
- $\mathfrak{l}_A^{twc}(\rho) = \begin{cases} N-1, & \text{if } \mathfrak{l}_A(\rho) < N-1, \\ 0, & \text{if } \mathfrak{l}_A(\rho) = N-1, \forall e \in A. \end{cases}$

The top weak complement of $\mho_A^{(3,13)}$ and $\mho_B^{(3,10)}$ as given in Example 3.2 by Eqs. (1) and (2) are given below in Eqs. (15) and (16), respectively.

$$(\mho_A^{(3,13)})^{twc} = \begin{pmatrix} \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 12,(.4,.6,.8;.5,.7,0.8;.1,.9,.7)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 12,(.7,.6,.5;.5,.4,.2;.4,.3,.4)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 12,(.6,.4,.2;.4,.3,.3;.5,.4,.5)\rangle \\ \langle 12,(.6,.7,.8;.7,.6.5;.2,.1,.2)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 12,(.8,.7,.6;.3,.2.1;.6,.5,.6)\rangle \\ \langle 12,(.5,.4,.3;.4,.3,.4;.3,.4,.5)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \end{pmatrix} \tag{15}$$

$$(\mho_B^{(3,10)})^{twc} = \begin{pmatrix} \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 12,(.3,.5,.7;.8,.9,.6;.7,.6,.4)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 12,(.7,.3,.6;.4,.3,.1;.2,.1,.3)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 12,(.5,.8,.9;.4,.6,.3;.8,.6,.5)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 12,(.8,.3,.7;.2,.3,.9;.1,.2,.4)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 12,(.8,.7,.5;.1,.2,.3;.2,.3,.6)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 12,(.7,.3,.1;.8,.7,.4;.1,.3,.4)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \end{pmatrix} \tag{16}$$

Proposition 4.1 Let $\mho_A^{(m,N)}$ be MPDNNS set, $\mho_\varnothing^{(m,0)}$ be null MPDNNS and $\mho_E^{(m,N-1)}$ absolute MPDNNS over X, then

(*i*) $(\mho_\varnothing^{(m,0)})^{twc} = \mho_E^{(m,N-1)}$.

(*ii*) $(\mho_E^{(m,N-1)})^{twc} = \mho_\varnothing^{(m,0)}$.

Remark 4.3. Again taking top weak complement of $\left(\mho_A^{(3,13)}\right)^{twc}$ given in Eq. (15) we have Eq. (17)

$$((\mho_A^{(3,13)})^{twc})^{twc} = \begin{pmatrix} \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(.1,.9,.7;.5,.3,0.2;.4,.6,.8)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(.4,.3,.4;.5,.6,.8;.7,.6,.5)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(.5,.4,.5;.6,.7,.7;.6,.4,.2)\rangle \\ \langle 0,(.2,.1,.2;.3,.4.5;.6,.7,.8)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(.6,.5,.6;.7,.8.9;.8,.7,.6)\rangle \\ \langle 0,(.3,.4,.5;.6,.7,.6;.5,.4,.3)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \end{pmatrix} \tag{17}$$

Hence from Eqs. (1) and (17) we conclude that $\left(\left(\mho_A^{(3,13)}\right)^{twc}\right)^{twc} \neq \mho_A^{(3,13)}$. Hence we have the following result:

Remark 4.4. Let $\mho_A^{(m,N)}$ be MPDNNS set, over X, then $((\mho_A^{(m,N)})^{twc})^{twc} \neq \mho_A^{(m,N)}$.

Definition 4.3. Let $\mho_A^{(m,N)}$ be MPDNNS set over X. Then their *bottom weak complement* is represented as $\left(\mho_A^{(m,N)}\right)^{bwc}$ and defined by

$$(\mho_A^{(m,N)})^{bwc} = \left\{\left(e,\left\{\frac{\langle\rho,\mathfrak{l}_A^{bwc}(\rho)\rangle}{(\mu_{1,A}^{bwc}(\rho),\mu_{2,A}^{bwc}(\rho),\cdots,\mu_{m,A}^{bwc}(\rho);\lambda_{1,A}^{bwc}(\rho),\lambda_{2,A}^{bwc}(\rho),\cdots,\lambda_{m,A}^{bwc}(\rho);\nu_{1,A}^{bwc}(\rho),\nu_{2,A}^{bwc}(\rho)\cdots,\nu_{m,A}^{bwc}(\rho))}\right\}\right) \middle| e\in A,\rho\in X,\mathfrak{l}_A^{bwc}(\rho)\in\mathfrak{L}\right\}$$

where,

- $\mu_{i,A}^{bwc}(\rho) = \nu_{i,A}(\rho),\ 1 \le i \le m,$
- $\lambda_{i,A}^{bwc}(\rho) = 1 - \lambda_{i,A}(\rho),\ 1 \le i \le m,$
- $\nu_{i,A}^{bwc}(\rho) = \mu_{i,A}(\rho),\ 1 \le i \le m$, and
- $\mathfrak{l}_A^{bwc}(\rho) = \begin{cases} N-1, & \text{if } \mathfrak{l}_A(\rho) = 0, \\ 0, & \text{if } \mathfrak{l}_A(\rho) > 0, \forall e \in A. \end{cases}$

Example 4.2. Consider $\mho_A^{(3,13)}$ and $\mho_B^{(3,10)}$ as given in Example 3.2 by Eqs. (1) and (2), respectively. Bottom weak complement of $\mho_A^{(3,13)}$ and $\mho_B^{(3,10)}$ are given below in Eqs. (18) and (19):

$$(\mho_A^{(3,13)})^{bwc} = \begin{pmatrix} \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(.4,.6,.8;.5,.7,0.8;.1,.9,.7)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(.7,.6,.5;.5,.4,.2;.4,.3,.4)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(.6,.4,.2;.4,.3,.3;.5,.4,.5)\rangle \\ \langle 0,(.6,.7,.8;.7,.6.5;.2,.1,.2)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(.8,.7,.6;.3,.2.1;.6,.5,.6)\rangle \\ \langle 0,(.5,.4,.3;.4,.3,.4;.3,.4,.5)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \end{pmatrix} \quad (18)$$

$$(\mho_B^{(3,13)})^{bwc} = \begin{pmatrix} \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(.3,.5,.7;.8,.9,.6;.7,.6,.4)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(.7,.3,.6;.4,.3,.1;.2,.1,.3)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(.5,.8,.9;.4,.6,.3;.8,.6,.5)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(.8,.3,.7;.2,.3,.9;.1,.2,.4)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(.8,.7,.5;.1,.2,.3;.2,.3,.6)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(.7,.3,.1;.8,.7,.4;.1,.3,.4)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \end{pmatrix} \quad (19)$$

Proposition 4.2. Let $\mho_A^{(m,N)}$ be MPDNNS set, $\mho_\varnothing^{(m,0)}$ be null MPDNNS and $\mho_E^{(m,N-1)}$ absolute MPDNNS over X, then

(*i*) $(\mho_\varnothing^{(m,0)})^{bwc} = \mho_E^{(m,N:\, amp:\, minus;1)}$.

(*ii*) $(\mho_E^{(m,N-1)})^{bwc} = \mho_\varnothing^{(m,0)}$.

Remark 4.5. Again taking bottom weak complement of $(\mho_A^{(3,13)})^{bwc}$ as given above in Eq. (18) we have Eq. (20) given below:

$$((\mho_A^{(3,13)})^{bwc})^{bwc} = \begin{pmatrix} \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 12,(.1,.9,.7;.5,.3,0.2;.4,.6,.8)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 12,(.4,.3,.4;.5,.6,.8;.7,.6,.5)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 12,(.5,.4,.5;.6,.7,.7;.6,.4,.2)\rangle \\ \langle 12,(.2,.1,.2;.3,.4.5;.6,.7,.8)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 12,(.6,.5,.6;.7,.8.9;.8,.7,.6)\rangle \\ \langle 12,(.3,.4,.5;.6,.7,.6;.5,.4,.3)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \\ \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle & \langle 0,(0,0,0;1,1,1;1,1,1)\rangle \end{pmatrix} \quad (20)$$

Hence from Eqs. (1) and (20) we conclude that $\left(\left(\mho_A^{(3,13)}\right)^{bwc}\right)^{bwc} \neq \mho_A^{(3,13)}$.

5 Relationships with Existing Models and Application in Multi-Attribute Group Decision-Making

The comparison with existing sets is shown in Tab. 1.

The following proposed algorithm for the choice values of MPDNNSs:

An Algorithm for the Choice Values of MPDNNSs

1: $S = \{\rho_1, \rho_2, \ldots, \rho_m\}$ is the universal set.

2: $T = \{e_1, e_2, \ldots, e_n\}$ set of attributes.

3: Input MPDFNSs $\psi_A^{(m,n)}$ with $N = \{0, 1, 2, \ldots, N - 1\}$.

4: Calculate $M_i^m = \left(\sum_{j=1}^m d_{\alpha ij}, \sum_{j=1}^m R_{a_{ij}}\right)$. Here,

$$d_{a_{ij}} = 1 - \frac{2\theta_a}{\pi}, \quad R_{a_{ij}} = \frac{1}{2} + r_a\left(\frac{1}{2} - \frac{2\theta_a}{\pi}\right),$$

$$r_a = \left(\mu_1^2 + \mu_2^2 + \cdots + \mu_m^2 + \lambda_1^2 + \lambda_2^2 + \cdots + \lambda_m^2 + \nu_1^2 + \nu_2^2 + \cdots + \nu_m^2\right)^{1/2},$$

$$\theta_a = Tan^{-1}\left(\frac{<(\lambda_1, \lambda_2, \cdots, \lambda_m), (\nu_1, \nu_2, \cdots, \nu_m)>}{||(\mu_1, \mu_2, \cdots, \mu_m)||}\right),$$

where, $<(\lambda_1, \lambda_2, \ldots, \lambda_m), (\nu_1, \nu_2, \ldots, \nu_m)> = \lambda_1\nu_1 + \lambda_2\nu_2 + \ldots, \lambda_m\nu_m$.

5: Calculate $M_t^m = max\left\{M_i^m\right\}$ with $i = 1, 2, \ldots, n$.

6: $M_t^m = max\left\{M_i^m\right\}$ can be chosen for any alternative.

Table 1: Comparison of proposed model with existing models

Sets	Truth membership	Falsity membership	Indeterminacy	Parametrization	Non-binary evaluation	Multi-polarity
Fuzzy set	✓	×	×	×	×	×
Intutionistic fuzzy set	✓	✓	×	×	×	×
Neutrosopic set	✓	✓	✓	×	×	×
Soft set	×	×	×	✓	×	×
N-soft set	×	×	×	✓	✓	×
Fuzzy N-soft set	✓	×	×	✓	✓	×
Intutionistic N-soft set	✓	✓	×	✓	✓	×
Neutrosopic N-soft set	✓	✓	✓	✓	✓	×
m-polar fuzzy set	✓	×	×	×	×	✓
m-polar diophantine neutrosopic N-soft set	✓	✓	✓	✓	✓	✓

Example 5.1. We assume that there are five candidates which appear in the interview and their interviews will be completed in three days. There are three judges which are observed to each candidate separately. The following data are elaborated in Tab. 2, of each candidate c_i, $i = 1, 2, 3, 4, 5$ which are observed by from judges j_i, $i = 1, 2, 3$.

Table 2: Data of interview of each candidates observed by judges

$\mho_C^{(3,11)}$	j_1	j_2	j_3
c_1	⟨4, (.3, .2, .5; .2, .7, .8; .1, .4, .6)⟩	⟨6, (.5, .4, .7; .6, .5, .4; .3, .6, .5)⟩	⟨7, (.3, .5, .5; .3, .6, .7; .6, .4, .3)⟩
c_2	⟨6, (.6, .4, .6; .3, .5, .7; .5, .4, .3)⟩	⟨5, (.2, .5, .6; .4, .6, .9; .1, .2, .3)⟩	⟨8, (.2, .6, .5;.5, .8, .6; .3, .1, .4)⟩
c_3	⟨7, (.4, .6, .5; .3, .5, .7; .1, .4, .3)⟩	⟨2, (.4, .5, .3; .5, .6, .2; .5, .4, .5)⟩	⟨3, (.5, .3, .5; .4, .3, .7; .4, .3, .6)⟩
c_4	⟨3, (.5, .3, .4; .4, .5, .6; .3, .2, .5)⟩	⟨8, (.3, .6, .4; .4, .9, .6; .5, .4, .3)⟩	⟨9, (.5, .4, .5; .3, .7, .6; .4, .3, .2)⟩
c_5	⟨1, (.6, .5, .4; .3, .5, .7; .3, .4, .4)⟩	⟨4, (.6, .5, .4; .4, .7, .1; .3, .5, .5)⟩	⟨7, (.4, .3, .5; .2, .5, .7; .4, .6, .4)⟩

In Tab. 3, we find the choice values of all candidates by using proposed algorithm.

From Tab. 4, the candidates c_5, c_2, c_2 are leading with respect to days first, second, and third, respectively.

The working of proposed algorithm is shown in Fig. 1.

Table 3: Tabular representation of choice value of $\psi_C^{(3,11)}$

$\mho_C^{(3,11)}$	j_1	j_2	j_3	M_i^1	M_i^2	M_i^3
c_1	⟨4, (.3, .2, .5; .2, .7, .8; .1, .4, .6)⟩	⟨6, (.5, .4, .7; .6, .5, .4; .3, .6, .5)⟩	⟨7, (.3, .5, .5; .3, .6, .7; .6, .4, .3)⟩	(17, 2.59680)	(17, 1.78017)	(17, 1.64489)
c_2	⟨6, (.6, .4, .6; .3, .5, .7; .5, .4, .3)⟩	⟨5, (.2, .5, .6; .4, .6, .9; .1, .2, .3)⟩	⟨8, (.2, .6, .5;.5, .8, .6; .3, .1, .4)⟩	(19, 2.30871)	(19, 2.36016)	(19, 2.24883)
c_3	⟨7, (.4, .6, .5; .3, .5, .7; .1, .4, .3)⟩	⟨2, (.4, .5, .3; .5, .6, .2; .5, .4, .5)⟩	⟨3, (.5, .3, .5; .4, .3, .7; .4, .3, .6)⟩	(12, 2.39995)	(12, 2.13054)	(12, 1.91588)
c_4	⟨3, (.5, .3, .4; .4, .5, .6; .3, .2, .5)⟩	⟨8, (.3, .6, .4; .4, .9, .6; .5, .4, .3)⟩	⟨9, (.5, .4, .5; .3, .7, .6; .4, .3, .2)⟩	(20, 2.32576)	(20, 2.05034)	(20, 1.89522)
c_5	⟨1, (.6, .5, .4; .3, .5, .7; .3, .4, .4)⟩	⟨4, (.6, .5, .4; .4, .7, .1; .3, .5, .5)⟩	⟨7, (.4, .3, .5; .2, .5, .7; .4, .6, .4)⟩	(12, 2.65388)	(12, 2.08484)	(12, 1.94623)

Table 4: Comparison table of MPDFNSs with pervious models

$\mho_C^{(3,11)}$	$NSS(\sigma_i)$	$FNSS(Q_i)$	$IFNSS(S_i)$	$PFNSS(H_i)$	M_i^1	M_i^2	M_i^3
c_1	17	(17, 0.89)	(17, −0.03)	(17, 1.7462)	(17, 2.59680)	(17, 1.78017)	(17, 1.64489)
c_2	19	(19, 0.79)	(19, 0.09)	(19, 1.6368)	(19, 2.30871)	(19, 2.36016)	(19, 2.24883)
c_3	12	(12, 0.99)	(12, 0.15)	(12, 1.8940)	(12, 2.39995)	(12, 2.13054)	(12, 1.91588)
c_4	20	(20, 1.09)	(20, 0.09)	(20, 1.5705)	(20, 2.32576)	(20, 2.05034)	(20, 1.89522)
c_5	12	(12, 1.22)	(12, 0.54)	(12, 1.9097)	(12, 2.65388)	(12, 2.08484)	(12, 1.94623)

5.1 Relationships with Existing Sets

In this subsection, we establish the relationship with existing sets.

Definition 5.1. Let n be a threshold lies between 0 and N for the level, the PFNSS and PFSS over X associated with $\mho_A^{(m,N)}$ and n, symbolized by $\mho_A^{(n,m,N)}$, defined by, for all $a \in A$,

$$\mho_a^{(n,m,N)} = \begin{cases} \langle x, (\mu(a), \nu(a)) \rangle, & \text{if } m = 1, \\ (x, d_a)(\mu(a), \nu(a)), & \text{if } (x, d_a) \in F(a), and d_a \geq n, \\ (0, 0.5), & \text{if } d_a/N \geq 0.5, \\ (0, 1), & \text{if } d_a/N < 0.5. \end{cases}$$

Definition 5.2. Let k be a threshold with $k \in [-1, 1]$ for the score function, the N-soft over X associated with $\mho_A^{(m,N)}$ and k, symbolized by $\mho_A^{(k,m,N)}$, defined by, for all $a \in A$,

$$\mho_a^{(k,m,N)} = \begin{cases} (x, d_a), & \text{if } (x, d_a) \in F(a), \text{ and } S_a(x) \geq k, m = 1, \\ 1, & \text{if } S_a(x) > 0, m = 1, \\ 0, & \text{if } S_a(x) \leq 0, m = 1. \end{cases}$$

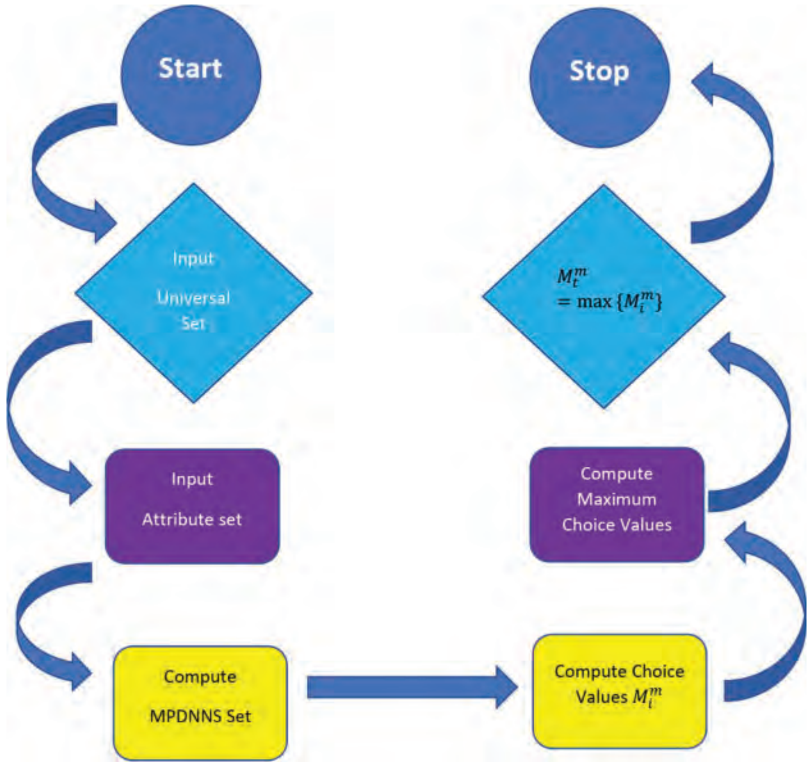

Figure 1: Flow chart of proposed algorithm

Definition 5.3. Let n be a threshold lies between 0 and N and threshold k with $k \in [-1, 1]$ for the score function, the soft over X associated with $\psi_A^{(m,N)}$ and (n, k), symbolized by $\mho_A^{((n,k),m,N)}$, defined by, for all $a \in A$,

$$\mho_A^{((n,k),1,N)} = \left\{x \in X | S_a^{\mho^{(n,1,N)}}(x) > k, \right\}$$

For soft set associated with $\psi_A^{(3,6)}$ and threshold $(n, k) = (3, 0.3)$ is $\mho_A^{((3,0.3),1,6)} = \{\ \}$. From Example 3.1, we have the following outcome which are elaborate in Tab. 5.

Table 5: Pythagorean fuzzy N-soft set associated with $\mho_A^{(3,6)}$ and $m = 1$

$\mho_A^{(1,7)}$	e_1	e_2	e_3
ρ_1	⟨4, (0.3, 0.1)⟩	⟨0, (0, 1)⟩	⟨0, (0, 1)⟩
ρ_2	⟨0, (0, 1)⟩	⟨0, (0, 1)⟩	⟨3, (0.2, 0.8)⟩
ρ_3	⟨0, (0, 1,)⟩	⟨0, (0, 1)⟩	⟨2, (0.5, 0.4)⟩
ρ_4	⟨5, (0.5, 0.3)⟩	⟨0, (0, 1)⟩	⟨0, (0, 1)⟩
ρ_5	⟨0, (0, 1)⟩	⟨0, (0, 1)⟩	⟨2, (0.4, 0.7)⟩
ρ_6	⟨0, (0, 1)⟩	⟨0, (0, 1)⟩	⟨0, (0, 1)⟩
ρ_7	⟨6, (0.6, 0.4)⟩	⟨0, (0, 1)⟩	⟨ 0, (0, 1)⟩

In Tabs. 6 and 7, we deduce Pythagorean fuzzy soft set and N-soft set from MPDFNSs.

Table 6: Pythagorean fuzzy soft set associated with $\mho_A^{(3,6)}$ and threshold $n = 3$

$\mho_A^{(3,1,6)}$	e_1	e_2	e_3
ρ_1	(0.3, 0.1)	(0, 1)	(0, 1)
ρ_2	(0, 1)	(0, 1)	(0.2, 0.8)
ρ_3	(0, 1,)	(0, 1)	(0, 1)
ρ_4	(0.5, 0.3)	(0, 1)	(0, 1)
ρ_5	(0, 1)	(0, 1)	(0, 1)
ρ_6	(0, 1)	(0, 1)	(0, 1)
ρ_7	(0.6, 0.4)	(0, 1)	(0, 1)

Table 7: N-soft set associated with $\mho_A^{(3,6)}$ and threshold $k = 0.3$

$\mho_A^{(0.3,1,6)}$	e_1	e_2	e_3
ρ_1	1	0	0
ρ_2	0	0	0
ρ_3	0	0	1
ρ_4	1	0	1
ρ_5	0	0	0
ρ_6	0	0	0
ρ_7	1	0	0

6 Conclusion

In this paper, we investigate a new set namely the m-polar Diophantine neutrosophic N-soft set which is based on neutrosophic set and soft set. We are discussed different types of compliments on the proposed set and elaborate these compliments with examples. The purposed set is a generalized form of fuzzy, soft, Pythagorean fuzzy, Pythagorean fuzzy soft, and Pythagorean fuzzy N-soft sets. Moreover, as an application, we proposed an algorithm for multi-attribute decision-making problems by defining the new score function. In future work, one can discuss algebraic structures and topological properties on m-polar Diophantine neutrosophic N-soft set. Moreover, ones can develop the concept of m-polar Diophantine neutrosophic N-soft graph and then discuss their properties.

