Neutrosophic Axiomatic System

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Abstract

In this paper, we introduce for the first time the notions of Neutrosophic Axiom, Neutrosophic Axiomatic System, Neutrosophic Deducibility and Neutrosophic Inference, Neutrosophic Proof, Neutrosophic Tautologies, Neutrosophic Quantifiers, Neutrosophic Propositional Logic, Neutrosophic Axiomatic Space, Degree of Contradiction (Dissimilarity) of Two Neutrosophic Axioms, and Neutrosophic Model. A class of neutrosophic implications is also introduced. A comparison between these innovatory neutrosophic notions and their corresponding classical notions is made. Then, three concrete examples of neutrosophic axiomatic systems, describing the same neutrosophic geometrical model, are presented at the end of the paper.

Keywords

Neutrosophic logic, Neutrosophic Axiom, Neutrosophic Deducibility, Neutrosophic Inference, Neutrosophic Proof, Neutrosophic Tautologies, Neutrosophic Quantifiers, Neutrosophic Propositional Logic, Neutrosophic Axiomatic Space.

1 Neutrosophic Axiom

A *neutrosophic axiom* or *neutrosophic postulate* (α) is a partial premise, which is t% true (degree of truth), i% indeterminate (degree of indeterminacy), and f% false (degree of falsehood), where <t, i, f> are standard or nonstandard subsets included in the non-standard unit interval]-0, 1+[.

The non-standard subsets and non-standard unit interval are mostly used in philosophy in cases where one needs to make distinction between “absolute truth” (which is a truth in all possible worlds) and “relative truth” (which is a truth in at least one world, but not in all possible worlds), and similarly for distinction between “absolute indeterminacy” and “relative indeterminacy”, and respectively distinction between “absolute falsehood” and “relative falsehood”.

But for other scientific and technical applications one uses standard subsets, and the standard classical unit interval [0, 1].

As a particular case of neutrosophic axiom is the classical axiom. In the classical mathematics an axiom is supposed 100% true, 0% indeterminate, and 0% false. But this thing occurs in idealistic systems, in perfectly closed systems, not in many of the real world situations.

Unlike the classical axiom which is a total premise of reasoning and without any controversy, the neutrosophic axiom is a partial premise of reasoning with a partial controversy.

The neutrosophic axioms serve in approximate reasoning.

The partial truth of a neutrosophic axiom is similarly taken for granting.

The neutrosophic axioms, and in general the neutrosophic propositions, deal with approximate ideas or with probable ideas, and in general with ideas we are not able to measure exactly. That’s why one cannot get 100% true statements (propositions).

In our life we deal with approximations. An axiom is approximately true, and the inference is approximately true either.

A neutrosophic axiom is a self-evident assumption in some degrees of truth, indeterminacy, and falsehood respectively.

2 Neutrosophic Deducing and Neutrosophic Inference

The neutrosophic axioms are employed in *neutrosophic deducing* and *neutrosophic inference* rules, which are sort of neutrosophic implications, and similarly they have degrees of truth, indeterminacy, and respectively falsehood.

3 Neutrosophic Proof

Consequently, a *neutrosophic proof* has also a degree of validity, degree of indeterminacy, and degree of invalidity. And this is when we work with not-well determinate elements in the space or not not-well determinate inference rules.

The neutrosophic axioms are at the foundation of various *neutrosophic sciences*.

The approximate, indeterminate, incomplete, partially unknown, ambiguous, vagueness, imprecision, contradictory, etc. knowledge can be neutrosophically axiomized.

4 Neutrosophic Axiomatic System

A set of neutrosophic axioms Ω is called *neutrosophic axiomatic system*, where the neutrosophic deducing and the neutrosophic inference (neutrosophic implication) are used.

The neutrosophic axioms are defined on a given space S. The space can be classical (space without indeterminacy), or neutrosophic space (space which has some indeterminacy with respect to its elements).

A neutrosophic space may be, for example, a space that has at least one element which only partially belongs to the space. Let us say the element x <0.5, 0.2, 0.3> that belongs only 50% to the space, while 20% its appurtenance is indeterminate, and 30% it does not belong to the space.

Therefore, we have three types of neutrosophic axiomatic systems:

[1] Neutrosophic axioms defined on classical space;

[2] Classical axioms defined on neutrosophic space;

[3] Neutrosophic axioms defined on neutrosophic space.

Remark:

The neutrosophic axiomatic system is not unique, in the sense that several different axiomatic systems may describe the same neutrosophic model. This happens because one deals with approximations, and because the neutrosophic axioms represent partial (not total) truths.

5 Classification of the Neutrosophic Axioms

[1] *Neutrosophic Logical Axioms*, which are neutrosophic statements whose truth-value is <t, i, f> within the system of neutrosophic logic. For example: (α or β) neutrosophically implies β.

[2] *Neutrosophic Non-Logical Axioms*, which are neutrosophic properties of the elements of the space. For example: the neutrosophic associativity $a(bc) = (ab)c$, which occurs for some elements, it is unknown (indeterminate) for others, and does not occur for others.

In general, a neutrosophic non-logical axiom is a classical non-logical axiom that works for certain space elements, is indeterminate for others, and does not work for others.

6 Neutrosophic Tautologies

A classical tautology is a statement that is universally true [regarded in a larger way, or lato sensu], i.e. true in all possible worlds (according to Leibniz's definition of "world"). For example, "M = M" in all possible worlds.

A *neutrosophic tautology* is a statement that is true in a narrow way [i.e. regarded in stricto sensu], or it is <1, 0, 0> true for a class of certain parameters and conditions, and <t, i, f> true for another class of certain parameters and conditions, where <t, i, f> $\neq$ <1, 0, 0>. I.e. a neutrosophic tautology is true in some worlds, and partially true in other worlds. For example, the previous assertion: "M = M".

If "M" is a number [i.e. the parameter = number], then a number is always equal to itself in any numeration base.

But if "M" is a person [i.e. the parameter = person], call him Martin, then Martin at time t1 is the same as Martin at time t1 [i.e. it has been considered another parameter = time], but Martin at time t1 is different from Martin at time t2 (meaning for example 20 years ago: hence Martin younger is different from Martin older). Therefore, from the point of view of parameters 'person' and 'time', "M = M" is not a classical tautology.

