LINFAN MAO

# COMBINATIRIAL THEORY ON THE UNIVERSE

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## Science: Local or Conditional Truth

As is known to all, science leads human activities and its function is to promote the material and spiritual civilization of humans, improve humans ability for surviving and benefit humans ourselves. In this way, believing in science and acting according to scientific laws is a basic principle of human's conduct. But *how many humans can understand this principle correctly?* The answer is unclear because most humans are standing on the humans side, understand this principle simply just for human benefits and ignore the intrusion of human activities on the nature. This is a narrow understanding of science because in the binary system consisting of humans and nature, the effect of nature on humans is immediate, visible by humans at once. However, the reaction of nature on humans caused by human's action on the nature is a delayed effect. It will appear only if the disturbance of nature caused by human activities accumulated to a certain amount, which will forms a disaster reaction on humans, i.e., produces the effect "*from quantitative change to qualitative change*" such as those destruction of the ozone layer, temperature rise, ice caps melt, extreme weather, drought and virus mutation, etc., also harmful to human development. It should be noted that this cumulative effects alone may not be visible to contemporary ones. Whence, one can not standing only on the humans side in response to scientific functions, can not only see the benefits of science to humans in present and allow the intrusion on the naturae without limitation. They should put the science with its application in the harmonious coexistence of humans with the nature and discuss its contribution and the harm to humans because science's benefits to humans should first guarantee the sustainability of human reproduction, i.e., not only the benefits to the present generation but also to the benefits of future generations. This ruler should be the basis or scientific motivation for the continuation of human civilization.

We should answer a basic question for recognition before exploring the science's function, i.e., *whether science is absolutely true or only local and conditional truth?* The answer lies in a famous fable, i.e., the blind men with an elephant. In this fable, *why did the blind men respectively perceived the shape of an elephant as a pillar, a rope, a radish, a big fan, a wall or a pipe?* Their answers are so different from the shape of an elephant in eyes of an ordinary man. Why do they so answer is because of the blind men lack of vision. They can only perceive the shape of an elephant by touching parts of the elephant's body with their hands and different parts of an elephant are bound to be different perception. Indeed, the blind men touch different parts of the elephant's body for perceiving the shape of an elephant. Similarly, science is the human recognition of things in the universe, which is similar to the situation of the blind men in this fable. It is human's local recognition of unknown things with known characteristics. Consequently, the scientific recognition is not the real face of things but a local or conditional knowing of things. This recognitive limitation comes from the limitation of "*six sense organs*", i.e., the eyes, ears, nose, tongue, body and mind of human in perceiving things [2-3]. Compared with an ordinary man, a blind man is lack of vision, only with five or less sense organs in the perception of things. This is the reason why it results in the different recognition of blind men on an elephant shape. So, how to solve the recognitive limitations of humans is an important question. The answer is what the sophist said to the blind men in the fable, namely, "*You are all right about the elephant! The reason why you think the elephant's shape different is because each of you touches the different part of the elephant's body. In fact, an elephant has those all characteristics that you are talking about!*" Notice that the sophist uses the "*six sense organs*" of human to arouse the recognition of the blind men only with five or less sense organs on the shape of an elephant, which is also applicable to the perception of things by the "*six sense organs*" of human. That is, the reality of a thing $T$ should be the combination of all local recognitions on $T$, i.e, the *combinatorial reality* and we should hold on the reality of things in the universe by the combinatorial notion. Certainly, the combinatorial notion on the reality of things is really a philosophical thought that humans follow the guidance of the sophist in the fable of the blind men with an elephant to solve the recognitive limitations of humans and then hold on the reality of things. A further generalization of this recognitive way that the sophist told the blind men is called the *Smarandache multispace* or *multisystem.* For example, the unified field theory, gauge field theory, electroweak theory and the standard model of particles are all Smarandache multisystems. Furthermore, a combinatorial model for the recognition of a thing $T$ can be established on its Smarandache multispace or multisystem by considering the intersection of spaces or systems to form a combinatorial structure, a combinatorial notion also. For example, the 12 meridians on which the vital energy are running is such a combinatorial model for recognizing the body of human in the traditional Chinese medicine, which are correspondent to the viscera organs of human body.

Usually, one needs to find the "*cause*" of a thing for its appearance "*effect*" and holds on the causal relationship of things. Among them, a systemic way is to decompose a thing by its appearance into the smallest units called *elements*, including the molecules, atoms, nuclei, leptons and quarks in material composition as well as the cells and genes in biological composition. Then, a behavior of the thing is assumed can be understood by its element behaviors, and the "*effect*" of its appearance is traced by the "*cause*" of the elements. This is the recognitive thought of reductionism. However, *is it possible to hold on anything in this way?* Certainly, it is an ideal model for systematically understand the causality of a thing. The difficulty lies in how to determine the elements from its appearance of things and characterize the action of elements. In this process, *how to determine the elements and how to characterize the behavior or the action of elements?* Such questions may become also the obstacle for holding on the truth of things. It is for this reason that our science is still a local knowledge of things rather than the reality of things. In this case, science with its applications needs to be evaluated under the harmonious coexistence of humans with the nature once again, to verify whether it is promoting human civilization rather than harming humans or excessive intrusion into the nature which will affects humans finally.

Realized this point, taking the quantitative recognition of things as the main line constraint on the ruler of harmonious coexistence of humans with the nature and using the combinatorial notion as the recognitive thought of things, it is a meaningful thing to systematically review and reflect on mathematical science and philosophy for the recognition of humans. In fact, I have been most concerned about the relationship between mathematical science and the recognition of things in the past ten years. Most of the topics that I reported in some academic conferences are related to the word *reality* because I think it is the most important that science needs to solve and it is necessary coming back to the philosophical thought of the combinatorial notion on the reality of things. Generally, a thing inherits a topological structure in space. We should establish an envelope mathematics, i.e., the "*mathematical combinatorics*" for solving the limitations of scientific recognition and gradually tend to the reality of things. This is the initial intention that I wrote the book *Combinatorial Theory on the Universe*. For this objective, I choose the contents and arrange the order of chapters. In its expression, I apply the dialogue of a father with his daughter in sections, also with some vivid images to help the reader understand easily this book. In fact, if one removing the mathematical formulas and deduction, this book can be used as a popular scientific book. Notice that the contents containing mathematical formulas with deduction, including the last two chapters on the philosophy of science in this book is to guide those who are interested in mathematical combinatorics and aim to help them further study on the literatures for related topics and then, realize the harmonious coexistence of humans with the nature.

There are 12 chapters in this book with main contents mentioned in the following.

Chapters 1 – 2 are an introduction to scientific recognition. Among them, Chapter 1 "*Ultimate Questions on the Universe*" presents the ultimate questions of the universe in the voice of a schoolboy, namely *where do we come from, where we will go?* including the celestial bodies, the earth, plants and animals and aims to briefly introduce scientific hypotheses or answers to such questions for children. Usually, a scientific answer often do not satisfy students in primary school because they do not believe in authority and like to repeatedly ask "*why*" for their frank desire of knowledge, which often leads to the disappearance of an adult's answer. But this frank attitude to study is exactly the quality of one engaged in scientific research. This chapter also introduces the answers to such ultimate problems in religion and legends in Chinese culture. It is not to propagate the superstition but in comparison because the answers in religious or the cultural legends are more vivid than scientific explanations and more accessible to primary school students. This is the object that the school education should pays attention to. Chapter 2 "*Perceptible Limitation on the Universe*" explains the limitation of human recognition of things, including the origin of human evolution and the legend of religion or God creation. Certainly, the six sense organs of human impact on the recognition. And in recognition, how to understand that "being out of non-being" and how to construct phenomenological theories by recognition of things such as grain cultivation and livestock rearing in the early period of humans are introduced. As an example, this chapter takes the double-slit experiment of physics in Figure 3 to summarize the limitations of human recognition, including the structural limitations of human eyes, ears, nose, tongue, body and mind which form three cases of perceptible unknown, unknowable and conditional limitation of humans. This chapter also introduces a few legends such as the Nuwa creation, Shennong taste herbs, the story of Adam and Eve ate forbidden fruits, the phenomenological theories such as the traditional Chinese medicine, the Kepler three laws of planets, etc.

Chapter 3 "*Combinatorial Notion on the Universe*" explains the combinatorial notion of systematic recognition of things on local recognitions and verifies the combinatorial notion by summarizing the methods of recognition in physics, chemistry and biology as examples. It begins with the famous fable of the blind men with an elephant as an introduction with graphs, labeled graphs and topological graphs in space and then, shows the reductionism establishes local sciences such as it had done in physics, chemistry or biology but we can establish an envelope science by the combinatorial notion, i.e., the science over inherited topological structures of things. For counting elements in a Smarandache multispace or multisystem, this chapter introduces the *inclusion-exclusion principle* and *pigeonhole principle*, also introduces a generalization of the pigeonhole principle, i.e., the existence of substructure with certain relationships among individuals if the number of individuals large enough, which affirmatively answers the Ramsey problem in combinatorics.

Certainly, the combinatorial notion of things derives easily the "*technological combination*" in theory. However, we can get this conclusion by simulating the decomposition of matter. In fact, the social development is reflected by the improvement of human's ability adapting to the nature such as those of the improvement and innovation of artificial appliances, devices and facilities. Among them, a technology is a kind of methods, techniques and means to make such appliances, devices or facilities possible. Similar to the structure of matter, a technology can be viewed as a combination of blocks which have their own combinatorial structure and the block is a combination of sub-blocks with a specific function, is the next level of technology. Similarly, the function sub-block is also a combination of next level sub-blocks of functions, which is the next level technology. In this way, the decomposition of a technology level by level will eventually reaches to the elementary components in the technical composition, likewise the elementary particles. Indeed, different technologies are all combination of the elementary components of "*function*" or "*effect*".

Chapters 4 – 6 are the basis of systematic recognition of things by reductionism. Among them, Chapter 4 "*Characterizing the Universe*" introduces the characterizing methods of things in a reference frame, including the reference frame to determine the position of a thing with change characterizing, Einstein's relativity principle, vector algebra, linear space with basis, Newton mechanics, $n$-body problem, and the application of Newton's law of universal gravitation in determining the first, second and third velocity of universe, Lorenz transform in Einstein's special relativity theory, etc., points out that the essence of Einstein's principle of general relativity is the mathematical display of the philosophical thought that objective things are not transferred by human's will. Chapter 5 "*Systemic Recognizing the Universe*" is designed to the combinatorial notion of systemic recognition and methods of things, including the system structure, combinatorial characteristics of system and the motifs in the systemic recognizing of things. This chapter introduces physical dimension and the measuring methods of distance objects and micro particles, involved such as the quality, time, system, also the state equation of a system, the solution and the solving methods, etc. Particularly, the stability of system and the Lyapunov's direct judgment, linear and hyperbolic nonlinear systems are discussed. And corresponding to the recognitive thought of combinatorial notion of things, a philosophical thought on mathematics is introduced also. That is the combinatorial conjecture for mathematical science, i.e., *mathematical combinatorics* which extends mathematics over topological structures in space. Chapter 6 "*System Synchronization*" aims to introduce an interesting synchronization phenomenon in the nature and explain the method of determining, regulating of system synchronization. It is pointed out that up to now, humans benefited from all systems simulating animal behavior such as automobile, train, ship, aircraft and other mechanical movements are based on the system synchronization and regulation of system elements. To this end, this chapter presents the major methods for determining the system synchronization, including the master function method, graph criteria as well as the introduction of error term converts a synchronization problem into the stability of system and the control of the system synchronization, explains the 2-matrix norm and Lyapunov index often used for determining system stability, system synchronization, etc.

Chapter 7 "*Contradictory Systems*" applies the combinatorial notion of things to explain and characterize contradictory phenomena of things in the eyes of human, which is extremely different from the textbook. It should be noted that the contradiction is caused by one's recognition, implied in the definition or named process that Laozi explained in his famous book of *Tao Te Ching*, i.e., replaces the reality of things by local recognitions, not the real face of things because he said that "*the heaven and the earth view all things as straw dogs*", i.e., all things are fair in the universe and the recognitive principle with "*logic consistency*" should be followed in this case. In this point, the allegory of Hanfeizi's contradiction of ancient China is essentially consistent with the living-death state or quantum collapse of the Schrodinger's cat. The problem lies in how to describe the living-death state or contradiction of Schrodinger's cat. For resolving the contradiction, the parallel space or 2-branch tree was introduced by H.Everett to explain the living-death of Schrodinger's cat covering both the living and the death states of the cat, which is in fact a special case of Smarandache contradictory systems, Smarandache multispaces or multisystems. To this end, this chapter explains the Smarandache contradictory systems, Smarandache geometries, Smarandache multispace or multisystem and the relationship of Smarandache denied axiom with them. On this basis, the application of Schrodinger cat's living-death state or quantum entanglement, quantum teleportation and the disentangling Smarandache multispaces and multisystems in the field of communication are discussed. Notice that the expression of Laozi's "*Name named not the eternal Name*" in the symbol deduction of mathematics is the limitation of mathematics, including the limitations of mathematical abstracting and deduction, which usually appears as a non-harmonious group or system of non-solvable equations, i.e., a system is unsolvable with contradictions but its each equation is solvable. It is worth noting that different from the equations in classical mathematics, the combinatorial solution of a non-harmonious group always exists, which provides the condition for characterizing such groups, including the sum stability and the product stability of non-harmonious groups.

Chapter 8 "*Complex Networks*" introduces the by-product in studying social phenomena, i.e., complex network which characterizes the social behavior of humans with certain randomness. Of course, a human's behavior is not completely random because he or she has a brain. So, it is only an assumption that human social behavior can be characterized by randomness. This chapter introduces some common random distribution, the law of large numbers, the central limit theorem and the network indexes. On this basis, the complete stochastic model introduced by Eröds and Rényi in the 1960s, the related network index and properties are introduced in details. In this field, the WS small-world network represents a breakthrough in the use of randomness to simulate social behavior, whose randomness is between the regular networks and the completely random models. At the same time, the BA scale-free network describes the connection of new sites with existing sites on the internet, corresponds to the phenomenon of "*the richer is more and more rich, the poorer is more and more poor*" in a society leading by the capitals. An extension of the BA scale-free network is the local world network. Different from the simple construction of differential equations to describe the spread of disease, the real spread of disease is carried out on the social network, which is related to one's social circle. Based on this situation, this chapter introduces also the application of various complex networks to simulate community networks, analyzes the SI, SIS, SIR models and describes the law of the spread of diseases on the social network.

Chapters 9 – 10 apply the combinatorial notion of things to generalize network flow to continuity flow, regard it as a new mathematical element for mathematics, which can be used as the mathematical model of things under the combinatorial notion. Among them, Chapter 9 "*Network Calculus*" begins with an introduction on some optimal problems in network and the methods operation on graphs, analyzes the possibility of simulating the behavior of things by network and constructs the operation system on network under the condition that the combinatorial structure of the network is unchanged. At this time, the network operations is similar to the operation of vectors, only need to keep the structure remains the same its in evolution, on which the metric can be introduced similar to that of the linear space and applying both the discrete or continuous model for simulating thing behaviors, namely the network sequences and the continuous networks, construct the algebraic operation, differential and integral operation on networks. In this way, the Newton-Leibniz theorem of integral operation on networks is generalized as an example. Chapter 10 "*Combinatorial Reality*" introduces the mathematical model that simulates the evolution of things on combinatorial notion, namely continuity flow. Certainly, the continuity flow is a generalization of network. That is, the labels of network vertices and edges are no longer limited to real numbers but vectors in Banach space. At the same time, operators in Banach space can be introduced on edge flows with requiring them to obey the law of flow conservation on vertices. This chapter begins with an introduction to Banach space, Hilbert space and some important results as well as three hypotheses of quantum behavior in quantum mechanics. Similar to network, continuity flow can also be regarded as a kind of mathematical element on which the operations such as the addition, subtraction, number multiplication and the Hadamard product can be defined, and the Banach space and Hilbert space on continuity flow, i.e., Banach or Hilbert flow space can be constructed and applied to the stability of continuity flow also. For describing the dynamic behavior of continuity flows, the Lagrange equation of continuity flows is obtained by the principle of least action. It should be noted that some important conclusions in functional analysis can be generalized on Banach flow spaces by using $G$-isomorphic operators. Particularly, the theorem of Fréchet and Riesz representation on Banach flow space holds which implies that the assumption of quantum behavior in quantum mechanics holds also. That is, whether a quantum has intrinsic structure or not it will not affects the conclusion of quantum mechanics.

Chapters 11 – 12 belong to the philosophy of science and discuss how science can promote human civilization in the combinatorial notion of things. Among them, Chapter 11 "*Chinese Knowledge*" aims to take Chinese civilization as an example to explain how the ancient Chinese perceived things and how the Chinese civilization formed under the notion of "*unity of humans with the heaven*", namely the principle of the harmonious coexistence of humans with the nature. This chapter also compares some scientific achievements of the Chinese with the western. Particularly, Laozi's explaining on the creation of the universe, the relationship between the heaven, the earth and humans in his *Tao Te Ching* is compared with the theory of big bang. It points out that the western science is a local recognition of the law of things, i.e, "*Tao*" while the ancient Chinese were a recognition on the whole life cycle of things and behaviors. At the same time, this chapter introduces two typical examples for applications of continuity flows. One is the surprised theory of 12 meridians and the relationship with the viscera organs of human body established on the Yin-Yang theory by the ancient Chinese. Another is the corresponding of the 64 hexagrams in *Change Book* to the continuity flows over circuits of order 6, which is essentially the soul of Chinese science and a scientific method for understanding objective things rather than a superstition. Chapter 12 "*Science's Philosophy*" aims to reaffirm that science is a kind of local recognition or conditional truth of things and to discuss how science promotes human civilization under the harmonious coexistence of humans with the nature. This chapter takes the *Theory of Everything* as an example, outlines the application of Smarandache multispace or multisystem in the development of science by combinatorial notion and points out also that humans should take the initiative to limit or end some fields or directions in science development. That is, science needs to have a limiting scale or standard while recognizing things in the universe for conducting the behaviour of humans and promotes humans harmonious coexistence with the nature, including those that affects the order of universe, destroys the biological diversity or affects the behavior of humans ourselves so as to realize the human activities guiding by science do not disturb the nature, which is the fundamental principle of human development.

Personally, I believe that the harmonious coexistence of humans with the nature is the most important objective in the development of science with the promotion of human civilization in the 21st century. In this process, the initiative to realize the harmonious coexistence of humans with the nature lies in humans ourselves. For this objective, the first is necessary to reflect on the immoral behavior in the past that humans excessively intrude on the nature, the second is to study the scientific programme of harmonious coexistence with the nature, to correct and eliminate the harm caused by human's excessive intrusion on the nature in the past and the third is the review and restraint of humans ourselves, including the immoral behavior in previous human activities and consciously harmonious coexistence with the nature for everyone. In this way and only in this way, the ultimate goal of our humans will comes true.

Finally, I would like to express my sincere thanks to my family, friends and colleagues who have helped or supported me. First of all, thanks to Prof.Smarandache. Over the years, Prof.Smarandache has always kept with me the discussions and exchanges on scientific topics and have been encouraging my research work. I would also like to express my gratitude to Editor D.Roounski and Dr.Preze Mihn for their encouraging and supporting. Without their encouragement and support, I would have given up my research many years ago and taken up other jobs for supporting my family. Secondly, I would like to express my gratitude to the teachers, classmates and friends who have guided or helped me over the years, especially the guiding of professors F.Tian, Y.P.Liu and M.Y.Xu, the academic discussions and exchanges with professors A.Bhattacharyya, P.S.K.Eddy, Q.Z.Ping, H.Ren, J.L.Cai, Y.Q.Huang, R.X.Hao, G.D.Liu, Y.Z.Chen, W.L.He, E.L.Wei and so on have enlightened my systematic research. Thirdly, thanks also to Yuan-Ling Data Group Co., Ltd for its financial support in past three years.

Linfan Mao

February 28, 2023

# Contents

$C_{\mathscr{M}}$
**Combin.Notion**

## Chapter 1

# Ultimate Questions on the Universe

*********************************************************************

**The Universe is Beauty but Never Showing,**

**All Seasons are Going on Laws without Disputing,**

**Everything Rules Inherently without Words Saying.**

**The Saint Knows the Truth Beyond the Beauty of Universe.**

—– *Chuang Tzu–Travel North*

§1. **The Origin of Universe**

The word *universe* is defined to be all of the times, space and their contents by humans ourselves, which includes two parts. One is the direction with contents including up and down, front and back, left and right and another is the time with contents from the ancient to the future including the past, present and the future. As a result, the universe is a real substance which can expands or extends both in space and times infinitely, i.e., it is a dynamic, not a static one and so, it is indeed a combination of the times, space with their contents in moving. We present a local view of the universe taken with a telescope in Figure 1.1($a$). It should be noted that even though it is a partial picture of the universe, it is enough to reveal the vastness, also the complexity of the universe.

**(a) A glimpse of Universe**

**(b) The Big Bang**

**Figure** 1.1

As a member of the universal family regardless of the different age, color or cultures, humans always wish to understand the real face of the universe for the survival and the developments ourselves such as where it comes from, which laws it should be conformed to and what is it's ending, etc., the ultimate questions on the universe. We have known different answers on the questions in different regions given by different cultures. For example, Chinese says Pangu split open the chaos by an axe, Christian talks about the God created the universe only in six days written in Bible and it is believed the Big Bang brought about the universe in scientific mainstream today. All of them are looking for the answer to the ultimate questions on the universe. Certainly, the understanding of humans still likes a kid's perception on things in the universe. It would be a long way off for finding the answers on these questions.

Dr.Ouyang is a well-known scientist on mathematics, physics and cosmology familiar with Chinese traditional culture. His research usually combines natural science with the humanity. Particularly, he gets the knowledge constantly from the ancient Chinese philosopher's notions and applies them to his research. Dr.Ouyang has a lovely daughter named Huizi, learning in an elementary school. She is in the age of innocence but with the most curious on things in the universe.

Now, let's look at the conversation of the father Dr.Ouyang with his daughter Huizi on the beginning of the universe.

**1.1.The Big Bang.** One day, Dr.Ouyang takes his daughter at the elementary school gate. On the road of going home, Huizi asks Dr.Ouyang a question:"*Daddy, could you tell me where did the universe come from?*" Dr.Ouyang asks her back:"*Why do you ask this question? It's very complex!*" Huizi says:"*My teacher told us that the universe cames from the Big Bang in class. But I always feel there is something wrong!*" Dr.Ouyang asks her further:"*What's wrong with that?*" Huizi answers:"*If this explaining is right, it's the same that concludes the beginning of the universe is likewise a firecracker on the New Year day. However, a firecracker is the gunpowder wrapped in a paper and it should be taken a fire to light the lead wire for its explosion. In this case, where was the paper and where did the paper and the explosive come from in the Big Bang, and who lighted the lead wire? I am unable to imagine how it happened.*"

Dr.Ouyang explains the Big Bang to her that an explosion is caused by the concentration of energy up to a certain extent which resulted in matters combining by elementary particles. An evidence that the universe came from the Big Bang in modern science is originated from a discovery found by an astronomer Edwin Hubble in 1929 when he studied the redshift of light in distant galaxies. He observed that no matter which direction you look, the distant galaxies were moving away from us and the nearby galaxies were approaching us, which implies that the universe is in a constantly expanding. Now, if we reverse this phenomenon and infer its reason, we will conclude that all stars were very close to each other in a long time ago, even they were in a same place which implies that there must be a moment, called the Big Bang that begun the universe.

However, to what extent energy needs to be accumulated to the Big Bang and why did it happen and particularly, what happened around the first seconds or minutes nearby the Big Bang, there are no scientific conclusion update. These question are the same that likewise who lighted the lead wire or what resulted in the explosion of a firecracker, Dr.Ouyang explains further to his daughter, there are no scientific conclusion on these ultimate questions. Huizi smiles as she heard her father's explanation and then asks:"*Daddy, at the time of the Big Bang happened, the universe was empty with nothing or like making firecrackers, wrapped in paper with explosives and lead wires?*" Dr.Ouyang answers:“*It should be seems empty in the eyes of humans, even likes a small point with infinite density.*” Huizi asks further:“*Then, how did these stars, earth and humans, mountains and trees appeared as what we see now?* ” They are created by that all beings were out of non-beings in the mechanism, Dr.Ouyang explains it further to Huizi. The scientific community believes that everything was created following in the first few seconds, dozens of seconds, dozens of minutes and hours to decades, hundreds or tens of billions of years after the Big Bang, i.e, all things appeared with the evolution of the universe accompanied with the high temperature generated in the Big Bang gradually cooled down and the spatial scale was constantly expanding and then, the collision and combination of elementary particles resulted in the Big Bang beginning to create atoms and molecules step by step. After that, the macromolecules and finally, everything in the world appeared gradually which formed you, me and him, stars, mountains, rivers and trees by the nature at a long last evolution.

Huizi still looks blank in her mind. She whisperingly what can not understand is that it is empty without anything, how can it have everything. *Is there an invisible unexpected power or hand continuously adding stars, mountains, rivers and trees to the universe from other places where can not be seen by humans?* She frankly speaks to her father:“*Daddy, I think the Big Bang isn't a right theory!*” Seeing Huizi thinking so, Dr.Ouyang explains to her:“*The Big Bang is merely a hypothesis on the beginning of the universe. But lots of collected datum by humans in observation in recent years show that it maybe a right theory.*” Considering his daughter still not believing the Big Bang, Dr.Ouyang asks her in replies:“*Then, how do you think the universe came into being?*” Heard father asked her thinking, Huizi refreshingly answers:“*I think the universe should be laid liking a hen falling an egg!*” Dr.Ouyang asks her:“*How do you think so?*” Huizi explains to her father:“*I think so because an egg can be hatched to a chicken and grow up gradually.*” Dr.Ouyang then asks her:“*Could you imagine how big a chicken that must be for the universe and where it lives?*” Huizi thinks for a while then answers:“*I can't imagine how big it is.*”

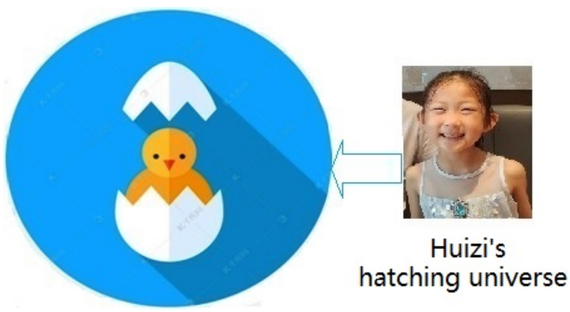


**Figure** 1.2. **A hatching universe**

Dr.Ouyang inspires his daughter further: “ *You can think it in this way. On the first, if there is really such a hen it should be larger than the universe because it's going to give birth to the universe, just like a hen to give birth to an egg and the egg is hatching to a* *chicken that is going to grow like a hen, but our universe is still in expanding. And second, we can't see the shape of such a hen because it's bigger than the universe. In other words, we can't determine what it's and where it's living because we are living in the universe, can't go out the universe in theory.*"

After listened to Dr.Ouyang's analysis, Huizi says: "*And then, Daddy, is I think the universe is an egg of a hen wrong or not?*" Dr.Ouyang tells her: "*It isn't sure to wrong also because there are no collected evidences to prove that the universe was similar to an egg of a hen falling until today.*" He continuously says: "*Yours opinion think the universe and its beginning is in a biological behavior, is living and reproducing according to the biological laws. However, to be a living thing in the universe ourselves, we can't go out the universe to observe the universe, can't decide the universe is a living or non-living thing likewise the chicken in an egg.*"

At this time, Dr.Ouyang asks another question to his daughter: "*What do you think the universe should look like? Is it look round or square? Is it like a tree, a mountain or like a rabbit or a horse?*" Huizi answers that she never thought about such questions. Dr.Ouyang says that a few scientists think the shape of the universe is elliptic. For example, for detecting the residual radiant heat of the universe after the Big Bang, the NASA developed a Wilkinson Microwave Anisotropy Probe (WMAP) and published its survey result in Figure 1.3 with an elliptic boundary in 2003. But in this way, *could we think that the shape of the universe is elliptic?* Certainly not! Dr.Ouyang explains to his daughter: "*The reason is similar to the previous because we can't go out of the universe to prove the universe is looking like an ellipse or rabbit, or other shapes!*" Then, *what is the elliptic boundary in this picture implying?* He then tells his daughter that it only implied that the WMAP was detecting datum flying along an elliptical orbits within a range of about 1.5 million kilometre away from the earth, not the whole universe.

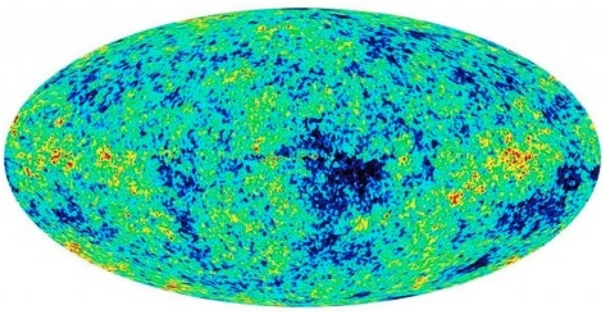

**Figure** 1.3. **WMAP**

Certainly, all researches on the universe depend on the development of space exploration technology. But whether or not humans can finally go out of the universe and know its truth, including its shape, the answers in the eastern and western cultures are not in consistent. At this time, Dr.Ouyang and his daughter already arrive at home. He asks his daughter to wash her hands first and then tells her the story of Pangu split open the chaos in Chinese culture and the creation story of God in the Bible after dinner.

**1.2.Pangu Split Open the Chaos.** After dinner, Dr.Ouyang tells his daughter the story of "*Pangu Split Open the Chaos*" in the Chinese traditional culture. He tells her that humans evolve themselves in the universe and imagine many theories of the beginning of the universe by the anthropomorphic means.

**Figure** 1.4. **Pangu's portrait**

For example, it is believed that Pangu created all things of the universe in Chinese traditional culture. Here, Pangu is one of the "*God of Chinese creation*" in legend, as shown in Figure 1.4. It is said that before the creation of the universe there was no division between the heaven and earth. At that time, the heaven and the earth were chaotic, and there was no differences between the day and night. It can not distinguished the up and down, the east and west, the southeast and northwest also. The world was a chaos liking a round mixed thing with a nucleus in the middle. It is said that Pangu, the ancestor of humans was born in the core of such a chaos and become a living thing after 18,000 years of gestation.

It is said that when Pangu was conscious, just like a "*baby*" born in that moment, he opened his eyes and looked around, but he was surrounded by "*chaos*", which was dark and he could see nothing. Anxiously, Pangu pulled out a tooth in his mouth, turned it into a mighty axe and swung it hard to chop the chaos. At this moment, the chaos split into two parts. One is the light and clear, floating upwards forming the sky and another is the heavy and turbid, falling down to the ground. Whence, Pangu was born between the heaven and the earth with his head on the sky and feet standing on the ground.

After his born, Pangu grew up between the heaven and the earth. His head was the God in the heaven and his feet were the saints on the earth. The sky rises by one foot every day, the earth thickens by one foot every day, and Pangu also grew by one foot every day. After eighteen thousand years, the sky became extremely high, the earth became extremely thick and the Pangu's body became extremely long. It is said that Pangu coexisted with the heaven and the earth for 1.8 million years, and when he fell, his head, feet, left arm, right arm and abdomen turned respectively into Taishan in Dongyue, Huashan in Xiyue, Hengshan in Nanyue, Hengshan in Beiyue and Songshan in Zhongyue of China. Since then, there have been sunshine and rain in the world, the rivers, lakes and seas beginning appeared and all things bred including humans on the earth. This is the legend of "*Pangu split open the chaos*" or "*Drawing a line opened the chaos*" in the Chinese traditional culture, and it is a picture of the creation of universe drawn by Chinese thinkers.

In three Kingdoms period of the ancient China, a taoist Xuzheng of Wu State wrote a detailed record on this legend as follows:

"*The heaven and the earth were chaotic like chickens in which Pangu was born. At the age of 18,000, the heavens and the earth were opened up, the Yang was clear, lighter upward forming the heaven and the Yin was heavy, turbidness falling down to the ground. Pangu was in it, and it changed nine times a day. The god was in the heaven and the saint was on the earth. The heaven was one foot higher, the earth was one foot thicker and Pangu was one foot longer every day. So, it was eighteen thousand years old which resulted in that the heaven was extremely high, the earth was extremely deep and Pangu was extremely long. After that, there was three emperors. Consequently, all things in the universe began in one chaos, formed in three characters of the heaven, human and the earth, consisted of five elements, i.e., the metal, wood, water, fire and the soil, prospered in cycling step seven and ended in its extremal state, the extremal number nine. Whence, the heaven is 90,000 miles away from the earth.*"

Dr.Ouyang asks his daughter if she can understand the Chinese creation story of the "*Pangu split open the chaos.*" Huizi answers:"*Yes, it's more intuitive and vivid than the Big Bang and it's similar to human's labor creation.*" Dr.Ouyang then asks her:"*Do you know the similarities and differences between the Big Bang and the story of Pangu split open the chaos?*" Huizi asks incomprehensively:"*Is there any difference?*" Dr.Ouyang explains to her: "*You can compare them like this. Firstly, before the creation of the universe, the Big Bang thought that the world was empty and all things were squeezed close to a point with infinite density while Pangu's story said that the world was a chaotic whole and so, where all things come from is no longer out of nothing again but from the mixed things, leaving a foreshadowing for the emergence of things in the universe. Is this the truth?*" Huizi replies:"*Yes, daddy! In this way, I don't have to wonder where the mountains or rivers come from. They are come all from things in the chaos!*"

Dr.Ouyang explains continuingly:"*Secondly, both of them assume that the universe had a beginning point. The Big Bang assumes that the universe originated from the moment of the Big Bang while the Pangu's story assumes that the universe originated from the* *moment that Pangu split the chaos. The two things are similar, isn't it?*" Huizi nods:"*Yes! It is the case!*"

Dr.Ouyang continues:"*And thirdly, both of them hold that everything was born after the birth of the universe. The Big Bang applies the heat energy generated by the explosion to lead to the expansion and cooling of the universe, and the collision, combination in elementary particles generated atoms, molecules and everything. Similarly, the Pangu's story says that Pangu opened the chaos with an axe and after then, the light ones in the chaos floated up to form the heaven while the heavy ones fell into the ground, and it's matters in the chaos formed the heavens and the earth, i.e., the universe. In this way, it naturally explains that everything come into beings from the initial chaos.*" As she heard this, Huizi has another question asking her father:"*Daddy, an explosion means throwing things around but a splitting is dividing the chaos into two parts from the whole. It isn't clear that things are produced everywhere in the universe.*" Dr.Ouyang explains further:"*Here, the word 'chaos' emphasizes that it's a mixed thing composed of clear and turbid. As for its shape, it isn't known whether it's round like a watermelon or a mound as we see today because it isn't the main in creation of the universe. At the same time, saying that Pangu split the chaos with an axe don't say whether Pangu split in one or several axes, which isn't also the main in the creation story. The main thing in the Pangu's story is his opening the chaos so that the chaos is no longer chaotic, the heaven and the earth come into beings, thus gave the birth to all things in the universe. Certainly, these philosophical conceptions of ancient Chinese can be realized further while you learn and understand the Change Book of China in the future.*"

Dr.Ouyang asks his daughter:"*Comparing the Big Bang with the Pangu split open the chaos, do you have some knowledge on the Big Bang?*" Huizi nods and then says:"*Yes, daddy! Although I still don't understand what the Big Bang said, it seems that I can hold on a little of its essence, and they are consistent in talking about the origin of the universe. However, I have a question. Why the Chinese doesn't study further on Pangu's story of split open the chaos but follow the foreigners to talk about the Big Bang?*" Dr.Ouyang inspires his daughter to say:"*That's a good question! On the way coming to home, you said that you thought the universe was an egg laid by a hen, attributed beginning of the universe to a hen. However, humans couldn't find such a big hen. Similarly, Pangu used one of his teeth as an axe to split the chaos. How big should Pangu be? Isn't it similar to the hen that you said? Humans can't find him also! So, this can only be the wisdom of ancient Chinese philosophers. It is myth rather than science!*" Dr.Ouyang tells his daughter that the purpose of the science is to study and discover the truth of things in the universe and then, apply it to human's practice.

**1.3.God's Six Days Creation.** Dr.Ouyang tells his daughter that this kind of anthropomorphic or deified creation exists in all nationalities of the world, and a few religions also incorporate the deified creation into their teachings. Among these creations, the six day creation of God recorded in the *Bible* is the most detailed, its whole creation process is as follows:

It is said that one day, the God was very dissatisfied with the boundless chaotic world and waved his hand gently saying: "*There must be light in the world.*" Then, there was light in the world. The God referred to the "*light*" as the "*day*" and the "*darkness*" as the "*night*". The light faded away while the darkness returned and the light rase then the darkness disappeared. From then on, there is an alternation of the day and night in the world. This is the first day of God's creation.

**Figure** 1.5. **God's creation**

The next day, when he opened his eyes, the God was dissatisfied with the empty world scene before him, waved his hand and said:"*The sky should be full of stars.*" Then, the universe was full of countless large and small stars in the sky, performing their duties including charging the day, night and the season of world.

On the third day, the God saw that the land was still chaotic and unhappy, saying:"*The water should be gathered together to expose the dry land.*" As a result, the water were gathered together to form rivers, and the dry lands were exposed one by one. God called the dry land as the "*land*" and the place of water gathered as the "*sea*".

The God said:**"** *The plants and all kinds of vegetables should be born on the land.***"** As a result, the grass, trees, fruits and vegetables were born in the earth which were full of vitality.

On the fourth day, the God said:"*There should be light bodies managing the charge of the day and night in the sky, marked to determine the season, day and year old, and also shine on the earth.*" So, the God made two light bodies and gave them a division of duties, i.e., the big one managing the day and the small one managing the night. They are the sun and the moon today.

On the fifth day, the God said:"*There must be life in the water, countless fish and* *insects and birds in the air, countless birds.*" As a result, all kinds of fishes and birds appeared in the world. The fishes swam freely in the water and birds soar freely in the sky. And then, the God said:"*There must be all kinds of animals on the earth.*" Then, all kinds of beasts and insects appeared on the earth. The beasts ran freely on the ground, the insects flew in flowers and plants.

On the sixth day, the God was very satisfied when he saw that the sun was shining, the earth was vast, the world was colorful, the animals jumped, the fishes swam and the birds sang. So he said:"*I want to make the human according to my appearance and let the human manages all things and animals on the earth.*" Then, the God made a clay figurine out of the mud and breathed a sigh of relief at the clay figurine. In a short time, a human was born in the hand of the God.

According to the *Bible*, the God created everything in the universe within five days and created the human in his own image on the sixth day. Seeing that things in the universe were all in order, he decided to take the seventh day as a rest day, which is the origin of "*Sunday*" in the "*Solar Calendar*".

After introducing his daughter to the record of "*God's six days of creation*" in the *Bible*, Dr.Ouyang asks his daughter:"*Which is more vivid and easier for children understanding between the stories of God's six days of creation and Pangu's spilt open the chaos?*" Huizi replies:"*Of course, it's the God's six days of creation! Although Pangu opened the chaotic world with an axe and could produce things, it didn't tell where everything came from in the story but it's clear that the God created everything in the God's six days of creation, which is better understood.*"

Dr.Ouyang fells it is necessary to establish a correct philosophy for his daughter. He says to her:"*God created a human in his own image on the sixth day. That is to say, a human looks the same as the God. So, how tall is God and who's the taller one in God and the hen you said?*" Huizi replies:"*It should be my hen because it gave the birth of the universe!*" Dr.Ouyang continues to ask:"*Good! Then, who's the stronger one in God and Pangu?*" Huizi says:"*Pangu split the chaos hardly with an axe but God gently waved his hand to make the world day and night, created mountains, trees and life. I feel that the God is the stronger.*" Dr.Ouyang explains to his daughter:"*This is the divine power in religion. In religion, God's power is boundless. Similarly, do you think that such an almighty God could be existed in the universe?*" Huizi seems a little confused and says:"*I can't think of it, shouldn't there be?*" Dr.Ouyang affirms:"*Of course not! The almighty God does not exist in the universe, which is the same as Pangu in Chinese traditional culture including the old hen you said. It's only the result of human's imagination when* *explaining the beginning of the universe, and can only be enshrined in human's mind. In religion, the God is worshiped by the followers because they can't imagine what causes the universe because they are in the universe themselves.*"

Dr.Ouyang tells his daughter that the stories of God's six days of creation and Pangu split open the chaos are a unclear worship of "*explaining itself*" because of the underdeveloped science and technology and an inadequate knowing on things in the universe at that time. With the progress of science and technology, humans realize that both of the Pangu's story in Chinese and the God's six days of creation in Christian need to know the reality of things in the universe from science which reveals the natural laws of the universe. Certainly, it is difficult and may be never know the true face of the universe for humans because we are all in the universe ourselves, may be never out of the universe but this is the ultimate goal of science, that is to understand and coexist with the universe.

## §2. The Drifting Galaxies

Generally, the "*stars*" in everyday words refers to those of celestial bodies visible to the naked eyes as a human stands on the earth. They are generally classified into stars, planets, satellites, dwarf planets and small celestial bodies.

One summer day, Dr.Ouyang takes his family to a mountain village in the suburbs for a vacation. After dinner, the family come to a stream in nearby the mountain to cool off. Breathing the fragrance of grass and flowers in the air, listening to the sound of streams, frogs in the water and cicadas in the trees, seeing fireflies flying in front of their eyes from time to time and the moon floating in the blue ocean with countless skylights like gems in the night sky, they felt very comfortable. Huizi plays nearby the stream for a while, then go up the dam and looks up into the sky. She says surprisingly:"*Daddy, the stars are all winking at me!*"

**Figure** 1.6. **Stars in the sky**

Feeling his daughter's happy in the nature, Dr.Ouyang wishes to enlighten his daughter's understanding of the nature, then asks to her:"*Do you know how the stars come from?*" Huizi replies:"*Didn't you told me that stars were created by the God on the second day?*" Indeed, the beginning of the universe that Dr.Ouyang told his daughter in the other day and explained that is in about $10^9$ seconds after the Big Bang, accompanied with the heat spreading and the temperature reduced gradually, small molecules began to synthesize macromolecules and then formed stars and planets in the universe after the Big Bang, but this explanation is too difficult for a pupil's understanding because she has only the abstract nouns on matters, atoms or molecules confined to her textbooks, no experience in the laboratory. However, it was the God's creation let her remembered. Dr.Ouyang considers on how to guide her a scientific understanding on the sun, the moon and the stars in situation.

The next morning, Dr.Ouyang inspires his daughter to understand the universe correctly and asks her:"*If there is no God, is there no the sun, moon and the stars in the universe?*" Huizi replies:"*Yes! If there is no God, not only no the sun, moon and the stars but no the humans also!*" Dr.Ouyang continues to guide her:"*Is the God taller or shorter than a human?*" Huizi answers:"*There are no words on this question in the God's creation. But the human was made by God, and the God can made the sky, the earth, mountains, rivers, animals, plants and the humans, no matter he is the taller or the shorter but he has really the divine power.*" Dr.Ouyang further asks his daughter:"*Do you think the god belongs to humans?*" Huizi answers affirmatively:"*Don't belong to because the human is created, not born by the God, only their appearance looks like the God!*" Dr.Ouyang asks her again:"*If the God doesn't belong to humans, nor animals, how can the God use his divine power to create the heaven and the earth, the stars, mountains, animals and the humans in six days?*" Huizi thinks for a while and then truthfully answers:"*I don't know how to answer this question. Is this divine power because you said that God's power is in boundless last time?*" Dr.Ouyang says:"*All right! This is the God's divine power. You can think it again. If the God doesn't belong to humans, where does the power of God came from?*" Seeing his daughter still in doubt, Dr.Ouyang gives her the answer directly:"*It's from the nature! It's the uncanny workmanship of the nature, i.e., it's nothing else but the natural power! In this opinion, one could even say that the God is the nature, and that a human reveres on God's powers is the same as his revering on the nature. This is the purpose of science, namely to find the natural laws of things in the universe but not idolizing to a superman, the God or other anthropomorphic charters.*"

Dr.Ouyang begin to tell his daughter about the origin of galaxies with legends.

**2.1.The Origin of Galaxies.** The humans appeared after the galaxies formed and stabilized in the universe. Whence, to find out their origins is difficult and perhaps impossible in some ways. However, scientists propose hypotheses and through data collection, experiments and theoretical deduction give a scientific explanation on the origin of galaxies in order to restore the process of the formation.

In the universe, any of a vast number of star systems held together by gravitational attraction in an asymmetric shape or more usually, in a symmetrical shape which is either a spiral or an ellipse is called galaxy, which may range in diameter from 1500 to 300000 light-years.

Generally, there are two forms of planetary rotation by observation. One is that many planets of small masses revolves around a central planet of large masses such as those shown in Figure 1.7. Another involves two or more stars orbiting each other around a common center of masses, which shows that the majority of galaxies belong to the former in the universe.

**Figure** 1.7. **Galaxy**

In the galaxies, there is a thin matter called plasma particles which are the intergalactic mediums. The total number of galaxies in the observable universe is more than 100 billion. With the help of astronomical telescopes, humans have found that galaxies can be divided into three classes, i.e., the elliptical galaxies, spiral galaxies and the irregular galaxies. It's observed that an elliptical galaxy has a bright appearance of the elliptic shape, a spiral galaxy is in the disk shape with curved spiral arms and an irregular galaxy is affected by its nearby galaxies, its shape is always in changing.

There are two hypotheses about how galaxies formed in the universe. One is based on hypothesis of the Big Bang, which shows the formation process of galaxies from the attraction and repulsion of elementary particles following the time evolution after the explosion. Another is there are so many particles of dust in the universe that galaxies form finally from them. A more likely explanation in scientific community is the Big Bang on the formation of galaxies. According to the Big Bang, the galaxies formed about a billion years after the Big Bang. At that time, the universe is filled with large amounts of hydrogen and helium gas. They attracted each other to form clouds. The clouds floated freely in the air, captured other substances, became clusters and collided each other in the air. Some of them inhaled or be inhaled by other clusters, became a more larger one, and some of them have been hit, bruised or crashed to the dust or small stars. They were floating in the air and then, combined to form a new star clusters. The clusters collided, regrouped and adjusting their orbits continually to avoid collisions again and gradually became stable to form the galaxies in the universe as we see today. It should be noted that either hypothesis is of course a human hypothesis for explaining the galaxies formed.

As Dr.Ouyang says here, Huizi interjects:"*It's a bit like children playing a game of mud. That's how the galaxies are formed either by making a lump of mud into a bigger one or by splashing a lump of mud on the floor.*" Dr.Ouyang then says:"*Playing mud needs the hand force of children, kneads a few pieces of small mud into a big one, a combination or divides a big mud into a few of small pieces, a subdivision. The galactic formation is in a more complicated mechanism than the mud playing of children, involving the character of gravitational and repulsive forces between star clusters, i.e., the gravity holds particles together but the repulsion keeps them apart, which is liking the attraction and repulsion in humans. However, the galaxies formed over years what we see today in this ways.*"

**2.2.Milky Way.** The milky way, like a shining river flowing in the sky, was called the *milky way* or *sky river* in the ancient times of China. It is a star system containing the solar system, including 150-400 billion stars, star clusters, nebulae and various types of interstellar gas and dust. Its total visible mass is about 210 billion times that of the sun in about. In the milky way, most stars are concentrated in a flat spherical space area with a discus shape, oblate spheroid middle section called "*nuclear bulge*" with a radius of about 7000 light-years. Surrounding the nuclear bulge is the extended bulge of stars that is nearly sphere in shape, known as the "*disk*". Outer the disk is the region with less stars and small density, called the "*stellar halo*" which extends from the nucleus bulge out to approximately 70000 light-years such as those shown in Figure 1.8.

**Figure** 1.8**. Milky way**

The milky way's nucleus bulge is a bulge at the center of the intersection of its axis of rotation and the galactic plane, is a bright sphere in about 20000 light-years across and 10000 light-years thick. The region is dominated by red stars that are more than 10 billion years old. There is a lot of interstellar dust between the nucleus and the solar system.

The milky way's disk is the main component of the milky way, including the sequence stars, giant stars, novas, nebulae, semi-regular variable stars, etc. It is said that the 90% detectable matter in the milky way lies within the disk. The disk is like a thin lens with an axial symmetric form distributed around the galactic center. Its central thickness is in about 10000 light-years, a circumference in about 2000 light-years, and a diameter of nearly 100000 light-years. Observation shows that except the nucleus in the range of 1000 parsecs rotates around the center as the fixed axis of rigid body, the other parts rotate around the galactic center and slower farther away from the center. Most of the material in the disk is in the form of stars, accounts for less than 10% of the galactic mass and is scattered throughout the disk. Apart from the ionized hydrogen, molecular hydrogen and various interstellar molecules, about 10% of the interstellar materials consist of the interstellar dust.

The milky way's halo is dispersed in a spherical region around the disk, which is about 98 million light-years in diameter and is populated by low density stars, mostly globular clusters of older stars. Outside the halo is a large spherical area of radio radiation called the "*corona*", which looks like a hat.

Dr.Ouyang tells his daughter that humans understanding of the galaxies due to the lack of modern observation instruments in the early days. Most of them were based on the naked eyes observation, giving the sun, moon, stars, stars and other planets and galaxies to worship or sacrifice for gods. For example, the sun god is called Helios, and then said that the sun god was replaced by Apollo who asked for Zeus this position, the moon goddess Selene, the star god Asteraios, the meteor goddess Asteria and so on in the ancient Greek mythology. In the ancient Chinese mythology, there are also the Fuxi's creation, Nvwa repairing the heaven and other myths in addition to Pangu's split open the chaos as well as some popular legends to help one understanding the galaxies.

Dr.Ouyang points to a bright star on one side of the milky way and tells his daughter who is called the "*Vega*" by the Chinese and then, he points to a star on the opposite side of the milky way and tells his daughter that it is called the "*Altair*", which is a bright star with two small stars on its both sides in the southeast direction, representing the two children of Altair and Vega. Dr.Ouyang tells his daughter there is a well-known story on Altair and Vega in China.

**[Altair and Vega's Story]** It's said that in the early eastern Zhou Dynasty of

China, there lived a family in a deep mountain. The old parents had all died away and there were only two brothers left. The eldest brother had married while the younger had no family and was always herding cattle. The villagers all called him *Niulang*. His elder brother's wife had a bad mind. She always picked, chose little matters scolded to Niulang. Sometimes, she drove even him out of the house and let him living in the cowshed.

Later, the elder brother's wife made a scene of separation the family properties and living apart. Niulang proposed that he only needed the old cow and the house with all furniture to his eldest brother family. He took the old cow to a hillside, built a shed with hay to live down. Niulang took the old cow to open the fields during the day and lived with the old cow at night. Sometimes, he talked with the old cow, and it was seemed that the old cow could understand him. One day, the old cow suddenly opened his mouth and told Niulang that it was once the Taurus Star in the heaven, said that there were seven fairies to take a bath in the lake on the other side of the mountain in the next evening and asked Niulang to hide the green fairy's clothes so that she could not flied away and would thrn become his wife.


**Figure** 1.9. **Altair and Vega**

In the next evening, Niulang did as the old cow told him. He actually saw a group of fairies bathing in the lake. According to the direction of the old cow, he hid the green clothes in the grass. It was the time that the fairies should flied back to the heaven, but the Vega could not find her clothes. Her sisters helped her also to look for a while but they did not find them. They did not dare to delay too long on the earth. So they flew away. At this time, Niulang took out the clothes to Vega. They sat down by the lake and talked for a while. When Vega heard about Niulang's suffering, she felt sorry for him. Seeing that Niulang could bear hardships, worked hard and had a good mind. And then, Vega decided to stay in the world as Niulang's wife. They lived together in male plow and female weaving, a sweet and happy life in the farming era of ancient China.

A few years later, they gave birth to a boy and a girl. The old cow became very old and told Niulang to keep its skin after it died so that he could put on if the emergency happens in the future. The jade emperor found out that Vega did not return to the palace on time, then sent the queen mother escorted Vega back to the heaven for trial. Niulang was anxious but could not catch up. Suddenly, he remembered what the old cow told him before it died. He put on the cow's skin, picked up a basket with their children and ran after the queen mother. It was seem that he was about to catch up with Vega. At this time, it was seen that the queen mother pulled out a hairpin from head, drew a line behind her. Suddenly, a river appeared in front of Niulang which was the milky way. Niulang could not fly across the river, which resulted in Niulang and Vega only being two sides in the milky way.

This is the origin of the Vega and Altair stars on either side of the milky way in Chinese legend. Dr.Ouyang says that the ancient Chinese believed that Pangu split open the chaos and then, the jade emperor was in charge of everything in the universe much like Zeus in ancient Greek mythology. In this story, the queen mother applied a hairpin to draw the milky way is similar to that Moses stretched out his cane to the red sea according to the God's command, and drew it across the river bed to the other side in the *Exodus* of the *Bible*. However, the myths and legends are all in the "*divine power*" recorded or created by some literati. Dr.Ouyang asks his daughter:"*Do you think the divine power will appear in humans world?*" Huizi replies:"*No way!*" Dr.Ouyang says:"*Even if it appeared it's nothing else but a natural power, not the divine power and the religion advocates the divine power to restrain the followers. However, the scientific research of the natural power is exploring the true face of the nature. This is the difference of science with the religion.*"

**2.3.Solar System.** The solar system is a celestial system bound together by the gravity of the sun in the milky way, including eight planets such as Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus and Neptune as well as 173 known moons, 5 recognized dwarf planets, hundreds of millions of small celestial bodies and Halley's comet in about. The sun is at the center of the solar system and other planets orbit around the sun such as those shown in Figure 1.10.

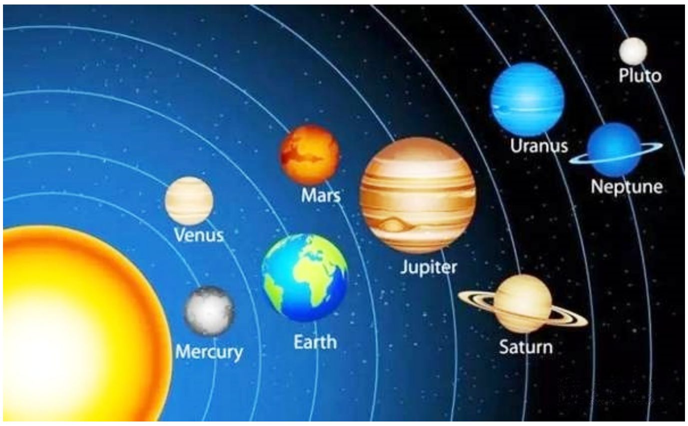


**Figure** 1.10**. Solar System**

The eight planets of the solar system move in nearly circular orbits in the same plane and move in the same direction around the sun. Except for Venus, all the planets spin in the same direction as their revolution. There is a nebula hypothesis in science for origin of the solar system, which holds that the solar system was formed in about 4.6 billion years ago while a huge molecular cloud collapsed in the universe, triggering the birth of several stars and the sun.

The sun bringing warmth and light to the earth is the source of growth for all things on the earth. Located about 26,000 light-years away from the center of the milky way and 23,000 light-years away from the edge of the milky way, the sun moves around the galactic center at a speed of 220km/s and makes a circle around the galactic center for about 250 million years, which results in periodic changes of the earth's climate in 250 million years. The interior of the sun can be divided into three layers, i.e., the core, radiation zone and the convection zone. The radius of the core region is in about 1/4 of the radius of the sun and its mass is more than half that of the sun.

The sun's core is so hot that it can reach to 15 million degrees Celsius. The forces in the core are so strong that causes the thermonuclear reactions, which fuse hydrogen into helium and release huge amounts of energy. After observation and research, the solar core energy through the radiation layer and the troposphere material transfers to the surface of the sun and then to the surrounding radiation. According to the analysis, the density of material in the central region of the sun is very high, up to $160g/cm^3$. The central region is in a state of the high density, high temperature and with strong acting force, which is the birthplace of solar energy.

The energy in the sun's core is mainly transmitted outward by radiation. Correspondingly, the radiation zone is outside the solar core with ranges in 0.25 solar radii to 0.71 solar radii from the top of the thermonuclear center. In the radiation zone of the sun, the temperature, density and the forces decrease from the inside out. It has been estimated that the volume of the radiation zone accounts for most of the volume of the sun. In addition to the radiation, there is also the convection. Its range reaches the solar surface from 0.71 solar radius of the sun. The sun's gas properties in this zone vary greatly, unstable, forming obvious upper and lower convective movement phenomenon, which is the outermost zone of the sun's inner structure.

According to the distance from the sun, the planets are respectively the Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus and the Neptune from near to far in order. Among them, the Mercury is the closest to the sun, the Venus is the brightest, the Earth is the most unusual with life on it, the Mars is a pale red, the Jupiter is the largest, the Saturn has seven beautiful rings, the Uranus is blue green and the Neptune is a pale blue gas planet. Certainly, the Mercury, like the Earth has mountains, plains and crags on its surface. It is the smallest planet in the solar system; the Venus has a relatively flat surface with about 60% of plain and covered by a thick yellow atmosphere but no blue oceans like the Earth's; the Earth has the oceans, mountains, plains and climate suitable for the growth of life. It is the only celestial star where life exists known to humans in the universe; the Mars is a desert planet, the second smallest planet in the solar system after the Mercury. The surface of Mars is covered with sand dunes, gravel, dust suspended in it, and there are perennial dust storms; the Jupiter is a large liquid hydrogen world. Its center is composed of silicates and iron and other materials, the upper atmosphere is composed of about 88-92% of the volume of hydrogen and about 8-12% of helium; the Saturn is also a gas giant. It's composed mainly of hydrogen, also with a small amount of helium and trace elements. The core includes meteorites and ice, and the outer is surrounded by metallic hydrogen and gas; the Uranus is composed mainly of rocks and water ice of various compositions with 83% hydrogen and 15% helium as its main elements. A standard model of the Uranus consists of three layers: a rocky center, a hot and dense fluid consisting of water, ammonia and other volatile materials, and a crust of hydrogen and helium in the outermost layer; the Neptune is a moderately large asteroid between the Earth, Jupiter and the Saturn. Its internal structure is similar to that of the Uranus with an atmosphere consisting of 80% hydrogen, 19% helium and a tiny amount of methane. It has violent storms with wind speed up to 2100km/h.

Dr.Ouyang tells Huizi that there were ten suns in the solar system once so that the ancestors worked hard in the ancient Chinese myth. It is Houyi who shot nine of them with arrows and then the Chinese could multiply and survive on the earth.

[**Houyi Shot Suns**] There is an important ancient book *Shan Hai Jing* filled with magical mystery, history recorded, geography, nationality, religion, mythology, biological and medical matters in the pre-Qin period of ancient China, whose author is unknown. Today, many amazing animals recorded in this book are extinct today but it provides unlimited imagination in order to promote students' interest in exploring things in the universe. There is a story of Houyi killing the giant "*Zoute*" with an arrow recorded in its *Wild South Longitude*. It is also said that Dijun and Yihe gave birth to ten sons, i.e., the suns in the ancient times and they all lived in a place called *Tang* valley in its *Wild East Longitude*. According to the rules of heaven, one of the ten sons took turns to shine in the sky and spread sunshine for everything while the others slept on the branches of a mulberry tree. At that time, although there were ten suns, humans saw only one sun on the day, worked with the sunrise and went to sleep after sunset which resulted in humans kept friendly with animals and all things thankful the sun for bringing them hours, light and joys.

One often says that the ten fingers are not even the same length, not to mention the conduct of humans. One day, the naughty youngest suns wanted to break the rules set by the emperor of heaven and begged his brothers to accompany him appearing in the sky. He thought it would be more fun than the normal. And so, when dawn came, the ten suns all appeared in the sky, like ten balls of fire. All things on the earth were afflicted. As the so called one day in the sky is the same length of one year in the earth, it then appeared that the forests were on fire, the animals were burnt to death, the rivers and seas were dried up, resulted in that fishes dead, crops and orchards withered and foods were cut off. Someone went out in search of foods but were burned alive in the heat of the sun or fed to animals.

Houyi saw humans lived in misery baked by the ten suns and determined to help them out of the misery by shooting out nine of them. It's said that Houyi took up a bow which was strong enough to hold ten thousands of pounds and sharp arrows, each weight of thousand pounds, climbed over ninety-nine mountains, crossed ninety-nine rivers and passed through ninety-nine canyons to the east Chinese sea. When Houyi reached the top of a mountain he pulled open his bow with his sharp arrow, aimed at the hottest sun in the sky and shot down the eldest of the ten suns. And then, he pulled open the bow again and took another sharp arrow to shoot at the remaining suns again. This time, the second and third elder suns were shotted down. Although three suns were shotted down, Houyi felt that the seven suns were still hot. So, he shotted out the third arrow. This time, the elder fourth, fifth, sixth and seventh suns were shotted down. In this way, the nine suns fell to the earth one by one after Houyi's shooting. The light and heat gave out disappeared in all the suns by shotted down.

**Figure** 1.11**. Houyi Shot Suns**

At this time, the youngest sun realized that he had violated the rules of the heaven and caused a great trouble. Being afraid of punished severely by the emperor, he hid himself in the sea at once. Then, there was no sun in the sky and it was dark on the earth. Everything were no nourished by the sunshine, the snakes and wild beasts run on the wild, and humans could not live on. They prayed for the emperor of heaven to call out the sun hiding in the sea to fulfill its duty so as to make all things surviving. At this time, the emperor realized that his ten sons had violated the rules of heaven and destroyed the order of the universe. He ordered the heavenly generals with soldiers to capture his youngest son and punished him to be on duty in the sky every day without a rest. His youngest son was fully aware of the severity of his own sin, received the punishment and faithfully action according to the punishment. He then rises from the sea in the east and sets down in the west, provides the world with sunshine, warmth and nourishment every day. Houyi was then awarded as a member of the heavenly general by the emperor because he saved the order of the unverse.

Dr.Ouyang tells his daughter that the story of Houyi shot suns is a popular myth in China. Most of its details are lost in *Shan Hai Jing* but it is written detailed in *Huai Nan Zi-Ben Jing Xun* by Liuan in the western Han dynasty of China. However, whether or not there were ten suns in the formation of the solar system and how those nine suns disappeared if they really existed is still a mystery. It will only be possible to tell the truth with further knowledge on the origin of the solar system.

## §3. The Rotating Earth

The earth is the cradle of humans, a cradle of known life in the universe which is in about 15 million kilometres away from the sun in the solar system and is the third farthest place in the solar system. On the earth, the highest surface temperature is 57.7 $^oC$ and the lowest surface temperature is $-89.2$ $^oC$, which covers 29.2% of the land and 70.8% of the water area. The perennial light, temperature, humidity, climate, geographical environment, water area and land $\cdots$ etc., provides the living environment for different kinds of livings. As the planet where humans live in the solar system, the origin of the earth is an ultimate question that humans need to understand thoroughly. Certainly, humans live on the earth. They know it much more than the other planets. The great leader Mr.Zedong Mao of Chinese once wrote "*sitting on the earth, traveling 80000 miles a day, watching thousand rivers afar in the sky*" in the *Send Plague Away*, one of his famous poems. Compared with other planets, there are more records on the origin of the earth and formation of the earth, and many phenomena in the historical records. For example, the structure and material distribution of the earth's interior, the causes of earthquakes, volcanoes, tsunamis and other natural disasters, etc.

**Figure** 1.12**. Earth**

On a summer weekend, Dr.Ouyang takes his family coming to a seaside for holiday. Seeing the boundless seawater, they are all released their nervous in working mood and enjoying the seawater clapping on their body. After dinner, the daughter Huizi asks Dr.Ouyang a question:"*Daddy, how did the earth come about?*" Dr.Ouyang asks her in reply:"*How do you think the earth came about?*" Huizi answers innocently:"*In the activities out of class, a few children said that the earth was a seed that fell from another planet accidentally and then grew up into the earth after being exposed to the sunshine and the wind.*" Dr.Ouyang tells her that although humans exist on the earth, there are no universally agreed conclusions about how the planet came to being. There are only a few hypotheses on the formation of earth in collected datum with exploration.

**3.1.Earth's Origin.** The origin of the earth and the origin of the solar system are the same question. There are two main hypotheses on the origin of solar system. One is the nebula formulation. It is said the solar system formed from a swirling mass of hot gas in its cooled and solidified gradually, i.e., in about 5 billion years ago, there was a swirling fireball consisting of dust, hydrogen and nitrogen in the milky way, known as the solar nebula. In the process of gravitational contraction, it grabbed floating objects continually in the universe. Then, its temperature and density gradually increased with its rotated, and the more closer to its axis the more so. Over time, the central fireball became the original sun and the remaining formed an envelope around the sun, which gradually expanded along the solar equator and increased the distance from the central fireball, forming a nebula disk. This nebula disk mainly consisted of three types of materials, namely the hydrogen and the helium accounting for 98% of the total mass, the ice, mainly consisting of the oxygen, carbon, nitrogen, chlorine, sulfur and other elements, nickel, argon and other elements accounting for 1.5% and the stone, mainly consisting of sodium, magnesium, argon, silicon, calcium, iron, nickel accounting for 0.5%. As the dust layer in the nebula's disk increased in density, the transparency of solar radiation gradually decreased, forming today's solar system; Another hypothesis is the capture formation. It is believed that there was only the sun of the solar system in the ancient times. In 6-7 billion years ago, the sun captured a large number of meteorite particles and dust when it passed through a huge dark nebula, Some of the captured materials combined within the sun while others gathered rotating around the sun, absorbed floating meteorites and dust continuously and then formed other planets of solar system.

**3.2.Earth's Structure.** The inner structure of the earth is a series of concentric circles from the center to the surface, successively called the *core, mantle* and *crust*, a combinatorial structure. It is formed step by step by the core capturing molten materials and a small amount of plastic materials, solid materials, gases and liquids such as those shown in Figure 1.13.

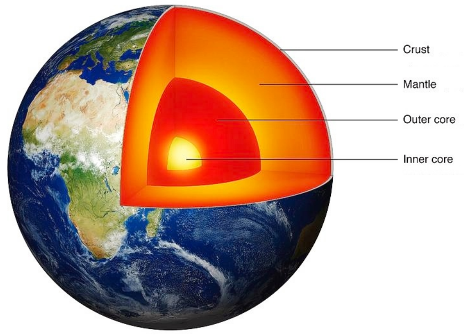


**Figure** 1.13. **Earth's Structure**

The core of the earth is also known as the iron-nickel core. Its material composition is mainly iron and nickel, divided into the core and the outer core. The core top interface is about 5100 kilometers from the earth's surface, and the outer core top interface is about 2900 kilometers from the earth's surface. It is thought that the outer core maybe consist of liquid iron while the inner core is consist of a highly rigid, solid alloy of Fe and Ni crystallized under extreme pressure. The core is thought to have a pressure of 3.5 million atmospheres and a temperature of up to 6000 degrees Celsius. The material in the core of earth has certain plasticity under the action of high temperature and pressure for a long time but harder than steel for a short time.

The mantle is nearly 2900 kilometers thick between the surface and the core of the earth with the most volume, and it is composed mainly of dense rock forming material and is part of the earth's interior. The part near the crust is mainly silicates while the part near the core is mainly iron and nickel metal oxides, close to the consisting of the core. Structurally, the mantle can be divided into the upper and the lower. The lower mantle top interface is about 1000 kilometers away from the surface with a density of $4.7g/cm^3$, and the upper mantle top interface is about 33 kilometers away with a density of $3.4g/cm^3$. Among them, there is an asthenosphere near the top of the upper mantle, which is a place where radioactive materials are concentrated. The temperature of the whole mantle is high enough to melt rocks which is the origin of the earth's magma. The temperature, pressure and density of the lower mantle are all increase, and its material becomes plastic and solid.

The crust is the solid just below the surface of the earth in about 17 kilometers thick on average. Its thickness is not uniform. An average thickness of the continental crust is 33 kilometers, the average thickness of the crust of the high mountains and plains is 60-70 kilometers, and the average thickness of the crust of the oceanic part is relatively thin, only 6 kilometers in about. In general, the crust is thicker at higher altitudes and thinner at lower altitudes. In addition to sedimentary rocks, most of the material composition of the earth's crust is granite and basalt. The granite is mostly distributed on the basalt, and mostly in the continental crust. The upper layer of the crust is composed of sedimentary rock and granite, mainly composed of silica-aluminum oxides, while the lower layer is composed of basalt or gabbro, mainly composed of silica-magnesium oxides. The granite in the oceanic crust is rare, generally covered in the basalt by a layer of sedimentary rock about 0.4 - 0.8 kilometers thicker. Generally, the crustal temperature increases gradually in about 30 $^oC$ with the depth increasing every 1 kilometers.

**3.3.Earth Drifting**. Dr.Ouyang asks his daughter:"*Do you think the earth is standing still in the space or it is constantly in the moving?*" Huizi answers:"*My teacher explained this question in class. It is rotating around its axis and the whole also revolves around the sun.*" Dr.Ouyang affirmed her answer:"*Right! All stars in the universe are in constant moving. Additionally, to its own rotation the sun along with other planets in the solar system revolve around the center of the milky way galaxy, taking about 226 million years to complete one orbit. Certainly, the milky way is only part of a larger local group of nearly 50 galaxies and it as a whole takes about 100 billion years to orbit the constellation Virgo in its local group.*" Huizi asks Dr.Ouyang:"*Then, why can't I feel it?*" Dr.Ouyang tells her:"*This is because the human eyes can't see very far and there are no relative objects for such an observation. At the same time, a human's life is too short compared with 226 million years, 100 billion years of the revolutions of the solar system and the milky way, i.e., it is normal that can not perceive the difference for a human.*"

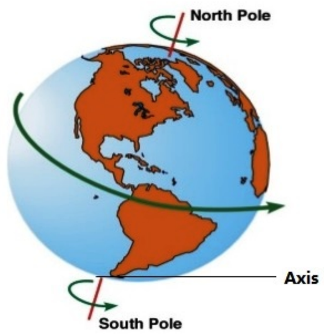


**Figure** 1.14**. Earth's rotation**

At this point, Huizi suddenly thinks a question and then asks Dr.Ouyang:"*Is the earth floating in space or is there a supporter below it? Otherwise, it would be fallen down!*" Dr.Ouyang replies her:"*It's floating!*" Huizi listens to surprisingly:"*Why does the earth not fall down? In class, the teacher said that Newton was reading under an apple tree, an apple fell from the tree and hit him head. If there were no a supporter under the earth the earth would be fell like an apple in the tree? Also, if there were no a supporter under the earth, the humans on the other side of the earth would be fell into the space.*" Dr.Ouyang then smilingly answers to her:"*That's the action of gravitation! Do you feel there's a hand holding you back from jumping?*" Huizi jumps and then says:"*That's*

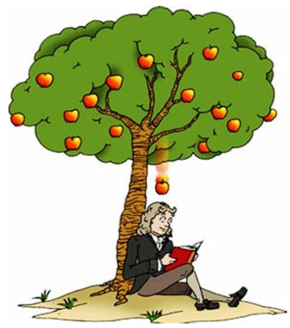

**Figure** 1.15**. Apple hit Newton**

*true! How's the gravitation generated?*" Dr.Ouyang tells her:"*The generation of the gravitation is complicated. Newton called it the universal gravitation,* *namely there will be a gravitational force between any two objects as long as they have masses and then, he put forward the law of universal gravitation, which concludes that the gravitational force is directly proportional to the product of the two masses and inversely proportional to the square of the distance between the two objects, or formally*

$$\mathbf{F} = G \cdot \frac{m_1 m_2}{r^2},$$

*where,* $G = 6.67 \times 10^{-11} N \cdot m^2/kg^2$, *called the the gravitational constant,* $m_1$ *and* $m_2$ *are respectively the masses of the two objects with distance* $r$. *By the way, you saw the rising tide at the seaside in the afternoon. Do you know what caused it?*" Huizi shakes her head:"*I don't know. It just feels like the seawater is rising fast and soon the rocks are gone over!*" Dr.Ouyang takes out a book named the *Philosophiae Naturalis Principia Mathematica* from his bag, opens a page to tell his daughter:"*This's a book on natural philosophy written by Newton. He summed up in this book that all experiments and astronomical observations show that objects around the earth are attracted to the earth's gravitation called the gravity which is proportional to the amount of masses that their respective contains and the same is happened for the moon. On the other hand, it also shows that our oceans are gravitated by the moon, all planets are gravitated towards each other and the comets are gravitated by the sun. It is because of this rule that we must generally admit that all objects whatever they may be, are endowed with the interaction principle of gravity and this leads to an argument on the existence of the universal gravitation.*"

Dr.Ouyang then explains the tide to his daughter. The tide is a kind phenomenon of periodic rise and fall of the seawater. Usually, the day is called the *Zhao* and the nigh evening the *Xi* in Chinese. Thus, the rise and fall of seawater is called the *tide*. The ancient Chinese were not clearly understand the cause of the tides. A few careful ones had found that the tides are related to the moon. In the Wild East Longitude of the book *Shan Hai Jing*, it is mentioned that there was a monster of white looking like an ox sitting on a mountain by the east Chinese sea. It had no horns but only one hoof. However, when it came in or out of the sea, it must be accompanied by the strong winds, heavy rain and also gave off light like the sun and moon which implies the relationship between tides with the sun and the moon. In the eastern Han dynasty of China, Wangchong clearly pointed out that "*waves rise and fall with the moon*" in his book *Lun Heng*, which revealed the relationship of tides with the waxing and waning of the moon and concluded that tidal phenomena were related to the moon. Based on the Newton's law of universal gravitation, Laplace mathematically proved that tides were indeed caused by the gravitational attraction of the sun and the moon, mainly due to the moon. Those humans who live near the sea can see the seawater rising and falling regularly twice a day, travelling from the east to the west taking half a lunar day, i.e., 12 hours and 25 minutes.

Although the tide force generated by the sun is not large, it affects the size of the tide. Sometimes, it creates a force with the moon and sometimes it repulses the force of the moon. Generally, the sun and the moon exert gravity in the same direction, producing a high tide in a new or full moon and a low tide in the first quarter of the moon. During the equinoxes of a year, if the earth, moon and sun are almost on the same plane, the generating force of the tide is greater than other months which results in two highest tides in spring and autumn of a year.

At this time, Huizi asks Dr.Ouyang: "*According to this statement, there is always gravitation between objects with respective masses. But why do we not feel it between humans? Is the love at first sight such an attraction for main characters in TV dramas?*" Dr.Ouyang explains to her: "*The gravitation is proportional to the product of the mass of the two objects and inversely proportional to the distance between the objects. However, what the gravitational value is also need to multiply by a constant* $G = 6.67 \times 10^{-11} N.m^2/kg^2$. *So, the actual gravitation between two humans is a very small one, can be completely ignored in our daily life. That's why we don't feel gravitation between us. However, you need to take the gravitational force into account in the case of stars because it's the gravitation forces them to move relatively. When it comes to love for main characters in TV dramas, it isn't because of the gravitation but the feelings or minds between humans.*"

Now, *why the gravitation appears between objects with mass?* Dr.Ouyang explains continuously. In fact, Newton did not explain himself why it should appears, only claimed it must being when he proposed the law of universal gravitation. Until the beginning of the 20th century, Einstein proposed the *general relativity* which offers an explanation that contradicts the common sense of humans, concluded that the physical space is in fact a curved one under the action of mass rather than the common belief. That is, the spacetime is curved, not a flat one and what a particle moving in space is liking a child sliding on a pulley from high to low as if there is a pull force below, which is the same with gravitation, even with the light.

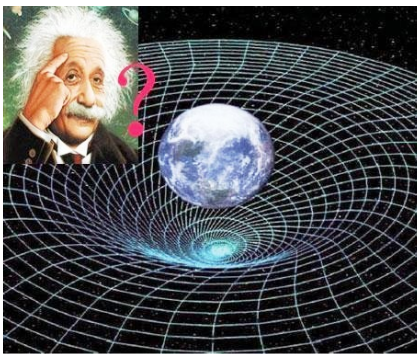

**Figure** 1.16. **Curved spacetime**

In 1916, Einstein put forward 3 experiments for verifying his general relativity, namely the precession of Mercury orbit perihelion, the deviation of light wave near the sun and the gravitational red shift of light wave. All of the experiments had been confirmed by humans and furthermore, the result of experiments showing light wave deflected near the sun was a big news, appeared in the front page of newspapers. This experiment used a total solar eclipse to look at stars near the sun carried out by Sir Arthur Eddington and others in 1919, which was impossible normally because of the strong light irradiation of the sun. Observations were made from both in the Ceara of Brazil and the Sao Tome, Principe in the west coast of Africa in the meantime. In this way, comparing the position of the background star approaching the sun measured the curved value of the light in that time. It is almost identical to the calculation of Einstein which brought about Einstein with his general relativity famous all over the world.

Dr.Ouyang says that a philosopher Archimedes of ancient Greece claimed that if you give him a fulcrum and a lever long enough, he could pry the earth in his book *Balance of Planes*. Dr.Ouyang then asks his daughter: "*Could you think how long and how strong that lever should be?*" Huizi answers: "*I have no ideas on that lever or fulcrum!*" Dr.Ouyang explains to her: "*Although his view is valid in theory, it's still absurd. Where could we find such a lever and where can we find such a fulcrum? We could't find them! There are no such a lever or fulcrum in the universe!*" Huizi asks: "*Such a lever may not exist but can the fulcrum not be on any planet outside the earth?*" Dr.Ouyang tells her that the quality of the earth is $5.965 \times 10^{24} kg$ in about. A planet becoming the fulcrum of Archimedes prying the earth must be dense with sufficiently larger masses than that of the earth because only in this way, the reaction when Archimedes prying the earth will not affect its orbit. Otherwise, it will disrupt the order of the universe running, bring a devastating to the universe, even to humans. However, there are no such a star and there are no such a lever in the universe, i.e., there are no way for Archimedes prying the earth. Although the Archimedes's claim is true in theory but it is fantasy, can only dependent on the divine power in the mind of human which implies that a scientific conclusion can not be only imagination and applied away from its conditions. However, the theory of universal gravitation founded the understanding of the universe. it is a tangible, not fantasy contribution of scientists to humans developing.

## §4. The Earth's Plants

Plants are one of the life forms that are fixed and autotrophic on the earth, including trees, shrubs, vines, grasses, ferns, green algae, lichens and other familiar creatures, which are directly nourished by the earth. An elementary estimation shows that there are about 350000 plant species live in the earth, more than 287655 species confirmed by humans, and there are 258650 species of flowering plants, about 16000 species of mosses, 11000 species of ferns and 8000 species of green algae. Plants are the basis of animal survival such as the food, medicine and vegetables for humans, and also to purify the ecological environment, to beautify the living environment, etc., which is a type of livings giving the necessary of humans in the universe.

One day, Huizi holds back the house a small bowl of green garlic from the balcony, says she will show to children in her school the next day. She tells Dr.Ouyang that it is the result of putting garlic clove by clove in a bowl with watering it twice a day for more than 10 days under the guidance of her mother.

**Figure** 1.17. **Garlic sprout**

Looking at the joy of his daughter's success in planting garlic, Dr.Ouyang wishes to teach some plant knowledge for his daughter, asks her: "*Did you know how garlic comes?*" Huizi replies: "*My mother bought it in the market.*" Dr.Ouyang asks her further: "*Then, how did the garlic in the vegetable market come from?*" Huizi replies: "*It's an uncle farmer took out his garlic planted in the field for selling!*" Dr.Ouyang inspires his daughter: "*Then, where the garlic come from that the uncle farmer planted?*" Huizi think for a while and then asks: "*Was his mother brought it for him in the market?*" Dr.Ouyang tells her that it could be his own garlic seeds left in the previous year or that his mother bought garlic seeds in the market. Growing vegetables need vegetable seeds, growing corns need corn seeds and growing trees need saplings. In this way, the grains, fruits and the vegetables are all made of the seed germination after grow up step by step. Some of them are growing up quickly, which is seasonal such as vegetables but some of them fructify after flowering, experiencing the "*spring sowing, summer glowing and autumn harvest*" such as the walnut. And all plants growing can not leave the sunshine, can not leave water or the moisture.

Dr.Ouyang asks his daughter where did the first seed come from, Huizi looks vacant in her brain: "*I don't know where did it come from?*" Dr.Ouyang tells her that according to Darwin's biological evolution, namely all living things evolve always from the low to the high and the origin of plants should be earlier than animals, more earlier than humans.

**4.1.Living's Origin.** The origin of livings, namely when, where and how livings born on the earth is an unsolved mystery until today. There have been many hypotheses or myths on the origin of livings. For example, the God's creation in the *Bible* or the Nvwa's creation in ancient Chinese mythology, etc. and there is also a kind of autogenesis hypothesis said that a living thing can be generated by non-living things or some other completely different substances similar to God's creation. In the Spring-Autumn period of ancient China, Laozi explained that "*Tao creates One, One creates Two, Two creates Three and Three creates all things*" in his famous book *Tao Te Ching*. He put aside God's creation and explained the origin of livings with a natural evolutionary rule, namely the "*generated rule*". Correspondingly, Darwin's theory proposed by the biological evolution in the 19th century brought a natural way to reveal the origin of living things by evolution.

**Figure** 1.18**. Marine livings**

The chemical origin hypotheses is a widely accepted on the origin of livings. It is said in this hypothesis that the origin of life is closely related to the origin and evolution of the universe, which is from inanimate matter evolved step by step through a complex chemical process in the long evolution process with the earth's temperature decline gradually because the constitute elements of a living body such as the carbon, hydrogen, oxygen, nitrogen, phosphorus and sulfur, etc., are all from the evolution of elements in the universe after the Big Bang. This evolution reveals the mystery of the origin of living things, namely some biological molecules such as amino acids, purine and pyrimidine maybe formed in the interstellar dust or condensed nebula in certain conditions after the explosion temperature gradually reduce or cooling of the Big Bang, and a few biological macromolecule such as the polypeptide and polynucleotide were synthesized through a series of biological evolution on the surface of the earth. The life with the primitive cell structure provided the basis for the biological evolution of the earth and then through the variation and evolution of single cell to multi-cell, simple life to complex life formed the complex life forms we see today. Dr.Ouyang tells his daughter that in order to prove the first seed evolved from nonliving matters, the biologists carried out a lot of simulation experiments, synthesizing biological macromolecules with some simple molecules in the laboratory. In order to restore the biology evolution process, they assumed that the early earth contains a large number of primitive atmosphere consisting of methane, ammonia, water, hydrogen, and marine, etc., if the lightning acts on the gas then it would gather up to many kinds of amino acids and then, these amino acids under atmospheric pressure can be further evolved into proteins, other polysaccharide and other polymer lipids after the local enrichment. In this way, life can be born under certain conditions. Of course, these studies are based on the hypothesis ourselves to restore the origin of life.

Dr.Ouyang concludes to his daughter that the chemical hypothesis on the origin of life such as those of garlic began with only one cell, namely the prokaryotic cells could not absorb nutrients through photosynthesis but could only absorb in the environment and then evolved into autotrophic plants such as the cyanobacteria and gradually, evolved into lower plants such as algae and hyphae of lichen, moss, plants and then evolved further, the seed plants appeared. But, *how do the plants evolve on the earth and be classified?*

**4.2.Plants Classification.** The number of plants on the earth, including trees, shrubs, vines, grasses, ferns and familiar organisms such as green algae and lichens is estimated at more than 500000 species. The first plants on the earth are the fungi and algae which once flourished. Around 438 million years ago, green algae broke free from their aquatic environment and landed on the land where they evolved into naked ferns and ferns. About 360 million years ago, the bare ferns and ferns gradually became extinct and the plants evolved into lycopsida, sphenosida, filicopsida and seed ferns, and began to form a certain scale of swamp forests. About 248 million years ago, ancient plants almost died out, the gymnosperms rose and some plants got rid of their dependence on water and forests began to form on the ground. About 140 million years ago, the angiosperms diverged from gymnosperms. After entering the Cenozoic era, angiosperms showed progress in heredity, development, stem and leaf structure especially in the flower propagator, which enabled them adapting to the changes of environmental conditions by genetic variation and differentiated into more plant types.

There are 8 types of common plants on the earth, including: ①Gymnosperm, a kind of paleontology, almost all over the continent including trees, shrubs, vines and other types; ②Angiosperm, a flowering plant that propagate their offspring through their stamens, accounting for about 50% of plant species; ③Bryophyte, a very wide distribution weather resistant plant which can be seen in the humid, the earth's polar, tropical and other places. Its general height is generally in 2cm-5cm with the highest is 30cm; ④Pteridophyte, a kind of leaf-watching plants growing in low-lying areas which has beautiful appearance and high ornamental value; ⑤Algae, a simple plant that produce energy through photosynthesis; ⑥Seed plant, a class of plants whose body consists of vascular tissue and can produce seeds for their reproduction, consisting of gymnosperms and angiosperms. For example, the pine, cypress, chestnut and so on; ⑦Lichen, a plant produced by the symbiosis of fungi and algae on the surface of the ground. They are very sensitive to air pollution and can be used as indicators of air pollution. According to the appearance of lichens, they are classified into shell-like, leaf-like, and clad-like lichens such as genus volvariella, xanthium on the grassland after rain, and the genera meici on the rocks or bark; ⑧Fungus plant, a large fungi formed by a colloid or fleshy fruiting body or river tissue and can be edible or medicinal fungi such as the ganoderma lucidum, agaric fungus, hericium ericii, bamboo fungus, mushroom and so on.

**4.3.Plants Structure.** As a form of life on the earth, there are generally six major organs in the combinatorial constituting of a plant, including its root, stem, leaf, flower, fruit and the seed. Among them, the roots are the vegetative organs of plants which generally take the root below the surface of the earth, absorb water and other nutrients in the soil, dissolve ions in it, supporting and storing synthetic organic materials. Generally, the plant roots are composed of parenchyma, vascular, protective, mechanical and meristematic tissues; its stem is the central axis of a plant carrying the nutrients, water for supporting the survival of leaves, flowers and fruits. Some of them also have the function of photosynthesis, storage of nutrients and reproduction; the leaves are organs that maintain plant nutrition which contains chlorophyll interacting with the sunshine for organic synthesis, for the root system to absorb water and mineral nutrients from the outside to provide the source power.

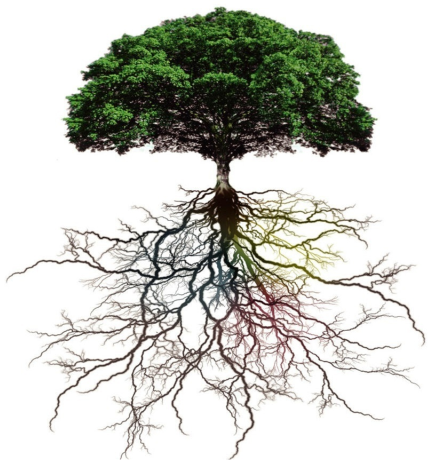

**Figure** 1.19**. Plant Structure**


**Figure** 1.20**. Plant pollination**

The flower of a plant grows on the receptacle with petal on the most outside, wrapped with stamen or pistil on the middle, which is the reproductive organ of the plant. Generally, the flowers and their bright colors attract insects to pollinate and carry on the family line.

Some grasses and trees have pale, unscented flowers and are pollinated by the wind.

The fruit of a plant is the medium of seed dispersal and develops from the pistil of a flower. The fruit of most plants contains the seeds. Some have one seed and some have many. As different fruits ripen, some are rich in water while others are dry. Typically, the water-bearing fruits are brightly colored and attract animals to eat them and carry their seeds far away. When the animals excrete the seeds, they take root and germinate. Some legumes spontaneously burst open when their fruits are ripen, scattering their seeds on nearby ground to germinate. Some plants have very light fruits which are carried by the wind to distant places and scattered on the ground. There are also plants which have burrs that stick to animals, and when they fall off, take root and germinate in the ground, providing conditions for the continuation of plant species. In this way, plants and animals in the process of long-term survival and development, formed a mutual adaptation, interdependent relationship. On the one hand, plants nourish animals and animals depend on plants for survival. On the other hand, animals help plants pollinate, spread fruits and seeds, help plants reproduce and expand their survival area.

Dr.Ouyang asks Huizi to summarize what are the seed and fruit of a garlic as a plant. Huizi thinks for a while and says:"*I planted garlic in a small bowl using garlic cloves. So, the garlic seed should be its cloves. But my garlic has not yet come to flower or bear fruit. I do not know what the fruit of my garlic is.*" Dr.Ouyang helps her answer:"*The fruit of garlic is still garlic. It starts with a clove and grows into a garlic which may be containing many cloves.*"

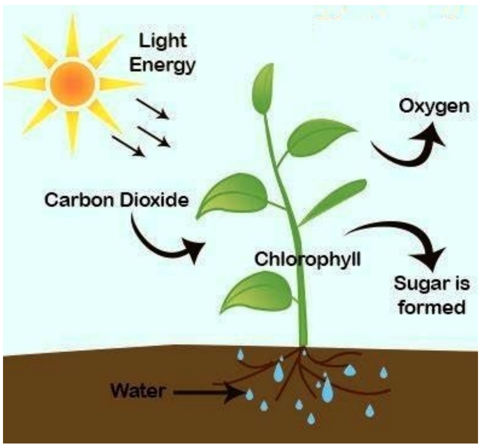


**Figure** 1.21. **Photosynthesis**

**4.4.Plants Function.** There are two main types of roles for plants participating in the natural cycle, i.e., photosynthesis and respiration, where the photosynthesis refers to the process in which green plants absorb light energy, synthesize carbon dioxide and water into the energy-rich organic compounds, and the release oxygen at the same time, including the absorption, transmission and the conversion of light energy, converting of light energy into the electrical energy which is then converted into the chemical energy. The photosynthesis is the most common and largest natural reaction process on the earth. It plays a natural regulatory role in organic synthesis, solar energy storage and air purification, and is the natural basis for human agricultural production and animals to provide oxygen.

The respiration is the hub of plant metabolism. All plants decompose substances through respiration to provide energy needed for various life activities in the body and synthesize important organic matter and simultaneously enhance the resistance of plants to diseases. Generally, the plants are divided into the aerobic breathing and the anaerobic breathing. The aerobic respiration is the dominant form in higher plants but the plants can also carry out anoxic respiration in special conditions and tissues in order to maintain the normal metabolism.

Dr.Ouyang concludes that the main function of the photosynthesis and respiration being: ①Conversion. The photosynthesis converts the carbon dioxide and water into oxygen and organic matter and the respiration decomposes oxygen and organic matter into carbon dioxide and water. This kind of photosynthesis plays an irreplaceable role in maintaining the carbon-oxygen balance in nature. At the same time, the photosynthesis provides a material guarantee for human and other organisms to coexist peacefully; ②Regulation. As we know, the matter in the nature is constantly moving and plants play an important role in the movement. For example, the water is the basis of life in the nature. Certainly, the plants increase air moisture by transpiration, absorb needed moisture from the soil and then, return to the environment in the form of gas in the nature. The natural water cycle is in a relatively balanced state. At the same time, the green plants in the process of photosynthesis, constantly release oxygen to enrich the animal respiration, combustion and other consumption caused by the lack of oxygen, maintain the balance of oxygen in the nature; ③Protection. Human activities on the earth's environmental pollution have been a serious threat to human survival, sulfur dioxide, carbon dioxide and other harmful gases in the air and water chemical reaction to produce acid rain, excessive emissions will lead to the death of trees, grain production and increase of human death and disease. The plants can effectively absorb dust in the air, transform harmful substances or reduce the content of toxic substances in the atmosphere, increase air humidity and precipitation through transpiration, play a role in maintaining ecological balance; ④Conservation. The branches and leaves of plants can effectively reduce the impact of rainwater on the soil, and the root system underground can hold the soil tightly to prevent it from being washed away; ⑤Antivirus and bacteriostasis. Some medicinal plants release volatile substances with certain bacteriostatic effect in their physiological process which can clear the bacteria, viruses in the air and enhance human immunity; ⑥Beautification. In human living environment, the flowers and other plants are the natural regulator of human physical and mental health, which can regulate human's mood and spirit, relieve the tension and anxiety, stabilize the emotions and make human happy. That is why human beautifies his or her living environment.

Certainly, humans are coexisting with plants on the earth. Looking out of the window at the moderately polluted air and poor sunshine again, Dr.Ouyang asks his daughter if she understands the cause of air pollution. Huizi replies with a question:"*Daddy, is it too little green plants but human output of pollutants too much?*" Dr.Ouyang affirms her answer:"*It's so! An element for maintaining clean air is that the pollutants discharged by humans are all absorbed by the green plants. However, humans blindly cut down forests and trees, reduced farmland and water and over hardened the ground in their living area over the years, which blocked the circulation of gases in the air and underground to some extent.*" In this sense, Dr.Ouyang says that the protection of the environment means that humans should maximally reduce the emission of pollutants to the nature in the first and then expand the area of forests and green vegetation to maintain the ecological equilibrium of the nature. Clearly, the plants are one of the most common forms of life on the earth.

**Figure** 1.22. **Ginseng**

Dr.Ouyang asks his daughter:"*Could a plant moves around the land at will just like animals?*" Huizi opens her eyes widely:"*Can't being! If so, it's the same as the animal!*" Dr.Ouyang says:"*Right! The most important characteristic of plants is that they are fixed in a certain point on the earth and can't move about. But there are also anthropomorphic legends to certain plants likewise the universe, galaxies or the earth. Have you heard the story of tied a ginseng by red rope once?*" Huizi answers:"*No, daddy! Tell me please. It must be very interesting.*"

**[Ginseng's Story]** The ginseng is a kind of precious Chinese medicine regarded as the king of grasses. Although it can not cure all diseases but it has the effect of invigorate vitality, improve self-immunity and strengthen of the body after a long-term consumption by human. There are many interesting stories on the ginseng.

It is said that there was a taoist temple in Chang Bai Sshang, a big mountain of China in which there were an old taoist and a boy taoist. Every day, the boy taoist had to go up the mountain to cut firewood, carry water, clean the temple, cook and wait on the old taoist besides chanting sutras. Even so, the old taoist would punish the boy taoist if he was not satisfied.

One day, the boy taoist went up the mountain to cut firewood with enduring pain on his body, crying while his cutting. At this time, a fat doll with a red bellyband ran in front of him:"*Little brother, why are you crying so and who's been bullying you?*" The boy taoist told the fat doll that he's living in the taoist temple from beginning to the end.

The fat doll looked sympathetically at the scar on the boy taoist and said:"*Just a moment, I'll get back you some medicine.*" In a short while, the fat doll took back a small red fruit in his hands, let the boy taoist swallow, said to help him cut wood and then play together. Strangely, the boy taoist felt no pain at all and became stronger after swallowing the little red fruit. The two boys cut a large bundle of wood, played happily until nearly noon and then, they were reluctant to leave with an engagement to play together in the afternoon. As soon as the boy taoist arrived at the place where he was cutting wood in the afternoon, he found that the fat doll had come again. They together cut a large bundle of wood, sang and danced with a great joy. This thing went on for more than a month. Every time, as long as the boy taoist was punished by the old taoist, the fat doll gave him a small red fruit to eat. The boy taoist became a lot of strong with a florid face and the scars are invisible on his body. The old taoist looked in his eyes but puzzled in his mind.

One day, the old taoist secretly followed the boy taoist in the mountain and found the fat doll. The old taoist asked the boy taoist where the doll come but the boy taoist said nothing. The old taoist hanged up the boy and tortured him, the boy taoist had to tell the truth. The next morning, the old taoist took out a red line spike to the boy taoist, let him pin on the top of the doll's bellyband. The boy taoist did not know what is his intention. While the two boys finished cutting wood to play, the boy taoist secretly pined the needle on the doll's bellyband. In that time, the old taoist hid on the side. He jumped out to grab the read rope, tied the doll back to the temple in the pot and let the boy taoist quickly lighting a fire. He wanted to eat the cooked doll. As a matter of fact, the old taoist knew that the doll is a thousand years of ginseng. He would become the immortal if he ate the doll.

The boy taoist of course could not let the old taoist to tie up the doll to cook for eating. By the old taoist not seeing, the boy taoist poured water on the wood so that the wood can not be fired and smoke billowing in the kitchen which resulted in the old taoist choked, can not stand and ran outside the kitchen shouting with abuse. At this time, the boy taoist took the doll out of the pot, removed the red rope tied on his body and let him running. The doll knew that if he ran away the old taoist would punish the boy taoist, maybe worried about his life. So the doll took the hand of the boy taoist and said:"*little brother, you can't learn anything from such a master. Let's go together!*" The boy taoist agreed. The doll dragging the boy taoist flew to the outside of the courtyard and then, travelling all over the world. The dream of the old taoist eating the ginseng was broken down by the boy taoist. This is the story that the ginseng tied by red rope but the taoist boy untied the rope to save the ginseng.

Then, *does a ginseng really have spirituality?* Dr.Ouyang explains to Huizi:"*Of course not!*" The story of ginseng tied by a red rope is only a folk because the ginseng in a mountain is too little. It is only a convention in the digging humans of ginseng, i.e., tells other ones that this ginseng belongs to the one who tied the read rope.

## §5. **The Earth's Animals**

All livings on the earth including plants and animals constitute a biosphere which constantly exchanges material and energy with the environment to establish an equilibrium. Among these, the plants are fixed to a point on the earth in the autotrophic while animals in the contrast, are not fixed to a point in the heterotrophic. Certainly, the animals and plants depend on each other with developing in the nature. For example, the animals help plants to spread powder, make plants reproduce offspring such as the bees to gather honey, birds to feed grass seeds which helps plants to spread fruits and seeds which expand the distribution areas, etc. As the consumers, the animals feed on plants directly or indirectly through the digestion and absorption, to absorb useful matter into the body, decompose substances in the body, digestion and can release a quantity to produce carbon dioxide, urine and feces applied by the producers to maintain the equilibrium of a ecosystem at the same time which enables the quantity of various kinds of ecosystem and proportion in a relatively stable state.

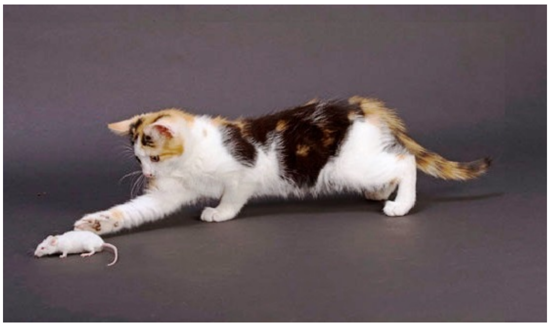

**Figure** 1.23. **Cat and mouse**

One day, Huizi embraces a kitten when she come home from outside door, tells Dr.Ouyang that a few classmate with her find a litter of newborn cats under the waste heap to meow at them with a very cute appearance. They each hold a home to raise. Dr.Ouyang helps Huizi washing the kitten and builds a nest by a box for it. Seeing his daughter holding a meat bone to amuse the cat, Dr.Ouyang asks his daughter why the cats likes to eat best. Huizi replies:"*The teacher told me that cats like to eat fishes but there are no a fish at home.*" Dr.Ouyang asks her again:"*There are lots of fishes in the river but why don't cats go to the river and catch fishes by themselves?*" Huizi says:"*All cats can't swim. If they go to the river to catch fishes, they will be drowned! By the way,* *daddy, why don't cats learn to swim?*" Dr.Ouyang replies:"*It's not that the cat can't swim. A cat can swim in the emergency. It is a land animal but likes to eat fishes. If the cats can enter the river at will, there will be no way for the fishes to live and other creatures will also kill the cats. In this way, the ecological equilibrium will be broken. This is the great of nature so that the cats live on the land and the fishes live in the water. Will the cats starve to death in this way?*" When she hears that, Huizi replies:"*No! Cats also like to eat mice, both of them living on the land but they are natural enemies.*" Dr.Ouyang continued to ask her:"*Why are cats and mice the natural enemies?*" Huizi thinks for a while and then says:"*My teacher didn't talk about it in class. He just said that cats catch mice whenever they see them and they are natural enemies in the eyes of humans.*" Dr.Ouyang explains to her:"*There are natural reasons for animals being enemies in the nature. All cats are agile, good at jumping who are nocturnal animals but if their body lack a substance called taurine, their night vision will be very weak and then, lose the ability to survive. However, the mice happen to be rich in the taurine. That's, cats eat mice because they are hungry and also because they need the taurine to survive in the nature. That's why the cats catch and eat mice in the eyes of humans.*"

**5.1.Food Chain.** Dr.Ouyang tells his daughter that the food chain in the ecosystem refers to a chain of eating and being eaten linked by food in which all kinds of organisms must feed on other organisms in order to maintain their own life. For example, the mouse is the food of cats, i.e., cats eat mouse in the cat-mouse relationship. But what do mice eat? The mouse is an omnivore and can eats all the food that of humans. This is the reason why mice steal crops like sorghum, corn, rice and peanuts as well as the processed foods of humans. A food chain is based on the biological population which links the energy and nutrient transferring relationship between different species in an area. For example, such a feeding relationship between organisms is shown in Figure 1.24. According to their role in the energy and matter moving in the ecosystem, different groups of organisms are classified into three categories, i.e., the *producers, consumers* and the *decomposers*. Here, the producers in the food chain are green plants which can produce nutrients from inorganic substances found in the nature. Generally, the producers

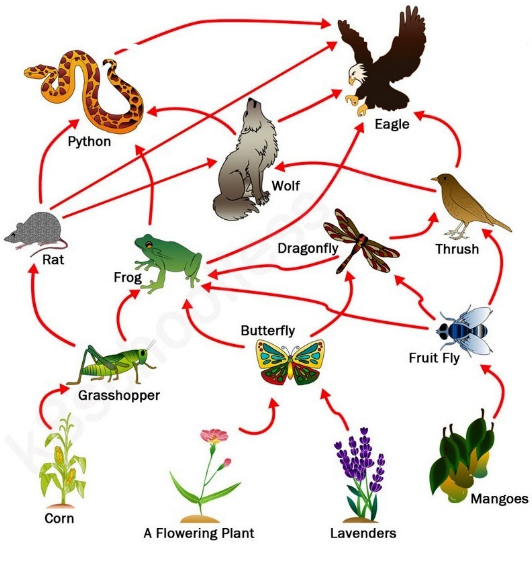


**Figure** 1.24. **A food chain**

get carbon dioxide and water from the natural environment and synthesize carbohydrates based on glucose under the action of sunlight or chemical energy, which become the source of life energy for consumers and decomposers. Notice that the producers are the basis of energy transmission in the ecosystem because the solar energy only through the synthesis of producers can be converted into biological energy and then, become the absorbed energy of the consumers and decomposers in the life activities of the biological population.

The consumers in food chain belong to the heterotrophic organisms in the ecosystem, i.e., those directly or indirectly eating plants for food, generally divided into two categories, i.e., the herbivores and the carnivores. Among them, the herbivores live on certain plants and get their food and energy by swallowing them. For example, the insects, rabbits, sheep, wild boars, elephants and so on; the carnivores prey on animals and survive by eating herbivores or other carnivores. For example, the foxes, weasels, tigers, lions and so on. In addition, there is a special consumer in the nature, namely the parasites and omnivorous consumers. Here, a parasite is an organism that lives on another organism which provides it with nutrients and shelter. For example, the bacteria, viruses, fungi and the parasitic worms; the omnivorous consumers are between the herbivores and the carnivores, eating both plants and animals such as the carp and the bears. Among these, the humans are the most advanced consumers, able to feed on both animals and plants for nutrients and energy.

The decomposers in the food chain are known also as the reducers of heterotrophy, which mainly refer to bacteria and fungi, include also some protozoa and animals that scavenge on the carcasses of dead animals such as deadwood eating beetles, termites, earthworms and some mollusks. These animals decompose the residues of plants and dead animals into simple compounds after eating, decompose them into inorganic substances and then release them into the nature so that the producers can reuse them, forming an ecological cycle in nature.

A basic law for living things in the food chain is also the survival of the fittest in natural selection. This fundamental law refers to the competition between species and within organisms, or species and the nature and then, a natural law by which those who can adapt to the nature are selected to survive. Dr.Ouyang tells his daughter a typical example of this is the extinction of the dinosaurs.

Historically, the world was ruled by reptiles more than 200 million years ago. At that time, the air was warm and humid, and the food was easy to find. The dinosaurs were the world's rulers. They were the largest and adapted to live in swamps and shallow lakes which ruled the earth for more than 100 million years but for some unknown reason, dinosaurs suddenly died out on the earth about 65 million years ago, leaving only a large number of dinosaur fossils.

**Figure** 1.25. **Dinosaurs**

[**Dinosaur's Extinction**] There have been a lot of biological species disappeared on the earth, which is inevitable in the biological evolution. For example, there are most of rare birds and animals in the Chinese book of *Shan Hai Jing* but they are no longer existing today. However, why a large ruling family likes the dinosaur suddenly disappeared from the earth is a matter of scientific debate. A few ones said that the earth began to build mountains on the flat land in about 65 million years ago. There were fewer marshes and the climate was no longer so wet and warm at that time which resulted in insufficient food for the dinosaurs and their breathing organs no longer adapt to the environment. Finally, it brought about the whole species extinction; Other ones said it was a cosmic supernova caused a precipitous change of the earth's climate that the dinosaurs no longer adapted to the earth's environment; And also someone said that it is the abrupt change in the earth's climate or the dramatic drop in temperature resulting in a drop in atmospheric oxygen as well as the continental drift, geomagnetic changes or species aging and eventually, went to the extinction.

A generally accepted explaining for the dinosaurs extinction is that a meteorite hit the earth and killed the dinosaurs. It's said that the sky suddenly appeared a dazzling white light, a meteorite about 10 kilometers in diameter fell from the sky in about 65 million years ago. It rushed into the sea at a speed of 40km/s, crashed a huge crater on the sea floor. Then, the seawater was rapidly vaporized and send a huge amounts of steam into the air in the high of tens of thousands meters. It followed by high tsunamis in high 5km at least with a great speed, which overflew the commanding heights of the land, triggering a powerful volcanic eruption on the Deccan Plateau which changed the moving direction of earth's plates. In the followed months and years, the volcanic ash filled the sky with temperatures plummeted, making the climate no longer suitable for the dinosaurs. At the same time, the heavy rain and mountain torrential floods triggered the debris flow, which swept the dinosaurs away and then buried them in the soil, namely the age of dinosaurs came to the end, following with a new era in the biological history.

**5.2.Animal Classification.** Dr.Ouyang asks his daughter: *how many kinds of animals are there on the earth?* Certainly, there are a variety of animal classification methods and generally, divided into two categories, i.e., the *invertebrate* and the *vertebrate*. It has been identified 46,900 vertebrate animals including the whales, sharks and other fish animals, snakes, lizards and other reptiles, the poultry, birds and the cattle, pigs, sheep, dogs and other mammals. However, the invertebrates are lower animals in grade. They are far exceed vertebrates in the species accounting for more than 90%, including the jellyfish, octopus in the ocean as well as the insects, parasites on land,$\cdots$, etc.

The vertebrates can be divided into the aquatic animals and the land animals according to their adapted living environment. There are also animals that can both live in water and land, called amphibians such as crocodiles, frogs, etc. It should be noted that if the near-earth airspace is considered as a habitat of animals, there are actually three places for animals, namely the water, land and the air. Among them, the fishes living in water and the tetrapods, humans living on land are only in one habitat, the crocodiles, frogs and others living both in water and land, the birds, bees and other flying insects living both on the land and air are amphibians. But an animal living on the land, air and the water at the same time ever been found on the earth. Someone think the dragonfly to be such a one because the dragonfly's larvae is founding in water and then living on the land and in the air as adults. But the dragonfly is actually amphibians. Dr.Ouyang tells his daughter that the three-dwelling animals appeared only in the myths or legends such as the Chinese dragon which could live in the water, land and air at the same time.

[**Chinese Dragon's Legend**] Chinese Dragon is a mythical creature that can make the clouds and rain, benefit all things in the universe. It can shrink into a silkworm and stretch out to cover the sky appearing in the clouds or disappearing into the abyss sometimes. Chinese call themselves as the "*descendants of the dragon*". The legend of the dragon can be found in almost every Chinese ancient book and there are countless legends and myths about the dragon. For example, there are records that say Zhurong and Houqi Xia respectively "*ride two dragons*" in the Overseas Southern or the Western Longitude in *Shan Hai Jing*, a Chinese ancient book.

In the folklore, most of the local mountains, lakes, canyons were anthropomorphized to dragons in good or evil characters. Among them, the story of "*two dragons playing with beads*" is the most widely spread. It's said that there is a deep pool in the heavenly pool of Chang Bai Shan, a great mountain in the northeast of China where two green dragons practiced. They lived in accordance with the agricultural customs and sow rain so that the local

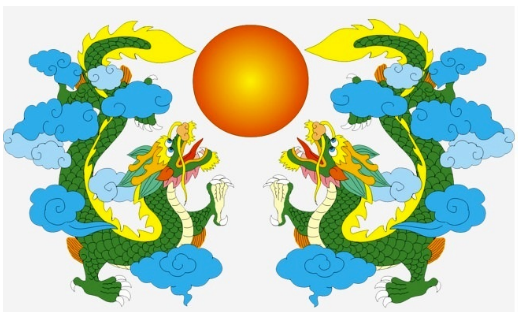

**Figure** 1.26. **Dragons with Beads**

humans lived in a life without anxious about the food and clothing. They were loved by the local humans.

It is said that this heavenly pool is the place where the celestial fairies take a bath, namely if there are few clouds with clean winds and the moon is clear in the sky, the fairies would go to the pool for taking a bath and playing at that day. One day, a hairy monster suddenly swam to the naked fairies in the pool when the fairies was washing in the pool. They were frightened and cried for help. The two green dragons heard the cries for help and rushed to the heavenly pool immediately with arms and armor. They saw that a bear monster was dallying with the fairies and then fought together to defeat the bear monster.

As the fairies returned to the heaven, they told the queen mother about the rescue of the dragons. The queen mother took out a bead from her gourd to reward their behavior so that they become immortal as soon as possible. However, there was only one golden bead, and neither of the dragons wanted to swallow it himself which resulted in a golden bead glittering darted up and down between the two dragons, which was very dazzling the eyes of human. This thing alerted the jade emperor and let Tai Bai Jin Xing, an immortal in the heaven to check it out.

After his inspection, the Tai Bai Jin Xing truthfully reported the concentration, kindness and loyalty of the moral character of the two dragons to the jade emperor. Heard the report, the jade emperor was very moved and took out a golden bead sending to the dragons. Then, the two dragons swallowed a golden bead each and became the immortal in the heaven controlling the sufferings of humans.

Dr.Ouyang tells Huizi that there had been dinosaurs on the earth because there are dinosaur fossils as evidence. However, whether the Chinese dragon ever existed is doubtful because it is only appearing in the written records but have no material evidence yet.

**5.3.Animal's Organs.** A cat has four claws, a mouth, a nose, two ears and eyes. *Do ordinary animals have these organs?* Dr.Ouyang asks his daughter and then answers that the animals also have these organs with similar functions such as the mouth for food and the eyes for vision. By the anatomizing, we know the combinatorial structure of animals.

Generally, there are 10 systems on an animal, namely its skin, skeletal, muscular, digestive, respiratory, circulatory, excretory, nervous, endocrine and reproductive systems. Each of the systems play different functions in the animal. Among them, ①The skin system has a sense of the outside, regulation, resistance and protection by the sweat glands respiration, absorption, excretion of metabolic wastes in the body; ②The skeletal system is supporting and maintaining the shape of the animal, providing muscle connections, assisting muscle movement and providing protection for the internal soft tissue structure. Some bones also have the ability to produce blood cells in the bone marrow; ③There are two main functions of muscles. One is the static effect so that the body to maintain a certain posture and achieve relative balance such as standing and other static actions. Another is the movement effect so that the body to complete a variety of actions such as running and other movement; ④The digestive system is responsible for breaking down food, converting energy and providing body with the needed nutrients. Most animals have one stomach but there are also animals have multiple stomachs. For example, the camels have three stomachs, the deers and the cows have four stomachs; ⑤The main role of the respiratory system is to provide the oxygen for animal's aerobic respiration and maintain its normal life activities; ⑥The circulation system ensures the blood circulation of animals, i.e., the blood flows from the left ventricle through the aorta and branches to the systemic capillary network and then, flows through the large and small veins to form the superior and inferior vena cava and finally returns to the right ventricle. At the same time, the blood from the right ventricle flows through the animal pulmonary artery, branches at all levels, flows through the animal alveolar wall capillary network and then through the pulmonary vein back to the left atrium. Most bloods of animals are red in color but there are also other colors. For example, the blood of shrimps and spiders is blue; ⑦The function of the excretory system is to bring the animal into being during the process of metabolism or during the intake of food, discharge toxic waste, regulate the metabolism of water, salt and acid-base balance in the animal and maintain the relative stability of the animal's internal environment; ⑧The nervous system is regulating the physiological function of the dominant system, to control and regulate the activities of various organs and to respond to environmental changes and adapt through the analysis and synthesis of the nervous system; ⑨The endocrine system is another regulation besides the nervous. It plays a role in maintaining the acid-base balance and temperature stability of the internal environment, regulating metabolism, ensuring growth, development and functional activities as well as the life continuation and reproduction. ⑩The functioning of the reproductive system is to secrete sex hormones, produce reproductive cells, reproduce offspring and perpetuate the species, divided into oviparous, viviparous and ovoviviparous. Here, the oviparous means

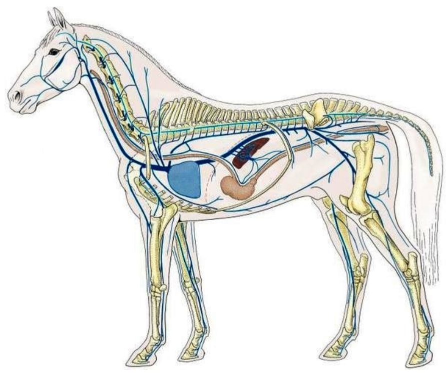

**Figure** 1.27. **Horse's anatomy**

that an animal becomes an animal after hatching an egg and its nutrition comes from the egg itself; the viviparous means that the baby of human or some animals developed in the mother to a certain stage before breaking away from the mother's body. For example, the birds, poultry for eggs but the cattle, sheep, pigs for viviparous. The viviparous animals are divided into one viviparous one and one viviparous more than one such as the cats and mice; the ovoviviparous animals between the egg and the viviparous, is fertilized eggs in the mother's developed into an individual before breaking away whose need of nutrition depends on the egg itself stored and other nutrients such as the pit viper snakes and guppies.

*Is the animal's behavior in isolation or divided into nations, regions or villages and tribes as humans society?* Dr.Ouyang tells his daughter that animal's behavior is individual or social depending on its grade or position in the food chain. Some animals like to live alone such as the snail and turtle, only find a temporary partner in the oestrus and after then rush things. But some animals such as bees and ants born in a crowded and noisy group interact with each other in the behavior as a kind of society. Generally, the lower down in the food chain an animal is the more social it is. The social animals follow the rules of working and hunting together to develop the advantage of community. Certainly, living in groups also makes animals compete for the food, living space and even mate the resource allocation which is easy to produce disputes, even fights within the group. Whence, for ensuring the constantly growth of population, different species should have their own rules on the behavior of individuals in general.

**5.4.Animal's Society.** Animal social behavior generally includes dominant hierarchy, communication, courtship, altruistic and homicidal behaviors. Among them, the dominant hierarchy is a kind of system in the community which means that animals compete for the corresponding position through established rules of the community on their own strength. Simply, the ranking is from the big to the small and from the strong to the weak. For example, the monkey king is the monkey who wins the fight with other monkeys in group and the other monkeys have to follow him; the communication refers to the interaction between animals such as the physical touch, vision or smell, to express friendship or sovereignty. For example, a dog urinating around outdoors is a declaration of its territorial sovereignty; the courtship behavior is a continuation behavior of species in animals. Its function is to attract the opposite sex, release energy, reproduce and extend the species; the altruistic behavior means that an individual sacrifices his or her own interests or life for the benefit of the group or other individual interests. For example, the male spider will offer his body as the nutrients of the female spider after mated with the female spider; the homicidal behavior is egoistic which is opposite to the altruistic behavior. For example, the winner of a lion contest will kill suckling lion cub. A tiger shark laid more than hundred eggs at a time but only one hatched because the other eggs are eaten by the hatched tiger shark. Dr.Ouyang tells Huizi that the animal society also has a strict division of labor and order so that the developing of the society is in order with natural laws but we do not know much about animal behaviors within the ability of humans.

[**Ant Community**] The ants are social insects of omnivorous. Usually, they can grow normally between $15-40\ ^oC$ with a best growth temperature $25-35\ ^oC$. They enter hibernation below $10\ ^oC$ in winter. Most species of ants like to build their nests underground, digging tunnels, chambers and dwellings. They pile the excavated soil, roots and leaves near the entrance to form a mound or use leaves, stems and stalks of plants to build nests and hang them on trees or rocks to protect them.

**Figure** 1.28 **Ant**

An ant community is a matriarchal group, consisting of a queen ant, several male ants, workers and soldiers, each with a clear division of responsibilities. Among them, the queen ant is in the reproductive ability of the female ant with the largest body size in the group, and also with a large abdomen, genitalia developed, antennae short, small chest feet, wings or without wings and other characteristics, responsible for ruling the whole ant community, laying eggs and breeding offspring. At the same time, in order to prevent workers from having the reproductive ability and deprived of their dominance, the queen ant will constantly secrete a scent to prevent the development of the genital ants in the ant nest. The duty of the male ant is to mate with the queen and lay eggs. It has developed external genitalia and has the characteristics of small round head, underdeveloped palate and elongated antennae. The worker ants are female ants without reproductive ability in the ant community. Their duties are to build and expand nests, collect food, feed young ants and queen ants, etc. They are characterized by wingless, small individuals, small compound eyes, well-developed upper jaw, antennae and three pairs of thorax feet and are good at running. The soldier ants are responsible for defense and against invasion. They are characterized by a large head, developed upper jaw and can crush hard food. They are weapons in wartime. Generally, an ant community is formed by the queen ant, through the marriage of the two sexes met as a starting point. They fall in love at first sight and then have sex during or after the flight. The male ants do not live long and die soon after mating. The queen takes off her wings and chooses a suitable place underground to build her nest and lay her eggs. As the ant eggs mature, larvae hatch out, the queen ant begin to busy by its mouth to mouth for feeding the larvae until the young ants grow into ants and can live independently. Once the first workers are established, they dig holes to the outside world for finding food, expand the nest area and provide housing for the members of the community. At this point, the queen begins to sit happily, become the head of the family but it will continue to mate to produce fertilized eggs to maintain the community reproduction with a life up to 15 years or so.

Dr.Ouyang concludes that the early humans was also a matriarchal one for the survival of populations, and had a strict division of roles and social order liking the ant or bee communities observed in the nature today. However, a main difference in the humans with other animals lies in that human activities seek sustainable development of humans ourselves on knowledge of the nature, which is a subjective behavior. For this objective, it's necessary not to intrude on the nature or minimize the intrusion because if the intrusion accumulates to a certain extent, the "*quantitative change*" will lead to the "*qualitative change*" finally, disturbs the universal order and finally, affects the survival and development of humans ourselves.

## §6. **Comments**

**6.1.** Exploring unknown things is one's eager in his or her inner, but with his or her growth it will gradually weaken and return to only care about the things around, which is normal because with the growth and the continuous enrichment of knowledge, one gradually enhances the sense of the security in dealing with things. This eagerness is especially reflected in the elementary school for the ultimate questions on the universe such as the origin of the universe, where we came from and where we will go, $\cdots$, etc. Even for the adults, there are no definitive answer on this kind of questions. For example, the Big Bang is only a scientific hypothesis on the origin of the universe but can not be verified. Similarly, there are also a few legends such as the Pangu's split open the chaos, the God's creation on the origin of the universe, see [MaL], [Mao2] and [Mao48], [Xie] and *Bible*, etc., and certainly, all of the beginnings on the universe are essentially for human's understanding. However, the Big Bang is hard understanding to pupils, especially the elementary particles such as the quarks, leptons and their interaction, can only be compared to a firecrackers lit after the explosion without a little childlike about for them. But the story of Pangu's split or the God's creation is interesting for pupils comparing with the Big Bang because pupils are innocence and their understanding are different from that of the adults. They are not superstitious about books, nor the authority. They like to understand the unknown with their known and ask "*why? why so?*" again and again, which is exactly the exploration enthusiasm that a researcher should have.

**6.2.** The humans view the sky and recognize things on the earth, and perceive everything from the perspective of humans ourselves. In this way, it is an inevitable means to think that the earth is the center and the humans is the master of the universe. The contribution of Newton and Einstein to human's knowing on the universe is to break the God's creation on the universe, not superstitious but objectively back to the truth of the universe. Newton's "*Philosophiae Naturalis Principia Mathematica*" [New] used a logical reasoning to successfully explain the ground moving and the celestial phenomena by a set of axioms and laws of mechanical moving and gravity, including the observed values and moving laws of the Mercury, Venus, Mars, Jupiter, Saturn and the moon, which unified the law of moving on the ground and the celestial. Notice that the essence of the Einstein's general relativity [Ein1]，[Ein2] is the reflection of the philosophical thought of "*dependent origination from emptiness*" in Buddhism on knowing things. Applying the general relativity to gravitational field, the conclusion that matter field is a curved space under the action of gravity, even the light is no exception has greatly impacted on human's recognitive intuition and formed a contrast between the local knowing with the reality.

**6.3.** All human's knowing on the earth's plants and animals are still in its infancy. In addition to the grain cultivation and livestock raising formed by humans in thousands of years, there are still lots of unknown matters should be explored. For example, the plants are generally at the bottom in the food chain but in the principle of survival of the fittest, their survival conception, communication, clock and symbiosis with other organisms have been studied and summarized in the literature *What Does the Plant Think* [Tas], which are worthy for further research. We have a few knowledge on the behavior of animals from anatomy, cells and genes but it is only limited to small animals or those populations that can be closely observed by humans, see [Jin], [Yan] and [Sha], and there are lots of unanswered questions even about humans ourselves. For example, the mechanism of human disease and its transmission or infection mechanism, human consciousness and behavior, etc. In this regard, the traditional Chinese medicine such as the *Emperor's Inner Canon - Plain Questions* and the *Emperor's Inner Canon - Lingshu Canon* have a complete understanding on the human body, corresponding to the ancient Chinese philosophy which should be further studied by researchers, see [Ren] and [Zha] for details.

$C_{\mathscr{M}}$

**Combin.Notion**

## Chapter 2

# Perceptible Limitation on the Universe

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**What You Know is What You Know,**

**What You Don't Know is What You Don't Know.**

**Then, It's Really What You Know.**

—– *Analects of Confucius – Govern*

## §1. The Origin of Humans

All things in the universe are recognized by humans ourselves, do not represent the recognition of other creatures, and maybe not the real face of things in the universe. A sounds familiar sentence says that the human is the spirit of all creatures in the universe from the words that "*the heaven and the earth are the parent of all creatures, and human is the spirit of all creatures*" in the *Book of Documents - Great Declaration*, where the word "*creatures*" refers to all living things in the universe; the word "*spirit*" refers to cleverness, dexterity and wisdom. Notice that every living thing has some intelligence to cope with the change of the environment. For example, there are many parenchyma cells on the mimosa plant. They will have a physiological reaction, causing its leaves to close if one touches it. There are some animals whose sense of touch, smell and hearing are far superior to humans. For example, before it began to rain, swallows fly low, lots of fishes jump, spiral and swimming in the water before an earthquake and so on, are animals adapting to the evolution of the survival as a result, but humans are almost not perceive to the natural environment change such as those of the wind, rain, snow, cold wave, earthquake, tsunami, etc., *why do we also say that human is the spirit of the universe in the situation?* The original intention of this sentence is to say that humans have the wisdom that other animals do not have, i.e., the humans have the ability of subjective observation, recognition, analysis, learning and creation, and could drive other animals to serve the human society ourselves.

**(a)Ancients hunting** **(b)Leopards hunt deer**

**Figure** 2.1. **Humans and animals**

However, similar to the origin of the universe, there are also ultimate questions about the origin of humans with the goal. For example, *where did the humans come from as the spirit of the universe? Did they evolve from creatures such as monkeys? Is it the seed that* *aliens cast or the criminal that aliens imprison on the earth?* Certainly, the solar energy will eventually run out, *would the human as the spirit of the universe will be destroyed along with other creatures?* All of these questions have no answers unless a few hypotheses. Among them, these are two typical answers on the origin of humans by hypothesises, i.e., one is the humans originated from the earth itself and another is the humans originated from other galaxies in the universe.

**1.1.Biological Evolution Theory.** The biological evolution theory holds that humans originated from the earth itself. One day before the dinner, Dr.Ouyang sees his daughter is not at home, then go to the community to look for her to go home for the dinner. He found Huizi sits alone in front of the trash can in the corner of community, looking at the trash can in a daze. He go up to her and asks:"*What are you doing beside the trash can? How dirty! Go home for the dinner!*" Huizi turns around and sees her father calling her. She comes to Dr.Ouyang and asks him:"*Daddy, could you tell me where I come from?*" Dr.Ouyang tells her that she had been given birth by her father and mother together but Huizi didn't believe him. Dr.Ouyang asks her:"*Why do you think you come from?*"Huizi replies:"*In the class, some students of mine said that humans are fish changed and some said that humans are monkeys changed, but my mother said that it is you picked me up from the trash can. I believe my mother and wish to pick up a brother. But so many days, the trash can never appears a little brother. You say it is strange or not?*" Dr.Ouyang asked her:"*Why do you think your mother's words are all right?*" Huizi replied:"*My mother never lies to me! I have raised fish, also went to see the monkey many times in the zoo but there is no sign that I am a fish or monkey changed, I do not believe!*" Dr.Ouyang says:"*you were born by mom and dad, not picked up in a trash can.*" Huizi asks:"*I was born by you. How was I born?*" Knowing that this is a more reluctant question to answer for most Chinese parents, Dr.Ouyang tells her daughter in a subtle way that her father and mother were together to have her, but she was so small at first that she needed to live in her mother's stomach for ten months and then, she grew up, came out of her mother's stomach, not from the trash can.

Seeing his daughter still looking confused, Dr.Ouyang changes the subject quickly and asks her a simple question:"*Why is* $1+1$ *equal to 2 but not* 3 *in mathematics?*" Huizi answers:"*Our teacher taught us that* $1+1=2$ *and we all memorized it!*" Dr.Ouyang explains that mathematically $1+1=2$ because the number originates from the counting. It is counted one by one. Consequently, there are the natural numbers $1,2,3,4,5,\cdots$. And, according to the rule of counting, $1+1$ is equal to the natural number 2 after 1. So, $1+1=2$ but not $1+1=3$. However, the law of biology is not constrained by this rule. As a man get together with a woman, they can produce a child. *Is that* $1$*(father)*$+1$*(mother)* $= 3$*(father, mother and child)?* Huizi listens to Dr.Ouyang's explanation and asks:"*Daddy, we found* 4 *kittens last time. Doesn't that make* $1 + 1 = 6$*?*" Dr.Ouyang replies:"*Yes! But here* $1 + 1 = 3$ *and* $1 + 1 = 6$ *are not mathematical rulers but biological laws. In the natural number system,* $1 + 1$ *must be* 2 *by the counting!*" By this time, they arrive at home. Dr.Ouyang asks his daughter to wash her hands for dinner first and then tells her about the origin of humans.

After dinner, Dr.Ouyang tells his daughter about the evolution of biology. He says that scientists have studied the origin of humans from the perspectives of anthropology, archaeology, history, biology and even chemistry, and have formed many theories on the origin of humans but there is no convincing explanation so far. The more objective view is that humans are the result of genetic variation in a species of apes because the apes are the most genetically similar species to humans on the earth.

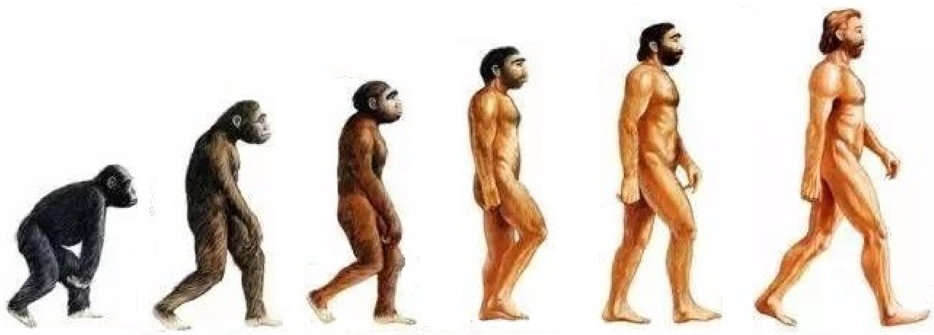


**Figure** 2.2. **Human evolution**

According to the theory of biological evolution, biological evolution follows the law from the lower to the higher, from the simpler to the complex and concludes that the human is the evolution of an already extinct ancient ape because of the survival of ape needs down to the ground for activities. Gradually, they learned to upright walk by two feet, liberating their forelimb for hands, learned to use stone or wood sticks and other natural tools, and developed to use hands for making tools. In response, their brain evolved also, gave rise to human's characters and further evolution resulted in the emergence of ape man, homo erectus, homo sapiens and finally modern man such as those shown in Figure 2.2. Among them, the homo erectus was living within about 1.7 - 1.5 million years ago to 2 - 3 hundred thousand years ago with brain volume of $775 - 1400ml$. Their skull was very thick with a prominent eyebrow ridge, low flat skull and ape-like characteristics but their leg bone was like humans, could walk upright. The homo sapiens is divided into the early and late homo sapiens. The early homo sapiens was close to modern humans in physique and morphology but still retained some original features such as the brain volume of about 1300 - 1750$ml$ with more complex brain tissue than homo erectus, more developed brow ridge than modern humans, low and obtrusive forehead, more prominent jaw, chin is not obvious, etc. The late homo sapiens also known as the new human appeared in about 40,000 years ago. Their physical shape was not much different from that of modern humans. Their forehead of skull was elevated, the brow ridge almost disappeared and the jaw was withdrawn and the chin was obvious. The late homo sapiens formed the modern humans and divided into three races in general, namely the Mongolians, Caucasians and the Nigro race.

Huizi hears here and then asks Dr.Ouyang:"*Daddy, says that humans are ape changed but I went to the zoo many times, why there is no sign of gibbon evolution of human?*" The answer is the evolution of apes into humans is estimated to have taken at least 1.2 million years according to the fossil findings. It is a long gradual process and unlikely that subtle changes in evolution can be observed in a few years or decades, Dr.Ouyang explains to her. Huizi still does not understand the explanation of Dr.Ouyang and says that if the evolutionary hypothesis is right, there should be the homo erectus and homo sapiens in the world but now, there are no homo erectus and homo sapiens, *are the emergence of humans like with the Big Bang, namely all apes on earth suddenly turned into homo erectus and then, suddenly turned into homo sapiens and finally, all became modern humans, caused the extinct of homo erectus and homo sapiens?* The biological evolution should be carried out step by step, not be the case that a population evolution appeared then the original population disappeared! In this regard, Dr.Ouyang says that the biological evolution, including genetic variation is only a hypothesis for human origin, put forward by humans based on archaeological discoveries and biological science research which can explain some doubts in archaeological and biological populations and partially, restore the origin of humans. Therefore, to say that man evolved from monkeys or fish would be both consistent with the hypothesis of biological evolution because the livings in the sea preceded the livings on the land. However, the biological evolution is only a theory. It is not clear whether humans really originated this way. It is for this reason that different regions and cultures, including religions and myths have put forward different creation theories on the origin of humans, which are more vivid than the biological evolution.

**1.2.Creation Theory.** The creation theory holds that there exists one or more creator in the universe, i.e., the "*Gods*" created humans like "*children playing game*". For example, it is said that the human is a God made clay in the Chinese ancient mythology and the Christian *Bible*, is called by God in Egyptian mythology, is a plant in Germanic mythology, is a lizard in Australian mythology, is a swan or a cow in Greek mythology. Among all the myths about the creation of human by Gods, the most fascinating one is the Chinese myth about the creation of Nuwa and the creation of God in the Bible.

[**Nuwa Creation**] Nuwa was a female emperor with a human face but a snake body, showed 70 changes every day in the Chinese ancient mythology. In the *Shan Hai Jing - Wild West Longitude*, there is a record that "*there are ten intestine of Nuwa transformed into Gods, lived in the field*" , in which the "*intestine of Nuwa*" is believed to be the stomach of Nuwa, which leads to the myth that Nuwa created humans. In the eastern Han dynasty, there is a paragraph concluded that "*as the saying goes, there is no citizen, Nuwa kneaded the loess in her hands to create humans. She was too tired that unable to create all humans. She drew a rope into the mud and then lifted the rope in a splash. The splashing mud points also became humans. Later, it was said that the rich and the poor were respectively created by kneading the loess or the mud of Nuwa's ropes*" in the *Popular Romance*, compiled by Yingshao, the supreme governor of Mount Tai, which deduced the process of Nuwa created humans in Chinese ancient mythology following.

After Pangu split open the chaos, there were only the heaven, the earth and Pangu himself, and no other one. After years passed of the death of Pangu, Nuwa was born in the universe. One day, she caught sight of herself on the surface of the water and thought it would be good if there were some humans talking with her. She unconsciously grabbed the loess on the ground with water and made a small clay figure according to her own appearance. She put it on the ground and blew a breath on the small clay figure. At this time, the small clay immediately turned into a human. Nuwa began to make human one by one. She felt a little tired and thought how to make more humans faster between the heaven and the earth to fill the vast land. She picked off a few rattan in the edge of the cliff into a rope, stirred it in the mud and threw with lots of mud spots on the ground. Accompanied by the wind blowing, all the mud spots became humans one by one. Nuwa kept waving the rope, the mud spots on the ground increased and the number of humans also increased. So that all humans appeared on the ground. Nuwa made humans in this way to give humans into the rich and the poor. It is said that the humans who knead the loess with her hands and blow a breath on the clay figures are endowed with wealth, while the humans transformed by the mud spots on the ground when Nuwa throws her rope are poor. This explains clearly why there are rich and poor humans in the world from its origin. It is because of the different mentality of Nuwa in creating humans. The rich humans were made and blew by Nuwa herself one by one but the poor humans were the mud points on the ground from Nuwa's rope, i.e., Nuwa's lack of dedication which gives an explaining for the fatalism.

In the myth of Nuwa creation, it is not clear why humans have different genders although there are high and low. Thus, humans created by Nuwa may all be men or women. So they cannot reproduce in the universe. However, this is clear in the myth of God's creation of humans in the Bible.

[**Adam and Eve Ate Forbidden Fruits**] Dr.Ouyang says that he has told Huizi that God in the Bible created humans on the sixth day last time, which is similar to the Nuwa creation in Chinese mythology, the God squeezed a man in the image of himself with clay and blew on it. Then, it became a living man named Adam. God decided to make a mate for Adam. He took a rib from Adam's side while he was sleeping and from that rib, God made a woman, named Eve. Here, the words "*Adam*" means "*dust*", i.e., that man came from the dust and "*Eve*" means "*ribs*". They are the legendary ancestors of humans. Meanwhile, God made a paradise for Adam and Eve in Eden on the east, where the ground was strewn with gold, pearls and agate, and where there grew strange flowers and trees whose fruits could be used as food. God gave Adam and Eve to live in the garden of Eden and keep watch over the garden with the command:"*You may eat fruits on the tree of the garden but you must not eat fruits on the wisdom tree. Otherwise, you will die if you eat.*" Adam and Eve lived naked in the garden.

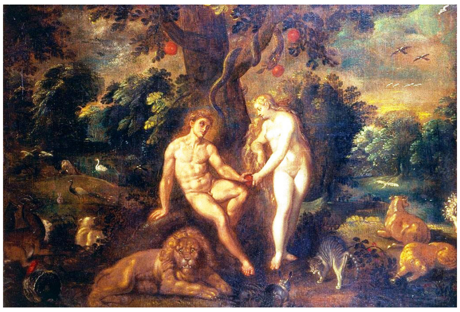


**Figure** 2.3. **Adam and Eve**

It is said that the snake is the most evil of all the animals in Eden garden. The snake asked Eve if she could eat any fruit she wants. Eve replied:"*Yes! We can eat whatever we want except the fruit of the wisdom tree. God told us that if we eat from the wisdom tree, we will die.*" The snake told Eve:"*God has deceived you. If you eat from the tree of the wisdom and know good from evil, you will be like the God.*" Eve looked at the wisdom tree and thought its fruits would make her wise. At last, she could bear it no longer, so she picked one and ate it herself and another one gave Adam to eat. At this time, they both had wisdom and shame, aware of the shame of male and female nudity and busy plucking some fig leaves to cover their bodies.

After dark God came into the garden and could not see Adam and Eve. He asked where Adam was and why he was hiding. Adam said he was scared when he heard the voice of God. God asked him if he had eaten the forbidden fruit. Adam confessed that it was Eve made him eat, and Eve said that it was the snake tempted her to eat, saying the snake told her that she would be wise to eat. Then, God punished the snake by cutting off its feet and condemning it to walk on its belly and eat dirt all its life, drove Adam and Eve out of the garden of Eden and made them multiply on the earth, saying that the woman would have pain in pregnancy and childbirth from now on, and the man would have sweat on his face until he was turned to dust.

Dr.Ouyang inspires his daughter to say that in the creation of Nuwa in the Chinese mythology and the God creation in western christianity, they were both made by making a clay figure in the image of Nuwa or God and blowing on the clay figure. Therefore, it can be concluded that Nuwa and God are similar in appearance to human. Dr.Ouyang asks:"*Could children understand the myth that Nuwa and God created humans?*"Huizi replies:"*Of course, we can learn to do it also!*"

The next day after school, Huizi dugs some soil in the grassland of the community, takes it home and make up with water. She squeezes two clay dolls in the shape of calabash babies and Chibi maruko-chan, and blows on them. But no matter how much Huizi blows, the clay dolls are still the clay dolls. When Dr.Ouyang come home, Huizi says with a depressed face:"*The creation of Nuwa and God are all liar! Look at the two clay dolls I have made. Whatever I blow them they are still clay dolls. They don't become humans!*" Dr.Ouyang says with smiling:"*It's because you don't have the divine power like Nuwa or God. No matter how you blow on them, they won't come back to livings. Therefore, Nuwa or God created humans can only be a myth.*" Dr.Ouyang tells his daughter that if there is really a divine power in the universe, such a power can only be the natural force. He then asks his daughter:"*Does that make the theory of biological evolution plausible to you?*"Huizi replies:"*That's right! Because I have proved by experiments that the creation of Nuwa and God can not be trusted.*" Then, Dr.Ouyang tells her about the second hypotheses about the origin of humans, which is in other galaxies outside the earth.

**1.3.Extraterrestrial Life.** Whether humans originate on the earth or in other galaxies in the universe is not universally agreed upon. The biological evolution proposed by the mainstream of science on the basis of the evolution of livings on the earth, and the theory of god's creation based on the divine power proposed by religion or mythology, etc., can not explain all the questions on humans which results in the hypothesis of the birth of extraterrestrial livings appears in science fiction.

**Hypothesis 1.** *Living began outside the earth and humans are the product of its evolution on the earth.*

This hypothesis holds that some of the oldest forms of livings existed as tiny, fragile single cells in the universe before the solar system formed. Thus, by the carrying of comets, asteroids or other space debris during interstellar travel, most of the primitive livings died during the long interstellar travel but a few lucky organisms were accidentally brought to the earth where livings could grow up. After hundreds of millions of years of integration and the evolution with the earth's environment, these livings formed genes different from other livings on the earth, namely humans. The evidence for this hypothesis is said to be the discovery of fossils of livings on the earth, which have been dated to about 6.5 billion years ago but the earth was only 4.5 billion years old.

**Hypothesis 2.** *Humans are livings imprisoned on the earth by higher civilizations.*

The essence of this hypotheses is that there are different levels of civilization in the universe. Among them, the first-level of civilization is the primary civilization, can break the gravity of their own planet into space; the second-level civilization can use stellar energy as a source of energy for space exploration and make contact with other extraterrestrial civilizations of the same rank; the third-level civilization is the galaxy civilization, can open the time travel for interstellar travel; the fourth-level civilization is a supergalactic civilization, which can carry out dimensional space shuttle and is no longer subject to the life ontology; the fifth-level civilization is a space civilization, which can live in multi-dimensional space and use resources of different dimensional space to make its own; the sixth-level civilization is the cosmic civilization, which can realize space travel and is no longer restricted by cosmic rules; the seventh-level civilization is the ultimate civilization, the creator of the universe and the rules of the universe. Although humans is the "*spirit of all creatures*", our civilization is the lowest which is only 0.75 in terms of the civilization level, not reaching the first-level in the cosmic civilization levels.

In this situation, a few ones suggested that humans were born on other planets in the universe but imprisoned on the earth by a higher civilization because they violated the rules of the heaven, which is similar to how banished some deposed ministers or scholars such as Su Dongpo, a writer in the northern Song dynasty and Hai Rui, a famous officer to Hainan island in ancient dynasty of China. In the view of highly civilized livings, humans are still in the development stage of the first civilization before they evolute to the second. They must be imprisoned on the earth as prisons controlled by the extraterrestrial civilizations. Someone provide an example that the meteorite strike over Russia in 2013 as an evidence for this view. It is reported that

**Figure** 2.4. **Extra-civilizations**

one day in 2013, an asteroid with a diameter of about $17km$ and weight of nearly $7 \times 10^6 kg$ flew at a speed of $18km/s$ toward Chelyabinsk of Russia. It left a $10km$ long burn trail in the sky. However, it was not detected by satellites until it entered the atmosphere of the earth. The damage that could be done to the earth is irreversible. Fortunately, the asteroid was hitted by an unidentified object coming from its behind and exploded about $30km$ above the ground. Most of the debris fell into the nearby Chebarkul Lake. The whole process took only a few seconds, which surprised the humans watching on the ground. The object that broke up the asteroid was estimated to have traveled at a speed of $36km/s$, which was beyond the reach of human technology at that time.

Dr.Ouyang tells his daughter that the question of human's origin is extremely complex, and that we might never know exactly how the first human came into being. Whether it's biological evolution, divine creation or human origin from other planet in the universe, what we can be sure is that we are the result of human reproduction, i.e., a $1+1$ combination of our parents (one male, one female), like you saw when a cat gave birth to kittens, absolutely not the matter plucked from the soil or scavenged from trash cans.

## §2. **All Things are Human's Perception**

The modern anatomy has proved that the human body is similar to the general structure of animal on the earth, which is composed of skin, bone, muscle, digestion, respiration, circulation, excretion, nerve, endocrine and reproductive systems. Among them, some are needed for human survival and some provide human with recognitive function to objective existence in the universe.

The human recognition of things is first tested by practice and then, rises to knowledge, forming the science. Notice that the appearance, color, sound, movement of things and the five tastes of sour, sweet, bitter, hot and salty are determined by one's own senses and consistent according to the recognitive standards common to human society. However, it is not clear whether human's recognition is the true nature of things. This is explained in the *Tao Te Ching* of Laozi, i.e.,"*Tao told is not the eternal Tao; Name named is not the eternal Name*", an answer to this question, i.e., the human's recognition is not necessarily the true face of things, and there is also an answer in the *Diamond Sutra* that Shakyamuni told Subodhi, i.e., there should be "*no images of the self being, human being, all beings and life being alike*" and claim that "*the color is empty, the empty is color*" in *Heart Sutra* of Buddhist philosophy. At the same time, it is not clear whether other animals have the same recognition on the same thing as humans, which is vividly demonstrated in "*Debate*

*Fish Fun* " in Zhuangzi with Huishi.

**[Debate Fish Fun]** In the autumn-water Chapter of *Zhuang Zi*, there is a record that Zhuangzi and Huishi were walking along the bank of Haoshui river. Zhuangzi casually said:"*These fishes swim easily in the river. They are really happy!*" Huishi aside asked:"*You are not a fish, how can you know the happiness of fish?*"Zhuangzi replied:"*You are not me. How do you know that I don't know the happiness of fish?*" Huishi asked again:"*I am not you, so I don't understand you. But you are not a fish, also should not know the happiness of fish!*" Zhuangzi replied calmly:"*I beg you to return to the beginning of the conversation. You just asked me that how do I know that fishes are happy? Since you ask me why fishes are happy, it means that you have already admitted that I know the fishes are happy, and now you ask me how I know fishes are happy. Then, I tell you that I know the fishes are happy on the bank of Haoshui!*"

According to Huishi, humans and fish are two different creatures from the perspective of human recognition. It is impossible to know whether the emotions of the two are the same. Zhuangzi is from the artistic point of view, i.e., the human's happy is the happy of fishes, is playing with the art of sophistry according to relativism. Then, *can Zhuangzi really perceive the happiness of fish?* The answer is No! Zhuangzi feels the happiness of fishes swimming in the water which is a subjective recognition of Zhuangzi himself. It is a behavior of "*empathy*", which is shared by the human and the fishes.

Dr.Ouyang tells Huizi that with the development of science and technology, human's perception of things has developed from the original direct perception to indirect perception, which is to use scientific instruments or facilities to sense things. This extends the range of human perception to a certain extent and enhances the ability to perceive things in the universe.

**2.1.Direct Perception.** The direct perception of humans is by the perception or reconditioned reflection of things by the six sensory organs, i.e., the eyes, ears, nose, tongue, body and mind of human. Notice that the human direct perception has certain limitations, not all the signals of things can be directly perceived by humans. Then, *what are the sensory organs and to what extent can perceive by humans?* Let's discuss the following in turn.

**(1) Eyes.** Human eyes are used by human to observe and perceive the size, brightness, color and moving or static of external objects for obtaining information about the survival of a visual organ. According to statistics, more than 80% of the external information obtained by humans is obtained through human vision. Thus, the vision is a human's most important perception. The human eye has a three-layer coat, which consists of three transparent structures. The outermost layer consists of the cornea and sclera, and the middle layer consists of the choroid, ciliary body and iris. The radius of the corneal segment is about 8mm and the radius of the sclera is about 12mm. The inner layer is the retina, which as seen with an ophthalmoscope, receives circulation from the blood vessels in the choroid. Inside these three layers are aqueous humor, vitreous body and flexible lens. Of these, the aqueous humor is a clear fluid contained in the exposed area of the lens and in the anterior chamber between the cornea and iris; the lens is a ciliary suspension ligament composed of transparent fine fibers. the vitreous and the posterior chamber of the eye are a clear jelly-like substance larger than the anterior chamber, distributed behind and in the rest of the lens, surrounds the sclera, zona and lens.

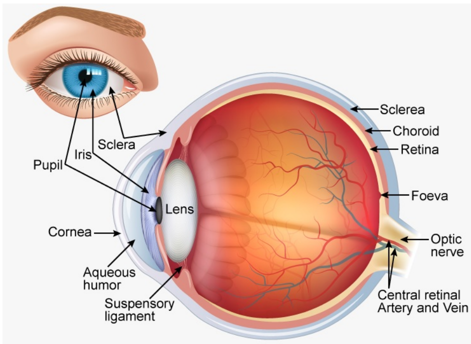



**Figure** 2.5. **Structure of eye**

The eyes are gradually grown up with the growth of a human. The eye size of a baby is about 16mm to 17mm when he or she is born. And at the age of 3, it grows to 22.5mm to 23mm, and it is with the size of 24mm, the volume of 6.5$ml$ and the weight of 7.5$g$ at the age of 13 years old, almost the same as an adult eye.

Generally, the vision of a human eye is about outward $95^o$, inward $60^o$, up $60^o$ and down $75^o$. The optic nerve defects or blind spot is located in the temporal $12^o - 15^o$, level down $1.5^o$, about $7.5^o$ high and $5.5^o$ wide. The normal resolution of human eyes is about 1/2000 to 1/5000. The eyes receive electromagnetic stimulation in a certain wavelength range in the external environment, encode and analyze it with the central part, and then obtain subjective sensation. Usually, the human eyes include photoreceptor cells of the retina and refraction system with suitable magnetic stimulation in $370 - 740nm$ wavelength, namely the visible part and about 150 kinds of color spreading through the optic nerve to the brain's visual center, can see the visual range of light or the reflector contour, shape, size, color, distance or surface details. Generally, it is difficult to see objects beyond the visible distance of 0.1$m$-4$km$ for human eyes. When the distance of the scene is greater than 500$m$, there is only a fuzzy image; when it is shortened to $250 - 270m$, the outline of the scene can be clearly seen and further discrimination should shorten the distance.

**(2) Ear.** The ear is the human hearing organ, consisting of three parts: the outer ear, the middle ear and the inner ear. The inner ear is a human's hearing and potential receptors, the outer ear includes auricle and external auditory canal. There are ear hairs and some glands on the skin of the external auditory canal. The secretions and ear hairs have a certain blocking effect on the outside dust. The outer ear includes three parts: auricle, external auditory canal and tympanic membrane. Among them, the function of the auricle and the external auditory canal is to collect sound waves, the tympanic membrane is an oval translucent film that separates the outer ear and the middle ear, and it can vibrate with the sound waves and conduct sound waves to the middle ear. The middle ear includes the tympanum and eustachian tube. The eustachian tube is the passage between the middle ear and the nasopharynx. The air pressure between the middle ear and the outside world is balanced by the eustachian tube. The tympanum cavity is located between the tympanic membrane and the inner ear which is a small cavity containing gas. There are listening ossicle and ligaments in the tympanum. The air in the tympanic chamber communicates with the outside air so that the air pressure inside and outside the tympanic membrane is balanced. In this way, the eardrum can vibrate well. Generally, when the sound wave vibrates the tympanic membrane, the serial movement of the auditory ossicle makes the stapes floor swing back and forth on the vestibular window, and then transmits the sound wave vibration to the inner ear. The inner ear, also known as the labyrinth, consists of the vestibule, semicircular canal and cochlea. The cochlea is where the auditory receptor is located.

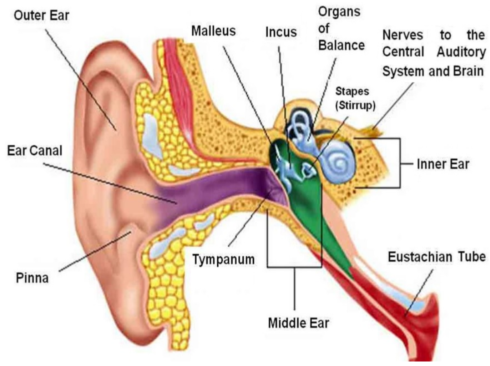



**Figure** 2.6. **Ear structure**

So, *how does hearing come about?* Certainly, the hearing is completed by the human ear, auditory nerve and auditory center. Sound waves pass through the external auditory canal, tympanic membrane and auditory ossicle to the inner ear so that the sound organs of the inner ear excite, transfer the sound waves into nerve impulses and then, through the auditory nerve into the human center to produce hearing. The human hearing is not very sensitive. The normal frequency range of hearing is $20Hz$ to $20,000Hz$ with the same intensity of sound at different frequencies differently. In fact, the human ear is most sensitive to sound at frequencies between $3000Hz - 4000Hz$. In addition, the ear is a major organ of balance in the body, i.e., an ear's problem can also cause the problem of balance with a human such as vertigo, difficulty walking or the feeling of floating in water as lying in bed, etc.

**(3) Nose.** The nose is the starting part of the human respiratory tract and is the gateway through which gases enter and exit. At the same time, the nose is the organ of smell. Among them, the nose olfactory nerve is formed by the collection of olfactory cell nerve fibers in the mucous membrane of the nasal olfactory area. The nose consists of the outer nose, nasal cavity and sinuses. The outer nose is located in the center of the face; the nasal cavity is located in the space between the two sides of the cranial cavity, its upper, posterior and sides are around with the left and right paired sinus and adjacent to the anterior fossa, the middle fossa, oral cavity and orbit, separated by a thin bone plate; the sinuses open in the nasal cavity and the mucous membranes of the two can move each other and become a whole.

The outer nose consisting of bone and cartilage is covered with soft tissue and skin. It is composed of nasal root, nasal tip, nasal bridge, nasal alar, nasal anterior foramina, nasal columella and other parts. The cartilage scaffold consists of the nasal bone, the nasal process of the frontal bone and the frontal process of the maxilla. It is the thicker skin at the tip, alar and vestibule of the nose, adhesion with subcutaneous tissue and perichondrium, rich sebaceous glands, sweat glands; the nasal cavity is located in the space between the two sides of the cranial cavity, based on the bony nasal cavity and cartilage, divided into the left and right cavities by the nasal septum through the nostril to the outside and the nasal posterior orifice to the pharyngeal cavity. Each nasal cavity can be divided into the nasal vestibule and the proper nasal cavity. The nasal cavity has an obvious warming effect on cold air. When the cold air of $0^oC$ enters the lungs of the human body through the nose and pharynx, the temperature can rise to $36^oC$ after adjustment which is basically close to the normal body temperature.