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Quadripartitioned Neutrosophic Pythagorean Soft Set

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Abstract

The aim of this paper is to introduce the new concept of Quadripartitioned Neutrosophic Pythagorean soft set with T, C, U, F as dependent neutrosophic components and have also discussed some of its properties.

Keywords: Neutrosophic pythagorean soft set, Quadripartioned Neutrosophic Pythagorean set and Quadripartioned Neutrosophic Pythagorean soft set .

1.Introduction

The fuzzy set was introduced by Zadeh [19] in 1965. The concept of neutrosophic set was introduced by Smarandache which is a mathematical tool for handling problems involving imprecise, indeterminancy and inconsistent data.

Smarandache introduced neutrosophic sets [14]. In neutrosophic sets, the indeterminacy membership function walks along independently of the truth membership or of the falsity membership. Neutrosophic theory has been widely explored by researchers for application purpose in handling real life situations involving uncertainty. Although the hesitation margin of neutrosophic theory is independent of the truth or falsity membership, looks more general than intuitionistic fuzzy sets yet. Recently, in Atanassov et al. [3] studied the relations between inconsistent intuitionistic fuzzy sets, picture fuzzy sets, neutrosophic sets and intuitionistic fuzzy sets; however, it remains in doubt that whether the indeterminacy associated to a particular element occurs due to the belongingness of the element or the non-belongingness. This has been pointed out by Chattejee et al. [4] while introducing a more general structure of neutrosophic set viz. quadripartitioned single valued neutrosophic set (QSVNS). The idea of QSVNS is actually stretched from Smarandache' s four numerical-valued neutrosophic logic and Belnap' s four valued logic, where the indeterminacy is divided into two parts, namely, “unknown” i.e., neither true nor false and “contradiction” i.e., both true and false. In the context of neutrosophic study however, the QSVNS looks quite logical. Also, in their study, Chatterjee [4] et al. analyzed a real-life example for a better understanding of a QSVNS environment and showed that such situations occur very naturally.

In 2018 Smarandache [17] generalized the soft set to the hyper soft set by transforming the classical uni-argument function F into a multi-argument function.

In 2016, Smarandache [14] introduced for the first time the degree of dependence between the components of fuzzy set and neutrosophic sets. The main idea of Neutrosophic sets is to characterize each value statement in a 3D – Neutrosophic space, where each dimension of the space represents respectively the truth membership, falsity membership and the indeterminacy, when two components T and F are dependent and I is independent then T+I+F≤ 2.

Radha and Stanis Arul Mary [10] introduced the concept of Quadripartitioned neutrosophic pythagorean set with dependent neutrosophic components.

If T and F are dependent neutrosophic pythagorean components then $T^2 + F^2 \leq 1$. Similarly, for U and C as dependent neutrosophic pythagorean components then $C^2 + U^2 \leq 1$. When combining both we get Quadripartitioned pythagorean set with dependent components as $T^2 + F^2 + C^2 + U^2 \leq 2$

Pabitra kumar Maji [9] had combined the neutrosophic set with soft sets and introduced a new mathematical model neutrosophic soft set. Arockiarani [1] introduced the new concept of fuzzy neutrosophic soft set. Yager introduced pythagorean fuzzy sets. Radha and tanis Arul Mary [11] introduced neutrosophic pythagorean soft set with T and F as neutrosophic dependent components.

In this we have to introduce the concept of introduced the concept of quadripartitioned neutrosophic pythagorean set with dependent components and establish some of its properties.

2.Preliminaries

Definition:2.1[14]

Let X be a universe. A neutrosophic set A on X can be defined as follows:

$$A = \{< x, T_A(x), I_A(x), F_A(x) >: x \in X\}$$

Where T_A , $I_A, F_A: U \to [0,1]$ and $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$

Here, $T_A(x)$ is the degree of membership, $I_A(x)$ is the degree of indeterminancy and $F_A(x)$ is the degree of non-membership.

*He*re, $T_A(x)$ and $F_A(x)$are dependent neutrosophic components and $I_A(x)$ is an independent component.

Definition:2.2[2]

Let U be the initial universe set and E be set of parameters. Consider a non-empty set A on E, Let P(U) denote the set of all neutrosophic sets of U. The collection (F, A) is termed to be neutrosophic soft set over U, where F is a mapping given by F: A → P(U).

Definition:2.3[11]

Let X be the initial universe set and E be set of parameters. Consider a non-empty set A on E, Let P(X) denote the set of all neutrosophic pythagorean sets of X. The collection (F, A) is termed to be neutrosophic pythagorean soft set over X, where F is a mapping given by F: A → P(X).

Definition:2.4[4]

Let X be a universe. A Quadripartitioned neutrosophic set A with independent neutrosophic components on X is an object of the form

$$A = \{< x, T_A(x), C_A(x), U_A(x), F_A(x) >: x \in X\}$$

$$and\ 0 \le T_A(x) + C_A(x) + U_A(x) + F_A\ (x) \le 4$$

Here, $T_A(x)$ is the truth membership, $C_A(x)$ is contradiction membership, $U_A(x)$ is ignorance membership and $F_A(x)$ is the false membership.

Definition:2.5[10]

Let X be a universe. A Quadripartitioned neutrosophic pythagorean set A with dependent neutrosophic components A on X is an object of the form

$$A = \{< x, T_A(x), C_A(x), U_A(x), F_A(x) >: x \in X\}$$

Where $T_A + F_A \le 1,\ C_A + U_A \le 1\ and\ 0 \le (T_A(x))^2 + (C_A(x))^2 + (U_A(x))^2 + (F_A\ (x))^2 \le 2$

Here, $T_A(x)$ is the truth membership, $C_A(x)$ is contradiction membership, $U_A(x)$ is ignorance membership and $F_A(x)$ is the false membership.

3.Quadripartitioned Neutrosophic Pythagorean Soft Set (QNPSS or QNPS Set)

Definition:3.1

Let X be the initial universe set and E be set of parameters. Consider a non-empty set A on E, Let P(X) denote the set of all Quadripartitioned neutrosophic pythagorean sets of X. The collection (F, A) is termed to be Quadripartitioned neutrosophic pythagorean soft set over X, where F is a mapping given by F: A → P(X).

Definition:3.2

A Quadripartitioned neutrosophic pythagorean soft set A is contained in another Quadripartitioned neutrosophic pythagorean soft set B (i.e) A$\subseteq B$ if $T_A(x) \le T_B(x), C_A(x) \le C_B(x), U_A(x) \ge U_B(x)$ and $F_A(x) \ge F_B(x)$

Definition:3.3

The complement of a Quadripartitioned neutrosophic pythagorean soft set (F, A) on X denoted by $(F, A)^c$ and is defined as

F^c(x)= $\{< x, F_A(x), U_A\ (x), C_A(x), T_A(x) >: x \in X\}$

Definition:3.4

Let X be a non-empty set, A = $< x, T_A(x), C_A(x), U_A(x), F_A(x) >$ and

B = $< x, T_B(x), C_B(x), U_B(x), F_B(x) >$ are Quadripartitioned neutrosophic pythagorean soft sets. Then

A ∪ B = $< x, max(T_A(x), T_B(x)), max(C_A(x), C_B(x)), min(U_A(x), U_B(x)), min(F_A(x), F_B(x)) >$

A ∩ B = $< x, min(T_A(x), T_B(x)), min(C_A(x), C_B(x)), max(U_A(x), U_B(x)), max(F_A(x), F_B(x)) >$

Definition:3.5

A Quadripartitioned neutrosophic pythagorean soft set (F, A) over the universe X is said to be empty neutrosophic pythagorean soft set with respect to the parameter A if

$T_{F(e)} = 0, C_{F(e)} = 0, U_{F(e)} = 1, F_{F(e)}$=1,$\forall x \in X, \forall e \in A$. It is denoted by 0 X

Definition:3.6

A Quadripartitioned neutrosophic pythagorean soft set (F, A) over the universe X is said to be universe neutrosophic soft set with respect to the parameter A if

$T_{F(e)} = 1, C_{F(e)} = 1, U_{F(e)} = 0, F_{F(e)}$=0,$\forall x \in X, \forall e \in A$. It is denoted by 1_X

Remark: 0 ${}^c{}_X$ = 1_X and 1 ${}^c{}_X$ = 0 X

Definition:3.7

Let A and B be two Quadripartitioned neutrosophic pythagorean soft sets on X then A\B may be defined as

A\B = $< x, min(T_A(x), F_B(x)), min(C_A(x), U_B(x)), max(U_A(x), C_B(x)), max(F_A(x), T_B(x)) >$

Definition:3.8

F_E is said to be absolute Quadripartitioned neutrosophic pythagorean soft set over X if F(e) = 1_X for any

$e \in E$. We denote it by X_E

Definition:3.9

F_E is said to be relative null Quadripartitioned neutrosophic pythagorean soft set over X if F(e) = 0_X for any

$e \in E$. We denote it by $\emptyset_E$

Obviously $\emptyset_E = X_E^c$ and $X_E = \emptyset_E^c$

Definition:3.10

The complement of a Quadripartitioned neutrosophic pythagorean soft set (F, A) over X can also be defined as $(F, A)^c = U_E \backslash F(e)$ for all $e \in A$.

Note: We denote X_E by X in the proofs of proposition.

Definition:3.11

If (F, A) and (G, B) be two Quadripartitioned neutrosophic pythagorean soft set then "(F, A) AND (G, B)" is a denoted by

(F, A) ∧(G, B) and is defined by (F, A)∧ (G, B) = (H, A ×B)

where H (a, b) = F(a) ∩ G(b) $\forall a \in A$ and $\forall b \in B$, where ∩ is the operation intersection of Quadripartitioned neutrosophic pythagorean soft set.

Definition:3.12

If (F, A) and j(G, B) be two Quadripartitioned neutrosophic pythagorean soft set then "(F, A) OR (G, B)" is a denoted by (F, A) ∨(G, B) and is defined by (F, A)∨ (G, B) = (K, A ×B) where K (a, b) = F(a) ∪ G(b) $\forall a \in A$ and $\forall b \in B$, where ∪ is the operation union of Quadripartitioned neutrosophic pythagorean soft set.

Theorem :3.13

Let (F, A) and (G, A) be Quadripartitioned neutrosophic pythagorean soft set over the universe X. Then the following are true.

(i) (F, A)⊆ (G, A) iff (F, A) ∩ (G, A) = (F, A)
(ii) (F, A)⊆ (G, A) iff (F, A) ∪ (G, A) = (F, A)

Proof:

(i)Suppose that (F, A)⊆ (G, A), then F(e) ⊆G(e) for all e ∈ A. Let (F, A) ∩ (G, A) = (H, A).

Since H(e) = F(e) ∩ G(e) =F(e) for all e ∈ A, by definition (H, A) = (F, A).

Suppose that (F, A) ∩ (G, A) = (F, A). Let (F, A) ∩ (G, A) = (H, A).

Since H(e) = F(e) ∩ G(e) =F(e) for all e ∈ A, we know that F(e) ⊆ G(e) for all e ∈ A.

Hence (F, A)⊆ (G, A).

(ii)The proof is similar to (i).

Theorem :3.14

Let (F, A), (G, A), (H, A), and (S, A) are Quadripartitioned neutrosophic pythagorean soft set over the universe X. Then the following are true.

(i) If (F, A) ∩ (G, A) =$\emptyset_A$, then (F, A)⊆ (G, A)^c
(ii) If (F, A)⊆ (G, A) and (G, A)⊆ (H, A) then (F, A)⊆ (H, A)
(iii) If (F, A)⊆ (G, A) and (H, A)⊆ (S, A) then (F, A) ∩ (H, A) ⊆ (G, A) ∩ (S, A)
(iv) (F, A) ⊆ (G, A) iff (G, A)^c ⊆ (F, A)^c

Proof:

(i)Suppose that (F, A) ∩ (G, A) =$\emptyset_A$.Then F(e)∩ G(e) = ∅.

So, F(e) ⊆ U\G(e)= $G^c(e)$ for all e ∈ A.

Therefore, we have (F, A)⊆ (G, A)^c

Proof of (ii) and (iii) are obvious.

(iv) (F, A)⊆ (G, A)⟺ F(e) ⊆ G(e) for all e ∈ A.

⟺ (G(e))^c ⊆ (F(e))^c for all e ∈ A.
⟺ $G^c(e) \subseteq F^c(e)$ $for\ all\ e \in A$.
⟺(G, A)^c ⊆ (F, A)^c

Definition:3.15

Let I be an arbitrary index $\{(F_i, A)\}_{i \in I}$ be a subfamily of Quadripartitioned neutrosophic pythagorean soft set over the universe X.

(i)The union of these Quadripartitioned neutrosophic pythagorean soft set is the Quadripartitioned neutrosophic pythagorean soft set (H, A) where H(e) = $\bigcup_{i \in I} F_i(e)$ for each e $\in A$.

$We\ write\ \bigcup_{i \in I}(F_i, A)$= (H, A)

(ii)The intersection of these Quadripartitioned neutrosophic pythagorean soft set is the Quadripartitioned neutrosophic pythagorean soft set (M, A) where M(e) = $\bigcap_{i \in I} F_i(e)$ for each e $\in A$.

$We\ write\ \bigcap_{i \in I}(F_i, A)$ = (M, A)

Theorem: 3.16

Let I be an arbitrary index set and $\{(F_i, A)\}_{i \in I}$be a subfamily of Quadripartitioned neutrosophic pythagorean soft set over the universe X. Then

(i) $(\bigcup_{i \in I}(F_i, A))^C = \bigcap_{i \in I}(F_i, A)^C$
(ii) $(\bigcap_{i \in I}(F_i, A))^C = \bigcup_{i \in I}(F_i, A)^C$

Proof:

(i) $(\bigcup_{i \in I}(F_i, A))^C$ = (H, A) ^{C}, By definition $H^C(e) = X_E \backslash H(e) = X_E \backslash \bigcup_{i \in I} F_i(e) = \bigcap_{i \in I}(X_E \backslash F_i(e))$ for all e $\in A$.
On the other hand, $(\bigcap_{i \in I}(F_i, A))^C$(K, A).
By definition, K(e)= $\bigcap_{i \in I} F_i^C(e) = \bigcap_{i \in I}(X - F_i(e))$ for all e $\in A$.
(ii) It is obvious.

Note: We denote $\emptyset_E$ by $\emptyset$ and X_E by X.

Theorem: 3.17

Let (F, A) be Quadripartitioned neutrosophic pythagorean soft set over the universe X. Then the following are true.

(i) $(\emptyset, A)^c$ = (X, A)
(ii) $(X, A)^c = (\emptyset, A)$

Proof:

(i) Let $(\emptyset, A)$ = (F, A)

Then $\forall$e $\in$ A,

F(e) = $\{< x, T_{F(e)}(x), C_{F(e)}(x), U_{F(e)}(x), F_{F(e)}(x) >: x \in X\}$

= $\{(x,0,0,1,1): x \in X\}$

$(\emptyset, A)^c = (F, A)^c$

Then $\forall$e $\in$ A,

(F(e) $)^c$ = $\{< x, T_{F(e)}(x), C_{F(e)}(x), U_{F(e)}(x), F_{F(e)}(x) >: x \in X\}^c$

= $\{< x, F_{F(e)}(x), U_{F(e)}(x), C_{F(e)}(x), T_{F(e)}(x) >: x \in X\}$

={(x1,1,0,0): $x \in X\}$ = X

Thus $(\emptyset,A)^c$ = (X, A)

(ii) Proof is similar to (i)

Theorem: 3.18

Let (F, A) be Quadripartitioned neutrosophic pythagorean soft set over the universe X. Then the following are true.

(i) (F, A)∪ (∅, A) = (F, A)

(ii) (F, A)∪ (X ,A) = (X, A)

Proof:

(i) (F, A) = {e, $(x, T_{F(e)}(x), C_{F(e)}(x), U_{F(e)}(x), F_{F(e)}(x)): x \in X\}$ ∀e ∈ A

(∅, A) = {e, (x,0,0,1,1): x ∈ X} ∀e ∈ A

(F, A)∪ (∅, A) = {e, $(x, max(T_{F(e)}(x),0), max(C_{F(e)}(x),0), min(U_{F(e)}(x),1) min(F_{F(e)}(x),1)): x \in X\}$ ∀e ∈ A

= {e, $(x, T_{F(e)}(x), C_{F(e)}(x), U_{F(e)}(x), F_{F(e)}(x)): x \in X\}$ ∀e ∈ A

= (F, A)

(ii) Proof is similar to (i).

Theorem: 3.19

Let (F, A) be Quadripartitioned neutrosophic pythagorean soft set over the universe X. Then the following are true.

(i) (F, A)∩ (∅, A) = (∅, A)

(ii) (F, A)∩ (X, A) = (F, A)

Proof:

(i) (F, A) = {e, $(x, T_{F(e)}(x), C_{F(e)}(x), U_{F(e)}(x), F_{F(e)}(x)): x \in X\}$ ∀e ∈ A

(∅, A) = {e, (x,0,0,1,1): x ∈ X} ∀e ∈ A

(F,A)∩(∅,A)={e, $(x, min(T_{F(e)}(x),0), min(C_{F(e)}(x),0), max(U_{F(e)}(x),1) max(F_{F(e)}(x),1)): x \in X\}$ ∀e∈ A

= {e, , $(x, 0,0,1,1): x \in X\}$ ∀e ∈ A

= (∅, A)

(ii) Proof is similar to (i).

Note: We denote T_F(x), $C_F(x), U_F(x)$ and $F_F(x)$ by T_F $C_F, U_F and\ F_F$

Theorem: 3.20

Let (F, A) and (G, A) be Quadripartitioned neutrosophic pythagorean soft set over the universe X. Then the following are true.

(i) (F, A)∪ (∅, B) = (F, A) iff B⊆A

(ii) (F, A)∪ (X, B) = (X, A) iff A⊆ B

Proof:

(i) We have for (F, A), F(e) = $\{(x, T_F, C_F, U_F, F_F): x \in U\}$ ∀e ∈ A

Also let (∅, B) = (G, B) then

G(e) = {(x, 0,0,1,1): x ∈ X} ∀e ∈ B

Let (F, A)∪ (∅, B) = (F, A)∪ (G, B) = (H, C) where C = A ∪B and for all e ∈ C

H(e) may be defined as

$$= \begin{cases} \{(x, T_{F(e)}, C_{F(e)}, U_{F(e)}, F_{F(e)}): x \in X\}\ if\ e \in A - B \\ \{(x, 0,0,1,1): x \in X\}\ if\ e \in B - A \\ \{(x, max(T_{F(e)}, 0), max(C_{F(e)}, 0), min(U_{F(e)}, 1) min(F_{F(e)}, 1)): x \in X\}\ if\ e \in A \cap B \end{cases}$$

$$= \begin{cases} \{(x, T_{F(e)}, C_{F(e)}, U_{F(e)}, F_{F(e)}): x \in X\}\ if\ e \in A - B \\ \{(x, 0,0,1,1): x \in X\}\ if\ e \in B - A \\ \{(x, T_{F(e)}, C_{F(e)}, U_{F(e)}, F_{F(e)}): x \in X\}\ if\ e \in A \cap B \end{cases}$$

Let B⊆A

$$\text{Then H (e)} = \begin{cases} \{(x, T_{F(e)}, C_{F(e)}, U_{F(e)}, F_{F(e)}(x)): x \in X\}\ if\ e \in A - B \\ \{(x, T_{F(e)}, C_{F(e)}, U_{F(e)}, F_{F(e)}): x \in X\}\ if\ e \in A \cap B \end{cases}$$

= F(e) $\forall e \in A$

Conversely Let $(F, A) \cup (\emptyset, B) = (F, A)$

Then A = A ∪B⟹ $B \subseteq A$

(ii) Proof is similar to (i)

Theorem: 3.21

Let (F, A) and (G, B) be Quadripartitioned neutrosophic pythagorean soft set over the universe X. Then the following are true.

(i) (F, A)∩ (∅, B) = (∅, A∩B)

(ii) (F, A)∩ (X, B) = (F, A∩B)

Proof:

(i) We have for (F, A)

F(e) = $\{(x, T_{F(e)}, C_{F(e)}, U_{F(e)}, F_{F(e)}): x \in X\}$ ∀e ∈ A

Also let (∅, B) = (G, B) then

G(e) = {(x, 0,0,1,1): x ∈ U} ∀e ∈ B

Let (F, A) ∩ (∅,B) = (F, A) ∩(G, B) = (H, C) where C = A ∩B and ∀e ∈ C

H(e) = $\{(x, min(T_{F(e)}, T_{G(e)}), min(C_{F(e)}, C_{G(e)}), max(U_{F(e)}, U_{G(e)}) max(F_{F(e)}, F_{G(e)})): x \in X\}$

= $\{ (x, min(T_{F(e)}, 0), min(C_{F(e)}, 0), max(U_{F(e)}, 1) max(F_{F(e)}, 1)): x \in X\}$

= $\{(x, 0,0,1,1): x \in X\}$

= (G, B) = (∅, B)

Thus (F, A)∩ (∅, B) = (∅, B) =(∅, A∩B)

(ii) Proof is similar to (i).

Theorem: 3.22

Let (F, A) and (G, B) be Quadripartitioned neutrosophic pythagorean soft set over the universe X. Then the following are true.