Similarly, we may have a proposition P which is true locally, but it is untrue non-locally.

A neutrosophic logical system is an approximate minimal set of partially true/indeterminate/false propositions.

While the classical axioms cannot be deduced from other axioms, there are neutrosophic axioms that can be partially deduced from other neutrosophic axioms.

7 Notations regarding the Classical Logic and Set, Fuzzy Logic and Set, Intuitionistic Fuzzy Logic and Set, and Neutrosophic Logic and Set

In order to make distinction between classical (Boolean) logic/set, fuzzy logic/set, intuitionistic fuzzy logic/set, and neutrosophic logic/set, we denote their corresponding operators (negation/complement, conjunction/ intersection, disjunction/union, implication/inclusion, and equivalence/equality), as it follows:

[1] For classical (Boolean) logic and set:

$$\neg \quad \wedge \quad \vee \quad \rightarrow \quad \leftrightarrow \tag{1}$$

[2] For fuzzy logic and set:

$$\underset{F}{\neg} \quad \underset{F}{\wedge} \quad \underset{F}{\vee} \quad \underset{F}{\rightarrow} \quad \underset{F}{\leftrightarrow} \tag{2}$$

[3] For intuitionistic fuzzy logic and set:

$$\underset{IF}{\neg} \quad \underset{IF}{\wedge} \quad \underset{IF}{\vee} \quad \underset{IF}{\rightarrow} \quad \underset{IF}{\leftrightarrow} \tag{3}$$

[4] For neutrosophic logic and set:

$$\underset{N}{\neg} \quad \underset{N}{\wedge} \quad \underset{N}{\vee} \quad \underset{N}{\rightarrow} \quad \underset{N}{\leftrightarrow} \tag{4}$$

8 The Classical Quantifiers

The classical *Existential Quantifier* is the following way:

$$\exists x \in A, P(x). \tag{5}$$

In a neutrosophic way we can write it as:

There exist *x<1, 0, 0>* in *A* such that *P(x)<1, 0, 0>*, or:

$$\exists x < 1,0,0 > \in A, P(x) < 1,0,0 >. \tag{6}$$

The classical *Universal Quantifier* is the following way:

$$\forall x \in A, P(x). \tag{7}$$

In a neutrosophic way we can write it as:

For any *x<1, 0, 0>* in *A* one has *P(x)<1, 0, 0>*, or:

$$\forall x < 1,0,0 > \in A, P(x) < 1,0,0 >. \tag{8}$$

9 The Neutrosophic Quantifiers

The *Neutrosophic Existential Quantifier* is in the following way:

There exist $x<t_x, i_x, f_x>$ in A such that $P(x)<t_P, i_P, f_P>$, or:

$$\exists x < t_x, i_x, f_x > \in A, P(x) < t_P, i_P, f_P >, \tag{9}$$

which means that: there exists an element x which belongs to A in a neutrosophic degree $<t_x, i_x, f_x>$, such that the proposition P has the neutrosophic degree of truth $<t_P, i_P, f_P>$.

The *Neutrosophic Universal Quantifier* is the following way:

For any $x<t_x, i_x, f_x>$ in A one has $P(x)<t_P, i_P, f_P>$, or:

$$\forall x < t_x, i_x, f_x > \in A, P(x) < t_P, i_P, f_P >, \tag{10}$$

which means that: for any element x that belongs to A in a neutrosophic degree $<t_x, i_x, f_x>$, one has the proposition P with the neutrosophic degree of truth $<t_P, i_P, f_P>$.

10 Neutrosophic Axiom Schema

A *neutrosophic axiom schema* is a neutrosophic rule for generating infinitely many neutrosophic axioms.

Examples of neutrosophic axiom schema:

[1] *Neutrosophic Axiom Scheme for Universal Instantiation.*

Let Φ(x) be a formula, depending on variable x defined on a domain D, in the first-order language L, and let's substitute x for $a \in$ D. Then the new formula:

$$\forall x \Phi(x) \to_N \Phi(a) \tag{11}$$

is $< t_{\to_N}, i_{\to_N}, f_{\to_N} >$-neutrosophically [universally] valid.

This means the following: if one knows that a formula Φ(x) holds $<t_x, i_x, f_x>$-neutrosophically for every x in the domain D, and for $x = a$ the formula $\Phi(a)$ holds $<t_a, i_a, f_a>$-neutrosophically, then the whole new formula (a) holds $< t_{\to_N}, i_{\to_N}, f_{\to_N} >$-neutrosophically, where $t_{\to_N}$ means the truth degree, $i_{\to_N}$ the indeterminacy degree, and $f_{\to_N}$ the falsehood degree –- all resulted from the neutrosophic implication $\to_N$.

[2] *Neutrosophic Axiom Scheme for Existential Generalization.*

Let Φ*(x)* be a formula, depending on variable *x* defined on a domain *D*, in the first-order language *L*, and let's substitute x for $a \in$ D. Then the new formula:

$$\Phi(a) \rightarrow_N \exists x \Phi(x) \tag{12}$$

is $< t_{\rightarrow_N}, i_{\rightarrow_N}, f_{\rightarrow_N} >$-neutrosophically [universally] valid.

This means the following: if one knows that a formula Φ*(a)* holds $<t_a, i_a, f_a>$-neutrosophically for a given x = a in the domain *D*, and for every *x* in the domain formula Φ(x) holds $<t_x, i_x, f_x>$-neutrosophically, then the whole new formula (b) holds $< t_{\rightarrow_N}, i_{\rightarrow_N}, f_{\rightarrow_N} >$-neutrosophically, where $t_{\rightarrow_N}$ means the truth degree, $i_{\rightarrow_N}$ the indeterminacy degree, and $f_{\rightarrow_N}$ the falsehood degree –- all resulted from the neutrosophic implication $\rightarrow_N$.

These are *neutrosophic metatheorems* of the mathematical neutrosophic theory where they are employed.

11 Neutrosophic Propositional Logic

We have many neutrosophic formulas that one takes as neutrosophic axioms. For example, as extension from the classical logic, one has the following.