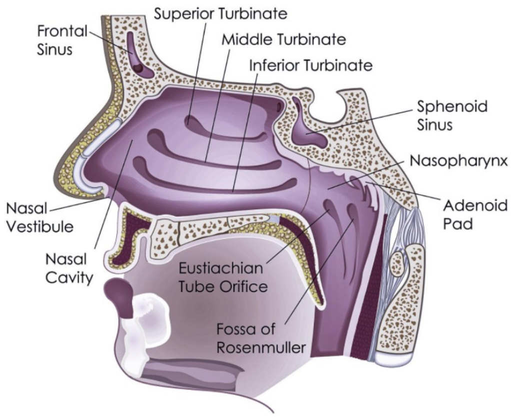



**Figure** 2.7. **Nose structure**

The olfactory nerve in the nose is composed of nerve fibers from olfactory cells in the mucosa of the nasal olfactory area, can smell incense, acid and odor. The sinuses consist of bony sinuses lined with mucous membranes. In this process, the mucous membrane continues with the mucous membrane of the nasal cavity through the opening of each sinus. It is the air-containing cavity in the skull and facial bone around the nasal cavity, which has a resonance effect on pronunciation and helps regulate the temperature and humidity of the inhaled air.

The function of the nose is mainly reflected in 8 aspects: ①Respiratory function. Air passing through the nose makes the presence of air perceived; ②Olfactory function. This function allows a human to sense odors; ③Cleaning function. This function has a filtration effect, allowing breathing air to become clean into the lungs; ④Heating and humidification function. This function can heat the cold air, humidify the dry air, adapt to the needs of the human body; ⑤Resonance effect. This feature makes everyone's voice colorful, clear, rounded and individual; ⑥Protect the head. The cavity structure of the nasal cavity and paranasal sinuses can reduce the impact of external impact on the human brain; ⑦Reflection effect. The reflex action of the nose makes the nose has the characteristic of periodic change, which is helpful to relieve the body fatigue. ⑧Excretion of tears. A small hole on the inside of the eye leads to the nose, called the nasolacrimal duct which drains excess tears.

**(4) Tongue.** The tongue is the human articulatory organ, which is one of the organs that helps to sound and communicate. But more importantly, the tongue is the taste organ, the main organ for sensing the sour, sweet, bitter, hot, salty and hemp tastes of foods. The human's vision, hearing and taste make a human perceive things but it is also easy to form the human's preferences and prejudices for different things. The twelfth chapter in *Tao Te Ching* says that the five colors could make human's blind, the five sounds could make human's deaf and the five flavors could make human's taste. Among them, the three words "*blind, deaf, taste*" are the generalization of the recognitive prejudice of humans towards things in the universe.

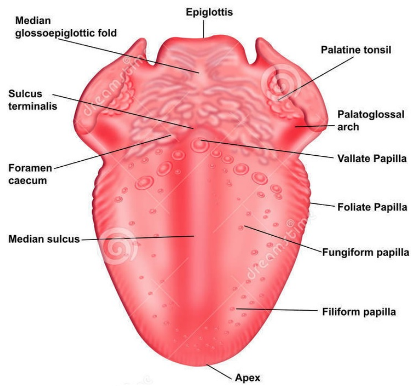



**Figure** 2.8. **Tongue structure**

The length of the human tongue from the oropharynx to the tip is 10cm in average. It is composed of skeletal muscle and divided into three parts, i.e., the base of the tongue, the back of the tongue and the tip of the tongue. The base of the tongue, the back of the tongue and the floor of the mouth are connected and only the tip of the tongue is free. The back of the tongue has many fine filamentous, fungiform, contoured and foliated papillae. Among them, except filamentous papilla, the other three have taste receptors which constitute the human taste buds. They are composed of sertoli cells and taste bud cells. The taste pores extend to the surface of the tongue and can feel the taste of foods in the mouth and the taste buds in different parts perceive different tastes. The bitter taste is the best to distinguish of human, then follows by sour, salty and the sweet the worst. Among them, both sides of the tongue tip are sensitive to sweet and salty, around the middle of the tongue is sensitive to spicy and both sides of the tongue are most sensitive to acid. The anterior part of tongue is sensitive to salt and the posterior part is sensitive to acid. The base of the tongue is sensitive to bitter taste.

**(5) Skin.** The skin of human covers the whole body with the weight accounts for 5% - 15% of body weight and area $1.5m^2$ - $2m^2$ with thickness $0.5-4mm$, which composes of epidermis, dermis and subcutaneous tissue, also contains sweat glands, sebaceous glands, nails, toenails and other accessory organs and blood vessels, lymphatic, nerve, muscle and other components, is the tissue of a human perception climate environment, especially warm and cold, dry or wet, heat prevention and reserving human energy with maintaining the stability of the human body environment, participating in the metabolic process of the human body and other functions.

The epidermis is the outermost layer of the skin with an average thickness of about $0.2mm$. From the outside to the inside, it can be divided into the stratum corneum, stratum hyaline, stratum granulosa, stratum spinosum and stratum basalis. The dermis is composed of fibers, stroma and cells. Close to the epidermis is the papillary layer, and below it is the reticular layer. There is no strict boundary between them. Among them, the fibrous layer is composed of collagen fiber, elastic fiber and reticular fiber. The matrix is an amorphous, uniform gelatinous substance which is packed between fiber bundles and cells. There are mainly three kinds of cells, including fibroblasts, histiocytes and mast cells. The subcutaneous tissue is in the lower part of the dermis, which has the function of preventing heat dissipation, reserving energy and resisting foreign mechanical impact. It is composed of loose connective tissue and fat lobules, and it is close to the muscle membrane. Accessory organs include small and large sweat glands, sebaceous glands, hair, finger or toe nails, blood vessels, lymphatics and nerves.

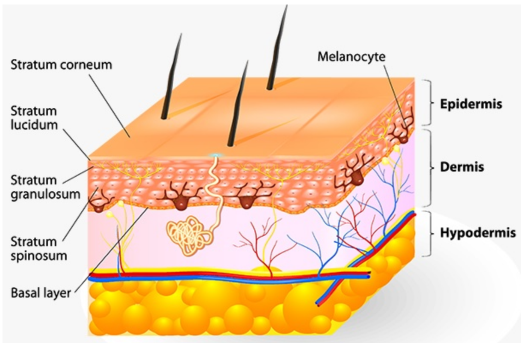



**Figure** 2.9. **Skin structure**

**(6) Brain.** A normal human brain has the functions of memory, regulation and information transmission. It is the source of human consciousness, the source of human body's collection of environmental information, the central nervous system, which makes decisions such as analysis and coping, controls a human's mental, visual, auditory, somatosensory and thinking functions. The main component of a human brain is water account for

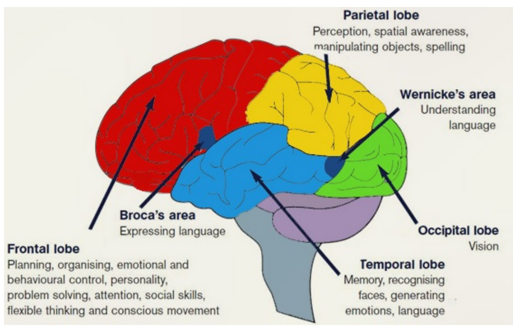



**Figure** 2.10. **Brain function**

about 80%, and its color is light pink. The human brain consists of telencephalon and diencephalon. Among them, the telencephalon from the olfactory nerve, diencephalon from the optic nerve. The telencephalon divided into the right and left hemispheres is the most advanced part of the central nervous system. The gully of the mediastinum separates the two hemispheres and the cerebral hemispheres have compartments of their lateral ventricles, which is connected with the third ventricle of the diencephalon, the cerebellum, the medulla and the fourth ventricle of the pons. The choroid plexus in the ventricles produces cerebrospinal fluid that circulates between each ventricle and the subarachnoid space. The diencephalon consists of the thalamus and hypothalamus. Among them, the thalamus connects with the cerebral cortex, brainstem, cerebellum and spinal cord to complete the relay and control of perception and the hypothalamus is associated with maintaining human constancy, controlling the autonomic nervous system and emotions.

The brain sections are divided into the gray and the white matters. Among them, the gray matter is the cerebral cortex, which covers the surface of the hemispheres and is where the cell bodies of neurons are concentrated. The differentiation of the cortex on the medial surface of the cerebral hemispheres is relatively simple, consisting of the molecular layer, pyramidal cell layer and polymorphic cell layer in general. The lateral neocortex of the cerebral hemispheres is highly differentiated, including molecular layer, outer granular layer, outer pyramidal cell layer, inner granular layer, inner pyramidal cell layer and polymorphic cell layer. The deep surface of the cerebral cortex is the white matter and beyond that there are gray matter nuclei. These nuclei near the base of the brain are called the basal ganglia, mainly the striatum.

For many years, the brain science has been an important field of scientific research. The researchers hope to clarify the memory, storage, calculation, judgment, decision-making and action, i.e., the consciousness process research, artificial intelligence simulation and practice in order to adapt the changing of circumstances and serve the development of human society. In fact, the memory, storage and operation of the computer is the preliminary simulation of human brain and then, the robot on the basis of the computer simulates the judgment, decision and action of human. Of course, whatever it is a computer or a robot, it is a simulation under the mode and action instruction set by human, may not really replace the human brain.

**2.2.Indirect Perception.** The indirect perception is the collection of data or information about things in the universe by means of instruments, meters or facilities. The collected data or information is analyzed and summarized by human and then the perception of thing in the universe is formed. Among them, the most are to extend human's vision and hearing, and the taste is the experience of humans ourselves.

**(1) Microscope.** The normal resolution of the human eye is about 1/2000 to 1/5000 with a minimum visual distance of about 10cm. Therefore, the human eye cannot directly observe the behavior of objects or microorganisms that are too small. In the 20th century, electron microscope, fluorescence microscope, super-resolution microscope and X-ray microscope were successively developed to extend the observation of different kinds of microscopic things.

An electron microscope works like light microscope, using electrons instead of light and electromagnets instead of glass lenses to achieve higher resolution which results in visible, clear images of small organelles, improves the ability for virus recognition and presents a more effective detection of pathogens in medicine.

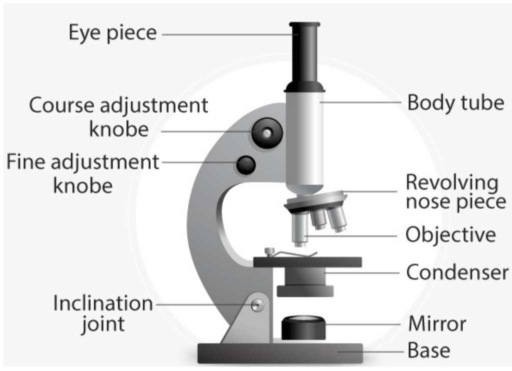



**Figure** 2.11. **Electron microscope**

The fluorescence microscopes use ultraviolet light as the light source, and most of them use $200W$ superpressure mercury lamp to irradiate the observation objects. The light filter cube table is placed above the objective lens and a camera is equipped to take photos, which are used to study the absorption, transport, distribution and location of chemical substances of cells in biology investigate. The super resolution microscopy is a further development of fluorescence microscope with the image resolution $20nm$, which can directly observe the structure of DNA and protein molecules at the molecular level. The X-ray microscope uses X-ray as the light source to observe the microscopic structure and figure that can not be resolved by the naked eyes. Its imaging principle is basically the same as that of the optical microscope.

The microscopes are generally used only to observe macromolecular structures. In the structure of atoms, the proton and neutron are only $1 \times 10^{-5}$ of the atom in size, and the electron is only $1 \times 10^{-9}$ of the atom in smaller size. It is necessary to construct Wilson cloud chamber, bubble chamber and Geiger-Miller counter for observing the trajectory of them without using a microscope in general.

**(2) Telescope.** The telescope uses a lens or mirror with other optical components to enlarge the opening angle of the distant objects, collects a much thicker than the pupil diameter beam of objects into the eye and then, makes the human eye can see smaller angular distance or the details of the objects can not find originally, which is a kind of optical instrument that extends the field of human vision, including the civil and military telescopes. Generally, the space exploration applies of the Hubble space telescope, Gemini telescope, solar telescope and the radio telescope. Among them, the Hubble space telescope is the first space telescope with a total length of more than $13m$, running on an orbit of about $600km$ above the ground.

A telescope is mainly measured by technical indicators such as the magnification, event horizon, exit pupil diameter, resolution, dusk coefficient, effective aperture and the light collecting force, where the magnification refers to the ratio of the focal length of the objective lens to the focal length of the eyepiece, which measures the degree of magnification of the telescope's visual angle; the visual angle refers to the range of objects visible to the telescope at $1000m$. For example, the event horizon $114m/1000m$ marked on a telescope means that the observer can observe the field of view within $114m$ at $1000m$ with the telescope; the exit pupil diameter is a parameter that roughly describes the image brightness; the larger the pupil diameter, the clearer the image will be; the resolution refers to the precision of the screen image, i.e., the display pixels; the twilight coefficient is squared by the product of the magnification and aperture of the telescope, which reflects the observation efficiency under dark light conditions; the part of the diameter of the objective that is not blocked by the frame and aperture is called the effective aperture; the multiple of the effective area of the objective relative to the pupil area is called the collecting force.

The optical telescopes extend the visual range of objects observed by humans, which is constrained by technical indicators such as magnification and horizon. A radio telescope is a special type of antenna and electromagnetic wave receiver used to receive electromagnetic waves from space.

**Figure** 2.12. **China's eye of heaven**

The radio telescopes work by reflecting the projected electromagnetic waves through a rotating parabolic mirror and reaching the common focus in the same phase. The radio waves projected from the celestial bodies detected by the receiver and collected to the focus of the telescope need to reach a certain power frequency, generally up to $10-20W$, and the power of the radio frequency signal at the focus is amplified $10-1000$ times. It is then transmitted by cable to a control room, where it is further amplified, detected and finally recorded, processed and displayed in a manner appropriate for a particular study.

The basic indicators of radio telescopes are sensitivity and spatial resolution, which reflect the ability to distinguish between radio point sources close to each other on two celestial bodies and the ability to detect weak radio sources, respectively. Here, the sensitivity refers to the lowest electromagnetic wave energy that can be measured by the radio telescope. Generally, the lower measured value of electromagnetic wave energy implies the higher the sensitivity and the resolution refers to the ability to distinguish between two identical point sources that are close to each other. Generally, the larger diameter of the antenna, the shorter the received radio wavelength and the higher the resolution. At present, the world's largest radio telescope is the eye of heaven FAST in China with a diameter of $500m$ and a diameter of $300m$ for receiving light. In theory, the FAST can receive electromagnetic signals from 13.7 billion light years away, which is close to the edge of the universe known by humans.

Hearing this, Huizi asks Dr.Ouyang: "*Daddy, is it possible to see everything in the universe with a microscope and radio telescope for humans?*" Of course not! Dr.Ouyang replies with an explanation, namely what the human eyes can perceive is the reflection of an object from a light source. The objects that do not emit or refract light are invisible to the human eyes in theory. For example, *why should we use a flashlight or other light source to illuminate an object in home at night if there is a power failure?* It is because we can see it only if the object reflects light into our eyes. The microscopes and telescopes only extend the human vision to a certain extent, which is far from the perception of the universe because this is only the human perception and furthermore, there are really stars that neither emit nor reflect light, and even do not emit electromagnetic waves in the universe. Certainly, we can not detect such kind of stars now.

## §3. **Being Out of Non-Being**

The human perception of the universe is a gradual process from the unknown to the know, i.e., "*being out of non-being*" in recognition. Here, the word "*non-being*" refers to things that exist objectively in the universe but have not been perceived yet by the six sensory organs of human, which is not the case of "*false*" or "*non-existence*"; the word "*being*" refers to things that has been perceived by the human's sensory organs.

**Figure** 2.13. **Perceive things**

One day, Huizi asks Dr.Ouyang what is the implications of "*being out of non-being*", says that she is always wondering why one can gets being from non-being. Dr.Ouyang asks her:"*Do you think what is being and what is non-being?*" Huizi replies:"*I think the being should be invisible or touchable and the non-being isn't visible or touchable!*" Dr.Ouyang continuously asks her:"*What that other one had seen is called being or non-being?*" Huizi says positively:"*Of course! It is being.*" Dr.Ouyang asks his daughters if she learned the Andersen's fairy tale of *Emperor's New Clothes*. Huizi nods and says:"*Yes! I learned!*" Dr.Ouyang inspires his daughter and says:"*The two swindlers lied to the emperor, saying they would weave and sew him the most beautiful clothes in the world, but there was nothing on the loom! Certainly, there were no the world's most beautiful cloth and no the world's most beautiful clothes to let the emperor wearing parade. The swindler cheated the emperor lots of money and also let the emperor happily wearing without clothes in the parade carriage until a child on the side said that he had nothing to wearing. Then, other humans also said he had nothing to wearing before he woke up and realized that he had been cheated by the two swindlers, but it was too late. Why is the case? If one can not see and touch is called non-being, everyone should find that the emperor has nothing wearing on?*" Huizi thinks for a while and says:"*Because the two swindlers said that anyone who was incompetent or stupid could not see the clothes they made! The officials sent by the emperor to inspect the loom clearly saw nothing but they did't want to be called incompetent or stupid and instead, they praised the beauty of the cloth made by the swindlers, even the emperor himself did so!*" Dr.Ouyang then asks his daughter:"*Do you think the emperor has new clothes or not?*" Huizi says:"*It's certainly nothing because he has nothing to wear on.*" Dr.Ouyang continuously asks her:"*But except for the child, everyone else said that the emperor was wearing a beautiful clothes including the emperor himself, is it not true?*" Huizi replies:"*Others lie because they are afraid of being called incompetent or stupid. Only the child spoke the truth!*" Dr.Ouyang confirms her daughter's answer and tells her that infants, adults and old one do not see and hear exactly the same. In addition, the vision and hearing of ordinary human and blind or deaf humans are also different. The perception of a thing does not refer to the perception of any one or several humans but refers to the objective existence, which is the real perception of the objective being. It should be noted that the objective being of things is not subject to human's will, and we can not conclude it does not exist by humans can not perceive it in the universe. This is why Laozi said in Chapter 40 of his *Tao Te Ching*, i.e., "*all things are created from being, and the being is out of non-being*" in the universe. That is to say, both of the "*being*" and "*non-being*" are terms of the human perception of things. It is the human's perception of things from the "*non-being*" to the perception of "*being*", not the fabrication of the swindler in the fairy tale of *Emperor's New Clothes*.

All beings come out of non-being. The beginning of the universe has no life, no plants or animals and of course, no humans, only reproduce the elementary particles of life. In process of the universe evolution, the basic particles combined to elements, gas and water, and under certain conditions such as the shining of the sun, the combination of elementary elements, the gas and water generated the single celled organisms and then, from the single cells to multiple cells, from plants to animals and finally, to humans. This evolution was a dynamic process. Dr.Ouyang tells his daughter that the human recognition on the universe is not the recognition of any one at a particular time but the accumulation of humans in the long developing, a collection of the wisdom of humans. In this way, being out of non-being of humans includes the recognition of predecessors, the recognition of improving ability and the recognition of variation of things in the universe.

**3.1.Being From Predecessors.** Most of the human recognition of things in the universe including clothing, food, housing and moving is not their own perception but comes into beings from the recognition of predecessors and from the cultural inheritance. The human recognition of things is usually the instinctive reaction of the survival, such as knowing which plants or their fruits can be fed, which can be used in medicine to cure one's diseases, which animals can be adapted for human use, which can be used as tools or weapons to resist the invasion of foreign enemies,$\cdots$, etc. All of these beings pass down from generation to generation and become the early experience of human perception of things, which is a kind of recognition already verified by humans ourselves.

**(1) Farming being.** Farming is a process out of the wild to the planned organization. Generally, plants include edible plants and medicinal plants. Here, the *edible plants* refer to plants that contain starch, protein, oil, sugar, vitamins, etc., needed by humans or can be eaten as vegetables, including the roots, stems, leaves, flowers, fruits and seeds of plants; the *medicinal plants* refer to plants used in medicine for disease prevention and treatment. Among them, which plants can be eaten and which can be used as medicine to cure patients need to be differentiated one by one, which is the result of the long-term practice of ancestors, also the gradual accumulation of experience. In Chinese ancient mythology, there was a Shennong, also known as Yan emperor and famous as the Youchao, Chert Ren, Fuxi and Xuanyuan emperors. He was the father of Chinese agriculture and medicine. He wrote the book of *Shen Nong Ben Cao Classics* containing 365 kinds of medicines, which is the earliest medical classics in China.

[**Shennong Tasted Herbs**] It is said that in the ancient times, the ancestors lived mainly by hunting. At that time, the hunting equipments used by hunters were sticks, stones and so on, which resulted in few prey and humans had to starve, countless humans died of starvation and disease. All of these were seen by Shennong day and day. He was very anxious because the grain and weeds grew together, medicine and flowers bloomed together in the wilderness. It was not clear which of them could be used as food or as medicine.

Shennong meditated this for three days. On the fourth day, he took a group of subjects from his hometown, the state of Lishan to the northwest mountains. After forty-nine days, they came to the foot of a mountain through a peak after a peak, a valley after a valley,which covered with fragrant flowers and plants, surrounded by cliffs and waterfalls, also covered with moss and slick but impossible to climb without a ladder.

**Figure** 2.14. **Shennong's portrait**

Shennong saw a few golden monkeys climbing up and down the old vines and the rotten wood on the cliff. He called his subjects to cut the poles, rattan and build a frame against the cliff. It took them the spring, summer, autumn and the winter of a year to reach the top of the mountain. Shennong led his humans up to the top of the mountain covered with colorful flowers and plants, and tried 100 herbs in here. He had his subjects planted rows of fir trees on the hill for a wall and built huts inside the wall for a living room. During the day, he led his subjects to the mountains to taste herbs in search of foods and medicines. At night, he had his subjects built bonfires and wrote down which grasses would satisfy their hunger and which were bitter, hot or cold.

Once, Shennong picked a piece of grass and put it in his mouth to chew. Just in a moment, he felt the world spinning around and fell down on the ground. He knew he had been poisoned but he could not speak. He pointed to a ganoderma lucidum plant in front of his subjects and then pointed to his mouth. His subjects put the red ganoderma into their mouths, chewed it and then fed it to him. Surprisedly, Shennong's poison was immediately removed, his head did not faint and could speak. In this way, Shennong tasted the flowers and plants of one mountain and then another. It took him forty-nine days to visit every mountain here. He tasted 365 herbal remedies and wrote the *Shen Nong Ben Cao Classics* for the benefit of humans. He tasted the seeds of plants such as rice, wheat and corn also, which could satisfy one's hunger. He asked his subjects to take the seeds back and began to cultivate crops in the wilderness, which began the farming age of China.

**(2) Domestic animals being.** From wild to domestic animals, the planned organization of feeding is a process out of wild animals. Here, the wild animals refer to all kinds of animals that are not raised in the artificial feeding and live in the natural environment; the domestic animals refers to the creation of artificial living environment for animals, providing food and other necessary living conditions for feeding in order to serve the production and life of humans.

In the primitive society, human's social production was mainly to obtain food. In that time, the women were engaged in picking while men went to catch wild animals to obtain meat and provide protein needed by the human body. However, due to the lack of effective equipment, the hunting success rate was very low and the combat casualty rate was high. Later, it was discovered that some wild animals could be domesticated and then cultivated in a planned way to provide the necessary meat for humans without the need of dangerous hunting. For example, the domestic pigs, sheep, chickens and other domestic animals can be domesticated by wild boar, wild goat, pheasant and so on as one's daily main meat; the cattle were domesticated from bison and used as meat at first and later, the cattle were found to be used for agricultural production and then, they were no longer used as the main source of meat; the horses were domesticated from wild horses as means of transportation; the dogs were domesticated by wolves and used for vigilance or hunting in primitive societies.

**(3) Time being.** The purpose of observing the sky and distinguishing the season is to summarize the laws of seasonal changes and guides the spring sowing, summer growing, autumn harvesting and winter storage in the farming era so as to realize the survival and reproduction of humans ourselves. In the ancients time, there are the words that "*don't go out if red sky appears in the dawn but can travel thousands of miles if it appears in the sunset; the grain without turning if the ichthyosis clouds appears in the sky*" and "*it will rain if the hook clouds appears in the sky; the thunderstorm is coming soon if the castle clouds appears in the sky*", etc. At that time, one noticed that the rising and setting of the sun, the waxing and waning of the moon gave rise to the conception of time and direction. Among them, a typical representation is the 24 solar terms of Chinese.

The 24 solar terms are an original creation of Chinese and also an important discovery in the history of astronomy. Among the 24 solar terms, the beginning of spring, spring equinox, beginning of summer, summer solstice, beginning of autumn, autumn equinox, beginning of winter and winter solstice reflect seasonal changes. Among them, the word "*beginning*" in the beginning of spring, beginning of summer, beginning of autumn and beginning of winter means the start of that season, which is the method of dividing the four seasons by the ancient Chinese. The spring equinox and autumn equinox are equal parts in the day and night. The summer solstice and winter solstice respectively refer to the arrival of hot summer and cold winter and the summer solstice has the longest day (Yang), the winter solstice has the shortest day with the longest night (Yin). The awakening of insects, pure brightness, grain full and grain in ear reflect phenological phenomena. Among them, the "*awakening of insects*" means it should be waking all animals in hibernation. The ancient Chinese believed that it was the spring thunder that woke hibernating animals. So, it was called awakening of animals; the pure brightness refers to the weather is clear and clean, green vegetation in season; the grain full refers to the crops began to set ear but the grain is not in the full or mature period; the grain in ear refers to the barley, wheat and other crops have been mature. And the rain water, grain rain, slight heat, great heat, limit of heat, white dew, cold dew, frost's descent, slight snow, great snow, slight cold and great cold reflect the climate change. Among them, the grain rain means that there is more rain, and the grain grows vigorously; the slight heat and great heat indicate the degree of hotness, i.e., the slight heat is when the hot begins and the great heat is the hottest time of one year; the white dew and cold dew are when the mist of the night condenses into white dew; the frost's descent is when water vapor condenses into white frost; the slight snow and great snow indicate the extent of snow at the beginning of winter; the slight cold and great cold indicate the degree of cold. Here, the slight cold refers to the early stage of cold but the great cold refers to the coldest time of one year.

Dr.Ouyang tells his daughter that the recognition of ancient Chinese was based on the notion of the "*one union of the heaven and humans*", which was around the basic necessities of human survival such as clothing, food, shelter and transportation. In addition to the plants and animals from wild to farming and livestock, and the 24 solar terms in astronomy, the ancient Chinese also invented gunpowder, compass, paper making, printing, weaving, seismograph and other technologies to serve human's production and life practice. At the same time, the ability of human recognition of things is gradually improved with the innovation and development of science and technology, especially the observation, detection and perception equipments. Therefore, to understand things and achieve being out of non-being is first to inherit the recognition of predecessors, namely the cultural inheritance and developing new recognition on that of the predecessors.

**3.2.Being by Extending Human's Ability.** The innovation of humans in the ability to perceive thing focuses on the perception of microscopic and macroscopic objects that can not be directly distinguished by human sensory organs such as molecules, atoms, nuclei, protons, neutrons and other microscopic particles, their combined DNA and other macromolecules as well as distant galaxies in the universe is by research and development of corresponding scientific instruments, equipment and facilities to assist human observation, collecting information and analysis to realize the recognition of these micro or macro things from unknown, i.e., out of non-being. This capability innovation mainly revolves around the microscope, telescope, supplemented with the necessary signal receiver, photographic equipment and analytical instruments to enhance the human observation ability. For different observation points, there are mainly two ways to innovate human observation ability. One way is the observation on the ground, i.e., applying specific scientific instruments on the earth to observe microscopic or macroscopic objects and realizes the recognition from scratch. Such scientific instruments are chiefly microscopes and telescopes; Another way is the observation in space, i.e., applying rockets to send observation instruments into the air near or far to collect extraterrestrial information such as those of the communications satellites, navigation satellites, meteorological satellites, scientific satellites, technology test satellites and so on.

The ancient Chinese watched the sky and determined the time for agricultural production, which depended on the perception of observers. However, the satellite makes meteorological observation of the atmosphere, which has the characteristics of large scope, timely and rapid, continuous and complete. A weather satellite is essentially a weather station suspended in space. It is a combination of more technologies such as space, remote sensor, computer, communication and control. Among them, remote sensor can receive and measure visible, infrared and microwave radiation from the earth and its atmosphere, converts into electrical signals and sent back to the earth. In response, the ground stations reconstruct the electrical signals into pictures of clouds, the earth's surface and the ocean, which can be analyzed and processed to predict weather patterns. There are mainly two types of meteorological satellites. One is polar-orbit meteorological satellites whose flight altitude is about $600 - 1500km$ and its orbital plane always keeps a relatively fixed intersection angle with the sun; Another is the synchronous meteorological satellite whose orbit altitude of about $35800km$ with its orbital plane and the earth's equatorial plane coincide.

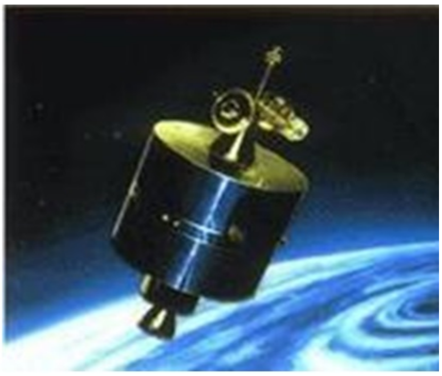


**Figure** 2.15. **Meteorological satellite**

The meteorological satellites carry a variety of meteorological remote sensors, receive and measure the visible light radiation, infrared radiation, microwave radiation and other information from the earth, ocean and the atmosphere. Its main contents of the observation include: ①Satellite cloud image shooting; ②Observations of cloud top temperature, conditions, cloud cover and phase of condensation within clouds; ③Observations of land surface conditions and changes such as snow, ice and aeolian sand, and the ocean surface conditions; ④Monitoring the distribution of total water vapor, humidity, rainfall area and precipitation in the atmosphere; ⑤Monitoring the content and distribution of ozone in the atmosphere; ⑥The incident radiation of the sun, the total reflectance of the earth-air system to the solar radiation and the infrared radiation monitoring of the earth-air system to space; ⑦Monitoring of space environment conditions such as the flux density of protons, $\alpha$ particles and electrons emitted by the sun. The observed data are then converted into electrical signals and transmitted to the ground receiving station for analysis and processing. And then, the medium and long term weather forecast is carried out to forecast typhoon, tornado, hurricane and other disaster weather, providing meteorological and hydrological data for navigation, aviation, agriculture, fishery and animal husbandry.

Similar to the remote sensors collecting the meteorological data from satellite, the space probe is equipped with established scientific detection, data acquisition instrument for the space of the earth, moon, planets and interstellar space exploration for studying the origin and current situation of the moon and the solar system, revealing the earth's atmosphere environment, i.e., the solar system's formation and evolution process, and exploring the origin and evolution of life, including: ①Long-range space exploration in low-earth orbit; ②Fly close by the moon or planets; ③Long-term observation of satellites of the moon or planets; ④Make a hard landing on the surface of the moon or planet and its satellites, and use the short time before landing to probe; ⑤Soft landing on the surface of the moon or planets and their satellites, conducting field investigation, sampling and sending back to the earth for research; ⑥Deep space flight, long-term acquisition of one or several indicators or data.

Up to now, humans have successively sent space probes to the moon and other planets in the solar system as well as the universe for deep investigation and exploration. Among them, the most detailed investigation on the Moon, Venus, Mars, not only took topographic maps but also launched an unmanned probe to land on their surface for investigation. Notice that the cosmic microwave background radiation which is the same in all directions is said to be the vestige of the Big Bang. And for that, the WMAP satellite ran on an orbit of the sun-earth system at a distance of 1.5 million kilometers, collected a large amount of basic data to reveal the origin of the universe.

**3.3.Being in Variation and Innovation.** The environment of living things is complex and diverse, changing and developing. Instead, livings must follow the natural law of survival of the fittest. The biological variation is to adapt to the change of living environment which is conducive to the evolution of the same species because the favorable variation will be continuously accumulated and strengthened by heredity so that a population is more adapted to its living environment, and new biological types will be generated by variation, realizing the evolution from simple to complex and from low to high. The innovation is a kind of creative behavior that humans carry out for our own survival. It is the use and recreation of beings in the universe.

**(1)Being in variation.** Change is the eternal law of things in the universe. The recognition of predecessors is correct in its time but may not be applicable to the present. In response, the ancient Greek philosopher Heracrit said:"*One can not step into the same river twice*" because the water is constantly flowing. The biological variation in nature is the behavior of organisms actively adapting to environmental changes, which in human eyes is a kind of creation out of non-being or a new discovery. In Figure 2.16, a bird with a dog's head is such an example.

**Figure** 2.16. **A bird's variation**

The biological variation is divided into heritable variation and non-heritable variation. Among them, there are three kinds of heritable biological variation:①Gene mutation. The gene mutation is the fundamental source of biological variation caused by changes in the internal structure of genes. Most of them are caused by DNA replication errors; ②Gene recombination. The genetic recombination refers to the recombination of the genes that make up a thing to produce the word "*being*", which is caused by the free combination of genes controlling different traits during meiosis or the exchange of non-sister chromatids between homologous chromosomes. Most biological variation is caused by the genetic recombination; ③Chromosomal variation. The genes are mainly located on chromosomes. The changes of chromosome structure and number will inevitably lead to the changes and recombination of gene number and sequence, which will lead to biological variation finally, including chromosome structure variation and chromosome number variation.

The biological variation is in fact the result of natural selection. It is the change of organisms to adapt to the environment. For the organism itself, it may change in a favorable direction, making the organism more able to adapt to the environment. It can also mutate in an unfavorable direction, making the organism less able to adapt to its environment or even die as a result. Generally, humans can apply the beneficial variation of organisms to serve their own development. For example, the hybrid rice is beneficial to improve rice quality and yield. The variation of dwarf and lodging resistance in wheat is beneficial to wheat growth and at the same time, to take the effective measure preventing from the biological unprofitable mutation harm to humans.

**(2)Being in innovation.** The word "*innovation*" means to create new things out of non-being instead of the "*minor change*". In fact, most of human innovation is centered on the liberation of human productivity by research and manufacture parts to simulate the function of biological basic functions, and then combine parts to simulate some action or ability of biological organisms including the innovation of production, transportation, loading and unloading machines and computers.

The instrument of production is the mainly embodiment of the social civilization such as those of: ①From the stone knife, stone axe and wooden stick and other natural tools to create hoe, sickle, plough and rake, and then to the application of machine tools, cutting machine, rice harvester, transplanter and other agricultural production tools innovation; ②From wearing animal skins, coir raincoat, thatch and leaves to artificial spinning and weaving, and then to the application of spinning machines, looms and other textile tools innovation; ③From being pulled and shouldered to tame animals, and then to the application of lifting machinery, conveyor, loading and unloading machinery, transport vehicles and storage equipment, and other transport tools innovation; ④From the manual hulling of rice, wheat, corn and sorghum to the rolling and pestle of stone mill, and then to the application of rice hulling machine, rice milling machine, milling machine and other grain processing tools innovation; ⑤From the primitive dwelling such as cave and tree branch to the basking stone nest, brick and lime, cement firing, and then to the cement ball mill, crane and other equipment innovation. All of these innovations provide humans with the improving on the survival conditions, i.e., the clothing, food, housing and transportation which is being in the innovation.

**Figure** 2.17. **Chinese space station**

The transportation is the embodiment of one travel ability in space, which emerges one's travel from walking, climbing, jumping to human or tame cattle, horse-drawn carts, bicycles, and then to the application of motorcycles, cars, trains and high-speed trains, from relying on wooden support, sampan to simulate fish travel invention paddle and wooden boat, and then to the application of steam engine, gas turbine, diesel engine or nuclear power unit driven submarines, ships and from the simulation of birds, dragonflies flying to the production of kites, balloons and rockets, and then to the application of air engines, aircraft, spacecraft, space shuttle and space station. All of these development break the restriction of humans living on the land, can carrying one travelling in the land, sea and the airspace.

The computer is a kind of electronic equipment that simulates human brain with a super storage, which can carry out calculation, logical judgment, decision and processes data at high speed according to the program. The application of computer, especially the artificial intelligence on human's consciousness, thinking and simulating the control process greatly extends the human's ability. And now, these technologies have been an important supplement of human abilities, including robotics, speech recognition, image recognition, natural language processing and expert system, etc., and all of them are being out of non-being.

Dr.Ouyang tells his daughter that since everything is the human's recognition, it must be a recognitive process from unknowing to knowing. That is, being out of non-being. Among them, the "*non-being*" is the case of existing but has not been recognized by humans or elements existing although "*non-being*" but they combined into "*being*" if the conditions are fulfilled. Dr.Ouyang asks his daughter: "*Do you remember what I told you about how you came to being last time?*" Huizi replied: "*Yes! I was born to you and my mother!*" Dr.Ouyang said: "*Good! Did you being before you were born?*" Huizi seems a little of blank and asks: "*How to know I was being or not before I was born. It should be non-being!*" Dr.Ouyang then says: "*Is there any you in the world now?*" Huizi smilingly says: "*Of course! I am being since I stand here.*" Dr.Ouyang concludes to his daughter: "*You see, you are a living example yourself of being out of non-being!*"

## §4. **Phenomenological Theory**

The first of one's perception on a object is the visual perception. That is, the pattern produced by the reflection of light into the human eyes, including its size, shape, speed of moving and direction change. The recognition of an object is obtained through the brain processing of human and then, take the measures for his survival, usually called the *phenomenological theory*. That is, when explaining a physical phenomenon, it does not investigate the causes of the phenomenon but obtains the laws from the empirical summary and generalization of observation facts and deductive inference to form the recognition theory on the phenomenon, i.e., an established theory only on patterns. It is the empirical generalization, summary and refinement of things or experimental phenomena and the embodiment of human survival instinct. The phenomenological theory is a modeling method commonly used in system science. That is, without considering the internal mechanism and constraints of the system, the evolution equation of the system is directly established by the observed properties or characteristics of the system at the macro level and the trend of the system is judged and predicted so as to serve humans. Notice that the word "*pattern*" here is an abstraction of the characteristics, referring to the perception of things by the six sensory organs of human, similar to the "*pattern*" in the *Diamond Sutra* or the "*color*" in the *Heart Sutra*, and also referring to the "*hexagram*" in *Change Book* of China.

Notice that the phenomenological theory is a main method by which humans perceive natural laws and form scientific theories on this basis. At the same time, the pattern is the test on the correctness of a scientific theory. For example, human's scientific recognition of plants, animals and the universe has been transformed from phenomenological approach to theoretical analysis. That is, to see the essence through manifestation and then through practical testing to form human's recognition on plants, animals and the universe. In fact, even into the 21st century, the human knowledge of things in the universe is still largely on patterns of things.

**4.1.Phenomenism in Humans.** Dr.Ouyang tells his daughter, a human from birth to grow up and living in society, the most uses is the phenomenological method for his survival, namely the intuition of things or as to its identification, analysis, judgment and then makes decisions, which is not necessarily back their cause after find out how to make decisions. *Why is this the case?* One reason is that the phenomenological method can make them responds in a shortest time to adapt to the natural changes, and another is that science is only the depiction of one or several characteristics of things, not necessarily the true face of things. At the same time, many situations require immediate reaction without allowed more time. And different humans have different scientific literacy, their reaction to things are not exactly the same. In this case, the phenomenological method is everyone's innate instinct to recognize things.

**(1)Babble's knowledge.** A baby's brain weight is about 25% of the adult with the vision of $40cm$ as he or she just born. They can hear the sound near but the brain function is worse than an adult. Their normal analysis, judgment and language abilities have not yet formed, and to things are mostly a kind of visual recognition, i.e., imitation, repetition and experience. Babies and young children learn their recognitive abilities through imitation, trial and errors. Generally, infants and young children are familiar with their mother's voice. Touching them can makes them happy and stabilize their mood. They can smell their mother's taste, imitate adult facial expressions, laugh or learn to do strange things.

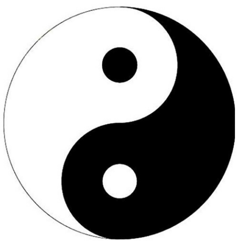
Different from adults, the infants' perception of external things is not stored in the brain as knowledge after understanding but stored in the brain by the subconscious mind as a kind of "*patterns*". It is a kind of graphical perception which is fast, happy, stress-free and unconscious. Generally, an adult remembers and extrapolates after understanding the meaning of words but low efficiency, the infants and young children do not ask the meaning of words. They learn words by word graphics direct perception, is similar to the rote learning and then set of words combined with the identification of life short. They can quickly understand different words and set different words, given different meanings, which is faster than adults.

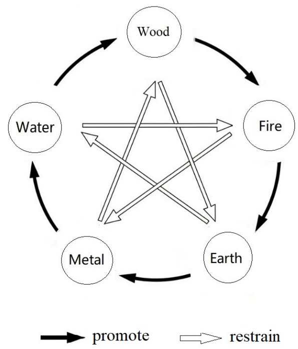
The babies learn language differently from adults. They store adult words first in their brains and use the whole sentence without grammar. Adults learn word meaning and grammar first, then sentences and texts. This is why lots of humans may never learn a foreign language in their lifetime while a normal baby can learn any language in three years or so because the infants and young children adopt a figurative approach similar to the famous words that "*if remembered 300 Tang poems by heart then can recite them although can't writing a poem*" in China.

**(2)Crafts from the master.** The most of professions in the social production such as those of the production and equipment operators, including lathe workers, foundry workers, blacksmithing workers, welders, car repairmen, equipment installers, maintenance electricians, carpenters, tile workers, rebar workers, drivers, boiler workers, hairdressers, etc., and also those who are engaged in the sculpture, printing and dyeing, brocade, ceramics, embroidery and other crafts can not master skills simply only learn through by books. They need to learn skills from the master in practice. That is, they need to apprentice observe, perceive and capture the master's knowledge and skills in the real workplace, i.e., the patterns in the master's operation skills or the "*draw a tiger with a cat as model*" style, learn the master's action and then, under the guidance of the master to operate and understand the operation skills. This kind of learning from masters is just a phenomenological method.

Chinese usually says that there are 72 professions and each of them has the champion itself. Among the 72 professions, there are essential differences in the professions but there is a close relationship within the master and the apprentice. At the same time, the mastery of master's skills varies from the apprentices to disciple, which is why apprentices taught by the same master have different skills. Generally, after the master's worship, the master will teach his apprentices the skills from his "*action's decomposition*" to "*how to use them for the result*", which is different from school education. In school, teachers teach students to follow the syllabus as a guide, in the form of large classes taught by one teacher but followed many students. Generally, the master teaches his apprentice alone or together with several apprentices and cultivates his apprentices so that the disciples do not go or less detours, achieve enlightenment, obtain the true teachings and then, carry forward the skill of this discipline. Why so is that there are many parts of the 72 professions that require the disciple to comprehend their essence by himself. For example, there is a famous dish called *Ma Po Dou Fu* in Sichuan cuisine restaurant of China. Its main ingredient is tofu but the auxiliary ingredients include the moderation of beef and green garlic shoots, the spices include the moderation of fermented black bean sauce, watercress, chili powder, Chinese pepper powder, salt and soy sauce. Then, "*what does the moderation mean?*" It means that the chef grasps the standard in the heart on his own. There is no a unified standard, only with the chef and common human's preferences shall prevail. This forms the competition pattern of Sichuanese cooking in China, i.e., different families or different chefs will cook out of different taste of this dish.

**(3)Chinese traditional medicine.** The traditional Chinese medicine is a phenomenological theory. However, it is a theory with the most profound recognition on the operation of human body. It abstracts the human body as a structure composed both of the vital energy and blood running and conforming to the balance of *Yin* and *Yang*, namely the 12 meridians and believes that the balance of Yin and Yang is the basis of a normal human's body. On the contrary, the imbalance of Yin and Yang will lead to the occurrence of human diseases and affect the normal life activities. Correspondingly, through the observation, listening, questioning and felling the pulse with "*patterns*", a doctor of Chinese traditional medicine explores the cause, nature and location of the disease, analyzes the pathogenesis and changes of the internal organs of the body, meridians and joints, the vital energy and blood fluid in the human body, determines the positive and negative growth, syndrome types and then, adopts the principle of syndrome differentiation and the treatment, i.e., sweat, vomiting, hypoxia, harmony, warmth, cleansing, tonic and elimination and applies the traditional Chinese medicine, acupuncture, massage, massage, cupping and diet therapy to restore the balance of Yin and Yang in the human body. Here, the meridians theory holds that there are 12 meridians running of the vital energy and blood in the human body, which play the role of communicating inside and outside and network the whole body. Under pathological conditions, the meridians present corresponding symptoms and signs, which can be used to diagnose and cure internal organs of the human body. In this process, the doctor is a purposeful observation of a patient's spirit, color, shape, state and tongue patterns so as to detect visceral lesions; the smell diagnosis is to listen to the patient's language breath of high and low, strong and weak, clear, slow or urgent, and smell the patient's smell to distinguish the disease's weakness, reality, cold and heat; the consultation is to ask the patient or his or her companion to understand the patient's condition and other conditions related to the disease; the felling pulse is a method by which the doctor can feel the pulse changes of the patient by pressing the pulsatile place of the posterior radial artery of the wrist with fingers and then, identifies the ups and downs of the functions of the internal organs as well as the stagnation of the vital energy and blood and other essence.






**(a)Taiji image** **(b)Five elements**

**Figure** 2.18. **Taiji diagram with five elements**

The traditional Chinese medicine is the embodiment of the *Yin-Yang theory* and the *five elements theory* in ancient Chinese philosophy. Among them, the Yin-Yang theory holds that everything in the universe, including the human body is made up of two parts: *Yin* and *Yang*, i.e., the Yin affirms the negative attribute of things but the Yang negates the negative attributes as the attribute of the Yang of things. The interaction between them includes Yin and Yang empathy, opposition restriction, mutual use, balance of growth and decline, mutual transformation. That is a dynamic pattern of opposites of things in the unity, i.e., the Taiji diagram shown in Figure 2.18(a). According to the five elements theory, all things are composed of five elements, i.e., the metal, wood, water, fire and the earth. And the interaction and transformation of things follow the dynamic law of the five elements as shown in Figure 2.18(b). Then, the internal organs of the human body are matched with the five elements in traditional Chinese medicine, which are used to explain the relationship between the internal organs of the human body and the mechanism of the occurrence of diseases so as to guide the treatment of diseases.

Notice that the essence of traditional Chinese medicine is to regulate the Yin, Yang and five elements of the human body and activate the body's own function, which can be seen from Figure 2.18(b). Among them, there are the kidney water to nourish the liver wood, the liver wood blood to generate the heart fire, the heart fire heat to warm the spleen soil, the spleen soil to produce water milkweed filling the lung gold, the lung gold down to help the kidney water. At the same time, the lung vital energy clear down to inhibit the liver Yang hyperactivity, i.e., the metal overcomes wood; the liver vital energy strip to dredge the stagnation of spleen soil, i.e., the wood overcomes the soil; the spleen transport to avoid the kidney water flooding, i.e., the soil overcomes the water; the kidney water moisture to prevent the heart fire hyper, i.e., the water overcomes the fire; the heart fire restricts the lung metal, i.e., the fire overcomes gold. In this way, the traditional Chinese medicine has formed it own medical system based on "*patterns*", which is a systematic conditioning of the vital energy and blood operation of the patient's body rather different than the "*headache cures only on the head, foot pain cures only on the foot*" of an isolated or local treatment in western medicine.

**4.2.Phenomenism in Plants and Animals.** The humans have established the system of food crops and livestock breeding for the survival after a long-term observation and imitating on wild plants, animals and distinguish which plant leaves, roots or seeds can dine for humans, to what can be domesticated animals and then, to grow food, raise livestock and poultry in an organized manner. It is nothing else but a kind of phenomenological method.

**(1)Corn.** The grain from the wild to the planned artificial culture is a simulation of the grain growth and its environmental by humans, including soil, sowing, fertilizing, watering, weeding, pest management and harvest such as the selection after seed germination, seeding the seedbed and when to go to a certain height of seedling transplanting, the field insect control, weed control, drought control and fertilization management.

**Figure** 2.19. **Paddy fields**

The rice is divided into three classes: the early rice, middle rice and the late rice. The early rice is usually sown in the late March or early April and harvested in the mid-to-late July; the middle rice is sown in the early April or late May and harvested in the mid-to-late September; the late rice is sown in the mid-to-late June and harvested in the early

October. It usually takes about 120 days including the rice from seed selection and seed with a blister, to be spread on the seedbed after the seed budding and then transferred to the field when the seedlings grow to a certain height for pest control, weeding, drought prevention and fertilization management, harvested, threshed, sun-dried and then stored.

The wheat grows period of 230 days to 270 days from sowing, seedling emergence, rooting, long leaves, jointing, booting, heading, flowering and fruiting, including seed selection, sowing, fertilization, weeding, watering, disease and pest control and other field management. Generally, the wheat is sown in the mid-September and reaped in May in the north of China or sown in the mid-November and harvested the following year from the early June to early July in the south of China.

The maize is divided into spring maize and autumn maize. The spring maize is sown in the late April to the early May and harvested in the late August. The autumn maize should be sown no later than the mid-July and harvested in the mid-to-late October. The growing period of maize includes the seedling stage, ear stage and flower grain stage. Among them, the seedling stage is the period of maize from the emergence to jointing, which is the vegetative growth stage of maize rooting, long leaves and differentiated stem nodes; the ear stage is maize from jointing to the ear flowering. In this period, there were vigorous growth of roots, stems and leaves, rapid differentiation and development of male and female ears. It was the key period to determine the number of ears, the size of ears and the number of grains per ear; the flowering stage is the growth of maize from the ear flowering to the grain maturity.

The sorghum can be sown from the mid-March to the late June every year corresponding to the ripening time from the early July to the mid-October. The growth of sorghum is divided into four stages: the seedling stage, jointing stage, heading and the flowering stage, the grain filling and maturity stage. The seedling stage refers to the period from the seed germination to the jointing which takes 25-30 days; the jointing stage is 30-40 days before the spike differentiation of sorghum and the flag leaf expansion; the heading and flowering period is 10-15 days when the spike is pulled out from the flag leaf sheath and blossoms successively from above to below; the grain filling maturity period refers to the grain expansion of sorghum 2-3 days after the flowering and pollination to the grain maturity, which takes 30-40 days.

The millet sowing time is from February to April every spring. Its growing period is generally about a year, divided into the early, medium and the late ripe three varieties. Among them, the growth period of early ripening varieties is 60-100 days, the growth period of medium millet is 101-120 days and the growth period of late millet is more than 121 days. Generally, the spring sowing are selected to the late varieties, the summer sowing to the medium varieties and the autumn sowing to the early varieties. The growth process of millet is divided into the seedling stage, booting stage and the maturity stage which is similar to the growth to seedling stage, jointing, heading and maturation process of the sorghum.

**(2)Melons, fruits and vegetables.** The transformation from wild to field cultivation of fruits and vegetables is the result of years of observation and simulation on their patterns of growth rules of humans, including seeds, growth cycles, edible parts and their living soil and environment. The fruits and vegetables contain a variety of minerals, vitamins and food fiber, are the main source of nutrition for providing the human body, including the vitamin C, carotene, minerals and the dietary fiber.

The melons, fruits and vegetables are harvested in season. According to the harvest period, they can be divided into three categories: annual, biennial and perennial. Among them, the annual fruits and vegetables in the sowing year flowering and bearing fruits such as tomatoes, peppers, cucumbers, cabbage, spinach and so on; the biennial fruits and vegetables in the sowing year to form fleshy roots, leaf balls and other storage organs after the winter coming year bolting, flowering and fruit; the perennial fruits and vegetables can be harvested for many years after a sowing such as day lily, asparagus, etc. Generally, the leafy vegetables need 25 to 45 days to harvest, fruits and vegetables need 80 to 120 days to harvest, and the spring, summer, autumn and winter all have seasonal vegetables.

The growth process of melons, fruits and vegetables is divided into seeds, germination, growth, flowering and fruit. For example, the growth of Chinese cabbage is divided into two stages: the vegetative growth and the reproductive growth. The vegetative growth is divided into the germination stage, seedling stage and rosette stage; the reproductive growth is the process of bolting, development, seed setting and wilting of Chinese cabbage. Here, the germination period is the process from seed germination to unearthing and revealing the true leaves, which takes 3-5 days; the seedling stage is the process from the appearance of the true leaves to the development of 5-8 leaves and the formation of the first leaf ring, which takes 12-18 days; the rosette stage refers to the emergence of two or more leaf rings after the end of the seedling stage when the plant appears encrusted, it takes 20-30 days.

**(3)Livestock and poultry.** The livestock and poultry refer to animals that have been bred and raised in a planned way after domestication to provide proteins, nutrients and cold-resistant materials needed for human survival, also assist human practical activities. So, which wild animals can be domesticated and what is the premise of domestication.

Certainly, whether an animal can be domesticated for humans depends on whether humans have mastered the feeding habits, habits and rehabilitation ability of animals, i.e., the patterns of different animals and then simulate, guide or change the feeding habits of animals to adapt to the living conditions provided by humans.

Animals in nature are divided into three categories according to their position in the food chain, namely the carnivores, omnivores and the herbivores. Among them, the carnivorous animals to catch other animals for food, such as tigers, lions and other beasts, birds for food; the herbivorous animals feed directly on plants such as horses, cattle and sheep; the virtually animals all in the food chain feed directly or indirectly on plants. By their feeding habits, birds and fish are divided into four types: the cereal, insectivorous, carnivorous and the omnivorous.

**Figure** 2.20. **Scalper**

There are great differences in animal habits. Some of them are nocturnal and some are diurnal, which is related to the behavior of their prey during day or at night. For example, the tigers and lions are carnivorous animals, like to rest during the day but activities in night. Each of them has its own exclusive territory, like to be solitary with no fixed nest; the horses are gentle, herbivorous animals, like to live in groups. They are timid and easily frightened with poor eyesight but good memory, and excellent hearing and smell; the cattle are herbivorous animals with multiple stomachs. The frequency and time of rumination are affected by the age, forage quality and health, and the daily rumination is 9-16 times; the sheep are herbivorous animals, like to live in groups, eat clean feed, drink cool and sanitary water. They are temperament tame, timid, smell and hearing sensitive, suitable for dry and cool environment, etc.; the wolves and dogs belong to the same family, are both carnivores. The wolves live in forests, deserts, mountains, boreal grasslands, coniferous forests, grasslands and other areas and usually activities at night. They are fast running in speed, acute sense of smell, good hearing and other characteristics; after the domestication, the daily life of the dog is basically the same as that of the human, i.e., the dog activities in daytime, not at night. The premise of animal domestication for humans is that humans can provide sufficient diet according to their living habits to make their populations multiply and have the characteristics of docility. This is also the reason why livestock and poultry are mostly herbivorous or gluttonous, insectivorous, omnivorous docile animals or birds and fish.

**4.3.Phenomenism for Time.** The ancient China was in an agrarian era. At that time, observing the sky, identifying the seasons and predicting the wind and rain were important agricultural matters. The ancient Chinese observed the sky with the help of astrological charts, including the sun, moon, earth and the five stars, i.e., Venus, Jupiter, Mars, Mercury, Saturn in the solar system and other 28 stars as well as the weather patterns of clouds, air, rainbows, wind, thunder, fog, frost and snow in the sky, which served agricultural production.

**(1)Observing sky for time.** The ancient Chinese needed to observe the sky for farming. After years of observing the orbit of the earth and the sun, the ancient Chinese found that it took 365 days for the earth to go around the sun. They divided the earth's orbit around the sun into 24 equal parts with each corresponding to a solar term, i.e., the "*24 solar terms*", representing 24 different positions of the earth in its orbit around the sun. It corresponds the change of climate, phenology and time. The 4 seasons of year begin respectively the beginning of spring, beginning of summer, beginning of autumn and the beginning of winter, represent the beginning of the spring, summer, autumn and winter. Each season has six solar terms. Among them, the six solar terms of spring are the beginning of spring, rain water, awakening of insects, spring equinox, pure brightness and grain rain; the six solar terms of summer are the beginning of summer, grain full, grain in ear, summer solstice, slight heat and great heat; the six solar terms of autumn are the beginning of autumn, limit of heat, white dew, autumn equinox, cold dew and frost's descent; the six solar terms of winter are the beginning of winter, slight snow, great snow, winter solstice, slight cold and great cold.

The 24 solar terms formed by observing the celestial phenomena in ancient China correspond both to the climate change and the agricultural production. For example, the beginning of spring is the time that all things recover and prepare for farming; the rain water is the beginning of tree planting, fruit grafting season; the awakening of insects is the spring ploughing season in some parts of China; the pure brightness is the season of spring ploughing seeds; the grain rain is the beginning of sorghum, corn and other autumn crops season; the grain full is the season when the grains of wheat and other crops begin to bear fruits and the grains are full but not yet ripe; the grain in ear is the beginning of the sowing of awn cereal crops such as late rice, millet, etc; the beginning of autumn refers to crops without hoeing; the white dew is the beginning of the harvest of sorghum and early corn season; the beginning of winter refers to the end of the year's field operations and all crops need to be collected after harvesting so as to prepare for the next year.

As the sun rose in the east and set in the west, the basic time unit *day* was invented by humans. In the Shang dynasty of China, Chinese had developed the concept of time such as dawn, morning, noon, afternoon, dusk, night and later, divided one day into twelve equal periods. Namely, the midnight, cock, daylight, sunrise, break first, corner middle, noon, sun falls, late afternoon, sun entry, twilight and the stillness of night, which corresponds to the operation of the vital energy and blood of a human body, also to the 12 heavenly stems and earthly branches in the ancient Chinese culture, whcih are respectively the Zishi (23:00 to 1:00 of the next day), Choushi (1:00 to 3:00), Yinshi (3:00 to 5:00), Maoshi (5:00 to 7:00), Chenshi (7:00 to 9:00), Yishi (9:00 to 11:00), Wushi (rising tone, 11:00 to 13:00), Weishi (13:00 to 15:00), Shenshi (15:00 to 17:00), Youshi (17:00 to 19:00), Wushi (falling tone, 19:00 to 21:00) and the Haishi (21:00 to 23:00). That is, the Zishi nourishes the gallbladder, the Choushi nourishes the liver, the Yinshi nourishes the lung, the Maoshi appropriates for eliminating toxicant, the Chenshi appropriate for breakfast, the Yishi appropriates for work, the Wushi appropriates to take a nap, the Weishi appropriates to drink plenty of water, the Shenshi appropriates for work, the Youshi appropriate to raise the kidney, the Wushi appropriates to relax and the Haishi appropriates sleep, and so on, matched the "*one union of the heaven and humans*" of the ancient Chinese philosophy.

**(2)Observing clouds for weather.** One often says that who knows which cloud will rains of clouds floating in the sky. That is to say that the predicting of rain is hard. The ancient Chinese observed clouds, summarized a set of methods for weather forecast, i.e., recognizing weather from cloud patterns such as the shape, direction, moving speed, thickness and color changes of clouds. For example, the appearance of cirrus is a precursor to rain and there will be continuous sunny days or frost after a rain or in the winter; the appearance of fort clouds means that the air is unstable and generally, there will be thunderstorms after 8-10 hours or so. And it will be wind gusts if there are low clouds to the east and rain heavily if low clouds to the west; the light rain will continues on the ground if the gray clouds appears in the sky and underground thunderstorm if the cotton cloud appears in the sky, and the cloud like law that it will be rain or not if there is the side bright or overhead light in the sky.

The rain is closely related to clouds with evaporating water on the land and the sea into steam, which rises to a certain height and cools to form small water droplets to form clouds. The small water droplets collide with each other in the cloud, merge into large water droplets bigger to the air can not hold with, and then they will fall from the cloud into rain. Dr.Ouyang tells his daughter that the ancient Chinese used phenomenological methods to conclude the relationship between cloud patterns and the wind and rain by observing, and then predicting the wind and rain. Nowadays, one uses weather satellites to collect cloud data and predict the wind and rain more accurately than the observation of clouds. The ancient prediction method is only a popular spread in Chinese.

**4.4.Phenomenism Knowing Matters.** The observation, collection and analysis of the distribution, state, composition, change and phenomenon of celestial bodies are major methods for humans using the phenomenological methods to recognize the celestial bodies, predict meteorology and climate on the earth, and then serve human practice, which is still applicable today in the highly developed science and technology.

**(1)Celestial body movement.** The human observation of celestial bodies over years shows that the celestial bodies are always in a state of rotation and orbiting a common center with other celestial bodies. The observation of the relative motion of celestial bodies has produced time and distance, and discovered the movement law of celestial bodies. For example, the rotation of the earth produces the day and time, the moon around the earth produces the month, the earth around the sun produces the concept of the year, etc. In particular, the comets are notable for their long, bright tails in the sky and different comets have different cycles. Some of them return once in years, decades or hundreds of years and some return once every thousands of years or never return to the earth at all.

**Figure** 2.21. **Halley's comet**

By observation, one has know the operation law of planets in the solar system. That is, the revolution period of Venus is 224.70 days with its rotation period 243.01 days; the revolution period of Mars is 686.98 days with its the rotation period 1.026 days; the revolution period of Mercury is 87.70 days with a rotation period of 58.65 days; the Jupiter has an orbit period of 4332.71 days and a rotation period of 0.41 days; the Saturn has a revolution period of 10759.5 days with a rotation period of 0.426 days while the Uranus has a revolution period of 30685 days and a rotation period of 0.426 days, and the revolution period of Neptune is 164.8 days and rotates in about 0.75 days.

The observations have also led to the discovery of a series of interesting astronomical phenomena. For example, there are the eclipse phenomena between celestial bodies such as the total solar eclipse, partial solar eclipse and the annular eclipse, the planetary conjunction, double star with the moon, the five star alignment, the Mercury transit, the Venus transit and the meteor cluster, etc. In response, the Chinese book of "*Book of History* (2137 BC)

**Figure** 2.22. **The five-star renju**

recorded various solar eclipses, and the heavenly palace chapter of *Historical Records* wrote various detailed celestial phenomena. Later, the ancient Chinese invented the armillary sphere in the eastern Han dynasty and the angular diameter of the sun and the moon is measured to be half a degree, the ecliptic angle is $24^o$, and the moonlight is proposed to be the albedo of sunlight and other astronomical phenomena.

The phenomenological method used by Kepler to discover his three laws of planetary motion is the deductive result based on Tycho's observation of planetary motion. The process is like this. It is said that the Danish astronomer Tycho observed the orbit of the planets for twenty-one years, discovered many astronomical wonders and recorded a large number of planetary movement data. On the deathbed he passed all of the datum to the German astronomer Kepler. After calculating Tycho's observation data, Kepler proposed that the reason for the inconsistency between the observation result and the previous calculation was that one had always wrongly assumed that the celestial body moved along a circular orbit. He believed that only along an elliptical orbit could the calculation result of Mars' movement be consistent with the observation. And then, Kepler presented his first law of planetary motion, i.e., "*the planets are moving in an elliptical orbit and the sun is located in one of its focus*". He found the rate of planetary motion changes related to the distance from the sun, i.e., the planet perihelion rate is maximum but the largest aphelion rate is minimum. He then summed up his second law of planetary motion, i.e., "*the planets' alignment with the sun sweeps the same area in the same amount of time*". Subsequently, Kepler studied the geometric relationship between the orbits of planets and discovered his third law, namely "*the square of the orbital period of a planet is equal to the cube of the semi-major axis of its orbit*" on planetary motion.

**(2)Center of the universe.** Dr.Ouyang tells his daughter that although the theory constructed by the phenomenological method can be verified by comparison of the observed phenomena, the theory constructed on this basis is likely to be also a local theory, not necessarily the truth of things because it relies on the human perception of observed patterns on objects. For example, it was the case in the early understanding on the center of the universe.

Living on the earth, one feels that all the extraterrestrial bodies are circling the earth. Thereby, Aristotle and others believed that the earth is stationary, the center of the universe and the universe is a limited sphere with the sun, the moon revolving around the earth and objects always fell to the ground. However, Copernicus thought the retrograde or anterograde of planets was an illusion that the earth and other planets around the sun in different periods through calculation, i.e., the earth was the center of the moon, all planets were in an orbit around the sun and the center of the universe should be near the sun, not the earth, which caused the revolution on the human recognition to the universe, namely the human's perception is not necessarily correct.

The modern science with technology have confirmed that the sun is the center of the solar system and the planets in the solar system revolve around the sun. Similarly, the Milky Way has a center, and all the stars in the Milky Way revolve around the center of the Milky Way. So, *does the universe really have a center? Is there a central point in the universe around which all the stars revolve just as the planets in our solar system revolve around the sun?* This question looks like finding a central point in the universe but it is actually similar to asking about the shape of the universe or where the boundaries of the universe are. The crux of such questions is whether the human recognition can look beyond the universe and ask questions such as *is the universe unique or are there parallel universes? Is the universe as we know it the real universe? Are there limits to human recognition and can we perceive the whole universe in which we live? Can humans travel in different universes?* $\cdots$, etc. In fact, if the universe is unique we can't leave the universe, we can't find the boundary of the universe because the universe is infinite without boundary. In other words, the only way we can see the universe in its entirety or its shape is there are multiple universes. Similarly, the only way we can find out if there is a center in the universe is there are multiple universes and we have the ability to travel between universes, i.e., to travel faster than the speed of light. However, this is a huge gap of universe traveling in front of humans.

Dr.Ouyang concludes to his daughter that if the Big Bang hypothesis is true there must be a center of gravity in the universe. That is the point where the Big Bang happened. And the center of gravity of the universe is the center of the universe because that is where everything came into being. Then, according to the expansion law of the Big Bang, the universe is a continuously expanding 3-dimensional sphere $\mathbb{B}^3$ bounded by the front end of the Big Bang, i.e., the sphere $\mathbb{S}^2$. However, we may never get out and discover the center of the universe, may not verify that the boundary of the universe is a sphere or other shapes. Therefore, the Big Bang is only a hypothesis established about the origin of the universe by phenomenological method, not a verifiable hypothesis by humans ourselves.

## §5. Human's Perceptible Limitation on the Universe

There is a fundamental question on the human recognition. That is, whether any objective existence in the universe can be recognized by humans. If we denote by $U$ the set of things that constitute the universe and $U^o(t)$ the set of things that humans already perceived at time $t$. Then, it is clear that $U^o(t)$ is a subset of $U$. The essence of this question is to ask whether there is such a moment $t_\infty$ that make $U^o(t_\infty) = U$ or have $U^o(t) \to U$ with $t \to \infty$. Notice that one's perception of an objective existence $T$ usually depends on the characteristics of $T$. Let's assume that the known characteristics on $T$ are $\mu_1, \mu_2, \cdots, \mu_n$ with unknown characteristics $\nu_k, k \geq 1$ at time $t$. Then, the recognition of $T$ is logically equivalent to

$$T = \left(\bigcup_{i=1}^{n}\{\mu_i\}\right) \bigcup \left(\bigcup_{k\geq 1}\{\nu_k\}\right)$$

and the previous question is transformed into whether each unknown feature $\nu_k$ of $T$ can be recognized by humans. If we assume the human recognition is infinite, the answer seems to be Yes but if the human recognition is finite, the answer must be No because humans are finite and the set of human recognition must so.

Scientifically, different answers to the fundamental question of the human recognitive ability have led to two opposing views. One is the knowability which holds that humans can understand the universe with its laws; Another is the agnosticism which holds that humans can not fully understand the universe with its laws. Then, *which one is right, the knowability or the agnosticism on the question of recognition of things in the universe?* The answer is that both of the views are slightly biased and incomplete! First of all, the correct recognition of an objective existence is by the six sensory organs of human. That is, the recognition is formed by consciousness and then tested by practice, i.e., practice $\to$ recognition $\to$ practice $\to$ recognition again $\to$ cognition again and then, to confirm the correct recognition. Secondly, the practical test is to compare the patterns perceived by humans to the recognition of the objective existence. That is, the phenomenological method on the premise that the patterns perceived by humans are correct or not. And then, *whether a pattern of human perception on objective existence is necessarily correct?* The answer is indeterminacy because humans may never be able to answer the ultimate questions on the universe. For example, the shape of the universe, the center of the universe and so on, and only a few hypotheses can be proposed to explain part of the observed phenomena. Therefore, although the theory is the foundation of scientific construction, it is maybe not completely correct! Then, *is the agnosticism true absolutely?* Of course not! Otherwise, *how can humans survive on the earth adapting to the natural changes?* Certainly, the premise that humans can adapt to natural changes and survive on the earth is the correct understanding of things in the universe. Notice that the correct understanding of the universe for human survive does not refers to the whole universe but to the part of the universe related to the survival of humans.

Notice that the perception by the six sensory organs, i.e, the sight, hearing, smell, touch, taste and consciousness of human is limited. For example, before the invention of microscope, the human eyes could not distinguish between small particles such as bacteria and viruses. So, *will microscopes make it possible to observe microscopic particles smaller than bacteria or viruses such as atoms, protons, neutrons, electrons or quarks?* The answer is certainly Not! Today, although one can observes the behaviour of atoms, protons, neutrons and electrons in science laboratory but still unable to capture the orbit of quark and the particles smaller than quark. That is to say, one can only observes the particles for sensing recognition, not outside the recognition of human. In other words, the recognition of human is partially but not completely knowable about things in the universe.

**5.1.Perceptible Unknown Limitation.** The knowable unknown limitation refers to the fact that the six sensory organs of humans have the ability to perceive an objective existence but cannot perceive its existence due to the limitation of conditions. One day, Huizi happily tells Dr.Ouyang that her teacher had arranged a new deskmate for her who had just transferred from another province of China. They were congenial and had a good time together. Dr.Ouyang asks his daughter:"*Did you know each other before?*" Huizi replies:"*No! She just transferred to our class!*" Dr.Ouyang says:"*It means you didn't know she existed before, did you?*" Huizi replies:"*Yes! We didn't know each other before!*" Dr.Ouyang asks his daughter:"*From this matter, do you think your recognitive ability is limited or unlimited?*" Huizi says:"*It should be limited because I didn't know she existed before!*" Dr.Ouyang confirms his daughter's answer, says that humans perceive everything with six sensory organs, and only those objective things in the universe who can be perceived by humans can be perceived. The knowability holds that the objective existence that can be perceived by the six sensory organs of human must be able to be recognized. But the point of this view is whether humans ability could really perceive the objective things. The answer is not necessarily Yes because one's vision is visible only in a certain range. Beyond the visual range of human, the observation is invisible or its observation of the image is blurred, and maybe even misleading conclusions. For example, *how can we know about distant galaxies that emit electromagnetic waves at frequencies beyond the sensitivity and spatial resolution of radio telescopes or that do not emit electromagnetic waves which can not be observed even with radio telescopes?* In fact, the objective existence that humans can see is a small part of the universe. Although it is only a small part, it is enough for humans to recognize the natural environment that they need to adapt for the survival and development. However, even in the visible part of humans, there are objective existences that cannot be clearly seen or discriminated. Among them, there are typical examples, i.e., the double slit experiment of microscopic particles and the temporary retention of human vision because we have no a scientific explaining on them unless the limitation of human vision until today.

[**Double Slit Experiment**] The double slit experiment is a famous experiment in physics to prove the wave-particle duality of microscopic particles. In this experiment, the microscopic particles emitted by the light source can travel from the light source to the recording screen through holes either $S_1$ or $S_2$ at the same time.

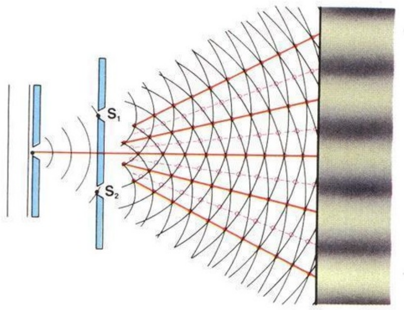



**Figure** 2.23. **Double slit experiment**

The photons are the smallest particles of light and have the properties of elementary particles. So, *do the photons still have wave properties?* If the photon is moving as a wave, it is going through the double slit and falling onto the board as a trail because there is interference between the waves. However, the experiment showed that even when photons were emitted one by one, streaks of light appeared on the screen proving that the fluctuating nature of light. *Why do the photons so?* The researchers set up a high-speed camera to photograph how photons interfere through the holes $S_1$ and $S_2$. At this time, a magic thing happened, i.e., there were only two stripes appearing on the screen, namely the emitting photons one after the other and there was no interference.

*Why is this case and why is it inconsistent with the results of not setting up a high-speed camera?* A few researchers think it is a horrible experiment, and it is tied to the human consciousness, i.e., it is human observation that causes the light to fluctuate. That is, a photon can not be both a particle and a wave, it can only be one or the other at a given moment. It is completely induced by human observation, i.e., if one observes it, it is fluctuating but it collapsed into a particle if no one observing. This explaining is certainly not convincing. Now, the question is, *how can one sure that a light source is emitting photons one after the other and not a blob or a beam of light? How can one sure that the magnetic field of the human body or device does not interfere with the motion of the* *photons?* Moreover, even if it could be determined that photons are emitted one by one, *what would be the internal structure of a photon or microscopic particle, a small sphere like a football?* The wave-particle duality of microscopic particles just shows that one's vision is limited. That is, one can not see the internal structure of microscopic particles clearly but think it is equivalent to an infinitesimally small sphere or particle maybe not right. It is necessary to determine the internal structure of microscopic particles further.

[**Visual Temporary Phenomenon**] Usually, one sits on the high-speed train and looks at the trees flying by out the window. He or she can not distinguish the variety or color of the trees, just a flash. The tree is still the tree. *Why is this so? Why can not see a tree and only a piece of wood flash by?* The reason is because the human eye have visual temporary retention, i.e., the human eye can still make the object image linger on the human retina for about 0.1-0.4 seconds after the disappearance of a certain visual image.

The film production applies the principle of visual retention of human eyes. Certainly, a film rotates at a constant speed of 24 frames per second during projection. Thus, a series of pictures will become a continuous visual action and entertainment human's life because of the temporary vision of human eyes. Essentially, the film is a continuous action to the discrete, relying on the visual retention of human eyes, a kind of illusion. Similarly, when one observes an object moving at high speed, the observation result is not necessarily the truth of the object movement because of the phenomenon of visual retention of human eyes. And they need to take pictures with the help of high-speed camera.

**Figure** 2.24. **Principles of film**

The knowability holds that the objective existence that can be perceived must be recognized by the six sensory organs of human. This view is based on the assumption that the development of the science and technology will eventually enable humans to recognize such objective existences. This assumption has its positive side but the drawback is that if the scientific nature of objective existence is not clear, how can the scientific principles produce a technology to enable humans recognizing this objective existence, which is wrong in logic. For example, if the universe really was created in the Big Bang, the universe boundary is the front end of the Big Bang. So, how is it possible for humans to invent an aircraft faster than light speed on the combustion of residues to fly with us to the edge of the universe and know the composition of matter at the edge of the universe, and how can we fly out of the universe and know the shape of the universe!

**5.2.Unknowable Limitation.** The unknowable limitation of humans recognition on objective existences refers to the limitation of human recognition caused by the fact that objective existences themselves do not possess the characteristics of the six sensory organs, i.e., the eyes, ears, nose, tongue, body and mind of human. Someone do not accept this recognitive limitation with the view that humans are the spirit of all things in the universe, conclude such things belong to the category of "*entirely unreal*" or "*superstition*" and can be not called the objective existence. They believe all objective existences should be recognized by humans, i.e., "*seeing is the reality*". In fact, this is affirming the objective reality equates with that of the human recognition, which hinders the human recognition of things in the universe to a certain extent.

There are 3 kinds of limitations of unknowable objective existences following.

**Class 1.** *The objective existence has no characteristics that humans can perceive.*

Certainly, an objective being is characterized by its natural characteristic and what can be perceived by humans is called objective reality, i.e., humans confirm this objective existence. However, what can not be perceived by humans is rarely discussed because it has no effect on human consciousness and can not be perceived by humans.

Then, *does the objective existence that can not be perceived by humans necessarily not existing in the universe?* The answer is No! Firstly, human sensory organs are consisted of the eyes, ears, nose, tongue, body and the mind. The objective existence that can be perceived by humans is that it can be perceived by human vision, hearing, taste, smell and touch. If a thing itself does not emit and reflect electromagnetic waves, does not emit and reflect sound, also tasteless or does not emit odor, does not contact with humans, humans must not perceive this objective existence and do not know what characteristics should be used to describe it. Certainly, the scientific instruments extend human perception but only to a certain extent; Secondly, it is far from enough to rely only on human vision, hearing, taste, smell and touch to perceive the characteristics of all things. At the same time, one's recognition of everything is a gradual process. For example, the knowledge of electromagnetic fields is an example. It should be noted that before the emergence of electromagnetic theory, few humans believe that it is an objective existence because a human's vision, hearing, taste, smell and touch of electromagnetic field can not perceive. But today, no one doubts that the electromagnetic fields are an objective existence. There are many other examples similar to the electromagnetic field in the universe. Whence, we should not simply or wrongly believe that the objective existence that humans can not perceive must not existing in the universe. Instead, we should break through the perceptual limitations of humans to recognize the objective existence in the universe and then developing in harmony with the nature.

**Class 2** *The objective existence has characteristics of perception but beyond humans.*

In this case, the objective existence has one or more characteristics perceived by the six sensory organs of human but the value of the characteristics is beyond the scope of human. For example, the electromagnetic wavelength visible to the human eye is $400-760nm$. That is, the ultraviolet and the infrared can not seen by the human eye because their wavelengthes are less than $400nm$ or greater than $760nm$, and need to use instruments to detect. However, a few animals such as the shrimp can seen the infrared, ultraviolet, X-ray, etc., only with its naked eye. Similarly, the human ear can hear sounds in the frequency range of $20Hz$ to $20000Hz$ and the most sensitive range is between $1000Hz-3000Hz$. The frequency greater than $20,000Hz$ is ultrasonic, less than $20Hz$ is infrasound, which is inaudible to human ears. Among them, the directivity of ultrasonic is good with a strong ability of reflection, can be applied to industry, agriculture, medicine, military and other fields, but the infrasonic wave has a strong penetrating force with a certain destructive, which interferes with the nervous system, and the dogs have a more 1200 times sensitive sense of smell than humans, which are often used in searching or rescuing humans or important articles that humans can not perceptible such as those search for explosives in security check as well as drug searches in drug prohibition. In fact, the tastes including the sour, sweet, bitter, hot and salty is the human perception but not necessarily any better than that of other animals. For example, the experiments have shown that the fish is more sensitive to sweetness than humans, which is 500 times of the human. Therefore, the comparison of human and animal perception shows that the sensitivity and range of human perception is less than some other animals, at least in sight, hearing, smell and taste.

**Figure** 2.25. **Police dog**

**Class 3.** *The objective existence both has the perceptibility and imperceptibility.*

In this situation, the humans can only make partial recognition of the objective existence but can not grasp its complete law. There are two kinds of situations: ①Some behaviors of the objective existence can not be described by the standards of vision, hearing, smell, taste and touch set by humans and they still do not know how to perceive them; ②The characteristics of the objective existence can be described by the standards of vision, hearing, smell, taste or touch but beyond the range of human perception. Certainly, it indicates that human recognition of this kind of objective existences is partial rather than holistic recognition in either case. For example, it is the case with the double-slit experiment. In fact, one observes the interference of particles through $S_1$ and $S_2$ on the screen in the experiment but there is no interference in the result of high-speed camera. *Why does this experiment feels weird by humans?* The answer is the results defy the common sense on particles. It is not clear that they either interfere or they don't. For examples, whether the human observation or the camera pictures objectively affect the particles or whether the common sense of a microscopic particle as an infinitesimally small sphere is itself a false assumption. This experiment shows that there are a lot of objective laws in the micro world that humans have not yet recognized. Whether is the human observation or the camera, i.e., the brain waves or camera magnetic field interferes with the behavior of micro particles has not been proved yet. Then, *can we conclude that the consciousness of human influences the objective existence?* Clearly, it can not holds in the macroscopic world because the objective existence in the macroscopic world is not subject to the human's will. The double-slit experiment is a strange phenomenon that occurs when observing the microscopic particles. Then, *can the consciousness affect the objective existence in the microscopic world?* Certainly, the microscopic particles have uncertainty in human eyes but whether this uncertainty comes from human observation, i.e., the effect of consciousness or the limitation of human vision has not been confirmed and needs to be explored further, seeing Chapter 11 for details also.

**5.3.Conditional Limitation.** Notice that the matter that can be recognized by humans only accounts for 4.6% of the total matter in the universe. In recognition of things in the universe, humans are applying the recognition of this 4.6% matter, i.e.,, a local recognition to infer all things in the universe. Of course, this method may not be able to get the correct recognition of everything and furthermore, the recognition of the 4.6% matter in the universe is mostly conditional referring to human's relative recognition on the objective existence under conditions with a paradigm that "*if A then B*", where A is the recognitive condition and B is the recognitive conclusion. In fact, the human recognition on things in the universe is mostly under conditions, which are in the earth or near earth space recognition because the speed of the aircraft invented by humans can not reach the speed of light, unable to catch up with the expansion of the universe if the Big Bang really happened and can not traverses the whole universe.

All experimental sciences are examples of conditional recognition in which a conclusion is made under certain conditions. Certainly, it can be confirmed or falsified by others under the same conditions but whether the experimental conclusion holds on other planets is unknown. Then, *is the theoretical science necessarily an absolute knowledge of every-*

*thing?* The answer is also No because all theoretical sciences are the recognitive system constructed by humans according to the logic under conditions. For example, *is the mathematical 1+1=2 absolute recognition?* Of course not because the equality of 1+1=2 is also conditional, i.e., the conclusion holds with a ring that includes the finite ring $3\mathbb{Z} = \{0, 1, 2\}$ as a subfield, maybe not necessarily holds in other mathematical systems. For example, $1 + 1 = 0$ in the finite-number field $\mathbb{Z}_2 = \{0, 1\}$, where 0 instead of 2. Similarly, $1 + 1 = 1$ in the truth operation of logic algebra, etc. The human recognition of things is humans ourselves recognition according to the recognitive ability and conditions, a gradual recognitive process. For example, there have been 5 billion years, 9 billion years, 11.5 billion years, 13.77 billion years and 13.82 billion years for the age of the universe, which are the human recognition under the conception of humans time. Certainly, we maybe not know the exact age of the universe forever but only deducing according to our local recognition on the universe.

Dr.Ouyang asks his daughter, if the universe was really born in the Big Bang of 13.82 billion years ago, *what was the world before the Big Bang?* Huizi thinks for a while and says:"*If the universe came into being after the Big Bang, it was supposed to be a chaotic world before that! You said it in the Pangu split open the chaos and God's creation.*" Dr.Ouyang explains to Huizi:"*Pangu split open the chaos and the God creation are both belong to the mythology. Whether humans can understand the world before the Big Bang is a puzzle. Logically, it should not be because humans appeared after the birth of the universe, which is also the value of the God creation. That is, to explain the doubts in the recognition of humans. By the way, do you know Stephen Hawking?*" Huizi says:"*Is the scientist who wrote the book of 'A Brief History of Time'? My teacher introduced him to us in class. His deed is very inspiring.*" Dr.Ouyang tells Huizi that Hawking once answered a reporter's question in a media interview, saying that you can not ask what the world was before the Big Bang because the time and space were created by the Big Bang, there was no time before the Big Bang and there are meaningless for asking such questions. Dr.Ouyang asks his daughter:"*Do you think the Hawking's answer is reasonable?*" Huizi replies:"*I am puzzled to his answer. If the time has a beginning point then there must be the beginning before. So, I don't think the Hawking's answer is reasonable because the time is a measure of the passage of time by humans. In his view, does it make no sense for us to talk about the pre-human universe, even the chaos?*" Dr.Ouyang confirms his daughter's answer and tells her that the point of the question is whether the universe as we know is the same as the objective universe, i.e., Hawking ignored the limitations of human recognition and concluded the universe of human recognition is the same of the objective universe. In fact, according to the Chinese philosophers' recognition of the universe, this question is easy to answer. That is, to admit the recognitive limitations of humans, told the questioner that the universe we say is the universe that humans can understand and the world before the Big Bang can not be recognized, which will not harms to science in the least. And then, we can objectively draw the conclusion that humans need self-restraint in getting along with things in the universe, so as to achieve the sustainable development with the nature.

## §6. **Comments**

**6.1.** The limitations of human recognition makes one only able to perceive part of things in the universe and unable to answer the ultimate questions on the universe such as those of where we came from and where we will to go. As we known, every human is born of his mother and father but it is not clear what the end result of this inversion is. For this reason, a few humans questioned Darwin's assertion in the biological evolution law that follows from the low to the high and from the simple to the complex, see document [Jin]. Particularly, humans evolved from a kind of extinct ancient apes. Notice that the Darwin's theory of biological evolution asserts the overall law of biological evolution but any one's life is limited, it is impossible to verify it in the lifetime. In contrast, it is not as easy for beginners to accept as the Nuwa creating humans in *Shan Hai Jing* or God creating humans in *Bible*. This is due to the recognitive limitations of humans but does not mean that this kind of recognitions is correct.
**6.2.** The limitation of human recognition comes from the "*six sensory organs*", namely the limitation of recognitive ability of the eyes, ears, nose, tongue, body and the mind of human. Of course, with the help of scientific instruments or equipments, the range of perception of the "*six sensory organs*" can be extended but the limitation remains, which results in the gradual process of human's recognition of things from "*out of non-being*" to the reality in a certain extent, included in the meanings of *Tao Te Ching*, *Diamond Sutra* and *Heart Sutra*, seeing [Luj1], [Luj2] and [Wrt] for details. At the same time, the human recognition of things rises from the perception to a theory, and lots of the theories are phenomenological theory because it is easier to verify and holding than an abstract theory. Certainly, the phenomenological theory is a theory that judges internal causes from phenomena and then, proposes coping strategies. The difficulty lies in how to accurately judge the internal causes from the phenomena such as the traditional Chinese medicine theory, etc. But the phenomenological theory formed in thousands of years has been enough to meet the survival of humans. We can not conclude it is not unscientific.

**6.3.** Many humans believe that the importance of double slit experiment in physics is to confirm the wave-particle duality of microscopic particles such as photons [Liu]. However, the double slit experiment has again confirmed the limitations of human recognition of things at the same time. That is, the human observation behavior can not be carried out independently of microscopic particles in the microscopic world. A fundamental question is whether the human "*six sensory organs*" perception can understand everything in the universe. In this regard, the Chinese ancient philosophy and the Buddhism both hold with a negative attitude, which is reflected in the first chapter of the book *Tao Te Ching*, i.e., "*Tao told is not the eternal Tao, Name named is not the eternal Name*", "*Dependent Origination out of Emptiness*" in Buddhism such as "*the color is the same as the emptiness, the emptiness is the same as the color; the color is empty, the empty is color and it is the same with the sensation, perception, decisions and consciousness*" in "*Heart Sutra*". And science is the local recognition of "*Tao*" or "*Dependent Origination*", see [Fur1]-[Fur3], [Luj1]-[Luj2] and [Wrt] for details, where, the taoist "*name*" is equivalent to the Buddhist "*color*" in terms of recognition of things. A further discussion of this topic leads to that science is a local knowledge or conditional truth of things in the universe by human's six sensory organs in the the discussion of Chapter 12.

*Neither believe nor reject anything, because any other person has rejected or believed it. Heaven has given you a mind for judging truth and error, use it.*

– by Thomas Jefferson, an American president

## Chapter 3

# Combinatorial Notion on the Universe

**************************************************************************

**It is Like a Range if You Look from the Front,**

**But It is Like a Peak if You Look at it Sideways,**

**And Shows Different Features in Different Levels Near or Far;**

**Why Do not Know the True Face of Lushan Mountain,**

**Only Because You Are in the Mountain Yourself.**

— *Writing Xilin Wall* by Sushi, a poet in Song dynasty of China

## §1. **Recognitive Models**

The limitation of human recognition on objective existence directly leads to the conclusion that the whole recognition should be a combined one, which is a combination of local recognitions on a spatial topology. Here, the combination refers to the relationship between the whole and the parts inherited in the objective existence. All parts that constitute the whole are also called the elements. That is, the whole is composed of elements according to certain relations in this existence. Dr.Ouyang tells Huizi that this notion could be learned from a thought model, namely, the famous fable of the blind men with an elephant. By this time, Huizi is already studying in a famous middle school.

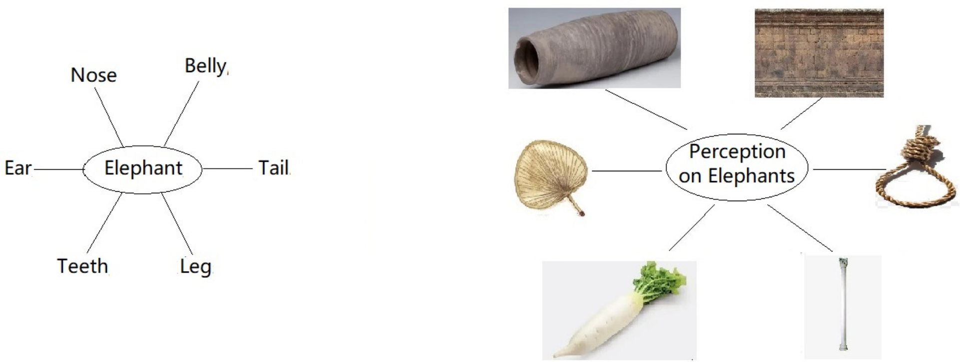
[**Blind Men with an Elephant**] This is a famous fable in Buddhist which contains the philosophy that the human recognition of thing is a combination of local recognitions because the human recognitive process of a thing is like a blind man on the shape of an elephant by feeling. It is said that there were six blind men heard about the wonder of elephants and wanted to know what an elephant looked like long ago. One day, one of their neighbors brought an elephant for perception of the blind men by feeling its body one by one under they repeated entreaties. The first felt the elephant's tooth and said:"*I know, an elephant is like a big, thick and smooth radish!*" . The second felt the elephant's trunk and said to the first: "*You are wrong! An elephant is like a tube!*" And then, the third felt the elephant's ear and said:"*No! I find an elephant is like a big fan!*" But the fourth felt the elephant's belly and said: "*How did you feel the touch? An elephant is clearly like a wall!*" The fifth felt the elephant's leg and said: "*You are all nonsense. An elephant is clearly like a big pillar!*" Finally, the sixth felt the elephant's tail and said: "*Oh, you are all wrong! An elephant is just like a piece of grass rope!*" They all believed that the perception themselves on the shape of elephant was correct and insisting on their opinions, kept quarrelling with each other. At this time, a sophist came forward and said:"*You are all right about the elephant!*" The sophist said continuously:"*The reason why you think the elephant's shape different is because each of you touches the different part of the elephant's*

**Figure** 3.1. **An elephant**

*body. In fact, an elephant has those all characteristics that you are talking about!"*

It should be noted that for a human with normal vision, the shape of an elephant is clear in his eyes as that shown in Figure 3.1 and can not help his jeers on the answer of the six blind men: *how can a living elephant become a pillar, a rope, a radish, a fan, a wall or a pipe in the perception of the blind men and why their perception of elephant is quite beside the point?* Certainly, the greatest regret of a blind man is that he can not discern the shape of things with his eyes compared with a sighted man. Dr.Ouyang asks his daughter: "*Do you think these blind men's answers are funny?*" Huizi replies: "*Is there a bit ridiculous! I've seen elephants in the zoos and I know what an elephant looks like.*" Dr.Ouyang tells his daughter that the philosophical thinking of this fable to a human is why the shape of an elephant becomes a pillar, a rope, a radish, a fan, a wall or a pipe in the perception of the blind and how this relates to the human recognition of things in the universe. In fact, the human recognition of things, especially the objective beings in the universe that have not yet been recognized by humans is the same as the situation of a blind man perceiving the shape of an elephant because if humans have no perception on an objective thing then the perception are local recognition also, just like the blind man feeling the body of an elephant!

And then, *what does the elephant shape really look like, what does the elephant shape look like in the eyes of the sophist?* The fable of the blind men with an elephant explains the limitation of human recognition of things or in other words, the scientific knowledge formed by human recognition is partial or unilateral, and also illustrates the importance of understanding the system or the wholeness of objective existence. Then, *how to correct the bias or incorrect recognition brought by human's local perception and determine the true face of objective existence?* The answer is to combine the local to form a complete recognition of the objective existence.




(a) (b)

**Figure 3.2. The blind men perception of elephant**

Dr.Ouyang tells Huizi that the words *"an elephant has those characteristics that you are talking about"* that the sophist told the blind in the fable of the blind men with an elephant are a combination of the six local perceptions of the blind men on the elephant. According to the different parts perceived by the six blind men, the elephant is decomposed into six perceptual parts, namely the elephant's legs, tail, trunk, ears, belly and teeth as shown in Figure 3.2($a$) and the results or perceptive features are the six patterns known to the blind, namely the pillar, rope, pipe, fan, wall or the radish as shown in Figure 3.2($b$). Thus, the sophist told the blind men that an elephant is

$$\text{An elephant} = \{4 \text{ Pillars}\} \bigcup \{1 \text{ Rope}\} \bigcup \{2 \text{ Branches}\} \bigcup \{ 2 \text{ Fans}\} \bigcup \{ 1 \text{ Wall}\} \bigcup \{ 1 \text{ Radish}\}.$$

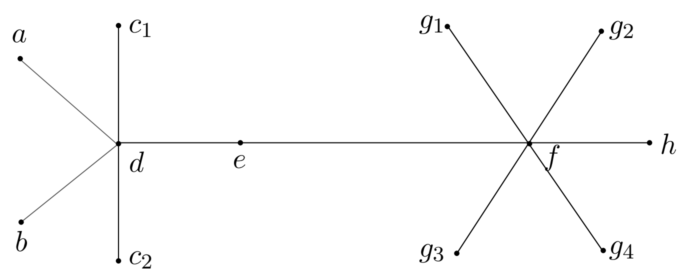



**Figure** 3.3. **Combinatorial structure of an elephant**

Furthermore, if the head, ears, nose, teeth, neck, belly, legs and tail of an elephant are abstracted respectively to a space point and if the different parts of the body are linking up such as the ears, nose and teeth with the head, the legs and tail with the belly, then the points are connected between the two spatial point by a line to show their relationship. In this way, the shape of the elephant can also be represented simply by a point-line diagram as shown in Figure 3.3. Here, the symbols $a = \{\text{teeth}\}$, $b = \{\text{nose}\}$, $c_1, c_2 = \{\text{ears}\}$, $d = \{\text{head}\}$, $e = \{\text{neck}\}$, $f = \{\text{belly}\}$, $g_1, g_2, g_3, g_4 = \{\text{legs}\}$, $h = \{\text{tail}\}$. Such a point-line diagram is called the combinatorial structure of an elephant's shape, the letters $a$, $b$, $c_1$, $c_2$, $d$, $e$, $f$, $g_1$, $g_2$, $g_3$, $g_4$ and $h$ are called the labels on the combinatorial structure. Notice that it is omitted the physical meanings of the points and lines in the diagram shown in Figure 3.3. For example, the point $b$ should corresponds to the trunk of an elephant in Figure 3.3 and so on.

Similarly, there are really objective existences in the universe from the universe to the microscopic particles in size described in Chapter 1 which influence and restrict each other and also with the corresponding combinatorial structures in their composition similar to the point-line structures shown in Figure 3.3. Therefore, it is beneficial for humans to study this kind of point-line combination structures in abstraction, which is noting else but the structure characters with evolution law of topological graphs for understanding things in the universe.

**1.1.Graph.** The human recognition on things is in fact a process that from the concrete to an abstract theory, verifying the theory and then, obtaining the recognition. Dr.Ouyang tells his daughter that there is an intuitive way for the recognition, i.e., to draw the relationship between *elements* and show the whole of thing on a piece of paper similar to Figure 3.3, where an element is represented by a point and a line connects two connected elements. A further abstraction of this idea is the concept of graph or abstract graph. A *graph* is a representation of a binary relation on a set. In general, let $V$ be a finite set of elements, $E \subset V \times V$, namely, the set of pairs of elements with an binary relations $(v, u)$ and $I : E \to V \times V$. Then, the 3-tuple $(V, E, I)$ is defined to be a graph $G$, and the set $V$ is called the vertex set of order $|V|$, $E$ the edge set of size $|E|$ and $I$ the incidence mapping of edges to vertices. Sometimes, these symbols are also denoted as $V(G)$, $E(G)$ and $I(G)$ respectively to emphasize that they are the vertex set, edge set or correspondences of the graph $G$. Usually, an orientation on an edge $(v, u) \in E(G)$ is defined as $v \to u$ and a graph so defined is called a *directed graph*. A graph $G$ is called *undirected* if there exist both of the directions $v \to u$ and $u \to v$ on every edge $(v, u) \in E(G)$. That is bidirectional and the edge $(v, u)$ is also denoted by $vu \in E(G)$ for convention. Notice that the graph defined here allows multiple edges appearing. If a graph $G$ does not allow multiple edges appearing, such a graph $G$ is called a *simple graph*.

A graph $H = (V_1, E_1; I_1)$ is called the *subgraph* of a graph $G = (V, E; I)$ if it is itself a graph and $V_1 \subseteq V$, $E_1 \subseteq E$ and $I_1 : E_1 \to V_1 \times V_1$, denoted by $H \prec G$. Furthermore, if $V_1 = V$, i.e., the graphs $H$, $G$ have the same vertex set, the subgraph $H$ is called a *spanning subgraph* of $G$, in which each vertex of $G$ appears in $H$ also.

Dr.Ouyang tells his daughter that a graph $G$ is characterized by its subgraph structures. A most intuitive way is to imagine that there is an elf walking on vertices and edges of graph $G$, and the structure or characters of $G$ are determined by the traces. Usually, a *trail* refers to a alternating sequence $u_1, e_1, u_2, e_2, \cdots, u_{n-1}, e_n, u_{n+1}$ of vertices and edges of $G$, where the edge $e_i = (u_i, u_{i+1})$, $1 \leq i \leq n$ and the integer $n$ is called the length of the trail. If a trail has $u_1 = u_{n+1}$, that is, back to the origin then it is said to be *closed*. Otherwise, the trail is *open*. Particularly, a trail of $G$ is called *walk* if none of its edges are repeated and a *path* if none of its vertices are repeated. A closed path is also called a *circuit* or *cycle*. For example, the trail $adefhfeda$ and $adefh$ are respectively a closed trail, a path in Figure 3.3.

So, *how can we clearly represent a graph $G$ on a plane?* Dr.Ouyang tells his daughter that for a given graph G, draw a point $p(v)$ on plane $\mathbb{R}^2$ corresponding to the vertex $v$, let $p(u) \neq p(v)$ if $u \neq v$ for vertices $u, v \in V(G)$ and connect points $p(u), p(v)$ by a curve if there is edge $(v, u) \in E(G)$. Such a drawing is called a *diagram* of graph $G$ in the plane $\mathbb{R}^2$ and the mapping $p : G \to \mathbb{R}^2$ is called a projection. For example, a diagram of an undirected graph $G = (V, E; I)$ is shown in Figure 3.4, where $V = \{v_1, v_2, v_3, v_4\}$, $E = \{e_i;\ 1 \leq i \leq 11\}$, $I(e_i) = (v_i, v_i), 1 \leq i \leq 4$; $I(e_5) = (v_1, v_2) = (v_2, v_1)$, $I(e_6) = I(e_7) = (v_2, v_3) = (v_3, v_2)$, $I(e_8) = (v_3, v_4) = (v_4, v_3)$, $I(e_9) = I(e_{10}) = (v_4, v_1) = (v_1, v_4)$, $I(e_{11}) = (v_2, v_4) = (v_4, v_2)$.

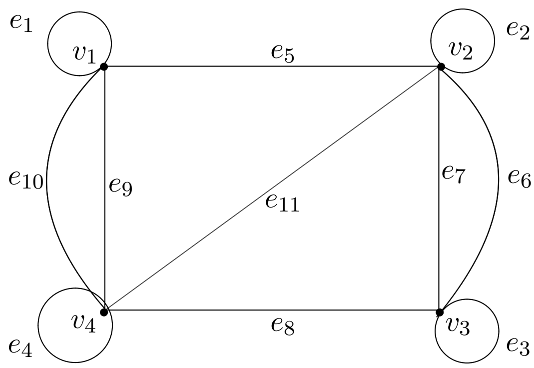



**Figure** 3.4. **Diagram of abstract graph**

Let $G_1 = (V_1, E_1; I_1), G_2 = (V_2, E_2; I_2)$ be two graphs. They are called to be *isomorphic*, denoted $G_1 \cong G_2$ if there exists a $1-1$ mapping $\phi : V_1 \to V_2$, $\phi : E_1 \to E_2$ such that $\phi I_1(e) = I_2 \phi(e)$ for $\forall e \in E_1$, i.e., $\phi(v, u) = (\phi(v), \phi(u))$ for $\forall (v, u) \in E\,(G_1)$. In fact, the difference of isomorphic graphs is their labels, the combinatorial structures are completely the same. For example, two isomorphic graphs are shown in Figure 3.5, where the isomorphic mapping $\phi : E_1 \bigcup V_1 \to E_2 \bigcup V_2$ is defined by $\phi(e_1) = f_2$, $\phi(e_2) = f_3, \phi(e_3) = f_1, \phi(e_4) = f_4$ and $\phi(v_i) = u_i$, $1 \leq I \leq 3$ and it is easy to verify that $\phi I_1(e) = I_2 \phi(e)$ for $\forall e \in E_1$.

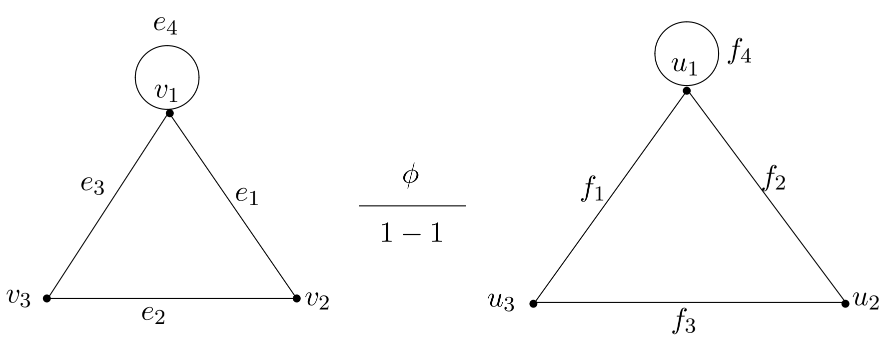



**Figure** 3.5. **An example of isomorphic graphs**

Notice that a graph can only characterizes the combinatorial relations between elements within an objective existence but can not represents the truth of the existence completely, Dr.Ouyang says. In fact, although the human recognition is by a local perception on an objective existence, it is the recognition on the combination of local perceptions, not only its combinatorial relationship. Recalling the implication of the sophist in the fable of the blind man with an elephant, the complete understanding of an objective being requires the restoration of the physical features abstracted by vertices, edges in the graph. This is nothing else but the *labeled graph* $G^L$ such as those shown in Figure 3.3. Then, *what is a labeled graph?* Let $G$ be a graph and $L$ be a set (physical or abstract set). A labeled graph $G^L$ is the graph $G$ with labels in $L$, i.e., there is a mapping $\phi : V(G)\bigcup E(G) \to L$. Usually, the labeling mapping $\phi$ is denoted by $L$ with no ambiguous in the context. For example, the labeled graph shown in Figure 3.6$(a)$ uses positive integers of $\mathbb{Z}^+$ to label vertices, and edges are labeled by the greatest common divisor of its end-vertex labeled numbers, but the labeled graph of Figure 3.6(b) uses cuboid to label the vertices $A$, $B$, $C$ and $D$, and uses rectangle to label the edge $a$, $b$, $d$, $d$ and $e$. Thus, it is a sequential connection of cuboid $A$, $B$, $C$ and $D$ with an intersection rectangle but the intersection of $A$ and $C$ is a straight line. Therefore, the physical meaning of Figure 3.6$(b)$ is a geometrical object.

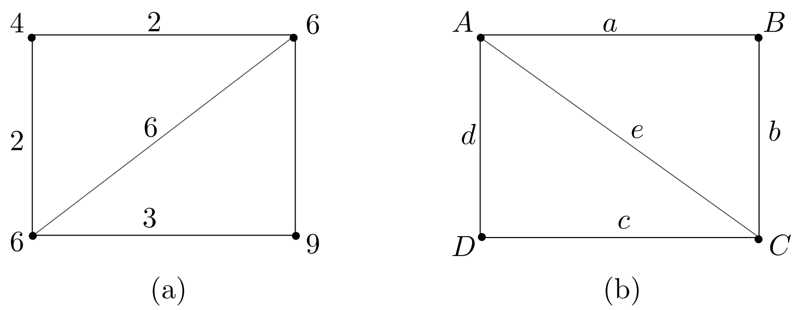



**Figure** 3.6. **Labelled graph**

**1.2.Typical Graphs.** Dr.Ouyang tells his daughter that there are typical graphs correspond to things with the characters of simple and clear structure, and easy to hold on the properties of graph. Some of typical graphs are listed in the following.

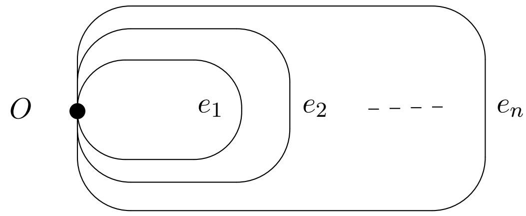



**Figure** 3.7. **Bouquet**

**Bouquet.** A *bouquet* $B_n = (V_b, E_b; I_b)$ is the description of a walk starting from and circulating back to a point $n$ times, i.e., its vertex set $V_b = \{O\}$, edge set $E_b = \{e_1, e_2, \cdots, e_n\}$ and the incidence mapping $I_b(e_i) = (O, O)$ for integers $1 \leq i \leq n$ such as those shown in Figure 3.7.

**Dipole.** A *dipole* $D_{s,l,t} = (V_d, E_d; I_d)$ is similar to the description of the magnetic line of a magnet, whose vertex set $V_d = \{O_1, O_2\}$, edge set $E_d = \{e_1, e_2, \cdots e_s, e_{s+1}, \cdots, e_{s+l}, e_{s+l+1}, \cdots, e_{s+l+t}\}$. If $1 \leq i \leq s$, $I_d(e_i) = (O_1, O_1)$; if $s+1 \leq i \leq s+l$, $I_d(e_i) = (O_1, O_2)$; if $s+l+1 \leq i \leq s+l+t$, $I_d(e_i) = (O_2, O_2)$ such as those shown in Figure 3.8.

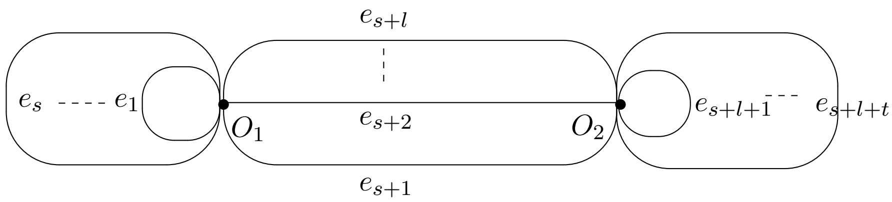



**Figure** 3.8. **A dipole**

**Path.** For any integer $n \geq 1$, a *path* $P_n$ of length $n-1$ is the description of a road passing through $n$ scenic spots with no return repeating. Its vertex set $V(P_n) = \{v_1, v_2, \cdots, v_n\}$, edge set $E(P_n) = \{(v_i, v_{i+1}); 1 \leq i \leq n-1\}$ and $I(v_i, v_{i+1}) = (v_i, v_{i+1})$, $1 \leq i \leq n-1$ as those shown in Figure 3.9.

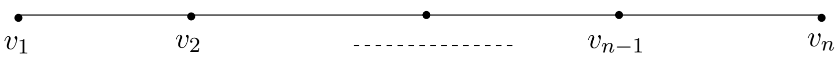



**Figure** 3.9. **Path**

**Cycle.** For any integer $n \geq 2$, a *cycle* $C_n$ is a closed path $P_{n+1}$ with $v_{n+1} = v_1$ such as those shown in Figure 3.10.

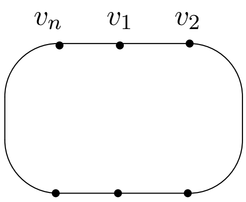
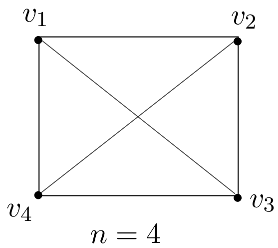



**Figure** 3.10. **Cycle**

**Tree.** A *tree* is a simulation of the trunk and branches of tree in the nature, which is an acyclic graph, namely there is only one path between its two vertices and there are no loops in a tree. For example, the combinatorial structure of an elephant in Figure 3.3 is a tree.

**Complete graph.** A *complete graph* $K_n = (V_c, E_c; I_c)$ is the description of interaction between elements. Its vertex set $V_c = \{v_1, v_2, \cdots, v_n\}$, edge set $E_c = \{e_{ij}, 1 \leq i, j \leq n \not i = j\}$ and $I_c(e_{ij}) = (v_i v_j)$. Two complete graphs in cases of $n = 4$ and $n = 6$ are shown in Figure 3.11.



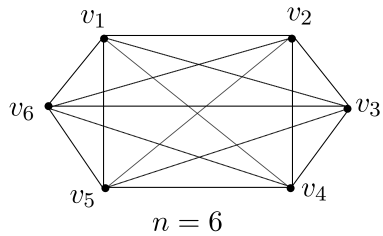




**Figure** 3.11. **Two complete graphs**

**Multi-partite graph.** For any integer $r \geq 1$, an *r-partite graph* $G = (V, E; I)$ means there is a division that could divides the vertex set $V(G)$ into $r$ subsets $V_1, V_2, \cdots, V_r$ holding with $\forall e, e \in V(G)$, $I(e) = (v_i, v_j)$ if $v_i \in V_i$, $v_j \in V_j$ and $i \neq j, 1 \leq i, j \leq r$. That is, all vertices in $V_i$ are not connected for any integer $1 \leq i \leq r$, correspondent to the elements do not affect each other. Generally, a 2-partite graph is called a *bipartite*.

**Regular graph.** For a vertex $v \in V(G)$, the number of edges adjacent with the vertex $v$ is called the *valency* or *degree* of $v$. For example, each vertex has degree 5 in a complete graph $K_6$; each vertex has degree 2 in a circle $C_n$; the vertex $O$ has degree $n$ in the bouquet $B_n$, etc. In general, a graph $G$ is called a *regular graph* if its every vertex has the same degree, which is a description of the case that any element is affected by other elements of the same number. For example, the complete graph $K_n$ is $(n-1)-$regular and the circle $C_n$ is 2-regular. A 3-regular graph with 10 vertices, called the *Petersen graph* is shown in Figure 3.12.

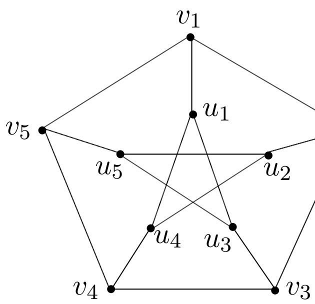



**Figure** 3.12. **Petersen graph**

**1.3.Embedded Graph.** A graph is characterizing the constituting relationship of elements. However, a real object is usually located in a 3-Euclidean space $\mathbb{R}^3$ in the human vision such as the graph of an elephant's shape and there is no crossing between different edges unless their end-points. Therefore, it is necessary to study the spatial embedding problem of graph or labeled graph in order to describe the state and change of objective existence in the universe. Generally, let $G$ be a graph and let $\mathcal{T}$ be a space (Euclidean space or topological space locally homeomorphic to a Euclidean space). A graph $G$ is said to be *embeddable* in $\mathcal{T}$ if there exists a continuous $1-1$ mapping $f : G \to \mathcal{T}$ such that if $p, q \notin V(G)$ then $f(p) \neq f(q)$, i.e., the mapped edges of graph $G$ maybe intersecting only at vertices. Particularly, a graph $G$ is *planar* if $\mathcal{T}$ is the Euclidean plane $\mathbb{R}^2$, where the *face* of a planar graph $G$ refers to the smallest area bounded by a finite number of edges in which any simple closed curve can continuously shrinks to a point on this face without leaving it, i.e., homeomorphic to a disk $\mathbb{B}^2 = \{(x_1, x_2)|x_1^2 + x_2^2 < 1\}$. For example, the Figure 3.13($a$) is an embedding of a graph in the plane with a face $v_1u_1u_2v_2v_1$ and ($b$) is the embedding of a regular hexahedron in the Euclidean space $\mathbb{R}^3$.

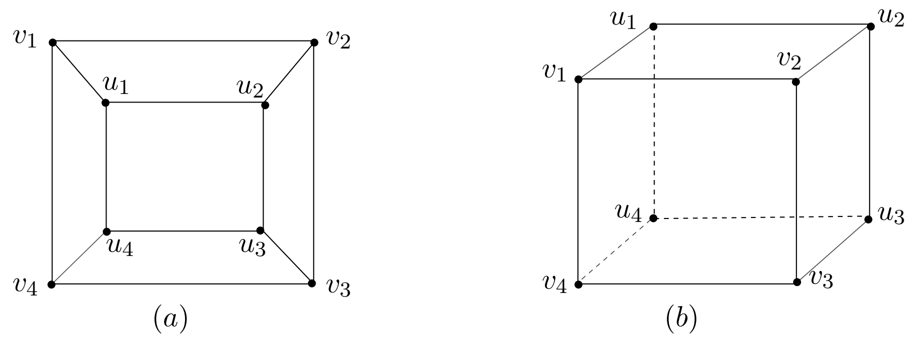



**Figure** 3.13. **Embedded graphs**

Dr.Ouyang tells his daughter that the embeddability of a graph $G$ or labeled graph $G^L$ in the 3-dimensional space $\mathbb{R}^3$ or $\mathbb{R}^n$ of dimension $n \geq 3$ is obviously true because any substance is what the human sees in the 3-dimensional space and the graph $G$ or label graph $G^L$ is only an abstraction of the real substance. However, one does not know whether a graph necessarily corresponds to a real substance and so, needs to verify that a graph $G$ can be really embedded in space $\mathbb{R}^n$ for any integer $n \leq 3$ because if $G$ can be embedded in a space $\mathcal{T}$, then the labeled graph $G^L$ and $G$ is only in the difference of labels, will not affect the embeddability of $G$, i.e., $G^L$ can be also embedded in $\mathcal{T}$. The answer on this question is positive. Furthermore, we can prove that a connected simple graph $G$ can be embedded not only into space but every two vertices of $G$ can be connected by a straight line in $\mathbb{R}^3$, i.e., each embedded edge of $G$ corresponds to a line segment in $\mathbb{R}^3$ following.

Firstly, let $\gamma = (t, t^2, t^3)$ be a spacial curve and we choose $n$ different real numbers points on the curve $\gamma$, i.e.,

$$(t_1, t_1^2, t_1^3), \ (t_2, t_2^2, t_2^3), \ \cdots, \ (t_n, t_n^2, t_n^3)$$

by setting parameters $t_1, t_2, \cdots, t_n$ differently. Applying the conclusion about collinearity of three points in analytic geometry, for integers $i, j, k, l$, $1 \leq i, j, k, l \leq n$ if there exists a line passing through the point $(t_i, t_i^2, t_i^3)$, $(t_j, t_j^2, t_j^3)$ and the point $(t_k, t_k^2, t_k^3)$, then there must be

$$\begin{vmatrix} t_k - t_i & t_j - t_i & t_l - t_k \\ t_k^2 - t_i^2 & t_j^2 - t_i^2 & t_l^2 - t_k^2 \\ t_k^3 - t_i^3 & t_j^3 - t_i^3 & t_l^3 - t_k^3 \end{vmatrix} = 0.$$

By the properties of determinant there must be integers $s, f \in \{k, l, i, j\}$, $s \neq f$ such that $t_s = t_f$, contradicting the assumption. Thus, any three points on the curve $\gamma$ must not be collinear.

Secondly, let the vertex set of graph $G$ be $V(G) = \{v_1, v_2, \cdots, v_n\}$. For any integer $1 \leq i \leq n$, define the mapping $\pi : G \to \mathbb{R}^3$ by $\pi(v_i) = (t_i, t_i^2, t_i^3)$ satisfying the condition that if $(v_i, v_j) \in E(G)$ then let $\pi(v_i, v_j)$ to be the line segment connecting points $(t_i, t_i^2, t_i^3)$ with $(t_j, t_j^2, t_j^3)$. Naturally, $\pi$ embeds $G$ into $\mathbb{R}^3$ and each embedded edge of $G$ is a line segment in $\mathbb{R}^3$.

In this way, the embeddable problem of a graph $G$ into space $\mathbb{R}^n$ leaves only the cases of $n = 1$ and 2 for discussing. Notice that $n = 1$ is an embedding of $G$ on a straight line. No graph can embedded in a line unless the path $P_n$ for an integer $n \geq 1$ or the union of paths. The case of $n = 2$ is the planar embedding which should be satisfied the *Euler's formula* $p - q + r = 2$, where $p, q, r$ are respectively the numbers of vertices, edges and faces of the embedded graph $G$ on plane $\mathbb{R}^2$.

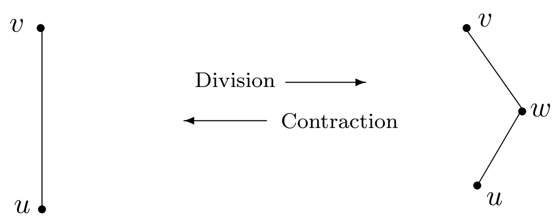



**Figure** 3.14. **Elementary subdivision on edge**

An *elementary subdivision* of a graph $G$ is defined as the result of a serial additions or deletions of vertices and edges on $G$, i.e., adding a new vertex $w$ in an edge $e = vu$, replacing the edge $e = vu$ by edges $vw, wu$ or shrinking a vertex $w$ of degree 2 and replacing edges $vw, wu$ by an edge $e = vu$ such as those shown in Figure 3.14. Correspondingly, a *subdivision* of a graph $G$ refers to a graph obtained by performing a series of elementary subdivisions on $G$. If a graph $H$ is the subdivision of $G$, denoted by $H \cong G$, called the graph $H$ *homeomorphic* to the graph $G$. For the planarity of graphs $G$, it is characterized further by Kuratowski. He proved a graph is plannable if and only if it contains no subgraphs homeomorphic to $K_5$ or $K(3,3)$. For example, $K(2,2,2,2)$ is not a planar graph because it contains the subgraph isomorphic to $K_5$.

Certainly, a graph $G$ or label graph $G^L$ is an abstraction of the constitutive relationship of elements. Then, *how should we recognize a 3-dimensional matter or macroscopic object by an embedded graph?* Generally, if we characterize only the external properties of a macroscopic object, i.e., the interaction of its elements is not considered, we need only to abstract this object to a geometrical point and then, constructs a phenomenological theory for characterizing its behavior. For example, the particle mechanics is such a theory. However, even for the behavior of macroscopic objects, a large number of them can not be described in this way. For example, the influence of gravity on the moving planets, the behavior of individual or group animals and the growth changes of plants can not be simply abstracted to a geometric point model but the interaction of elements that constitutes the whole should be considered. In this case, the recognition of objective existence requires a general study on the mutual influence or relationship of elements. Generally, the embedded label graph is applied to describe the change or state of the research matters or objects with rules, which is the subject of *Continuity Flow Theory* established on embedded label graphs.

## §2. **Combinatorial Structure of Matters**

The human recognition is a combination of recognition based on the local recognition of matters from the big planets to the small elementary particles. Dr.Ouyang reminds his daughter that there is a priori assumption that if we could hold on the characteristics or changes of the smallest constituent elements of matter, we then could hold on the matter and determines which or several constituent elements were responsible for an observed behavior of the matter. This idea, known as the *reductionism* has been around the human recognition for thousands of years and even today, it remains a major method for understanding the universe. *So does this idea of combined recognition apply to the micro world as well?* The answer is positive! Certainly, the recognition of the micro world is affected by the limitation of human recognition to a certain extent and its structure is not clarified. Even so, some combined recognitive models are put forward.

**2.1.Planetary Systems.** The understanding of planetary systems has always been a gradual process because we do not know exactly how many planets there are in the universe today but it is certain that there is a gravitational interaction between the planets and each of them moves in a circular moving, i.e., rotation or revolution.

**Earth-Sun system.** The earth-sun system is a moving system that describes the relative relationship between the earth and the sun locally, also known as the 2-body system, namely the $K_2^L$ labeled graph of the earth's rotation and revolution around the sun. Here, the vertices of $K_2^L$ are labeled by the sun and the earth and its edges are labeled by the gravitational forces between them, i.e., $\mathbf{F}_{es} = G \cdot \frac{m_{\text{earth}} \cdot m_{\text{sun}}}{|\mathbf{r}_{\text{earth}} - \mathbf{r}_{\text{sun}}|^2}$, see Figures 3.15$(a)$ and $(b)$, which constitutes the earth-sun moving system, where $\mathbf{r}_{\text{earth}}, \mathbf{r}_{\text{sun}}$ denotes the position vectors of the earth and the sun in a coordinate system, respectively.

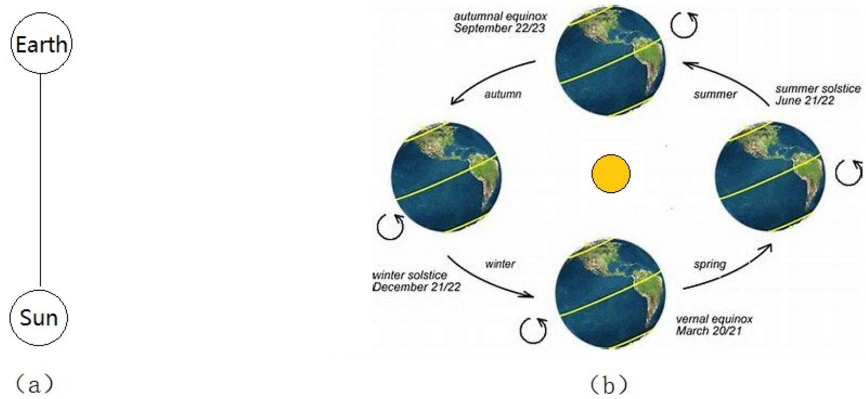



**Figure** 3.15. **The earth-moon system**

**Sun-Earth-Moon system.** The sun-earth-moon system is a moving system that locally depicts the relative relationship between the sun, the moon and the earth, also known as the 3-body system, namely the $K_3^L$ labeled graph of the moon's rotation and its revolution around the earth, the earth's rotation with the moon around the sun, where the vertex labels are respectively the sun, the earth and the moon and its edge labels are the gravitational forces $\mathbf{F}_{es}$, $\mathbf{F}_{em}$ and $\mathbf{F}_{ms}$ between them such as shown in Figure 3.16$(a)$. The moving system constituted by the sun, the earth and the moon is also shown in Figure 3.16$(b)$.

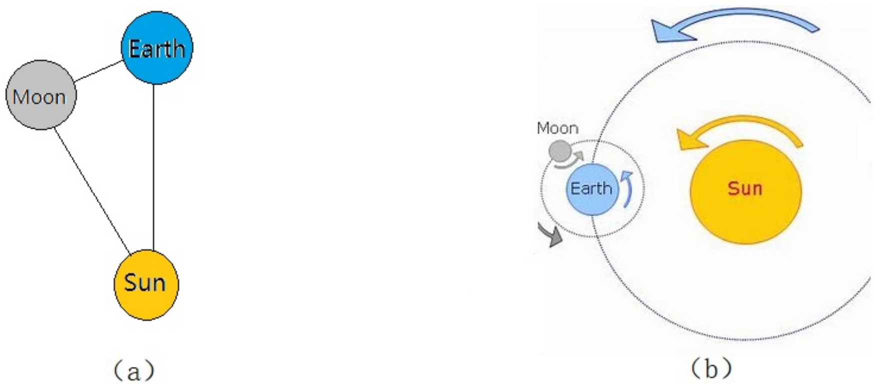
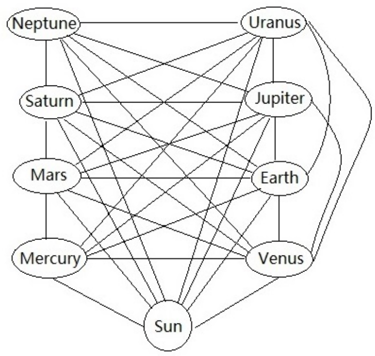



**Figure** 3.16. **The sun-earth-moon system**

At this time, if one ignores the gravitation between the moon and the sun due to its relatively small, the sun-earth-moon system can also be approximated characterized by the labeled graph $P_3^L$.

**Solar planetary system.** The solar planetary system is a moving system that locally describes the relative relationship in the solar system, i.e., the sun with other planets, also known as the 9-body system as shown in Figure 3.17$(a)$. That is a labeled graph $K_9^L$ of the planets of Mercury, Venus, Earth, Mars, Jupiter, Saturn, Uranus, Neptune rotation themselves and also around the Sun, i.e., the vertex labels are respectively the sun and the eight planets and its edge labels are the gravitation $\mathbf{F}_{ij}$ between them, where the indexes $i, j$ are any two planets of the solar system as shown in Figure 3.17$(b)$.

(a)




(b)

**Figure** 3.17. **Solar planetary system**

Generally, a moving system composed of $n$ planets can be characterized by a labeled graph $K_n^L$ on which the vertices are labeled by the planets and the edges are labeled by the gravitational forces $\mathbf{F}_{s_1s_2} = G \cdot \frac{m_{s_1} \cdot m_{s_2}}{|\mathbf{r}_{s_1} - \mathbf{r}_{s_2}|^2}$ between two planets $s_1$ and $s_2$ in this system, where $\mathbf{r}_{s_1}, \mathbf{r}_{s_2}$ are the position vector of the planets $s_1$ and $s_2$ in the coordinate system.

**2.2.Matter Structure.** In the process of recognition of things in the universe, humans always take the notion that "*any thing is composed of elements*" as the basis for recognition of the natural things. The ancient Greek philosophers believed that the origin of the universe was atoms and the void, i.e., any kind of matter is composed of atoms and the void is the place where atoms move. They thought that all atoms are essentially the same, have no "*internal structure*" and the different substances are caused by their differences in the number, shape and spacial arrangement of their constituting atoms, where an atom is believed to be an indivisible particle of matter, approximated to a sphere of diameter infinitesimally small. Of course, limited by the technical conditions, the ancient Greek philosophers' "*no longer be subdivided*" of atoms was similar to molecules in the eyes of human today, which is the objects composed of atoms arranged in a certain bond order in space. For example, the molecular structure of water molecule with its solid state (ice) is shown in Figure 3.18($a$), and the quartz crystal structure with its corresponding molecular structure is shown in Figure 3.18($b$). Clearly, they are all labeled graphs $G^L$.

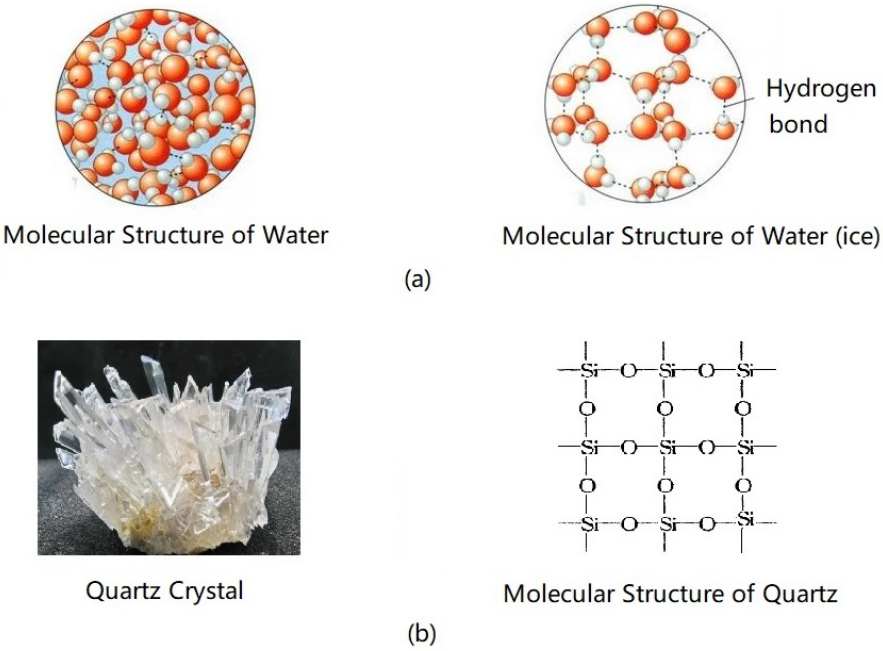



**Figure** 3.18. **The molecular structure of water**

Certainly, the notion that "*atoms can no longer be subdivided*" of the Greek philosophers, i.e., an atom is approximated to a geometrical point is no longer true with the progress of science and technology. In the late 1800s and the early 1930s, some experiments showed that the atom was not a tight entity which can be not equal to a geometrical point. For describing the structure of atoms, Rutherford proposed the *watermelon model* which consists of a nucleus with revolving electrons around it, similar to the seeds of a watermelon. In this model, the nucleus has positive charges which are equal to the negative charges on the electrons, and similarly, Bohr proposed the *Bohr model* by analogy with the planetary motion.

**(1)Bohr atomic model.** Bohr imagined both the nucleus and the electron of atom as an infinitesimally small 3-dimensional sphere and proposed a model of atom by analogy with the motion of the planets. That is, the nucleus is located in the center like the sun in the solar system and the electron is located in different energy levels like the planet moving around the nucleus in a circle. However, it is different from the case of planetary motion in Figure 3.17, i.e., the force between electrons in an atom is nearly 0 repulsive, which is negligible in general. In this way, an atom is a labeled central tree $T^L$ or $K^L(1,n)$ that labels the central vertex by the nucleus, the leaf vertex by the electron and labels the edges by the electromagnetic force $\mathbf{F}_{he_i}, 1 \leq i \leq n$, where $\mathbf{F}_{he_i} = K \cdot \frac{q_1 \cdot q_2}{|\mathbf{r}|^2}$, $K = 8.9880 \times 10^9$,

$q_1 = 1.6021892 \times 10^{-19}$ is the charges of electron, $q_2$ is the charges of the nucleus, $\mathbf{r}$ is the vector from the nucleus to the electron and $n$ is the number of electrons. For example, an atomic model for $n = 8$ is shown in Figure 3.19. Among them, the electron 1 and electron 2, electron 3 and electron 4, electron 5 and electron 6 and electron 7 and electron 8 are both in the same energy level orbit.

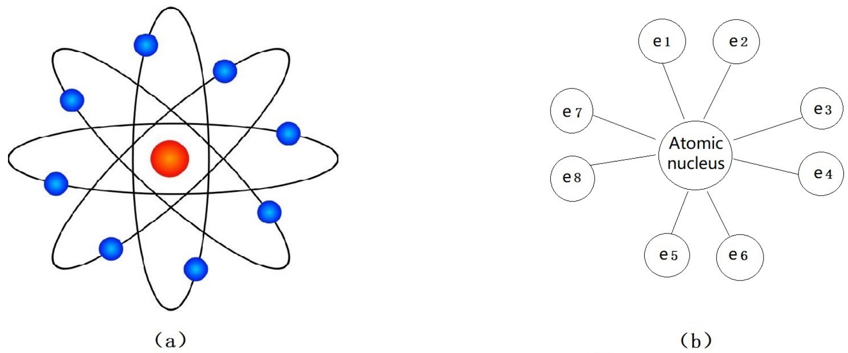



**Figure** 3.19. **Atomic structure**

Many humans believe that the Bohr's atomic model is not correct because humans can not simultaneously know the position and velocity of a microscopic particle such as the electrons. That is, the uncertainty of a particle position is greater than or equal to the Planck's constant $\hbar = 6.6 \times 10^{-34} J/s$ divided by $4\pi$. For this reason, the particle is artificially assumed to be accompanied by volatility, and is characterized by a probability function $\Psi(\mathbf{x}; t)$, namely the probability $|\Psi(\mathbf{x}; t)|^2$ of particle appeared in the position $\mathbf{x} = (x_1, x_2, x_3)$ at time $t$. Notice that a particle must appear somewhere in space $\mathbb{R}^3$, the probability $\Psi(\mathbf{x}; t)$ needs to satisfy the normalization condition, i.e., the $\int_{\mathbb{R}^3} |\Psi(\mathbf{x}; t)|^2 \, d\mathbf{x} = 1$, referred to as a *particle field*. It should be noted also that applying the probability of particle in the spacetime to characterize the behaviour of particle is essentially on the appearing possibility of particle in spacetime. So, we can not conclude that the recognition of microscopic particles is completed.

Then, *whether the electron and other microscopic particles have no a precise position or velocity in spacetime?* Of course Not! It is because of the limitation of human recognition that one is unable to measure the position and velocity of electron or other microscopic particles at the same time, i.e., unknowable! Certainly, the Bohr's model of atom presents the structure of atoms in combination but the particle field is characterized by the probability of a particle historically appearing in the space. Both of them are not the true face but characterized the different sides of particles. In contrast, the Bohr's model of atom is more intuitive to understand, which is similar to depicting the appearance of an elephant by the blind men in Figure 3.3 and the particle field equates a particle to the spacetime $(\mathbf{x}, t)$ endowed with its appearing probability, which is somewhat similar to the way that Chinese use the 64 hexagrams in the *Change Book* to predict the good or bad luck and the trend of an event.

In fact, the structure of atoms is not very clear even today and all of the known on atom are local. In such a case, *which of the two models is more conducive to human recognition, the Bohr model or the particle fields?* Although the mainstream science prefers the particle fields but there are no conclusive conclusion. It is further necessary to find a new recognitive method on the microscopic world. For example, *why do a few humans consider the double slit experiment to be incredible and introduce particle waves?* The answer is because the experimental result violates the common sense of human, i.e., launching a particle is like firing a bullet either through the small holes $S_1$ or $S_2$ but can not pass simultaneously through $S_1$ and $S_2$. However, the result of the double-slit experiment is that the particle can pass simultaneously through $S_1$ and $S_2$ in the eyes of human. *Where did it go wrong?* Certainly, the particle wave hypothesis could explains the incredible in the double-slit experiment and then, instead a microscopic particle by a particle field $\Psi(\mathbf{x}; t)$ with the assumption that the history of a particle can fill the entire space $\mathbb{R}^3$. However, *can the particle defined so still be regarded as the element or atom of matter in the human recognition?* Indeed, there is a little inconsistence among them in logic.

To break this inconsistence it is necessary to reflect on whether the electron and other particles are visually equivalent to an infinitesimally small 3-dimensional sphere, which is the basic hypothesis of electron and other particles. The original intention of the particle field did not deny this hypothesis but since the position and velocity of a microscopic particle are uncertain, the probability or history of such a particle appearing in spacetime is used to describe the particle, and its volatility is endowed to explain the phenomenon that the particle could pass through $S_1$ and $S_2$ holes at the same time. However, this explaining denies the hypothesis of the particle to a certain extent. The particle is no longer an infinitesimally small 3-dimensional sphere, contradicts to its original intention. So, *what does the particle really look like, as the blind men with an elephant have sought?* Certainly, it's not an infinitesimally small 3-dimensional sphere in $\mathbb{R}^3$ used in the particle field characterization. But *what is it?* Notice that humans live in a 3-dimensional space and have no way for knowing the dimensions other than the 3-dimensions of their lived. Thus, *is a point or infinitesimally small sphere of human vision in the 3-dimensional space still a point or infinitesimally small sphere in a 4 dimension or higher space?* The answer is not necessarily. It may or may not be because what confirms of the double-slit experiment is that a particle such as a photon is not an infinitesimally small sphere.

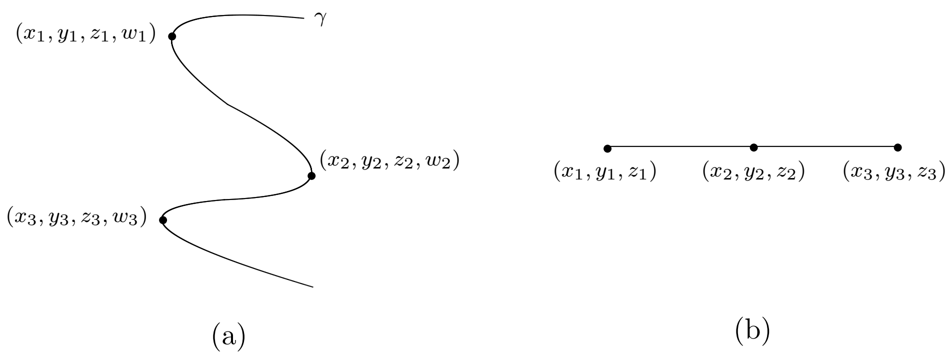



**Figure** 3.20. **Combinatorial structure of particles**

So, *how does mathematically characterize such a kind of geometry?* This is simple to assume that the dimension of a microscopic particle is lager than 3. They have the same dimension in the 3-dimension space of human lived but have different coordinates in the extra dimensions. For example, assuming a particle $P$ simultaneously appears at three locations $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$ and $(x_3, y_3 z_3)$ in 3-dimensional space of human lived, choose a geometrical body $\mathscr{P}$ including $\mathbb{R}^3$ with $\mathscr{P} \bigcap \mathbb{R}^3 = \{(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3 z_3)\}$. Then, the particle $P$ is still an infinitely small sphere in the human vision but it can appears in three locations at the same time. This idea can even be implemented by adding just one more dimension $w$ to the space $\mathbb{R}^3$ of human lived. For example, assume that the particles $P$ is a $\gamma$ curve in the plane $(z, w)$ with $\gamma \bigcap \mathbb{R}^3 = \{(x_1, y_1, z_1), (x_2, y_2, z_2), (x_3, y_3 z_3)\}$ such as those shown in Figure $3.20(a)$, then the particle $P$ is represented by three points of space $\mathbb{R}^3$ in the human vision and its corresponding combination structure is a labeled path $P_3^L$, which makes the curve $\gamma$ passes simultaneously through points $(x_1, y_1, z_1, w_1)$, $(x_2, y_2, z_2, w_2)$ and $(x_3, y_3 z_3, w_3)$ in $\mathbb{R}^4$ as shown in Figure $3.20(b)$, where $w_1, w_2, w_3$ are three coordinates on the extra dimension of curve $\gamma$ out of $\mathbb{R}^3$.

**(2)Atomic structure.** Bohr's model of atom depicts an atom consisting of a nucleus with electrons revolving around it. In general, the diameter of an atom is about $2 \times 10^{-10} m$ with a nucleus of density about $10^{17} kg/m^3$ but diameter about $2 \times 10^{-15} m$, only $10^{-5}$ of the size of the atom. Thus, there exists a large amount of airspace in an atom. In this case, *how do we explore the nucleus structure and discover the composition of matters?*

Dr.Ouyang tells Huizi that although the microscope extends the human vision to a certain extent indeed but it is still unable to observe microscopic matters like atomic nuclei. Generally, a particle beam is used to bombard the nucleus and analyze the trajectory of the particles after the collision to detect the structure of the nucleus. Beginning from

1909, Rutherford used the $\alpha$ particle beam emitted from radium to bombard the nitrogen nucleus, discovered the proton and then, he proposed that a nucleus was composed of protons with positive charges equal to the negative charges of the electrons revolving around the nucleus, i.e., an atom should be electrically neutral. Certainly, this is true for hydrogen atom which has only one proton and an electron. However, according to the ruler of "*Like repel, Opposites attract*" in electric charges, if the nucleus is composed only of protons, the protons can not be combined together. Rutherford proposed that there should be neutral particles without charge distributed between the protons so to make the nucleus a compact substance. In 1932, B.Chadwick discovered a kind of new particles in the nucleus. They are uncharged, heavy nearly as the proton, i.e., the neutron which leads to the theory that the nucleus is composed of protons and neutrons such as those shown in Figure 3.21($a$). Today, it is known that the mass of a proton is $1.672621637 \times 10^{-27} kg$ and that of a neutron is $1.6749286 \times 10^{-27} kg$. In addition, some atoms have different masses but have the same number of electrons or protons and different numbers of neutrons, called their *isotopes*. For example, the hydrogen atom and deuterium atom both contain a proton and an electron but the deuterium nucleus has a neutron and the hydrogen nucleus has no neutron, see Figure 3.21($b$) for details.

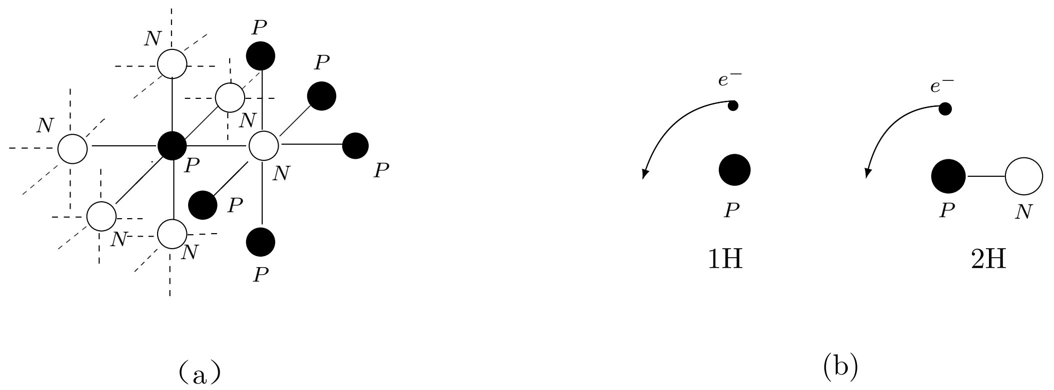



**Figure** 3.21. **Proton-neutron gluing and hydrogen isotopes**

The $\alpha$ particles emitted from the radioactive materials used by Rutherford have relatively low energies, typically less than $7MeV$. In order to generate the high-energy particle beams, researchers have developed high-energy accelerators such as the electron collider and hadron collider to generate particle beams for exploring the structure of matter. For examples, the PEPII, an electron-positron collider built by the United States at the SLAC accelerator center has an energy of $9GeV \times 3.1GeV$ $(1GeV = 10^9 eV)$; the KEKB, an electron-positron collider built by Japan at KEK has an energy of $8GeV \times 3.5Gev$ and the LHC hadron collider built at CERN with a $26,640m$ long underground tunnel near the Swiss and France border has beams of protons with energies as high as $7TeV + 7TeV$ $(1TeV = 10^{12}eV)$. The energy acceleration pipeline and the collision core components of LHC hadron collider are shown in Figure 3.22$(a)$ and $(b)$, respectively.

(a) (b)

**Figure** 3.22. **LHC large hadron collider**

**(3)Quark model.** The quark is the smaller constituent element proposed by researchers in studying the structure of nuclear for describing the behavior of hadrons such as protons, neutrons and mesons. Certainly, they have explored the existence of quarks in experiments only along with the observation of hadrons but have no free quarks been found so far. Notice that the quark is a type of particle proposed as a constituent of protons, neutrons and mesons such as those shown in Figure 3.23, where $(a), (b)$ and $(c)$ show respectively the composition of protons, neutrons and mesons, correspondent to two isomorphic labeled graphs $K_3^L$ and one $K_2^L$ by quarks as shown in Figure 3.24.

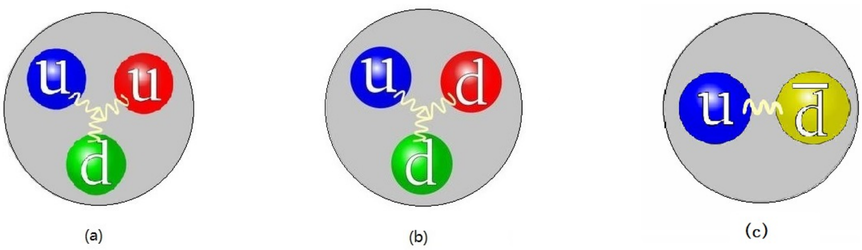
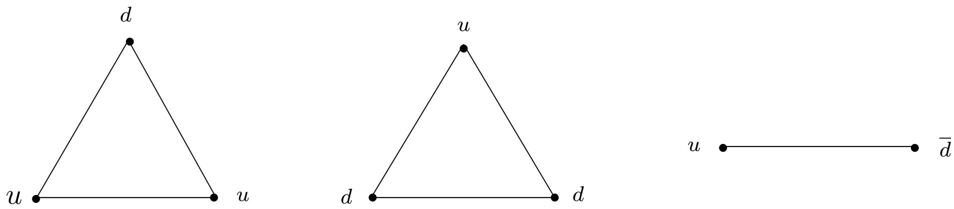



**Figure** 3.23. **Quark models**




(a) (b) (c)

**Figure** 3.24. **Combinatorial structure of hadrons**

Most researchers consider quarks to be elementary particles but some criticize the use of quarks as elementary particles, who argue that quarks are merely hypothetical entities proposed for further explaining the behavior of hadrons.

**2.3.Elementary Particles.** The elementary particles refer to the smallest elements of matter that can be recognized without changing the properties of matter. Until today, the main way that one finds elementary particles is to search them in cosmic rays, atomic nuclei bombarded by high-energy particle beams and the nuclear reactors. Notice that the elementary particle is only a relative concept of the composition of matter because any particle can be further subdivided under certain technical conditions, including the elementary particles.

As everyone known, all matters are made up of atoms, an atom is formed by atomic nuclei and electrons, and the nucleus is made up of protons and neutrons. Gradually, researchers discovered the electron, photon, proton and the neutron, and then also found the positron, neutrino, meson, hyperon, lepton, quarks and bosons, more than 700 kinds of microscopic particles. Certainly, the microscopic particle family will continuously expand with the progress of science and technology. Indeed, it's a beautiful picture of how things combined together.

There are three main moving of microscopic particles, namely the *decay, scattering* and *radiation*, where the decay is a spontaneous decomposition of one particle into other particles. For example, the $\beta$ decay $n \to p + e^- + \overline{\nu}_e$, a neutron spontaneously decomposes into a proton, an electron and an antineutrino; the scattering occurs in the case of collision between particles, energy excitation, etc. usually with an equation $a_1 + a_2 + \cdots + a_n \to b_1 + b_2 + \cdots + b_m$. For example, $p + \gamma \to n + e^+ + \nu_e$, i.e., a photon collides with a proton to produce a neutron, a positron and a neutrino; the radiation refers to the process by which electrons jump from a higher energy orbit to a lower energy orbit with emitting photons such as $e^- \to e^- + \gamma$.

Then, *what can the characteristics of microscopic particles be characterized by?* Similar to the blind man feeling the elephant, one describes the microscopic particles by six characteristics: ①Mass, refers to the static mass. For example, the electron mass is 0.511MeV, the $\mu$ mass is 105.7MeV and the proton mass is 938.3MeV; ②Charge, the charge of a proton or a electron $e = 1.6 \times 10^{-19}C$ is taken as the basic unit for measuring the number of charge carried by a particle, which is generally an integer times of the electron charge and strictly follows the law of charge conservation; ③Lifetime and decay, with the exception of a few stable particles such as $\gamma$, $e^{\pm}$, $p$, $\overline{p}$, $\nu$, etc., most particles are unstable and decay after a period of time. Their decaying process are restricted by the conservation law of matter. The lifetime of a particle is generally measured by the average lifetime of a large number of particles at the static; ④Spin $\mathbf{s}$, like the moving of planets, the microscopic particles have spin with the property of angular momentum. In general, one denotes by ↑ the right-handed particle and ↓ the left-handed particle. The spins of common particles are shown in Table 3.1. According to the spin, particles are divided into two classes, those with half-integer spins are called *Fermions* and those with integer spins are called *Bosons*; ⑤Chirality $\lambda$, namely the projection of an angular momentum of particle spin $\mathbf{s}$ on the direction of $\lambda = \mathbf{k}\cdot\mathbf{s}/|\mathbf{k}|$, referred to as the direction of the left hand or the right hand. For example, the electron is the right hand if $\lambda = +1/2$ and the electron is the left handed if $\lambda = -1/2$; ⑥Magnetic moment $\boldsymbol{\mu}$, refers to measure a charged particle parameters in spin angular momentum $\mathbf{s}$, namely the $\boldsymbol{\mu} = g\cdot\frac{e}{2m}\mathbf{s}$ where $e$ is the charge carried by and $m$ is the mass of the particle. For a point particle field, the factor $g$ satisfies the relation $gs = 1$. For example, $g = 2$ for a spin particle with $s = 1/2$. Notice that the value is generally $gs-1$ measured in the experiment which is called the *anomalous magnetic moment* $g/2$. For example, the anomalous magnetic moment is 1.0011596522209(31) of an electron with spin 1/2.

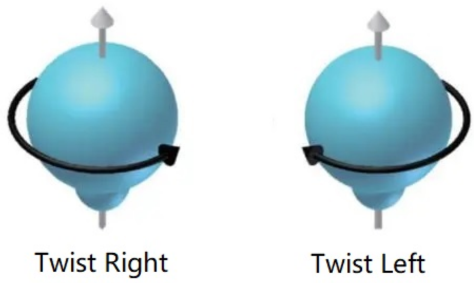



**Figure** 3.25. **Chirality**

| Particle | $e$ | $\mu$ | $p$ | $n$ | $\pi$ | $\gamma$ | $W^{\pm}$ | $Z$ |
|---|---|---|---|---|---|---|---|---|
| Spin | 1/2 | 1/2 | 1/2 | 1/2 | 0 | 1 | 1 | 1 |

**Table** 3.1. **Particle spins**

All researches shows that there are four kinds of interactions in composition and moving of matters in the universe, namely the gravitation, electromagnetic force, strong interaction and the weak interaction. The relationship of the four interactions with the structure of matter is: *gravity* ↔ *universe/galaxy/planet*, *electromagnetic forces* ↔ *molecules/atoms*, *strong force* ↔ *nucleus/hadron/quark*, *weak force* ↔ *leptons*. So, all microscopic particles can be classified to leptons, quarks, mesones, hadrons and Higgs particles according to their properties.

(1)**Lepton.** Leptons participate in the weak but not in the strong interactions including electronic $e^-$, $\mu$, $\tau$ and the neutrino $\nu_e$, $\nu_\mu$, $\nu_\tau$, are divided into three generations. Among them, electronic $e^-$, $\mu$ and $\tau$ bear a charge each but the neutrinos $\nu_e$, $\nu_\mu$ and $\nu_\tau$ without charge. Combined with antiparticles $e^+$, $\mu^+$, $\tau^+$, $\overline{\mu}_e$, $\overline{\nu}_\mu$ and $\overline{\nu}_\tau$, there are 12 kinds of leptons altogether, which are divided into 6 pairs following

$$\{e^-,\nu_e\},\ \{\mu^-,\nu_\mu\},\ \{\tau^-,\nu_\tau\},\ \{e^+,\overline{\nu}_e\},\ \{\mu^+,\overline{\nu}_\mu\},\ \{\tau^+,\overline{\nu}_\tau\}.$$

(2)**Quarks.** Quarks are the building elements of hadrons. There are altogether 6 kinds of quarks, i.e., the upper quarks $u$, lower quarks $d$, strange quarks $s$, charm quarks $c$, bottom quarks $b$ and the top quarks $t$ in three generations. Combined with antiquarks $\overline{u}$, $\overline{d}$, $\overline{s}$, $\overline{c}$, $\overline{b}$, $\overline{t}$, there are 12 kinds of quarks, which are divided into 6 pairs, i.e., $\{u,d\}$, $\{s,c\}$, $\{b,t\}$, $\{\overline{u},\overline{d}\}$, $\{\overline{s},\overline{c}\}$, $\{\overline{b},\overline{t}\}$.

By assumption the quarks $d$, $s$ and $b$ carry a charge of $-1/3$, $u$, $u$ and $t$ carry a charge of $+2/3$. They are both participating the strong and weak interactions. Remarkably, a free quark has not be discovered until today. They appear only in groups of three or pairs bounded together. For this reason, the hypothesis of *quark confinement* has been proposed which claims that the three quarks or the positive and negative quarks can not be broken by humans.

**(3)Meson.** Mesons are particles that transmit interaction forces. According to the interaction between particles in the universe, there are 4 kinds of mesons, namely *gravitational meson* $\rightarrow$ *gravitons* $g$*, electromagnetic meson* $\rightarrow$ *photons* $\gamma$*, strongly acting meson* $\rightarrow$ *gluons* $g^k, 1 \leq k \leq 8$*, weakly acting meson* $\rightarrow$ *intermediate boson* $W^{\pm}$*,* $Z$. Notice that the graviton, photon, gluon and intermediate boson $Z$ carry no charge while $W^+$ and $W^-$ carry a positive or negative charge, respectively.