(i) ((F, A)∪ $(G$, B))^C ⊆ (F, A)^C ∪ (G, B)^C

(ii) (F, A)^C ∩ (G, B)^C ⊆((F, A)∩ $(G$, B))^C

Proof:

Let (F, A)∪(G, B) = (H, C) Where C = A∪B and∀ e∈ C

H(e) may be defined as

$$\begin{cases} \{(x, T_{F(e)}, C_{F(e)}, U_{F(e)}, F_{F(e)}): x \in X\}\ if\ e \in A - B \\ \{(x, T_{G(e)}, C_{G(e)}, U_{G(e)}, F_{G(e)}): x \in X\}\ if\ e \in B - A \\ \{(x, max(T_{F(e)}, T_{G(e)}), max(C_{F(e)}, C_{G(e)}), min(U_{F(e)}, U_{G(e)}), min(F_{F(e)}, F_{G(e)})): x \in X\}\ if\ e \in A \cap B \end{cases}$$

Thus (F,A)∪(G,B))^C = (H,C)^C Where C = A∪B and∀ e∈ C

$$(H(e))^C = \begin{cases} (F(e))^C\ if\ e \in A - B \\ (G(e))^C\ if\ e \in B - A \\ (F(e) \cup G(e))^C if\ e \in A \cap B \end{cases}$$

$$= \begin{cases} \{(x, F_{F(e)}, U_{F(e)}, C_{F(e)}, T_{F(e)}): x \in X\}\ if\ e \in A - B \\ \{(x, F_{G(e)}, U_{G(e)}, C_{G(e)}, T_{G(e)}): x \in X\}\ if\ e \in B - A \\ \{(x, min(F_{F(e)}, F_{G(e)}), min(U_{F(e)}, U_{G(e)}), max(C_{F(e)}, C_{G(e)}), max(T_{F(e)}, T_{G(e)}): x \in X\}\ if e \in A \cap B \end{cases}$$

Again (F, A) ^{C} ∪ (G, B) ^{C} = (I, J) say J = A∪B and ∀ e∈ J

$$I(e) = \begin{cases} (F(e))^C \; if \; e \in A - B \\ (G(e))^C \; if \; e \in B - A \\ (F(e) \cup G(e))^C if \; e \in A \cap B \end{cases}$$

$$= \begin{cases} \{(x, F_{F(e)}, U_{F(e)}, C_{F(e)}, T_{F(e)}) : x \in X\} if \, e \in A - B \\ \{(x, F_{G(e)}, U_{G(e)}, C_{G(e)}, T_{G(e)}) : x \in X\} if \, e \in B - A \\ \{(x, min(F_{F(e)}, F_{G(e)}), min(U_{F(e)}, U_{G(e)}), max(C_{F(e)}, C_{G(e)}), max(T_{F(e)}, T_{G(e)}), max(F_{F(e)}, F_{G(e)})) \\ \quad : x \in X\} \; if \; e \in A \cap B \end{cases}$$

So, C⊆ J ∀ e∈ J, (H(e))^C ⊆I(e)

Thus (F, A)∪(G, B)) ^{C} ⊆ (F, A) ^{C} ∪ (G, B) ^{C}

(ii) Let (F, A)∩(G, B) = (H, C) Where C = A∩B and∀ e∈ C

H(e) = F(e)∩G(e)

$$=\{(x, min(T_{F(e)}, T_{G(e)}), min(C_{F(e)}, C_{G(e)}), max(U_{F(e)}(x), U_{G(e)}(x)), max(F_{F(e)}, F_{G(e)})\}$$

Thus ((F, A)∩(G, B)) ^{C} = (H, C)^C Where C = A∩B and∀ e∈ C

$$(H(e))^C = \{(x, min(T_{F(e)}, T_{G(e)}), min(C_{F(e)}, C_{G(e)}), max(U_{F(e)}, U_{G(e)}), max(F_{F(e)}, F_{G(e)})\}^C$$

$$=\{(x, max(F_{F(e)}, F_{G(e)}), max(U_{F(e)}, U_{G(e)}), min(C_{F(e)}, C_{G(e)}), min(T_{F(e)}, T_{G(e)})\}$$

Again (F, A) ^{C}∩ (G, B) ^{C} = (I, J) say where J= A∩ B and ∀ e∈ J

I(e) = (F(e)) ^{C}∩ (G(e)) ^{C}

$$=\{(x, min(F_{F(e)}, F_{G(e)}), min(U_{F(e)}, U_{G(e)}), max(C_{F(e)}, C_{G(e)}), max(T_{F(e)}, T_{G(e)})\}$$

We see that C = J and ∀ e∈ J, I(e) ⊆(H(e)) ^{C}

Thus (F, A) ^{C} ∩ (G, B) ^{C} ⊆((F, A)∩ $(G$, B))^C

Theorem :3.23

Let (F, A) and (G, A) are two Quadripartitioned neutrosophic pythagorean soft sets over the same universe X. We have the following

(i) ((F, A)∪ $(G$, A)) ^{C} = (F, A) ^{C} ∩ (G, A) ^{C}

(ii) ((F, A)∩ $(G$, A)) ^{C} = (F, A) ^{C}∪ (G, A) ^{C}

Proof:

(i) Let (F, A) ∪ (G, A) = (H, A) ∀ e∈ A

H(e) = F(e)∪G(e)

$=\{(x, max(T_{F(e)}, T_{G(e)}), max(C_{F(e)}, C_{G(e)}), min(U_{F(e)}, U_{G(e)}), min(F_{F(e)}, F_{G(e)})\}$

Thus (F, A) ∪ (G, A))^C = (H, A)^C ∀ e∈ A

(H(e)) ^{C} = (F(e)∪G(e)) ^{C}

$=\{(x, max(T_{F(e)}, T_{G(e)}), max(C_{F(e)}, C_{G(e)}), min(U_{F(e)}, U_{G(e)}), min(F_{F(e)}, F_{G(e)})\}^C$

$=\{(x, min(F_{F(e)}, F_{G(e)}), min(U_{F(e)}, U_{G(e)}), max(C_{F(e)}, C_{G(e)}), max(T_{F(e)}, T_{G(e)})\}$

Again (F, A) ^{C}∩ $(G$, A) ^{C} = (I, A) where$\forall\ e \in\ A$

I(e) = (F(e)) ^{C}∩ (G(e)) ^{C}

$=\{(x, \mathrm{min}(F_{F(e)}, F_{G(e)}), \mathrm{min}(U_{F(e)}, U_{G(e)}), \mathrm{max}(C_{F(e)}, C_{G(e)}), \mathrm{max}(T_{F(e)}, T_{G(e)})\}$

Thus ((F, A)∪ $(G$, A)) ^{C} = (F, A) ^{C} ∩ (G, A) ^{C}

(ii) Let (F, A) ∩ (G, A) = (H, A) ∀ e∈ A

H(e) = F(e)∩G(e)

$=\{(x, min(T_{F(e)}, T_{G(e)}), min(C_{F(e)}, C_{G(e)}), \mathrm{max}(U_{F(e)}, U_{G(e)}) max(F_{F(e)}, F_{G(e)})\}$ ∀ e∈ A

Thus (F, A) ∩ (G, A)) ^{C} = (H, A) ^{C}

(H(e)) ^{C} = (F(e)∩G(e)) ^{C}

$=\{(x, min(T_{F(e)}, T_{G(e)}), min(C_{F(e)}, C_{G(e)}), \mathrm{max}(U_{F(e)}, U_{G(e)}), max(F_{F(e)}, F_{G(e)})\}^C$

$= \{(x, min(T_{F(e)}, T_{G(e)}), max(U_{F(e)}, U_{G(e)}) min(I_{F(e)}(x), I_{G(e)}(x)), max(F_{F(e)}(x), F_{G(e)}(x))\ \}^C$

$=\{(x, max(F_{F(e)}, F_{G(e)}), max(U_{F(e)}, U_{G(e)}), min(C_{F(e)}, C_{G(e)}), min(T_{F(e)}, T_{G(e)})\}\forall\ e \in\ A$

Again (F, A) ^{C}∪ $(G$, A) ^{C} = (I, A) where$\forall\ e \in\ A$

I(e) = (F(e)) ^{C}∪ (G(e)) ^{C}

$=\{(x, max(F_{F(e)}(x), F_{G(e)}(x)), max(1 -\ I_{F(e)}(x), 1 -\ I_{G(e)}(x)), min(T_{F(e)}(x), T_{G(e)}(x))\}$

$=\{(x, max(F_{F(e)}, F_{G(e)}), 1 - min(I_{F(e)}, I_{G(e)}), min(T_{F(e)}, T_{G(e)})$

Thus ((F, A)∩ $(G$, A)) ^{C} = (F, A) ^{C} ∪ (G, A) ^{C}

Theorem: 3.24

Let (F, A) and (G, A) are two neutrosophic pythagorean soft sets over the same universe X. We have the following

(i) ((F, A)∧ $(G$, A)) ^{C} = (F, A) ^{C} ∨ (G, A) ^{C}

(ii) ((F, A)∨ $(G$, A)) ^{C} = (F, A) ^{C}∧(G, A) ^{C}

Proof:

Let (F, A) ∧ (G, B) = (H, A ×B) where H (a, b) =F(a) ∩G(b) $\forall\, a \in A$ and $\forall\, b \in B$ where ∩ is the operation intersection of QNPS.

Thus H (a, b) = F(a) ∩G(b)

$$= \{(x, min(T_{F(a)}, T_{G(b)}, min(C_{F(a)}, C_{G(b)}), max(U_{F(a)}, U_{G(b)}), max(F_{F(a)}, F_{G(b)})\}$$

((F, A) ∧ (G, B)) ^{C} = (H, A ×B) ^{C} $\forall(a, b) \in A \times B$

Thus (H (a, b)) ^{C}=$\{(x, min(T_{F(a)}, T_{G(b)}), min(C_{F(a)}, C_{G(b)}), max(U_{F(a)}, U_{G(b)}), max(F_{F(a)}, F_{G(b)})\}^C$

$$=\{(x, max(F_{F(a)}, F_{G(b)}), max(U_{F(a)}, U_{G(b)}), min(C_{F(a)}, C_{G(b)}), min(T_{F(a)}, T_{G(b)})\}$$

Let (F, A) ^{C} ∨ (G, A) ^{C} = (R, A ×B) where R (a, b) =(F(a)) ^{C} ∪ (G(b)) ^{C} $\forall\, a \in A$ and $\forall\, b \in B$ where ∪ is the operation union of NPSS.

R (a, b) = $\{(x, max(F_{F(a)}, F_{G(b)}), max(U_{F(a)}, U_{G(b)}), min(C_{F(a)}, C_{G(b)}), min(T_{F(a)}, T_{G(b)})\}$

Hence ((F, A)∧ $(G$, A)) ^{C} = (F, A) ^{C} ∨ (G, A) ^{C}. Similarly we can prove (ii).

5. Conclusion

In this paper, we have introduced the idea of Quadripartitioned neutrosophic pythagorean soft set with dependent neutrosophic compnents and discussed some of its properties.We have put forward some theorems based on this new notion. In future, this paper will leads us to develop QNPS topological space. Further, we can study on QNPSS to carry out a general framework for this application in day today life.

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Some New Classes of Neutrosophic Minimal Open Sets

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Abstract: This article focuses on N_m-β-open, β-interior and β-closure operators using neutrosophic minimal structures. We investigate properties of such concepts and we introduced the concepts of N_m-β-continuous, N_m-β-closed graph, N_m-β-compact and almost N_m-β-compact. Finally, we introduced the concepts of N_m-regular-open sets and N_m-π-open sets and investigate some properties.

Key words: N_m-β-continuous, N_m-β-closed graph, N_m-β-compact, almost N_m-β-compact, N_m-regular-open and N_m-π-open

1. Introduction

Zadeh's [17] Fuzzy set laid the foundation of many fields such as intuitionistic fuzzy, neutrosophic set, rough sets. Later, researchers developed K. T. Atanassov's [4] intuitionistic fuzzy set theory in many fields such as differential equations, topology, computerscience and so on. F. Smarandache [15, 16] found that some objects have indeterminacy or neutral other than membership and non-membership. So he coined the notion of neutrosophy. V. Popa & T. Noiri [12] introduced the notions of minimal structure which is a generalization of a topology on a given nonempty set. We introduced the concepts of $\mathcal{M}$-continuous maps. M. Karthika et al [11] studied neutrosophic minimal structure spaces. S. Ganesan and F. Smarandache [9] studied N_m-semi-open in neutrosophic minimal structure spaces. S. Ganesan et al [10] studied N_m-pre-continuous maps. This article focuses on N_m-β-open, β-interior and β-closure operators using neutrosophic minimal structures. We investigate properties of such concepts and we introduced the notions of N_m-β-continuous, N_m-β-closed graph, N_m-β-compact and almost N_m-β-compact and investigate some properties for such concepts. Finally, we introduced N_m-regular-open, N_m-π-open sets and investigate fundamental properties.

2. Preliminaries

Definition 2.1. [15, 16] Neutrosophic set (in short ns) K on a set $G \neq \emptyset$ is defined by $K = \{\prec a, P_K(a), Q_K(a), R_K(a) \succ : a \in G\}$, where $P_K : G \to [0,1]$, $Q_K : G \to [0,1]$ and $R_K : G \to [0,1]$ denotes the membership of an object, indeterminacy and non-membership of an object, for each a on G to K, respectively and $0 \leq P_K(a) + Q_K(a) + R_K(a) \leq 3$ for each $a \in G$.

Proposition 2.1. *[13] For any ns S, then the following conditions are holds:*

1. *$0_\sim \leq S$, $0_\sim \leq 0_\sim$.*
2. *$S \leq 1_\sim$, $1_\sim \leq 1_\sim$.*

Definition 2.2. [13] Let K = $\{\prec$ a, P_K(a), Q_K(a), R_K(a) $\succ$: a $\in$ G$\}$ be a ns.

1. A ns K is an empty set i.e., K = $0_\sim$ if 0 is membership of an object and 0 is an indeterminacy and 1 is an non-membership of an object respectively. i.e., $0_\sim$ = {g, (0, 0, 1) : g $\in$ G}
2. A ns K is a universal set i.e., K = $1_\sim$ if 1 is membership of an object and 1 is an indeterminacy and 0 is an non-membership of an object respectively. $1_\sim$ = {g, (1, 1, 0) : g $\in$ G}
3. $K_1 \cup K_2$ = {a, max $\{P_{K_1}$(a), P_{K_2}(a)$\}$, max $\{Q_{K_1}$(a), Q_{K_2}(a)$\}$, min $\{R_{K_1}$(a), R_{K_2}(a)$\}$: a $\in$ G}
4. $K_1 \cap K_2$ = {a, min $\{P_{K_1}$(a), P_{K_2}(a)$\}$, min $\{Q_{K_1}$(a), Q_{K_2}(a)$\}$, max $\{R_{K_1}$(a), R_{K_2}(a)$\}$: a $\in$ G}
5. K_1^C = $\{\prec$ a, R_K(a), 1 $-$ Q_K(a), P=P_K(a) $\succ$: a $\in$ G$\}$

Definition 2.3. [13] Neutrosophic topology (nt) in Salama's sense on a nonempty set G is a family τ of ns in G satisfying three conditions:

1. Empty set ($0_\sim$) and universal set ($1_\sim$) are members of τ.
2. $K_1 \cap K_2 \in \tau$ where $K_1, K_2 \in \tau$.
3. $\cup K_\delta \in \tau$ for every $\{K_\delta : \delta \in \Delta\} \leq \tau$.

Definition 2.4. [11] The neutrosophic minimal structure space over a universal set G be denoted by N_m. N_m is said to be neutrosophic minimal structure space (in short, nms) over G if it satisfying following the axiom: $0_\sim, 1_\sim \in N_m$. A family of neutrosophic minimal structure space is denoted by (G, N_{mG}).
Note that neutrosophic empty set and neutrosophic universal set can form a topology and it is known as neutrosophic minimal structure space.

Remark 2.1. *[11] Each ns in nms is neutrosophic minimal open set (in short, nmo).*
Complement of nmo is neutrosophic minimal closed set (in short, nmc).

Definition 2.5. [11] A is N_m-closed if and only if $N_m$cl(A) = A. Similarly, A is a N_m-open if and only if $N_m$int(A) = A.

Definition 2.6. [11] Let N_m be any nms and A be any neutrosophic set. Then

1. Every A $\in N_m$ is open and its complement is N_m closed.
2. N_m-closure of A = min {F : F is a nmc and F $\geq$ A} and it is denoted by $N_m$cl(A).
3. N_m-interior of A = max {F : F is a nmo and F $\leq$ A} and it is denoted by $N_m$int(A).

In general $N_m$int(A) is subset of A and A is a subset of $N_m$cl(A).

Proposition 2.2. *[11] Let R and S are any ns of nms N_m over G. Then*

1. *N_m^C = {0, 1, R_i^C} where R_i^C is a complement of ns R_i.*

2. *$G - N_m int(S) = N_m cl(G - S)$.*
3. *$G - N_m cl(S) = N_m int(G - S)$.*
4. *$N_m cl(R^C) = (N_m cl(R))^C = N_m int(R)$.*
5. *N_m closure of an empty set is an empty set and N_m closure of a universal set is a universal set. Similarly, N_m interior of an empty set and universal set respectively an empty and a universal set.*
6. *If S is a subset of R then $N_m cl(S) \leq N_m cl(R)$ and $N_m int(S) \leq N_m int(R)$.*
7. *$N_m cl(N_m cl(R)) = N_m cl(R)$ and $N_m int(N_m int(R)) = N_m int(R)$.*
8. *$N_m cl(R \vee S) = N_m cl(R) \vee N_m cl(S)$.*
9. *$N_m cl(R \wedge S) = N_m cl(R) \wedge N_m cl(S)$.*

Definition 2.7. Let (G, N_{mG}) be a nms and S $\leq$ G is said to be

1. N_m-semi-open set (in short, N_m-so) [9] if S $\leq$ $N_m$cl($N_m$int(S)).
2. N_m-pre-open set (in short, N_m-po) [10] if S $\leq$ $N_m$int($N_m$cl(S)).
   The complement of above N_m-open set is called an N_m-closed set.

Definition 2.8. [11] Let (G, N_{mG}) be nms.

1. Arbitrary union of nmo in (G, N_{mG}) is nmo. (Union Property).
2. Finite intersection of nmo in (G, N_{mG}) is nmo. (intersection Property).

Definition 2.9. [11] A function f: (G, N_{mG}) $\to$ (H, N_{mH}) is called neutrosophic minimal continuous map iff f^{-1}(V) $\in$ N_{mG} whenever V $\in$ N_{mH}.

Definition 2.10. [11] let A be a ns in nms (G, N_{mG}). Then Y is said to be neutrosophic minimal subspace if (H, N_{mH}) = {A $\cap$ U : U $\in$ N_{mH}}.

3. N_m-β-open sets

Definition 3.1. (G, N_{mG}) be a nms & S $\leq$ G is said to be N_m-β-open set (in short, N_m-βo) if S $\leq$ $N_m$cl($N_m$int($N_m$cl(S))).
The complement of an N_m-βo is called an N_m-β-closed set(in short, N_m-βc)

Remark 3.1. *(G, $\mathcal{T}$) be a nt & $S \leq G$ is said to be $\mathcal{N}$-β-open set [3] if $S \leq \mathcal{N}cl(\mathcal{N}int(\mathcal{N}cl(S)))$. If the nms N_{mG} is a topology, clearly an N_m-βo is $\mathcal{N}$-β-open.*

Above definition of 3.1, trivially the following statement are obtained.

Lemma 3.1. *Consider (G, N_{mG}) be a nms.*

1. *Every N_m-open is N_m-βo.*
2. *S is an N_m-βo iff $S \leq N_m cl(N_m int(N_m cl(S)))$.*

3. *Every N_m-closed set is N_m-β-closed.*
4. *S is an N_m-β-closed set iff $N_m int(N_m cl(N_m int(S)) \leq S$.*

Theorem 3.1. *(G, N_{mG}) be a nms. Any union of N_m-βo is N_m-βo.*

Proof. Suppose A_δ be an N_m-βo for $\delta \in \Delta$. Above definition 3.1 and Proposition 2.2(6), $A_\delta \leq N_m cl(N_m int(N_m cl(A_\delta)))$ $\leq N_m cl(N_m int(N_m cl(\bigcup A_\delta)))$. This implies $\bigcup A_\delta \leq N_m cl(N_m int(N_m cl(\bigcup A_\delta)))$. Hence $\bigcup A_\delta$ is an N_m-βo. □

Remark 3.2. *Consider (G, N_{mG}) be a nms. Intersection of any 2 N_m-βo may not be N_m-βo.*

Example 3.1. *Consider $G = \{a\}$ with $N_m = \{0_\sim, P, Q, R, S, 1_\sim\}$ and $N_m^C = \{1_\sim, I, J, K, L, 0_\sim\}$ where*
$P = \prec (0.5, 0.6, 0.6)\succ$; $Q = \prec (0.4, 0.6, 0.8)\succ$
$R = \prec (0.4, 0.7, 0.9)\succ$; $S = \prec (0.5, 0.7, 0.6)\succ$
$I = \prec (0.6, 0.4, 0.5)\succ$; $J = \prec (0.8, 0.4, 0.4)\succ$
$K = \prec (0.9, 0.3, 0.4)\succ$; $L = \prec (0.6, 0.3, 0.5)\succ$
We know that $0_\sim = \{\prec g, 0, 0, 1 \succ : g \in G\}$, $1_\sim = \{\prec g, 1, 1, 0 \succ : g \in G\}$ and $0_\sim^C = \{\prec g, 1, 1, 0 \succ : g \in G\}$, $1_\sim^C = \{\prec g, 0, 0, 1 \succ : g \in G\}$.
Now we define the two N_m-βos as follows:
$A = \prec (0.6, 0.7, 0.9)\succ$; $B = \prec (0.5, 0.8, 0.4)\succ$
Here $N_m cl(A) = 0_\sim^C$, $N_m int(N_m cl(A)) = 1_\sim$, $N_m cl(N_m int(N_m cl(A))) = 0_\sim^C$ and
$N_m cl(B) = 0_\sim^C$, $N_m int(N_m cl(B)) = 1_\sim$, $N_m cl(N_m int(N_m cl(A))) = 0_\sim^C$. But $A \wedge B = \prec (0.5, 0.7, 0.9)\succ$ is not a N_m-βo in G.

Proposition 3.1. *Let (G, N_{mG}) be a nms.*

1. *If S is a $N_m so$ then it is a N_m-βo.*
2. *If S is a N_m-po then it is a N_m-βo.*

Proof. (1) The proof is straightforward from the definitions.
(2) The proof is straightforward from the definitions. □

Definition 3.2. Let (G, N_{mG}) be a nms.

1. N_m-β-closure of A = min {S : S is N_m-β-closed set and S $\geq$ A} and it is denoted by N_m-βcl (A).
2. N_m-β-interior of A = max {V : V is N_m-βo and V $\leq$ A} and it is denoted by N_m-βint(A).