Let $P<t_P, i_P, f_P>$, $Q<t_Q, i_Q, f_Q>$, $R<t_R, i_R, f_R>$, $S<t_S, i_S, f_S>$ be neutrosophic propositions, where $<t_P, i_P, f_P>$ is the neutrosophic-truth value of *P*, and similarly for *Q, R,* and *S*. Then:

a) Neutrosophic modus ponens (neutrosophic implication elimination):

$$P \rightarrow_N (Q \rightarrow_N P) \tag{13}$$

b) Neutrosophic modus tollens (neutrosophic law of contrapositive):

$$((P \rightarrow_N Q) \wedge_N \neg_N Q) \rightarrow_N \neg_N P \tag{14}$$

c) Neutrosophic *disjunctive syllogism* (neutrosophic disjunction elimination):

$$((P \vee_N Q) \wedge_N \neg_N P) \rightarrow_N Q \tag{15}$$

d) Neutrosophic *hypothetical syllogism* (neutrosophic chain argument):

$$((P \rightarrow_N Q) \wedge_N (Q \rightarrow_N R)) \rightarrow_N (P \rightarrow_N R) \tag{16}$$

e) ***Neutrosophic constructive dilemma*** (neutrosophic *disjunctive* version of *modus ponens*):

$$(((P \to_N Q) \wedge_N (R \to_N S)) \wedge_N (P \vee_N R)) \to_N (Q \vee_N S) \tag{17}$$

f) ***Neutrosophic distructive dilemma*** (neutrosophic *disjunctive* version of *modus tollens*):

$$(((P \to_N Q) \wedge_N (R \to_N S)) \wedge_N (\neg_N Q \vee_N \neg_N S)) \to_N (\neg_N P \vee_N \neg_N R) \tag{18}$$

All these neutrosophic formulae also run as *neutrosophic rules of inference*.

These neutrosophic formulas or neutrosophic derivation rules only *partially* preserve the truth, and depending on the neutrosophic implication operator that is employed the indeterminacy may increase or decrease.

This happens for one working with approximations.

While the above classical formulas in classical proportional logic are classical tautologies (i.e. from a neutrosophical point of view they are *100%* true, *0%* indeterminate, and *0%* false), their corresponding neutrosophic formulas are neither classical tautologies nor neutrosophical tautologies, but ordinary neutrosophic propositions whose $< t, i, f >$ – neutrosophic truth-value is resulted from the $\underset{N}{\to}$ neutrosophic implication

$$A < t_A, i_A, f_A > \underset{N}{\to} B < (t_B, i_B, f_B) >. \tag{19}$$

12 Classes of Neutrosophic Negation Operators

There are defined in neutrosophic literature classes of neutrosophic negation operators as follows: if $A(t_A, i_A, f_A)$, then its negation is:

$$\underset{N}{\neg} A(f_A, i_A, t_A), \tag{20}$$

or $\underset{N}{\neg} A(f_A, 1 - i_A, t_A)$, (21)

or $\underset{N}{\neg} A(1 - t_A, 1 - i_A, 1 - f_A)$, (22)

or $\underset{N}{\neg} A(1 - t_A, i_A, 1 - f_A)$, etc. (23)

13 Classes of Neutrosophic Conjunctive Operators.

Similarly: if $A(t_A, i_A, f_A)$ and $B(t_B, i_B, f_B)$, then

$$A \overset{\wedge}{_N} B = \langle t_A \overset{\wedge}{_F} t_B, i_A \overset{\vee}{_F} i_B, f_A \overset{\vee}{_F} f_B \rangle, \tag{24}$$

or $A \overset{\wedge}{_N} B = \langle t_A \overset{\wedge}{_F} t_B, i_A \overset{\wedge}{_F} i_B, f_A \overset{\vee}{_F} f_B \rangle$, (25)

or $A \overset{\wedge}{_N} B = \langle t_A \overset{\wedge}{_F} t_B, i_A \overset{\wedge}{_F} i_B, f_A \overset{\wedge}{_F} f_B \rangle$ (26)

or $A \overset{\wedge}{_N} B = \langle t_A \overset{\wedge}{_F} t_B, \frac{i_A + i_B}{2}, f_A \overset{\vee}{_F} f_B \rangle$, (27)

or $A \overset{\wedge}{_N} B = \langle t_A \overset{\wedge}{_F} t_B, 1 - \frac{i_A + i_B}{2}, f_A \overset{\vee}{_F} f_B \rangle$, (28)

or $A \overset{\wedge}{_N} B = \langle t_A \overset{\wedge}{_F} t_B, |i_A - i_B|, f_A \overset{\vee}{_F} f_B \rangle$, etc. (29)

14 Classes of Neutrosophic Disjunctive Operators

And analogously, there were defined:

$$A \overset{\vee}{_N} B = \langle t_A \overset{\vee}{_F} t_B, i_A \overset{\wedge}{_F} i_B, f_A \overset{\wedge}{_F} f_B \rangle, \tag{30}$$

or $A \overset{\vee}{_N} B = \langle t_A \overset{\vee}{_F} t_B, i_A \overset{\vee}{_F} i_B, f_A \overset{\wedge}{_F} f_B \rangle$, (31)

or $A \overset{\vee}{_N} B = \langle t_A \overset{\vee}{_F} t_B, i_A \overset{\vee}{_F} i_B, f_A \overset{\vee}{_F} f_B \rangle$, (32)

or $A \overset{\vee}{_N} B = \langle t_A \overset{\vee}{_F} t_B, \frac{i_A + i_B}{2}, f_A \overset{\wedge}{_F} f_B \rangle$, (33)

or $A \overset{\vee}{_N} B = \langle t_A \overset{\vee}{_F} t_B, 1 - \frac{i_A + i_B}{2}, f_A \overset{\wedge}{_F} f_B \rangle$, (34)

or $A \overset{\vee}{_N} B = \langle t_A \overset{\vee}{_F} t_B, |i_A - i_B|, f_A \overset{\vee}{_F} f_B \rangle$, etc. (35)

15 Fuzzy Operators

Let $\alpha, \beta \in [0, 1]$.