**(4)Hadrons.** Hadrons are divided into the baryons and the mesons. Notice that a protons or a neutron is made up of three quarks and a meson is made up of a positive and a negative quarks. It is know that the hadrons are the largest family of particles, hundreds of which have been discovered, and they participate in both strong and weak interactions.

**(5)Higgs boson.** The Higgs boson is a boson predicted by the standard model of particles, which is assumed to give mass to the intermediate bosons in weak interaction. In July 2012, the European Center for Nuclear Research (CERN) announced the discovery of Higgs boson after experiments at the large hadron collider (LHC).

We have listed the statistics characters of 61 particles in Table 3.2 that have been discovered until today.

| Particle | Category | Generation | Antiparticle | Chromatin | Number |
|---|---|---|---|---|---|
| lepton | 2 | 3 | pairs | Colorless | 12 |
| quark | 2 | 3 | pairs | 3 | 36 |
| gluon | 1 | 1 | own | 8 | 8 |
| $W^{\pm}$ | 1 | 1 | pairs | Colorless | 2 |
| $Z$ | 1 | 1 | own | Colorless | 1 |
| V | 1 | 1 | own | Colorless | 1 |
| Higgs boson | 1 | 1 | own | Colorless | 1 |

**Table** 3.2. **Statistics table of particles**

## §3. Chemical Combination of Matters

Different from the combination of matters by elementary particles in physics, the chemistry studies the combination structure, properties, generation and change rules of matter at the level of molecules and atoms, including the chemical elements, molecules, ions (groups) and chemical bonds. It is the main method for humans to understand the world of matter and then, create new matters. Dr.Ouyang tells Huizi that the recognitive limitation of humans naturally generates the science of humans can only hold under conditions and can be only applied under the conditions holding with the scientific conclusion for serving the development of humans. Thus, its excess of application will inevitably cause harms to the nature in a certain extent, threaten the survival of humans and other creatures. For example, those wastes undegradable in the nature from chemical production will inevitably affect the environment if they are discharged into the nature because up to now, humans hold only on local recognitions, do not know the operation of universe completely.

**3.1.Molecular Structure.** A molecule is the smallest unit that can exist independently and relatively stable for maintaining the physical and chemical properties of the substance. It is a whole combined by atoms according to certain bond order and spatial arrangement. This kind of bond order, spatial arrangement involving the spatial position of atoms is usually called the *molecular structure*, which largely affects the reactivity, polarity, phase shape, color, magnetic and biological activity of chemical substances. It is related to the type of chemical bonds, including bond length, bond angle and dihedral angle between three adjacent bonds. Any molecule is an embedded labeled graph with the vertex labels of atoms and edge labels of bonds between atoms similar to Figure 3.24. The combination and structure of atoms in molecules are quite complex in space. Some molecules are composed of only one atom, called *monatomic molecules* such as helium, neon, xenon and so on. A monatomic molecule is both an atom and a molecule, and some molecules consist of two atoms. For example, a molecule of oxygen consists of two oxygen atoms and a molecule of carbon monoxide consists of a carbon atom and an oxygen atom. Certainly, most of the molecules involved in human survival are made up of more than two atoms. For example, a benzene molecule contains six carbon atoms and six hydrogen atoms, a biological macromolecule consists of thousands to hundreds of thousands of atoms, etc.

**(1)Periodic law of elements.** A chemical element is a general term for a class of atoms with the same nuclear charge number. In fact, the properties of elements show a periodic change law with the increase of the atomic number of elements, called the *periodic law of elements*. It provides conditions and analysis basis for discovering all elements of substances and compiling the periodic table of elements. Notice that the elements in the periodic table consist of the labeling set of the vertices in the labeled graph of molecules. It is Mendeleev, a chemist of Russia who contributed the greatest to the periodic law of elements. He put forward the first periodic table of elements and stated also the basic points of the periodic law of elements, i.e., ①the elements are arranged in order of atomic weight, and the elements show obvious periodicity in nature; ②the atomic weight determines the characteristics of elements; ③the unknown elements can be predicted according to the periodic law of elements; ④the atomic weight of the element can be modified if one element of the element's family is known.

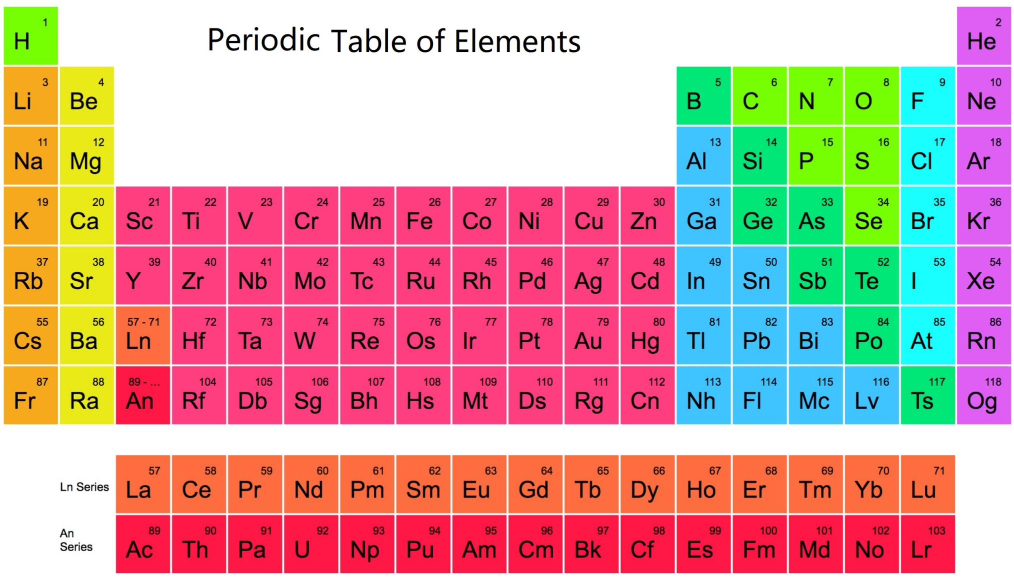



**Figure** 3.26. **Periodic table of elements**

Certainly, the periodic table of elements provides an objective basis for the structure of matter or the combinatorial theory of things in the universe, and also provides a positive clue for predicting the structure and properties on new elements. For example, there were four places where only the atomic weight were marked but with a question mark replaced the symbols of elements while Mendeleev arranged the elements in order of increasing atomic weight because he had concluded that titanium should belong to the same family as silicon, there should be an element between calcium and titanium and then, he left a space. Similarly, the spaces should be left between zinc and arsenic, barium and tantalum. He predicted the atomic weights 45, 68, 70 and properties of the three unknown elements named boron-like, aluminoid and silicone-like by him. Indeed, these three elements were discovered in 1875, 1879 and 1886, respectively.

**(2)Atoms bind.** Atoms or ions are bound together to form the molecules depending on the binding force between them, i.e., the *chemical bonds*, where the *ion* refers to a charged particle formed by the loss or gain of one or more electrons from an atom or atomic group such as those shown in Figure 3.27($a$) and ($b$); the *chemical bond* refers to the force between adjacent atoms or ions in a molecule or crystal including the ionic bond, covalent bond and the metal bond. Among them, the *ionic bonds* are formed by electrostatic interaction through the transfer of electrons between atoms to form positive or negative ions. It should be noted that the essence of ionic bond is the electrostatic interaction, including the electrostatic attraction between anion and cation, and the electrostatic repulsion between electrons and nuclei; the *covalent bond* are formed by the sharing of one or more pairs of electrons between the atoms. If an atom has a few unpaired electrons, it can be paired up with a few electrons with opposite spins to form a bond. According to the number of shared electron pairs, the covalent bonds are divided into the single bond, double bond, triple bond, etc.; the *metal bonds* are mainly found in metals. They are modified covalent bonds formed by electrostatic attraction between free electrons and metal ions arranged in a lattice.

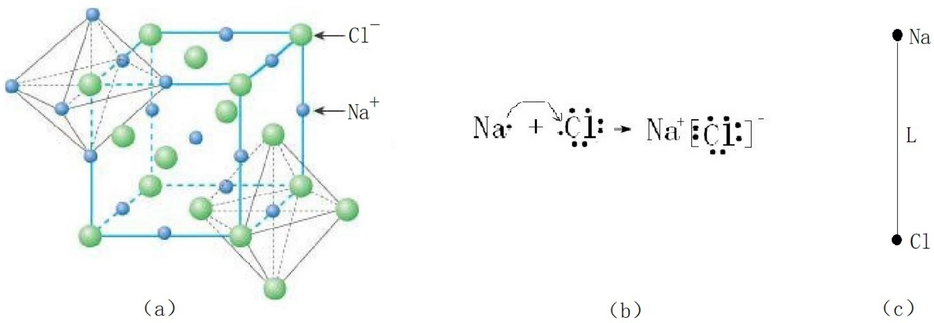



**Figure** 3.27. **Ion structure of NaCl**

Usually, an ion is denoted in the upper right corner of an element symbol, namely $X^{n+}$ or $X^{n-}$ to indicate the number of positive or negative charges, where the notation $X$ denotes the symbol of an element or the chemical formula of a group, the number $n+$ or $n-$ indicates the number of electrons gained or lost and the sign "+" or "-" designates the ion that has a positive or negative charges. For example, a potassium atom loses an electron to become a potassium ion $K^+$ and a chlorine atom gains an electron to become a chloride ion $Cl^-$; the chemical bonds are represented by labels on the edges connecting atoms or ions such as the shown in Figure 3.27($c$).

**(3)Molecular formulas and quantities.** A molecular formula is a chemical formula that expresses the composition and relative molecular mass of a molecule using elemental symbols. A molecular formula represents not only the composition of a substance but also a molecule of a substance with its components, the number of each element in the molecule, the molecular weight and the weight ratio of each component. For example, the molecular formula of oxygen is $O_2$ which means that an oxygen molecule consists of two oxygen atoms; the molecular formula of water is $H_2O$ which means that a water molecule is composed of two hydrogen atoms and one oxygen atom; the molecular formula $C_6H_6$ of the benzene ring indicates that the benzene ring is composed of six carbon atoms and six hydrogen atoms with the corresponding ring structure shown in Figure 3.28.

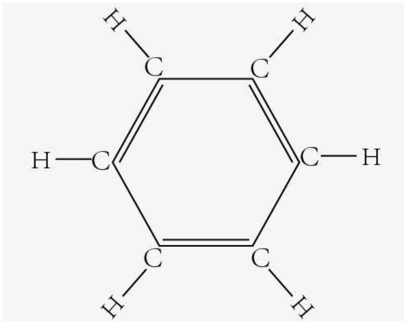


**Figure** 3.28. **Benzene Ring**

After determination, the masses of a hydrogen atom, an oxygen atom and a carbon 12 atom are respectively $1.674\times10^{-27}kg$, $2.657\times10^{-26}kg$ and $1.993\times10^{-26}kg$. Consequently, the mass of an oxygen molecule is $2\times2.657\times10^{-26} = 5.314\times10^{-26}kg$, the mass of a water molecule is $2\times1.674\times10^{-27}kg + 2.657\times10^{-26}kg = 3.0018\times10^{-26}kg$ and the mass of a benzene ring molecule is $6\times1.674\times10^{-27}kg+6\times1.993\times10^{-26}kg = 1.29624\times10^{-25}kg$. As a result, the mass of an atom or molecule is so small that it is troublesome for calculation.

For convenience, one usually describes an atomic mass in terms of its relative mass. Notice that the relative mass of an atom is defined as $1/12$ of the mass of the carbon 12, also called its atomic weight for short. For example, the relative mass of hydrogen is $12\times(1.674\times10^{-27}/1.993\times10^{-26})\approx1$, the relative mass of oxygen is $12\times(2.657\times10^{-26}/1.993\times10^{-26})\approx16$ and the relative mass of a carbon atom is approximately 12. Accordingly, the relatively molecular weight of oxygen $O_2$ is approximately 32, that of water $H_2O$ is approximately 18, the carbon dioxide $CO_2$ is approximately 44 and that of the benzene ring $C_6H_6$ is approximately 78.

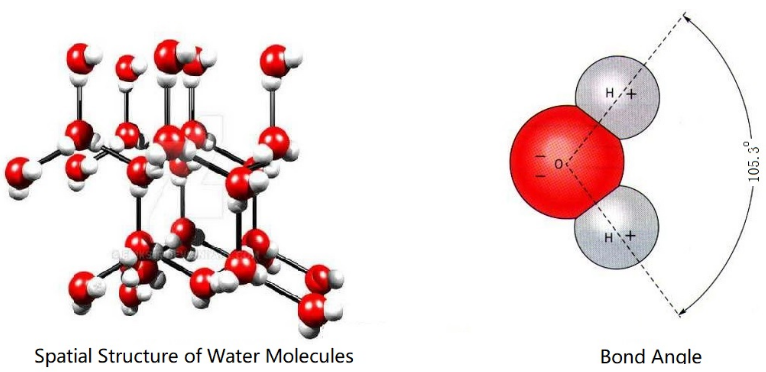



**Figure** 3.29. **Composition of water molecules**

It should be noted that the perceptive limitation of human brings about there are no techniques for directly observing internal structure of microscopic particles including the atoms and molecules, moving or changing, which can only be obtained indirectly by experiment data such as the spectral data combining with mathematical analysis on some simple molecular structures, for instance the water molecular shape with judgment on its spatial structure and prediction. At the same time, each part in a molecule is continuously moving dependent on the temperature. In addition, the size of molecules are also different in the states of solid, liquid, gas, dissolved in solution or adsorbed on a material surface. For example, the bond spacial structure of water molecule in liquid, the bond shape of two hydrogen atoms $H$ and one oxygen atom $O$ in a water molecule $H_2O$ such as the bond angle $105.3^o$ are shown in Figure 3.29.

**3.2.Chemical Reaction.** The chemical reaction is a process in which molecules break into atoms under certain conditions and atoms recombine in space to form new molecules and produce new substances. Its essence is the process of chemical bond breaking and formation again. Generally, according to the types of chemical reactants and products, the chemical reactions can be classified into 4 categories, namely the combinatorial reactions, decomposition reactions, displacement reactions and metathesis reactions. Among them, the *combinatorial reaction* is the reaction in which two or more substances form a new substance with the formula $A + B \to AB$. For example, if hydrogen is burned to form water, the reaction equation is $2H_2 + O_2 \to 2H_2O$; the *decomposition reaction* is the decomposition $AB \to A + B$ of a compound $AB$ into two or more simpler substances or compounds $A, B$ under specific conditions. For example, the decomposition reaction $2H_2O \to 2H_2 + O_2$ of water under direct current; the *displacement reaction* is the reaction in which one substance and one compound give rise to another substance and another compound using the formula $A + BC \to B + AC$. For example, the displacement reaction of iron in dilute hydrochloric acid is $Fe + 2HCl \to FeCl_2 + H_2$: the *metathesis reaction* is usually with the chemical formula $AB + CD \to AD + CB$ in which the two compounds $AB, CD$ exchange components to form two new compounds $AD, CB$. For example, the acid-base neutralization reaction $HCl + NaOH \to NaCl + H_2O$ is such a metathesis reaction.

**(1)Conservation law of matters.** Generally, the eternal law of matters is the conservation law of matters, refers to the physical nature of the infinite, eternal, absolute of matters. That is, no matter what happens of the matter in eyes of human, its composition elements would not disappear, also would not produce out of new elements but only by one combination of elements into another kind of combination, and conversion forms from one to another. The performance of the eternal law of matters in chemical reactions is the conservation principle that the types of atoms of reactants participating in a reaction do not change, the number of atoms does not increase or decrease and the mass of atoms does not change before and after the reaction. From this, one can formulates a chemical reaction equation by $A_1 + A_2 + \cdots + A_k = B_1 + B_2 + \cdots + B_s$, where, $A_i,\ 1 \leq i \leq k$ is the reactants involved in the reaction and $B_j,\ 1 \leq j \leq s$ is the output of the reaction which needs to be satisfied three equations following

$$\sum_{i=1}^{k} m\left(A_i\right) = \sum_{j=1}^{s} m\left(B_j\right), \quad \sum_{i=1}^{k} n\left(A_i\right) = \sum_{j=1}^{s} n\left(B_j\right), \quad \sum_{i=1}^{k} o\left(A_i\right) = \sum_{j=1}^{s} o\left(B_j\right),$$

where, $m(A), n(A)$ and $o(A)$ represent the mass, number of atoms and the class of substance $A$, respectively. For example, there is one iron $Fe$ atom, two hydrogen atoms $H$ and two chlorine atoms $Cl$ before the reaction, and one iron $Fe$ atom, two hydrogen atoms $H$ and two chlorine atoms $Cl$ remain after the reaction in the substitution reaction $Fe + 2HCl = FeCl_2 + H_2$.

**(2)Material innovation.** The improvement of human living conditions is related to the recognition of chemical reactions. It is exactly the recognition with application of chemical combination of atoms (ions) to create substances for improving the living conditions. Here, a few examples of chemical combination for improving the living conditions of humans with chemical equations are discussed.

**1) Fire.** The fire is the origin, also the symbol of human civilization which is one of the most important discoveries of chemical reactions in the early time, known as the *burning reaction*, i.e., the chemical reaction between the twigs or hay and the oxygen $O_2$ in the air to make a fire, barbecue food or keep warm. Here, the main composition of the twigs or hay is the carbon $C$, and the carbon dioxide $CO_2$ or the carbon monoxide $CO$ is generated after a sufficient or insufficient combustion. Their reaction equations are $C + O_2 = CO_2$ and $2C + O_2 = 2CO$, respectively.

At the same time, the fire is also a kind of combustion energy which can provides power source for human's social practice. For example, the chemical equations of methane and alcohol burning in air are respectively $CH_4 + 2O_2 = CO_2 + 2H_2O$, $C_2H_6O + 3O_2 = 2CO_2 + 3H_2O$, and the gasoline burning in air is $2C_8H_8 + 25O_2 = 16CO_2 + 18H_2O$, which provide power for engines and extend the travelling ranges of human in land, sea and airspace.

**2)Material.** Generally, the materials divided into the metal, alloy and the non-metallic materials are essentially the basic and important strategic materials for development of the national economy, daily life, industry and the science with technology such as the farm tools, machinery parts, home supplies, aircraft, missiles, satellites, television, communication, radar and computer, all of them can not leave the metal. Certainly, the metals are extracted from collected ore by a series of recombination of atoms, known as the chemical reactions. For example, the coke is used as the fuel in the iron making including the programming of iron making and slag removal with a complete process: ①Coke combustion $C+O_2 = CO_2$ increases the furnace temperature; ②Reduction of the coke combustion to the carbon monoxide $CO_2+C = 2CO$; ③Iron making $Fe_2O_3+3CO = 2Fe+3CO_2$; ④Removing the miscellaneous slag $CaCO_3 = CaO + CO_2$, $CaO + SiO_2 = CaSiO_3$, etc.

The synthetic materials such as the plastics, rubber and fibers provide convenience for producing and living but at the same time, they also have a certain impact on the survival environment of humans. Among them, the main ingredient of *plastic* is resin which is an indispensable necessity in the producing and living of humans such as those of the plastic pipe, plate, plastic film, plastic basin, chair, shoes, raincoat, bottle, etc.; the *synthetic rubber* is a kind of linear polymer with special elasticity synthesized by low molecular substances, which is widely used in tire and shoe industry, etc.; the *synthetic fibre* is a linear polymer made by polymerization of some low molecular substances such as the nylon, polyester and artificial wool (polyacrylonitrile) with the characteristics that of wear resistance, corrosion resistance and no shrinkage. As an example, the common product of PVC plastic with its reaction equation in daily life is shown in Figure 3.30.

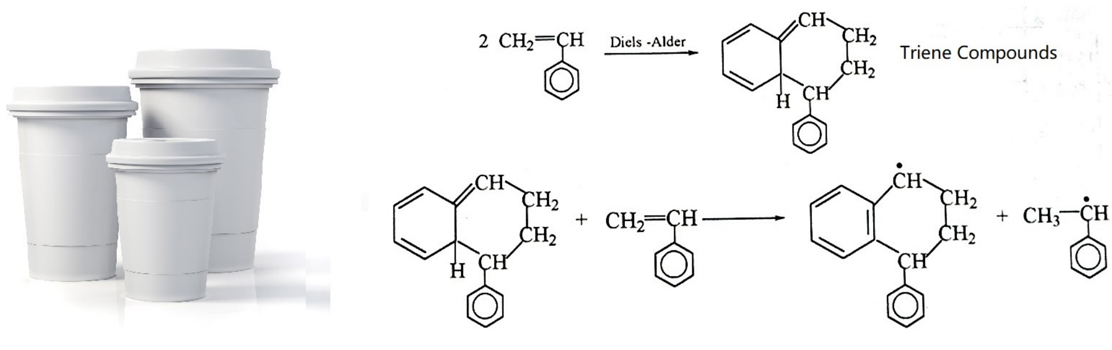



**Figure** 3.30. **PVC plastic**

The construction materials including the steel, wood, cement, brick, tile, sand, stone, ceramics, glass, plastic with composite materials such as those of finished or semi-finished products, etc., have greatly improved the living condition of humans. Among these, the chemical reaction of the cement with the sand and water forming the concrete in a certain strength constitutes an indispensable component in a construction project. For example, the reaction equations of the main clinker with water after the Portland cement mixed with the sand and water are respectively: $3CaO \cdot SiO_2 + 6H_2O = 3CaO \cdot 2SiO_2 \cdot 3H_2O + 3Ca(OH)_2$, $2CaO{\cdot}SiO_2{+}4H_2O = 3CaO{\cdot}2SiO_2{\cdot}3H_2O{+}Ca(OH)_2$, $3CaO{\cdot}Al_2O_3{+}6H_2O = 3CaO \cdot Al_2O_3 \cdot 6H_2O$ and $4CaO \cdot Al_2O_3 \cdot Fe_2O_3 + 7H_2O = 3CaO \cdot Al_2O_3 \cdot 6H_2O + CaO \cdot Fe_2O_3 \cdot H_2O$. And finally, the concrete is produced which is one of the main components of a building.

**3)Drug.** All drugs are chemicals that affect the physiological, biochemical and pathological processes of the body of human and are used to prevent, diagnose, treat diseases and family planning. Usually, the chemical drugs are divided into phytochemicals, chemical synthetic drugs, antibiotics, semi-synthetic antibiotics, biochemical drugs, etc. The production process of chemical drugs consists of bulk drugs and preparations. Certainly, different drugs have different production characteristics and the general process of PHA production is shown in Figure 3.31. It is also necessary to treat waste water, waste gas and other wastes harmlessly to protect the ecological environment in the pharmaceutical process.

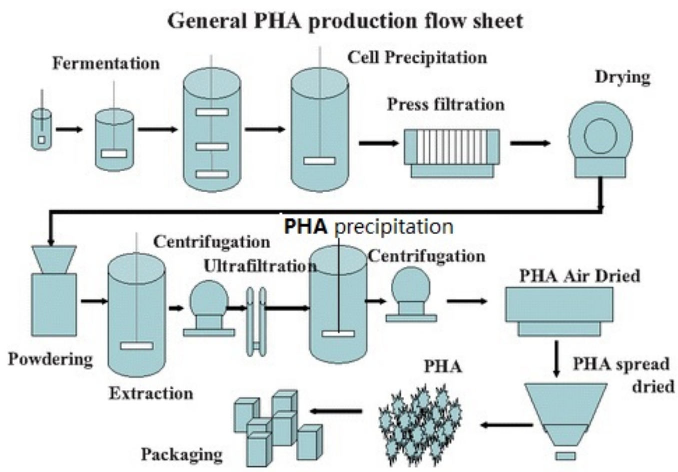



**Figure** 3.31. **PHA production**

The *fertilizer* including nitrogen fertilizer, phosphorus fertilizer, potassium fertilizer, micro fertilizer and compound fertilizer has characteristics such as the simple composition, high nutrient content, fast fertilizer effect, strong fertilizer strength, improving the fertility of soil and then, increasing the crop yield; the *pesticide* is one of the most important prevention and control measures to protect crops, control diseases, insects, grass and mice that occur in the growth of crops and reduce the production. Certainly, it is an auxiliary product in agricultural production. However, the use of chemical fertilizers and pesticides will lead to the increase of heavy metals and toxic elements such as the $Zn$, $Cu$, $Co$ and $Cr$ in soil to some extent, which will cause the soil oxidation loss, the increased acidification and the residue of pesticides on the crops harms to human health and also with other side effects. So, it is necessary to improve further the components, standardize with limiting the applying scope of fertilizer and pesticides.

## §4. **Biological Combination of Livings**

Generally, the livings refer to the living bodies through chemical reactions generated with the ability to survive and reproduce under the natural conditions, relative to non-living organisms. They are all responding to the external stimuli, promote each other with the external environment and have the characteristics of heredity and variation, where the *heredity* refers to the similarity of traits between the parents with their children or offspring. That is, the characters can be passed from the parents to offspring. Dr.Ouyang tells Huizi that the skin color, height, fat and thin, eyes, nose, jaw and ear shape of the parents are generally passed to their offspring, which implies that the offsprings are similar to their parents in these characteristics. Therefore, the biological combination is more about studying biological elements on the basis of microscopic particles or chemical combination, namely the *cell combination* and genetic genes, i.e., the *gene combination* so as to analyze, judge and predict biological characters.

**4.1.Biological Macrosystem.** The hierarchy of biological structures from small to large is the cell, tissue, organ, system, individual, population, biome, ecosystem and biosphere. Among them, the cell is the smallest element of biological composition and those of visible to human eyes such as the tissue, organs in anatomy and the population, biome, ecosystem and biosphere are all called the *macroscopic system* of biology. Notice that the premise of harmonious development of humans with the nature is the harmonious coexistence of humans with other species. To this end, it is necessary to understand the structure of organisms from the macro to the micro including the plants and animals, protect the ecological environment and then, maintain the home of humans.

**(1)Biome.** A *biome* is the combination of biological populations which refers to the collection of various organisms such as the plants, animals and microorganisms living in a region. It is an organic combined whole formed by the species relations of mutual benefit, symbiosis, competition, parasitism and predation among different organisms. For example, all the plants in a certain forest provide shelter and food for the animals that inhabit it, some animals feed on other animals and a large number of microorganisms live in the soil by breaking down the remains of fallen leaves, etc., i.e., all of these livings make up the a biome of the area. Notice that a biome can be formed from a bare ground, namely an area where no plants have ever grown due to the changes in topography, climate, biological action or human influence, etc., but

**Figure** 3.32. **Biome**

can also being out of non-being or some existing biomes. Generally, after the formation of a biome there will be a process of growth, change and development, including the adaptation to the season, climate change and the population renewal. The structure of biome can be divided into three types, i.e., the spatial structure, temporal composition and the species structure. The *spatial structure* refers to the plant stratification, i.e. the stratification of the trunk, branches and leaves above ground and the roots formed by different plants living together with their vegetative organs arranged at different heights or depths underground; the *temporal pairing* refers to the complementarity of biome organisms in time. For example, some plants grow in spring or fall but some grow in the summer, some animals are active during the day, some like at dusk but some are used to night activities, etc.; the *species structure* is the composition of relatively fixed populations. Among them, each species occupies its own living environment and plays an important role in improving the environmental conditions and utilizing environmental resources of the biome.

**(2)Population.** A *population* is a combination of individual organisms consisting of the same species occupying a certain space. It is the basic unit of biological evolution, which maintains the population through the reproduction and transmits population genes to the offspring. The density or quantity of a certain organism in a region is generally determined by the reproductive ability of the organism, the route of seed transmission, the resources and the conditions permitted by the living space of the organism in that region, and the relative stability is maintained through the self-regulation within the population.

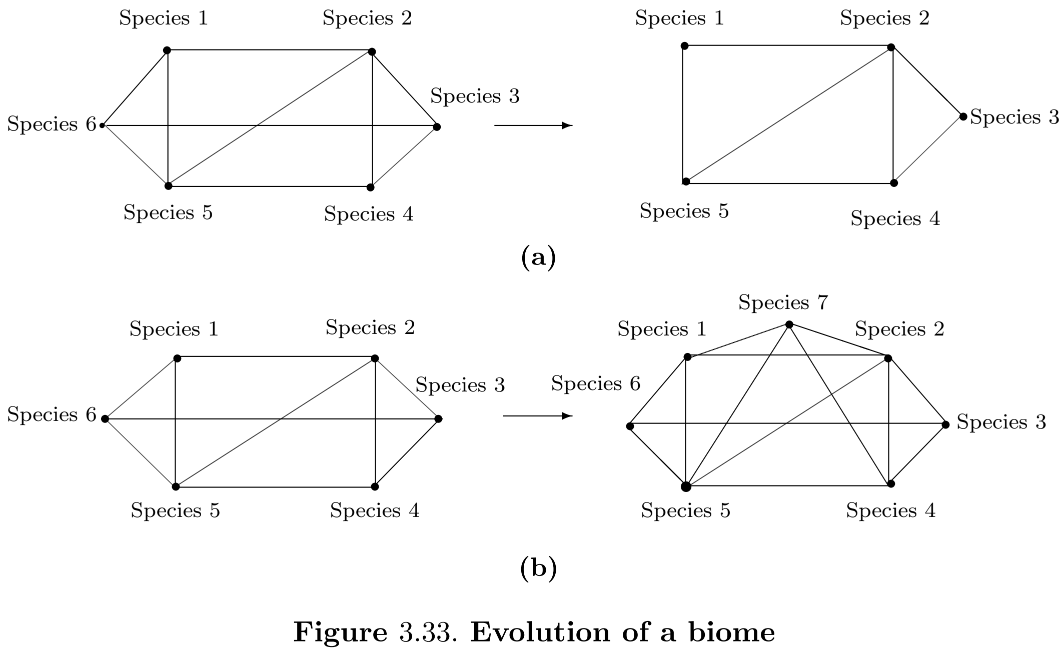



**Figure** 3.33. **Evolution of a biome**

Notice that the relationship of populations with the biome can be represented by a labeled graph. That is, the vertices represent the populations in the biome and an edge between vertices represents two interdependent and mutually influencing populations. A biome then is a labeled graph $G^L$. At the same time, the development and change process of the biome can be described by the dynamic behaviors of the labeled graph $G^L$, which is a dynamical process in the natural evolution of the biome. For example, the appearance or the extinction of a species corresponds to the addition or deletion of a vertex corresponding to that species on the labeled graph such as those shown in Figure 3.33. Clearly, the labeled graphs $G^L$ in Figure 3.33$(a)$ and $(b)$ characterize the evolution process of the biome, respectively. For example, $(a)$ has initially 6 species in the biome but the evolution of the biome results in the extinction of the species 6, $(b)$ is the case that a new species 7 appears and its relationship with other organisms in the biome along with time. Certainly, both of them are a dynamical process of biological evolution.

Furthermore, the characteristics of the species can be labeled on each vertex such as the multivariate tuple $\mathbf{v}$ composed of the number, height, population area with other quantitative indicators, and the changes of the biome can be used to describe the changes of $\mathbf{v}$ on each vertex of $G^L$, etc. Similarly, the relationship between populations and individuals can be characterized also by the labeled graph $G^L$. In this case, the vertex set is the set of constituting individuals of the population and the edge set is the interdependent relations between in individuals such as the relations of mate, parents with offsprings, etc.

**(3)Individual.** The biological individuals including animals, plants and microorganisms are the combined products of various tissues, organs and systems within living organisms. Among them, the appearance of plants or animals is a tree structure, relatively simple. Observation for years as well as the dissection of biological individual showed that the growth of individual plants is usually composed of the seed development to form the seedlings, seedlings after a period of time to the growth of a plant with root, stem and leaf of vegetative organs of plants, plant growth and begin to flower bud formation at a certain time, then blossom and bear fruit completing the plant breeding process, where the formation of flower bud is the beginning of the reproductive growth of the plant and the blossom, fruit show that it has the function of the reproduction of next generation.

Notice that there are two ways delivered the plant nutrients. One is from the bottom up, i.e., by the plant roots absorbing water and nutrients for supplying the branches and leaves from the bottom up; another is from the top down, i.e., through the photosynthesis of plant leaves to produce carbohydrates supplying the branches, roots and other nutrients from the top down. At the same time, the oxygen produced by the process is released. For example, the nutrient delivery of a tree is done by the core and bark of the tree. That is, the bark transports water and nutrients from the bottom up and the core of the tree transports products by the photosynthesis of green leaves, a process that can be depicted in a labeled tree such as those shown in Figure 3.34, where $(a)$ is the distribution of roots, trunks, branches and leaves of a tree in the ground and underground, and $(b)$ is its nutrient delivery system, which is nothing else but a labeled tree $T^L$.

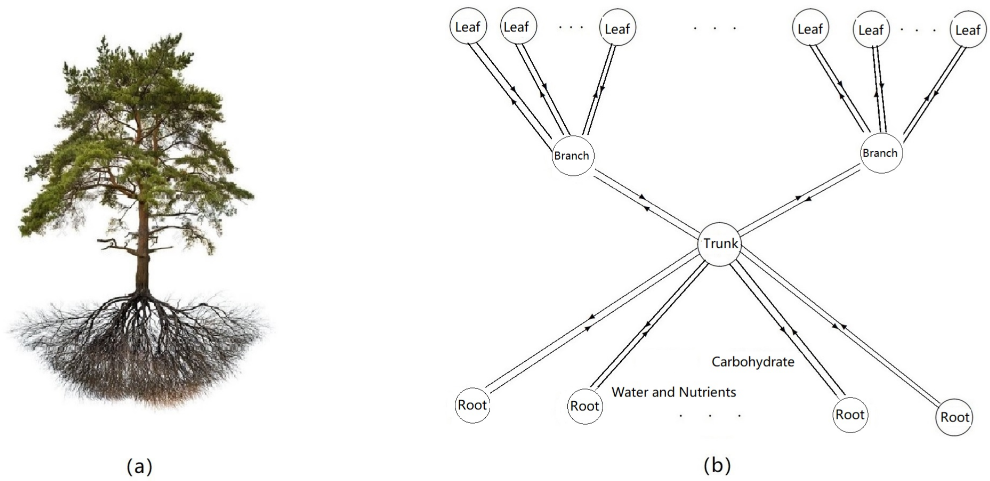



**Figure** 3.34. **Plant nutrient absorbing and recycling**

The appearance of an animal as the elephant shown in Figure 3.3 is a labeled tree. However, as a living creature walking on the earth, animals need to have the function of moving. Of course, its structure is not simple so.

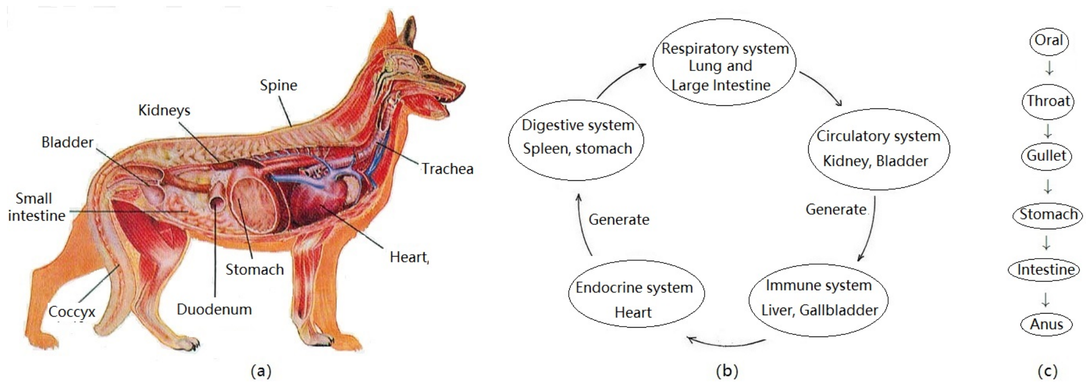



**Figure** 3.35. **Nutrient absorption and circulation of animals**

Certainly, the structure of an animal is closely related to the functions of digestive system, respiratory system, circulatory system, immune system and endocrine system in the body. In this regard, the nutrient absorption process of animals is different from the plants, i.e., the water and nutrients absorption by the roots, and the carbohydrate absorption by leaves. Certainly, the nutrient absorption process of animal is described by the food entering the stomach through the oral cavity and then, absorbing the nutrients, discarding wastes through the stomach and intestines, which can be depicted by a labeled graph $P_6^L$ as shown in Figure 3.35$(c)$, where, $(a)$ is the visceral distribution diagram of a dog and $(b)$ is the circulation of nutrients in the digestive system (spleen, stomach), respiratory system (lung, large intestine), circulatory system (kidney, bladder), immune system (liver, bile) and endocrine system (heart), which can be depicted by a labeled cycle $C_5^L$ with 5 vertices.

All microorganisms visible to the eyes of human such as mushrooms and ganoderma lucidum are composed of two parts: the mycelium and the fruiting body. Among them, the *mycelium* is a vegetative organ formed by the conjugated mycelium with transverse septum, elongated by the apical growth and mostly with the white color, slender and woolly; the *fruiting body* is the reproductive organs that open like a small umbrella at maturity. In terms of structure, a mature mushroom or ganoderma lucidum is composed of mycelium, stalk, ring and cover, etc., which is a labeled tree as shown in Figure 3.36.

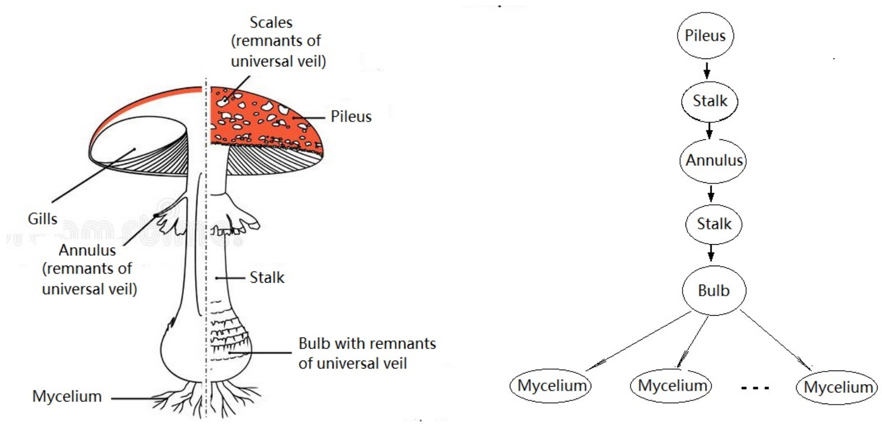
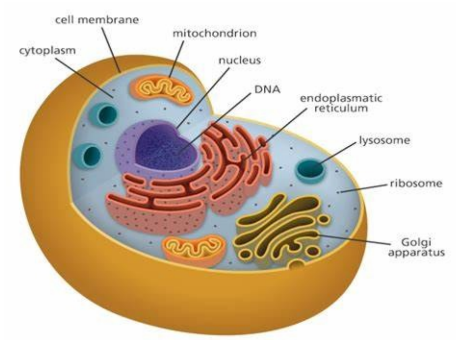



**Figure** 3.36. **Mushroom structure**

**4.2.Biological Microsystems.** All macroscopic matters are composed of molecules or atoms in the eyes of human and the livings are no exception. They are also the combination of molecules or atoms. However, all livings have the life and memory, also with a life cycle of the birth, growth, expansion and the extinction, which is the typical difference of livings from the rest of the universe. Therefore, the biology can not regards a living simply as a combination of molecules or atoms but needs to explore the elements of livings with memory, i.e., the cells and genes to understand them on the combination of molecules or atoms, including humans ourselves.

**(1)Cell.** Similar to the elementary particles that constituent matters, a *cell* is the smallest functional unit that constitutes an organism, which has all basic functions of a living including the material transport, energy conversion, information transduction, cell recognition, cell support and movement, cell digestion and the defense. Besides the viruses, all organisms are combination of cells with a certain spatial structure such as those shown in Figure 3.37($a$). Generally, cells are divided into the prokaryotic cells and the eukaryotic cells. Among them, the *prokaryotic cells* have no a typical nucleus but all *eukaryotic cells* have a nuclear membrane that separates the cytoplasm and the nucleus into two parts.

(a)




(b)

**Figure** 3.37. **Cellular structure**

Observing under the electron microscope, an eukaryotic cell can be divided into three parts, i.e, the cell membrane, cytoplasm and nucleus such as those shown in Figure 3.37($b$), where, ①the *cell membrane* is a very thin membrane surrounding the cell surface, mainly composed of lipids and proteins; ②the *cytoplasm* is composed of matrix, organelles and inclusions, lies within the cell membrane and outside the nucleus, it is translucent and homogeneous with low viscosity, where the matrix is a homogeneous translucent gelatinous material in the cytoplasm except organelles with the main components of water, inorganic salts, ions, esters, sugars, amino acids, nucleotides, proteins, lipoproteins, polysaccharides, RNA and enzymes; the organelles are organic components with certain morphological structure and function in the cytoplasm, including the mitochondria, ribosomes, endoplasmic reticulum, golgi, lysosomes, peroxisomes, centrioles, microtubules, microfilaments and medium fibers, etc.; ③the *cell nucleus* is the storage site of biological genetic information which consists of nuclear membrane, nucleolus, chromatin, nuclear matrix and nuclear skeleton. Constitutionally, the nuclear membrane is composed of two layers of single membranes; the nucleolus is a dense spherical structure without membrane and its main chemical components are the RNA, protein and a small amount of DNA; the chromatin is a fine fiber that carries the genetic information; the nuclear skeleton is a network of non-histone fibers in eukaryotic cells that except the nuclear envelope, chromatin and the nucleolus.

Why an organism may have life and grow from small to a large one is related to the proliferation and differentiation of cells in its body. That is, the cell division produces new individuals or new cells to increase and replenish the aging or dead cells in the body.

Generally, the *cell division* appears in three ways, i.e., the mitotic, amitotic and the meiotic. Among them, the mitosis is a common mode in eukaryotes such as humans, animals, plants and fungi, which is the main mode of eukaryotic cell proliferation; the *cell differentiation* refers to the process in which cells of the same origin produce different cell groups with different morphology, structure and functional characteristics, resulting in differences in time and space between cells and their former state.

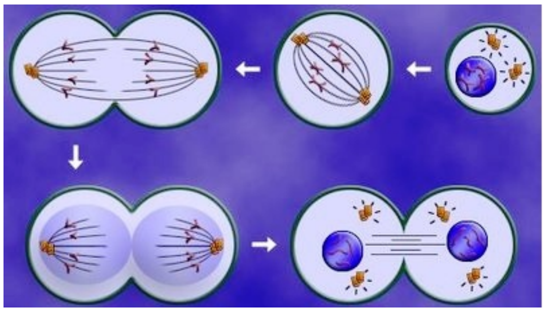


**Figure** 3.38. **Celluar mitotic**

Notice that the cell senescence is the emerging of the decline of cell proliferation and differentiation ability and the physiological function such as the nuclear enlargement, chromatin condensation, pyknosis and fragmentation, plasma membrane viscosity increase and fluidity decrease, mitochondria number decrease and volume increase, nuclear membrane invagination and other manifestations. Accordingly, the death of cell is the end of life, including the active cell death, programmed cell death, apoptosis and the passive death of cell, all of which are the necessary biological process for maintaining the function of tissue and morphology.

Usually, there are about $4 \times 10^{14} - 6 \times 10^{14}$ cells in an adult body, $1.2 \times 10^{10}$ cells in a brain and $4.9 \times 10^{6}$ cells in human olfactory. In the contrast, why the dog has a more sensitive sense of smell than a human is because the olfactory cells of a dog are about $2.205 \times 10^{8}$, which is 45 times that of a human.

**(2)Gene.** The *gene* is the basic unit of biological inheritance that exists on the chromosome of a cell. It is a fragment of DNA on the chromosome that carries genetic information. In the biological phenomena, the reason of why the children look like their parents and why some diseases are passed from the parents to the children is because of the genes between them. Generally, the genes store the life information such as the race, blood type, gestation, growth and apoptosis, etc. They have two characteristics. Among them, one is to reproduce themselves faithfully and keep the basic characteristics of their parents and another is the ability to "*mutation*" which changes the combinatorial structure of the basic pairs or

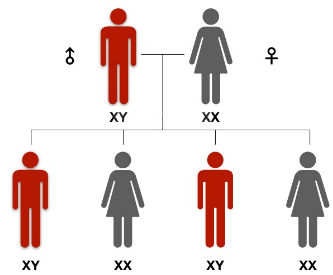
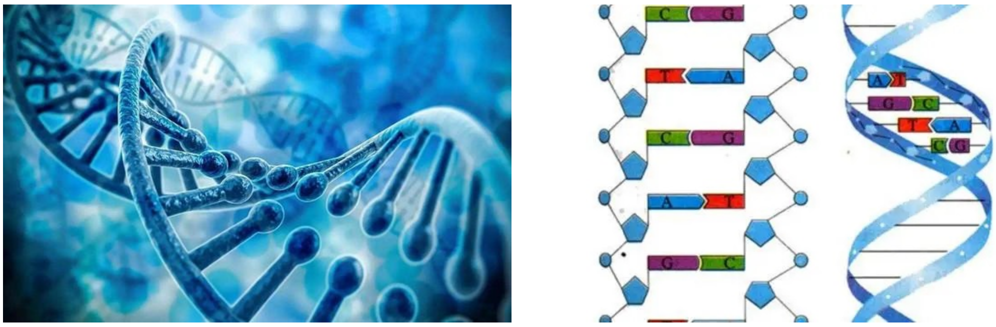



**Figure** 3.39. **Gene inheritance**

their arrangement order, resulting in the appearance of traits in the offspring that never be seen in the parents.



(a) (b)

**Figure** 3.40. **Genetic structure**

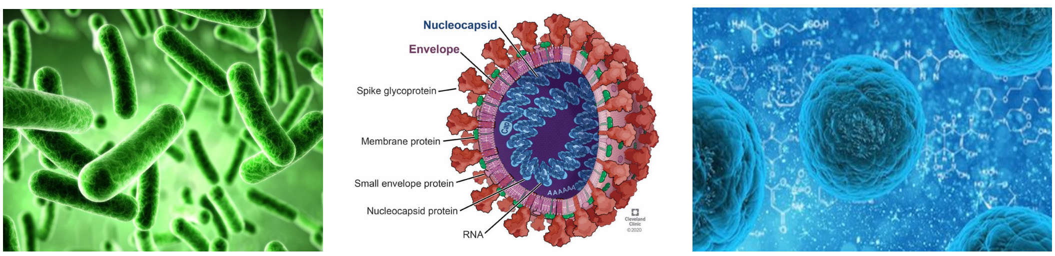
The human genome is a spatial double helix structure consisting of 23 pairs (46 chromosomes in total) such as those shown in Figure 3.40($a$) and ($b$), where each chromosome contains only one or two DNA molecules, each DNA molecule contains multiple genes and each containing hundreds or thousands of deoxynucleotides. At the same time, there is a gene segment between genes that may contain a regulatory sequence and the non-coding DNA. In fact, there are 24 chromosomes of humans. Among them, there are 22 chromosomes which are the somatic chromosomes and the other two, i.e., $X$ and $Y$ chromosomes determine the sex. The numbering sequence of chromosomes from 1 to 22 roughly corresponds to their size from largest to smallest. The largest chromosome contains about 250 million base pairs while the smallest has about 38 million base pairs. Correspondingly, there are 32 pairs of chromosomes in the somatic cells of horses, 31 pairs of chromosomes in donkeys and 19 pairs of chromosomes in pigs. All of which are double helix labeled graphs $(K_2 \times P_n)^L$ with $n = 23, 32, 31$ or 19.

The genes can be simply understood as the causes of life traits, which can be divided into three types, i.e., the genes of encoding proteins, genes without translation products and the DNA segments that are not transcribed. Each gene has its own specific location on the chromosome and stores the genetic information for life traits. Among them, the genes that occupy the same position but have different forms are called *alleles*. The predominant allele in the natural population is called the *wild-type gene*. In contrast to the wild-type gene, other alleles are called the *mutant genes*.

Notice that the gene is the unit of biological inheritance in life trait. It is bound to be directly or indirectly related to some human diseases such as albinism, hypertension, diabetes, coronary heart disease, epilepsy, asthma and so on. Everyone is born with genes that carry the "*inherited causes*" of certain diseases. Then, *is it possible to draw the human* *genome and then, master the law of birth, aging, disease and death, disease diagnosis and treatment, achieve human health development?* This was the *human genome project*, which ultimately determined the sequence of nucleotides that consist of 3 billion base pairs that make up the human chromosome, identified the genes contained in it and their sequences, found their positions on the chromosome.

The human genome project was proposed by American scientists in 1985, with the participation of many countries and was officially launched in 1990. In 2003, the human genome map was formally completed which has exerted an important influence on the development of biology, medicine and even the whole life science. For example, the disease genes with unknown biochemical functions can be searched through positional cloning and then, research and developing the targeted drugs, screening the drug targets, carry out the targeted therapy on the tumors related oncogenes and tumor suppressor genes, etc.

**(3)Microorganisms.** The microscopic organisms that are difficult to directly observe with the naked eyes of humans are called *microorganisms*, including bacteria, viruses, fungi and a few algae. Microorganisms are related to the production and life of humans, play an important role in the change of the earth's climate. Particularly, they are closely related to the human health. For example, some foods, industrial products and medicines depend on the microbial fermentation. It can be said that if there are no microorganisms, a large number of livings will lose their essential sources of nutrition, the remnants of animals and plants will not be able to decompose and there will be no prosperity and order in the nature.




(a)Bacteria (b)COVID-19 Virus (c)Fungi

**Figure** 3.41. **Microorganisms**

Among the microorganisms, the *bacteria* is a kind of unicellular prokaryotes with short thin cells, simple structure, tough cell wall and strong hydrobiosis, which is mainly propagated by binary division such as shown in Figure 3.41($a$); the *virus* is the lowest form of life that has no cellular structure and can only live by inhabiting certain living cells. Generally, a virus is composed of one or more nucleic acid molecules with a protective coat of protein or lipoprotein, called capsid which is composed of nucleic acid and capsid proteins. More complex viruses have an outer membrane of lipids and glycoproteins. Among them, the *corona virus* belongs to the family of corona viruses in classification. Like corona, it has an envelope and there is spines on the envelope. Certainly, different corona viruses have obvious differences in spines, and the main human diseases caused by them are respiratory infections. The novel corona virus Covid-19 spread globally since 2019 is shown in Figure 3.41$(b)$. It is worth noting that in the process of fighting Covid-19, most of the virus samples were sequenced to understand the source and variant of the virus so as to formulate targeted defense measures; the *fungi* is a group of eukaryotes whose cells do not contain chloroplasts or plastids. They are heterotrophic organisms including parasitic and saprophytic organisms. That is, they absorb and decompose organic matter from living animals and plants and their excretions, dead animals and plants, and the soil humus. They are divided into three categories, i.e., the yeast, mold and the macrofungi. Usually, a large fungi such as mushrooms, agaric fungus and dictyophora can be visible by the eyes of human; the *algae* is a primitive group of aquatic eukaryotes, which can be divided into the planktonic algae, floating algae and the benthic algae. Most algaes need oxygen in living and use their chloroplast molecules for photosynthesis.

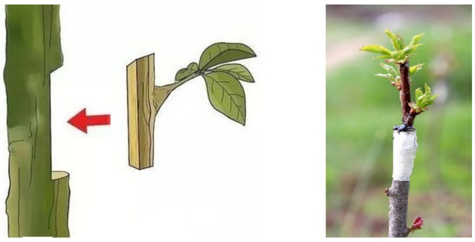
**4.3.Species Improvement.** The species improvement is a kind of biological genetic traits that applies the biological genetic performance to improve the normal index of growth, environmental conditions or genetic sequence such as improving the postnatal care, hybridization, genetically modified (GM) and other methods which enhances the capacity of rice lodging resistance and resists the disease for achieving the purpose of increase production, protecting the bio-diversity, etc.

**(1)Hybridization.** The *hybridization* is a method by which animals and plants of different species, genera or varieties are mated to produce offspring from the recombination of parental genes. Notice that the heterosis is a common phenomenon in the biological world. But, how to use this advantage to improve the yield and quality of crops is the pursuit of agriculture. Generally, the essential work for this objective is the breeding, including the process of the selection of hybrid parents, experimental planting and planting of hybrid parents and then, the selection of breeding with the high yield, high quality, good characters and others.

The hybrid rice is a typical example of increasing crop yield by the hybridization. Unlike other crops, the rice is self-pollinated with the female and male stamens growing in the same spikelet. To hybridize different varieties of rice, it is needed to emasculate or kill the stamens of one variety and then, transfer the stamen pollen of another variety to the emasculated variety. For example, the two-line hybrid rice uses the photo temperature-sensitive sterile rice because this kind of rice is male sterile in summer or under long sunshine and high temperature conditions but becomes normal in autumn, short sunshine and low temperature propagating itself.

In addition, the hybrid technology is also widely applied in breeding of the maize, sorghum, rice, sugar beet, onion, spinach, sunflower, broccoli and other crops for increasing production.

**(2)Grafting.** The *grafting* is a cultivation method in which branches or buds of superior varieties are transferred to another plant so that they can grow together to form an independent new plant. In general, the branch or bud that is grafted is called the *scion*, the plant that bears the scion is called the *rootstock*. The principle of grafting is depending on the regeneration ability of parenchyma cells formed at the joint site of scion and rootstock, forming a healing tissue so that the scion and rootstock are closely combined, then make the scion and the original rootstock transport tissue connected so that the nutrients and water of the two communicate up and down, forming a new plant. The common grafting method of flowers and trees is divided into three kinds such as the branch grafting, bud grafting and the root grafting. After grafting, the plant enhances the resistance to disease, improves the ability to withstand low temperature, help to overcome the harm of continuous cropping, expands the absorption range and capacity of the root system, and also improves the yield of crops to meet the survival needs of humans.



(a) (b)

**Figure** 3.42. **Grafting**

**(3)Gene technology.** Once the genome map of an organism has been drawn, a natural ideas is to improve the species by modifying its gene sequence. Indeed, this idea has been practiced and led to the development of gene technologies such as the *transgene* and the *gene editing*, which essentially improve the gene sequence, i.e., the combination structure of genes of a species according to a given goal.

The transgenic technology is a biotechnology in which a gene of one organism is transferred to a fragment of DNA from another organism. When a fragment of DNA was transferred to another organism and the restructuring of the genomes of organisms themselves cause the change of biological traits and decoration with heritable. One can then acquires an individual with a genetic trait of stable performance from the artificial recombination, develops the crop with varieties of high-yield, high-quality, antivirus, resistance to the insect, cold, drought, flood, saline-alkali and herbicide, etc.

The gene editing technology is a kind of technology that modifies specific DNA fragments by manipulating CRISPR/Cas9 scissors to delete or add a small piece of DNA on the living gene sequence, like cutting a slip of paper so as to edit the specific target genes in the genome of an organism, as shown in Figure 3.43, where the CRISPR/Cas9 is an adaptive immune defense that developed during the long evolution of bacteria and archaea to fight the invading viruses and foreign DNA.

**Figure** 3.43. **Gene editing**

Certainly, the technical difference between the transgene and the gene editing lies in which and where section of DNA is cut or added. It is clear and controllable in the gene editing technology. However, it is not clear where should be added the DNA fragment of another organism in the transgene, whose uncontrollable degree is high. In this sense, the gene editing technology has advantages over the transgenic technology. However, no matter what kind of genetic technology, it is essentially to adjust or modify the genetic sequence formed by the nature. Dr.Ouyang tells his daughter, the genetic topology comes into being dependent on a local or current recognition of humans on genetic sequence of species. It should be tested in a long period of practice. Otherwise, it may cause irreversible harms to species and the nature, including humans ourselves.

## §5. **Counting Things in the Universe**

In the nature, any kind of things do not appear or exist in isolation but are generally connected or intertwined with others. In this case, to hold on the whole picture of things is the only way to recognize the truth of things. However, the perceptible limitation of humans on things causes that one often can not see the whole picture of a thing but only the measuring characteristics of things from a local recognition of the whole such as those of the individual number, living area or spatial scale of a species. So, *is there a unified way for holding on the overall characteristics of such measures by local recognition?* If so, humans can extend these local recognition to the whole and then hold on the truth of things to a certain extent. In response, Dr.Ouyang tells Huizi that there are two important principles in combinatorics, called respectively the *inclusion-exclusion principle* and the *pigeonhole principle*, which can generally solve the problem of counting things from local to the whole and determine the existence of a subgroup with certain properties of a population.

**5.1.Population Counting.** The counting is also known as numbering, i.e., counting objects one by one with the fingers or using the natural numbers $1, 2, 3, 4, \cdots$ to mark the counting objects one by one and then, constructing an object sequence $x_1, x_2, x_3, \cdots, x_n, \cdots$. Given a population or a set $S$, one of the most basic questions is to know how many members are in $S$, denoted by $|S|$ and also known as the *cardinality of set* $S$. For example, let $S$ be the tigers in a zoo. If there are 3 tigers raised in the zoo, then we have $|S| = 3$ in this case.

Certainly, the animals raised in a zoo can be counted out one by one. However, the wild animals can only rely on the observing, discrimination and the forecast in a relatively complex situation because the wild animals walk without boundary limitation. Besides, any kind of animal has the reproducing ability, it needs on a long-term observation for determining the size of the population. Among this kind of counting problems, the Fibonacci's rabbit problem is a famous one.

[**Fibonacci Problem**] *If a farmer raises a pair of baby rabbits (one male, one female) initially which will grow into rabbits with fertility in two months and a pair of mating rabbits produce a pair of baby rabbits (one male, one female) every month, then how many pairs of rabbits will the farmer have at the 12th month?*

**Figure** 3.43. **A pair of rabbits**

Under the conditions, we try to calculate the rabbit pairs step by step following:

Firstly, suppose the farmer has a pair of baby rabbits at the first month. Then, at the second month the rabbits do not grow up with the fertility and the farmer only has a pair of rabbits. At the third month, the pair of rabbits grow up and will give birth to a pair of baby rabbits, the farmer has $1 + 1 = 2$ pair of rabbits. Similarly, at the fourth month, the new born pair of rabbits have not grown up, there are only a pair of baby rabbits and a pair of new born rabbits. So, the farmer will have $2 + 1 = 3$ pair of rabbits and at the fifth month, the rabbits born in the first three months grow up and become fertile, giving birth to two pairs of rabbits. At this point, the farmer should have $3 + 2 = 5$ pair of rabbits at the fifth month. Continuously, the rabbit born at the first fourth month will grow up with fertile and able to produce three pairs of rabbits, totaling $5 + 3 = 8$ pairs at the sixth month.

We can calculate the number of pair rabbits step by step, i.e., if the rabbits born in the first two months of the current month grow up and become fertile, the number of pair rabbits owned by the farmer in the current month is the number of rabbits in the previous two months plus the number of rabbits in the previous month. That is, there are $5+8=13$ pairs of rabbits at the seventh month, $8+13=21$ pairs of rabbits at the eighth month, $13+21=34$ pairs of rabbits at the ninth month, $21+34=55$ pairs of rabbits at the 10th month, $34+55=89$ pairs of rabbits at the 11th month and finally, $55+89=144$ pairs of rabbits at the 12th month. Consequently, we know that the farmer has 144 pairs of rabbits at the 12th month.

Generally, *could we find out how many pairs of rabbits the farmer have at the nth month for any integer $n \geq 3$?* Suppose the farmer has $F_n$ pair of rabbits at the month $n$. The previous calculation shows that $F_1 = F_2 = 1$, i.e., the number of rabbit pairs at the 1st month and the 2th month is 1, respectively. But beginning from the 3nd month, the number of rabbit pairs $F_n$ at the $n$th month is the sum of the number of rabbit pairs in the previous two months $F_{n-1}$ and $F_{n-2}$, satisfying the recursive relations

$$F_n = F_{n-1} + F_{n-2}. \tag{3.1}$$

Then, *how to solve the equation* (3.1) *for finding $F_n$ in general?* Assume that

$$F(t) = \sum_{i\geq 0} F_i t^{i-1} = F_0 + F_1 t + F_2 t^2 + \cdots + F_n t^n + \cdots \tag{3.2}$$

with $F_0 = 0$. Calculation shows that

$$\begin{aligned} F(t) - tF(t) - t^2F(t) &= \left(F_1 t + F_2 t^2 - F_1 t^2\right) \\ &\quad + \sum_{n\geq 3} \left(F_n - F_{n-1} - F_{n-2}\right) t^{n-1} \\ &= \left(F_1 t + F_2 t^2 - F_1 t^2\right) = t. \end{aligned}$$

We consequently get that

$$\begin{aligned} F(t) &= \frac{t}{1-t-t^2} = \frac{t}{\left(1 - \frac{1+\sqrt{5}}{2}t\right)\left(1 - \frac{1-\sqrt{5}}{2}t\right)} \\ &= \frac{1}{\sqrt{5}}\left(\frac{1}{1-\frac{1+\sqrt{5}}{2}t} - \frac{1}{1-\frac{1-\sqrt{5}}{2}t}\right) \\ &= \frac{1}{\sqrt{5}} \sum_{n\geq 1} \left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] t^n. \end{aligned} \tag{3.3}$$

Comparing the equality (3.2) with that of (3.3), we immediately know that

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]. \tag{3.4}$$

Notice that the formula (3.4) is interesting because $F_n$ is an integer of rabbit pairs but it is on the face of irrational numbers $(1+\sqrt{5})/2$ and $(1-\sqrt{5})/2$.

It should be noted that the Fibonacci problem does not consider the birth and death of the rabbits, also ignores the carrying capacity of the resources. That is, it is assumed that rabbits can grow without any limitation which is certainly not in accordance with the law of the nature. Assume the number of rabbit pairs can be approximated by a continuous function $F(t)$ and let $K$ be the maximum number of population members permitted by the resources, namely the resource carrying capacity and let $R$ be the growth rate of rabbit pairs. Then, the growth rate $R$ of rabbit pairs is $rF\left(1-\frac{F}{K}\right)$. So, we have the Logisitic equation

$$\frac{dF}{dt}=rF\left(1-\frac{F}{K}\right)=\frac{rF(K-F)}{K} \quad\Rightarrow\quad \left(\frac{1}{F}+\frac{1}{K-F}\right)dF=rdt. \tag{3.5}$$

Integrating both sides of (3.5), we have

$$\ln|F|-\ln|K-F|=rt+C \ \Rightarrow\ \ln\left|\frac{F}{K-F}\right|=rt+C \ \Rightarrow\ \left|\frac{F}{K-F}\right|=C_1e^{rt},$$

where $C_1=e^C$ and $C$ is a constant. Therefore,

$$F(t)=\frac{KC_1e^{rt}}{1+C_1e^{rt}}. \tag{3.6}$$

Let $F(0)=F_0$ at time $t=0$. We know that $C_1=\frac{F_0}{K-F_0}$ by (3.6), i.e.,

$$F(t)=\frac{KF_0e^{rt}}{(K-F_0)+F_0e^{rt}}. \tag{3.7}$$

Certainly, the constants $r, K$ can be measured experimentally for a given region. For example, if $K=1.2\times10^6$, $F_0=1$ and $r=0.2311$ in measurement, then the farmer should have

$$F(t)=\frac{1.2\times10^6e^{0.2311t}}{(1.2\times10^6-1)+e^{0.2311t}}. \tag{3.8}$$

pair of rabbits at time $t$.

**5.2.Inclusion-Exclusion Principle.** Generally, most population counting needs to be calculated jointly with counting of other populations. Dr.Ouyang asks his daughter that if there are 14 students scores more than 95 points in Chinese, 18 students scores more than 95 points in mathematics and 12 students more than 95 points on both of the two courses in the final examination of your class. Then, *how many students scored 95 or more in one of the two courses?* Huizi thinks for a while and replies:"*there are* 32 *students because there are* 14 *students scores more than* 95 *points in Chinese,* 18 *students scores more than* 95 *points in mathematics.* Dr.Ouyang reminds her:"*Among the 32 students,* *there are* 12 *students scored above* 95 *in both subjects included in the* 14 *students who scored above* 95 *in Chinese and the* 18 *students who scored above* 95 *in mathematics. So, these* 12 *students have been counted twice and you need to get rid of the twice counted out of your 32 answers.*" Huizi answers:"*It is* 20 *students! But why it is* 20*, not* 32 *students?*" Dr.Ouyang tells Huizi that it is a matter of counting elements in sets.

Generally, the "*union*" operation $A\bigcup B$ defined on sets $A$ and $B$ is analogical to the plus operation of numbers. That is, the elements of two sets are put together to form a new set, i.e., $A\bigcup B$ is defined by $A\bigcup B = \{x \mid x\in A \text{ or } x\in B\}$, as shown in Figure 3.45. So, *is* $|A\bigcup B| = |A|+|B|$ *or not?* In fact, if $A$ and $B$ are both numbers, the sum of $A$ and $B$ equal to the sum of each number is a basic rule. But in the case of set operation, $|A\bigcup B| = |A|+|B|$ is conditional because an elements $x$ of the set $A\bigcup B$ may be an element of $A$, may be an element in $B$ and may also appears in both sets $A$ and $B$. The last case is called the intersection $A\bigcap B$ of $A$ and $B$, denoted by $A\bigcap B=\{x|x\in A \text{ and } x\in B\}$, as shown in Figure 3.45. Generally, it holds with the equality $|A\bigcup B| = |A|+|B|-|A\bigcap B|$ because the elements of $A\bigcap B$ are repeatedly counted in $|A|+|B|$ for counting the elements in $A\bigcup B$ which should be removed. Consequently, $|A\bigcup B| = |A|+|B|$ holds only with $|A\bigcap B|=0$, namely the intersection of $A$ and $B$ is an empty set. For the counting problem in the beginning of this subsection, let $A$ and $B$ be the sets of students in scores above 95 points in Chinese or mathematics, respectively. Then, $|A|=14$, $|B|=18$ and $A\bigcap B$ are the students scores more than 95 points both in Chinese and mathematics, i.e., $|A\bigcap B|=12$ and $A\bigcup B$ is the students scores above 95 points in one of the Chinese and mathematics. We have the answer $|A\bigcup B| = |A|+|B|-|A\bigcap B| = 14+18-12 = 20$.

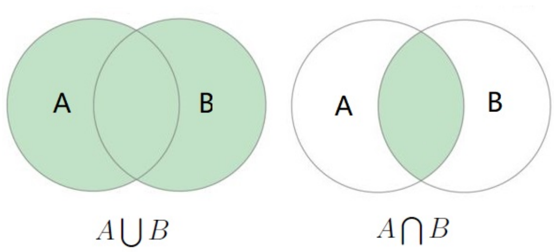



**Figure** 3.45. **Union and intersection**

Then, *what is about the general case?* Firstly, let $A$, $B$ and $C$ be 3 sets. We can consider the condition that hold with the equality $|A\bigcup B\bigcup C| = |A|+|B|+|C|$ in general. Similar to the case of 2 sets, there must be the intersections $A\bigcap B$, $A\bigcap C$ and $B\bigcap C$ are empty sets. Thus, $|A\bigcap B| = |A\bigcap C| = |B\bigcap C| = 0$ because we have $|A\bigcup B\bigcup C| = |A|+|B|+|C|-(|A\bigcap B|+|A\bigcap C|+|B\bigcap C|)+|A\bigcap B\bigcap C|$ in

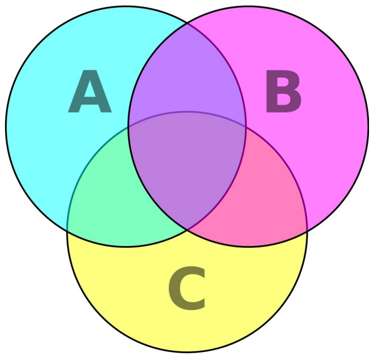



**Figure** 3.46. **Three sets**

general. *Why is this so?* By the definition of union and intersection of sets, there are

$$A\bigcup B\bigcup C=\left(A\bigcup B\right)\bigcup C,\ \ \left(A\bigcap B\right)\bigcup C=\left(A\bigcup C\right)\bigcap\left(B\bigcup C\right).$$

In this way, according to the counting formula of two set, there is

$$\begin{aligned}\left|A\bigcup B\bigcup C\right| &= \left|\left(A\bigcup B\right)\bigcup C\right|=\left|A\bigcup B\right|+|C|-\left|\left(A\bigcup B\right)\bigcap C\right|\\ &= \left|A\bigcup B\right|+|C|-\left|\left(A\bigcap C\right)\bigcup\left(B\bigcap C\right)\right|\\ &= \left|A\bigcup B\right|+|C|-\left(\left|A\bigcap C\right|+\left|B\bigcap C\right|-\left|\left(A\bigcap C\right)\bigcap\left(B\bigcap C\right)\right|\right)\\ &= |A|+|B|+|C|-\left(\left|A\bigcap B\right|+\left|A\bigcap C\right|+\left|B\bigcap C\right|\right)+\left|A\bigcap B\bigcap C\right|.\end{aligned}$$

Generally, the *inclusion-exclusion principle* is the counting formula in the case of $n$ sets, i.e., the equality

$$\left|\bigcup_{i=1}^{n}X_i\right|=\sum_{s=1}^{n}(-1)^{s+1}\sum_{1\leq i_1<i_2<\cdots<i_s\leq n}\left|X_{i_1}\bigcap X_{i_2}\bigcap\cdots\bigcap X_{i_s}\right| \tag{3.9}$$

holds for any integer $n\geq 2$. For the cases of $n=2$ or $n=3$, we have proved in the previous and the method of $n=3$ can be applied for the general case of $n$ sets with an integer $n\geq 3$. In fact, if the equality (3.9) holds with the integer $n=k$ then for the integer $n=k+1$, we have

$$\begin{aligned}\left|\bigcup_{i=1}^{k+1}X_i\right| &= \left|\left(\bigcup_{i=1}^{k}X_i\right)\bigcup X_{k+1}\right|\\ &= \left|\bigcup_{i=1}^{k}X_i\right|+|X_{k+1}|-\left|\left(\bigcup_{i=1}^{k}X_i\right)\bigcap X_{k+1}\right|\\ &= \left|\bigcup_{i=1}^{k}X_i\right|+|X_{k+1}|-\left|\bigcup_{i=1}^{k}\left(X_i\bigcap X_{k+1}\right)\right|\\ &= \sum_{s=1}^{k}(-1)^{s+1}\sum_{1\leq i_1<i_2<\cdots<i_s\leq k}\left|X_{i_1}\bigcap\cdots\bigcap X_{i_s}\right|+|X_{k+1}|\\ &\quad -\sum_{s=1}^{k}(-1)^{s+1}\sum_{1\leq i_1<i_2<\cdots<i_s\leq k}\left|\left(X_{i_1}\bigcap X_{k+1}\right)\bigcap\cdots\bigcap\left(X_{i_s}\bigcap X_{k+1}\right)\right|\\ &= \sum_{s=1}^{k}(-1)^{s+1}\sum_{1\leq i_1<i_2<\cdots<i_s\leq k}\left|X_{i_1}\bigcap\cdots\bigcap X_{i_s}\right|+|X_{k+1}|\\ &\quad -\sum_{s=1}^{k}(-1)^{s+1}\sum_{1\leq i_1<i_2<\cdots<i_s\leq k}\left|\left(X_{i_1}\bigcap\cdots\bigcap X_{i_s}\right)\bigcap X_{k+1}\right|\\ &= \sum_{i=1}^{k+1}|X_i|+\sum_{s=2}^{k+1}(-1)^{s+1}\sum_{1\leq i_1<i_2<\cdots<i_s\leq k+1}\left|X_{i_1}\bigcap\cdots\bigcap X_{i_s}\right|\\ &= \sum_{s=1}^{k+1}(-1)^{s+1}\sum_{1\leq i_1<i_2<\cdots<i_s\leq k+1}\left|X_{i_1}\bigcap X_{i_2}\bigcap\cdots\bigcap X_{i_s}\right|,\end{aligned}$$

namely the equality (3.9) hold with the integer $n=k+1$. By the mathematical induction, we then know that the equality (3.9) hold for all integers $n\geq 2$.

Let $n$ be an integer with $n\geq 2$. We can define the *intersection graph* $G\left[X_1^n\right]$ on the sets $X_1,X_2,\cdots,X_n$ with respectively the vertex and edge sets following

$$\begin{aligned} V\left(G\left[X_1^n\right]\right) &= \{X_1,X_2,\cdots,X_n\};\\ E\left(G\left[X_1^n\right]\right) &= \left\{(X_i,X_j)\,|X_i\bigcap X_j\neq\emptyset, 1\leq i,j\leq n\right\}.\end{aligned}$$

For an index subset $\{i_1,i_2,\cdots,i_s\}\subset\{1,2,\cdots,n\}$ if

$$X_{i_1}\bigcap X_{i_2}\bigcap\cdots\bigcap X_{i_s}\neq\emptyset,$$

then the induced subgraph of $X_{i_1},X_{i_2},\cdots,X_{i_s}$ in $G\left[X_1^n\right]$ is a complete subgraph of order $s$. And then, the inclusion-exclusion principle (3.9) can be also expressed by completes subgraphs of $G\left[X_1^n\right]$ as follows

$$\left|\bigcup_{i=1}^n X_i\right| = \sum_{s=1}^n(-1)^{s+1}\sum_{K_s\simeq G[X_{i_1}^{i_s}]\prec G[X_1^n]}\left|X_{i_1}\bigcap X_{i_2}\bigcap\cdots\bigcap X_{i_s}\right|. \qquad (3.10)$$

Notice that if there is a complete subgraph $K_m$ in the intersection graph $G\left[X_1^n\right]$, there must be all complete subgraphs of order$\leq m$. Particularly, if there are no subgraphs isomorphic to $K_3$ in $G\left[X_1^n\right]$, i.e., $G\left[X_1^n\right]$ is $K_3$-free then the equality (3.9) and (3.10) can be simplified further to

$$\left|\bigcup_{i=1}^n X_i\right| = \sum_{i=1}^n|X_i| - \sum_{(X_i,X_j)\in E\left(G\left[X_1^n\right]\right)}\left|X_i\bigcap X_j\right|. \qquad (3.11)$$

**5.3.Pigeonhole Principle.** The pigeonhole principle asserts the existence of substructure with certain relationship among individuals of a population while the number of individuals has reached a certain number and it has many important generalizations. Usually, this principle has three simple expressions as follows:

**Figure** 3.47. **Pigeonhole**

**Principle** 5.1 *If there are $n+1$ pigeons fly into $n$ holes then there is one hole containing two or more pigeons at least .*

The correctness of Principle 5.1 is obvious because if one pigeon hole flies into one pigeon then the $n$ holes can fly inton pigeons at most. However, there are $n+1$ pigeons flying in now. So, there must been one pigeon hole with two or more pigeons flying in.

**Principle** 5.2 *If put $n(r-1)+1$ balls into $n$ boxes then there is one box containing $r$ balls at least.*

If the number of balls in each box is at most $r-1$, then there are $n(r-1)$ balls at most. But now, there are $n(r-1)+1$ balls, i.e., there must be a box with at least $r$ balls.

**Principle** 5.3 *If put infinite balls into $n$ boxes, there is one box containing an infinite number of balls at least.*

The proof of Principle 5.3 is similar to that of Principle 5.2. If there are only a finite number of balls in each box, then the total number of balls in the $n$ boxes must be finite too. But now, there are infinite balls in $n$ boxes, there must be a box with an infinite number of balls.

Many interesting results can be obtained by applying the pigeonhole principle. For example, when a theatre with a capacity of 1500 seats is full, there exist 5 audiences at least born on the same day of the same month because if there are 4 audiences born on the same day of the same month at most, there would be a maximum of $4 \times 366 = 1464$ individuals. But now $1500 > 1464$. So, if the theatre is full, there are minimum 5 individuals born on the same day of the same month. For another example, there are few humans know how many hairs they have on their heads. But the pigeonhole principle can be used to conclude that there are at least 14,000 humans with the same number of hairs in the mainland of China. *Why is it so?* According to the national statistics of 2021, the total population on the mainland of China is 1411.178 million and the average number of hairs for yellow humans is about 100,000. It is obvious that $141178 > 14000 \times 10$. So, there are at least 14,000 humans in the mainland of China with the same amount of hairs.

**Ramsey problem.** The pigeonhole principle describes a trait of individuals that if the number of pigeons is larger than the average number of pigeons in pigeonholes times the number of pigeonholes, there exists a pigeonhole with pigeons more than the average number of pigeons. Similarly, there are a lot of these traits. In the June -July, 1958 issue of the *American Mathematical Monthly*, there is an interesting question appeared, in which the author Ramsey gave an interesting conclusion that *if there are at least 6 humans at a party then there must exist groups of three humans among them either all know each other or have never met before*. Later, it was called the *Ramsey problem*.

This conclusion can be easily proved by applying the pigeonhole principle. Firstly, let's represent these six humans as six points on the plane. If two humans know each other, connect a red line between them; if they don't know each other, connect a green line. Pick any one point $v$ of these six points. According to the pigeonhole principle, the point $v$ is known or never met with at least three of the other five humans. Without loss of generality, assume that $v$ is known to $u$, $w$ and $x$. That is, they are connected by red lines, as shown in Figure 3.48($a$). At this time, if there is a red line between three points $u$, $w$ and $x$, assuming it is between $u$ and $w$ then the lines between $v$, $u$ and $w$ are all red lines. That is, $v$, $u$ and $w$ know each other as shown in Figure 3.48($b$). On the contrary, the three points $u$, $w$ and $x$ are all green lines, which implies that the three humans $u$, $w$ and $x$ do not know each other. And this proves Ramsey's conclusion.

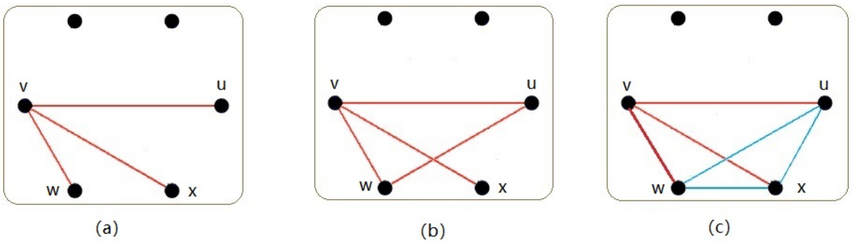



**Figure** 3.48. **Ramsey problem**

Notice that the integer 6 is minimum in the Ramsey problem which implies that as long as there are 6 humans at least, there must be 3 humans who know each other or 3 humans who do not know each other but this conclusion would not work if there are 5 or less humans. Generally, *for any integer* $m, n \geq 3$*, is there a minimum integer* $R(m, n)$ *such that for any* $R(m, n)$ *individuals, there are* $m$ *individuals at least know each other or there are* $n$ *individuals at least do not know each other?* The answer is Yes! At the same time, the Ramsey number is symmetrical, i.e., $R(m, n) = R(n, m)$ and also

$$R(m, n) \leq R(m-1, n) + R(m, n-1). \tag{3.12}$$

Then, the existence of $R(m, n)$ can be known from the Ramsey numbers $R(m, 2) = m$ and $R(2, n) = n$.

However, *how do we prove the inequality* (3.12)*?* the prove is similar to the case of proving $R(3, 3)$. We apply points on the plane to represent humans and use red edge or green edge to indicate that two humans know each other or not. Firstly, choose any human $v$ from the $R(m-1, n) + R(m, n-1)$ individuals. According to the pigeonhole principle, $v$ is known with at least $R(m-1, n)$ humans or is not know with at least $R(m, n-1)$ humans in the other humans $R(m-1, n) + R(m, n-1) - 1$, respectively. The discussion is divided into 2 cases: ①If $v$ is known at least $R(m-1, n)$ individuals, there are $m-1$ humans known to each other in $R(m-1, n)$ individuals, i.e., $m$ humans known each other including the human $v$ by definition or there are $n$ individuals who do not know each other; ②If $v$ is not known with at least $R(m, n-1)$ individuals, there exist $m$ individuals who know each other or $n-1$ individuals who do not know each other in the $R(m, n-1)$ individuals by definition, which combined with $v$ implies that there are $n$ individuals do not know each other. In summary, there is the conclusion that $R(m,n) \le R(m-1,n) + R(m,n-1)$, i.e., for any integers $m, n \ge 2$, the Ramsey number $R(m,n)$ is existing.

By the graph terminologies, the existence of $R(m,n)$ indicates that if one dyes a complete graph $K_l$ with two colors such as red and green with $l = R(m,n)$, then a complete graph $K_m$ or $K_n$ in monochrome must appear. Notice that any graph $G$ is a subgraph of the complete graph $K_{|G|}$. Let $G, H$ be any two graphs. Then, the existence of number $R(G,H)$ can be directly obtained from the existence of $R(m,n), m, n \ge 2$. That is, if two colors are used to dye a complete graph of order $R(G,H)$, a monochromatic graph $G$ or $H$ must be appearing.

However, *can one determines the exactly value of* $R(m,n)$*?* Dr.Ouyang tells Huizi that this is a difficult problem in combinatorics. Ramsey himself only determined that $R(3,3) = 6$. Later, researchers have successively determined some values of $m, n \le 5$ and the upper or lower bounds of $R(m,n)$ but it is still an opened problem to determine $R(m,n)$ for the integer $m, n \ge 5$. Some already known values of $R(m,n)$ are listed in Table 3.3, where the dashed lines represents the parts that have not yet been determined.

| $n \backslash m$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 |
|---|---|---|---|---|---|---|---|
| 3 | 6 | 9 | 14 | 18 | 23 | 28 | 36 |
| 4 | 9 | 18 | 25 | | | | |
| 5 | 14 | 25 | | | | | |
| 6 | 18 | | | | | | |
| 7 | 23 | | | | | | |
| 8 | 28 | | | | | | |
| 9 | 36 | | | | | | |

**Table** 3.3. **Ramsey number** $R(m,n)$

Although it is difficult to determine $R(m,n)$ in general, it is sometimes relatively easy to determine $R(G,H)$ for some graph families $G, H$. Hearing this, Huizi is a little confused and then asks Dr.Ouyang: "*Dad, the inclusion-exclusion principle and the pigeon-hole principle are very interesting but what effect do they have on scientific recognition of populations?* Dr.Ouyang explains to her that the population counting is count individuals without repeat and presents the method of counting if individuals appear with cross and overlap, which is very important in scientific research. For example, let $(k,n)$ denote the greatest common divisor of the integers $k$ and $n$. Dr.Ouyang tells Huizi that there is a Euler function $\varphi(n) = |\{k \in \mathbb{Z} | (k,n) = 1, \quad 0 < k \le n\}|$ in number theory introduced by Euler himself. Certainly, if $n$ is a smaller, $\varphi(n)$ can be easily determined but if $n$ is a larger, to determine $\varphi(n)$ is complicated by its definition. However, there's a famous formula for $\varphi(n)$, namely

$$\varphi(n) = n\prod_{i=1}^{l}\left(1-\frac{1}{p_i}\right), \tag{3.13}$$

where, $p_1, p_2, \cdots, p_l$ are primes of the integer $n$. We can immediately prove this formula by applying the inclusion-exclusion principle. Firstly, for any integer $i, 1 \leq i \leq l$, define the set $X_i = \{k \in \mathbb{Z}|\ (k,n) = p_i, 0 < k \leq n\ \}$. Then, by the inclusion-exclusion principle (3.9) we get that

$$\begin{aligned}
\varphi(n) &= |\{k \in \mathbf{Z}|(k,n) = 1, 0 < k \leq n\ \}| = \left|\{1,2,\cdots,n\} \setminus \left(\bigcup_i^l X_i\right)\right| \\
&= n - \sum_{s=1}^{n}(-1)^{s+1}\sum_{1\leq i_1<\cdots<i_s\leq l}\left|X_{i_1}\bigcap X_{i_2}\bigcap\cdots\bigcap X_{i_s}\right| \\
&= n\left[1-\sum_{1\leq i\leq l}\frac{1}{p_i}+\sum_{1\leq i,j\leq l}\frac{1}{p_ip_j}-\cdots+(-1)^l\frac{1}{p_1p_2\cdots p_l}\right] \\
&= n\left(1-\frac{1}{p_1}\right)\left(1-\frac{1}{p_2}\right)\cdots\left(1-\frac{1}{p_l}\right) = n\prod_{i=1}^{l}\left(1-\frac{1}{p_i}\right).
\end{aligned}$$

Certainly, the pigeonhole principle is judging on certain traits while the number of individuals is large to a certain extent and the Ramsey problem can be regarded as a generalization of the inclusion-exclusion principle. Dr.Ouyang tells Huizi that in the daily life, the face recognition technology is an application of the pigeonhole principle by scanning, memorizing and comparing features of the face such as those of the shape and change of the face, eyes, nose, mouth and the teeth. Furthermore, an interesting conclusion on daily life can be obtained by the existence of the Ramsey number $R(m,n)$, i.e., if the number of comparing individuals in the crowd is sufficiently large, there must be two identical humans. Huizi is a little surprisedly and then asks Dr.Ouyang:"*Are you sure, why do I not met anyone who looks exactly like me?*" Dr.Ouyang smilingly tells his daughter that it is because the number of humans that she had met is not sufficiently large! We have known that there are more than 7.26 billion humans on the earth. *Is the number* 7.26 *billion qualify as a Ramsey number?* If the answer is affirmative, Huizi will surely meet someone who looks just like her and not her twin sister. But she has never met such one! *What does this fact implies?* Dr.Ouyang tells Huizi that this can be only explained as the number of 7.26 billion is not large enough for the required Ramsey number. In other words, the Ramsey number of two humans who look exactly alike is too large, can not be properly feeded by resources on the earth.

## §6. **Comments**

**6.1.** In the recognition of things in the universe, the combinatorial notion is a philosophical thought for solving the limitation of human recognition. That is, to combine different recognitive results of a thing together to form a complete recognition of the thing, contained in the words that the sophist told the blind men in the fable of the blind men with an elephant because the human recognition is similar to the blind men for unknown things, both of them are based on a local perception of the unknown things in the universe. Historically, this kind of combinational notion was called "*reductionism*", reducing the recognition of things to the smallest constituent unit, i.e., the atom which belongs to the scope of the philosophy of science, including the theory of material composition, the law of atomic transformation and reorganization in the transformation of matter, etc. In modern science, a further subdivision of the proton, neutron or meson into more subdivided particles, i.e., the quark is based on the topological structure $K_3, K_2$ inherited in the proton, neutron and the meson as described in [Nam], [Mat] and [XiL]. Similarly, the reduction of organisms to cells, genes and DNA in biology and particularly, the recognition of the double helix structure of DNA are such recognitive thoughts, see [Jin] and [Yan]. Under the guidance of this thought, the qualitative and quantitative methods with techniques in topology, combinatorics, graph theory and topological graph theory are the basic methods in the recognition of things, including the abstraction of things, the relationship between components, the simulation and the characterizing of component behaviors, see [Arm], [ChL], [GrT], [GrW], [Liu1]-[Liu2] and [MoT], etc. Of these, the most common models are labeled topological graphs (see references [Mao10], [Mao16]-[Mao17], [Mao23], and [Mao39]).

**6.2.** Technology is the knowledge, experience, technique and means accumulated continuously in order to meet the needs of humans and complies with laws of the nature in adapting to the nature, which is the sum of the methods, technique and the means of humans. So, *how do we view the nature of technology under the combinatorial notion?* Certainly, the social development is reflected by the improvement of human's ability adapting to the nature such as those of the improvement and innovation of artificial appliances, devices and facilities. Among them, a technology is a kind of methods, techniques and means to make such appliances, devices or facilities possible. Similar to the structure of matter,

a technology can be viewed as a combination of blocks which have their own combinatorial structure and the block is a combination of sub-blocks with a specific function, is the next level of technology. Similarly, the function sub-block is also a combination of the next level sub-blocks of functions, which is the next level technology. In this way, the decomposition of a technology level by level will eventually reaches to the elementary components in technical composition, likewise the elementary particles. Indeed, different technologies are different combinations of the elementary components of function or effect, i.e., *technology combination*, see W.B.Arthur's [Art]. Correspondingly, the relationship of science with technology is a dialectic relationship that promotes each other under the combinatorial structure because the science, on the one hand provides principles for technology and the combination of different principles corresponds to different technologies with different functions and the technology, on the other hand is the product by science leading, which promotes the progress of human material civilization. At the same time, it proposed problems need to be solved further in the development of science, including how to realize the harmonious coexistence of humans with the nature under the combination of different structure, the effect of different combination of technologies led by the combination of different scientific principles because the purpose of science with technology is to achieve the ultimate goal of human, i.e., the coexistence with the nature harmoniously.
**6.3.** Determining the size of a biological population is a typical counting problem of combinatorics. That is, to determine the number of certain kind of elements or with a particular structure in the set, which is generally solved by recursive formula. Certainly, there are two basic counting principles in combinatorics. The first one is the inclusion-exclusion principle which is essentially a principle to determine the number of elements in an overlapping sets, see reference [GrW]. The inclusion-exclusion principle, commonly known as the *elimination principle* can immediately deduces some important formulas such as the Euler's formula (3.13) in number theory, etc., see [Hua]; the second principle is the pigeonhole principle, also known as the drawer principle which determines the smallest number of pigeons in a pigeonhole while the number of pigeons entering the pigeonholes exceeds the number of pigeonholes. As a generalization of the pigeonhole principle, the Ramsey problem generally answers the existence of particular subsets while the number of elements in a set is large to a certain extent, see literature [GrW] and obtains counting conclusions on interesting populations.

## Chapter 4

# Characterizing the Universe

***********************************************************************

**Grasses do not Thank Spring Wind,**

**Woods do not Complain About Autumn Falling.**

**Who Drives the Four Season Moving?**

**All Things Rise and Rest Naturally.**

– *Sunrise Traveling* by Libai, a poet in Tang dynasty of China

## §1. **Relative Characterization**

All things are constantly moving in the universe. That is, the moving of an objective thing is absolute but the stationary is relative. However, the limitation of human recognition leads to a kind of combined recognition which applies the combinatorial notion to recognize objective things and holds on objective laws. In this context, *how can humans quantify the characteristics exhibited by an objective thing?* On a night of light cloudy, Dr.Ouyang points to the moon in the sky and asks Huizi: "*You see, the moon seems to be moveless in the sky. So, is it stationary or moving?*" Huizi replies: "*It is moving! The teacher told us that the moon is a satellite revolving around the earth in the class of science. At the same time, it keeps turning itself.*" Dr.Ouyang continuously asks her: "*Is the earth under our feet is stationary or moving?*" Huizi replies: "*It is moving! The earth rotates on its own axis as well as around the sun.*" Dr.Ouyang wishes to inspire his daughter for thinking and then further asks her: "*We stand here moveless, are we stationary or moving?*" Huizi smilingly says: "*We are not moving. Of course we are stationary!*" Dr.Ouyang asks her again: "*You should think it once more. We seem not in moving but the earth is in moving! Are we standing on the earth stationary or moving?*" While Dr.Ouyang asks her once more, Huizi is a little confused and says: "*I have no idea because you are stationary in my eyes but if the earth is taken into account, you and I are both in moving. Dad, how can a same thing be both in the stationary and moving?*" Dr.Ouyang tells Huizi that this is the *relativity* of objective things in moving. That is, while one says that a thing is in moving or at rest, the premise is that it is in moving or at rest with respect to a reference frame, which is the benchmark for discussing the moving of things. For example, why is Dr.Ouyang stationary in Huizi's eyes is because it is referring to Huizi as a benchmark or reference. Similarly, why Dr.Ouyang is in moving is because it is referring a reference to a planet such as the sun or a point outside the earth. This is the reason why we set up the coordinate system $\{O; x, y, z\}$ for characterizing change of things in the universe, i.e., assume the measuring of things in three directions of $x$, $y$ and $z$ under a unified frame of reference, also known as the measuring coordinates $(x, y, z)$ of change behavior. For example, if a point $P$ has respectively values 5, 4 and 6 in the $x$, $y$

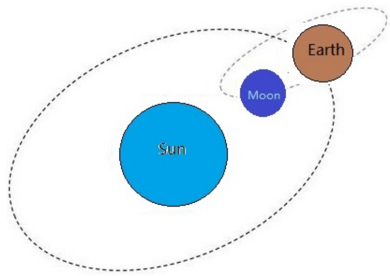



**Figure** 4.1. **Relative moving**

and $z$-axis, then the coordinates of the point $P$ is $(5,4,6)$, as shown in Figure 4.2. Here, the directed line segment $\overrightarrow{OP}$ from the origin $O$ of the reference system to the point $P$ is called the position vector of point $P$, denoted $\mathbf{r}_P$. In this way, the point $P$ can be characterized by its coordinates $(x,y,z)$ or the position vector $\mathbf{r}_P$. Furthermore, it is meaningful to establish such a reference frame in the universe for describing the behavior of things so that humans can easily describe and hold on the behavior of things. Notice that all humans live on the earth. In a sense, the reference frame established with the earth as the origin should be the most convenient for humans to recognize and examine other things in the universe. However, the earth itself is also in moving. It not only rotates around the sun at a rate of $v = 29.8km/s$ but also rotates at a rate of $\omega = 0.466km/s$ as shown in Figure 4.3.

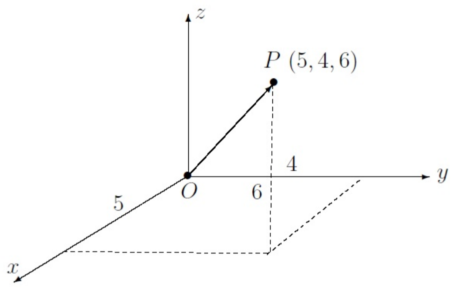



**Figure** 4.2. **Coordinate system**

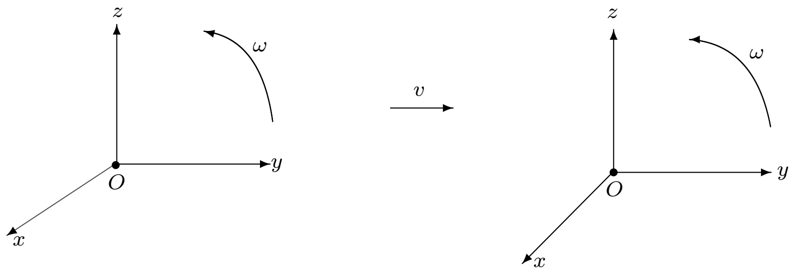



**Figure** 4.3. **Coordinate system in moving**

Notice that it is quite complicated to depict the loci of other planets, not as simple as the moon going round the earth in the reference frame with the earth as the origin. Then, *is there a reference frame that is immutable with respect to everything, i.e., describes things in the universe based on a fixed references frame* $\{O;\ x,y,z\}$, *called the absolute references frame?* Certainly, such an absolute reference frame should be existed in theory because one can choose a fixed point in the universe as the origin $O$ and any three spatial directions selected that are not in the same plane to be the coordinate axes $x$, $y$ and $z$, respectively. However, an absolute references frame determined by such a way is only a theoretical choice, has no practical significance because the moving of things is absolute but all the observations of humans are carried out in a relative coordinate system, which can not be verified in an absolute references frame and is not good for holding on the behavior of things.

**1.1.Affine Coordinates.** Generally, the coordinate system characterizing the behavior of things in space is called an *affine coordinate system.* In an affine coordinate system, the axes are not necessarily perpendicular to each other as long as they are not all in a lower dimensional space, i.e., 2-dimensional plane. The visual space of the human eyes is a 3-dimensional $\mathcal{T}^3$. That is, one can chooses 3 axes that are not in a plane and uniquely determine points in the space. At this point, one can chooses any point in space to be the origin $O$ and any three non-coplanar directions $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ with length $|\mathbf{e}_1| = |\mathbf{e}_2| = \mathbf{e}_3| = 1$ for the axis directions and then construct a reference frame $\{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, i.e., the *coordinate system* satisfying the existence of the unique coordinate values $x$, $y$ and $z$ for any point $P$, i.e., the geometric point

$$P \in \mathcal{T}^3 \ \leftrightarrow \ (x, y, z) \text{ or denoted by } \mathbf{r}_P = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 \tag{4.1}$$

is a $1-1$ mapping, where $x$, $y$ and $z$ are the projected values of the point $P$ onto the $x$-axis, $y$-axis and $z$-axis, respectively. Particularly, if the axis directions $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ are perpendicular to each other, the coordinate system $\{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is said to be a rectangular coordinate system as shown in Figure 4.2, which is the most commonly used coordinate system in practice. Notice that this idea of building a coordinate system can be generalized to extend the visual range of 3-dimension to $n$-dimension $\mathcal{T}^n$ for an integer $n \geq 2$, i.e., let $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$ be $n$ unit directions and then construct an $n$-dimensional coordinate system $\{O; \mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$. At this time, the coordinates of a point $P \in \mathcal{T}$ in this coordinate system is $(x_1, x_2, \cdots, x_n)$, namely there is an $1-1$ mapping

$$P \in \mathcal{T}^n \leftrightarrow (x_1, x_2, \cdots, x_n) \text{ or denoted by } \mathbf{r}_P = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \cdots + x_n\mathbf{e}_n. \tag{4.2}$$

Dr.Ouyang tells Huizi that any geometrical body is a set of geometric points in this way and it can be represented by the set of points in the rectangular coordinate system using the $1-1$ correspondence between geometric points and coordinates (4.1) and (4.2).

**(1)Line segment.** A line segment is the segment $AB$ between two points $A$, $B$ on a curve $\gamma$ in space. In particular, if $\gamma$ is a straight line, it is called a *straight line segment*, see Figure 4.4($a$) and ($b$).

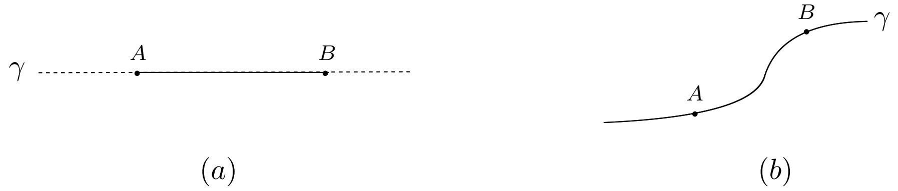



**Figure** 4.4. **Line segment**

Let $\gamma$ be a curve on an independent variable $t$, i.e., $\gamma(t)$ with $-\infty < t < +\infty$ and $A = \gamma(a), B = \gamma(b), b \geq a$. Then, the line $AB$ can be represented as $AB = \{x \in \gamma | a \leq t \leq b\}$. Particularly, if $\gamma$ is a straight line, one can chooses $\gamma$ to be the coordinate axis such as the $x$ axis to establish a coordinate system $\{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and assumes that the coordinates of points $A$ and $B$ are respectively $(a, 0, 0)$ and $(b, 0, 0)$ with $b \geq a$. Then, the line segment $AB = \{(x, 0, 0) | a \leq x \leq b\}$.

**(2)Plane.** A plane is the set of points in space in which lines passes through a fixed point $P_0 = (x_0, y_0, z_0)$ is perpendicular to a fixed direction $(A, B, C)$. In a coordinate system $\{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, a plane can be expressed by a point set $\{P = (x, y, z) \in \mathbb{R}^3 | P_0P \perp (A, B, C)\}$, where the symbol $\perp$ denotes that the line segment $P_0P$ is perpendicular to the fixed direction $(A, B, C)$. Dr.Ouyang tells Huizi that there is a more concise expression, i.e., $\{(x, y, x) \in \mathbb{R}^3 | Ax + By + Cz = D\}$ of plane if learned the vector inner product, where $D$ is a constant and the equation $Ax + By + Cz = D$ is called the plane equation and generally, $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ is called a hyperplane equation in Euclidean space $\mathbb{R}^n$ and its essence is a Euclidean space of dimension $n - 1$.

**(3)Cuboid.** A typical character a cuboid $\Delta$ is its each face is a rectangle and the intersection of faces is a straight line, called the edges of the cuboid. The points where the edges meet are called vertices of the cuboid. Notice that the intersecting faces are perpendicular to each other and the intersecting edges are perpendicular to each other also in a cuboid. Therefore, the intersection point of three edges can be taken as the origin $O$ and the three edges are respectively the $x$, $y$ and $z$ axes for establishing a coordinate system, called the *Cartesian coordinate system*, as shown in Figure 4.5. Assuming that the coordinates of point $C_1$ is $(a, b, c)$. Then, the cuboid can be represented by a point set

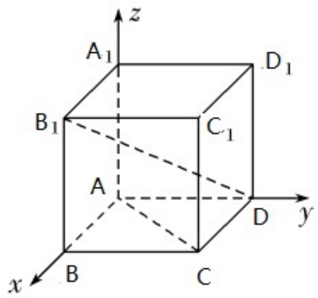


**Figure** 4.5. **Cuboid**

$$\{(x, y, z) \mid 0 \leq x \leq a, 0 \leq y \leq b, 0 \leq z \leq c\} \tag{4.3}$$

in this coordinate system. If the $x$, $y$ and $z$ axes are not the three intersecting edges of the cuboid but three directions $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ parallel to the three intersecting edges of the cuboid, assuming the coordinates of point $A$ are $(a_1, b_1, c_1)$ and point $C_1$ are $(a_2, b_2, c_2)$ in this coordinate system such as those shown in Figure 4.5, then the point set of cuboid $\Delta$ in this coordinate system can be denoted by

$$\{(x, y, z) \mid a_1 \leq x \leq a_2, b_1 \leq y \leq b_2, c_1 \leq z \leq c_2\}. \tag{4.4}$$

It should be noted that the point set representation of (4.4) can be extended to cuboid in an $n$-dimensional Cartesian coordinate system, i.e., a cuboid is represented by

$$\{(x_1, x_2, \cdots, x_n) \mid a_1 \leq x_1 \leq a_2, b_1 \leq x_2 \leq b_2, \cdots, c_1 \leq x_n \leq c_2\}. \tag{4.5}$$

in the $n$-dimensional Cartesian coordinate system for an integer $n \geq 3$. Particularly, if $a_2 - a_1 = b_2 - b_1 = \cdots = c_2 - c_1$, i.e., all lengths of edges in the cuboid are equal, such a cuboid (4.5) is said to be a *cube* of dimension $n$.

**(4)Sphere.** A geometrical subspace consisting of points in space whose distance from a fixed point $O$ is equal to a fixed length is called a *spherical surface*, and a subspace enclosed by a spherical surface in space is called a *sphere* or *ball*. In 3-dimension space, a typical feature of sphere $\mathbb{S}^2$ is the points $P$ on a fixed distance to the centre $O$, namely $|OP| = R$ for a constant $R > 0$. Thus, it is important to determine the distance between two points for holding on a geometrical sphere. But, *how do we determine the distance between two points* $(x_1, y_1, z_1)$ *and* $(x_2, y_2, z_2)$ *in space?* The length of a line segment $AB$ can be easily determined on the number axis, i.e., the distance between two points $A$ and $B$ on the number axis. Without loss of generality, let the coordinates of $A$ and $B$ on the number axis is $x_1, x_2$ with $x_2 \geq x_1$. Then, the length of the line $AB$ is $|AB| = |x_2 - x_1|$, as shown in Figure 4.7($a$). So the next, *how do we determine the distance between two points $A$ and $B$, i.e., the length of the line segment $|AB|$ in a 3-dimensional space?*

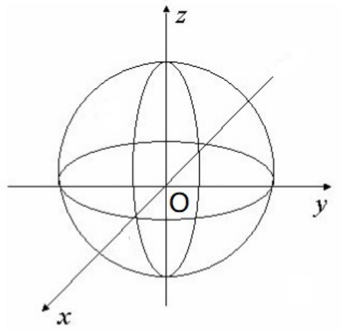



**Figure** 4.6. **Sphere**

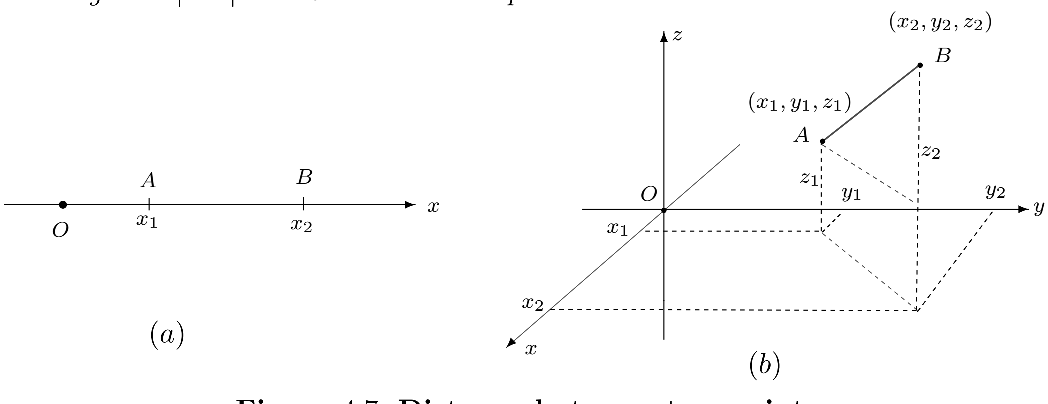



**Figure** 4.7. **Distance between two points**

According to the Pythagorean theorem in elementary geometry, i.e., the hypotenuse length of a right triangle is the square root of the sum of the length squares of its right angle sides, which can be easily known from Figure 4.7($b$) that

$$|AB| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \tag{4.6}$$

Thus, the point sets of a spherical surface $\mathbb{S}^2$ and a ball $\mathbb{B}^3$ in the coordinate system established with the center $O$ of sphere as the origin are respectively

$$\begin{aligned}\mathbb{S}^2 &= \left\{ (x,y,z) \mid x^2+y^2+z^2=R^2 \right\},\\ \mathbb{B}^3 &= \left\{ (x,y,z) \mid x^2+y^2+z^2\leq R^2 \right\}\end{aligned} \tag{4.7}$$

by definition. Similarly, the set of points of a circle with radius $R$ can be denoted by $\mathbb{S} = \left\{(x,y)|x^2+y^2=R^2\right\}$ and a disc $\mathbb{B}^2 = \left\{(x,y)|x^2+y^2\leq R^2\right\}$. And generally, the point set of a spherical surface and a sphere are respectively

$$\begin{aligned}\mathbb{S}^{n-1} &= \left\{ (x_1,x_2,\cdots,x_n) \mid x_1^2+x_2^2+\cdots+x_n^2=R^2\right\},\\ \mathbb{B}^n &= \left\{ (x_1,x_2,\cdots,x_n) \mid x_1^2+x_2^2+\cdots+x_n^2\leq R^2\right\}.\end{aligned} \tag{4.8}$$

in an $n$-dimensional Cartesian coordinate system, $n\geq 3$. Furthermore, if the origin of the coordinate system is not at the center of the sphere, let the coordinates of the center be $(x_1^0,x_2^0,\cdots,x_n^0)$. Then, the point set of the spherical surface and sphere are respectively

$$\begin{aligned}&\left\{ (x_1,x_2,\cdots,x_n) \mid \left(x_1-x_1^0\right)^2+\left(x_2-x_2^0\right)^2+\cdots+\left(x_n-x_n^0\right)^2=R^2\right\},\\ &\left\{ (x_1,x_2,\cdots,x_n) \mid \left(x_1-x_1^0\right)^2+\left(x_2-x_2^0\right)^2+\cdots+\left(x_n-x_n^0\right)^2\leq R^2\right\}.\end{aligned} \tag{4.9}$$

Particularly, the point set of a circle and a disk on the plane are respectively

$$\begin{aligned}&\left\{ (x,y) \mid \left(x-x^0\right)^2+\left(y-y^0\right)^2=R^2\right\},\\ &\left\{ (x,y) \mid \left(x-x^0\right)^2+\left(y-y^0\right)^2\leq R^2\right\}\end{aligned} \tag{4.10}$$

and the point sets of the 2-dimensional spherical surface and 3-dimensional sphere are respectively

$$\begin{aligned}&\left\{ (x,y,z) \mid \left(x-x^0\right)^2+\left(y-y^0\right)^2+\left(z-z^0\right)^2=R^2\right\},\\ &\left\{ (x,y,z) \mid \left(x-x^0\right)^2+\left(y-y^0\right)^2+\left(z-z^0\right)^2\leq R^2\right\},\end{aligned} \tag{4.11}$$

where the $(x^0,y^0)$ and $(x^0,y^0,z^0)$ are the coordinates of the 2-dimensional center and 3-dimensional center of sphere, respectively.

**(5)Cylinder.** A cylinder $\mathcal{C}$ is a polyhedron in which two faces are the parallel congruence, and the intersection lines of the remaining adjacent faces are parallel, as shown in Figure 4.5. Geometrically, a cylinder can be viewed as a translation of a 2-dimensional polygon in the plane $(x,y)$ on the third dimensional direction $z$ of space. In this way, a cylinder can be represented generally as a set of points

$$\mathcal{C} = \{ (x,y,z) \mid F(x,y)\leq 0, a\leq z\leq b\},$$

where $F(x, y) = 0$ is a 2-dimensional polygon in the plane $(x, y)$ and $a, b$ are constants. For example, if $F(x, y) = \left(x - x^0\right)^2 + \left(y - y^0\right)^2 - R^2 = 0$ is a 2-dimensional disk in the plane $(x, y)$ then

$$\{\ (x, y, z) \mid \left(x - x^0\right)^2 + \left(y - y^0\right)^2 \le R^2, a \le z \le b\} \tag{4.12}$$

is the point set of the cylinder, where $\left(x^0, y^0\right)$ is the coordinates of the center of the 2-dimensional disc $F(x, y)$.

**1.2.Coordinate Transformation.** Certainly, there are many methods for constructing an affine coordinate system $\{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, including the choice of different origin $O$ and different axis directions $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. Among them, a natural question is that *for any two affine coordinate systems* $\{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ *and* $\{O'; \mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$, *whether there is a* $1-1$ *transformation*

$$T : \{O;\ \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\} \ \leftrightarrow\ \left\{O';\ \mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\right\},$$

*for coordinates* $(x, y, z)$ *and* $(x', y', z')$ *of a point* $P$ *respectively in the coordinate systems* $\{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ *and* $\{O'; \mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ *holding with*

$$\begin{cases} (x', y', z') & = & T(x, y, z), \\ (x,\ y,\ z) & = & T^{-1}(x', y', z'), \end{cases}$$

*such as those shown in Figure* 4.8*?*

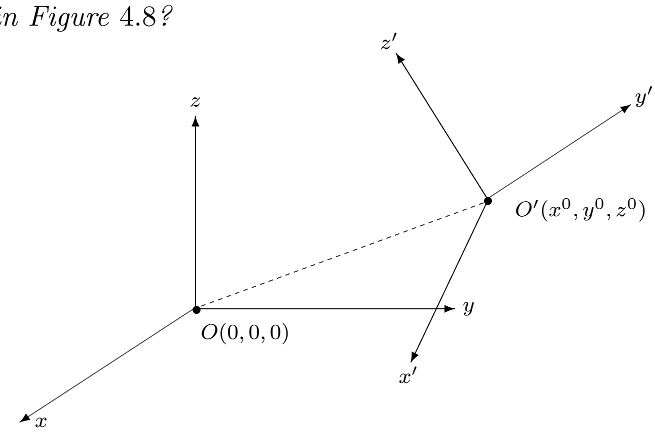



**Figure** 4.8. **Coordinate transformation**

Dr.Ouyang tells Huizi that it is geometrically easy to obtain such a transformation $T$. Firstly, according to Figure 4.8, one can translates the origin $O$ and $O'$ of the two coordinate systems coincide together. Without loss of generality, assume the coordinate of $O'$ is $(x^0, y^0, z^0)$ in the coordinate system $\{O, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and define a transformation

$$T_p = \begin{cases} x' & = & x + x^0, \\ y' & = & y + y^0, \\ z' & = & z + z^0; \end{cases} \tag{4.13}$$

Secondly, rotate the axis $\mathbf{e}_1'$ overlap with $\mathbf{e}_1$, keeping the axes $\mathbf{e}_2$, $\mathbf{e}_3$ unchanged and denote by $T_1$ the transformation obtained; Third, rotate the axis $\mathbf{e}_2'$ overlap with $\mathbf{e}_2$, keeping the axes $\mathbf{e}_1', \mathbf{e}_3$ unchanged and denote the obtained transformation by $T_2$; Finally, rotate the axis $\mathbf{e}_3'$ overlap with $\mathbf{e}_3$, keeping the axes $\mathbf{e}_1', \mathbf{e}_2'$ unchanged and denote the obtained transformation by $T_3$. In this way, we get a transformation $T$ through by $T_p$ and $T_1, T_2, T_3$ which transforms $\{O', \mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'\}$ to $\{O, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, i.e.,

$$T = T_3 T_2 T_1 T_p. \tag{4.14}$$

Notice that each step in the previous construction of transformations $T_p, T_1, T_2$, $T_3$ is reversible. Thus, $T$ is also reversible, i.e., $T^{-1}$ exists. Furthermore, Dr.Ouyang tells Huizi that an algebraic representation of the transformation $T$ could be deduced. Firstly, assume that the transformations $T_1, T_2$ and $T_3$ are respectively

$$\begin{cases} T_1 : \mathbf{e}_1' = a_{11}\mathbf{e}_1 + a_{12}\mathbf{e}_2 + a_{13}\mathbf{e}_3, \\ T_2 : \mathbf{e}_2' = a_{21}\mathbf{e}_1 + a_{22}\mathbf{e}_2 + a_{23}\mathbf{e}_3, \\ T_3 : \mathbf{e}_3' = a_{31}\mathbf{e}_1 + a_{32}\mathbf{e}_2 + a_{33}\mathbf{e}_3. \end{cases} \tag{4.15}$$

where, $a_{ij},\ 1 \le i, j \le 3$ are the transformation coefficients of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$; Secondly, let the coordinates of point $P$ be $(x, y, z)$ and $(x', y', z')$ respectively in the coordinate system $\{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{O'; \mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'\}$. Then, by the geometric transformations $T_p, T_1, T_2, T_3$ in Figure 4.8, we get that

$$\begin{aligned} x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3 &= \left(x^0\mathbf{e}_1 + y^0\mathbf{e}_2 + z^0\mathbf{e}_3\right) + \left(x'\mathbf{e}_1' + y'\mathbf{e}_2' + z'\mathbf{e}_3'\right) \\ &= x^0\mathbf{e}_1 + y^0\mathbf{e}_2 + z^0\mathbf{e}_3 + x'\left(a_{11}\mathbf{e}_1 + a_{21}\mathbf{e}_2 + a_{31}\mathbf{e}_3\right) \\ &\quad + y'\left(a_{12}\mathbf{e}_1 + a_{22}\mathbf{e}_2 + a_{32}\mathbf{e}_3\right) + z'\left(a_{13}\mathbf{e}_1 + a_{23}\mathbf{e}_2 + a_{33}\mathbf{e}_3\right) \\ &= \quad \left(x^0 + a_{11}x' + a_{12}y' + a_{13}z'\right)\mathbf{e}_1 \\ &\quad + \left(y^0 + a_{21}x' + a_{22}y' + a_{23}z'\right)\mathbf{e}_2 \\ &\quad + \left(z^0 + a_{31}x' + a_{32}y' + a_{33}z'\right)\mathbf{e}_3. \end{aligned}$$

Consequently,

$$\begin{cases} x = x^0 + a_{11}x' + a_{12}y' + a_{13}z', \\ y = y^0 + a_{21}x' + a_{22}y' + a_{23}z', \\ z = z^0 + a_{31}x' + a_{32}y' + a_{33}z' \end{cases}$$

or a matrix expression

$$(x, y, z)^t = (x^0, y^0, z^0)^t + (a_{ij})_{3\times 3}\,(x', y', z')^t \tag{4.16}$$

of transformation, where

$$(x,y,z)^t=\begin{pmatrix} x\\ y\\ z\end{pmatrix},\quad (a_{ij})_{3\times 3}=\begin{pmatrix} a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{pmatrix},\quad (x',y',z')^t=\begin{pmatrix} x'\\ y'\\ z'\end{pmatrix}$$

are respectively the transpose of the coordinate of point $P$ in the coordinate system $\{O;\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$, the transformation matrix of coefficients and the coordinate tranpose of point $P$ in the coordinate system $\{O';\mathbf{e}_1',\mathbf{e}_2',\mathbf{e}_3'\}$ with the addition and multiplication of matrix defined as follows

$$\begin{aligned}(a_{ij})_{3\times 3}+(b_{ij})_{3\times 3} &= (a_{ij}+b_{ij})_{3\times 3}\,,\\ (a_{ij})_{3\times 3}\ (b_{ij})_{3\times 3} &= \begin{pmatrix} \sum\limits_{i=1}^{3} a_{1i}b_{i1} & \sum\limits_{i=1}^{3} a_{1i}b_{i2} & \sum\limits_{i=1}^{3} a_{1i}b_{i3}\\ \sum\limits_{i=1}^{3} a_{2i}b_{i1} & \sum\limits_{i=1}^{3} a_{2i}b_{i2} & \sum\limits_{i=1}^{3} a_{2i}b_{i3}\\ \sum\limits_{i=1}^{3} a_{3i}b_{i1} & \sum\limits_{i=1}^{3} a_{3i}b_{i2} & \sum\limits_{i=1}^{3} a_{3i}b_{i3}\end{pmatrix}.\end{aligned}$$

Particularly, Dr.Ouyang tells Huizi that if the coordinate systems $\{O;\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ and $\{O';\mathbf{e}_1',\mathbf{e}_2',\mathbf{e}_3'\}$ both are the rectangular coordinate system, it is easy to verify the matrix $A=(a_{ij})_{3\times 3}$ in (4.16) satisfying $AA^t=I_{3\times 3}$. Such a mattrix $A$ is called the *orthogonal matrix* because it comes from the transformation of rectangular coordinate systems or the Cartesian coordinate systems.

Notice that the numbers $x,y,z$ in the coordinates $(x,y,z)$ are respectively the projective distance of the point from the origin $O$ on the $x$-axis, $y$-axis and $z$-axis in the rectangular coordinate system. But in general, the numbers $x,y,z$ can not be always measured by length. For example, the polar coordinates look like this. In fact, the polar coordinates are extended on the rectangular coordinates.

Usually, assume $P$ to be a point in the Euclidean space $\mathbb{R}^3$, connect $P$ with the origin $O$ and let $P'$ be the orthogonal projection point of point $P$ on the $xOy$ plane. Let $\alpha$ $(0\leq\alpha<2\pi)$ be the angle between $OP'$ and $Ox$ and let $\beta$ $(-\pi/2\leq\leq\beta\leq\pi/2)$ be the angle of $OP$ with the plane $xOy$, $\rho$ denote the length of $OP$. Then, the polar coordinates of the point $P$ is $(\rho,\alpha,\beta)$, as shown in Figure 4.9. And then, *how do we determine the relationship between the rectangular coordinates* $(x,y,z)$ *of a point* $P$ *with its polar coordinates* $(\rho,\alpha,\beta)$*?* Notice that for a point $P$ in a rectangular coordinate system as shown in Figure 4.9, there are $|OP|=\rho$ and angles $\alpha,\beta$. It is easy to determine the relationship of the coordinates $(x, y, z)$ with $(\rho, \alpha, \beta)$ by

$$\begin{cases} x = \rho\cos\alpha\cos\beta, \\ y = \rho\sin\alpha\cos\beta, \\ z = \rho\sin\beta. \end{cases} \tag{4.17}$$

Certainly, it is also a coordinate transformation, i.e., $T : (x, y, z) \leftrightarrow (\rho, \alpha, \beta)$.

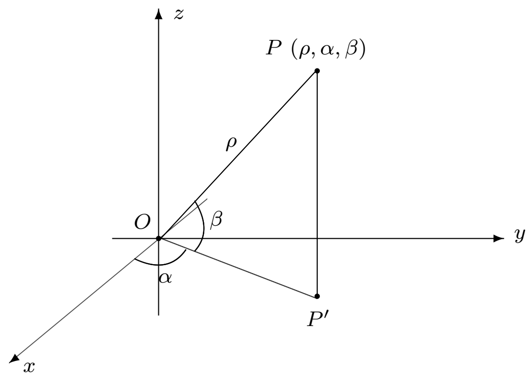



**Figure** 4.9. **Polar coordinate system**

**1.3.Relativity Principle.** Generally, different coordinate systems describe the behavior of the same thing with different equations. For example, if the origin $O$ or the point $(x^0, y^0, z^0)$ in the coordinate system is chosen as the center of the sphere, the spherical surface and sphere are described by the equations (4.7) or (4.11), respectively. Indeed, they have different expressions. However, the spherical surface is the same spherical surface and the sphere is also the same sphere, they ares not really different. Thus, the difference between the equations (4.7) and (4.11) is the difference of equation forms caused by the different coordinate systems rather than the difference of the spherical surface or sphere depicted, i.e., it is only different expressions but not the different truths of things. In other words, the equations used to characterize the behavior of things intensely depend on the coordinate system, which are relative to the coordinate system established by humans.

Dr.Ouyang tells Huizi that this epistemically involves a fundamental philosophical question, i.e., *is an objective thing subject to the human's will?* If so, humans can control all things and then become the supreme power holder in the universe whatever he or she wants! This is also a theme expressed in lots of science fictions and films. However, Laozi said that *the heaven and the earth view all things as straw dogs* in his *Tao Te Ching*, where the words "*the heaven*" and "*the earth*" refer to the universe. However, the civilization history of thousands of years shows that humans not only can't to dominate the universe but also need to imitate and subject to the heaven and the earth. That is, the law of human development needs to follow, i.e., the words that "*human follows earth, earth* *follows heaven, heaven follows Tao and Tao follows nature*" said by Laozi in his *Tao Te Ching* because humans are nothing else but equivalent to the "*ants*" in front of the nature, which are very insignificant. In this case, it is very important to eliminate the idolatry or fantasy on the nature in human recognition, including some auxiliary means introduced by human recognition such as the reference frame of relativity so as to truly hold on the behavior of things. Then, *how do we eliminate the different equations that looks different in describing the behavior of things due to the different coordinate systems and restore the true nature of things?* Dr.Ouyang tells Huizi that the prerequisite for answering this question is a correct understanding of various aids introduced by humans, such as the coordinate systems.

In response, Einstein's *general relativity*, also known as the *covariation principle* published in 1916, is a reflection on the aids that humans introduced for understanding things. Its essence is the embodiment of the philosophical thought "*objective things are not subject to human will*" in the recognition of things, specifically expressed as follows:

**Relativity Principle** *An equation that describing the evolving law of an objective thing should have the same form in all coordinate systems.*

Then, *how do we understand the general relativity or the relativity principle of Einstein?* Dr.Ouyang says that a proper understanding of Einstein's general relativity involves two things. One is the form of equation that describes the evolving laws of objective things; Another is the connotation of the evolving law of objective things. For the first, Einstein proposed the equation that describing the physical laws should be in the same form under the action of coordinate transformations $T$, i.e, it should be in the covariant form. As an example, Einstein proposed the gravitational field equations in the covariant form on the hypothesis of the equivalence of gravitation with the inertial interaction. Here, the covariant form used by Einstein is actually a generalization of the invariance of the differential form. So, *what is the invariance of differential form?* Dr.Ouyang explains to his daughter that let $y = f(x)$, $x = \phi(t)$, i.e., $y = f(\phi(t))$ for a differentiable function $f$. By the chain differential rule of composite functions, we get that

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} \ \Leftrightarrow \ dy = \frac{dy}{dx}\frac{dx}{dt}dt = \frac{dy}{dx}dx = \frac{dy}{dt}dt, \tag{4.18}$$

i.e., the differential form $dy = y'_x dx$ is the same as the differential form $dy = y'_t dt$, not affected by the coordinate transformation $x = \phi(t)$. Generally, the transformation invariance of (4.18) can be extended to $n$-dimensional Euclidean space for any integer $n \geq 3$, obtain the *n-dimensional tensor space* and then discuss the geometrical properties, which is nothing else but the *Riemannian geometry*. After then, the covariant form is adopted to rewrite the known evolving equation of things such as the equation of particle moving, the equation of electromagnetic field and so on, which became the main work once by Einstein himself and many of his followers in the 20th century. Certainly, these covariant forms of equations must satisfy the relativity principle of Einstein.

But, *is it a formal revolution or a recognitive improvement of things by using the relativity principle to rewrite the equation of the known evolving laws of things?* Dr.Ouyang tells Huizi that it can be definitely concluded that this is only a revolution on the equation's form, not improving on the human recognitive ability, which is essentially to hide the coordinates of points in its neighboring domain with a compact and neat form but no changing or expanding the human's perception of things law. Then, *how do we understand or evaluate the function of coordinate systems in the recognition of things?* Notice that the general relativity proposed by Einstein does not deny the function of coordinate system in the recognition of things but it reaffirms that the evolution of objective things does not subject to human's will. Thus, we should not conclude the different laws of things based on different expression forms caused by different coordinate systems, without any denying on the meaning of coordinate system because the coordinate system is the benchmark framework for human quantitatively describing the evolution of things. Therefore, Dr.Ouyang says that the relativity principle can naturally lead to the equal-right principle following, which is beneficial to human recognition of things in the universe.

**Equal-Rights Principle** *All coordinate systems are equally important for understanding things in the universe.*

The equal-right principle shows that in the recognition of things in the universe, the important thing lies in the recognition of the evolution of things, not limited to the expression form. Therefore, a new understanding on the Einstein's general relativity of coordinate systems is not to abandon the reference frames, only get the covariant forms of evolving equations in the human recognition of things but needs to build such sorts of reference frames which are advantageous for observation and inspection of the results obtained by humans according to the characteristics of thing's evolution. That's what we introduce the coordinate system to do.

## §2. Vector Operations

Notice that most things in the nature are developing and changing along a certain direction. For example, all of the flow of a river, the wind with its direction, the forces, velocities and displacements that characterize the moving of objects have direction. Such a quantity both with size and direction is called "*vector*", which is an important conception for describing the changing behavior of things. Usually, it is represented by a line segment with an arrow $\overrightarrow{AB}$ or boldface letters $\mathbf{a}, \mathbf{b}, \mathbf{c}, \cdots$, and those quantities that have only magnitude but without direction such as the number of planets, field area, reservoir capacity, etc., are called *scalars*.

**Figure** 4.10. **Tug of War**

One day after school, Huizi happily tells Dr.Ouyang that they had beaten Class two in the tug-of-war on the playground that afternoon and the teacher had given a red flag to each contestant in her class. Dr.Ouyang wishes to teach his daughter a little vector knowledge and then asks her: "*In the tug of war, the students to pull the rope in which direction?*" Huizi replies:"*Of course, each student pulls hard in her own direction because only the red silk hanging from the rope enters her side of the field is the winner!*" Dr.Ouyang continuously asks her:"*Does anyone in your class pulls the rope in the direction of the other side?*" Huizi is taken aback:"*Of course not! such one must be a traitor because she helps the other side win?!*" Dr.Ouyang explains that the force each student pulls on a rope has magnitude and direction, which is a kind of vector. If a contestant pushes in the opposite direction, the tug of war on his or her side will be cancelled out. Only if all the contestants in her class are pulling towards their own side, i.e., the direction of the force is in the same direction, the resultant force is equal to the sum of the tug-of-war force of each contestant, and only if the resultant force of the tug of war is greater than the other side, they can win the game. This is the condition of winning the tug of war.

In general, it is agreeable that a directed line segments with the same length and direction represents a same vector $\mathbf{a}$. That is, the vector $\mathbf{a}$ can move parallel and start at any point in the space. For a vector $\mathbf{a}$, its the length is defined to be the magnitude of $\mathbf{a}$, denoted by $|\mathbf{a}|$, and the vector of length 0 is called the *zero vector*, denoted $\mathbf{0}$. Notice that the direction of the zero vector is uncertain. A vector of length 1 is called the *unit vector*. For example, the unit directions $\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n$ in (4.2) and (4.1) are all unit vectors. A vector of the same length but in the opposite direction as the vector $\mathbf{a}$ is called the *inverse* of vector $\mathbf{a}$ and denoted $-\mathbf{a}$.

Dr.Ouyang tells Huizi that the main feature of the inverse is its actions with the positive vector is the zero vector. For example, if one of her classmates pulls the rope in the other's direction, it will cancel out their own tug of war and maybe lose the game. So, *can we generalize the action of such forces, summarize the actions by analogy with that* *of the numbers and characterize the action or relationship in things?* The answer is Yes! Dr.Ouyang says that such a generalization is the vector addition, scalar multiplication, the inner and outer products of vectors.


**Figure** 4.11. **Vector addition**

**2.1.Vector Addition.** Adding one number to another such as $2 + 3 = 5$ is pure quantity addition because the numbers 2 and 3 have no direction. But if add one vector to another, Dr.Ouyang tells Huizi that she has to consider both the quantity and the direction to ensure that the sum is still a vector. Notice that a vector can start at any point in space. So, for vectors $\mathbf{a}$ and $\mathbf{b}$, we can assume that the vector $\mathbf{b}$ starts at the end point of the vector $\mathbf{a}$ and define the addition vector $\mathbf{a} + \mathbf{b}$ is such a vector that from the starting point of the vector $\mathbf{a}$ to the end point of the vector $\mathbf{b}$, as shown in Figure 4.11($a$). Of course, it is also possible to assume that the vector $\mathbf{a}$ and the vector $\mathbf{b}$ have the same starting point $O$ and construct a parallelogram with $\mathbf{a}$ and $\mathbf{b}$ as adjacent edges. Then, the diagonal line in the parallelogram starting at $O$ is the vector $\mathbf{a} + \mathbf{b}$, as shown in Figure 4.11($b$).

So, by the triangle inequality we generally have $|\mathbf{a} + \mathbf{b}| \leq |\mathbf{a}| + |\mathbf{b}|$ and the equality holds if and only if the vectors $\mathbf{a}$ and $\mathbf{b}$ are in the same direction. For the tug of war organized by Huizi's school, the direction of pulling force in each team is in the same direction, the resultant force of the team is the summation of all their forces, namely $|\mathbf{a}_1| + |\mathbf{a}_2| + \cdots + \mathbf{a}_s|$, where $s$ is the number of participants in the tug-of-war for each team. Then, if someone pulls the force in the opposite direction to their own pair, i.e., in the opposite direction of the other team, its pulling force is an inverse vector $-\mathbf{F}$ and the effect is to offset rather than contribute to the tug of war force of themselves team, need to subtract $\mathbf{F}$ from the tug of war force in the previous. Particularly, the sum of a vector $\mathbf{a}$ with the vector $-\mathbf{a}$, i.e., $\mathbf{a} + (-\mathbf{a}) = \mathbf{0}$ is a zero vector.

Generally, Dr.Ouyang says that for vectors $\mathbf{a}$ and $\mathbf{b}$, define the vector $\mathbf{a}-\mathbf{b} = \mathbf{a}+(-\mathbf{b})$. In this way, the sum of a vector $\mathbf{a}$ with its inverse vector $-\mathbf{a}$ can be also written as $\mathbf{a}-\mathbf{a} = \mathbf{0}$.

For any vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$, we can easily obtain the following by the definition of vector addition:

(1) Commutative law, namely, $\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a}$;

(2) Associative law, namely, $(\mathbf{a} + \mathbf{b}) + \mathbf{c} = \mathbf{a} + (\mathbf{b} + \mathbf{c})$;

(3) Unique zero, namely, there is a unique vector $\mathbf{0}$ for an arbitrary vector $\mathbf{a}$ holding with $\mathbf{a}+\mathbf{0}=\mathbf{a}$;

(4) Unique inverse, namely, there is a unique vector $-\mathbf{a}$ for an arbitrary vector $\mathbf{a}$ holding with $\mathbf{a}+(-\mathbf{a})=\mathbf{0}$.

Among them, the commutative law and associative law of vector addition can be immediately obtained from Figure 4.12($a$) and ($b$), respectively.

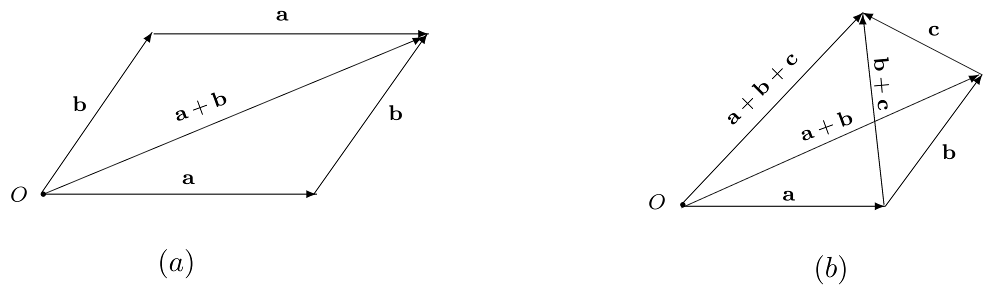



**Figure** 4.12. **Addition laws of vectors**

Then, *how do we know the uniqueness of the zero vector* $\mathbf{0}$*?* Dr.Ouyang tells Huizi that if there is a vector $\mathbf{x}$ holding with $\mathbf{a}+\mathbf{x}=\mathbf{a}$, we can minus the vector $\mathbf{a}$ on both sides and then get $\mathbf{x}=\mathbf{0}$, namely such a vector $\mathbf{x}$ must be $\mathbf{0}$. Similarly, if there is a vector $\mathbf{y}$ holding with $\mathbf{a}+\mathbf{y}=\mathbf{0}$, we can minus the vector $\mathbf{a}$ on both sides and get $\mathbf{a}-\mathbf{a}+\mathbf{y}=\mathbf{0}-\mathbf{a}$, i.e., $\mathbf{y}=-\mathbf{a}$, namely the inverse of vector $\mathbf{a}$ must be the vector $-\mathbf{a}$. This concludes the uniqueness of the vector $\mathbf{0}$ and the uniqueness of the inverse $-\mathbf{a}$ for a given vector $\mathbf{a}$.

**2.2.Scalar Multiplication.** Usually, the scalar multiplication of vector is similar to multiplying numbers by numbers. Dr.Ouyang says that is, for a given real number $\lambda$ and vector $\mathbf{a}$ define $\lambda\mathbf{a}$ to be a vector with length $|\lambda\mathbf{a}|=|\lambda||\mathbf{a}|$, i.e., if $\lambda>0$, this vector has the same direction as the vector $\mathbf{a}$ but if $\lambda<0$, this vector is in the opposite direction to the vector $\mathbf{a}$.

For any real number $\lambda$ and the zero vector $\mathbf{0}$, there are $0|\mathbf{a}|=0|\mathbf{a}|=0$ by definition. So, $0\mathbf{a}=\mathbf{0}$. Similarly, there is $\lambda\mathbf{0}=\mathbf{0}$. Generally, for any vector $\mathbf{a}$, $\mathbf{b}$ and real numbers $\lambda$, $\mu$, there are equalities following:

(1) $1\mathbf{a}=\mathbf{a},\ (-1)\mathbf{a}=-\mathbf{a}$;

(2) $\lambda(\mu\mathbf{a})=(\lambda\mu)\mathbf{a}$;

(3) $(\lambda+\mu)\mathbf{a}=\lambda\mathbf{a}+\mu\mathbf{a}$;

(4) $\lambda(\mathbf{a}+\mathbf{b})=\lambda\mathbf{a}+\lambda\mathbf{b}$。

Dr.Ouyang tells Huizi that with the conception of vector addition and the scalar multiplication, the position vectors $\mathbf{r}_P=\mathbf{e}x_1+\mathbf{e}y_2+\mathbf{e}z_3$ and $\mathbf{r}_P=\mathbf{e}_1+x_2\mathbf{e}_2+\cdots+\mathbf{e_n}x_n$ in the (4.2) and (4.1) are no longer a formal expression of point $P$ or $\mathbf{r}_P$ but the position vectors or $\overrightarrow{OP}$. Consequently, let $\mathbf{r}_{P_1} = x_1\mathbf{e}_1 + y_1\mathbf{e}_2 + z_1\mathbf{e}_3$, $\mathbf{r}_{P_2} = x_2\mathbf{e}_1 + y_2\mathbf{e}_2 + z_2\mathbf{e}_3$ be to two vectors in the 3 - dimensional affine coordinates. Then, we get that

$$\begin{aligned} \mathbf{r}_{P_1} + \mathbf{r}_{P_2} &= (x_1\mathbf{e}_1 + y_1\mathbf{e}_2 + z_1\mathbf{e}_3) + (x_2\mathbf{e}_1 + y_2\mathbf{e}_2 + z_2\mathbf{e}_3) \\ &= (x_1 + x_2)\,\mathbf{e}_1 + (y_1 + y_2)\,\mathbf{e}_2 + (z_1 + z_2)\,\mathbf{e}_3 \end{aligned}$$

by definition. That is, the coordinates of the vector $\mathbf{r}_{P_1} + \mathbf{r}_{P_2}$ is the coordinates of

$$(x_1, y_1, z_1) + (x_2, y_2, z_2) = (x_1 + x_2, y_1 + y_2, z_1 + z_2)\,. \tag{4.19}$$

Notice that the (4.19) is a representation of vector addition or the parallelogram rule in the coordinate system. That is, *the sum of coordinates is equal to the sum of coordinate components*, which holds also for any $n$-dimensional space of $n \geq 2$.

It is worth noted that there is an angle $\theta$ between vectors $\mathbf{a}$ and $\mathbf{b}$, as shown in Figure 4.13. Using the angle $\theta$, the projection of vector $\mathbf{b}$ on the direction of vector $\mathbf{a}$ is $\mathbf{b}\cos\theta$. Meanwhile, the projection of $\mathbf{b}$ on the direction perpendicular to the vector $\mathbf{a}$ is $\mathbf{b}\sin\theta$. This reflects the influence of vector $\mathbf{b}$ on vector $\mathbf{a}$ to some extent, and describes the interaction behavior of things. Dr.Ouyang tells Huizi that it is necessary to introduce the inner and outer product of vectors.

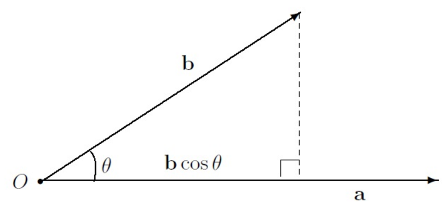


**Figure** 4.13. **Vector projection**

**2.3.Vector Inner Product.** The inner product $\mathbf{a} \cdot \mathbf{b}$ of vectors $\mathbf{a}$ and $\mathbf{b}$ is a scalar, corresponding to the product of the lengths of vector $\mathbf{a}$ and vector $\mathbf{b}\cos\theta$, i.e., $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$. From the definition of the inner product of vectors $\mathbf{a}$ and $\mathbf{b}$, it can be deduced that $\cos\theta = \mathbf{a} \cdot \mathbf{b}/|\mathbf{a}||\mathbf{b}|$. Whence, the vector inner product can solve the problems related to the length of vectors or the angle between vectors. Particularly, if the vector $\mathbf{a}$ is perpendicular to the vector $\mathbf{b}$, i.e., $\mathbf{a}\perp\mathbf{b}$ or $\mathbf{a}$ and $\mathbf{b}$ are orthogonal, we have $\theta = 90°$ or $\cos\theta = 0$, get that $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta = 0$, and vice versa. Therefore, we know that

$$\mathbf{a}\perp\mathbf{b} \quad \Leftrightarrow \quad \mathbf{a} \cdot \mathbf{b} = 0. \tag{4.20}$$

The inner product of any vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ has the properties following:

(1) Positive definiteness, namely $\mathbf{a} \cdot \mathbf{a} \geq 0$ and the equality holds only if $\mathbf{a} = \mathbf{0}$;

(2) Symmetry, namely $\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$;

(3) Linearity, namely $(\lambda\mathbf{a}) \cdot \mathbf{b} = \lambda(\mathbf{a} \cdot \mathbf{b})$ for any real number $\lambda$;

(4) Distributive law, namely $\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$.

Among them, the equations $(1)-(3)$ can be immediately obtained from the definition of vector inner product. For the distributive (4), it should be noted that the projection of $\mathbf{b}+\mathbf{c}$ on the direction of the vector $\mathbf{a}$ is equal to the sum of the projection $\mathbf{b}\cos\alpha$ of $\mathbf{b}$ on the direction of vector $\mathbf{a}$ and the projection $\mathbf{c}\cos\alpha$ of the vector $\mathbf{c}$ on the direction of vector $\mathbf{a}$ such as those shown in Figure 4.14.

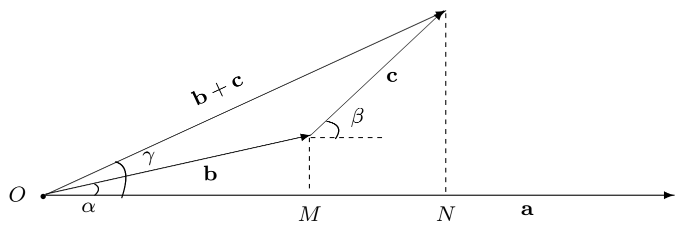



**Figure** 4.14. **Distributive laws**

Whence, we have

$$\begin{aligned}\mathbf{a}\cdot(\mathbf{b}+\mathbf{c}) &= |\mathbf{a}||\mathbf{b}+\mathbf{c}|\cos\gamma = |\mathbf{a}|(|OM|+|MN|)\cos\gamma\\ &= |\mathbf{a}|(|\mathbf{b}|\cos\alpha+|\mathbf{c}|\cos\beta) = |\mathbf{a}||\mathbf{b}|\cos\alpha+|\mathbf{a}||\mathbf{c}|\cos\beta\\ &= \mathbf{a}\cdot\mathbf{b}+\mathbf{a}\cdot\mathbf{c}.\end{aligned}$$

Similarly, there are also $\mathbf{a}\cdot\lambda\mathbf{b}=\lambda(\mathbf{a}\cdot\mathbf{b})$, $(\mathbf{a}+\mathbf{b})\cdot\mathbf{c}=\mathbf{a}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{c}$, etc.

Notice that $\mathbf{e}_1\perp\mathbf{e}_2$, $\mathbf{e}_1\perp\mathbf{e}_3$, $\mathbf{e}_2\perp\mathbf{e}_3$, i.e., $\mathbf{e}_1\cdot\mathbf{e}_2=0$, $\mathbf{e}_1\cdot\mathbf{e}_3=0$, $\mathbf{e}_2\cdot\mathbf{e}_3=0$ in a rectangular coordinate system $\{O;\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$, there is a simpler way to write the inner product. For example, let $\mathbf{a}=x_1\mathbf{e}_1+y_1\mathbf{e}_2+z_1\mathbf{e}_3$, $\mathbf{b}=x_2\mathbf{e}_1+y_2\mathbf{e}_2+z_2\mathbf{e}_3$ be two vectors in $\{O;\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$. Then, the property of the inner product of vectors implies that

$$\begin{aligned}\mathbf{a}\cdot\mathbf{b} &= (x_1\mathbf{e}_1+y_1\mathbf{e}_2+z_1\mathbf{e}_3)\cdot(x_2\mathbf{e}_1+y_2\mathbf{e}_2+z_2\mathbf{e}_3)\\ &= x_1x_2\mathbf{e}_1\cdot\mathbf{e}_1+y_1y_2\mathbf{e}_2\cdot\mathbf{e}_2+z_1z_2\mathbf{e}_3\cdot\mathbf{e}_3+x_1y_2\mathbf{e}_1\cdot\mathbf{e}_2\\ &\quad +x_1z_2\mathbf{e}_1\cdot\mathbf{e}_3+y_1x_1\mathbf{e}_2\cdot\mathbf{e}_1+y_1z_2\mathbf{e}_2\cdot\mathbf{e}_3+z_1x_2\mathbf{e}_3\cdot\mathbf{e}_1+z_1y_2\mathbf{e}_3\cdot\mathbf{e}_2\\ &= x_1x_2\mathbf{e}_1\cdot\mathbf{e}_1+y_1y_2\mathbf{e}_2\cdot\mathbf{e}_2+z_1z_2\mathbf{e}_3\cdot\mathbf{e}_3 = x_1x_2+y_1y_2+z_1z_2,\end{aligned}$$

namely, the coordinates of the vector $\mathbf{a}\cdot\mathbf{b}$ are

$$(x_1,y_1,z_1)\cdot(x_2,y_2,z_2)=x_1x_2+y_1y_2+z_1z_2. \tag{4.21}$$

Notice that the equality (4.21) is the vector inner product, i.e., *the vector inner product equals to the sum of the coordinate products of vectors* in a rectangular coordinate system. Particularly, we know that $\mathbf{a}\perp\mathbf{b}$ *if and only if the sum of coordinate products of vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *is* 0. For example, the vector $\mathbf{a}=(3,0,5)$ is orthogonal to $\mathbf{b}=(0,6,0)$ because $3\times0+0\times6+5\times0=0$. Generally, for any integer $n\geq2$, (4.21) holds for any

$n$-dimensional Cartesian coordinate system, i.e

$$(x_1, x_2, \cdots, x_n) \cdot (x_1', x_2', \cdots, x_n') = x_1x_1' + x_2x_2' + \cdots + x_nx_n'. \tag{4.22}$$

Now, Dr.Ouyang tells Huizi that it can show why the hyperplane equation in an Euclidean space $\mathbb{R}^n$ is the first-order equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$. Firstly, let's look at the case of $n = 3$. Choose a point $P_0 = (x_0, y_0, z_0)$. Let $P = (x, y, z)$ be any point in the plane $\mathbb{R}^2$ and let $\mathbf{v} = (A, B, C)$ be a given direction vector. According to the definition of plane, there must be $\overrightarrow{P_0P} \perp \mathbf{v}$ and we can therefore deduce the equation of plane in the Euclidean space $\mathbb{R}^3$ as follows:

$$\begin{aligned} \overrightarrow{P_0P} \perp \mathbf{v} \quad &\Rightarrow \quad (x - x_0, y - y_0, z - z_0) \perp (A, B, C) \\ &\Rightarrow \quad (x - x_0, y - y_0, z - z_0) \cdot (A, B, C) = 0 \\ &\Rightarrow \quad Ax + By + Cz = D. \end{aligned}$$

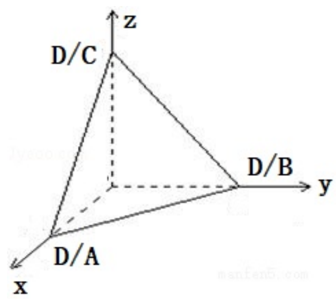



**Figure** 4.15. **A plane**

That is, the plane equation of 3-dimensional Euclidean space is a linear equation $Ax + By + Cz = D$, where $D = Ax_0 + By_0 + Cz_0$ is a constant such as those shown in Figure 4.15.

Similarly, let $P_0' = (x_{10}, x_{20}, \cdots, x_{n0})$ be another point respect to the point $P' = (x_1, x_2, \cdots, x_n)$ in the $n$-dimensional Euclidean space $\mathbb{R}^n$. Then, for a given vector $\mathbf{v}' = (a_1, a_2 \cdots, a_n)$ direction we can also deduce the hyperplane equation is a linear equation $a_1x_1 + a_2x_2 + \cdots + a_nx_n = b$ in $\mathbb{R}^n$, where $b = a_1x_{10} + a_2x_{20} + \cdots + a_nx_{n0}$ is a constant.

**2.4.Vector Outer Product.** The vector outer product $\mathbf{a} \times \mathbf{b}$ of two vectors $\mathbf{a}$ and $\mathbf{b}$ is still a vector with the length $|\mathbf{a}||\mathbf{b}| \sin\theta$ and vector direction perpendicular to both vectors $\mathbf{a}$, $\mathbf{b}$ and forming a right-handed system $\{\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}\}$. Then, *what is the right-handed system in here and what is a left-handed system?* Dr.Ouyang asks Huizi to hold out her right hand so that the forefinger is directed to the vector $\mathbf{a}$, the middle finger is directed to the vector $\mathbf{b}$ and the angle between the two fingers is less than $180^o$. Then, the thumb is directed to the vector $\mathbf{a} \times \mathbf{b}$, as shown in Figure 4.16. Such a system $\{\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}\}$ is called a *right-handed system*. Otherwise, if the three vectors are consistent with the forefinger, middle finger and thumb in the situation of the left hand, it is called a *left-handed system*. Here, $\theta$ is the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$.

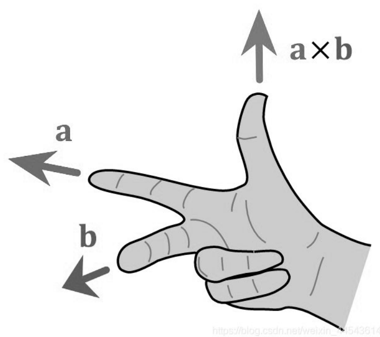



**Figure** 4.16. **Right-hand system**

By definition, if $\mathbf{a}\times\mathbf{b}=\mathbf{0}$, i.e., the zero vector we must have $|\mathbf{a}||\mathbf{b}|\sin\theta=\mathbf{0}$. So, we get that $|\mathbf{a}|=0$, $|\mathbf{b}|=\mathbf{0}$ or $\sin\theta=0$. Now if $|\mathbf{a}|=0$ or $|\mathbf{b}|=0$ then the vector $\mathbf{a}$ or vector $\mathbf{b}$ must be the zero vector and if $\sin\theta=0$, then the angle between the vectors $\mathbf{a}$ and $\mathbf{b}$ must be $0^o$, i.e., the vector $\mathbf{a}$ is parallel or collinear with the vector $\mathbf{b}$, denoted by $\mathbf{a}/\!/\mathbf{b}$. Whence, we know that

$$\mathbf{a}\times\mathbf{b}=\mathbf{0} \quad\Leftrightarrow\quad \mathbf{a}=\mathbf{0},\ \mathbf{b}=0 \text{ or } \mathbf{a}/\!/\mathbf{b}. \tag{4.23}$$

Notice that if the vector $\mathbf{a}$ is not parallel to the vector $\mathbf{b}$, we can translate them starting at the same point $O$ such that the vector $\mathbf{a}$ and $\mathbf{b}$ over the edge of a parallelogram and the length $\mathbf{a}\times\mathbf{b}$, i.e., $|\mathbf{a}\times\mathbf{b}|=|\mathbf{a}||\mathbf{b}|\sin\theta$ is exactly the area of this parallelogram, and the vector $\mathbf{a}\times\mathbf{b}$ is perpendicular to this parallelogram. So, by the definition of inner product of vectors, we must have $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{a}=\mathbf{a}\cdot(\mathbf{a}\times\mathbf{b})=0$, $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{b}=\mathbf{b}\cdot(\mathbf{a}\times\mathbf{b})=0$.