Theorem 3.2. *Suppose (G, N_{mG}) be a nms and $R, S \leq G$. Then*

1. *N_m-$\beta int(0_\sim) = 0_\sim$.*
2. *N_m-$\beta int(1_\sim) = 1_\sim$.*
3. *N_m-$\beta int(R) \leq R$.*
4. *If $R \leq S$, then N_m-$\beta int(R) \leq N_m$-$\beta int(S)$.*

5. *R is N_m-β o iff N_m-β int(R) = R.*
6. *N_m-β int(N_m-β int(R)) = N_m-β int(R).*
7. *N_m-β cl (G − R) = G − N_m-β int(R).*

Proof. (1), (2) are Obvious.
(3), (4) are Obvious.
(5) It follows from Theorem 3.1.
(6) It follows condition from (5).
(7) For R ≤ G, G − N_m-βint(R) = G − max {U : U ≤ R, U is N_m-βo} = min { G − U : U ≤ R, U is N_m-βo} = min {G - U : G - R ≤ G - U}, U is N_m-βo} = N_m-βcl (G − R). □

Theorem 3.3. *Let (G, N_{mG}) be a nms and R, S ≤ G. Then*

1. *N_m-β cl ($0_\sim$) = $0_\sim$.*
2. *N_m-β cl ($1_\sim$) = $1_\sim$.*
3. *R ≤ N_m-β cl (R).*
4. *If R ≤ S, then N_m-β cl (R) ≤ N_m-β cl (S).*
5. *R is N_m-β c iff N_m-β cl (R) = R.*
6. *N_m-β cl (N_m-β cl (R)) = N_m-β cl (R).*
7. *N_m-β int(G − R) = G − N_m-β cl (R).*

Proof. It is similar to the proof of above Theorem 3.2. □

Theorem 3.4. *Let (G, N_{mG}) be a nms and S ≤ G. Then*

1. *g ∈ N_m-β cl (S) iff S ∩ V ≠ ∅ for every N_m-β o V containing g.*
2. *g ∈ N_m-β int(S) iff there exists an N_m-β o U such that U ≤ S.*

Proof. (1) Suppose there is an N_m-βo V containing g such that S ∩ V = ∅. Then G − V is an N_m-βc such that S ≤ G − V, g ∉ G − V. This implies g ∉ N_m-βcl (S).

The reverse relation is obvious.
(2) Obvious. □

Lemma 3.2. *Let (G, N_{mG}) be a nms and S ≤ G. Then*

1. *N_m int(N_m cl(N_m int(S))) ≤ N_m int(N_m cl(N_m int(N_m-β int(S))) ≤ N_m-β int(S).*
2. *N_m-β cl (S) ≤ N_m cl(N_m int(N_m cl(N_m-β cl(S))) ≤ N_m cl(N_m int(N_m cl(S))).*

Proof. (1) For S ≤ G, by Theorem 3.3, N_m-βcl (S) is an N_m-βc set. Hence from Lemma 3.1, we have $N_m$int($N_m$cl($N_m$int(S))) ≤ $N_m$int($N_m$cl($N_m$int(N_m-βint(S)))) ≤ N_m-βint(S).
(2) It is similar to the proof of (1). □

4. N_m-β-continuous map

Definition 4.1. Map f : (G, N_{mG}) → (H, N_{mH}) is said to be N_m-β-continuous if $f^{-1}(O)$ is a N_m-βo in G, for each N_m-open O in H.

Theorem 4.1. 1. *Every neutrosophic minimal continuous is N_m-β-continuous but not conversely.*

2. *Every N_m-semi-continuous is N_m-β-continuous but not conversely.*
3. *Every N_m-pre-continuous is N_m-β-continuous but not conversely.*

Proof. (1) The proof follows from [Lemma 3.1 (1)].
(2) The proof follows from [Proposition 3.1 (1)].
(3) The proof follows from [Proposition 3.1 (2)]. □

Theorem 4.2. *Map f : G → H be a function on 2 nms (G, N_{mG}) and (H, N_{mH}). Then the following statements are equivalent:*

1. *f is N_m-β-continuous.*
2. *f^{-1}(O) is an N_m-βo, for each N_m-open set O in H.*
3. *f^{-1}(S) is an N_m-βc set, for each N_m-closed S in H.*
4. *f(N_m-βcl (R)) ≤ $N_m$cl(f(R)), for R ≤ G.*
5. *N_m-βcl (f^{-1}(S)) ≤ f^{-1}($N_m$cl(S)), for S ≤ H.*
6. *f^{-1}($N_m$int(S)) ≤ N_m-βint(f^{-1}(S)), for S ≤ H.*

Proof. (1) ⇒ (2) Let O be an N_m-open in H and g ∈ f^{-1}(O). By hypothesis, there exists an N_m-βo U_g containing g such that f(U) ≤ O. This implies g ∈ U_g ≤ f^{-1}(O) for all g ∈ f^{-1}(O). Hence by Theorem 3.1, f^{-1}(O) is N_m-βo.
(2) ⇒ (3) Obvious.
(3) ⇒ (4) For R ≤ G, f^{-1}($N_m$cl(f(R))) = f^{-1}(min {F ≤ H : f(R) ≤ F and F is N_m-closed}) = min {f^{-1}(F) ≤ G : R ≤ f^{-1}(F) and F is N_m-βc} ≥ min {K ≤ G : R ≤ K and K is N_m-βc} = N_m-βcl (R). Hence f(N_m-βcl (R)) ≤ $N_m$cl(f(R)).
(4) ⇒ (5) For R ≤ G, from (4), it follows f(N_m-βcl (f^{-1}(R))) ≤ $N_m$cl(f(f^{-1}(R))) ≤ $N_m$cl(R). Hence we get (5).
(5) ⇒ (6) For S ≤ H, from $N_m$int(S) = Y − $N_m$cl(H − S) and (5), it follows: f^{-1}($N_m$int(S)) = f^{-1}(Y − $N_m$cl(H − S)) = G − f^{-1}($N_m$cl(H − S)) ≤ G − N_m-βcl (f^{-1}(H − S)) = N_m-βint(f^{-1}(S)). Hence (6) is obtained.
(6) ⇒ (1) Let g ∈ G and O an N_m-open set containing f(g). Then from (6) and Proposition 2.2, it follows g ∈ f^{-1}(O) = f^{-1}($N_m$int(O)) ≤ N_m-βint(f^{-1}(O)). So from Theorem 3.4, we can say that there exists an N_m-βo U containing g such that g ∈ U ≤ f^{-1}(O). Hence f is N_m-β-continuous. □

Theorem 4.3. *Map f : G → H be a function on 2 nms (G, N_{mG}) and (H, N_{mH}). Then the following statements are equivalent:*

1. *f is N_m-β-continuous.*
2. *$f^{-1}(O) \leq N_m cl(N_m int(f^{-1}(O)))$, for each N_m-open O in H.*
3. *$N_m int(N_m cl(f^{-1}(F))) \leq f^{-1}(F)$, for each N_m-closed set F in H.*
4. *$f(N_m int(N_m cl(R))) \leq N_m cl(f(R))$, for $R \leq G$.*
5. *$N_m int(N_m cl(f^{-1}(S))) \leq f^{-1}(N_m cl(S))$, for $S \leq H$.*
6. *$f^{-1}(N_m int(S)) \leq N_m cl(N_m int(f^{-1}(S)))$, for $S \leq H$.*

Proof. (1) ⇔ (2) It follows from Theorem 4.2 and Definition of N_m-βos.
(1) ⇔ (3) It follows from Theorem 4.2 and Lemma 3.1.
(3) ⇒ (4) Let $R \leq X$. Then from Theorem 4.2(4) and Lemma 3.2, it follows $N_m int(N_m cl(R)) \leq N_m$-$\beta cl(R)$ $\leq f^{-1}(N_m cl(f(R)))$. Hence $f(N_m int(N_m cl(R))) \leq N_m cl(f(R))$.
(4) ⇒ (5) Obvious.
(5) ⇒ (6) From (5) and Proposition 2.2, it follows: $f^{-1}(N_m int(S)) = f^{-1}(H - N_m cl(H - S)) = G - f^{-1}(N_m cl(H - S)) \leq G - N_m int(N_m cl(f^{-1}(H - S)))$
$= N_m cl(N_m int(f^{-1}(S)))$. Hence, (6) is obtained.
(6) ⇒ (1) Let O be an N_m-open in H. Then by (6) and Proposition 2.2, we have $f^{-1}(O) = f^{-1}(N_m int(O)) \leq N_m cl(N_m int(f^{-1}(O)))$. This implies $f^{-1}(O)$ is an N_m-βo. Hence by (2), f is N_m-β-continuous. □

Definition 4.2. [10] (G, N_{mG}) be a nms. Then G is said to be N_m-T_2 if for each distinct points g and h of G, there exist two disjoint N_m-open U, V such that $g \in U$ and $h \in V$.

Definition 4.3. (G, N_{mG}) be a nms. Then G is said to be N_m-β-T_2 if for any distinct points g and h of G, there exist disjoint N_m-βo C, D such that $g \in C$ and $h \in D$.

Theorem 4.4. *Map $f : G \to H$ be a map on two nms (G, N_{mG}) and (H, N_{mH}). If f is an injective and N_m-β continuous map and if H is N_m-T_2, then G is N_m-β-T_2.*

Proof. Obvious. □

Theorem 4.5. *Map $f : G \to H$ be a map on two nms (G, N_{mG}) and (H, N_{mH}). If f is an injective and N_m-β continuous map with an N_m-β-closed graph, then G is N_m-β-T_2.*

Proof. Suppose g_1 and g_2 be any distinct points of G. Then $f(g_1) \neq f(g_2)$, so $(g_1, f(g_2)) \in (G \times H) - L(f)$. Since the graph L(f) is N_m-βc, there exist an N_m-βo containing g_1 and $D \in N_{mH}$ containing $f(g_2)$ such that $f(C) \cap D = \emptyset$. Since f is N_m-βcontinuous, $f^{-1}(D)$ is an N_m-βo containing g_2 such that $C \cap f^{-1}(D) = \emptyset$. Hence G is N_m-β-T_2. □

Definition 4.4. [10] (G, N_{mG}) be a nms and $S \leq G$, S is called N_m-compact (respectively, almost N_m-compact) relative to S if every collection $\{U_i : i \in \Delta\}$ of N_m-open subsets of G such that $S \leq \max \{U_i : i \in \Delta\}$, there exists a finite subset Δ_0 of Δ such that $S \leq \max \{U_j : j \in \Delta_0\}$ (respectively, $S \leq \max \{N_m cl(U_j) : j \in \Delta_0\}$). (G, N_{mG}) be a nms and $S \leq G$, S is said to be N_m-compact (respectively, almost N_m-compact) if S is N_m-compact (respectively, almost N_m-compact) as a neutrosophic minimal subspace of G.

Definition 4.5. (G, N_{mG}) be a nms and S $\leq$ G, S is called N$_m$-β-compact (respectively, almost N$_m$-β-compact) relative to S if every collection $\{U_\delta : \delta \in \Delta\}$ of N$_m$-β-open subsets of G such that S $\leq$ max $\{U_\delta : \delta \in \Delta\}$, there exists a finite subset Ω of Δ such that S $\leq$ max $\{U_\omega : \omega \in \Omega\}$ (respectively, S $\leq$ max $\{N_m \beta cl(U_\omega) : \omega \in \Omega\}$). (G, N_{mG}) be a nms and S $\leq$ G, S is said to be N$_m$-β-compact (respectively, almost N$_m$-β-compact) if S is N$_m$-β-compact (resp. almost N$_m$-β-compact) as a neutrosophic minimal subspace of G.

Theorem 4.6. *Map f : G $\to$ H be a map on 2 nms (G, N_{mG}) and (H, N_{mH}). If S is an N_m-β-compact set, then f(S) is N_m-compact.*

Proof. Obvious. ☐

5. N$_m$-regular open

We introduce following definitions

Definition 5.1. (G, N_{mG}) be a nms and A $\leq$ G, A is called N$_m$-regular open (in short, N$_m$-ro) if A = N$_m$int(N$_m$cl(A)).

Theorem 5.1. *Any N_m-ro is N_m-open.*

Proof. If A is N$_m$-ro in (G, N_{mG}), A = N$_m$int(N$_m$cl(A)). Then N$_m$int(A) = N$_m$int(N$_m$int(N$_m$cl(A))) = N$_m$int(N$_m$cl(A)) = A. That is, Ais N$_m$-open in (G, N_{mG}). ☐

Example 5.1. *$G = \{a\}$ with $N_m = \{0_\sim, P, 1_\sim\}$ and $N_m^C = \{1_\sim, Q, 0_\sim\}$ where*
$P = \prec (0.5, 0.5, 0.5)\succ$; $Q = \prec (0.5, 0.5, 0.5)\succ$
Now we define the N_m-ro sets as follows:
$A = \prec (0.5, 0.5, 0.5)\succ$
Here $N_m cl(A) = Q$, $N_m int(N_m cl(A)) = P$ is a N_m-ro in G.

Definition 5.2. (G, N_{mG}) be a nms and S $\leq$ G, S is said to be N$_m$-π-open set if S is the finite union of N$_m$-ro.

Remark 5.1. *For a subset of A of an nms (G, N_{mG}), we have following implications:*

$$N_m\text{-regular open} \Rightarrow N_m\text{-}\pi\text{-open} \Rightarrow N_m\text{-open}$$

Diagram-I

Example 5.2. *$G = \{a\}$ with $N_m = \{0_\sim, P, L, 1_\sim\}$ and $N_m^C = \{1_\sim, M, N, 0_\sim\}$ where*
$P = \prec (0.1, 0.5, 0.1)\succ$; $L = \prec (0.5, 0.5, 0.5)\succ$
$M = \prec (0.1, 0.5, 0.1)\succ$; $N = \prec (0.5, 0.5, 0.5)\succ$
Now we define the two N_m-ro sets as follows:
$A = \prec (0.1, 0.5, 0.1)\succ$
$B = \prec (0.5, 0.5, 0.5)\succ$
Here $N_m cl(A) = M$, $N_m int(N_m cl(A)) = P$; $N_m cl(B) = N$, $N_m int(N_m cl(B)) = L$ is a N_m-ro set in G. Here, $A \vee B = \prec (0.5, 0.5, 0.1)\succ$ is a N_m-π-open sets but it is not a N_m-ro.

Example 5.3. *$G = \{a\}$ with $N_m = \{0_\sim, A, 1_\sim\}$ and $N_m^C = \{1_\sim, B, 0_\sim\}$ where*
$A = \prec (0.6, 0.7, 0.3) \succ$; $B = \prec (0.3, 0.3, 0.6) \succ$
Now we define the N_m-ro sets as follows:
$R = \prec (0, 0, 1) \succ$; $S = \prec (1, 1, 0) \succ$
Here $R \vee S \prec (1, 1, 0) \succ$ is a N_m-π-open set in G. Here, $A = \prec (0.6, 0.7, 0.3) \succ$ is N_m-open but it is not a N_m-π-open.

Conclusion

We presented several definitions, properties, explanations and examples inspired from the concept of N_m-β-open, N_m-regular-open and N_m-π-open. The results of this study may be help in many reserches.

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Neutrosophic Fuzzy Goal Programming Algorithm for Multi-level Multiobjective Linear Programming Problems

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1 Introduction

Most often, the mathematical programming problems consist of only one decision maker who takes the decisions all alone. Apart from that, many decision-making problems involve hierarchical decision structures, each with independent, and most often contradictory in nature. Such decision-making scenarios are termed as decentralized planning problems. Thus, the hierarchical decision-making texture of the problem is formulated as multi-level programming problems (MLPPs). If there are only two decision makers, then it becomes bi-level programming problems, tri-level for three decision makers, and so on. The fundamental concepts behind the MLPPs optimization techniques are that the leader-level decision maker defines his/her goals/target and then seeks the optimal solution from each subordinate level of the organization that has calculated individually. The follower-level decisions are then submitted and satisfied by the leader-level in view of overall benefit of the organizations. There may be more than one linear objective function that are to be optimized by different levels in MLPPs, then such kind of decentralized decision-making problems are termed as multi-level multiobjective linear programming problems (ML-MOLPPs).

There are several research works available in the literature that contribute to the domain of multi-level multiobjective linear programming problems. Based on fuzzy set theory, [1, 6, 19, 20, 22] presented fuzzy programming and fuzzy goal programming approaches to bi-level decision-making problems. Furthermore, [8, 10, 12, 14, 15] suggested the fuzzy-based solution procedure for ML-MOLPPs. Later on, intuitionistic fuzzy set theory [7] is also introduced to solve the ML-MOLPPs under intuitionistic fuzzy environment. Recently, [9, 21] discussed the intuitionistic fuzzy techniques to solve the ML-MOLPPs by considering the membership as well as non-membership functions of all objectives at each level. Furthermore, the extension and generalization of fuzzy and intuitionistic fuzzy sets are presented and named as neutrosophic set. First, [18] proposed the neutrosophic set, and later on it was extensively used in the field of mathematical programming problems and their optimization techniques. Only few research work is available that captures the neutrosophic decision set theory in ML-MOLPPs. Only two research articles are cited below that have contributed to neutrosophic ML-MOLPPs domain. Maiti et al. [11] investigated neutrosophic goal programming strategy for ML-MOLPPs with neutrosophic parameters. Pramanik and Dey [13] also suggested a goal programming technique for neutrosophic ML-MOLPPs where the parameters have been taken as triangular or trapezoidal neutrosophic numbers. Thus, this chapter provides more emphasis toward the neutrosophic ML-MOLPPs research area and laid down a concrete base for neutrosophic ML-MOLPPs optimization domain.

In this chapter, the neutrosophic fuzzy goal programming (NFGP) algorithm is introduced to solve the multi-level multiobjective linear programming problems. Two different NFGP procedures based on neutrosophic fuzzy decision set are presented for ML-MOLPPs. To formulate any of these two proposed NFGP models of the ML-MOLPPs, the neutrosophic fuzzy goals of the objectives are determined by finding individual optimal solutions. Marginal evaluations of each objective functions are then depicted by the associated membership functions under neutrosophic environment. These membership functions are converted into neutrosophic fuzzy flexible membership goals by means of incorporating over- and under-deviational variables and assigning highest truth membership value (unity), indeterminacy value (half), and a falsity value (zero) as aspiration levels to each of them. To determine the membership functions of the decision variable vectors monitored by any level decision maker, the optimal solution of the corresponding MOLPP is separately solved. A marginal relaxation of the decisions is prescribed to avoid decision deadlock.

The first proposed NFGP solution algorithm provides an extension of the work presented by [1, 8, 16] under neutrosophic environment, which deals with bi-level linear single-objective programming problems. It also extends the work of [14] by introducing the NFGP algorithm to multi-level programming problems with a multiple linear objective at each level. The final fuzzy model groups the membership functions for the defined neutrosophic fuzzy goals of the objective functions and the decision variable vectors at all levels, which are determined separately for each level except the follower level of the multi-level problem.

The second proposed NFFGP algorithm may be seen as a method for solving multi-level multiobjective programming problems. First, it develops the NFGP model of the leader-level problem to obtain a satisfactory solution to the leader-level decision maker's problem. A marginal relaxation of the leader-level decision maker's decisions is taken into account to avoid a decision deadlock. These decisions of the leader-level decision makers are depicted by membership functions of neutrosophic fuzzy set theory and transferred to the second-level DM (SLDM) as an additional constraint. Then, the SLDM modeled its NFGP model that considers the neutrosophic fuzzy membership goals of the objectives and decision variable vectors of the leader-level decision makers. Afterward, the achieved solution is passed to the third-level DM (TLDM) who seeks the solution in a similar fashion. The same process is carried out until the follower level reaches. Thus, this procedure may be assumed as an extension of the fuzzy mathematical programming algorithm of [16, 17] under the neutrosophic environment.

The remaining part of the chapter is summarized as follows: in Sect. 2, the preliminaries regarding neutrosophic set have been discussed, while Sect. 3 discusses the formulations of multi-level multiobjective programming problems. The proposed neutrosophic fuzzy goal algorithm is developed in Sect. 4, whereas in Sect. 5, a numerical example is presented to show the applicability and validity of the proposed approaches. Finally, conclusions and future scope are discussed based on the present work in Sect. 6.

2 Preliminaries

Some basic preliminaries regarding neutrosophic set are presented in the following section.

Definition 1 ([4]) Let Y be a universe discourse such that $y \in Y$, then a neutrosophic set A in Y is defined by three membership functions namely, truth $\mu_A(y)$, indeterminacy $\lambda_A(y)$, and a falsity $\nu_A(y)$ and is denoted by the following form:

$$A = \{< y,\ \mu_A(y),\ \lambda_A(y),\ \nu_A(y) > \mid y \in Y\},$$

where $\mu_A(y),\ \lambda_A(y)$ and $\nu_A(y)$ are real standard or non-standard subsets belong to $]0^-, 1^+[$, also given as, $\mu_A(y) : Y \rightarrow]0^-, 1^+[$, $\lambda_A(y) : Y \rightarrow]0^-, 1^+[$, and $\nu_A(y) : Y \rightarrow]0^-, 1^+[$. There is no restriction on the sum of $\mu_A(y),\ \lambda_A(y)$ and $\nu_A(y)$, so we have

$$0^- \leq\ sup\ \mu_A(y) + \lambda_A(y) +\ sup\ \nu_A(y) \leq 3^+.$$

Definition 2 ([4]) A single-valued neutrosophic set A over universe of discourse Y is defined as

$$A = \{< y,\ \mu_A(y),\ \lambda_A(y),\ \nu_A(y) > \mid y \in Y\},$$

where $\mu_A(y),\ \lambda_A(y)$, and $\nu_A(y) \in [0, 1]$ and $0 \leq \mu_A(y) + \lambda_A(y) + \nu_A(y) \leq 3$ for each $y \in Y$.

Definition 3 ([4]) The complement of a single valued neutrosophic set A is represented as $c(A)$ and defined by $\mu_{c(A)}(y) = \nu_A(y)$, $\lambda_{c(A)}(y) = 1 - \nu_A(y)$ and $\nu_{c(A)}(y) = \mu_A(y)$, respectively.

Definition 4 ([4]) Let A and B be the two single-valued neutrosophic sets, then the union of A and B is also a single-valued neutrosophic set C, that is, $C = (A \cup B)$, whose truth $\mu_C(y)$, indeterminacy $\lambda_C(y)$, and falsity $\nu_C(y)$ membership functions are given by

$$\mu_C(y) = \max\left(\mu_A(y),\ \mu_B(y)\right)$$
$$\lambda_C(y) = \max\left(\lambda_A(y),\ \lambda_B(y)\right)$$
$$\nu_C(y) = \min\left(\nu_A(y),\ \nu_B(y)\right) \text{ for each } y \in Y.$$

Definition 5 ([4]) Let A and B be the two single-valued neutrosophic sets, then the intersection of A and B is also a single-valued neutrosophic set C, that is, $C = (A \cap B)$, whose truth $\mu_C(y)$, indeterminacy $\lambda_C(y)$, and falsity $\nu_C(y)$ membership functions are given by

$$\mu_C(y) = \min\left(\mu_A(y),\ \mu_B(y)\right)$$
$$\lambda_C(y) = \min\left(\lambda_A(y),\ \lambda_B(y)\right)$$
$$\nu_C(y) = \max\left(\nu_A(y),\ \nu_B(y)\right) \text{ for each } y \in Y.$$

Definition 6 A solution set $Y^* \in S$ is said to be an efficient solution to the MLMOPPs if and only if there does not exist any other $Y \in S$ such that $O_{ij} \geq O^*_{ij}$ for all $i = 1, 2, \ldots, t;\quad j = 1, 2, \ldots, m_t$, respectively.