15.1. The *Fuzzy Negation* has been defined as $\neg_F \alpha = 1 - \alpha$. (36)

15.2. While the class of *Fuzzy Conjunctions* (or *t-norm*) may be:

$$\alpha \overset{\wedge}{_F} \beta = \min\{\alpha, \beta\}, \tag{37}$$

or $\alpha \overset{\wedge}{_F} \beta = \alpha \cdot \beta$, (38)

or $\alpha \overset{\wedge}{_F} \beta = \max\{0, \alpha + \beta - 1\}$, etc. (39)

15.3. And the class of *Fuzzy Disjunctions* (or *t-conorm*) may be:

$$\alpha \overset{\vee}{_F} \beta = \max\{\alpha, \beta\}, \tag{40}$$

$$\text{or } \alpha \overset{\vee}{_F} \beta = \alpha + \beta - \alpha\beta, \tag{41}$$

$$\text{or } \alpha \overset{\vee}{_F} \beta = \min\{1, \alpha + \beta\}, \text{ etc.} \tag{42}$$

15.4. Examples of *Fuzzy Implications* $x \underset{F}{\rightarrow} y$, for $x, y \in [0, 1]$, defined below:

- Fodor (1993): $I_{FD}(x, y) = \begin{cases} 1, \text{ if } x \leq y \\ \max(1 - x, y), \text{ if } x > y \end{cases}$ (43)
- Weber (1983): $I_{WB}(x, y) = \begin{cases} 1, \text{if } x < y \\ y, \text{if } x = 1 \end{cases}$ (44)
- Yager (1980): $I_{YG}(x, y) = \begin{cases} 1, \text{if } x = 0 \text{ and } y = 0 \\ y^x, \text{if } x > 0 \text{ or } y > 0 \end{cases}$ (45)
- Goguen (1969): $I_{GG}(x, y) = \begin{cases} 1, \text{if } x \leq y \\ \frac{y}{x}, \text{if } x > y \end{cases}$ (46)
- Rescher (1969): $I_{RS}(x, y) = \begin{cases} 1, \text{if } x \leq y \\ 0, \text{if } x > y \end{cases}$ (47)
- Kleene-Dienes (1938): $I_{KD}(x, y) = \max(1 - x, y)$ (48)
- Reichenbach (1935): $I_{RC}(x, y) = 1 - x + xy$ (49)
- Gödel (1932): $I_{GD}(x, y) = \begin{cases} 1, \text{if } x \leq y \\ y, \text{if } x > y \end{cases}$ (50)
- Lukasiewicz (1923): $I_{LK}(x, y) = \min(1, 1 - x + y)$, (51)

according to the list made by Michal Baczyński and Balasubramaniam Jayaram (2008).

16 Example of Intuitionistic Fuzzy Implication

Example of *Intuitionistic Fuzzy Implication* $A(t_A, f_A) \underset{IF}{\rightarrow} B(t_B, f_B)$ is:

$$I_{IF} = \left(\left[(1 - t_A) \overset{\vee}{_F} t_B\right] \overset{\wedge}{_F} \left[(1 - f_B) \overset{\vee}{_F} f_A\right], f_B \overset{\wedge}{_F} (1 - t_A) \right), \tag{52}$$

according to Yunhua Xiao, Tianyu Xue, Zhan'ao Xue, and Huiru Cheng (2011).

17 Classes of Neutrosophic Implication Operators

We now propose for the first time *eight new classes of neutrosophic implications* and extend a ninth one defined previously:

$$A(t_A, i_A, f_A) \underset{N}{\to} B(t_B, i_B, f_B),$$

in the following ways:

17.1-17.2. $I_{N1}\left(t_A \xrightarrow[F/IF]{} t_B, i_A \overset{\wedge}{_F} i_B, f_A \overset{\wedge}{_F} f_B\right),$ (53)

where $t_A \xrightarrow[F/IF]{} t_B$ is any fuzzy implication (from above or others) or any intuitionistic fuzzy implication (from above or others), while $\overset{\wedge}{_F}$ is any fuzzy conjunction (from above or others);

17.3-17.4. $I_{N2}\left(t_A \xrightarrow[F/IF]{} t_B, i_A \overset{\vee}{_F} i_B, f_A \overset{\wedge}{_F} f_B\right),$ (54)

where $\overset{\vee}{_F}$ is any fuzzy disjunction (from above or others);

17.5-17.6. $I_{N3}\left(t_A \xrightarrow[F/IF]{} t_B, \frac{i_A+i_B}{2}, f_A \overset{\wedge}{_F} f_B\right);$ (55)

17.7-17.8. $I_{N4}\left(t_A \xrightarrow[F/IF]{} t_B, \frac{i_A+i_B}{2}, \frac{f_A+f_B}{2}\right).$ (56)

17.9. Now we extend another neutrosophic implication that has been defined by S. Broumi & F. Smarandache (2014) and it was based on the classical logical equivalence:

$$(A \to B) \leftrightarrow (\neg A \vee B). \quad (57)$$

Whence, since the corresponding neutrosophic logic equivalence:

$$\left(A \underset{N}{\to} B\right) \underset{N}{\leftrightarrow} (\underset{N}{\neg} A \overset{\vee}{_N} B) \quad (58)$$

holds, one obtains another *Class of Neutrosophic Implication Operators* as:

$$(\underset{N}{\neg} A \overset{\vee}{_N} B) \quad (59)$$

where one may use any neutrosophic negation $\underset{N}{\neg}$ (from above or others), and any neutrosophic disjunction $\underset{N}{\vee}$ (from above or others).

18 Example of Neutrosophic Implication

Let's see an *Example of Neutrosophic Implication*.

Let's have two neutrosophic propositions $A\langle 0.3, 0.4, 0.2\rangle$ and $B\langle 0.7, 0.1, 0.4\rangle$. Then $A \underset{N}{\rightarrow} B$ has the neutrosophic truth value of $\underset{N}{\neg} A \underset{N}{\vee} B$, i.e.:

$$\langle 0.2, 0.4, 0.3\rangle \underset{N}{\vee} \langle 0.7, 0.1, 0.4\rangle,$$

$$\text{or } \langle \max\{0.2, 0.7\}, \min\{0.4, 0.1\}, \min\{0.3, 0.4\}\rangle,$$

$$\text{or } \langle 0.7, 0.1, 0.3\rangle,$$

where we used the neutrosophic operators defined above: $\underset{N}{\neg}\langle t, i, f\rangle = \langle f, i, t\rangle$ for neutrosophic negation, and $\langle t_1, i_1, f_1\rangle \underset{N}{\vee} \langle t_2, i_2, f_2\rangle = \langle \max\{t_1, t_2\}, \min\{i_1, i_2\}, \min\{f_1, f_2\}\rangle$ for the neutrosophic disjunction.