Furthermore, by the definition of the outer product of vectors, if $\{O;\mathbf{e}_1,\mathbf{e}_2,\mathbf{e}_3\}$ is a right-handed Cartesian coordinate system, i.e., $\mathbf{e}_1\perp\mathbf{e}_2$, $\mathbf{e}_1\perp\mathbf{e}_3,\mathbf{e}_2\perp\mathbf{e}_3$, Dr.Ouyang tells Huizi that there must be

$$\mathbf{e}_1\times\mathbf{e}_2=\mathbf{e}_3,\ \mathbf{e}_2\times\mathbf{e}_3=\mathbf{e}_1,\ \mathbf{e}_3\times\mathbf{e}_1=\mathbf{e}_2. \tag{4.24}$$

Generally, for any vectors $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ and a real number $\lambda$, there are laws of the outer vector product of vectors following:

(1) Anti-commutative law, namely $\mathbf{a}\times\mathbf{b}=-\mathbf{b}\times\mathbf{a}$. *Why is it so?* Because $\mathbf{a}\times\mathbf{b}$ and $\mathbf{b}\times\mathbf{a}$ are the same length but they go in opposite directions in the right-hand system;

(2) Linearity, namely $(\lambda\mathbf{a})\times\mathbf{b}=\mathbf{a}\times(\lambda\mathbf{b})=\lambda(\mathbf{a}\times\mathbf{b})$. *Why is it so?* Because if assume the angle between vectors $\mathbf{a}$ and $\mathbf{b}$ is $\theta$, then $|(\lambda\mathbf{a})\times\mathbf{b}|=|\lambda\mathbf{a}||\mathbf{b}|\sin\theta=|\lambda||\mathbf{a}\times\mathbf{b}|$ by the definition of outer product and then, if $\lambda>0$ the direction of $\lambda\mathbf{a}$ is the same as the vector $\mathbf{a}$, we know the direction of $\lambda\mathbf{a}\times\mathbf{b}$ is the same as the vector $\lambda(\mathbf{a}\times\mathbf{b})$, i.e., $\lambda\mathbf{a}\times\mathbf{b}=\lambda(\mathbf{a}\times\mathbf{b})$; if $\lambda<0$ then the direction of $\lambda\mathbf{a}$ is the negative direction as the vector $\mathbf{a}$, we know the direction of $\lambda\mathbf{a}\times\mathbf{b}$ is the negative direction as the vector $\mathbf{a}\times\mathbf{b}$ and then the direction of $\lambda\mathbf{a}\times\mathbf{b}$ is the same as the vector $\lambda(\mathbf{a}\times\mathbf{b})$. Whence, $(\lambda\mathbf{a})\times\mathbf{b}=\lambda(\mathbf{a}\times\mathbf{b})$. Similarly, we know that $\mathbf{a}\times(\lambda\mathbf{b})=\lambda(\mathbf{a}\times\mathbf{b})$;

(3) Distributive law, namely $\mathbf{a}\times(\mathbf{b}+\mathbf{c})=\mathbf{a}\times\mathbf{b}+\mathbf{a}\times\mathbf{c}$ (Left-distributive law), $(\mathbf{a}+\mathbf{b})\times\mathbf{c}=\mathbf{a}\times\mathbf{c}+\mathbf{b}\times\mathbf{c}$ (Right-distributive law)。

Without loss of generality, assume $\mathbf{a}\perp\mathbf{b}$ and $\mathbf{a}\perp\mathbf{c}$ in the left-distributive law by (4.23). Otherwise, decompose the vectors $\mathbf{b}$ and $\mathbf{c}$ to be $\mathbf{b}=\mathbf{b}^{\perp}+\mathbf{b}^{/\!/}$, $\mathbf{b}=\mathbf{c}^{\perp}+\mathbf{c}^{/\!/}$, $\mathbf{b}+\mathbf{c}=(\mathbf{b}+\mathbf{c})^{\perp}+(\mathbf{b}+\mathbf{c})^{/\!/}$ along the direction of vector $\mathbf{a}$ and its vertical direction, where $\perp$ and $/\!/$ denote the component that perpendicular or parallel to the vector $\mathbf{a}$ in the decomposition. According to (4.23), we have $\mathbf{a}\times\mathbf{b}^{/\!/}=\mathbf{0}$, $\mathbf{a}\times\mathbf{c}^{/\!/}=\mathbf{0}$ and $\mathbf{a}\times(\mathbf{b}+\mathbf{c})^{/\!/}=\mathbf{0}$. Whence, we only verify the distributive law in the case of $\mathbf{a} \times (\mathbf{b}+\mathbf{c})^{\perp} = \mathbf{a} \times \mathbf{b}^{\perp} + \mathbf{a} \times \mathbf{c}^{\perp}$, see Figure 4.17. Furthermore, we assume $\mathbf{a}$ is the unit vector, i.e., $|\mathbf{a}| = 1$ in the distributive law. Otherwise, because $\mathbf{a} = |\mathbf{a}|\mathbf{a}^*$, where $\mathbf{a}^*$ is a unit vector, i.e., $|\mathbf{a}^*| = 1$ and then, by the linearity (2) of outer product, if $\mathbf{a}^* \times (\mathbf{b}+\mathbf{c}) = \mathbf{a}^* \times \mathbf{b} + \mathbf{a}^* \times \mathbf{c}$, there must be $\mathbf{a} \times (\mathbf{b}+\mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$.

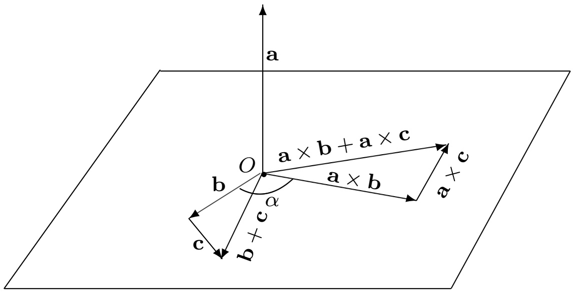



**Figure** 4.17. **Distributive law of outer product**

So, by the definition of outer product of vectors, $\mathbf{a} \times \mathbf{b}$ is the vector obtained by $\mathbf{b}$ right-rotating $90^o$ around $\mathbf{a}$ and $\mathbf{a} \times \mathbf{c}$ is the vector obtained by $\mathbf{c}$ right-rotating $90^o$ around $\mathbf{a}$ because $|\mathbf{a}| = 1,\ \sin 90^o = 1$. By definition, the vectors $\mathbf{b}, \mathbf{c}, \mathbf{b}+\mathbf{c}$ form a triangle $\triangle$. So, its right-rotating $90^o$ around $\mathbf{a}$ is also a triangle $\triangle'$ with two progressive edges of vectors $\mathbf{a} \times \mathbf{b}$ and $\mathbf{a} \times \mathbf{c}$ but the vector on its third edge is the vector obtained by $\mathbf{b}+\mathbf{c}$ right-rotating $90^o$ around $\mathbf{a}$, i.e., $\mathbf{a} \times (\mathbf{b}+\mathbf{c})$, see Figure 4.17. Notice that under the assumption that $\mathbf{a}$ is a unit vector, the triangle $\triangle'$ is the result of $\triangle$ right-rotating $90^o$ around $\mathbf{a}$, i.e., the vector on the third edge in the triangle $\triangle'$ is $\mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$ by the vector addition. Whence, we get that $\mathbf{a} \times (\mathbf{b}+\mathbf{c}) = \mathbf{a} \times \mathbf{b} + \mathbf{a} \times \mathbf{c}$.

The right-distributive law $(\mathbf{a}+\mathbf{b}) \times \mathbf{c} = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}$ can be immediately verified by the left distributive law because

$$\begin{aligned}(\mathbf{a}+\mathbf{b}) \times \mathbf{c} &= -\mathbf{c} \times (\mathbf{a}+\mathbf{b}) = -(\mathbf{c} \times \mathbf{a} + \mathbf{c} \times \mathbf{b}) \\ &= -(\mathbf{c} \times \mathbf{a} + (-\mathbf{c} \times \mathbf{b}) = \mathbf{a} \times \mathbf{c} + \mathbf{b} \times \mathbf{c}.\end{aligned}$$

Dr.Ouyang tells Huizi that the outer product of vectors in rectangular coordinates could be expressed in simpler coordinates. For example, let $\{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be a rectangular coordinate system and vectors $\mathbf{a} = x_1\mathbf{e}_1 + y_1\mathbf{e}_2 + z_1\mathbf{e}_3$, $\mathbf{b} = x_2\mathbf{e}_1 + y_2\mathbf{e}_2 + z_2\mathbf{e}_3$. Then, it can be known that

$$\begin{aligned}\mathbf{a} \times \mathbf{b} &= (x_1\mathbf{e}_1 + y_1\mathbf{e}_2 + z_1\mathbf{e}_3) \times (x_2\mathbf{e}_1 + y_2\mathbf{e}_2 + z_2\mathbf{e}_3) \\ &= x_1x_2\mathbf{e}_1 \times \mathbf{e}_1 + y_1x_2\mathbf{e}_2 \times \mathbf{e}_1 + z_1x_2\mathbf{e}_3 \times \mathbf{e}_1 + x_1y_2\mathbf{e}_1 \times \mathbf{e}_2 \\ &\quad + y_1y_2\mathbf{e}_2 \times \mathbf{e}_2 + z_1y_2\mathbf{e}_3 \times \mathbf{e}_2 + x_1z_2\mathbf{e}_1 \times \mathbf{e}_3 + y_1z_2\mathbf{e}_2 \times \mathbf{e}_3 + z_1z_2\mathbf{e}_3 \times \mathbf{e}_3 \\ &= (x_1y_2 - x_2y_1)\,\mathbf{e}_1 \times \mathbf{e}_2 + (z_3x_1 - x_1z_3)\,\mathbf{e}_3 \times \mathbf{e}_1 + (x_2z_3 - z_3x_2)\,\mathbf{e}_2 \times \mathbf{e}_3\end{aligned}$$

$$= \quad (x_2z_3 - z_3x_2)\,\mathbf{e}_1 + (z_3x_1 - x_1z_3)\,\mathbf{e}_3\mathbf{e}_2 + (x_1y_2 - x_2y_1)\,\mathbf{e}_3,$$

by the linearity of vector outer product, distributive law and (4.24), i.e., the coordinates of the vector $\mathbf{a} \times \mathbf{b}$ are

$$(x_1, y_1, z_1) \times (x_2, y_2, z_2) = (x_2z_3 - z_3x_2, z_3x_1 - x_1z_3, x_1y_2 - x_2y_1)_{\circ} \qquad (4.25)$$

## §3. **Linear Space**

A vector has both length and direction which is an abstraction for humans to characterize the changing behavior of things. Dr.Ouyang tells Huizi that human recognition of things follows the general process, i.e., from the concrete to the abstract then rising to a theory, and then come back to guide the practice of humans by the theoretical recognition. For example, the addition and scalar multiplication operations of vectors follow the linear property, i.e., the piling or growth on the unit scale and the translation does not change the space. Such a space can be abstracted to a typical space, namely the linear space that characterizes vectors with relations and then, recognize the behavior of things in the universe. In fact, this is the general process of "*practice → recognizing → more practice → more recognizing*" in the human recognition of things. For example, the essence of a point $P$ corresponding to a vector $\overrightarrow{OP} = x\mathbf{e}_1 + y\mathbf{e}_2 + z\mathbf{e}_3$ or a geometrical body corresponding to a set of points $P$ in the coordinate system $\{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is based on the origin $O$, the units vectors $\mathbf{e}_1$, $\mathbf{e}_2$ and $\mathbf{e}_3$ for determining the coordinates $(x, y, z)$, where the coordinate values $x$, $y$ and $z$ are respectively the projections of point $P$ on the directions of $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$. It is used to characterize a point $P$ or the geometry of a thing by the linear combination or accumulation respectively with the unit vectors $\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ as a reference frame on the origin $O$, as shown in Figure 4.18. Whence, a vector can be viewed as a kind of abstract conception with both size and the direction for characterizing things in the universe following the basis of linear characteristics of vectors. Then, *for a given family of vector elements, can we obtain all vectors from the linearity to characterize the behavior of things?* The answer is Yes, i.e., the linear space introduced to solve!

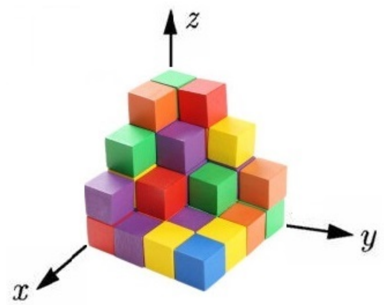



**Figure** 4.18. **Linear Combination**

**3.1.Numbers and Fields.** Unlike the vectors, Dr.Ouyang tells Huizi that the numbers, including natural numbers, integers, rational numbers, real numbers and complex numbers are a purely quantities that characterizes all things, also known as the *scalars*. Here, the natural number is the result of the counting, the integer is an extension of the natural number including 0 and negative natural numbers; a rational number is an extension of integers that can be expressed as the ratio$m/n$ of two integers $m, n$. Particularly, a rational number is an integer if its denominator $m = 1$. It should be noted that the early humans believed that the things could be characterized only by the natural numbers, integers and the rational numbers. For example, the Pythagorean school in the ancient Greece complied with the *general reductive principle*, which is the Pythagorean assertion that "*all things are numbers*", see Figure 4.19. Here, the "number" of Pythagorean refers to the rational number. Accordingly, the ratio between any two geometric quantities $Q$ and $V$ can be expressed as the ratio $m/n$ of two integers $m$ and $n$.


**Figure** 4.19. **Hippasus disaster of** $\sqrt{2}$

However, a scholar named Hippasus of Pythagorean school discovered that the diagonal length of a square with side length 1 is $\sqrt{2}$ but $\sqrt{2}$ can not be expressed as the ratio of two integers, because if there exist two such integers $m$ and $n$ with $\sqrt{2} = m/n$, without loss of generality we can assume that the greatest common divisor between $m$ and $n$ is 1. If not, we can make them coprime, i.e., dividing both $m$ and $n$ by the greatest common divisor between them. Notice that if the $\sqrt{2} = m/n$ we must have $2m^2 = n^2$. In this case, $m$ must be even. Without loss of generality, assume $m = 2s$. Then, we get that $4s^2 = 2n^2$, i.e., $2s^2 = n^2$. So, $n$ must also be even. In this way, there is at least one common divisor 2 between integers $m$ and $n$, contradicting to the assumption that the greatest common divisor is 1. Whence, such integers $m$ and $n$ can not exist. This discovery of Hippasus destroyed the assertion that "*all things are numbers*" and confirmed the existence of irrational numbers. At the same time, Hippasus' discovery shook the foundations of the Pythagorean school and brought about himself death. It is said that for maintaining the dignity of the school, the loyal Pythagorean disciples captured Hippasus after then and sank him into the Mediterranean sea to punish for his disloyalty to Pythagorean school.

A typical characteristic of integers $\mathbb{Z}$ is that the addition "+" and multiplication "×" of integers are still integers. That is, the integer set $\mathbb{Z}$ is closed under the additive and multiplicative operations. Meanwhile, there exist respectively the addition and multiplicative units 0 and 1, the additive inverse $-k$ of $k$ and multiplicative inverse $k^{-1}$ of an integer $k \neq 0$, holding with $k+0=0+k=k$, $k\times 1=1\times k=k$ and $k+(-k)=0$, $k\times k^{-1}=1$ which form the *integer field* $\mathbb{Z}$. Furthermore, Dr.Ouyang tells Huizi that this property of integers $\mathbb{Z}$ would lead to the conception of *number field* or abstract *field*. Generally, for any subset $S$ of real numbers $\mathbb{R}$, if it is closed on the additive and multiplicative operations, contains the additive identity 0 and multiplicative identity 1 and if for any element $\lambda$ of $S$ there exists an additive inverse $-\lambda$ and a multiplicative inverse $\lambda^{-1}$ for $\lambda \neq 0$ further, such an $S$ is called a *field*. For example, the set of rational numbers $\mathbb{N}$ or the set of real numbers $\mathbb{R}$ satisfies these properties under the addition "+" and multiplication "$\times$", which are called the *rational number field* and *real number field*, respectively.

Notice that the integer field and the real number field both have infinitely many elements, called the *infinite field*. So, *is there a field only with finite elements?* Dr.Ouyang tells Huizi that the answer is Yes! For example, let $\mathbb{Z}_2=\{0,1\}$ be a set only with two elements. Define the addition and multiplication on the 2 elements following

$$\begin{aligned} &0+0=0,\ 0+1=1,\ 1+0=1,\ 1+1=0;\\ &0\times 0=0,\ 0\times 1=0,\ 1\times 0=0,\ 1\times 1=1. \end{aligned} \tag{4.26}$$

Then, the elements 0 and 1 are respectively the additive and multiplicative units of $\mathbb{Z}_2$, and $-0=0, -1=1$, $1^{-1}=1$. So, the 2-set $\mathbb{Z}_2$ is a finite field of 2 elements under the additive and multiplicative operations defined in (4.26).

Generally, let $p$ be a prime number and let $p\mathbb{Z}=\{\overline{0},\overline{1},\cdots,\overline{p-1}\}$ be the set consisting of the remainders of integers divided by $p$. Define the additive operation $\overline{k}+\overline{l}=\overline{k+l}$, the multiplicative operation $\overline{k}\times\overline{l}=\overline{kl}$, i.e, the remainder of the operation result divided by $p$ such as $\overline{1}+\overline{p-1}=\overline{0},\overline{2}+\overline{p-1}=\overline{1},\cdots$ and $\overline{1}\times\overline{p-1}=\overline{p-1},\overline{2}\times\overline{p-1}=\overline{p-2},\cdots$ etc. Then, the set $p\mathbb{Z}$ is a finite field with $p$ elements whose additive identity is $\overline{0}$ and the multiplicative identity is $\overline{1}$, and the inverse $\overline{x}$ of $\overline{k}\in p\mathbb{Z}$ is unique as the smallest positive integer $x$ satisfying the Diophantine equation $kx=pl+1$. For example, let $p=3$. Then, the inverses of $\overline{1}$ and $\overline{2}$ are respectively the elements $\overline{1}$ and $\overline{2}$, i.e., $\overline{1}^{-1}=\overline{1}$, $\overline{2}^{-1}=\overline{2}$ because the smallest positive integer solutions of the Diophantine equations $1x=3l+1$ and $2x=3l+1$ are respectively $x=1$ and $x=2$.

**3.2.Linear Spanning.** Dr.Ouyang tells Huizi that the linear spanning of vectors is applying the vector addition and scalar multiplication to find a new vector based on the given vectors $\mathbf{a},\mathbf{b},\mathbf{c},\cdots$, where the scalar $\lambda$ can be an integer, a rational number, a real number, a complex number or an element of the field $\mathscr{F}$. Generally, denote the set consisting of vectors $\mathbf{a},\mathbf{b},\mathbf{c}\cdots$ by $\mathcal{C}$, called the *spanning set* and each element $\lambda\in\mathscr{F}$ is a scalar. Then, a vector spanning by vectors in $\mathcal{C}$ is

$$\lambda_1\mathbf{a} + \lambda_2\mathbf{b} + \lambda_3\mathbf{c} + \cdots, \tag{4.27}$$

where, vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}\cdots \in \mathcal{C}$ and $\lambda_1, \lambda_2, \lambda_3, \cdots \in \mathscr{F}$. Usually, one can write down all vectors spanned by the vectors in $\mathcal{C}$ as $\langle\mathcal{C}\rangle$, i.e

$$\langle\mathcal{C}\rangle = \{\ \lambda_1\mathbf{a} + \lambda_2\mathbf{b} + \lambda_3\mathbf{c} + \cdots \mid \mathbf{a}, \mathbf{b}, \mathbf{c}\cdots \in \mathcal{C}, \lambda_1, \lambda_2, \lambda_3, \cdots \in \mathcal{F}\ \}. \tag{4.28}$$

Generally, if $\mathcal{C}$ or $\mathscr{F}$ is an infinite set, then the vector spanning $\langle\mathcal{C}\rangle$ must be an infinite set; Conversely, if neither $\mathcal{C}$ nor $\mathscr{F}$ are infinite sets, then the vector spanning $\langle\mathcal{C}\rangle$ must be finite. For example, if the spanning set $\mathcal{C}$ consists of three orthogonal unit vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ and the field $\mathcal{F}$ is the integer field $\mathbb{Z}$, then the vector spanning $\langle\mathcal{C}\rangle$ is an infinite set. Without loss of generality, let the origin of vectors $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ be at the point $O$. Then, the linear spanning $\langle\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\rangle$ is the geometrical set consisting of all lattice points $P$, including the origin $O$ in space, i.e., the coordinates of all position vectors $\overrightarrow{OP}$ are integers. On the other hand, if the field $\mathscr{F}$ is finite, then there are only finite vectors spanned by vectors in $\mathcal{C}$. For example, let $\mathscr{F} = \mathbb{Z}_2$. Then, the scalars $\lambda = 0$ or 1. By (4.28) there are only 8 vectors in the linear spanning $\langle\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\rangle$ such as those shown in Figure 4.20, namely

$$\langle\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\rangle = \{\mathbf{0}, \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1+\mathbf{e}_2, \mathbf{e}_1+\mathbf{e}_3, \mathbf{e}_2+\mathbf{e}_3, \mathbf{e}_1+\mathbf{e}_2+\mathbf{e}_3\}.$$

Dr.Ouyang explains that there are two kinds of problems should be solved in the linear spanning of vectors in this way. One is that given the scalar $\lambda_1, \lambda_2, \lambda_3, \cdots$ in (4.27) to determine the spanning vector $\lambda_1\mathbf{a} + \lambda_2\mathbf{b} + \lambda_3\mathbf{c} + \cdots$, which is a simple computational problem. Another is to determine how a given vector $\boldsymbol{\alpha}$ is spanned by vectors in $\mathcal{C}$. That is, to determine scalars $\lambda_1$, $\lambda_2, \lambda_3, \cdots$ in the field $\mathscr{F}$ holding with $\boldsymbol{\alpha} = \lambda_1\mathbf{a} + \lambda_2\mathbf{b} + \lambda_3\mathbf{c} + \cdots$.

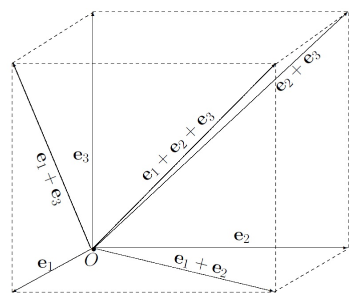



**Figure** 4.20. **Spanning vector**

Certainly, both of these two problems can be transformed to the coordinate calculation in a given coordinate system $\{O; \mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$, $n \geq 2$. For example, let the spanning vectors be respectively

$$\mathbf{e}_1 = (1, \underbrace{0, \cdots, 0}_{n-1}), \mathbf{e}_2 = (0, 1, \underbrace{0, \cdots, 0}_{n-2}), \cdots, \mathbf{e}_n = (\underbrace{0, \cdots, 0}_{n-1}, 1).$$

Then, there must be

$$\lambda_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2 + \cdots + \lambda_n\mathbf{e}_n = (\lambda_1, \lambda_2, \cdots, \lambda_n), \tag{4.29}$$

and the coordinates of the spanning vector are $(\lambda_1, \lambda_2, \cdots, \lambda_n)$, which is the solution for the first problem. For the second problem, assume that the coordinates of vector $\boldsymbol{\alpha}$ is $(\lambda_1, \lambda_2, \cdots, \lambda_n)$ in the coordinate system $\{O; \mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$. Then, there are

$$\boldsymbol{\alpha} = (\lambda_1, \lambda_2, \cdots, \lambda_n) = \lambda_1 \mathbf{e}_1 + \lambda_2 \mathbf{e}_2 + \cdots + \lambda_n \mathbf{e}_n. \tag{4.30}$$

**3.3.Linear Space.** Notice that the linear spanning $\langle \mathcal{C} \rangle$ of vectors in $\mathcal{C}$ is closed over the field $\mathscr{F}$. That is, for any vectors $\boldsymbol{\alpha}, \boldsymbol{\beta}$ in $\langle \mathcal{C} \rangle$ and scalars $\lambda, \mu$ in the field $\mathscr{F}$, the linear combination $\lambda\boldsymbol{\alpha} + \mu\boldsymbol{\beta}$ is an vector in $\langle \mathcal{C} \rangle$. And so, all vectors in $\langle \mathcal{C} \rangle$ are linearly spanned by vectors in $\mathcal{C}$ and we can describe the spanning relationship of vectors quantitatively and hold on the characteristics of things. Dr.Ouyang tells Huizi that a generalization of this notion further leads to the conception of linear space following.

A *linear space* $(V; \mathscr{F})$ is called the linear space over the field $\mathscr{F}$ or simply the linear space $V$ if the field $\mathscr{F}$ is not emphasized, and generally consists of the following:

$(a)$ A set of vectors $V$;

$(b)$ A field $\mathscr{F}$;

$(c)$ The addition "+" is defined on $V$. That is, for any two vectors $\boldsymbol{\alpha}, \boldsymbol{\beta}$ in $V$, there exists a sum vector $\boldsymbol{\alpha} + \boldsymbol{\beta}$ in $V$ that satisfies the following property:

$(c1)$ Commutative law, namely $\boldsymbol{\alpha} + \boldsymbol{\beta} = \boldsymbol{\beta} + \boldsymbol{\alpha}$;

$(c2)$ Associative law, namely $(\boldsymbol{\alpha} + \boldsymbol{\beta}) + \boldsymbol{\gamma} = \boldsymbol{\alpha} + (\boldsymbol{\beta} + \boldsymbol{\gamma})$, where $\boldsymbol{\gamma} \in V$ is a vector;

$(c3)$ There exists a unique vector $\mathbf{0}$, called the *zero vector* such that for any vector $\boldsymbol{\alpha}$ in $V$ holding with $\boldsymbol{\alpha} + \mathbf{0} = \boldsymbol{\alpha}$;

$(c4)$ For any vector $\boldsymbol{\alpha}$ in $V$, there exists a unique vector $-\boldsymbol{\alpha}$ in $V$, called the *inverse* of $\boldsymbol{\alpha}$ holding with $\boldsymbol{\alpha} + (-\boldsymbol{\alpha}) = \mathbf{0}$;

$(d)$ Scalar multiplication, namely for any vector $\boldsymbol{\alpha}$ in $V$ and a scalar $\lambda$ in the field $\mathscr{F}$, there exists a multiplicative vector $\lambda\boldsymbol{\alpha}$ in $V$ holding with:

$(d1)$ For any vector $\boldsymbol{\alpha}$ in $V$, $1\boldsymbol{\alpha} = \boldsymbol{\alpha}$;

$(d2)$ For any scalars $\lambda_1, \lambda_2$ in $\mathscr{F}$, $(\lambda_1\lambda_2)\, \boldsymbol{\alpha} = \lambda_1 (\lambda_2 \boldsymbol{\alpha})$;

$(d3)$ For any two vectors $\boldsymbol{\alpha}, \boldsymbol{\beta}$ in $V$ and a scalar $\lambda$ in the field $\mathscr{F}$, $\lambda(\boldsymbol{\alpha} + \boldsymbol{\beta}) = \lambda\boldsymbol{\alpha} + \lambda\boldsymbol{\beta}$;

$(d4)$ For any vector $\boldsymbol{\alpha}$ in $V$ and scalars $\lambda_1, \lambda_2$ in the field $\mathscr{F}$, $(\lambda_1 + \lambda_2)\, \boldsymbol{\alpha} = \lambda_1 \boldsymbol{\alpha} + \lambda_2 \boldsymbol{\alpha}$.

Some well-known linear spaces are explained in the follow examples.

**Example** 3.1 (Field) For a given field $\mathscr{F}$ such as those of the integer field $\mathbb{Z}$, the rational number field $\mathbb{N}$, and the real number field $\mathbb{R}$, whose elements satisfy all the conditions of linear spaces. So, it is a linear space $(\mathscr{F}; \mathscr{F})$ over the field $\mathscr{F}$.

**Example** 3.2 ($n$-Tuple) For an integer $n \geq 2$ and a field $\mathscr{F}$, let $\mathscr{F}^n$ be a set consisting of the $n$-tuples $(x_1, x_2, \cdots, x_n)$ with each entry $x_i \in \mathscr{F}, 1 \leq i \leq n$. Define the addition of $n$-tuples by

$$(x_1, x_2, \cdots, x_n) + (y_1, y_2, \cdots, y_n) = (x_1 + y_1, x_2 + y_2, \cdots, x_n + y_n)$$

and the scalar multiplication $\lambda(x_1, x_2, \cdots, x_n) = (\lambda x_1, \lambda x_2, \cdots, \lambda x_n)$, where $\lambda$ is a scalar in the field $\mathscr{F}$. Then, $(\mathscr{F}^n; \mathscr{F})$ is a linear space. Particularly, let $\mathscr{F} = \mathbb{R}$. Then, $\mathbb{R}^n$ is the linear space consisted of points $(x_1, x_2, \cdots, x_n)$ in the rectangular coordinate system $\{O; \mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$, called the *Euclidean space* $\mathbb{R}^n$.

**Example** 3.3 (Matrix) For an integer $n \geq 2$ and a field $\mathscr{F}$, let $\mathscr{F}^{n\times n}$ be the set consisting of matrixes $(a_{ij})_{n\times n}$ with entries in the $\mathscr{F}$ and let $A = (a_{ij})_{n\times n}, B = (b_{ij})_{n\times n}$ be two matrixes, a scalar $\lambda \in \mathscr{F}$. Define

$$A + B = (a_{ij} + b_{ij})_{n\times n}, \quad \lambda A = (\lambda a_{ij})_{n\times n}.$$

Then, $\mathscr{F}^{n\times n}$ is a linear space. Particularly, $\mathscr{F}^{1\times n} = \mathscr{F}^n$ which is the linear space consisting of $n$-tuples with entries in the field $\mathscr{F}$.

**Example** 3.4 (Mapping) Let $\mathscr{F}$ be a field with two subsets $S_1$ and $S_2$ closed under operations of addition "+" and multiplication "·". Denoted by $F(S_1, S_2)$ the set of all mappings $f : S_1 \to S_2$, namely mappings from $S_1$ to $S_2$. For any two mapping $g, h \in F(S_1, S_2)$ and an element $\lambda$ in $\mathscr{F}$, define

$$(g + h)(x) = g(x) + h(x), \quad (\lambda g)(x) = \lambda g(x),$$

where $x$ is an element in $S_1$. Then $F(S_1, S_2)$ is a linear space. Particularly, let $S_1 = S_2 = \mathbb{R}$, then $F(\mathbb{R}, \mathbb{R})$ is nothing else but the real function space on $\mathbb{R}$.

Dr.Ouyang tells Huizi that although linear space originates from vectors in the nature such as the force and velocity, it is essentially an abstract generalization on linear combination and a universal conception or tool for describing the behavior of things. In fact, the human recognition of things nearly all starts from the construction of the linear space model or the introduction of different measures on linear space for characterizing the behavior of things. The inner product space introduced below is such an example.

Let $V$ be a linear space over the real numbers field $\mathbb{R}$. For any two vectors $\boldsymbol{\alpha}, \boldsymbol{\beta}$ in $V$, we introduce a metric $\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle$ between them, called the *inner product*, namely $\{\boldsymbol{\alpha}, \boldsymbol{\beta}\} \to \langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle$. Certainly, an inner product is a scalar and satisfies conditions following for the vectors $\boldsymbol{\alpha}, \boldsymbol{\beta}, \boldsymbol{\gamma}$ in $V$ and the real number $\lambda$ in $\mathbb{R}$:

(1) $\langle \boldsymbol{\alpha}, \boldsymbol{\alpha} \rangle \geq 0$ with the equality hold if and only if $\boldsymbol{\alpha} = \mathbf{0}$;

(2) $\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle = \langle \boldsymbol{\beta}, \boldsymbol{\alpha} \rangle$;

(3) $\langle \lambda\boldsymbol{\alpha}, \boldsymbol{\beta} \rangle = \lambda \langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle$;

(4) $\langle \boldsymbol{\alpha}+\boldsymbol{\beta}, \boldsymbol{\gamma} \rangle = \langle \boldsymbol{\alpha}, \boldsymbol{\gamma} \rangle + \langle \boldsymbol{\beta}, \boldsymbol{\gamma} \rangle$.

A linear space $V$ whose inner product satisfies the conditions (1) $-$ (4) is called the *inner product space*. Notice that $\langle \boldsymbol{\alpha}, \boldsymbol{\alpha} \rangle \geq 0$. So, the length of a vector $\boldsymbol{\alpha}$ can be defined as $|\boldsymbol{\alpha}| = \sqrt{\langle \boldsymbol{\alpha}, \boldsymbol{\alpha} \rangle}$, also called the *norm* of vector $\boldsymbol{\alpha}$.

There is a well-known inequality in the inner product space, called the *Cauchy-Schwartz inequality*. That is, for any two vectors $\boldsymbol{\alpha}, \boldsymbol{\beta}$ in $V$ holds with

$$\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle^2 \leq \langle \boldsymbol{\alpha}, \boldsymbol{\alpha} \rangle \langle \boldsymbol{\beta}, \boldsymbol{\beta} \rangle \tag{4.31}$$

and the equality only holds if $\boldsymbol{\alpha}, \boldsymbol{\beta}$ is linearly dependent. So, *how could we find this inequality?* Firstly, if $\boldsymbol{\alpha}, \boldsymbol{\beta}$ is linearly dependent, there must be $\boldsymbol{\alpha} = \mathbf{0}$ or $\boldsymbol{\alpha} \neq \mathbf{0}$, $\boldsymbol{\beta} = \lambda \boldsymbol{\alpha}$ for a real number $\lambda \in \mathbb{R}$. In this case, we can directly verify that there are $\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle^2 = \langle \boldsymbol{\alpha}, \boldsymbol{\alpha} \rangle \langle \boldsymbol{\beta}, \boldsymbol{\beta} \rangle$; Secondly, if $\boldsymbol{\alpha}, \boldsymbol{\beta}$ is linearly independent, i.e., for any real number $t$, $t\boldsymbol{\alpha}+\boldsymbol{\beta} \neq \mathbf{0}$, we have

$$\langle t\boldsymbol{\alpha}+\boldsymbol{\beta}, t\boldsymbol{\alpha}+\boldsymbol{\beta} \rangle \geq 0 \ \Rightarrow \ t^2 \langle \boldsymbol{\alpha}, \boldsymbol{\alpha} \rangle + 2t \langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle + \langle \boldsymbol{\beta}, \boldsymbol{\beta} \rangle \geq 0. \tag{4.32}$$

Notice that (4.32) is a quadratic inequality holding with any real number $t$. We therefore know that its discriminant must be less than zero, i.e.,

$$\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle^2 \leq \langle \boldsymbol{\alpha}, \boldsymbol{\alpha} \rangle \langle \boldsymbol{\beta}, \boldsymbol{\beta} \rangle,$$

which is nothing else but the Cauchy-Schwartz inequality (4.31).

For vectors $\boldsymbol{\alpha}, \boldsymbol{\beta}$ in an $n$-Euclidean space $\mathbb{R}^n$, the calculation of inner product by the properties (1)$-$(4) is consistent with (4.22), i.e., if $\boldsymbol{\alpha} = (a_1, a_2, \cdots, a_n), \boldsymbol{\beta} = (b_1, b_2, \cdots, b_n)$, then

$$\langle \boldsymbol{\alpha}, \boldsymbol{\beta} \rangle = (a_1, a_2, \cdots, a_n) \cdot (b_1, b_2, \cdots, b_n) = a_1 b_1 + a_2 b_2 + \cdots + a_n b_n.$$

Particularly, $\langle \boldsymbol{\alpha}, \boldsymbol{\alpha} \rangle = a_1^2 + a_2^2 + \cdots + a_n^2$, i.e., the length of vector $\boldsymbol{\alpha}$ in $\mathbb{R}^n$ is

$$|\boldsymbol{\alpha}| = |(a_1, a_2, \cdots, a_n)| = \sqrt{a_1^2 + a_2^2 + \cdots + a_n^2}, \tag{4.33}$$

which is the same as the geometrical distance from the origin $O$ to the point $(a_1, a_2, \cdots, a_n)$. Accordingly, the Cauchy-Schwartz inequality is simplified to

$$(a_1 b_1 + a_2 b_2 + \cdots + a_n b_n)^2 \leq \left(a_1^2 + a_2^2 + \cdots + a_n^2\right)\left(b_1^2 + b_2^2 + \cdots + b_n^2\right), \tag{4.34}$$

called the *Cauchy inequality*.

By the Cauchy inequality (4.34), the *triangle inequality*, i.e.,

$$|\boldsymbol{\alpha}+\boldsymbol{\beta}| \leq |\boldsymbol{\alpha}| + |\boldsymbol{\beta}| \tag{4.35}$$

can be further obtained, which is a well-known inequality in geometry, i.e., the length of any two sides of a triangle is greater than the length of the third side. So, *how do we get this inequality from the Cauchy inequality?* Firstly, we write the inequality (4.35) in the coordinate form, i.e.,

$$\begin{aligned}&\sqrt{(a_1+b_1)^2+(a_2+b_2)^2+\cdots+(a_n+b_n)^2}\\&\qquad\qquad\le\sqrt{a_1^2+a_2^2+\cdots+a_n^2}+\sqrt{b_1^2+b_2^2+\cdots+b_n^2}\end{aligned}$$

and simultaneous square at both sides, we have

$$\begin{aligned}&(a_1+b_1)^2+(a_2+b_2)^2+\cdots+(a_n+b_n)^2\\&\qquad\qquad\le\left(a_1^2+a_2^2+\cdots+a_n^2\right)+\left(b_1^2+b_2^2+\cdots+b_n^2\right)\\&\qquad\qquad+2\sqrt{a_1^2+a_2^2+\cdots+a_n^2}\cdot\sqrt{b_1^2+b_2^2+\cdots+b_n^2}.\end{aligned}$$

To simplify it further, we get that

$$(a_1b_1+a_2b_2+\cdots+a_nb_n)^2\le\left(a_1^2+a_2^2+\cdots+a_n^2\right)\left(b_1^2+b_2^2+\cdots+b_n^2\right),$$

i.e., the Cauchy's inequality. However, we already know that the Cauchy's inequality (4.34) holds and each step of the reasoning process is reversible. In this way, the triangle inequality (4.35) can be obtained by working backwards step by step.

## §4. **Basis and Dimension**

In a linear space, some vectors are linearly spanned by other vectors and some are not. Dr.Ouyang tells his daughter that a natural idea for characterizing a linear space $V$ is to find those vectors in $V$ that could not be spanned by other vectors, namely the spanning set $\mathcal{C}$ of vectors. Generally, we would like to find such a spanning set of vectors in two criteria: the first is that there are no redundant vectors $\mathbf{x}$ in $\mathcal{C}$, i.e., removing these vectors $\mathbf{x}$ from $\mathcal{C}$ will not affect the vector spanning $\langle\mathcal{C}\rangle$, which essentially requires that $\mathcal{C}$ is the least or in the most economic and the second, it is easy for calculating or determining the spanning by the relationship of vectors.

For example, the $n$ spanning vectors $\mathbf{e}_1,\mathbf{e}_2,\cdots,\mathbf{e}_n$ with $|\mathbf{e}_i|=1,\ \mathbf{e}_i\perp\mathbf{e}_j,\ 1\le i\ne j\le n$ of an $n$-dimensional Euclidean space $\mathbb{R}^n$ accord with both of these two criteria. *Why do we say so?* Firstly, there are no redundant vectors in the generation set $\mathbf{e}_1,\mathbf{e}_2,\cdots,\mathbf{e}_n$ because the any removing in the spanning vectors $\mathbf{e}_1,\mathbf{e}_2,\cdots,\mathbf{e}_n$ will not span the Euclidean space $\mathbb{R}^n$; Secondly, the conditions $|\mathbf{e}_i|=1,\ \mathbf{e}_i\perp\mathbf{e}_j,\ 1\le i\ne j\le n$ concludes that the vector spanning result calculation in $\mathcal{C}$ is equivalent to the coordinates $(\lambda_1, \lambda_2, \cdots, \lambda_n)$, i.e.,

$$\mathbf{a} = \lambda_1\mathbf{e}_1 + \lambda_2\mathbf{e}_2 + \cdots + \lambda_n\mathbf{e}_n \quad \Leftrightarrow \quad \mathbf{a} = (\lambda_1, \lambda_2, \cdots, \lambda_n), \quad \lambda_1, \lambda_2, \cdots, \lambda_n \in \mathbb{R}$$

which results in that a vector $\mathbf{a} \in \mathbb{R}^n$ can be immediately determined by the real number set $\mathbb{R}$.

Generally, for a given vector spanning set $\mathcal{C}$, *is there really a spanning subset $\mathcal{C}'$ that conforms to these two criteria?* Dr.Ouyang explains that such a spanning set $\mathcal{C}'$ should exist intuitively because each vector $\mathbf{x}$ in $\mathcal{C}$ can be verified one by one for redundancy. In fact, if it is redundancy we can remove it from $\mathcal{C}$ and consider the same problem on the spanning set $\mathcal{C}\backslash\{\mathbf{x}\}$ until the remaining vectors are no longer redundant, and called the remaining vectors a *basis* of $\langle\mathcal{C}\rangle$. Meanwhile, the basis of Euclidean space shows that it can greatly simplify the calculation if the spanning vectors are mutually orthogonal. So, *how do we make the basis elements orthogonal to each other?* This needs the orthogonalization on elements in the spanning basis $\mathcal{C}'$. Geometrically, this is easy to do. For example, we can choose one basis vector and rotate others orthogonal to it for getting two orthogonal basis vectors. And then, rotate the other basis vectors enable them orthogonal to each other until all the basis vectors are orthogonal to each other such as those shown in Figure 4.21, which gives us an orthogonal spanning basis $\mathcal{C}'$ finally.

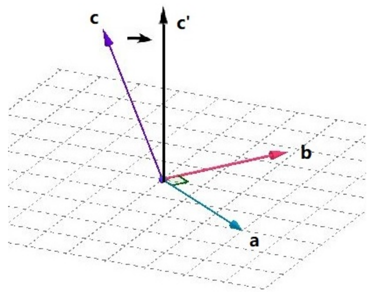



**Figure** 4.21. **Rotating vector**

**4.1.Linear Independence.** The linear independence or dependence of vectors is an important conception in determining the basis of linear space. Let $(V.\mathscr{F})$ be a linear space with a spanning set $\mathcal{C}$. If there is a redundancy vector $\mathbf{x}$ in $\mathcal{C}$, there are must be vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_k$ in $\mathcal{C}\backslash\{\mathbf{x}\}$ and scalars $\lambda_1, \lambda_2, \cdots, \lambda_k$, not all 0 in field $\mathscr{F}$ such that

$$\mathbf{x} = \lambda_1\mathbf{a}_1 + \lambda_2\mathbf{a}_2 + \cdots + \lambda_k\mathbf{a}_k, \tag{4.36}$$

namely, the spanning role of vector $\mathbf{x}$ can replaced by vector $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_k$, where $k \geq 1$ is an integer. Conversely, if the spanning role of vector $\mathbf{x}$ can not be replaced by vectors in $\mathcal{C}\backslash\{\mathbf{x}\}$, then $\mathbf{x}$ is not a redundancy vector in $\mathcal{C}$. Dr.Ouyang tells Huizi that a generalization of this notion brings about the conceptions of linear dependence and linear independence.

For an integer $n \geq 1$, the vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ in a linear space $V$ are called *linear dependent* if there exist scalars $\lambda_1, \lambda_2, \cdots, \lambda_n$, not all 0 in the field $\mathscr{F}$ such that

$$\lambda_1\mathbf{a}_1 + \lambda_2\mathbf{a}_2 + \cdots + \lambda_n\mathbf{a}_n = \mathbf{0}. \tag{4.37}$$

Conversely, if there are no scalars $\lambda_1, \lambda_2, \cdots, \lambda_n$, not all 0 in the field $\mathscr{F}$ holding on with (4.37), namely (4.37) sets up only if $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$, then the vectors $\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n$ are called *linear independent.*

For example, the vectors

$$\mathbf{a}_1 = (3, 0, -3),\ \mathbf{a}_2 = (-1, 1, 2),\ \mathbf{a}_3 = (4, 2, -2)$$

are linearly dependent in the Euclidean space $\mathbb{R}^3$ because there are real numbers $\lambda_1 = 2, \lambda_2 = 2, \lambda_3 = -1$ such that

$$\begin{aligned} 2\mathbf{a}_1 + 2\mathbf{a}_2 - \mathbf{a}_3 &= (6, 0, -6) + (-2, 2, 4) - (4, 2, -2) \\ &= (6 - 2 - 4, 0 - 2 + 2, -6 + 4 + 2) = (0, 0, 0), \end{aligned}$$

but the vectors

$$\mathbf{b}_1 = (1, 1, 1),\ \mathbf{b}_2 = (0, 1, 1),\ \mathbf{b}_3 = (0, 0, 1)$$

are linearly independent because if there are real numbers $\lambda_1, \lambda_2, \lambda_3$ holding with

$$\lambda_1\mathbf{b}_1 + \lambda_2\mathbf{b}_2 + \lambda_3\mathbf{b}_3 = (0, 0, 0) \ \Rightarrow\ (\lambda_1, \lambda_1 + \lambda_2, \lambda_1 + \lambda_2 + \lambda_3) = (0, 0, 0),$$

we then get a system of linear equation

$$\begin{cases} \lambda_1 &= 0, \\ \lambda_1 + \lambda_2 &= 0, \\ \lambda_1 + \lambda_2 + \lambda_3 &= 0 \end{cases}$$

which only has the solutions $\lambda_1 = \lambda_2 = \lambda_3 = 0$, i.e., the vectors $\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3$ are liner independent.

From the definitions of linearly dependent and linearly independent of vectors, the conclusions following are easily obtained:

(1) Any vector set containing linearly dependent vectors is linearly dependent;

(2) Any subset of linearly independent vectors is also linearly independent;

(3) Any vector set containing the zero vector $\mathbf{0}$ is linearly dependent because for any scalar $\lambda$ in the field $\mathscr{F}$, $\lambda\mathbf{0} = \mathbf{0}$;

(4) If a vector set $S$ is linearly independent then every finite subset of $S$ is linearly independent, and vice versa. That is, $S$ is linearly independent if every finite subset of $S$ is linearly independent.

**4.2.Basis and Dimension.** Dr.Ouyang tells Huizi that finding the basis of a linear space $V$ under the condition of linear spanning is an effective way for characterizing vectors in $V$. The *spanning basis* of a linear space $V$ refers to a linearly independent subset $\mathcal{C}$ of $V$ whose elements linearly span $V$. According to the spanning basis $\mathcal{C}$ of linear space containing finite or infinite elements, the linear spaces are divided into two categories, namely the *finite dimensional linear space*, i.e., $|\mathcal{C}| < +\infty$ and the *infinite dimensional linear space.* Here, we mainly discuss the properties of finite dimensional linear space.

Firstly, we prove that *the cardinality of any spanning basis in a finite-dimensional linear spaces $V$ is equal, no matter how many spanning basis it has.* If not, suppose $\mathcal{C}_1 = \{\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_n\}$ and $\mathcal{C}_2 = \{\mathbf{b}_1, \mathbf{b}_2, \cdots, \mathbf{b}_m\}$ are two basis of $V$, $m > n$ and $\mathbf{b}_j = \gamma_{1j}\mathbf{a}_1 + \gamma_{2j}\mathbf{a}_2 + \cdots + \gamma_{nj}\mathbf{a}_n$, where, $1 \leq j \leq m$. Then, for $n$ scalars $x_1, x_2, \cdots, x_m$ in $\mathscr{F}$, we know that

$$\begin{aligned} x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_m\mathbf{b}_m &= \sum_{j=1}^{m} x_j\mathbf{b}_j = \sum_{j=1}^{m} x_j \sum_{i=1}^{n} \gamma_{ij}\mathbf{a}_i \\ &= \sum_{j=1}^{m}\sum_{i=1}^{n} (\gamma_{ij}x_j)\,\mathbf{a}_i = \sum_{i=1}^{n}\left(\sum_{j=1}^{m}\gamma_{ij}x_j\right)\mathbf{a}_i. \end{aligned}$$

Notice that $m > n$, namely the number of equations in the system

$$\sum_{j=1}^{m} \gamma_{ij}x_j = 0, \qquad 1 \leq i \leq n \tag{4.38}$$

is less than the number of variables. So, the system (4.38) has non-zero solutions $\lambda_1, \lambda_2, \cdots, \lambda_m$ such that

$$\lambda_1\mathbf{b}_1 + \lambda_2\mathbf{b}_2 + \cdots + x_m\mathbf{b}_m = \mathbf{0},$$

contradicts to that $\mathcal{C}_2$ is a spanning basis of $V$. Therefore, there can not be $m > n$, i.e., $m \leq n$ only. Similarly, replacing $\mathcal{C}_1$ by $\mathcal{C}_2$ in the previous, we also know $n \leq m$. Thus, there must be $m = n$, i.e., the number of elements in any spanning basis of $V$ is equal, which is called the *dimension* of linear space $V$, denoted by $\dim V$. For example, $\dim\mathbb{R}^n = n$ or the Euclidean space $\mathbb{R}^n$ is an $n$-dimensional linear space.

Clearly, for a linearly independent subset $W_0$ of $V$ there must be $|W_0| \leq \dim V$. So, *can we extend $W_0$ to form a basis of $V$?* Dr.Ouyang tells Huizi that the answer is Yes! Firstly, if $W_0$ spans $V$ the conclusion is self-evident. Otherwise, there must be an element in $V$ that can not be spanned by elements in $W_0$. Let $\boldsymbol{\alpha}_1$ be such a element and let $W_1 = S_0 \bigcup\{\boldsymbol{\alpha}_1\}$. Similarly, if $W_1$ spans $V$, the desired conclusion is obtained. Otherwise, there must be an element in $V$ that can not be spanned by elements in $W_1$. Denote such an element by $\boldsymbol{\alpha}_2$ and let $W_2 = W_0 \bigcup\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2\}$. Continue this process again and again. Notice that $\dim V$ is finite. That is, this programm would be ended after $s$ steps with $s \leq \dim V - 1$, i.e., there are no elements in $V$ that can not be spanned by elements in

$W_s$, which enables us to get a set of linear independent vectors

$$W_s = W_0 \bigcup \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_s\} \tag{4.39}$$

spanning the linear space $V$. Whence, we get a spanning basis $W_s$ of $V$ by adding vectors $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_s$ to the linear independent set $W_0$. Certainly, there must be $|W_0|+s = \dim V$.

Generally, for any two subsets $U_0, W_0$ of the linear spanning set $\mathcal{C}$ of $V$, by the inclusion-exclusion principle in Chapter 3 there are

$$|U_0| + |W_0| = \left|U_0 \bigcup W_0\right| + \left|U_0 \bigcap W_0\right|. \tag{4.40}$$

Now, if the sets $U_0$, $W_0$ linearly span the spaces $U$ and $W$, respectively, we must have $|U_0| = \dim U$, $|W_0| = \dim W$. Let $U + W = \{\mathbf{a} + \mathbf{b} | \mathbf{a} \in U, \mathbf{b} \in W\}$, $U \bigcap W = \{\mathbf{c} | \mathbf{c} \in U \bigcap W\}$. Dr.Ouyang asks Huizi that *wether there is a dimension equality*

$$\dim U + \dim W = \dim (U + W) + \dim \left(U \bigcap W\right) \tag{4.41}$$

*in the linear space $V$?* Dr.Ouyang tells Huizi that the answer is affirmative because $U_0 \bigcup W_0$ is linear independent and any vector of $U + W$ can be linearly spanned by vectors in $U_0 \bigcup W_0$, we must have $\dim(U + W) = |U_0 \bigcup W_0|$. Similarly, we also have $\dim (U \bigcap W) = |U \bigcap W|$. Consequently, the dimension equality (4.41) can be immediately obtained by the inclusion-exclusion principle (4.40). Of course, the equality (4.41) can also be obtained by extending the elements in $U_0 \bigcap W_0$ to a basis of $V$ but its essence is the application of the inclusion-exclusion principle (4.40).

In order to eliminate the personal differences in vector representation caused by different vector notations of linear spaces, Dr.Ouyang explains that there is also a classification problem for the linear spanning of vectors. That is, to determine whether the linear spaces $V$ and $W$ spanned by different spanning basis $\mathcal{C}$ and $\mathcal{C}'$ are consistent over a field $\mathscr{F}$. And to do that, we need to define what it means that two linear spaces here are consistent. Since we are talking about linear spaces, the meaning of "*consistence*" should be defined as the corresponding between elements of the two linear spaces that maintain linear structure consistence, i.e.,"*isomorphism*". Generally, the linear spaces $V$ and $W$ are said to be "*isomorphic*" refers to that there exists a $1-1$ mapping $T: V \to W$ to maintain a linear structure in the two spaces, i.e., for any vector $\mathbf{a}, \mathbf{b}$ in the linear space $V$, hold with

$$T(\lambda\mathbf{a} + \mu\mathbf{b}) = \lambda T(\mathbf{a}) + \mu T(\mathbf{b}) \tag{4.42}$$

for any scalars $\lambda, \mu$ in the field $\mathscr{F}$.

For two linear spaces $V$ and $W$, whether $V$ and $W$ are isomorphic can be generally determined by whether the dimension is the same. That is, *if* $\dim V = \dim W$ *then* $V$

*and $W$ are isomorphic!* So, *how do we get this conclusion by linear spanning?* Firstly, assume that $\dim V = \dim W = n$ and the linear bases of $V$ and $W$ are respectively $\mathcal{C} = \{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_n\}$ and $\mathcal{C}' = \{\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_n\}$; Secondly, define a mapping $T$ holding with the linear relationship (4.42) and satisfying

$$T(\boldsymbol{\alpha}_1) = \boldsymbol{\beta}_1,\ T(\boldsymbol{\alpha}_2) = \boldsymbol{\beta}_2,\ \cdots,\ T(\boldsymbol{\alpha}_n) = \boldsymbol{\beta}_n.$$

Then, the mapping $T$ is a $1-1$ linear mapping, i.e.,

$$T : x_1\boldsymbol{\alpha}_1 + x_2\boldsymbol{\alpha}_2 + \cdots + x_n\boldsymbol{\alpha}_n \ \leftrightarrow x_1\boldsymbol{\beta}_1 + x_2\boldsymbol{\beta}_2 + \cdots + x_n\boldsymbol{\beta}_n \tag{4.43}$$

and we know that *if* $\dim V = n$ *then* $(V;\mathscr{F})$ *is isomorphic to the linear space* $\mathscr{F}^n$ *consisted of the n-tuples with entries in* $\mathscr{F}$ *by* (4.43) and meanwhile, we can define the coordinates of a vector $\boldsymbol{\alpha}$ by

$$\boldsymbol{\alpha} = x_1\boldsymbol{\alpha}_1 + x_2\boldsymbol{\alpha}_2 + \cdots + x_n\boldsymbol{\alpha}_n \ \leftrightarrow\ (x_1, x_2, \cdots, x_n), \tag{4.44}$$

in a coordinate system $\{O; \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_n\}$, i.e., the coordinates of $\boldsymbol{\alpha}$ are $(x_1, x_2, \cdots, x_n)$ in the coordinate system $\{O; \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_n\}$, which is the same as those in (4.2).

**4.3.Basis Orthogonalization.** Similar to that of (4.43), there is a $1-1$ correspondence between a basis $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_n\}$ of linear space $V$ with the coordinate system $\{O; \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_n\}$, i.e., the coordinates $(x_1, x_2, \cdots, x_n)$ of vector $\boldsymbol{\alpha}$ are uniquely determined by (4.43) in $\{O;\ \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_n\}$. Dr.Ouyang tells Huizi that the simplest coordinate systems is the coordinate system $\{O; \mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ of $n$-Euclidean space $\mathbb{R}^n$, called the *normal coordinate system* with a *normal orthogonal basis* $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$, where

$$\mathbf{e}_1 = (1, \underbrace{0, \cdots, 0}_{n-1}),\ \mathbf{e}_2 = (0, 1, \underbrace{0, \cdots, 0}_{n-2}),\ \cdots,\ \mathbf{e}_n = (\underbrace{0, \cdots, 0}_{n-1}, 1)$$

and the elements $\mathbf{e}_i, 1 \leq i \leq n$ are all vectors with $|\mathbf{e}_i| = 1$ and

$$\mathbf{e}_i \cdot \mathbf{e}_j = (\underbrace{0, \cdots, 0}_{i-1}, 1, \underbrace{0, \cdots, 0}_{n-i}) \cdot (\underbrace{0, \cdots, 0}_{j-1}, 1, \underbrace{0, \cdots, 0}_{n-j}) = 0 + 0 + \cdots + 0 = 0,$$

i.e., $\mathbf{e}_i \perp \mathbf{e}_j$ or the elements $\mathbf{e}_i$ and $\mathbf{e}_j$ are orthogonal for any integer $i, j$, $1 \leq i \neq j \leq n$.

Notice that it is easy to do for unitizing any vector $\boldsymbol{\alpha} \in V$ because

$$\left|\frac{\boldsymbol{\alpha}}{|\boldsymbol{\alpha}|}\right| = \frac{|\boldsymbol{\alpha}|}{|\boldsymbol{\alpha}|} = 1 \tag{4.45}$$

shows that $\boldsymbol{\alpha}/|\boldsymbol{\alpha}|$ is a unit vector. Then, *is it possible to deduce an orthogonal basis from a basis* $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_n\}$ *of linear space* $V$*?* Dr.Ouyang tells his daughter that the answer is Yes! This is the *Grim-Schmidt orthogonalization* on the basis of linear space $V$, which is essentially the geometric intuition of the orthogonalization process shown in Figure 4.21 with steps as follows:

**Step 1.** Choose $\boldsymbol{\beta}_1 = \boldsymbol{\alpha}_1$. Then $\boldsymbol{\beta}_1 \neq \mathbf{0}$ and $\boldsymbol{\beta}_1$ is a linear combination of $\boldsymbol{\alpha}_1$;

**Step 2.** Define a vector

$$\boldsymbol{\beta}_2 = \boldsymbol{\alpha}_2 - \frac{\langle \boldsymbol{\alpha}_2, \boldsymbol{\beta}_1 \rangle}{\langle \boldsymbol{\beta}_1, \boldsymbol{\beta}_1 \rangle} \boldsymbol{\beta}_1,$$

i.e., $\boldsymbol{\beta}_2$ is a linear combination of vectors $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$. Notice that the vectors $\boldsymbol{\alpha}_1$ and $\boldsymbol{\alpha}_2$ are linear independent. Whence, $\boldsymbol{\beta}_2 \neq \mathbf{0}$ and

$$\langle \boldsymbol{\beta}_2, \boldsymbol{\beta}_1 \rangle = \langle \boldsymbol{\alpha}_2, \boldsymbol{\beta}_1 \rangle - \frac{\langle \boldsymbol{\alpha}_2, \boldsymbol{\beta}_1 \rangle}{\langle \boldsymbol{\beta}_1, \boldsymbol{\beta}_1 \rangle} \langle \boldsymbol{\beta}_1, \boldsymbol{\beta}_1 \rangle = 0,$$

i.e., $\boldsymbol{\beta}_2$ is orthogonal to $\boldsymbol{\beta}_1$.

**Step 3.** In general, if the pairwise orthogonal vectors $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_k$have been constructed for integers $1 \leq k \leq n$ and each vector $\boldsymbol{\beta}_i$ is a linear combination of vectors $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_i$, where $1 \leq i \leq k$. Define a vector

$$\boldsymbol{\beta}_{k+1} = \boldsymbol{\alpha}_{k+1} - \frac{\langle \boldsymbol{\alpha}_{k+1}, \boldsymbol{\beta}_1 \rangle}{\langle \boldsymbol{\beta}_1, \boldsymbol{\beta}_1 \rangle} \boldsymbol{\beta}_1 - \cdots - \frac{\langle \boldsymbol{\alpha}_{k+1}, \boldsymbol{\beta}_k \rangle}{\langle \boldsymbol{\beta}_k, \boldsymbol{\beta}_k \rangle} \boldsymbol{\beta}_k. \tag{4.46}$$

Clearly, the vector $\boldsymbol{\beta}_{k+1}$ is a linear combination of vectors $\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_{k+1}$. Notice that the vectors $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_k$ are pairwise orthogonal. So, for any integer $i, 1 \leq i \leq k$ we have

$$\left\langle \boldsymbol{\beta}_{k+1}, \boldsymbol{\beta}_i \right\rangle = \langle \boldsymbol{\alpha}_{k+1}, \boldsymbol{\beta}_i \rangle - \frac{\langle \boldsymbol{\alpha}_{k+1}, \boldsymbol{\beta}_i \rangle}{\langle \boldsymbol{\beta}_i, \boldsymbol{\beta}_i \rangle} \langle \boldsymbol{\beta}_i, \boldsymbol{\beta}_i \rangle = 0,$$

i.e., $\boldsymbol{\beta}_{k+1}$ is orthogonal to the vector $\boldsymbol{\beta}_i$. Whence, the vectors $\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_{k+1}$ consist of a pairwise orthogonal set of vectors.

**Step 4.** If $k+1 < n$, repeat Step 3 until $k+1 = n$, the programming ends.

By applying Steps $1-4$ on the basis $\{\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \cdots, \boldsymbol{\alpha}_n\}$, we get a pairwise orthogonal basis $\{\boldsymbol{\beta}_1, \boldsymbol{\beta}_2, \cdots, \boldsymbol{\beta}_n\}$ of linear space $V$. Furthermore, applying the unitization (4.45) of vectors $\boldsymbol{\beta}_i$ to the units $\boldsymbol{\beta}'_i$ for integers $1 \leq i \leq n$, we get a normal coordinate system $\left\{O; \boldsymbol{\beta}'_1, \boldsymbol{\beta}'_2, \cdots, \boldsymbol{\beta}'_n\right\}$ of the linear space $V$ finally.

## §5. **Particle Equations**

Dr.Ouyang tells Huizi that if the scale of an object is so small that tends to 0 compared with the distance of other neighboring objects or if the moving of all components consisting of the object is the same, the moving of any component in the object could represents the moving of the whole object, i.e., the object can be abstracted to a geometric point without considering its internal structure at this time and the mass of the object is concentrated on this geometric point for abstraction. Such a geometrical point is called a *particle*, an abstract point that has mass but no volume or shape. Usually, the *particle model* can be applied for recognizing the moving behavior of such kind of objects in a coordinate system. For example, while one looks at the sky standing on the earth, the stars and clouds show colorful appearances in the eyes of human. Some of them look like swans flying with wings but some look like scorpions with tails in the air with different characters such as big or small, bright or dark sometimes, such as those shown in Figure 4.22. However, if the observations are scaled up further such as on the 200 million, 300 million or further more, for instance on the 1 billion light years, i.e, the cosmological scale, then the planets of clouds in the universe are indistinguishable at any point and in any direction. At this time, the size of a planet is so small that can be negligible compared with the spacing between the planets, i.e., all planets can be regarded as a particle $P$. In this way, we can first construct a normal coordinate system $\{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and apply the position vectors

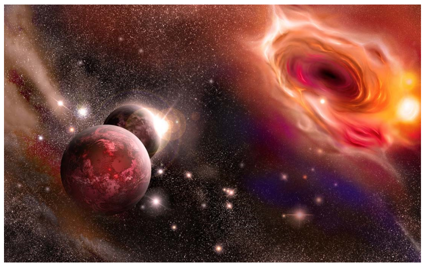


**Figure** 4.22. **Starry sky**

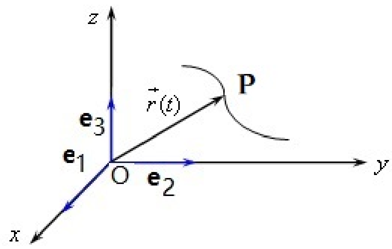



**Figure** 4.23. **Particle miotion**

$$\mathbf{r} = \mathbf{r}(t) \quad \text{or} \quad \mathbf{r} = x(t)\mathbf{e}_1 + y(t)\mathbf{e}_2 + z(t)\mathbf{e}_3. \tag{4.47}$$

for characterizing the moving of planets $P$ in the coordinate system. Here, the particle moving equation (4.47) can also be written in scalar or parametric form

$$x = x(t), \quad y = y(t), \quad x = z(t). \tag{4.48}$$

In this way, the spatial curve described by the end $P$ of the vector $\overrightarrow{OP}$ varies on the time $t$ is a complete description of the moving locus of the planet $P$, as shown in Figure 4.23. So, the vector equation (4.47) or the parametric equation (4.48) is called the *moving equation* of the particle $P$. Certainly, the parameter $t$ can be also eliminated from the parametric equation (4.48) to obtain the curvilinear equation in general form. For example, the elimination of the parameter $t$ in the vector equation $\mathbf{r} = r\sin t\mathbf{e}_1 + r\cos t\mathbf{e}_2 + s\mathbf{e}_3$

or the parameter equation $x(t) = R\sin t,\ y(t) = R\cos t,\ z(t) = c$ provides the general equation

$$x^2(t) + y^2(t) = R^2, \quad z(t) = c \tag{4.49}$$

which is nothing else but a circle of radius $R$ on the $z = s$ plane in the space $\mathbb{R}^3$. Here, $0 \le t \le 2\pi$ and $c$ is a constant. Particularly, if the locus of a particle $P$ is a straight line, it's moving is called a *straight line moving*. Otherwise, it is called in a *curvilinear moving*.

Notice that the equation (4.47) or (4.48) is the moving equation of a particle $P$ in a stationary coordinate system $\{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. So, the coordinate system $\{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is also known as the reference frame of moving. In general, the moving equation of a particle $P$ may have different expressions in different reference frames. For example, an object $P$ falls freely on a train moving at a uniform speed in a straight line, i.e., the distance traveled by the train per unit time is a constant value. If the reference frame is taken as the train, the moving locus of $P$ is a vertical straight line. However, if the reference frame is on the ground, the moving locus of object $P$ is a parabola. According to the relativity principle or the equal-rights principle in Section 1 of this chapter, these equations are not essentially different in describing the moving of the particle $P$. However, Dr.Ouyang tells Huizi that the moving equation of $P$ would have a relatively simple form in the reference frame of the train, which is a linear equation of the first order.

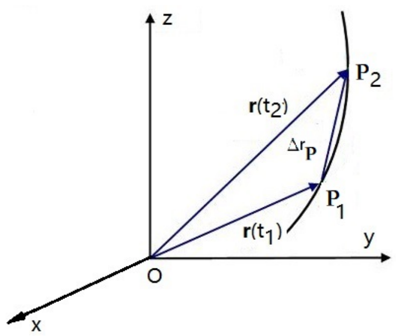



**Figure** 4.24. **Moving displacement**

**5.1.Moving Displacement.** The essence of an object moving lies in its change of spatial position. Clearly, the moving of a particle $P$ relative to a reference frame $\{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the change of the position vector $\mathbf{r}_P$ or coordinates of $P$ on different time $t$, as shown in Figure 4.24. With out loos of generality, assume there is a moving particle $P$ on time $t$ with the position $P_1$ or coordinates $(x_1, y_1, z_1)$ at time $t_1$ and the position $P_2$ or coordinates $(x_2, y_2, z_2)$ at time $t_2$ in the reference frame. Define the *moving displacement* of particle $P$ from times $t_1$ to $t_2$ to be the difference vector

$$\Delta\mathbf{r}_P = \mathbf{r}_P(t_2) - \mathbf{r}_P(t_1) = (x_2 - x_1, y_2 - y_1, z_2 - z_1) \tag{4.50}$$

such as the vector $\overrightarrow{P_1P_2}$ shown in Figure 4.24. Now, if the displacement rate of particle $P$ is a constant vector $v_c = (c_x c_y, c_z)$, namely the displacements per unit time of $P$ on the $x$-axis, $y$-axis and $z$-axis directions are respectively constants $c_x, c_y$ and $c_z$, there is a relationship between $v_c$, moving displacement and time $t$ following

$$v_c = \frac{\Delta \mathbf{r}_P}{t_2 - t_1} = \left( \frac{x_2 - x_1}{t_2 - t_1}, \frac{y_2 - y_1}{t_2 - t_1}, \frac{z_2 - z_1}{t_2 - t_1} \right) = (c_x, c_y, c_z), \tag{4.51}$$

where $v_c$ is called the *moving speed* of the particle $P$. However, the displacements per unit time of the particle $P$ in the $x$-axis, $y$-axis and $z$-axis are not constants in general. So, *how do we determine the displacement rate or velocity of a particle $P$?* Dr.Ouyang tells Huizi that this problem can be considered like this, i.e., assume the time difference $\Delta t = t_2 - t_1$ gradually decreases or the time $t_2$ infinitely closes to $t_1$, namely $t_2 \to t_1$. In this way, the displacement difference $\Delta \mathbf{r}_P$ of the particle $P$ will also decrease as the time $t_2$ approaches $t_1$. As the time difference $\Delta t$ is small enough it can be approximately considered that the displacement rate of the particle $P$ over this time period $\Delta t$ is a constant vector. Whence, the equation (4.51) should still hold if substituting the moving displacement and time by their differentiation. So, the *velocity* of the particle $P$ at time $t$ is defined by

$$\mathbf{v}(t) = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{r}_P}{\Delta t} = \dot{\mathbf{r}}_P = (\dot{x}, \dot{y}, \dot{z}) = (v_x, v_y v_z), \tag{4.52}$$

where, $(x, y, z)$ is the coordinates of particle $P$ at time $t$, $\dot{\mathbf{r}}_P = d\mathbf{r}_P/dt, \dot{x} = dx/dt, \dot{y} = dy/dt, \dot{z} = dz/dt$. By (4.33), the magnitude for the velocity of particle $P$ is

$$v(t) = |\mathbf{v}(t)| = \sqrt{v_x^2 + v_y^2 + v_z^2} = \sqrt{\dot{x}^2 + \dot{y}^2 + \dot{z}^2}. \tag{4.53}$$

For example, assume the position vector of the particle $P$ is $\mathbf{r}_P = (r \sin t, r \cos t, c)$, $r > 0$. Then, its velocity vector $\mathbf{v}(t) = (R \cos t, -R \sin t, 0)$ and

$$v(t) = \sqrt{(R \sin t)^2 + (-R \sin t)^2 + 0^2} = R.$$

Generally, the velocity vector $\mathbf{v}(t)$ is not necessarily a constant vector. So, *how do we characterize the varying of the velocity vector?* Dr.Ouyang tells Huizi that it is similar to the introducing of velocity vector, i.e., consider the varying of the velocity vector $\mathbf{v}(t)$ at time $t$ and $t + \Delta t$, and let the time period $\Delta t \to 0$ infinitely. In this way, as the time difference $\Delta t$ is small to a certain extent, $\mathbf{v}(t)$ can be approximately considered as a constant vector in this period of time and then, put a vector

$$\mathbf{a}(t) = \lim_{\Delta t \to 0} \frac{\Delta \mathbf{v}(t)}{\Delta t} = \dot{\mathbf{v}}(t) = (\ddot{x}, \ddot{y}, \ddot{z}) \tag{4.54}$$

to measure the varying rate of the velocity vector $\mathbf{v}(t)$, which is called the *accelerating vector* of particle $P$ with a magnitude

$$a(t) = |\mathbf{a}(t)| = \sqrt{\ddot{x}^2 + \ddot{y}^2 + \ddot{x}^2}, \tag{4.55}$$

where, $\dot{\mathbf{v}} = d\mathbf{v}/dt, \ddot{x} = d^2x/dt^2, \ddot{y} = d^2y/dt^2, \ddot{z} = d^2z/dt^2$. For example, if the position vector of particle $P$ is $\mathbf{r}_P = (R\sin t, R\cos t, c)$, then its acceleration vector is $\mathbf{a}(t) = (-R\sin t, -R\cos t, 0)$ with the magnitude $a(t) = |\mathbf{a}(t)| = R$.

**5.2.Newtonian Mechanics.** The displacement change of a particle is the outward phenomena of the particle moving, which does not reveal the real cause of the particle moving. So, *why do we say that the moving of all things is eternal and what causes the moving of objects to move?* Dr.Ouyang tells Huizi that the Newtonian mechanics answers this question by introducing the conceptions of force $\mathbf{F}$ and the mass $m$ of particles to reveal the cause of moving. So, *what is the force?* A force $\mathbf{F}$ is an action in objects, which is a vector with an active position, magnitude and direction. And then, *what is the mass?* The mass $m$ is the intrinsic quantity of an object, which is equal to the density of the object times its volume and also a measure on the inertia of the object. For example, a barbell has a certain mass which causes its gravity under the gravitation of the earth with a downward direction. So, lifting a barbell with one hand of human must overcome the gravity and exert an upward force on the barbell. Certainly, a barbell can be lifted successfully only if the upward force of the athlete is greater than the weight of the barbell, as shown in Figure 4.25. Here, the earth's downward attraction and the athlete's upward lifting force are both actions on the barbell and the athlete's lifting the barbell is an outward presentation of the action result.

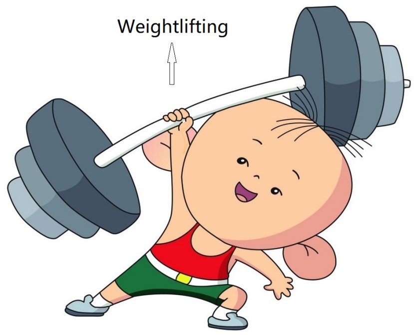



**Figure** 4.25. **Force**

**(1)Fundamental dynamic laws.** The three fundamental laws of Newtonian reveal the relationship of force with the particle moving in dynamics, which are presented in the following.

**Newton's 1st Law**(Inertia law). Every object will remain at rest or in uniform moving in a straight line unless compelled to change its state by the action of an external action. Such a tendency to resist the state changes of moving is called the *inertia* of the object.

**Newton's 2nd Law.** The acceleration $\mathbf{a}$ of an object acted by an external force $\mathbf{F}$ is proportional to the amount of the external force $\mathbf{F}$ and inversely proportional to the mass $m$ of the object with the same direction of the external force action on.

If the units of force, mass and the acceleration are chosen appropriately such as the unit of mass $m$ is the "*kilogram*"($kg$), the unit of acceleration $\mathbf{a}$ is "*meter/second*$^2$"

$(m/s^2)$ and the unit of force is "*kilogram.meter/second*$^2$"$(kg.m/s^2)$, also known as the "*Newton*"$(N)$, the Newton's second law can be expressed as

$$\mathbf{F} = m\mathbf{a}. \tag{4.56}$$

**Newton's 3rd Law** (Action & reaction law). Whenever one object exerts a force on a second object, the second object exerts an equal and opposite force on the first, i.e., the interaction forces between two objects are always equal but with opposite directions along the same line.

For example, the athlete's reaction force $-\mathbf{W}$ and the barbell's gravity $\mathbf{W}$ are equal in magnitude but in opposite directions on a straight line orthogonal to the earth's surface in Figure 4.24. Consequently, the weightlifting force of the athlete $\mathbf{F}$ acts on the barbell in the opposite direction to the gravity of the barbell. So, the condition of the athletes lifting the barbell is $|\mathbf{F} - \mathbf{W}| > 0$, namely his lifting force is greater than the barbell's gravity. Notice that the gravity of the object is the gravitational force on the object at the same point. Assume there are $k$ objects of masses $m_1, m_2, \cdots, m_k$, $k \geq 1$ at the same point and the earth's gravity on them are $\mathbf{W}_1, \mathbf{W}_2, \cdots, \mathbf{W}_k$, respectively. By the (4.56) formula, there should be $\mathbf{W}_1 = m_1\mathbf{a}_1, \mathbf{W}_2 = m_2\mathbf{a}_2, \cdots, \mathbf{W}_k = m_k\mathbf{a}_k$. However, the experiment verified that $|\mathbf{a}_1| = |\mathbf{a}_2| = \cdots = |\mathbf{a}_k| =$ a constant $g = 9.80665m/s^2$, namely

$$\left|\frac{\mathbf{W}_1}{m_1}\right| = \left|\frac{\mathbf{W}_2}{m_2}\right| = \cdots = \left|\frac{\mathbf{W}_k}{m_k}\right| = g = 9.80665m/s^2. \tag{4.57}$$

Whence, an object $P$ of mass $m$ has a weight of $|\mathbf{W}| = mg$.

If an object $P$ moves on a straight line, its force and acceleration are both along this straight line and so, the relationship of the force with the acceleration (4.55) can be written in a scalar form $|\mathbf{F}| = m|\mathbf{a}|$ or simplified to $F = ma$. However, if the object $P$ moves on a curve, it can not be so simplified. In this case, Dr.Ouyang tells Huizi that a normal coordinate system $\{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ could be set up and assumes that $\mathbf{F}_x, \mathbf{F}_y, \mathbf{F}_z$ are the components of force $\mathbf{F}$ along the $x$-axis, $y$-axis and $z$-axis in this coordinate system, which are all functions dependent on the variables $x, y, z$, time $t$ and their differentials. From the component forms of acceleration (4.54), the Newton's second law (4.56) can be written as the component forms of force $\mathbf{F}$ along the coordinate axes as follows

$$\begin{cases} \mathbf{F}_x(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) &= m\ddot{x}, \\ \mathbf{F}_y(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) &= m\ddot{y}, \\ \mathbf{F}_z(x, y, z, \dot{x}, \dot{y}, \dot{z}, t) &= m\ddot{z}, \end{cases} \tag{4.58}$$

called the *dynamic equation of particles*. Notice that this is a second order system of ordinary differential equations in which the unknown functions are the position vector

$\mathbf{r}_P(t) = x(t)\mathbf{e}_1 + y(t)\mathbf{e}_2 + z(t)\mathbf{e}_3$ or the coordinates $(x(t), y(t), z(t))$ of particle $P$. In this way, the locus of particle $P$ can be obtained by solving the system (4.58) of differential equations.

For example, if a particle $P$ of mass $m$ falls vertically from the air without initial velocity, the coordinate system $\{O; x\}$ is established with the origin $O$ as the initial position and the downward position as the positive direction of $x$-axis in the coordinate system, as shown in Figure 4.26. So, *how do we determine the locus of particle $P$ in this coordinate system?* Dr.Ouyang explains that first, assuming that there is no air resistance, i.e., $P$ falls freely with an initial velocity $v_0 = 0$ in vacuum.

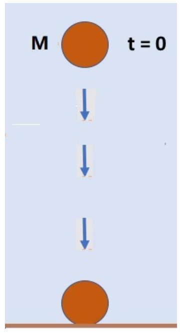


**Figure** 4.26. **Free fall**

According to the equation (4.58), the dynamic equation of particle $P$ is

$$\begin{cases} m\ddot{x} = mg, \\ x(0) = 0, \ \ \dot{x}|_{t=0} = 0. \end{cases}$$

Directly integrating the first equation gets $x(t) = mgt^2/2 + c_1 t + c_2$, where, $c_1, c_2$ are both constants. By the initial conditions $x(0) = 0$, $\dot{x}|_{t=0} = 0$, we know the constants $c_1 = 0, c_2 = 0$ and so, $x(t) = gt^2/2$.

Secondly, consider the case of existing air resistance. Assume that the air resistance is proportional to the falling velocity with a resistance coefficient $k$, then the particle $P$ will be subjected to a resistance force $-mk\dot{x}$ when it falls. According to the equation (4.58), the dynamic equation of particle $P$ falling is

$$\begin{cases} m\ddot{x} = mg - mk\dot{x}, \\ \\ x(0) = 0, \ \ \dot{x}|_{t=0} = 0 \end{cases}$$

in this case. Let $\dot{x} = u + g/k$. Then, the first equation is simplified to $\dot{u} + ku = 0$, i.e., $du/u = -kdt$. Integrating this equation on both sides we get $\ln|u| = -kt + c$, i.e., $u = c_1 e^{-kt}$ where $c, c_1 = e^c$ are constants. So, $\dot{x} = c_1 e^{-kt} + g/k$. By the initial condition $\dot{x}|_{t=0} = 0$ we get $c_1 = -g/k$. Whence,

$$\dot{x} = -\frac{g}{k}e^{-kt} + \frac{g}{k} = \frac{g}{k}\left(1 - e^{-kt}\right). \tag{4.59}$$

Reintegrating the equation (4.59) on both sides, we get that

$$x(t) = \frac{g}{k}t + \frac{g}{k^2}e^{-kt} + c_2,$$

where $c_2$ is a constant. By the condition $x(0) = 0$ we know that $c_2 = -g/k^2$ and

$$x(t) = \frac{g}{k}t - \frac{g}{k^2}\left(1 - e^{-kt}\right).$$

Using the relation of the arc length with that of the arc angle, i.e., the arc length $L$ is equal to the product $R\theta$ of the arc angle $\theta$ and the radius, it can be known that the angular velocity $\omega = \theta/t = 2\pi/T$ and the linear velocity $v = 2\pi R/T$ i.e., the linear velocity $v = R\omega$ for a particle $P$ in uniform circular moving, where $T$ is the period of the moving. For determining the direction of the linear velocity, let the time interval $\Delta t \to 0$ similar to the curve moving of a particle. Then, the centripetal velocity $\Delta v$ points to the center of the circle, as shown in Figure 4.27. Whence, the centripetal velocity $dv = vd\theta$ and the centripetal acceleration $a(t) = \dot{v} = v\dot{\theta} = v\omega = v^2/R$. So, the amount of the centripetal force $\mathbf{F}$ is

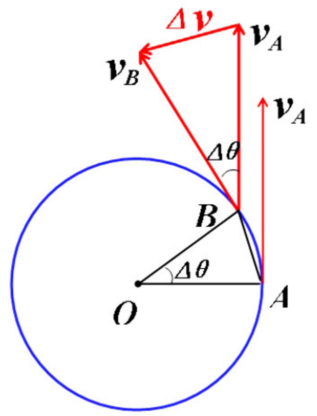



**Figure** 4.27. **Circular moving**

$$|\mathbf{F}| = F = ma(t) = \frac{mv^2}{R} \tag{4.60}$$

for a circular particle with mass $m$ in general.

**(2)Conservative field.** A force $\mathbf{F}$ acting on an object $P$ causes $P$ to produce a displacement $\Delta\mathbf{r}$, the acting effect of which can be measured by the magnitude of the *work* $W$, i.e

$$W = \mathbf{F} \cdot \Delta\mathbf{r} = |\mathbf{F}||\Delta\mathbf{r}|\cos\theta, \tag{4.61}$$

where, $\theta$ is the angle between the action direction of $\mathbf{F}$ and the direction of displacement $\Delta\mathbf{r}$. An object that has the ability to do work is said to have energy, including both the kinetic and potential energies. By definition, the *kinetic energy* $\mathcal{T}$ is such an energy that an object possessed because it has a certain velocity, namely if a particle $P$ of mass $m$ has the velocity $v$, then its kinetic energy is $mv^2/2$ and the *potential energy* $V$ is the energy possessed by the change of an object $P$ in a relative position. Usually, the sum of kinetic and potential energies possessed by a particle $P$ is called the *mechanical energy* $E$.

A force $\mathbf{F}$ is said to be a *conservative force* if its acting work is only related to its position, i.e., independent of the intermediate paths. Such an acting field is called the *conservative field*. In this case, there exists a potential function $\mathcal{V}$ such that $\mathbf{F} = \mathcal{V}(x_0, y_0, z_0) - \mathcal{V}(x_1, y_1, z_1)$, where $(x_0, y_0, z_0)$, $(x_1, y_1, z_1, )$ are respectively the coordinates of the starting and the end points while the force $\mathbf{F}$ acts on particle $P$. For example, lifting a particle $P$ from the height $z_1$ to $z_2$ on the earth will result in a work $-mg(z_2 - z_1)$ by the lifting force, which can be characterized by a scalar function $mgz$, i.e., the work of a conservative force on a particle is equal to the reduction of the scalar function $\mathcal{V}$ at different positions. Now, let the mass of the particle $P$ subject to a conservative force $\mathbf{F}$ be $m$ and its velocities at the starting and end points be $v_0$ and $v_1$, respectively. Notice that $\mathbf{v} \cdot d\mathbf{v} = vdv$. By the Newton's 2nd law, we know that

$$\ddot{\mathbf{r}} = \mathbf{F} \ \Rightarrow \ m\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} = \mathbf{F} \cdot \dot{\mathbf{r}} \ \Rightarrow \ m\dot{\mathbf{v}} \cdot \mathbf{v} = \mathbf{F} \cdot \dot{\mathbf{r}},$$

i.e.,

$$d\left(\frac{1}{2}mv^2\right) = \mathbf{F} \cdot d\mathbf{r}. \tag{4.62}$$

It should be noted that the force $\mathbf{F}$ is conservative, i.e., it's action is independent on the intermediate path. Whence, integrating the equation (4.62) on both sides enables us to get

$$\frac{1}{2}mv_1^2 - \frac{1}{2}mv_0^2 = \int_{(x_0,y_0,z_0)}^{(x_1,y_1,z_1)} \mathbf{F} \cdot d\mathbf{r} = \mathcal{V}\left(x_0, y_0, z_0\right) - \mathcal{V}\left(x_1, y_1, z_1\right),$$

namely, the mechanical energy

$$\frac{1}{2}mv_0^2 + \mathcal{V}\left(x_0, y_0, z_0\right) = \frac{1}{2}mv_1^2 + \mathcal{V}\left(x_1, y_1, z_1\right) = E \tag{4.63}$$

is a constant, which is called the *conservative law of mechanical energy*, i.e., the mechanical energy of a particle $P$ in a conservative field is always conservative.

**(3)$n$-Body problem.** Dr.Ouyang tells Huizi that a major area of application of Newtonian mechanics is to determine the orbits of planets. Certainly, there exists a gravitational force $\mathbf{F}$ between two objects $P_1, P_2$ with respectively masses $m_1, m_2$ in the universe, whose amount is proportional to the product of masses $m_1, m_2$ and inversely proportional to the square of their distance, i.e., $|\mathbf{F}| = G \cdot m_1 \cdot m_2/r^2$. This is the famous *gravitational law* of Newton, where $G = 6.67 \times 10^{-11} N \cdot m^2/kg^2$ is the gravitational constant and $r$ is the distance between objects $P_1$ and $P_2$. Notice that in the moving of celestial bodies, all other actions compared with planetary gravity are trivial. And so, we can assume that the force between the planets is mainly gravitational. Now, let $\{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ be the coordinate system of $\mathbb{R}^3$. Then, *given the initial positions $\mathbf{r}_{P_i}(t_0)$, velocities $\mathbf{v}(t_0)$ and the masses $m_i$ of the $n$ planets $P_i, 1 \leq i \leq n$, how do we get the moving state of the $n$ planets in a subsequent time $t$?* This is the famous *$n$-body problem* in celestial mechanics. In fact, by applying Newton's 2nd law and the gravitational law to the $i$th planet, we can get a differential equation

$$m_i \ddot{\mathbf{r}}_{P_i} = \sum_{j \neq i} \frac{G m_i m_j \left(\mathbf{r}_{P_j} - \mathbf{r}_{P_i}\right)}{\left|\mathbf{r}_{P_j} - \mathbf{r}_{P_i}\right|^3},$$

where, $\mathbf{r}_{P_i}$ and $m_i$ are respectively the position vector and the mass of planet $P_i$, $1 \leq i \leq n$. In this way, the $n$-body problem corresponds to an initial value problem of differential equations

$$\begin{cases} \ddot{\mathbf{r}}_{P_i} = \sum\limits_{j \neq i} \dfrac{G m_j \left(\mathbf{r}_{P_j} - \mathbf{r}_{P_i}\right)}{\left|\mathbf{r}_{P_j} - \mathbf{r}_{P_i}\right|^3}, \\ \mathbf{r}_{P_i}\big|_{t=t_0} = \mathbf{r}^0_{P_i}, \ \dot{\mathbf{r}}_{P_i} = \mathbf{r}^1_{P_i}, \ 1 \leq i \leq n. \end{cases} \tag{4.64}$$

Generally, it is very difficult to get an analytical solution of the system (4.64) for $n \geq 3$. Even if $n = 3$, the system (4.46) is solved only in special cases until today. However, for the 2-body problem, the system (4.64) is

$$\ddot{\mathbf{r}}_{P_1} = \frac{G m_2 \left(\mathbf{r}_{P_2} - \mathbf{r}_{P_1}\right)}{\left|\mathbf{r}_{P_2} - \mathbf{r}_{P_1}\right|^3}, \quad \ddot{\mathbf{r}}_{P_2} = \frac{G m_1 \left(\mathbf{r}_{P_1} - \mathbf{r}_{P_2}\right)}{\left|\mathbf{r}_{P_1} - \mathbf{r}_{P_2}\right|^3}. \tag{4.65}$$

Let $\mathbf{r} = \mathbf{r}_{P_2} - \mathbf{r}_{P_1}$. Then, by the equation (4.65) we know that

$$\begin{aligned} \ddot{\mathbf{r}} &= \ddot{\mathbf{r}}_{P_2} - \ddot{\mathbf{r}}_{P_1} \\ &= \frac{G m_1 \left(\mathbf{r}_{P_1} - \mathbf{r}_{P_2}\right)}{\left|\mathbf{r}_{P_1} - \mathbf{r}_{P_2}\right|^3} - \frac{G m_2 \left(\mathbf{r}_{P_2} - \mathbf{r}_{P_1}\right)}{\left|\mathbf{r}_{P_2} - \mathbf{r}_{P_1}\right|^3} \\ &= -G\left(m_1 + m_2\right) \frac{\mathbf{r}}{|\mathbf{r}|^3}, \end{aligned}$$

i.e.,

$$\ddot{\mathbf{r}} = -G\left(m_1 + m_2\right) \frac{\mathbf{r}}{|\mathbf{r}|^3}. \tag{4.66}$$

Whence, we can first get an analytical solution of equation (4.66) for $\mathbf{r}$ and then, substituting $\mathbf{r}$ into the equation (4.65), we get respectively the equations of $\mathbf{r}_{P_1}$ or $\mathbf{r}_{P_2}$ following

$$\ddot{\mathbf{r}}_{P_1} = G m_2 \frac{\mathbf{r}}{|\mathbf{r}|^3}, \quad \ddot{\mathbf{r}}_{P_2} = -G m_1 \frac{\mathbf{r}}{|\mathbf{r}|^3}. \tag{4.67}$$

Notice that each differential equation in (4.67) can be integrated separately. We therefore get that

$$\mathbf{r}_{P_1}(t) = G m_2 \int \left( \int \frac{\mathbf{r}}{|\mathbf{r}|^3} dt \right) dt, \ \ \mathbf{r}_{P_2}(t) = -G m_1 \int \left( \int \frac{\mathbf{r}}{|\mathbf{r}|^3} dt \right) dt, \tag{4.68}$$

which is an analytical solution to the 2-body problem.

So, *how do we get analytical solutions to the general $n$-body problem?* Dr.Ouyang tells Huizi that it is almost impossible to solve the system (4.64) analytically in general by the classical conception of solving equations. That is, we have to think about this problem with different views. Notice that the combinatorial structure of $n$ planets under the gravitational force is a complete graph $K_n$, and a solution of the $n$-body problem is in fact a graph $K_n^L$ labelled for each vertex $v \in V(K_n)$. For example, the sun, earth and the moon constitute the sun-earth-moon system in which the gravitational relationship is nothing else but the 3-body problem with a labeled complete graph $K_3^L$ in Figure 3.16$(a)$ of Chapter 3.

Notice that the forces acts independently, namely the acceleration caused by several forces applied to the same object is equal to the sum of the vectors of the acceleration caused by each individual force applied to the object, which is called the *independent law of force action.* According to this law, the force of gravitation on the $i$th planet is additive, equal to the sum of the forces exerted on it by each of the other planet. Similarly, the vectors are also additive and $\mathbf{r}_{P_i}$ is the sum of the position vectors $\mathbf{r}_{P_i}$ determined by the gravitation acting on them by each other planet, i.e.,

$$\begin{aligned}\mathbf{F}_i &= \mathbf{F}_1 + \cdots + \mathbf{F}_{i-1} + \mathbf{F}_{i+1} + \cdots + \mathbf{F}_n,\\ \mathbf{r}_{P_i} &= \mathbf{r}_{P_1} + \cdots + \mathbf{r}_{P_{i-1}} + \mathbf{r}_{P_{i+1}} + \cdots + \mathbf{r}_{P_n}.\end{aligned} \tag{4.69}$$

Certainly, the additive property of vectors may be applied to the $n$-body problem. That is, the gravitational action on the planets $v$ can be decomposed into $n-1$ 2-body problems, namely $\{v, u_1\}, \{v, u_2\}, \cdots, \{v, u_{n-1}\}$, where $v, u_1, u_2, \cdots, u_{n-1}$ are the $n$ vertices of $K_n$. In this way, the position vector of vertex $v_i,\ 1 \leq i \leq n$ can be obtained by using the solution of 2-body problem, i.e.,

$$\mathbf{r}_{P_i}(t) = G\sum_{j\neq i} m_j \int\left(\int \frac{\mathbf{r}_{ji}}{|\mathbf{r}_{ji}|^3}dt\right)dt, \tag{4.70}$$

where, $\mathbf{r}_{ji} = \mathbf{r}_{P_j} - \mathbf{r}_{P_i}$, $j \neq i$, $1 \leq j \leq n-1$.

**(4)Cosmic velocity.** Certainly, the purpose of studying the $n$-body problem is to hold on the laws of celestial bodies and then, serve our human society. However, Dr.Ouyang tells Huizi that it is very difficult to solve the equation (4.64) of $n$-body moving in general, which shows the limitation or there are always things that can not be recognized by humans in the universe. Even so, the complete solution of the 2-body problem enables humans also to simulate the operation of small celestial bodies and launch the satellites to serve production and living practices of humans. Among them, the fundamental question is to determine what an initial velocity of a satellite is needed to go around the earth, what an initial velocity is needed to go around the sun escaped from the earth gravitation and what an initial velocity is needed to fly out of the solar system, completely escaped from the gravitation of the sun, which are respectively called the *1st cosmic velocity* $v_1$, *2nd cosmic velocity* $v_2$ *and the 3rd cosmic velocity* $v_3$, see Figure 4.28 for details. Such a problem is equivalent to the emission of an object $P$ with mass $m$ from a point on the earth's surface at a horizontal angle $\alpha$, velocity $v_i, 1 \le i \le 3$. So, *how do we determine the values of the 1st, 2nd and the 3rd cosmic velocities* $v_i, 1 \le i \le 3$*?*

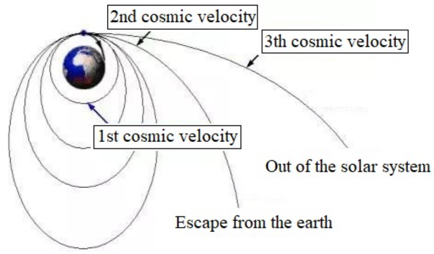



**Figure** 4.28. **Flying into space**

First of all, a satellite going around the earth in height $h$ from the ground is in a circular moving and its required centripetal force is provided by the gravitation of the earth. Whence there must be

$$\frac{mv^2}{R+h} = G\frac{mm_{earth}}{(R+h)^2} \quad \Rightarrow \quad v = \sqrt{\frac{Gm_{earth}}{R+h}}. \tag{4.71}$$

by (4.60), where $R = 6400km$, $m_{earth} = 5.965 \times 10^{24}kg$ are respectively the radius and the mass of earth. Notice that a satellite flies generally near the ground, namely $h$ is far less than $R$, i.e., $h \ll R$. So, $R+h$ can be approximately replaced by $R$. In this way, substituting the values of $R$, $m_{earth}$ and the constant $G$ into the equation (4.71), we get that

$$v_1 = \sqrt{\frac{Gm_{earth}}{R}} = \sqrt{\frac{6.67 \times 10^{-11} \times 5.965 \times 10^{24}}{6.4 \times 10^6}} \approx 7.9km/s,$$

which is the 1st cosmic velocity.

For calculating the 2nd cosmic velocity $v_2$, it is assumed that the kinetic energy generated by the satellite after its launch can escape from the earth's gravitation and fly to the infinity. So, we have

$$\frac{1}{2}mv_2^2 \ge \int_{x=R}^{\infty} G\frac{mm_{earth}}{x^2}dx = G\frac{mm_{earth}}{R},$$

i.e.,

$$v_2 \ge \sqrt{\frac{2Gm_{earth}}{R}} = \sqrt{2} \times \sqrt{\frac{Gm_{earth}}{R}} = \sqrt{2}v_1.$$

Whence, we get the second cosmic velocity $v_2 \ge \sqrt{2}v_1 = 11.2km/s$ by substituting $v_1 \ge 7.9km/s$ into this formula with the minimum $7.9km/s$.

For the 3rd cosmic velocity $v_3$, we first calculate the launch velocity needed to launch a satellite on the earth such that it can escape from the sun's gravitation. This is actually in the calculation of 2nd cosmic velocity by replacing $m_{earth}$ with the sun mass $m_{sun}$ and the earth radius $R$ with the distance $R'$ between the earth and the sun. It is known that the mass of the sun is $333,400$ times of the earth's mass and the average radius of the earth's orbit around the sun is about $23,400$ times of the earth's radius. Thus, by the calculation of the 2nd cosmic velocity the 3rd cosmic velocity should be satisfied

$$\frac{1}{2}mv_3^2 \geq \frac{Gmm_{sun}}{R'} \Rightarrow v_3 \geq v_2\sqrt{\frac{m_{sun}/m_{earth}}{R'/R}} = 11.2 \times \sqrt{\frac{333400}{23400}} \approx 42.27km/s$$

which is a considerable launch velocity. However, it should be noted that the revolution speed of the earth around the sun is $29.8km/s$. If the launching direction of the satellite is chosen to be consistent with the revolving direction of the earth, the launch velocity can be reduced to $42.27 - 29.8 = 12.47km/s$. At the same time, the launch should be escaped also from the earth gravitation, i.e., the launch of the initial velocity should be satisfied

$$v_3^2 - \frac{2Gm_{earth}}{R} \geq 12.47^2 \Rightarrow v_3 \geq \sqrt{2 \times 7.9^2 + 12.47^2} \approx 16.7km/s,$$

namely, $v_3 \geq 16.7km/s$ can be escaped from the sun's gravitation, which is the 3rd cosmic velocity with the minimum $16.7km/s$.

**5.3.Time Dilation.** A coordinate system is the benchmark for measuring the evolution of things. Whence, the coordinate transformation assumes that the origin $O$ of one coordinate system is in another coordinate system with fixed direction of coordinate axes so as to determine the coordinates of the same point $P$ in the two coordinate systems, i.e., the coordinate transformation (4.16), which is a linear transformation. However, the moving is eternal which concludes the fixation of the origin and direction of the axes is only relative. If one coordinate system is moving with respect to another coordinate system, *how do we determine the transformation relationship between the two coordinates?* Dr.Ouyang asks Huizi thinking about this question and tells her that we can assume that the moving speed of a coordinate system $\{O'; \mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ in $\{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is a uniform speed $v$ along its $x$-axis direction such as the shown in Figure 4.29. So, *how do we determine the coordinates* $(x, y, z)$ *in the coordinate system* $\{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ *for a point* $(x', y', z')$ *in the coordinate system* $\{O'; \mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$*?* Suppose one can simultaneously measure the coordinates of a particle $P$ in the coordinate systems of $\{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{O'; \mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$. Then, this problem can be solved by such a consideration, i.e., we have $y' = y, z' = z$ for the particle $P$ moving in both coordinate systems at time $t$. Notice that $P$ moves along the $x$-axis, the coordinate value $x$ consists of two

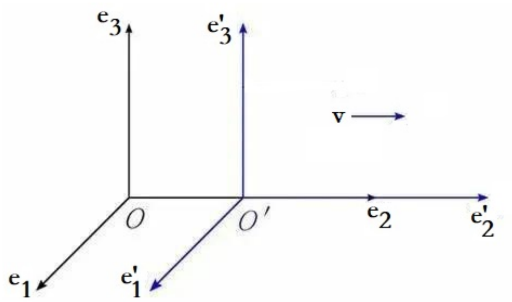


**Figure** 4.29. **Coordinates moving**

parts, i.e., one is the coordinate $x'$ in $\{O'; \mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'\}$ and another is the coordinates of the origin $O'$ of the coordinate system $\{O'; \mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'\}$ in the coordinate system $\{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, namely its moving distance $vt$, i.e., $x = x' + vt$. So, the coordinates of the particle $P$ in the coordinate system $\{O'; \mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'\}$ are respectively

$$x' = x - vt, \ y' = y, \ z' = z \tag{4.72}$$

at time $t$. Certainly, if the two coordinate systems $\{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ and $\{O'; \mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'\}$ are only different in the spatial location, this calculation is absolutely right. However, *how do we ensure that the measured time is the same and the measured length is the same in the two coordinate systems with one $\{O'; \mathbf{e}_1', \mathbf{e}_2', \mathbf{e}_3'\}$ moving in another $\{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$?* Dr.Ouyang asks Huizi if she had ever heard of the saying that "*one day in the heaven equal to a year on the earth*". Huizi replies:"*Yes! There is such a saying in Journey to the West.*" Dr.Ouyang reminds her further:"*Does this implies that the running of a clock in the heaven is slower than it on the earth?*" Huizi thinks for a while and says:" *It seems like this, Dad!*" After a pause, Huizi asks Dr.Ouyang back:"*But if it like this, is it implies that one lives longer in the heaven than on the earth? For example, if we send an astronaut into the space, let he lives two months in the sky and then returns, which is equivalent to 60 years that he lives on the earth and maybe younger than his little daughter. How is this possible?*" Dr.Ouyang tells her that the science fiction written in this way is showing the difference life in the heaven and on the earth. In fact, the length of the measuring time depends on the moving speed of the clock, namely the faster the clock moves the slower the time is measured. This phenomenon is known as the *time dilation*. For example, the running speed of a clock on the earth is $29.8km/s$ with the earth's revolution. Certainly, the measured time is essentially the earth time. Moreover, even placing two clocks in two different places on the earth and one of them is in moving, there is no consistency in the measuring time because the measurement is the reflection of the time on the clock transmitted to the human eyes and the medium is the light with speed $c \approx 3 \times 0^5 km/s$ in vacuum. Whence, there is a time difference between the transmission and reception which causes the time dilation. Similarly, the length has also this dilation effect even if the length is only a product in the chosen reference frame. Generally, the relative coordinates of a particle $P$ are not determined by equation (4.72) in a moving coordinate system but they are

$$x' = \frac{x - vt}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}, \quad y' = y, \quad z' = z, \quad t' = \frac{t - \frac{v}{c^2}x}{\sqrt{1 - \left(\frac{v}{c}\right)^2}}, \tag{4.73}$$

called the *Lorenz transform*, i.e., the dilation of length and time is related to the coefficient

$\sqrt{1-(v/c)^2}$, where $v$ is the moving speed of $P$. Then, *why do we not perceive the dilation of length or time in our daily life?* It is because the moving speed $v$ of objects in daily life is far less than the speed of light, i.e., $v \ll c$ which can be neglected and then obtain the calculation result of (4.72).

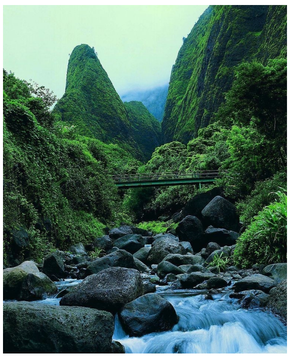
For further inspiring Huizi's understanding of the relativity of human recognition, Dr.Ouyang asks her:"*Did you ever feel that sometimes time passed very slowly and sometimes it passed very quickly in school?* Huizi replies:"*Yes! If the teacher's lecture was too boring or I was not interested in the topic, I felt the time is passed too slow and looking forward to finish the class earlier. But after class, I played games with my classmates and then felt the time is passed too fast, soon the bell rang. Dad, is this time inflation?*" Dr.Ouyang tells her that this situation is due to her psychological effects rather than the time dilation, which is related to the personal preferences rather than an objective passage of time.

## §6. **Comments**

**6.1.** Usually, we recognize a thing by its characters and hold on its character vector or "*name*" as the recognition, which needs to be carried out in a fixed reference frame. Generally, it is relatively simple for characterizing things in linear space, particularly in the Cartesian coordinates, see [Hok], [Qiu] and [Zho]. Usually, the different researchers stand on different perspectives and apply different reference frames to get different forms of vectors for describing things. It should be noted that the different characterizing forms are caused by the reference frames, not from the reality of things. So, *how do we solve the formal difference of characterizing vectors caused by different reference frames?* The answer is the general relativity of Einstein, i.e., an equation that describing the evolving law of an objective thing should have the same form in all coordinate systems, see [Car], [Ein1]-[Ein2] and [Fei], etc. because the evolving law of an objective thing does not different due to the different reference frame of human. That is, the different equations obtained in different reference frames are only different in forms, not the reality of things. However, after Einstein proposed the general relativity, more researchers follow Einstein's example to adjust the existing physical laws to covariant forms. It should be paid attention to this trend because the essence of finding a covariant form is a mathematical game, i.e., invariant under the coordinate transformation, not an objective law of things. In fact, a physical equation should be describing the physical law of things, no matter in what kind of reference frame, not necessarily limited to the covariant form. That is why the equal-rights principle was proposed in [Mao52].

**6.2.** Obviously, the elements in a spanning basis of a finite linear space is also finite. Whence, it is different from the usual proof of the dimension formula (4.41) of linear space in general textbook of linear algebra, it is more enlightening to find and obtain the formula (4.41) by applying the inclusion-exclusion principle.

**6.3.** Newtonian mechanics is characterizing the macroscopic objects moving with low-speed in a Cartesian coordinate system, see references [New1]-[New2] and [Zhy]. Among them, asking the physical equation to meet the invariance under Lorenz transformation is the Einstein's special relativity. Notice that the "*low speed*" and "*macro*" conditions in Newtonian mechanics indicate that it is a local characterization of object moving, including Newton's law of universal gravitation. However, the limited recognition is more willing to accept by humans and would like to apply this kind of local laws because it is consistent with human perception, which is why we still use Newton's law of universal gravitation rather than Einstein's gravitational field equations to calculate the cosmic velocities if launching satellites from the ground. But the Newtonian mechanics is indeed a local theory on macroscopic objects moving with low-speed.



*Nature never deceives us; it is we who deceive ourselves.*

– By Rousseau, a French philosopher

## Chapter 5

# Systemic Recognizing the Universe

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**All Things Play Own Role in Universe,**
**Unless an Unintentional Cloud Without Mind;**
**Even Quietly Living Alone,**
**But Close to Mountain and Clouds Every Day;**
**See Osmanthus Comparing with Grasses,**
**Wearing Orchids to Show Pureness;**
**Why Not Need to Wash in Water,**
**There Are no Dust in Mind.**

— *Send Zhu Sheng* by Xiu Ouyang, a writer in Song dynasty of China

## §1. **Systemic Recognition**

Things in the universe do not exist in isolation but are all interdependent and interrelated. So, we need to apply a holistic or a systematic view for understanding a thing. By this time, Huizi has developed her habit of independent thinking. However, she still puzzles about why the blind man who had respectively felt the legs, tail, nose, ears, stomach and the teeth of the elephant but described respectively the elephant as looking like a pillar, a rope, a radish, a fan, a wall or a pipe in the fable. Dr.Ouyang tells her, the reason why the blind men perception on the shape of an elephant is incorrect when they touch the body of the elephant is that the blind men's perception of the shape of the elephant is local, and what the sophist told the blind men that the shape of the elephant should be the sum of all their individual perceptions is a systematic or a holistic recognition of a thing.

The essence of the recognition of a thing is to find the causal relationship, i.e., to trace the "*cause*" from its appearing "*effect*" of a thing to the elements, combination structure and the dynamic law of thing, and then hold on the behavior of thing, including the physical and the event recognitions, whose main work lies in the characterizing of the behavior of things. Notice that the essence of systematic recognition of a thing is to find the "*causes*" at the micro level, namely the elements, which is to equate a thing with a system composed of elements in a spatial combination structure. This is undoubtedly correct from the perspective of the physical combination of things because everything in the universe is composed of elementary particles. But, *do the elements constitute the thing necessarily the elementary particles in physical assemblage?* Dr.Ouyang tells Huizi that the answer is No! This has been demonstrated in biological systems at least because the elements of livings are cells and genes or bits of DNA of biological macromolecules, namely the compositions of elements in livings are not just the elementary particles themselves. From the perspective of human's need to know everything, a system recognition should be more in line with the human's recognitive needs because the elements of a system can be selected according to human's recognitive needs so as to bridge the gap of uncertainty in the recognition of elementary particles. For example, by anatomical studying, the human recognition of bovine digestion is a systematic recognition of its feeding, rumination, absorption and excretion, as shown in Figure 5.1.

**1.1.System Features.** The system is an important conception for understanding things in the universe, from a celestial body in the universe to a grain of sand in a river. It includes two main types, i.e., the *closed system*, no any exchanging of the energy and information with the outside of the system and the *open system* influenced by the outside world. Dr.Ouyang told Huizi that all systems discussed by humans should be open because they are subsystems of the system consisting of things in the universe. This is also a reason that the human recognition of thing is local. And then, *under what conditions that we can adopt the closed system for understanding the behavior of thing?* Dr.Ouyang tells Huizi that it can only be used when other things and the external environment have a little influence or effect on the behavior of the thing that can be completely ignored. That is to say, the closed system recognizes a thing is an approximation rather than necessarily the real face of thing. Because of this, the human recognition of thing in most cases adopts the open system model with certain input, i.e., the effect of other related things, the external environment on the thing and output, namely the externally observable case, as shown in Figure 5.2.

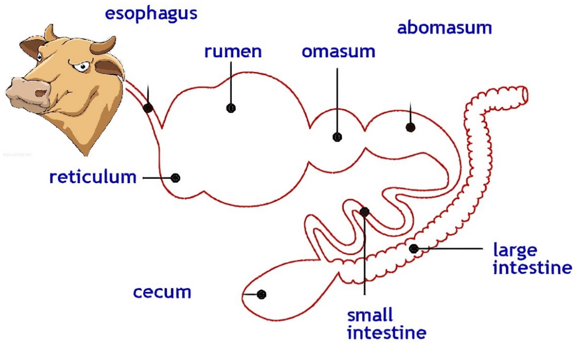



**Figure** 5.1. **Cattle's digestive system**

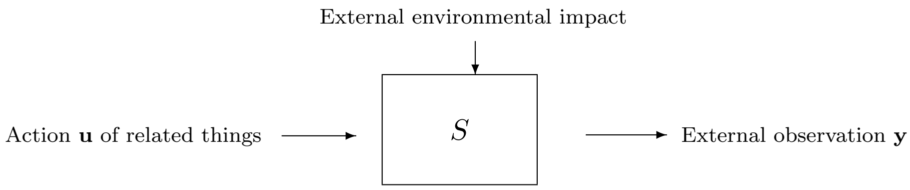



**Figure** 5.2. **System input and output**

Notice that if a system $S$ is regarded as a *black box* in Figure 5.2, i.e., it only relies on the action $\mathbf{u}$ of related things, external environmental influences and the external observation results $\mathbf{y}$ to hold on the changing behavior of the system $S$ without considering the interaction between elements in system $S$. Such a theory established is nothing else but the "*phenomenological theory*" discussed in Chapter 2. Generally, we hope to hold on the behavior of system $S$ with knowing the interaction of elements in $S$. This can only be an ideal state in the recognition of thing, which is the simulation of a natural system, namely the artificial system by humans.

Abstractly, Dr.Ouyang tells Huizi that if a set $S$ satisfies conditions: ①$S$ consists of a set of elements with $|S| \geq 2$; ②Elements in $S$ are relating and acting on each other according to a rule $R$; ③$S$ has a functional appearance that can be observed by humans, then $S$ is said to be a *system* and elements in $S$ are the elements of system $S$. Notice that the abstract definition of system can be further visualized as a labeled graph $G^L[S]$, i.e

$$V\left(G^L[S]\right) = \{\ v \mid v \in S\},\ \ E\left(G^L[S]\right) = \{\ (v,u) \mid (v,u) \in R\},$$
$$L : v \in S \to v, \quad (v,u) \to v \overset{R}{\sim} u. \tag{5.1}$$

Generally, if the action of $v$ on $u$ can be quantified, the action function $f(v,u)$ can be introduced, then the notation $v \overset{R}{\sim} u$ can be replaced by $f(v,u)$ in the labeled graph $G^L[S]$, as shown in Figure 5.3. In this way, a system $S$ is equivalent to a label graph $G^L[S]$. Accordingly, the systematic recognition of thing is equivalent to two situations, i.e., one is the description of the dynamic behavior of the system labeled graph $G^L[S]$ itself and another is the dynamic behavior of the system labeled graph $G^L[S]$ under the external action, namely the dynamic characterization $G^L[S](t)$.

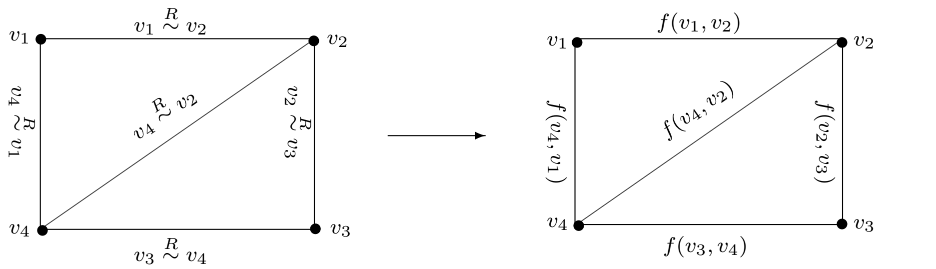



**Figure** 5.3. **Labeled graph of system**

A system $S$ is characterized by three features: ①A system is composed of a set of elements which may be the microscopic particles, cells or the subsystems in a relatively higher order system such as the technological systems indicated in the comment of Chapter 3. Generally, the different choice of observation or standing point will directly affect whether the observation result is an isolated or a systematic behavior of thing $M$. If the observer stands outside $M$ and observes the behavior of $M$ from a macroscopic perspective then only the external manifestation or behavior of things $M$ can be observed. It can only be assumed or inferred by the human brain for the internal structure and interaction of elements in $M$. At this time, the object $M$ can be abstracted as a point or an element for studying. However, if the observer stands inside the object $M$ and observes the behavior of $M$ from a microscopic perspective, the internal structure of $M$ will be displayed in front of the observer. That is, $M$ itself contains a combination structure that needs to be studied. In this case, an element becomes a subsystem relative to the interior. For example, the human body system is composed of the moving subsystem, nervous subsystem, endocrine subsystem, circulatory subsystem, respiratory subsystem, digestive subsystem, urinary subsystem and the reproductive subsystem; ②A system may have an intrinsic combinatorial structure which is the topological structure inherited in its constituent elements, i.e., all elements are interrelated, mutually restricted and there is a certain organizational order and the out-of-control relationship, i.e., the internal structure of the system. For example, the elephant shape tree structure shown in Figure 3.3, the intrinsic combination structure of hadrons shown in Figure 3.24 reflect the mutual influence and interaction between elements, which determine the state of a system; ③Function or purpose of system. Here, the function of system refers to the character, capability and the role shown in the action or relation between the system with the external environment. Notice that the function of a system is only had if the elements are combined into the system over its topological structure. Otherwise, it does not have the function and characterization of the system. For example, although the moving, nervous, endocrine, circulatory, respiratory, digestive, urinary and the reproductive subsystems are subsystems of a human body but only the functions of these subsystems are respectively the subsystems of a human, also with the characteristic of a human such as the consciousness, the system then maybe a human. Otherwise, although the eight systems could be combined into a living, it would only be called an animal but not a human.

Now that a system is consisted of its elements. The recognition on the element behaviors and the change of its combinational structure is a useful way for holding on the behavior and then tracing the "*cause*" from the appearing of a thing.

**1.2.System Element.** The recognition of system elements generally ignores its internal composition and structure, abstracts them as geometrical points labeled by vectors consisting of corresponding features in the space, similar to the particle points as shown in Figure 5.4, where the notations $t_1, t_2, \cdots, t_s$ denote the quality or characteristic of an element $v$ including the physical properties such as the position, mass, velocity, electric charge, spin angular momentum and the isospin of $v$, the chemical properties such as the acid-base, flammability, stability, oxidation and the corrosion, and the biological characteristics such as the growth, reproduction, metabolism, heredity and the variation. So, *how do we describe the behavior of an element?* Generally, one uses an explicit function $f(t_1, t_2, \cdots, t_s)$ to describe the change of an element $v$, i.e., establish an equation

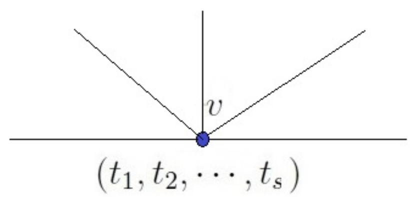



**Figure** 5.4. **System element**

$$v(t_1, t_2, \cdots, t_s) = f(t_1, t_2, \cdots, t_s) \tag{5.2}$$

on its behavior. For example, the element $v$ moves along a straight line on the $f(t) = 3t+2$ in a plane $\mathbb{R}^2$ or on the surface of $f(t_1, t_2, t_3) = t_1^2+t_2^2+t_3^2+t_1t_2+t_2t_3$ in the 3-dimensional $\mathbb{R}^3$

and so on. But in general, it is difficult to directly obtain $f(t_1, t_2, \cdots, t_s)$ because one can only simulates and deduces the behavior of $v$ based on the observed data. Dr.Ouyang tells Huizi that each observed quantity of the feature $t_i, 1 \leq i \leq s$ is a relative quantity of the feature $t_i$ changing, namely the difference $\Delta t_i = t_i - t_{i0}$ on an initial value $t_{i0}$ rather than an absolute quantity. At the same time, the observation results are usually the discrete values. Whence, no matter which method is used to simulate the change rule $t_i$ of element $v$, it can only be an approximate describing. Furthermore, it is necessary to assume that the change of element $v$ satisfies certain mathematical conditions and the corresponding mathematics can be applied to obtain the simulation function $v(t_1, t_2, \cdots, t_s)$ in advance. In general, the implicit equation is obtained by taking partial derivatives on both sides of the equation (5.2) with respect to $t_i$, i.e.,

$$\frac{\partial}{\partial t_i} v(t_1, t_2, \cdots, t_s) = h\left(\frac{\partial v}{\partial t_1}, \frac{\partial v}{\partial t_2}, \cdots, \frac{\partial v}{\partial t_s}\right) \tag{5.3}$$

and then, the change $v(t_1, t_2, \cdots, t_s)$ of element $v$ corresponds to an initial value problem on a first order partial differential equation

$$\begin{cases} \dfrac{\partial}{\partial t_i} v(t_1, t_2, \cdots, t_s) = h\left(\dfrac{\partial v}{\partial t_1}, \dfrac{\partial v}{\partial t_2}, \cdots, \dfrac{\partial v}{\partial t_s}\right), \\ v|_{t_i = t_{i0}} = v_{i0}, \ 1 \leq i \leq s. \end{cases} \tag{5.4}$$

Similarly, taking the second partial of $t_i$ on both sides of the equation (5.2) yields the initial value problem on the second partial differential equation corresponding to the change of element $v$, i.e.,

$$\begin{cases} \dfrac{\partial^2}{\partial t_i^2} v(t_1, t_2, \cdots, t_s) = g\left(\dfrac{\partial v}{\partial t_1}, \dfrac{\partial v}{\partial t_2}, \cdots, \dfrac{\partial v}{\partial t_s}, \dfrac{\partial^2 v}{\partial t_1^2}, \dfrac{\partial^2 v}{\partial t_2^2}, \cdots, \dfrac{\partial^2 v}{\partial t_s^2}\right) \\ v|_{t_i = t_{i0}} = v_{i0}, \ \left.\dfrac{\partial v}{\partial t_i}\right|_{t_i = t_{i0}} = v_{i1}, \ 1 \leq i \leq s. \end{cases} \tag{5.5}$$

For example, suppose that the element $v$ is a micro particle field. Its change $v(\mathbf{r}, t)$ follows a $\psi(\mathbf{r}, t) = \psi(\mathbf{r}, 0)e^{-i\omega t}$ rule, where $\mathbf{r}$ is the position vector of $v$ and $\omega$ is the frequency. Differentiating it on time $t$ we get

$$\frac{\partial v}{\partial t} = \frac{\partial \psi}{\partial t} = \psi(\mathbf{r}, 0)(-i\omega)e^{-i\omega t} = -i\omega\left(\psi(r, 0)e^{-i\omega t}\right) = -i\omega v,$$

i.e., the change behavior of element $v$ corresponds to an initial value problem on the first order partial differential equation

$$\begin{cases} \dfrac{\partial v}{\partial t} = -i\omega v \\ v(\mathbf{r}, t)|_{t=0} = \psi(\mathbf{r}, 0). \end{cases} \tag{5.6}$$

However, there is a relation $E = \hbar\omega$ for energy $E$ and frequency $\omega$ in a microscopic particle field, where $\hbar$ is the Planck's constant. We therefore have

$$Ev = -i\hbar(-i\omega\psi) = -i\hbar\frac{\partial\psi}{\partial t} = -i\hbar\frac{\partial v}{\partial t}. \tag{5.7}$$

Differentiating with respect to $t$ on both sides of (5.7) and applying the first equation in (5.6), we get that

$$i\hbar^2\frac{\partial^2 v}{\partial t^2} + E^2 v = 0.$$

In this way, the initial value problem (5.5) is equivalent to the initial value problem

$$\begin{cases} i\hbar^2\dfrac{\partial^2 v}{\partial t^2} + E^2 v = 0 \\ v(r,t)|_{t=0} = \psi(\mathbf{r},0), \ \dfrac{\partial v}{\partial t}\Big|_{t=0} = \psi'(\mathbf{r},0) \end{cases} \tag{5.8}$$

on a second order partial differential equation.

Dr.Ouyang tells Huizi the problem now is that we do not know the explicit function $v(t_1, t_2, \cdots, t_s)$, only by assuming that $v(t_1, t_2, \cdots, t_s)$ complies with the formal calculation and determining

$$h\left(\frac{\partial v}{\partial t_1}, \frac{\partial v}{\partial t_2}, \cdots, \frac{\partial v}{\partial t_s}\right), \quad g\left(\frac{\partial v}{\partial t_1}, \frac{\partial v}{\partial t_2}, \cdots, \frac{\partial v}{\partial t_s}, \frac{\partial^2 v}{\partial t_1^2}, \frac{\partial^2 v}{\partial t_2^2}, \cdots, \frac{\partial^2 v}{\partial t_s^2}\right),$$

in the (5.4) or (5.5) by observing data, namely the relationship between differentials

$$\frac{\partial v}{\partial t_1}, \frac{\partial v}{\partial t_2}, \cdots, \frac{\partial v}{\partial t_s} \quad \text{or} \quad \frac{\partial v}{\partial t_1}, \frac{\partial v}{\partial t_2}, \cdots, \frac{\partial v}{\partial t_s}, \frac{\partial^2 v}{\partial t_1^2}, \frac{\partial^2 v}{\partial t_2^2}, \cdots, \frac{\partial^2 v}{\partial t_s^2}$$

for establishing the equations (5.4) or (5.5) and the next is how to solve the equations (5.4) or (5.5) to get the explicit function $v(t_1, t_2, \cdots, t_s)$.

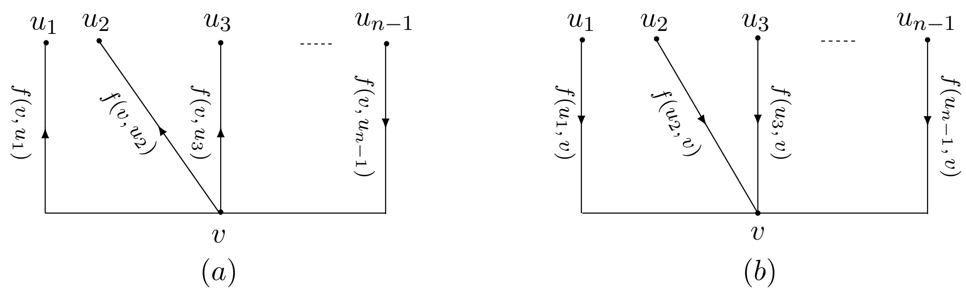



**Figure** 5.5. **Action graph of elements**

**1.3.System Structure.** In the human recognition of system $S$, an element is abstracted as a point in space. In turn, the system structure is a labeled graph $G^L[S]$ embedded in space, where, $G$ is the combinatorial structure between elements and $L$ is the labels mapping to the vertices and edges of $G$. Ignoring all labels on the labeled graph $G^L[S]$, we get an abstract graph $G$ which is the combinatorial structure inherited in elements of system $S$. For example, the complete graph $K_n$, complete bipartite graph $K(n,m)$ and the circle $C_n$, etc, where $n = |G|$.

So, *how do we determine the combinatorial structure of a system?* Dr.Ouyang tells Huizi that the combinational structure $G^L[S]$ inherited in a system $S$ is determined by the interaction of elements, i.e., the action function $f(v,u)$ of element $v$ on element $u$ for $v, u \in V(G[S])$. In fact, there are two ways for determining the combinatorial relationship between elements. One is the abstract graph $G$, namely there are edges $(v,u) \in E(G)$ or $(u,v) \in E(G)$ if there exists the action $f(v,u)$ of elements $v$ on $u$ or the action $f(u,v)$ of elements $u$ on $v$. Another is by using the action functions $f(v,u)$, $f(u,v)$ and the complete graph $K_n$, $n = |G|$, and views the complete labeled graph $K_n^L$ is the labeled graph $G^L$ if removes all $\mathbf{0}$ labels edge in $K^L$, as shown in Figure 5.5. Among them, $(a)$ is the effect of $v$ changes to other elements $u_1u_2, \cdots, u_{n-1}$, $(b)$ is the effect of other elements $u_1, u_2, \cdots, u_{n-1}$ change to the element $v$ and $f(v,u_i) \neq \mathbf{0}$ or $= \mathbf{0}$ for integers $1 \leq i \leq n-1$.

Notice that by using the second way it is no longer necessary to tangle for determining the specific structure $G$ of $S$. Whence, it can be uniformly assumed that the combinatorial structure inherited in a system is a complete $K_n$, where $n$ is the number of elements in the system.

A main work for determining the combination structure of system $S$ is to determine the interaction of one element $v$ action on other elements and then find the "*cause*" on an appearance "*effect*" $T$ of system $S$, namely the "*causality*", i.e., looks for the elements $v$ that causes the appearance $T$, establishes the equation for determining the causality $\{v\} \to T$. Notice that such elements $\{v\}$ may be the effect of one or multiple ones $\{v\}$ acting together. Then, *how do we view the action $f(v,u)$ of elements $v$ on $u$ in Figure* 5.5 *and then establish the equation?* Certainly, there is an intuitive way, i.e., view the action $f(v,u)$ of elements $v$ on $u$ as a flow on edge $(v,u)$ that transfers from $v$ to $u$, called an *action flow* from $v$ to $u$. For example, the flow of water in a large river, the current in a cable, etc., i.e, a flow quantity, not a stationary one. Generally, there are two main types of action flow, namely the *energy flow* and the *information flow* in the nature.

**Figure** 5.6. **River flows**

**(1)Energy flow.** As is known to all, things are all in moving and the energy measures the transformation of the moving of matter, is a measure of the ability of a physical system to do work. Thus, the energy can describe the process of change of an element $v$. Correspondingly, the energy of a system can be defined as the sum of the work converted from a zero-energy state to current state of the system. In this sense, the element $v$ applies work on the element $u$, transfers the energy $f(v, u)$ from elements $v$ to $u$ by moving matters because the mass of matter is equivalent to the energy. Certainly, the earliest energy experienced by humans is the "*fire*" or combustion energy, the physical and chemical changes that occur if materials burn or encounter fire. No matter in the Big Bang at the origin of the universe or in today's observations, most of the energies in the universe are the energy released by the nuclear fusion reaction, as shown in Figure 5.7. For example, the solar energy is the nuclear energy released by the fusion of hydrogen and helium between hydrogen atoms inside the sun. In fact, the vast majority of the energy needed by plants, animals and humans on the earth comes directly or indirectly from the solar energy.

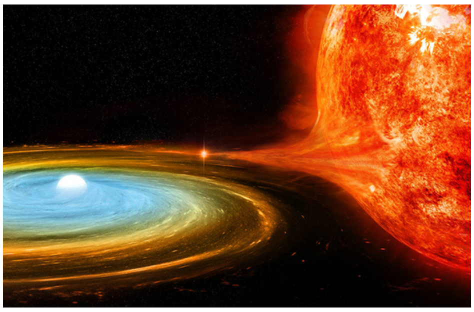


**Figure** 5.7. **Nuclear fusion**

Dr.Ouyang tells Huizi that the energy can shows in many forms, including the mechanical energy, heat energy, electric energy, chemical energy and the nuclear energy. Among them, the mechanical energy includes the kinetic energy and the potential energy. We have shown in Chapter 4 that an object of mass $m$ moving in speed $v$ has a kinetic energy $mv^2/2$ and the potential energy is the relative energy that an object dues to its position in space. If we choose the ground as the zero position of potential energy, then the gravitational potential energy of an object at height $h$ is $mgh$, where $g = 9.80665m/s^2$ is the acceleration of gravity $mg$. The heat energy is the kinetic energy of the thermal moving of molecules inside a substance. Similarly, the electricity is the energy corresponding to the moving of charged particles, the light energy is the energy corresponding to the moving of photons and the chemical energy is the energy released or absorbed in a chemical reaction, which is the energy absorbed or released by the change of electron binding energy caused by the change of electron in the outer layer of an atom. Finally, the nuclear energy is the binding energy $E$ of protons and neutrons in the nucleus, which can be released in the nuclear fission or fusion. In this regard, the Einstein's mass-energy equation $E = mc^2$ reveals the relationship of nuclear energy with that of the mass. Here, $c$ is the speed of light in vacuum. Generally, the moving properties of matters can be described by the physical or chemical quantities if they have the same form of moving.

For example, the mechanical motion of an object can be described in terms of speed, acceleration, momentum, etc., and the current can be described in terms of current intensity, voltage, power, etc. But if the form of moving of matter is not the same, the moving of matter can only be described by the change of energy. So, the energy is the common property of all moving matter and can be converted from one form to another, as shown in Figure 5.8. Certainly, the common words "*to drill a wood for fire*" of ancient Chinese is essentially the converting of energy forms, which refers to the use of friction between objects to generate combustion energy from kinetic energy. For a closed system $S$, the total energy remains unchanged in the process of energy conversion because the energy of the system is transferred among its elements, i.e., if a certain form of energy of an element decreases there must be other elements to obtain this part of energy, and the amount of decrease and increase must be equal, i.e., the total amount of energy does not change. This is the law of *conservation of energy*. The law of conservation of energy is also called the *law of continuous flow* in a closed system $S$. That is, the difference between the energy flowing in and the flowing out of an element $v$ is equal to the energy remained in the element $v$ for any element $v$ in the system, i.e., for any element $v$ of the system, there must be

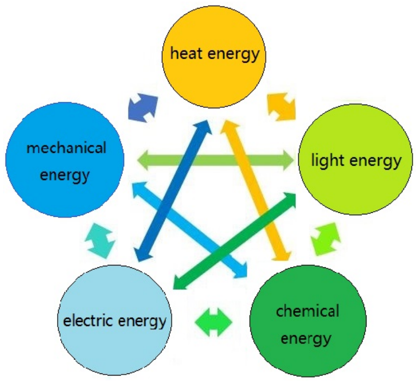



**Figure** 5.8. **Energy conversion**

$$\sum_{i=1}^{n-1} f(u_i, v) - \sum_{i=1}^{n-1} f(v, u_i) = v(t_1, t_2, \cdots, t_s), \tag{5.9}$$

where, $n$ is the number of elements in the system $S$ and $v(t_1, t_2, \cdots, t_s)$ is the energy function at the element $v \in S$.

**(2)Information flow.** Generally, the information refers to all the contents communicated in human society, including the audio, message, transmission and processing matters in a communication system. For example, when a human expresses his thoughts, opinions, feelings and facts through the language, expressions, letters and the correspondence, he is in the transmitting information to others, namely an information is the reflection of moving state and change of various things in the objective world by humans in order to adapt to the nature, also known as the *data*. Whence, the information represents the interrelation

**Figure** 5.9. **Data transmitting**

and interaction between things. That is, the essence of the state of moving and change of things. Notice that the information is not material. The characteristic of information is that it must rely on the information transmitting and can not exist independently. Certainly, the transmission of information requires a certain amount of energy but the information itself does not have any energy.

The unit of information is bit which measures an information as bits. For example, a packet of 100 bits contains 100 bits of information. The information transmission generally refers to the information transmitted from one base point $v$ to another base point $u$, as shown in Figure 5.10$(a)$, where $f(v,u)$ indicates the amount of information sent by the base point $v$. Even if one base point $v$ is transferred to multiple points $u_1, u_2, \cdots, u_k$, it also refers to the symbols carrying information sent by $v$, as shown in Figure 5.10$(b)$ and how much information these symbols can carry is related to the probability of occurrence of these symbols.

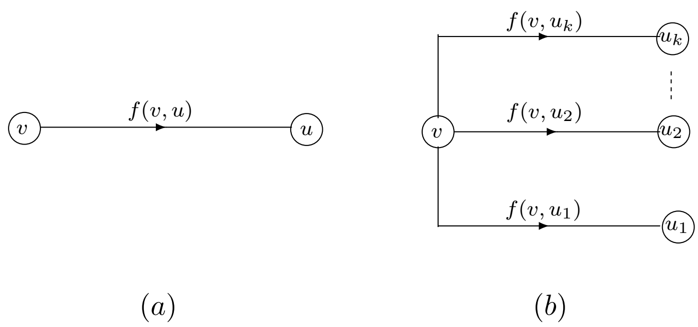



**Figure** 5.10. **Information transmitting**

For any receiver $u_i, 1 \leq i \leq k$, the probability of these symbols occurring is the same. That is, the information transmitted by the base point $v$ is satisfied with

$$f(v,u_1) = f(v,u_2) = \cdots = f(v,u_k) \tag{5.10}$$

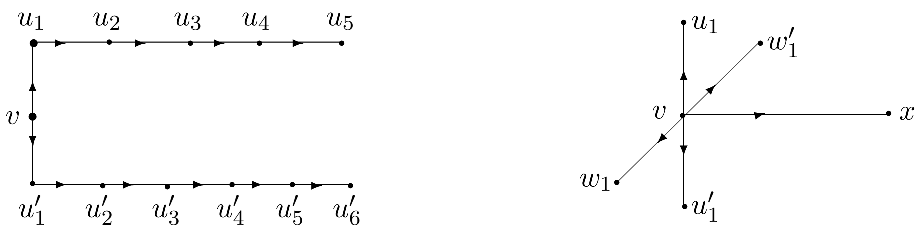
and the conservation law (5.9) is no longer holding on, i.e., the amount of information flowing into a basis point $v$ is not necessarily equal to the amount of information flowing out of $v$.

If the number $n$ of elements in the system is relatively small, there is little difference between the number of actions of element $v$ on others and the value $n-1$. However, if $n$ is very large, for instance the number of human cells in an adult body is $4 \times 10^{14} - 6 \times 10^{14}$ and if the cells are used as the elements, the number of elements in the human body is $4 \times 10^{14} - 6 \times 10^{14}$, which is not economical to calculate the number of primitive actions by $n-1$ because a cell $v$ only affects and transmits information to its neighbors. In addition, even if the number of element $n$ is not very large, for instance the birds' flock changes from scattered to gathered and then synchronously flies out a beautiful geometric pattern as shown in Figure 5.11($a$), or geese fly in V-shape as shown in Figure 5.11($b$), etc., which all involve the information transmission. Generally, a bird $v$ only communicates with its neighbors similar to the spatial structure shown in Figure 5.11($c$).


($a$) ($b$) ($c$)

**Figure** 5.11. **Neighbor information transmitting**

Usually, the head bird leads the flock and decides the flight direction, speed and landing site. Other birds keep only a certain distance from neighboring birds by discriminating the flight direction and speed of the neighboring birds, follow the synchronized flight of the head bird. If the head bird is not strong, it is replaced by other birds. The piloting function is the same as the head bird. That is, the head bird piloting determines the direction and speed of the flight, other birds observe the flight direction and speed of the top, bottom, left, right and the front birds, determine their own flight direction, speed, and the distance from the neighboring birds, as shown in Figure 5.12($a$) and ($b$), where ($a$) is the information transmission of the geese flying in the shape of V and ($b$) is the flight information transmission of bird $v$ in the flock of birds in Figure 5.11($c$).




($a$) ($b$)

**Figure** 5.12. **Flock's information transmitting**

Dr.Ouyang tells Huizi that a virus is a non-cellular living form that colonizes and replicates inside a living cell. Generally, a virus expelled from the source of an infected body also needs the transmitting medium such as the air, water, food or contact for it into a new susceptible cell but not needs the energy, i.e., the protein coat of virus adsorbs on the surface of living cell, injects the genetic material into the host cell and issues the instruction in the living cell for synthesizing a new generation of virus, which is similar to the information transmitting in Figure 5.12($b$).

## §2. Physical Dimensions

The dimension is a kind of natural property on the characteristics of things, which is a reflection of characteristics in linear combination of things. On this basis, the recognition of a thing is measuring its quantity of the characteristics by comparing it with the unit, namely the linear combination relationship and then, analyzes the causal relationship between the independent variable and the dependent variable. For example, the length, width, height, moving time, mass, speed, force, kinetic energy and the work on an object, etc. So, the physical dimension is the basis for human discussing the causality of things.

**Figure** 5.13. **Carrying wife**

Since dimensions reflect linear properties in measuring, only those features or things of the same dimension can be added or subtracted. Dr.Ouyang tells Huizi:"*You have known the arithmetic formula* $1 + 1 = 2$. *Using this formula, you can get* 1 *apple* + 1 *apple* = 2 *apples and* 1 *pear* + 1 *pear* = 2 *pears regardless of the size of apples or pears.*" Dr.Ouyang then asks his daughter:"*So,* 1 *apple* + 1 *pear* = 2 *what? Also, in Journey to the West, there is a scene that Zhu Ba Jie carried his wife, which is a pig and a woman. So* 1 *human* + 1 *pig* = 2 *what? Is it two humans or two pigs?*" Huizi asks Dr.Ouyang with wide-eyed doubts:"*Dad, they can't be added together, they are not the same kind of things!*" Dr.Ouyang confirms her answer:"*Right! The addition of one thing to another is generally the addition of the same things but it may also be the addition on a common feature of things. For example,* 1 *apple* + 1 *pear* = 2 *fruits in the previous question because both of the apples and pears are fruits. Similarly,* 1 *human* + 1 *pig* = 2 *animals because both of the humans and pigs are animals. Here, the apples, pears, fruits and the humans, pigs or other animals are all abstract features or dimensions of things. If we take one of the humans or pigs to be the dimension, then* 1 *human* + 1 *pig doesn't make any sense because they have different dimensions. But if you take animal as their dimension, you can add them together because the humans and pigs are both of animals despite their very different looks.*"

Dr.Ouyang tells Huizi that the way for measuring the characteristics of thing is to use the metrological tools and the quantities measured are called the physical quantities. Certainly, different physical values are measured if we use different standards but there is a close relationship between them, i.e., the fundamental or the derived quantities. Here, the *fundamental quantity* is such a quantity that has an independent physical quantity and the *derived quantity* is such a physical quantity that can be expressed as a combination of fundamental dimensions. For example, the length and weight were the fundamental quantities but the volume was a derived quantity in the ancient Chinese. In the *International System of Units* (SI), the system of quantities is composed of seven fundamental quantities, selected from which other physical quantities can be derived. The dimensions of these seven fundamental quantities are respectively the length L, mass M, time T, current intensity I, temperature Θ, substance amount $n$ and the light intensity J. Then, the dimension $\widehat{D}$ of any physical quantity $D$ can be expressed by the power of the fundamental dimensions as

$$\widehat{D} = L^{\alpha}T^{\beta}M^{\gamma}I^{\delta}\Theta^{\varepsilon}n^{\zeta}J^{\eta}, \tag{5.11}$$

i.e., a combination of the fundamental dimensions. Among these, $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ and $\eta$ are called the *dimensional indices*. For example, $\alpha = 1, \beta = -1, \gamma = \delta = \varepsilon = \zeta = \eta = 0$ in the dimension $LT^{-1}$ of the velocity $v = ds/dt$ of object.

**2.1.Fundamental Dimensions.** The fundamental dimensions of SI system includes seven dimensions, i.e., the length, mass, time, current intensity, temperature, substance amount and the light intensity, respectively.

**(1)Length.** The *length* is a one-dimensional metric on the size of a geometric body. That is, the straight-line distance from one point to another. In the agricultural society, measuring the area of a cultivated land was one of the most important tasks in the land trading, sowing and the estimating yields. For this purpose, each country issued its own unit of length measurement in that period. For example, the American countries generally used the length unit of foot but the ancient China had the foot, Zhang(3.3333 metres), Li (500 metres) and other units as the length unit.

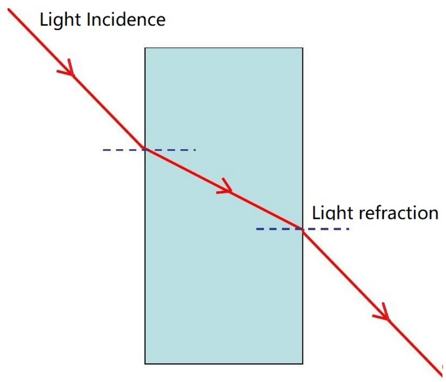



**Figure** 5.14. **Light travelling**

First of all, *are there standards on the measuring conditions including the temperature, humidity and other requirements?* Dr.Ouyang tells Huizi that the answer is Yes! The length of an object is measured under a standard atmospheric pressure and the ambient temperature is 20$^o$C, etc. Notice that the light propagates being a straight line in the same medium but refracts in different media, as shown in Figure 5.14. Consequently, the light is used to define the reference length. In the SI system, the benchmark for the prescribed length "*1 meter (m)*" is defined by "*the distance of light travelling in a time interval of* 1/299792458 *second in vacuum*". Secondly, *how do we measure the length of an object or the distance between* *two objects?* Dr.Ouyang tells Huizi that one usually measures or calculates the distance between two points by measuring ruler, measuring instrument with visual distance device, photoelectric rangefinder, etc., so as to meet one's daily production and living practice on the earth, see Figure 5.15($a$).

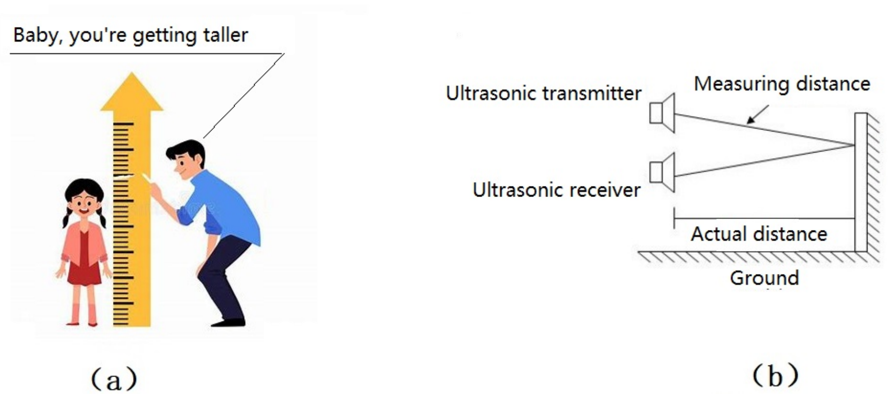



**Figure** 5.15. **Length measurement**

Among these, the photoelectric rangefinder uses the electromagnetic wave or the light wave as the medium to measure the distance between two points by getting the round-trip propagation time of light wave between the two ends of the measuring line and the speed of light wave in the atmosphere. Its principle is shown in Figure 5.15($b$). However, all of these measuring ways can not solve the micro measurement problem on the electron, atomic and the molecular spacing or trajectory, behavior capture and can not solve the super-distance measurement such as the space measurement, planetary spacing measurement and other problems also, which needs to further develop the new tools and instruments with innovative ways.

**Particle measurement.** The capture of particle behavior is known as particle track measurement. Dr.Ouyang explains to his daughter that a general way for an electric particle is measuring the particles track in the detector when it passed through and action with the specific substances in the detector such as the ionization, excitation and other processes. For example, the particle tracks can be scanned, measured and read by the eye through a microscope in the early use of nuclear latex and solid track detectors. After then, the cloud room, bubble room, spark room, flow room (tube) and other track room are developed for research, which can carry out the three-dimensional photos, film scanning and the automatic processing, computer data reading. The principle of this method applied to particle track observation is that the particles can make a track formed by multiple particles appear in the latex after imaging, and the lower energy of the particles, the more particles in the track. In this way, the particle track can be divided into a number of small segments, only a few microns in length and the number of particles on the track can be counted by a small segment as a particle.

**Astrometry.** The distances between the planets in the universe are all very distant. Dr,Ouyang tells Huizi that the astrometry generally identifies the distances between celestial bodies by different measuring ways based on their distance from the earth: ①Distance between the earth and the moon. In theory, the speed of radar signals is equal to the speed of light. If the radar is aimed at the moon to send out a short pulse signal and the round-trip time of the signal is measured to be 2.56 seconds. So, the distance between the earth and the moon can be found to be about $1.28 \times 3 \times 10^5 = 3.84 \times 10^5 km$ by $s = vt/2$; ②Distance between the earth and the sun. Using the transit of Venus, i.e., the sun, Venus and the earth are moving in a straight line, the distance between the earth and the sun is estimated to be about $1.5 \times 10^8 km$ by the triangle parallax principle; ③Distance between the sun and other planets can be inferred by the vision difference or by Kepler's law $T^2/d^3$ =constant. Here, $T$ is the planet's orbital period and $d$ is the planet's average distance from the sun. Assuming that the distance between the earth and the sun is 1AU (1 astronomical unit, about $1.496 \times 10^8 km$) and the earth's revolution period is 1 sideral year, the Kepler's law is simplified to be $T^2 = d^3$. For example, if the Mercury has an orbital period of 0.2411 years, the distance from Mercury to the sun is 0.3874AU or about $5.79 \times 10^7 km$; ④The size of the milky way galaxy has been determined by stellar spectroscopy which is about 105 light years in diameter and 104 light years in thickness; ⑤The distances between the milky way and neighbouring galaxies are inferred by the variable star method. Here, the variable stars refer to stars whose apparent brightness varies over time and are classified into the regular or the irregular. One of the regular variable stars is a pulsating variable whose luminosity changes in cycles. From these periodic luminosity curves and spectra, the distance between the milky way and its neighbors can be deduced. For example, the milky way is about 2 million light-years away from the Andromeda nebula. For the galaxies farther away from the milky way, the Doppler effect or redshift is used to measure how fast the galaxy is moving away from the earth. For example, the departure velocity of Virgo is about $1210km/s$, etc.

Thirdly, everything is in continuously moving. So, *is it possible for human to accurately measure the length of an object?* Dr.Ouyang tells Huizi, if assuming a coordinate system $K' : \{O'; \mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$ moves with a uniform speed $v$ along the $\mathbf{e}_1$ direction in the coordinate system $K : \{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$, the starting and the end coordinates of an object $P$ along $\mathbf{e}_1$ direction are respectively $x'_1, x'_2$, then the static length of $P$ is $L_0 = x'_2 - x'_1$ in $K'$. At the same time, assuming the starting and the end coordinates of the object $P$

along the direction of the $\mathbf{e}_1$ in $K$ are $x_1$ and $x_2$, respectively, then the length $L = x_2 - x_1$ is a moving length of $P$. By the Lorentz transformation (4.72), if $t_1 = t_2$ we get that

$$L_0 = \frac{x_2 - vt_2}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{x_1 - vt_1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{x_2 - x_1}{\sqrt{1 - \frac{v^2}{c^2}}} = \frac{L}{\sqrt{1 - \frac{v^2}{c^2}}}, \tag{5.12}$$

i.e., the moving length $L$ of the object $P$ along the direction of $\mathbf{e}_1$ is

$$L \;=\; L_0 \cdot \sqrt{1 - \frac{v^2}{c^2}}, \tag{5.13}$$

where $c$ is the speed of light in vacuum. So, $L \leq L_0$. Now that all things are in moving, *is the length L of an object measured by human in the practice the static length of object?* Dr.Ouyang tells Huizi that the answer is No because the earth rotates at about $29.8km/s$. Thus, according to (5.13), the measured length $L$ of an object should be

$$L = L_0\sqrt{1 - \frac{v^2}{c^2}} = L_0\sqrt{1 - \frac{29.8^2}{300000^2}} = L_0 \times 0.99994\cdots \approx L_0,$$

they are almost identical. Similarly, the microscopic particles and the planets are in continuously moving and the measurement results of the distance between the microscopic particles and the celestial bodies are relative or approximate, rather than their absolute distance.

**(2)Time.** Clearly, the *time* is an objective existence which is a parameter set by human on the passage of time to describe the process of material moving or the occurrence of events. Dr.Ouyang tells Huizi that the ancient Chinese divided the rotation of the earth into twelve equal parts and each part was further divided into two equal subparts, i.e., the *beginning* and the *full*. That is to say, the length of one day is equally divided into twenty-four parts which is the "*hour*" now, and the length of one hour is further divided into four quarters by ancient Chinese. For example, we usually read the words "beheaded at Wushi three quarters" in literatures, where the "*Wushi three quarters*" refers to the time "*11:45 pm*" because the Wushi refers to the times from 11:00 to 13:00 and one quarter is equal to 15 minutes in ancient China. Whence, the Wushi three quarters is the time 11:00pm past $3 \times 15$ minutes $= 45$ minutes.

Usually, the humans in different regions of the earth may see the sun rise and set at different times. The international standard of Greenwich time, also known as the universal time is based on the time in Greenwich, which divides the earth's surface into 24 time zones according to the longitude such that the time difference between the adjacent areas is 1 hour, i.e., the time difference seeing the sun rise in the east and the west of the same area is about an hour apart. In addition, each country or region has its own set time standards. For example, the Chinese time standard "*Beijing time*" refers to the meridian of $120^o$ east longitude, the standard time in the zone where Beijing is located.

So, *how do we measure time in a certain region or area?* Dr.Ouyang tells Huizi that generally, we use a timer such as a mechanical alarm clock, stopwatch, hourglass, pocket watch, pendulum clock, quartz clock, etc., to measure the time or moment, to meet the needs of human's production and life. In the SI system, the basic unit of time is second (s). In this way, the time can be derived by one minute, one hour and one day. So, "*How long is a second?*" Dr.Ouyang says that the international cesium atomic clock is adopted as the time measuring standard, which stipulates 1 second (s) to be "*the duration of 9192631770 cycles of radiation corresponding to the two superfine energy levels of the atomic ground state of undisturbed cesium Cs-133*" under the measuring condition that "*the cesium atom is stationary at the absolute zero and the environment on the ground is zero magnetic field*", namely the determination of the 9192631770 cycles of cesium-133 oscillating in the atomic ground state is the standard time of 1 second to meet the requirement of a few high-precision fields such as the space exploration and the national defense construction.

Notice that the standard time of 1 second is measured by the cesium atomic clock at the static. So, *if the cesium atomic clock is moving at a speed of v, does it still measures the same time as it is at the static?* Dr.Ouyang tells Huizi that the answer is No because time has an inflationary effect. Assuming the coordinates $K'$ is moving in a uniform speed $v$ along the $\mathbf{e}_1$ direction in the coordinate system $K$ with $K' : \{O'; \mathbf{e}'_1, \mathbf{e}'_2, \mathbf{e}'_3\}$, $K : \{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. We put a cesium atomic clock at each of the origins $O$ and $O'$. Without loss of generality, let $t_1, t_2$ be the two times shown on the atomic clock at $O$ with corresponding times $t'_1, t'_2$ on the cesium atomic clock at $O'$, respectively. Then, the static time length $t_0 = t'_2 - t'_1$ and the moving time length $t = t_2 - t_1$. Notice that $x' = 0$, the moving distances of cesium atomic clock at $O'$ in times $t_1, t_2$ along $\mathbf{e}_1$ direction are respectively $x_1 = vt_1$ and $x_2 = vt_2$. By the Lorentz transformation (4.72), we know that

$$t_0 = \frac{t_2 - \frac{v}{c^2}x_2}{\sqrt{1 - \frac{v^2}{c^2}}} - \frac{t_1 - \frac{v}{c^2}x_1}{\sqrt{1 - \frac{v^2}{c^2}}} = (t_2 - t_1)\frac{1 - \frac{v^2}{c^2}}{\sqrt{1 - \frac{v^2}{c^2}}} = t\sqrt{1 - \frac{v^2}{c^2}}, \tag{5.14}$$

i.e.,

$$t = \frac{t_0}{\sqrt{1 - \frac{v^2}{c^2}}}. \tag{5.15}$$

So, $t \geq t_0$ in theory. Now that the earth's rotation speed is about $29.8km/s$. By the formula (5.16) the measured time on the earth is

$$t = \frac{t_0}{\sqrt{1 - \dfrac{v^2}{c^2}}} = \frac{t_0}{\sqrt{1 - \dfrac{29.8^2}{300000^2}}} = \frac{t_0}{0.99994\cdots} \approx t_0,$$

i.e., the measured time is almost identical to the static time.

**(3)Mass.** The *mass* $m_0$ is a natural property of an object which is a measurement of its static mass and a scalar quantity equal to the product of its density and volume. Notice that the mass and weight are two different conceptions, i.e., the weight is the gravitation of an object with mass from the earth, which is a vector but the mass is a scalar quantity.

Certainly, the weight and mass are closely related, i.e., the weight of an object on the earth with a mass of $m$ is $mg$, where $g$ is the acceleration of gravity in vacuum. In this way, the object can be weighed for determining its weight $W$ and then, its mass is $W/g$ in fact. This is the general way for knowing the mass of an object on the earth. Usually, the mass is measured in kilograms ($kg$), i.e. the same as the unit of weight defined by that "*1 kg (kg)*" is the mass of an object derived from the Planck's constant at a length of $1m$ in the time of $1s$, almost the same as the weight of one litre water. To this end, humans have made various "*scales*" using the lever principle to weigh objects. For example, a balance commonly used in laboratories is shown in Figure 5.16. Its basic principle is to place standard weights in one plate and object in another plate. if the upper bar is horizontal or the vertical pointer is vertical, the sum of standard weights is the weight of the object.

**Figure** 5.16. **Balance**

So, *how do we determine the masses of the microscopic particles and the celestial bodies?* Dr.Ouyang tells Huizi that a few special methods are needed for determining the masses of the microscopic particles and the celestial bodies.