Definition 7 For any ML-MOPPs, an efficient solution selected by the decision makers is the best compromise optimal solution which is chosen on the basis of decision makers' explicit and implicit criteria.

3 Description of ML-MOLPPs

Assume that a t- level multiobjective programming problem with minimization-type objective functions at different level. Consider that DM_i represents the i-th level decision maker and controls over the decision variable $\mathbf{y}_i = (y_{i1},\ y_{i2},\ \ldots,\ y_{in_i}) \in R^{n_i}$ for all $i = 1,\ 2,\ \ldots,\ t$, where $\mathbf{y} = (\mathbf{y}_1,\ \mathbf{y}_2,\ \ldots,\ \mathbf{y}_t) \in R^n$ such that $n = n_1 +_2 + \ldots + n_t$. Furthermore, we assume that

$$O_i(\mathbf{y}) = O_i(\mathbf{y}_1,\ \mathbf{y}_2,\ \ldots,\ \mathbf{y}_t) : R^{n_1} \times R^{n_2} \times \cdots \times R^{n_t} \ \rightarrow\ R^{m_i},\ \ \forall\ i = 1,\ 2,\ \ldots,\ t \qquad (1)$$

represents the vector-set of a well-defined linear objective function to the i-th decision makers, $i = 1,\ 2,\ \ldots,\ t$. The equivalent mathematical expressions for the ML-MOLPPS with minimization-type objectives can be stated as follows:

[1st level]

$$\underset{y_1}{\text{Min}}\ O_1(\mathbf{y}) = \underset{y_1}{\text{Min}}\ \left(o_{11}(\mathbf{y}),\ o_{12}(\mathbf{y}),\ \ldots,\ o_{1m_1}(\mathbf{y})\right)$$

where $\mathbf{y}_2,\ \mathbf{y}_3,\ \ldots,\ \mathbf{y}_t$ solves

[2nd level]

$$\underset{y_2}{\text{Min}}\ O_2(\mathbf{y}) = \underset{y_2}{\text{Min}}\ \left(o_{21}(\mathbf{y}),\ o_{22}(\mathbf{y}),\ \ldots,\ o_{2m_2}(\mathbf{y})\right) \tag{2}$$

$\ldots$

where $\mathbf{y}_t$ solves

[t-th level]

$$\underset{y_t}{\text{Min}}\ O_t(\mathbf{y}) = \underset{y_t}{\text{Min}}\ \left(o_{t1}(\mathbf{y}),\ o_{t2}(\mathbf{y}),\ \ldots,\ o_{tm_t}(\mathbf{y})\right)$$

subject to

$$\begin{aligned}\mathbf{y} \in \mathbf{S} = \{&\mathbf{y} \in R^n | G_1 y_1 + G_1 y_1 + \cdots + G_t y_t (\leq\ or\ =\ or \geq)\ \mathbf{q},\ \mathbf{y} \geq 0,\\ &\mathbf{q} \in R^m\} \neq \phi,\end{aligned} \tag{3}$$

where

$$\begin{aligned} o_{ij}(\mathbf{y}) =& c_1^{ij}\mathbf{y}_1 + c_2^{ij}\mathbf{y}_2 + \cdots + c_t^{ij}\mathbf{y}_t,\ \ i = 1,\ 2,\ \ldots,\ t,\ \ j = 1,\ 2,\ \ldots,\ m_i \\ =& c_{11}^{ij}\mathbf{y}_{11} + c_{12}^{ij}\mathbf{y}_{12} + \cdots + c_{1n_1}^{ij}\mathbf{y}_{1n_1} + c_{21}^{ij}\mathbf{y}_{21} + c_{22}^{ij}\mathbf{y}_{22} + \cdots \\ &+ c_{2n_2}^{ij}\mathbf{y}_{2n_2} + \cdots c_{t1}^{ij}\mathbf{y}_{t1} + c_{t2}^{ij}\mathbf{y}_{t2} + \cdots + c_{tn_t}^{ij}\mathbf{y}_{tn_t} \end{aligned} \tag{4}$$

such that $\mathbf{S}$ is the multi-level convex constraints in feasible decision set under multi-level multiobjective programming problems. The notation $m_i,\ \ i = 1,\ 2,\ \ldots,\ t$ denotes the number of objective function under i-th decision maker, m is the number of constraints, $c_k^{ij} = \left(c_{k1}^{ij},\ c_{k2}^{ij},\ \cdots,\ c_{kn_k}^{ij}\right),\ k = 1,\ 2,\ \ldots,\ t,\ c_{kn_k}^{ij}$ are constants, and the coefficient matrices of size $m \times n_i$ are depicted as $G_i,\ \forall\ i = 1,\ 2,\ \ldots,\ t$.

4 Proposed Neutrosophic Fuzzy Goal Programming Techniques

In the past few decades, it has been observed that the situation may arise in real-life decision-making problems where the indeterminacy or neutral thoughts about element into the feasible set exist. Indeterminacy/neutral thoughts are the region of the negligence of a proposition's value and lie between truth and a falsity degree. Therefore, the further generalization of fuzzy set (FS) [20] and intuitionistic fuzzy set (IFS) [7] is presented by introducing a new member into the feasible decision set. First, [18] investigated the neutrosophic set (NS) which comprises three membership functions, namely truth (degree of belongingness), indeterminacy (degree of belongingness up to some extent), and a falsity (degree of non-belongingness) functions of the element into the neutrosophic decision set (see [2, 3, 5]).

In ML-MOLPPs, if an imprecise aspiration level under neutrosophic environment is assigned to each of the objectives at each level of the ML-MOLPPs, then such neutrosophic objectives are termed as neutrosophic goals and dealt with neutrosophic decision-making techniques. Hence, the marginal evaluation of each neutrosophic goals is characterized through three different membership functions, namely truth, indeterminacy, and a falsity membership functions by defining the tolerance limits for attainment of their respective aspiration levels.

4.1 Characterization of Different Membership Functions Under Neutrosophic Environment

In multi-level decision-making problems, each DM intends to minimize its own objectives in each level over the same feasible region depicted by the system of constraints; hence, the individual optimal solutions are obtained by them and can be regarded as the aspiration levels of their associated neutrosophic goals.

Assume that $\mathbf{y}^{ij} = (\mathbf{y}_1^{ij}, \mathbf{y}_2^{ij}, \ldots, \mathbf{y}_t^{ij})$ and $o_{ij}^{\min}$, $i = 1, 2, \ldots, t$, $j = 1, 2, \ldots, m_i$ be the best individual optimal solutions of each DMs at each level, respectively. Furthermore, consider that $l_{ij} \geq o_{ij}^{\min}$ denotes the aspiration level assigned to the ij-th objective $o_{ij}(\mathbf{y})$ (where ij means that when $i = t$ for t-th level decision makers then $j = 1, 2, \ldots, m_i$). Moreover, also consider that $\mathbf{y}^{i*} = (\mathbf{y}_1^{i*}, \mathbf{y}_2^{i*}, \ldots, \mathbf{y}_t^{i*})$, $i = 1, 2, \ldots, t-1$, be the optimal solutions for the t-th level decision makers of ML-MOLPPs. Consequently, the neutrosophic goals of each objective function at each level and the vector-set of neutrosophic goals for the decision variables monitored by t-th level decision makers can be stated as follows:

$$o_{ij}(\mathbf{y}) \tilde{\lesssim} l_{ij}, \quad i = 1, 2, \ldots, t, \quad j = 1, 2, \ldots, m_i \text{ and}$$
$$\mathbf{y}_i \tilde{\cong} \mathbf{y}_i^{i*}, \quad i = 1, 2, \ldots, t-1,$$

where $\tilde{\lesssim}$ and $\tilde{\cong}$ represent the degree of neutrosophy of the aspiration levels.

One can note that the solutions $\mathbf{y}^{ij} = (\mathbf{y}_1^{ij}, \mathbf{y}_2^{ij}, \ldots, \mathbf{y}_t^{ij})$; $i = 1, 2, \ldots, t$, $j = 1, 2, \ldots, m_i$ are probably different due to the conflicting nature of the objective functions at each level for all the decision makers. Therefore, it can be obvious to consider that the values of $o_{gm}(\mathbf{y}_1^{gm}, \mathbf{y}_2^{gm}, \ldots, \mathbf{y}_t^{gm}) \geq$

$o_{ij}^{\min}$, $g = 1, 2, \ldots, t$, $m = 1, 2, \ldots, m_i$, and $\forall\ ij \neq gm$ with all values greater than $o_{gm}^{u} =$ max [o_{ij} $(\mathbf{y}_1^{gm}, \mathbf{y}_2^{gm}, \ldots, \mathbf{y}_t^{gm})$, $i = 1, 2, \ldots, t$, $j = 1, 2, \ldots, m_i$ and $ij \neq gm$] are absolutely unacceptable to the objective function $o_{gm}(\mathbf{y}) = o_{gm}(\mathbf{y}_1, \mathbf{y}_2, \ldots, \mathbf{y}_t)$. As a result, $o_{gm}(\mathbf{y})$ can be taken as the upper tolerance limit $u_{gm}(\mathbf{y})$ of the neutrosophic goal to the objective functions. The upper and lower bounds for ij-th objective function under the neutrosophic environment can be obtained as follows:

$$U_{ij}^{\mu} = u_{ij}, \quad L_{ij}^{\mu} = l_{ij} \qquad \text{for truth membership}$$

$$U_{ij}^{\lambda} = L_{ij}^{\mu} + a_{ij}, \quad L_{ij}^{\lambda} = L_{ij}^{\mu} \qquad \text{for indeterminacy membership}$$

$$U_{ij}^{\nu} = U_{ij}^{\mu}, \quad L_{ij}^{\nu} = L_{ij}^{\mu} + b_{ij} \qquad \text{for falsity membership,}$$

where a_{ij} and $b_{ij} \in (0, 1)$ are predetermined real numbers.

Thus, the different membership functions, namely truth $\mu_{o_{ij}}(o_{ij}(\mathbf{y}))$, indeterminacy $\lambda_{o_{ij}}(o_{ij}(\mathbf{y}))$, and a falsity $\nu_{o_{ij}}(o_{ij}(\mathbf{y}))$ membership functions for the ij-th neutrosophic goals can be stated as follows:

$$\mu_{o_{ij}}(o_{ij}(\mathbf{y})) = \begin{cases} 1 & \text{if } o_{ij}(\mathbf{y}) \leq L_{ij}^{\mu} \\ 1 - \frac{o_{ij}(\mathbf{y}) - L_{ij}^{\mu}}{U_{ij}^{\mu} - L_{ij}^{\mu}} & \text{if } L_{ij}^{\mu} \leq o_{ij}(\mathbf{y}) \leq U_{ij}^{\mu} \\ 0 & \text{if } o_{ij}(\mathbf{y}) \geq U_{ij}^{\mu} \end{cases} \tag{5}$$

$$\lambda_{o_{ij}}(o_{ij}(\mathbf{y})) = \begin{cases} 1 & \text{if } o_{ij}(\mathbf{y}) \leq L_{ij}^{\lambda} \\ 1 - \frac{o_{ij}(\mathbf{y}) - L_{ij}^{\lambda}}{U_{ij}^{\lambda} - L_{ij}^{\lambda}} & \text{if } L_{k}^{I} \leq o_{ij}(\mathbf{y}) \leq U_{ij}^{\lambda} \\ 0 & \text{if } o_{ij}(\mathbf{y}) \geq U_{ij}^{\lambda} \end{cases} \tag{6}$$

$$\nu_{o_{ij}}(o_{ij}(\mathbf{y})) = \begin{cases} 1 & \text{if } o_{ij}(\mathbf{y}) \geq U_{ij}^{\nu} \\ 1 - \frac{U_{ij}^{\nu} - o_{ij}(\mathbf{y})}{U_{ij}^{\nu} - L_{ij}^{\nu}} & \text{if } L_{ij}^{\nu} \leq o_{ij}(\mathbf{y}) \leq U_{ij}^{\nu} \\ 0 & \text{if } o_{ij}(\mathbf{y}) \leq L_{ij}^{\nu}. \end{cases} \tag{7}$$

To construct the different membership functions for the decision variables monitored by i-th decision makers, first, the optimal solution for the t-th level MOLPPs, $\mathbf{y}^{i*} = (\mathbf{y}_1^{i*}, \mathbf{y}_2^{i*}, \ldots, \mathbf{y}_t^{i*})$, $i = 1, 2, \ldots, t-1$, should be carried out by using any appropriate method for MOLPPs optimization techniques.

Suppose that $T_k^{i\alpha}$ and $T_k^{i\beta}$, $i = 1, 2, \ldots, t-1$, $k = 1, 2, \ldots, n_i$ be the maximum negative and positive tolerance limits on the decision variables imposed by the i-th level decision makers. Usually, the tolerances T_{ik}^{-} and T_{ik}^{+} may not be equal. The upper and lower bounds for ik-th decision variable vectors under the neutrosophic environment can be stated as follows:

$$\mu^U_{y_{ik}} = y^*_{ik} + T^{i\beta}_k,\ \ \mu^L_{y_{ik}} = y^*_{ik} - T^{i\alpha}_k \qquad \text{for truth membership}$$
$$\lambda^U_{y_{ik}} = \mu^L_{y_{ik}} + a_{ik},\ \ \lambda^L_{y_{ik}} = \mu^L_{y_{ik}} \qquad \text{for indeterminacy membership}$$
$$\nu^U_{y_{ik}} = \mu^U_{y_{ik}},\ \ \nu^L_{y_{ik}} = \mu^L_{y_{ik}} + b_{ik} \qquad \text{for falsity membership,}$$

where a_{ik} and $b_{ik} \in (0, 1)$ are predetermined real numbers.

For each of the n_i components of the decision variable vector $\mathbf{y}^*_{ik} = (\mathbf{y}^*_{i1}, \mathbf{y}^*_{i2}, \ldots, \mathbf{y}^*_{in_i})$ controlled by the leader $(t-1)$-th level decision makers, the different linear membership functions under neutrosophic environment such as truth $\mu_{y_{ik}}(y_{ik})$, indeterminacy $\lambda_{y_{ik}}(y_{ik})$, and a falsity $\nu_{y_{ik}}(y_{ik})$ can be furnished as follows:

$$\mu_{y_{ik}}(y_{ik}) = \begin{cases} \frac{y_{ik}-\mu^L_{y_{ik}}}{y^*_{ik}-\mu^L_{y_{ik}}} & \text{if } \mu^L_{y_{ik}} \le y_{ik} \le y^*_{ik} \\ \frac{\mu^U_{y_{ik}}-y_{ik}}{\mu^U_{y_{ik}}-y^*_{ik}} & \text{if } y^*_{ik} \le y_{ik} \le \mu^U_{y_{ik}} \\ 0 & \text{otherwise.} \end{cases} \tag{8}$$

$$\lambda_{y_{ik}}(y_{ik}) = \begin{cases} \frac{y_{ik}-\lambda^L_{y_{ik}}}{y^*_{ik}-\lambda^L_{y_{ik}}} & \text{if } \lambda^L_{y_{ik}} \le y_{ik} \le y^*_{ik} \\ \frac{\lambda^U_{y_{ik}}-y_{ik}}{\lambda^U_{y_{ik}}-y^*_{ik}} & \text{if } y^*_{ik} \le y_{ik} \le \lambda^U_{y_{ik}} \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

$$\nu_{y_{ik}}(y_{ik}) = \begin{cases} \frac{y_{ik}-\nu^L_{y_{ik}}}{y^*_{ik}-\nu^L_{y_{ik}}} & \text{if } \nu^L_{y_{ik}} \le y_{ik} \le y^*_{ik} \\ \frac{\nu^U_{y_{ik}}-y_{ik}}{\nu^U_{y_{ik}}-y^*_{ik}} & \text{if } y^*_{ik} \le y_{ik} \le \nu^U_{y_{ik}} \\ 1 & \text{otherwise.} \end{cases} \tag{10}$$

Also, it should be noted that the range of y_{ik} may be shifted according to the decision makers' choices.

In a neutrosophic decision environment, the neutrosophic goals comprising the decision makers' objective functions at different level and the neutrosophic goals of the decision variable vectors are monitored by leader $(t-1)$-th level decision makers. The attainment degrees to their aspiration levels to the extent possible are virtually achieved by the possible achievement of their respective memberships, namely truth, indeterminacy, and a falsity membership functions to their utmost degrees. Obviously, this aspect of neutrosophic fuzzy programming approach enables a neutrosophic fuzzy goal programming technique as a justified approach for solving the leader t-th level MOLPPs and consequently ML-MOLPPs.

4.2 Neutrosophic Fuzzy Goal Programming Approach

In neutrosophic programming approaches, the neutrosophic membership degrees can be transformed into neutrosophic membership goals according to their respective maximum degrees of attainment. The highest degree of truth membership function that can be achieved is unity (1), the indeterminacy membership function is neutral and independent with the highest attainment degree half (0.5), and the falsity membership function can be achieved with the highest attainment degree zero (0). Now, the transformed membership goals under the neutrosophic environment can be expressed as follows:

$$\left.\begin{array}{l} \mu_{o_{ij}}(o_{ij}(\mathbf{y})) + d_{ij\mu}^{-} - d_{ij\mu}^{+} = 1, \\ \lambda_{o_{ij}}(o_{ij}(\mathbf{y})) + d_{ij\lambda}^{-} - d_{ij\lambda}^{+} = 0.5, \\ \nu_{o_{ij}}(o_{ij}(\mathbf{y})) + d_{ij\nu}^{-} - d_{ij\nu}^{+} = 0, \end{array}\right\} \forall\ i = 1,\ 2,\ \ldots,\ t,\quad j = 1,\ 2,\ \ldots,\ m_i \tag{11}$$

$$\left.\begin{array}{l} \mu_{y_{ik}}(y_{ik}) + d_{ik\mu}^{-} - d_{ik\mu}^{+} = 1, \\ \lambda_{y_{ik}}(y_{ik}) + d_{ik\lambda}^{-} - d_{ik\lambda}^{+} = 0.5, \\ \nu_{y_{ik}}(y_{ik}) + d_{ik\nu}^{-} - d_{ik\nu}^{+} = 0, \end{array}\right\} \forall\ i = 1,\ 2,\ \ldots,\ t-1,\quad k = 1,\ 2,\ \ldots,\ n_i \tag{12}$$

or equivalently represented as follows:

$$\left.\begin{array}{l} 1 - \frac{o_{ij}(\mathbf{y}) - L_{ij}^{\mu}}{U_{ij}^{\mu} - L_{ij}^{\mu}} + d_{ij\mu}^{-} - d_{ij\mu}^{+} = 1, \\ 1 - \frac{o_{ij}(\mathbf{y}) - L_{ij}^{\lambda}}{U_{ij}^{\lambda} - L_{ij}^{\lambda}} + d_{ij\lambda}^{-} - d_{ij\lambda}^{+} = 0.5, \\ 1 - \frac{U_{ij}^{\nu} - o_{ij}(\mathbf{y})}{U_{ij}^{\nu} - L_{ij}^{\nu}} + d_{ij\nu}^{-} - d_{ij\nu}^{+} = 0, \end{array}\right\} \forall\ i = 1,\ 2,\ \ldots,\ t,\quad j = 1,\ 2,\ \ldots,\ m_i \tag{13}$$

$$\left.\begin{array}{l} \frac{y_{ik} - \mu_{y_{ik}}^{L}}{y_{ik}^{*} - \mu_{y_{ik}}^{L}} + d_{ik\mu}^{\alpha-} - d_{ik\mu}^{\alpha+} = 1, \\ \frac{\mu_{y_{ik}}^{U} - y_{ik}}{\mu_{y_{ik}}^{U} - y_{ik}^{*}} + d_{ik\mu}^{\beta-} - d_{ik\mu}^{\beta+} = 1, \\ \frac{y_{ik} - \lambda_{y_{ik}}^{L}}{y_{ik}^{*} - \lambda_{y_{ik}}^{L}} + d_{ik\lambda}^{\alpha-} - d_{ik\lambda}^{\alpha+} = 0.5, \\ \frac{\lambda_{y_{ik}}^{U} - y_{ik}}{\lambda_{y_{ik}}^{U} - y_{ik}^{*}} + d_{ik\lambda}^{\beta-} - d_{ik\lambda}^{\beta+} = 0.5, \\ \frac{y_{ik} - \nu_{y_{ik}}^{L}}{y_{ik}^{*} - \nu_{y_{ik}}^{L}} + d_{ik\nu}^{\alpha-} - d_{ik\nu}^{\alpha+} = 0, \\ \frac{\nu_{y_{ik}}^{U} - y_{ik}}{\nu_{y_{ik}}^{U} - y_{ik}^{*}} + d_{ik\nu}^{\beta-} - d_{ik\nu}^{\beta+} = 0, \end{array}\right\} \forall\ i = 1,\ 2,\ \ldots,\ t-1,\quad k = 1,\ 2,\ \ldots,\ n_i, \tag{14}$$

where $d_{ik}^{-} = (d_{ik}^{\alpha-},\ d_{ik}^{\beta-})$, $d_{ik}^{+} = (d_{ik}^{\alpha+},\ d_{ik}^{\beta+})$; $d_{ij.}^{-},\ d_{ij.}^{+},\ d_{ik}^{\alpha-},\ d_{ik}^{\beta-},\ d_{ik}^{\alpha+},\ d_{ik}^{\beta+} \geq 0$; and $d_{ik}^{\beta-} \times d_{ik}^{\beta+} = 0,\ \forall\ i = 1,\ 2,\ \ldots,\ t-1,\ k = 1,\ 2,\ \ldots,\ n_i$ are the over and under deviations for truth, indeterminacy, and a falsity membership goals from their respective aspiration levels under neutrosophic environment.