Using different versions of the neutrosophic negation operators and/or different versions of the neutrosophic disjunction operators, one obtains, in general, different results. Similarly as in fuzzy logic.

18.1. Another Example of Neutrosophic Implication.

Let A have the neutrosophic truth-value (t_A, i_A, f_A) , and B have the neutrosophic truth-value (t_B, i_B, f_B), then:

$$\left[A \underset{N}{\rightarrow} B\right] \underset{N}{\leftrightarrow} [(\underset{N}{\neg} A) \underset{N}{\vee} B], \tag{60}$$

where $\underset{N}{\neg}$ is any of the above neutrosophic negations, while $\underset{N}{\vee}$ is any of the above neutrosophic disjunctions.

19 General Definition of Neutrosophic Operators

We consider that the most general definition of neutrosophic operators shall be the followings:

$$A(t_A, i_A, f_A) \underset{N}{\oplus} B(t_B, i_B, f_B) = A \underset{N}{\oplus} B\langle u(t_A, i_A, f_A, t_B, i_B, f_B), \\ v(t_A, i_A, f_A, t_B, i_B, f_B),\ w(t_A, i_A, f_A, t_B, i_B, f_B)\rangle \tag{61}$$

where $\underset{N}{\oplus}$ is any binary neutrosophic operator, and

$$u(x_1, x_2, x_3, x_4, x_5, x_6), v(x_1, x_2, x_3, x_4, x_5, x_6),$$

$$w(x_1, x_2, x_3, x_4, x_5, x_6) \colon [0,1]^6 \rightarrow [0,1].$$

Even more, the neutrosophic component functions u, v, w may depend, on the top of these six variables, on hidden parameters as well, such as: $h_1, h_2, \dots, h_n$.

For a unary neutrosophic operator (for example, the neutrosophic negation), similarly:

$$\underset{N}{\neg}A(t_A, i_A, f_A) = \langle u'(t_A, i_A, f_A), v'(t_A, i_A, f_A), w'(t_A, i_A, f_A) \rangle, \qquad (62)$$

where $u'(t_A, i_A, f_A), v'(t_A, i_A, f_A), w'(t_A, i_A, f_A): [0,1]^3 \to [0,1]$,

and even more u', v', w' may depend, on the top of these three variables, of hidden parameters as well, such as: $h_1, h_2, \dots, h_n$.

{Similarly there should be for a *general definition of fuzzy operators* and *general definition of intuitionistic fuzzy operators*.}

As an example, we have defined [6]: (63)

$$A(t_A, i_A, f_A) \underset{N}{\wedge} B(t_B, i_B, f_B) \\ = \langle t_A t_B, i_A i_B + t_A i_B + t_B i_A, t_A f_B + t_B f_A + i_A f_B + i_B f_A \rangle$$

these result from multiplying

$$(t_A + i_A + f_A) \cdot (t_B + i_B + f_B) \qquad (64)$$

and ordering upon the below pessimistic order:

truth $\prec$ indeterminacy $\prec$ falsity,

meaning that to the *truth* only the terms of t's goes, i.e. $t_A t_B$,

to *indeterminacy* only the terms of t's and i's go, i.e. $i_A i_B + t_A i_B + t_B i_A$,

and to *falsity* the other terms left, i.e. $t_A f_B + t_B f_A + i_A f_B + i_B f_A + f_A f_B$.

20 Neutrosophic Deductive System

A *Neutrosophic Deductive System* consists of a set $\mathcal{L}_1$ of neutrosophic logical axioms, and a set $\mathcal{L}_2$ of neutrosophic non-logical axioms, and a set $\mathcal{R}$ of neutrosophic rules of inference – all defined on a neutrosophic space $\mathcal{S}$ that is composed of many elements.

A neutrosophic deductive system is said to be neutrosophically complete, if for any neutrosophic formula φ that is a neutrosophic logical consequence of $\mathcal{L}_1$, i.e. $\mathcal{L}_1 \underset{N}{\vDash} \varphi$, there exists a neutrosophic deduction of φ from $\mathcal{L}_1$, i.e. $\mathcal{L}_1 \underset{N}{\vdash} \varphi$, where $\underset{N}{\vDash}$ denotes neutrosophic logical consequence, and $\underset{N}{\vdash}$ denotes neutrosophic deduction.

Actually, everything that is neutrosophically (partially) true [i.e. made neutrosophically (partially) true by the set $\mathcal{L}_1$ of neutrosophic axioms] is neutrosophically (partially) provable.

The neutrosophic completeness of set $\mathcal{L}_2$ of neutrosophic non-logical axioms is not the same as the neutrosophic completeness of set $\mathcal{L}_1$ of neutrosophic logical axioms.

21 Neutrosophic Axiomatic Space

The space $\mathcal{S}$ is called *neutrosophic space* if it has some indeterminacy with respect to one or more of the following:

a. Its *elements*;

1. At least one element x partially belongs to the set $\mathcal{S}$, or $x(t_x, i_x, f_x)$ with $(t_X, i_X, f_X) \neq (1, 0, 0)$;

2. There is at least an element y in $\mathcal{S}$ whose appurtenance to $\mathcal{S}$ is unknown.

b. Its *logical axioms*;

1. At least a logical axiom $\mathcal{A}$ is partially true, or $\mathcal{A}(t_A, i_A, f_A)$, where similary $(t_A, i_A, f_A) \neq (1, 0, 0)$;

2. There is at least an axiom $\mathcal{B}$ whose truth-value is unknown.

c. Its *non-logical axioms*;

1. At least a non-logical axiom $\mathcal{C}$ is true for some elements, and indeterminate or false or other elements;

2. There is at least a non-logical axiom whose truth-value is unknown for some elements in the space.

d. There exist at least two neutrosophic logical axioms that have some degree of contradiction (strictly greater than zero).

e. There exist at least two neutrosophic non-logical axioms that have some degree of contradiction (strictly greater than zero).