**Micro particle mass.** The mass of a micro particle is generally measured by the moving law of the micro charged particle in the electromagnetic field for getting the mass of the micro particle. Notice that charged particles are respectively subject to the Coulomb force in an electric field and the Lorentz force in the magnetic field, which enable the microscopic particles have a certain acceleration. Depending on the different masses of particles, the acceleration and trajectory of the particles projected onto the film will be different. Thus, the masses of some microscopic particles can be deduced by studying their trajectories. For example, according to the definition of atomic mass unit, the humans choose the mass spectrum line of ${}^{12}_{6}C$ as the standard reference for comparison, and determine the mass $0.0005485799111(12)amu$ of an electron by the ratio of the cyclotron rate of electron $e$ and ion ${}^{12}_{6}C^{+}$ in the ion trap, multiplying by the conversion factor 931.49432 to get the electron mass $m_e = (0.51099907 \pm 0.00000015)MeV/c^2$. In addition, the particle mass can be also determined according to the particle generation threshold and the excitation curve of the particle resonance state, etc.

**Celestial mass.** The celestial bodies are so large that it is impossible to weight their mass because there are no such a scale that big enough, can be only obtained by theoretical derivation. For example, the mass of the earth can be deduced from Newton's law of gravitation. Let the object $P$ be placed on the surface of the earth with a mass of $m$. According to the universal law of gravitation, there is

$$F = G \cdot \frac{mm_{earth}}{R^2} = mg \quad \Rightarrow \quad m_{earth} = \frac{gR^2}{G}. \tag{5.16}$$

The gravitational constant $G$, gravitational acceleration $g$ and the earth radius $R$ are all known numbers. Thus, the mass of the earth can be calculated to be about $5.965 \times 10^{24} kg$. In fact, if the radius of the celestial body $R$ and the gravitational acceleration on the celestial surface $g$ are known, the mass of the celestial body can be directly calculated from the formula (5.16). In addition, the more massive the star, the greater the luminosity and the luminosity or gravitational redshift can be used to calculate the mass of an object. For example, there is a gravitational redshift

$$z = \frac{Gm}{c^2 R} \quad \Rightarrow \quad m = \frac{c^2 Rz}{G}, \tag{5.17}$$

where $G$ is the gravitational constant, $c$ is the speed of light in vacuum, $R$ is the radius of the celestial body and $z$ is the spectral line offset which can be measured. In this way, the mass of the celestial body $m$ can be obtained by the formula (5.17).

Dr.Ouyang tells Huizi that the size of the mass should not vary with the speed of the object. However, the moving particles have the effect of time dilation. The time to observe a moving particle is $t$ in the coordinate system but the time is $\tau$ on the coordinate system established on the observed moving particle. From (5.15) we have $dt = \sqrt{1 - v^2/c^2} d\tau$. So, for a particle with static mass $m_0$, let the speed of the moving particle be $\mathbf{v}$, $|\mathbf{v}| = v$. Then, by the Newton's second law we know that

$$\mathbf{F} = m_0 \mathbf{a} = m_0 \cdot \frac{d\mathbf{v}}{dt} = \frac{m_0}{\sqrt{1 - \frac{v^2}{c^2}}} \cdot \frac{d\mathbf{v}}{d\tau}, \tag{5.18}$$

in which there is a factor $1/\sqrt{1 - \frac{v^2}{c^2}}$. If we introduce the moving particle mass $m = m_0/\sqrt{1 - v^2/c^2}$, then the equality (5.18) shows that a moving particle still obeys the

Newton's second law $\mathbf{F} = m\mathbf{a} = m \cdot d\mathbf{v}/d\tau$ of motion. Notice that the particle moving mass $m$ is similar to the moving length and time of a particle and the moving mass of a particle on the earth is

$$m = \frac{m_0}{\sqrt{1 - \dfrac{v^2}{c^2}}} = \frac{m_0}{\sqrt{1 - \dfrac{29.8^2}{300000^2}}} = \frac{m_0}{0.99994\cdots} \approx m_0,$$

i.e., the moving mass of a particle is almost identical to its static mass. Thus, $m_0$ can be replaced by $m$ for a particle moving at a low speed, i.e. $v \ll c$.

**(4)Other dimensions.** The other four fundamental dimensions in the SI system are the current intensity I, temperature $\Theta$, substance amount $n$ and the light intensity J. A brief introduction on the four dimensions is as follows:

**Current intensity.** The current is the flow manifestation of electric charges. So, the *current intensity* I is defined to be the amount of charges passing through a section of the wire per unit time. In the SI system of units, it is accepted that two infinite parallel straight wires apart in 1 meter in vacuum are provided with equal constant current. If the action force on the wire is $2 \times 10^{-7}N$ per meter, the current on each wire is defined to be 1 ampere (A). There is a famous relationship, i.e., the *Ohm's law* $I = V/R$ between the current intensity I, voltage V and the resistance R, where, the unit voltage of V is volts (V) which is the energy difference of unit charge in electrostatic field due to the difference of electric potential and the unit resistance of R is ohms ( $\Omega$ ), i.e., the resistance of the wire to the flow of charges.

Generally, the current intensity I in a wire is measured by an ammeter. Certainly, there is a magnet in the ammeter for generating a magnetic field which provides with a coil and a spring at each end of the coil connected with the terminal post of the ammeter. A rotating shaft is arranged between the spring and the coil, also provided with a pointer at the front end of the rotating shaft. In this way, the current passes through the wire along the spring and the rotating shaft through the magnetic field and cuts the magnetic sensing line so that the coil deflects and drives the rotating shaft and the pointer on the ammeter to deflect. Usually, the deflection degree of the pointer is determined by the size of the current and then, the current can be measured by the scales on the current dial corresponding to the pointer, as shown in Figure 5.17.

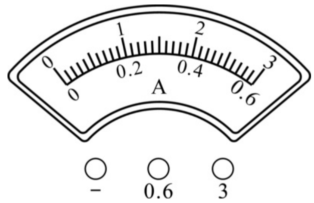



**Figure** 5.17. **Ammeter dial**

**Temperature.** The *temperature* $\Theta$ is a physical quantity that reflects the degree of heat and cold of an object, which is a manifestation of the translational kinetic energy between molecules within the object. For example, the faster molecules move the hotter they are, feel more hotter and the slower the molecules move the colder they are, fell more colder by hands. Generally, the temperature is divided into the Celsius temperature, the Kelvin temperature and the Fahrenheit temperature, measured by thermometer. Notice that the scale units of both the Celsius $^oC$ and the Kelvin K are completely consistent. That is, 1 degree on the Kelvin thermometer is equal to 1 degree on the Celsius thermometer with the only difference in the location of $0^oC$. Generally, the Celsius thermometer takes the freezing point of water to be $0^oC$ but the corresponding Kelvin temperature is 273.15K, i.e., the Kelvin temperature $K =$ the Celsius temperature+273.15 in the value of temperature, as shown in Figure 5.18 in which the column $^oC$ denotes the Centigrade degree and $^oF$ denotes the Fahrenheit degree with $^oF = (9/5)\ ^oC + 32$. However, the temperature is Kelvin in the SI system.

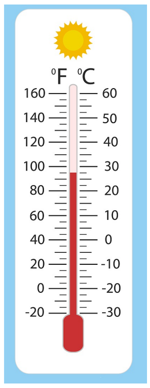



**Figure** 5.18. **Thermometer**

**Substance amount.** The *substance amount* $n$ is the ratio of the number of particles in a substance ($N$) to the Avogadro's constant ($N_A$), i.e., $n = N/N_A$ with a unit mole (mol). Here, the number of Avogadro's constant is the carbon atoms in $^{12}C$ of mass $0.012kg$, i.e., $N_A \approx 6.02 \times 10^{23}$. The mass of one unit substance amount is called the molar mass, namely the relative atomic mass of substance is numerically equal to the molar mass of one mole. Whence, the relation among the substance amount (n), mass (m) and the molar mass (M) is $n = m/M$. For example, there are $3 \times 6.02 \times 10^{23}$ $Na^+$ ions, $1.5 \times 6.02 \times 10^{23}$ $CO_2^{2-}$ ions and $15 \times 6.02 \times 10^{23}$ $H_2O$ molecules in $1.5mol$ of $Na_2CO_3{\cdot}10H_2O$ and there are $N = 2 \times 6.02 \times 10^{23} = 1.204 \times 10^{24}$ $H$ atoms in $2mol$ of $H_2$, etc.

**Light intensity.** The *light intensity* is defined as the luminous flux in a unit solid angle, namely the amount of light emitted by the light source in one unit time. The light intensity of a specific light source is measured by a special light intensity instrument.

Generally, assume that the luminous flux transmitted by the light source within a solid angle $d\Omega$ in direction is $d\Phi$. Then, the light intensity is $d\Omega/d\Phi$, where the luminous flux $d\Phi$ and the solid angle $d\Omega$ are shown in Figure 5.19 which can also be understood as the differentiation of the solid angle and the luminous flux. In the SI system, the unit of light intensity is candela (cd) and 1 candela is

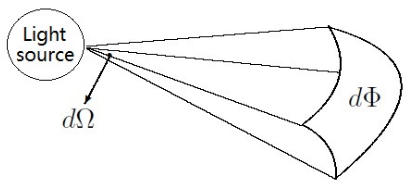



**Figure** 5.19. **Luminous flux**

defined as the light intensity of a monochromatic light with a frequency of $540 \times 10^{12} HZ$ and a radiation intensity of $1/683$ watt in direction. In the SI system, the luminous flux is measured in lumens (lm) and the solid angle is measured in sr, i.e., $1cd$ is equal to $1lm/sr$.

**2.2.Dimensional Combination.** A physical quantity $D$ in type (5.11) shows the dimension of $\widehat{D}$. If there are two of the indices $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta$ not 0 at least, then the dimension $\widehat{D}$ is called a *combinatorial dimension* or *derived dimension*. Notice that the combinatorial dimensions are not arbitrary combinations of the fundamental dimensions but are determined by the laws of physics. For example, we know the acceleration $\mathbf{a} = d\mathbf{v}/dt$ of particle and the dimension of speed $\mathbf{v}$ is $LT^{-1}$. So, the dimension of acceleration is $\mathbf{a}$ $LT^{-2}$. Similarly, by the Newton's second law $\mathbf{F} = m\mathbf{a}$ and the dimension of mass is $M$, the dimension of acceleration is $LT^{-2}$ we then know the dimension of force is $MLT^{-2}$. Corresponding, the dimension of work $W = \mathbf{F} \cdot \Delta\mathbf{r}$ is $ML^2T^{-2}$, etc. On the other hand, the dimension is a formal verifying for determining whether the formula of a physical law is correct. That is, a correct formula of a physical law follows the rules that the dimension on both sides of the formula must be the same. For example, the dimensions of the kinetic energy and the potential energy both are $ML^2T^{-2}$. Let $P$ be a particle of mass $m$ throwing with a velocity $v$ along the horizontal direction $\mathbf{e}_1$-axis at the origin $O$ in a normal coordinate system $\{O; \mathbf{e}_1, \mathbf{e}_2\}$, where $\mathbf{e}_2$ is the vertical downward direction. Then, a formula

$$\frac{1}{2}mv^2 + mgh = m\frac{d}{dt}\sqrt{x^2 + y^2}$$

on particle $P$ at the point $(x, y)$ can not be correct because the dimension of left-hand side of the equation is $ML^2T^{-2}$ but the right-hand side is $MLT^{-1}$.

## §3. System State

The system state is the collection of state performance or moving information of the system on time $t$. Dr.Ouyang tells Huizi that her class is like a system and the state of the students in class such as the physical fitness, academic performance, hobbies and moral character are the system state of her class, which changes from time to time. So, *how do we describe the system state of $S$?* Generally, there are two ways for human systematically perceiving things in the universe. One is the elements in $S$ that is completely unknown to humans, namely the *black box* system. In this case, $S$ can be simply regarded as a particle and then, establish a phenomenological theory by the particle equation characterizing the behavior of system $S$. The advantage of this way is that the system state obtained is consistent with the observed results but the disadvantage is that the cause of inducing the system state is not clear. Another is that all elements of $S$ are known, at least at the level of the elementary particles, cells and genes. At this time, the system state of $S$ is determined by the influences or actions $a(v, u)$ in elements and the system state of $S$ can be described by (5.1), i.e., the labeled graph $G^L[S]$ where $v, u \in S$ are the elements of system $S$. The advantage of this way is that it can not only obtain the system state but also get the causes of inducing the state. The disadvantage is that if the number of system elements is extremely large, the human's computing ability may not be able to know or determine the system state. However, for the number of elements that humans can calculate such as the learning state of the students in Huizi's class, it is still an effective way to systematically recognize things.

Among these, the function $a(v, u)$ is completely known as the *white system* while the function $a(v, u)$ is not completely known as the *grey system*. However, even for the white system, the labeled graph $G^L[S]$ only shows the relationship between the elements but not reflects the strength or quantity of the interaction between elements, and can not reflect the development of the system state. Then, *how do we represent the role of elements in the labeled graph $G^L[S]$ and describe the dynamic behavior of system $S$?* Dr.Ouyang explains, we can label the vertices and edges in $G^L[S]$ by variables or vectors at this time. That is, $L : v \in V\left(G^L[S]\right) \to \mathbf{x}_v$ to describe the change of element $v$, $L : (v, u) \in E\left(G^L[S]\right) \to a(v, u)$ and $a(u, v)$, denotes the element actions of $v$ on $u$ and $u$ on $v$, respectively, as shown in Figure 5.20. Such a labeled graph depending on the time $t$ is denoted by $G^L[S](t)$.

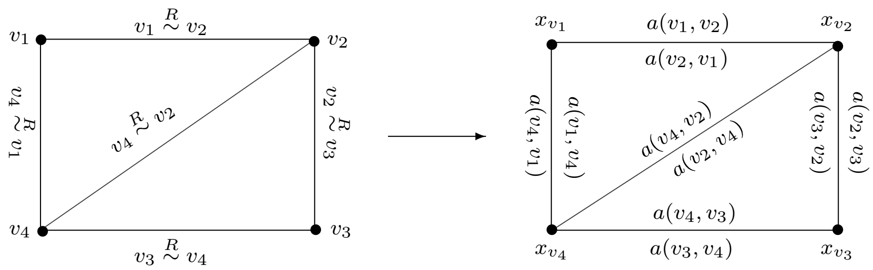



**Figure** 5.20. **System state**

**3.1.State Space.** The variable vector $\mathbf{x}_v$ depends on time $t$ and can describe the change trajectory of element $v$, which is called the state variable of element $v$. In this way, the state collection of elements in system $S$ at time $t$ is known as the *system state* $S(t)$, i.e., $S(t) = \{\mathbf{x}_v(t)|v \in S\}$. Now, let $S = \{v_1, v_2, \cdots, v_n\}$, where $n \geq 1$. So, *how do we characterize the status of variable* $\mathbf{x}_{v_1}, \mathbf{x}_{v_2}, \cdots, \mathbf{x}_{v_n}$*?* Dr.Ouyang tells Huizi that a natural way is establishing a coordinate system $\{O; \mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ with $\mathbf{x}_{v_i}, 1 \leq i \leq n$ as the axis directions $\mathbf{e}_i, 1 \leq i \leq n$, i.e., an $n$- dimensional Euclidean space $\mathbb{R}^n$ and applies the point coordinates in the $n$-dimensional Euclidean space to characterize the system state $S(t)$ of $S$ in time $t$, i.e.,

$$\textit{State } S(t) \quad \longrightarrow \quad \textit{Coordinates } (\mathbf{x}_{v_1}, \mathbf{x}_{v_2}, \cdots, \mathbf{x}_{v_n}) \textit{ of points in } \mathbb{R}^n, \qquad (5.19)$$

called the system's *state variable*. In this way, the system state $S(t)$ corresponds to a point subset $\mathcal{S}_P$ of the $n$-dimensional Euclidian space $\mathbb{R}^n$ and the system state of $S$ can be characterized by properties of the point set $\mathcal{S}_P$. For example, a system $S$ is defined to be bounded or unbounded dependent on that of the point set $\mathcal{S}_P$ to be bounded or unbounded.

Then, *can we directly obtain the system state $S(t)$ by observing the elements in system $S$?* Dr.Ouyang tells Huizi that the observation of an element $v$ in system $S$ could not directly obtain the trajectory of $v$ because we can only observe the state of element $v$ in a certain period of time but not the whole information of its trajectory. So, the change rate of element $v$ over time $t$ can only be calculated and predicted based on the observed information, namely differential $\dot{\mathbf{x}}_v(t)$ or furthermore, 2-order differential $\ddot{\mathbf{x}}_v(t)$, etc. At this time, the label of vertex $v$ in the labeled graph $G^L[S](t)$ is not $x_v$ but instead of differential $\dot{\mathbf{x}}_v$ or $\ddot{\mathbf{x}}_v$, as shown in Figure 5.21$(a)$ and $(b)$. Certainly, any observation on an element $v$ must have a beginning time, denoted by $t_0$ and the beginning state $v$ at time $t_0$ is called the *initial state* of $v$, denoted by $\mathbf{x}_v(t_0)$. Correspondingly, all initial states of elements constitute a collection of $\{\mathbf{x}_v(t_0); v \in S\}$, called the *initial state* of system $S$, denoted by $S(t_0)$.

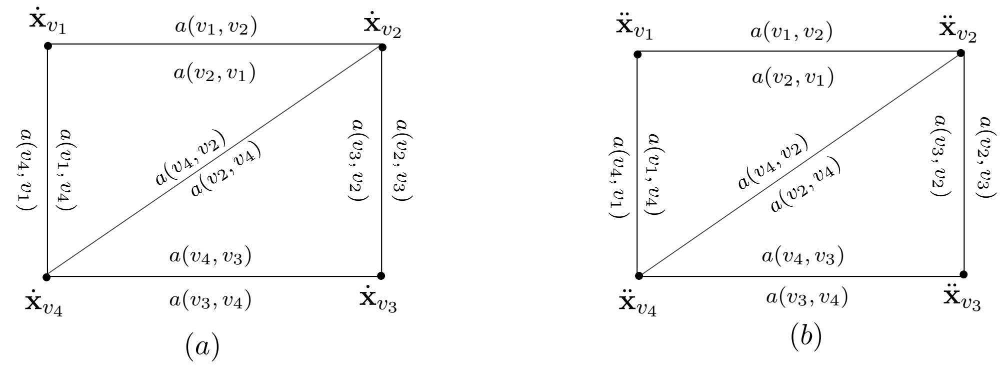



**Figure** 5.21. **Differential on elements**

Thus, the essence of system state of $S$ can be determined by observation on a premise that the initial condition $S(t_0)$ and the interactions $a(v, u)$ between elements are known for any $v, u \in S$ and then by the 1-order differentials $\dot{\mathbf{x}}_v$ to determine $\mathbf{x}_v, v \in S$, i.e., the system state $S(t)$, or by the 2-order differentials $\ddot{\mathbf{x}}_v$ to determine the state system $S(t)$ of $S$ under the initial conditions $S(t_0)$, $\dot{\mathbf{x}}_v(t_0), v \in S$ and $a(v, u)$, $v, u \in S$ known. Among these, the existence of the system state $S(t)$ is self-evident by definition, which is a simple relation that of "*the cause leading to the effect*". However, the purpose of human recognition of things lies in "*tracing the cause on the effect*", which essentially lies in answering the question that *whether one's observation on the changes* $\mathbf{x}_v$ *of element* $v \in S$ *with its interaction* $a(v, u)$ *is complete and whether the system state* $S(t)$ *can be uniquely determined in the recognition?* Dr.Ouyang explains it to Huizi that except for special cases such as the linear systems, the answer to this question is uncertain in general and the element state $v(t), v \in S$ that give rise to the system state $S(t)$ can not be determined or can not be uniquely determined by humans in most cases, which is also the cause of complex systems, namely the complexity of interactions or information of a system. For example, the human body is a system consisting of cells and the cancer is one of the top three diseases threatening human health in the 21st century. In theory, the cancer is such a disease that the normal cells of human body are transformed into cancer cells under the action of carcinogenic factors. As long as the targeted treatment is carried out on the cancer cells to causes them to death, a cancer patient will certainly recover from the disease such as those shown in Figure 5.22.

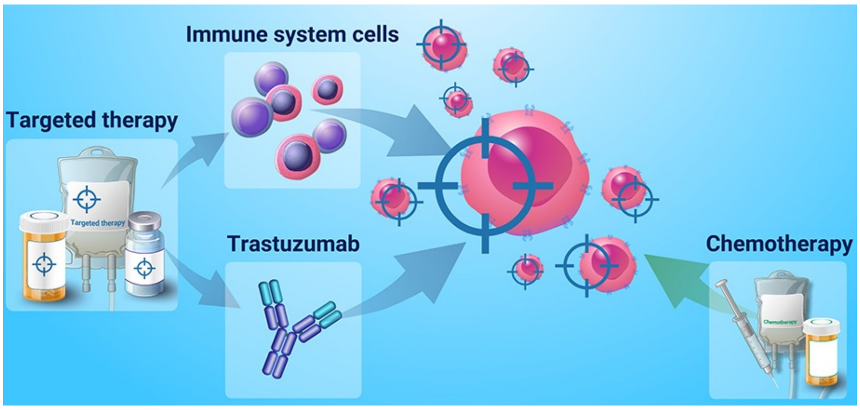



**Figure** 5.22. **Targeted cancer therapy**

However, the number of cells in a human body is $4 \times 10^{14} - 6 \times 10^{14}$ which is quite large. Thus, whether we can find those carcinogenic cells and then how to achieve the targeted therapy is a worldwide problem in technology. Clearly, this problem is from the cancer "*effect*" to trace the genes "*cause*", i.e., the genes that caused the cancer cells. However, the genes that caused the cancer can not be found so that the rehabilitation of cancer patients can not be carried out in most cases.

**3.2.State Equation.** The *state equation* of a system $S$ is a system of first-order differential equations deduced from the equal relations in system with the input variables. Dr.Ouyang tells Huizi, assume that the elements of $S$ are respectively $v_1, v_2, \cdots, v_n$ and only elements $v_{i_1}, v_{i_2}, \cdots, v_{i_l}$ are affected by the input variables $\mathbf{u}_1(t), \mathbf{u}_2(t), \cdots, \mathbf{u}_s(t)$, as shown in Figure

5.23. Thus, the first-order differential equations describing the state of the system are

$$\begin{cases} \dot{\mathbf{x}}_{v_{i_j}} = \mathbf{f}_{i_j}\left(\mathbf{x}_{v_1}(t), \cdots, \mathbf{x}_{v_n}(t);\ \mathbf{u}_1(t), \cdots, \mathbf{u}_l(t)\right), \quad 1 \le j \le l, \\ \dot{\mathbf{x}}_{v_k} \ = \mathbf{f}_k\left(\mathbf{x}_{v_1}(t), \mathbf{x}_{v_2}(t), \cdots, \mathbf{x}_{v_n}\right), \quad v_k \in S \setminus \{v_{i_1}, v_{i_2}, \cdots, v_{i_l}\}, \end{cases} \tag{5.20}$$

where, $\mathbf{f}_{i_j}\left(\mathbf{x}_{v_1}(t), \cdots, \mathbf{x}_{v_n};\ \mathbf{u}_1(t), \cdots, \mathbf{u}_l(t)\right)$ denotes the function of element $v_{i_j}$ acted both by elements $v_{i_s}, s \neq j$ and the inputs $\mathbf{u}_r, 1 \le r \le l$, $\mathbf{f}_k\left(\mathbf{x}_{v_1}(t), \mathbf{x}_{v_2}(t), \cdots, \mathbf{x}_{v_n}\right)$ denotes the function of the element $v_k$ acted only by elements $v_i, i \neq k$.

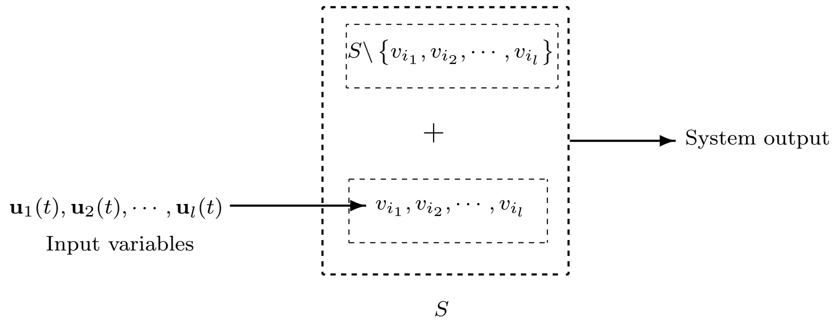



**Figure** 5.23. **System input-output**

Notice that if the interactions $a(v, u), v, u \in S$ between elements follow the vector rule and are completely known by humans, then the action function $\mathbf{f}_i$ is

$$\mathbf{f}_i = \sum_{(u, v_i) \in E(G^L S)} a(v_j, v_i) \tag{5.21}$$

and particularly, if all elements in a system $S$ are affected or not affected by input variable $\mathbf{u}_1(t), \mathbf{u}_2(t), \cdots, \mathbf{u}_s(t)$, the equations (5.20) are respectively simplified into the equations

$$\begin{cases} \dot{\mathbf{x}}_{v_1} = \mathbf{f}_1\left(\mathbf{x}_{v_1}(t), \cdots, \mathbf{x}_{v_n}(t);\ \mathbf{u}_1(t), \cdots, \mathbf{u}_l(t)\right), \\ \dot{\mathbf{x}}_{v_2} = \mathbf{f}_2\left(\mathbf{x}_{v_1}(t), \cdots, \mathbf{x}_{v_n}(t);\ \mathbf{u}_1(t), \cdots, \mathbf{u}_l(t)\right), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ \dot{\mathbf{x}}_{v_n} = \mathbf{f}_n\left(\mathbf{x}_{v_1}(t), \cdots, \mathbf{x}_{v_n}(t);\ \mathbf{u}_1(t), \cdots, \mathbf{u}_l(t)\right) \end{cases} \tag{5.22}$$

and

$$\begin{cases} \dot{\mathbf{x}}_{v_1} = f_1\left(\mathbf{x}_{v_1}(t), \cdots, \mathbf{x}_{v_n}(t)\right), \\ \dot{\mathbf{x}}_{v_2} = f_2\left(\mathbf{x}_{v_1}(t), \cdots, \mathbf{x}_{v_n}(t)\right), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ \dot{\mathbf{x}}_{v_n} = f_n\left(\mathbf{x}_{v_1}(t), \cdots, \mathbf{x}_{v_n}(t)\right). \end{cases} \tag{5.23}$$

Certainly, the equation (5.22) is the usual state equation of system but the equation (5.23) characterizes such a system $S$ that not affected by the external environment, i.e., an isolated system. Generally, a classical problem for applying the equation (5.20) or (5.22) to characterize the system state of $S$ is the well-posedness problem of equations, i.e., whether the equation (5.20) or (5.22) is solvable, whether the solution is unique if it is solvable and then, the solution $\mathbf{x}_{v_1}(t), \mathbf{x}_{v_2}(t), \cdots, \mathbf{x}_{v_n}(t)$ of equation (5.20) or (5.22) can be applied to characterize the system state of $S$. Dr.Ouyang tells Huizi that the essence of this idea is in the solvability of equation (5.22) or (5.20), i.e., if it is solvable we can use the solution $\mathbf{x}_{v_1}(t), \mathbf{x}_{v_2}(t), \cdots, \mathbf{x}_{v_n}(t)$ to characterize the system state of $S$. Obviously, by the causal relationship any solution $\mathbf{x}_{v_1}(t), \mathbf{x}_{v_2}(t), \cdots, \mathbf{x}_{v_n}(t)$ of (5.20) or (5.22) corresponds to one state of system $S$. However, the inverse problem, i.e., *is any state of the system $S$ must correspond to a solution of the equation* (5.20) *or* (5.22)*?* Particularly, if the system (5.20) or (5.22) is unsolvable, i.e., there are no solution in the classical sense, *is it meaningless or not correspond to the system state of $S$?* Dr.Ouyang tells Huizi that the answer is No. Even if the system (5.20) or (5.22) of equations has no a classical solution, it does not implies that the system state of $S$ is not existing but the view that the system state of $S$ must correspond to a solution of the equation (5.20) or (5.22) is incorrect. This topic will be further discussed on contradictory systems, especially how the system state of $S$ corresponds to the solution of equations in Chapter 7. But we still assume that the system state of $S$ corresponds to the solution of equation (5.20) or (5.22) in here.

Now that *how can we establish the state equation* (5.20) *or* (5.22) *of a system?* Dr.Ouyang explains that if the action is viewed as a vector action similar to a force, it corresponds to the flow of energy. At this time, the state equation of the system can be determined by the vertex conservation equation (5.9) of system $S$, combined with the interaction in the system and the external environment. Among these, the simplest case is that all interactions are linear with known interactions $a(v,u)$ for any elements $v,u \in S$ and the input variables $\mathbf{u}_i(t), 1 \le i \le l$. In this case, the system $S$ can be characterized by the equation (5.22) as

$$\dot{\mathbf{x}} = A[t]\mathbf{x} + B[t]\mathbf{u}, \tag{5.24}$$

where, $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_l)^t$ is an input vector, $\mathbf{x} = (\mathbf{x}_{v_1}, \mathbf{x}_{v_2}, \cdots, \mathbf{x}_{v_n})^t$ is the state vector of system $S$ and $A[t]$, $B[t]$ are the known function matrixes, i.e

$$\mathbf{A}[t] = \begin{pmatrix} a_{11}(t) & a_{12}(t) & \cdots & a_{1n}(t) \\ a_{21}(t) & a_{22}(t) & \cdots & a_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ a_{n1}(t) & a_{n2}(t) & \cdots & a_{nn}(t) \end{pmatrix}, \ \mathbf{B}[t] = \begin{pmatrix} b_{11}(t) & b_{12}(t) & \cdots & b_{1l}(t) \\ b_{21}(t) & b_{22}(t) & \cdots & b_{2l}(t) \\ \vdots & \vdots & \vdots & \vdots \\ b_{nl}(t) & b_{n2}(t) & \cdots & b_{nl}(t) \end{pmatrix}.$$

Particularly, if the matrixes $A[t]$ and $B[t]$ both are respectively the constant matrixes $[a_{ij}]_{n\times n}$ and $[b_{ij}]_{n\times n}$, the equation (5.24) is a system of linear differential equations of first order with constant coefficients, i.e.,

$$\dot{\mathbf{x}} = [a_{ij}]_{n\times n}\,\mathbf{x} + [b_{ij}]_{n\times n}\,\mathbf{u}. \tag{5.25}$$

For example, let $\mathbf{x}$ be a 1-dimensional variable. A linear system $S = \{v_1, v_2, v_3, \ v_4\}$ with constant coefficients is shown in Figure 5.24,

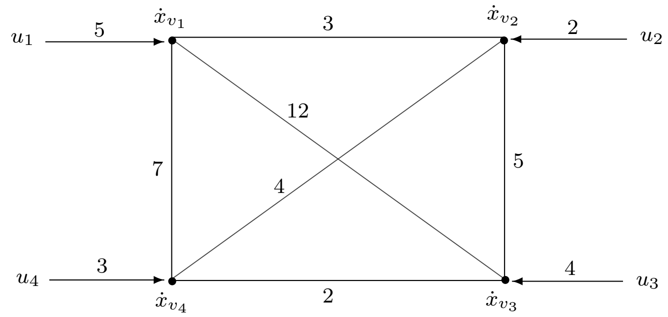



**Figure** 5.24 **A linear system with constant coefficients**

where $G^L[S]$ is a complete graph $K_4$ of order 4, the edge and the number on the input vector represent the action strength which is symmetry on edges, i.e., $a(v_1, v_2) = a(v_2, v_1) = 3$, etc. Then, the system state corresponds to a system of linear differential equations following

$$\begin{cases} \dot{x}_{v_1} = 3x_{v_2} + 12x_{v_3} + 7x_{v_4} + 5u_1 \\ \dot{x}_{v_2} = 3x_{v_1} + 5x_{v_3} + 4x_{v_4} + 2u_2 \\ \dot{x}_{v_3} = 12x_{v_1} + 5x_{v_2} + 2x_{v_4} + 4u_3 \\ \dot{x}_{v_4} = 7x_{v_1} + 4x_{v_2} + 2x_{v_3} + 3u_4 \end{cases}$$

i.e., a matrix equation

$$\dot{\mathbf{x}} = [a_{ij}]_{4\times 4}\,\mathbf{x} + [b_{ij}]_{4\times 4}\,\mathbf{u}$$

where the matrixes $[a_{ij}]_{4\times 4}$ and $[b_{ij}]_{4\times 4}$ are respectively

$$[a_{ij}]_{4\times 4} = \begin{pmatrix} 0 & 3 & 12 & 7 \\ 3 & 0 & 5 & 4 \\ 12 & 5 & 0 & 2 \\ 7 & 4 & 2 & 0 \end{pmatrix}, \quad [b_{ij}]_{4\times 4} = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 3 \end{pmatrix}.$$

Certainly, the system state $\mathbf{x}_{v_1}(t), \mathbf{x}_{v_2}(t), \cdots, \mathbf{x}_{v_n}(t)$ of a linear system $S$ can be determined by solving the state equation and then, hold on the behavior of system. Furthermore, there is a natural question, i.e., *is there a nonlinear system in the universe?* If the answer is No, all things can be characterized and understood by linear systems, we can therefore hold on the behavior of things in the universe.

However, Dr.Ouyang tells Huizi that the non-linear systems exist abundantly but the linear systems are only an approximate characterization on the system state of things in the universe. For example, let a binary system consist of cats and mice, i.e., a predator-prey system $S$ as shown in Figure 5.25 of Tom and Jerry with a labeled graph $K_2^L[S]$ or $P_2^L[S]$ and let $x$ and $y$ be the respective population sizes i.e. the number of cats and mice in this binary system. Assume that the average growth rate of cats without mice is $-\mu$, the existence of mice increases the average growth rate of cats from $-\mu$ to $-\mu+cx$. Meanwhile, assume that the average growth rate of mice without cats is $\lambda$ and the appearance of cats changes the average growth rate of mice from $\lambda$ to $\lambda - by$, where $b$ and $c$ are both assumed to be constants. Then, the numbers of cats and mice in this binary system are characterized by variables $x(t), y(t)$ with a state equation

**Figure** 5.25. **Tom and Jerry**

$$\left\{\begin{array}{lcl} \dot{x} & = & x(-\mu + cy) \\ \dot{y} & = & y(\lambda - bx) \end{array}\right. \tag{5.26}$$

called the *Lotka-Volterra predator-prey model* on population, which is obviously a nonlinear system of differential equations.

**3.3.System Equation.** The *system equation* is a system of equations followed by adding the output equation of vector $\mathbf{y}$ of the system on the state equation of system. It is a system of equations that completely characterize the input, output and the state of system. Dr.Ouyang tells Huizi that for a system $S$ with $l$ input variables, $n$ state variables and $m$ output variables, a general form of the system equation is

$$\left\{\begin{array}{l} \dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) \\ \mathbf{y}(t) = \mathbf{g}(\mathbf{x}(t), \mathbf{u}(t)) \end{array}\right. \tag{5.27}$$

where $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_l)^t \in \mathbb{R}^l$ is the input variable vector, $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n)^t$, $\mathbf{f} = (\mathbf{f}_1, \mathbf{f}_2, \ \cdots, \mathbf{f}_n)^t \in \mathbb{R}^n$, $\mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_m)^t$, $\mathbf{g} = (\mathbf{g}_1, \mathbf{g}_2, \cdots, \mathbf{g}_m)^t \in \mathbb{R}^m$. Usually, the first equation is called the *state equation* and the second is the *output equation* of system $S$. Notice that the system state is the change of the interaction in elements varying with time but not directly included the time $t$ in $\mathbf{f}$ and $\mathbf{g}$ as the equation $\dot{\mathbf{x}} = [a_{ij}]_{4\times 4}\mathbf{x} + [b_{ij}]_{4\times 4}\mathbf{u}$ has no the single term $h(t)$ on time in Figure 5.24. Such a kind of equation is called the *autonomy equation* in general. Particularly, if the state equation and the output equation are both linear equation in (5.27), i.e., there exist matrixes $\mathbf{A}[t], \mathbf{B}[t], \mathbf{C}[t], \mathbf{D}[t]$ holding with

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}[t]\mathbf{x}(t) + \mathbf{B}[t]\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}[t]\mathbf{x}(t) + \mathbf{D}[t]\mathbf{u}(t) \end{cases} \tag{5.28}$$

such a system $S$ is called the *linear time-varying system*, where the matrixes $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are the same as in the equation (5.24), the matrixes $\mathbf{C}(t)$ and $\mathbf{D}(t)$ are introduced in the following

$$\mathbf{C}[t] = \begin{pmatrix} c_{11}(t) & c_{12}(t) & \cdots & c_{1n}(t) \\ c_{21}(t) & c_{22}(t) & \cdots & c_{2n}(t) \\ \vdots & \vdots & \vdots & \vdots \\ c_{m1}(t) & c_{m2}(t) & \cdots & c_{mn}(t) \end{pmatrix}, \ \mathbf{D}[t] = \begin{pmatrix} d_{11}(t) & d_{12}(t) & \cdots & d_{1l}(t) \\ d_{21}(t) & d_{22}(t) & \cdots & d_{2l}(t) \\ \vdots & \vdots & \vdots & \vdots \\ d_{m1}(t) & d_{m2}(t) & \cdots & d_{mn}(t) \end{pmatrix}.$$

Similarly, if the matrixes $\mathbf{A}(t), \mathbf{B}(t), \mathbf{C}(t)$ and $\mathbf{D}(t)$ are constant matrixes, then the system $S$ is called a *linear time-invariant system*. For a linear time-invariant system with a single input and output, i.e., the input $\mathbf{u}(t)$ and the output $\mathbf{y}(t)$ both are one-dimensional, the system equation (5.28) can be simplified as

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}[t]\mathbf{x}(t) + \mathbf{B}u(t) \\ y(t) = \mathbf{C}[t]\mathbf{x}(t) + du(t) \end{cases} \tag{5.29}$$

where $\mathbf{B} = (b_{11}, b_{21}, \cdots, b_{n1})^t$ is a column vector, $\mathbf{C} = (c_{11}, c_{21}, \cdots, c_{n1})$ is a row vector and $d$ is a constant.

Certainly, the equation (5.29) indicates that the output variable $\mathbf{y}(t)$ of the system $S$ has a causal relationship with the system state affected by both the system state variables and the input variables of the system. So, *what is the role of output vector* $\mathbf{y}(t)$ *for characterizing the behavior of system $S$?* Dr.Ouyang tells Huizi that the system state variables $\mathbf{x}_v(t), v \in S$ are not necessarily for human observation but the system output variable $\mathbf{y}(t)$ is the external expression of the system state on the first, which can be observed by human and secondly, the output variable of the system has a certain consequential relationship with the input variable. Even for a black box system, the causal relationship between input and output can be established similar to (5.29), namely finding a vector function $\mathbf{g}$ such that

$$\mathbf{y}(t) = \mathbf{f}(t) + \mathbf{g}(\mathbf{u}(t), t) \text{ or a linear relationship } \mathbf{y}(t) = \mathbf{f}(t) + \mathbf{D}[t]\mathbf{u}(t), \tag{5.30}$$

where $\mathbf{f}(t)$ is introduced for establishing the relationship between the input variable $\mathbf{u}$ and the output variable $\mathbf{y}$. It can also be regarded as the influence of external environment. Conversely, if the state variables $\mathbf{x}_v(t), v \in S$ of the system are all known, we can determine the input variable $\mathbf{u}(t)$ in order to obtain a specific result of the output variable $\mathbf{y}(t)$, called the *feedback regulation* which is a common application of "*tracing the cause by the effect*" in the system control.

Usually, there is a forward channel from the input to the output and also a feedback channel from the output to the input in the information transmission of feedback regulation, which form a closed loop, i.e., a directed circle of information transmission. For example, Dr.Ouyang tells Huizi that the population size is governed by the feedback regulation in a natural way on the 2-system consisting of cats and mice in Figure 5.25, namely the increase or decrease of mice feeds back to the increase or decrease of cats, which naturally regulates the increase or decrease in the number of living cats and then, the number of mice predation by cats decreases or increases, thus achieving the ecological balance between the mice and the cats.

## §4. **Solving Equation**

Huizi asks Dr.Ouyang that the evolution of a system $S$ is characterized by its states $x_v(t), v \in S$ but the observation can not directly obtain $x_v(t), v \in S$, which leads to the differential equation or difference equation and then, from these equations to get the solution $\mathbf{x}_v(t), v \in S$ of (5.27) recognizing system state, holding on its evolution. However, Huizi says:"*The equation* (5.27) *is extremely complicated. How do we determine it is solvable or not? If we can't solve it, we're not going anywhere, Dad!*" Huizi continuously asks Dr.Ouyang:"*Even if an equation is solvable, how can we find its solution and hold on the system state of* $S$ *such as whether the direction of change is to the left or to the right, and how fast?*" Dr.Ouyang explains to her:"*The questions you have asked are very meaningful which are the foundation for humans holding on the state of things!*" Dr.Ouyang then tells Huizi: "*I have said before that the dimensions are the units of understanding the physical characteristics of things. Correspondingly, the measuring of a characteristic is a multiple one of this dimension. For example, suppose that the length of an object* $x$ *is* $8.63m$*. What is the* $8.63m$ *meanning here? It means that the length* $x$ *of the object is* $8.63$ *times of meters (m), denoted as* $x = 8.63m$*. In addition, use a* $4m$ *ruler to measure this object, the ruler is not long enough and need to measure three times, the measuring results are respectively* $4m, 4m$ *and* $0.63m$*, then* $x = 4 + 4 + 0.63 = 8.63m$*. This is the linear addition of dimension. As can be seen from this example, the equation satisfying linear characteristics is easier to solve than other equations!*" Dr.Ouyang explains further to his daughter that one can expand a nonlinear function $\mathbf{f}$ by Taylor expansion in (5.27), choose the first two terms of the expansion and ignore the terms after the third term to get a linear equation, which approximately characterizes $\mathbf{f}$ at this time. Certainly, this way needs certain conditions to do so. For simplicity, we assume that the variable $\mathbf{x}_v$ is a 1-dimensional variable $x_v, v \in S$ in this section.

Generally, assuming $(\mathbf{x}_0, \mathbf{u_0})$ is a zero-point of $\mathbf{f}$, i.e., $\mathbf{f}(\mathbf{x}_0, \mathbf{u}_0) = \mathbf{0}$, we can expand $\mathbf{f}$ with respect on $\mathbf{x}, \mathbf{u}$ by Taylor expansion as follows:

$$\mathbf{f}(\mathbf{x}, \mathbf{u}) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{(\mathbf{x}_0, \mathbf{u}_0)} (\mathbf{x} - \mathbf{x}_0)^t + \left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_{(\mathbf{x}_0, \mathbf{u}_0)} (\mathbf{u} - \mathbf{u}_0)^t + \boldsymbol{\alpha}(\Delta \mathbf{u}, \Delta \mathbf{x}), \tag{5.31}$$

where, $\partial \mathbf{f}/\partial \mathbf{x}$ and $\partial \mathbf{f}/\partial \mathbf{u}$ are called the *Jacobian matrixes* of $\mathbf{f}$ respectively respect to $\mathbf{x}$ and $\mathbf{u}$, i.e.,

$$\frac{\partial \mathbf{f}}{\partial \mathbf{x}} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}, \quad \frac{\partial \mathbf{f}}{\partial \mathbf{u}} = \begin{pmatrix} \frac{\partial f_1}{\partial u_1} & \frac{\partial f_1}{\partial u_2} & \cdots & \frac{\partial f_1}{\partial u_l} \\ \frac{\partial f_2}{\partial u_1} & \frac{\partial f_2}{\partial u_2} & \cdots & \frac{\partial f_2}{\partial u_l} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_n}{\partial u_1} & \frac{\partial f_n}{\partial u_2} & \cdots & \frac{\partial f_n}{\partial u_l} \end{pmatrix}$$

and $\boldsymbol{\alpha}(\Delta \mathbf{u}, \Delta \mathbf{x})$ is the the remainder of order $\geq 2$ in the expansion of $\mathbf{f}$ on $\mathbf{x}$ and $\mathbf{u}$, where $\Delta \mathbf{x} = \mathbf{x} - \mathbf{x}_0$ and $\Delta \mathbf{u} = \mathbf{u} - \mathbf{u}_0$. If we ignore the remainder $\boldsymbol{\alpha}(\Delta \mathbf{u}, \Delta \mathbf{x})$ with order$\geq 2$ in (5.31), then the state equation in (5.27) can be simplified as

$$\dot{\mathbf{x}}^t = \mathbf{f}(\mathbf{x}, \mathbf{u}) = \left.\frac{\partial \mathbf{f}}{\partial \mathbf{x}}\right|_{(\mathbf{x}_0, \mathbf{u}_0)} (\mathbf{x} - \mathbf{x}_0)^t + \left.\frac{\partial \mathbf{f}}{\partial \mathbf{u}}\right|_{(\mathbf{x}_0, \mathbf{u}_0)} (\mathbf{u} - \mathbf{u}_0)^t. \tag{5.32}$$

Notice that $(\partial \mathbf{f}/\partial \mathbf{x})|_{(\mathbf{x}_0, \mathbf{u}_0)}$ and $(\partial \mathbf{f}/\partial \mathbf{u})|_{(\mathbf{x}_0, \mathbf{u}_0)}$ are constant matrixes, denoted respectively by $\mathbf{A}$ and $\mathbf{B}$. Then, the equation (5.32) is a linear differential equation

$$\dot{\mathbf{x}}^t = \mathbf{A}(\mathbf{x} - \mathbf{x}_0)^t + \mathbf{B}(\mathbf{u} - \mathbf{u}_0)^t \tag{5.33}$$

of first order with constant coefficients.

**4.1.Algebraic Equation.** Notice that a linear equation is the product that accompanies with the physical dimension, Dr.Ouyang says that it should be solvable theoretically and know the state $S(t)$ of such a system $S$.

**(1)Linear equation.** A general form of linear algebraic equation is

$$\mathbf{A}\mathbf{x}^t = \mathbf{B}, \tag{5.34}$$

where $\mathbf{A}$ and $\mathbf{B}$ are both matrixes of constant, i.e.,

$$\mathbf{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}, \quad \mathbf{x}^t = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

So, *how do we solve the equation* (5.34)*?* Dr.Ouyang tells Huizi that we can start with some simple cases.

1) $n = 1$, i.e., $ax = b$, there is only one variable. In this case, as long as $a \neq 0$ there is a solution $x = b/a$. Otherwise, if $a = 0$ but $b \neq 0$, the equation is not valid and if $b = 0$, any element in a field $\mathscr{F}$ is the solution of equation, where $\mathscr{F}$ is a field in which solves the equation $ax = b$ and can be finite or infinite, likes the fields of integers, real numbers or complex numbers. For example, the equation $ax = b$ has an integer solution if $b$ is divisible by $a$ such as the equation $2x = 4$ has an integer solution $x = 2$ but the equation $2x = 3$ has no integer solution.

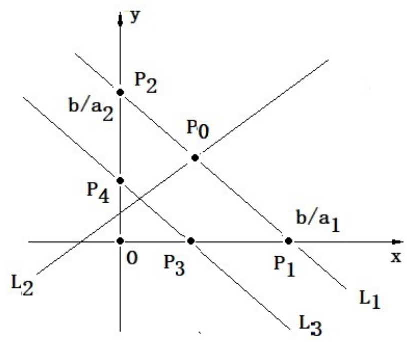


**Figure** 5.26. **Straight lines**

2) $n = 2$, i.e., $a_1x + a_2y = b$, there are two variables $x$ and $y$. The point set of this equation in Euclidean plane $\mathbb{R}^2$ is a straight line $L_1$ intersecting $x$-axis and $y$-axis with points $(b/a_1, 0)$ and $(0, b/a_2)$ respectively, as shown in Figure 5.26. In other words, all points on the line $L_1$ are solutions of the equation $a_1x + a_2y = b$. Notice that $L_2$ is another straight line with equation $a_1'x + a_2'y = b'$ in Figure 5.26. We see that the line $L_1$ has an intersection $P_0$ with $L_2$. Let $(x_0, y_0)$ be the coordinates of the point $P_0$. So, the coordinates $x_0, y_0$ should not only satisfy the equation $a_1x + a_2y = b$ but also the equation $a_1'x + a_2'y = b'$, i.e., $x_0, y_0$ is a solution of the system of equations $a_1x + a_2y = b$, $a_1'x + a_2'y = b'$. Meanwhile, it can also be seen from Figure 5.26 that the system has only the solutions $x_0, y_0$ because the lines $L_1$ and $L_2$ have only one intersection $P_0$.

So, *can we get the solution* $x_0, y_0$ *directly from the two equations?* The answer is Yes because if we multiply both sides of the first equation by $a_2'$ and minus the second equation by $a_2$, we have $(a_1a_2' - a_2a_1')x = ba_2' - b'a_2$. Similarly, multiply both sides of the first equation by $a_1'$ minus the second equation by $a_1$ on both sides, we have $(a_1'a_2 - a_2'a_1)y = ba_1' - b'a_1$. Now that if $a_1a_2' - a_2a_1' \neq 0$, we get that

$$x_0 = \frac{ba_2' - b'a_2}{a_1a_2' - a_2a_1'}, \qquad y_0 = \frac{ba_1' - b'a_1}{a_2a_1' - a_2'a_1},$$

which is the coordinates $(x_0, y_0)$ of point $P_0$.

Now, let's look at the lines $L_1$ and $L_3$ in Figure 5.26. Clearly, they are parallel, namely they have no intersection. Let the equation of $L_3$ be $\widehat{a}_1x + \widehat{a}_2y = \widehat{b}$ which intersects respectively with $x$-axis and $y$-axis at points $P_3 = (\widehat{b}/\widehat{a}_1, 0)$ and $P_4 = (0, \widehat{b}/\widehat{a}_2)$. Notice that $L_1 /\!/ L_2$, the triangle $\triangle OP_1P_2$ is similar to that of $\triangle OP_3P_4$, we have

$$\frac{\widehat{b}}{\widehat{a}_1}\Big/\frac{b}{a_1} = \frac{\widehat{b}}{\widehat{a}_2}\Big/\frac{b}{a_2} \quad \Rightarrow \quad \frac{a_1}{\widehat{a}_1} = \frac{a_2}{\widehat{a}_2},$$

i.e., the coefficients of $x$ and $y$ in the equation of $L_1$ and $L_3$ are equal in proportion.

3) $n \geq 3$, let's look at the case of $n = 3$ that we have three variables $x, y, z$ with an equation $a_1x + a_2y + a_3z = b$. Certainly, the point set of this equation is a plane in 3-dimensional Euclidean space $\mathbb{R}^3$ such as those shown in Figure 4.15 where the equation is $Ax+By+Cz = D$, respectively intersecting the $x$-axis, $y$-axis and the $z$-axis with points $(D/A, 0, 0)$, $(0, D/B, 0)$, $(0, 0, D/C)$. In other words, the coordinates $x, y, z$ of any point on this plane are one solution of this equation. So, there are infinitely many solutions of this equation. Geometrically, if two planes have intersection, the intersection points must be a straight line. If this straight line intersects with the third plane then the intersection must be a point. For example, the plane $P_1$ intersects $P_2$ with the straight line $L_1$, the plane $P_3$ intersects $P_2$ with the straight linear $L_2$, etc., as shown in Figure 5.27.

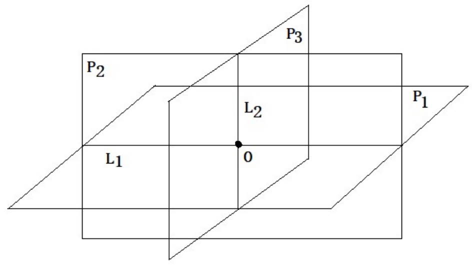


**Figure** 5.27. **Plane intersection**

In this way, if a system consisting of equations $a_{11}x + a_{12}y + a_{13}z = b_1$ and $a_{21}x + a_{22}y+a_{23}z = b_2$ has solutions, move the terms $a_{13}z, a_{23}z$ on the left side to the right side, get the equations $a_{11}x + a_{12}y = b_1 - a_{13}z$, $a_{21}x + a_{22}y = b_2 - a_{23}z$ and view $z$ as a known quantity, then it is going to the case of $n = 2$. By the previous discussion, there can only be one solution $x_0, y_0$. But since $z$ can be chosen as any number in $\mathbb{R}$. The elimination of parameter $z$ will yield a linear equation of $x_0, y_0$, i.e., a straight line on the plane. So, *under what conditions are two planes $P_1, P_2$ parallel, i.e., have no intersection?* The answer is analogous to that of two lines parallel in $\mathbb{R}^2$, i.e., if and only if the coefficients of $x, y$ and $z$ are equal in proportions in the equations of planes $P_1$ and $P_2$.

In the case of three planes, such as the planes $P_1, P_2$ and $P_3$ intersect at the point $O$ in Figure 5.27, which is the case that the linear equations

$$\begin{cases} (P_1) \ : \ a_{11}x + a_{12}y + a_{13}z = b_1, \\ (P_2) \ : \ a_{21}x + a_{22}y + a_{23}z = b_2 \\ (P_3) \ : \ a_{31}x + a_{32}y + a_{33}z = b_3 \end{cases}$$

of three planes$P_1, P_2$ and $P_3$ have a unique solution, i.e., the coordinates $(x_0, y_0, z_0)$ of point $O$. So, *can we directly calculate the values of $x_0, y_0, z_0$ from the three equations?* Dr.Ouyang tells Huizi that there is a standard algorithm for obtaining the solution $x_0, y_0, z_0$ of linear equations, called the *elimination method*, namely turn the coefficient of the variable eliminated to 1, subtract the first equation from the other equations in sequence, i.e., the other equations no longer contain this variable, which results in the system of equations in the following equivalent form

$$\left\{\begin{array}{r} x+c_{12}y+c_{13}z=d_1 \\ y+c_{23}z=d_2 \\ z=d_3 \end{array}\right. \tag{5.35}$$

step by step. Thus, $z$ can be obtained from the last equation of (5.35), $y$ can be obtained by substituting $y$ into the second equation and $x$ can be obtained by substituting $y, z$ into the first equation. For example, if the equations of the planes $P_1, P_2$ and $P_3$ are respectively $3x + 9y + 6z = 6$, $3x + 6y + 9z = 9$ and $6x + 12y + 24z = 30$, the eliminating process is shown as follows:

$$\left\{\begin{array}{l} 3x+9y+6z=6 \\ 3x+6y+9z=9 \\ 6x+12y+24z=30 \end{array}\right. \longrightarrow \left\{\begin{array}{l} x+3y+2z=2 \\ x+2y+3z=3 \\ x+2y+4z=5 \end{array}\right.$$

$$\longrightarrow \left\{\begin{array}{r} x+3y+2z=2 \\ -y+z=1 \\ -y+2z=3 \end{array}\right. \longrightarrow \left\{\begin{array}{r} x+3y+2z=2 \\ y-z=-1 \\ z=2 \end{array}\right.$$

Thus, we can easily get the solution $x = -5, y = 1$ and $z = 2$.

Dr.Ouyang tells Huizi that the previous method could be applied to the general equation (5.34) which enables one to get a similar result as (5.35), i.e

$$\left\{\begin{array}{l} a_{11}x_1+a_{12}x_2+\cdots+a_{1n}x_n=b_1 \\ a_{21}x_1+a_{22}x_2+\cdots+a_{2n}x_n=b_2 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \\ a_{m1}x_1+a_{m2}x_2+\cdots+c_{mn}x_n=b_n \end{array}\right. \longrightarrow \left\{\begin{array}{r} x_1+c_{12}x_2+c_{13}x_3+\cdots+c_{1n}x_n=d_1 \\ x_2+c_{23}x_3+\cdots+c_{2n}x_n=d_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x_l+c_{li+1}x_{l+1}+\cdots+c_{ln}x_n=d_l \\ 0=d_{l+1} \\ \cdots\cdots \\ 0=d_m \end{array}\right.$$

where, the integers $l \leq m$. Thus, the solution of the linear equation (5.34) is divided into two cases: ①$l = m$ or $l < m$ and $d_{l+1} = d_l = \cdots = d_n = 0$, there are infinitely many solutions of the system in this case and these solutions constitute a $(n-l)$-dimensional Euclidean subspace $\mathbb{R}^{n-l}$. Particularly, there is a unique solution if $l = n$, which can be obtained from the bottom in order to get $x_n, x_{n-1}, \cdots, x_1$ liking the case of (5.35); ②$l < m$ but $d_{l+1}, d_{l+2}, \cdots, d_m$ are not all to be 0, the equations are contradicting with no solution in this case.

**(2)Nonlinear equation.** A nonlinear algebraic equation is such a algebraic equation that order $n \geq 2$. For example, the $n$th-order equation

$$a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 = 0 \tag{5.36}$$

in one variable $x$ and the equation of several variables such as $xyz + x^2 + y = 0$, etc.

Dr.Ouyang explains that there is a famous conclusion for the $n$th-order equation in one variable, i.e., *for any integer $n \geq 1$, an nth-order equation in one variable has n solutions in the complex field $\mathbb{C}$ (the multiplicity of roots is calculated by multiplicity).* However, this conclusion does not necessarily hold in the real and integer fields. For example, the equation $2x = 3$ has the solution $x = 1.5$ in the real field $\mathbb{R}$ but has no solution in the integer field $\mathbb{Z}$ because if there is a solution $x$, then $2x$ is even but 3 is odd, namely the equality $2x = 3$ can not be valid in $\mathbb{Z}$.

A more general case of solving an equation in the real or complex field is to determine the functional relation $F$ between $y$ and $x$ by a functional equation $F(x, y) = 0$, i.e., to determine conditions for solving $y$ in the equation $F(x, y) = 0$ or answer the question that *under what conditions there is a solution $y = f(x)$ and under what conditions there is a continuous solution or continuous differentiable solution $y = f(x)$?* Dr.Ouyang tells Huizi that the question had been completely solved by the *implicit function theorem*, i.e., if $U \subset \mathbb{R} \times \mathbb{R}$ and $F : U \to \mathbb{R}$ is continuously differentiable so that for any point $(x_0, y_0) \in U$ holds with

$$F(x_0, y_0) = 0 \quad \text{but} \quad \left.\frac{\partial F}{\partial y}\right|_{(x_0, y_0)} \neq 0,$$

then there exists an open interval $I \subset \mathbb{R}$, $x_0 \in I$ and a continuously differentiable function $\varphi : I \to \mathbb{R}$ such that for any $x \in I$ there is $(x, \varphi(x)) \in U, \varphi(x_0) = y_0$ and $F(x, \varphi(x)) = 0$, i.e., $y = \varphi(x)$ and

$$\frac{d\varphi}{dx} = -\left. \frac{\partial F}{\partial x} \middle/ \frac{\partial F}{\partial y} \right..$$

For example, if $xy + x^2 + y = 0$, let $F(x, y) = xy + x^2 + y$. Obviously, $F(0, 0) = 0$ and

$$\frac{\partial F}{\partial y} = x + 1 \quad \Rightarrow \quad \left.\frac{\partial F}{\partial y}\right|_{(0,0)} = 1 \neq 0,$$

by the implicit function theorem there is an interval $I$ of 0, i.e, $0 \in I \subset \mathbb{R}$ and a continuously differentiable function $y = f(x)$ on $I$ holding with the equation $F(x, y) = 0$. Furthermore, by

$$xy + x^2 + y = 0 \quad \Rightarrow \quad y = f(x) = -\frac{x^2}{x+1}$$

and it is easy to verify that $y = f(x)$ is continuously differentiable at any point $x \neq -1$ on $\mathbb{R}$. Generally, Dr.Ouyang tells Huizi that for $m$ function equations

$$F_i(x_1, x_2, \cdots, x_n; y_1, y_2, \cdots, y_m) = 0, \quad 1 \leq i \leq m,$$

finds solutions $y_1, y_2, \cdots, y_m$ holding with certain properties, i.e.,

$$\left\{\begin{array}{l} y_1 = \varphi_1(x_1, x_2, \cdots, x_n) \\ y_2 = \varphi_2(x_1, x_2, \cdots, x_n) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ y_m = \varphi_m(x_1, x_2, \cdots, x_n) \end{array}\right. \Rightarrow \left\{\begin{array}{l} F_1(x_1, x_2, \cdots, x_n; y_1, y_2, \cdots, y_m) = 0 \\ F_2(x_1, x_2, \cdots, x_n; y_1, y_2, \cdots, y_m) = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ F_m(x_1, x_2, \cdots, x_n; y_1, y_2, \cdots, y_m) = 0 \end{array}\right. \tag{5.37}$$

is a basic problem in function equation of serval variables. However, the problem (5.37) has been completely solved by the implicit function theorem in general similar to that of one variable case, i.e., let

$$\mathbf{x}^0 = \left(x_1^0, x_2^0, \cdots, x_n^0\right),\ \mathbf{y}^0 = \left(y_1^0, y_2^0, \cdots, y_m^0\right),\ F_i(\mathbf{x}^0; \mathbf{y}^0) = 0,\ 1 \leq i \leq m$$

and replacing the previous condition "*the partial differential $\partial F/\partial y$ is not equal to* 0 *at the point* $(x_0, y_0)$" in the one variable case with the "*Jacobian matrix $J_{\mathbf{y}|\mathbf{x}}$ is reversible on a neighborhood $U$ of point* $\left(\mathbf{x}^0; \mathbf{y}^0\right)$". Then, the solutions $y_1, y_2, \cdots, y_m$ at the neighborhood $U$ of point $(\mathbf{x}^0; \mathbf{y}^0)$ are not only existing but also continuously differentiable, where the *Jacobian matrix* $J_{\mathbf{y}}|\mathbf{x}$ is an $m \times n$ matrix defined by

$$J_{\mathbf{y}|\mathbf{x}} = \begin{pmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_1}{\partial x_2} & \cdots & \frac{\partial y_1}{\partial x_n} \\ \frac{\partial y_2}{\partial x_1} & \frac{\partial y_2}{\partial x_2} & \cdots & \frac{\partial y_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial y_m}{\partial x_1} & \frac{\partial y_m}{\partial x_2} & \cdots & \frac{\partial y_m}{\partial x_n} \end{pmatrix} \tag{5.38}$$

which is important in the transformation of functions and particularly, useful in the transformation of coordinate systems.

Notice that the complete differential of the function $y_i$ is

$$dy_i = \frac{\partial y_i}{\partial x_1} dx_1 + \frac{\partial y_i}{\partial x_2} dx_2 + \cdots + \frac{\partial y_i}{\partial x_n} dx_n,$$

i.e., the entries of $i$th row in the Jacobian matrix are the coefficients of the complete differential of $y_i$. Particularly, the Jacobian matrix $J_{\mathbf{y}}|\mathbf{x}$ is an $n$ square if $m = n$, which is the transformation matrix between local frames of dual tangent vector spaces at points of an $n$-dimensional manifold, holds with the invariance under the coordinate transformation, i.e., a generalization of differential invariance of (4.18).

**4.2.Differential Equation.** Certainly, the differential equations constructed by the observing data of system are all autonomous equations. Dr.Ouyang tells Huizi that they should be solvable in theory because the differential equations properly characterize the elements and these element states constitute the system state. However, solving the differential equations to obtain the state of elements is not simple. In practice, there are a lots of differential equations that the explicit or implicit $\mathbf{x}$ with respect to time $t$ can not be solved by elementary method but only the function series used to approximate the solution $\mathbf{x}$, characterizing the system state.

**(1)Solution existence.** The system state $x_v(t)$, $v \in S$ is the state of all elements of $S$ on time $t$. The solution existence of a differential equation is to determine conditions of the solvability by the properties of functions appeared in the equation. Dr.Ouyang explains that this is similar to the implicit function theorem for solving a functional equation, which is essentially to find the conditions for solvability on the initial value problem

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) \\ \mathbf{x}|_{t=t_0} = \mathbf{x}^0 \end{cases} \quad \Leftrightarrow \quad \begin{cases} \dot{x}_1 = F_1(x_1, x_2, \cdots, x_n) \\ \dot{x}_2 = F_2(x_1, x_2, \cdots, x_n) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dot{x}_n = F_n(x_1, x_2, \cdots, x_n) \\ x_1(t_0) = x_1^0, \cdots, x_n(t_0) = x_n^0 \end{cases} \tag{5.39}$$

of differential equation, where $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ and $\mathbf{F} = (F_1, F_2, \cdots, F_n)$.

So, *how do we know the existence of the solution of* $\mathbf{x}$ *by the property of* $\mathbf{F}$*?* Dr.Ouyang explains that we can start with the simplest case, i.e., $m = 1$ and $n = 1$. In this case, the equation (5.39) becomes $\dot{x} = F(x)$, $x(t_0) = x^0$ and the *existence theorem of solution* states that if there are real numbers $a, b \in \mathbb{R}$ such that $F(t, x)$ is continuously in a rectangle

$$\mathcal{L}: \ t_0 - a \leq t \leq t_0 + a, \quad x^0 - b \leq x \leq x^0 + b$$

and there is a constant $N$ such that for any two points $(t, x)$ and $(t, x')$ on $\mathcal{L}$ holding with the *Lipschitz condition*, i.e.,

$$|F(x) - F(x')| \leq N|t - t'|,$$

then there is a unique solution $x = \varphi(t)$ of equation (5.39) on the interval $x^0 - h_0 \leq x \leq x^0 + h_0$ satisfied with the initial condition $\varphi(t_0) = x^0$, where $h_0 = \min\{a, b/M\}$ and $M = \max\limits_{(t,x)\in\mathcal{L}}\{F(x)\}$.

So, *how do we get this theorem by that* $F(x)$ *satisfies the Lipschitz condition?* Actually, if $x = \varphi(t)$ is a solution to the equation $\dot{x} = F(x)$, then there is $\dot{\varphi}(t) = F(\varphi(t))$ and

$\varphi(t_0) = x^0$. Consequently, there is

$$\varphi(t) = x^0 + \int_{t_0}^{t} F(\varphi(s))ds \tag{5.41}$$

by integral operation, called the *Picard approximation* which can successive approximating the solution of equation (5.39), i.e., the first by substituting $\varphi(t) = x^0$ in (5.41) to get $\varphi_1(t)$ and then substituting $\varphi_1(t)$ in (5.41) to get $\varphi_2(t), \cdots$. Generally, for any integer $k \geq 0$ we substitute $\varphi_k$ in (5.41) to get

$$\varphi_{k+1}(t) = x^0 + \int_{t_0}^{t} F(\varphi_k(s))ds$$

and construct a sequence $x^0, \varphi_1(t), \varphi_2(t), \cdots, \varphi_n(t), \cdots$.

So, *what are the properties of sequence* $x^0, \varphi_1(t), \cdots, \varphi_n(t), \cdots$*?* Dr.Ouyang tells Huizi that it is actually such a sequence of gradually approximating to the solution of equation (5.39). *Why is it so?* Because it is assumed that $F(x)$ satisfies the Lipschitz condition, there must be

$$|\varphi_{k+1}(t) - \varphi_k(t)| \leq N \int_{t_0}^{t} |\varphi_k(s) - \varphi_{k-1}(s)|\, ds$$

and we can then derive that

$$|\varphi_{k+1}(t) - \varphi_k(t)| \leq MN^n \frac{|t - t_0|^{n+1}}{(n+1)!}$$

in general. Whence, the function

$$\varphi(t) = \lim_{n \to +\infty} \left( \varphi_0(t) + \sum_{n=1}^{n} (\varphi_n(t) - \varphi_{n-1}(t)) \right) \tag{5.42}$$

exists which is the solution of equation (5.39).

Similarly, the method of Picard approximation can generically obtain the existence of solutions to the equation (5.39) for $m = n \geq 2$. Dr.Ouyang explains that if $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable, then it can be proved that $\mathbf{F}$ satisfy the Lipschitz condition and there is an interval $(t_0 - a, t_0 + a)$ for a real number $a$ such that there is a unique solution

$$\mathbf{x} : (t_0 - a, t_0 + a) \to \mathbb{R}^n \quad \Rightarrow \quad \dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) \tag{5.43}$$

of equation (5.39) holding with the initial condition $\mathbf{x}(t_0) = \mathbf{x}^0$.

In practice, the conditions in the solvability theorem of equation (5.39) are easily verified. For example, the function $F(x) = 2x^2$ is continuous on the rectangular neighborhood

$$\mathcal{L}: \ -1 \leq t \leq 1, \ -1 \leq x \leq 1$$

of the origin $(0, 0)$ in equation $\dot{x} = 2x^2$ and for any points $x$ and $x'$ in $\mathcal{L}$,

$$|F(x) - F(x')| = \left|x^2 - x'^2\right| = |(x + x')(x - x')| \leq 2|x - x'|,$$

i.e., it holds on the Lipschitz condition with $N = 2$. By the solution existence theorem of equation (5.39) in the case of $m = n = 1$, there must be a solution $x = \varphi(t)$ on the interval $-1/2 \leq x \leq 1/2$ with $x(0) = 0$, where $h_0 = 1/2$ because the maximum of $F(x)$ is 2 in the interval $\mathcal{L}$, i.e., $h_0 = \min\{1, 1/2\} = 1/2$.

**(2)System solving.** As we have known the existence of solution to the equation (5.39), a more realistic problem is how to use an elementary method to solve it. For example, Dr.Ouyang asks Huizi: "*how do we obtain the solution of binary system*

$$\begin{cases} \dot{x} &= x(-\mu + cy), \\ \dot{y} &= y(\ \lambda\ - bx). \end{cases}$$

*formed by the cats and mice only, i.e., solving the differential equation* (5.26)*?*" For solving this equation, divide the first equation by the second we know that

$$\frac{\dot{y}}{\dot{x}} = \frac{dy}{dx} = \frac{y(\lambda - bx)}{x(-\mu + cy)}. \tag{5.44}$$

Separating variables of the equation (5.44) and then integrating, we get that

$$\begin{aligned} \frac{-\mu + cy}{y} dy = \frac{\lambda - bx}{x} dx \quad &\Rightarrow \quad \mu \ln|y| + \lambda \ln|x| = cy + bx + C \\ &\Rightarrow \quad y^{\mu} x^{\lambda} = C_1 e^{cy + bx}, \end{aligned} \tag{5.45}$$

where, $C_1 = e^C$ and $C$ is an integral constant.

Then, *can we solve all equations* (5.39) *by an elementary method?* The answer is Not! Dr.Ouyang explains that even for the nonlinear differential equations in the simple form such as the *Riccati equation* $\dot{x} = p(t)x^2 + q(t)x + r(t)$, we can not solve it by an elementary method. However, Dr.Ouyang tells Huizi that any linear differential equation, especially linear differential equations with constant coefficients could be solved by an elementary method as follows:

Firstly, consider a linear differential equation with one variable

$$x^{(s)} + p_1(t)x^{(s-1)} + \cdots + p_{s-1}(t)x' + p_s(t)x = 0, \tag{5.46}$$

where, $s \geq 1, x^{(s)} = d^s x/dt^s$ and $p_k(t)$ is a function of $t$. Without loss of generality, let $x_1, x_2, \cdots, x_s$ be $s$ solutions of the equation (5.46) and let $C_1, C_2, \cdots, C_s$ be $s$ constants. We then have

$$\left(\sum_{i=1}^{n} C_i x_i\right)^{(s)} + p_1(t)\left(\sum_{i=1}^{s} C_i x_i\right)^{(s-1)} + \cdots + p_s(t)\left(\sum_{i=1}^{s} C_i x_i\right)$$
$$= \sum_{i=1}^{s}\left(x_i^{(s)} + p_1(t)x_i^{(s-1)} + \cdots + p_{s-1}(t)x_i' + p_s(t)x_i\right) = 0$$

by the linear property of differential, i.e., $C_1x_1 + C_2x_2 + \cdots + C_sx_s$ is also a solution of the equation (5.46). Furthermore, if $x_1(t), x_2(t), \cdots, x_s(t)$ are $s$ linearly independent solutions of the equation (5.46), then any solution $x(t)$ is a linear combination of $x_1(t), x_2(t), \cdots, x_s(t)$ by the spanning of linear space, namely there are constants $C_1, C_2, \cdots, C_s$ such that

$$x(t) = C_1x_1(t) + C_2x_2(t) + \cdots + C_sx_s(t), \tag{5.47}$$

called a *general solution* of equation (5.46). In this case, if $x^{(s-1)}(t_0), \cdots, x'(t_0)$ and $x(t_0)$ are all known, there are $s$ linear algebraic equations deduced from (5.47) on constants $C_1, C_2, \cdots, C_s$, which can be obtained by the elimination method and then, get the solution of the initial value problem

$$\begin{cases} x^{(s)} + p_1(t)x^{(s-1)} + \cdots + p_{s-1}(t)x' + p_s(t)x = 0 \\ x^{(s-1)}(t_0) = x^0_{s-1}, x^{(s-2)}(t_0) = x^0_{s-2}, \cdots, x'(t_0) = x^0_1, x(t_0) = x_0. \end{cases}$$

Particularly, if $p_1(t), p_2(t), \cdots, p_n(t)$ are all constants $p_1, p_2, \cdots, p_n$ and the characteristic equation of (5.46) is

$$\lambda^n + p_1\lambda^{n-1} + \cdots + p_{n-1}\lambda + p_n = 0. \tag{5.48}$$

Without loss of the generality, assume the roots of equation (5.48) distinct one by one are $\lambda_1$ (multiple $m_1$), $\lambda_2$ (multiple $m_2$),$\cdots$,$\lambda_s$(multiple $m_s$) and $m_1 + m_2 + \cdots + m_s = n$. Then, the elements

$$\begin{cases} e^{\lambda_1 t} & te^{\lambda_1 t} & \cdots & t^{m_1-1}e^{\lambda_1 t} \\ e^{\lambda_2 t} & te^{\lambda_2 t} & \cdots & t^{m_2-1}e^{\lambda_2 t} \\ \cdots & \cdots & \cdots & \cdots \\ e^{\lambda_s t} & te^{\lambda_s t} & \cdots & t^{m_s-1}e^{\lambda_s t} \end{cases} \tag{5.49}$$

consist of a linear spanning basis of the solution space of equation (5.46). For example, there are two roots $\lambda_1 = 2, \lambda_2 = 3$ in the characteristic equation $\lambda^2 - 5\lambda + 6 = 0$ of the second order linear homogeneous equation $\ddot{x} - 5\dot{x} + 6x = 0$. Whence, a linear spanning basis of its solution space is $\left\{e^{\lambda_1 t}, e^{\lambda_2 t}\right\}$ and we therefore get its general solution $x(t) = C_1 e^{\lambda_1 t} + C_2 e^{\lambda_2 t}$, where $C_1, C_2$ are constants.

Secondly, consider the linear differential equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + B$ with coefficient matrixes of constants in the case of $m = n \geq 2$, where the matrixes $\mathbf{A}, \mathbf{B}$ are the same matrixes $\mathbf{A}, \mathbf{B}$ as in the equation (5.34). Similarly, assuming the roots of its characteristic equation $\det(\mathbf{A} - \lambda I_n) = 0$ are respectively $\lambda_1$ (multiple $m_1$), $\lambda_2$ (multiple $m_2$),$\cdots$,$\lambda_s$ (multiple $m_s$) and $m_1 + m_2 + \cdots + m_s = n$. Then, the elements

$$\begin{pmatrix} p_{11}(t) \\ p_{12}(t) \\ \vdots \\ p_{1n}(t) \end{pmatrix} e^{\lambda_1 t}, \quad \begin{pmatrix} p_{21}(t) \\ p_{22}(t) \\ \vdots \\ p_{2n}(t) \end{pmatrix} e^{\lambda_2 t}, \cdots, \begin{pmatrix} p_{s1}(t) \\ p_{s2}(t) \\ \vdots \\ p_{sn}(t) \end{pmatrix} e^{\lambda_s t} \tag{5.50}$$

consist of a spanning basis of the solution space of equation (5.34), where for any integers $1 \leq i \leq s, 1 \leq j \leq n$, $p_{ij}(t)$ denotes a polynomial of $t$ with the maximum order$\leq m_i - 1$, $I_n$ is an $n \times n$ unit matrix and $\det(a_{ij})_{n\times n}$ denotes the *determinant* of square $(a_{ij})_{n\times n}$ defined by

$$\det(a_{ij})_{n\times n} = \sum_{i=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij}),$$

where $A_{ij}$ represents the submatrix corresponding to the entry $a_{ij}$ of $A$ by deleing $i$th row and $j$th column of the matrix $(a_{ij})_{n\times n}$ for integers $1 \leq i, j \leq n$. For example, let $n = 2$ or $n = 3$, we respectively know that

$$\begin{aligned}
\det(a_{ij})_{2\times 2} &= a_{11}a_{22} - a_{12}a_{21}, \\
\det(a_{ij})_{3\times 3} &= (-1)^{1+1} a_{11} \det \begin{pmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{pmatrix} \\
&\quad + (-1)^{1+2} a_{12} \det \begin{pmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{pmatrix} + (-1)^{1+3} a_{13} \det \begin{pmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{pmatrix} \\
&= a_{11}(a_{22}a_{33} - a_{23}a_{32}) \\
&\quad - a_{12}(a_{21}a_{33} - a_{23}a_{31}) + a_{13}(a_{21}a_{32} - a_{22}a_{31}) \\
&= a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} \\
&\quad + a_{31}a_{23}a_{12} - a_{31}a_{22}a_{13} - a_{21}a_{12}a_{33} - a_{11}a_{23}a_{32}.
\end{aligned}$$

Generally, by applying (5.50) we can solve the linear differential equations $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + B$ with coefficient matrixes of constants. For example, the values $\lambda_1 = 0, \lambda_2 = \lambda_3 = 1$ are three roots of equation

$$|A - \lambda I_3| = -\lambda(1 - \lambda)^2 = 0$$

which is the characteristic equation of the differential equation

$$\begin{cases} \dot{x}_1 = x_2 + x_3, \\ \dot{x}_2 = x_1 + x_2 - x_3, \\ \dot{x}_3 = x_2 + x_3. \end{cases}$$

So, the elements

$$\begin{pmatrix} 2 \\ -1 \\ 1 \end{pmatrix}, \quad \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} e^t, \quad \begin{pmatrix} t \\ 1 \\ t \end{pmatrix} e^t$$

consist of a spanning basis of its solution space by (5.50).

## §5. **Systemic Stability**

A system is stable or unstable depends on the dynamic behavior of the system. For example, Dr.Ouyang asks Huizi:"*why does water run to the lower?*" The answer is because the higher the water is the greater the kinetic energy and the lower the kinetic energy results in the less. Similarly, a ball moving in an arc is always subject to a downward action of force $G$. Then, it swings to the left and right on the arc and finally stops at the midpoint $A$ of arc, as shown in Figure 5.28. Generally, for a differential equation that constructed on a system $S$, if it is nonlinear the solution of this equation can not be found by an elementary method. In this case, the idea that characterizes the system state of $S$ by the solution $x_v, v \in S$ of its equation may be disappointed. So, *can we determine the system state without solving the differential equation of system $S$?* Dr.Ouyang tells Huizi that this is the problem of system stability. Usually, one perceives a thing by observing its characteristics in practice and builds a mathematical model to hold on the system state by observed data. For the differential equation of a system $S$, as long as the observed data correctly reflect the moving law of the system, the constructed differential equation must contain sufficient information about the stability of system, and does not necessarily require the solution of the system state $x_v, v \in S$ in theory. Dr.Ouyang tells Huizi that there are two topics need to be discussed. One is the dependence of the solution on the initial values, i.e., a little observing error should not lead to a big difference in the system state because an observation always has some error. For this reason, the system state is required to be

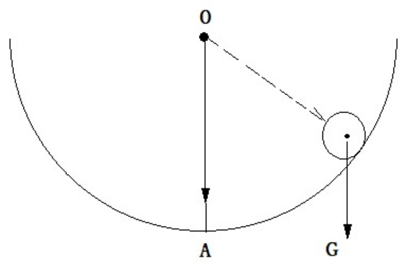



**Figure** 5.28. **Ball moving on arc**

continuously dependent on the observed initial value, namely the solution of the equation is required to be continuous in a neighborhood of the initial value. Another is the observation error of initial value should not affect one's judgment on the system state too much, which includes that if the initial observation difference is small, the deviation of the system state should be also small and they should be consistent in a long run on time, namely the results of judging the system state should be consistent for different observations of the system. Otherwise, it is impossible to hold on the system state of $S$, which is the stability problem of system.

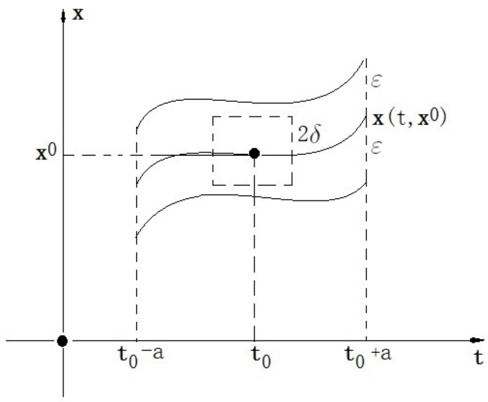



**Figure** 5.29. **Continuous dependence**

**5.1.Continuous Dependence.** Generally, a solution $\mathbf{x}(t)$ of equation (5.39) said to be *continuously depend* on the initial value $\mathbf{x}^0$ refers to the continuousness of $\mathbf{x}(t)$ at the point $\mathbf{x}^0$ on an interval $(t_0 - a, t_0 + a)$, where $a$ is a real number, i.e., for any number $\varepsilon > 0$ there always exists a number $\delta\left(\varepsilon, t_0, \mathbf{x}^0\right) > 0$ such that for any numbers $t_0', \mathbf{x}'^0$ holding with $|t_0' - t_0| < \delta$ and $\left|\mathbf{x}_0' - \mathbf{x}^0\right| < \delta$, there must be such a solution $\mathbf{x}'\left(t, \mathbf{x}'^0\right)$ on the interval $(t_0 - a, t_0 + a)$ that

$$\mathbf{x}'\left(t_0, \mathbf{x}'^0\right) = \mathbf{x}'^0 \text{ and } \left|\mathbf{x}'\left(t, \mathbf{x}'^0\right) - \mathbf{x}\left(t, \mathbf{x}^0\right)\right| < \varepsilon \text{ for } t \in (t_0 - a, t_0 + a). \tag{5.51}$$

Notice that the essence of the solution of equation (5.39) continuously depends on the initial value is the requiring that the solution to be within the set deviation if the deviation of the initial observed value is small or within an interval containing $t_0$, then the solution $\mathbf{x}\left(t, \mathbf{x}'^0\right)$ falls between two curves that are $2\varepsilon$ apart, as shown in Figure 5.29. Only if the recognitive system meets this condition, the measuring of the system state is meaningful and then, the differential equation constructed by the measured data can be compatible with the system state well.

So, *can we only by the behaviors of* $\mathbf{F}(\mathbf{x})$ *in equation of* (5.39) *to determine the solution* $\mathbf{x}\left(t, \mathbf{x^0}\right)$ *is continuous at the initial point* $\mathbf{x}^0$ *or not?* Dr.Ouyang tells his daughter that the answer is Yes! First of all, if $n = 1$ the equation of (5.39) is $\dot{x} = F(x)$ with $x(t_0) = x^0$. In this case, if $F(x)$ on a subset $\mathcal{D} \subset \mathbb{R} \times \mathbb{R}$ is continuous and satisfies the Lipschtz condition on the square with side length $2\delta$ in Figure 5.29, i.e., there is a constant $N$ that for any $(t, x), (t', x') \in \mathcal{D}$ holding with $|F(x') - F(x)| \leq N|x' - x|$ and $(t_0, x^0) \in \mathcal{D}$, there must be a constant $a > 0$ and a solution $x = \varphi(t, x^0)$ on the interval $[t_0 - a, t_0 + a]$ which is continuously dependent on the initial value $x^0$. Particularly, if the condition of $F(x)$ holding with Lipschitz condition for $x$ is further enhanced by the continuously differentiable of $F(x)$ on a neighborhood of the point $t_0$,the conclusion still holds because $F(x)$ is bounded on $\mathcal{D}$ and we can obtain a general criterion on the solution of equation (5.39) to be continuously dependent on the initial value.

Generally, if $m = n \geq 2$, as long as the function $\mathbf{F}(\mathbf{x})$ is continuously differentiable on a neighborhood $U \subset \mathbb{R}^n$ of point $\mathbf{x}^0$, the solution $\mathbf{x}(t)$ of equation of (5.39) must be continuously on the initial value $\mathbf{x}^0$. Furthermore, let $\mathbf{x}(t)$ be a solution of equation (5.39) on the interval $[t_0, t_1]$ and let $\mathbf{x}'(t)$ be another solution of (5.39) satisfying $\mathbf{x}'(t_0) = \mathbf{x}'^0$ and $\mathbf{x}'^0 \in U$, then there must be such a constant $K$ that holds with $|\mathbf{x}'(t) - \mathbf{x}(t)| \leq K \left|\mathbf{x}'^0 - \mathbf{x}^0\right| e^{K(t-t_0)}$ for any $t \in [t_0, t_1]$, where $|\cdot|$ denotes the length or norm of a vector in $\mathbb{R}^n$, i.e., the formula (4.33).

**5.2.System Stability.** Dr.Ouyang explains that the system stability is such a property of system that determines the proximity of a system's initial value to its subsequent state. Usually, a solution $\mathbf{x}$ such that $\mathbf{F}(\mathbf{x}) = \mathbf{0}$ in an autonomous system of equation (5.39) is referred to the *equilibrium point* of system. Apparently, the zero point $\mathbf{0} \in \mathbb{R}^n$ is an equilibrium point of autonomous equation (5.39) and the phase portrait around $\mathbf{0}$ is shown in Figure 5.30. Generally, we can only consider the stability of initial value $\mathbf{0}$ of the equation (5.39) because if the $\mathbf{x}^0 \neq \mathbf{0}$, we can introduce a transformation $\mathbf{x} = \mathbf{x}' - \mathbf{x}^0$ in equation (5.39) and get a new equation $\dot{\mathbf{x}}' = \mathbf{F}(\mathbf{x}')$. So, $\mathbf{0}$ must be an equilibrium point of the equation $\dot{\mathbf{x}}' = \mathbf{F}(\mathbf{x}')$.

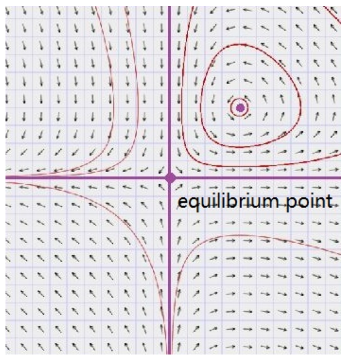



**Figure** 5.30. **Phase portrait**

Generally, let $\mathbf{x}^0 \in \mathbb{R}^n$ be an equilibrium point of the equation $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$. If for any neighborhood $\mathcal{O} \subset \mathbb{R}^n$ of point $\mathbf{x}^0$, there are always a neighborhood $\mathcal{O}_1 \subset \mathcal{O}$ of point $\mathbf{x}^0$ such that the solution $\mathbf{x}(t)$ of equation $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ holding with $\mathbf{x}(t_0) = \mathbf{x}'^0 \in \mathcal{O}_1$ and $\mathbf{x}(t) \in \mathcal{O}$ for $t \geq t_0$, then the solution $\mathbf{x}(t)$ is said to be *stable*, as shown in Figure 5.31($a$).

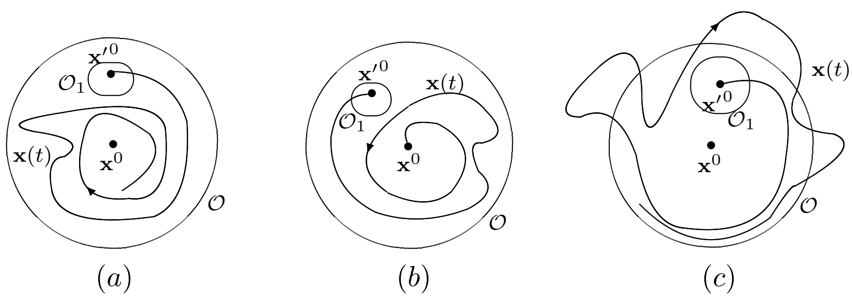



**Figure** 5.31. **Stable or unstable equilibrium point**

Furthermore, if the neighborhood $\mathcal{O}_1$ satisfies $\lim_{t\to\infty} \mathbf{x}(t) = \mathbf{x}^0$, the equilibrium point $\mathbf{x}^0$ is said to be *asymptotically stable*, as sown in Figure 5.31($b$). Conversely, if there is such a neighborhood $\mathcal{O} \subset \mathcal{R}^n$ of point $\mathbf{x}^0$ that for any neighborhood $\mathcal{O}_1$ included in $\mathcal{O}$ of point $\mathbf{x}^0$, there is a solution $\mathbf{x}(t)$ at least holding with $\mathbf{x}(t_0) \in \mathcal{O}_1$ but for $t \geq t_0$, $\mathbf{x}(t)$ is not entirely in the neighborhood $\mathcal{O}$. Such a point $\mathbf{x}^0$ is said to be *unstable*, as shown in Figure 5.31($c$).

Dr.Ouyang explains to his daughter that the cases of $(a) - (c)$ in Figure 5.31 are in fact the cases of solutions to equation (5.39) when $n = 2$, i.e., plane curves. In this case, we can define the stable and unstable points $(x_1(t), x_2(t)) \in \mathbb{R}^n$ of solution $x_1(t), x_2(t)$ in a more accurate way. Let $x_1(t), x_2(t)$ be the solution of equation (5.39) for $t \geq t_0$, and $x_1(t_0) = x_1^0, x_2(t_0) = x_2^0$. By the plane coordinates $(x_1, x_2)$ and rectangular neighborhoods, the stable or asymptotically stable cases of an equilibrium point can be defined as there is a number $\delta = \delta(t_0, \varepsilon)$ for any number $\varepsilon > 0$ such that for two given initial values ${x'}_1^0, {x'}_2^0$ holding with $\left|{x'}_1^0 - x_1^0\right| \leq \delta$, $\left|{x'}_2^0 - x_2^0\right| \leq \delta$ and $t \geq t_0$, there exists solution $x_1'(t), x_2'(t)$ holding with $|x_1'(t) - x_1(t)| < \varepsilon$, $|x_2'(t) - x_2(t)| < \varepsilon$ or

$$\lim_{t\to\infty} \big(x_1'(t) - x_1(t)\big) = 0 \quad \text{and} \quad \lim_{t\to\infty} \big(x_2'(t) - x_2(t)\big) = 0.$$

Certainly, the unstable case of an equilibrium point corresponds to such a number $\delta(t_0, \varepsilon)$ not existing. Dr.Ouyang tells Huizi that the curve of solution $\mathbf{x}(t)$ over time $t$ in space is called the *solution track*. Thus, the solution of the equation (5.39) is stable or unstable like the behavior of tracks in Figure 5.31, i.e., whether it is liking ($a$), ($b$) always inside the neighborhood $\mathcal{O}$ or ($c$) that runs outside $\mathcal{O}$.

So, *how can we determine the solution* $\mathbf{x}(t)$ *of the equation* (5.39) *is stable, asymptotically stable or unstable at an equilibrium point from the properties of function* $\mathbf{F}(\mathbf{x})$*?* Generally, there are two main methods answering this equation. Among them, one is by the properties of function $\mathbf{F}$ such as the Lyapunov criterion and another is by the properties of the equation solution, which is only applicable to the case that the equation (5.39) can be solved by the elementary method.

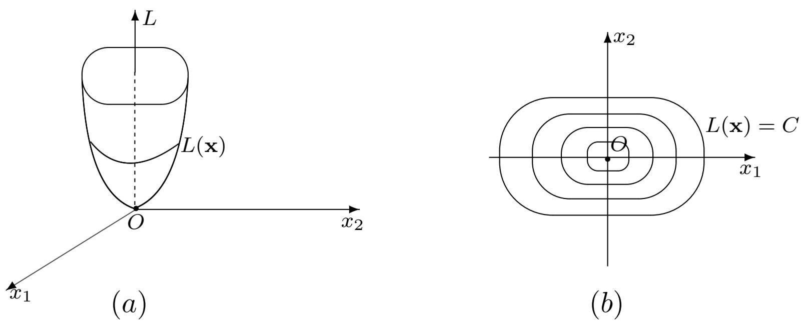



**Figure** 5.32. **Neighborhood of the origin**

**(1)Lyapunov criterion.** Assumes that $\mathbf{x}^0 = \mathbf{0}$ is an equilibrium point of equation $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ and $\Omega : |\mathbf{x}| < h, h > 0$ is a neighborhood of $\mathbf{0}$. The *Lyapunov criterion* is firstly constructed a continuously differentiable function $L(\mathbf{x})$ holding with $L(\mathbf{0}) = 0$ by the properties of $\mathbf{F}$ and such a function $L(\mathbf{x})$ on $\Omega$ is said to be *non-negative (non-positive)* or *positive definite (negative definite)* if for any $\mathbf{x} \in \Omega$, $L(\mathbf{x})$ holds with $L(\mathbf{x}) \geq 0$ ($\leq 0$) or $> 0$ ($< 0$). At this time, by the derivative $\dot{L}(\mathbf{x})$ of $L(\mathbf{x})$ on time $t$ along the solution $\mathbf{x}$ is positive or negative, then determine the stability of equilibrium point $\mathbf{x}^0$.

Dr.Ouyang explains that due to $L(\mathbf{0}) = \mathbf{0}$ and $L(\mathbf{x})$ continuously differentiable on $\Omega$ at this time $t$. In this way, the derivatives $\dot{L}(\mathbf{x}) \leq 0$ or $\geq 0$ can characterize the solution behavior of $\mathbf{x}(t)$ in $\Omega$. For example, let $L(\mathbf{x})$ be a positive definite function on $\Omega$ and let $C > 0$ be a number. Then, $L(\mathbf{x}) = C$ is a neighborhood surrounded the equilibrium point of $\mathbf{0}$, as the case of $n = 2$ shown in Figure 5.32. By the arbitrariness of number $C$, we can let $C > 0$ but sufficiently small in order to meet the curve $L(\mathbf{x}) = C$ in the neighborhood $\Omega$ entirely. Now, if $\mathbf{x}(t)$ is a solution of the equation $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ with an initial value of $\mathbf{x}^0$, then $L(\mathbf{x}) = \mathbf{x}^0$ is also a neighborhood contains $\mathbf{0}$. At this time, applying the chain rule for the derivative, if

$$\frac{dL}{dt} = \sum_{i=1}^{n} \frac{dL}{dx_i}\frac{dx_i}{dt} \leq 0, \tag{5.52}$$

then $L(\mathbf{x})$ is a non-increasing function on $t$ in the neighborhood contained by $L(\mathbf{x}) = L(\mathbf{x}^0)$. Consequently, $L(\mathbf{x}) \leq L(\mathbf{x}^0)$ if $t \geq t_0$, namely $\mathbf{x}(t)$ is entirely in the neighborhood contained by $L(\mathbf{x}) = L(\mathbf{x}^0)$. By this way, we know that the solution $\mathbf{x}(t)$ is stable. Otherwise, if $L(\mathbf{x})$ is not always positive or always negative in $\Omega$, there must be points $\mathbf{x}'$ such that $\dot{L}(\mathbf{x}') > 0$ in any neighborhood of point $\mathbf{0}$. At this time, if $\dot{L}(\mathbf{x}) > 0$, $L(\mathbf{x})$ is a strictly increasing function, can always greater than a positive number, i.e., the $\mathbf{x}(t)$ can not close to $\mathbf{0}$. Whence, the zero-solution $\mathbf{0}$ of equation is unstable.

Following this thinking, Dr.Ouyang tells Huizi that Lyapunov obtained the criterion for the stability of zero-solution $\mathbf{0}$ of $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$, namely, ①*If there exists a positively definite function* $L(\mathbf{x})$ *on* $\Omega$ *and* $\dot{L}(\mathbf{x}) \leq 0$*, the zero-solution* $\mathbf{0}$ *of the equation is stable.* Furthermore, *if* $\dot{L}(\mathbf{x}) < 0$*, then the zero-solution* $\mathbf{0}$ *of the equation is asymptotically stable*; ②*If there exists a always negative (positive) function* $L(\mathbf{x})$ *on* $\Omega$ *and* $\dot{L}(\mathbf{x}) > 0$*, the zero-solution* $\mathbf{0}$ *of the equation is unstable.* In practice, the difficulty of applying the Lyapunov criterion for determining the stability of zero-solution $\mathbf{0}$ of a system equation lies in how to construct the function $L(\mathbf{x})$. Surely, we have no a general method for constructing $L(\mathbf{x})$ but can only get $L(\mathbf{x})$ for a few specific system equation until today. For example, we often choose the system's energy function or the quadratic function $L(\mathbf{x}) = \mathbf{x}A\mathbf{x}^t$ as the function $L(\mathbf{x})$, where $A$ is a positive definite matrix.

For example, replace the ball moving along the arc in Figure 5.28 with a simple pendulum and then, the system equation is $\ddot{\theta} + (g/l)\sin\dot{\theta} = 0$, where $l$ is the length of the pendulum and $g$ is the acceleration of gravity. By introducing a parameter $\omega = \dot{\theta}$, we have

$$\ddot{\theta} + \frac{g}{l}\sin\dot{\theta} = 0 \quad \Rightarrow \quad \begin{cases} \dot{\theta} = \omega, \\ \dot{\omega} = -\dfrac{g}{l}\sin\omega. \end{cases}$$

Clearly, the zero-solution $\mathbf{0}$, i.e., $\theta = \omega = 0$ is a solution of this equation. For the lowest, i.e., the equilibrium point $O$ of the pendulum, let

$$L(\theta, \omega) = \frac{1}{2}\omega^2 + \frac{g}{l}(1 - \cos\theta),$$

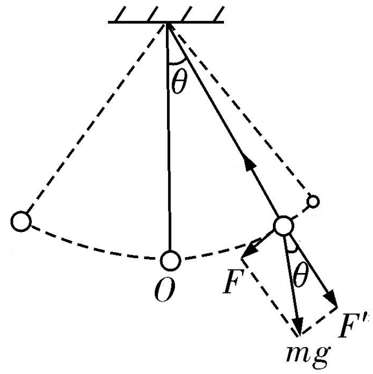



**Figure** 5.33. **Pendulum moving**

i.e., the mechanical energy of the pendulum. Then, it is easily to know that $L(\theta,\omega) > 0$ if $\theta \neq 0$ or $\omega \neq 0$ in the small neighborhood of $(0,0)$ and

$$\frac{dL}{dt} = \omega\dot{\omega} + \frac{g}{l}\dot{\theta}\sin\theta = -\omega\cdot\frac{g}{l}\sin\theta + \frac{g}{l}\omega\sin\theta = 0.$$

Whence, the zero-solution $\mathbf{0}$ or $\theta = \omega = 0$ is a stable point of the system by the Lyapunov criterion, which is consistent with the experimental result that the pendulum is stable at the lowest point $O$.

**(2)Linear system criterion.** For a system $S$ characterized by a linear autonomous equation $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, the Lyapunov criterion $L(\mathbf{x})$ can be similarly constructed to determine whether the equilibrium point $\mathbf{0}$ of the system is stable. However, Dr.Ouyang tells Huizi that there is a more simpler way because by the linear spanning basis (5.49) and (5.50) of solution space, we can obtain the general state of the system $x_v(t), v \in S$ for the linear system $S$.

Notice that for any complex number $\lambda$, let $\lambda = \mathrm{Re}(\lambda) + i\mathrm{Im}(\lambda) = \alpha + i\beta$. By the Euler formula $e^{i\theta} = \cos\theta + i\sin\theta$, we have $p(t)e^{\lambda t} = p(t)e^{\alpha t}\cdot e^{i\beta t} = p(t)e^{\alpha t}\cos(\beta t) + ip(t)e^{\alpha t}\sin(\beta t)$, where $p(t)$ is a polynomial of $t$. Now that

$$\lim_{t\to\infty} p(t)e^{\lambda t} = \lim_{t\to\infty}\left(p(t)e^{\alpha t}\cos(\beta t) + ip(t)e^{\alpha t}\sin(\beta t)\right), \tag{5.53}$$

we know that if $\alpha > 0$, $\lim\limits_{t\to\infty} p(t)e^{\lambda t} \neq 0$, i.e., the equilibrium point $\mathbf{0}$ is unstable and conversely, if $\alpha < 0$, then by applying the L'Hopital's rule in calculus for any integer $n > 0$, we have

$$\lim_{t\to\infty}\frac{t^n}{e^{\alpha t}} = \lim_{t\to\infty}\frac{(t^n)'}{(e^{\alpha t})'} = \frac{n}{\alpha}\lim_{t\to\infty}\frac{t^{n-1}}{te^{\alpha t}} = \cdots = \frac{n!}{\alpha^n}\lim_{t\to\infty}\frac{1}{t^n e^{\alpha t}} = 0. \tag{5.54}$$

Consequently, for any solution $\mathbf{x}(t)$ linear spanned by elements in a basis of (5.50) or (5.49) of the solution space, there must be $\lim_{t\to\infty}\mathbf{x}(t)=\mathbf{0}$, namely the equilibrium point $\mathbf{0}$ is stable. This fact enables us to get the criterion for the stability of linear system, i.e., by the roots of characteristic equation of a linear system $|\mathbf{A}-\lambda I_n|=0$ to directly determine the stability of the equilibrium point $\mathbf{0}$, namely ①*if all the real parts of roots of equation* $|\mathbf{A}-\lambda I_n|=0$ *are negative then the equilibrium point* $\mathbf{0}$ *is asymptotically stable*; ②*if the equation* $|\mathbf{A}-\lambda I_n|=0$ *exists roots of positive real part, the equilibrium point* $\mathbf{0}$ *is unstable*; ③*if there are no positive unless* 0 *real part roots in the equation* $|\mathbf{A}-\lambda I_n|=0$*, the equilibrium point* $\mathbf{0}$ *may be stable or unstable. In this case, if the multiplicity of the root of* 0 *real part is equal to the dimension of its eigenvector space, the equilibrium point* $\mathbf{0}$ *is stable. Otherwise, the equilibrium point* $\mathbf{0}$ *is unstable.*

Generally, *can the stability of an equilibrium point* $\mathbf{x}^*$ *of nonlinear equation* $\dot{\mathbf{x}}=\mathbf{F}(\mathbf{x})$ *be determined by the stability of* $\mathbf{x}^*$ *in its linear approximation* (5.55)*?* In some cases, the answer is Yes! Usually, for a nonlinear systems (5.39), i.e., the equation $\dot{\mathbf{x}}=\mathbf{F}(\mathbf{x})$, the stability of an equilibrium point $\mathbf{x}^*$ can apply its stability in its linear approximation on the neighborhood of (5.33) at $\mathbf{x}^*$, namely

$$\dot{\mathbf{x}}=\mathbf{F}(\mathbf{x}) \quad \Rightarrow \quad \dot{\mathbf{x}}=\left.\frac{\partial \mathbf{F}}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}^*}(\mathbf{x}-\mathbf{x}^*), \tag{5.55}$$

where, $(\partial\mathbf{F}/\partial\mathbf{x})|_{\mathbf{x}=\mathbf{x}^*}$ is the Jacobian matrix at $\mathbf{x}^*$ of $\mathbf{F}$ respect to $\mathbf{x}$, which is a matrix of constants, denoted by $\Lambda_{\mathbf{x}^*}$.

Dr.Ouyang tells Huizi that an equilibrium point $\mathbf{x}^*$ of the equation (5.39) is siad to be *hyperbolic* if the characteristic equation of $|\Lambda_{\mathbf{x}^*}-\lambda I_n|=0$ on this point without roots of 0 real parts. In this case, the stability of equilibrium point $\mathbf{x}^*$ in a nonlinear system (5.39) is consistent with its stability in its linear approximation (5.55). Particularly, *if all real parts of roots in the equation* $|\Lambda_{\mathbf{0}}-\lambda I_n|=0$ *are negative, the equilibrium point* $\mathbf{0}$ *in the nonlinear system* (5.39) *is asymptotically stable.* Conversely, *if the equation* $|\Lambda_{\mathbf{0}}-\lambda I_n|=0$ *has roots of positive real part, the equilibrium point* $\mathbf{0}$ *in* (5.39) *is unstable.* Thus, one can know the stability of the equilibrium point $\mathbf{0}$ from its linear approximation of a nonlinear system. For example, assume that the linear approximation of a nonlinear system is

$$\begin{cases} \dot{x}_1=-2x_1-x_3 \\ \dot{x}_2=-2x_2+x_3 \\ \dot{x}_3=-x_1-x_2-2x_3 \end{cases}$$

with a characteristic equation

$$|\mathbf{A} - \lambda I_3| = \begin{vmatrix} -2-\lambda & 0 & -1 \\ 0 & -2-\lambda & 1 \\ -1 & -1 & -2-\lambda \end{vmatrix} = -(\lambda+2)^3 = 0$$

that only has a root $\lambda = -2$ of multiple 3. So, the equilibrium point $\mathbf{0} = (0,0,0)$ is a stable point of this system.

Dr.Ouyang tells Huizi that if the continuity and stability of a system at the equilibrium point have been determined, the system can be then observed, simulated and controlled for serving the human society. Hearing in this, Huizi asks his father with questions:"*isn't it easier to directly observe the external appearances of things? Why do we need to discuss the systemic recognition? Doesn't it complicate our understanding of things?*" Dr.Ouyang explains that observing the external appearances of things can only establish that of a phenomenological theory and can not predict or control how things will behave in the next.

And then, Dr.Ouyang asks his daughter that if she remembered the fable about the blind man with an elephant. Huizi replies:"*Yes! But there's still one thing I don't understand. In this fable, it is clear that the blind men perceived a wrong shape of the elephant but why did the sophist told the men that what they perceived were all correct? Is this a kind of deception or a white lie?*" Dr.Ouyang tells her:"*You just asked a good question! What the implication of the blind men with an elephant is the limitation of human recognition on things. What the sophist told the blind men was that the shape of an elephant had all characters by each of their senses, i.e., the perception of the elephant's shape should be a combination of their different perceptions on the elephant's shape. Similarly, humans need to adopt this combination notion to understand things, i.e., combine all local recognitions together, systematically understand things and then hold on the true face of things.*" Huizi nods if understand and then says:"*Indeed, a systematic recognition has these advantages which can hold on the dynamic behavior of thing as a whole and know the cause and the effect.*"

Dr.Ouyang continuously explains:"*The system recognition is the fundamental, also correct way for human understanding things. However, it is not easy to establish the systematic knowledge of a thing! First of all, we need to know what a thing is made of and secondly, we need to know all interactions between its elements. It's the only way that we can establish the system equation of a thing. Certainly, we can use elementary particles, cells or genes as the elements of a system. But if we use elementary particles, cells or genes as system elements, the number of elements is very large which results in the system* *constructed is extremely complex. How to solve such a state equation of system? Clearly, it is all good if the state equation can be solved by an elementary method but what if it can't do so or if the equations of the system contradict each other and have no solution? The system is the same system. How do we characterize such a state of system? If these problems are not solved effectively, the notion of humans using systems to understand a thing has its limitation also. Even so, it is a useful way for understanding things in the universe.*"

Dr.Ouyang asks Huizi if she heard the *combinational conjecture for mathematical science*, also known as the "*CC conjecture*" on science before. Huizi answers that she never heard of it. Dr.Ouyang tells her that the combinatorial conjecture for mathematical science is a philosophical thought on science by the combinatorial notion of things, which concludes that the mathematics with all sciences based on mathematics could be combinatorially reconstructed or generalized by choice a finite number of combinatorial rules on elements so as the tools or methods for human recognition, i.e., abstracting a spatial 1-dimensional topological structure or embedded graph inherited in a thing. It endows elements in classical mathematics such as the number, array, function, vector, matrix and other mathematical element on the graph, regards them as the energy flow on the topological structure, follows the conservation law of matter and uses this weighted topological graph as a mathematical element to extend the classical mathematics, called the *mathematical combinatorics*. This is the way to solve the limitations of human recognition of thing and realizes the recognition from the part to the whole which is pointed out by the enlightenment of the fable that the blind man with an elephant because the process of human recognition of thing is equivalent to the blind man feeling the elephant, which is a realizing process of humans out from non-being to being.

## §6. **Comments**

**6.1.** The system recognition of thing is the representative of reductionism, i.e., a thing is reduced to a system composed of elements and the behavior of the thing is characterized by the behavior of elements. Notice that the *elements* are not necessarily the elementary particles in the building blocks of matter or the cells and genes in the building blocks of living things but they are the smallest units needed to characterize how things behave. In this way, a thing is equivalent to a system and the system structure corresponds to a labeled graph composed of elements. The equation of element states in a unified dimension constitutes the state equation of system and the system equation is formed by combining the input and output equations of the system, see [Mia] and [YGH].

**6.2.** The solution of the system equation is equivalent to solving a system consisting of state equation, input and output equations. Generally, the purpose of solving the system equation is to hold on the system state by solution of the equation. But there are no a unified method for solving nonlinear equations, only the linear systems or a few special equations can be found the exact solution in general. For the nonlinear system equations, one often applies the multiple Taylor series to expand the nonlinear terms, take the linear part of approximate simulation and then judge the properties of solution. Certainly, the stability of the system can also be directly determined by Lyapunov criterion and so on, see literatures [Gof], [BrG], [HSD], [Lyn] and [Mia], etc., particularly the [HSD] for details.

**6.3.** The combinational conjecture for mathematical science is the mathematical representation of the perceiving things by systems, first announced in the foreword to Chapter 5 of the first edition of my post-doctoral report for the *Chinese Academy of Sciences* [Mao1] or [Mao21] in 2005. Later, by the 2-cell embedding of graphs on surfaces [Liu1], [MoT], [Whi] and surface subdivision [MaT1] as special cases, I finished the literature [Mao8] that proposed the combinatorial conjecture for mathematical science, reported in the *Second National Conference on Combinatorics and Graph Theory of China* (Tianjing, August 16-19, 2006) and discussed with the participating scholars, also communicated with several foreign professors on the internet after then, which was affirmed by the professor Lovasz who was the president of the *International Union of Mathematicians* in that time. And then, I had written a series of papers, such as [Mao9]-[Mao20], [Mao23], [Mao24] -[Mao25] and [Mao37] by this conjecture, contributed to the spread of combinatorial notion in the development of mathematical science, which motivated the foundation of the *International Journal of Mathematical Combinations* in United States by proposal of my friend in 2007. Notice that one chapter on this conjecture is included in the second edition of [Mao21] and my book *Combinatorial Geometry and Field Theory* [Mao23] is motivated by this conjecture which includes the combinatorial geometry, algebra, topology and fields, etc., received widespread attention in the mathematical community after its publication. Particularly, it is worth mentioning that [Mao8] become a reference later for interpreting the terminology "*kombinatorika*" in the Wikipédia, which is the term of "*combinatorics*" in Hungarian.

Certainly, the combinatorial conjecture for mathematical science is not so much a mathematical conjecture name as a philosophical thought corresponding to the combinatorial notion of things [Mao56], aiming at promoting human recognition of things in the universe and developing science. Notice that this conjecture was formally proposed in literature [Mao8] with the example of the 2-cell embedding of graphs on surfaces, namely the combinatorial subdivision of surface and asserted that "*any mathematical science can be reconstructed from or made by combinatorialization*", which leads in a developing direction of mathematical sciences for us, i.e., reconstruct and extend classical mathematics by selecting finite combinatorial rules or axioms with elements, establish an envelope mathematics called the *mathematical combinatorics* which contains different branches of mathematics over a combinatorial structure or topological graph for realizing the combinatorialization or reconstruction of science and then, to promote the progress of human civilization.

*There are many wonderful things in nature, but the most wonderful of all is man.*

– By Sophocles, an ancient Greek dramatist

## Chapter 6

# System Synchronization

**********************************************************************

**Waves Seem Intentionally as Snows Layer by Layer,**

**Peach's and Plum's Flowers Show Spring Silently Row by Row,**

**Hands Hold Bamboo Pole with Wine Pot Hanging on Body,**

**How Many Persons Happy so as me in World?**

— *Fisherman* by Li Yu, a poet in Tang dynasty of China

************ *Linfan Mao. Combinatorial Theory on the Universe* *************

## §1. **Synchronization**

Generally, one describes the "*synchronization*" as a neat and consistent phenomenon in which different things keep a fixed relative relationship in their evolution or carrying out at the same time. Dr.Ouyang tells Huizi that the synchronization is universal in the nature and human society which has a wide range of practical implications. For example, the neat and coordinated posture, arms holding, alignment, lift foot, pace, slogans and marching speed in the honour guard of "*army, navy, air force*" during the parade reflect the strict discipline and well-trained army. Certainly, the neat and consistent coordination phenomenon of synchronization is a series of pictures reflected in the eyes of human such as $(a)$ the martial arts performance, $(b)$ the flight show and $(c)$ the rice that are about to mature shown in Figure 6.1, which are all beautiful patterns.

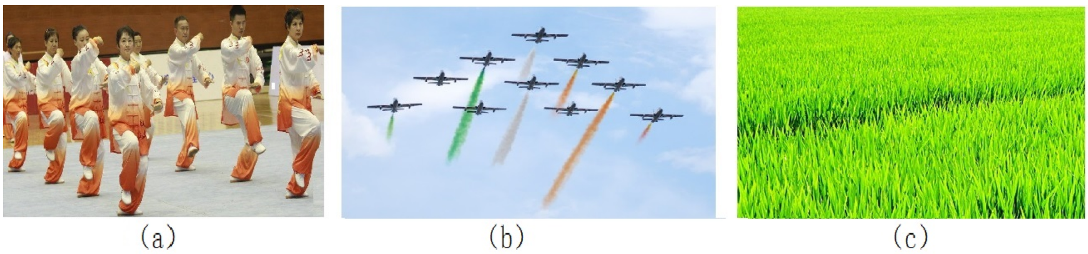



**Figure** 6.1. **Synchronization phenomenon**

Usually, the *system synchronization* refers to that the system elements maintain a relatively fixed relationship or state in the process of evolution such as the moving coordination and formation in Figure 6.1$(a)$ and $(b)$, and the consistency of height, shape, maturity and the color of each rice plant in $(c)$. Dr.Ouyang asks Huizi: "*Do you want to applaud when the school leaders finish their speech at the commendation conference?*" Huizi replies: "*Certainly, it's what the teacher asked before the conference.*" Dr.Ouyang continuously asks her: "*Then, how do you know in what time or what situation that you need to clap?*" Huizi answers: "*Generally, at the end of a speech there are closing words or actions, then it is good to clap. But it is difficult to judge when I need to clap in the progress of speech. I can only follow the students next to me.*" Dr.Ouyan then asks her: "*Follow your classmates next to clap? Is that a round of applause?*" Huizi tells her father: " *Usually, the applause is uneven in the first few seconds but the applause is very neat after then.*" Dr.Ouyang then tells her that if we view the set consisting of students and teachers in the conference as a system, each student is an element of the system and so each teacher. So, a nest applauding of the teachers and students at the commendation conference is a kind of system synchronization. For obtaining this effect, when some important conferences are organized, a few secretaries will be specially arranged to applaud on purpose so as to promote the unanimous applause and form a synchronized applause from the audience, as shown in Figure 6.2. So *how do we characterize the phenomena that of system synchronization?* Dr.Ouyang explains to Huizi that this requires a scientific definition of system synchronization in first.

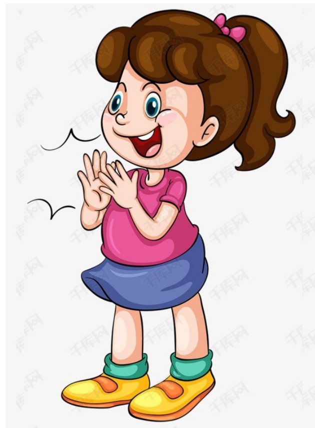


**Figure** 6.2. **Applause**

**1.1.Synchronous Tracks.** A system $S$ consists of elements. In this way, the evolution of system $S$ with time $t$, namely the system state is characterized by the element track $x_v(t), v \in S$. Correspondingly, the synchronization is a special state of elements in system $S$, which likes the previous applause for a leader's speech, the elements may not be consistent at the beginning but with a short passage of time, the evolution of elements tends to be consistent, i.e., for any $v \in S$, $\mathbf{x}_v(t) = \mathbf{F}(\mathbf{x}_v(t))$, where $\mathbf{F}(\mathbf{x}_v)(t)$ is the synchronous function of element $v$ state after the time $t_0$, as shown in Figure 6.3.

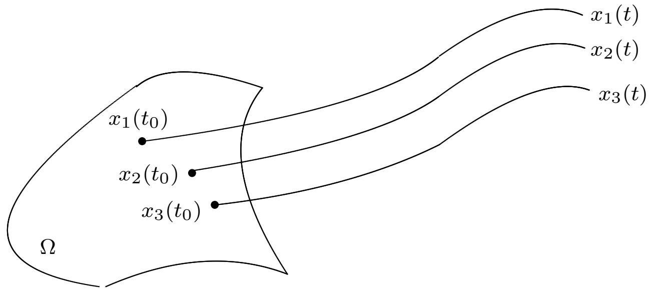



**Figure** 6.3. **Synchronous tracks**

If the tracks of system elements are synchronized, then for any two elements $v, u \in S$, $\mathbf{x}_u(t) - \mathbf{x}_v(t) = \mathbf{x}_u(t_0) - \mathbf{x}_v(t_0) = \mathbf{c}$ is a constant vector, i.e., $\mathbf{x}_u(t) = \mathbf{x}_v(t) + \mathbf{c}$ or $\mathbf{x}_u(t)$ is a translation of track $\mathbf{x}_v(t)$ by $\mathbf{c}$. Thus, the state function of any element in the system can be expressed as

$$\mathbf{x}_u(t) = \mathbf{x}_v(t) + \mathbf{c}_u + \mathbf{o}_u(t), \quad u \in S, \tag{6.1}$$

where, $\mathbf{o}_v(t) \to \mathbf{0}$ if $t \to \infty$ which is a high order infinitesimal of $t$, $\mathbf{c}_u$ is a constant vector dependent on the element $u$. Notice that the synchronous track $\mathbf{x}_v(t), v \in S$ is the action result of system elements. Accordingly, the synchronous track can be generally constructed as: *for a system* $S$, *if*

$$\mathbf{x}_u = \mathbf{x}_v(t) + \mathbf{c}_u + \mathbf{o}_v(t)$$

*for any element* $u \in S$, $u \neq v$, *the tracks of elements in* $S$ *consist of a synchronous tracks of system* $S$. For example, let

$$\begin{aligned} x_{v_1}(t) &= 1 + t^5 \sin t + \frac{\cos^2 t}{t}, \\ x_{v_2}(t) &= 3 + t^5 \sin t + te^{-t}, \\ x_{v_3}(t) &= 5 + t^5 \sin t + \frac{6}{t^2}, \\ x_{v_4}(t) &= 7 + t^5 \sin t + 3t^5 e^{-2t} \end{aligned}$$

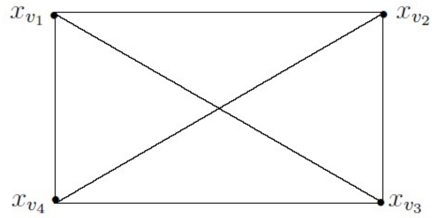



**Figure** 6.4. **A** 4**-system**

be a 4-system $S$ shown in Figure 6.4. Then, for integers $2 \leq i \leq 4$ there are

$$\begin{aligned} x_{v_2}(t) &= 2 + x_{v_1}(t) + \left( te^{-t} - \frac{\cos^2 t}{t} \right) = 2 + x_{v_1}(t) + o(t), \\ x_{v_3}(t) &= 4 + x_{v_1}(t) + \left( \frac{6}{t^2} - \frac{\cos^2 t}{t} \right) = 4 + x_{v_1}(t) + o(t), \\ x_{v_4}(t) &= 6 + x_{v_1}(t) + \left( 3t^5 e^{-2t} - \frac{\cos^2 t}{t} \right) = 6 + x_{v_1}(t) + o(t). \end{aligned}$$

Whence, the element tracks in $S$ consist of a synchronous tracks by definition. However, if the state functions of elements in Figure 6.3 are chosen to be

$$\begin{aligned} \mathbf{x}_{v_1}(t) &= t \sin t + 2t^2, \quad \mathbf{x}_{v_2}(t) = t \sin t + te^{-t}, \\ \mathbf{x}_{v_3}(t) &= t \sin t + t^2 e^{-t}, \quad \mathbf{x}_{v_4}(t) = t \sin t + e^{-2t} \cos t, \end{aligned}$$

then, $|\mathbf{x}_{v_1}(t) - \mathbf{x}_{v_2}(t)| = \left|2t^2 - te^{-t}\right| \to \infty$, $|\mathbf{x}_{v_1}(t) - \mathbf{x}_{v_3}(t)| = \left|2t^2 - t^2 e^{-t}\right| \to \infty$ and $|\mathbf{x}_{v_1}(t) - \mathbf{x}_{v_4}(t)| = \left|2t^2 - e^{-2t} \cos t\right| \to \infty$ if $t \to \infty$, i.e, the element tracks do not consist of a synchronous track of $S$.

The previous examples show that if the states of the system elements $\mathbf{x}_v(t), v \in S$ are known, it is not difficult to determine whether the elements of system $S$ constitute a synchronous track. The problem lies in that we observe the system state of $S$ in a certain or several periods of time and can not directly know the element state of $\mathbf{x}_v(t)$. So, it is necessary to solve the state equation of system established by the observed data. However, for a nonlinear system of state equations, we can not usually apply an elementary method for obtaining solution $\mathbf{x}_v(t), v \in S$. In this case, *how can we tell if a system is synchronous or not?* Dr.Ouyang explains that the structural characteristics of the equation are usually applied to analyze. Usually, assume that $S$ consists of elements $v_1, v_2, \cdots, v_m$, $|S| = m$.

Then, a typical feature of synchronous tracks is that there is such a time $t^*$ that makes $\dot{\mathbf{x}}_v(t) = \mathbf{F}(\mathbf{x}_n(t))$, $v \in S$ if $t \geq t^*$, namely all elements have the same state equation after time $t^*$. In this case, the system $S$ can be characterized by the state of any element and we can abstract the system $S$ as a particle with its equation as the state equation of $S$, namely if $t \geq t^*$, the state equation of system $S$ is given by

$$\begin{cases} \dot{\mathbf{x}}_{v_1}(t) = \mathbf{f}(\mathbf{x}_{v_1}(t)) + \mathbf{y}_{v_1}(t) \\ \dot{\mathbf{x}}_{v_2}(t) = \mathbf{f}(\mathbf{x}_{v_2}(t)) + \mathbf{y}_{v_2}(t) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dot{\mathbf{x}}_{v_m}(t) = \mathbf{f}(\mathbf{x}_{v_m}(t)) + \mathbf{y}_{v_m}(t) \end{cases} \Rightarrow \begin{cases} \dot{\mathbf{x}}_{v_1}(t) = \mathbf{f}(\mathbf{x}_{v_1}(t)) \\ \dot{\mathbf{x}}_{v_2}(t) = \mathbf{f}(\mathbf{x}_{v_2}(t)) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \dot{\mathbf{x}}_{v_m}(t) = \mathbf{f}(\mathbf{x}_{v_m}(t)) \end{cases}$$

or shortly denoted by

$$\dot{\mathbf{x}}_v(t) = \mathbf{f}(\mathbf{x}_v(t)), \quad v \in S. \tag{6.2}$$

where, $\mathbf{f} = \dot{\mathbf{F}}$, $\mathbf{y}_v(t) = d(\mathbf{c}_v + \mathbf{o}_v(t))/dt = \dot{\mathbf{o}}_v(t)$ and $\mathbf{y}_v(t) \to \mathbf{0}$ if $t \to \infty$ which is a higher order infinitesimal of $t$. For any element $v \in S$, the equation (6.2) corresponds to the state equation of elements $v \in S$. As the system elements are in a synchronous track, there is no interaction between the tracks of elements. In this case, the equation (6.2) actually corresponds to a dynamical node equation, namely

$$\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)). \tag{6.3}$$

where, $\mathbf{f} \in \mathbb{R}^m, \mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_m), \mathbf{x}_i \in \mathbb{R}^n$. Then, *does the general solution* $\overline{\mathbf{x}}$ *of synchronous system* (6.2) *satisfy the equation* (6.3)*?* Consider a linear system as an example. For any selected $m$ constants $C_1, C_2, \cdots, C_m$, a general solution $\overline{\mathbf{x}}(t) = C_1\mathbf{x}_1(t) + C_2\mathbf{x}_2(t) + \cdots + C_m\mathbf{x}_m(t)$. In this case, if $\mathbf{f}$ is a linear function, by the equation

$$\begin{cases} \dot{\mathbf{x}}_{v_1}(t) = \mathbf{f}(\mathbf{x}_{v_1}(t)) \\ \dot{\mathbf{x}}_{v_2}(t) = \mathbf{f}(\mathbf{x}_{v_1}(t)) \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \dot{\mathbf{x}}_{v_m}(t) = \mathbf{f}(\mathbf{x}_{v_m}(t)) \end{cases} \tag{6.4}$$

we have

$$\dot{\overline{\mathbf{x}}}(t) = \sum_{i=1}^{m} C_i\dot{\mathbf{x}}_i(t) = \mathbf{f}\left(\sum_{i=1}^{n} mC_i\mathbf{x}_i(t)\right) = \mathbf{f}(\overline{\mathbf{x}}(t)), \tag{6.5}$$

i.e., the general solution $\overline{\mathbf{x}}(t)$ is also a solution of equation (6.3). Conversely, let $\mathbf{x}_1(t), \mathbf{x}_2(t), \cdots, \mathbf{x}_m(t)$ be $m$ linearly independent solutions of equation (6.3), namely they are all satisfying the equation (6.4) and let $\mathbf{x}_1(t), \mathbf{x}_2(t), \cdots, \mathbf{x}_m(t)$ be the state function of elements $v_1, v_2, \cdots, v_m$ in system $S$. Then, they satisfy the equation (6.2), i.e., the tracks of elements constitute a synchronous track of $S$. In other words, add $m$ infinitesimal vectors

$\mathbf{o}_1(t), \mathbf{o}_2(t), \cdots, \mathbf{o}_m(t)$ respectively on the basis of $m$ independent solutions of equation (6.3), i.e., $\mathbf{x}_1(t)+\mathbf{o}_1(t), \mathbf{x}_2(t)+\mathbf{o}_2(t), \cdots, \mathbf{x}_m(t)+\mathbf{o}_m(t)$, we get a synchronous track of $S$ consisting of element tracks of $S$ by (6.1), where $\mathbf{o}_i(t)$ denotes a higher order infinitesimal vector of $t$ for integers $1 \leq i \leq m$.

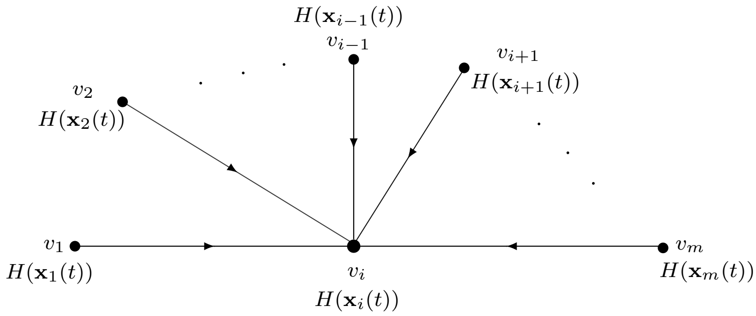



**Figure** 6.5. **Element action**

**1.2.Synchronous Equation.** Let $v_1, v_2, \cdots, v_m$ be $m$ elements consisting of a synchronous system $S$ with respective states $\mathbf{x}_1(t), \mathbf{x}_2(t), \cdots, \mathbf{x}_m(t)$. Dr.Ouyang tells Huizi that the characteristic of a synchronous track is that the state of each element $v$ can be decomposed into two parts, namely the node function $\mathbf{f}\left(\mathbf{x}_v(t)\right)$ and an infinitesimal vector $\mathbf{o}_v(t)$. Accordingly, for any integer $1 \leq i, j \leq m$, the effect of element $v_j$ acting on $v_i$ can be also decomposed into two parts, i.e., one part is its contribution to $\mathbf{f}\left(\mathbf{x}_{v_i}(t)\right)$, another part is its contribution to the infinitesimal vector $\dot{\mathbf{o}}_{v_i}(t)$. At this time, if we view the action of an element on another element is analogical to the potential energy, then the acting strength of $v_j$ on $v_i$ can be described by $\mathbf{H}(\mathbf{x}_j(t)) - \mathbf{H}(\mathbf{x}_i(t))$, as shown in Figure 6.5. Thus, the state equation of elements in the system $S$ can be generally characterized by equations

$$\dot{\mathbf{x}}_i(t) = \mathbf{f}(\mathbf{x}_i(t)) + c\sum_{j=1}^{m} a_{ij}\left(\mathbf{H}(\mathbf{x}_j(t)) - \mathbf{H}(\mathbf{x}_i(t))\right), \quad 1 \leq i \leq m \tag{6.6}$$

where, $c > 0$ is the coupling strength of network, $a_{ij}$ is the entry at the position $(i,j)$, $1 \leq i, j \leq m$ in the adjacency matrix $M(G) = (a_{ij})_{m\times m}$ of labeled graph $G^L[S]$, defined by

$$\begin{cases} a_{ij} = 1, & (v_i, v_j) \in E\left(G^L[S]\right), \\ a_{ij} = 0, & (v_i, v_j) \notin E\left(G^L[S]\right). \end{cases} \tag{6.7}$$

For example, for an integer $m \geq 1$, the adjacency matrixes $M(\overrightarrow{C}_m), M(K_m)$ of a directed cycle $\overrightarrow{C}_m$, a complete graph $K_m$ are respectively

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 1 & \cdots & 1 & 0 & 1 \\ 1 & 1 & \cdots & 1 & 1 & 0 \end{pmatrix}.$$

Among them, $M(K_m)$ is a symmetrical matrix, namely for any integers $1 \leq i, j \leq m$, there are $a_{ij} = a_{ji}$ and generally, the adjacency matrix of a graph $G$ without direction must be a symmetrical matrix by definition. However, for an asymmetric directed graph $\overrightarrow{G}$, namely there are vertices $v, u \in V(\overrightarrow{G})$ such that $(v, u) \in E(\overrightarrow{G})$ but $(u, v) \notin E(\overrightarrow{G})$, such a property no longer holds, i.e., $M(\overrightarrow{G})$ is an asymmetric matrix, such as the adjacency matrix $M(\overrightarrow{C}_m)$ of the directed cycle $\overrightarrow{C}_m$.

Particularly, for a system $S$ whose inherited combinatorial structure of elements is $\overline{C}_m$ with coupling strength $c = 1$ or $K_m$ with coupling strength $c = 2$, the state equation (6.6) of system is

$$\begin{cases} \dot{\mathbf{x}}_1(t) & = & \mathbf{f}(\mathbf{x}_1(t)) + \mathbf{H}\left(\mathbf{x}_2(t)\right) - \mathbf{H}\left(\mathbf{x}_1(t)\right) \\ \dot{\mathbf{x}}_2(t) & = & \mathbf{f}(\mathbf{x}_2(t)) + \mathbf{H}\left(\mathbf{x}_3(t)\right) - \mathbf{H}\left(\mathbf{x}_2(t)\right) \\ \cdots & \cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dot{\mathbf{x}}_m(t) & = & \mathbf{f}(\mathbf{x}_m(t)) + \mathbf{H}\left(\mathbf{x}_1(t)\right) - \mathbf{H}\left(\mathbf{x}_m(t)\right) \end{cases}$$

or

$$\begin{cases} \dot{\mathbf{x}}_1(t) & = & \mathbf{f}(\mathbf{x}_1(t)) + 2\sum\limits_{j=1}^{m} \mathbf{H}\left(\mathbf{x}_j(t)\right) - 2m\mathbf{H}\left(\mathbf{x}_1(t)\right) \\ \dot{\mathbf{x}}_2(t) & = & \mathbf{f}(\mathbf{x}_2(t)) + 2\sum\limits_{j=1}^{m} \mathbf{H}\left(\mathbf{x}_j(t)\right) - 2m\mathbf{H}\left(\mathbf{x}_2(t)\right) \\ \cdots & \cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \dot{\mathbf{x}}_m(t) & = & \mathbf{f}(\mathbf{x}_m(t)) + 2\sum\limits_{j=1}^{m} \mathbf{H}\left(\mathbf{x}_j(t)\right) - 2m\mathbf{H}\left(\mathbf{x}_m(t)\right) \end{cases}$$

Formally, the equation (6.6) can be further simplified by introducing the negative Laplacian matrix $L = (l_{ij})_{m\times m}$, i.e., for any integers $1 \leq i, j \leq m$, the entry at $l_{ij}$ is determined by

$$l_{ij} = \begin{cases} a_{ij}, & i \neq j \\ -\sum\limits_{j\neq i} a_{ij}, & i = j \end{cases}$$

then the equation (6.6) can be simplified as

$$\dot{\mathbf{x}}_i(t) = \mathbf{f}(\mathbf{x}_i(t)) + c\sum_{j=1}^{m} l_{ij}\mathbf{H}(\mathbf{x}_j(t)), \quad 1 \leq i \leq m. \tag{6.8}$$

**1.3.Completely Synchronized.** Generally, let $S$ be a system consisting of $m$ elements $v_1, v_2, \cdots, v_m$ characterized by $\mathbf{x}_1(t), \mathbf{x}_2(t), \cdots, \mathbf{x}_m(t)$. For any integers $1 \leq i \neq j \leq m$ and any initial point in the solution domain $\Omega$, if the tracks of elements of $S$ tend to be the same when $t \to \infty$, i.e.,

$$\mathbf{x}_i(t) - \mathbf{x}_j(t) \to \mathbf{0} \quad \Leftrightarrow \quad \lim_{t\to\infty} |\mathbf{x}_i(t) - \mathbf{x}_j(t)| = 0, \tag{6.9}$$

such a system $S$ is said to be *completely synchronized*, where $|\mathbf{a}|$ denotes the length of vector $\mathbf{a}$ or norm (4.33). Notice that the synchronous track depends on the initial value and it is the difference in the starting points that leads to the difference in the element tracks. But the complete synchronization of system is a convergence property of the system entirely, which only depends on the general solution of the equation. In this case, if define a set

$$\mathrm{Sta}[S] \;=\; \{\; (\mathbf{x}_1(t), \mathbf{x}_2(t), \cdots, \mathbf{x}_m(t)) \mid \mathbf{x}_1(t) = \mathbf{x}_2(t) = \cdots = \mathbf{x}_m(t)\} \tag{6.10}$$

of points, then the general solution of equation (6.6) converges to the point set $\mathrm{Sta}[S]$, i.e., $\lim\limits_{t\to\infty} (\mathbf{x}_1(t), \mathbf{x}_2(t), \cdots, \mathbf{x}_m(t)) \in \mathrm{Sta}[S]$. Dr.Ouyang explains further that this is a meaningful abstraction because one can derives the synchronous property of system $S$ from the property of the point set $\mathrm{Sta}[S]$.

For a linear system (6.4), we have verified in (6.5) that its general solution satisfies the node equation (6.3). So, *is there a combination* $\overline{\mathbf{x}}$ *of solutions* $\mathbf{x}_i(t)$, $1 \leq i \leq m$ *for a nonlinear systems* (6.6) *that satisfies the node equation* (6.3)*?* Certainly, different from the case of linear systems, we need $\mathbf{f}_i(\mathbf{x}_i(t)), 1 \leq i \leq m$ satisfying certain conditions. Notice that the Laplace matrix $L$ has a row sum of 0, i.e., its row vectors are linearly dependent. At this time, its characteristic equation $|L - \lambda I_m| = 0$ exists a solution $\lambda = 0$. Furthermore, there are non-negative and non-zero solutions $\boldsymbol{\xi} = (\xi_1, \xi_2, \cdots, \xi_m)$, namely $\xi_i \geq 0,\ 1 \leq i \leq m$ and not all equal to 0 to the linear equation $\mathbf{X} \cdot L = \mathbf{0}$ if there are $m-1$ linearly independent rows in $L$. Without loss of generality, assume that

$$\xi_1 + \xi_2 + \cdots + \xi_m = 1$$

because if $\xi_1 + \xi_2 + \cdots + \xi_m = K > 1$ we can replace $\xi_1, \xi_2, \cdots, \xi_m$ by $\xi_1/K, \xi_2/K, \cdots, \xi_m/K$ and then, there must be

$$\frac{\xi_1}{K} + \frac{\xi_2}{K} + \cdots + \frac{\xi_m}{K} = 1.$$

Define the weighted average of $\mathbf{x}_1(t), \mathbf{x}_2(t), \cdots, \mathbf{x}_m(t)$ by

$$\overline{\mathbf{x}} \;=\; \sum_{k=1}^{m} \xi_k \mathbf{x}_k(t). \tag{6.11}$$

Now that if $t \to \infty$, we have

$$\begin{aligned} 0 \le |\mathbf{x}_i(t) - \overline{\mathbf{x}}| &= \left| \mathbf{x}_i(t) - \sum_{k=1}^{m} \xi_k \mathbf{x}_k(t) \right| \\ &= \left| \sum_{k=1}^{m} \xi_k \left( \mathbf{x}_i(t) - \mathbf{x}_{v_k}(t) \right) \right| \le \sum_{k=1}^{m} \xi_k \left| \mathbf{x}_i(t) - \mathbf{x}_{v_k}(t) \right| \to 0, \end{aligned}$$

i.e., $\lim_{t\to\infty} |\mathbf{x}_i(t) - \overline{\mathbf{x}}| = 0$. Conversely, if $\lim_{t\to\infty} |\mathbf{x}_i(t) - \overline{\mathbf{x}}| = 0$ for integers $1 \le i \le m$, define

$$N_{\max} = \max\left\{ \ |\mathbf{x}_i(t) - \overline{\mathbf{x}} \ || \ 1 \le i \le m \right\}.$$

Then, $\lim_{t\to\infty} N_{\max} = 0$. Meanwhile, applying the triangle inequality (4.35), i.e., $|\boldsymbol{\alpha} + \boldsymbol{\beta}| \le |\boldsymbol{\alpha}| + |\boldsymbol{\beta}|$ we know that

$$\begin{aligned} |\mathbf{x}_i(t) - \mathbf{x}_j(t)| &= |\mathbf{x}_i(t) - \overline{\mathbf{x}}(t) + \overline{\mathbf{x}}(t) - \mathbf{x}_j(t)| \\ &\le |\mathbf{x}_i(t) - \overline{\mathbf{x}}(t)| + |\overline{\mathbf{x}}(t) - \mathbf{x}_j(t)| \le 2N_{\max}. \end{aligned}$$

Thus, there must be $|\mathbf{x}_i(t) - \mathbf{x}_j(t)| \to 0$ if $t \to 0$ in the previous inequality. We therefore get an equivalent condition on system synchronization, i.e., *a system $S$ synchronizes if and only if it has*

$$\lim_{t\to\infty} |\mathbf{x}_i(t) - \overline{\mathbf{x}}(t)| = 0 \tag{6.12}$$

*for any integer $1 \le i \le m$.*

Dr.Ouyang tells Huizi that by the equivalent condition (6.12) of system synchronization we could obtain the condition for the solution of node equation (6.3). Similar to the stability discussing of an equilibrium point, by the property of function $\mathbf{f}$ in the node equation, it can be divided into two cases following:

**(1)Linear synchronization.** A synchronous system is said to be *linear* if the function $\mathbf{f}$ in the equation (6.8) is linearly homogeneous and has $\dot{\overline{\mathbf{x}}}(t) = \mathbf{f}(\overline{\mathbf{x}}(t))$. So, *why the weighted average $\overline{\mathbf{x}}$ of $\mathbf{x}_1(t), \mathbf{x}_2(t), \cdots, \mathbf{x}_m(t)$ is a solution of the node equation* (6.3)*?* At this time, because the $\boldsymbol{\xi}$ is a non-zero solution of linear algebraic equation $\mathbf{X} \cdot L = \mathbf{0}$, namely $\boldsymbol{\xi} \cdot L = \mathbf{0}$, we have

$$\begin{aligned} \dot{\overline{\mathbf{x}}}(t) &= \sum_{i=1}^{m} \xi_i \dot{\mathbf{x}}_i(t) = \sum_{i=1}^{m} \xi_i \left( \mathbf{f}(\mathbf{x}_i(t)) + c \sum_{j=1}^{m} l_{ij} \mathbf{H}(\mathbf{x}_j(t)) \right) \\ &= \sum_{i=1}^{m} \xi_i \mathbf{f}(\mathbf{x}_i(t)) + c \sum_{i=1}^{m} \sum_{j=1}^{m} \xi_i l_{ij} \mathbf{H}(\mathbf{x}_j(t)) \\ &= \mathbf{f}\left( \sum_{i=1}^{m} \xi_i \mathbf{x}_i(t) \right) + c\left(\xi_1, \xi_2, \cdots, x_m\right) L \left( \mathbf{H}\left(\mathbf{x}_1(t)\right), \cdots, \mathbf{H}\left(\mathbf{x}_m(t)\right) \right)^t \end{aligned}$$

$$\begin{aligned} &= \mathbf{f}\left(\sum_{i=1}^{m}\xi_i\mathbf{x}_i(t)\right) + c\cdot(\boldsymbol{\xi}\cdot L)\cdot\left(\mathbf{H}\left(\mathbf{x}_1(t)\right),\cdots,\mathbf{H}\left(\mathbf{x}_m(t)\right)\right)^t \\ &= \mathbf{f}\left(\sum_{i=1}^{m}\xi_i\mathbf{x}_i(t)\right) + c\cdot\mathbf{0}\cdot\left(\mathbf{H}\left(\mathbf{x}_1(t)\right),\cdots,\mathbf{H}\left(\mathbf{x}_m(t)\right)\right)^t = \mathbf{f}(\overline{\mathbf{x}}(t)), \end{aligned}$$

by the equation (6.8), $\overline{\mathbf{x}}$ is a solution of the node equation, i.e., $\dot{\overline{\mathbf{x}}}(t) = \mathbf{f}(\overline{\mathbf{x}}(t))$.

**(2)Lipschitz synchronization.** A synchronous system is called *Lipschitz* if the function $\mathbf{f}$ in equation (6.8) satisfies the Lipschitz condition, i.e., there exists a constant $N$ such that $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| \leq N|\mathbf{x} - \mathbf{y}|$. At this time, there must be $\lim\limits_{t\to\infty}\left|\dot{\overline{\mathbf{x}}}(t) - \mathbf{f}\left(\overline{\mathbf{x}}(t)\right)\right| = 0$, namely $\overline{\mathbf{x}}(t)$ is a solution of the node equation (6.3). *How can we get this conclusion?* Notice that for any integer $1 \leq i \leq m$, by (6.12) $|\mathbf{x}_i(t) - \overline{\mathbf{x}}(t)| \to 0$ if $t \to \infty$. So,

$$0 \leq \left|\mathbf{f}\left(\mathbf{x}_i\right) - \mathbf{f}\left(\overline{\mathbf{x}}\right)\right| \leq N\left|\mathbf{x}_i(t) - \overline{\mathbf{x}}(t)\right| \to 0.$$

Now that by the equation (6.8) with applying $\boldsymbol{\xi}\cdot L = \mathbf{0}$, if $t \to \infty$ we have

$$\begin{aligned} 0 \;&\leq\; \left|\dot{\overline{\mathbf{x}}}(t) - \mathbf{f}\left(\overline{\mathbf{x}}(t)\right)\right| = \left|\sum_{i=1}^{m}\xi_i\dot{\mathbf{x}}_i(t) - \mathbf{f}\left(\overline{\mathbf{x}}(t)\right)\right| \\ &= \left|\sum_{i=1}^{m}\xi_i\left(\mathbf{f}(\mathbf{x}_i(t)) + c\sum_{j=1}^{m}l_{ij}\mathbf{H}(\mathbf{x}_j(t))\right) - \mathbf{f}\left(\overline{\mathbf{x}}(t)\right)\right| \\ &= \left|\sum_{i=1}^{m}\xi_i\mathbf{f}(\mathbf{x}_i(t)) + c\sum_{i=1}^{m}\sum_{j=1}^{m}\xi_i l_{ij}\mathbf{H}(\mathbf{x}_j(t)) - \mathbf{f}\left(\overline{\mathbf{x}}(t)\right)\right| \\ &= \left|\sum_{i=1}^{m}\xi_i\mathbf{f}(\mathbf{x}_i(t)) + c\cdot(\boldsymbol{\xi}\cdot L)\cdot\left(\mathbf{H}\left(\mathbf{x}_1(t)\right),\cdots,\mathbf{H}\left(\mathbf{x}_m(t)\right)\right)^t - \mathbf{f}\left(\overline{\mathbf{x}}(t)\right)\right| \\ &= \left|\sum_{i=1}^{m}\xi_i\mathbf{f}(\mathbf{x}_i(t)) + c\cdot\mathbf{0}\cdot\left(\mathbf{H}\left(\mathbf{x}_1(t)\right),\cdots,\mathbf{H}\left(\mathbf{x}_m(t)\right)\right)^t - \mathbf{f}\left(\overline{\mathbf{x}}(t)\right)\right| \\ &= \left|\sum_{i=1}^{m}\xi_i\left(\mathbf{f}\left(\mathbf{x}_i(t)\right) - \mathbf{f}\left(\overline{\mathbf{x}}(t)\right)\right)\right| \leq \sum_{i=1}^{m}\xi_i\left|\mathbf{f}\left(\mathbf{x}_i(t)\right) - \mathbf{f}\left(\overline{\mathbf{x}}(t)\right)\right| \\ &\leq\; N\sum_{i=1}^{m}\xi_i\left|\mathbf{x}_i(t) - \overline{\mathbf{x}}(t)\right| \to 0. \end{aligned}$$

So, there must be $\left|\dot{\overline{\mathbf{x}}}(t) - \mathbf{f}\left(\overline{\mathbf{x}}(t)\right)\right| \to 0$ if $t \to \infty$ , i.e., $\lim\limits_{t\to\infty}\left|\dot{\overline{\mathbf{x}}}(t) - \mathbf{f}\left(\overline{\mathbf{x}}(t)\right)\right| = 0$.

For example, the state equation of the star 4-system shown in Figure 6.6 is

$$\begin{cases} \dot{x}_1(t) = -x_1(t) + 2x_4(t) = x_1(t) - 2\,(x_1(t) - x_4(t)) \\ \dot{x}_2(t) = -x_2(t) + 2x_4(t) = x_2(t) - 2\,(x_2(t) - x_4(t)) \\ \dot{x}_3(t) = -x_3(t) + 2x_4(t) = x_3(t) - 2\,(x_3(t) - x_4(t)) \\ \dot{x}_4(t) = -x_4(t) + 2x_1(t) = x_4(t) - 2\,(x_4(t) - x_1(t)) \end{cases}$$

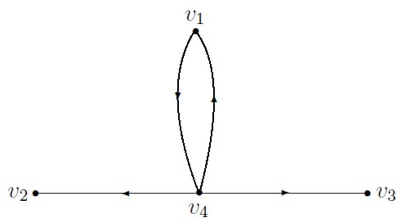



**Figure** 6.6. **A star 4-system**

where, $\mathbf{f}(\mathbf{x}(t)) = \mathbf{x}(t)$ with coupling constant $c = 2$.

The adjacency matrix of this system is $A = (\boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2, \boldsymbol{\alpha}_3, \boldsymbol{\alpha}_4)^t$ with row vectors $\boldsymbol{\alpha}_1 = (-1, 0, 0, 1)$, $\boldsymbol{\alpha}_2 = (0, -1, 0, 1)$, $\boldsymbol{\alpha}_3 = (0, 0, -1, 1)$, $\boldsymbol{\alpha}_4 = (1, 0, 0, -1)$ and a general solution $x_1(t) = C_1e^t - C_2e^{-3t}$, $x_2(t) = C_1e^t - C_2e^{-3t} + C_3e^{-t}$, $x_3(t) = C_1e^t - C_2e^{-3t} + C_4e^{-t}$, $x_4(t) = C_1e^t + C_2e^{-3t}$ with constants $C_1, C_2, C_3, C_4$. Correspondingly, its node equation is $\dot{x}(t) = xt$ with a general solution $x(t) = Ce^t$ and the left eigenvector corresponding to the eigenvalue 0 of matrix $A$ is $\boldsymbol{\xi} = (1/2, 0, 0, 1/2)$. Certainly, for any integer $1 \leq i, j \leq 4$ it is easy to verify that $|x_i(t) - x_j(t)| \to 0$ if $t \to \infty$, i.e., the star 4-system is synchronized and $|x_i(t) - \overline{x}(t)| \to 0$ with $\overline{x}(t) = 0.5x_1(t) + 0.5x_4(t) = C_1e^t$.

## §2. **Synchronous Control**

The condition for a driver to run fast with a cart is that he moves in step with the cart. Here, the initial state of the cart is stationary and needs to be pulled along. Once the cart has a certain speed, i.e., the inertia, the driver pulling the cart only needs to maintain the balance of the cart and run forward without too much force, namely he running forward can synchronize with the cart, as shown in Figure 6.7. Similarly, for an out of synchronize system $S$, *can we manually intervene on some of its element states to achieve system synchronization?* The answer is Yes under certain conditions! Dr.Ouyang tells Huizi that similar to determining whether a system is stable at an equilibrium point or not, it is possible to consider only the solution behavior of node equation $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$ near the equilibrium point $\mathbf{0}$. In this case, when $t \to \infty$ there is $\mathbf{x}(t) \to \mathbf{0}$. Correspondingly, a system $S$ composed of elements $v_1, v_2, \cdots, v_m$ is not dependent on the initial value but just asking for any integer $1 \leq i, j \leq m$, there are $\left|\mathbf{x}_{v_i}(t) - \mathbf{x}_{v_j}(t)\right| \to 0$ if $t \to 0$ which is the same near the equilibrium point $\mathbf{0}$, is consistent with the solution of equation $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$ stable or not because we can choose a neighborhood $\mathcal{O}$ containing the point set Sta$[S]$ at the equilibrium point $\mathbf{0}$. In

**Figure** 6.7. **Driver cart**

this way, it is the same for asking a solution track $\mathbf{x}_{v_i}$ convergent in the point set Sta$[S](t)$ and $\mathbf{x}_{v_i}(t)$ being asymptotically stable at the equilibrium point $\mathbf{0}$.

Certainly, there is a general criterion for determining the $\mathbf{0}$ solution of $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$ being asymptotically stable or not, called the *LaSalle principle of invariance*, i.e., *let $D \subset \mathbb{R}^n$ be a defining domain included $\mathbf{0}$ of function $\mathbf{f}$. If there is a continuously differentiable positive definite function $L : D \to \mathbb{R}$ but $\dot{L}(t)$ is not positive on $D$ and $S = \left\{\mathbf{x} \in D|\dot{L}(t) = 0\right\}$ contains no other solutions except $\mathbf{0}$ of the equation, then the equilibrium point $\mathbf{0}$ of the equation is asymptotically stable.* For example, let $L(x_1, x_2) = \left(x_1^2 + x_2^2\right)/2$ for equation $\dot{x}_1 = -x_1 - x_2 + x_2(x_1 + x_2)$, $\dot{x}_2 = x_1 - x_1(x_1 + x_2)$. Then,

$$\begin{aligned} \dot{L} = x_1\dot{x}_1 + x_2\dot{x}_2 = x_1(-x_1 - x_2 &+ x_2(x_1 + x_2)) \\ &+ x_2(x_1 - x_1(x_1 + x_2)) = -x_1^2 \leq 0 \end{aligned}$$

is non-positive and $x_1 = 0$ if $\dot{L} = 0$, namely the set $S$ no longer contains solutions $\mathbf{x} \neq \mathbf{0}$ of this equation. So, the equilibrium point $\mathbf{0}$ of this equation is asymptotically stable by the LaSalle principle of invariance.

**2.1.Matrix 2-Norm.** Let $A = (a_{ij})$ be an $n \times n$ matrix. The *norm* $|A| : A \to \mathbb{R}$ of matrix $A$ is a mapping from the matrix $A$ to the real numbers $\mathbb{R}$ satisfying

(1) $|A| \geq 0$ with the equality hold only if $A = 0_{n\times n}$;

(2) for any constant $c \in \mathbb{R}$, $|cA| = c|A|$;

(3) $|A + B| \leq |A| + |B|$ and $|AB| \leq |A||B|$, where $A, B$ are both $n \times n$ matrixes.