In goal programming strategy, the over- and/or under-deviational variable vectors are considered in the objective function to minimize them and solely depend on the nature of objective function that is being optimized. In the proposed neutrosophic goal programming technique, the over-deviational variables for neutrosophic goals of each objective function, $d^{+}_{ij.}\ \ \forall\ \ i = 1,\ 2,\ \ldots,\ t,\ \ j = 1,\ 2,\ \ldots,\ m_i$ and the over and under-deviational variables for the neutrosophic fuzzy goals of the decision variable vectors, $d^{\alpha+}_{ik},\ d^{\alpha-}_{ik},\ d^{\beta+}_{ik}$ and $d^{\beta-}_{ik}\ \ \forall\ \ i = 1,\ 2,\ \ldots,\ (t-1)$, $k = 1,\ 2,\ \ldots,\ n_i$ are needed to be minimized to attain the neutrosophic fuzzy goals.

4.3 Neutrosophic Fuzzy Goal Programming Approach for ML-MOLPPs

The proposed neutrosophic fuzzy goal programming (NFGP) algorithm for solving the multi-level multiobjective linear programming problems (ML-MOLPPs) is presented, and the two new algorithms are suggested under neutrosophic environment.

4.3.1 The First NFGP Algorithm for ML-MOLPPs

First of all, the first NFGP algorithm proposed in this chapter groups over the different membership functions for the prescribed neutrosophic fuzzy goals of the objective functions at each levels; it also groups the different membership functions of the neutrosophic fuzzy goals of the decision variable vector of the t-th leader-level problems that are optimized individually under neutrosophic environment. Thus, by assuming the goal attainment problems at the same preference level, the equivalent proposed neutrosophic fuzzy multi-level multiobjective linear goal programming model of the ML-MOLPPs under neutrosophic environment can be expressed as follows:

$$\begin{aligned}
\text{Min F} = & \sum_{j=1}^{m_1} w^{+}_{1j\mu} d^{+}_{1j\mu} + \sum_{j=1}^{m_2} w^{+}_{2j\mu} d^{+}_{2j\mu} + \cdots + \sum_{j=1}^{m_t} w^{+}_{tj\mu} d^{+}_{tj\mu} \\
& + \sum_{j=1}^{m_1} w^{+}_{1j\lambda} d^{+}_{1j\lambda} + \sum_{j=1}^{m_2} w^{+}_{2j\lambda} d^{+}_{2j\lambda} + \cdots + \sum_{j=1}^{m_t} w^{+}_{tj\lambda} d^{+}_{tj\lambda} \\
& - \sum_{j=1}^{m_1} w^{+}_{1j\nu} d^{-}_{1j\nu} - \sum_{j=1}^{m_2} w^{+}_{2j\nu} d^{-}_{2j\nu} - \cdots - \sum_{j=1}^{m_t} w^{+}_{tj\nu} d^{-}_{tj\nu} \\
& + \sum_{k=1}^{n_1} \left(w^{\alpha}_{1k\cdot} (d^{\alpha-}_{1k\cdot} + d^{\alpha+}_{1k\cdot}) + w^{\beta}_{1k\cdot} (d^{\beta-}_{1k\cdot} + d^{\beta+}_{1k\cdot}) \right)
\end{aligned}$$

$$+\sum_{k=1}^{n_2}\Big(w^{\alpha}_{2k\cdot}(d^{\alpha-}_{2k\cdot}+d^{\alpha+}_{2k\cdot})+w^{\beta}_{2k\cdot}(d^{\beta-}_{2k\cdot}+d^{\beta+}_{2k\cdot})\Big)\quad\cdots\quad\cdots\quad\cdots$$

$$+\sum_{k=1}^{n_{t-1}}\Big(w^{\alpha}_{t-1k\cdot}(d^{\alpha-}_{t-1k\cdot}+d^{\alpha+}_{t-1k\cdot})+w^{\beta}_{t-1k\cdot}(d^{\beta-}_{t-1k\cdot}+d^{\beta+}_{t-1k\cdot})\Big)$$

subject to

$$\mu_{o_{1j}}(o_{1j}(\mathbf{y}))+d^{-}_{1j\mu}-d^{+}_{1j\mu}=1,\quad j=1,\ 2,\ \ldots,\ n_1$$

$$\mu_{o_{2j}}(o_{2j}(\mathbf{y}))+d^{-}_{2j\mu}-d^{+}_{2j\mu}=1,\quad j=1,\ 2,\ \ldots,\ n_2$$

$$\ldots$$

$$\mu_{o_{tj}}(o_{tj}(\mathbf{y}))+d^{-}_{tj\mu}-d^{+}_{tj\mu}=1,\quad j=1,\ 2,\ \ldots,\ n_t$$

$$\lambda_{o_{1j}}(o_{1j}(\mathbf{y}))+d^{-}_{1j\lambda}-d^{+}_{1j\lambda}=0.5,\quad j=1,\ 2,\ \ldots,\ n_1$$

$$\lambda_{o_{2j}}(o_{2j}(\mathbf{y}))+d^{-}_{2j\lambda}-d^{+}_{2j\lambda}=0.5,\quad j=1,\ 2,\ \ldots,\ n_2$$

$$\ldots$$

$$\lambda_{o_{tj}}(o_{tj}(\mathbf{y}))+d^{-}_{tj\lambda}-d^{+}_{tj\lambda}=0.5,\quad j=1,\ 2,\ \ldots,\ n_t$$

$$\nu_{o_{1j}}(o_{1j}(\mathbf{y}))+d^{-}_{1j\nu}-d^{+}_{1j\nu}=0,\quad j=1,\ 2,\ \ldots,\ n_1$$

$$\nu_{o_{2j}}(o_{2j}(\mathbf{y}))+d^{-}_{2j\nu}-d^{+}_{2j\nu}=0,\quad j=1,\ 2,\ \ldots,\ n_2$$

$$\ldots$$

$$\nu_{o_{tj}}(o_{tj}(\mathbf{y}))+d^{-}_{tj\nu}-d^{+}_{tj\nu}=0,\quad j=1,\ 2,\ \ldots,\ n_t$$

$$\mu_{y_{1k}}(y_{1k})+d^{-}_{1k\mu}-d^{+}_{1k\mu}=1,\quad k=1,\ 2,\ \ldots,\ n_1$$

$$\mu_{y_{2k}}(y_{2k})+d^{-}_{2k\mu}-d^{+}_{2k\mu}=1,\quad k=1,\ 2,\ \ldots,\ n_2$$

$$\ldots$$

$$\mu_{y_{t-1k}}(y_{t-1k})+d^{-}_{t-1k\mu}-d^{+}_{t-1k\mu}=1,\quad k=1,\ 2,\ \ldots,\ n_{t-1}$$

$$\lambda_{y_{1k}}(y_{1k})+d^{-}_{1k\lambda}-d^{+}_{1k\lambda}=0.5,\quad k=1,\ 2,\ \ldots,\ n_1$$

$$\lambda_{y_{2k}}(y_{2k})+d^{-}_{2k\lambda}-d^{+}_{2k\lambda}=0.5,\quad k=1,\ 2,\ \ldots,\ n_2$$

$$\ldots$$

$$\lambda_{y_{t-1k}}(y_{t-1k})+d^{-}_{t-1k\lambda}-d^{+}_{t-1k\lambda}=0.5,\quad k=1,\ 2,\ \ldots,\ n_{t-1}$$

$$\nu_{y_{1k}}(y_{1k})+d^{-}_{1k\nu}-d^{+}_{1k\nu}=0,\quad k=1,\ 2,\ \ldots,\ n_1$$

$$\nu_{y_{2k}}(y_{2k})+d^{-}_{2k\nu}-d^{+}_{2k\nu}=0,\quad k=1,\ 2,\ \ldots,\ n_2$$

$$\ldots$$

$$\nu_{y_{t-1k}}(y_{t-1k}) + d^{-}_{t-1k\nu} - d^{+}_{t-1k\nu} = 0, \quad k = 1,\ 2,\ \ldots,\ n_{t-1}$$
$$G_1 y_1 + G_1 y_1 + \cdots + G_t y_t (\leq \ or \ = \ or \geq) \, \mathbf{q}, \ \mathbf{y} \geq 0$$
$$d^{-}_{ij\cdot},\ d^{+}_{ij\cdot} \geq 0 \text{ and } d^{-}_{ij\cdot} \times d^{+}_{ij\cdot} = 0, \quad \forall\ i = 1,\ 2,\ \ldots,\ t, \quad j = 1,\ 2,\ \ldots,\ m_i$$
$$d^{-}_{ik\cdot},\ d^{+}_{ik\cdot} \geq 0 \text{ and } d^{-}_{ik\cdot} \times d^{+}_{ik\cdot} = 0, \quad \forall\ i = 1,\ 2,\ \ldots,\ t-1, \quad k = 1,\ 2,\ \ldots,\ n_i \tag{15}$$

Now the above model in Eq. (15) can be represented as follows:

$$\begin{aligned}
\text{Min F} = &\sum_{j=1}^{m_1} w^{+}_{1j\mu} d^{+}_{1j\mu} + \sum_{j=1}^{m_2} w^{+}_{2j\mu} d^{+}_{2j\mu} + \cdots + \sum_{j=1}^{m_t} w^{+}_{tj\mu} d^{+}_{tj\mu} \\
&+ \sum_{j=1}^{m_1} w^{+}_{1j\lambda} d^{+}_{1j\lambda} + \sum_{j=1}^{m_2} w^{+}_{2j\lambda} d^{+}_{2j\lambda} + \cdots + \sum_{j=1}^{m_t} w^{+}_{tj\lambda} d^{+}_{tj\lambda} \\
&- \sum_{j=1}^{m_1} w^{+}_{1j\nu} d^{-}_{1j\nu} - \sum_{j=1}^{m_2} w^{+}_{2j\nu} d^{-}_{2j\nu} - \cdots - \sum_{j=1}^{m_t} w^{+}_{tj\nu} d^{-}_{tj\nu} \\
&+ \sum_{k=1}^{n_1} \left(w^{\alpha}_{1k\cdot} (d^{\alpha-}_{1k\cdot} + d^{\alpha+}_{1k\cdot}) + w^{\beta}_{1k\cdot} (d^{\beta-}_{1k\cdot} + d^{\beta+}_{1k\cdot}) \right) \\
&+ \sum_{k=1}^{n_2} \left(w^{\alpha}_{2k\cdot} (d^{\alpha-}_{2k\cdot} + d^{\alpha+}_{2k\cdot}) + w^{\beta}_{2k\cdot} (d^{\beta-}_{2k\cdot} + d^{\beta+}_{2k\cdot}) \right) \quad \cdots \quad \cdots \quad \cdots \\
&+ \sum_{k=1}^{n_{t-1}} \left(w^{\alpha}_{t-1k\cdot} (d^{\alpha-}_{t-1k\cdot} + d^{\alpha+}_{t-1k\cdot}) + w^{\beta}_{t-1k\cdot} (d^{\beta-}_{t-1k\cdot} + d^{\beta+}_{t-1k\cdot}) \right)
\end{aligned}$$

subject to

$$1 - \frac{o_{ij}(\mathbf{y}) - L^{\mu}_{ij}}{U^{\mu}_{ij} - L^{\mu}_{ij}} + d^{-}_{tj\mu} - d^{+}_{tj\mu} = 1, \quad i = 1,\ 2,\ \ldots,\ t, \quad j = 1,\ 2,\ \ldots,\ m_i$$

$$1 - \frac{o_{ij}(\mathbf{y}) - L^{\lambda}_{ij}}{U^{\lambda}_{ij} - L^{\lambda}_{ij}} + d^{-}_{tj\lambda} - d^{+}_{tj\lambda} = 0.5, \quad i = 1,\ 2,\ \ldots,\ t, \quad j = 1,\ 2,\ \ldots,\ m_i$$

$$1 - \frac{U^{\nu}_{ij} - o_{ij}(\mathbf{y})}{U^{\nu}_{ij} - L^{\nu}_{ij}} + d^{-}_{tj\nu} - d^{+}_{tj\nu} = 0, \quad i = 1,\ 2,\ \ldots,\ t, \quad j = 1,\ 2,\ \ldots,\ m_i$$

$$\frac{y_{ik} - \mu^{L}_{y_{ik}}}{y^{*}_{ik} - \mu^{L}_{y_{ik}}} + d^{-}_{1k\mu} - d^{+}_{1k\mu} = 1, \quad i = 1,\ 2,\ \ldots,\ t-1, \quad k = 1,\ 2,\ \ldots,\ n_i$$

$$\frac{\mu^U_{y_{ik}} - y_{ik}}{\mu^U_{y_{ik}} - y^*_{ik}} + d^-_{1k\mu} - d^+_{1k\mu} = 1, \;\; i = 1,\ 2,\ \ldots,\ t-1,\;\; k = 1,\ 2,\ \ldots,\ n_i$$

$$\frac{y_{ik} - \lambda^L_{y_{ik}}}{y^*_{ik} - \lambda^L_{y_{ik}}} + d^-_{1k\lambda} - d^+_{1k\lambda} = 0.5, \;\; i = 1,\ 2,\ \ldots,\ t-1,\;\; k = 1,\ 2,\ \ldots,\ n_i$$

$$\frac{\lambda^U_{y_{ik}} - y_{ik}}{\lambda^U_{y_{ik}} - y^*_{ik}} + d^-_{1k\lambda} - d^+_{1k\lambda} = 0.5, \;\; i = 1,\ 2,\ \ldots,\ t-1,\;\; k = 1,\ 2,\ \ldots,\ n_i$$

$$\frac{y_{ik} - \nu^L_{y_{ik}}}{y^*_{ik} - \nu^L_{y_{ik}}} + d^-_{1k\nu} - d^+_{1k\nu} = 0, \;\; i = 1,\ 2,\ \ldots,\ t-1,\;\; k = 1,\ 2,\ \ldots,\ n_i$$

$$\frac{\nu^U_{y_{ik}} - y_{ik}}{\nu^U_{y_{ik}} - y^*_{ik}} + d^-_{1k\nu} - d^+_{1k\nu} = 0, \;\; i = 1,\ 2,\ \ldots,\ t-1,\;\; k = 1,\ 2,\ \ldots,\ n_i$$

$$G_1 y_1 + G_1 y_1 + \cdots + G_t y_t (\leq \;\; or \;\; = \;\; or \; \geq)\, \mathbf{q},\;\; \mathbf{y} \geq 0$$

$$d^-_{ij\cdot},\ d^+_{ij\cdot} \geq 0 \text{ and } d^-_{ij\cdot} \times d^+_{ij\cdot} = 0,\;\; \forall\;\, i = 1,\ 2,\ \ldots,\ t,\;\; j = 1,\ 2,\ \ldots,\ m_i$$

$$d^{\alpha-}_{ik\cdot},\ d^{\alpha+}_{ik\cdot} \geq 0 \text{ and } d^{\alpha-}_{ik\cdot} \times d^{\alpha+}_{ik\cdot} = 0,\;\; \forall\;\, i = 1,\ 2,\ \ldots,\ t-1,\;\; k = 1,\ 2,\ \ldots,\ n_i$$

$$d^{\beta-}_{ik\cdot},\ d^{\beta+}_{ik\cdot} \geq 0 \text{ and } d^{\beta-}_{ik\cdot} \times d^{\beta+}_{ik\cdot} = 0,\;\; \forall\;\, i = 1,\ 2,\ \ldots,\ t-1,\;\; k = 1,\ 2,\ \ldots,\ n_i, \tag{16}$$

where F represents the neutrosophic achievement function comprising the weighted over-deviational variables $d^+_{ij\cdot},\ \forall\; i = 1,\ 2,\ \ldots,\ t,\;\; j = 1,\ 2,\ \ldots,\ m_i$ of the neutrosophic goals l_{ij} and the under-deviational and over-deviational variables $d^{\alpha-}_{ik\cdot},\ d^{\alpha+}_{ik\cdot},\ d^{\beta-}_{ik\cdot}$ and $d^{\beta+}_{ik\cdot},\ \forall\; i = 1,\ 2,\ \ldots,\ t-1,\;\; k = 1,\ 2,\ \ldots,\ n_i$ for the neutrosophic goals of all the decision variable vectors for the leader $t-1$-th levels. The corresponding weights $w^+_{ij\cdot},\ w^\alpha_{ik\cdot}$ and $w^\beta_{ik\cdot}$ depict the relative importance of attaining the aspired levels of the respective neutrosophic goals under the given constraints in the hierarchical decision-making scenarios.

To assign the different relative importance of the neutrosophic goals adequately, we have suggested the weighting scheme with the aid of u_{ij} and l_{ij}. The weighting scheme to each weight $w^+_{ij\cdot},\ w^\alpha_{ik\cdot}$, and $w^\beta_{ik\cdot}$ has been stated as follows:

$$w^+_{ij\cdot} = \frac{1}{u_{ij} - l_{ij}},\;\; \forall\;\, i = 1,\ 2,\ \ldots,\ t,\;\; j = 1,\ 2,\ \ldots,\ m_i \tag{17}$$

$$w^\alpha_{ik\cdot} = \frac{1}{T^{i\alpha}_k} \text{ and } w^\beta_{ik\cdot} = \frac{1}{T^{i\beta}_k},\;\; \forall\;\, i = 1,\ 2,\ \ldots,\ t-1,\;\; k = 1,\ 2,\ \ldots,\ n_i. \tag{18}$$

The NFGP model (16) gives the most satisfactory solution for the decision makers at all levels by attaining the aspired level of different neutrosophic membership goals at utmost possible in neutrosophic decision environment. The solution method is quite simple and demonstrated with the help of numerical examples in Sect. 5.

The step-wise solution procedure for the first NFGP algorithm for solving ML-MOLPPs can be stated as follows:

1. Solve each objectives individually for all levels under given constraints in order to find the maximum and minimum values of each objectives at all levels.
2. Depict the goals and upper tolerance limits—u_{ij}, l_{ij} ; $\forall\ i = 1, 2, \ldots, t,\ j = 1, 2, \ldots, m_i$—for each objectives at all levels.
3. Calculate the weights, $w^+_{ij\cdot} = \dfrac{1}{u_{ij} - l_{ij}},\ \forall\ i = 1, 2, \ldots, t,\ j = 1, 2, \ldots, m_i$ and set $g = 1$.
4. Evaluate the different membership functions $\mu_{o_{gj}}(o_{gj}(\mathbf{y})),\ \lambda_{o_{gj}}(o_{gj}(\mathbf{y}))$ and $\nu_{o_{gj}}(o_{gj}(\mathbf{y})),\ j = 1, 2, \ldots, m_g$ for each objective function under neutrosophic environment.
5. Develop the model given in Eq. (22) for the g-th level MOLPPs.
6. Obtain the value of $\mathbf{y}^{g*} = (\mathbf{y}_1^{g*}, \mathbf{y}_2^{g*}, \ldots, \mathbf{y}_t^{g*})$ by solving model given in Eq. (22).
7. Impose the maximum negative and positive tolerance limits on the decision variable vectors $\mathbf{y}_g^{g*} = (\mathbf{y}_{g1}^{g*}, \mathbf{y}_{g2}^{g*}, \ldots, \mathbf{y}_{gn_g}^{g*})$, $T_k^{g\alpha}$ and $T_k^{g\beta}$; $k = 1, 2, \ldots, n_g$.
8. Calculate the weights $w^\alpha_{gk\cdot} = \dfrac{1}{T_k^{g\alpha}}$ and $w^\beta_{gk\cdot} = \dfrac{1}{T_k^{g\beta}},\ k = 1, 2, \ldots, n_g$.
9. Evaluate the different membership functions $\mu_{y_{gk}}(y_{gk})$, $\lambda_{y_{gk}}(y_{gk})$ and $\nu_{y_{gk}}(y_{gk})$ for the decision variable vectors $\mathbf{y}_g^{g*} = (\mathbf{y}_{g1}^{g*}, \mathbf{y}_{g2}^{g*}, \ldots, \mathbf{y}_{gn_g}^{g*})$ given in Eq. (12).
10. If $g > t - 1$, then proceed to step 11, otherwise go to step 4.
11. Depict the different membership functions $\mu_{o_{tj}}(o_{tj}(\mathbf{y})),\ \lambda_{o_{tj}}(o_{tj}(\mathbf{y}))$ and $\nu_{o_{tj}}(o_{tj}(\mathbf{y})),\ j = 1, 2, \ldots, m_t$ for each objective function at the p-th level under neutrosophic environment.
12. Calculate the weights, $w^+_{tj\cdot} = \dfrac{1}{u_{tj} - l_{tj}},\ \forall\ j = 1, 2, \ldots, m_t$.
13. Formulate the model given in Eq. (16) under neutrosophic environment and solve it to get the satisfactory solution of the ML-MOLPPs.

4.3.2 The Second NFGP Algorithm for ML-MOLPPs

In the first NFGP algorithm, the final model contains the different membership functions for the neutrosophic goals of the decision variable vectors monitored by $t - 1$ levels, which separately solves for the i-th level MOLPPs. The second NFGP algorithm solves t MOLPPs that considers the decisions of the leader levels. After initialization steps 1 to 3 in first algorithm, the solution methods initiate with MOLPP of the first decision maker obtaining the compromise solution. A marginal evaluation of the first decision maker's decisions is taken into account to get rid of the decision deadlock. Hence, decisions of the first decision maker are depicted by the different membership functions under neutrosophic environment and sent to the second decision maker as additional auxiliary constraints. Afterward, the second decision maker considers the neutrosophic membership goals of the objectives as well as decision variable vectors of the first decision maker. After that, the achieved solution is passed to the third decision maker who tries to find out the optimal solution in a similar fashion. The processes of finding the optimal solution are repeated until the follower level is reached and consequently, the process is terminated.

The step-wise solution procedure for the second NFGP algorithm for solving ML-MOLPPs can be stated as follows:

1. Follow the same procedure from steps 1 to 9 as discussed in the first NFGP algorithm.
2. Formulate the model given in Eq. (16) for the ML-MOLPPs with $t = g$ under neutrosophic environment.
3. Solve the model given in Eq. (16) to get $\mathbf{y}^{g*} = (\mathbf{y}_1^{g*},\ \mathbf{y}_2^{g*},\ \ldots,\ \mathbf{y}_t^{g*})$.
4. Establish $g = g + 1$.
5. If $g > t$, then terminating with a satisfactory solution results $\mathbf{y}^{g*} = (\mathbf{y}_1^{g*},\ \mathbf{y}_2^{g*},\ \ldots,\ \mathbf{y}_t^{g*})$ to the ML-MOLPPs, otherwise proceed to step 7 of the first NFGP algorithm.

According to the solution priority, the second NFGP algorithm can be used to obtain the direct solution of the ML-MOLPPs to decisions of the first-level decision maker. After that, it directs the solutions to the decisions of second-level decision maker by preserving the solution closer to the decisions of first-level decision maker. Thus, the process goes on until the last level of the ML-MOLPPs preserving the solution closer to the decision of the leader levels.