22 Degree of Contradiction (Dissimilarity) of Two Neutrosophic Axioms

Two neutrosophic logical axioms $\mathcal{A}_1$ and $\mathcal{A}_2$ are contradictory (dissimilar) if their semantics (meanings) are contradictory in some degree d_1, while their neutrosophic truth values $<t_1, i_1, f_1>$ and $<t_2, i_2, f_2>$ are contradictory in a different degree d_2 [in other words $d_1 \neq d_2$].

As a particular case, if two neutrosophic logical axioms $\mathcal{A}_1$ and $\mathcal{A}_2$ have the same semantic (meaning) [in other words $d_1 = 0$], but their neutrosophic truth-values are different [in other words $d_2 > 0$], they are contradictory.

Another particular case, if two neutrosophic axioms $\mathcal{A}_1$ and $\mathcal{A}_2$ have different semantics (meanings) [in other words $d_1 > 0$], but their neutrosophic truth values are the same $<t_1, i_1, f_1> = <t_2, i_2, f_2>$ [in other words $d_2 = 0$], they are contradictory.

If two neutrosophic axioms $\mathcal{A}_1$ and $\mathcal{A}_2$ have the semantic degree of contradiction d_1, and the neutrosophic truth value degree of contradiction d_2, then the total degree of contradiction of the two neutrosophic axioms is $d = |d_1 - d_2|$, where $|\ \ |$ mean the absolute value.

We did not manage to design a formula in order to compute the semantic degree of contradiction d_1 of two neutrosophic axioms. The reader is invited to explore such metric.

But we can compute the neutrosophic truth value degree of contradiction d_2. If $\langle t_1, i_1, f_1 \rangle$ is the neutrosophic truth-value of $\mathcal{A}_1$ and $\langle t_2, i_2, f_2 \rangle$ the neutrosophic truth-value of $\mathcal{A}_2$, where $t_1, i_1, f_1, t_2, i_2, f_2$ are single values in $[0, 1]$, then the neutrosophic truth value degree of contradiction d_2 of the neutrosophic axioms $\mathcal{A}_1$ and $\mathcal{A}_2$ is:

$$d_2 = \frac{1}{3}(|t_1 - t_2| + |i_1 - i_2| + |f_1 - f_2|), \qquad (65)$$

whence $d_2 \in [0, 1]$.

We get $d_2 = 0$, when $\mathcal{A}_1$ is identical with $\mathcal{A}_2$ from the point of view of neutrosophical truth values, i.e. when $t_1 = t_2, i_1 = i_2, f_1 = f_2$. And we get $d_2 = 1$, when $\langle t_1, i_1, f_1 \rangle$ and $\langle t_2, i_2, f_2 \rangle$ are respectively equal to:

$\langle 1, 0, 0 \rangle$, $\langle 0, 1, 1 \rangle$;

or $\langle 0, 1, 0 \rangle$, $\langle 1, 0, 1 \rangle$;

or $\langle 0, 0, 1 \rangle$, $\langle 1, 1, 0 \rangle$;

or $\langle 0, 0, 0 \rangle$, $\langle 1, 1, 1 \rangle$.

23 Neutrosophic Axiomatic System

The neutrosophic axioms are used, in neutrosophic conjunction, in order to derive neutrosophic theorems.

A *neutrosophic mathematical theory* may consist of a neutrosophic space where a neutrosophic axiomatic system acts and produces all neutrosophic theorems within the theory.

Yet, in a *neutrosophic formal system*, in general, the more recurrences are done the more is increased the indeterminacy and decreased the accuracy.

24 Properties of the Neutrosophic Axiomatic System

[1] While in classical mathematics an axiomatic system is consistent, in a neutrosophic axiomatic system it happens to have *partially inconsistent (contradictory) axioms*.

[2] Similarly, while in classical mathematics the axioms are independent, in a neutrosophic axiomatic system they may be *dependent in certain degree*.

[3] In classical mathematics if an axiom is dependent from other axioms, it can be removed, without affecting the axiomatic system.

[4] However, if a neutrosophic axiom is partially dependent from other neutrosophic axioms, by removing it the neutrosophic axiomatic system is affected.

[5] While, again, in classical mathematics an axiomatic system has to be complete (meaning that each statement or its negation is derivable), a neutrosophic axiomatic system is *partially complete and partially incomplete*. It is partially incomplete because one can add extra partially independent neutrosophic axioms.

[6] The *neutrosophic relative consistency* of an axiomatic system is referred to the neutrosophically (partially) undefined terms of a first neutrosophic axiomatic system that are assigned neutrosophic definitions from another neutrosophic axiomatic system in a way that, with respect to both neutrosophic axiomatic systems, is neutrosophically consistent.

25 Neutrosophic Model

A *Neutrosophic Model* is a model that assigns neutrosophic meaning to the neutrosophically (un)defined terms of a neutrosophic axiomatic system.

Similarly to the classical model, we have the following classification:

[1] *Neutrosophic Abstract Model*, which is a neutrosophic model based on another neutrosophic axiomatic system.

[2] *Neutrosophic Concrete Model*, which is a neutrosophic model based on real world, i.e. using real objects and real relations between the objects.

In general, a neutrosophic model is a *<t, i, f>-approximation*, i.e. *T%* of accuracy, *I%* indeterminacy, and *F%* inaccuracy, of a neutrosophic axiomatic system.

26 Neutrosophically Isomorphic Models

Further, *two neutrosophic models are neutrosophically isomorphic* if there is a neutrosophic one-to-one correspondence between their neutrosophic elements such that their neutrosophic relationships hold.

A neutrosophic axiomatic system is called *neutrosophically categorial* (or *categorical*) is any two of its neutrosophic models are neutrosophically isomorphic.

27 Neutrosophic Infinite Regressions

There may be situations of neutrosophic axiomatic systems that generate neutrosophic infinite regressions, unlike the classical axiomatic systems.