For an $n \times m$ matrix $A = (a_{ij})$, the product of its transpose $A^t$ with $A$ is

$$A^tA = (a_{ji})_{m\times n}(a_{ij})_{n\times m} = (\widehat{a}_{ij})_{m\times m} = \left(\sum_{k=1}^{n} a_{ik}a_{kj}\right)_{m\times m} \tag{6.13}$$

which is an $m \times m$ matrix. For example,

$$\begin{pmatrix} 2 & -1 & 0 \\ 0 & 2 & 3 \\ 1 & 2 & 0 \end{pmatrix}^t \begin{pmatrix} 2 & -1 & 0 \\ 0 & 2 & 3 \\ 1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 2 & 2 \\ 0 & 3 & 0 \end{pmatrix} \begin{pmatrix} 2 & -1 & 0 \\ 0 & 2 & 3 \\ 1 & 2 & 0 \end{pmatrix} = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 9 & 6 \\ 0 & 6 & 9 \end{pmatrix}.$$

By definition, the matrix transpose has properties $\left(A^t\right)^t = A$ and $(AB) = B^tA^t$. So, there must be $\left(A^tA\right)^t = (A^t)^tA^t = AA^t$, namely $A^tA$ is a symmetrical matrix. Furthermore, if $A$ is a matrix of real numbers, then for any $n$-dimensional column vector $\mathbf{x}$, we have $\mathbf{x}^tA^tA\mathbf{x} = \left(\mathbf{x}^tA^t\right)(A\mathbf{x}) = (A\mathbf{x})(A\mathbf{x})^t \geq 0$.

Now, let $\lambda$ be a root of the characteristic equation $\det\left(A^tA - \lambda I_m\right) = 0$ and let $\boldsymbol{\xi}$ be its characteristic vector, namely the solution of equation $\left(A^tA\right)\boldsymbol{\xi} = \lambda\boldsymbol{\xi}$. Then, we know that

$$0 \leq \boldsymbol{\xi}^t A^t A \boldsymbol{\xi} = \lambda \boldsymbol{\xi}^t \boldsymbol{\xi}.$$

Notice that $\boldsymbol{\xi}^t \boldsymbol{\xi} > 0$. There must be $\lambda \geq 0$, i.e., *all eigenvalues of $A^tA$ are non-negative real numbers.* Assume the eigenvalues of $A^tA$ are respectively $\lambda_1 \geq 0, \lambda_2 \geq 0, \cdots, \lambda_m \geq 0$ with a maximum root $\lambda_{\max}\left(A^tA\right) = \max\{\lambda_1, \lambda_2, \cdots, \lambda_m\}$. The *2-norm of matrix $A$* is defined by

$$|A|_2 = \sqrt{\lambda_{\max}(A^tA)}, \tag{6.14}$$

i.e., the square root of the maximum characteristic root of matrix $A^tA$.

For example, the characteristic equation of matrix $A^tA$ in previous example is

$$\begin{aligned} 0 &= \left|A^tA - \lambda I_3\right| = \left|\begin{pmatrix} 5 & 0 & 0 \\ 0 & 9 & 6 \\ 0 & 6 & 9 \end{pmatrix} - \lambda I_3\right| = \begin{vmatrix} 5-\lambda & 0 & 0 \\ 0 & 9-\lambda & 6 \\ 0 & 6 & 9-\lambda \end{vmatrix} \\ &= (5-\lambda)(9-\lambda)^2 - 36(5-\lambda) \\ &= (3-\lambda)(5-\lambda)(15-\lambda), \end{aligned}$$

i.e., its maximum characteristic root $\lambda_{\max} = 15$. By definition, we then know that $|A|_2 = \sqrt{\lambda_{\max}(A^tA)} = \sqrt{15}$.

Certainly, the 2-norm $|A|_2$ of a matrix $A$ is a generalization of the norm $|\mathbf{a}|$, namely (4.33) of vector $\mathbf{a}$. Generally, for any $\mathbf{x} \in \mathbb{R}^n$, there is

$$|A|_2 = \max_{\mathbf{x} \neq \mathbf{0}} \frac{|A\mathbf{x}|}{|\mathbf{x}|}. \tag{6.15}$$

Notice that $|A|_2 = \sqrt{\lambda_{\max}(A^tA)}$ by definition. So, there must be $|A\mathbf{x}| \leq |A|_2|\mathbf{x}|$. Generally, the 2-norm $|A(\mathbf{x})|_2 : D \to \mathbb{R}$ of matrix $A(\mathbf{x})$ of functions defined on $\mathbf{x} \in D \subset \mathbb{R}^n$ refers to the 2-norm $\left|A(\mathbf{x}^0)\right|_2$ at point $\mathbf{x}^0$ in $D$, i.e., the 2-norm of matrix $A(\mathbf{x}^0)$. In other words, $|A(\mathbf{x})|_2$ is a non-negative function from $D$ into $\mathbb{R}$, i.e., $|A(\mathbf{x})|_2 \geq 0$ holding with conditions (1) − (3). For example, let $n = 3$ and $f_1(\mathbf{x}) = x_1^2 - x_2^2/2, f_2(\mathbf{x}) = x_2^2 + 3x_3^2/2, f_3(\mathbf{x}) = x_1^2/2 + x_2^2$, then the Jacobian matrix in (5.31) is

$$J_{\mathbf{f}|\mathbf{x}} = \begin{pmatrix} 2x_1 & -x_2 & 0 \\ 0 & 2x_2 & 3x_3 \\ x_1 & 2x_2 & 0 \end{pmatrix} \Longrightarrow$$

$$J_{\mathbf{f}|\mathbf{x}}^t J_{\mathbf{f}|\mathbf{x}} = \begin{pmatrix} 2x_1 & 0 & x_1 \\ -x_2 & 2x_2 & 2x_2 \\ 0 & 3x_3 & 0 \end{pmatrix} \begin{pmatrix} 2x_1 & -x_2 & 0 \\ 0 & 2x_2 & 3x_3 \\ x_1 & 2x_2 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 5x_1^2 & 0 & 0 \\ 0 & 9x_2^2 & 6x_2x_3 \\ 0 & 6x_2x_3 & 9x_3^2 \end{pmatrix}.$$

Correspondingly, the characteristic equation of $J_{\mathbf{f}|\mathbf{x}}^t J_{\mathbf{f}|\mathbf{x}}$ is

$$\left| J_{\mathbf{f}|\mathbf{x}}^t J_{\mathbf{f}|\mathbf{x}} - \lambda I_3 \right| = \left(5x_1^2 - \lambda\right)\left(45x_2^2x_3^2 - 9\left(x_2^2 + x_3^2\right)\lambda + \lambda^2\right) = 0. \tag{6.16}$$

Thus, solving the equation (6.16) we can get $\lambda_{\max}$. For example, the equation (6.16) appears respectively $(5-\lambda)\lambda^2 = 0$, $(5-\lambda)(-9+\lambda)\lambda = 0$ and $(3-\lambda)(5-\lambda)(15-\lambda) = 0$ at points $\mathbf{x}^0 = (1,0,0), (1,1,0)$ and $(1,1,1)$, which has respectively the maximum root $\lambda_{\max} = 5, 9$ or $15$. Whence, the 2-norm $\left|J_{\mathbf{f}|\mathbf{x}}\right|_2$ of the Jacobian matrix $J_{\mathbf{f}|\mathbf{x}}$ are $\sqrt{5}, 3$ and $\sqrt{15}$ at points $\mathbf{x}^0 = (1,0,0), (1,1,0)$ and $(1,1,1)$, respectively.

**2.2.Control Equation.** Dr.Ouyang explains that the control equation of a system refers to input variables $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_m$ acting on elements $v_1, v_2, \cdots, v_m$ for finding the synchronous condition. In this case, the system equation (6.8) becomes

$$\dot{\mathbf{x}}_i(t) = \mathbf{f}\left(\mathbf{x}_i(t)\right) + h_i\left(\mathbf{x}_1(t), \mathbf{x}_2(t), \cdots, \mathbf{x}_m(t)\right) + \mathbf{u}_i(t),\ 1 \le i \le m, \tag{6.17}$$

where

$$h_i\left(\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_m\right) = c\sum_{j=1}^m l_{ij}\mathbf{H}(\mathbf{x}_j(t)),$$

called the *coupling term* and $\mathbf{u}_i(t) \in \mathbb{R}^n$ the *control term.*

Now, let $\mathbf{s}(t)$ be the solution of node equation $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$. For any integer $1 \le i \le m$, the *error vector* is defined by $\boldsymbol{\epsilon}_i(t) = \mathbf{x}_i(t) - \mathbf{s}(t)$. Then, the goal of system control of synchronization is to find the condition on the control terms $\mathbf{u}_i(t)$, $1 \le i \le m$ that can realize the synchronization of system (6.17). At this point,

$$\lim_{t\to\infty} |\boldsymbol{\epsilon}_i(t)| \ = \ 0, \quad 1 \le i \le m, \tag{6.18}$$

namely for any integer $1 \le i \le m$, the error vector $\boldsymbol{\epsilon}_i(t)$ is an infinitesimal vector. By $\dot{\mathbf{s}}(t) = \mathbf{f}(\mathbf{s}(t))$ and the equation (6.17), we know that the error term $\boldsymbol{\epsilon}_i(t)$ satisfies the equation

$$\dot{\boldsymbol{\epsilon}}_i(t) = \overline{\mathbf{f}}(\mathbf{x}_i(t), \mathbf{s}) + \overline{\mathbf{h}}_i(\mathbf{x}(t), \mathbf{s}) + \mathbf{u}_i, \quad 1 \le i \le m, \tag{6.19}$$

where,

$$\begin{aligned}\overline{\mathbf{f}}(\mathbf{x}_i(t),\mathbf{s}) &= \mathbf{f}(\mathbf{x}_i(t))-\mathbf{f}(\mathbf{s}(t)),\\ \overline{\mathbf{h}}_i\left(\mathbf{x}(t),\mathbf{s}\right) &= \mathbf{h}_i\left(\mathbf{x}_1(t),\mathbf{x}_2(t),\cdots,\mathbf{x}_m(t)\right)-\mathbf{h}_i(\mathbf{s},\mathbf{s},\cdots,\mathbf{s}).\end{aligned}$$

Generally, there are two useful assumptions for synchronous control of system:

**Condition 1.** *There is such a constant $\alpha \geq 0$ that the 2-norm of the Jacobian matrix $J_{\mathbf{f}|\mathbf{s}}$ satisfies $\left|J_{\mathbf{f}|\mathbf{s}}\right|_2 \leq \alpha$.*

**Condition 2.** *For any integers $1 \leq i, j \leq m$ there are constants $\gamma_{ij} \geq 0$ such that*

$$\left|\overline{\mathbf{h}}_i\left(\mathbf{x}(t),\mathbf{s}\right)\right| \leq \sum_{j=1}^{m} \gamma_{ij}\left|\boldsymbol{\epsilon}_i\right|.$$

Among them, Dr.Ouyang tells Huizi that the Condition 1 is a strong constraint which needs the function $\left|J_{\mathbf{f}|\mathbf{s}}\right|\mathbf{x} \leq \alpha|\mathbf{x}|$, i.e., the changed norm can not exceed $\alpha$ times of norm $|\mathbf{x}|$ and the Condition 2 is the upper norm bound of the function $\overline{\mathbf{h}}_i, 1 \leq i \leq m$ by the error functions $\boldsymbol{\epsilon}_i, 1 \leq i \leq m$. In this way, under the assumption that system (6.17) satisfies Conditions 1 and 2, the control conditions of nonlinear system synchronization can be obtained.

**2.3.Local Synchronous Control.** The local synchronization of a system refers to that the system is synchronized near the equilibrium point. At this point, Dr.Ouyang explains that we can expand the nonlinear terms of $\overline{\mathbf{f}}(\mathbf{x}_i(t),\mathbf{s}), 1 \leq i \leq m$ in equation (6.19) near at the equilibrium point $\boldsymbol{\epsilon}_i = \mathbf{0}$ by (5.31) and choose the linear terms to be its approximation, i.e.,

$$\dot{\boldsymbol{\epsilon}}_i = J_{\mathbf{f}|\mathbf{x}}\boldsymbol{\epsilon}_i + \overline{\mathbf{h}}_i(\mathbf{x}(t),\mathbf{s}) + \mathbf{u}_i, \quad 1 \leq i \leq m. \tag{6.20}$$

So, *if the system* (6.17) *satisfies Conditions* 1 *and* 2, *let* $\mathbf{u}_i = -d_i\boldsymbol{\epsilon}_i$, *where* $\dot{d}_i = k_i\boldsymbol{\epsilon}_i^t\boldsymbol{\epsilon}_i = k_i\left|\boldsymbol{\epsilon}_i\right|^2$, $k_i \geq 0, 1 \leq i \leq m$, *then* $\boldsymbol{\epsilon}_i \to \mathbf{0}$ *if* $t \to \infty$, *namely the system* (6.20) *is stable near the equilibrium point and the system* (6.17) *is locally synchronized.* So, *why is system* (6.17) *locally synchronized at this time?* The key to answer this question is to determine that the solution of equation (6.20) is stable at the equilibrium point $\mathbf{0}$ and the Lyapunov criterion can be applied. Let the Lyapunov function

$$L(t) = \frac{1}{2}\sum_{i=1}^{m}\boldsymbol{\epsilon}_i^t\boldsymbol{\epsilon}_i + \frac{1}{2}\sum_{i=1}^{m}\frac{(d_i-\widehat{d_i})^2}{k_i}, \tag{6.21}$$

where $\widehat{d_i} > 0, 1 \leq i \leq m$ are numbers to be determined. If $\boldsymbol{\epsilon}_i = \mathbf{0}$ for any integer $1 \leq i \leq m$, then the system (6.17) is already a synchronous system. Otherwise, by the definition (6.21) of $L(t)$, there is one integer $1 \leq i \leq m$ at least such that $\boldsymbol{\epsilon}_i \neq \mathbf{0}$, i.e.,

$L(t) > 0$ is a positive definite function. So, *is $\dot{L}(t)$ non-negative?* Dr.Ouyang tells Huizi that we should calculate $\dot{L}(t)$ first and then see under what conditions it is non-negative.

Notice that if $\mathbf{g} = (g_1, g_2, \cdots, g_n)^t$, then $\mathbf{g}^t\mathbf{g} = \left(g_1^2, g_2^2, \cdots, g_n^2\right)$ and $\dot{\boldsymbol{\epsilon}}_i^t = \boldsymbol{\epsilon}_i^t J_{\mathbf{f}|\mathbf{x}}^t + \overline{\mathbf{h}}_i^t(\mathbf{x}(t), \mathbf{s}) + \mathbf{u}_i^t$, $1 \leq i \leq m$. Denoted by $\overline{J}_{\mathbf{f}|\mathbf{s}} = \left(J_{\mathbf{f}|\mathbf{s}} + J_{\mathbf{f}|\mathbf{s}}^t\right)/2$. Now, differentiate on both sides of (6.21) with substituting (6.20). By the hypothesis, we have

$$\begin{aligned}
\dot{L}(t) &= \frac{1}{2}\sum_{i=1}^{m}\left(\dot{\boldsymbol{\epsilon}_i}^t\boldsymbol{\epsilon}_i + \boldsymbol{\epsilon}_i^t\dot{\boldsymbol{\epsilon}}_i\right) + \sum_{i=1}^{m}\frac{(d_i - \widehat{d}_i)\dot{d}_i}{k_i} \\
&= \frac{1}{2}\sum_{i=1}^{m}\left(\boldsymbol{\epsilon}_i^t J_{\mathbf{f}|\mathbf{x}}^t + \overline{\mathbf{h}}_i^t(\mathbf{x}(t), \mathbf{s}) - d_i\boldsymbol{\epsilon}_i^t\right)\boldsymbol{\epsilon}_i \\
&\quad + \frac{1}{2}\sum_{i=1}^{m}\boldsymbol{\epsilon}_i^t\left(J_{\mathbf{f}|\mathbf{x}}\boldsymbol{\epsilon}_i + \overline{\mathbf{h}}_i(\mathbf{x}(t), \mathbf{s}) - d_i\boldsymbol{\epsilon}_i\right) + \sum_{i=1}^{m}\left(d_i - \widehat{d}_i\right)\boldsymbol{\epsilon}_i^t\boldsymbol{\epsilon}_i \\
&= \sum_{i=1}^{m}\boldsymbol{\epsilon}_i^t\left(\overline{J}_{\mathbf{f}|\mathbf{x}} - \widehat{d}_i I_n\right)\boldsymbol{\epsilon}_i + \sum_{i=1}^{m}\boldsymbol{\epsilon}_i^t\overline{\mathbf{h}}_i(\mathbf{x}(t), \mathbf{s}) \\
&\leq \sum_{i=1}^{m}\boldsymbol{\epsilon}_i^t\left(J_{\mathbf{f}|\mathbf{x}} - \widehat{d}_i I_n\right)\boldsymbol{\epsilon}_i + \sum_{i=1}^{m}\sum_{j=1}^{m}\gamma_{ij}|\boldsymbol{\epsilon}_i||\boldsymbol{\epsilon}_j| \\
&\leq \sum_{i=1}^{m}\left(\alpha - \widehat{d}_i I_n\right)|\boldsymbol{\epsilon}_i|^2 + \sum_{i=1}^{m}\sum_{j=1}^{m}\gamma_{ij}|\boldsymbol{\epsilon}_i||\boldsymbol{\epsilon}_j| \\
&= \boldsymbol{\epsilon}^t\left(\Gamma + \mathrm{diag}(\alpha - \widehat{d}_1, \alpha - \widehat{d}_2, \cdots, \alpha - \widehat{d}_m)\right)\boldsymbol{\epsilon},
\end{aligned}$$

where $\Gamma = (\gamma_{ij})_{m\times m}$, $\boldsymbol{\epsilon} = (|\boldsymbol{\epsilon}_1|, |\boldsymbol{\epsilon}_2|, \cdots, |\boldsymbol{\epsilon}_m|)^t$, $\mathrm{diag}(\alpha - \widehat{d}_1, \alpha - \widehat{d}_2, \cdots, \alpha - \widehat{d}_m)$ is a diagonal matrix $(d_{ij})_{m\times m}$, i.e., for any integers $1 \leq i \neq j \leq m$, there are $d_{ij} = 0$, $d_{ii} = \alpha - \widehat{d}_i, 1 \leq i \leq m$.

Notice that *for an $m \times m$ symmetrical matrix $A$, there must be such a real number $b < 0$ that $a + bI_{m\times m}$ is a negative definite matrix.* Without loss of generality, for any integers $1 \leq i, j \leq m$ we can assume the matrix $\Gamma$ to be a symmetric matrix because if $\gamma_{ij} \geq \gamma_{ji}$, then $\gamma_{ji}$ can be replaced by $\gamma_{ij}$ in Condition 2 without affecting the assumption. In this way, we can always choose real numbers $\widehat{d}_i, 1 \leq i \leq m$ appropriately such that the matrix $\Gamma + \mathrm{diag}(\alpha - \widehat{d}_1, \alpha - \widehat{d}_2, \cdots, \alpha - \widehat{d}_m)$ is a negative definite matrix. Then, by the LaSalle principle of invariance, every solution of equation (6.19) converges to the $\mathbf{0}$ solution, i.e., $\boldsymbol{\epsilon}_i \to \mathbf{0}, 1 \leq i \leq m$ if $t \to \infty$, the system (6.17) is synchronized near the equilibrium point $\mathbf{0}$.

Particularly, if all $h_i\left(\mathbf{x}_1(t), \mathbf{x}_2(t), \cdots, \mathbf{x}_m(t)\right), 1 \leq i \leq m$ are linear functions in system (6.17), i.e.,

$$h_i\left(\mathbf{x}_1(t), \mathbf{x}_2(t), \cdots, \mathbf{x}_m(t)\right) = \sum_{j=1}^{m} a_{ij}\mathbf{x}_j(t), \quad \sum_{j=1}^{m} a_{ij} = 0, \quad 1 \leq i \leq m$$

with system equation

$$\dot{\mathbf{x}}_i(t) = \mathbf{f}\left(\mathbf{x}_i(t)\right) + \sum_{j=1}^{m} a_{ij}\mathbf{x}_j(t) + \mathbf{u}_i(t),\ 1 \le i \le m, \tag{6.22}$$

which naturally satisfies Condition 2. In this case, *if system* (6.22) *further satisfies Condition 1, let* $\mathbf{u}_i = -d_i\boldsymbol{\epsilon}_i$ *and* $\dot{d}_i = k_i\boldsymbol{\epsilon}_i^t\boldsymbol{\epsilon}_i = k_i\left|\boldsymbol{\epsilon}_i\right|^2,\ k_i \ge 0, 1 \le i \le m,$ *there must be that system* (6.22) *is synchronized near the equilibrium point* $\mathbf{0}$. This conclusion can be generalized further. For example, suppose that

$$h_i\left(\mathbf{x}_1(t), \mathbf{x}_2(t), \cdots, \mathbf{x}_m(t)\right) = \sum_{j=1}^{m} a_{ij}p\left(\mathbf{x}_j(t)\right),\quad \sum_{j=1}^{m} a_{ij} = 0,\quad 1 \le i \le m$$

and the function $p$ is differentiable to $\mathbf{x}$ holding with $\left|J_{p|\mathbf{x}}\right|_2 \le \beta$, we have

$$\left|\overline{h}_i\left(\mathbf{x}_1(t), \mathbf{x}_2(t), \cdots, \mathbf{x}_m(t)\right)\right| \le \sum_{j=1}^{m}\left|a_{ij}\right|\left|p\left(\mathbf{x}_j(t)\right) - p(\mathbf{s})\right| \le \sum_{j=1}^{m}\left|a_{ij}\right|\beta\left|\boldsymbol{\epsilon}_i\right|.$$

In this case, we can let $\mathbf{u}_i = -d_i\boldsymbol{\epsilon}_i$ and $\dot{d}_i = k_i\boldsymbol{\epsilon}_i^t\boldsymbol{\epsilon}_i = k_i\left|\boldsymbol{\epsilon}_i\right|^2$, $k_i \ge 0, 1 \le i \le m$. Then, the system

$$\dot{\mathbf{x}}_i(t) = \mathbf{f}\left(\mathbf{x}_i(t)\right) + \sum_{j=1}^{m} a_{ij}p\left(\mathbf{x}_j(t)\right) + \mathbf{u}_i(t),\ 1 \le i \le m \tag{6.23}$$

is stable at the equilibrium point $\mathbf{0}$ if it further satisfies Condition 1, i.e., synchronized near the point $\mathbf{0}$.

**2.4.Global Synchronous Control.** The global synchronous cotrol of system refers to the system synchronization in the whole domain of $\mathbf{x}_i(t), 1 \le i \le m$. Then, *can we obtain a control condition that system $S$ synchronizes globally on Conditions 1 and 2?* Dr.Ouyang explains that the function $\mathbf{f}$ in the node equation $\dot{x}(t) = \mathbf{f}(\mathbf{x}(t))$ must be divided into linear and nonlinear parts at this time, namely $\dot{\mathbf{x}} = B\mathbf{x}(t) + \mathbf{g}(\mathbf{x}(t))$, where $B \in \mathbb{R}^{m\times m}$ is a matrix of constants, $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^m$ is a nonlinear differentiable mapping. In this case, the system equation (6.17) becomes

$$\dot{\mathbf{x}}_i(t) = B\mathbf{x}_i(t) + \mathbf{g}(\mathbf{x}_i(t)) + \mathbf{h}_i(\mathbf{x}_1(t), \mathbf{x}_2(t), \cdots, \mathbf{x}_m(t)) + \mathbf{u}_i(t) \tag{6.24}$$

and accordingly, the error equation (6.19) becomes

$$\dot{\boldsymbol{\epsilon}}_i(t) = B\boldsymbol{\epsilon}_i(t) + \overline{\mathbf{g}}(\mathbf{x}_i(t), \mathbf{s}) + \overline{\mathbf{h}}_i(\mathbf{x}(t), \mathbf{s}) + \mathbf{u}_i(t), \tag{6.25}$$

where, $1 \le i \le m$ and $\overline{\mathbf{g}}(\mathbf{x}_i(t), \mathbf{s}) = \mathbf{g}(\mathbf{x}_i(t)) - \mathbf{g}(\mathbf{s}(t))$.

**Condition 3.** *There exists a non-negative constant* $\mu$ *such that* $|\overline{\mathbf{g}}(\mathbf{x}_i(t), \mathbf{s})| \leq \mu\,|\boldsymbol{\epsilon}_i|\,, 1 \leq i \leq m$.

Now that if the function $\overline{\mathbf{g}}(\mathbf{x}_i(t), \mathbf{s}), 1 \leq i \leq m$ satisfies Condition 3, the global synchronization of system (6.24) is similar to that of the local synchronized situation near the equilibrium point $\mathbf{0}$, namely *if the system* (6.24) *satisfies Conditions 2 and 3, similarly, let* $\mathbf{u}_i(t) = -d_i\boldsymbol{\epsilon}_i(t)$ *and* $\dot{d}_i = k_i\boldsymbol{\epsilon}_i^t(t)\boldsymbol{\epsilon}_i(t) = k_i\,|\boldsymbol{\epsilon}_i(t)|^2$*, then the system* (6.24) *holds with* $\boldsymbol{\epsilon}_i(t) \to \mathbf{0}$ *if* $t \to \infty$*, i.e., namely the system is globally synchronized, where* $k_i > 0, 1 \leq i \leq m$ *are all constants.* So, *how do we get this conclusion?* Dr.Ouyang explains that first of all, it should be noted that $B$ is a constant matrix. By the definition of 2-norm of matrix, there is such a constant $\delta$ that $|B|_2 \leq \delta$. Denoted by $\overline{B} = \left(B + B^t\right)/2$. Then it is easily to know that $\left|\overline{B}\right|_2 \leq \delta$.

Similar to the discussing on local synchronization of system (6.17), Dr.Ouyang tells Huizi that it is also necessary to define a positive definite function $L(t)$ and then verify that $\dot{L}(t)$ is non-negative at this time. To do this, we can choose $L(t)$ to be the same as in (6.21) and calculate the derivative $\dot{L}(t)$ of $L(t)$ to $t$, namely

$$\begin{aligned}
\dot{L}(t) &= \frac{1}{2}\sum_{i=1}^{m}\left(\dot{\boldsymbol{\epsilon}_i}^t\boldsymbol{\epsilon}_i + \boldsymbol{\epsilon}_i^t\dot{\boldsymbol{\epsilon}}_i\right) + \sum_{i=1}^{m}\frac{(d_i - \widehat{d}_i)\dot{d}_i}{k_i}\\
&= \sum_{i=1}^{m}\boldsymbol{\epsilon}_i^t\left(\overline{B} - \widehat{d}_iI_n\right)\boldsymbol{\epsilon}_i + \sum_{i=1}^{m}\boldsymbol{\epsilon}_i^t\overline{\mathbf{g}}(\mathbf{x}_i(t), \mathbf{s}) + \sum_{i=1}^{m}\boldsymbol{\epsilon}_i^t\overline{\mathbf{h}}_i(\mathbf{x}(t), \mathbf{s})\\
&\leq \sum_{i=1}^{m}\boldsymbol{\epsilon}_i^t\left(\delta + \mu - \widehat{d}_i\right)|\boldsymbol{\epsilon}_i|^2 + \sum_{i=1}^{m}\sum_{j=1}^{m}\gamma_{ij}|\boldsymbol{\epsilon}_i||\boldsymbol{\epsilon}_j|\\
&= \boldsymbol{\epsilon}^t\left(\Gamma + \mathrm{diag}(\delta + \nu - \widehat{d}_1, \delta + \nu - \widehat{d}_2, \cdots, \delta + \nu - \widehat{d}_m)\right)\boldsymbol{\epsilon}_\circ
\end{aligned}$$

Similarly, without loss of generality, we assume that $\Gamma$ is a symmetric matrix. In this case, since $\delta, \mu$ and $\gamma_{ij}, 1 \leq i, j \leq m$ are all non-negative constants, we can appropriately choose constants $\widehat{d}_i, 1 \leq i \leq m$ such that the matrix $\Gamma + \mathrm{diag}(\delta + \nu - \widehat{d}_1, \delta + \nu - \widehat{d}_2, \cdots, \delta + \nu - \widehat{d}_m)$ is a negative definite matrix. Then, applying the LaSalle principle of invariance, there are $\boldsymbol{\epsilon}_i \to \mathbf{0}, 1 \leq i \leq m$ if $t \to \infty$, i.e., the system (6.24) is global synchronization.

Particularly, for a system (6.22) that all $\mathbf{h}(\mathbf{x}_1, \mathbf{x}_12, \cdots, \mathbf{x}_n)$ are linear or a system (6.23) that $p$ is differentiable to $\mathbf{x}$ with $\left|J_{p|\mathbf{x}}\right|_2 \leq \beta$ both hold with Condition 3, we can further require the systems (6.22) and (6.23) satisfying Condition 2. In this case, let $\mathbf{u}_i(t) = -d_i\boldsymbol{\epsilon}_i(t)$ and $\dot{d}_i = k_i\boldsymbol{\epsilon}_i^t(t)\boldsymbol{\epsilon}_i(t) = k_i\,|\boldsymbol{\epsilon}_i(t)|^2$. Then, the system (6.22) and (6.23) are both globally synchronized.

For example, let $S$ be a system with state equation

$$(\dot{x}_{i1}, \dot{x}_{i2}, \dot{x}_{i3})^t = B\,(x_{i1}, x_{i2}, x_{i3})^t + (0, -x_{i1}x_{i3}, x_{i1}x_{i2})^t + \left(\widehat{f}_1, \widehat{f}_2, \widehat{f}_3\right)^t - d_i\boldsymbol{\epsilon}_i,$$

where,

$$1 \le i \le 50 \quad \begin{cases} \widehat{f_1} = f_1(x_i) - 2f_1(x_{i+1}) + f_1(x_{i+2}), \\ \widehat{f_2} = 0, \\ \widehat{f_3} = f_2(x_i) - 2f_2(x_{i+1}) + f_2(x_{i+2}), \\ f_1(x_i) = r_1(x_{i2} - x_{i1}), \\ f_2(x_i) = x_{i1}x_{i2} - r_2 x_{i3}. \end{cases}$$

Then, the system $S$ satisfies Conditions 2 and 3. In this case, let $\dot{d}_i = k_i|\boldsymbol{\epsilon}_i|^2$ with constants $k_i$ for any integer $1 \le i \le 50$. Then, the system $S$ is globally synchronized.

Notice that the essence of previous discussion for determining whether a system $S$ is synchronized is converting the system synchronization into that the error functions $\boldsymbol{\epsilon}_i$ is stable at the equilibrium point, namely whether there is $\boldsymbol{\epsilon}_i \to \mathbf{0}$ if $t \to \infty$ for integers $1 \le i \le m$. In this way, the known results of whether the system is stable at the equilibrium point such as the Lyapunov criterion can be applied and then to get whether the system is synchronized near the equilibrium point or not. Among these, it is relatively simple to determine the stability of linear systems. A natural idea is to expand the function $\mathbf{f}(\mathbf{x}_i(t))$ near the equilibrium point and approximate by its linear part. By limiting the upper bound of the nonlinear part and $\mathbf{h}_i(\mathbf{x}_1(t), \mathbf{x}_2(t), \cdots, \mathbf{x}_m(t))$ etc., the stability of the system at the equilibrium point is obtained by the stability of the linear part, i.e., the synchronous condition. Dr.Ouyang tells Huizi that this idea can be further developed to find the master stability function for system synchronization.

## §3. Master Stability Function

For an autonomous linear system $S$ with state equation $\dot{\mathbf{x}}(t) = A\mathbf{x}(t)$, where $A$ is an $n \times n$ matrix of constants, let the solutions of its characteristic equation $\det(A - \lambda I_m) = 0$ be $\lambda_1$ (multiple $k_1$), $\lambda_2$ (multiple $k_2$),$\cdots$,$\lambda_s$ (multiple $k_s$) with $k_1 + k_2 + \cdots + k_s = n$. We have known that any solution of this equation can be obtained by linear spanning of elements $p_l(t)e^{\lambda_l}$, where $p_l(t)$ is a polynomial of $t$ with order of less or equal to $k_l - 1$ for any integer $1 \le l \le m$. In this case, when the parameter $t \to \infty$, if $\mathrm{Re}(\lambda_l) < 0$ then $p_l(t)e^{\lambda_l} \to 0$ and if $\mathrm{Re}(\lambda_l) > 0$ then $p_l(t)e^{\lambda_l} \to \infty$. Thus, the characteristic root $\mathrm{Re}(\lambda) \le 0$ is a necessary condition for determining the stability of a linear system solution.

Now, add a nonlinear term on the linear system, we get a general system $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)$. So, *whether the behavior of linear approximation* $\dot{\mathbf{x}}(t) = \mathbf{J}_{\mathbf{f}|\mathbf{x}}(\mathbf{x} - \mathbf{x}^*)$ *of system* $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)$ *be applied to determine its stability in the neighborhood of an equilibrium point* $\mathbf{x}^*$*?* If the answer is Yes, Dr.Ouyang tells Huizi that it will undoubtedly provides a convenient condition for determining the stability of the nonlinear system at the equilibrium point and the system synchronization near the equilibrium point. However, a positive answer to this question is only in some special cases. Particularly, when the real part of eigenvalue of its Jacobian matrix $\mathbf{J}_{\mathbf{f}|\mathbf{x}}$ are all not 0, namely the function $\mathbf{f}:\mathbb{R}^n\to\mathbb{R}^n$ is hyperbolic, the eigenvalues of Jacobian matrix $\mathbf{J}_{\mathbf{f}|\mathbf{x}}$ enables us to determine the local synchronization of nonlinear system.

**3.1.Lyapunov Exponent.** If the solution of an autonomous system $\dot{\mathbf{x}}(t)=\mathbf{f}(\mathbf{x}(t))$ is asymptotically stable, it does not depend on the choice of initial value of $\mathbf{x}(t_0)$, namely we can assume that solutions $\mathbf{x}(t)$ and $\mathbf{x}'(t)$ corresponds the initial value $\mathbf{x}(t_0)$ and $\mathbf{x}'(t_0)$, respectively then $\mathbf{x}(t)\to\mathbf{x}'(t)$ or $|\mathbf{x}(t)-\mathbf{x}'(t)|\to 0$ if $t\to\infty$. Conversely, if solution $\mathbf{x}(t)$ is not stable, then $|\mathbf{x}(t)-\mathbf{x}'(t)|\not\to 0$ if $t\to\infty$, namely some solution $\mathbf{x}(t)$ does not depend on the initial value while others depend strongly on the initial value, as shown in Figure 6.8. So, *why do we introduce the Lyapunov exponent and what problem can it solve?* Dr.Ouyang tells Huizi that the matrix eigenvalues play an important role in characterizing the stable property of linear systems. However, the Lyapunov exponent is such a similar index that introduced for nonlinear systems.

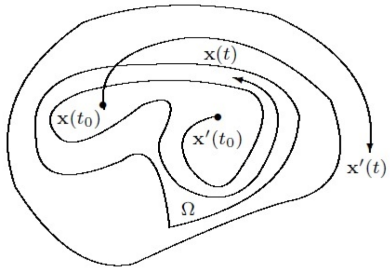


**Figure** 6.8. **Initial value with track**

**(1)1-dimensional Lyapunov exponent.** Let $f$ be a continuously differentiable function. By the iterative equation $x_{i+1}(t)=f(x_i(t)), i\geq 1$, we define a 1-dimensional point sequence $\{x_i\}_0^\infty=\{x_0,x_1,\cdots,x_n,\cdots\}$. Firstly, let the initial $x_0=x(t_0), x_0'=x'(t_0)$ and let $x_0'$ be sufficiently close to that of $x_0$, i.e., $|x_0-x_0'|\to 0$. Then, by the Taylor expansion of function $f$, we have an estimation

$$\begin{aligned}
&|x_1-x_1'|\approx\left|\frac{df}{dx}\bigg|_{x_0}\right||x_0-x_0'|,\\
&|x_2-x_2'|\approx\left|\frac{df}{dx}\bigg|_{x_1}\right||x_1-x_1'|\approx\left|\frac{df}{dx}\bigg|_{x_1}\frac{df}{dx}\bigg|_{x_0}\right||x_0-x_0'|,\\
&\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\
&|x_k-x_k'|\approx\left|\frac{df}{dx}\bigg|_{x_{k-1}}\right||x_{k-1}-x_{k-1}'|\approx\cdots\approx\left|\prod_{i=0}^{k-1}\frac{df}{dx}\bigg|_{x_i}\right||x_0-x_0'|,
\end{aligned}$$

i.e., the varying of distance $|x_k-x_k|$ between the points $x_k,x_k'$ after $k$ iteration is closely related to $\left|\prod_{i=0}^{k-1}\frac{df}{dx}\Big|_{x_i}\right|$. Dr.Ouyang explains to his daughter that the $x(t)$ and $x'(t)$ are closed or separated from each other. In order to measure them as a whole, representing it by an exponential form, i.e., $e^{kL_k} = \left|\prod_{i=0}^{k-1} \frac{df}{dx}\Big|_{x_i}\right|$ we introduce the exponent $L_k$ by

$$L_k \;=\; \frac{1}{k}\ln\left|\prod_{i=0}^{k-1}\frac{df}{dx}\bigg|_{x_i}\right| = \frac{1}{k}\sum_{i=0}^{k-1}\ln\left|\frac{df}{dx}\right|_{x_i} \tag{6.26}$$

for characterizing the average distance between tracks $x_k(t)$ and $x'_k(t)$ after iterations of $k$ times, i.e., $|x_k - x'_k| \approx e^{kL_k}|x_0 - x'0|$. Generally, let $k \to \infty$, namely discuss the distance between tracks $x_k(t), x'_k(t)$ after iterations of infinite times. In this case, if the limit $\lim\limits_{k\to\infty} L_k$ in (6.26) exists, the *Lyapunov exponent* $L$ of the 1-dimensional system $S$ is defined by

$$L \;=\; \lim_{k\to\infty} L_k \;=\; \lim_{k\to\infty}\frac{1}{k}\sum_{i=0}^{k-1}\ln\left|\frac{df}{dx}\right|_{x_i}. \tag{6.27}$$

Notice that the Lyapunov exponent has nothing to do with the initial point $x_0$ because it is a property with $k \to \infty$. For example, a tent map $T : [0,1] \to [0,1]$ is defined by

$$T(x) = \begin{cases} \mu x, & 0 \le x \le \dfrac{1}{2}; \\ \mu(1-x), & \dfrac{1}{2} \le x \le 1, \end{cases}$$

where $0 \le \mu \le 2$ with a corresponding iterative equation $x_{i+1} = T(x_i)$, $dT/dx = \pm\ln\mu$. So,

$$L \;=\; \lim_{k\to\infty} L_k = \lim_{k\to\infty}\frac{1}{k}\sum_{i=0}^{k-1}\ln\left|\frac{dT}{dx}\right|_{x_i} = \lim_{k\to\infty}\frac{1}{k}\times k\ln\mu = \ln\mu.$$

**(2)$n$-dimensional Lyapunov exponent.** So, *how do we generalize Liapunov exponent* (6.27) *of 1-dimension to the n-dimensional system?* Dr.Ouyang explains to his daughter that for an $n$-dimensional system $S$ characterized by an equation $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$, we can apply linear approximation of $\mathbf{f}$ by (5.31) at the equilibrium point $\mathbf{x}^*$, i.e., $\dot{\mathbf{x}} = \frac{d\mathbf{f}}{d\mathbf{x}}\Big|_{\mathbf{x}^*}(\mathbf{x}-\mathbf{x}^*)+\mathbf{x}^*$, where $\frac{d\mathbf{f}}{d\mathbf{x}} = J_{\mathbf{f}|\mathbf{x}}$ is the Jacobian matrix of $\mathbf{f}$ to $\mathbf{x}$ and $J_{\mathbf{f}|\mathbf{x}}\big|_{\mathbf{x}^*}$ is the value of $J_{\mathbf{f}|\mathbf{x}}$ at point $\mathbf{x}^*$ which is a matrix of constants. Similarly, by the iterative equation $\mathbf{x}_{i+1} = J_{\mathbf{f}|\mathbf{x}}\big|_{\mathbf{x}_i}\mathbf{x}$, $i \ge 1$, define a point sequence $\{\mathbf{x}_i\}_0^\infty$, namely $\mathbf{x}_1 = J_{\mathbf{f}|\mathbf{x}}\big|_{\mathbf{x}_0}\mathbf{x}_0$, $\mathbf{x}_2 = J_{\mathbf{f}|\mathbf{x}}\big|_{\mathbf{x}_1}\mathbf{x}_1$, $\cdots$, $\mathbf{x}_k = J_{\mathbf{f}|\mathbf{x}}\big|_{\mathbf{x}_{k-1}}\mathbf{x}_{k-1}, \cdots$. Notice that although each term of $J_{\mathbf{f}|\mathbf{x}}\big|_{\mathbf{x}_0}$, $J_{\mathbf{f}|\mathbf{x}}\big|_{\mathbf{x}_1}$, $\cdots$, $J_{\mathbf{f}|\mathbf{x}}\big|_{\mathbf{x}_k}$, $\cdots$ is a matrix of constants but they may be different matrixes, i.e., different from a linear system that $J_{\mathbf{f}|\mathbf{x}}$ is the same matrix $A$

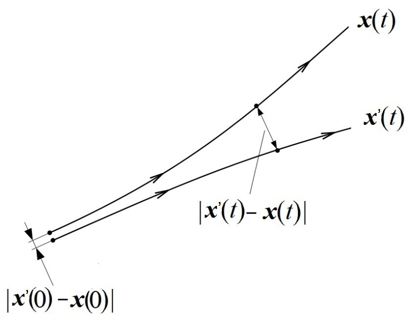



**Figure** 6.9. **Track spacing**

of constants. Similarly, for solutions $\mathbf{x}(t), \mathbf{x}'(t)$ at two different initial values $\mathbf{x}_0, \mathbf{x}'_0$ of the equation $\dot{\mathbf{x}} = J_{\mathbf{f}|\mathbf{x}}\big|_{\mathbf{x}^*}\mathbf{x}$, the distance $|\mathbf{x}'(t) - \mathbf{x}(t)|$ between them is shown in Figure 6.9. Now, $\mathbf{x}_k = \left(J_{\mathbf{f}|\mathbf{x}}\big|_{\mathbf{x}_{k-1}} \cdots J_{\mathbf{f}|\mathbf{x}}\big|_{\mathbf{x}_1} J_{\mathbf{f}|\mathbf{x}}\big|_{\mathbf{x}_0}\right)\mathbf{x}_0$. So,

$$\left|\mathbf{x}'_k - \mathbf{x}_k\right| \approx \left|J_{\mathbf{f}|\mathbf{x}}\big|_{\mathbf{x}_{k-1}} \cdots J_{\mathbf{f}|\mathbf{x}}\big|_{\mathbf{x}_1} J_{\mathbf{f}|\mathbf{x}}\big|_{\mathbf{x}_0}\right| \left|\mathbf{x}'_0 - \mathbf{x}_0\right|_{\circ}$$

Denoted by $\mathbf{J}_{[k]} = J_{\mathbf{f}|\mathbf{x}}\big|_{\mathbf{x}_{k-1}} \cdots J_{\mathbf{f}|\mathbf{x}}\big|_{\mathbf{x}_1} J_{\mathbf{f}|\mathbf{x}}\big|_{\mathbf{x}_0}$ and let $k \to \infty$. If the matrix $\mathbf{J}_{[k]}$ is limited, i.e., the limitation $\lim\limits_{k\to\infty} \mathbf{J}_{[k]}$ exists, we can define the *Lyapunov exponent* similar to that of 1-dimensional system (6.27), i.e.,

$$L_i = \lim_{k\to\infty} \frac{\ln\left|\lambda_i(\mathbf{J}_{[k]})\right|}{k} \quad \text{or} \quad L_i = \lim_{k\to\infty} \frac{\ln\left|\lambda_i\left(\mathbf{J}^t_{[k]}\mathbf{J}_{[k]}\right)\right|}{2k} \tag{6.28}$$

and obtain the *Lyapunov exponent sequence* $L_1 \leq L_2 \leq \cdots \leq L_n$, where $\lambda_i(A)$ denotes the $i$th eigenvalue of matrix $A$ for integers $1 \leq i \leq n$. Notice that $\mathbf{J}^t_{[k]}\mathbf{J}_{[k]}$ in the second formula of (6.28) is a symmetrical matrix. We know that for any mapping $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$, $\lambda_i\left(\mathbf{J}^t_{[k]}\mathbf{J}_{[k]}\right), k \geq 1$ are all non-negative real numbers.

Generally, we apply the first formula in (6.28) to calculate the Lyapunov exponent $L_i, 1 \leq i \leq n$ for characterizing the contraction or divergence rate of a dynamic system near an equilibrium point, namely the contraction of $L_i < 0$ in the $\boldsymbol{\xi}_i$ direction or divergence of $L_i > 0$ in the $\boldsymbol{\xi}_i$ direction. If $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_n$ are the eigenvectors corresponding to eigenvalues $\lambda_1(\mathbf{J}), \lambda_2(\mathbf{J}), \cdots, \lambda_n(\mathbf{J})$, respectively, then the distance between the solution tracks $\mathbf{x}(t), \mathbf{x}'(t)$ in vector $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_n$ directions are respectively in the rate $L_1$, $L_2, \cdots, L_n$ of contraction or divergence, i.e., the exponent rate $|\mathbf{x}(t) - \mathbf{x}'(t)|$ of convergence or divergence when $t \to \infty$, which is essentially the change extent of the function $\mathbf{f}$ at a point $\mathbf{x} \in \mathbb{R}^n$. In this way, we can also define the maximum Lyapunov exponent

$$L_{\max} = \lim_{k\to\infty} \sup \frac{\ln\left|\mathbf{J}_{[k]}\right|}{k} \tag{6.29}$$

and then, by the definition of matrix norm, $|\mathbf{f}(\mathbf{x})| \approx \left|\mathbf{J}_{\mathbf{f}|\mathbf{x}}\mathbf{x}\right| \leq \left|\mathbf{J}_{\mathbf{f}|\mathbf{x}}\right| |\mathbf{x}| \leq e^{L_{\max}}|\mathbf{x}|$ for any point $\mathbf{x} \in \mathbb{R}^n$ and integer $k \geq 1$, i.e., each contraction or divergence of $\mathbf{x}$ will not exceed $e^{L_{\max}}$ by one iteration of function $\mathbf{f}$.

**(3)Linear system exponent.** Notice that the Lyapunov exponent is the result of limiting the eigenvalues of Jacobian matrix by $k \to \infty$ for the nonlinear system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$. Then, *what is the Lyapunov exponent of a linear system and what is its relation to the matrix eigenvalues of a linear system?* Dr.Ouyang explains to his daughter that if the nonlinear item is $\mathbf{0}$ in the equation $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$, then the system can be characterized by a linear system $\dot{\mathbf{x}}(t) = A\mathbf{x}(t), \mathbf{x}(t_0) = \mathbf{x}_0$. In this case, let the eigenvectors $\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_n$

correspond respectively to the $n$ characteristic roots $\lambda_1, \lambda_2, \cdots, \lambda_n$ (double root double counting) in $\det(A - \lambda I_n) = 0$ of the matrix $A$, i.e., $A\boldsymbol{\xi}_i = \lambda_i I_n$ for integers $1 \le i \le n$. Particularly, *if the characteristic roots of n are different two by two, there exists a reversible linear transformation* $T : \mathbb{R}^n \to \mathbb{R}'^n$ *such that*

$$\dot{\mathbf{x}}' = \left(T^{-1}AT\right)\mathbf{x}' = \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & \cdots & 0 & 0 & \lambda_n \end{pmatrix} \quad \Rightarrow \quad \begin{cases} \dot{x}_1' = \lambda_1 x_1', \\ \dot{x}_2' = \lambda_2 x_2', \\ \cdots\cdots, \\ \dot{x}_n' = \lambda_n x_n' \end{cases}$$

namely the linear equation $\dot{\mathbf{x}} = A\mathbf{x}$ is equivalent to $n$ independent equations $\dot{x}_1' = \lambda_1 x_1'$, $\dot{x}_2' = \lambda_2 x_2'$, $\cdots$, $\dot{x}_n' = \lambda_n x_n'$ with respective solutions $x_1(t) = C_1 e^{\lambda_1 t}$, $x_2(t) = C_2 e^{\lambda_2 t}$, $\cdots$, $x_n(t) = C_n e^{\lambda_n t}$, where, $C_1, C_2, \cdots, C_n$ are constants. In this case, the average change rate of distances between tracks $\mathbf{x}(t), \mathbf{x}'(t)$ along the direction of vector $\boldsymbol{\xi}_i$ is $\sum\limits_{i=0}^{k-1} \ln e^{\lambda_i}/k$ after $k$ iterations.

Generally, the solution of linear equation $\dot{\mathbf{x}} = A\mathbf{x}, \mathbf{x}(t_0) = \mathbf{x}_0$ is $\mathbf{x}(t) = e^{tA}\mathbf{x}_0$, where

$$e^{tA} = I_n + A + \frac{A^2}{2!} + \cdots + \frac{A^k}{k!} + \cdots.$$

For example, let $A = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix}$. We have

$$e^{tA} = I_2 + t\begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} + \frac{t^2}{2!}\begin{pmatrix} 2^2 & 0 \\ 0 & 3^2 \end{pmatrix} + \cdots + = \begin{pmatrix} e^{2t} & 0 \\ 0 & e^{3t} \end{pmatrix}.$$

Now that for an initial point $\mathbf{x}_0 \in \mathbb{R}^n$ and an iterative equation $\mathbf{x}_{i+1} = e^{tA}\mathbf{x}_i, i \ge 1$, define a point sequence $\{\mathbf{x}_i\}_0^\infty$ by $\mathbf{x}_1 = \mathbf{x}_0, \mathbf{x}_2 = e^A\mathbf{x}_1 = e^{A^2}\mathbf{x}_0, \cdots, \mathbf{x}_k = e^A\mathbf{x}_{k-1} = \cdots = e^{A^k}\mathbf{x}_0$ for any integer $k \ge 1$. It should be noted that the differential $d\mathbf{f}/d\mathbf{x}$ or the Jacobian matrix is the matrix $e^A$ of constants and

$$\left.\frac{d\mathbf{f}}{d\mathbf{x}}\right|_{x_0} \left.\frac{d\mathbf{f}}{d\mathbf{x}}\right|_{x_1} \cdots \left.\frac{d\mathbf{f}}{d\mathbf{x}}\right|_{x_{k-1}} = \prod_{i=0}^{k-1} \left.\frac{d\mathbf{f}}{d\mathbf{x}}\right|_{x_i} = \prod_{i=0}^{k-1} e^A = e^{A^k}, \tag{6.30}$$

namely the distance of solution tracks $\mathbf{x}(t)$ and $\mathbf{x}'(t)$ grows $\left|e^{A^k}\right|$ times on the initial distance $|\mathbf{x}(t_0) - \mathbf{x}'(t_0)|$ after $k$ iterations. Now, let $k \to \infty$. Then, the Lyapunov exponent along the vector $\boldsymbol{\xi}_i$ direction is

$$L_i = \lim_{k\to\infty} \frac{\ln e^{\lambda_i(A^k)}}{k} = \lim_{k\to\infty} \frac{k \ln e^{\lambda_i(A)}}{k} = \lambda_i(A), \quad 1 \le i \le n \tag{6.31}$$

by (6.30), i.e., the Lyapunov exponent of a linear system $\dot{\mathbf{x}} = A\mathbf{x}$ agrees with the eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_n$ of the matrix $A$.

For example, the characteristic equation $\det(A - \lambda I_3) = \lambda^3 - 11\lambda^2 + 36\lambda - 36 = 0$ of the 3-dimensional linear system

$$\begin{cases} \dot{x}_1 = 3x_1 - x_2 + x_3 \\ \dot{x}_2 = -x_1 + 5x_2 - x_3 \\ \dot{x}_3 = x_1 - x_2 + 3x_3 \end{cases} \Rightarrow \quad A = \begin{pmatrix} 3 & -1 & 1 \\ -1 & 5 & -1 \\ 1 & -1 & 3 \end{pmatrix}$$

has 3 roots, i.e., $\lambda_1 = 2, \lambda_2 = 3, \lambda_3 = 6$. By (6.31) we know that the Lyapunov exponents of this linear system are respectively $L_1 = \lambda_1(A) = 2, L_2 = \lambda_2(A) = 3$ and $L_3 = \lambda_3(A) = 6$.

Therefore, for a linear system $S$ it is consistent for determining the stability of system $S$ by the Lyapunov exponents or by the matrix eigenvalues at an equilibrium point $\mathbf{x}^*$. Furthermore, we know the solution track of an autonomous system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$ near the equilibrium point $\mathbf{0}$ as follows:

(1) *If there are integers $i, 1 \leq i \leq n$ such that $L_i > 0$, the system $S$ is unstable at the equilibrium point $\mathbf{0}$ and if $L_i < 0$ for all integers $1 \leq i \leq n$, the system $S$ is asymptotically stable at the equilibrium point $\mathbf{0}$ (which is consistent with the result determined by matrix eigenvalues);*

(2) *If $L_1 = L_2 = \cdots = L_s = 0,\ L_i < 0,\ s+1 \leq i \leq n$, then the system $S$ may be stable or unstable. In this case, the track $\mathbf{x}(t)$ tends to a stable $m$-ring space. Particularly, $\mathbf{x}(t)$ tends to a stable limit cycle or a double-ring space when $m = 1$ or $m = 2$.*

**3.2.Master Stability Function.** The master stability function is a function defined for a system $S$ based on the maximum Lyapunov exponent $L_{\max}$ because it can be used to determine the stability of system $S$ at the equilibrium point and then, the system synchronization. Here, it is assumed that the system $S$ is composed of $m$ elements $v_1, v_2, \cdots, v_m$ and the state equation is

$$\dot{\mathbf{x}}_i(t) = \mathbf{f}(\mathbf{x}_i(t)) + c\sum_{j=1}^{m} l_{ij} H(\mathbf{x}_j), \quad 1 \leq i \leq m$$

i.e., the equation (6.8), where, the node dynamics equation is $\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t))$, $\mathbf{x}_i \in \mathbb{R}^n$ is the state variable of $i$th element of the system, $\mathbf{H} : \mathbb{R}^n \to \mathbb{R}^n$ is the action function between elements, $c > 0$ is the coupling strength of the network and $(l_{ij})_{m\times m}$ is the negative Laplian matrix.

Assumes that $\mathbf{s}(t)$ is the solution of node equation $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t))$ and $\boldsymbol{\xi}_i(t) = \mathbf{x}_1(t) - \mathbf{s}(t), 1 \leq i \leq m$. Furthermore, assume the system $S$ satisfies the following conditions, i.e., ①Its dynamic behaviors of elements $v_1, v_2, \cdots, v_m$ are identical; ②Its action functions of elements are identical, i.e., the function $\mathbf{H}(\mathbf{x}_j), 1 \leq i \leq m$; ③The point set

$\text{Sta}[S] = \{(\mathbf{x}_1(t), \mathbf{x}_2(t), \cdots, \mathbf{x}_m(t))\,|\mathbf{x}_1(t) = \mathbf{x}_2(t) = \cdots = \mathbf{x}_m(t)\}$ is an *invariable synchronized manifold*, namely for a point $\mathbf{x}_0 \in \text{Sta}[S]$ there always exists such a time $t^*$ that there is a solution $\mathbf{x}(\mathbf{x}_0, t)$ of system $S$ which is close enough to points in $\text{Sta}[S]$ if $t \geq t^*$; ④Its system state can be linearized around the synchronized manifold $\text{Sta}[S]$. Under these four assumptions, one can differentiate the equation (6.8) of system $S$ with respect to $\mathbf{s}$, called the *variation* of the system equation to $\mathbf{s}$, where $\mathbf{s}$ is a function of $t$ and then get the error equation

$$\dot{\boldsymbol{\xi}}_i(t) = \mathbf{J}_{\mathbf{f}|\mathbf{s}}\boldsymbol{\xi}_i(t) + c\sum_{j=1}^{m} l_{ij}\mathbf{J}_{\mathbf{H}|\mathbf{s}}\boldsymbol{\xi}_i(t), \quad 1 \leq i \leq m, \tag{6.32}$$

where, $\mathbf{J}_{\mathbf{f}|\mathbf{s}}, \mathbf{J}_{\mathbf{H}|\mathbf{s}}$ are respectively the Jacobian matrixes of $\mathbf{f}(\mathbf{x})$ and $\mathbf{H}(\mathbf{x})$ respect to $\mathbf{s}$. Let $\boldsymbol{\xi} = (\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \cdots, \boldsymbol{\xi}_m)$. Then, the equation (6.32) can be rewritten as

$$\dot{\boldsymbol{\xi}} = \mathbf{J}_{\mathbf{f}|\mathbf{s}}\boldsymbol{\xi} + c\mathbf{J}_{\mathbf{H}|\mathbf{s}}\boldsymbol{\xi}L^t. \tag{6.33}$$

If the matrix $L$ can be further diagonalized with eigenvalues $\lambda_1, \lambda_2, \cdots, \lambda_m$, there must exist such a reversible matrix $P$ that holds with $P^{-1}LP = \text{diag}\,(\lambda_1, \lambda_2, \cdots, \lambda_m)$. Assume that $\boldsymbol{\eta} = \boldsymbol{\xi}P = (\boldsymbol{\eta}_1, \boldsymbol{\eta}_2, \cdots, \boldsymbol{\eta}_m)$. Then, there are

$$\dot{\boldsymbol{\eta}}_k = \left(\mathbf{J}_{\mathbf{f}|\mathbf{s}} + c\lambda_k\mathbf{J}_{\mathbf{H}|\mathbf{s}}\right)\boldsymbol{\eta}_k, \quad 1 \leq k \leq m. \tag{6.34}$$

In this way, if the system $S$ is synchronized then the solution of equation (6.34) is asymptotically stable at the equilibrium point $\mathbf{0}$ and the Lyapunov exponent $L_i(\mathbf{0}) < 0, 1 \leq i \leq n$ can guarantee the stability of system $S$ at the point $\mathbf{0}$ or synchronization of system $S$ near the point $\mathbf{0}$. Notice that for an asymmetric matrix, its eigenvalues may be real or complex. Thus, $c\lambda_k$ in the equation (6.34) can be generally expressed as a complex number, i.e., $c\lambda_k = \alpha_k + i\beta_k, i^2 = -1$. Accordingly, the equation (6.34) can be rewritten as

$$\dot{\boldsymbol{\eta}}_k = \left(\mathbf{J}_{\mathbf{f}|\mathbf{s}} + (\alpha_k + i\beta_k)\,\mathbf{J}_{\mathbf{H}|\mathbf{s}}\right)\boldsymbol{\eta}_k, \quad 1 \leq k \leq m. \tag{6.35}$$

And then, *what is the master stability function of system?* Dr.Ouyang explains that since the Lyapunov index being negative is the stability or synchronization of system (6.32) at or near the equilibrium point $\mathbf{0}$, the maximum Lyapunov index $L_{\max}$ can be chosen for determining the system synchronization. At this point, $L_{\max}$ is a function of $L_{\max}(\alpha + i\beta)$ of $\alpha + i\beta$, called the *master stability function* and the region $SR$ constituted by $L_{\max}(\alpha + i\beta) < 0$ is called the *synchronized region* of system $S$, i.e.,

$$SR = \{\alpha + i\beta = c\lambda_k, 1 \leq\leq kn | L_{\max}(\alpha + i\beta) < 0\}, \tag{6.36}$$

i.e., the synchronous ability of system $S$ can be determined by whether $c\lambda$, the product of eigenvalue $\lambda$ of the adjacency matrix of label graph $G^L[S]$ and the coupling strength $c$ falls into the synchronized region $SR$. Notice that the stable function $L_{\max}$ can be viewed as a plane $\mathbb{R}^2 = \left\{(\alpha,\beta)|\alpha,\beta \in \mathbb{R}^2\right\}$. If $L_{\max}$ is negative or positive at a point on this plane, it shows the stability or instability of the system $S$ at that point. Particularly, if $L_{\max}$ is negative at all points on the plane, it implies that the system is synchronized as a result of the interaction between the coupling constant $c$ and the elements.

Now, we assume that all elements are equal roles then the labeled graph $G^L[S]$ of system $S$ is an undirected graph. In this case, the Laplacian matrix $L$ is symmetric with the minimum eigenvalue 0 and all other eigenvalues are real numbers greater than 0, namely $0 = \lambda_1(L) < \lambda_2(L) < \cdots < \lambda_m(L)$, i.e., $c\lambda_k = \alpha_k$ is a real number and the master stability function $L_{\max}$ is the function of $\alpha$, namely the synchronized region (6.36) is $SR = \{\alpha = c\lambda(L)|L_{\max}(\alpha) < 0\}$. Particularly, if there is $c\lambda_k \in SR$ for any integer $2 \leq k \leq m$, the system $S$ is locally synchronized.

By definition, the synchronized region $SR$ of system $S$ consists of interval segments of $\mathbb{R}$, which can be divided into the following four types:

**Type I.** $SR$ is an unbounded interval $(\alpha_1, \infty)$, namely $L_{\max} < 0$ if $0 < \alpha_1 < c\lambda_2 \leq c\lambda_3 \leq \cdots \leq c\lambda_m$. In this case, there is $c\lambda_2 > \alpha_1$ or $c \geq \alpha_1/\lambda_2$. Notice that $\lambda_2$ is the smallest non-zero eigenvalue of the Laplacian matrix $L$ and so, the larger $\lambda_2$ is the smaller coupling constant $c$ is needed, which implies the stronger synchronous ability of system.

**Type II.** $SR$ is a bounded interval $(\alpha_1, \alpha_2)$, $-\infty < \alpha_1 \leq \alpha_2 < \infty$, namely $L_{\max} < 0$ if $\alpha_1 < c\lambda_2 \leq \cdots \leq c\lambda_m < \alpha_2$. Then, the coupling constant $c$ satisfies $\alpha_1/\lambda_2 < c < \alpha_2/\lambda_m$. Define $R = \lambda_m/\lambda_1 < \alpha_2/\alpha_1$. The smaller the $R$ is, closer to 1 further the stronger the system synchronous ability is.

**Type III.** $SR$ is a union of finite intervals, namely $SR = (\alpha_1, \alpha_2) \bigcup (\alpha_3, \alpha_4) \bigcup \cdots \bigcup (\alpha_{2k-1}, \alpha_{2k})$ or $SR = (\alpha_1, \alpha_2) \bigcup (\alpha_4, \alpha_3) \bigcup \cdots \bigcup (\alpha_{2k-1}, +\infty)$. In this case, the system synchronization requires $c\lambda_k \in SR$ to be valid for any integer $2 \leq k \leq m$, which is related to the system coupling constant $c$ and the Laplacian matrix $L$ of interaction in elements. It is needed to adjust the coupling strength of system and the interactions of elements for the system synchronization.

**Type IV.** $SR$ is an empty set $\emptyset$, i.e., for any number $\alpha \in \mathbb{R}$, $L_{\max}(\alpha) \geq 0$ there are no real numbers $\alpha$ such that $L_{\max} < 0$. In this case there is no system synchronization holding with assumptions ①-④ regardless of how changing the interaction or coupling strength between the system elements but needs to adjust the assumptions.

So, if all the numbers $c\lambda_k, 2 \leq k \leq m$ do not fall in the synchronized region $SR$ of system, *whether the system must be out of synchronization?* The answer is No! Dr.Ouyang explains to his daughter, this case only implies that the condition is not met the synchronization of assumptions ①-④ but not lead to the conclusion that the system is necessarily not synchronized.

For example, the adjacency matrix of star system $S_{1,m-1}$ shown in Figure 6.10 is

$$A\left(S_{1,m-1}\right)=\begin{pmatrix} 0 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}$$

with eigenvalues

$$\lambda_1=0, \lambda_2=\cdots=\lambda_{m-1}=1, \lambda_m=m,$$

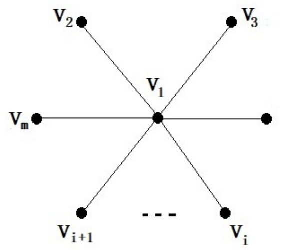



**Figure** 6.10. **Star system**

which belongs to Type II. In this case, if $m \to \infty$ then $R=\alpha_2/\alpha_1 \to \infty$ instead of tending to 1. By the method of master stability function, the system is not easy to realize the synchronization if the number $m$ of system elements is large enough. However, this is only a conclusion in the case of system synchronization holding with the assumptions ①-④, not that it can not be synchronized. For example, let the state functions of elements be $\dot{x}_{v_i}(t)=x_{v_i}^3(t)+t^i/e^t$. Then, the system will eventually be synchronized to the node function $\dot{s}^3(t)=s(t)$.

**3.3.Linear Coupled Synchronization.** If the interaction function $\mathbf{H}:\mathbb{R}^n \to \mathbb{R}^n$ between elements is linear, which can be expressed as an $n \times n$ matrix $H$ on a spanning basis $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ and the state equation (6.8) can be divided into the sum of the node function $\mathbf{f}(\mathbf{x}(t))$ and the linear items $H\mathbf{x}_j, 1 \le j \le m$, i.e.,

$$\dot{\mathbf{x}}_i(t)=\mathbf{f}\left(\mathbf{x}(t)\right)+c\sum_{j=1}^{m} l_{ij}H\mathbf{x}_j, \quad 1 \le j \le m, \tag{6.37}$$

a few interesting criterions for system synchronization can be obtained. Particularly, if the matrix $H$ is a diagonal matrix $\Gamma$, a more convenient criterion for system synchronization can be found which enables one to get the criterion on system synchronization by graph comparison in next section. Generally, *if there is such a diagonal matrix* $\Gamma$ *of positive real numbers and the constants* $\omega_0>0, \tau_0>0$ *that for any positive number* $\omega \ge \omega_0$*,*

$$c\lambda_2 \ge \omega_0, \quad \left(\mathbf{J}_{\mathbf{f}|\mathbf{x}}\big|_{\mathbf{s}}+\omega H\right)^t\Gamma+\Gamma\left(\mathbf{J}_{\mathbf{f}|\mathbf{x}}\big|_{\mathbf{s}}+\omega H\right) \le -\tau I_n, \tag{6.38}$$

*then the system* (6.37) *is stable at the point* $\mathbf{0}$*, i.e., synchronized near the point* $\mathbf{0}$*,* where $\mathbf{J}_{\mathbf{f}|\mathbf{x}}\big|_{\mathbf{s}}$ denotes the matrix value of Jacobian matrix $\mathbf{J}_{\mathbf{f}|\mathbf{x}}$ at $\mathbf{s}$.

So, *how do we get this conclusion from the Lyapunov function of system?* First of all, since $\lambda_2$ is the smallest non-zero eigenvalue of the Laplacian matrix, it is true $c\lambda_k \ge \omega_0$

for any integer $2 \le k \le m$. Thus, for any integer $2 \le k \le m$,

$$\left(\mathbf{J}_{\mathbf{f}|\mathbf{x}}\big|_{\mathbf{s}} + c\lambda_k H\right)^t \Gamma + \Gamma\left(\mathbf{J}_{\mathbf{f}|\mathbf{x}}\big|_{\mathbf{s}} + c\lambda_k H\right) \le -\tau I_n$$

by the assumption. Let the Lyapunov function $L(t) = \boldsymbol{\eta}_k^t \Gamma \boldsymbol{\eta}_k$, where the vector $\boldsymbol{\eta}_k$ is characterized by the equation (6.34). Notice that any diagonal entry in the diagonal matrix $\Gamma$ is positive, we know that $L(t) > 0$ and

$$\begin{aligned}\dot{L}(t) &= \dot{\boldsymbol{\eta}}_k^t \Gamma \boldsymbol{\eta}_k + \boldsymbol{\eta}_k^t \Gamma \dot{\boldsymbol{\eta}}_k \\ &= \boldsymbol{\eta}_k^t \left(\left(\mathbf{J}_{\mathbf{f}|\mathbf{s}}^t + c\lambda_k H\right)\Gamma + \boldsymbol{\eta}_k^t \Gamma\left(\mathbf{J}_{\mathbf{f}|\mathbf{s}} + c\lambda_k H\right)\right)\boldsymbol{\eta}_k \le -\tau \boldsymbol{\eta}_k^t I_n \boldsymbol{\eta}_k < 0,\end{aligned}$$

i.e., the system (6.37) is stable at the equilibrium point $\mathbf{0}$ or synchronized near $\mathbf{0}$.

Particularly, if the elements $v_i \in S$ are all in the free state, namely for any integer $1 \le i_1, i_2 \le m$, there are no interaction between $v_{i_1}$ and $v_{i_2}$ and the labeled graph $G^L[S] = \bigcup_{i=1}^{m} B_1^L$ of system $S$ is the union of $m$ bouquet graphs of order $m$ as shown in Figure 6.11.

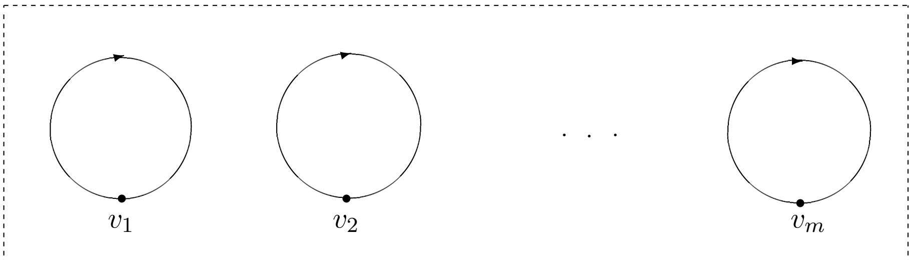



**Figure** 6.11 **Free action system of elements**

In this case, the adjacency matrix of system $S$ is $A = I_m$ with the Laplacian matrix $L = -I_m$. Let $H = I_n$, $\Gamma = I_n$. Then, the diagonal entry of the matrix $\left(\mathbf{J}_{\mathbf{f}|\mathbf{x}}\big|_{\mathbf{s}} + c\lambda_k H\right)^t \Gamma + \Gamma\left(\mathbf{J}_{\mathbf{f}|\mathbf{x}}\big|_{\mathbf{s}} + c\lambda_k H\right)$ is $2\left(L_k - c\lambda_k\right)$, $1 \le k \le n$, namely

$$\left(\mathbf{J}_{\mathbf{f}|\mathbf{x}}\big|_{\mathbf{s}} + \omega H\right)^t \Gamma + \Gamma\left(\mathbf{J}_{\mathbf{f}|\mathbf{x}}\big|_{\mathbf{s}} + \omega H\right) \le -\tau I_n$$

is equivalent to that $L_k - c\lambda_k \le -\tau/2$ for any integer $1 \le k \le n$. In this case, let the constant $\omega_0 = L_{\max}$ and $\tau = 2\min\{L_{\max} - L_k, 1 \le k \le n\}$. If the system $S$ satisfies $c\lambda_2 \ge L_{\max}$ then there must be $c\lambda_k - \tau/2 \ge L_k, 1 \le k \le n$ and at the same time, $\omega \ge L_{\max}$ hold with the condition

$$\left(\mathbf{J}_{\mathbf{f}|\mathbf{x}}\big|_{\mathbf{s}} + \omega H\right)^t \Gamma + \Gamma\left(\mathbf{J}_{\mathbf{f}|\mathbf{x}}\big|_{\mathbf{s}} + \omega H\right) \le -\tau I_n$$

in (6.38), namely for a free action system $S$ of elements, if $c\lambda_2 \ge L_{\max}$ then the system $S$ is synchronized in a neighborhood of point $\mathbf{0}$.

## §4. Graph Comparison

Generally, the distinguishing of nonlinear system synchronization is converted to determining the stability of system at equilibrium points. Whence, it is necessary to construct a Lyapunov function by the features of system equation but it is a little complicated. Huizi asks Dr.Ouyang:" *is there an intuitive and computationally convenient way for determining the system synchronization?*" Dr.Ouyang tells her that for a system with a simple acting functions **H** on elements, i.e

$$\mathbf{x}_i = \mathbf{f}\left(\mathbf{x}_i\right) + \sum_{j=1}^{m} \varepsilon_{ij}(t) H \mathbf{x}_j, \quad 1 \leq i \leq m, \tag{6.39}$$

there is an operation on graphs for calculating and determining the system synchronization on graph $G^L[S]$, where $H = \mathrm{diag}\left(h_1, h_2, \cdots, h_n\right)$ is a diagonal matrix, $h_i \geq 0$ for integers $1 \leq i \leq s$ but $h_i = 0$ for integers $s+1 \leq i \leq n$ and $\varepsilon_{ij}(t)$ is the coupling strength of $v_j$ to $v_i$ with $\varepsilon_{ij}(t) = \varepsilon_{ji}(t)$ and

$$\varepsilon_{ii}(t) = -\sum_{j=1, j\neq i}^{m} \varepsilon_{ij}(t), \ 1 \leq i \leq m$$

similar to Laplacian matrix. In this case, the vertex set, edge set and labels of the system labeled graph $G^L[S]$ are respectively

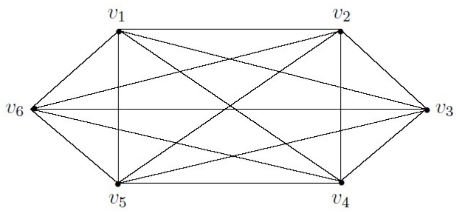



**Figure** 6.12. **Graph** $K_6$

$$V\left(G^L[S]\right) = \{v_1, v_2, \cdots, v_m\}, \quad E\left(G^L[S]\right) = \{(v_i, v_j) |\ \varepsilon_{ij}(t) \neq 0, 1 \leq i, j \leq m\}$$

and $L:\ v_i \to \dot{\mathbf{x}}_i(t),\ \ (v_i, v_j) \to \varepsilon_{ij}(t),\ 1 \leq i, j \leq m$. Particularly, if $\varepsilon_{ij} \neq 0, 1 \leq i, j \leq m$, then $G^L[S]$ is a complete graph $K_m^L$. For example, the label of vertex $v_1$ of $K_6$ is $\dot{\mathbf{x}}_{v_1}$ in Figure 6.12, the label on edge $(v_1, v_2)$ is $\varepsilon_{12}(t)$. However, if $\varepsilon_{12}(t) = 0$, i.e., the label on edge $(v_1, v_2)$ is **0**, we can let the edge $(v_1, v_2)$ off in $K_6^L$ to get $G^L[S]$.

**4.1.Walk Covering Index.** Assume that a system $S$ consists of $m$ elements $v_1,\ v_2,\ \cdots,$ $v_m$ and $G^L[S]$ is a undirected graph. A walk in an undirected graph is defined to be an alternating sequence of vertex-edge in the graph that have no repeating edges. For example, it is easy to see that the vertex edge alternating sequence $\mathcal{P} = v_1 a v_2 f v_6$ is a walk of $v_1 - v_6$ in Figure 6.13. So, *what is walk covering?* Dr.Ouyang tells Huizi that a *walk covering* $\mathscr{P}$ of a system $S$ is such a set of walks on the

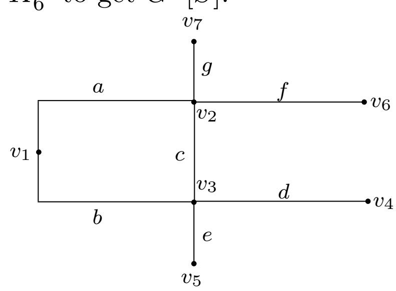



**Figure** 6.13. **Unicycle graph**

graph $G^L[S]$ that for any integers $1 \leq i,\ j \leq m$ there is a unique walk $\mathcal{P}_{ij} \in \mathscr{P}, j > i$ from $v_i$ to $v_j$. Now, let $e \in E\left(G^L[S]\right)$ and $\mathscr{P}$ be a walk covering of $G^L[S]$. The index $b_e[\mathscr{P}]$ is defined by the sum of lengths of all walks $\mathcal{P}_{ij}$ passing through the edge $e$ with $j > i$ in the walk covering $\mathscr{P}$. Among these, the maximum number of indexes $b_e[\mathscr{P}], e \in E(G^L[S])$ is called the *walk covering index* of $\mathscr{P}$, denoted by $b_{\max}[\mathscr{P}]$, i.e.,

$$b_{\max}[\mathscr{P}] = \max\left\{\ b_e[\mathscr{P}] \mid e \in E\left(G^L[S]\right)\right\}. \tag{6.40}$$

Furthermore, let $\mathscr{P}_1, \mathscr{P}_2, \cdots, \mathscr{P}_s$ be all walk coverings on the graph $G^L[S]$ of system $S$. Define the *walk covering index* of system $S$ to be

$$b_{\min}[S] = \min\left\{\ b_{\max}\left[\mathscr{P}_i\right]\ ;\ 1 \leq i \leq s\ \right\}, \tag{6.41}$$

i.e., the minimum index $b_{\max}[\mathscr{P}]$ in all walk covering of system $S$.

So, *why do we introduce walk covering index and how to determine the number $b_{\min}[S]$?* Dr.Ouyang explains that the introduction of the walk covering index is the basis for determining the stability or synchronization of system (6.39) by using the operation on graph. Certainly, the walk covering index can be determined by the definition (6.40) and (6.41). First of all, we can calculate the walk covering index $b_{\max}[\mathscr{P}]$ for a given covering $\mathscr{P}$ of $G^L[S]$ and then, choose the minimum number in walk covering indexes $b_{\max}[\mathscr{P}]$ for all walk covering $\mathscr{P}$ of $G^L[S]$, which is nothing else but the $b_{\min}[S]$. Here, it is easy to calculate $b_{\max}[\mathscr{P}]$ for a walk covering $\mathscr{P}$. For example, a walk covering $\mathscr{P}$ determined by the shortest walks on the graph in Figure 6.13 is

$$\begin{aligned}
&\mathcal{P}_{12} = v_1av_2,\ \mathcal{P}_{13} = v_1bv_3,\ \mathcal{P}_{14} = v_1bv_3dv_4,\ \mathcal{P}_{15} = v_1bv_3ev_5, \mathcal{P}_{16} = v_1av_2fv_6,\\
&\mathcal{P}_{17} = v_1av_2gv_7,\ \mathcal{P}_{23} = v_2cv_3,\ \mathcal{P}_{24} = v_2cv_3dv_4,\ \mathcal{P}_{25} = v_2cv_3ev_5,\ \mathcal{P}_{26} = v_2fv_6,\\
&\mathcal{P}_{27} = v_2gv_7,\ \ \mathcal{P}_{34} = v_3dv_4,\ \ \mathcal{P}_{35} = v_3ev_5,\ \ \mathcal{P}_{36} = v_3cv_2fv_6,\ \ \mathcal{P}_{37} = v_3cv_2gv_7,\\
&\mathcal{P}_{45} = v_4dv_3ev_5,\ \mathcal{P}_{46} = v_4dv_3cv_2fv_6,\ \mathcal{P}_{47} = v_4dv_3cv_2gv_7,\ \mathcal{P}_{56} = v_5ev_3cv_2fv_6,\\
&\mathcal{P}_{57} = v_5ev_3cv_2gv_7,\ \mathcal{P}_{67} = v_6fv_2gv_7
\end{aligned}$$

with respective lengthes

$$\begin{aligned}
&|\mathcal{P}_{12}| = |\mathcal{P}_{13}| = |\mathcal{P}_{23}| = |\mathcal{P}_{26}| = |\mathcal{P}_{27}| = |\mathcal{P}_{34}| = |\mathcal{P}_{35}| = 1,\\
&|\mathcal{P}_{14}| = |\mathcal{P}_{15}| = |\mathcal{P}_{16}| = |\mathcal{P}_{17}| = |\mathcal{P}_{24}| = |\mathcal{P}_{25}| = |\mathcal{P}_{36}| = 2,\\
&|\mathcal{P}_{37}| = |\mathcal{P}_{45}| = |\mathcal{P}_{67}| = 2,\ |\mathcal{P}_{46}| = |\mathcal{P}_{47}| = |\mathcal{P}_{56}| = |\mathcal{P}_{57}| = 3.
\end{aligned}$$

By the definition of $b_{kl}[\mathscr{P}]$, we know that

$$\begin{aligned}
b_{12}[\mathscr{P}] &= |\mathcal{P}_{12}| + |\mathcal{P}_{16}| + |\mathcal{P}_{17}| = 1 + 2 + 2 = 5,\\
b_{13}[\mathscr{P}] &= |\mathcal{P}_{13}| + |\mathcal{P}_{14}| + |\mathcal{P}_{15}| = 1 + 2 + 2 = 5,
\end{aligned}$$

$$\begin{aligned}
b_{23}[\mathscr{P}] &= |\mathcal{P}_{23}| + |\mathcal{P}_{24}| + |\mathcal{P}_{25}| + |\mathcal{P}_{36}| + |\mathcal{P}_{37}| \\
&\quad + |\mathcal{P}_{46}| + |\mathcal{P}_{47}| + |\mathcal{P}_{56}| + |\mathcal{P}_{57}| \\
&= 1+2+2+2+2+3+3+3+3+3 = 21, \\
b_{26}[\mathscr{P}] &= |\mathcal{P}_{16}| + |\mathcal{P}_{26}| + |\mathcal{P}_{36}| + |\mathcal{P}_{46}| + |\mathcal{P}_{56}| + |\mathcal{P}_{67}| \\
&= 2+1+2+3+3+2 = 13, \\
b_{27}[\mathscr{P}] &= |\mathcal{P}_{27}| + |\mathcal{P}_{37}| + |\mathcal{P}_{47}| + |\mathcal{P}_{57}| + |\mathcal{P}_{67}| \\
&= 1+2+3+3+2 = 11, \\
b_{34}[\mathscr{P}] &= |\mathcal{P}_{24}| + |\mathcal{P}_{34}| + |\mathcal{P}_{45}| + |\mathcal{P}_{46}| + |\mathcal{P}_{47}| \\
&= 2+1+2+3+3 = 11, \\
b_{35}[\mathscr{P}] &= |\mathcal{P}_{15}| + |\mathcal{P}_{25}| + |\mathcal{P}_{35}| + |\mathcal{P}_{45}| + |\mathcal{P}_{56}| + |\mathcal{P}_{57}| \\
&= 2+2+1+2+3+3 = 13
\end{aligned}$$

and $b_{\max}[\mathscr{P}] = b_{23}[\mathscr{P}] = 21$ by (6.40). So, *is its length definitely greater than* 21 *if a walk covering* $\mathscr{P}$ *obtained by not traveling on the shortest walk of graph in Figure* 6.13*?* The answer is No! For example, if we replace $\mathcal{P}_{23}$ in the walk covering $\mathscr{P}$ by $\mathcal{P}'_{23} = v_2 a v_1 b v_3$ and leave other walks unchanged, we get a walk covering $\mathscr{P}'$ of $G^L[S]$ of system $S$. In this case, the calculation results on $b_{12}[\mathscr{P}'], b_{13}[\mathscr{P}']$ and $b_{23}[\mathscr{P}']$ will be replaced respectively by

$$\begin{aligned}
b_{12}[\mathscr{P}'] &= b_{13}[\mathscr{P}'] = \left|\mathcal{P}'_{12}\right| + \left|\mathcal{P}'_{16}\right| + \left|\mathcal{P}'_{17}\right| + \left|\mathcal{P}'_{23}\right| \\
&= 1+2+2+2 = 7, \\
b_{23}[\mathscr{P}'] &= \left|\mathcal{P}'_{24}\right| + \left|\mathcal{P}'_{25}\right| + \left|\mathcal{P}'_{36}\right| + \left|\mathcal{P}'_{37}\right| + \left|\mathcal{P}'_{46}\right| + \left|\mathcal{P}'_{47}\right| + \left|\mathcal{P}'_{56}\right| + \left|\mathcal{P}'_{57}\right| \\
&= 2+2+2+2+3+3+3+3+3 = 20
\end{aligned}$$

with an index $b_{\max}[\mathscr{P}'] = 20 < b_{\max}[\mathscr{P}]$, which shows that the index $b_{\max}[\mathscr{P}]$ of walk covering obtained by the shortest walks is not necessarily the $b_{\min}[S]$ of system $S$ because $b_{\min}[S]$ is taken over all walk covering indexes $b_{\max}[\mathscr{P}]$ on $G^L[S]$ for the minimum. This is the difficulty in determining $b_{\min}[S]$ of system $S$.

**4.2.Walk Covering Criterion.** And then, *how can we apply the index* $b_{\max}[S]$ *or* $b_{\min}[S]$ *to characterize the stability of system and obtain the system synchronization?* Dr.Ouyang explains that for any integer pair $1 \le i, j \le m$, we introduce variables $\mathbf{X}_{ij} = \mathbf{X}_j - \mathbf{X}_i$ and then, the system (6.39) becomes

$$\dot{\mathbf{X}}_{ij} = \mathbf{f}(\mathbf{x}_j) - \mathbf{f}(\mathbf{x}_i) + \sum_{k=1}^{m} \left(\varepsilon_{jk} H \mathbf{X}_{jk} - \varepsilon_{ik} H \mathbf{X}_{ik}\right). \tag{6.42}$$

By the properties of definite integral and Jacobian matrix $\mathbf{J}_{\mathbf{f}|\mathbf{x}}$, we know that

$$\mathbf{f}(\mathbf{x}_j) - \mathbf{f}(\mathbf{x}_i) = \left(\int_0^1 \mathbf{J}_{\mathbf{f}|\mathbf{x}}(\tau\mathbf{x}_j + (1-\tau)\mathbf{x}_i)\, d\tau\right)\mathbf{X}_{ij}, \tag{6.43}$$

which enables us to construct an auxiliary system of equations

$$\dot{\mathbf{X}}_{ij} = \left(\int_0^1 \mathbf{J}_{\mathbf{f}|\mathbf{x}}(\tau\mathbf{x}_j + (1-\tau)\mathbf{x}_i)\, d\tau - A\right)\mathbf{X}_{ij},\ 1 \le i, j \le m \tag{6.44}$$

for determining the stability of system (6.42), where $A = \mathrm{diag}(a_1, a_2, \cdots, a_n)$ is complete determined by the matrix $H$ and $a_i \ge 0$ if $1 \le i \le s$; $a_i = 0$ if $s+1 \le i \le n$. Notice that applying the Lyapunov criterion for determining the synchronization of system (6.39) is equivalent to that of the stability of system (6.44) at the equilibrium point $\mathbf{0}$. In this case, we need an assumptions following for constructing the Lyapunov function $L(t)$:

**Condition 4.** *For any integers $1 \le i, j \le m$, assume that there are functions $\omega_{ij}(t) = \frac{1}{2}\mathbf{X}_{ij}^t D\mathbf{X}_{ij}$, where $D = \mathrm{diag}(d_1, d_2, \cdots, d_s, D_1)$ for constants $d_i, 1 \le i \le s$, $D_1$ is an $(n-s)\times(n-s)$ positive definite matrix and the derivative*

$$\dot{\omega}_{ij}(t) = \mathbf{X}_{ij}^t D\left(\int_0^1 \mathbf{J}_{\mathbf{f}|\mathbf{x}}(\tau\mathbf{x}_j + (1-\tau)\mathbf{x}_i)\, d\tau - A\right)\mathbf{X}_{ij} < 0 \tag{6.45}$$

*works to $t$ for any vector $\mathbf{X}_{ij} \ne \mathbf{0}$, then $\omega_{ij}(t)$ has the property of Lyapunov function.*

Notice that $H$ and $A$ are both diagonal matrices and we can let $A = a_0 H$ for a constant $a_0$. Now that by (6.43), if $\mathbf{f}(\mathbf{x})$ is continuously differentiable in system (6.39), then (6.45) can be simplified to the condition

$$(\mathbf{x}_j - \mathbf{x}_i)^t D\left((\mathbf{f}(\mathbf{x}_j) - \mathbf{f}(\mathbf{x}_i)) - a_0 H(\mathbf{x}_j - \mathbf{x}_i)\right) < 0. \tag{6.46}$$

On the other hand, the condition $\varepsilon_{ij}(t) = \varepsilon_{ji}(t)$ implies that the edges $(v_{i_k}, v_{j_k})$ and $(v_{j_k}, v_{i_k})$ exist simultaneously in graph $G^L[S]$, which can be combined into one edge $e_k$. Without loss of generality, assume that there are $M$ edges $e_1, e_2, \cdots, e_M$ with respective coupling strengthes $\varepsilon_1(t), \varepsilon_2(t), \cdots, \varepsilon_M(t) \ne 0$ on the labeled graph $G^L[S]$. Now that *if $|\mathbf{x}_i(t)| < \infty$ for integers $1 \le i \le m$ and the system* (6.39) *holds with Condition* 4,

$$\sum_{k=1}^{M} \varepsilon_{i_k j_k} |\mathbf{X}_{i_k j_k}|^2 > \frac{a_0}{m} \sum_{i=1}^{m-1} \sum_{j>i}^{m} |\mathbf{X}_{ij}|^2, \tag{6.47}$$

*then the system* (6.39) *is asymptotically stable at the equilibrium point* $\mathbf{0}$. Dr.Ouyang tells Huizi that the inequality (6.47) seems complicated but it is actually a condition introduced for the existence of Lyapunov functions. Firstly, define a function

$$L(t)=\frac{1}{4}\sum_{i,j=1}^{m}\mathbf{X}_{ij}^{t}D\mathbf{X}_{ij}.$$

Then, by the definition of $D$, $L(t)\geq 0$ and $\dot{L}(t)<0$ can be further verified, namely

$$\begin{aligned}\dot{L}(t) &= \frac{1}{2}\sum_{i,j=1}^{m}\dot{\omega}_{ij}(t)+\frac{1}{2}\sum_{i,j=1}^{m}\mathbf{X}_{ij}^{t}DA\mathbf{X}_{i}j\\ &\quad -\frac{1}{2}\sum_{i,j,k=1}^{m}\left(\varepsilon_{jk}\mathbf{X}_{ji}^{t}DP\mathbf{X}_{jk}+\varepsilon_{ik}\mathbf{X}_{ik}^{t}DP\mathbf{X}_{ij}\right)=\Sigma_1+\Sigma_2+\Sigma_3.\end{aligned}$$

Notice that $\Sigma_1<0$ by Condition 4. Then, because of $\varepsilon_{ij}(t)=\varepsilon_{ji}(t), 1\leq i,j\leq m$ and the assumption (6.47), we know that $|\mathbf{X}_{ij}|^2=|\mathbf{X}_{ji}|^2, |\mathbf{X}_{ii}|^2=0$. Thus,

$$\Sigma_2+\Sigma_3=\sum_{i=1}^{m-1}\sum_{j>i}^{n}\mathbf{X}_{ij}^{t}D\left(A-m\varepsilon_{ij}H\right)\mathbf{X}_{ij}<0.$$

So, there will be $\dot{L}(t)<0$, i.e., $L(t)$ is a Lyapunov function on system $S$. By the Lyapunov criterion we know that the system (6.42) is asymptotically stable at the point $\mathbf{0}$, i.e., the system (6.39) is synchronized in a neighborhood of point $\mathbf{0}$.

Then, *how do we use the walk covering index of system to further clarifying the condition* (6.47) *and obtain the condition that the coupling strength $\varepsilon_k(t)$ satisfies on edge $e_k$ if the system is synchronized?* Firstly, if $e_k$ is an edge of $G^L[S]$, denoted by $\overline{\mathbf{X}}_k=\mathbf{X}_{i_kj_k}$ for integers $1\leq k\leq M$. Now, for each pair of elements $\{v_i,v_j\}$, $1\leq i,j\leq m$, we get a walk covering $\mathscr{P}$ of system $S$ by selecting a walk $\mathcal{P}_{ij}$ from $v_i$ to $v_j$. Assume that $\mathcal{P}_{ij}=v_ie_1v_{i_1}e_2\cdots e_{|\mathcal{P}_{ij}|}v_j$. By the definition of $\mathbf{X}_{ij}$, there must be

$$\mathbf{X}_{ij} = \mathbf{X}_{ii_1}+\mathbf{X}_{i_1i_2}+\cdots+\mathbf{X}_{j-1j}=\overline{\mathbf{X}}_1+\overline{\mathbf{X}}_2+\cdots+\overline{\mathbf{X}}_{|\mathcal{P}_{ij}|}$$

For any integer $n\geq 1$, let $(b_1,b_2,\cdots,b_n)=(1,1,\cdots,1)$ in Cauchy-Schwartz inequality (4.33). We then know that $(a_1+a_2+\cdots+a_n)^2\leq n\left(a_1^2+a_2^2+\cdots+a_n^2\right)$, i.e., the inequality

$$|\mathbf{X}|_{ij}^2=\left|\sum_{k=1}^{|\mathcal{P}_{ij}|}\overline{\mathbf{X}}_k\right|^2\leq\left(\sum_{k=1}^{|\mathcal{P}_{ij}|}\left|\overline{\mathbf{X}}_k\right|\right)^2\leq|\mathcal{P}_{ij}|\sum_{k=1}^{|\mathcal{P}_{ij}|}\left|\overline{\mathbf{X}}_k\right|^2.$$

Substituting it into the right hand side of (6.47) we get that

$$\sum_{i=1}^{m-1}\sum_{j>i}^{m}|\mathbf{X}_{ij}|^2\leq\sum_{k=1}^{m}\left(\sum_{i=1}^{m-1}\sum_{j>i,e_k\in\mathcal{P}_{ij}}^{m}|\mathcal{P}_{ij}|\right)\left|\overline{\mathbf{X}}_k\right|^2.$$

Whence, if

$$\varepsilon_{i_kj_k}(t) > \frac{a_0}{m}\sum_{j>i, e_k\in\mathcal{P}_{ij}}^{m}|\mathcal{P}_{ij}| = \frac{a_0}{m}b_{e_k}[\mathscr{P}],\ 1\leq k\leq M,$$

the condition (6.47) must be true. Thus, *if the coupling strengthes* $\varepsilon_1(t), \varepsilon_2(t), \cdots, \varepsilon_k(t)$ *respectively on the the edges* $e_1, e_2, \cdots, e_M$ *of system graph in* (6.39) *hold with*

$$\varepsilon_k(t) > \frac{a_0}{m}b_{e_k}[\mathscr{P}]\ or\ furthermore\ \varepsilon_k(t) > \frac{a_0}{m}b_{\min}[S],\ 1\leq k\leq M, \tag{6.48}$$

*the system* (6.39) *is asymptotically stable at the point* $\mathbf{0}$*, i.e., the system is synchronized in a neighborhood of point* $\mathbf{0}$.

So, *how do we apply the coupling condition of* (6.48) *to construct a synchronized system?* Dr.Ouyang tells his daughter that the Lorenz system could be used as an example. Usually, there is an abstract convection model in the meteorology, i.e., to study the dynamic behavior of the fluid by heating below and cooling above the heat convection tube. In this model, the fluid velocity is characterized by a variable $x_1$, the horizontal and vertical temperature differences are characterized by variables $x_2$ and $x_3$, respectively. Based on this model, it is Lorenz, an American meteorologist proposed an autonomous equation

$$\begin{cases} \dot{x}_1 = a(x_2 - x_1) \\ \dot{x}_2 = bx_1 - x_2 - x_1x_3 \\ \dot{x}_3 = x_1x_2 - cx_3 \end{cases},$$

of dimension 3 for the state of fluid, called the *Lorenz system* in 1963. Among them, $a$ and $b$ are constants related to the known constants of the fluid, and $c$ is a constant related to the fluid space. Particularly, if $a = 10.0$, $b = 28.0$, $b = 8/3$, the system is in a chaotic state and sensitive to the selection of initial points. For a numerical simulation of Lorenz attractor, see Figure 6.14 as an example.

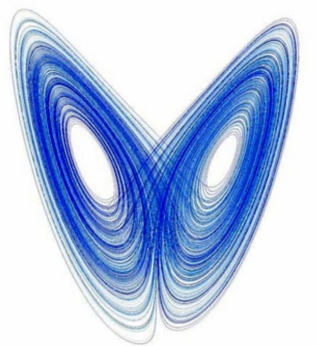


**Figure** 6.14. **Lorenz attractor**

In this case, the coupling condition (6.48) can be applied to construct a synchronized system. For example, choose $m$ Lorenz systems as the elements to form a system $S$, namely the behavior of each element follows Lorenz system and the system graph $G^L[S]$ is a star system shown in Figure 6.10. Then, for any integer $1 < i \leq m$ there exist only one walk $\mathcal{P}_{1i}$ between the element $v_i$ and $v_1$ with $|\mathcal{P}_{1i}| = 1$. Meanwhile, for any integer $1 < i, j \leq m$, any walk from the element $v_i$ to element $v_j$ must pass through the element $v_1$, i.e., $|\mathcal{P}_{ij}| = 2$. In this way, $\mathscr{P} = \{\mathcal{P}_{1i}, \mathcal{P}_{ij}; 1 < i, j \leq m\}$ is a walk covering of the graph

$G^L[S]$ and correspondingly, the sum of lengths passing through the edge $v_1v_i$ is

$$\sum_{j>i, e_k\in\mathcal{P}_{ij}}^{m} |\mathcal{P}_{ij}| = 1 + 2(m-1) = 2m-3.$$

By the condition (6.48), if the coupling strengthes on edges hold with

$$\varepsilon_k(t) > \frac{a_0}{m} b_{e_k}[\mathscr{P}] = a_0\left(2 - \frac{3}{m}\right),$$

then the system $S$ is synchronized, where $a_0$ is a non-zero constant.

**4.3.Graph Comparison.** For two $n\times n$ matrixes $A, B$, if $A-B$ is positive semi-definite, namely for any $n$-dimensional column vector $\mathbf{x}$, $\mathbf{x}^t(A-B)\mathbf{x} \geq 0$, it is said that the matrix $A$ is non-inferior to $B$, denoted by $A \succeq B$. In this case, there must be such a reversible matrix $P$ that $P^{-1}(A-B)P = \mathrm{diag}\,(\lambda_1\lambda_2, \cdots, \lambda_s, 0\cdots, 0)$ with $\lambda_i \geq 0, 1 \leq s \leq n$. Particularly, if there is at least one $\lambda_i > 0, 1 \leq s \leq n$, i.e., $A-B$ is positively definite, denoted by $A \succ B$. In this way, a partial order can be established on the $n\times n$ matrixes. Similarly, for two graphs $G, H$ of order $n$, if their corresponding Laplacian matrix $L(G), L(H)$ satisfies $L(G) \succeq L(H)$ or $L(G) \succ L(H)$, we define $G \succeq H$ or $G \succ H$ for determining the order of graphs. By definition, *for any constant $c \neq 0$, $cG \succeq H$ or $cG \succ H$ if and only if $c\lambda_k(G) > \lambda_k(H)$ or $c\lambda_k(G) \geq \lambda_k(H)$ for integers $1 \leq k \leq n$.*

Let the edge set of $G^L[S]$ be $E = \{e_1, e_2, \cdots, e_M\}$. By the definition of system (6.39), for any elements $v_{i_k}, v_{j_k}, 1 \leq k \leq M$, there is an non-zero edge $e_k = (v_{i_k}, v_{j_k})$ in graph $G^L[S]$ if and only if the coupling strength $\varepsilon_{e_k}(t) \neq 0$ and we can therefore characterize the behavior of $G^L[S]$ by edges of $\varepsilon_{e_k}(t) \neq 0$. Now, define an *elementary Laplacian matrix* $L_{e_k} = \left(l_{ij}^k\right)_{M\times M}$ of edge $e_k = (v_{i_k}, v_{j_k}) \in E\left(G^L[S]\right)$ by

$$l_{ij}^k = \begin{cases} -1, & e = (v_{i_k}, v_{j_k}) \text{ or } (v_{j_k}, v_{i_k}), \\ 1, & e = (v_{i_k}, v_{i_k}) \text{ or } (v_{j_k}, v_{j_k}) \\ 0, & e \neq (v_{i_k}, v_{j_k}), (v_{j_k}, v_{i_k}), (v_{i_k}, v_{i_k}) \text{ and } (v_{j_k}, v_{j_k}). \end{cases}$$

In other words, the elementary Laplacian matrix $L_{e_k}$ is an $M\times M$ matrix with entries $-1$ in positions $(i_k, j_k)$, $(j_k i_k)$ and 1 in positions $(i_k, i_k)$, $(j_k, j_k)$ but 0 in other positions. Then, by the definition of elementary Laplacian matrix we easily have a decomposition

$$L_{G^L[S]} = \sum_{e\in E} \varepsilon_e(t) L_e \tag{6.49}$$

of the Laplacian matrix $L_{G^L[S]}$.

Now that arranging the elementary Laplacian matrixes $L_{e_1}, L_{e_2}, \cdots, L_{e_M}$ in a sequence, then *for any numbers $c_1, c_2, \cdots, c_{M-1} > 0, M \geq 2$ there must be*

$$c\sum_{i=1}^{M-1} c_i L_{e_i} \succeq L_{e_M} \quad \Leftrightarrow \quad \sum_{i=1}^{M-1} c_i L_{e_i} \succeq \frac{1}{c} L_{e_M}, \tag{6.50}$$

*where* $c = \sum_{i=1}^{M} 1/c_i$. We prove the inequality (6.50) by the mathematical induction.

First of all, the inequality (6.50) must hold for $M = 2$ because of the left hand side $= c_1 L_{e_1} = 1/(1/c_1)L_{e_1} =$ the right hand side of (6.50) in this case. For $M = 3$, let $\mathbf{x} = (x_1, x_2, x_3)^t$ be a column vector. Then, the inequality of (6.50) is

$$c_1 \mathbf{x}^t L_{e_1} \mathbf{x} + c_2 \mathbf{x}^t L_{e_2} \mathbf{x} \ \geq \ \frac{1}{\frac{1}{c_1} + \frac{1}{c_2}} \mathbf{x}^t L_{e_3} \mathbf{x}$$

or

$$c_1 \mathbf{x}^t \begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mathbf{x} + c_2 \mathbf{x}^t \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \mathbf{x} \geq \frac{1}{\frac{1}{c_1} + \frac{1}{c_2}} \mathbf{x}^t \begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \mathbf{x}$$

in the matrix form, namely

$$c_1 (x_1 - x_2)^2 + c_2 (x_2 - x_3)^2 \geq \frac{c_1 c_2}{c_1 + c_2} (x_1 - x_3)^2 .$$

Now, let $a = x_1 - x_2, b = x_2 - x_3$ with $a + b = x_1 - x_3$ and substituting them in the previous formula, there must be

$$c_1 a^2 + c_2 b^2 \geq \frac{c_1 c_2}{c_1 + c_2} (a + b)^2 \quad \Rightarrow \quad (c_1 a - c_2 b)^2 \geq 0$$

which holds obviously and each of the step is reversible in the previous derivation. So, the formula (6.50) is true for $M = 3$.

General, assuming that the (6.50) is true for $3 \leq M \leq k$ we consider the case of $M = k + 1$. In this case, by the induction hypothesis we have

$$\begin{aligned} \sum_{i=1}^{k} c_i L_{e_i} &= \sum_{i=1}^{k-1} c_i L_{e_i} + c_k L_{e_k} \\ &\succeq \frac{L_{e_{k-1}}}{\sum\limits_{i=1}^{k-1} \frac{1}{c_i}} + c_k L_{e_k} \succeq \frac{L_{e_{k+1}}}{\sum\limits_{i=1}^{k-1} \frac{1}{c_i} + \frac{1}{c_k}} = \frac{L_{e_{k+1}}}{\sum\limits_{i=1}^{k} \frac{1}{c_i}}, \end{aligned}$$

i.e., the formula (6.50) also holds for $M = k+1$. According to the mathematical induction, we know that the formula (6.50) is true for any integer $M \geq 2$. Particularly, let $c_1 = c_2 = \cdots = c_{M-1} = 1$ in (6.50), we get that

$$(M - 1) \sum_{i=1}^{M-1} L_{e_i} \ \succeq \ L_{e_M}. \tag{6.51}$$

Usually, the following assumption is needed for determining the system synchronization by graph comparison.

**Condition 5.** *For any vectors $\mathbf{x}_i, \mathbf{x}_j \in \mathbb{R}^n$, the inequality*

$$(\mathbf{x}_j - \mathbf{x}_i)^t \left( (\mathbf{f}(\mathbf{x}_j) - \mathbf{f}(\mathbf{x}_i)) - \varepsilon H \left( \mathbf{x}_j - \mathbf{x}_i \right) \right) \leq -c \left| \mathbf{x}_j - \mathbf{x}_i \right|^2$$

*always holds for numbers $\varepsilon > a > 0$, where $c$ is a positive number and $H$ is a diagonal matrix, i.e., $H = \mathrm{diag}(h_1, h_2, \cdots, h_s, 0, \cdots, 0)$.*

Let $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_m$ be $m$ column vectors and denoted by $\mathbf{x} = \left(\mathbf{x}_1^t, \mathbf{x}_2^t, \cdots, \mathbf{x}_m^t\right)^t$ an $nm$-dimensional column vector. Similarly, $\mathbf{f}(\mathbf{x}) = \left(\mathbf{f}_1(\mathbf{x}_1)^t, \mathbf{f}_2(\mathbf{x}_2)^t, \cdots, \mathbf{f}_m(\mathbf{x}_m)^t\right)^t$ is a column vector of dimension $nm$. Let $A = (a_{ij})_{m\times n}$ and $B = (b_{kl})_{p\times q}$ be two matrixes. The *Kronecker product* $A \otimes B$ of matrixes $A$ with $B$ is defined by $A \otimes B = (a_{ij}B)_{mp\times nq}$. For example,

$$\begin{pmatrix} 3 & 1 & 4 \\ 0 & 1 & 0 \\ 4 & 1 & 3 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 1 & 4 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 4 & 1 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & 4 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 4 & 1 & 3 \end{pmatrix}$$

By the Kronecker product of matrixes, the system (6.39) can be written in a more compact form, i.e.,

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) - \left( L_{G^L[S]} \otimes H \right) \mathbf{x}. \tag{6.52}$$

Now that if Condition 5 is true for the system and there exists an undirected labeled graph $G_0^L$ on the vertex set of $G_{G^L[S]}$ such that there is

$$L_{G_0^L} L_{G^L[S]} \succ a_0 L_{G_0^L}, \tag{6.53}$$

on time $t$ then $\mathbf{x}^t \left( L_{G_0^L} \otimes I_n \right) \mathbf{f}(\mathbf{x}) \leq a_0 \mathbf{x}^t \left( L_{G_0^L} \otimes H \right) \mathbf{x}$. Thus, if we define $L(t) = \frac{1}{2}\mathbf{x}^t \left( L_{G_0^L} \otimes I_n \right) \mathbf{x}$, then we know that $\dot{L}(t) \leq 0$ with equality holds if and only if $\mathbf{x}_i = \mathbf{x}_j, 1 \leq i, j \leq m$ by equation (6.52), i.e., all $\mathbf{x}_i, 1 \leq i \leq m$ are in set $\mathrm{Sta}[S]$. In other words, $L(t)$ is a Lyapunov function, i.e., *for a system* (6.39) *that satisfies Condition 5, if there is a graph $G_0^L$ satisfying the graph comparison* (6.53) *then the system* (6.39) *is synchronized.*

For the labeled graph $G^L[S]$ on system $S$ consisting of elements $\{v_1, v_2, \cdots, v_m\}$, if there are coupling action and reaction in elements which comply with the property of Newton's 3rd law, i.e., the action and reaction have the same magnitude but with opposite direction, the labelled graph of system is a complete graph $K_m^L$. Then, the minimum walk covering is $\mathscr{P} = \{\, \mathcal{P}_{ij} \mid \mathcal{P}_{ij} = v_{i_k} e_k v_{j_k}, 1 \leq k \leq m(m-1)/2\}$, i.e., $b_{\min}[S] = 1$. In this case, by (6.48) if the coupling strength $\varepsilon_k(t) > a_0/m$ on respective edges $e_k, 1 \leq k \leq$

$m(m-1)/2$, the system (6.39) is synchronized. Thus, a natural idea is to let $G_0^L = K_m^L$ in (6.53) and take the condition (6.53) as a comparative relation between graphs. Notice that $L_{K_m^L} = mI_m - J$, where $J$ is an $m \times m$ matrix with all entries 1 and $L_{K_m^L} L_{G^L[S]} = mI_m L_{G^L[S]} = mL_{G^L[S]}$ and the condition (6.53) then turns to

$$mL_{G^L[S]} \succ a_0 L_{K_m^L} \quad \Rightarrow \quad G^L[S](t) \succ \frac{a_0}{m} K_m(t). \tag{6.54}$$

By applying the condition (6.54), the condition (6.53) can also be stated as $\lambda_2\left(G^L[S]\right) > a_0$ by the minimum non-zero eigenvalue because all eigenvalues of the Laplacian matrix of complete graph $K_m^L$ are 0 and $m$ (multiple $m-1$), i.e., $\lambda_2\left(K_m^L[S]\right) = m$, Therefore, as long as $\lambda_2\left(G^L[S]\right) > a_0$ we get the formula (6.54).

Similarly, we can choose labeled graphs $G_0^L$ for other graphs with known $b_{\min}[S]$ or $\lambda_2\left(G_0^L\right)$ to get the graph comparison conditions for synchronization of system (6.39) by applying the formula (6.53) with removing the conditions $\varepsilon_{ij}(t) = \varepsilon_{ji}(t), 1 \leq i, j \leq m$, i.e., the assumption of coupling symmetry in (6.39) to obtain the condition of system synchronization similar to that of (6.48) or (6.53). Notice the assumption that $H$ is a diagonal matrix in system (6.39) implies that the coupling in system elements is an isolated linear action, i.e., for any integer $1 \leq i \leq m$, the element $v_j, 1 \leq j \leq m$ acts on $v_i$ can be expressed as a linear function of $\mathbf{x}_j$, i.e., $h_j \mathbf{x}_j$ independent on other actions of elements $v_k, k \neq j$ on $v_i$.

## §5. Synchronized Benefits with Harms

Certainly, the synchronization is an important character of things reflected in human eyes which is the basis for humans to systematically understand things in the universe and hold on the law of thing evolutions, benefit ourselves. Dr.Ouyang tells Huizi that in the middle of the 17th century, a few persons in European countries were keen on voyages and exploration. At that time, the angle between the sun and the sea level had been learned to determine the latitude. However, there was no an effective way to measure longitude. Among them, the difficulty in determining the exact position of a ship was that there was no a clock that kept accurate time. To this end, Huygens and his collaborators had tested on a voyage with a mechanical clock. In this experiment, two clocks were placed at the same time in case the ship's turbulence stopped one of them. The experiment was a success, predicting the ship's position with relative accuracy. But, *how can two clocks on a ship keep the same time so that if one clock stops, it can be reset according to the working clock of the other?* Huygens brought in two old backchairs, placed a stick on the back of each and hung two heavy wooden swings on the stick to simulate the moving of a clock suspended from the cabin during the voyage. But what this experiment really surprised Huygens was that *no matter what the initial position of the two wooden pendulums, after only about 30 minutes the two wooden pendulums always swing at the same frequency and the phase of the two clocks always in the opposite position, realizing reverse phase synchronization.* Huygens wrote to the Royal Society about the synchronisation of the pendulum but it did not attract much attention. Certainly, Huygens himself could not to give a theoretical explanation for the observed phenomenon mathematically but he explained the physical mechanism of synchronization in his *Pendulum Theory*, namely the reason for the synchronization of pendulums lies in the energy exchange, information transmission and interaction between the two pendulums by swinging through the wooden beam and while the two pendulums are in the reverse phase synchronization, the net force acting on the wooden beam is zero so that the whole system is in an equilibrium state. So, *how can we characterize mathematically the pendulum synchronization phenomenon?* Many scholars have established mathematical models to study the pendulum synchronization theoretically. Among them, the more influential is the phase synchronization model proposed by Kuramoto in 1980s. Then, *what is phase synchronization?* Assuming the behavior of two oscillators $O_1$, $O_2$ can be characterized by their phase $\phi_1(t), \phi_2(t)$ if there are such integers $n, m$ or a sufficiently small positive number $\varepsilon$ that

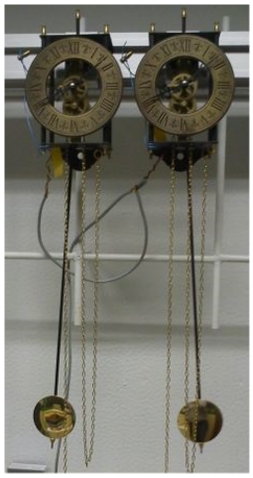


**Figure** 6.15. **Pendulum**

$$|n\phi_1(t) - m\phi_2(t)| = 0 \text{ or } |n\phi_1(t) - m\phi_2(t)| \le \varepsilon, \tag{6.55}$$

i.e., there is a fixed proportional relation between the phases or the phases are close under a fixed proportional relation, it is called the oscillators $O_1, O_2$ in the *phase synchronization*. This is actually a weakened synchronization, weaker than that of the system synchronization defined by (6.9) in Section 1 of this chapter because the two oscillators $O_1$ and $O_2$ may have different amplitudes even if they are in phase synchronization.

In order to characterize the phase synchronization of pendulums, Kuramoto assumed that a group of oscillators $O_i, 1 \le i \le m$ constituted a system $S$ and there was coupling between the oscillators with the same coupling strength. In this case, the corresponding labelled graph of system $S$ is $K_m^L$ and the system equation is

$$\dot{\phi}_i(t) = \omega_i(t) + \frac{c}{m}\sum_{j=1}^{m} \sin\left(\phi_j(t) - \phi_i(t)\right), \quad 1 \le i \le m, \tag{6.56}$$

where $\phi_i$ is the phase of the $i$th oscillator, $\dot{\phi}_i$ is its angular frequency, $\omega_i$ is the natural frequency when the oscillator is isolated, i.e. without coupling and $c$ is the coupling strength. Particularly, for $m = 2$, i.e., the system labelled graph is $K_2^L$, the equation (6.55) simplifies to

$$\begin{cases} \dot{\phi}_1(t) = \omega_1(t) + \frac{c}{2}\sin\left(\phi_2(t) - \phi_1(t)\right), \\ \dot{\phi}_2(t) = \omega_2(t) + \frac{c}{2}\sin\left(\phi_1(t) - \phi_2(t)\right). \end{cases} \tag{6.57}$$

In this case, if the system phase is synchronized, we substitute the phase of oscillator $O_2$ by $\phi_2'(t) = (m/n)\phi_2(t)$. Without lose of generality, we still denote $\phi_2'(t)$ by $\phi_2(t)$. Then, $\phi_1(t) - \phi_2(t) =$ constant, i.e., $\dot{\phi}_1(t) - \dot{\phi}_2(t) = 0$. Substituting it into the equation (6.57) we get that

$$\omega_1(t) - \omega_2(t) - c\sin\left(\phi_1(t) - \phi_2(t)\right) = 0,$$

which shows that the oscillators $O_1, O_2$ satisfy the condition of phase synchronization if the coupling strength $c \geq |\omega_1(t) - \omega_2(t)|$. Particularly, the closer the natural frequencies of $\omega_1(t)$ and $\omega_2(t)$ in the oscillators $O_1$ and $O_2$ is, the easier the phase synchronization will be. Similarly, the mean field method can also be used for a general discussion on the phase synchronization of equation (6.56) of system $S$.

Notice that the condition of system synchronization (6.9) is determined that the element tracks converge to a same track after a certain time $t^*$, which is too strong. Dr.Ouyang explains that this kind of synchronization occurs very rare in the nature because it is only a kind of approximate synchronization in most time with certain rules to create beautiful pictures, as the birds, peacocks and the fish shoal shown in Figure 6.16.

(a) (b) (c)

**Figure** 6.16. **Approximating synchronous phenomenon**

At this time, it is necessary to relax appropriately the condition of system synchronization (6.9), namely introduce the *pan-synchronization* similar to that of the phase synchronization (6.55). That is, for any integer $1 \leq i, j (i \neq j) \leq m$, there is a regulating function $g_{ij}(\mathbf{x}_j(t))$ such that if $t \to \infty$ then

$$\mathbf{x}_i(t) - g_{ij}(\mathbf{x}_j(t))\mathbf{x}_j(t) \to \mathbf{0} \quad \Leftrightarrow \quad \lim_{t\to\infty} |\mathbf{x}_i(t) - g_{ij}(\mathbf{x}_j(t))\mathbf{x}_j(t)| = 0. \tag{6.58}$$

Particularly, if $g_{i1}(\mathbf{x}(t)) = g_{i2}(\mathbf{x}(t)) = \cdots = g_{im}(\mathbf{x}(t))$, namely all system regulating functions are one given function $g(\mathbf{x}(t))$, the synchronized mechanism is more needed to clarify theoretically, to conduct data simulation and control for the service of human developing. However, everything has two sides, benefits and harms which result in the application of synchronization no exception also.

**5.1.Synchronized Benefits.** From the view point of system dynamics, Dr.Ouyang tells Huizi that most systems that can serve humans are synchronized or pan-synchronized systems because only for such systems, humans can carry out the automatic control design and then, control or adjust the system state to meet the desired state by the principle that of "*seek benefits and avoid harms*" of humans, including from the small household appliances to the large space shuttle or spaceship.

**(1)Synchronized moving.** In simulating the natural phenomena, the most important thing is moving to extend human's activity area and space. For example, Newtonian mechanics characterizes the moving of a particle or a rigid body. Notice that the moving of rigid body is a generalization of the synchronization of particle moving. *Why do we say so?* Because if a rigid body is divided into its elements, no matter how big or small the elements are, the elements must move according to the node equation, i.e., are moving in synchronization. Here, the node equation of the rigid body is nothing else but the equation of particle abstracted from the rigid body. But the same can not be applied for the flexible or near flexible bodies such as the living or self-organizing systems. Dr.Ouyang asks Huizi:"*if we divide the body of man into five parts, i.e., the trunk, two hands and two feet. So, can he walk with one foot forward but one foot back?*" Huizi replies immediately:"*Of course not! If one of his foot forward but one foot back, he can't move at all because if he moves so, his two foot forces would be completely cancelled out.*" Dr.Ouyang confirms her answer and then tells her:"*Right! Both of one's feet must move forward if this human wants to move forward, i.e., synchronized with the trunk!*" Huizi then says with a deep feeling: "*It seems this case. His two feet and body need to be a synchronized moving!*" Dr.Ouyang then asks her: "*Can you move your feet and hands such that one forward but another backward in synchronization?*" Huizi thinks for a while and then says: "*The direction is not the same if one forward but another backward, they should not be in synchronization. However, they are following the trunk in walk as in synchronization!*" Seeing his daughter

**Figure** 6.17. **Walking**

is indistinct on this question, Dr.Ouyang explains that the feet and hands move forward and back in order to balance of the body in moving, which can be roughly thought of as counter-synchronization. The feet move forward synchronously with the body and the hands swing back to maintain the balance of body in moving. Huizi nods her head if she understood. Dr.Ouyang further explains that if you multiply the state function of the human hand by a regulating function $-1$, then the human hand and foot are moving in synchronization with the body. So, the five parts of human body in (6.58) are in the case of pan-synchronous moving. Generally, if an animal is divided into feet, hands and trunk, the moving of the animal must be pan-synchronous because if the trunk's moving relative to the earth is assumed to be $\mathbf{x}_1 = \mathbf{f}(\mathbf{x}_1)$, the hands and feet relative to the trunk's moving are respectively $\mathbf{x}_2 = \mathbf{g}(\mathbf{x}_2)$ and $\mathbf{x}_3 = \mathbf{h}(\mathbf{x}_3)$. Then, if $t \to \infty$, the moving equation

**Figure** 6.18. **Wheel of history**

$$\begin{cases} \mathbf{x}_2 = \mathbf{f}(\mathbf{x}_2) + \mathbf{g}(\mathbf{x}_2) \to g_{21}(\mathbf{x}_2)\,\mathbf{f}(\mathbf{x}_2) \\ \mathbf{x}_3 = \mathbf{f}(\mathbf{x}_3) + \mathbf{h}(\mathbf{x}_3) \to g_{31}(\mathbf{x}_3)\,\mathbf{f}(\mathbf{x}_3) \end{cases}$$

namely $\mathbf{x}_1 - \mathbf{x}_2/g_{21}(\mathbf{x}_2) \to \mathbf{0}$, $\mathbf{x}_1 - \mathbf{x}_3/g_{31}(\mathbf{x}_3) \to \mathbf{0}$, the moving of animal's hands and feet to the earth coordinate system is pan-synchronized. Dr.Ouyang explains further, *why does the wheel of history move forward as shown in Figure* 6.18 *and not back?* It is because the human eyes can only see in front and not behind! Thus, assuming there is a uniformly time-passing parameter, namely time $t$, then we have $t > 0$ and obey the additive group $(\mathbb{R};+)$, i.e., for any two times $t_1, t_2$ has the property of $t_1 + t_2 > t_1$. Accordingly, the evolving of thing $T$ in human eyes with the passage of time $t$ can be then characterized by a state equation $\mathbf{x}_T = \mathbf{f}(\mathbf{x}(t))$, as that shown in Figure 6.19. In fact, this is also a kind of synchronous hypothesis, which assumes that the development and change of things are synchronously with the time $t$, namely

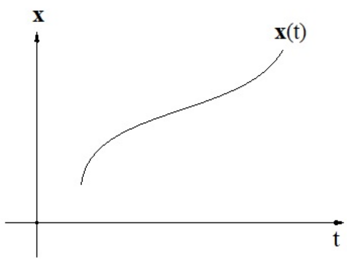



**Figure** 6.19. **Evolution**

$$\mathbf{x}_T - \mathbf{f}(\mathbf{x}(t)) = \mathbf{x}_T - \frac{1}{t} \cdot \mathbf{f}(\mathbf{x}(t)) \cdot t = \mathbf{0}$$

if $t \to \infty$, where the regulating function $g(t) = \mathbf{f}(\mathbf{x}(t))/t$.

Dr.Ouyang tells Huizi that the most common scenario in the applications of synchronization is wheel moving, including rickshaws, cars and trains. Among them, the number of wheels of large flatbed trucks and trains is dozens or hundreds. So, *how do we keep these* *wheels moving in synchronization?* This is especially important in train design. Similar to Huygens' explanation of the synchronized pendulum, a group of three is made of girders or steel frames to ensure the synchronized moving of wheels. In response to the flatness difference, gap or turning needs of the track, the part of wheel contact with the rail is designed as an inclined plane with a small diameter of the outer ring but a large inner ring. In this way when a train turns, the middles wheel on both the left and right contact the rail, the big and small wheels suspended on the rail do not work. Certainly, the small wheel walks in a small circle and the big wheel walks in a big circle.

**(2)Synchronized information.** Usually, the information in mechanical moving such as force distribution, etc., is automatically transmitted by component perception. Similarly, the information transmission is the moving of locations within an organ through the transmission of text, language, code, images, color, light and smell carriers to reach the audience, which is one of the basic functions of the ecosystem, including the sending and receiving ends usually. Indeed, a synchronized transmission means that the receiving ends simultaneously receive the information or instructions sent by the sending end at a time. Among them, a well-known example of the application of synchronization in communication is the synchronized satellite, namely the artificial satellite operating in the geosynchronous orbit transmits various television programs and enriches human's cultural life. Usually, a synchronized satellite operates at an altitude of $36000km$ above the earth and is launched by a multi-stage rocket, as shown in Figure 6.20. Generally, launching a synchronized satellite to its obit needs two accelerations in sky, i.e., the satellite is first put into the low earth or initial orbit and while the satellite flies near the earth's equator, the command on ground controls the rocket fire again to accelerate firstly the satellite flying in an elliptical orbit on the equatorial plane. Then, while the satellite reaches the farthest point on the orbit, the command on ground controls the rocket fire again to accelerate secondly and finally, put the satellite into a relative geostationary orbit.

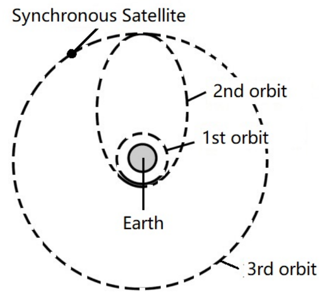


**Figure** 6.20. **Synchronized satellite**

Dr.Ouyang tells his daughter why such a satellite is called a *geostationary satellite* is that when it is observed on the ground, the relative position between the satellite and the earth remains unchanged, i.e., the satellite is in a relatively stationary state to the earth. However, the satellite is in moving if observe on other planets. Certainly, its operating direction is the same as the rotation direction of the earth with a circular orbit located on the equatorial plane of the earth and its period is the same as the time of one rotation of the earth, i.e., 23 hours, 56 minutes and 4 seconds in speed about $3.08km/s$, equal to the angular speed of the earth's rotation, i.e., a geostationary satellite is such a satellite that moves in synchronization with the earth.

**(3)Synchronized with nature.** Certainly, the human is one of the members of things in the universe. Dr.Ouyang tells Huizi that it is the human who assumes that everything is in synchronization with time $t$ in recognition and then, characterizes the evolution of things with time $t$. If the human is regarded as independent on other things or the nature, the human and the nature naturally constitute a binary system, as shown in Figure 6.21. Among them, the Yang and Yin actions in the Qian and Kun hexagrams, namely the Qian hexagram and the Kun hexagram constitute a coupling effect between humans and the nature. It forms a real picture of the Qian element "*striving upward*" and Kun element "*cautiously moving forward*", which is a natural scene of Taiji diagram in the ancient Chinese culture, similar to that of the ecological relationship formed by the cat and mouse, which can be approximately characterized by the Lotka-Volterra model, also give rise to the thought of "*harmonious coexistence of humans with the nature*", namely the words that "*return is the moving of Tao and weakness is the usefulness of Tao*" in Chapter 40 of *Tao Te Ching*, i.e., the evolution of thing is in a cyclic moving. Dr.Ouyang explains that this is exactly the "*one union of the heaven and humans*" in Chinese philosophy, namely the synchronized thought of "*harmonious symbiosis*" of humans with the nature.

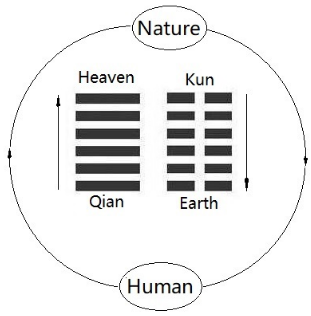



**Figure** 6.21. **Humans with the nature**

The thought of harmonious coexistence of humans with the nature was more clearly expressed in *Wen Yan Zhuan* written by Confucius for the *Change Book*. He proposed that a great human should be "*with the same Te as the heaven and the earth, appearing or showing the same as the sun and the moon, order the same as the four seasons and well or evil the same as the god or the ghosts*, where the word "*same*" is a synonym of the synchronization, which requires one's behavior to synchronize with the heaven and earth, the sun and the moon, the four seasons of a year, the God or the ghosts because the "*Te*" refers to the harmony of human with the nature, i.e., a synchronized cultivation, which is the pursuit of ancient Chinese education. Correspondingly, the "*great human*" is such a human that has "*Te*", also has a official position. Notice that while humans depend on the nature for survival in the ecological binary system of humans with the nature, the subjective consciousness of humans will also react against the nature. Among them, the behavior that conforms to the law of natural evolution is the symbiotic of humans with the nature. Otherwise, if the intrusion or destruction of humans on the nature are not restrained to a certain extent, the results will cause a qualitative change of the nature and lead to the nature's acting back on humans, which is manifested in the frequent outbreak of extreme natural disasters, affects the survival and development of humans. Certainly, it is based on this ecological relationship of humans with the nature that the ancient Chinese characterized the intrusive or destructive behavior on the nature as a rejected conduct of "*Te non-being*". They asked all humans of China to synchronize with the nature and coexist with the nature, namely "*Te being*". This is consistent with the "*law of unity of opposites*" notion embodied in ecological binary systems such as the cat and mouse or the Lotka-Volterra model.

Similarly, different from the natural systems observed by humans, the societal organizations of humans can be viewed as the biological subsystems composed of a few humans. They are collectives or groups combinatorially formed by mutual cooperation of humans to achieve certain goals, restricted by their subjective consciousness to a certain extent. In this case, the organization also needs to scientifically recognize and synchronize with the nature, namely the organizational program should guides the orderly developing of the organization over time, including: ①Relatively stable organizational structure; ②Organizational programmes and norms of conduct universally recognized and followed by its members;③Conflicts of interest within organizations can be reconciled, etc. Therefore, if the members of an organization are regarded as elements, the members play a role of mutual influence or coupling within the organization. The ideal state of organization then should be also a synchronized system in coordination with the nature, including its established programs and goals. This is essentially the societal structure for the sustainable development of humans by the ancient Chinese philosophers.

**5.2.Synchronzied Harms.** As a component of things in the universe, the human recognition of things is like that of the blind men feeling the shape of elephant which maybe quite different if stand on different viewpoints, contained in the "*name named is not the eternal name*" of first chapter in *Tao Te Ching* of Laozi, also implied in words that "*why do not know the true face of Lushan mountain, only because you are in the mountain yourself*" in *Writing Xilin Wall*, a famous poem written by Sushi, namely the human recognition of things is human's own local recognition on things in the universe. If we look at the human's recognition of things in the 2-system of humans with the nature, it is beneficial to humans but maybe harms to the nature at the same time. Dr.Ouyang explains further that the perceptible limitations of humans often brings about unable to see the harmful of scientific applications to the nature. So, it is necessary for humans to adopt a combination notion for recognition on things, put all scientific achievements into the consideration of the 2-system of humans with the nature and then for the benefit of humans. Certainly, the historical experience shows that ignoring the impact of scientific application on the nature will not only disturb the nature but also have the reacting, result in accidents or disasters on humans with no exception of the synchronization, for instance the ants and locust swarms biting and destroying food or other crops simultaneously.

**[Crossing Bridge in March].** This is a real incident of walking across the bridge in march which caused a tragedy. In 1831, a fully dressed group of Napoleonic soldiers marched under the command of the commander across a cable-stayed bridge in Angers of France. Looking at the neat and majestic posture of the soldiers, many residents unconsciously followed the team and stepped onto the bridge deck. Unfortunately, the bridge suddenly had a strong vibration when the team approached the middle of the bridge. Only a short time later, the bridge cableway broke, the bridge deck collapsed, many officers, soldiers and the pedestrians on the bridge fell into the river, caused lots of them loss of their life finally. After the field investigation and research by some professional organizations, the cause of this tragedy is that the soldiers' walking in march resonates with the natural frequency of the suspension bridge. It is the frequency of the soldiers' walking in march happens to be consistent with the natural frequency of the bridge, resulting in the superposition effect of the bridge's vibration so that the amplitude exceeds the maximum compressive capacity, which resulted in the collapse of the bridge of Angers finally. This incident has then prompted professionals to think the phenomenon of synchronization once again. Dr.Ouyang tells Huizi that many countries have then changed the "*walking in march*" to "*walking on route step*" when large groups of humans are crossing bridges to avoid the similar incidents.

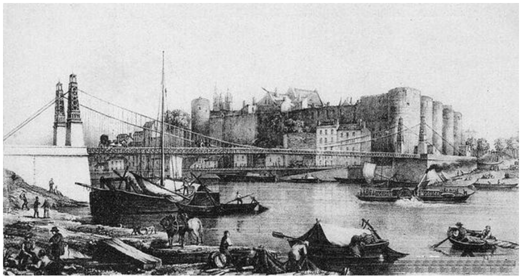


**Figure** 6.22. **March crossing**

The event of Napoleon's soldiers crossing the bridge in march in France lies in the superposition effect between the frequency of soldiers' march and the natural frequency of the bridge. Similarly, the combination of high winds and bridge frequencies can also cause bridges to collapse. For example, a Tacoma Narrows bridge built in Washington State of the United States in 1938 collapsed on November 7, 1940, which is due to the strong torsional vibration of the bridge deck under the strong wind, only in about 70 minutes later the bridge deck broke off and fell into the canyon. Although the wind was about force 8 at that time but the frequency of the wind and the natural frequency of the bridge had a superimposed effect, which resulted in the bridge collapse.

In addition, the wavy crests and troughs caused by a pebble dropped into a pond are a synchronous phenomenon, forming successive waves on the pond which makes one linger. This synchronization can also be harmful. For example, Dr.Ouyang explains that during a flood, the crest of a wave lashes against a farmhouse, causing it to collapse and soaking crops for a long time. Similarly, the locusts flying over the sky landed in the fields, a wave of gnawing crops resulting in crops no harvest and the destruction of buildings caused by earthquake waves, etc., none of them is not the harmful case of synchronized moving to humans ourselves.

Usually, a rumor refers to the remarks that are deliberately fabricated by someone with ulterior motives to achieve their personal goals and spread to the public through modern mediums such as the internet or we-media without facts. For example, fabricate the death, divorce or illegitimate child information of a public figure and hype it on the public network to attract viewers or fans for realizing the commercial purposes. Similar to the viruses, the rumors spread at a fast speed especially on the internet or mobile phone terminals and the audience will be magnified layer by layer, which is no less harmful to the disruption of society than the spread of viruses. So, *how do we quantify the range of rumour propagation?* Assume that the propagation index of a rumor is $R$, namely a monger spreads a rumor to $R$ individuals on average. Thus, the number of humans at level $k$ who simultaneously receive the rumor is $R^k$. For example, the number of individuals received the rumor at the same time is respectively 3 individuals at level 1, $3^2 = 9$ individuals at level 2, $3^3 = 27$ at level 3, $\cdots$, $3^k$ at level $k$ if $R = 3$, as shown in Figure 6.23. In this way, the rumor goes on and on until that the positive news is opened. Here, each level of the rumor propagation constitutes a crest. So, *why do a few humans believe rumors and also spread them level by level?* The answer is the absence of the positive news. Certainly, it natural leaves the ground for rumors to spread if there are no positive news. At the same time, not all rumors are harmful to society. In fact, a few kind of rumors can play a recreational or entertaining role in situations. For example, some rumor mongers

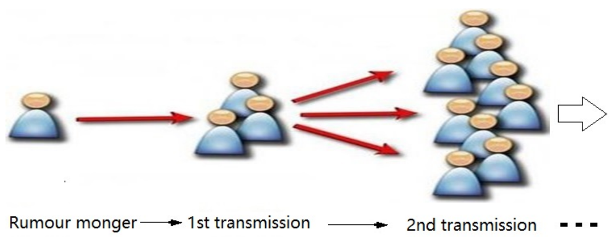



**Figure** 6.23. **Rumour spreading**

in the entertainment circle are often the leading actors or the beneficiaries behind the rumors, just a hype on purpose. The audience also regards such rumors as a kind of entertainment, only as the fun to chat or show off after dinner.

## §6. **Comments**

**6.1.** The synchronization is an interesting moving phenomenon, which is both beneficial and harmful to human society. It started from the study of synchronized pendulum by Huygens at the earliest and then gradually attracted ones' attention. In fact, the human recognition of things has an unavoidable assumption, namely the human recognition of a thing is synchronized with the real behavior of the thing. Otherwise, it is impossible for humans to recognize this thing and achieve the synchronization with everything. Therefore, the synchronization is the basis for humans to simulate other livings or non-livings to benefit humans and then improve the survival ability of humans because only if the components of system progress synchronously in a coordinated direction the set goals can be finally achieved. Otherwise, the goal may not be achieved, even brings about the apparatus to be destroyed with harmful to humans.
**6.2.** Generally, the system synchronization characterizes a phenomenon in which the system elements may not be synchronized at the beginning but tend to be consistent with the developing of time. For example, the audience applause for a performance is generally inconsistent in the first few seconds but the applause is then consistent or synchronized gradually. In the field of automatic control, one needs to take the initiative to synchronously control for preventing the system from appearing out of synchronized state. Usually, subtracting the final synchronized function from each element state of a synchronized system, the error vector of the element state must be asymptotically stable at the equilibrium point, particularly, the zero vector $\mathbf{0}$ for an autonomous system. In this way, the problem of system synchronization can be transformed into the problem of system stability and the conclusion of system synchronization can be obtained from the stability of system. See the literatures [HLW], [CLW] and [LLC], etc.
**6.3.** The determining of system synchronization by the master stability function was proposed by Pecora and Carrol in literature [PeC]. Its essence is to assume that all acting functions of elements are equal, apply the condition that the Lyapunov's maximum exponent is negative, assume the asymptotic stability of system at the point $\mathbf{0}$ and then, determine the synchronized region of system, see [PeC] and [LLC]. Further discussion of Lyapunov exponent can be found in the literature related to dynamical systems such as

[BrG] and [Lyn], etc. The graph comparison method of system synchronization is aimed at a class of relatively special nonlinear systems, which is proposed and studied by Igor Belykh et al in the literature [BBH1] and [BBH2]. Certainly, there is one chapter in literature [LLC] to discuss this method. In this chapter, we introduce the concept of walk covering index by using the term of graph theory, generalize and simplify the method of graph comparison for system synchronization.

*There are two sides to every story, $\cdots$, at least.*

– By A.N. Whitehead, a British mathematician

$C_{\mathscr{M}}$

**Combin.Notion**

## Chapter 7

# Contradictory System

*********************************************************************

**Who is the Busybody Planted Plantains?**

**Rain Falls in Morning and Night Without Stopping.**

**It is You Too Boring in Mind Yourself,**

**Planted Plantains Then Complain About Plantains.**

— *Autumn Lamp Memories* By Jiangtan, a poet in Qing dynasty of China

************ *Linfan Mao. Combinatorial Theory on the Universe* *************

## §1. **Logical Consistency**

There is a fundamental hypothesis in humans' recognition of things, namely the evolution of things follow the law of causality, the humans can hold on the causal relationship of things and then predict the future behavior of things by observing on things at present. This hypothesis requires humans to comply with the "*logical consistency*" in their recognition of things, namely there is no inconsistency or logical error caused by violating the rules of logic, i.e., the same subject can not make an inconsistent or contradictory recognition on a thing. For example, there is a fable on contradiction in the *Paradox 1* of *Han Fei Zi*, written by Hanfeizi, a philosopher in the warring states period of China. It was said that there was a merchant in the state of Chu selling his shield and spear in the market. He boasted to the passers-by about the shield in his left hand and said: "*This shield in my hand is so strong that nothing can penetrate it!*" After then, he began to boast about the spear in his right hand and said: '*This spear in my hand is so sharp that it can pierce even the strongest thing!*" At this time, a man in the crowd could not stand his words any longer and asked him: "*Your just claimed that your shield is so strong and your spear is so sharp! Please, use the spear in your right hand to pierce the shield in your left hand. Let's see what happens.*" Hearing these words, this merchant was stunned and stood tongue-tied, unable to say any words. This is where the term "*contradiction*" comes from because a spear that pierces everything can not exist with a shield that can be pierced nothing at the same time in world, i.e. his words are logically contradictory.

**Figure** 7.1. **Spear and Shield**

Certainly, the humans have been trying to characterize the causal relationship of evolution of things in the recognition on the fundamental hypothesis of logical consistency. They believe that this process from the "*cause*" to the "*effect*" is unique and can be characterized quantitatively. Dr.Ouyang tells Huizi that this hypothesis seems to be true for non-livings on the appearance but not for livings because all animals have a subjective awareness of the nature and all plants seem to have this subjective awareness too, only with a few geographical limitations. Dr.Ouyang asks Huizi: "*If we have not fully understood the evolving mechanism of a thing, how can we assume that it is non-biological, has no the* *ability of subjective consciousness and its behavior follows the logical consistency?*" Huizi does not know the implication of this question and asks Dr.Ouyang vaguely: "*Why can't a substance like a piece of earth or ore be assumed to be non-living?*" Dr.Ouyang explains to her: "*Whether a substance is biological or non-biological is defined by humans ourselves because we have not yet understood the origin of the consciousness and then simulated it. If the distinction is purely by the moving and rest, what matter is not in moving? Even a piece of ore that you see it at rest is relative only because its atoms, nuclei and electrons are always in moving at the microscopic level.*" Huizi nods and then says: "*It is indeed so! Moving is absolute but the rest is relative. But then, how can we understand a thing if it behaves without in a logical consistency?*" Dr.Ouyang replies: " *That's a good question! Notice that it is always believed that the evolution of a thing complies with the principle of logical consistency in classical science. But, who determines the consistent or inconsistent in here?*" Huizi thinks for a while and answers: "*It should be humans ourselves! But then, is it the same to assume the logical consistency in evolution of a thing as it is assuming that one can perceive it?*" Dr.Ouyang affirms her answer: "*Yes! It is the humans ourselves that determine this uniformity, characterize the evolution of a thing and prays that the evolution of everything is in synchronization with our recognition. The recognitive subject is not other one but the humans ourselves. However, contradiction exists everywhere and all the time in human's eyes, i.e., there are always the case of logical inconsistency in the evolution of any thing. Then, what is the implication on the recognition of humans?*" Huizi can not hold on the essence of question and then cautiously asks Dr,Ouyang: "*Does this implies the limitation of human recognition on things which should follow in a gradual progress?*" Dr.Ouyang confirms her answer and then tells her that it is only part answer on the question. More importantly, it shows that it is impossible for humans to recognize everything in the universe by adhering to the principle of logical consistency without progressing. It is necessary to constantly extend the recognitive method of humans from the logical consistency in classical science to including contradiction of things, namely the "*logical inconsistency*" and then establish a new "*logical consistency*" because it is just like the above fable on spear and shield, we need to realize the self-logical consistency in the recognition and get the truth of a thing rather than an illusion by assuming the synchronization of recognition of thing in the universe with evolving and then hold on it.

**1.1.Recognitive Synchronization.** A main purpose of scientific recognition is to establish the knowledge of things in the universe. Among these, there are two things should be discussed. One is that human recognition is gradual, it is impossible to hold on the truth of things in one day at a time. Another is the limitation of human recognition. It may take thousands of generations to hold on the truth of things or it may always be on the way of recognition.

**(1)Illusion true.** The humans have known things by the "*six sensory organs*", namely the eyes, ears, nose, tongue, body and the mind, which form a consensus of knowledge on everything around humans. For the importance of six sensory organs in the human recognition, there is a usually saying that the "*hearing maybe the false but seeing must be believing*", which means that the hearing may necessarily be not true but what you have seen with your own eyes is true and reliable. However, *is the seeing always believing?* The answer is not necessarily! Certainly, the sight may be also an illusion but it can also stir up the shock and resonance of a human heart.

(a) (b) (c)

**Figure** 7.2. **Out of place**

For example, the reason why most ones think that the Penrose triangle shown in Figure 7.2($a$) is impossible is that it is mentally impossible to appear on a Euclidean plane. In fact, it is actually a misaligned projection of a three-dimensional form. Dr.Ouyang tells Huizi that some photographes usually choose the different positions for capturing many strange and incredibly beautiful images of objects. For example, *is the person in Figure* 7.2($b$) *really holding up a cloud in the sky? Can the little girl in Figure* 7.2($c$) *really lift her father up with two fingers?* Of course not, it is simply the result of misplaced photography where objects that are are projected onto a plane but not really together.

So, *can we get the truth by setting different humans standing on different places and then observing?* Huizi asks Dr.Ouyang and continuously says such as setting different persons between the man and cloud, the little girl and her father will result in different shots in Figures 7.2($b$) and ($c$). Dr.Ouyang explains that changing the location of the shot can get the actual result of the object and solve the problem of visual misalignment within the limits of human ability. So, *what happens if it is beyond the reach of humans and can we obtain the truth of things by the recognition of humans?* Notice that the structure of human body is nearly the same. Even if all observers see the same scene, i.e., all humans' recognitive synchronization of a thing, it does not necessarily represent the truth of the thing because the six sensory organs of human are nearly the same, the result of perceiving things may also tend to be the same but it may still be an illusion rather than the truth of the thing. Among them, a famous example is the problem of finding the center of universe. Indeed, one worked hard at sunrise and rested at sunset in the agrarian society, and almost everyone believed that the earth was the center of the universe in that time. This view was accepted until Copernicus published his *De Revolutionibus Orbium Coelestium* in 1543 and proposed the heliocentric theory. However, his view was not accepted until 1609, i.e., Galileo invented the telescope and discovered some astronomical phenomena that supported the heliocentric theory, a few humans began to pay attention to the fact that the earth moved around the sun and that the sun was the center of the universe. Today, the progress in astronautics science and technology has made it clear that the sun, including the center of the milky way is not the center of universe. So, *where is the center of universe?* Certainly, the answer to this question needs constantly revising, not only on the synchronization of humans' recognition must be the right answer.

**(2)Local true.** Notice that everyone perceives a thing through by his or her six sensory organs. Then, *why do humans usually quarrel over their recognitive conclusions?* Dr.Ouyang explains that the reason lies in the different recognitive conclusions on a same thing. Here, although the structure of human body is nearly the same but different humans have different reflecting and perception ability to a thing, including the innate and acquired abilities. This difference in perception ability of things leads naturally to the difference in recognition on a same thing, and then the difference in recognitive conclusions. For example, in the fable of blind men with an elephant, *why the results of the six blind men's perceptions on an elephant's shape were respectively a pillar, a rope, a radish, a fan, a wall or a pipe which are very different from the common recognition?* The answer is because the blind men have no vision. They can only feel an elephant by touching its body with their hands. Certainly, different blind men touched different parts of the elephant's body and then, resulted in different recognitive conclusions in this fable, namely their recognitive conclusions are not synchronized.

Then, *what is the difference in recognition between a normal human and a blind one?* Dr.Ouyang tells Huizi that the difference is that a blind human lacks the "*vision*", i.e., visual recognition ability in perception comparing with a normal one. He or she only has five or less perceptive ability to things around them, which is the fundamental causing a blind one limited recognition of things. In this fable, the blind men have only a local recognition on the elephant's appearance in absence of the visual state. Similarly, all livings on the earth have only a local perception of things in the universe also. For example, all ants live in a 2-dimensional space in the eyes of humans, i.e., on a plane without the stereoscopic perception. So, they can only be slaughtered by aardvarks. They may never have the ability for recognizing the face of a human. In this way, a natural question is the reflection on human's recognitive ability, namely "*is the perception by the six sensory organs of human on a thing its really face of the thing?*" If the answer is affirmative, one should not be the helpless in front of the natural disasters such as the earthquake, tsunami but should be able to control the occurrence of these natural disasters and become the "*ruler*" of the universe. However, the developing history of humans for billions of years shows that humans with the nature in the binary system, humans are still at such a low level that need to recognize the laws of nature and then adapt to the nature. They do not have a clear perception of natural laws even though a little far away from the earth. How to they become the ruler of universe. Therefore, the human perception of things in the universe is only local. It is only the human's six sensory organs definition on things similar to other animals such as ants defining an object as a flat thing, not necessarily the real face of things which is then back to Laozi's interpretation in his *Tao Te Ching*, i.e., *Tao told is not the eternal Tao, Name named is not the eternal Name.*".

**Figure** 7.3. **Tsunami**

Hearing this, Huizi has a question in her mind and then asks Dr.Ouyang: "*If the human's recognition of things is local, why do all kinds of equipments and facilities created by humans really serving the humans such as the cars, trains, planes and the ships?*" Dr.Ouyang explains that there is a basic question, namely "*is a local recognition an incorrect recognition?*" Of course not! A local recognition is a conclusion that holds true under certain constraints. Of course, beyond the limits set by humans, a local recognition may no longer be correct. Therefore, although the recognition within the human ability or control is local, it will not cause harm to the human society as long as it is applied under the recognitive conditions. However, this point is often ignored by the users. For example, using $C$ in the twigs or hay to make a fire consumes the oxygen $O_2$ and will produce carbon dioxide $CO_2$. Its chemical law is $C + O_2 = CO_2$, i.e., one carbon molecule combines with one oxygen molecule will produce one carbon dioxide molecule, which is a local recognition. So, *what is its hypothesis condition?* The answer is such a reaction should be in a closed system that the carbon and oxygen molecules match each other under the combustion conditions. However, things in the universe are influence and restrict each other. The history of human developing shows that instead of applying this chemical reaction in a closed system, humans always assumed that the carbon dioxide produced by the combustion would be released into the nature which would be able to absorb the carbon dioxide emitted by humans without limits, which finally breaks the equilibrium or conservation state of carbon dioxide in the nature to some extent today. Accordingly, it natural returning to the conservation state of carbon dioxide needs a spontaneous adjustment of this imbalance, which is the reason for the frequent occurrence of natural disasters with impact on human survival. Certainly, this is only one but a typical example of local recognition not conforming to its the restrictive conditions for applications.

**1.2.Self-Organization System.** The human recognition of things is to characterize the evolving of everything in human recognition, namely the human recognition is synchronized with the evolution of everything. In theory, such a synchronization can be achieved for non-livings without subjective consciousness as long as it is within one's ability to objectively characterize the evolving. Then, *is the human recognition of a living with subjective consciousness the really face of its evolution?* Dr.Ouyang tells Huizi that humans can hold on the evolution of a living only if the human recognition is synchronized with its evolving. But this requirement is almost impossible in theory because livings, especially those with subjective consciousness will hardly evolve and change according to the human subjective consciousness because their subjective consciousness is independent that of humans. This fact explains partially why a few of humanity's scientific achievements fail to apply to biology in some extent because all livings are living with a certain subjective consciousness that is not necessarily in synchronization with the human recognition. Generally, such a system with the ability of self-regulation is called the *self-organization*, namely without outside intervention, the system can spontaneously adjust its elements with interactions and realize the transformation from one state to another. It is not difficult to find such a trace of self-organizing process in the evolution of species, evolution of human thinking and the developing of social organization.

**(1)Element initiative.** The self-organizing system has a self-regulation or repair ability independent of the human consciousness, which is essentially a subjective consciousness ability. It is difficult to use the deterministic recognition to hold on the evolving of individual or group animals. For example, in response to the hypothesis of quantum state "*collapse*" suggested by the Copenhagen school, a worrying question on the living and death of a kitten put forward by Schrödinger, which indicates the impossibility of human recognition to be logically consistent or synchronized with the real of things and one can not even give a logical consistent answer.

**[Schrödinger's cat]** *The Schrödinger's cat is essentially a thought experiment. Place a cute kitten into an opaque box with a toxic gas switch at the bottom and closed the lid as shown in Figure* 7.4. *Of course, the cat is alive and can walk around freely in the box. If it accidentally presses the gas switch, the toxic gas will soon fill the box and the cat will die. But, if the cat does not step on the gas switch, the cat is alive.* However, there is no way of knowing whether the cat is living or death until one lifts the lid. Schrödinger then asked: "*Is the lovely kitten dead or still alive?*"

**Figure** 7.4. **Schrödinger's cat**

It is a very weird question, namely that neither the answer to the cat "*alive*" nor the "*dead*" is complete. Both of them face endings that may or may not be correct. However, once the box is opened, the cat is alive or dead is the only certainty. *Why does such an ambiguous answer exists?* The answer is because between the closing and opening the lid of box, the cat's information is lost in the opaque box and we can not establish a "*logically consistent*" causal relationship. It is impossible to tell whether the cat is "*alive*" or "*dead*" except by guessing. In this case, the best way is to put a camera inside the box, observe the state of cat at all times and then establish a causal relationship and answer the question of whether the cat is alive or dead, namely you have to open the box to determine whether the cat is alive or dead. However, "*how can a cat's living or death depend objectively on human observation?*" The answer is certainly not because it is the human to determine the living and death of cat but we lost such a piece of information about the cat in the recognition that can not establish the causal relationship between the cat from "*living*" to "*living*" or "*death*".

So, "*how do we characterize mathematically the state of living and death on the Schrodinger's cat?*" Dr.Ouyang explains that let $\mathbf{L}$, $\mathbf{D}$ denote respectively the cat's state of living and death. Then, the Schrödinger cat's state can be expressed as $\mathbf{L}+\mathbf{D}$, which may be a living state $\mathbf{L}+\mathbf{D}\to\mathbf{L}$, may also be a death state $\mathbf{L}+\mathbf{D}\to\mathbf{D}$. The question is how one know the alive or dead state of Schrödinger's cat. In fact, recognizing the living or death of Schrodinger's cat we need to lift the lid for seeing the cat's living or death. However, *how do we explain the state* $\mathbf{L}+\mathbf{D}$ *of cat from* $\mathbf{L}+\mathbf{D}$ *to* $\mathbf{L}$ *or* $\mathbf{D}$ *when one open the lid for an instant?* In order to give a logically consistent explanation, the Copenhagen school led by Bohr et al proposed the *state collapse hypothesis* of Schrodinger's cat in the 1830s, namely $\mathbf{L}+\mathbf{D}\to\mathbf{L}$ or $\mathbf{L}+\mathbf{D}\to\mathbf{D}$ dependent on the human observing, i.e., if the cat is not observed, its state is $\mathbf{L}+\mathbf{D}$ but the cat state becomes $\mathbf{L}$ or $\mathbf{D}$ at the moment of observing of human. This seems to be logically consistent only in form, explaining the living and death of the cat but it leaves a hidden danger in human recognition of things, namely the Schrodinger's cat living or death depends on observing of human. It is the human observing that determines the living or death of the cat, which does not accord with human recognition law of objective things, i.e., they are not subject to the will of human.