5 Numerical Illustrations

The following numerical example consisting of tri-level multiobjective linear programming problems is presented to show the validity and applicability of the proposed NFGP optimization algorithms.

[1st level]

$$\underset{y_1}{\text{Min}}\ O_1(\mathbf{y}) = \underset{y_1}{\text{Min}}\ \left(o_{11}(\mathbf{y}) = y_1 - y_2 - 4y_3,\ o_{12}(\mathbf{y}) = -y_1 + 3y_2 - 4y_3\right),$$

where $\mathbf{y}_2$ and $\mathbf{y}_3$ solve

[2nd level]

$$\underset{y_2}{\text{Min}}\ O_2(\mathbf{y}) = \underset{y_2}{\text{Min}}\ \left(o_{21}(\mathbf{y}) = 2y_1 - y_2 + 2y_3,\ o_{22}(\mathbf{y}) = 2y_1 + y_2 - 3y_3,\right.$$

$$\left. o_{23}(\mathbf{y}) = 3y_1 - y_2 + y_3\right),$$

where $\mathbf{y}_3$ solves

Table 1 Individual minimum and maximum values for each objectives

	1st level		2nd level			3rd level	
	o_{11}	o_{12}	o_{21}	o_{22}	o_{23}	o_{31}	o_{32}
$\min_S o_{ij}$	−2.5	−3.5	−1	−1	−1	−0.5	0
$\max_S o_{ij}$	1	3	4	2	5	8.5	2

$$\text{[3rd level]}$$

$$\underset{y_3}{\text{Min}}\ O_3(\mathbf{y}) = \underset{y_3}{\text{Min}}\ \left(o_{31}(\mathbf{y}) = 7y_1 + 3y_2 - 4y_3,\ o_{32}(\mathbf{y}) = y_1 + y_3\right)$$

subject to

$$y_1 + y_2 + y_3 \leq 3, \quad y_1 + y_2 - y_3 \leq 1,$$

$$y_1 + y_2 + y_3 \geq 1, \quad -y_1 + y_2 + y_3 \leq 1,$$

$$y_3 \leq 0.5, \quad y_1,\ y_2,\ y_3 \geq 0.$$

The individual minimum and maximum values of each objective function for all the three levels of MOLPP under the given constraints S is furnished in Table 1. To apply the proposed NFGP algorithms, the aspiration levels and leader tolerance limits to the objective functions may be taken as the minimum and maximum individual optimal solutions.

The first NFGP algorithm can be elaborated through the solution method of the second NFGP algorithm. Thus, the following is the proposed first NFGP algorithm to tri-level multiobjective linear programming problem with the step-wise solution procedures.

$$First - level\ decision\ maker's\ NFGP\ model:$$

$$\text{Min } F_1 = 0.286d^+_{11\mu} + 0.154d^+_{12\mu} + 0.286d^+_{11\lambda} + 0.154d^+_{12\lambda} - 0.286d^-_{11\nu} - 0.154d^-_{12\nu}$$

subject to

$$-0.286y_1 + 0.286y_2 + 1.143y_3 + d^-_{11\mu} - d^+_{11\mu} = 0.714$$

$$-0.286y_1 + 0.286y_2 + 1.143y_3 + d^-_{11\lambda} - d^+_{11\lambda} = 0.143$$

$$-0.286y_1 + 0.286y_2 + 1.143y_3 + d^-_{11\nu} - d^+_{11\nu} = 0.03$$

$$0.154y_1 - 0.154y_2 + 0.62y_3 + d^-_{12\mu} - d^+_{12\mu} = 0.54$$

$$0.154y_1 - 0.154y_2 + 0.62y_3 + d^-_{12\lambda} - d^+_{12\lambda} = 0.21$$

$$0.154y_1 - 0.154y_2 + 0.62y_3 + d^-_{12\nu} - d^+_{12\nu} = 0.07$$

$$
\begin{aligned}
& y_1 + y_2 + y_3 \leq 3, \quad y_1 + y_2 - y_3 \leq 1, \\
& y_1 + y_2 + y_3 \geq 1, \quad -y_1 + y_2 + y_3 \leq 1, \\
& y_3 \leq 0.5, \quad y_1, \; y_2, \; y_3 \geq 0, \\
& d_{ij\cdot}^{-}, \; d_{ij\cdot}^{+} \geq 0 \text{ and } d_{ij\cdot}^{-} \times d_{ij\cdot}^{+} = 0, \;\; \forall \; i = 1, \;\; j = 1, \; 2.
\end{aligned} \tag{19}
$$

With the help of optimizing software, the optimal solution of the problem given in Eq. (19) is $\mathbf{y}^{1*} = (0.5, 0, 0.5)$. Assume that the first-level decision maker assigns $y_1^{1*} = 0.5$ along with the negative and positive tolerances $T_1^{1\alpha} = T_1^{1\beta} = 0.5$ and with the weights $w_{11\cdot}^{\alpha} = w_{11\cdot}^{\beta} = \dfrac{1}{0.5} = 2$, respectively.

$$Second - level \ decision \ maker's \ NFGP \ model:$$

$$
\begin{aligned}
\text{Min } F_1 = {} & 0.286 d_{11\mu}^{+} + 0.154 d_{12\mu}^{+} + 0.286 d_{11\lambda}^{+} + 0.154 d_{12\lambda}^{+} - 0.286 d_{11\nu}^{-} \\
& - 0.154 d_{12\nu}^{-} + 0.2 d_{21\mu}^{+} + 0.33 d_{22\mu}^{+} + 0.167 d_{23\mu}^{+} + 0.2 d_{21\lambda}^{+} + 0.33 d_{22\lambda}^{+} \\
& + 0.167 d_{23\lambda}^{+} - 0.2 d_{21\nu}^{-} - 0.33 d_{22\nu}^{-} - 0.167 d_{23\nu}^{-} + 2 \left[d_{11\cdot}^{-\alpha} + d_{11\cdot}^{+\alpha} + d_{11\cdot}^{-\beta} + d_{11\cdot}^{+\beta} \right]
\end{aligned}
$$

subject to

$$
\begin{aligned}
& -0.4 y_1 + 0.2 y_2 - 0.4 y_3 + d_{21\mu}^{-} - d_{21\mu}^{+} = 0.2 \\
& -0.4 y_1 + 0.2 y_2 - 0.4 y_3 + d_{21\lambda}^{-} - d_{21\lambda}^{+} = 0.13 \\
& -0.4 y_1 + 0.2 y_2 - 0.4 y_3 + d_{21\nu}^{-} - d_{21\nu}^{+} = 0.04 \\
& -0.667 y_1 - 0.33 y_2 + y_3 + d_{22\mu}^{-} - d_{22\mu}^{+} = 0.33 \\
& -0.667 y_1 - 0.33 y_2 + y_3 + d_{22\lambda}^{-} - d_{22\lambda}^{+} = 0.18 \\
& -0.667 y_1 - 0.33 y_2 + y_3 + d_{22\nu}^{-} - d_{22\nu}^{+} = 0.02 \\
& -0.5 y_1 + 0.167 y_2 - 0.167 y_3 + d_{23\mu}^{-} - d_{23\mu}^{+} = 0.17 \\
& -0.5 y_1 + 0.167 y_2 - 0.167 y_3 + d_{23\lambda}^{-} - d_{23\lambda}^{+} = 0.09 \\
& -0.5 y_1 + 0.167 y_2 - 0.167 y_3 + d_{23\nu}^{-} - d_{23\nu}^{+} = 0.01 \\
& 2 y_1 + d_{11\mu}^{-\alpha} - d_{11\mu}^{+\alpha} = 1, \;\; 2 y_1 + d_{11\mu}^{-\beta} - d_{11\mu}^{+\beta} = 1, \\
& 2 y_1 + d_{11\lambda}^{-\alpha} - d_{11\lambda}^{+\alpha} = 0.5, \;\; 2 y_1 + d_{11\lambda}^{-\beta} - d_{11\lambda}^{+\beta} = 0.5, \\
& 2 y_1 + d_{11\nu}^{-\alpha} - d_{11\nu}^{+\alpha} = 0, \;\; 2 y_1 + d_{11\nu}^{-\beta} - d_{11\nu}^{+\beta} = 0,
\end{aligned}
$$

constraints (19)

$$d_{ik\cdot}^{\alpha-},\ d_{ik\cdot}^{\alpha+} \geq 0 \text{ and } d_{ik\cdot}^{\alpha-} \times d_{ik\cdot}^{\alpha+} = 0,\ \ \forall\ i = 1,\ 2\ \ k = 1. \tag{20}$$

$$d_{ik\cdot}^{\beta-},\ d_{ik\cdot}^{\beta+} \geq 0 \text{ and } d_{ik\cdot}^{\beta-} \times d_{ik\cdot}^{\beta+} = 0,\ \ \forall\ i = 1,\ 2\ \ k = 1.$$

The optimal solution for the second-level NFGP model in Eq. (20) is obtained as $\mathbf{y}^{2*} = (0.5, 0, 0.5),\ (0.5, 0.998, 0.5),\ (0.5, 0.5, 0)$. Suppose that second-level decision maker finalizes $y_1^{2*} = 0.998$ along with the negative and positive tolerances $T_1^{2\alpha} = 0.75$, and $T_1^{2\beta} = 0.25$ and with weights $w_{21\cdot}^{\alpha} = \dfrac{1}{0.75} = 1.333$, and $w_{21\cdot}^{\beta} = \dfrac{1}{0.25} = 4$, respectively.

$$Third - level\ decision\ maker's\ NFGP\ model:$$

$$\text{Min } F_1 = 0.286d_{11\mu}^{+} + 0.154d_{12\mu}^{+} + 0.286d_{11\lambda}^{+} + 0.154d_{12\lambda}^{+} - 0.286d_{11\nu}^{-}$$
$$- 0.154d_{12\nu}^{-} + 0.2d_{21\mu}^{+} + 0.33d_{22\mu}^{+} + 0.167d_{23\mu}^{+} + 0.2d_{21\lambda}^{+} + 0.33d_{22\lambda}^{+}$$
$$+ 0.167d_{23\lambda}^{+} - 0.2d_{21\nu}^{-} - 0.33d_{22\nu}^{-} - 0.167d_{23\nu}^{-} + 2\left[d_{11\cdot}^{-\alpha} + d_{11\cdot}^{+\alpha} + d_{11\cdot}^{-\beta} + d_{11\cdot}^{+\beta}\right]$$
$$+ 1.33(d_{21\cdot}^{-\alpha} + d_{21\cdot}^{+\alpha}) + 4(d_{21\cdot}^{-\beta} + d_{21\cdot}^{+\beta})$$

subject to

$$- 0.78y_1 + 0.33y_2 + 0.44y_3 + d_{31\mu}^{-} - d_{31\mu}^{+} = 0.06$$
$$- 0.78y_1 + 0.33y_2 + 0.44y_3 + d_{31\lambda}^{-} - d_{31\lambda}^{+} =$$
$$- 0.78y_1 + 0.33y_2 + 0.44y_3 + d_{31\nu}^{-} - d_{31\nu}^{+} =$$
$$- 0.5y_1 - 0.5y_3 + d_{32\mu}^{-} - d_{32\mu}^{+} = 0$$
$$- 0.5y_1 - 0.5y_3 + d_{32\lambda}^{-} - d_{32\lambda}^{+} = 0$$
$$- 0.5y_1 - 0.5y_3 + d_{32\nu}^{-} - d_{32\nu}^{+} = 0$$
$$1.33y_2 + d_{21\mu}^{-\alpha} - d_{21mu}^{+\alpha} = 1.33,\ \ 4y_1 + d_{21\mu}^{-\beta} - d_{21\mu}^{+\beta} = 3.99,$$
$$1.33y_2 + d_{21\lambda}^{-\alpha} - d_{21\lambda}^{+\alpha} = 0.94,\ \ 4y_1 + d_{21\lambda}^{-\beta} - d_{21\lambda}^{+\beta} = 2.35,$$
$$1.33y_2 + d_{21\nu}^{-\alpha} - d_{21\nu}^{+\alpha} = 0.35,\ \ 4y_1 + d_{21\nu}^{-\beta} - d_{21\nu}^{+\beta} = 1.86,$$

constraints (20)

$$d_{ik\cdot}^{\alpha-},\ d_{ik\cdot}^{\alpha+} \geq 0 \text{ and } d_{ik\cdot}^{\alpha-} \times d_{ik\cdot}^{\alpha+} = 0,\ \ \forall\ i = 1,\ 2,\ 3\ \ k = 1,\ 2.$$
$$d_{ik\cdot}^{\beta-},\ d_{ik\cdot}^{\beta+} \geq 0 \text{ and } d_{ik\cdot}^{\beta-} \times d_{ik\cdot}^{\beta+} = 0,\ \ \forall\ i = 1,\ 2,\ 3\ \ k = 1,\ 2. \tag{21}$$

Table 2 Comparison of optimal solutions and satisfactory degrees of the given example

Proposed NFGP algorithm	Baky approach	Abo-Sinna approach	Shih approach
$(o_{11},\ \mu_{11})$ = (−2.499, 0.999)	(−2.498, 0.99)	(−2.21, 0.92)	(−2.21, 0.92)
$(o_{12},\ \mu_{12})$ = (0.4941, 0.399)	(0.494, 0.39)	(−0.569, 0.55)	(−0.569, 0.55)
$(o_{21},\ \mu_{21})$ = (1.002, 0.59)	(1.002, 0.6)	(1.88, 0.56)	(1.88, 0.56)
$(o_{22},\ \mu_{22})$ = (0.498, 0.50)	(0.498, 0.5)	(−0.09, 0.7)	(−0.09, 0.7)
$(o_{23},\ \mu_{23})$ = (1.002, 0.67)	(1.002, 0.67)	(1.09, 0.65)	(1.09, 0.65)
$(o_{31},\ \mu_{31})$ = (4.491, 0.47)	(4.493, 0.45)	(2.62, 0.65)	(2.62, 0.65)
$(o_{32},\ \mu_{32})$ = (1, 0.50)	(1, 0.50)	(0.899, 0.55)	(0.899, 0.55)
$\mathbf{y}*$ = (0.5, 0.9975, 0.5)	(0.5, 0.998, 0.5)	(0.339, 0.61, 0.5)	(0.339, 0.61, 0.5)

Table 3 Theoretical comparison of proposed NFGP algorithms with others

Proposed NFGP approach	Other approaches
Proposed approach considers the indeterminacy degree in decision-making process.	Abo-Sinna [1], Baky [8], and Shih et al. [16] cannot deal with indeterminacy in decision-making processes.
The overall satisfactory degree is achieved by attaining the neutrosophic fuzzy goals.	In [1, 8, 16] approaches, satisfactory degree is achieved by attaining the fuzzy goals.
It characterizes neutrosophic membership functions for both objectives and decision variables under neutrosophic environment.	Abo-Sinna [1], Baky [8], and Shih et al. [16] do not cover this aspects.
Additional predetermined parameters in indeterminacy and falsity degrees make the decisions more flexible according to decision makers' choices.	This facility is not provided in Abo-Sinna [1], Baky [8], and Shih et al. [16]

The final optimal solution for the ML-MOLPPs given in Eq. (21) is obtained as $\mathbf{y}^{3*} = (0.5, 0.9975, 0.5)$ with the different objectives values $o_{11} = -2.499$, $o_{12} = 0.4941$, $o_{21} = 1.002$, $o_{22} = 0.498$, $o_{23} = 1.002$, $o_{31} = 4.491$, and $o_{31} = 1$, along with membership functions $\mu_{11} = 0.999$, $\mu_{12} = 0.399$, $\mu_{21} = 0.590$, $\mu_{22} = 0.50$, $\mu_{23} = 0.67$, $\mu_{31} = 0.47$, and $\mu_{31} = 0.50$, respectively. A comparative study is performed among the proposed NFGP algorithm and presented in the Table 2. Other approaches reveal that the solution results are very close to [8], whereas [1, 16] give the same solution results for the presented numerical examples. Furthermore, the theoretical contributions in the domain of ML-MOLPPs are also summarized in Table 3.

6 Conclusions

This chapter proposes two different neutrosophic fuzzy goal programming algorithms for the solutions of ML-MOLPPs. The neutrosophic goal programming model is constructed to minimize the group tolerance of satisfactory degree of all the decision makers and to attain the highest degree for truth (unity), indeterminacy (half), and a falsity (zero) of each kind of the defined membership functions' goals to the utmost possible by minimizing their respective deviational variables and so that obtain the optimal solution for all decision makers. The primary advantages of the proposed two different neutrosophic fuzzy goal programming algorithms that the chances of refusing the solution repeatedly by the leader-level decision maker and reevaluation of the problem again and again by restating the defined membership functions required to reach the optimal solution would not arise.

The first NFGP algorithm considers the different membership functions for the defined neutrosophic goals of the objective functions at all levels as well as the different membership functions for the neutrosophic goals for the decision variable vectors at each level except the follower level of the ML-MOLPPs. The second NFGP algorithm solves the MOLPPs of the ML-MOLPPs by taking into account the decisions of the MOLPPs for the leader level only. A numerical example is presented to show the validity and applicability of the proposed NFGP algorithms with the fact that the degree of indeterminacy may arise in the hierarchical decision-making processes and can be overcome by utilizing the proposed algorithms. In future, it can be applied to real-life applications such as transportation, assignment, vendor selection, inventory control, supply chain, etc. and problems in multi-level decision-making scenarios.

Appendix

The NFGP approach to solve the single-level MOLPPs is presented to facilitate the achievement function $\mathbf{y}^{g*} = (\mathbf{y}_1^{g*},\ \mathbf{y}_2^{g*},\ \ldots,\ \mathbf{y}_t^{g*}),\ \ g = 1,\ 2,\ \cdots,\ t-1$. By using the same notations and symbols of this chapter, the NFGP model can be formulated for any g-th level MOLPPs and can be stated as follows:

$$
\begin{aligned}
&\text{Min } F = \sum_{j=i}^{m_g} w_{gj\mu}^{+} d_{gj\mu}^{+} + w_{gj\lambda}^{+} d_{gj\lambda}^{+} - w_{gj\nu}^{+} d_{gj\nu}^{-} \\
&\text{subject to} \\
&c_1^{gj}\mathbf{y}_1 + c_2^{gj}\mathbf{y}_2 + \cdots + c_t^{gj}\mathbf{y}_t + d_{gj\mu}^{-} - d_{gj\mu}^{+} = 1, \quad j = 1,\ 2,\ \ldots,\ m_g \\
&c_1^{gj}\mathbf{y}_1 + c_2^{gj}\mathbf{y}_2 + \cdots + c_t^{gj}\mathbf{y}_t + d_{gj\lambda}^{-} - d_{gj\lambda}^{+} = 0.5, \quad j = 1,\ 2,\ \ldots,\ m_g \\
&c_1^{gj}\mathbf{y}_1 + c_2^{gj}\mathbf{y}_2 + \cdots + c_t^{gj}\mathbf{y}_t + d_{gj\nu}^{-} - d_{gj\nu}^{+} = 0, \quad j = 1,\ 2,\ \ldots,\ m_g \\
&G_1 y_1 + G_1 y_1 + \cdots + G_t y_t (\leq\ or\ =\ or\ \geq)\ \mathbf{q},\ \mathbf{y} \geq 0 \\
&d_{gj\cdot}^{-},\ d_{gj\cdot}^{+} \geq 0 \text{ and } d_{gj\cdot}^{-} \times d_{gj\cdot}^{+} = 0, \quad j = 1,\ 2,\ \ldots,\ m_g
\end{aligned}
\tag{22}
$$

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M-generalised q-neutrosophic extension of CoCoSo method

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Abstract

Nowadays fuzzy approaches gain popularity to model multi-criteria decision making (MCDM) problems emerging in real-life applications. Modern modelling trends in this field include eval-uation of the criteria information uncertainty and vagueness. Traditional neutrosophic sets are considered as the effective tool to express uncertainty of the information. However, in some cases it cannot cover all recently proposed cases of the fuzzy sets.

The m-generalized q-neutrosophic sets (mGqNNs) can effectively deal with this situation. The novel MCDM methodology CoCoSomGqNN is presented in this paper. An illustrative example presents the analysis of the effectiveness of different retrofit strategy selection decisions for the application in the civil engineering industry.

Keywords: Multi-criteria decision making, CoCoSo, Neutrosophic sets, retrofit strategy.

1 Introduction

Multi-criteria decision making (MCDM) approaches introduce efficient tools to solve conflict situations for the managers. The MCDM process can be distinguished into the following steps:

a) Defining management system governing criteria which are associated with the compulsory decision-making goals;

b) Establishing the discrete alternative systems to reach the prescribed necessary goals;

c) Performing multicriteria analysis by any MCDM method;

d) Acknowledging one alternative as the best by the ranking results. Therefore, the application of the MCDM techniques facilitates the decision-makers to choose the best alternative, which provides a suitable compromise between all possibly conflicting criteria.

Therefore, the application of the MCDM techniques facilitates the decision-makers to embrace the alternative of the highest quality, which provides a suitable compromise between all possibly conflicting criteria. During the last decade, MCDM methods have actively been applied for the solution of the various real-life problems. In [5], the authors point out the following the most popular MCDM methods, such as SAW, WASPAS, VIKOR, TOPSIS, ELECTRE PROMETHEE methods. Recently, in [28] was proposed a new MCDM technique, which composes a combined compromise solution (CoCoSo). This method constructs the compromise solution applying various aggregation approaches. This method was under intensive research during the last years. In [39], the authors solved the problem of the ideal sustainable supplier choice by combining two methods: CoCoSo and the BWM method. In [14] was proposed a hybrid MCDM approach which is constructed by combining the CoCoSo and CRITIC approaches and applied to evaluate the 5G industries. In [25], the authors combined SWARA and CoCoSo methods to solve the drug cold chain logistics supplier evaluation and the location selection for a logistics center problems. This relatively new (CoCoSo) technique was applied to study the sustainability aspects in the OPEC countries [3]. The extension of the CoCoSo approach was proposed in [4]. The essence of this extension is the modelling environment in the form of normalized weighted geometric Bonferroni mean functions. This approach was employed to solve the supply chain problem. In [11], stochastic multi-criteria acceptability analysis was performed within the multi-criteria decision-making framework, namely the CoCoSo approach, to handle the problem of the renewable energy investments with stochastic information. This hybrid approach is based on the DIBR method, which defines interrelationships between the ranked criteria, and the D'CoCoSo approach, which is modelled under fuzzy Dombi environment. This technique was applied to solve prioritizing aspects of the circular economy concepts for urban mobility [12]. The two-stage decision-making technique, which was realized by the data envelopment analysis (DEA) and the RCoCoSo method, which was modelled under rough full consistency (R-FUCOM) environment, was discussed in [30]. The tourism attraction selection problem was utilised by applying the probabilistic linguistic term set. For the solution, IDOCRIW and CoCoSo approaches are implemented [9]. Different versions of the hybrid MCDM proposals based on the CoCoSo technique have been presented in [23], [10], [24], [6].