28 Neutrosophic Axiomatization

A *Neutrosophic Axiomatization* is referred to an approximate formulation of a set of neutrosophic statements, about a number of neutrosophic primitive terms, such that by the neutrosophic deduction one obtains various neutrosophic propositions (theorems).

29 Example of Neutrosophic Axiomatic System

Let's consider two neighboring countries M and N that have a disputed frontier zone Z:

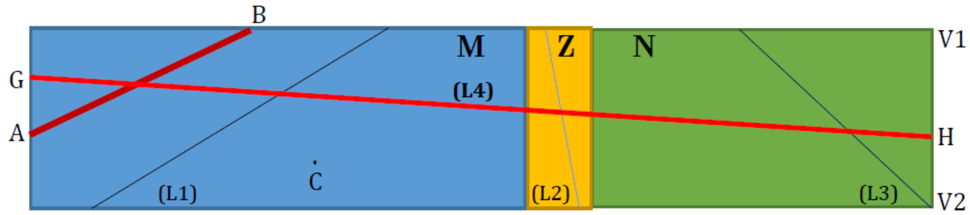

Figure 1: A Neutrosophic Model.

Let's consider the universe of discourse $U = M \cup Z \cup N$; this is a *neutrosophic space* since it has an indeterminate part (the disputed frontier).

The *neutrosophic primitive notions* in this example are: neutrosophic point, neutrosophic line, and neutrosophic plane (space).

And the *neutrosophic primitive relations* are: neutrosophic incidence, and neutrosophic parallel.

The four boundary edges of rectangle Z belong to Z (or Z is a closed set). While only three boundary edges of M (except the fourth one which is common with Z) belong to M, and similarly only three boundaries of N (except the fourth one which is common with Z) belong to N. Therefore M and N are neither closed nor open sets.

Taking a classical point P in U, one has three possibilities:

[1] $P \in M$ (membership with respect to country M);
[2] $P \in Z$ (indeterminate membership with respect to both countries);
[3] or $P \in N$ (nonmembership with respect to country M).

Such points, that can be indeterminate as well, are called *neutrosophic points*.

A *neutrosophic line* is a classical segment of line that unites two neutrosophic points lying on opposite edges of the universe of discourse U. We may have:

[1] determinate line (with respect to country M), that is completely into the determinate part M {for example ($L1$)};
[2] indeterminate line, that is completely into the frontier zone *{*for example *(L2)}*;
[3] determinate line (with respect to country N), that is completely into the determinate part N *{*for example *(L3)}*;

[4] or mixed, i.e. either two or three of the following: partially determinate with respect to *M*, partially indeterminate with respect to both countries, and partially determinate with respect to *N {*for example the red line *(L4)}.*

Through two neutrosophic points there may be passing:

[1] only one neutrosophic line {for example, through *G* and *H* passes only one neutrosophic line (*L4*)};
[2] no neutrosophic line {for example, through *A* and *B* passes no neutrosophic line, since the classical segment of line *AB* does not unite points of opposite edges of the universe of discourse *U*}.

Two *neutrosophic lines* are *parallel* is they have no common neutrosophic points.

Through a neutrosophic point outside of a neutrosophic line, one can draw:

[1] infinitely many neutrosophic parallels {for example, through the neutrosophic point *C* one can draw infinitely many neutrosophic parallels to the neutrosophic line *(L1)*};
[2] only one neutrosophic parallel {for example, through the neutrosophic point *H* that belongs to the edge *(V1V2)* one can draw only one neutrosophic parallel (i.e. V1V2) to the neutrosophic line *(L1)*};
[3] no neutrosophic parallel {for example, through the neutrosophic point *H* there is no neutrosophic parallel to the neutrosophic line *(L3)*}.

For example, the neutrosophic lines (*L1), (L2)* and (*L3)* are parallel. But the neutrosophic line *(L4)* is not parallel with *(L1),* nor with *(L2)* or *(L3)*.

A *neutrosophic polygon* is a classical polygon which has one or more of the following indeterminacies:

[1] indeterminate vertex;
[2] partially or totally indeterminate edge;
[3] partially or totally indeterminate region in the interior of the polygon.

We may construct several neutrosophic axiomatic systems, for this example, referring to incidence and parallel.

a) First neutrosophic axiomatic system

α1) Through two distinct neutrosophic points there is passing a single neutrosophic line.

> {According to several experts, the neutrosophic truth-value of this axiom is *<0.6, 0.1, 0.2>,* meaning that having two given neutrosophic points, the chance that only one line (that do not intersect the indeterminate zone *Z*) passes through them is *0.6,* the chance that line that passes through them intersects the indeterminate zone *Z*) is *0.1*, and the chance that no line (that does not intersect the indeterminate zone *Z*) passes through them is *0.2*.}

α2) Through a neutrosophic point exterior to a neutrosophic line there is passing either one neutrosophic parallel or infinitely many neutrosophic parallels.

> {According to several experts, the neutrosophic truth-value of this axiom is *<0.7, 0.2, 0.3>,* meaning that having a given neutrosophic line and a given exterior neutrosophic point, the chance that infinitely many parallels pass through this exterior point is *0.7*, the chance that the parallels passing through this exterior point intersect the indeterminate zone *Z* is *0.2*, and the chance that no parallel passes through this point is *0.3.*}

Now, let's apply a first neutrosophic deducibility.

Suppose one has three non-collinear neutrosophic (distinct) points *P, Q,* and *R* (meaning points not on the same line, alike in classical geometry). According to the neutrosophic axiom *(α1),* through *P, Q* passes only one neutrosophic line {let's call it *(PQ)*}, with a neutrosophic truth value *(0.6, 0.1, 0.2).* Now, according to axiom *(α2)*, through the neutrosophic point *R,* which does not lie on *(PQ),* there is passing either only one neutrosophic parallel or infinitely many neutrosophic parallels to the neutrosophic line (PQ), with a neutrosophic truth value *(0.7, 0.2, 0.3).*

Therefore,

$$(\alpha 1) \underset{N}{\wedge} (\alpha 2) = <0.6,\ 0.1,\ 0.2> \underset{N}{\wedge} <0.7,\ 0.2,\ 0.3> = <\min\{0.6,\ 0.7\}, \max\{0.1,\ 0.2\}, \max\{0.2,\ 0.3\}>= <0.6, 0.2, 0.3>, \tag{66}$$

which means the following: the chance that through the two distinct given neutrosophic points *P* and *Q* passes only one neutrosophic line, and through the exterior neutrosophic point *R* passese either only one neutrosophic parallel or infinitely many parallels to *(PQ)* is *(0.6, 0.2, 0.3),* i.e. *60%* true, *20%* indeterminate, and *30%* false.