Thus, in order to give a logically consistent explanation of the Schrödinger's cat, it is necessary to examine what is the cat state $\mathbf{L}+\mathbf{D}$ in the box. Generally, it can be interpreted as the sum of two vectors, namely the superposition of the cat's living and dead states. In this regard, the Copenhagen school led by Bohr et al determined living or death by human observing on the cat. They believed that the living or death of the Schrodinger's cat was determined by the human observing. Then, *can the human fully observes the cat state* $\mathbf{L}+\mathbf{D}$ *or both the living* $\mathbf{L}$ *and the death* $\mathbf{D}$*?* If the human recognition of things has certain limitations, it is impossible to observe both the living $\mathbf{L}$ and the dead $\mathbf{D}$ of Schrodinger's cat at the same time but can only see one of them, i.e., the cat is alive or dead at the moment of opening the box. In fact, the problem is not the "*state collapse*" at the moment of observing but the human recognitive limitations. Along this view, a doctoral student H.Everett of the physicist Wheeler submitted his doctoral thesis *Relative State Formulation of Quantum Mechanics* to the doctoral committee in 1956, in which he proposed a "*multiverse*" explanation of the living and death states of the Schrodinger's cat. It is believed that "*living*" and "*death*" of Schrodinger's cat exist in two different spaces, i.e., living $\mathbf{S}_1$ and death $\mathbf{S}_2$ respectively, as shown in Figure 7.5. However, one can only enter one of the spaces $S_1$ or $S_2$ to observe the life and death of the cat at a specific time, namely the cat state space is the union $S_1\bigcup S_2$ of $S_1$ and $S_2$ with $S_1\bigcap S_2=\emptyset$ in the eyes of human, which is a *parallel space*.

**Figure** 7.5. **Multiverse of cat state**

Usually, the academic research needs to constantly deny itself, innovate recognitive views, methods or means and promote the coordinated development of humans with the the nature. H.Everett's multiverse explanation of the living and death on the Schrodinger's cat is undoubtedly a breakthrough of the existing recognitive views of humans but it also challenges the academic mainstream at that time, namely the explanation of "*state collapse*" advocated by the Copenhagen school. His view was regarded as a "*weird view*" and rejected by the committee. As a result, he failed to defend his doctoral thesis in 1956. Relying on Wheeler's influence and guidance in the academic circle, Everett had to revise and made a minor modification on his doctoral thesis, passed the dissertation defense and then received a doctoral degree in the next year. This thing was greatly shocking the research of Everett. He felt extremely disappointed and depressed after then and would not talk about physics again. He had to work for a defense department of United States in order to survive. He died in a depressed state in 1982. It was until 1970s that the academic community gradually accepted the multiverses explanation as a logically consistent theory when B.DeWitt with his students published articles supporting the Everett's multiverse explanation of living and death states of Schrodinger's cat. Today, it is generally applied to the interpretation of quantum states in quantum mechanics. In 2007, the *Scientific American*, a well-known American journal published a special issue to commemorate the fiftieth anniversary of Everett's multiverse explanation of Schrodinger's cat and his contribution to the breakthrough of human recognition.

**(2)Systematic activeness.** In the systematic recognition of a thing, the system is composed of a combination of elements. Therefore, it is important to determine what kind of components or subsets of the system are the elements. Certainly, the selection of system elements should be based on the principle that the minimum number of elements to the system behaviors. So, it should not be subdivided without limitation. For example, Dr.Ouyang explains that the cat as a whole is the system element in the Schrodinger's cat. However, a cat as an animal has 10 subsystems including its skin, bone, muscle, digestion, respiration, circulation, excretion, nervous system, endocrine system and the reproduction system. It is composed of about $10^{27}$ atoms and has about 250 million nerve cells, about 4 billion brain cells and billion individual cells. So, *can we characterize a cat with the* 10 *systems, billions of cells or with about* $10^{27}$ *atoms?* The answer is Yes but besides adding complexity to the question, the result of this way does not help answer the Schrodinger question itself because the living and death of a subsystem or cell does not mean the living and death of a cat. In this way, the question on the living or death of Schrodinger's cat only needs to be taken the whole cat as the element without further subdivision in the depiction. Similarly, the behavior of a group of animals can be characterized by each animal as an element but it is necessary to construct a self-organizing system $S$ on the 10 subsystems or cells to depict the impact or damage of microorganisms or viruses on human body.

At this time, the system label graph $G^L[S]$ is similar to the general system label graph with the vertex set formed by the elements $v$ and if there is an action between vertices $v$ and $u$, defines an edge $(v, u)$ or $(u, v)$ in $G^L[S]$. Accordingly, the vertex label is defined as the element $v$ state and the edge $(v, u)$ label as the strength of interaction between the elements $v$ and $u$ in $S$. So, *how do we use the system labeled graph $G^L[S]$ to characterize this self-organizing system state?* Dr.Ouyang explains that there is only one cat in the system of Schrodinger's cat and the corresponding system labeled graph is the graph $K_1$ of order 1, on which the state labels is the union of *living* $S_1$ and *death state* $S_2$, namely a parallel space $S_1 \bigcup S_2$.

Generally, whether it is a microcosmic system $S$ such as a brain that consisted of cells or a macro system $S$ such as a group that composed of animal populations, we always assume that $S$ is consisted of $m$ elements and each element has $n$ states $s_1, s_2, \cdots, s_n$. Then, *a simple question is how many states $n(S)$ does the system $S$ exhibit?* Certainly, the states of system $S$ are related to the states of each element in general, namely the different states of elements correspond to the different states of system. So, applying the multiplication rule in combinatorics, there are $n(S) = \prod\limits_{i=1}^{m} n = n^m$, i.e., $n^m$ different states. For example, for the Schrödinger's cat there are only two states, i.e., $n(S) = 2$, the living or death states. For a family of $m$ cats, if one asks about the states or situation of this family similar to Schrödinger, there are

**Figure** 7.6. **A cat family**

$$\binom{m}{0} + \binom{m}{1} + \cdots + \binom{m}{k} + \cdots + \binom{m}{m} = 2^m$$

states, where the integers $0, 1, \cdots, k, \cdots, m$ in the binomial coefficients are the number of cats possiblely surviving in the family.

**1.3.Contradictory System.** The multiverse of the living and death of Schrödinger's cat is used to explain the "*contradiction*" appeared in the human recognition and also the contradiction of spear and shield in the merchant's hands, i.e., whether the spear is sharper than the shield or the shield is stronger than the spear in Hanfeizi's fable. In fact, the claims that "*the spear sharper than the shield*" and "*the shield stronger than the spear*" are both existed as the visible states to human eyes before the merchant stabbed the shield in his left hand with the spear in his right hand, which will result in that of two parallel spaces $S_1$ and $S_2$ but the human eyes can only see one of them, namely "*the spear is sharper than the shield*" or "*the shield is stronger than the spear*" similar to that of the living or death of Schrodinger's cat in Figure 7.5. Certainly, there are many such phenomena in human's life. For example, in the fable of "*the tortoise and the hare race*", whether the rabbit wins or the tortoise wins is a question before the contest although the tortoise is usually slower than the rabbit running. However, it is not impossible for the tortoise to win if the rabbit is lazy or has other conditions on its legs. This is the philosophical implication of "*the tortoise and the hare race*", namely we should look at a problem from all sides and it is not necessarily correct to recognize an unknown thing only according to local recognitions. Similarly, the blind men's perception on the shape of an elephant is not correct to an ordinary human in the fable that of the blind men with an elephant. However, *why did the blind men insist on arguing over it?* Dr.Ouyang explains that is because none of them admitted that the perception of elephant was local himself but everyone believed that his perception of elephant was correct. Similarly, the same is true on the human recognition of things that are beyond our ability. An ideal situation for solving the dispute in human recognitions can be dong similar to the sophist in the blind men with an elephant, namely there appears such a sophist with more than six sensory organs, i.e., more than seven sensory organs who can tell the human that of his whole recognition on the thing. However, even if there is such a sophist with seven or more than seven sensory organs in the universe, *must his recognition of things necessarily be the true face of things?* Logically, his perception on things in the universe remains also local unless the sophist is the omnipotent god who lives with the universe. This is the philosophical implication of the "*name named is not the eternal name*" in the first paragraph of Laozi's *Tao Te Ching*.

So, *how do we solve the contradictions produced by human's local recognition and follow the principle of logical consistency in the recognition of things?* The answer is to construct a recognitive system that can accommodate contradictions because in the multiverse interpretation $\mathbf{L}+\mathbf{D}$ for Schrodinger's cat suggested by Everett, the living $\mathbf{L}$ and the death $\mathbf{D}$ are a pair of contradictions but they are in the same cat state $\mathbf{L}+\mathbf{D}$.

In the recognitive system of humans, an axiom is such a fundamental proposition that accepted by humans without further proof according to the facts of humans' long-term practice. Then, a proposition is the recognitive conclusion that humans deduce according to the axioms and the principle of logical consistency. Such propositions are compatible without contradiction in general. In this way, the construction of a logical system containing contradictions needs to contain contradictions in the axioms so as to recognize the truth of things.

**Smarandachely Denied System** *An axiom is said Smarandachely denied if in the same system the axiom behaves differently, i.e., validated and invalided, or only invalidated* *but in at least two distinct ways. Correspondingly, a system $S$ containing a Smarandachely denied axioms is called a Smarandachely denied system.*

So, *are there Smarandachely denied systems which contains contradiction in general?* The answer is Yes! For example, the *Smarandache geometry* contains the Smarandachely denied axioms. As we known, the Euclidean geometrical axiom system consists of five axioms, i.e., ①There is a straight line between any two points; ②A finite straight line can produce an infinite straight line continuously; ③Any point and a distance can describe a circle; ④All right angles are equal; ⑤If a straight line falling on two straight lines makes the interior angles on the same side less than two right angles, then the two straight lines if produced indefinitely, must meet on that side on which are the angles less than the two right angles. Notice that the fifth axiom is usually replaced by ⑤*For a given straight line and a point exterior this straight line, there is one line parallel to this line.

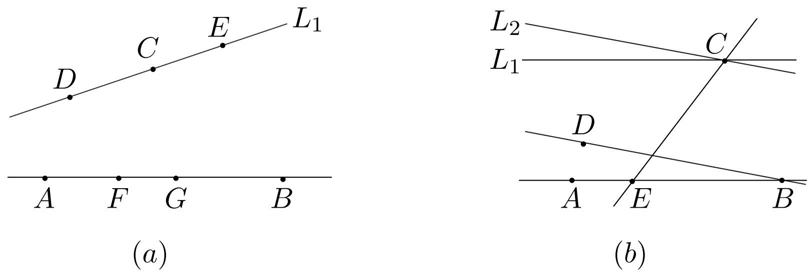



**Figure** 7.7. **An Example of Smarandache geometry**

Now, let $\mathbb{R}^2$ be a Euclidean plane, $A, B, C$ are three non-collinear points in $\mathbb{R}^2$. Define the $s$-points $\mathcal{P}_s$ to be all points on $\mathbb{R}^2$ and $s$-lines $\mathcal{L}_s$ to be all such straight lines that passes through one and only one of points $A, B, C$, as shown in Figure 7.7. Then, all these $s$-points with $s$-lines on $\mathbb{R}^2$ constitute a point-line set $\{\mathcal{P}_s; \mathcal{L}_s\}$, which is a Smarandache geometry on $\mathbb{R}^2$.

So, *why is the point-line set $\{\mathcal{P}_s; \mathcal{L}_s\}$ on $\mathbb{R}^2$ a Smarandache geometry?* Firstly, the axiom ① in Euclidean geometry is now replaced by *one $s$-line* or *no $s$-line* because through any two distinct $s$-points $D, E$ collinear with one of $A, B$ or $C$ there is one $s$-line passing through them. But, through any two distinct $s$-points $F, G$ lying on $AB$ or non-collinear with one of $A, B, C$, there are no $s$-lines passing through them such as those shown in Figure 7.7$(a)$. Secondly, the axiom ⑤ is now replaced by *one parallel* or *no parallel* because if we let $L_1, L_2$ be two $s$-lines passing through $C$ with $L_1$ parallel but $L_2$ not parallel to $AB$ in the Euclidean sense, then through any $s$-point $D$ not lying on $AB$ there are no $s$-lines parallel to $L_1$ but there is one $s$-lines parallel to $L_2$ if one of $DB$,$DA$ or $DC$ happens parallel to $L_2$. Otherwise, there are no $s$-lines passing through $D$ parallel to $L_2$, see Figure 7.7$(b)$ for details.

Notice that if we define the propositions A: "*the cat is alive*" and B: "*The cat is dead*" in the multiverse state $\mathbf{L}+\mathbf{D}$ explanation of the Schrodinger's cat living and death, then both of the propositions A and B are Smarandacely denied axioms, namely $\mathbf{L}+\mathbf{D}$ is a Smarandache denied system. According to the logical consistency in classical science, the cat state $\mathbf{L}+\mathbf{D}$ can be spatially decomposed into the cat living state $\mathbf{L}$ or the dead state $\mathbf{D}$ with $\mathbf{L}\bigcap\mathbf{D}=\emptyset$ by propositions A and B, namely $\mathbf{L}$ is parallel to $\mathbf{D}$. In this caes, such an operation "+" in $\mathbf{L}+\mathbf{D}$ is a special kind addition of vector spaces, called the *direct sum* and denoted the $\mathbf{L}+\mathbf{D}$ by $\mathbf{L}\bigoplus\mathbf{D}$. A generalization of this cat state $\mathbf{L}+\mathbf{D}$ is the *Smarandache multispace* or *Smarandache multisystem* defined as follows:

**Smarandache Multisystem** *For an integer $n\geq 1$, let $(\mathcal{S}_1,\mathcal{R}_1),(\mathcal{S}_2,\mathcal{R}_2),\cdots,$ $(\mathcal{S}_n,\mathcal{R}_n)$ be $n$ mathematical systems in the classic, distinct two by two, i.e., for any two spaces $(\mathcal{S}_i;\mathcal{R}_i)$ and $(\mathcal{S}_j;\mathcal{R}_j)$, $\mathcal{S}_i\neq\mathcal{S}_j$ or $\mathcal{S}_i=\mathcal{S}_j$ but $\mathcal{R}_i\neq\mathcal{R}_j$, $1\leq i,j\leq n$. A Smarandache multisystem $(\mathscr{S};\mathscr{R})$ is defined by*

$$(\mathscr{S};\mathscr{R})=\bigcup_{i=1}^{n}(\mathcal{S}_i,\mathcal{R}_i) \tag{7.1}$$

*where, $\mathscr{S}=\bigcup\limits_{i=1}^{n}\mathcal{S}_i$, $\mathscr{R}=\bigcup\limits_{i=1}^{n}\mathcal{R}_i$.*

Notice that $\mathbf{L}\bigcap\mathbf{D}=\emptyset$ in the multiverse $\mathbf{L}+\mathbf{D}$ of the Schrodinger's cat but there are no $(\mathcal{S}_i,\mathcal{R}_i)\bigcap(\mathcal{S}_j,\mathcal{R}_j)=\emptyset$, i.e., the intersection maybe not an empty set on Smarandache multisystem in general for integers $1\leq i,j\leq n$. Dr.Ouyang tells Huizi that *a Smarandachely denied system must be a Smarandache multisystem and vice versa* by definition. So, *how do we prove this fact?* Generally, let $S$ be a Smarandachely denied system. Dr.Ouyang tells Huizi that the proof is divided into two cases:

**Case 1.** *There is a Smarandachely denied axiom $A$ which is meanwhile validated and invalided in the system $S$.*

In this case, define $\Sigma_1=\{x\in S|\ x$ holds with the axiom $A\}$, $\Sigma_2=\{y\in S|\ y$ not holds with axiom $A\}$. Then, $S=\Sigma_1\bigcup\Sigma_2$, namely $S$ is a Smarandache multisystem.

**Case 2.** *There is a Smarandachely denied axiom $A$ which is invalided in cases of* $W_1$, $W_2,\cdots,W_s, s\geq 2$ *in the system $S$.*

In this case, for any integer $1\leq i\leq s$, let $\Sigma_i=\{x\in S|x$ not holds with axiom $A$ in case $W_i\}$ and $\Sigma_0=S\setminus\bigcup\limits_{i=1}^{s}\Sigma_i$, respectively, where the set $\Sigma_0$ may be an empty set. Then, there are

$$S\ =\ \left(\bigcup_{i=1}^{s}\Sigma_i\right)\bigcup\Sigma_0,$$

i.e., $S$ is a Smarandache multisystem.

Conclusively, a Smarandachely denied system $S$ is equivalent to a Smarandache multisystem $(\mathscr{S};\mathscr{R})$ by definition. Conversely, a Smarandache multisystem $(\mathscr{S};\mathscr{R})$ is obviously a Smarandachely denied system because an element in $\mathcal{S}_i$ can be viewed as a Smarandachely denied axiom $W_i$ for integers $1 \leq i \leq n$. We can therefore let $(\mathcal{S}_i, \mathcal{R}_i), 1 \leq i \leq n$ be elements of the Smarandache multisystem which consist of a labeled graph $G^L[S]$ with vertex and edge sets

$$V\left(G^L[S]\right) = \{\mathcal{S}_1, \mathcal{S}_2, \cdots, \mathcal{S}_n\},$$
$$E\left(G^L[S]\right) = \{(\mathcal{S}_i, \mathcal{S}_j)\,|\mathcal{S}_i \bigcap \mathcal{S}_j \neq \emptyset,\ 1 \leq i \neq j \leq n\}$$

and for any integers $1 \leq i \neq j \leq n$, the labels on the vertex $\mathcal{S}_i$ and edge $(\mathcal{S}_i, \mathcal{S}_j)$ are respectively

$$L:\ \mathcal{S}_i \to L(\mathcal{S}_i) = \mathcal{S}_i \ \text{ and }\ L:\ (\mathcal{S}_i, \mathcal{S}_j) \to L(\mathcal{S}_i, \mathcal{S}_j) = \mathcal{S}_i \bigcap \mathcal{S}_j.$$

For example, let $\mathcal{S}_1 = \{a, b, c\}$, $\mathcal{S}_2 = \{a, b, e\}$, $\mathcal{S}_3 = \{b, c, e\}$ and $\mathcal{S}_4 = \{a, c, e\}$, i.e., $n = 4$ and $\mathcal{R}_i = \emptyset, 1 \leq i \leq 4$ in a Smarandache multisystem $(\mathscr{S};\mathscr{R})$. Then,

$$S = \bigcup_{i=1}^{4} \mathcal{S}_i = \{a, b, c, d, e\}$$

is a Smarandache multiset with a labeled graph $G^L[S]$ shown in Figure 7.8, which is the complete graph $K_4^L$ of order 4.

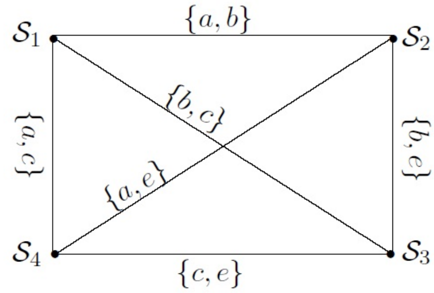



**Figure** 7.8. **Multiset**

Dr.Ouyang tells Huizi that since a Smarandache denied system is equivalent to a Smarandache multisystem $S$, we need to characterize the state of Smarandache multisystem $S$ or its corresponding labeled graph $G^L[S]$ according to the guidance of sophist to the blind men in the fable of the blind men with an elephant for holding on a thing in accordance with the principle of logical consistency and then, recognize its evolving of the thing. Certainly, his guidance is such a recognition that by the combination notion.

## §2. **Quantum Entanglement**

Generally, the essence of human recognition of a thing $S$ is to characterize the Smarandache multisystem corresponding to $S$. The system elements may or may not comply with the synchronization rules. Now, let $S = \bigcup\limits_{i=1}^{n} (\mathcal{S}_i, \mathcal{R}_i)$ be a Smarandache multisystem with a labeled graph $G^L[S]$. A mapping $\iota_v : G^L[S] \to v \in V\left(G^L[S]\right)$, i.e., $\iota_i : S \to (\mathcal{S}_i, \mathcal{R}_i)$ is said to be a *collapse mappings*, which is a mapping from the real face $G^L[S]$ of thing $S$

to a local recognition, i.e., system $(\mathcal{S}_i, \mathcal{R}_i)$ on $v_i$ for an integer $1 \leq i \leq n$ and characterize the recognition process of humans. Among them, for any integers $1 \leq i \leq n$, the system $(\mathcal{S}_i, \mathcal{R}_i)$ is dependent on other systems $(\mathcal{S}_j, \mathcal{R}_j)$, namely its elements are overlapped with that of systems $\mathcal{S}_j, 1 \leq j \neq i \leq n$ if $\mathcal{S}_i \bigcap \mathcal{S}_j \neq \emptyset$ for integers $1 \leq i, j \leq n$, i.e., the system elements on vertex $v_i$ are overlapped with those systems on vertices adjacent to $v_i$ in the labeled graph $G^L[S]$.

Mathematically, a collapse mapping can be seen as a projection from the whole $G^L[S]$ of a thing to the local $v_i \in V\left(G^L[S]\right)$, as shown in Figure 7.9 for the orthographic projection of a cube. Now, a natural question is *whether such a collapse mapping exists objectively?* The answer is certainly Yes! For example, the Schrödinger cat's living and death state $\mathbf{L} \bigoplus \mathbf{D}$ is a Smarandache multisystem with a labeled graph $P_2^L$. In this case, there are two human's local recognitive states, namely the living $\iota_L : \mathbf{L} \bigoplus \mathbf{D} \to \mathbf{L}$ and the death $\iota_D : \mathbf{L} \bigoplus \mathbf{D} \to \mathbf{D}$. If one opens the lid of box and finds the cat in living state $\mathbf{L}$, namely "*the cat is alive*" then "*the cat is dead*" is not valid and deduces $\mathbf{D} = \emptyset$ and vice versa. Therefore, the living and death states $\mathbf{L} \bigoplus \mathbf{D}$ of Schrodinger's cat is a multisystem with $n = 2$. At the same time, the living or death of a cat is exclusive to humans, namely the state of a cat can be only one of the alive or dead but not the both.

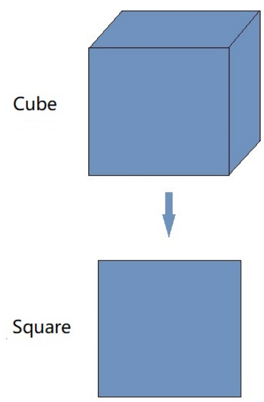



**Figure** 7.9. **Projection**

Certainly, it is a gradual process from human's local recognition of a thing $T$ to the whole, which is the expression

$$T = \left(\bigcup_{i=1}^{n} \{\mu_i\}\right) \bigcup \left(\bigcup_{k \geq 1} \{\nu_k\}\right)$$

appeared in Section 5 of Chapter 2. It is actually a Smarandache multisystem, where $\mu_1, \mu_2, \cdots, \mu_n$ are known characters of things $T$ at time $t$ and $\nu_k, k \geq 1$ are unknown characters. So, *how do we characterize the collapse mappings of a Smarandache multisystem?* The answer of this question lies in how to understand $\{\mu_i, 1 \leq i \leq n\}$ and $\{\nu_k\}, k \geq 1$ in the formula. Dr.Ouyang tells Huizi that if $T$ is equivalent to a Smarandache multisystem $\widetilde{T}$ of elements, i.e., $\widetilde{T} = \{a_\lambda | \lambda \in \Lambda\}$, where $\Lambda$ is the index set of elements in $T$. Assume the character of an element $a_\lambda$ in $T$ is $\chi(a_\lambda)$. Then, the characters of $\mu_i$ and $\nu_k$ are essentially equivalent to the classification of elements in $T$ according to the characteristics of $\mu_i$ and

$\nu_k$, i.e.,

$$\{\mu_i\} = \{\ a_\lambda \in T \mid \chi(a_\lambda) = \mu_i\} \ \text{ and } \ \{\nu_k\} = \{\ a_\lambda \in T \mid \chi(a_\lambda) = \nu_k\}, \tag{7.2}$$

namely for any integer $1 \leq i \leq n$ or $k \geq 1$, the set $\{\mu_i\}$ and $\{\nu_k\}$ are respectively consisted of elements with character $\mu_i$ or $\nu_k$ in $T$. In this case, the collapse mappings are nothing else but the mappings

$$\mu_i : T \to \{\mu_i\} \quad \text{and} \quad \nu_k : T \to \{\nu_k\}, \tag{7.3}$$

of $T$ known by humans. Similar to the observation of the living or death of Schrodinger's cat, for any integer $1 \leq i \leq n$ or $k \geq 1$, the character $\mu_i$ or $\nu_k$ here is the observing result of human on thing $T$. But unlike the observation of Schrodinger's cat, the characters here $\{\mu_i\}, 1 \leq i \leq n$ and $\{\nu_k\}, k \geq 1$ are no longer exclusive for $n \geq 2$.

Hearing this, Huizi puzzlingly asks Dr.Ouyang: "*Dad, the word 'character' feels a bit abstract. Can you give me a popular example for explaining the collapse mapping?* Dr.Ouyang asks her: "*If there is a basket full with apples and pears. How do you determine how many apples and pears there are in the basket?*" Huizi replies: "*It is simple. I can jump out the apples or pears one by one from the basket. Then, I can count them again and know how many apples or pears there are in the basket.*" Dr.Ouyang asks her again:"*Why did you pick the apples or pears out of the basket one by one?*" Huizi answers: "*Because the apples and pears are different fruits.*" Dr.Ouyang continuously asks her: "*Why you believe the apples and pears are different fruits?*" Huizi replies: "*because of their difference in texture, taste, sugar, water and acid, etc!*" Dr.Ouyang nods and tells her that the apples or pears are the distinguishing characters of fruit in the basket here. Accordingly, the process of picking out the apples or pears one by one from the basket is a collapse mapping from the fruits in the basket to the apple or pear.

**2.1.Quantum Entanglement.** Notice that the classical mechanics characterizes the state of a particle using its position and momentum. For example, suppose A and B are two elastic balls with masses of $m_1$ and $m_2$ respectively. They are moving with velocities of $v_1$ and $v_2$ after their collision at the origin $O$ in vacuum and travelling in reverse direction along a straight line, as shown in Figure 7.10. In this case, assuming that the velocities of A and B after the collision are respectively $v_1'$ and $v_2'$, by the law of conservation of momentum

$$m_1v_1 + m_2v_2 = m_1v_1' + m_2v_2'$$

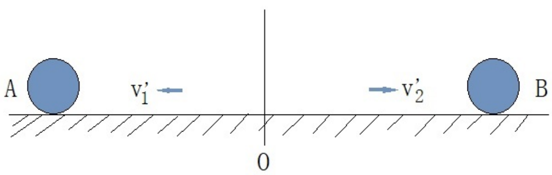



**Figure** 7.10. **Ball collision**

in classical mechanics we then know that the velocity of ball B is

$$v_2' = \frac{m_1 (v_1 - v_1') + m_2 v_2}{m_2},$$

namely if the velocity of one ball in A and B can be known by measuring, the velocity of the other ball can be immediately obtained after they elastically collided. Similarly, we also have this experience in our lives. For example, someone has a pair of socks in one red and another white. One night he put on his socks and then go to the lamppost, looks at his left foot and finds that it is red. Then, he does not need to look at the right foot to infer immediately that it is white on his right foot because there is only one red and one white socks. However, these two examples are similar to the phenomenon of quantum entanglement but none of them is the quantum entanglement in the microcosmic world because they are not random and can be deduced straightforward in logic.

As is known to all, one of the weirdest properties of a microscopic particle is its uncertainty. Dr.Ouyang explains that it is impossible to accurately measure the position and momentum of a microscopic particle at the same time, which means that it moves randomly because of the uncertainty principle. In this case, different from the macrocosmic world, humans can not use the causality to predict the particle state because the double-slit experiment shows that in the 2-element system composed of humans with the microscopic particles, the measuring behavior of human will interfere with the state of microscopic particle and the particle state freely on the human measuring can not be found. Thus, the way to characterize macrocosmic particles can not be applied to the microscopic particles. For this reason, it is Born, a physicist who introduced the wave function for characterizing the state of microscopic particles and believed that a microscopic particle is a complex function $\Psi(x,y,z,t)$ of space and time, called the "*wave function*" and interpreted the square $|\Psi(x,y,z,t)|^2$ of the norm of wave function $\Psi(x,y,z,t)$ to be the probability of the particle appeared in the nearby volume of point $(x,y,z)$ per unit at moment $t$ which follows the wave superposition principle, i.e., if $\Psi_1(x,y,z,t)$, $\Psi_2(x,y,z,t)$ are two possible wave functions of systems state, then

$$\Psi(x,y,z,t) = c_1\Psi_1(x,y,z,t) + c_2\Psi_2(x,y,z,t) \tag{7.4}$$

is also the wave function of the system and hold with the Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = -\frac{\hbar^2}{2m}\left(\frac{\partial^2\Psi}{\partial x^2} + \frac{\partial^2\Psi}{\partial y^2} + \frac{\partial^2\Psi}{\partial x^2}\right) + U\Psi,$$

where, $m$ is the mass of the particle, $c_1, c_2$ are two constants, $\hbar = 6.582 \times 10^{-22} MeVs$ is the Planck constant and $U$ is the potential energy of the particle field. For example, let $\Psi_1(x,y,z,t)$ and $\Psi_2(x,y,z,t)$ be respectively the living and death states of the

Schrödinger's cat. Then, the collapse mappings are respectively $\Psi(x, y, z, t) \to \Psi_1(x, y, z, t)$ and $\Psi(x, y, z, t) \to \Psi_2(x, y, z, t)$, i.e., the wave collapse mappings.

So, *what is quantum entanglement?* Certainly, the *quantum entanglement* is a phenomenon of microscopic particles. That is, if two entangled particles are separated and one measured, the other particle will immediately occurs a corresponding state change. In 1935, Einstein and two other physicists published a paper in *Physical Review* known as EPR paper later, discussing the pinball game of two microscopic particles and arguing that if the description of a system by state vectors is complete then the definite values of quantities could be inferred from a state vector for the system itself or from a state vector for a composite of which the system is a part, called the *completeness criterion*. And so, the eigenvalue corresponding to the eigenstate of the system is a value determined by the real physical state of that system, called the *reality criterion*, namely if one can predict the value of a physical quantity with certainty without any interference with the system, there must be an element of physical reality corresponding to that quantity and if the systems are separated in space, the measuring of one system does not directly affect the reality that pertains to the others, called the *local hypothesis*, which states that if the two systems do not interact when measured, then no thing needed be done to the first system will cause any real change to the second system but the quantum mechanics is incomplete without satisfying these criteria. For example, whenever one of the two separated entangled particles A and B is measured, as long as the spin direction of measured A is up, then the spin direction of B must be down. Conversely, if the spin direction of A is measured downward, then the spin direction of B must be upward, as shown in Figure 7.11. Generally, the similar particle properties include position, momentum, and so on. In other words, the measurement results of these properties of particle B are contrary to those of particle A and the properties of particle B can be known by measuring only one of the two particles such as the particle A, which is similar to the situation in classical mechanics where the state of the other ball can be known only by measuring one particle after the collision of two elastic balls. However, the certainty of the microscopic particles here is different from that of the ball in classical mechanics because the microscopic particles are random.

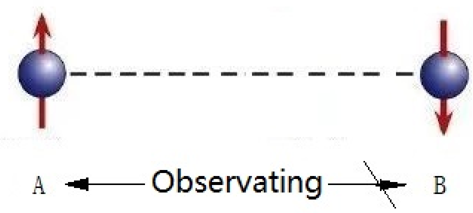



**Figure** 7.11. **Entangled particles**

Certainly, one will not affect the moving state of A when measuring the collision ball A in classical mechanics but the measuring on particle A will affect the moving state of A in the microscopic world. In this case, it seems that there is a transmission that informs instantaneously particle B of the measuring of particle A. But how it is transmitted one struggles to find the medium but has no result. The phenomenon of particle entanglement has no choice but to be classified as the "*action at a distance*", which certainly violates the laws of classical mechanics. Einstein even called the quantum entanglement as "*spooky action at a distance*" because it is similar to the double-slit experiment and can not be explained by the laws of classical mechanics. This also shows the limitation of human recognition from one side again.

So, *how should we interpret the phenomenon of quantum entanglement?* In 1957, One of the Einstein's followers, the physicist Bohm came up with a hidden variable interpretation that Einstein admired, namely the particles still follow a precise continuous trajectory but this trajectory is not only determined by the forces found in the macroscopic world but also affected by the quantum potential in the quantum world. This quantum potential is generated by the wave function of the particle, which provides the dynamic information to the particle's environment at any time, guides the direction of the particle's moving and results in the magical behaviors of particle that is different from the macroscopic particle. In Bohm's hidden variable interpretation of quantum entanglement, the nature of quantum system does not belong to the system itself, the evolution of particles depends on the system and also depends on the measuring instrument, the statistical distribution of the measuring results of hidden variables will be different with different measuring apparatus. This ensures that Bohm's hidden variable interpretation is exactly the same as the quantum mechanical prediction of the measuring results. However, although Bohm provided an interpretation of quantum entanglement consistent with quantum mechanics, the principle of the position and velocity determined by the particle may never be known because of the hidden variables of particle and humans may never know the true trajectory of particle. This is why Bohm's hidden variable interpretation is unacceptable to someone.

Mathematically, it is easy to characterize the hidden variable interpretation of quantum entanglement. Assuming that $S$ and $S'$ are two sets, if the mapping $f : S \to S'$ and $f' : S' \to S$ are both given onto, i.e., $f(S) \supseteq S'$ and $f'(S') \supseteq S$, then $S$ and $S'$ are called a *entangled pair.* In this case, one only needs to know one of sets $S$, $S'$ and the other set can be known by the mapping $f$ or $f'$ action. For example, $f$ or $f'$ is a $1-1$ mapping or $f : \{S, S'\} \to \{0, +1\}$ is a 2-value function and meet the exclusivity, i.e., for any $x \in S, y \in' S$ holds with $f(x) \neq f(y)$, then $S$ and $S'$ must be an entangled pair. Furthermore, it is possible to give algebraic structures one sets $S$ and $S'$ such as the groups, rings or fields, which by their algebraic properties can easily be deduced from one set to another. For example, if $S$ and $S'$ are two isomorphic finite fields, an entangled pair $S$ and $S'$ can be immediately obtained. Similarly, the living **L** and death **D** of Schrodinger's cat as well as particles A and B in the quantum entanglement are 2-valued function cases if the exclusivity is satisfied, which also constitute entangled pairs. So, *can an entangled pair of $S$ and $S'$ definitely be observed by human eyes?* The answer is not necessarily! Dr.Ouyang tells Huizi that not all natural phenomena can be recognized by humans because we have certain limitations in the recognition of things. However, a question that induced by these examples is that "*can we not accept the Bohm's hidden variable interpretation just because the hidden variables can not be seen to be true?*" Of course not! Dr.Ouyang explains that the hidden variable interpretation is indeed a scientific interpretation for quantum entanglement.

**2.2.Action at a Distance.** Notice that the Bohm's hidden variable interpretation of quantum entanglement aims to return to determinism or causality in the microscopic world, namely the microscopic particles still move along an exact continuous trajectory but humans can not accurately measure on particles due to the hidden variables. So, *why does the Bohm's hidden variable interpretation of quantum entanglement not agree with someone?* The reason why the hidden variable is "*hidden*" is that no such a physical reality can be found for human measuring unless the "*action at a distance*", namely the instantaneous transmission of information has not been explained clearly. This can be also explained by examining the life of ants on a plane $\mathbb{R}^2$ which is one dimension less than that of the human living space $\mathbb{R}^3$. Certainly, the ants live in a world of length and width but no height to humans. For example, let $A$ and $B$ be two points on the expanded plane $I = \{(x, y)|x, y \in \mathbb{R}\}$ in Figure 7.12$(a)$ but the coordinate on $z$-axis is sufficiently small, i.e, $z \to 0$, as shown in Figure 7.12$(b)$. Assume that an ant at $A$ can only transmit a message to the ant at $B$ through the line segments $AB$ consisting of a series of points $(x, y)$ on the plane $I$. But to humans, there is a more closer route, i.e., along the $z$-axis direction, as shown in Figure 7.12$(b)$. In this way, although $A$ and $B$ are in different positions on the expanded plane $I$, the distance between them in the direction of $z$-axis is nearly 0. So, an instantaneous transmission along the direction of $z$-axis can definitely be realized. However, it is an incredible event for the ants compared with humans other than an effect of the spooky.

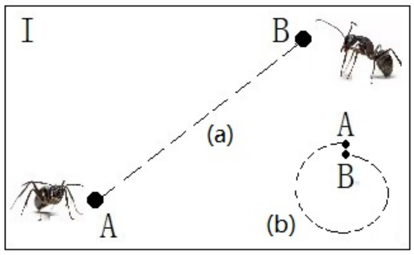



**Figure** 7.12. **Ants experience**

For an information transmitting, an important question is that *how many are there* *spatial dimensions needed for an information transmitting? Do we really need three dimensions, i.e., spatial medium on transmission?* Dr.Ouyang tells Huizi that the humans usually understand the transmission of information by analogy to that of the transmission process of observed forces, which is the most convenient situation for human perception. It is precisely based on such a perception process that when they understand the information transmission. They believe that there must be a visible medium for information transmission and mistakenly interpret the 3-dimensional space of the medium as the transmitting space. Then, *are the visible transmission made by humans necessarily a series of* 3*-dimensional spaces?* The answer is not necessarily! For example, it is true that the transmission of energy, heat and the light requires the transmission of visible medium but *is the gravity or magnetism transmitted through the medium of a series of* 3*-dimensional space?* In fact, the gravity is regarded as an action at a distance in the Newtonian mechanics; Einstein objectively interpreted gravity as the action of space bending and asserted that the space in which humans living is not flat but a curved one. It is a bending effect of space that forms the gravity and put forward the equation of gravitational field accordingly; Maxwell explained the phenomenon of magnetic force as the property of field, i.e., the field action and no transfer medium was found but later a few of humans assumed that there is a magnetic medium transferring magnetic force. Here, neither the gravity nor magnetic transmission has found to be human visible transmission medium. Therefore, there are no evidence to regard the transmission space of gravity and magnetism as a series of 3-dimensional spaces. Similarly, it is assumed that the voice and signal are transmitted through the transmission medium of cables or optical cables but the transmission mechanism of wireless signals is not clear, which can only be realized according to the Maxwell's electromagnetic field theory but the spatial dimension required by the information transmission medium is not clear even so. Particularly, the information is understood as data or codes in the human recognition and the information can be transmitted as 1-dimensional space instead of 3-dimensional space with transmitting channels everywhere in all directions. So, *why do we think that an information transmission needs a series of* 3*-dimensional spaces or fields?* Under the assumption that the transmission medium is visible, humans think that its transmission needs a physical field, namely a 3-dimensional space that is weighted at each point. However, humans have not observed the process of gravity, magnetism or information transmission in the experiment so far and can only interpret them as the effect of space.

If this point is explained clearly, the hidden variables in Bohm's interpretation of quantum entanglement can be explained clearly and then, we can understand why the quantum entanglement seems as a kind of "*action at a distance*" or "*spooky action*" in human vision. Certainly, the humans have a prior assumption in their recognition of things, namely *an information must be transmitted through the visible medium in the visible* 3*-dimensional space of humans*, which is similar to the ants' being confined to the plane recognition situation. In Figure 7.12($a$), they can not realize that A and B can also transmit information along $z$-axis direction in Figure 7.12($b$). Especially, when the distance between A and B in the $z$-axis direction tends to 0, an instantaneous transmission of information can be realized. If the ants had any recognitive ability, they could only interpret it as action at a distance because the transmission of information from points A to B can only be achieved through a set of sequentially connected line segment AB on the plane I standing on the ant's view but it is only the result of the distance between A and B approaching 0 along the $z$-axis direction on the human's view.

Then, *where are the hidden variables of quantum entanglement located in Bohm's interpretation, are they still in the human visual space?* If so, the humans will eventually find these hidden variables. But if the hidden variables are not in the human visual space, the six sensory organs of human may never discover the physical reality pursued by Einstein and others in the EPR papers. In this case, "*do we deny its existence if we can not see or can not clearly see?*" Of course Not because the human recognition of things is limited and restricted by the six sensory organs of human. In fact, the recognition that complies with the principle of logical consistency is still a kind of scientific recognition and also belonging to science.

There is another question with this question, i.e., *Why do humans accept Einstein's interpretation that the gravity is a bending effect of space and the Maxwell's interpretation that magnetism is an action effect of magnetic fields but not accept that the information transmission in quantum entanglement is the action of hidden variables?* Dr.Ouyang explains that the reason of humans accepted the Einstein's interpretation of gravity and the Maxwell's interpretation of magnetism is because that although the space bending or acting effects of magnetic fields were still not seen but they could imagine the bending space, the action of magnetism in their daily life and so accepted the two interpretations. Accordingly, the Bohm's interpretation of quantum entanglement by hidden variables is rejected because of the word "*hidden*", i.e., one can not imagine how this variable behind which quantum information transmission is realized, i.e., ①They believe that an information transmission requires the 3-dimensional media and ②They do not know the embedded structures of different biospaces. For the item ①, as stated above the dimensions required for information transmission are 1-dimensional rather than 3-dimensional and for the item

②, we need to introduce the embedded structures of biospaces for understanding the action at a distance. So, *what is a biospace?* A *biospace* is defined as follows:

**Biospace** *A biospace is a Euclidian space* $\mathbb{R}^k, k \geq 2$ *where the species of organisms live and know things in the universe.*

For example, the biospaces of ants and humans are respectively the Euclidean spaces $\mathbb{R}^2$ and $\mathbb{R}^3$. Certainly, the biospace of livings that are more recognitive ability than humans are $\mathbb{R}^4, \mathbb{R}^5, \cdots$, etc. So, *how do a biospace of lower ability stacks up in the biospace of higher ability?* Usually, the same populations have the same recognitive ability, their biospaces should be the same. Then, a lower dimensional biospace has the same value in a higher dimensional biospace unless the extra dimension from the perspective of a living that is only one dimension higher. For example, the ant's biospace is a plane $\mathbb{R}^2$ embedded in the human's biospace $\mathbb{R}^3$ and the coordinate $z$ in the third dimension is the same. In this way, $\mathbb{R}^2 \subset \mathbb{R}^3 \subset \mathbb{R}^4 \cdots$. It is not chaotic but has a certain rule to follow. This rule is that the bottom biospace is embedded into the last biospace and the increased coordinate is the same, represented as

$$\{(x, y, 0, \cdots, 0)\} \subset \{(x, y, z, 0 \cdots, 0)\} \subset \{(x, y, z, w, 0 \cdots, 0)\} \subset \cdots, \tag{7.5}$$

where the added dimension coordinates of the bottom biospace is expressed as 0 for the sake of simplification. For example, the recognitive spatial relationship between ants and humans is $\{(x, y, 0)\} \subset \{(x, y, z)\}$, etc.

Now, it is possible to explain the *action at a distance* in quantum entanglement in which an information between the entangled particles is transmitted through the fourth dimension $w$ because the spatial dimension required for an information transmission is 1-dimensional. So, *how does achieve this action at a distance?* Dr.Ouyang explains that because the two entangled particles A and B remain in the human biospace $\mathbb{R}^3$ after being separated under human vision but their fourth dimension coordinates have the same value. In this way, if we measure particle A, the particle B will know at the same time without any need for medium transmission. In other words, all entangled particles known by humans are entangled in such a way that the fourth coordinates are the same. Otherwise, they are not entangled if they have no the same coordinate in its four coordinates of $\mathbb{R}^4$.

Generally, we can also embed the systems of a Smarandache multisystem $S = \bigcup\limits_{i=1}^{n} (\mathcal{S}_i, \mathcal{R}_i)$ $(\mathcal{S}_i, \mathcal{R}_i), 1 \leq i \leq n$ the same as (7.5) and get an embedding relationship in $S$, i.e.,

$$\mathcal{S}_1 \subset \mathcal{S}_1 \bigcup \mathcal{S}_2 \subset \mathcal{S}_1 \bigcup \mathcal{S}_2 \bigcup \mathcal{S}_3 \subset \cdots \subset S \tag{7.6}$$

holding with $\mathcal{S}_i \bigcap \mathcal{S}_j = T_{ij} \subset \mathbb{R}^{k_{ij}}, 1 \leq i \neq j \leq n$ and then, for two given systems

$(\mathcal{S}_i, \mathcal{R}_i), (\mathcal{S}_j, \mathcal{R}_j)$ if there is such an integer sequence $ii_1 \cdots i_s j$ that $k_{ii_1}, k_{i_1 i_2}, \cdots, k_{i_s j} \geq 1$, we can certainly realize the information transmitting between systems $(\mathcal{S}_i, \mathcal{R}_i) \to (\mathcal{S}_j, \mathcal{R}_j)$ in theory. Furthermore, if there are $k_{ii_1}, k_{i_1 i_2}, \cdots, k_{i_s j} \geq 3$ we can then realize the 3-dimensional object migration from systems $(\mathcal{S}_i, \mathcal{R}_i)$ to $(\mathcal{S}_j, \mathcal{R}_j), 1 \leq i \neq j \leq n$ in the 3-dimensional biospace $\mathbb{R}^3$ of humans.

**2.3.Lucky Number Game.** Suppose that there are two boxes X and Y in which each contains one red ball and one blue ball. The game is played in a team consisting of three humans A, B and C to draw a ball from boxes X and Y in order with a rule that all players draw a ball from the X box or one player draws from the X but the other two draw ball from Y and after drawing, the player should put the ball back in the original box for the next player to draw again and write a lucky number 1 or $-1$ on a piece of paper. The answer is kept secret from the rest of other players and the winning condition is that the product of lucky numbers is $-1$ if all players draw the ball in X or 1 if one player draws the ball in X and two other ones draw the ball in Y. For example, if the lucky number answered by A is $-1$ and the lucky number answered by B and C is 1, then the product of the lucky numbers drawn by the three players is $-1 \times 1 \times 1 = -1$ and the team wins.

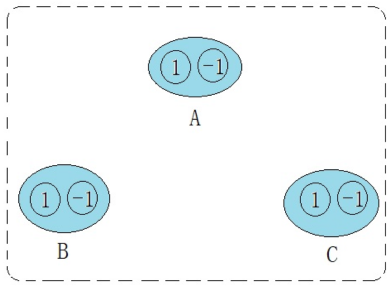


**Figure** 7.13. **Draw ball game**

Now, let $X_A$, $Y_A$, $X_B$, $Y_B$, $X_C$, $Y_C$ be respectively the answering lucky numbers of A, B and C after drawing ball in X or Y boxes, where each of these quantities such as $X_A$ or $Y_A$ is either 1 or $-1$ values, as shown in Figure 7.13. So, *how can a team* $\{A, B, C\}$ *wins the game?* According to the game rule, the team must satisfies one of the following conditions, i.e.,

$$X_A X_B X_C = -1, \quad X_A Y_B Y_C = 1, \quad Y_A X_B Y_C = 1, \quad Y_A Y_B X_C = 1. \tag{7.7}$$

For example, let $X_A = 1, Y_B = 1, Y_C = 1$, then $Y_A Y_B X_C = 1$. Notice that if $X_A, Y_A, X_B$, $Y_B, X_C$ and $Y_C$ take the same value in each formula of (7.7), then the product of these four equations is $(X_A Y_A X_B Y_B X_C Y_C)^2 = -1$, which can not be true on $\{-1, 1\} \subset \mathbb{R}$. Therefore, $X_A$, $Y_A$, $X_B$, $Y_B$, $X_C$ and $Y_C$ are not necessarily the same in different equation of (7.7). Thus, by solving $X_A, Y_A, X_B, Y_B, X_C, Y_C$ in equation (7.7) to get the win condition of team will not work.

Dr.Ouyang explains, if one player is allowed to tell the other two players his lucky number answered after drawing the ball, the team can win easily by control their answering numbers. For example, if A draws the red ball in X and answers the lucky number $-1$, the team will win if B and C both answer the lucky numbers 1 or $-1$ after drawing the ball. Similarly, if A draws a red ball from Y for a lucky number of $-1$, B and C draw a red ball from X and Y for a lucky number of 1 and if B and C draw a red ball from Y for a lucky number of $-1$, the team wins without having to look at the color of the ball drawn. However, the game rule is that A, B and C are not allowed to tell the lucky number answered after drawing the ball to others in the team, which certainly blocks the information transmission among the team members.

So, *is there such a rule that the three players can always win by drawing a ball to answer the lucky number* 1 *or* $-1$*?* The answer is Yes by using the quantum entanglement without the confused of color of the ball. In the way, there are three microscopic particles of spin entanglement such as the electrons or photons need to be prepared in advance. After the entanglement is disentangled, the each player of A, B and C saves a microscopic particle. Now, if A draws the ball in box X, he measures the spin of the particle in his hand along the $x$-axis, counting 1 or $-1$ as his lucky number. Similarly, if A draws the ball in box Y, measures the spin of the particle in his hand along the $y$-axis with a value of 1 or $-1$ as his lucky number. In this case, if all three players in the team draw the ball in the X box, the product of the lucky numbers must be $-1$; If one player draws in box X and two players draw the ball in box Y, the product of the lucky numbers in the answers must be 1. *Why is this so?* The answer is because the microscopic particles in the hands of A, B and C are entangled and separated microscopic particles, the spin direction is measured along the $x$-axis and the measuring result is one $-1$ but two 1 and similarly, the spin directions measured along the $x$-axis is $-1$ and the spin directions measured along the $y$-axis are one 1 and one $-1$. In this way, if the players A, B and C draw the ball in box X, the product of their lucky numbers is -1 and if one player in box X and two players in box Y drawing the ball, the product of the lucky number is 1, which satisfies the winning condition.

**2.4.Quantum Teleportation.** Notice that all humans live in a 3-dimensional Euclidean space $\mathbb{R}^3$ and can certainly migrate 3-dimensional objects freely in $\mathbb{R}^3$ without the need of higher dimensions. But, how to realize the migration or transmission of low-dimensional objects and media without the help of 3-dimensional medium is still an unsolved problem because the dimension of objects in human eyes is 3-dimensional. In this case, the radio transmission relies on the electromagnetic waves by applying the Maxwell's electromagnetic field, namely the 3-dimensional transmission of information rather than the 1-dimensional transmission of information. Dr.Ouyang tells Huizi that the quantum entanglement is enlightening for humans to realize 1-dimensional transmission of information, namely an invisible transmission.

In the lucky numbers game, a team wins the game by using the quantum entanglement of "*action at a distance*" or "*teleportation*", i.e, wins by using the quantum teleportation. Generally, it is assumed that the quantum state $\Psi$ of particle $\alpha$ needs to be transmitted between humans A and B in quantum communication. So, *how can a transmitting information on the internet without disclosure?* A classic way is to encrypt the transmitted information and allocate the pre-determined key between A and B for encryption and decryption. At this time, if the key is stolen the information is leaked. However, it is said that the quantum teleportation can effectively prevent the information leakage because each measuring result will be different after the entangled quantum separated. So, *how does the quantum teleportation work in communication?* Generally, a quantum teleportation process in which A transmits an information $\alpha$ to B as shown in Figure 7.14.

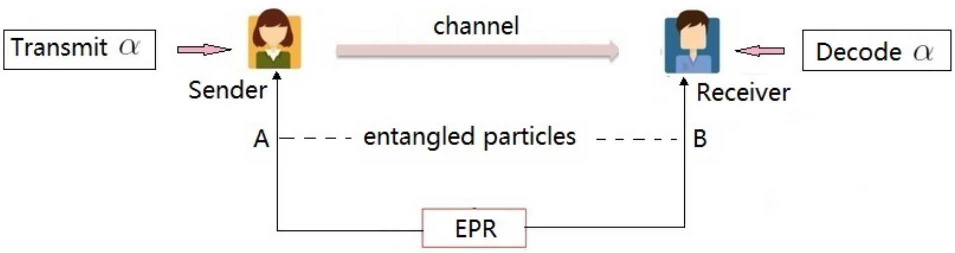



**Figure** 7.14. **Quantum teleportation**

Here, an EPR source refers to such a device or system that can generate and separate the entangled particles. So, *how do we transmit a quantum information $\alpha$?* Firstly, the sender and receiver need to be allocated respectively an entangled and then separated particle A and B; Secondly, the sender makes a joint measuring on particles $\alpha$ and A which will cause the collapse of the particle system composed of $\alpha$, A and B. At this time, due to the entanglement of particles A and B, the "*action at a distance*" transmits the transformation of quantum state $\Psi$ of $\alpha$ from A to B and the receiver measures B will know the measuring result of particle A by the sender. In this way, as the receiver receives the measuring result transmitted through the information channel, i.e., the measuring result of transformed quantum state $\Psi$ and $\alpha$, it can be decoded to the quantum state $\Psi$ of particle $\alpha$ by an inverse transformation on the measuring result and then realize the teleportation of particle $\alpha$.

Dr.Ouyang tells Huizi that the main idea of quantum teleportation is the action at a distance in the separated entangled particles, which is a information transmission from 3-dimensional medium reduced to the 1-dimensional, namely the application of the fourth dimension of space embedding (7.5) for instantaneous transmission of information.

However, we do not know the mechanism of 1-dimensional information transmission until today. It is just a phenomenological application based on the quantum entanglement. Notice that the quantum entanglement is only a case of Smarandache entangled pairs with the key generators of quanta and what the mysterious feeling of humans on the quantum entanglement is not because of the encoding or the key itself but the mystery and uncertainty of quantum behavior. In fact, similar to the quantum teleportation in Figure 7.14, a Smarandache multisystem and Smarandache entangled pair $S, S'$ can be applied also, i.e., to be the encoding and decoding keys in general. In this case, suppose that there is an information $I$ transmitted, the sender chooses a Smarandache multisystem $\widetilde{A} = \bigcup\limits_{i=1}^{m} A_i$, some sets of $A_i, 1 \leq i \leq m$ and a Smarandache entangled pair $S$ to encode and encrypt the information $I$ to obtain an encrypted information $\widetilde{A}'$, encodes or encrypts the information $\widetilde{A}'$ by the public digital encoding to get $I(\widetilde{A}')$ and then, transmits $I(\widetilde{A}')$ on the classical channel. And while the information $I(\widetilde{A}')$ is received, the receiver can decode $I(\widetilde{A}')$ by using the public digital encoding key and $S'$ in Smarandache entangled pair to obtain $\widetilde{A}'$, $\widetilde{A}$ and then get $A_i, 1 \leq i \leq m$ from $\widetilde{A}$ by applying the collapse mappings of Smarandache multisystem $\widetilde{A}$ to obtain the information $I$. Of course, it is necessary to have a key generator that carries the characters of Smarandache entangled pairs and multisystems so as to generalize the application of action at a distance in information transmission similar to that of the EPR source in quantum entanglement.

## §3. **Limited Mathematics**

As everyone knows, mathematics is a formal science describing the space with quantity relationships. Its limitation comes so as the science of humans. Essentially, mathematics discusses the artificially conceptions defined by humans with relationship. Its main function is to provide an abstract tool for other sciences to characterize things quantitatively and to hold on the macrocosm and microcosm matters. On the one hand, mathematics is the product of abstract thinking of human and can be then regarded as a science over science. On the other hand, because matter determines the consciousness of human. So, mathematics is only the reflection and abstraction of things in the human brain. Certainly, mathematics comes from the human recognition of things for solving various problems in social practice, which needs to carry out formal abstraction, logical deduction, promotion and rise to an abstracting science of understanding the natural reality of things by summarizing the methods and skills of predecessors. Generally, the pattern of mathematical formation is: Observation with experiment→Scientific hypothesis→Mathematical model→Hypothesis and experiment→Induction and deduction →Mathematical theory.

Usually, things are complex, even it overlaps sometimes with other things, more or less. So, *what is the natural reality of a thing?* The natural reality of a thing is its past, present and future state of being in the world, which is independent of the will of human. This definition means that human recognition and holding on the natural reality is partial and gradual, and the natural reality understood by human at a time is not necessarily the natural reality of that thing. So, the recognizing and holding the natural reality of things become a major scientific goal in human developing. Notice that mathematics is a form of deductive science constructed by humans for recognizing things on the principle of logical consistency. *Can a mathematical conclusion really reflect the natural reality of things?* The answer is not necessarily because the mathematics is a product based on the abstraction and deduction of human's observing on things. Among them, there are two things that need to be distinguished in the first. One is that the human's observing on a thing is limited, not necessarily "*seeing is the truth*". For example, the famous *Ames room* shown in Figure 7.15 is a trapezoid rather than a rectangle on the plane but a rectangle viewed at the observing point. At the same time, the position A in the upper left corner is not in the upper left corner of the room but in the upper left corner of the visual rectangle on the observer's view point, namely the human observation of the position of A in the observation point is a dislocation. Secondly, things are not always logically consistency in the eyes of human, namely the logical consistency principle of mathematics can not necessarily characterize all things, which is especially true if a thing is characterized by abstract symbols or restricted in an abstract space. For example, the Schrodinger's cat would either be the "*living*" or "*death*" in human recognition. However, the cat state may be "it living" and "*death*" before one opens the lid of box, which is not conform to the principle of logic consistency and needs to be characterized by a Smarandache multispace $\mathbf{L}\bigoplus\mathbf{D}$.

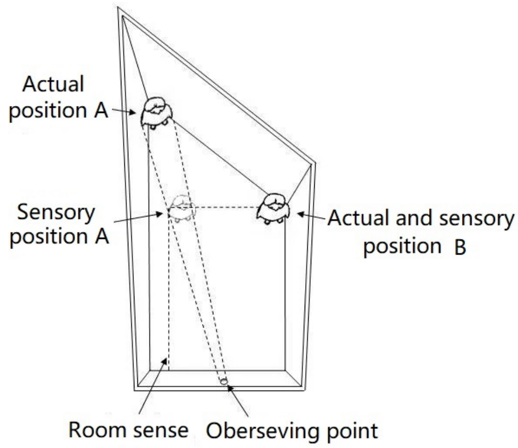



**Figure** 7.15. **Ames room**

Now that mathematics is a formal science with a main work that characterizes everything with formal symbols. Here, a symbol is a unified name of things or their parts that corresponds to the thing or local character of things which. So, *does the thing correspond locally to the same as the formal symbol?* Dr.Ouyang tells Huizi that when a thing is defined by a formal symbol, the formal symbol essentially limits or broadens the connotation of the thing itself, which is the philosophical implication of "*Name named is not the enteral Name*" in *Tao Te Ching*. Furthermore, the operations, axioms or corresponding between symbols form a mathematical system following the principle of logical consistency and the deduced theorems or propositions are the embodiment of overlapping relations in formal symbols. Indeed, the overlapping relationship in a mathematical system is the product of symbols, operations, axioms or corresponding and the product of defined symbols, operations, axioms or describing things of humans, not necessarily the truth of thing.

As we known, the ultimate goal of mathematics is mapping all things into formal symbols and then, any proposition of the relationship between things can be abstractly verified or falsified by formal symbols so as to hold on the behavior of things in the universe. Certainly, if this notion is true, the humans can use formal symbols to hold on all laws of the universe. However, a famous mathematical logician Godel come from the Danube river proved that such a mathematical system does not exist in 1931, namely the formal symbol system is not equivalent to things under the principle of logical consistency, which is known as the Godel's incompleteness theorem, i.e., "*for any compatible formal system S so long as it contains the Peano's axioms of arithmetic, it must be possible to construct a statement that is neither verifiable nor falsifiable in S*" similar to the assertion that the existence of God can not be proved in Figure 7.16. Here, the Peano's axioms of arithmetic are the set of natural numbers $\mathbb{Z}^+$ with five axioms on its structure, i.e., ①1 is a natural number; ②Every natural number $n$ has a definite successor $n'$ which is also a natural number. In other words, the successor of number $n$ is the number $n+1$ that comes after $n$. For example, $1' = 2, 2' = 3, \cdots$, etc.; ③If $m$ and $m'$ are the successors of a natural number $k$, then $m = m'$; ④The number 1 is not the successor of any natural number; ⑤A proposition $P$ is true for all natural numbers if it is shown that $P$ is true for the natural number 1 and that $P$ is true for $n'$ if it is assumed that $P$ is true for the natural number $n$, then the proposition $P$ is true for all natural numbers which is generally called the mathematical induction.

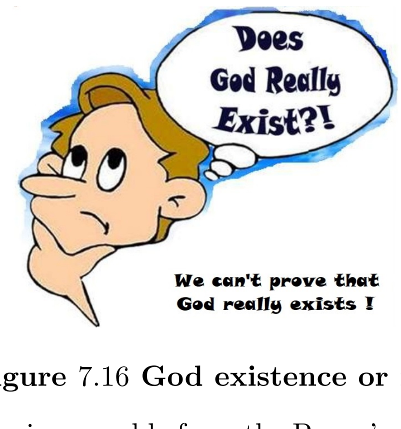



**Figure** 7.16 **God existence or not**

Notice that the proof steps of any statement are inseparable from the Peano's axioms of arithmetic. Therefore, according to the Godel's incompleteness theorem, we can never find such an all-purpose axiom system that can proves all the mathematical truths, namely a logical theory that we can recognize things through formal symbols in human's own thoughts. In fact, mathematics is more limited than the whole science of humans. This can also be confirmed from the mathematical recognition method, namely in mathematical abstraction and deduction of mathematical recognition on everything.

**3.1.Limitated Mathematical Abstracting.** Generally, we need to extract the common and essential attributes or characters, abandon those of other non-essential attributes or characters, establish a mathematical model for simulating the evolution, analyze and then hold on the things, which is a universal way of recognizing things mathematically, called the "*mathematical abstraction*", as shown in Figure 7.17. Certainly, the mathematical abstraction can be briefly summarized into 4 types, i.e., ①Weak abstractions, namely establish such a abstraction from a thing that has reduced their connotation, weakened their structure or expanded their extension, i.e., a mathematical models obtained covers a wider range of the thing than itself; ②Strong abstraction, namely introduce additional characters into the model so that the connotations or structures of things characterized by the model become stronger, i.e., shrink the extension of the thing, connect some seemingly unrelated mathematical concepts and obtain a more abundant mathematical structure than the structure of thing itself; ③Conformational abstraction, namely by the need of logic to establish such a model that can not be directly abstracted by the real thing but the idealized mathematical element can be added to such a mathematical system that make it complete. ④Axiomatic abstraction, namely select a few self-evident propositions as axioms and extrapolate to other conclusions in order to make the mathematical system conforms to the principle of logical consistency. For example, the group theory, ring theory, field theory and ideal theory in algebra, the Euclidean geometry, Riemannian geometry and Lobachevsky geometry are all products of the axiomatic abstractions.

**Figure** 7.17. **Abstraction**

The purpose of mathematical abstraction is to use abstract symbols representing a concrete thing and choose an effective mathematical system or method to establish the mathematical models on thing. Accordingly, the mathematical abstraction requires that the mathematical system should be consistent with the behavior of thing so that the mathematical system can be applied for simulating the behavior of thing. However, this situation will not usually appear in practice because the behavior of thing does not depend on human's will, namely the reason why we need the weak or strong abstraction accompanied by the conceptual or the axiomatic in abstracting and establishing a mathematical model is the approximate simulation of the behavior of thing.

So, *why is a mathematical model only the approximate simulation of behavior of a thing?* Dr.Ouyang tells Huizi that there are the reason of human recognition for a thing and also the reason that mathematics has limitations in characterizing things as a human science. First of all, the human's recognition on the main attributes or characters of a thing, namely the answer to whether the humans can accurately hold on the main attributes or characters of a thing is not necessarily positive and there are no criteria for determining the main attributes or characters of a thing. And so, if we classify the attributes or characters of a thing into the primary and the secondary, *can we surely approve the secondary attributes or characters necessarily not affecting the behavior of thing?* The answer is certainly No! For example, *why does a human sometimes behave irrationally or even violate public order and good customs?* The answer is that the public order and good custom correspond to the general norms of behavior such as social order, public interest, social morality and social fashion. However, there must be special or particular if there is general. In a self-regulating social system such as human or animal, there are no the individual synchronization law, only a majority of individuals can be required to be synchronized in a certain behavior or a period of time to guide the developing of society.

Secondly, the Godel's incompleteness theorem has shown that there is no such a mathematical system that can represent everything conforming to the principle of logical consistency, namely any mathematical system can only characterize the behavior of thing by characterizing its one or several attributes or characters in a period of time. It is a local characterizing of the behavior of thing by using a system of symbols and rules of humans ourselves.

At the same time, there are no such a mathematical method that can describes all behavior of things. Thus, establishing a mathematical model of thing the humans have to assuming the behavior state of the main attributes or characters of thing in order to adapt to the selected mathematical system. And so, the conclusion obtained from the mathematical model is only an approximate characterizing of the behavior of thing under certain constraints. Certainly, if the real conditions exceed the constraints set by humans, the behavior of thing may no longer conform to the mathematical laws that human obtained in the mathematical system assumed priorly. So, *how do the artificial constraints display when we apply the mathematical system to simulate the behavior of a thing?* Dr.Ouyang tells Huizi that this artificial constraint can be easily found in the predator-prey model

$$\begin{cases} \dot{x} &= x(-\mu + cy) \\ \dot{y} &= y(\ \lambda\ - bx) \end{cases} \tag{7.8}$$

of Lotka-Volterra on 2-system (5.26) consisting of the cats and mice as an example for further discussion. Here, the Lotka-Volterra's predator-prey model only characterizes the population size in the 2-system. Certainly, the cats and mice represent respectively the predators and prey in this 2-system and the population size is denoted by $x, y$ which are variables of the model characterizing. For establishing this model, it is assumed that the average growth rate of cats is $-\mu$ if there are no mice and the average growth rate of mice is $\lambda$ if there are no cats. And there are constants $b$ and $c$, namely the appearance of the mice causes the average growth rate of cats increase from $-\mu$ to $-\mu + cx$ which is $c$ times the size of cat population. Certainly, the appearance of cats reduces the average growth rate of mice from $\lambda$ to $\lambda - by$ which is $d$ times the size of mouse population. Solving the state equation (5.26) we get $y^{\mu}x^{\lambda} = C_1 e^{cy+bx}$ of the predator $x$ and prey $y$, i.e., the solution (5.45) with a constant $C_1$.

**Figure** 7.18. **Predator-prey model**

So, *how do we know the limitation of mathematical abstraction in the establishing of Lotka-Volterra's predator-prey model?* Firstly, it is an artificial assumption that the natural death rate $-\mu$ of cats or the natural growth rate $\lambda$ of mice are constants, namely it is assumed that the growth of cats and mice in the natural state follows the Logsitic equation (3.5), i.e., $\dot{x}(t) = -\mu(t)$ and $\dot{y} = \lambda(t)$. Notice $\mu(t)$ and $\lambda(t)$ are functions of $t$ that vary over time and are not necessarily constants in general. In other words, even under the natural conditions, the average growth rates $\dot{x}$ and $\dot{y}$ of the cat and mouse population sizes are not necessarily constants but variables related to the innate quality, food supply and living environment. It is only an artificial assumption that their natural average growth rates are both constants because it is necessary for abstracting and simplifying the problem and establishing the equation (7.8). Secondly, the appearance of mice causes the average rate of cats increasing $cx$ and the appearance of cats causes the average rate of mice decreasing $-by$ are also the artificial assumptions, which are made respectively on the size $x, y$ of cats and mice with constants $c, b$. However, $c, b$ should be also the functions of $t$. In fact, the average amount of food eaten per unit time assumed to be a constant regardless of that how much cats eat over time, the average increase rate of cats and mice are all artificially assumed to be a linear function of the population sizes, regardless of the size of the big or small of mouse's body. Similarly, most mathematical models are established by assuming a few constants or the evolving that follows is an artificial assumption, i.e., all things follow a uniform evolving rule set by humans. This is the limitation of mathematical abstraction. Otherwise, we might not be able to establish or solve the state equations of things.

**3.2.Limitated Mathematical Deducing.** The *mathematical deduction* refers to the process of deducing to a specific statement or mathematical conclusion from a general premise or axioms by deduction. It can be classified in forms as the syllogism, disjunctive reasoning, hypothetical reasoning and the relational reasoning. Among them, ①Syllogism is a common pattern of deduction, Generally, it is composed of "*general premise*" (general principle or conclusion P known), "*minor premise*" (special case or condition S studying) and "*conclusion*" (judgment C of special case by general principle or conclusion). Formally, the conclusion of $S \to C$ is deduced from the propositions $P \Rightarrow C$ and $S \subset P$. For example, by "*P: a human has two eyes*" and "*S: John is a human*" must have the conclusion "*C: John has two eyes*"; ②Hypothetical reasoning is divided to the sufficient and the necessary conditional hypotheses. The principle of sufficient conditional hypothesis is that the minor premise affirms or negates the antecedent of the major premise and the conclusion affirms or negates the latter. For example, if the major premise is that all numbers with the last even digit can be divisible by 2, the minor premise is that the number with the last digit 6. Then, the conclusion is that such numbers must be divisible by 2. The principle of necessary conditional hypothesis is that the minor premise affirms the latter of the major premise and the conclusion needs to affirm the former of the major premise; the minor premise negates the antecedents of the major premise and the conclusion negates the latter. For example, the major premise is that the Chinese cabbage can grow well only if it is watered adequately and the minor premise is that the Chinese cabbage in this field grows well. Then, the conclusion must be the Chinese cabbage in the field is watered adequately; ③Disjunctive reasoning is on disjunctive judgment, divided into the compatible or incompatible cases. Generally, the principle of a compatible hypothesis is that the major premise is a compatible hypothesis, the minor premise negates one or some of the hypotheses and then, the conclusion needs to affirm the remaining choice; the principle of incompatible selection is that the major premise is an incompatible selection, the minor premise affirms one of the choices and then, the conclusion needs to negate the others and if the minor premise negates all but one of the choices and the conclusion needs to affirm the remaining choice; ④Relational reasoning is the form of reasoning in which at least one of the premises is a relational proposition such as symmetric relations, antisymmetric relations or transitive relations. For example, $x = y \Rightarrow y = x$, $x \geq y \Rightarrow y \leq x$ and $x \geq y, y \geq z \Rightarrow x \geq z$, etc.

Notice that each step of reasoning needs to follow the corresponding restrictive conditions in mathematical deduction. Otherwise, it may get a ridiculous deductive conclusion. For example, the error proof for "1 =2" following:

*On the first, let* $a, b > 0$ *be two real numbers and assume* $a = b$. *Multiplying both sides of* $a = b$ *by number* $b$ *we have* $a \times b = b \times b$. *Subtracting* $a \times a$ *from both sides we know that* $a \times b - a \times a = b \times b - a \times a$. *Because of* $a \times b - a \times a = b \times b - a \times a$ *we factorize this equality on both sides and then get an equality* $(b - a)a = (b - a)(b + a)$. *Dividing by* $b - a$ *on its both sides we obtain* $a = b + a$. *Now that* $a = b$ *by assumption, we therefore get* $a = a + a = 2a$, *i.e.,* $a = 2a$. *Applying the assumption that* $a > 0$ *and dividing* $a = 2a$ *by* $a$ *on both sides we finally get the conclusion* $1 = 2$.

Of course, this is a false proof because 1 and 2 are two different numbers in the set of natural numbers $\mathbb{Z}^+$. So, *what is wrong with the reasoning previous?* Dr.Ouyang tells Huizi that it is caused in assuming $a = b$, i.e., $a - b = 0$. As a result, *could we divide either side in equation* $a = (b + a)(b - a)$ *by number* $a - b$? Of course Not because $a - b = 0$, we can not use 0 as a divisor. Huizi asks Dr.Ouyang: "*Dad, why can't* 0 *be a divisor in mathematics?*" Dr.Ouyang explains that $c = a/b$ means $c = a \times 1/b$, where $1/b$ is the multiplicative inverse of number $b$, namely $b \times 1/b = 1$. Thus, $a/0$ means $a \times 1/0$, i.e., there is a number of $1/0$ such that $0 \times 1/0 = 1$. However, by the multiplication rule, the product of any number with 0 must equal to 0. Thus, $0 \times 1/0 = 1$ can not be true if $1/0$ is really a number or $1/0$ does not exist in mathematics. That is why 0 can not be used as a divisor in mathematics. Huizi nods and then suddenly asks Dr.Ouyang: "*Surely,* $1/0$ *has no meaning in mathematics. But we can find the limitation of type* $0/0$ *in higher mathematics. So, does* $0/0$ *make sense in mathematics?*" Dr.Ouyang explains to her, $0/0$ means $a \times 0 = 0$ which is true for any number $a$, namely $0/0$ can be equal to any number in form but also has no mathematical significance because it violates the principle of uniqueness of operation result. As for the limitation $0/0$ in higher mathematics, its numerator and denominator are two infinitesimally small quantities that tend to 0 infinitely but never equal to 0, which does not violate the operation rule that 0 can not be the divisor.

**(1)Limited deducing.** Generally, mathematics pursues the perfect properties of unity, symmetry and simplicity of systems. Among them, the unity reflects the formal commonality, correlation or consistency of mathematics, represented by the unity of mathematical concepts, rules, methods, theories and the unity of mathematics with other sciences; the symmetry is a special symbolic expression of the harmony of things and it is a kind of enjoyment of mathematical beauty, including the systematism, closure and equivalence of mathematics; the simplicity is shown in the mathematical structure, methods and expression of the concise, easy to understand and master. Among these, there is no deductive method such as not knowing how to solve an established differential equation model or not getting the concise deductive conclusions which leads to the limitations of mathematical deduction.

It should be noted that most things in the universe are uneven, asymmetrical, even chaotic or complex and their dynamic behavior can not be characterized by a unified, symmetrical, uniform or simple symbol system in contrast to the pursuit of perfection or characterizing of mathematics. Because of this, Dr.Ouyang explains that the mathematical deductions are mostly "*conditional truth*" rather than the absolute truth. They are symbolic truth under certain conditions and do not necessarily correspond to the real state of things' behavior. The pursuit of the perfection of system tends easily leading to a deviation on the original intention of mathematical abstract description of things, which is often appeared in: ①The more abstract the mathematical concepts (many concepts simply add conditions to already defined or only change symbols or terminologies) and the more branches of mathematics are the more disconnected become that the mathematics from characterizing of properties and states of things, which result in the mathematical deduction to pursue symbolic perfection rather than the quantitative characterization or recognition of things; ②The more concise the mathematical conclusion is in form, the more artificial conditions are set in premise which results in the narrower the behaviors or states characterized and the fewer behaviours of things that can be characterized; ③In the human's system of recognition, everything is a combination of finite elements but mathematics lies in pursuing the infinite or unbounded case, i.e., giving the conclusion of the quantity $n \to \pm\infty$ or time $t \to \pm\infty$ for the concise of mathematical conclusion. However, such a situation can not being in the finite living of any human unless a tendency; ④Introducing the probability or possibility to the behaviour of things which the six sensory organs can not accurately perceive or define such as the behaviour of microscopic particles, is usually characterized by a vague approach in characterizing. The conclusion can only be the "*possibility*", i.e., a vague rather than definitive conclusion. At the same time, it increases the complexity of human recognition of things to a certain extent. For example, the cat's living or death can be known at a glance after opening the box in Schrodinger's cat; if the merchant pierced the *spear* in his right hand into the *shield* in his left hand, he and everyone would know whether his spear was sharp or his shield was strong in the fable of the spear and shield of *Han Fei Zi*. Here, what lack of the recognition in Schrodinger's cat is the "*cause*" between the cat putting in the box and opening the lid and what missing in the fable of spear and shield is also the real situation or "*cause*" that the spear in his right hand pierces the shield in his left hand, namely both of them are concluding in the state of black box for determining the "*effect*", i.e., the living and death on the cat, the sharp of spear or the firm of shield. Mathematically, it is a discontinuous function, namely the limitation of human recognition which is only a vague perceptible on the thing but claim definitely and so, can only get a vague conclusion, i.e., two possibilities of 50% each.

Einstein once complained about the tendency of mathematics to pursue formal perfection rather than the original intension of recognizing things in an address *Geometry and My Experience* on January 27, 1921 at the Prussian Academy of Sciences in Berlin, said that: "*As far as the laws of mathematics refer to reality, they are not certain; and as far as they are certain, they do not refer to reality.*" Meanwhile, with the developing of human society and the recognition of things, more and more problems appeared in practice can not find an applicable mathematics, do not know how to characterize its behavior or characters to hold on the truth of things and the mathematics limitations become increasingly serious day and day. In this situation, Dr.Ouyang tells Huizi that an important thing is to reconsider how mathematics can resolve or accommodate the contradictions in the eyes of human with the principle of logical consistency and other sciences to promote the ability of mathematics for characterizing things, i.e., holding on the laws of things rather than the only pursuit of formal perfection in mathematics.

**(2)Deductive contradiction.** Certainly, the aim of mathematical deduction is to reach the interesting conclusions and conclude mathematical laws with concise formulas, symbols or conclusions. So, *do they necessarily imply that the things or behaviors of things not exist if there are inconsistencies or contradictions in the system?* Dr.Ouyang tells Huizi that we can not conclude the existence or non-existence of things or their behavior in this case because all things are in here, they must exist! What we really need is verifying whether the symbol abstraction of characters, i.e., the connotation and denotation of things and the mathematical model established is appropriate for characterizing, whether the deductive process is logical with the correct theory, theorems and formulae.

For example, let $X_A$, $Y_A$, $X_B$, $Y_B$, $X_C$ and $Y_C$ be respectively the symbols of lucky numbers that the members A, B and C of team answered after drawing the ball from boxes X or Y in the lucky number game of previous section. If the team wins it must be satisfied with

$$X_A X_B X_C = -1, \quad X_A Y_B Y_C = 1, \quad Y_A X_B Y_C = 1, \quad Y_A Y_B X_C = 1,$$

i.e., the equation (7.7). However, solving the equation (7.7) to obtain numbers $X_A, Y_A, X_B$, $Y_B, X_C, Y_C$ for wining the game will not work because the product of formulas in (7.7) results in $(X_A Y_A X_B Y_B X_C Y_C)^2 = 1$, which is contradict in $\{-1, 1\}$. Then, Dr.Ouyang asks Huizi: "*Can you conclude so that the team can not win the game no matter what case?*" Huizi replies: "*Of course Not because the team can always win if the members A, B and C answer the lucky numbers by measuring the spin direction of three separated particles that of entangled initially!*" Dr.Ouyang continuously asks his daughter: "*Do you know where is incorrect in the previous deduction?*" Huizi thinks for a while and answers honestly: "*I don't know because there is no problem at each step but I get the contradictory equation* $(X_A Y_A X_B Y_B X_C Y_C)^2 = -1$ *finally, which is incredible!*" Dr.Ouyang tells her that she should analyze what is the meaning of solving the equations

$$\begin{cases} X_A X_B X_C & = & -1 \\ X_A Y_B Y_C & = & 1 \\ Y_A X_B Y_C & = & 1 \\ Y_A Y_B X_C & = & 1 \end{cases}$$

and what are the implicit assumptions in the first and secondly, shows a reasonable explanation to the contradiction equation $(X_A Y_A X_B Y_B X_C Y_C)^2 = -1$ by considering the difference of the symbol hypothesis with the game rules. Accordingly, the members of team will either draws a ball in box X and answer the lucky number or one will draw a ball in box X but two will draw balls in box Y. Dr.Ouyang explains that in this case, solving the equation (7.7) implies that whether a member of the team draws a ball in boxs X or Y, the lucky number answered by one player should always remain the same, i.e., not change. However, this situation is not consistent with the game rule. Certainly, the winning situation of drawing the ball in boxs X or Y should be distinguished at least, i.e., solving the equation

$$① \ X_A X_B X_C = -1 \text{ and } ② \begin{cases} X_A Y_B Y_C = 1 \\ Y_A X_B Y_C = 1 \\ Y_A Y_B X_C = 1 \end{cases}$$

separately. Even so, *what does it mean for solving equation ②?* Dr.Ouyang explains that this is also the case that solves the equations ② in assuming the same of answering the lucky number each time after drawing a ball in box Y. Otherwise, there are no meaning for solving equation ②. However, the game rule is clear that $X_A, Y_A, X_B, Y_B, X_C$ and $Y_C$ can be selected at will in set $\{-1, 1\}$ without the relation of solving (7.7) or ② and the contradiction $(X_AY_AX_BY_BX_CY_C)^2 = -1$ is caused by the artificial assumption that $X_A, Y_A, X_B, Y_B, X_C$ and $Y_C$ have the same values in every equation, which is inconsistent with the game rule. Hearing this, Huizi says if she understand: "*Dad, so that's it! But how do we solve the equation* (7.7) *consistently with the game rule in this case?*" Dr.Ouyang tells her that we should independently solve each equation of $X_AX_BX_C = -1$, $X_AY_BY_C = 1$, $Y_AX_BY_C = 1$ and $Y_AY_BX_C = 1$ in the range of $\{-1, 1\}$ and then, get the solutions $X_A, Y_A, X_B, Y_B, X_C$ and $Y_C$ that are the same as the game rule.

## §4. Contradictory Reality

Usually, the perceptible limitation of humans leads to the local or partially recognition of things which results in the contradiction because of the coexistence of contradictory or local truths as the implication in the fable of the blind men with an elephant. However, a mathematical system follows the principle of logical consistency and is applied to the characterizing of the behavior of things, which holds on the truth of things still from a local side. Certainly, the contradiction in the deduction does not necessarily indicate the non-existence of things but may be caused by the limitation of the connotation or denotation of formal symbols, or may be caused by the improper selection of a mathematical system with rules. This case is most evident in the lucky number game where the improper deduction results in a contradiction $(X_AY_AX_BY_BX_CY_C)^2 = -1$ caused by the improper simulation of symbols and can be eliminated by assuming that each draw in the game is appeared in different spaces, namely replacing variables $X_A, Y_A, X_B, Y_B, X_C, Y_C$ by $X_A^1, X_B^1, X_C^1, X_A^2, Y_B^2, Y_C^2, Y_A^3, X_B^3, Y_C^3$ and $Y_A^4, Y_B^4.X_C^4$. Dr.Ouyang tells Huizi that if there are contradictions in the deduction of a mathematical system, there must be incorrect deduction in steps because a mathematical system is an autonomous system by the principle of logical consistency. However, when a mathematical system is used to simulate the behavior of a thing, if there are contradictions in the deduction, except for the incorrect deduction in process, most of them are caused by the improper application of a mathematical system, namely the characterizing behavior of mathematical system is not completely consistent with that of the thing but the assumption that the thing follows the rules in the mathematical system. This is especially true when using mathematics to model a self-regulating system, such as the behavior of animal group or livings.

Now, let $A = \{H_1, H_2, H_3, H_4\}$ and $B = \{H_1', H_2', H_3', H_4'\}$ be two groups of four horses, respectively, as shown in Figure 7.19. Notice that each horse has its own thoughts and actions. All horses in group $A$ or $B$ consist of a system which is not necessarily in synchronization but a self-regulating biological system and they are not necessarily coordinated in their actions.

**Figure** 7.19. **Horses galloping**

In this case, the running behavior of horses in groups $A$ or $B$ can not be simply characterized by a system of solvable equations. For example, suppose each horse in the group $A$ and $B$ runs along four straight lines of $(LES^N)$ or $(LES^S)$ on the Euclidian plane $\mathbb{R}^2$, as shown in Figure 7.20.

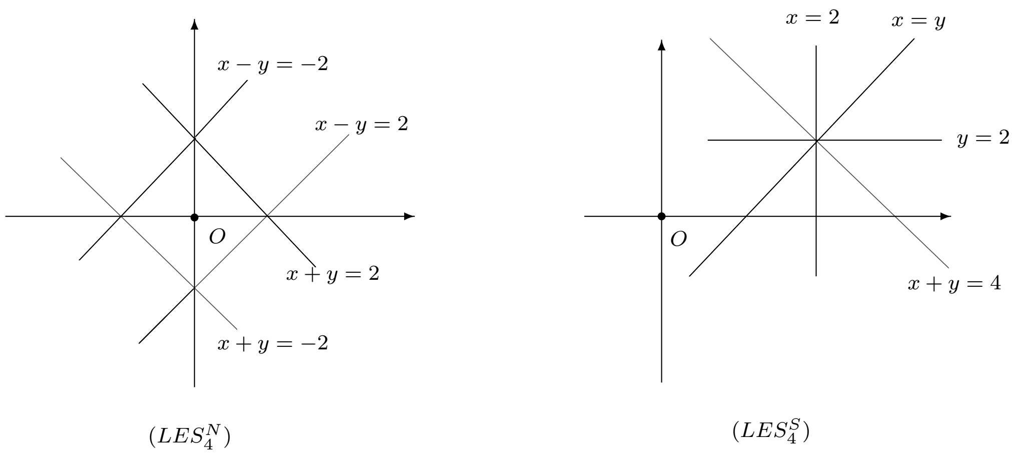



**Figure** 7.20. **Horse running**

Now, by the equations of running line of horses in Figure 7.20, the 4 horses running line in groups $A$ or $B$ respectively form two systems of linear equations, i.e

$$(LES_4^N)\quad \left\{\begin{array}{rcr} x+y &=& 2\\ x+y &=& -2\\ x-y &=& -2\\ x-y &=& 2 \end{array}\right. \qquad\qquad (LES_4^S)\quad \left\{\begin{array}{l} x=y\\ x+y=4\\ x=2\\ y=2 \end{array}\right.$$

Dr.Ouyang asks Huizi: "*How should you characterize the moving behavior of horses* *in groups $A$ or $B$ on the plane?*" Huizi is a little at a loss of answer. Dr.Ouyang explains to her that a natural idea is to solve the two systems of linear equations. Obviously, the first system has no solution and the second one has a solution $(2,2)$. Then, *can we conclude from this solution that horses in $A$ are nothing unless an empty set and horses in $B$ are all stay at point $(2,2)$ without moving?* Of course Not because the horses in group $A$ and $B$ really run in the four straight lines on Euclidean plane $\mathbb{R}^2$. So, *why do we come to such absurd conclusions by solving the equations $(LES_4^N)$ and $(LES_4^S)$ as the blind men made about the shape of elephant?* Dr.Ouyang explains that this is similar to the situation of the lucky number game in which the variables $x,y$ of each equation may be the same or not. In fact, the essence of solving the equations $(LES_4^N)$ and $(LES_4^S)$ is to assume that the variables $x,y$ in each system of equations are the same. However, the variables $x,y$ in different equations correspond to the plane points $(x,y)$ that the horse runs through in a straight line, which should theoretically be completely different. Consequently, the solving of equations $(LES_4^N)$ and $(LES_4^S)$ has no significance for holding on the behavior of horses in groups A or B.

In fact, a running track of horses in groups A or B on the Euclidian plane $\mathbb{R}^2$ can indeed be characterized by the equation solution of running line but definitely not the solution of four running equations of horses in $A$ or $B$. In this case, the tracks $\mathrm{Orb}(A)$ or $\mathrm{Orb}(B)$ of groups A or B should be the union of points on plane, i.e.,

$$\begin{aligned}
\mathrm{Orb}(A) &= \{(x,y):\ x+y=2\}\bigcup\{(x,y):\ x+y=-2\}\\
&\quad \bigcup\{(x,y):\ x-y=2\}\bigcup\{(x,y):\ x-y=-2\},\\
\mathrm{Orb}(B) &= \{(x,y):\ x=2\}\bigcup\{(x,y):\ x+y=4\}\\
&\quad \bigcup\{(x,y):\ x=y\}\bigcup\{(x,y):\ x=2\},
\end{aligned}$$

where the track $\mathrm{Orb}(A)$ and $\mathrm{Orb}(B)$ are exactly the Smarandache multisystems defined in Section 1 of this chapter.

So, *what objective fact are we seeking to solve the system $(LES_4^N)$ or $(LES_4^S)$ of equations?* Now that the essence of solving the equation systems $(LES_4^N)$ or $(LES_4^S)$ is to find out whether the running lines of four horses in groups $A$ or $B$ have intersection, whether two horses need to be prevented from running into each other. Certainly, the solutions enables us to know that the running along the four lines in $(LES_4^N)$ will not but the running along the line in $(LES_4^S)$ may collide at point $(2,2)$. So, it is necessary to adjust the setting of running line or adjust the running speed of four horses in the case of $(LES_4^S)$ to avoid the collision of horses at point $(2,2)$. All of these show that a system of contradiction equations can also be a natural behavior of a thing, i.e., the behavior of group animals. Furthermore, a generalization of characterizing behavior of group animals is the non-harmonious group corresponding to the systems of equations with contradiction in mathematics.

**4.1.Combinatorial Solution of Non-Harmonious Group.** Certainly, the state equation of system assumes that the state of an element $v \in S$ of system $S$ follows the differential operations characterized by its state equation. Generally, if the system equation (5.20) is solvable the solution $\mathbf{x}_v(t)$ of the element $v$ can be completely characterized by the state of element $v$. However, *if the state equation of system $S$ is a contradictory system of equations, can it be inferred that system $S$ does not exist?* The answer is certainly No! Dr.Ouyang explains that because the system $S$ and its elements are an objective existence, i.e., their existence can not be questioned. So, *why is the system of equations obtained by contradictory ones?* The answer is the spatial overlap effect of symbols in characterizing or mathematical abstraction of element behavior in the system $S$. Such a system $S$ is generally called *non-harmonious group*, defined as follows:

**Non-Harmonious Group** *A non-harmonious group $\mathcal{S}$ is such a system $\mathcal{S}$ consisting of elements $P_i$, $1 \leq i \leq p, p \geq 2$ with interactions that $P_i$ is constrained on equation $\mathscr{F}_i(\mathbf{x}, \mathbf{y}) = 0$ at time $t$, where $\mathscr{F}_i(\mathbf{x}^0, \mathbf{y}^0) = 0$ and $\mathscr{F}_i$ holds with the existence condition of the implicit function theorem in a neighborhood $U$ of point $(\mathbf{x}^0, \mathbf{y}^0)$ for integers $1 \leq i \leq p$.*

By this definition, the state equation of a non-harmonious group $S$ in the $n$-dimensional Euclidean space $\mathbb{R}^n$ is

$$S = \begin{cases} \mathscr{F}_{v_1}(\mathbf{x}, \mathbf{y}) = 0 \\ \mathscr{F}_{v_2}(\mathbf{x}, \mathbf{y}) = 0 \\ \cdots\cdots\cdots\cdots \\ \mathscr{F}_{v_p}(\mathbf{x}, \mathbf{y}) = 0 \end{cases} \quad \text{or vector equation} \quad \overline{\mathscr{F}}(\mathbf{x}, \mathbf{y}) = \mathbf{0}. \tag{7.9}$$

Now, assume the functions $\mathscr{F}_{v_1}, \mathscr{F}_{v_2}, \cdots, \mathscr{F}_{v_p}$ in equation (7.9) satisfying all conditions in the implicit function theorem, i.e., exist a solution manifold $S_{\mathscr{F}_i} \subset \mathbb{R}^n$ with $\mathscr{F}_i : \ S_{\mathscr{F}_i} \to 0$ for any integer $1 \leq i \leq p$. Then, the condition that the system (7.9) geometrically has no solution or has a solution is

$$\bigcap_{i=1}^{p} S_{\mathscr{F}_i} = \emptyset \quad \text{or} \quad \bigcap_{i=1}^{p} S_{\mathscr{F}_i} \neq \emptyset. \tag{7.10}$$

In this case, *how do we explain that system* (7.9) *have or have no solution?* Notice that the solution of system (7.9) represents the overlap state of elements $v_1, v_2, \cdots, v_p$ at time $t$, not the state of elements $v_1, v_2, \cdots, v_p$ because the state of element $v_i$ is the solution manifold $S_{\mathscr{F}_i}, 1 \leq i \leq p$ which is similar to the multispace of living or death state of Schrödinger's cat. Accordingly, the non-solvable case of system (7.9) indicates only that there is no overlap state in system elements, not that the they do not exist because the states of elements $v_1, v_2, \cdots, v_p$ are described by the solution manifold $S_{\mathscr{F}_i}$ for integers $1 \leq i \leq p$.

Certainly, this discussion implies that the characterization of system behavior should be the set $\bigcup_{i=1}^{p} S_{\mathscr{F}_i}$ for a non-harmonious group $\mathcal{S}$, not the usual set $\bigcap_{i=1}^{p} S_{\mathscr{F}_i}$ in classical mathematics. In other words, the solution of system (7.9) can be applied to the system recognition of a thing only if each element state is unchanged in its evolving, namely holds with $S_{\mathscr{F}_i} = S_{\mathscr{F}_j}, 1 \leq i, j \leq p$ or

$$\bigcup_{i=1}^{p} S_{\mathscr{F}_i} = \bigcap_{i=1}^{p} S_{\mathscr{F}_i}, \tag{7.11}$$

i.e., the system state should use $\bigcup_{i=1}^{p} S_{\mathscr{F}_i}$ instead of $\bigcap_{i=1}^{p} S_{\mathscr{F}_i}$ for characterizing a non-harmonious group $\mathcal{S}$ in general. It is worth noted that $\bigcup_{i=1}^{p} S_{\mathscr{F}_i} \subset \mathbb{R}^n$ is a union that can characterize the state of system $S$ to some extent but still can not completely reflect the state of system elements, which still leaves some differences in holding on the state of system $S$. So, *how should we describe the state of system $S$ in this case?* Dr.Ouyang tells Huizi that the conception of *combinational solution* on contradictory system (7.9) should be introduced in this time, namely the $G$-solution of system (7.9) defined by solution manifolds of elements as follows:

**Combinatorial Solution of Non-Harmonious Group** *For any integer $p \geq 1$, the combinatorial solution or $G$-solution to the system* (7.9) *of non-harmonious group is a labeled graph $G^L$ whose vertex and edge sets are defined by*

$V\left(G^L\right) = \{S_{\mathscr{F}_i},\ 1 \leq i \leq p\}$;

$E\left(G^L\right) = \left\{\ (S_{\mathscr{F}_i}, S_{\mathscr{F}_j}) \mid if\ S_{\mathscr{F}_i} \bigcap S_{\mathscr{F}_j} \neq \emptyset\ for\ integers\ 1 \leq i, j \leq p\ \right\}$.

*Accordingly, the labels on vertices and edges of $G^L$ are defined by*

$$L:\ S_{\mathscr{F}_i} \to S_{\mathscr{F}_i}, \qquad (S_{\mathscr{F}_i}, S_{\mathscr{F}_j}) \to S_{\mathscr{F}_i} \bigcap S_{\mathscr{F}_j}.$$

Notice that in classical mathematics, if $\bigcap_{i=1}^{p} S_{\mathscr{F}_i} = \emptyset$, the system of (7.9) is regarded as containing contradiction and meaninglessness in practice. However, the contradiction of equations in (7.9) is mostly due to the overlap of element states rather than the non-existence of system or elements, namely it is far more meaningful to study the combinatorial solution of the contradictory system (7.9) than the classical solution, i.e.,

**Existence of Combinational Solutions** *For any integer $p \geq 1$, the combinational solution $G^L$ for the system* (7.9) *of a non-harmonious group always exists.*

In this way, it is necessary no longer to discuss whether the system of (7.9) is solvable or not for characterizing a non-harmonious group because its combinatorial solution must exist, which can properly characterize the behavior of system's elements. For example, denote the points of a horse running on a straight line $ax + by = c$ by a set

$$L_{a,b,c} = \{(x, y)|ax + by = c, ab \neq 0\}$$

in Figure 7.20, the line intersections of $(LES_4^N)$ for horses in groups $A$ by

$$\begin{aligned} u_1 &= L_{1,-1,-1}\bigcap L_{1,1,2}, \quad u_2 = L_{1,1,2}\bigcap L_{1,-1,2}, \\ u_3 &= L_{1,-1,2}\bigcap L_{1,1,-2}, \quad u_4 = L_{1,1,-2}\bigcap L_{1,-1,-2} \end{aligned}$$

and the line intersection of $(LES_4^S)$ for horses in group $B$ by $v$. Then, the combinatorial solutions of systems $(LES_4^N)$, $(LES_4^S)$ for horses in groups $A$ and $B$ are respectively shown in Figure 7.21, i.e., a labeled graph $C_4^L$ and a labelled complete graph $K_4^L$ of order 4.

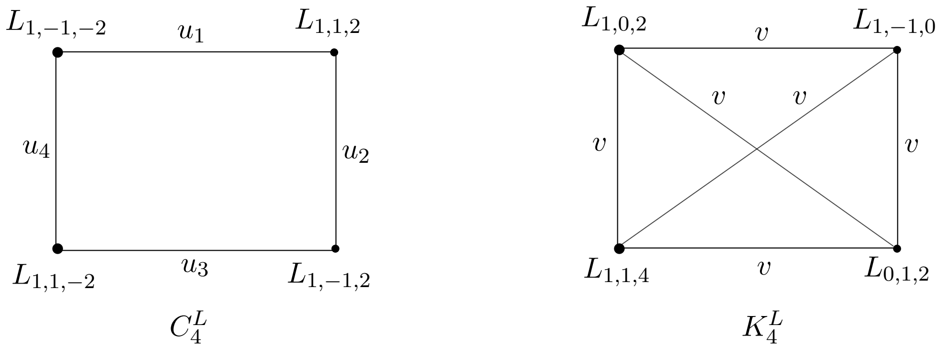



**Figure** 7.21. **Combinatorial solutions of** $(LES_4^N)$ **and** $(LES_4^S)$

**4.2.Combinatorial Characters of Non-Harmonious Group.** Certainly, the system (7.9) is an algebraic system of equations. Differentiating on its both sides we get a system of differential equations following

$$\begin{cases} \dot{\mathscr{F}}_{v_1}(\mathbf{x}, \mathbf{y}, \dot{\mathbf{y}}) = 0 \\ \dot{\mathscr{F}}_{v_2}(\mathbf{x}, \mathbf{y}, \dot{\mathbf{y}}) = 0 \\ \cdots\cdots\cdots\cdots\cdots \\ \dot{\mathscr{F}}_{v_p}(\mathbf{x}, \mathbf{y}, \dot{\mathbf{y}}) = 0 \end{cases} \tag{7.12}$$

and abbreviate it as $(DES_p)$. Notice that the combinatorial solution of the algebraic system (7.9) is always existing. Whence, the combinatorial solution of differential system (7.12) must exist also. In this way, the characters of non-harmonious groups (7.9) or (7.12) can be characterized by applying the labeled graph $G^L$ of combinatorial solution and then obtain the characters of system.

**(1)Algebraic equations.** Geometrically, for any integers $1 \leq i \neq j \leq p$ if the intersection of solution sets of the $i$th equation and the $j$th equation in algebraic system (7.9) is an empty set, i.e., $S_{\mathscr{F}_i} \bigcap S_{\mathscr{F}_j} = \emptyset$, they are said to be *parallel*. By definition, there must be $\left(S_{\mathscr{F}_i}, S_{\mathscr{F}_j}\right) \notin \left(E(G^L)\right.$ or $G \not\simeq K_p$ in this case; Conversely, if the algebraic system (7.9) has a solution, then there must be $G \simeq K_p$. Furthermore, the equations in the algebraic system (7.9) can be classified by the property of *parallel*, namely the equations included in the same class are parallel to each other. Assuming that the classification of equations in system (7.9) is $\mathscr{C}_1, \mathscr{C}_2, \cdots, \mathscr{C}_s$, $1 \leq s \leq p$, then by the definition of combinatorial solution $G^L$ of a non-harmonious groups (7.9), $G \simeq K_{n_1, n_2, \cdots, n_s}$, where $n_i = |\mathscr{C}_i|, 1 \leq i \leq s$ with $n_1 + n + 2 + \cdots + n_s = p$.

Now that the parallel algebraic equations can be further visualized by the linear equation. For example, let $a_{11}x_1 + a_{12}x_2 + \cdots a_{1n}x_n = b_1$ and $a_{21}x_1 + a_{22}x_2 + \cdots a_{2n}x_n = b_2$ be two linear algebraic equation. Then, the condition that they do not intersect or they are parallel is that the coefficients of corresponding variables are proportional, i.e., holding with the condition

$$c = a_{j1}/a_{i1} = a_{j2}/a_{i2} = \cdots = a_{jn}/a_{in} \neq b_j/b_i. \tag{7.13}$$

So, these two linear algebraic equations must be contradict each other. In this way, if there are no solution to a contradictory linear system (7.9), there must be two linear algebraic equations, without loss of generality assuming that the variable coefficients of $i$th and $j$th equations satisfy the condition (7.13). Then, *can we further characterize an non-solvable system of linear algebraic equations* (7.9) *with combinatorial notion?* The answer is Yes! Dr.Ouyang tells Huizi that let $n = 2$ as an example. Then, the combinatorial characters of non-solvable linear equations could be further characterized. At this time, the system (7.9) of equations is

$$AX = (b_1, b_2, \cdots, b_p)^T, \tag{7.14}$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ \cdots & \cdots \\ a_{p1} & a_{p2} \end{bmatrix}, \qquad X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

and $p \geq 2$. In this case, the point set of each equation in (7.14) is a straight line on Euclidean plane $\mathbb{R}^2$ and the combinatorial solution $G^L$ is a planar graph $K_{n_1, n_2, \cdots, n_s}$, namely we can characterize such a system by a planar graph $G^L_{leq2}$ of intersection points in 2 variables without solving the system (7.14) and know that *the system* (7.14) *is non-solvable if and only if its combinatorial solution is a planar graph $H$ of order$\geq 2$ with a* *straight line segment for each of its edges*, where the intersection graph $G^L_{leq2}$ as a point subset of Euclidean plane $\mathbb{R}^2$ is respectively defined by

$$V\left(G_{leq2}\right)=\left\{\left(x_1^{ij},x_2^{ij}\right)\Big|\, a_{i1}x_1^{ij}+a_{i2}x_2^{ij}=b_i,\ a_{j1}x_1^{ij}+a_{j2}x_2^{ij}=b_j, 1\leq i,j\leq p\right\};$$

$$E\left(G_{leq2}\right)=\left\{(v_{ij},v_{il})\right|\text{ Points on straight segment from }\left(x_1^{ij},x_2^{ij}\right)\text{ to }\left(x_1^{il},x_2^{il}\right)\}$$

Among them, the vertex $v_{ij}=(x_1^{ij},x_2^{ij})$ is for any integer $1\leq i,j\leq p$. For example, there is a non-harmonious group consisting of 5 elements $\{v_1,v_2,v_3,v_4,v_5\}$ with an planar graph $G_{\substack{L\\leq2}}$ shown in Figure 7.22.

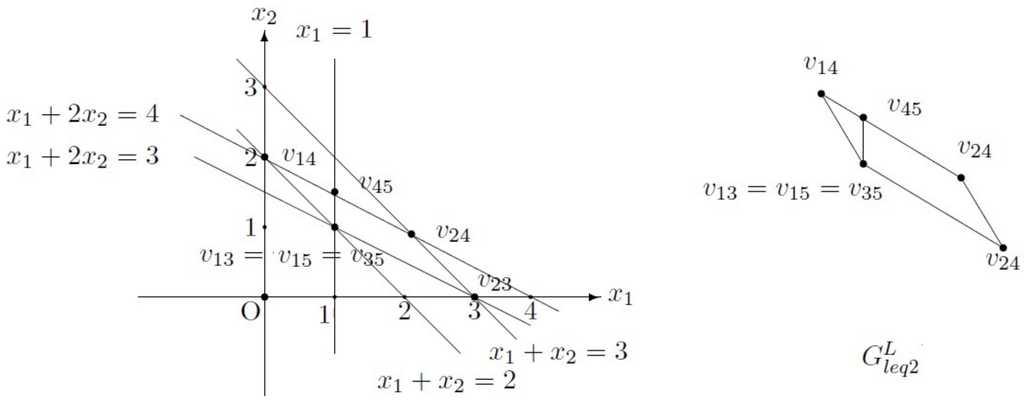



**Figure** 7.22. **A planar graph of non-solvable equations**

In this case, each element $v_i, 1\leq i\leq 5$ follows a linear evolving rule, i.e

$$v_1:\ x_1+x_2=2,\qquad v_2:\ x_1+x_2=3,$$
$$v_3:\ x_1+2x_2=3,\quad v_4:\ x_1+2x_2=4,\quad v_5:\ x_1=1.$$

**(2)Differential equations.** For any integers $p,\ n\geq 1$, let

$$\dot{\mathbf{x}}=\mathscr{F}_1(\mathbf{x}),\ \dot{\mathbf{x}}=\mathscr{F}_2(\mathbf{x}),\cdots,\dot{\mathbf{x}}=\mathscr{F}_p(\mathbf{x}) \qquad (DES^1_p)$$

be a system of first-order ordinary differential equations and let $\mathscr{F}_i:\mathbb{R}^n\to\mathbb{R}^n$ be continuous with $\mathscr{F}_i(\overline{0})=\overline{0}$ for integers $1\leq i\leq p$. Particularly, let

$$\dot{\mathbf{x}}=A_1\mathbf{x},\ \dot{\mathbf{x}}=A_2\mathbf{x},\ \cdots,\ \dot{\mathbf{x}}=A_p\mathbf{x} \qquad (LDES^1_p)$$

be a system of first-order linear differential equations and furthermore, let

$$\begin{cases} x^{(n)}+a^{[0]}_{11}x^{(n-1)}+\cdots+a^{[0]}_{1n}x=0\\ x^{(n)}+a^{[0]}_{21}x^{(n-1)}+\cdots+a^{[0]}_{2n}x=0\\ \cdots\cdots\cdots\cdots\\ x^{(n)}+a^{[0]}_{p1}x^{(n-1)}+\cdots+a^{[0]}_{pn}x=0\end{cases} \qquad (LDE^n_p)$$

be a system of differential equations of order $n$ with constant coefficients, where

$$A_k = \begin{bmatrix} a_{11}^{[k]} & a_{12}^{[k]} & \cdots & a_{1n}^{[k]} \\ a_{21}^{[k]} & a_{22}^{[k]} & \cdots & a_{2n}^{[k]} \\ \cdots & \cdots & \cdots & \cdots \\ a_{p1}^{[k]} & a_{p2}^{[k]} & \cdots & a_{pn}^{[k]} \end{bmatrix}, \qquad \mathbf{x} = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \cdots \\ x_p(t) \end{bmatrix}$$

and for any integers $0 \leq k \leq p,\ 1 \leq i, j \leq n$, $a_{ij}^{[k]}$ is real number.

According to the spanning bases (5.49) and (5.50) of solution space in Chapter 5, each differential equation in the systems $(DES_p^1), (LDES_p^1)$ and $(LDE_p^n)$ can be solved. Certainly, they are all special cases of the non-harmonious group (7.12) and the combinatorial solutions $G^L\left(DES_p^1\right)$, $G^L\left(LDES_p^1,\right)$ and $G^L\left(LDE_p^n\right)$ of respective equations $(DES_p^1)$, $(LDES_p^1)$ and $(LDE_p^n)$ can be discussed in general. Now, denote the solution space of $i$th equation in systems $(DES_p^1)$, $(LDES_p^1)$ and $(LDE_p^n)$ by $S_i$. If the intersection of solution space of the $i$th equation with that of $j$th equation in $(DES_p^1)$, $(LDES_p^1)$ or $(LDE_p^n)$ is $S_i \bigcap S_j = \emptyset$, then the $i$th equation is said to be *parallel* to the $j$ equation. Similarly, the differential equations are then classified according to whether they are parallel and the combinatorial solutions $G^L$ of equations $(DES_p^1)$, $(LDES_p^1)$ and $(LDE_p^n)$ are determined by $G \simeq K_{n_1,n_2,\cdots,n_s}$ similar to the case of linear algebraic equations, where $n_i = |\mathscr{C}_i|$ and $n_1 + n + 2 + \cdots + n_s = p$. It should be noted that the labeling $L : v \to L(v)$ and $L : (u, v) \to L(u, v)$ are not only point sets but also linear spaces, spanned by linear bases, respectively. Whence, all labels on the vertices and edges of $G^L$ can be simplified to the corresponding bases of linear solution spaces. Such a kind of labeled graph is called a *basis graph*.

For example, let $p = 6$. Then, the 6 bases of solution spaces of equation $(1) - (6)$ in system of equations

$$\begin{cases} \ddot{x} - 3\dot{x} + 2x = 0 & (1) \\ \ddot{x} - 5\dot{x} + 6x = 0 & (2) \\ \ddot{x} - 7\dot{x} + 12x = 0 & (3) \\ \ddot{x} - 9\dot{x} + 20x = 0 & (4) \\ \ddot{x} - 11\dot{x} + 30x = 0 & (5) \\ \ddot{x} - 7\dot{x} + 6x = 0 & (6) \end{cases}$$

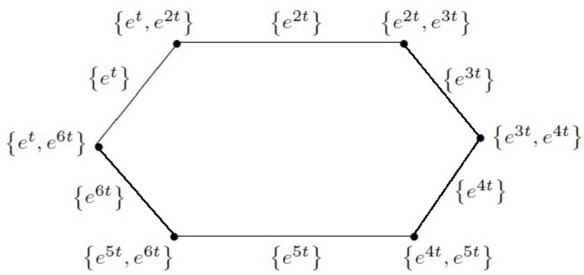



**Figure** 7.23. **Bisis graph**

are respectively

$$\{e^t, e^{2t}\}, \{e^{2t}, e^{3t}\}, \{e^{3t}, e^{4t}\}, \{e^{4t}, e^{5t}\}, \{e^{5t}, e^{6t}\}, \{e^{6t}, e^t\}$$

with the combinatorial solution $G^L$ or $G$-solution shown in Figure 7.23.

Generally, let $G_1^{L_1}, G_2^{L_2}$ be two combinatorial solutions of respective differential equations $(LDES_m^1)$ and $(LDES_m^1)'$ (or differential equations $(LDE_m^n)$ and $(LDE_m^n)'$). If there is a graph isomorphism $\varphi : G_1^L \to G_2^L$, namely a $1-1$ mapping $\varphi : G_1^{L_1} \to G_2^{L_2}$ such that $\varphi L_1(x) = L_2\varphi(x)$ for $\forall x \in V(G_1)\bigcup E(G_1)$, the two systems of differential equations are called the *combinatorial equivalent*, denoted by $(LDES_m^1) \overset{\varphi}{\simeq} (LDES_m^1)'$ (or $(LDE_m^n) \overset{\varphi}{\simeq} (LDE_m^n)'$). Notice that the combinatorial equivalence in differential equations characterizes the same non-harmonious group, i.e., there are no essentially difference but only in mathematical symbols.

So, *how do we determine that two systems of differential equations are combinatorially equivalent?* Dr.Ouyang tells Huizi that it should be noted that two linear spaces are isomorphic if and only if they have the same dimension. Thus, a natural idea is to simplify the linear space labeled on vertices and edges of a combinatorial solution $G^L$ to the corresponding dimension of linear space, introduce the conception of *positive integer label graph* $G^I$, $I : V(G)\bigcup E(G) \to \mathbb{Z}^+$ and then, determine the combinational equivalence of two systems of differential equations by the isomorphism of integer labeled graphs. For example, there are 3 integer labeled graphs $G_1^{I_\theta}, G_2^{I_\tau}$ and $G_3^{I_\sigma}$ on graph $K_4 - e$ in Figure 7.24. Although they are all integer labeled graphs on $K_4 - e$, we know that $G_1^{I_\theta} \simeq G_2^{I_\tau}$ but $G_1^{I_\theta} \not\simeq G_3^{I_\sigma}$ by definition.

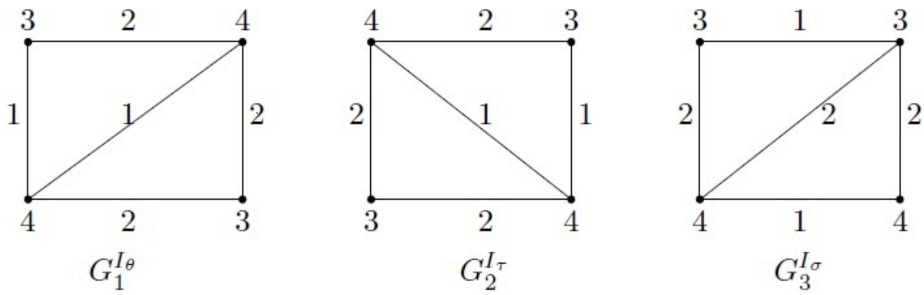



**Figure** 7.24. **Integer labeled graphs**

Now that there is a criterion for determining the combinatorial equivalence of differential equations, namely *the systems of* $(LDES_m^1)$ *and* $(LDES_m^1)'$ *(or* $(LDE_m^n)$ *and* $(LDE_m^n)'$*) are combinatorial equivalent if and only if their corresponding integer label graphs are isomorphic, i.e.,*

$$G^{I_1}\left(LDES_m^1\right) \simeq G^{I_2}\left(LDES_m^1\right)' \quad \text{or} \quad G^{I_1}\left(LDE_m^n\right) \simeq G^{I_2}\left(LDE_m^n\right)'.$$

**4.3.Geometry on Non-Harmonious Group.** For any integer $1 \le i \le p$, the solution space $S_{\mathscr{F}_i}$ of $i$th equation in system (7.9) is usually called a *solution manifold* because $S_{\mathscr{F}_i}$ is really an *n-dimensional manifold* in geometry, namely there is a neighborhood $U(v)$ of point $v$ in the solution space $S_{\mathscr{F}_i}$ and a homeomorphic mapping $\varphi : U(v) \leftrightarrow \mathbb{R}^n$, where $\varphi$ is a homeomorphic mapping means that $\varphi$ and its inverse $\varphi^{-1}$ are both continuous. For example, the sphere and ring in Euclidean space $\mathbb{R}^3$ are 3-dimensional manifolds. Meanwhile, the surface of ball and ring are 2-dimensional manifold, etc., as shown in Figure 7.25. Similarly, for any integer $1 \le i \le p$, the solution space $S_{\mathscr{F}_i}$ of every equation in the system (7.9) is a subset of the Euclidean space $\mathbb{R}^n$, i.e., an $n$-dimensional manifold.

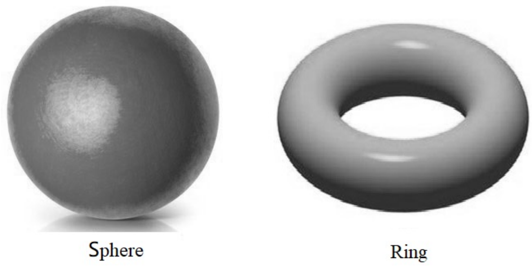



**Figure** 7.25 **Sphere and torus**

So, *what geometry does the combinatorial solution $G^L$ of a non-harmonious group* (7.9) *correspond to?* Dr.Ouyang tells Huizi that it is interesting that the combinatorial solution $G^L$ just corresponds to the combinatorial solution manifold of each equation in system (7.9), called *combinatorial manifold*, namely if all vertices of graph $G$ are $v_1, v_2, \cdots, v_p$, the vertex $v_i$ is nothing else but the solution manifold of $i$th equation and the edge $(v_i, v_j)$ is the intersection of solution manifold $S_{\mathscr{F}_i}$ of $i$th equation and solution manifold $S_{\mathscr{F}_j}$ of $j$th equation in non-harmonious group (7.9). Here, the integers $1 \le i, j \le p$.

For a given function $\varphi : \mathbb{R}^m \to \mathbb{R}^n$, the graph of function $\mathbf{y} = \varphi(\mathbf{x})$ is defined by

$$\Gamma[\mathbf{x}, \mathbf{y}] = \{(\mathbf{x}, \mathbf{y}) \in \mathbb{R}^m \times \mathbb{R}^n \mid \mathbf{y} = \varphi(\mathbf{x})\ \}. \tag{7.15}$$

For example, the graphs of functions $y = x \sin \frac{1}{x}$ when $m = 1, n = 1$ and $z = x^2 - y^2$ or $z = 2x \ln \frac{y}{x} - x(y, x) + 2x(1 - y)$ when $m = 1, n = 2$ are respectively shown in Figure 7.26($a$), ($b$) and ($c$), where ($a$) is a curve and ($b$), ($c$) are both surfaces.

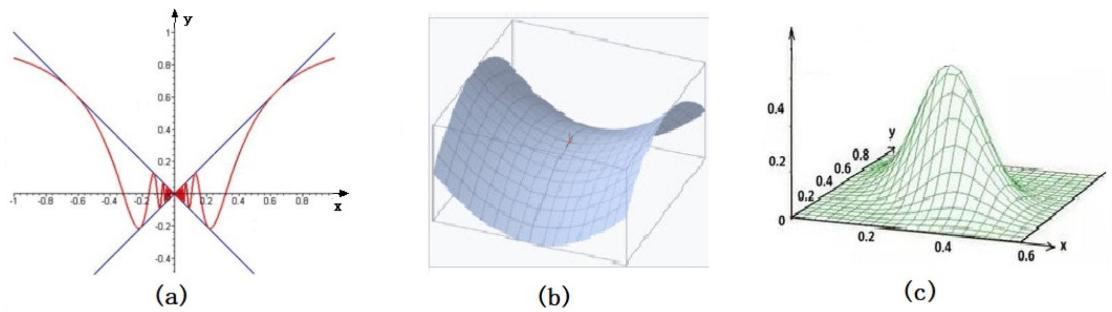



**Figure** 7.26. **Examples of function graphs**

Generally, for an integer sequence $0 < n_1 < n_2 \cdots < n_p,\ p \ge 1$, a *combinatorial manifold* $\widetilde{M}$ is such a Hasudorff space that there exist neighborhoods $U_v, U_u$ for any two different points $v, u \in \widetilde{M}$ and a homeomorphic mapping, $\varphi_v : U_v \to \widetilde{\mathbb{R}}(n_1(p), \cdots, n_{s(v)}(v))$

holding with $U_v \bigcap U_u = \emptyset$ and

$$\{n_1(v), \cdots, n_{s(v)}(v)\} \subseteq \{n_1, \cdots, n_p\}, \bigcup_{v \in \widetilde{M}} \{n_1(v), \cdots, n_{s(v)}(v)\} = \{n_1, \cdots, n_p\}_{\circ}$$

Particularly, a combinatorial manifold $\widetilde{M}$ is called a *finitely combinatorial manifold* if it consists of only a finite number of manifolds, i.e. $p < +\infty$ and neither of which is a submanifold of the other manifolds or their union. An example of two finite combined manifolds is shown in Figure 7.27, where $B^1$, $T^2$ and $M^3$ denote a 1-dimensional ring, a 2-dimensional torus and a 3-dimensional manifolds, respectively.

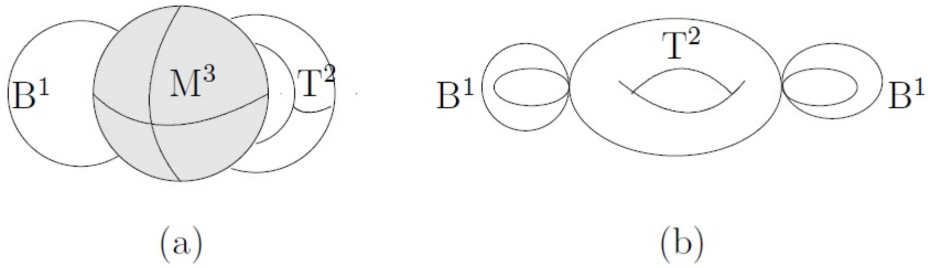



**Figure** 7.27. **Examples of finitely combinatorial manifolds**

Notice that there is a natural topological or graph structure inherited in a finite combinatorial manifold $\widetilde{M}$. Let $\widetilde{M} = \bigcup_{i=1}^{m} M_i$, $p < +\infty$ be a finitely combinatorial manifold. Its natural inherited combinatorial structure $G^L[\widetilde{M}]$ is defined by

$$\begin{aligned} V\left(G^L[\widetilde{M}]\right) &= \{M_1, M_2, \cdots, M_m\}, \\ E\left(G^L[\widetilde{M}]\right) &= \left\{ (M_i, M_j) \mid M_i \bigcap M_j \neq \emptyset, 1 \leq i, j \leq n \right\}, \end{aligned}$$

and labels the vertices in $G^L[\widetilde{M}]$ by manifolds, labels each edge by the intersection of two manifolds at its end-vertex, namely the labeling mapping is defined by

$$L: \ M_i \to M_i, \quad L: \ (M_i, M_j) \to M_i \bigcap M_j$$

for any integers $1 \leq i, j \leq m$. Then, the labeled graph $G^L[\widetilde{M}]$ truly reflects the characters of combinational manifold $\widetilde{M}$. Thus, the geometry corresponding to the combinatorial solution $G^L$ of non-harmonious group (7.9) is exactly a combinatorial manifold $\widetilde{S}_{\mathscr{F}}$, where $\widetilde{S}_{\mathscr{F}} = \bigcup_{i=1}^{p} S_{\mathscr{F}_i}$. Similarly, Dr.Ouyang tells Huizi that the finitely combinational manifold $\widetilde{M}$ can be combinatorially classified by integer labeled graphs $G^I$, regardless of whether the system is an algebraic system (7.9) or a differential system (7.12) because they are both corresponding to the combinatorial equivalence of non-harmonious group (7.9).

## §5. **Group-System Stability**

The system stability is solving the measuring error problem of element state $\mathbf{x}_v, v \in S$ of a system $S$, namely its subsequent change of the system near the initial value. Certainly, for a stable system as long as the measuring error is within a certain extent, it does not affect the subsequent changes of system. Dr.Ouyang tells Huizi that a systemic recognition needs us to establish a stable system for holding on a thing, which has a priori assumption, namely the state variable of element must be the same connotation and denotation in all equations, i.e., synchronization with time $t$ whether the system equations are algebraic or differential. And then, the solution to the system of equations corresponds to the element states. Of course, this assumption is no longer true for non-harmonious groups because there are overlapping states in such systems, i.e., the states of several elements may overlap in mathematical abstraction, namely the connotation and denotation of the state variable $\mathbf{x}_v$ are not necessarily consistent or synchronized in all equations. In this case, although the non-harmonious groups is still composed of elements but the system state is similar to that each element constitutes its own systems such as the food chain composed of the owl, snake, bird, mouse, rabbit, herbivorous insects and the green plants shown in Figure 7.28. Its stability can not be characterized by the method of classical mathematics but should be determined by the the whole food chain or the group.

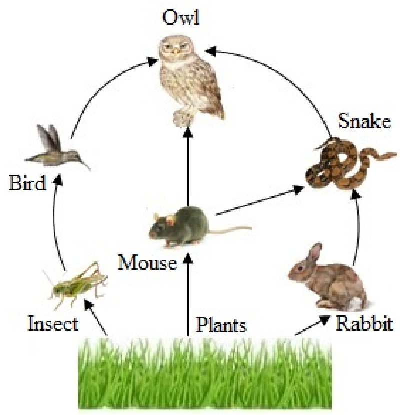


**Figure** 7.28. **A food chain**

So, *how do we determine the stability of a non-harmonious group?* Dr.Ouyang asks this question to Huizi, Huizi thinks for a while and then answers:"*For a non-harmonious group the system stability can be achieved by the stability of each of its element.*" Dr.Ouyang confirms her answer by saying: "*Right! But if one element in the system is unstable, can we conclude that the whole system is unstable?*" Huizi replies: "*It should be so because one unstable element will cause the whole system to be unstable!*" Dr.Ouyang continuously asks her: "*Why does one unstable element only will cause the instability of whole system?*

Seeing Huizi can not answer this question, Dr.Ouyang tells her that the stability of a non-harmonious group could not be determined simply by the stability of a single element. For example, assume the elements $v$ and $u$ constitute a 2-system with state variables $x_v(t), x'_v(t)$ and $x_u(t), x'_u(t)$ of elements $v, u$ as follows:

$$x_v(t) = e^{2t} + 3e^t, \qquad x'_v(t) = e^{2t} + 2e^t,$$
$$x_u(t) = e^{2t} - 3e^t + \frac{t}{e^t}, \quad x'_u(t) = e^{2t} - 2e^t$$

on initial values $t = t_0$ and $t = t'_0$, respectively. Then, if $t \to \infty$, there is $|x_v(t) - x'_v(t)| = e^t \to \infty$ and $|x_u(t) - x'_u(t)| = -e^t + \frac{t}{e^t} \to \infty$, i.e., tend to infinity, the elements $v$ and $u$ are both unstable. However, $|(x_v(t) + x_u(t)) - (x'_u(t) + x'_u(t))| = t/e^t \to 0$, i.e., the element $u + v$ or the 2-element system presents a stable output state in observing. This example shows that although one element is unstable but the output of non-harmonious group may still be stable. So, the stability of element can not be simply applied.

Now that the stability of elements can not be simply applied. *How can we determine the stability of non-harmonious groups?* Dr.Ouyang explains that the stability of state variable $\mathbf{x}_v$ on element $v$ is an approach to measure two states $\mathbf{x}_v, \mathbf{x}'_v$ is consistent or not if their initial values $t_0, t'_0$ are close enough, namely the distance $|\mathbf{x}_v - \mathbf{x}'_v| \to 0$ between them. Surely, this idea in classical mathematics is assuming that the element state $\mathbf{x}_v$ is known, namely the state equation is solvable. However, if the state equation is unsolvable or contradictory, it is usually considered meaningless in classical mathematics and will not be discussed further, which results in the lack of methods for determining the stability of non-harmonious group or contradictory system (7.9), (7.12) and the limitation of systematic recognition of things to a certain extent. For out of this dilemma, it is necessary to extend the solution of non-harmonious group to be a combinatorial solution in first. This problem has been solved in the previous section, namely the combinatorial solution of non-harmonious group is introduced and there is always a combinatorial solution $G^L$ for a non-harmonious group (7.9) or (7.12). And the second is extending the distance between two state vectors $\mathbf{x}_v, \mathbf{x}'_v$ to the combinatorial solutions $G^{L_1}$, $G^{L_2}$, namely define a measure $\omega : G^L \to \mathbb{R}^+$ on $G^L$. Then, we can introduce the $\omega$-*stability* of non-harmonious group similar to the element stability in general, i.e., for any given integer $\varepsilon > 0$ there is a number $\delta(\varepsilon)$ such that if the initial value of $\omega\left(G^L(t_0) - G^{L'}(t_0)\right) < \delta(\varepsilon)$, there must be

$$\omega\left(G^L(t) - G^{L'}(t)\right) < \varepsilon \quad \text{or} \quad \lim_{t\to\infty} \omega\left(G^L(t) - G^{L'}(t)\right) = 0,$$

called respectively the $\omega$-*stability* and $\omega$-*asymptotic stability* of the combinatorial solution $G^L$ of a non-harmonious group, where

$$G^L(t) - G^{L'}(t) = G^{L-L'}(t). \tag{7.16}$$

For further embodiment of the $\omega$-metric, two positive definite functions on the combinatorial solution of non-harmonious group are simply introduced on the basis of element metric, i.e., the sum or product stability of a non-harmonious group (7.9) or (7.12).

**5.1.Sum-Stability.** On the first, consider the case of non-harmonious group characterized by linear differential equations. In this case, assume the labeled graphs $G^L[LDES_p^1]$ and $G^L[LDE_p^n]$ are respectively the combinatorial solutions of homogeneous linear differential equations $(LDES_p^1)$ in initial values $\mathbf{x}_v(0)$, $v \in S$ or differential equations $(LDE_p^n)$ in initial values $\mathbf{x}_v(0), \mathbf{x}_v^{(1)}(0), \cdots, \mathbf{x}_v^{(n-1)}(0)$. Then, the system $(LDES_p^1)$ or $(LDE_p^n)$ is said to be *sum-stable* or *asymptotically sum-stable* if there is such real numbers $\delta_v, v \in S$ for a given real number $\varepsilon > 0$ that

$$\left|\sum_{v\in V(G)} \mathbf{x}_v(t) - \sum_{v\in V(G)} \mathbf{x}'_v(t)\right| < \varepsilon \quad \text{or} \quad \lim_{t\to 0}\left|\sum_{v\in V(G)} \mathbf{x}_v(t) - \sum_{v\in V(G)} \mathbf{x}'_v(t)\right| = 0, \tag{7.17}$$

if $|\mathbf{x}_v(0) - \mathbf{x}'_v(0)| < \delta_v$ for $v \in S$, where $\mathbf{x}_v(t), \mathbf{x}'_v(t)$ denote respectively the state function of element $v, v \in S$ of systems $(LDES_p^1)$ or $(LDE_m^n)$ with initial values $\mathbf{x}_v(0)$ or $\mathbf{x}_v(0), \mathbf{x}_v^{(1)}(0), \cdots, \mathbf{x}_v^{(n-1)}(0)$.

Notice that

$$\left|\sum_{v\in V(G)} h_v(t)\mathbf{x}_v(t) - \sum_{v\in V(G)} h_v(t)\mathbf{x}'_v(t)\right| \leq \sum_{v\in V(G)} |h_v(t)|\left|\mathbf{x}_v(t) - \mathbf{x}'_v(t)\right|,$$

$$\lim_{t\to 0}\left|\sum_{v\in V(G)} h_v(t)\mathbf{x}_v(t) - \sum_{v\in V(G)} h_v(t)\mathbf{x}'_v(t)\right| \leq \sum_{v\in V(G)} |h_v(t)| \lim_{t\to 0}\left|\mathbf{x}_v(t) - \mathbf{x}'_v(t)\right|$$

by the triangle inequality, where $h_v(t)$ is a polynomial. According to these two inequalities, *if each equation of a homogeneous system of linear differential equations* $(LDES_p^1)$ *in initial* $\mathbf{x}_v(0), v \in S$ *or* $(LDE_p^n)$ *in initial* $\mathbf{x}_v(0), \mathbf{x}_v^{(1)}(0), \cdots, \mathbf{x}_v^{(n-1)}(0)$ *is stable or asymptotic stability at the point* 0*, then the combinatorial solution* $G^L[LDES_p^1]$ *or* $G^L[LDE_p^n]$ *is sum-stable or asymptotic sum-stability at the point* 0.

Usually, for the linear differential equations $(LDES_1^1)$ and $(LDE_1^n)$, denote its maximum characteristic root $\gamma = \max\{\mathrm{Re}\lambda||A - \lambda I_{n\times n}| = 0\}$. By the linear spanning basis (5.50) and (5.49) of solution space, we know that the systems of equations $(LDES_1^1)$ and $(LDE_1^n)$ are not stable at the point 0 if $\mathrm{Re}\gamma > 0$ and they are asymptotically stable at the point 0 if $\mathrm{Re}\gamma < 0$. But in the case of $\mathrm{Re}\gamma = 0$, the system of equations $(LDES_1^1)$ or $(LDE_1^n)$ is stable at the point 0 if and only if $m'(\lambda) = m(\lambda)$, where $m(\lambda)$ is the multiplicity of characteristic root $\lambda$ and $m'(\lambda)$ denotes the dimension of eigenvector space corresponding to $\lambda$. Applying this conclusion to non-harmonious groups (7.12), we get that *the combinatorial solution of a linear system of differential equations* $(LDES_p^1)$ *(or* $(LDE_p^n)$*) is asymptotic sum-stable at point* 0 *if and only if the solution basis* $\overline{\beta}_v(t)e^t\alpha_v$ *in* *the solution space $\mathscr{B}_v$ of system $(LDES_1^1)$ holds with* $\mathrm{Re}\alpha_v < 0$ *(or the basis $t^{l_v}e^{\lambda_v t}$ in the solution space $\mathscr{C}_v$ of system $(LDE_m^1)$ holds with* $\mathrm{Re}\lambda_v < 0$*) for vertices $v \in V(G)$.*

So, *why is there such a conclusion?* Dr.Ouyang explains that its sufficiency of the condition is obvious by definition. For its necessity of the condition, assume that there exists such a vertex $v \in V(G)$ that $\mathrm{Re}\alpha_v \geq 0$ or $\mathrm{Re}\lambda_v \geq 0$, then if $\overline{\beta}_v(t) \neq$ constant, it is easily know that $\lim\limits_{t\to\infty} \overline{\beta}_v(t)e^{\alpha_v t} \to \infty$ and $\lim\limits_{t\to\infty} t^{l_v}e^{\lambda_v t} \to \infty$, i.e., the systems of equations $(LDES_p^1)$ and $(LDE_p^n)$ are unstable at the point 0.

Now that the stability of nonlinear contradictory differential equations (7.12) can be determined by a generalized Lyapunov criterion, Dr.Ouyang explains. First of all, for $\forall v \in V(G)$ a point $\mathbf{x}_v^* \in \mathbb{R}^n$ is called an *equilibrium point* of state equation of $v$ in (7.12) if there is $\mathscr{F}_v(\mathbf{x}_v^*, \mathbf{y}(\mathbf{x}_v^*), \dot{\mathbf{y}}(\mathbf{x}_v^*)) = 0$. Generally, denote $\mathbf{x}^* = \sum_{v\in V(G)} \mathbf{x}_n^*$, called a *equilibrium sum-point*; Secondly, the sum-stability of linear system $(LDES_p^1)$ or $(LDE_p^n)$ can be extended to nonlinear systems. Generally, a nonlinear system (7.12) is called *sum-stable* or *asymptotically sum-stable* if there exists a real number $\delta_v > 0$ for any number $\varepsilon > 0$, vertex $v \in V(G)$ and there are two solutions $\mathbf{x}_v(t)$, $\mathbf{x}'_v(t)$ with respective initial values $\mathbf{x}_v(0)$, $\mathbf{x}'_v(0)$ such that

$$\left|\sum_{v\in V(G)} \mathbf{x}_v(t) - \sum_{v\in V(G)} \mathbf{x}'_v(t)\right| < \varepsilon \quad \text{or} \quad \lim_{t\to 0}\left|\sum_{v\in V(G)} \mathbf{x}_v(t) - \sum_{v\in V(G)} \mathbf{x}'_v(t)\right| = 0, \tag{7.18}$$

if $|\mathbf{x}_v(0) - \mathbf{x}'_v(0)| < \delta_v$, then the combinatorial solution $G^L[DES_p]$ of system (7.12) is said to be *sum-stable* or *asymptotically sum-stable*.

Then, *how do we introduce the Lyapunov function on a combinatorial solution of system* (7.12)*?* Dr.Ouyang explains that $G^L[DES_p]$ is assumed to be a combinatorial solution of the system of differential equations (7.12) if there exists a differentiated function $L : \mathscr{O} \to \mathbb{R}$ on an opened neighborhood $\mathscr{O} \subset \mathbb{R}^n$ of the equilibrium sum-point $\mathbf{x}^*$ satisfies:

(1) $L(\mathbf{x}^*) = 0$ and if $\sum\limits_{v\in V(G)} \mathbf{x}_v(t) \neq \mathbf{x}^*$ then $L\left(\sum\limits_{v\in V(G)} \mathbf{x}_v(t)\right) > 0$;

(2) For any $\displaystyle\sum_{v\in V(G)} \mathbf{x}_v(t) \neq \mathbf{x}^*$ there must be $\dot{L}\left(\sum\limits_{v\in V(G)} \mathbf{x}_v(t)\right) \leq 0$,

then $L$ is called a *Lyapunov sum-function* on the non-harmonious group (7.12).

By applying the Lyapunov sum-function on non-harmonious group (7.12) similar to that of the nonlinear equation, we can obtain the stability criterion on the combinatorial solution $G^L(DES_p)$ of non-harmonious group (7.12) at the equilibrium sum-point $\mathbf{x}^*$, namely *if there exists a Lyapunov sum-function $L : \mathscr{O} \to \mathbb{R}$ on a neighborhood of the equilibrium sum-point $\mathbf{x}^*$, then the combinatorial solution $G^L(DES_p)$ of non-harmonious* *group* (7.12) *is sum-stable at the equilibrium sum-point* $\mathbf{x}^*$. *Furthermore, if*

$$\sum_{v\in V(G)} \mathbf{x}_v(t) \neq \mathbf{x}^* \Rightarrow \dot{L}\left(\sum_{v\in V(G)} \mathbf{x}_v(t)\right) < 0, \tag{7.19}$$

*the combinatorial solution* $G^L(DES_p)$ *of non-harmonious group* (7.12) *is asymptotically sum-stable at the equilibrium sum-point* $\mathbf{x}^*$.

**5.2.Prod-Stability.** Notice that the essence of sum-stability of group-system is to determine the stability of non-harmonious group by summing the element states $\mathbf{x}_v(t)$. Similarly, the product of $\mathbf{x}_v(t), v \in S$ can be also used to determine the stability of non-harmonious groups. Firstly, for the case that the non-harmonious group (7.12) is characterized by a linear system of differential equations, similar to the sum-stability we can introduce the *prod-stable* and *asymptotically prod-stable* on systems $(LDES_p^1)$ or $(LDE_p^n)$, namely if for any given real number $\varepsilon > 0$ exist real numbers $\delta_v, v \in S$ such that if $|\mathbf{x}_v(0) - \mathbf{x}'_v(0)| < \delta_v, v \in S$ there must be

$$\left|\prod_{v\in V(G)} \mathbf{x}_v(t) - \prod_{v\in V(G)} \mathbf{x}'_v(t)\right| < \varepsilon \text{ or } \lim_{t\to 0}\left|\prod_{v\in V(G)} \mathbf{x}_v(t) - \prod_{v\in V(G)} \mathbf{x}'_v(t)\right| = 0, \tag{7.20}$$

then the combinatorial solution $G^L(LDES_p^1)$ (or $G^L(LDE_p^n)$) of differential equations $(LDES_p^1)$ (or $(LDE_p^n)$) is called the *prod-stable* or *asymptotically prod-stable* of system $(LDES_p^1)$ (or $(LDE_p^n)$). Similarly, for a linear non-harmonious group (7.12), we know that *the combinatorial solution of a linear system of differential equations* $(LDES_p^1)$ *(or* $(LDE_p^n)$*) is asymptotic prod-stable at point* 0 *if and only if the solution basis* $\overline{\beta}_v(t)e^t\alpha_v$ *in the solution space* $\mathscr{B}_v$ *of system* $(LDES_1^1)$ *holds with* $\sum_{v\in V(G)} \mathrm{Re}\alpha_v < 0$ *(or the basis* $t^{l_v}e^{\lambda_v t}$ *in the solution space* $\mathscr{C}_v$ *of system* $(LDE_m^1)$ *holds with* $\sum_{v\in V(G)} \mathrm{Re}\lambda_v < 0$*) for vertices* $v \in V(G)$.

Dr.Ouyang tells Huizi that different from the sum-stability of linear non-harmonious group (7.12), the condition of prod-stability here is

$$\sum_{v\in V(G)} \mathrm{Re}\alpha_v < 0 \text{ or } \sum_{v\in V(G)} \mathrm{Re}\lambda_v < 0,$$

i.e., the system may be global stable even though some elements are unstable. For example, the labeled graph shown in Figure 7.29 is the combinatorial solution with 6 vertices

$$\begin{aligned} v_1 &= \{e^{-2t}, e^{-3t}, e^{3t}\}, \quad v_2 = \{e^{-3t}, e^{-4t}\},\ v_3 = \{e^{-4t}, e^{-5t}, e^{3t}\}, \\ v_4 &= \{e^{-5t}, e^{-6t}, e^{-8t}\}, \quad v_5 = \{e^{t}, e^{-6t}\}, \quad v_6 = \{e^{t}, e^{-2t}, e^{-8t}\} \end{aligned}$$

of an inhomogeneous system $(LDES_m^1)$ or $(LDE_m^n)$. Among these elements, $v_1, v_3, v_5, v_6$ are unstable but the global system is prod-stable, namely the system is in a stable state observing by humans standing out the system, or in other words, the instability can be offset by interaction of elements in system.

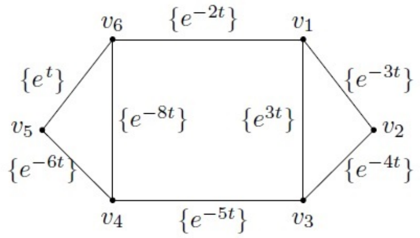



**Figure** 7.29. **Prod-stability**

Generally, denote $\mathbf{y}^* = \prod_{v\in V(G)} \mathbf{x}_n^*$, called the *equilibrium prod-point* of non-harmonious group (7.12). Then, the system (7.12) is said to be *prod-stable* or *asymptotic prod-stable* if there exist such a real number $\delta_v > 0$ for any integer $\varepsilon > 0, v \in V(G)$ and two solutions $\mathbf{y}_v(t)$, $\mathbf{y}'_v(t)$ with respective initial values $\mathbf{y}_v(0)$ and $\mathbf{y}'_v(0)$ that

$$\left|\prod_{v\in V(G)} \mathbf{y}_v(t) - \prod_{v\in V(G)} \mathbf{y}'_v(t)\right| < \varepsilon, \quad \text{or} \quad \lim_{t\to 0}\left|\prod_{v\in V(G)} \mathbf{y}_v(t) - \prod_{v\in V(G)} \mathbf{y}'_v(t)\right| = 0. \tag{7.21}$$

if $|\mathbf{y}_v(0) - \mathbf{y}'_v(0)| < \delta_v$. Now, assume that $G^L[DES_p]$ is a combinatorial solution of an inhomogeneous system (7.12), if there is a differentiated function $L : \mathscr{O} \to \mathbb{R}$ on an open neighborhood $\mathscr{O} \subset \mathbb{R}^n$ of the equilibrium prod-point $\mathbf{y}^*$ satisfies

(1) $L(\mathbf{y}^*) = 0$ and if $\prod\limits_{v\in V(G)} \mathbf{y}_v(t) \neq \mathbf{y}^*$ then $L\left(\prod\limits_{v\in V(G)} \mathbf{y}_v(t)\right) > 0$;

(2) For any $\prod\limits_{v\in V(G)} \mathbf{y}_v(t) \neq \mathbf{y}^*$ there must be $\dot{L}\left(\prod\limits_{v\in V(G)} \mathbf{y}_v(t)\right) \leq 0$,

then $L$ is called a *Lyapunov prod-function* on the non-harmonious group (7.12).

By applying the Lyapunov prod-function on the non-harmonious group (7.12), we can determine the stability of combinatorial solution $G^L(DES_p)$ of the non-harmonious group (7.12) at the equilibrium prod-point $\mathbf{y}^*$, namely *if there exists a Lyapunov prod-function $L : \mathscr{O} \to \mathbb{R}$ in the neighborhood of the equilibrium prod-point $\mathbf{y}^*$, the combinatorial solution $G^L(DES_p)$ of non-harmonious group* (7.12) *is stable at the point $\mathbf{y}^*$. Furthermore, if*

$$\prod_{v\in V(G)} \mathbf{y}_v(t) \neq \mathbf{y}^* \ \Rightarrow\ \dot{L}\left(\prod_{v\in V(G)} \mathbf{y}_v(t)\right) < 0, \tag{7.22}$$

*the combinatorial solution $G^L(DES_p)$ of non-harmonious group* (7.12) *is asymptotically prod-stable at the equilibrium prod-point $\mathbf{y}^*$.*

Listening here, Huizi asks Dr.Ouyang: "*Dad, the solvability and stability of differential equations have already been solved in classical mathematics, why are we discussing the sum-stability and prod-stability of group-systems here?*" Dr.Ouyang explains that determining whether the solution of a differential equation is stable or unstable depends on the fact that the differential equation is solvable and suitable for such a uniform system that the system's elements follow a uniform and consistent rule in evolving. Generally, humans do not discuss contradictory systems of equations in classical mathematics, which are considered only meaningless but result in the classical mathematics is only suitable for partially describing such uniform systems and can do nothing about non-harmonious group to a certain extent, namely the conclusion recognized by classical mathematics can be only local. However, when we apply the equation (7.9) or (7.12) to characterize the state of a non-harmonious group, the resulting equations are mostly contradictory one because the state variable $\mathbf{x}_v(t)$ of an element $v$ in this kind of equation does not necessarily have the same meaning in different equations. However, what humans need to characterize in recognition of a thing is its state of the whole system rather than a single element. So, it seems that the only conclusion of classical mathematics that has no solution to the contradictory equations is no longer valuable for humans. It is necessary to introduce the combinatorial solution $G^L$ of non-harmonious group to describe the behavior of things.

Huizi nods and continuously asks Dr.Ouyang: "*Indeed, the introduction of combinatorial solution of contradictory equations can characterize those of non-harmonious group. However, it seems that the combinatorial solution $G^L$ of non-harmonious group can not be completely used to characterize the stability of system.*" In response to Huizi's question, Dr.Ouyang further explains to her the recognitive gap between the formal symbols of humans and the reality of things, tells her that the combinational solution $G^L$ of non-harmonious group is a labeled graph on the intrinsic structure $G$ of elements and can reflect the states of all elements. However, there are no a method for determining the stability of a combinatorial solution $G^L$ or a group of element states in classical mathematics. In this case, it is necessary to look for or discover simple metric mappings that characterize the overall states of system, i.e., $\omega : G^L \to \mathbb{R}^+$. It is this reason that the sum-stability and prod-stability are introduced on non-harmonious groups. Of course, there are other ways to achieve this goal. Generally, any metric mapping in Euclidean space such as the length mapping

$$\rho : (x_1, x_2, \cdots, x_n) \to \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \tag{7.23}$$

combining with the structure of graph $G$ and characters of system can be applied for determining the behavior of system state $\{\mathbf{x}_v, v \in S\}$, including the stability of combinatorial solution $G^L$ for a non-harmonious group. Certainly, a most important work is to choose which metric mapping is more consistent with the appearance or state of system so that holding on things rather than only the formal increase of recognitive complexity.

## §6. **Comments**

**6.1.** Clearly, a contradiction is out of synchronization or logical inconsistency in developing of things, which is a universal existence in human's eyes. In the human recognition, a mathematical system should follows the principle of logical consistency and can not contradict itself on one hand. But on the other hand, contradiction exists everywhere among things and the logical consistency of mathematical system must lead to the limited mathematics on characterizing things, including the incompleteness of formal logic of itself such as the Godel's incompleteness theorem, etc. So, the mathematical reality of human recognition is definitely less than the natural reality. For example, the living and death of the Schrodinger's cat can not be simply determined by classical mathematics, see references [Gri1]-[Gri2] and [Mao45]. Thus, the recognition or understanding of things in the universe requires the transformation of the contradictory into the compatible and establishes a pan-mathematical system, which is the Smarandachely denied system, including the Smarandache geometry, Smarandache multispace or multisystem and furthermore, mathematical combinatorics established over topological graphs inherited in things, see [Del], [Sma1] - [Sma2], [Sma4], [SCF], [VaS1] - [VaS2], [Ise], [KuA], [Mao3] - [Mao6], [Mao9], [Mao13] - [Mao23], [Mao53] and [Mao55], etc. Certainly, the essence of a thing $T$ in the human recognition is a Smarandache multispace or multisystem $\mathcal{S}=\bigcup_{i=1}^{m}(\mathcal{S}_i,\mathcal{R}_i)$ because the multilateral property of a thing $T$ and the mathematical consistence determine that such a recognition is only local, not the whole reality on $T$, which leads to a Smarandache multispace or multisystem. So, *how do we characterize the spaces or systems $(\mathcal{S}_i,\mathcal{R}_i)$ in $\mathcal{S}$?* Indeed, we can characterize $(\mathcal{S}_i,\mathcal{R}_i)$ by its all characters and view them as a vector $(\mu_1,\mu_2,\cdots,\mu_n;\nu_1,\nu_2,\cdots)$. However, the characters $\nu_j, j\geq 1$ are unknown. In this case, we can endow each element $x\in\mathcal{S}_i$ with a ternary array $(T,I,F)$, i.e., a *neutrosophic set* and characterize $x\in\mathcal{S}_i$ by $x(T,I,F)$ for integers $1\leq i\leq m$, where $T,I,F\subseteq[0,1]$ are respectively the *confident set* $T$, *indefinite set* $I$ and the *fail set* $F$, see [Sma1].

**6.2.** Generally, a Smarandachely denied axiom is different from the axiom in classical mathematics in that it can be validate and invalidate both, or only invalidate but in at least two distinct ways. Such a mathematical axiom is always regard as beneath one's notice in classical mathematics for believing that an axiom behaves validate and invalidate simultaneously in a mathematical system must violates the principle of logical consistency but it is an effective way to unify different systems, including the contradict systems. Certainly, the Smarandachely denied axiom is the basis of Smarandache geometry, Smarandache multispaces or multisystems originally from the discovery of Smarandache plane geometry or 2-dimensional manifolds and Smarandache bialgebra, see references [KuA], [Ise] and [VaS1]-[VaS2]. In [Mao10]-[Mao12] and [Mao23], I have applied the combinatorial notion to construct Smarandache $n$-dimensional manifolds and geometry in general. Among them, a basic question similar to the "*being*" and "*non-being*" of Taoist or the "*color*" and "empty" of Buddhist, *is whether a contradictory system necessarily implies such a system or thing does not exist?* The answer is No because most of them are formal contradictions of humans in characterizing rather than the natural reality, including the lucky number game, also see [GuG].

**6.3.** Now that *how can we construct the mathematical systems which satisfy the axiom of Smarandacely denied axiom on the principle of logical consistency?* After I completed my monograph on combinatorial geometry with field theory [Mao23], I realized suddenly that one of the most immediately case of Smarandachely denied axiom in mathematics should be the non-solvable system of equations, which characterizes the group behavior or the combinatorial behavior of non-harmonious group and then, I completed the first paper on non-solvable systems of linear algebraic equations [Mao26]. In July 2012, Professor Smarandache visited Guangdong University of Technology. I was on a business trip in Guangzhou for two days in the month and visited him. I told him the main thought of paper [Mao26] about the Smarandachely denied axiom with non-solvable system of algebraic equations. He was a little surprised because he never realized that the Smarandachely denied axiom could be related with mathematics and the recognition of things in this way. As I went back to the hotel he sent me an email that evening, in which he called me "*a great researcher*". With his encouragement, I completed a series papers on characterizing non-solvable systems of the algebraic homogeneous equations, ordinary differential equations and the partial differential equations [Mao27]-[Mao32], [Mao35] by applying combinatorial notion to those equations considered meaningless in classical mathematics, which aroused the international reaction but very dull in home. Finishing the papers [Mao26]-[Mao30] and [Mao35], there is a question can not refrain to ask me myself: "*Why didn't my works characterizing non-solvable equations get much attention in China? Is it really true that blooming inside but perfuming outside the yard?*" On the New Year's day in 2014, I finally figured out an answer to this question while I was walking in the Beihai Park of Beijing, which is because that my paper is discussing non-solvable system of equations but most of researchers only care about the classical mathematics. They are generally unwilling to accept such research from their heart because in their opinion, the mathematical systems should follow the principle of logical consistency and can not contain contradictions. Of course, all these humans ignore the intension of mathematics, namely it is a quantitative tool for recognizing things rather than a game of symbols and the non-harmonious groups really correspond to non-solvable system states of things. We have to further research such kind of groups in mathematics if we would like to know the truth of things. In this regard, I wrote the paper [Mao31] which explained how to transform a non-mathematics system in classical mathematics into a mathematics over a topological structure $G^L$ by a combinatorial approach for contributing the recognition of things.

**6.4.** Certainly, a non-harmonious group corresponds to a system of non-solvable equations but each equation has a solution itself, namely there are no solution of the system in classical mathematical sense. In fact, such systems exist in large numbers in the nature such as the biological populations, cell systems, genes and other self-organizing systems, see [BrC], [LuW] and [Mur]. Such systems, especially the stability of such systems can not be characterized by classical mathematics because they are unsolvable in the sense of classical mathematics but they can be characterized by combinatorial solutions, including the sum-stability, prod-stability and the asymptotic sum-stability, asymptotic prod-stability and other group-system stability because any non-harmonious group must have a combinatorial solution which is the value of them. Geometrically, the representation of a non-harmonious group in space $\mathbb{R}^n$ is exactly the combinatorial manifold defined in [Mao10] and [Mao23], reflecting a philosophical thought that all roads must lead to Rome.

**6.5.** The quantum entanglement is an unknown phenomenon closely related to quantum collapse or the Schrodinger's cat, which Einstein called the "*spooky action at a distance*" and especially, caused the debate between Einstein et al and Bohr et al on physical reality after the publication of EPR paper. In recent years, a few scientists have carried out active exploration on the generation, preservation and application of quantum entanglement such as photon entanglement pairs with applications to the secure communication and obtained many commendable results, see [GuG] for details. In fact, if a quantum effect is regarded as a quantum field, the quantum entanglement corresponds only to the case of Smarandache multispace or multisystem in 2 spaces or systems. It is a very meaningful work for studying the entangling and disentangling of Smarandache multispace or multisystem in general, provides the entanglement mechanism theoretically, explain the word "*hidden*" in Bohm et al's interpretation on the quantum entanglement [BoA] and apply it to encoding and decoding of information. This is the motivation of [Mao55] for discussing the disentangling Smarandache multispaces and multisystems.