Modern research in the area of multi-criteria decision-making tries to model the indeterminacy and inconsistency of the initial decision making information, which prevails in most cases of real-life applications. Therefore, different fuzzy set variations have been proposed starting width the pioneering work in [33]. Existing challenges and future development trends of the fuzzy modelling importance in the decision-making field have been discussed in [27]. During the last years have been proposed the different extensions of the CoCoSo method applying the environments of the different fuzzy sets.

In [29], the authors modelled the best supplier problem in construction management applying grey numbers and solving this problem with the DEMATEL, BWM, and CoCoSo methods. In [26] was considered the problem of the third party logistics service providers applying a hesitant linguistic fuzzy set environment to construct the extent of the CoCoSo method. In [13], the authors applied a q-rung orthopair fuzzy set to develop a hybrid MCDM extension including CoCoSo and CRITIC methods and tested this approach on the financial risk evaluation problem. In [1], the authors suggested a novel interval-valued intuitionistic set model for CoCoSo method extension and solved the problem of the sustainable evolution in the fabrication region by means of green growth indicators. In [17], was proposed a CoCoSo method extension applying a single-valued neutrosophic set. This technique was applied to deal with the problem of the equipment selection for the waste management field including sustainability aspects. Various fuzzy set environments have been implemented to model multi-criteria decision-making problems [15], [16], [31], [32], [37], [21], [7], [7]. [8], [35], [38].

In 1999, Smarandache [19] proposed the notion of the neutrosophic sets. Neutrosophic sets (NS) are based on the generalization of the fuzzy logic that takes into account the knowledge of neural thought. Therefore, the greater amount of uncertainty can be analyzed. Each parameter of the analyzed problem can be represented by the scope of the truth (T), the scope of the indeterminacy (I) and the scope of the falsity (F) using the NS logic. In 2019, a notion of neutrosophic sets, which can be considered as the generalization of the following fuzzy sets: intuitionistic fuzzy (IFS), q-rung orthopair fuzzy and Pythagorean fuzzy sets was proposed by Smarandache [20]. Based on this concept, the new extensions of the classical MCDM methods, like MULTIMOORA, WASPAS, PROMETHEE [34], [18], [2], were developed by applying the environment of the m-generalized q-neutrosophic sets (mGqNSs).

The structure of the paper is established as follows. The second chapter presents the basic foundations of the m-generalised q- neutrosophic sets. The third chapter provides the proposed extension of the CoCoSo method under the environment m-generalised q- neutrosophic set. The problem formulation together with the solution procedure of the case study are presented in the fourth chapter. The paper ends with conclusions.

2 Foundations of the m-generalised q-neutrosophic set

First, the m-generalised q-neutrosophic set (mGqNS) prevailing definitions and the algebraic operations are presented. The algebraic operation presentation is constructed by applying m-generalised q-neutrosophic numbers (mGqNNs). These neutrosophic numbers form the basis of the proposed CoCoSo method extension, namely CoCoSo-mGqNS.

2.1 Definitions

Definition 1. *The modelled objects form the set X, in which every single object is $x \in X$. In the present research, the objects are represented by a neutrosophic set. Given that X is a set of criteria modelled under the m-generalised q-neutrosophic environment, x is a single criterion value.*

Let $q \geq 1$, $m = 1 \parallel 3$. The three membership functions define m-generalised q-neutrosophic set A_{mq}:

$$T_{mq}, I_{mq}, F_{mq} : X \to [0, r],\ \textit{here}\ (0 \leq r \leq 1)$$

Given T_{mq} is the m-generalised function representing the scope of the truth, I_{mq} is the m-generalised indeterminacy membership function and F_{mq} is the m-generalised falsity membership function. In such a way, the m-generalised q-neutrosophic set is prescribed by the following expression:

$$A_{mq} = \{\langle T_{mq}(x), I_{mq}(x), F_{mq}(x)\rangle : x \in X\}$$

These three membership functions also must comply with the following requirements:

$$0 \leq T_{mq}(x), I_{mq}(x), F_{mq}(x) \leq 1,\ x \in X;$$
$$0 \leq (T_{mq}(x))^q + (I_{mq}(x))^q + (F_{mq}(x))^q \leq \frac{3}{m},\ x \in X;$$

The different m and q values represent different fuzzy sets.

Definition 2. *The m-generalised q-neutrosophic number, mGqNN, is represented by the following expression:*

$$N_{mq} = \langle t, i, f\rangle$$

Definition 3. *The summation of the two mGqNNs, when* $N_{mq_1} = \langle t_1, i_1, f_1\rangle$ *and* $N_{mq_2} = \langle t_2, i_2, f_2\rangle$ *are the m-generalised q-neutrosophic numbers, we can designate as follows:*

$$N_{mq_1} \oplus N_{mq_2} = \langle (1-(1-t_1^q)(1-t_2^q)^{\frac{1}{q}}, i_1 i_2, f_1 f_2\rangle \tag{1}$$

The multiplication between the two mGqNNs can be performed as follows:

$$N_{mq_1} \otimes N_{mq_2} = \langle t_1 t_2, (1-(1-i_1^q)(1-i_2^q))^{\frac{1}{q}}, (1-(1-f_1^q)(1-f_2^q))^{\frac{1}{q}}\rangle \tag{2}$$

The multiplication operation of the m-generalised q-neutrosophic number and a real number $\lambda \geq 0$ *is performed as follows:*

$$\lambda \cdot N_{mq_1} = \langle (1-(1-t_1^q)^{\lambda})^{\frac{1}{q}}, i_1^{\lambda}, f_1^{\lambda}\rangle \tag{3}$$

When $\lambda \geq 0$*, the power function of mGqNN can be determined by:*

$$N_{mq_1}^{\lambda} = \langle t_1^{\lambda}, (1-(1-i_1^q)^{\lambda})^{\frac{1}{q}}, (1-(1-f_1^q)^{\lambda})^{\frac{1}{q}}\rangle \tag{4}$$

The complementary m-generalised q-neutrosophic numbers is calculated as:

$$N_{mq}^c = \langle f_1, 1-i_1, t_1\rangle \tag{5}$$

Definition 4. *For the deneutrosophication step, the score value* $S(N_{mq})$ *can be calculated as follows:*

$$S(N_{mq}) = \frac{3+3t^q-2i^q-f^q}{6} \tag{6}$$

In this case, for $N_{mq_1} = \langle t_1, i_1, f_1\rangle$ *and* $N_{mq_2} = \langle t_2, i_2, f_2\rangle$ *the comparison operations will be completed by the following condition:*

$$\begin{aligned} &\text{If } S(N_{mq}) \geq S(N_{mq}) \text{ , then } N_{mq} \geq N_{mq} \\ &\text{If } S(N_{mq}) = S(N_{mq}) \text{ , then } N_{mq} = N_{mq} \end{aligned} \tag{7}$$

3 Combined compromise solution (CoCoSo-mGqNN) method

A new extension of the CoCoSo method under m-generalised q-neutrosophic set environment, namelly CoCoSo-mGqNN, is designed. The main steps of this approach can be expressed as follows:

(1) Solution procedure of the CoCoSo method starts with the constructed initial decision-making matrix which can be defined as follows:

$$x_{ij} = \begin{bmatrix} x_{11} & x_{12} & \cdots & x_{1l} \\ x_{21} & x_{22} & \cdots & x_{2l} \\ \vdots & \vdots & \ddots & \vdots \\ x_{k1} & x_{k,2} & \cdots & x_{kl} \end{bmatrix}; i = 1,2,\ldots,k; j = 1,2,\ldots,l. \tag{8}$$

(2) The compromise normalization approach is applied to normalize the decision-making matrix elements [36]:

$$r_{ij} = \frac{x_{ij} - \min_i x_{ij}}{\sqrt{k}(\max_i x_{ij} - \min_i x_{ij})} \text{ for maximised criteria,} \tag{9}$$

$$r_{ij} = \frac{\max_i x_{ij} - x_{ij}}{\sqrt{k}(\max_i x_{ij} - \min_i x_{ij})} \text{ for minimised criteria.} \tag{10}$$

The standard compromise normalization equation is slightly modified to make it be applicable for neutrosophic algebra.

(3) Next, the neutrosophication of the decision matrix should be accomplished. This means that crisp values r_{ij} of the decision matrix elements are replaced by the values $(N_{mq})_{ij}$ of m-generalised truth membership, m-generalised indeterminacy membership and m-generalised falsity membership functions. This action is performed applying the standard crisp-to-neutrosophic mapping. The novel m-generalised neutrosophic decision matrix is created at this step.

(4) At this step, the weighted comparability series and the power weighted comparability series for each alternative are constructed. These series are denoted by R_i and P_i, respectively:

$$R_i = \sum_{j=1}^{l} (w_j (N_{mq})_{ij}) \tag{11}$$

the R_i series reflect weighted sum model:

$$P_i = \sum_{j=1}^{l} ((N_{mq})_{ij})^{w_j} \tag{12}$$

the P_i series correspond to the WASPAS multiplicative component.

(5) Three score assessment strategies are the basis for the calculations of the relative weights. These strategies are expressed by the following equations (13)-(15).

$$k_{ia} = \frac{S(P_i) + S(R_i)}{\sum_{i=1}^{k} (S(P_i) + S(R_i))}, \tag{13}$$

$$k_{ib} = \frac{S(R_i)}{\min_i S(R_i)} + \frac{S(P_i)}{\min_i S(P_i)}, \tag{14}$$

$$k_{ic} = \frac{\lambda S(R_i) + (1-\lambda) S(P_i)}{\lambda \max_i S(R_i) + (1-\lambda) \max_i S(P_i)}; 0 \leq \lambda \leq 1. \tag{15}$$

By equation (13), arithmetic mean values of the weighted sum model (WSM) and weighted product model (WPM) are calculated. By equation (14), relative scores of WSM and WPM are determined. These relatives scores are determined with respect to minimum values of $S(P_i)$ and $S(R_i)$. By equation (15), the balanced compromise scores of WSM and WPM models are calculated. In our case, value for $\lambda = 0.5$ is chosen.

(6) The final ranking function is constructed as follows:

$$k_i = (k_{ia} k_{ib} k_{ic})^{\frac{1}{3}} + \frac{1}{3}(k_{ia} + k_{ib} + k_{ic}). \tag{16}$$

4 Case study

A secondary school is the object of the research [22]. The school's frame is a three-story reinforced concrete frame (constructed in the eighth decade of the twentieth century). The 2478.60 m2 building has 240 mm thick aerated concrete slab exterior walls and the 200 mm thick reinforced concrete block plinth. Under part of the building was a local diesel boiler house. The building has plastered and painted external and external surfaces. The local municipality decided to renovate the structure according to the prepared project. They provide the following thermal insulation: 200 mm thick mineral wool boards for walls and 200 mm thick polystyrene boards for the plinth. The project envisages the roof insulated with 250 mm thick extruded polystyrene panels and an upper 20 mm thick mineral wool panels. Besides, old windows and doors need replacement as planned.

The most significant part of construction projects are multifaceted, and decision-makers must take into account tangible and intangible characteristics of plans, opinions, negotiations and dispute resolution and projects.

Decisions in the construction industry need compromise solutions [28], [29], [35]. Project managers should select among feasible discrete alternatives in a specific construction site and specific objects taking into account international and local perspectives. Designers presented nine possible project implementation scenarios. The building renovation process could involve several phases and be selected from other options with different employees (20, 40 or 60 for retrofitting at different speeds), see Table 1.

The project implementation contractor offered the following project implementation choices:

1. **One-phase retrofitting process**: The general contractor shall carry out all modernisation works in one phase:
   - Replacement of windows, exterior and interior doors; b) Insulation of basement and external walls and roof;
   - Installation of the roof;
   - Renovation of heating, ventilation, air conditioning and electrical systems;
   - Installation of a new lighting system.

     The modification process may affect the efficiency of the project due to the multi-tasking of the retrofitting methods.
2. **Two-phase retrofitting process**: The upgrading process takes place in two phases:
   - The first phase includes replacing windows and exterior doors, repairing the roof, and installing thermal insulation of exterior walls and the basement;
   - The second phase includes installing thermal insulation of ground-level floors, updating heating, ventilation, air conditioning and electric lighting systems.

     There is a nine months break between the first and the second phases. The employees work during the summer seasons (summer holidays).

     There is a risk (depending on the number of employees) that contractor not all works will finish in time, so construction work can adversely affect physical performance.
3. **Three-phase retrofitting** :
   - The first phase includes replacing windows and exterior doors;
   - The second phase includes repairing the roof and installing thermal insulation of exterior walls and the basement;
   - The third phase includes installing thermal insulation of ground-level floors, updating heating, ventilation, air conditioning and electric lighting systems.

     The contractor will make a one-year break between the first and the second phases. The employees work during the summer seasons (summer holidays).

     There is a risk (depending on the number of employees) that contractor not all works will finish in time, so construction work can adversely affect physical performance.

Many of the contradictory features reflect the solution to a complex problem. The authorities usually determine the winners based on a single indicator, the lowest price at the local level. This attitude of civil servants incorrectly describes the real economic benefits of renovating buildings and other features of modernised buildings. After three steps of the Delphi method, the decision-makers selected five key characteristics that assessed the renovation's economic benefits and the impact of the construction process on workers and visitors.

Costs with VAT, [€]: The construction works costs interlinks with the amount of work required to complete the project, the current price level in the country (salary, prices of materials and equipment), the number of potentially competitive contractors and their competitive advantages, construction time and project implementation time. Longer project implementation time and an increasing number of

Table 1: Initial decision-making matrix

Alternatives	Price with VAT, €	Number of working days (8 working hours per day)	Renovation's payback time, [years]	Energy savings through ten years, [MWh]	People's satisfaction, [scores]
A_1	1309.50	800	21.86	3656.0	6.66
A_2	1309.50	510	21.86	3656.0	11.38
A_3	1309.50	320	21.86	3656.0	12.89
A_4	1379.0	920	21.86	3524.4	7.46
A_5	1379.0	640	22.60	3524.4	12.18
A_6	1415.0	440	22.60	3524.4	13.69
A_7	1415.0	980	23.18	3188.7	8.27
A_8	1415.0	700	23.18	3188.7	12.98
A_9	1415.0	500	23.18	3188.7	14.49
Optimality	min, ↓	min, ↓	min, ↓	max, ↑	max, ↑

stages of construction processes increase construction costs. Longer-term constructions are expensive because the project requires household and storage facilities to bring and take out bio-toilets, temporary construction site's fencing, lighting, and need to implement protection measures.

Duration of project implementation [working days]: depends on the number of outputs (person-hours) and the number of workers on the construction site. This study examines the project implementation options (twenty, forty or sixty employees employed on the construction site). The contractors comply with all the necessary construction technology and safety requirements. They cannot perform the work faster than the established technological requirements in terms of time. For instance, it cannot paint the facade until three weeks have elapsed since the end of the plastering. The mineral plaster is undergoing chemical processes at that time.

Renewal payback period, [years]: The customers calculate this time according to the investment required for the construction work and the energy savings. They consider the efficiency of the work performed (such as replacing exterior doors and windows). The fuel's calorific value, the boiler's efficiency, the level of fuel prices and laboratory tests of the research object are the basis for calculating the economic payback time. Laboratory research shows actual energy savings by implementing specific design modernisation solutions.

Energy savings over ten years, [MWh], determine the expected economic benefits and return on investment. Calculating the value of this indicator over a more extended period is very difficult or even impossible due to the significant change in inflation, the change in energy prices and the price of energy resources, the changing rate requirements that govern the environmental impact and many other reasons. The changing climate is also contributing to its contribution. Determining energy savings by calculating energy consumption [MWh] rather than monetary value [€] partially reduces this uncertainty.

People Satisfaction, [points]: Survey of members of interest groups (building staff and visitors) is the basis for calculating this characteristic. Twenty-five stakeholders gave nine points for the best available option and one for the worst. Decision-makers multiplied the final scores by 100 per cent at the end. Determining weights of attributes is one of the most important and critically acclaimed issues in multi-attribute decision-making problems. The decision-makers chose and applied a systematic procedure for the study: the SWARA (Step-wise Weight Assessment Ratio Analysis) method.

The formulated initial decision-making matrix is presented in Table 1, and the normalized values of the weights are w=(0.37, 0.12, 0.36, 0.1, 0.05). At the following step the proposed extension CoCoSo-mGqNN was applied to perform the final ranking of the alternatives. The normalization of the decision-making matrix was performed applying equations (9-10). The elements of the initial decision-making matrix after the normalization step are presented in Table 2. The results of the neutrosophication step are presented in Table 3. The intermediate results of the proposed extension CoCoSo-mGqNN together with final ranking results are presented in Table 4.

Table 2: Initial decision-making matrix

Alternatives	X_1	X_2	X_3	X_4	X_5
A_1	0.3333	0.0909	0.3333	0.3333	0
A_2	0.3333	0.2374	0.3333	0.3333	0.2009
A_3	0.3333	0.3333	0.3333	0.3333	0.2652
A_4	0.1137	0.0303	0.3333	0.2395	0.0341
A_5	0.1137	0.1717	0.1465	0.2395	0.2350
A_6	0	0.2727	0.1465	0.2395	0.2993
A_7	0	0	0	0	0.0685
A_8	0	0.1414	0	0	0.2691
A_9	0	0.2424	0	0	0.3333
Optimality	min, ↓	min, ↓	min, ↓	max, ↑	max, ↑

Table 3: Neurosophic decision-making matrix

Alternatives	X_1	X_2	X_3	X_4	X_5
A_1	(0.3333, 0.7167, 0.6667)	(0.0909, 0.9091, 0.9091)	(0.3333, 0.7167, 0.6667)	(0.3333, 0.7167, 0.6667)	(0.0, 1.0, 1.0)
A_2	(0.3333, 0.7167, 0.6667)	(0.2374, 0.8126, 0.7626)	(0.3333, 0.7167, 0.6667)	(0.3333, 0.7167, 0.6667)	(0.2009, 0.8491, 0.7991)
A_3	(0.3333, 0.7167, 0.6667)	(0.3333, 0.7167, 0.6667)	(0.3333, 0.7167, 0.6667)	(0.3333, 0.7167, 0.6667)	(0.2652, 0.7848, 0.7348)
A_4	(0.1137, 0.8931, 0.8863)	(0.0303, 0.9697, 0.9697)	0.3333, 0.7167, 0.6667)	(0.2395, 0.8105, 0.7605)	(0.0341, 0.9659, 0.9659)
A_5	(0.1137, 0.8931, 0.8863)	(0.1717, 0.8641, 0.8283)	(0.1465, 0.8768, 0.8535)	(0.2395, 0.8105, 0.7605)	(0.2350, 0.8150, 0.7650)
A_6	(0.0, 1.0, 1.0)	(0.2727, 0.7773, 0.7273)	0.1465, 0.8768, 0.8535)	(0.2395, 0.8105, 0.7605)	(0.2993, 0.7507, 0.7007)
A_7	(0.0, 1.0, 1.0)	(0.0, 1.0, 1.0)	(0.0, 1.0, 1.0)	(0.0, 1.0, 1.0)	(0.0685, 0.9315, 0.9315)
A_8	(0.0, 1.0, 1.0)	(0.1414, 0.8793, 0.8586)	(0.0, 1.0, 1.0)	(0.0, 1.0, 1.0)	(0.2691, 0.7809, 0.7309)
A_9	(0.0, 1.0, 1.0)	(0.2424, 0.8076, 0.07576)	(0.0, 1.0, 1.0)	(0.0, 1.0, 1.0)	(0.3333, 0.7167, 0.6667)
Optimality	min, ↓	min, ↓	min, ↓	max, ↑	max, ↑

Table 4: Intermediate results of the CoCoSo-mGqNN and the final ranking of the alternatives

Alternatives	R	P	k_a	k_b	k_c	k	*Rank*
A_1	(0.3139, 0.7498, 0.7061)	(0.9728, 0.0548, 0.0425)	0.1241	60.8264	0.9529	22.5646	3
A_2	(0.3201, 0.7337, 0.6837)	(0.9960, 0.0163, 0.0106)	0.1288	63.7319	0.9895	23.6271	2
A_3	(0.3306 0.7199 0.6699)	(0.9973, 0.0123, 0.0080)	0.1302	66.0184	1.0000	24.4313	1
A_4	(0.2488, 0.8285, 0.7998)	(0.9786, 0.0485, 0.0404)	0.1168	45.1218	0.8715	17.0566	4
A_5	(0.1628, 0.8712, 0.8478)	(0.9914, 0.0342, 0.0255)	0.1135	35.2094	0.8715	13.5805	5
A_6	(0.1848, 0.8932, 0.8690)	(0.9937, 0.0364, 0.0270)	0.1115	30.6951	0.8561	11.9850	6
A_7	(0.0253, 0.9965, 0.9965)	(0.8746, 0.4295, 0.4295)	0.0778	2.0000	0.5973	1.3446	9
A_8	(0.1098, 0.9726, 0.9666)	(0.9689, 0.1596, 0.1405)	0.0969	9.3949	0.7439	4.2899	8
A_9	(0.1533, 0.9586, 0.9478)	(0.9793, 0.1249, 0.1048)	0.1006	13.7132	0.7727	5.8837	7

5 Conclusion

Modern modelling trends in this field include evaluation of the uncertainty and vagueness of the initial information. Traditional neutrosophic sets are considered as the effective tool to express uncertainty of the information. However, in some cases, it cannot cover all recently proposed cases of the fuzzy sets. The m-generalized q-neutrosophic sets were recently proposed to deal with this situation. The m-generalized q-neutrosophic sets can be considered as the generalisation of fuzzy set, Pythagorean fuzzy set, intuitionistic fuzzy set, q-rung orthopair fuzzy set, single-valued neutrosophic set, single-valued n-hyperspherical neutrosophic set and single-valued spherical neutrosophic set. In this paper, the CoCoSo method extension under the environment of the m-generalized q-neutrosophic numbers (mGqNN) is proposed. This novel extension has been tested for the selection of the best retrofit strategy. The numerical example also showed that the CoCoSo-mGqNN extension provides a robust approach that can be applied to deal with different fuzzy sets within the same MCDM framework.

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