Herein we have used the simplest neutrosophic conjunction operator $\underset{N}{\wedge}$ of the form <*min, max, max*>, but other neutrosophic conjunction operator can be used as well.

A second neutrosophic deducibility:

Again, suppose one has three non-collinear neutrosophic (distinct) points *P, Q,* and *R* (meaning points not on the same line, as in classical geometry).

Now, let's compute the neutrosophic truth value that through *P* and *Q* is passing one neutrosophic line, but through *Q* there is no neutrosophic parallel to *(PQ)*.

$$\alpha 1 \underset{N}{\wedge} (\underset{N}{\neg} \alpha 2) = <0.6, 0.1, 0.2> \underset{N}{\wedge} (\underset{N}{\neg} <0.7, 0.2, 0.3>) = <0.6, 0.1, 0.2> \underset{N}{\wedge} <0.3, 0.2, 0.7> = <0.3, 0.2, 0.7>. \tag{67}$$

b) Second neutrosophic axiomatic system

β1) Through two distinct neutrosophic points there is passing either a single neutrosophic line or no neutrosophic line. {With the neutrosophic truth-value *<0.8, 0.1, 0.0>*}.

β2) Through a neutrosophic point exterior to a neutrosophic line there is passing either one neutrosophic parallel, or infinitely many neutrosophic parallels, or no neutrosophic parallel. {With the neutrosophic truth-value *<1.0, 0.2, 0.0>*}.

In this neutrosophic axiomatic system the above propositions W1 and W2:

W1: Through two given neutrosophic points there is passing only one neutrosophic line, and through a neutrosophic point exterior to this neutrosophic line there is passing either one neutrosophic parallel or infinitely many neutrosophic parallels to the given neutrosophic line; and W2: Through two given neutrosophic points there is passing only one neutrosophic line, and through a neutrosophic point exterior to this neutrosophic line there is passing no neutrosophic parallel to the line; are not deducible.

c) Third neutrosophic axiomatic system

γ1) Through two distinct neutrosophic points there is passing a single neutrosophic line.

{With the neutrosophic truth-value *<0.6, 0.1, 0.2>*}.

γ2) Through two distinct neutrosophic points there is passing no neutrosophic line.

{With the neutrosophic truth-value *<0.2, 0.1, 0.6>*}.

δ1) Through a neutrosophic point exterior to a neutrosophic line there is passing only one neutrosophic parallel.

{With the neutrosophic truth-value *<0.1, 0.2, 0.9>*}.

δ2) Through a neutrosophic point exterior to a neutrosophic line there are passing infinitely many neutrosophic parallels.

{With the neutrosophic truth-value *<0.6, 0.2, 0.4>*}.

δ3) Through a neutrosophic point exterior to a neutrosophic line there is passing no neutrosophic parallel.

{With the neutrosophic truth-value *<0.3, 0.2, 0.7>*}.

In this neutrosophic axiomatic system we have contradictory axioms:

- *(γ1)* is in *100%* degree of contradiction with *(γ2);*
- and similarly *(δ3)* is in *100%* degree of contradiction with *[(δ1)* together with *(δ2)].*

Totally or partially contradictory axioms are allowed in a neutrosophic axiomatic systems, since they are part of our imperfect world and since they approximately describe models that are - in general - partially true.

Regarding the previous two neutrosophic deducibilities one has: (68)

$$\gamma 1 \underset{N}{\wedge} (\delta 1 \underset{N}{\vee} \delta 2) = <0.6,\ 0.1,\ 0.2> \underset{N}{\wedge} \left(<0.1, 0.2, 0.9> \underset{N}{\vee} <0.6, 0.2, 0.4>\right) = <0.6, 0.1, 0.2> \underset{N}{\wedge} <\max\{0.1,\ 0.6\},\ \min\{0.2,\ 0.2\}, \min\{0.9, 0.4\}> = <0.6, 0.1, 0.2> \underset{N}{\wedge} <0.6,\ 0.2, 0.4> = <0.6, 0.2, 0.4>,$$

which is slightly different from the result we got using the first neutrosophic axiomatic system *<0.6, 0.2, 0.3>,* and respectively*:*

$$\gamma 1 \underset{N}{\wedge} \delta 3 = <0.6,\ 0.1,\ 0.2> \underset{N}{\wedge} <0.3, 0.2, 0.7> = <0.3,\ 0.2,\ 0.7>, \qquad (69)$$

which is the same as the result we got using the first neutrosophic axiomatic system.

The third neutrosophic axiomatic system is a refinement of the first and second neutrosophic axiomatic systems. From a deducibility point of view it is better and easier to work with a refined system than with a rough system.

30 Conclusion

This paper proposes a new framework to model interdependencies in project portfolio. NCM representation model is used for modeling relation among risks.

In many real world situations, the spaces and laws are not exact, not perfect. They are inter-dependent. This means that in most cases they are not 100% true, i.e. not universal. For example, many physical laws are valid in ideal and perfectly closed systems. However, perfectly closed systems do not exist in our heterogeneous world where we mostly deal with approximations. Also, since in the real world there is not a single homogenous space, we have to use the multispace for any attempt to unify various theories.

We do not have perfect spaces and perfect systems in reality. Therefore, many physical laws function approximatively (see [5]). The physical constants are not universal too; variations of their values depend from a space to another, from a system to another. A physical constant is t% true, i% indeterminate, and f% false in a given space with a certain composition, and it has a different neutrosophical truth value <t', i', f'> in another space with another composition.

A neutrosophic axiomatic system may be dynamic: new axioms can be added and others excluded.

The neutrosophic axiomatic systems are formed by axioms than can be partially dependent (redundant), partially contradictory (inconsistent), partially incomplete, and reflecting a partial truth (and consequently a partial indeterminacy and a partial falsehood) - since they deal with approximations of reality.

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