$C_{\mathscr{M}}$

**Combin.Notion**

Chapter 8

# Complex Networks

**********************************************************************

**Never Plucks String on a Lute for Sorrow Pour Out,**

**Love Will Never Die if Sky Not Old;**

**Heart Like Pair of Silk Nets with Thousands of Knots,**

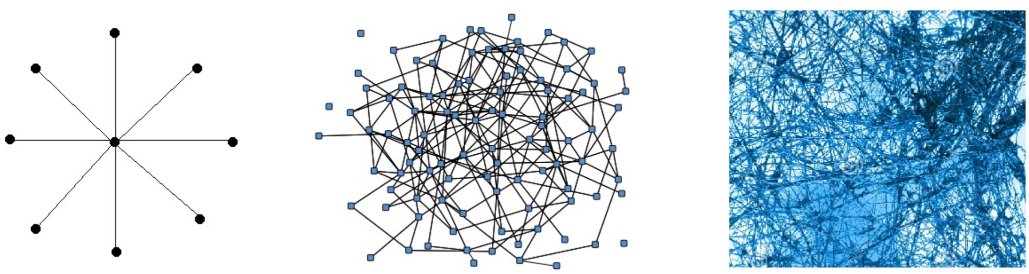
**Night Passed but no White on East, Remains a Waning Moon.**

— *Years - Cuckoo Sounds* By X.Zhang, a poet in Song dynasty of China

## §1. Network Structure

A network is such a specially labeled graph $G^L$ that simplifies the action of system elements. Here, the labelling mappings $L : G \to \mathbb{R}$ usually corresponds to the actual network of transportation, power transmission, information transfer, etc. It is worth noted that some of the labeled graphs $G^L$ obtained by systematic recognition of things are simple and clear while others are complex and unpredictable. For example, the internal network consisting of cells is far more complex than that of the ten systems of an animal. Generally, Dr.Ouyang tells Huizi that the networks one sees in the macroscopic world are simpler than those made of microscopic particles. Certainly, we can recognize and characterize easily the evolving behavior of simple networks but for these complexes, it needs us continuously to recognize the truth for serving the human society under conditions.

So, *what is a complex network?* Dr.Ouyang tells Huizi that the word "*complex*" in the *complex networks* is such a case that relative to the simple structure, behavior and a uniform evolving, which is mainly manifested in following aspects: ①Structure complexity, namely the number of nodes such as the human cells are huge in about $4 \times 10^{14} - 6 \times 10^{14}$ with different characteristics in its structures; ②Connecting complexity, namely it may be differ in the connecting weighted of nodes with directional changes; ③Evolving complexity, namely the emergence and disappearance of nodes or connecting are random somewhat, the network structure changes frequently and does not follow the linear law, which results in the complexity of system recognition or characterizing, as shown in Figure 8.1.



(a) (b) (c)

**Figure** 8.1. **Networks**

Dr.Ouyang asks Huizi: "*What's your intuitive feeling about observing the three networks in Figure* 8.1*?*" Huizi looks for a while and then answers: "*Among the three networks, the structure of* (*a*) *is simple and clear. It is a central node connecting with eight other* *unconnected nodes, like a blooming flower. The number of nodes in network* ($b$) *is large, its connecting is complicated but the law is obvious. However, the nodes and connecting of network* ($c$) *are too complex to see in detail.*" Dr.Ouyang confirms her observation and continuously asks her: "*So, which one or more of these three networks do you think are complex network?*" Huizi replies: "*The details in network* ($c$) *are not clear in observing which should be a complex one. Network* ($a$) *is a simple network with only nine nodes and an obvious structural relationship. Network* ($b$) *is between that of* ($a$) *and* ($c$). *Although the number of its nodes is quite much but we can still count them only with the naked eyes. And its connecting structure is also clear. I think it should be a simple network!*" Dr.Ouyang tells her that all of these three networks would become complex networks under certain conditions. Among them, ($a$) is a star network with nine nodes. Certainly, its connecting is clear but the random changes of nodes and connecting will cause a complicated change even if it is a star network with only nine nodes. The nodes and connecting of network ($b$) are not complicated, which is similar to ($a$), i.e., as long as its nodes or connecting show uncertainty, it will become a complex network also. Clearly, network ($c$) is a typical complex network with a large number of nodes, complex connecting and we can not clearly observing its details. So, it is necessary to enlarge the local structure technically of network ($c$) for observing its local connecting, structures and then, we can recognize ($c$). In fact, a complex network refers to those networks in which humans can not accurately hold on their behavior state at a certain time, including the complex structure of ($c$) and the complex behavior of ($a$) or ($b$).

Although it seems that humans can systematically recognize but it is truly impossible to characterize and hold on all things in the universe because of the recognition limitation of humans. This is exactly the philosophical thought contained in the words that "*Name named is not the eternal Name*" of *Tao Te Ching*. In this situation, we can only combine all local recognitions on thing similar to that of the network ($c$) in Figure 8.1, which needs to be continuously deepened or modified along with the progress of humans' technology and an infinite process approaching the truth. Then, *what local characters of system recognition or its corresponding complex network can be proposed to characterizing the system of thing within humans' ability?* Dr.Ouyang explains that a natural way would be to characterize the statistical properties of complex networks show before the eyes of human.

**1.1.Random Distribution.** A deterministic event is such an event that the *cause* is clear in the causal relationship and the *effect* can be uniquely determined by the developing process. On the other hand, if the *cause* or the process is unclear, an event may or may not happen is called a random event. For example, tossing a coin into the air is called a test. If there is a high-speed camera continuously shooting the coin from throwing to landing, it can be clearly known whether the coin will land heads up or tails up. However, if there is no such a camera to observe the coin landing process and a period of change or evolving information before the coin landing is lost, then both of the heads up or sides up may appear, which is an approximately random event. In the process of human recognition, there are a lot of random events with unclear cause or process. So, it is necessary to use random model to analyze the possibility of thing behaviors.

In a repeated test $\Omega$, each possible outcome is called the *sample point* $\omega(\Omega)$ and all sample points $\omega_1(\Omega)$,$\omega_2(\Omega), \cdots, \omega_n(\Omega)$ constitute the sample space of test $\Omega$, i.e

$$\Omega = \{\omega_1(\Omega), \omega_2(\Omega), \cdots, \omega_n(\Omega)\}. \tag{8.1}$$

Accordingly, such a test is called that of a *randomized test*. Obviously, if $n = 1$, i.e. $\Omega$ has only one outcome, then (8.1) is a deterministic test and not a randomized one. Thus, for a randomized test there must be $n \geq 2$ in (8.1), Dr.Ouyang tells Huizi. Now assume that the sample point $\omega_k$ appears $\mu_k$ times in the $n$ outcomes of $\Omega$, where $1 \leq k \leq n$, define the *frequency* of $\omega_k$ occurring in the $n$ tests of $\Omega$ by $F_n(\omega_k) = \mu_k/n$ which satisfies: ①Non-negativity, namely $0 \leq F_n(\omega_k) \leq 1$; ②Normality, namely $F_n(\Omega) = 1$; ③Additivity, namely for two independent events $\omega_i$ and $\omega_j$, i.e., $\omega_i \bigcap \omega_j = \emptyset$, $F_n(\omega_i \bigcup \omega_j) = F_n(\omega_i) + F_n(\omega_j)$, where $1 \leq i, j \leq n$.

Generally, $F_n(\omega_k)$ is called the *probability* of event $\omega_k$, denoted by $P(\omega_k)$ if the limitation of $F_n(\omega_k$ exists when $n \to \infty$, i.e.,

$$P(\omega_k) = \lim_{n\to\infty} F_n(\omega_k) = \lim_{n\to\infty} \frac{\mu_k}{n}. \tag{8.2}$$

Accordingly, by the properties of limitation, the probability $P(\omega_k)$ also satisfies the non-negativity, normality and the additivity.

The limitation of human recognition implies that the humans' recognition conclusions are only valid under certain conditions. Similarly, Dr.Ouyang explains that the possibility of events occur under certain condition, namely the *condition probability*, i.e., under the condition that "*A occurred*" to determine the probability of events $\omega$ occurring, denoted by $P(\omega|A)$. In this case, by the multiplication formula the probability that condition A and $\omega$ occur simultaneously is

$$P(A\omega) = P(A)P(\omega|A) \quad \Rightarrow \quad P(\omega|A) = \frac{P(A\omega)}{P(A)}. \tag{8.3}$$

For example, let A and B be two cities in the southwest China. According to the meteorological data, the number of rainy days in A city accounts for 20% in a year and the number of rainy days in both cities accounts for 16%. Then, the probability of rain in B city under the rain in A city is a conditional probability. According to the formula (8.3), it is $0.16/0.2 = 0.8 = 80\%$.

Generally, let $(\Omega, \mathbb{R}, P)$ be a triple, where $\Omega$ is the sample space, $\mathbb{R}$ is the set of real numbers and $P$ denotes the set of probability and let $\xi : \Omega \to \mathbb{R}$ be a real function defined on $\Omega$. For any real number $x$, if $\{\omega : \ \xi(\omega) < x\}$ is an event, then $\xi(\omega)$ is called a *random variable* to distinguish it from the usual variables. Usually, if $\xi$ is a random variable, then $P(\xi < x)$ is meaningful for any real number $x$ amd defining $F_\xi(x) = P(\xi < x)$ as the distribution function of random variable $\xi$ on $-\infty < x < \infty$. Certainly, there are two distribution functions as follows:

**Discrete Random Model.** *A random variable $\xi$ is said to be discrete if it has only a finite or listed number of outcomes $x_1, x_2, \cdots, x_n, \cdots$, denoted by $P(\xi_i = x_i) = p_i, i \geq 1$ and the sequence $\{p_i\}$ is called the distribution of $\xi$, i.e.*

| $\xi$ | $x_1$ | $x_2$ | $\cdots$ | $x_n$ | $\cdots$ |
|---|---|---|---|---|---|
| $P(\xi = x_i)$ | $p_1$ | $p_2$ | $\cdots$ | $p_n$ | $\cdots$ |

where $p_i \geq 0$ for integers $i \geq 1$ satisfy $\sum\limits_{i \geq 1} p_i = 1$.

**Continuous Stochastic Model.** *If the distribution function $F(x)$ of a random variable $\xi$ can be expressed as the integral of a non-negative integrable function $f(x)$, i.e*

$$F(x) = \int_{-\infty}^{x} f(t)dt, \quad -\infty < x < +\infty, \tag{8.4}$$

*Then, $\xi$ is called the continuous random variable and $f(x)$ is its density function.*

By the definition of continuous random variables and the integral properties, we are easily knowing the properties of continuous distribution following.

(1)The density function $f(x)$ satisfies: ①Non-negativity, namely for any variable $x$, $-\infty < x < +\infty$ there is $f(x) \geq 0$; ②Normality, namely $\int_{-\infty}^{+\infty} f(t)dt = 1$.

(2)The probability of random variable $\xi \in [a, b]$ is

$$P(a \leq \xi \leq b) = \int_{a}^{b} f(t)dt. \tag{8.5}$$

(3)The probability of random variable $\xi$ continuous at any point $a$ is 0, namely $P(\xi = a) = 0$ because $P(\xi = a) = F(a + 0) - F(a)$ and $F(x)$ is continuous at point $a$.

Examples of some common probability distribution functions are listed as follows:

(1) **Binomial distribution.** The *binomial distribution* is for such cases that there are only two possible outcomes, i.e., an event A occurs or does not occurs, also known as the *Benouli test*. Assume that the probability of event A occurring is $p$, $q = 1 - p$ and let $\xi$ be the number of times of event A occurring in $n$ independent repeated Bernoulli tests. Then, the distribution of the random variable $\xi$ is

$$P(\xi = i) = \begin{pmatrix} n \\ i \end{pmatrix} p^i q^{n-i}, \ 1 \leq i \leq n. \tag{8.6}$$

Generally, (8.6) is called the *Benouli binomial distribution*, denoted as $B(n, p)$.

(2) **Poisson distribution.** Assume that the possible outcomes of a random variable $\xi$ are $0, 1, 2, \cdots$ and

$$(\xi = k) = P\frac{\lambda^k}{k!}e^{-\lambda}, \tag{8.7}$$

then the random variable $\xi$ is said to be the *Poisson distribution* with parameter $\lambda$. By definition, it is easy to know that $P(\xi = k) \geq 0$ and by the expansion of exponential function $e^x$, there is

$$\sum_{i=0}^{\infty} \frac{\lambda^i}{i!} e^{-\lambda} = e^{-\lambda} \sum_{i=0}^{\infty} \frac{\lambda^i}{i!} = e^{-\lambda}e^{\lambda} = 1,$$

i.e., Poisson distribution satisfies the non-negativity and the normality with a form similar to that of the binomial distribution. Generally, assume that a random variable $\xi_n$ ($n = 1, 2, \cdots$) obey the binomial distribution (8.6), namely $np_n = \lambda$ is constant, $q_n = 1 - p_n$. Then, for any integer $k \geq 1$, there are

$$P(\xi_n = i) = \begin{pmatrix} n \\ i \end{pmatrix} p_n^i q_n^{n-i}, \ 1 \leq i \leq n$$

and

$$\lim_{n\to\infty} P(\xi_n = k) = \frac{\lambda^k}{k!} e^{-\lambda}, \tag{8.8}$$

i.e., the Bernoulli binomial distribution $P(\xi = k)$ is the limitated case that of Poisson $P(\xi_n = k)$ in $n \to \infty$.

(3) **Uniform distribution.** A *uniform distribution* is such a probability distribution that the points are uniformly cast within an interval $[a, b]$, namely the probability of a point falling in any position of interval $[a, b]$ is the same. In this case, the probability distribution function

$$F(x) = P(\xi < x) = \begin{cases} 0, & x \leq a, \\ \dfrac{x-a}{b-a}, & a < x < b, \\ 1, & x \geq b, \end{cases} \tag{8.9}$$

with a density function

$$f(x) = F'(x) = \begin{cases} \dfrac{1}{b-a}, & x \in (a,b), \\ 0, & x \notin (a,b). \end{cases} \tag{8.10}$$

(4) **Exponential distribution.** The density function of *exponential distribution* for a random variable $\xi$ is

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & x \geq 0 \\ 0, & x < 0 \end{cases} \tag{8.11}$$

with $\lambda > 0$. In this case, it is easily verified that $f(x) \geq 0$ and

$$\int_{-\infty}^{+\infty} f(x)dx = \int_0^{+\infty} \lambda e^{-\lambda x} dx = 1,$$

i.e., it holds with the normality and

$$P(a \leq \xi \leq b) = \lambda \int_a^b f(x)dx = e^{-\lambda a} - e^{-\lambda b}. \tag{8.12}$$

(5) **Normal distribution.** A *normal distribution* is also known as the *Gaussian distribution*, its density function for a random variable $\xi$ is

$$f(x) = \frac{1}{\sqrt{2\pi}\sigma}\, e^{-\frac{(x-a)^2}{2\sigma^2}}, \quad -\infty < x < +\infty, \tag{8.13}$$

where $a, \sigma > 0$ are both constants and its distribution function is

$$F(x) = \frac{1}{\sqrt{2\pi}\sigma} \int_{-\infty}^{x} e^{-\frac{(t-a)^2}{2\sigma^2}} dt, \quad -\infty < x < +\infty, \tag{8.14}$$

called usually the random variable $\xi$ follows the normal distribution in parameters $a, \sigma^2$, denoted by $\xi \sim N(a, \sigma^2)$, as shown in Figure 8.2. Particularly, if parameters $a = 0,\ \sigma = 1$, the normal distribution $N(a, \sigma^2)$ is called a *standard normal distribution*, denoted by $N(0,1)$, which is a statistical model commonly used in the study of a large number of appearances in the nature, human society, psychology and the education. In this case, the density function $\varphi(x)$ and the distribution function $\Phi(x)$ of the standard normal distribution are respectively

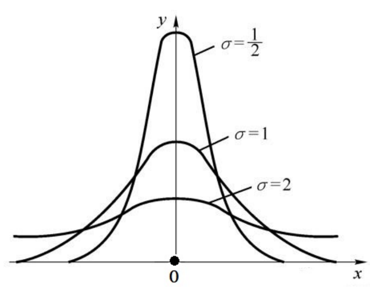


**Figure** 8.2. **Normal distribution**

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}}, \quad \Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt,$$

where $-\infty < x < +\infty$.

Notice that an important property of Poisson distribution is that when $n \to \infty$, there is $P(\xi_n = k) \to P(\xi = k)$, i.e., the Benoulli binomial distribution. A generalization of this property is that for any sequence of random variables $\{\xi_n\}$ $(n = 1, 2, \cdots)$, if there exists a random variable $\xi$ such that for any given number $\varepsilon > 0$ there is always

$$\lim_{n\to\infty} P\{|\xi_n - \xi| < \varepsilon\} \;=\; 1, \tag{8.15}$$

Then, it is said that the random variable sequence $\{\xi_n\}$ *probability converges* to the random variable $\xi$, denoted by

$$\lim_{n\to\infty} \xi_n \;\overset{P}{=}\; \xi. \tag{8.16}$$

Now that for a random variable sequence $\{\xi_n\}$ and an arbitrary integer $n \geq 1$, let $P(\xi_n = x_i) = p_i$ be the probability distribution of variable $\xi_n$. If the series $\sum\limits_{i=1}^{\infty} x_i p_i$ converges absolutely to a number $E(\xi_n)$, called the *expectation* of random variable $\xi_n$. Similarly, if $E\left\{(\xi_n - E(\xi_n))^2\right\}$ exists, then $D(\xi_n) = E\left\{(\xi_n - E(\xi_n))^2\right\}$ is called the *variance* and $\sqrt{D(\xi_n)}$ the *mean square error* of $\xi_n$. Define $\overline{\xi}_n = \dfrac{1}{n}\sum\limits_{i=1}^{n} \xi_i$ to be the arithmetic average of the first $n$ random variables. In this case, if

$$\lim_{n\to\infty} \left(\overline{\xi}_n - E(\overline{\xi}_n)\right) \;\overset{P}{=}\; 0, \tag{8.17}$$

then the random variables sequence $\{\xi_n\}$ is said to be satisfied the *law of large numbers*.

Notice that the law of large numbers is an important property of random variables which has many manifestations. For example, Chebyshev proved that *for given $n$ independent random variables, if their expectations are the same of finite and so are their variances, then the arithmetic mean of $n$ random variables is almost a constant if $n$ is sufficiently large, i.e*

$$\lim_{n\to\infty} P\left\{\left|\frac{1}{n}\sum_{i=1}^{n} \xi_i - \mu\right| < \varepsilon\right\} \;=\; 1, \tag{8.18}$$

*where $\mu$ is the expectation of these random variables.* Notice that randomized tests of humans can only be done in a limited times to get frequencies. In contrast, the probability is the limiting case that the number of test times approaches to the infinity. So, "*what is the relationship between the frequency and probability?*" Dr.Ouyang explains that Bernoulli mathematically proved that *if $n$ is sufficiently large, the probability that the frequency of event $A$ occurring different greatly from the probability $p$ of event $A$ is very small, namely for any positive number $\varepsilon > 0$, there is*

$$\lim_{n\to\infty} P\left\{\left|\frac{n_A}{n} - p\right| < \varepsilon\right\} = 1 \tag{8.19}$$

and Poisson further proved that *for any number $\varepsilon > 0$, there is*

$$\lim_{n\to\infty} P\left\{\left|\frac{n_A}{n} - \frac{1}{n}\sum_{i=1}^{n} p_i\right| < \varepsilon\right\} = 1, \tag{8.20}$$

where $n_A, p$ are respectively the occurring times of event A in $n$ tests, the probability of event A in a test and $p_i$ is the occurring probability of event A in the $i$th test.

Notice that the normal distribution is the most common distribution in application such as the length or weight of an organism, the error of measuring the same object, the weight of seeds and so on. Generally, assuming that $\{\xi_n\}$ is an independent sequence of random variables with finite expectations and variances, namely for integers $i = 1, 2, \cdots$, $E(\xi_i) = \mu_i, D(\xi_i) = \sigma_i^2$, define $B_n^2 = \sum\limits_{i=1}^{n} \sigma_i^2$, $Y_n = \sum\limits_{i=1}^{n} (\xi_i - \mu_i)/B_n,\ n = 1, 2, \cdots$. If for any $x \in (-\infty, +\infty)$ there is

$$\lim_{n\to\infty} P\{Y_n \leq x\} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{2}} dt, \tag{8.21}$$

then the sequence $\{\xi_n\}$ of random variables is said to obey the *central limit theorem*, denoted by $\{\xi_n\} \overset{L}{\sim} N(0,1)$ usually.

So, *what kind of random variable sequences obey the central limit theorem?* Dr.Ouyang explains that if the random variable sequence $\{Y_n^{[1]}\}$ is defined on a random variable sequence $\{\xi_n^{[1]}\}$ obeying the Bernoulli binomial distribution $B(n,p)$ or the random variable sequence $\{Y_n^{[2]}\}$ defined on a random variable sequence $\{\xi_n^{[2]}\}$ obeying that for any integer $i \geq 1$, the random variables $\xi_i^{[2]}$ are independent of each other, follow the same distribution with finite expectations and variances, i.e., $E(\xi_i^{[2]}) = \mu,\ D(\xi_i^{[2]}) = \sigma^2 \neq 0$ as follows

$$Y_n^{[1]} = \frac{\xi_n - np}{\sqrt{npq}} \quad \text{or} \quad Y_n^{[2]} = \frac{\sum\limits_{i=1}^{n} \xi_i - n\mu}{\sqrt{n}\sigma}, \tag{8.22}$$

then the sequences $\{Y_n^{[1]}\}$ and $\{Y_n^{[2]}\}$ both obey the central limit theorem.

**1.2.Valency Distribution.** Let $G$ be a graph. For any vertex $v \in V(G)$, all vertices adjacent to $v$ in $G$ is denoted by $N_G(v)$ and the number $\rho_G(v) = |N_G(v)|$ is called the *valency* or *degree* of vertex $v$ in $G$. By definition, $\rho_G(v)$ is the same as the number of incident edges with $v$ in $G$. A *valency sequence* is a number sequence $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_{|G|}$ arranged in the quantity sizes of vertex valencies of graph $G$.

So, *under what condition that a non-negative integer sequence is the valency sequence of a graph?* First of all, Dr.Ouyang explains that we have known a recursive relationship of valency sequences, namely *a non-negative integer sequence $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_p$ is the* *valency sequence of a graph if and only if the integer sequence* $\rho_2 - 1, \rho_3 - 1, \cdots, \rho_{\rho_1+1} - 1, \rho_{\rho_1+2}, \cdots, \rho_p$ *is the valency sequence of a graph, where* $\rho_1 \geq 1$, $p \geq 2$; Secondly, there is a criterion for an integer sequence being that of valency sequence of a graph, i.e., *a non-negative integer sequence* $\rho_1 \geq \rho_2 \geq \cdots \geq \rho_p$ *is the valency sequence of a graph if and only if* $\sum\limits_{i=1}^{p} \rho_i$ *is even and for any integer* $n, 1 \leq n \leq p-1$ *there is*

$$\sum_{i=1}^{n} \rho_i \leq n(n-1) + \sum_{i=n+1}^{p} \min\{n, \rho_i\}. \tag{8.23}$$

The average valency $\langle\rho\rangle_G$ and the maximum valency $\Delta(G)$ in a given graph $G$ are respectively defined by

$$\langle\rho\rangle_G = \frac{\sum\limits_{v\in V(G)} \rho_G(v)}{|G|}, \quad \Delta(G) = \max\{\rho_G(v) | v \in V(G)\}. \tag{8.24}$$

Usually, the vertices with maximum valency in a graph $G$ is called its *coreness* of $G$ and the *coreness* $C_\rho(v)$ of a vertex $v \in V(G)$ is defined by

$$C_\rho(v) = \frac{\rho_G(v)}{|G|-1}. \tag{8.25}$$

A sequence $C_\rho(v), v \in V(G)$ arranged in the quantity sizes is called the *coreness sequence* of graph $G$. For example, the coreness sequence of path $P_n$ is $1/(n-1), 1/(n-1), 2/(n-1), \cdots, 2/(n-1)$ and that of a complete graph $K_n$ is $1, 1, \cdots, 1$, etc.

The *valency distribution* $P(k)$ of a graph $G$ is defined as the probability that the degree of any vertex is exactly $k$. For example, $P(k) = 1$ in a $k$-regular graph $G$ and $P(i) = 0$ if $i \neq k$. Particularly, $P(n-1) = 1$ and $P(i) = 0$ for a complete graph $K_n$ if $i \neq n-1$. Obviously, for a graph $G$ there is

$$\sum_{i=0}^{\infty} P(i) = \sum_{i=0}^{\Delta(G)} P(i) = 1 \tag{8.26}$$

and

$$\langle\rho\rangle_G = \frac{\sum\limits_{v\in V(G)} \rho_G(v)}{|G|} = \sum_{v\in V(G)} \frac{\rho_G(v)}{|G|} = \sum_{i=0}^{\Delta(G)} iP(i). \tag{8.27}$$

Clearly, the probability of a choosing vertex with degree $\geq k$ in $G$ is

$$P_k = \sum_{i=k}^{\Delta(G)} P(i), \tag{8.28}$$

and if there is constant at $\sigma, \gamma > 0$ such that $P(k) \propto e^{k/\sigma}$ or $P(k) \propto k^{-\gamma}$ then

$$P_k = \sum_{i=k}^{\Delta(G)} e^{-i/\sigma} \propto e^{-k/\sigma} \quad \text{or} \quad P_k = \sum_{i=k}^{\Delta(G)} k^{-\gamma} \propto k^{-(\gamma-1)}. \tag{8.29}$$

where $P_k \propto f(k)$ implies that there is a constant $C > 0$ such that $P_k = Cf(k)$.

Usually, the probability of an edge $(u, v) \in E(G)$ with valencies $\rho_G(u) = k_1, \rho_G(v) = k_2$ in a graph $G$ is called the *joint probability*, denoted by $P(k_1, k_2)$. So, by definition of the probability, we know that

$$P(k_1, k_2) = \frac{|E_G(k_1, k_2)|}{|E(G)|}, \tag{8.30}$$

where $E_G(k_1, k_2)$ denotes the set of edges with one end vertex of degree $k_1$ and another of degree $k_2$ in graph $G$.

Notice that there are famous words that "*there must be one of my teacher in any three humans, following those of good and improving those of bad characters*" of Confucius in his *Confucius Analects – Narrated* and the Ramsey's assertion that "*if there are at least 6 humans at a party then there must exist groups of three humans among them either all know each other or have never met before*", both of them indicate that the number "*three*" is an important quantifier in crowd characterizing. So, *how do we characterize this three-human group or cluster character in the crowd relationship network $G^L$?* Dr.Ouyang explains that this needs to introduce the *cluster coefficient*, i.e., the frequency of the triangle $K_3$ occurring in graph $G$. Generally, for any vertex $v \in V(G)$, the cluster coefficients of vertex $v$ and graph $G$ are respectively determined by

$$N_3(v) = \frac{2|E\langle N_G(v)\rangle|}{\rho_G(v)(\rho_G(v)-1)}, \qquad N_3(G) = \frac{1}{|G|}\sum_{v\in V(G)} N_3(v), \tag{8.31}$$

where $E\langle N_G(v)\rangle$ denotes the edge set of subgraph spanned by the neighborhood $N_G(v)$ of vertex $v$ in $G$. For example, for any integer $n \geq 3$ the valency of any vertex $v$ in complete graph $K_n$ is $n-1$ and the spanned subgraph of its neighborhood is $K_{n-1}$. Thus, the cluster coefficient of the vertex $v$ in graph $K_n$ is

$$N_3(v) = \frac{(n-1)(n-2)}{(n-1)(n-2)} = 1, \quad N_3(G) = n.$$

However, for a tree $T, v \in V(T)$, $N_3(G) = N_3(v) = 0$ because by definition, there are no circuits in a tree $T$, i.e., no subgraph $K_3$. So, a complete graph is the case with the most groups of three but a tree is the case with the least, i.e., no such a subgraph.

For any integer $k$, denote $R_G(k) = \{v \in V(G) | \rho_G(v) > k\}$, i.e., all vertices of valency greater than $k$, called the *$k$-core* of graph $G$. Usually, define

$$\Phi_G(k) = \frac{|E\langle R_G(k)\rangle|}{\left|E\left(K_{|R_G(k)|}\right)\right|} = \frac{2\,|E\langle R_G(k)\rangle|}{|R_G(k)|\,(|R_G(k)|-1)} \tag{8.32}$$

for measuring the proportion of the $k$-core, called the *$k$-core coefficient* of graph $G$. Here, $E\langle R_G(k)\rangle$ denotes the edge set of subgraph spanned by the $k$-core $R_G(k)$ in $G$ and $K_{|R_G(k)|}$ is a complete graph of order $|R_G(k)|$.

For example, a path $P_n$ has no vertices of degree$> 2$ for any integer $n \geq 3$ but with $n-2$ vertices of degree 2 whose spanned subgraph is a path $P_{n-2}$. Thus, $\Phi_{P_n}(k) = 0$ if $n \geq 3,\ k \geq 2$ and

$$\Phi_{P_n}(1) = \frac{2\,|E\langle R_G(k)\rangle|}{|R_G(k)|\,(|R_G(k)|-1)} = \frac{2(n-3)}{(n-2)(n-3)} = \frac{2}{n-2}$$

if $n \geq 3$, $k = 1$.

**1.3.Distance Distribution.** Let $G$ be a graph with $u, v \in V(G)$. The distance between $u$ and $v$ in $G$ is defined by the length $d_G(u,v)$ of shortest path connecting vertices $u$ and $v$ in $G$, namely

$$d_G(u,v) = \min\{\ |P| - 1 \mid P \in \mathcal{P}_G(u,v)\ \}, \tag{8.33}$$

where $\mathcal{P}_G(u,v)$ denotes the set of paths connecting vertices $u, v$ in graph $G$. Similarly, the distance of the vertices $u, v$ in labeled graph $G^L$ is defined by

$$d_G^L(u,v) = \min \sum_{P \in \mathcal{P}_G(u,v)} \sum_{(v_1,v_2) \in E(P)} L(v_1,v_2). \tag{8.34}$$

Clearly, (8.34) degenerates to (8.33) if $L(v_1,v_2) = 1$ for any edge $(v_1,v_2) \in E(P)$ in (8.34) and the *average distance* $\langle d\rangle_G$ in a connected graph $G$ is defined as

$$\langle d\rangle_G = \frac{1}{|G|(|G|-1)} \sum_{u,v \in V(G)} d_G(u,v). \tag{8.35}$$

Certainly, a large average distance in a graph $G$ implies that a large number of steps from one vertex to another in $G$. Similarly, the network efficiency of graph $G$ is characterizing the difficulty of passage by distances between vertices on a graph, i.e.,

$$\langle e\rangle_G = \frac{1}{|G|(|G|-1)} \sum_{u,v \in V(G)} \frac{1}{d_G(u,v)}, \tag{8.36}$$

which is the average sum of reciprocal distances between any two vertices in graph $G$.

Similarly, replacing $d_G(u,v)$ by $d_G^L(u,v)$ in (8.34) with (8.36) and 8.35), we can introduce the indexes such as the average distance, network efficiency on a labeled graph or network $G^L$. Certainly, analyzing the coreness (8.25) of a connected graph or network can characterize the centrality of vertices in the graph or network. Similarly, the distance can also be applied to characterize the centrality of a vertex in the network, which is more closely related to the real network as follows:

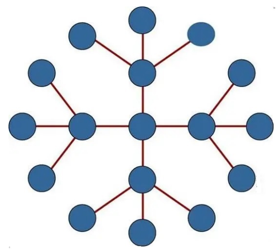


**Figure** 8.3. **Centrality**

**Type 1.** *Average Closeness*, namely the central vertices are such vertices with the closeness to other vertices. Accordingly, the *closeness distribution* of vertex $v \in V(G)$ is determined by

$$C_B^1(v) = \frac{|G|-1}{\sum\limits_{u \in V(G)} d(v,u)}. \tag{8.37}$$

**Type 2.** *Betweenness*, namely the central vertices are such vertices with the maximum sum of betweenness to other vertices. Accordingly, the *betweenness* of a vertex $v \in V(G)$ is determined by

$$C_B^2(v) = \frac{2 \sum\limits_{u,w \in V(G)} |\mathcal{P}(u,w)(v)}{(|G|-1)(|G|-2)|\mathcal{P}(u,w)|}, \tag{8.38}$$

where $\mathcal{P}(u,w)$ is the set of all shortest pathes between vertices $u,w \in V(G)$ and $\mathcal{P}(u,w)(v)$ is all shortest pathes between vertices $u,w \in V(G)$ passing through the vertex $v$.

**Type 3.** *Flow-Betweenness*, namely the central vertices are such vertices with the maximum sum of flow-betweenness to other vertices. Accordingly, the *flow-betweenness* of a vertex $v \in V(G)$ is determined by

$$C_B^3(v) = \sum_{u,w \in V(G)} \frac{|\mathcal{P}'(u,w)(v)}{|\mathcal{P}(u,w)|}, \tag{8.39}$$

where $\mathcal{P}'(u,w)$ is the set of all pathes between vertices $u,w \in V(G)$ and $\mathcal{P}'(u,w)(v)$ is the set of all pathes between vertices $u,w \in V(G)$ passing through the vertex $v$.

Notice that the centrality defined by (8.25) and the three centrality of Types $1-3$ are characterizing the importance of a vertex in a graph or a network and how much influence of one vertex on other vertices in network. So, *how do we compare the different centralities between graphs?* Dr.Ouyang explains that comparing the centralizing degree between graphs leads to introduce the *centralizing degree* $C_A(G)$ of a graph by

$$C_A(G) = \frac{\sum\limits_{v \in V(G)} (C_A^* - C_A(v))}{(|G|-1)\max\left(C_A^* - C_A(v)\right)}, \tag{8.40}$$

where $C_A^* = \max C_A(x), x \in V(G)$ is the maximum centrality. In this case, if the centrality of each vertex in the graph is the same then there must be $C_A(G) = 0$ by (8.38), i.e., there are no center in $G$. For example, let $G$ be an $r$-regular graph. Then, we know that $C_{\rho(G)}(v) = r/(|G|-1)$ for $v \in V(G)$ by (8.25), namely $C_A(G) = 0$ or there are no center in an $r$-regular graph $G$.

## §2. **Random Network**

A graph $G$ is essentially a definite binary relation $E(G) \subset V(G) \times V(G)$ on the vertex set $V(G)$, analogous to that "1 *is* 1 *and* 2 *is* 2" in the integer set $\mathbb{Z}^+$, i.e., an invariable state but each thing is in evolving. So, *how can we characterize the evolving state of a thing by graph?* Dr.Ouyang tells Huizi that a simple case of evolving state in graph is its random change in connecting between vertices $u$ and $v$, i.e., an edge $(u,v) \in E(G)$ of a graph changes with a probability $p$ to two possible results $(u,v) \in E(G)$ or $(u,v) \notin E(G)$. Accordingly, there are two commonly random models to characterize such changes in graphs for two integers $m \geq 0, n \geq 1$. One is the set $\Phi(n,m)$ of all graphs with $m$ edges, $n$ vertices as a probability space with the same probability $p$ for each graph, where

$$p = 1 \Bigg/ \begin{pmatrix} N \\ m \end{pmatrix}, \quad N = \begin{pmatrix} n \\ 2 \end{pmatrix} = \frac{n(n-1)}{2}$$

proposed by Gilbert in 1959. Another is determining the edge connecting between two vertices with the same probability $p$ and characterize the generating process of all graphs with $m$ edges and $n$ vertices similar to that of Benoulli test in which the probability of event A occurring is $p$ and not occurring is $q = 1 - p$ until a graph of order $n$ with size $m$ appeared, i.e., the set $\mathscr{G}(n,m) = \mathscr{G}(n, P(\text{edge}) = p)$ proposed by Eröds and Rényi in 1960, called the *ER random network*. Essentially, these two random graph models are consistent because they only difference are the probability introduced on the perspective.

**Random Graph Model.** *For any integers* $n \geq 1$, $0 \leq m \leq |E(K_n,)| = n(n-1)/2$, *generate a graph in* $\mathscr{G}(n,m)$ *by the programming following:*

**Step 1.** *Select* $n$ *independent vertices arbitrarily;*

**Step 2.** *Select pairs on the* $n$ *vertices step by step and each pair of vertices is selected only once, connecting edges between the selected vertices with probability* $p \in (0,1)$, *where,* $p = 2m/n(n-1)$*;*

**Step 3.** *Repeat Step 2 until a graph* $\mathscr{G}(n,m)$ *with* $m$ *edges produced and the program terminates.*

Dr.Ouyang asks Huizi to think about the difference between the graph $\mathscr{G}(n,m)$ generated by a random graph programming and an abstract graph of order $n$ with $m$ edges.

Huizi thinks for a while and says somewhat uncertainly: "*An abstract graph of order* $n$ *with* $m$ *edge is definite but the the* $m$ *edges in the random graph* $\mathscr{G}(n,m)$ *seems to be uncertain, may or may not appear in a pair of vertices.*" Dr.Ouyang suggests to her: "*This question can be asked in another way, namely is the random graph programming can only generate one graph?*" Huizi then understand Dr.Ouyang's question and replies: "*Of course, there can only be one graph of* $n$ *vertices with* $m$ *edges generated by the random graph programming!*" Dr.Ouyang continuously asks her: "*Then, how does the probability* $p$ *affect when an edge is connected in the random graph programming?*" As Dr.Ouyang asks so, Huizi gets a little confused about $\mathscr{G}(n,m)$ and replies: "*Yes, Dad! There may be or not an edge between two vertices* $u$ *and* $v$ *in the random graph rather than the definite existence of an edge or not as I understand. Then, how do we understand the probability* $p$ *in this programming?*" Dr.Ouyang tells Huizi that why she can not understand the role of probability $p$ in selected pair of vertices $u$ and $v$ of the random graph $\mathscr{G}(n,m)$ is because she views the probability $p$ as 1 or 0, namely there must be or not exist an edge between $u$ and $v$, i.e., as an abstract graph of order $n$ with $m$ edges.

Listening to the explaining of Dr.Ouyang, Huizi was still puzzled and then asks Dr.Ouyang: "*Dad, your explaining doesn't looks right! Is it possible that the random graph programming generates not a graph of* $n$ *vertices with* $m$ *edges but something else? Does the termination condition not generate a graph with* $m$ *edges on* $n$ *vertices?*" Dr.Ouyang explains: "*Yes! The random graph programming really terminates on a graph with* $m$ *edges on* $n$ *vertices generated. However, under the condition that a pair of vertices* $u$ *and* $v$ *are connected according to the probability* $p$*, a graph of order* $n$ *with* $m$ *edges is generated according to Step 3, whether or not* $u$ *and* $v$ *are connected. In this way, the result is not one graph but multiple graphs on* $n$ *vertices, namely* $\mathscr{G}(n,m)$ *is a set of graphs of order* $n$ *with* $m$ *edge.*" At this moment, Huizi seems to comprehend the role of probability $p$ in generating $\mathscr{G}(n,m)$ and says to herself: "*Oh,* $\mathscr{G}(n,m)$ *turns out to be not an abstract graph but a set of graphs of order* $n$ *with* $m$ *edges! But how do we know that the probability of each edge occurring is* $p$ *and not some other values?*" Dr.Ouyang explains, there are total of $n(n-1)/2$ methods for choice vertices $u$ and $v$, a graph in $\mathscr{G}(n,m)$ has $pn(n-1)/2$, i.e., $m$ edges. Conversely, any graph of order $n$ with $m$ edges can be generated in the $n(n-1)/2$ pairs of vertices according to the probability $p = 2m/n(n-1)$ because $p$ satisfies $0 < p \leq 1$, we can obtain

$$\sum_{i=1}^{\frac{n(n-1)}{2}} \left( \begin{array}{c} \frac{n(n-1)}{2} \\ i \end{array} \right) p^i (1-p)^{\frac{n(n-1)}{2}-i} = (p+1-p)^{\frac{n(n-1)}{2}} = 1 \tag{8.41}$$

by the multiplication principle, namely it satisfies the normality. Consequently, there are

$$|\mathscr{G}(n,m)| = \binom{\frac{n(n-1)}{2}}{m} = \binom{\frac{n(n-1)}{2}}{\frac{pn(n-1)}{2}} \tag{8.42}$$

graphs of order $n$ with $m = pn(n-1)/2$ edges in $\mathscr{G}(n,m)$. For example, let $n = 3, m = 2$ and $p = 1/3$. Then, the random graph $\mathscr{G}(3,2)$ contains 3 graphs of order 3 with 2 edges shown in Figure 8.4, which are isomorphic each other.

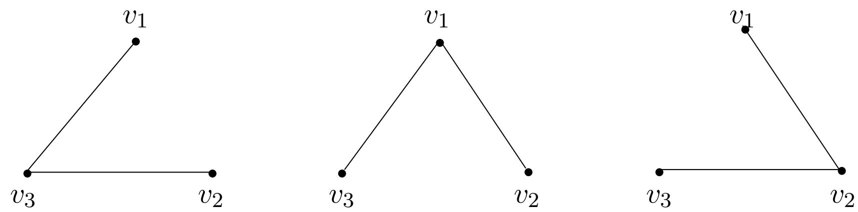



**Figure** 8.4. **Random graph** $\mathscr{G}(3,2)$

Furthermore, the numbers of graphs in random graphs $\mathscr{G}(n,2)$ are listed in Table 8.1 for integers $3 \le n \le 10$, which is enlightening.

| $n$ | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|
| $\lvert\mathscr{G}(n,2)\rvert$ | 3 | 15 | 45 | 105 | 210 | 378 | 630 | 990 |

**Table 8.1. Number of graphs in** $\mathscr{G}(n,2)$

**2.1.**$\mathscr{G}(n,m)$ **character distribution.** The average valency $\langle\rho\rangle_{\mathscr{G}(n,m)}$, average distance $\langle d\rangle_{\mathscr{G}(n,m)}$ and the clustering coefficient $N_3(v)$ of random graph $\mathscr{G}(n,m)$ can be calculated by definition. First of all, a vertex $v$ in the random graph $\mathscr{G}(n,m)$ is adjacent to other vertices $n-1$ with probability $p$ by definition. so, if $n$ is sufficiently large or $n \to \infty$, the average valency

$$\langle\rho\rangle_{\mathscr{G}(n,m)} = p(n-1) \approx pn. \tag{8.43}$$

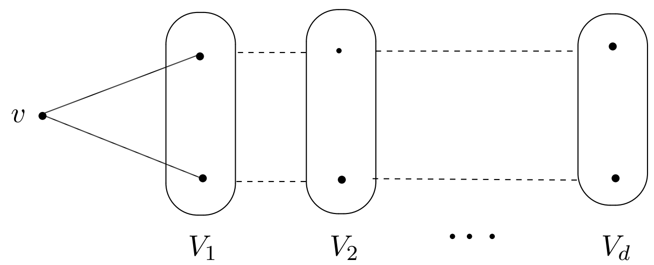



**Figure** 8.5. **Vertex partition on distance**

Secondly, Let $G$ be a graph, $v \in V(G)$. All vertices in vertex set $V(G)\setminus\{v\}$ can be classified into $V_1, V_2, \cdots, V_d$ by the distance of a vertex to $v$, namely the vertices in $V_1$ are adjacent to vertex $v$, the vertices in $V_2$ are 2 away from vertex $v$ and in general, the vertices in $V_d$ are $d$ away from vertex $v$, as shown in Figure 8.5. So, by the definition of random graph, a vertex $v$ of $\mathscr{G}(n,m)$ with adjacent vertices $\langle\rho\rangle_{\mathscr{G}(n,m)}$ in average, with average vertices $\langle\rho\rangle_{\mathscr{G}(n,m)}\langle\rho\rangle_{\mathscr{G}(n,m)} = \langle\rho\rangle^2_{\mathscr{G}(n,m)}$ distance 2 from $v$, with average vertices $\langle\rho\rangle^2_{\mathscr{G}(n,m)} \times \langle\rho\rangle_{\mathscr{G}(n,m)} = \langle\rho\rangle^3_{\mathscr{G}(n,m)}$ distance 3 from $v$, etc., and generally, with average vertices $\langle\rho\rangle^{d-1}_{\mathscr{G}(n,m)}\langle\rho\rangle_{\mathscr{G}(n,m)} = \langle\rho\rangle^d_{\mathscr{G}(n,m)}$ distance $d$ from $v$. Additionally, by definition the average distance between two vertices in $\mathscr{G}(n,m)$ is $\langle d\rangle_{\mathscr{G}(n,m)}$, namely $v$ can be arrived by $\langle d\rangle_{\mathscr{G}(n,m)}$ average steps at a vertex in the random graph $\mathscr{G}(n,m)$. Whence, we know that

$$n \propto \langle\rho\rangle^{\langle d\rangle_{\mathscr{G}(n,m)}}_{\mathscr{G}(n,m)} \quad\Rightarrow\quad \langle d\rangle_{\mathscr{G}(n,m)} \propto \frac{\ln n}{\ln\langle\rho\rangle_{\mathscr{G}(n,m)}}. \tag{8.44}$$

Thirdly, for calculating the cluster coefficient choose any vertex $v$ and in the remaining vertices of $\mathscr{G}(n,m)$, there are $\begin{pmatrix} n-1 \\ 2 \end{pmatrix}$ methods for selecting vertices adjacent to $v$ with probability $p$ in which the probability that two selected vertices are adjacent is $p$. So, there are $p^2 \times p \times \begin{pmatrix} n-1 \\ 2 \end{pmatrix}$ triangles and the cluster coefficient of vertex $v$ in the random graph $\mathscr{G}(n,m)$ is determined by

$$N_3(v) = p^2 \times p \times \begin{pmatrix} n-1 \\ 2 \end{pmatrix} \Big/ p^2 \times \begin{pmatrix} n-1 \\ 2 \end{pmatrix} = p \propto \frac{\langle\rho\rangle_{\mathscr{G}(n,m)}}{n}. \tag{8.45}$$

Fourthly, the valency of random graph $\mathscr{G}(n,m)$ follows the Poisson distribution. Dr.Ouyang explains that by the definition of valency distribution, $P(k)$ is the probability that the valency of a selected vertex $v$ is $k$ in $\mathscr{G}(n,m)$. Notice that there are $\begin{pmatrix} n-1 \\ k \end{pmatrix}$ methods for selecting $k$ vertices adjacent to $v$ in the $n-1$ remaining vertices with probability $p$ and there are $n-1-k$ vertices not adjacent to $v$ in probability $1-p$. Therefore, the probability that a selected vertex $v$ adjacent to $k$ vertices in $\mathscr{G}(n,m)$ is

$$\begin{aligned} P(k|n) &= \begin{pmatrix} n-1 \\ k \end{pmatrix} p^k(1-p)^{n-1-k} \\ &= \frac{(n-1)!}{k!(n-1-k)!}\left(\frac{\mu}{n-1}\right)^k\left(1-\frac{\mu}{n-1}\right)^{n-1-k}, \end{aligned} \tag{8.46}$$

where $\mu = p(n-1)$ by(8.43). Now let $n \to \infty$ we have

$$\begin{aligned} P(k) &= \lim_{n\to\infty} P(k|n) \\ &= \lim_{n\to\infty} \frac{(n-1)(n-2)\cdots(n-1-k)}{(n-1)^k}\frac{\mu^k}{k!} \\ &\quad \times\left(1-\frac{\mu}{n-1}\right)^{n-1}\left(1-\frac{\mu}{n-1}\right)^{-k} = \frac{\mu^k}{k!}e^{-\mu}, \end{aligned} \tag{8.47}$$

i.e., the valency $P(k)$ of random graph $\mathscr{G}(n,m)$ follows the Poisson distribution.

**2.2.Almost All Property.** For a given graph property $\mathscr{P}$, $G \in \mathscr{P}$ denotes that the graph $G$ posses the property of $\mathscr{P}$. Now that for the random graph $\mathscr{G}(n,m)$, if

$$\lim_{n\to\infty} P\left(G \in \mathscr{G}(n,m) : G \in \mathscr{P}\right) \;=\; 1, \tag{8.48}$$

i.e., the probability that graphs in $\mathscr{G}(n,m)$ possing the property $\mathscr{P}$ tends to 1, we say that *almost all graphs* in $\mathscr{G}(n,m)$ poss the property $\mathscr{P}$.

For example, let $k, h$ with $h < k$ be two given natural numbers, $0 < p < 1, q = 1 - p$. Then, almost all graphs in $\mathscr{G}(n,m)$ satisfies: for any $k$ vertex sequence $v_1, v_2, \cdots, v_k$, there must be such a vertex $v$ that $vv_i \in E(G)$ if $1 \leq i \leq h$ and $vv_i \notin E(G)$ if $h+1 \leq i \leq k$ because the probability of a vertex $v \in V(G) \setminus \{v_1, v_2, \cdots, v_k\}$ satisfies the property is $p^h q^{k-h}$. Whence, the probability of a vertex that does not satisfy the property in the vertex set $V(G) \setminus \{v_1, v_2, \cdots, v_k\}$ is $\left(1 - p^h q^{k-h}\right)^{n-k}$. However, there are $\begin{pmatrix} n \\ k \end{pmatrix}$ methods for selecting sequence $v_1, v_2, \cdots, v_k$ in $n$ vertices and $\begin{pmatrix} n \\ k \end{pmatrix}\left(1 - p^h q^{k-h}\right)^{n-k} \to 0$ if $n \to \infty$. So, by definition almost every graph in $\mathscr{G}(n,m)$ must have such a vertex $v$. Similarly, there are also conclusions such as *for a given integer $k$, almost every graph is $k$-connected* and *almost every graph has a diameter of* 2.

Eröds and Rényi found in 1960 that an important property of random graphs is that with $n \to \infty$, most graph property $\mathscr{P}$ emerges suddenly with the probability $p \to 1$, namely almost all graphs have the graph property $\mathscr{P}$. For example, there exists such a threshold $p_c$ for graph connectivity that the probability of connectivity of random graph $\mathscr{G}(n,m)$ tends to 0 if the probability $p \leq p_c$ but it suddenly becomes 1 once $p > p_c$. Furthermore, let $p = c \ln n/n$. Pösa proved in 1976 that *for two vertices $u, v$ in random graph $\mathscr{G}(n,m)$, almost all graphs contain a $u-v$ path $P_n$ if $c > 3$ and if $c > 9$, almost all graphs have further a path $P_n$ between any two vertices in random graph $\mathscr{G}(n,m)$.*

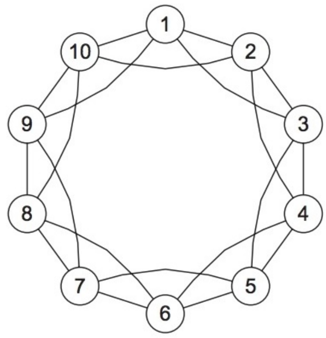


**Figure** 8.6. **Network** $N(10,4)$

**2.3.WS Small-World Network.** A small-world network is a random network operated on a $k$-regular network with a probability $p$, namely reconnecting every edge of the $k$-regular network in such a way that one of its end is fixed but another is adjacent to other vertices with a probability $p$. So, *what is a $k$-regular network?* Dr.Ouyang tells Huizi that a $k$-regular network is a symmetrical $k$-regular graph $N(n,k)$ of order $n$ in which each vertex is connected to its $k$ nearest neighbors, as shown in Figure 8.6 for $n = 10$ and $k = 4$. By definition, the valency distribution of regular network is $P(k) = \delta_{ik}$, namely $P(i) = 1$ if $i = k$, $P(i) = 0$ if $i \neq k$ with an average valency $\langle \rho \rangle_G = k$.

Obviously, the maximum distance in a regular network $N(n, k)$ is $d_{\max} = \frac{n/2}{k/2} = n/k$ and its average distance is $\langle d \rangle_G = d_{\max}/2 = n/2k \propto n$ if $n \to \infty$.

Notice that the *WS small-world network* is introduced by D.J.Watts and S.H.Strogatz in 1998 by defining the reconnecting operation on edges of the regular network $N(n, 2k)$ with a programming following:

**Step 1.** *For any integers $n, k > 0$, let $N(n, 2k)$ be a regular network with each vertex adjacent to $k$ vertices on either side;*

**Step 2.** *For any adjacent vertices of $\{u, v\}$ in $N(n, 2k)$, keep vertex $u$ or $v$ fixed such as $u$ but the other end $v$ disconnected with probability $p$ and then reconnected it to a vertex randomly selected in $N(n, k)$;*

**Step 3.** *Perform Step 2 on every edge in $N(n, 2k)$ until the reconnecting has operated on all edges of $N(n, 2k)$, resulting in a WS small-world network $\mathscr{N}(n, k)$.*

Certainly, a regular network $N(n, k)$ is an abstract graph but the WS small-world network $\mathscr{N}(n, k)$ is a family of graphs with statistical properties following:

(1) **Cluster coefficient.** Clearly, if $p = 0$, $\mathscr{N}(n, k)$ degenerates to a network $N(n, 2k)$ with valency $2k$ of each vertex $v$. In this case, we know that the adjacent vertices of a vertex $v$ are $k$ vertices next to each other in sequence on on either side of $v$. Without loss of generality, let the neighborhood of $v$ be $\left\{v_k^-, \cdots, v_1^-, v_1^+, \cdots, v_k^+\right\}$, as the distribution shown in Figure 8.6. Notice that the number of left edges after removing those edges with one end $v$ in this neighborhood is $(2k-2) + (2k-3) + \cdots + (k-1) = 3k(k-1)/2$. the So, the vertex $v$. So, the cluster coefficient of $v$ is $N_3(v) = 3(k-1)/2(2k-1)$ in $N(n, k)$. Now that if $p > 0$, the triangle formed with the vertex $v$ is still a triangle if the probability $p = 1$. Otherwise, the edge of triangle will be disconnected in probability $p$, i.e., still be an edge of $\mathscr{N}(n, k)$ in probability $1 - p$. Thus, the probability of a triangle in $N(n, k)$ still being a triangle of $\mathscr{N}(n, k)$ is $(1-p)^3$. So, the cluster coefficient of a vertex $v \in V(\mathscr{N}(n, k))$ is

$$N_3(v) = \frac{3(k-1)}{2(2k-1)}(1-p)^3. \tag{8.49}$$

(2) **Average distance.** The average distance of a WS small-world network is

$$\langle d \rangle_G = \frac{2n}{k} \times f\left(\frac{2np}{k}\right), \tag{8.50}$$

where the function $f(x)$ is

$$f(x) = \begin{cases} c & x \ll 1, \\ \dfrac{\ln x}{x} & x \gg 1, \end{cases}$$

for example, let $c = 1/4$, etc.

**(3) Valency distribution.** The valency distribution of WS small-world network is

$$P(i) = \sum_{s=0}^{\min\{i-k,k\}} \begin{pmatrix} k \\ s \end{pmatrix} (1-p)^s p^{k-s} \frac{(kp)^{i-k-s}}{(i-k-s)!} e^{-kp} \quad (i \geq k) \tag{8.51}$$

but $P(i) = 0$ if $i < k$.

Dr.Ouyang tells Huizi that the system recognition of a thing is on its element states with interactions. Certainly, the human's systematic recognition of a thing is a gradual process. For example, the regular network characterizes such a state that the interaction in elements is unchanged, corresponding to the initial recognition of a thing; the random graph characterizes such a state that the interaction in elements is completely random, which is on the assumption that the interaction in elements is complying with random laws. However, the randomness of WS small-world network is between the regular network and random graph, characterizing the state in which the role of elements changes according to a probability, as shown in Figure 8.7.

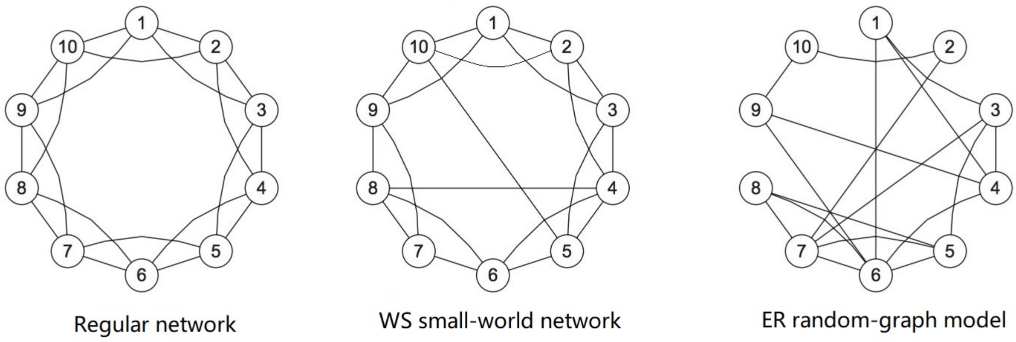



**Figure** 8.7. **Networks** $N(10,4)$**,** $\mathscr{N}(10,2)$ **and** $\mathscr{G}(10,17)$

Among them, the vertices in a regular network are symmetrically connected without any randomness which is similar to the moving of rigid body; the random graph is randomly connected in pair of vertices with a probability $p$ which reflects the complete randomness of connecting edges and the WS small-world network reconnects edges one by one with probability $p$ on a regular network, i.e., local or part randomness. However, the WS small-world network really characterizes a kind of randomly evolving of thing because the state evolving of a thing is mostly random in its part, not the whole.

**§3. Scale-Free Network**

Notice that all systems characterized by the random graphs or the WS small-world networks have a common character, namely the system elements only randomly act on the other elements but the size or elements of system does not change all the time, which is called the "*scale limited*". Dr.Ouyang asks Huizi: "*On the microscopic level, when does this situation happen?*" Huizi replies: "*This situation should be appeared with all the compositions of matters such as the atoms in molecules, protons, neutrons and the electrons in atoms!*" Dr.Ouyang nods and says: "*Yes! This is the conservation law of matter. For example. In a chemical reaction, the atoms in a molecule simply recombine to form a new substance but the atoms involved are neither be produced nor destroyed but only remain in a state that never changes.*" Dr.Ouyang continuously asks Huizi: "*Can we then assume that there are no change occurring in the evolving process of a thing or its elements by the conservation law of matter for the human recognition?*" Huizi replies: "*Of course Not because matter is made up of molecules, the elements must be changing when a chemical reaction happens if we view at the molecular level!*" Dr.Ouyang confirms Huizi's answer: "*You are right! This case appears particularly in biological systems. For example, the human body metabolizes 3.8 million cells every second. So, it is not necessarily correct to assume that the elements do not change in the evolving process. Instead, it is necessary to introduce a kind of network characterizing the changes of elements, which is the situation that a scale-free network hope to characterize.*"

**Figure** 8.8. **Chemical reaction**

So, *what is a scale-free network?* Dr.Ouyang explains that a scale-free network refers to a network in which the number of vertices and edges increases freely without restriction, which is abundant in the actual networks of human society such as the internet, actor cooperative network and researchers cooperative network, etc.

**3.1.Matthew Effect.** The *Matthew effect* refers to a pattern in which "*those who begin with advantage accumulate more advantage over time and those who begin with disadvantage become more disadvantaged over time*". Dr.Ouyang explains that there is a fable about a king who once went on a long journey in *Bible - New Testament: Gospels of Matthew*. It is said that before he left, he gave each of the three servants a mina of silver and asked them to earn money with it and he would reward them with the amount of silver each had earned when he returned. Six months later, the king returned to his palace. The first servant reported to the king: "*My dear master, I have traded with the mina you gave me and have earned ten minas.*" Heard this, the king thought the servant was very smart and awarded the servant ten cities; the second servant reported: "*My dear master, I used the mina you gave me to organize men working and earned five minas.*" The king thought this servant was good too and awarded him five cities and then, the third servant reported: "*My dear master, I worried about losing of the mina that you gave me. So, I hide it in my underwear. Now I can return it to you.*" When the king heard the third report, he was very unhappy and decided to award the mina to the first servant. He then issued a command that "*those who have much will receive more and they will have more than they need. But as for those who don't have much, even the little bit they have will be taken away from them.*" This command later developed into the *Matthew effect*, which reflects the unfair distribution of "*the rich getting more richer and the poor getting more poorer*" and the nature of "*capital profit-seeking*" in an economic society, as shown in Figure 8.9. Certainly, a typical expression of Matthew effect is that the 20% humans hold with the 80% social wealth while 80% humans only hold with the 20% social wealth, as shown in Figure 8.9. Similarly, *why are banks more would like to provide a loan to the rich than to the poor?* Dr.Ouyang explains that is because all banks assume that the rich will not only repay their loans on time but also earn an amount of money to the bank. However, the poor may not even repay the loans, which is another manifestation of the nature of capital profit-seeking.


**Figure** 8.9. **Matthew effect**

R.K.Merton, an American historian of science first presented the term *Matthew effect* in 1968 aiming to an overview the social phenomenon in which famous scientists are generally obtained more credit than those of unknown researchers even if they achieved similar results. Similarly, the reputation on a project is often given to researchers who are already well known. He summed up such a kind of social phenomenon as for any individual, group or region, success and progress in a certain aspect such as the wealth, fame, status, etc., will produce an accumulated advantage, follow by more opportunities to achieve greater success and progress.

**3.2.BA Scale-Free Network.** Then, *how is the Matthew effect reflected in the evolving or changing of network?* Dr.Ouyang explains that the Matthew effect is more common in evolving of the internet. For example, new websites are added to the internet every day and they are usually willing to connect with those well-known websites, i.e., the Matthew effect that "*the rich are getting more richer and the poor are getting more poorer*" reflects in the evolving of network by the valency, namely the valency of a vertex is used to measure the importance of vertex, i.e., the greater valency of a vertex, the more important the vertex is in network evolving and the greater chance of the newly added vertex is as its neighbors in the network evolving. This is the *BA scale-free network* introduced by A-L.Barabasi and R.Alebert in 1999 with a generating programming as follows:

**Step 1.** *Start with a connected network* $G_0$ *of size* $m_0 \geq 1$*;*

**Step 2.** *Add a new vertex* $v \notin V(G_0)$ *according to the probability*

$$P(u) = \frac{\rho(u)}{\sum\limits_{w \in V(G_0)} \rho(w)} \tag{8.52}$$

*connecting with* $m$ *vertices* $u \in V(G_0)$, *where* $1 \leq m \leq m_0$*;*

**Step 3.** *Denote the network obtained in Step 2 by* $G_1$. *Replace* $G_0$ *in Step 1 with* $G_1$ *and repeat Step 2 until a stable evolving state. After adding* $s \geq 1$ *vertices by Step 2, the numbers of vertices and edges in network are respectively* $N = m_0 + s$, $ms + |E(G_0)|$. *For example, let* $G_0$ *be a circuit* $C_{m_0}$ *with* $m_0$ *vertices,* $E(G_0) = m_0$. *Then, adding s vertices to the initial network the edge number of network is* $ms + m_0$.

An example of BA scale-free network evolving process is shown in Figure 8.10. Notice that the BA scale-free network is open and its number of vertices is constantly adding and evolving. Meanwhile, the probability that a vertex is adjacent to a new vertex depends on its valency in the network. This is consistent with the Matthew effect that the rich getting more richer and the poor getting more poorer in social phenomena, i.e., the accumulated advantages of a role.

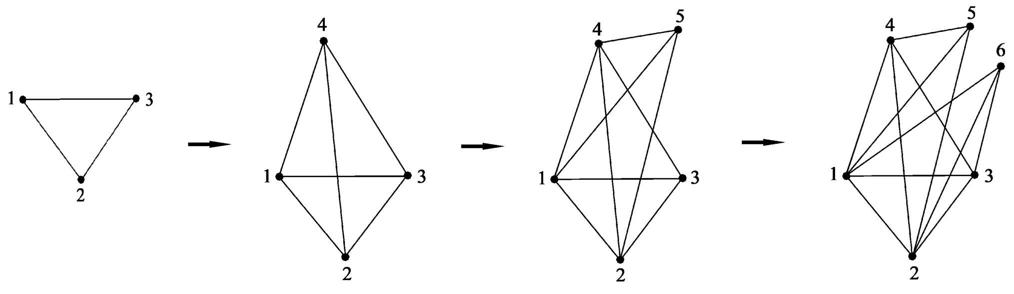



**Figure** 8.10. **A scale-free network evolving**

The statistical properties of BA scale-free network are as follows:

(1) **Cluster coefficient.** After $t$ steps evolving, the average clustering coefficient of

BA scale-free network is

$$N_3(G) = \frac{m^2(m+1)^2}{4(m-1)}\left(\ln\left(1+\frac{1}{m}\right) - \frac{1}{m+1}\right)\frac{\ln^2 t}{t}. \tag{8.53}$$

**(2) Average distance.** The average distance of BA scale-free network is varying on the number of network vertices $N$, i.e

$$\langle d\rangle_G \ \propto \ \frac{\ln N}{\ln\ln N}. \tag{8.54}$$

**(3) Valency distribution.** The average valency $\langle\rho\rangle_G$, the valency distribution $P(k)$ of a BA scale-free network are respectively approximating $2m$ and

$$P(k) \ \propto \ 2m^2k^{-3}. \tag{8.55}$$

Dr.Ouyang tells Huizi that the formula (8.55) could be deduced in the following way. Firstly, for any new vertex $v$ with $\rho_G(v) = k_v$, assume that $v$ is the vertex added in the evolving of step $t_v$ and $k_v$ is a continuous variable. Accordingly, the probability $P(k)$ in Step 2 of the programming is also a continuous variable. By the generating programming of a BA scale-free network, the added vertex $v$ should only be adjacent to the $m$ vertices existing in the network. After that, each new added vertex is adjacent to a vertex according to the importance of vertex $v$ in the network, namely the probability $P(v)$. Thus, the valency varying of vertex $v$ follows the law

$$\frac{\partial k_v}{\partial t} = mP(v) = m \times \frac{k_v}{\sum\limits_{u\in V(G_0)} \rho(u)} \tag{8.56}$$

in the evolving of network and $\sum\limits_{u\in V(G_0)} \rho(u) = 2mt - |E(G_0)|$ at $t$ steps. Furthermore, if $t\to\infty$ or $t$ is sufficiently large, there is $\sum\limits_{u\in V(G_0)} \rho(u) \approx 2mt$, the equation (8.56) then becomes $\partial k_v/\partial t = k_v/2t$. In this case, solving this equation under the initial assumption that the new added vertex $v$ is adjacent to the existing $m$ vertices, i.e., $k_v(t_v) = m$, we get that

$$k_v = m\sqrt{\frac{t}{t_v}} \quad\Rightarrow\quad t_v = \frac{m^2t}{k_v^2}.$$

On the other hand, assume that the new vertices are added in equal time periods, namely the time variable $t_v$ follows the uniform distribution $P(t_v) = 1/(t+m_0)$ and by the probabilistic properties, we know that

$$P(k_v < k) = P\left(t_v > \frac{m^2t}{k^2}\right) = 1 - P\left(t_v \le \frac{m^2t}{k^2}\right) = 1 - \frac{m^2t}{k^2}\cdot\frac{1}{t+m_0}.$$

Let $t \to \infty$. We get that

$$\lim_{t\to\infty} P(k_v(t) = k) = \lim_{t\to\infty} \frac{\partial P(k_v(t) < k)}{\partial k} = \lim_{t\to\infty} \frac{2m^2 t}{t + m_0} \cdot k^{-3} = 2m^2 k^{-3},$$

i.e., the formula (8.55) on the valency distribution of BA scale-free network.

Usually, a random event $\xi$ is said to satisfy the *power law distribution* if its probability distribution $P(k) \propto k^{-\gamma}$. Thus, the valency distribution $P(k)$ of BA scale-free network satisfies the power law distribution by formula (8.55) with $\gamma = 3$.

Furthermore, R.Alebert and A-L.Barabasi introduced reconnecting operation on BA scale-free network in 2000 and extended the BA scale-free network to that of *EBA scale-free network* through by following generating programming:

**Step 1.** *Start with a connected network $G_0$ of size $m_0 \geq 1$;*

**Step 2.** *Add a new vertex $v \notin V(G_0)$ adjacent to $m$ vertices $u \in V(G_0)$ in probability $p$ and in probability*

$$P(u) = \frac{\rho(u)}{\sum\limits_{w \in V(G_0)} \rho(w)} \tag{8.57}$$

*to select the other ends, where $1 \leq m \leq m_0$. And then, randomly select $m$ existing edges, reconnect one of its end to another vertex in probability $q = 1 - p$ and obtain a network denoted by $G_1$.*

**Step 3.** *Replace $G_0$ in Step 1 with $G_1$ and repeat Step 2 until a stable evolving state.*

After $s$ steps adding vertices randomly, the average number of vertices and edges in the EBA scale-free networks are respectively $m_0 + sp$, $ms(p, q) + |E(G_0)|$ and if $p = 1$, i.e., there are no the reconnecting operation, the generating programming of EBA scale-free network is nothing else but the BA scale-free network.

Generally, the valency distribution $P(k)$ of EBA scale-free network satisfies

$$P(k) \propto (k + 1 + A(p, q, m))^{-(1+B(p.q,m))}, \tag{8.58}$$

where

$$A(p, q, m) = (p - q)\left(\frac{2m(1 - q)}{1 - p - q} + 1\right), \quad B(p, q, m) = \frac{2m(1 - q) + 1 - p - q}{m}$$

and $0 \leq p < 1, 1 \leq q < 1 - p$. Meanwhile, it is easy to know that the EBA scale-free network follows the power-law distribution with $2 < \gamma < 3$ by the expression of $B(p, q, m)$.

Notice that the scale-free network model of BA or EBA includes two growth factors, i.e., the vertex growth and the edge optimizing, which is an opened or scale-free complex network. Dr.Ouyang explains that the total number of vertices of the network is increasing and the probability of the new vertex connecting to the existing vertices depends on the valencies of existing vertices in the network, namely the optimal connecting is made by the valencies of existing vertices. And so, different existing vertex optimization methods may result in different scale-free networks. For example, let $\kappa, 0 \leq \kappa \leq 1$ be a constant and the adding new vertex $v \notin V(G_0)$ is adjacent to $m$ existing vertices by the probability

$$P(u) = \frac{(1-\kappa)\rho(u)+\kappa}{\sum\limits_{w\in V(G_0)}((1-\kappa)\rho(w)+\kappa)} \tag{8.59}$$

in Step 2 of the generating programming of BA scale-free network. In this case, as long as $\kappa \neq 0$, the evolving result is a complex network different from that of the BA scale-free network. However, if $\kappa = 0$, the evolving result is the BA scale-free network.

**3.3.Local-World Network.** Certainly, the network adjacency optimization of BA or EBA scale-free network is selecting in the existing vertices and the new added vertex is optimized to connect with the existing vertices according to a probability. So, *can the evolving process be optimized in a subclass of existing vertices?* The answer is Yes! Dr.Ouyang tells Huizi that this is the *local-world network* proposed by X.Li and G.Chen studying the world trade network in 2003. Usually, it is expressed in the multilateral structure of trade in economic developing in which many countries turn to strength their cooperating with the economic organizations in their regions such as the APEC, NAFTA, EU and so on. Generally, a country first chooses the organization and then chooses the countries or regions for cooperating in economic and trade for itself developing, and does not cooperating with the countries or regions outside the organization. For this kind of cooperating, if we denote the countries by vertices, the economic and trade cooperating between countries by edges, then the corresponding network is constantly evolving. Surely, its evolving result is different from that of the BA scale-free network but a local-world network with limited vertex adjacency.

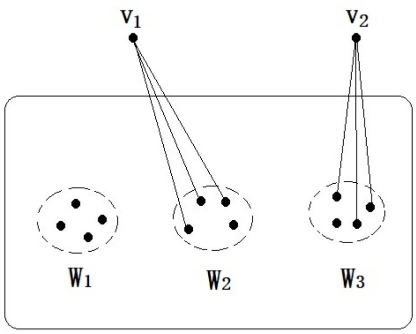


**Figure** 8.11. **Optimial connecting**

**Local-World Model.** *A local-world network evolving through the following two steps:*

**Step 1.** *Start with a connected network $G_0$ with $m_0 \geq 1$ vertices and $e_0$ edges;*

**Step 2.** *Repeats the following (a) and (b) steps in $t$th step evolving, where $t = 0, 1, \cdots$.*

(*a*) *Randomly select subset $W \subset V(G_0), |W| = M \leq m_0$ in existing vertices $V(G_0)$ as the adjacency optimization range of adding new vertices;*

(*b*) *Add a new vertex* $v \notin V(G_0)$ *adjacent to* $m \le e_0$ *vertices* $u \in W$ *with probability*

$$P_W(u) \ = \ \frac{M}{m_0 + t} \frac{\rho(u)}{\sum\limits_{w \in W} \rho(w)} \tag{8.60}$$

*until a stable evolving state, where* $1 \le m \le M$.

Denote the average valency by $k = \langle \rho \rangle_G$. Then, the valency distribution of local-world network can be discussed in the following 3 cases:

**Case 1.** $M = m$, namely the new vertex is adjacent to all vertices in the adjacency preferred range $W$. In this case, the average valency can be characterized by the equation $\partial k/\partial t = m/(m_0 + t)$ and the valency distribution $P(k) \propto e^{-k/m}$ can be obtained similar to that of the BA scale-free network.

**Case 2.** $M = t + m_0$, namely the preferred adjacency range $W$ of new vertices is the entire existing vertices. In this case, the evolving result is a BA scale-free network, the average valency follows $\partial k/\partial t = k/2t$ with the valency distribution $P(k) \propto 2m^2k^{-3}$.

**Case 3.** $m < M < m_0 + t$. In this case, if $M$ is fixed as a constant much larger than $m$ then the valency distribution is $P(k) \propto 2m^2k^{-3}$ similar to that of the BA scale-free network.

Notice that the adjacency probability between a new vertex and the existing vertices is determined by the valencies of existing vertices in a BA scale-free network and the EBA scale-free network is obtained by improving the generating programming of BA scale-free network, namely the new vertex is randomly determined with probability $p$ and the edge is randomly selected in probability $q, 0 \le q < 1 - p$ for reconnecting. Certainly, the local-world network uses an adjacent preferred vertex set $W$ to replace all vertices adjacent to the newly added vertex in network which is a generalization of BA scale-free network. So, *is there such an evolving network that satisfies all characters of the BA, EBA scale-free network and the local world network in evolving?* Dr.Ouyang tells Huizi that the answer is Yes! This is the *multi-local-world network* proposed by G.Chen, X.Li et al in 2005.

**Multi-Local-World Model.** *A multi-local-world network is evolving through the following two steps:*

**Step 1.** *Start with a set composed of* $m$ *disjointed connected subnetworks* $\Omega$, *where each subnetwork* $\Omega$ *has* $m_0 \ge 1$ *vertices,* $e_0$ *edges as a local-world and* $1 \le m_0 \le m, 0 \le e_0 \le m_0(m_0 - 1)/2$;

**Step 2.** *Choose numbers* $0 < q < 1$, $0 < p_1, p_2, p_3, p_4 \le 1$ *and* $p_1 + p_2 + p_3 + p_4 + q = 1$. *Performs one of the following four operations at each evolving step* $t$:

$(a)$ Add vertices and edges. *Randomly select a subnetwork* $\Omega$ *from existing subnetworks in probability* $p_1$. *Similar to the EBA scale-free network, add a vertex* $v$ *according to probability* $q$ *which is adjacent to* $m_1$ *vertices* $u \in \Omega$ *in probability*

$$P_1(u) = \frac{\rho(u) + \alpha}{\sum\limits_{w \in V(G_0)} (\rho(w) + \alpha)}, \tag{8.61}$$

*where* $\alpha > 0$ *is a constant,* $1 \le m_1 \le m$*;*

$(b)$ Connect subnetwork $\Omega$ with other vertices. *A subnetwork* $\Omega$ *is randomly selected in probability* $p_2$ *and then, randomly select an edge whose one end is randomly selected and the other is randomly selected from* $\Omega$ *by the probability determined in* (8.61), *repeat* $m_2$ *times.*

$(c)$ Disconnecting in existing network. *A subnetwork* $\Omega$ *is randomly selected with a probability* $p_3$ *and then, an edge* $(v, u)$ *is randomly selected from existing edges whose one end* $v$ *is randomly selected, the other end* $u$ *is determined in probability*

$$P_2(u) = \frac{1}{|\Omega| - 1}\left(1 - \sqrt{\frac{t}{t_v}}\right) \tag{8.62}$$

*and repeat* $m_3$ *times, where* $t_v$ *is the moment that the vertex* $v$ *is added to the network;*

$(d)$ Connect edges between subnetworks. *Select randomly a subnetwork* $\Omega$ *in probability* $p_4$ *and connect a randomly selected vertex* $v$ *in* $\Omega$ *with probability determined in* (8.61) *adjacent to another vertex* $v'$ *in subnetwork* $\Omega'$ *randomly selected in probability determined by* (8.61). *This connecting process is repeated* $m_4$ *times.*

*Repeat the steps* $(a) - (d)$ *until a stable state obtains in evolving.*

Obviously, the multi-local-world networks cover all characters of the BA scale-free networks, EBA scale-free networks and the local-world network in evolving with a valency distribution $P(k)$ approximately

$$P(k) = \frac{t}{a(3m + t(1 + 2p_1))}\left(m_1 + \frac{b}{a}\right)^{\frac{1}{a}}\left(k + \frac{b}{a}\right)^{-\left(1 + \frac{1}{a}\right)} \tag{8.63}$$

in step $t$, where

$$\begin{aligned}
a &= \frac{qm_1}{c} + \frac{p_2 m_2 (q + m_0 p_1 - p_1)}{(q + m_0 p_1)c} + \frac{p_3 m_3 p_1}{(q + m_0 p_1)c} + \frac{2p_4 m_4}{c}, \\
b &= \frac{q\alpha m_1}{c} + \frac{p_2 m_2}{q + m_0 p_1} + \frac{p_2 m_2 (q + m_0 p_1 - p_1)\alpha}{(q + m_0 p_1)c} \\
&\quad + \frac{p_3 m_3 p_1 \alpha}{(q + m_0 p_1)c} - \frac{2p_3 m_3}{q + m_0 p_1} + \frac{2p_4 m_4 \alpha}{c}, \\
c &= 2(p_1 e_0 + qm_1 + p_2 m_2 - p_3 m_3 + p_4 m_4) + q\alpha_\circ
\end{aligned}$$

Particularly, a few interesting cases of multi-local-world networks are discussed in the following:

**Case 1.** $m = q = 1$, $p_1 = p_2 = p_3 = p_4 = 0$. In this case, there is only one connected subnetwork $\Omega$ and the multi-local-world network degenerates to a BA scale-free network in evolving. Its valency distribution $P(k)$ satisfies the power-law distribution with $\gamma = 3 + \alpha/m_1$.

**Case 2.** $m = 1, p_1 = p_3 = p_4 = 0$. In this case, there is only one connected subnetwork $\Omega$ and the multi-local-world network degenerates to local-world network in evolving. Also, its valency distribution $P(k)$ satisfies the power-law distribution with $\gamma = 3 + \alpha q\,/((m_1 - m_2)q + m_2)$.

**Case 3.** $p_1 = p_2 = p_3 = 0$. In this case, there are a finite number of connected subnetworks $\Omega$ but no edges are added or subtracted in the same subnetwork and a multi-local-world network is evolved. The valency distribution $P(k)$ satisfies the power law distribution with $\gamma = 2 + (m_1 + \alpha)q\,/((m_1 - 2m_4) + 2m_4)$.

**Case 4.** $p_2 m_2 = 2P - 3m_3$. In this case, $b = \alpha a$ and the valency distribution $P(k) \propto (k + \alpha)^{-\gamma}$, which is different from $P(k) \propto k^{-\gamma}$ in BA scale-free network because the new vertex is preferably adjacent to vertices in the subnetwork $\omega$ according to (8.61) in the evolving process of multi-local-world network, where the parameter $\alpha$ can be considered as the correction coefficient on the importance of a vertex in network.

Dr.Ouyang tells Huizi that the essence of recognizing a thing systematically is to hold on the behavior of elements with interactions. This is a simple thing for those of known white systems but it is quite difficult for the grey or black systems because we are not very clear about the role of elements in a grey system or completely not clear about the role of elements in a black system. They can only be speculated or assumed by the possible elements with possible actions between elements according to the external appearance of a thing and then, characterize the possible states of system. It is in this sense that the random networks provide a structural simulation for characterizing the possible states of grey or black systems.

## §4. **Epidemic Network**

Generally, the infectious diseases are a class of diseases caused by pathogens that can spread among humans, animals or humans with animals such as the plague, cholera, hepatitis, AIDS, SARS and the Covid-19, etc., which can be divided into the virus, bacterial, fungal or parasitic infection by different pathogens. For example, being infected with the norovirus can cause the symptoms such as the fever, vomiting, diarrhea, nausea and the abdominal pain. In China, the infectious diseases are divided into three categories, i.e., classes A, B and C. Here, the class A only includes the plague and cholera; the class B includes 26 kinds of infectious diseases such as the SARS, AIDS, viral hepatitis, etc. and the class C includes 11 kinds of infectious diseases such as the influenza and the leprosy. Among them, the infectivity is the main character that distinguishes a infectious disease from the others, namely the pathogen is excreted from the host and transmit to another susceptible human through the transmission routes such as the droplets generated by the coughing, sneezing or speaking, contact transmission, aerosol transmission show the infectivity in the air, in which the number, symptoms of the infected are related to the type, quantity, toxicity of pathogens and immunity of the susceptible. Dr.Ouyang tells Huizi that there are three factors for an infectious disease spreading in a population: the first is the source of infection, namely the human or animal that can expel the pathogen; the second is the transmission chains, i.e., through it the pathogen can spread to others such as respiratory tract, digestive tract, blood or sexual contact, etc.; thirdly, the susceptible population have low immunity to the infectious disease. Certainly, as long as one of the three factors is controlled, the occurrence and spread of the infectious disease can be prevented in the population. This is why the infected population should be quarantined and their living area should be closed in case of pathogen infection for stopping the transmission of pathogen. On the contrary, once an infectious disease spread largely in the population, it will not only affect the living and death of the infected individuals but affect the survival of whole humans, which is a kind of public health event.

[**Cholera Pandemic**] The cholera is an infectious disease caused by the Vibrio cholerae, including hundreds of strains that produce cholera toxin, cause the cells in the human intestinal wall to release more water, result in the diarrhoea and rapid loss of fluid and salt in the human body.

Generally, there are about 80% individuals infected with the bacteria do not develop cholera symptoms, about 20% develop severe symptoms, including diarrhea, vomiting and leg cramps that can cause the dehydration, septic shock and even the death within a few hours.

**Figure** 8.12. **Ganges river**

Since the first outbreak of cholera in Jesore of India in 1817, humans have been fighting against the cholera for more than 200 years but it has never been completely eradicated and always posed a threat to human health. According to the estimation of WHO experts, there are 1.3 million to 4 million individual cases of cholera worldwide each year and about 21,000 to 143,000 deaths.

Some researchers have traced an outbreak of cholera in India to contaminated rice in 1817. So, *how did it spread to other areas after the outbreak?* It was mainly in drinking water and food. Dr.Ouyang explains that the Gangaes river is a big river in India mentioned by Sakyamuni Buddha in his *Diamond Sutra* as a reference. It is the custom of Indians to bury a person in water when he die. In the year of cholera outbreak, the Gangaes river was flooded and the pathogens carried by diseased bodies spread rapidly in the lower of Gangaes river, which contaminated the water and aquatic products such as fishes and shrimps in the river. Thus, it spread rapidly through the human digestive tract. At the same time, the unprotected contacting and the mosquitoes, flies also contributed to the epidemic spreading among individuals. This epidemic affected the whole Indian continent and then, spread to the Bangkok, Thailand, Philippines and other places. In 1821, it was spread to the southeast coast of China, causing a pandemic of cholera in Asia and triggered the reflection of humans and then, carried out the research on cholera and other infectious diseases from the aspects of infectious mechanism, treatment and epidemic prevention, etc.

Huizi asks Dr.Ouyang: "*Dad, healing the wounded and rescuing the dying is the responsibility of a doctor. What does it have to do with the system recognition here?*" Dr.Ouyang tells Huizi that the infectious diseases generally have no available medical records for referring and they should be treated dialectically according to the patient's symptoms. And meanwhile, the treatment of infectious diseases is not only about the patients but also on the source of disease, transmission mode and its possible chains, including person-to-person or animal-to-person transmission and whether it is through water, food, air, contact or other special transmission routes in our daily life so as to implement public policies to block the source of infection, cut off the infecting chain and then, protect the health of more individuals.

**4.1.Warehouse Model.** A fundamental question in the spreading of infectious disease is to determine how the number of infected individuals in a population changes over time. For solving this question, one of the simplest ideas for a particular infectious disease is dividing the population into three groups, namely the susceptible population, infected population and the recovered population. This is the *warehouse model* in infectious disease research, as shown in Figure 8.13. Usually, the number $S(t)$ of susceptible population denotes the number of such individuals that have not been infected by the disease at time $t$ but are at the risk of being infected by the disease at any time, the number $I(t)$ of infected population refers to the number of individuals infected with the disease at time $t$ and the number $R(t)$ of recovered population is the number of such individuals that removed from the infected population, including those who have developed antibodies or died from the disease. Thus, the number of total population is $N(t) = S(t) + I(t) + R(t)$.

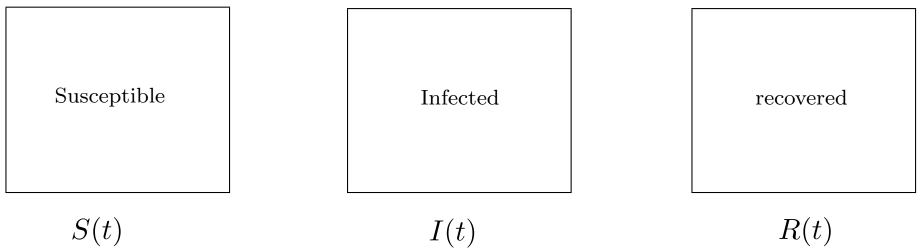



**Figure** 8.13. **Warehouse model**

Generally, we divide $S(t), I(t), R(t)$ by $N(t)$ respectively to obtain the respective ratio or "*density*" of susceptible population, infected population and the recovered population in the total population, which are respectively called the susceptible density $S(t)$, infected density $I(t)$ and the recovered density $R(t)$ with $S(t) + I(t) + R(t) = 1$. Usually, there are three hypothesises following on disease transmission in the warehouse model:

$C1$. *The contact and infection rate is the same in population, namely assume that the susceptible population and the infected population are in the same area, the number of individuals in contact with other individuals per unit time is proportional to the total population with a proportion coefficient $\beta$. And meanwhile, the individuals are uniformly mixed in the population, the probability of contact and infection is the same.*

$C2$. *The cure ability of disease is the same, namely assume that the number of individuals removed from the infected population per unit time is proportional to the total number of infected population with a proportion coefficient $\alpha$.*

$C3$. *The total population remains the same, namely assume that there are no moving in and out individuals, and the new births and deaths would be negligible in the population.*

According to the hypothesis $C3$ if there are enough individuals in each warehouse, the ratio $S(t), I(t)$ and $R(t)$ can be approximately assumed to be continuous differentiable functions of time $t$. So, by the hypothesis $C1$ the probability of random contact between susceptible and infected individuals is $S/N$ and the number of each infected individual infects to others is $\beta NS/N = \beta S$ per unit time. Thus, the number of individuals infected to susceptible individuals per unit time, i.e., the number of reduced susceptible individuals is $I\beta NS/N = \beta SI$. At the same time, it can be considered that the contact probability between the susceptible population and the infected individuals is $I/N$ and the infection probability of each susceptible individual is $(\beta N)I/N = \beta I$. So, the increase in the number of infected individuals is $\beta IS$ per unit time. Now that combining with the hypothesis $C3$, the disease transmission follows the differential equation

$$\begin{cases} \dot{S}(t) = -\beta S(t)I(t), \\ \dot{I}(t) = \beta S(t)I(t) - \alpha I(t), \\ \dot{R}(t) = \alpha I(t) \end{cases} \tag{8.64}$$

called the *SIR model*. Dr.Ouyang explains that different infectious characters can be discussed under the following circumstances:

**SI Model.** In this case, $R(t) = 0$ and there are no curing and death in population, a patient can be recovery by himself or herself. Assume that the initial infectious density is $I_0$. By $S(t) + I(t) = 1$ we know that $S(t) = 1 - I(t)$. The equation (8.64) simplified as the initial value problems of Logistic equation (3.5) following

$$\begin{cases} \dot{I}(t) = \beta(1 - I(t))I(t), \\ I(0) = I_0 \end{cases} \tag{8.65}$$

where $K = 1, F(t) = I(t), r = \beta$. According to the formula (3.7), the solution of (8.65) is

$$I(t) = \frac{I_0 e^{\beta t}}{(1 - I_0) + I_0 e^{\beta t}}. \tag{8.66}$$

Obviously, the infectious density $I(t) \to 1$ if $t \to \infty$, namely everyone will eventually contract the disease.

**SIS Model.** In this case, $R(t) = 0$ and the cure rate is $\delta$, namely the cured patients with the cure rate $\delta I$ per unit time and do not have immunity to the infectious disease, and they are return to the susceptible population. Assume that the infectious density is $I_0$ at the beginning of infection, the SIR equation (8.64) could be reduced to the initial value problem of Logistic equation (3.5) following

$$\begin{cases} \dot{I}(t) = \beta(1 - I(t))I(t) - \delta I(t) \\ I(0) = I_0 \end{cases} \Rightarrow \begin{cases} \dot{I}(t) = \beta\left(1 - \dfrac{\delta}{\beta} - I(t)\right)I(t) \\ I(0) = I_0 \circ \end{cases} \tag{8.66}$$

In this case, $K = 1 - \delta/\beta, F(t) = I(t), r = \beta(1 - \delta/\beta) = \beta - \delta$. Now that by the solution of Logistic equation (3.7) we know that

$$I(t) = \frac{\left(1 - \frac{\delta}{\beta}\right) I_0 e^{(\beta-\delta)t}}{(1 - \frac{\delta}{\beta} - I_0) + I_0 e^{(\beta-\delta)t}} \tag{8.67}$$

and conclusions that ①if $\delta \geq \beta$, namely the cure rate is greater than or equal to the infection rate, then we know that $I(t) \to 0$ by (8.67) if $t \to \infty$, i.e., the infectious disease will be finally eradicated by humans; ②if $\delta < \beta$, namely the cure rate is less than the infection rate, then $I(t) \to 1 - \delta/\beta$ by (8.67) if $t \to \infty$, which implies that there are approximately $(1 - \delta/\beta)N$ of infected individuals in proportion of the population. Particularly, if the cure rate is very low, i.e. $\delta$ is much smaller than the infection rate $\beta$, then $I(t) \approx 1$ if $t \to \infty$, i.e. almost everyone will be infected.

**SIR Model.** In this case, $R(t) \neq 0$ and the recover moves out of the susceptible and infected population. Assume that the susceptible density is $S_0$ and the infectious density is $I_0$ at the beginning of infection. Now that $R(t) = 1 - S(t) - I(t)$, the SIR equation (8.64) is transformed to an initial value problem

$$\begin{cases} \dot{S}(t) = -\beta S(t)I(t), \\ \dot{I}(t) = \beta S(t)I(t) - \alpha I(t), \\ S(0) = S_0,\ I(0) = I_0. \end{cases} \tag{8.68}$$

Usually, the initial value problem (8.68) can not be solved by an elementary method in general but we can qualitatively analyze the trend of susceptible or infectious density, namely whether they tend to stable states if $t \to \infty$. Notice that the equilibrium point of equation (8.68), i.e., the points that let the right side of equation (8.68) equal to 0 are $(S, I) = (S, 0)$, $(\alpha/\beta, I)$ and only such points that $S(t) \geq 0$, $I(t) \geq 0$ are meaningful. By equation (8.68), $\dot{S}(t) < 0$ for any time $t$ and $\dot{I}(t) > 0$ only if $S(t) > \alpha/\beta$. So, ① $\dot{I}(t) < 0$ if $S_0 < \alpha/\beta$ and the infectious density $I(t)$ is a decreasing function on $t$ with $I_\infty(t) = 0$ if $t \to \infty$; ②the infectious density $I(t)$ is an increasing function if $S_0 > \alpha/\beta$ but $\dot{S}(t) < 0$, i.e., $S(t)$ is a decreasing function and turns to $S_\infty$ when $t \to \infty$. In this case, $I(t)$ will eventually decrease as the infected individuals are cured, namely the function $I(t)$ will increase first, reaching the maximum value at $S(t) = \alpha/\beta$ and then decrease to $I_\infty = 0$, i.e., there will be no more infected individuals. This pattern implies that the parameter $\beta S_0/\alpha$ is an important threshold in SIR Model.

Generally, the number

$$\mathscr{R}_0 \ = \ \beta S_0/\alpha$$

is defined to be the *reproduction rate* of infectious diseases. If $\mathscr{R}_0 < 1$, the disease will die by itself but if $\mathscr{R}_0 > 1$, the disease will spread in a large scale of population, which needs the intervention of humans. As the susceptible density $S(t)$ is a decreasing function, the initial value $S_0$ is the maximum but $I(t)$ is in opposite with the minimum at the initial. Thus, we can assume $S_0 \approx 1$ from $S_0 + I_0 = 1$ or consider all infected individuals as the visitors, not from the susceptible population, i.e., $S_0 = 1$ and view the reciprocal $1/\alpha$ of $\alpha$ as the *average duration of illness.*

Adding the two equations in (8.68) and integrating them from 0 to $\infty$, we know that

$$\int_0^\infty \left(\dot{S}(t) + \dot{I}(t)\right) dt = -\alpha \int_0^\infty I(t)dt = S_\infty - (S_0 + I_0) = S_\infty - N. \tag{8.69}$$

Dividing the first equation at both ends by $S(t)$ in (8.68) and integrating it from 0 to $\infty$, we get that

$$\ln \frac{S_0}{S_\infty} = \int_0^\infty \frac{d(S(t))}{S(t)} = \beta \int_0^\infty I(t)dt = \frac{\beta}{\alpha}(N - S_\infty) = \mathscr{R}_0 \left(1 - \frac{S_\infty}{N}\right), \tag{8.70}$$

which gives the relationship of the reproduction rate with that of the infectious disease scale $N(1 - S_\infty)$.

Similarly, by adding the two equations in (8.68), dividing the both ends of the first equation by $S(t)$ and integrating it from 0 to $t$, there are respectively

$$\begin{aligned} \int_0^t \left(\dot{S}(t) + \dot{I}(t)\right) dt &= -\alpha \int_0^t I(t)dt = S(t) + I(t) - N, \\ \ln \frac{S_0}{S(t)} = \int_0^t \frac{d(S(t))}{S(t)} &= \beta \int_0^t I(t)dt = \frac{\beta}{\alpha}(N - S(t) - I(t)) \end{aligned}$$

and we get the relationship

$$S(t) + I(t) - \frac{\alpha}{\beta} \ln S(t) = N - \frac{\alpha}{\beta} \ln S_0 \tag{8.71}$$

of functions $S(t)$ and $I(t)$. Notice that the maximum of infectious density $I(t)$ is reached at $S(t) = \alpha/\beta$. By applying the (8.71), we are easily get that the maximum of infectious density to be

$$I_{\max} = N - \frac{\alpha}{\beta} \ln S_0 - \frac{\alpha}{\beta} + \frac{\alpha}{\beta} \ln \frac{\alpha}{\beta}. \tag{8.72}$$

Usually, the formula (8.71) can also be used for estimating the size of infection rate $\beta$, namely assuming $S_0 = 1, I_0 = 0$ at beginning of the epidemic, then there is

$$S_0 - \frac{\alpha}{\beta} \ln S_0 = \lim_{t\to\infty} (S(t) + I(t)) - \lim_{t\to\infty} \frac{\alpha}{\beta} \ln S = S_\infty - \frac{\alpha}{\beta} \ln S_\infty.$$

So, we have

$$\beta = \alpha \cdot \frac{\ln S_0 - \ln S_\infty}{1 - S_\infty} = -\frac{\alpha \ln S_\infty}{1 - S_\infty}. \tag{8.73}$$

Among them, $S_\infty$ can be estimated through the serum study of susceptible individuals before and after they are infected and then, the infection rate $\beta$ can be obtained by formula (8.73), the reproduction rate $\mathscr{R}_0$ by formula (8.70). Consequently, we can predict the trend of infectious disease by $\mathscr{R}_0 < 1$ or $\mathscr{R}_0 > 1$ and then prevent it spreading in population, including: ①take such measures as the isolating or vaccinating to reduce the initial susceptibility $S_0$; ②reduce the infection rate $\beta$ of an individual with others; ③intensify treatment to reduce the average duration $1/\alpha$ of patients, etc. For example, if the immunization rate of population is $p, 0 \leq p \leq 1$, then the reproduction rate $\mathscr{R}_0$ is reduced from the original $\beta S_0/\alpha$ to $(1-p)\beta S_0/\alpha$.

**4.2.Network Infectious Model.** Notice that the assumption of warehouse infectious model is that the three warehouses, i.e., the susceptible population, infected population and the recovered population are large enough and the mixing of individuals follows the uniform distribution. It must be a large-scale statistical model. However, it is inconsistent with the actual situation of population because everyone only frequently contacts with his familiar relatives, colleagues or friends and the probability of contacting with strangers is very small. In this case, an individual's contacting with others follows an inherent network of relationships, namely the infectious diseases are transmitted through the contacting of person-to-person in the network of humans. Therefore, it is necessary to construct a network infectious model on human relationship network according to the warehouse model of infection, namely the infection model on the relationship network $G^L$ for characterizing the law of disease transmission.

So, *what is the relationship network of humans?* Dr.Ouyang explains that the relationship network $G^L$ is established on the relationship of humans, i.e., a vertex represents a natural human and an edge between two vertices represents the contact relationship between the two natural humans such as the consanguinity shown in Figure 8.14. In this way, the spread of disease in the relationship network is carried out along the edge between vertices, i.e., from one vertex to another vertex with a direction. Certainly, the relationship between humans is constantly changing, not invariant. That is, the relationship network is also changing from time to time, which can be approximately regarded as a random network and divided into the uniform network and non-uniform network for characterizing.

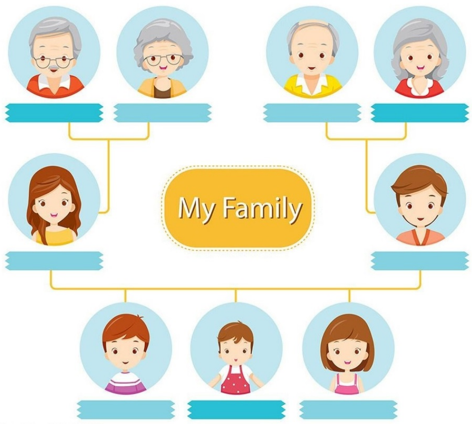



**Figure** 8.14. **Consanguinity**

(1) **Uniform network.** A *uniform network* refers to the exponential decay of the

valency of network vertices near a certain average value, namely the range of value of the valency distribution $P(k)$ is not large and it can be approximately regarded as a complex network that follows the uniform distribution of valency of vertices such as the regular network or small-world network. Dr.Ouyang tells Huizi that the valency of each vertex is approximately equal to the average valency of vertices in the uniform network $G^L$, namely $\rho(v) \approx \langle\rho\rangle_G = \rho$ for $v \in V(G)$. In this case, an infected vertex is adjacent to $\rho$ susceptible vertices on average and the infection probability is $\beta$. Thus, there are $\beta\rho I(t)$ susceptible vertices become infected vertices in each step $t$. Also, assume that an infected vertex is cured with probability $\alpha$, i.e., $\dot{R}(t) = \alpha I(t)$. Notice that $S(t), I(t)$ and $R(t)$ are not independent because of $S(t) + I(t) + R(t) = 1$, i.e., $R(t) = 1 - S(t) - I(t)$. We only need to research the SIR model on $S(t)$ and $I(t)$, i.e., the equation

$$\begin{cases} \dot{S}(t) = -\beta\rho S(t)I(t), \\ \dot{I}(t) = \beta\rho S(t)I(t) - \alpha I(t), \\ S(0) = S_0, \ I(0) = I_0. \end{cases} \tag{8.74}$$

Notice that the spreading process of disease on a uniform network is similar to that of the warehouse model except that the susceptible individuals is not the entire susceptible population but the vertices adjacent to infected nodes. In this case, the equilibrium points of equation (8.76) are $(S, I) = (S, 0)$ and $(\alpha/(\rho\beta), I)$. Certainly, there must be $S(t) > 0$ and $I(t) > 0$ in the actual transmission. In this way, there must be $\dot{S}(t) < 0$ at any time $t$ by equation (8.74), namely $S(t)$ is a decreasing function and $\dot{I}(t) < 0$ if $S(t) < \alpha/(\rho\beta)$; $\dot{I}(t) > 0$ if $S(t) > \alpha/(\rho\beta)$ or $I(t)$ is increase if $S(t) > \alpha/(\rho\beta)$, decrease to $I_\infty = 0$ if $S(t) < \alpha/(\rho\beta)$ and reach the maximum at $S(t) = \alpha/(\rho\beta)$. At this time, the reproduction rate $\mathscr{R}_0 = \beta\rho S_0/\alpha$ by equation (8.74), namely if $\mathscr{R}_0 < 1$, the disease decays to 0 and if $\mathscr{R}_0 > 1$, the disease will transmit to other susceptible individuals, i.e., the human intervention is needed to block the infectious source and cut off the chain of transmission.

Particularly, assuming $\alpha = 1$, namely the infected individuals can be cured in 100% and they immediately return to the susceptible population. In this case, it can be approximately assumed that $R(t) = 0, S(t)+I(t) = 1$ and the infectious density $I(t)$ is determined by the initial value problem

$$\begin{cases} \dot{I}(t) &= \beta\rho I(t)(1 - I(t)) - I(t) \\ I(0) &= I_0 \end{cases}, \tag{8.75}$$

i.e., the case of $\delta = 1$ in (8.66) and replacing $\beta$ with $\beta\rho$. In this case, the solution of

Logistic equation (3.7) is known by

$$I(t) = \frac{\left(1 - \frac{1}{\rho\beta}\right) I_0 e^{(\rho\beta-1)t}}{(1 - \frac{1}{\rho\beta} - I_0) + I_0 e^{(\rho\beta-1)t}}. \tag{8.76}$$

Thus, the threshold of (8.76) can be defined as $\beta_c = 1/\rho$. From the solution (8.76) of equation (8.75), we can know that if $\beta < \beta_c$, the infectious density will decay exponentially until $I_\infty = 0$ and the disease will not spread to other vertices on a large scale. Otherwise, if $\beta > \beta_c$, the disease will spread to the whole network and the infectious density will finally stabilize at $I_\infty = 1 - \beta_c/\beta$.

However, we can not directly determine the infectious density $I(t)$ by using the elementary methods for a large number of SIR or SIS models. Then, *can the trend of disease transmission on the uniform network be determined without solving the equation?* Dr.Ouyang tells Huizi that the equilibrium points of equation (8.75) could be considered for determining the stability of equation solution and get the trend of disease transmission in this case. For example, the equilibrium points of equation (8.75), i.e., the solutions $I(t) = 0$ or $I(t) = 1 - 1/\beta\rho$ of $\beta\rho I(t)(1 - I(t)) - I(t) = 0$. So, if $t \to \infty$, there must be

$$I_\infty = \lim_{t\to\infty} I(t) = \begin{cases} 0, & \beta < \beta_c, \\ 1 - \beta_c/\beta, & \beta \geq \beta_c, \end{cases}$$

we can also get the conclusion that the infectious density $I(t)$ will decay until $I_\infty = 0$ with $t \to \infty$ if $\beta < \beta_c$ and the disease will spread to the whole network on time $t$ with a stabilizer at $I_\infty = 1 - \beta_c/\beta$ if $\beta > \beta_c$.

(2) **Non-uniform network.** A *non-uniform network* refers to the complex network whose valency distribution $P(k)$ of vertices follows the power law or whose valency distribution $P(k)$ has a large value range such as the BA scale-free network or local-world network. In this case, the disease transmission depends on the edges between vertices with transmission efficiency relating to the distribution $P(k)$ of vertices. Certainly, different valency distributions lead to different cases of the disease transmission, as shown in Figure 8.15.

**Figure** 8.15. **Relationship web**

**SIS model.** Firstly, assume that two vertices with different valencies have no influence on each other in the network transmission of disease and each independently spreads according to the infection rate, $\alpha = 1$ and the patient returns to the susceptible population immediately after being cured. It is similar to the SIS equation (8.75) and all vertices of valency $k$ are divided into the susceptible $S_k(t)$, infectious $I_k(t)$ and the recovered $R_k(t)$. Notice that the changing of infectious density $I_k(t)$ is characterized by

$$\dot{I}_k(t) = -I_k(t) + \beta k(1 - I_k(t)) \sum_{k'} P_{k'}(k'|k) I_{k'}(t), \tag{8.77}$$

where $\sum_{k'} P_k(k'|k)$ is the average probability of infection that a vertex of valency $k$ is adjacent to a vertex of valency $k'$. Denoted by $P_k(t) = \sum_{k'} P_{k'}(k'|k) I_{k'}(t)$. Then, the equation (8.77) turns to

$$\dot{I}_k(t) = -I_k(t) + \beta k(1 - I_k(t)) P_k(t). \tag{8.78}$$

Let $\dot{I}_k(t) = 0$ in (8.78). We know that

$$I_k(t) = \frac{k\beta P_k(t)}{1 + k\beta P_k(t)}. \tag{8.79}$$

Dr.Ouyang tells his daughter that for a non-uniform network, the probability $P(k'|k)$ of a vertex of valency $k$ adjacent to a vertex of valency $k'$ is $kP(k)/\langle\rho\rangle_G$. Thus, $P_k(t)$ can be rewritten as

$$P_k(t) = \sum_{k'} P_{k'}(k'|k) I_{k'}(t) = \frac{1}{\langle\rho\rangle_G} \sum_{k=1}^{N} kP(k) \frac{k\beta P_k(t)}{1 + k\beta P_k(t)} \tag{8.80}$$

from (8.79). Substituting the nontrivial solution $P_k(t)$ of (8.80) into (8.79), the susceptible density $I_k(t)$ can be known for integer $k$. Obviously, $P_k(t) = 0$ is a trivial solution of (8.80). For finding the nontrivial solutions of (8.80), differentiate (8.80) at both ends respect to $P_k(t)$ and let $P_k(t) = 0$, we have

$$\frac{d}{dP_k(t)} \left( \frac{1}{\langle\rho\rangle_G} \sum_{k=1}^{N} kP(k) \frac{k\beta P_k(t)}{1 + k\beta P_\infty} \right)\Bigg|_{P_k(t)=0} \ = \ 1|_{P_k(t)=0} \ = \ 1, \tag{8.81}$$

i.e., if the left hand side of (8.81) satisfies

$$\frac{d}{dP_k(t)} \left( \frac{1}{\langle\rho\rangle_G} \sum_{k=1}^{N} kP(k) \frac{k\beta P_k(t)}{1 + k\beta P_k(t)} \right)\Bigg|_{P_k(t)=0} \ > \ 1,$$

such a solution $P_k(t)$ of (8.80) must be a nontrivial solution. In this case, the left hand side of (8.81) is

$$\frac{d}{dP_k(t)} \left( \frac{1}{\langle\rho\rangle_G} \sum_{k=1}^{N} kP(k) \frac{k\beta P_k(t)}{1 + k\beta P_\infty} \right)\Bigg|_{P_k(t)=0} = \sum_{k=1}^{N} \frac{\beta k^2 P(k)}{\langle k\rangle} = \beta \frac{\langle\rho^2\rangle_G}{\langle\rho\rangle_G} \ > \ 1.$$

Notice that the infectious density $I(t) = \sum_k P(k)I_k(t)$ in the network infection model. Define the threshold of disease spreading by

$$\beta_c = \frac{\langle\rho\rangle_G}{\langle\rho^2\rangle_G} \tag{8.82}$$

for scale-free networks. Now if the valency distribution $P(k)$ holds with $P(k) \propto k^{-\gamma}$ and $\gamma \leq 3$, then $\langle\rho^2\rangle_G \geq \sum_k 1/k \to \infty$, i.e., $\beta_c \to 0$ if $N \to \infty$, namely the disease is easy to spread in a large scale of population. For example, let $G^L$ be a BA scale-free network. We have known that

$$\langle\rho\rangle_G = \int_m^\infty kP(k)dk = 2m, \quad P(k) \propto 2m^2k^{-3}.$$

Applying the formula (8.80), we get that

$$P_k(t) \propto m\beta P_k(t)\int_m^\infty \frac{1}{k}\frac{dk}{1+k\beta P_k(t)} = m\beta P_k(t)\ln\left(1+\frac{1}{m\beta P_k(t)}\right),$$

i.e.,

$$P_k(t) = \frac{e^{-1/m\beta}}{m\beta}\left(1-e^{-1/m\beta}\right)^{-1}. \tag{8.83}$$

Thus, the final infectious density $I(t)$ on the network can be known from the formula (8.79), namely the limitation of $I(t)$ when $N \to \infty, t \to \infty$ is

$$\begin{aligned} I_\infty &= \sum_{k=1}^\infty P(k)I_k(t) = 2m^2\beta P_k(t)\int_m^\infty \frac{1}{k^2}\frac{dk}{1+kP_k(t)} \\ &= 2m^2\beta P_k(t)\left(\frac{1}{m}+\beta P_k(t)\ln\left(1+\frac{1}{m\beta P_k(t)}\right)\right) \propto 2e^{-1/(m\beta)} \end{aligned}$$

and $I_\infty = 0$ only if $\beta = 0$, corresponding to the threshold $\beta_c = 0$. Dr.Ouyang explains that the transmission rate $\beta = 0$ occurs only if the disease is a noncommunicable diseases. In this case, if the relationship network is really a BA scale-free network, no matter how small the infection rate is, as long as it outbreak it will exist in the population permanently. This is a frightening conclusion that humans can not cure any infectious disease. Fortunately, the relationship network of humans is not a BA scale-free network, an epidemic disease can be limited in population by changing the structure of relationship network for eradicating infectious diseases.

**SIR model.** Similarly, the vertices of valency $k$ on the network are divided into the susceptible, infected vertices and the recovered vertices, corresponding to density respectively $S_k(t)$, $I_k(t)$, $R_k(t)$ with $S_k(t) + I_k(t) + R_k(t) = 1$, infectious and cure rate are respectively $\beta$ and $\alpha = 100\%$. In this case, the SIR model is similar to the warehouse model, i.e.,

$$\begin{cases} \dot{S}_k(t) = -k\beta S_k(t)I_k(t)P_k(t), \\ \dot{I}_k(t) = k\beta S_k(t)I_k(t)P_k(t) - I_k(t), \\ \dot{R}_k(t) = I_k(t) \circ \end{cases} \tag{8.84}$$

Assume that the initial conditions are $R_k(0) = 0, I_k(0) = I_k^0,\ S_k(0) = 1 - I_k^0$ and approximately $I_k^0 \approx 0, S_k(0) \approx 1$ if $I_k^0 \to 0$. In this case, integrate the first and third equations in (8.84) we know that

$$S_k(t) = e^{-k\beta\phi(t)}, \qquad R_k(t) = \int_0^t I_k(s)ds, \tag{8.85}$$

where the function $\phi(t)$ is

$$\phi(t) = \int_0^t P_k(s)ds = \frac{1}{\langle\rho\rangle_G}\sum_k kP(k)\int_0^t I_k(s)ds = \frac{1}{\langle\rho\rangle_G}\sum_k kP(k)R_k(t).$$

Differentiating on both ends of the previous formula we have that

$$\begin{aligned} \dot{\phi}(t) &= \frac{1}{\langle\rho\rangle_G}\sum_k kP_k(t)\dot{R}_k(t) = \frac{1}{\langle\rho\rangle_G}\sum_k kP_k(t)I_k(t) \\ &= \frac{1}{\langle\rho\rangle_G}\sum_k kP(k)\left(1 - R_k(t) - S_k(t)\right) \\ &= \frac{1}{\langle\rho\rangle_G}\sum_k kP(k) - \frac{1}{\langle\rho\rangle_G}\sum_k kP(k)R_k(t) - \frac{1}{\langle\rho\rangle_G}\sum_k kP(k)S_k(t) \\ &= 1 - \phi(t) - \frac{1}{\langle\rho\rangle_G}\sum_k kP(k)e^{-k\beta\phi(t)}. \end{aligned} \tag{8.86}$$

Notice that (8.86) is an ordinary differential equation on $\phi(t)$. As long as the valency distribution of $P(k)$ is known, we can can get $\phi(t)$ by solving equation (8.86) and known the susceptible, recovered density $S_k(t), R_k(t)$ by (8.85).

Usually, the spreading tend of an infectious disease in population can be analyzed on the final result of the susceptible, infected and the recovered populations by allowing $t \to \infty$. Now, let $\phi_\infty = \lim_{t\to\infty}\phi(t)$, $S_\infty = \lim_{t\to\infty}\sum_k S_k(t)$, $I_\infty = \lim_{t\to\infty}\sum_k I_k(t)$ and $R_\infty = \lim_{t\to\infty}\sum_k R_k(t)$. According to $I_\infty = 0$ and (8.85) we know that

$$R_\infty = \lim_{t\to\infty}\sum_k R_k(t) = \sum_k P(k)\left(1 - e^{-k\beta\phi_\infty}\right) \tag{8.87}$$

and meanwhile, let the right hand side of equation (8.86) equal to 0. We get the equation

$$\phi_\infty = 1 - \frac{1}{\langle\rho\rangle_G}\sum_k kP(k)e^{-k\beta\phi_\infty} \tag{8.88}$$

on $\phi_\infty$. Similar to the discussion of SIS equation (8.75), a non-zero solution of equation (8.88) can be obtained, i.e., let

$$\frac{d}{d\phi_\infty}\left(1-\frac{1}{\langle\rho\rangle_G}\sum_k kP(k)e^{-k\beta\phi_\infty}\right)\Bigg|_{\phi_\infty=0} > 1$$

which results in

$$\frac{1}{\langle\rho\rangle_G}\sum_k P(k)k\beta = \beta\frac{\langle\rho^2\rangle_G}{\langle\rho\rangle_G} > 1$$

and the threshold $\beta_c = \langle\rho\rangle_G / \langle\rho^2\rangle_G$ of SIR model, which can be used to determine the spreading trend of infectious disease and whether it is necessary to take the intervention or quarantine measures for infected individuals. This threshold is the same as that of (8.82) in the SIS model.

## §5. **Social Network**

Surely, a society is a complex system composed of biological individuals, i.e., elements with the characters of social, economic and the cultural relationship owned by individuals. Generally, the elements form a social community under the relationship and then, the social communities form a society also through the relationship. In a society, the spreading of diseases, products, view points or attitudes is strongly dependent on the social network that an individual belongs to, namely the network formed in a region or country according to the social relationship. Here, the biological individuals are vertices in the network $G^L$ and if there is an established social relationship between two individuals, an edge is then connected between the two vertices corresponding to the individuals. Dr.Ouyang tells Huizi that the conception of graph or network is originated in characterizing the behavior of social community. Certainly, the evolving laws introduced in the random networks, WS small-world networks, BA scale-free networks and the multi-local-world networks are all random simulating of the developing and changing behavior of social community.

Usually, a social community behavior has its subjective initiative and is restricted by the individual consciousness to a certain extent. So, it can not be characterized simply by a known network or random network. For example, a social community will continuously add new individuals along with its evolving. Indeed, it has unchanging substructures such as the family relationship of father, son and brother which is similar to the small-world network. But there are also changing relationship in individuals such as the friend relationship, superior or subordinating relationship and the cooperating relationship, etc. Accordingly, the evolving of social community includes the increase of new vertices, the expanding of network scale by connecting new vertices with the existing vertices and the possible operation of disconnecting or reconnecting of existing edges. However, these changes are not random events, nor do they have a unified evolving mode but they are a complex network in a real sense. In this case, applying a fixed graph or a random network to characterize the behavior of social community can be only an approximate simulation. All of these show that although we can systematically characterize the genome of human but the recognizing on the dynamic behavior of biological groups is still in the infancy. For this reason, the social community is related to the individual psychology, culture, customs and the social, economic, geographical characters, different from the invariant of known social substructures such as the clans or blood relations. Certainly, the existing social communities studied by scholars are only a few of small-scale communities with special relationships at a certain time, namely the local or fragment of social communities.

Notice that the vertex set of social community is composed of biological individuals within a certain range. So, the social relationship in individuals must be the condition for establishing the social network. Then, *what is the social relationship?* Dr.Ouyang tells Huizi that the sum of all relationships in individuals of community is called the *social relationship*, including the relationships in individuals, individual with community, community with community, etc. It is the basis for analyzing the community state and interactions, including the intimating, imitating, adapting, assimilating, exchanging, co-operating, competing, conflicting and the coercing. Among them, a community refers to such a community consisting of different individuals, including families, enterprises, social groups, religious or political organizations, etc.

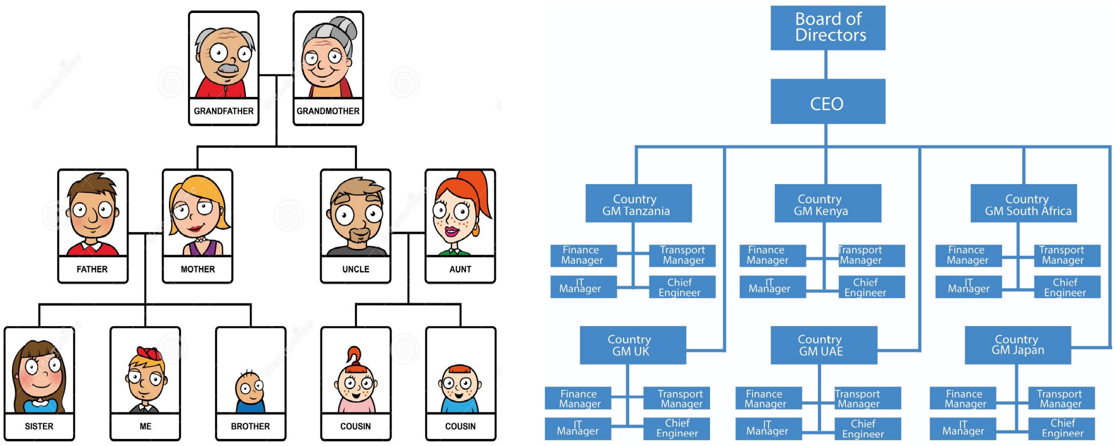



**Figure** 8.16. **Structures in a family or a company**

Generally, the social relationship consists of the blood relationship, geographical relationship and the industrial relationship. Here, the blood relationship refers to marriage, family and the kinship or non-direct blood relationship such as the husband, wife, nephew, uncle, aunt, cousin, son-in-law, brother-in-law and the sister-in-law, etc., as shown in Figure 8.16($a$); the geographical relationship is based on one's living space or geographical location, which is one of the earliest social relationship formed by humans such as the neighborhood, township and the same village, etc.; the business relationship refers to that one belongs exclusively to a certain occupation or organization on the social division of labor, like an enterprise, a political party or church, as shown in Figure 8.16($b$). Usually, for a given social relationship, a *social network* $G^L$ corresponding to a social community must exist and can be constructed on which the behavior of individual and group can be theoretically characterized.

Notice that the social relationship has a cohesive character, i.e., the birds of a feather flock together but humans together in like or preference, namely the individuals with the same interesting will have more communication and interaction, corresponding to a kind of typical subnetwork in which a particular vertex relationship exists in the social community. For example, the "*tree*" or "*clique*" structure often occurs in social network, where the former refers to the network shaped like a tree such as the family and company structure shown in Figure 8.16, the latter refers to a relationship existing in all vertices, corresponding to a complete network $K_n^L$ on complete graph $K_n$ for integer $n \geq 3$. For example, each family has a $K_4^L$ or $K_3^L$ structure in the relationship and there is a friendship or interaction among members of the 4 families $C_1, C_2, C_3, C_4$ shown in Figure 8.17.

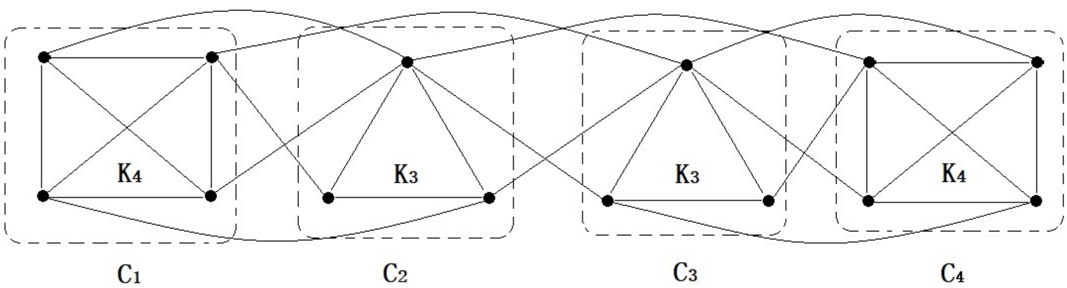



**Figure** 8.17. **Friendship**

**5.1.Community Role.** As we all known the famous words of W.Shakespeare on the role of human in his *As You Like It*, i.e., "*all the world's a stage, and all the men and women merely players; they have their exits and their entrances; and one man in his time plays many parts, his acts being seven ages.*" So, *what are a social role?* Dr.Ouyang explains that the social role is an individual behavior pattern associated with a certain social position, namely the social status and identity that an individual is endowed with in a social community and the functions that this identity should play. Notice that unlike playing on the opera stage, the social role of an individual is the social recognition which was identified in the long-term social practice rather than the braggart of the individual himself.

So, *why are actors classified into the famous and non-famous in social life?* The answer is because of the public's review on the actor's previous performances. Similarly, *why did Martin Luther King wave just his arm and then hundreds of thousands of Americans followed him to fight for the civil rights of African American?* The answer is because Martin Luther King has a high reputation or the recognition among Americans. In this case, *how do we measure and evaluate the different social roles?* Dr.Ouyang tells Huizi that the social community can be used to measure and classify individuals, communities so as to analyze their influence in society.

(1) **Individual influence.** In a social community constructed according to a given social relationship, there are two indexes that can reflect the influence of an individual within the community. Among them, one is the number of individuals adjacent to the individual, reflecting the number of individuals directly affected by the individual; another is the distance from an individual to other individuals, which reflects the difficulty or distance of its influence on other members of the community. Accordingly, there are the coreness and betweenness for characterizing the influence of an individual in a community or society.

**(a) Coreness.** A most direct way for measuring the influence of an individual $v$ in a community is applying the valency $\rho_G(v)$ or other related parameters of vertex $v$ in the corresponding social community $G^L$, including: ①Valency, namely use the valency $\rho_G(v)$ to measure the influence of individual $v$ in community $G^L$, i.e., characterizing the rights, status and the influence distribution of individual $v$ by the valency of $v$ in network $G^L$, corresponding to the valency sequence $\rho_1 \geq \rho_2 \cdots \geq \rho_p, p = |G|$ of network $G^L$. Usually, the higher the valency is, the higher influence of the individual with more effectively control or influence on other individuals in community. Accordingly, the lower the valency is, the more marginalized the individual with less influence to other individuals in community; ②Coreness, namely use the coreness $C_\rho(v)$ of vertex $v$ to measure the influence of individual $v$ in community. Notice that using the valency to measure the influence of an individual ignoring the size $|G|$ of network or community which can only apply to compare the influences of individuals in the same size of communities, namely it is limited to analyze communities of different sizes only from the valency because if two individuals have the same valency in groups of different sizes, then the individual in the large community must have relatively low influence in the common sense. So, it needs to apply the coreness $C_\rho(v) = \rho_G(v)/(p-1)$ in (8.25) rather than the valency $\rho_G(v)$ for measuring the individual influence in communities of different size. For example, the two individuals $v, u$ with $\rho_G(v) = \rho_G(u) = 5$ are in communities of size 11 and 101 respectively. Certainly, their valencies are the same but their coreness are $C_\rho(v) = 0.5$ and $C_\rho(u) = 0.05$ respectively, namely the influence of individual $v$ in community of size 11 is higher than that of individual $u$ in community of size 101.

**(b) Betweenness.** In the social relationship, an individual $v$ directly affects another individual $u$ in social activities is characterized by that there is an edge $(v, u)$ in social community $G^L$ and for these individuals who do not directly influence each other, a usual way is through the intermediaries who spread to and influence on others indirectly. Of course, there may be one or more such intermediaries.

So, *can one individual in community necessarily be established an indirect connection to another through intermediaries?* The answer is Yes to the connected subnetwork in the social network $G^L$! Dr.Ouyang explains that it is finding a path from one individual to another in the corresponding connected subnetwork of social network. Notice that an individual is not exists in isolation, we can assume the social network is connected and then, such a path must exist in theory! Now that the issue needs to further discussing its efficiency, i.e., *does such a path connecting two individuals that needs to go through all other vertices in social network?* Dr.Ouyang tells Huizi that there is a surprising result in the social community, i.e., the average maximum of 6 intermediaries can establish the relationship between two different individuals, which is the *six degrees of separation theory* or the *small-world theory* in sociology.

**Figure** 8.18. **Humans' separation**

**[Separation Experiment]** It is said that a German newspaper accepted a challenge to help a Turkish kebab shop owner finding a connection with his idol, the movie star Malone Brando. It was then discovered that no more than six humans between him and Malone Brando to connect after months of searching by the newspaper's staff. This is the early version of the six degrees of separation theory.

Later, a professor Stanley Milgram in the department of psychology of Harvard university designed a letter sending experiment to study the minimum needs of intermediaries in 1967. His experiment asked for the first one sending a letter to the second one, the second one sending a letter to the third one and so on, until the letter reached the final goal of the experiment. He recruited a group of volunteers from Nebraska and Kansas, and randomly selected 160 of them to complete the mailing experiment. They were asked to mail a letter to a stockbroker living in Boston. Certainly, none of the volunteers knew the stockbroker in Boston and the letters did not go directly to him. Instead, S.Milgram asked the volunteers to send the letter to one of their friends that they thought who were most likely to connect with the stockbroker and asked each one who forwarded the letter also sending a letter back to him. The end result was somewhat surprising, namely most letters reached the stockbroker with handled in an average of 6.2 times. Summarizing the results of this experiment, S.Milgram put forward the theory of six degrees of separation, believing that it needs at most six intermediaries to establish a connection between two humans in the world.

In sociology, it is necessary to pay attention to the propagating path and distance $d_G(v,u)$ of individuals $v,u$ on the social community for studying the diffusion or spreading of diseases, products, view points or attitudes, etc., so as to evaluate the propagation efficiency. Certainly, the six degree of separation theory provides a statistical approximation for measuring the average distance between individuals in population, i.e., $\langle d\rangle_G \approx 6$. In this case, the influence of individual in community can be evaluated from the propagating efficiency, i.e., introducing respectively the average closeness $C_B^1(v)$ defined by (8.37), the betweenness defined by (8.38) and the flow-betweenness defined by (8.39), namely

$$\begin{aligned} C_B^1(v) &= \frac{|G|-1}{\sum\limits_{u\in V(G)} d(v,u)}, \quad C_B^3(v) = \sum_{u,w\in V(G)} \frac{|\mathcal{P}'(u,w)(v)}{|\mathcal{P}(u,w)|} \\ C_B^2(v) &= \frac{2\sum\limits_{u,w\in V(G)} |\mathcal{P}(u,w)(v)}{(|G|-1)(|G|-2)|\mathcal{P}(u,w)|} \end{aligned}$$

and then, to evaluate the influence of an individual $v$ in community by the value of vertex $v$ in sequence $C_{B1}^1 \geq C_{B2}^1 \geq \cdots C_{Bp}^1$, $C_{B1}^2 \geq C_{B2}^2 \geq \cdots C_{Bp}^2$ or $C_{B1}^3 \geq C_{B2}^3 \geq \cdots C_{Bp}^3$.

(2) **Subgroup influence.** A subgroup $H^L$ of social community $G^L$ refers to such an individual community $H^L \prec G^L$ in which two or more individuals are linked together in a certain way for achieving the common goal in society. Generally, a typical character of subgroup is that all of its members have a common, i.e., synchronized goal with the sense of identity, the sense of belonging, an internal structure and the common values in subgroup. Accordingly, the subgroup influence is the influence of a subgroup on other individuals or groups in social community.

($a$) **Subgroup density.** The subgroup density $\rho_{\text{dens}}\left(H^L\right)$ of subgraph $H^L \prec G^L$ characterizes the closeness of individual association in group, which is defined as the ratio between the number of edges of subgroup $H^L$ in the social community $G^L$ and the maximum number of possible edges, i.e

$$\rho_{\text{dens}}\left(H^L\right) = \frac{2\left|E(H^L)\right|}{|H^L|\left(|H^L|-1\right)}. \tag{8.89}$$

Certainly, there must be $0 \leq \rho_{\text{dens}}\left(H^L\right) \leq 1$ for a subgroup $H^L$ of community network $G^L$ by definition (8.89). Particularly, if the density of $H^L$ is 0, the subgroup $H^L$ is a network composed of isolated vertices, namely there are no established relationship in individuals, i.e., any information can not be transmitted in this subgroup in which the efficiency is the lowest; if $\rho_{\text{dens}}\left(H^L\right) = 1$, the subgroup $H^L$ is a complete network $K_s^L$ with $s = \left|H^L\right|$, namely all individuals have been established a connection and the efficiency of transmission information is highest in subgroup. Meanwhile, for a subgroup whose structure is already known, its density can be easily calculated by the definition of (8.89). For example, if the subgroup $H^L$ is a tree $T^L$ of order $n$, then there must be $\rho_{\text{dens}}\left(T^L\right) = 2(|T|-1)/(|T|(|T|-1)) = 2/|T|$ because of $\left|E\left(T^L\right)\right| = |T|-1$.

($b$) **Subgroup influence.** The subgroup influence is the ability that other individuals accept the influence of subgroup, changing the view points and actions themselves in community. For measuring the influence of subgroup, the distance between subgroups should be introduced in social communitys, i.e., ①Subgroup valency $\rho_G\left(H^L\right)$. For a subgroup $H^L \prec G^L$, the subgroup valency $\rho_G\left(H^L\right)$ refers to the edge number of $H^L$ adjacent to other vertices in social community $G^L$, i.e.,

$$\begin{aligned} \rho_G\left(H^L\right) &= \left|E\left(G^L\right)\right| - \left|E\left(H^L\right)\right| - \left|E\left(G^L \setminus H^L\right)\right| \\ &= \left|E_G\left(H^L, G^L \setminus H^L\right)\right|, \end{aligned} \tag{8.90}$$

which is the number of edges between subgroups $G^L$ and $G^L \setminus H^L$. So that if we contract the subgroup $H^L$ to a vertex $v_H$ in the social network $G^L$ and denote the contracted network by $(G/H)^L$, then the subgroup valency $\rho_G\left(H^L\right)$ is the valency of vertex $v_H$ in network $(G/H)^L$, i.e., $\rho_G\left(H^L\right) = \rho_{G/H}\left(v_H\right)$; ②Subgroup distance $d_G\left(H_1^L, H_2^L\right)$. For any two subgroups $H_1^L, H_2^L \prec G^L$, the distance $d_G\left(H_1^L, H_2^L\right)$ between subgroups $H_1$ and $H_2^L$ is defined as the minimum distance between all vertices in $H_1$ and $H_2^L$, i.e.,

$$d_G\left(H_1^L, H_2^L\right) = \min\left\{ d_G(v_1, v_2) \middle| \ v_1 \in V\left(H_1^L\right), v_2 \in V\left(H_2^L\right)\right\}. \tag{8.91}$$

Similarly, if we contract the subgroups $H_1^L, H_2^L$ to vertices $v_H^1, v_H^2$ in the social network $G^L$ and denote the contracted network by $(G/H_1H_2)^L$, the subgroup distance $d_G\left(H_1^L, H_2^L\right)$ is the distance of vertices $v_H^1$, $v_H^2$ in $(G/H_1H_2)^L$, i.e., $d_G\left(H_1^L, H_2^L\right) = d_{G/H_1H_2}\left(v_1, v_2\right)$.

For example, the induced subgroups by individuals $v_4, v_5, v_6$ or $u_5, u_6, u_7, u_8$ are respectively $\langle v_4, v_5, v_6\rangle_G = C_3^L$, $\langle u_5, u_6, u_7, u_8\rangle_G = C_4^L$ in the network shown in Figure 8.19 with subgroup valencies $\rho_G\left(C_3^L\right) = 4$, $\rho_G\left(C_4^L\right) = 5$ and the distance between the two subgroups is $d_G\left(C_3^L, C_4^L\right) = 1$ because there is an edge $(v_6, u_7)$ between the two subgroups.

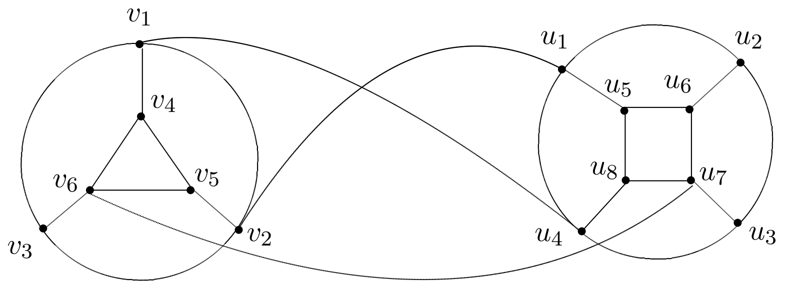



**Figure** 8.19. **Subgroup distance**

In this way, similar to the characterizing of individual influence, the measuring of subgroup influence can apply the subgroup coreness and betweenness, namely use the influence of vertex $v_H$ in the contracted network $(G/H)^L$ as the influence of subgroup $H^L \prec G^L$, i.e., the influence of subgroup $H^L$ in $G^L$ is equal to the influence of vertex $v_H$ in the contracted network $(G/H)^L$. So, the calculating of subgroup influence should replace the vertex $v$ in $G^L$ by the vertex $v_H$ in $(G/H)^L$, replace $d_G(v,u)$ in $G^L$ by $d_{G/H}(v_H, u)$ and meanwhile, the length of path $\mathcal{P}(u,w)$ in contracted network $(G/H)^L$, i.e.,

$$
\begin{aligned}
C_\rho(v_H) &= \frac{\rho_{G/H}(v_H)}{|G/H|-1}, \quad C_B^1(v_H) = \frac{|G/H|-1}{\sum\limits_{u\in V(G/H)} d_{G/H}(v_H,u)},\\
C_B^2(v_H) &= \frac{2\sum\limits_{u,w\in V(G/H)} |\mathcal{P}(u,w)(v_H)}{(|G/H|-1)(|G/H|-2)|\mathcal{P}(u,w)|},\\
C_B^3(v_H) &= \sum_{u,w\in V(G/H)} \frac{|\mathcal{P}'(u,w)(v_H)}{|\mathcal{P}(u,w)|}
\end{aligned}
$$

for obtaining the values $C_\rho(v_H), C_B^1(v_H), C_B^2(v_H)$ and $C_B^3(v_H)$ of subgroup $H^L$, arrange the influence values $C_\rho(v_H), C_B^1(v_H), C_B^2(v_H)$ or $C_B^3(v_H)$ of subgroups $H^L$ in network $G^L$ and then, evaluate the influence of subgroup $H^L \prec G^L$ in social community.

For a real number $k > 0$, the subnetwork $G_{ke}^L$ induced by all vertices of valency not less than $k$ in social community $G^L$ is called the *k-coreness*. Notice that if $k$ is less than the minimum valency $\delta\left(G^L\right)$ of vertices in $G^L$, the $k$-coreness $G_{ke}$ is nothing else but the network $G^L$ itself and if $k$ is more than the maximum valency $\Delta\left(G^L\right)$ of vertices in $G^L$, the $k$-coreness $G_{ke}^L$ is an empty set $\emptyset$, i.e., only if $k$ satisfies $\delta\left(G^L\right) \leq k \leq \delta\left(G^L\right)$, the

$k$-coreness $G_{ke}$ is a nontrivial subnetwork of $G^L$ with the average valency

$$\begin{aligned}\langle\rho\rangle_{G_{ke}} &= \frac{1}{|G_{ke}|}\sum_{v\in V\left(G_{ke}^L\right)}\rho_G(v) = \frac{|G^L|}{|G_{ke}^L|}\sum_{v\in V\left(G_{ke}^L\right)}\frac{\rho_G(v)}{|G^L|}\\ &= \frac{|G^L|}{|G_{ke}^L|}\sum_{i\geq k} iP(i) = \frac{\sum\limits_{i\geq k} iP(i)}{|G_{ke}^L|\,/\,|G^L|} = \frac{\sum\limits_{i\geq k} iP(i)}{\sum\limits_{i\geq k} P(i)}. \qquad (8.92)\end{aligned}$$

**5.2.Random Simulating.** As is known to all, there is a transitive hypothesis on friend relationship in sociology, namely the friend of friend may likely become a new friend which implies that the evolving of social community should obey the law of small-world in some extent. Notice that the word is only "*may*", not the "*must*", namely the hypothesis is not completely confirmed. Dr.Ouyang tells Huizi that the character of social community lies in that it has a certain subjective initiation, can not apply the regular network to characterize and its newly added vertices and edges are not completely random, can not be characterized by a ER random network, namely the social community should be characterized or simulated on the organic combination of the certainty and randomness.

(1) **Small-World Network.** Notice that a regular network has the given numbers of vertices and edges but a WS small-world network $\mathscr{N}(n,k)$ is a local random network obtained by introducing reconnecting operation on existing edges in regular network with a probability $p$, which characterizes the random change of individual adjacency constraint on the community size and adjacency number in probability $p$. Certainly, this is not a common character in social community but it is the first locally random network between the regular network and ER completely random network, which implies that the randomness of social community should lie between the *certainty* and the *complete randomness*.

Clearly, the valency of regular network $N(n,k)$ is a constant $k$ and the valency distribution of WS small-world network $\mathscr{N}(n,k)$ is determined by (8.51), i.e., if $i\geq k$, then

$$P(i) = \sum_{s=0}^{\min\{i-k,k\}}\begin{pmatrix} k\\ s\end{pmatrix}(1-p)^s p^{k-s}\frac{(kp)^{i-k-s}}{(i-k-s)!}e^{-kp}$$

but if $i<k$ then $P(i)=0$. Applying (8.27) and (8.92), we know respectively the average valencies of WS small-world network and its $h$-coreness as follows:

$$\langle\rho\rangle_G = \sum_{i\geq 0} iP(i), \qquad \langle\rho\rangle_{G_{ke}} = \sum_{i\geq h} iP(i).$$

**(2) Scale-Free network.** In order to go beyond the growth limitation of the community size, a BA scale-free network can increase arbitrarily in size. It is suitable for characterizing the situation that new individuals are constantly added in community and tend to establish the relationships with those individuals of great influence, which simulates the real heterogeneity such as the non-uniform, non-balance, complex and the ordered characters of social community to some extent. Its valency distribution follows the power law $P(k) \propto 2m^2k^{-3}$ respectively with the average valencies

$$\langle\rho\rangle_G \propto \sum_k k2m^2k^{-3} = 2m^2\sum_k k^{-2} \propto 3\pi^2m^2$$

itself and

$$\langle\rho\rangle_{G_{ke}} \propto \sum_{k\geq h} k2m^2k^{-3} = 2m^2\sum_{k\geq h} k^{-2} \propto 3\pi^2m^2 - 2m^2\sum_{k=1}^{h-1} k^{-2} < 3\pi^2m^2$$

of its $h$-coreness.

**(3) Multi-Local-World network.** Notice that a multi-local-world network is adding new vertices with optimal selection in existing vertices for connecting and the reconnecting operation on existing edges, which characterizes the size evolving of an open community, optimal connecting of new individuals with existing individuals and the changing or adjusting of existing relationship in individuals. So, it is closer to the real evolving process of community. Certainly, its valency distribution $P(k)$ satisfies the power-law $P(k) \propto k^{-\gamma}$ with a constant $\gamma > 2$. And the average valency

$$\langle\rho\rangle_G \propto \sum_k kk^{-\gamma} = \sum_k k^{1-\gamma}$$

of social community is a convergent series, denoted by $\langle\rho\rangle_G^\infty$. Accordingly, the average number

$$\langle\rho\rangle_{G_{ke}} \propto \sum_k k^{1-\gamma} - \sum_{k<h} k^{1-\gamma} = \langle\rho\rangle_G^\infty - \sum_{k<h} k^{1-\gamma} < \langle\rho\rangle_G^\infty$$

of $h$-cores in community, i.e., $\langle\rho\rangle_{G_{ke}}$ is a finite number with significance in practice.

Dr.Ouyang tells Huizi that the evolving of the real social community does not necessarily follow the established random laws, including the probability of edge connecting between the new vertices and existing vertices, the probability of disconnecting in existing edge and so on, are not constants in general. The reason why they are assumed to be constants in the random models is to simulate and simplify the behavior of social community. So, we know that using random networks to simulate the uncertainty in the evolving of social community is only an approximation to the average evolving law of social community, which confirms the complexity of biological group behavior and the limitations of human recognition once again.

## §6. **Comments**

**6.1.** Certainly, the element behavior of natural system is not all certainty in the eyes of human. In fact, there are a large number of uncertain behaviors of elements, even elements themselves of some natural systems can not be determined, which is the complexity shown in the eyes of human, including the large number of system elements, complex relationship between elements, especially the randomness of elements and the interaction between elements. Generally, it is difficult to discover the deterministic analytical methods for such uncertainty behaviors. Surely, the rise of complex networks was born in such a situtation, comes from the paper [WaS] published in *Nature* by two American physicists Watts and Strogatz in 1998, which introduced the randomness and firstly analyzed the dynamic behavior of small-world networks and also the paper [BaA] published in *Science* by Barabaśi and Albert in 1999, they simulated the Matthew effect, namely "*the rich getting more richer, the poor gettig more poorer*", in which they defined random connecting for a scale-free network and analyzed its dynamic growth mode. It is believed that complex systems can be analyzed and simulated by complex networks, see [CWL] and [HLW].

**6.2.** A random graph is a kind of random model which introduced edge probability between vertices by the Hungarian Eröds and Rényi in 1960, namely the vertex pairs are connected with a constant probability $p$ and the result is not only a graph but a graph family. They found that a lots of graph properties occur to the increase of probability $p$ suddenly, namely there is a threshold $p_c$ so that if $p \leq p_c$ they do not appear but as long as $p > p_c$ they appear immediately, see Bollobás [Bol1] and [Bol2]. Certainly, Watts and Strogatz's small-world network introduces the reconnecting on existing edges of regular network or $m$-regular graph according to a probability $p$, which can characterize the changes of interpersonal relations in social community. The network randomness obtained is between the $m$-regular graph (without randomness) and the random graphs (complete randomness) of Eröds and Rényi. Clearly, the number of vertices in these two types of networks is fixed. But Barabaśi and Albert simulated the growth of internet customers for a scale-free network in which the new adding vertex connects in an optimal probability defined by valencies of existing vertices, without limiting the growth of the network size. A similar idea was used by X.Li and G.Chen et al., who proposed that the local-world network simulated the multilateral trade structure in world, see [CWL] for details.

**6.3.** In the spreading of epidemic disease, the warehouse model divides the population into the susceptible, infected and the recovered population, which is the basis for studying the law of disease spreading. By assuming spreading conditions on the population, 3 models for infectious diseases can be established for studying the law of disease spreading in view of patients' self-recovery, patients with no immunity after cured and those with immunity, namely the SI model, SIS model and the SIR model. Furthermore, a human always has a community in social living. Thus, a disease must be transmitted in one's social network, not any route. It is necessary to study the law of disease transmitting in social networks so as to obtain the SI model, SIS model and SIR model of social network, see [BrC], [ChL], [HLW], [LuW] and [Mur] for details. Similarly, the spreading models of diseases can be also used to characterize the spreading of computer viruses on the internet.

**6.4.** The social network is a complex network composed of social individuals with relationship, which is the most difficult one to simulate by humans because the biggest difference between humans and other animals is that humans have thoughts or subjective awareness. Their behaviors do not completely conform to *Tao* or the natural laws. With that in mind, the random networks, small-world networks, scale-free networks and the local-world networks can all be used to simulate the evolving of social community but they are all characterizing the communities in local, and can not completely characterize the evolving of national or global community until today. Certainly, there are some computing software available for quantitative analysis such local communities or regional social networks, see [Sco]. Similarly, although the intelligence of other animals is lower than that of humans, the developing law of animal social networks are poorly understood by humans and most of them are even blank to humans.

*If winter comes, can spring be far behind?*

– By P.B.Shelley, a British poet.

## Chapter 9

# Network Calculus

********************************************************************

**One Piece, Two Piece, Three and Four Pieces,**

**Five, Six Pieces, Eight, Nine and Ten Pieces;**

**Thousand, Ten Thousand and Countless Snow Pieces,**

**All Flying in Sky Missing into the Plum Blossom.**

— *Ode to Snow By X.Zheng, a poet in Qing dynasty of China*

§1. **Network Reality**

Notice that everything inherits a topological structure shows that the human recognition of a thing should be established on a system recognition over its topological structure, which suggests that the perception of a thing as the fragment or isolate is not the goal of human pursuit in some extent. This fact implies that characterizing the network evolving and then holding on a thing should be a systemic way for recognizing things in the universe. So, *what is a network and what is a network flow?* Dr.Ouyang tells Huizi that the network is an abstraction of "*net*" visible by human eyes in the nature or human society and the network flow is a special labeled graph $G^L$ such as those shown in Figure 9.1, where the *network flow* $G^L$ is with such a labeling $L$ that maps vertices and edges of $G$ into the real number $\mathbb{R}$ or $\mathbb{R}^+$, which is the number characters $L(v)$ and $L(v,u)$ for elements $v, u \in V(G)$ with interactions in $\mathbb{R}$ or $\mathbb{R}^+$. In most cases, the real number $L(v,u)$ is viewed as a continuous flow on edge $(v,u)$, similar to the water flow in that of the urban water supply pipeline and satisfies the conservation laws

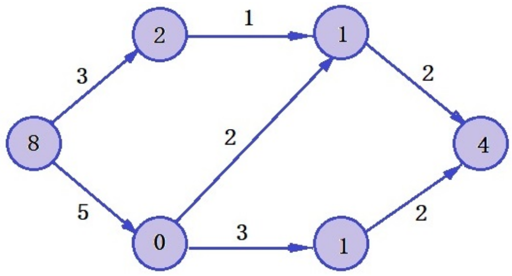


**Figure 9.1. Network flow**

$$\sum_{u\in N_G^-(v)} L(u,v) - \sum_{u\in N_G^+(v)} L(v,u) = L(v) \qquad (9.1)$$

at any vertex or edge crossing section $v \in V(G)$, namely the difference between the inflow and outflow on a vertex or edge crossing section $v$ is equal to the inventory flow on $v$, also known as the *conservation of continuous flow*. For example, the vertex labeled 0 in Figure 9.1 has an input flow of 5 and an output flow of $3+2=5$, i.e., the inventory of vertex 0 is $5-3-2=0$.

Notice that the moving is absolute but the stationary is relative for things in the universe. Generally, a network is the combination of internal structure and the continuous flow moving, which is a model that can simultaneously characterize the inner cause and outer effect of a thing by observing the states of elements at a microscopic level. Hearing this, Huizi is a little puzzling and then asks Dr.Ouyang: "*Dad, what we can see on a network is the flow of continuous flow between elements. What does it have to do with the evolving of a thing?*" Dr.Ouyang tells Huizi that at the microscopic level, every thing is a network composed of elements as vertices and the dynamic action among elements is a continuous flow here. But the network reflects outer effect of a thing itself on the macroscopic level. It is the state reflection of system recognition of the thing. Huizi seems to have understood Dr.Ouyang's ideas and asks: "*Then, does this implies that humans can recognize the universe only if they hold on the networks?*" Dr.Ouyang replies: "*You have just asked a very good question. It's a question that needs to be thought for solving the systemic perception of a thing in general. Certainly, the networks have this function in theory but the network flows, including the complex networks do not correspond to things in the universe but only a few of them! Although the network flow shows theoretically elements with interactions of a thing, it is not certain whether it systematically characterizes the thing itself. For example, the vertices may still represent the local rather than all elements of the thing and what makes one concludes that the interaction in elements of a thing can be characterized by real numbers, not by an array of multi-factors or other non-quantitative feature? The answer is that one is subjectively defined as such when introducing the network flow! Then, why do one define the network flow in such a way? There are two reasons. One is to solve a few of practical problems such as the transportation problems and the Chinese postman problem on the roads, railways and other real transportation networks. Another is to simplify the problem for applying mathematical methods. However, this will inevitably lead to the limitations of few things that can be characterized by the network flow model, which can not correspond to all things in the universe.*" After a short pause, Dr.Ouyang continuously explains that although humans are interested in some economic optimization problems on network flows, the first thing to be clear is what kind of analytical model of network flows can provide, whether it is qualitative or quantitative, local or global characterizing and so on. Only in this way we can effectively play the contribution of networks in the recognition of everything.

**1.1.Network Flow Simulation.** The fact that everything has an inherited combinatorial structure implies that the network can simulate anything in the universe, especially the group state of elements. According to the flow medium on network, things that can be theoretically simulated by the network include: ①Group state, including the transferring process of social relations between individuals, the moving of celestial bodies, material composition, the flow of atoms and microscopic particles in the synthesis and decomposition of material, cell metabolism, energy transferring and conversion such as various social communities, galaxies, material and biological composition, chemical reactions of material synthesis and decomposition,

**Figure** 9.2. **E-commerce logistics**

energy conversion and transferring, etc.; ②Information transmission, including the signal packets, viruses and the rumours spreading, which is not generally complying with the law of energy conservation, including the internet, the radio, television, telephone networks and the wireless transmission channels such as the naturally formed frequency channels and the transmission of hidden or unknown variables in quantum entanglement; ③Matter flow, namely the stable matter composed of atoms or molecules flows in the naturally formed or artificially established channels, which generally complies with the law of energy conservation. For example, the carbon dioxide in the atmosphere, blood in the animal body, the water, current, gas and heat flows in urban systems of water, power, gas, heat supply and the logistics of goods by land, sea and air. For example, a logistics common process of commodity e-commerce is shown in Figure 9.2.

It should be noted that the categories ①-③ of things that network flows can modeling cover almost all of things in the universe. So, *can the network flow model provides a true simulation of things in ①-③?* Dr.Ouyang explains that if the network flows can really provide a model for these things, it is nothing else but an assertion of the consistency of the natural reality of things with that of the reality of networks, which is certainly contradictory to Lao Tzu's assertion in *Tao Te Ching*, namely "*Name named is not the eternal Name*" because a network flow is the *name* of the thing that it simulating. The answer on this question is to analyse whether or under what conditions that network flows can model the evolving states of things in categories ①-③.

Firstly, the labeling $L : V(G)\bigcup E(G) \to \mathbb{R}$ or $\mathbb{R}^+$ in the network flow $G^L$ indicates that the network flow can only simulates the action between elements $v$ and $u$ of things in case that the action $L(v,u)$ can be characterized as a real number, i.e., $L(v,u) \in \mathbb{R}$ or $\mathbb{R}^+$, which actually assumes that the interaction between elements is uniform. As a result, for those things that the interaction between elements are not uniform or can not be expressed as a real number, the results by network flow simulating must be inconsistent with the truth or only an approximation on things. Secondly, *what an action of elements in categories ①-③ is uniform and can be characterized by a real number?* Dr.Ouyang tells Huizi that among the objects of ①, the action of gravity in the moving of celestial bodies, the composition of matter and the strong, weak and electromagnetic forces of atomic nucleus or microscopic particles in the synthesis and decomposition of matter, the metabolic actions of cells, the transfer and transformation of energy are all not uniform. Among these, the gravity and the electromagnetic force are long-range forces, which obey the inverse square law; the strong and weak nuclear forces in nucleus are explained by the theory of meson exchanging and the interaction between cells is a biological force but the mechanism is not clear and can not be characterized by a real number. The action or affection between individuals is determined by individual subjective consciousness in social community, which is only a qualitative but lacks quantitative description, i.e., there are no theory for characterizing an individual subjective consciousness. The only thing we can conclude is that the individual subjective decision is generally subject to multiple factors. For example, there is an elusively subjective behavior of "*love at first sight*" between a boy and a girl shown in Figure 9.3 which is certainly a complex multi-factors rather than a single factor decision, and can not be characterized by a real number. What is the essence of information in category of ② has not been fully revealed in science but only experientially information can bring to mind the various messages of knowledge. Generally, an information is characterized by the amount of information transmission. Dr.Ouyang tells Huizi that there is a game to guess the winner of world cup in which a person not watching the world cup could ask a question to a person watching the world cup but should pay a dollar for each question. So, *what is the minimum amount of dollars the person has to pay for knowing which team is the champion of the world cup?* Surely, Dr.Ouyang explains that one could ask team by team, then this person would have to pay the other one 32 dollars. But, *is that the most economical way to do so?* Of course Not! The most economical way to do is to number the teams respectively by 1-32 and asks: "*Is the champion in teams* $1-16$*?*" If the other one answers "*Yes*" (positive, mark by 1), then continuously asks: "*Is the champion in teams* $1-8$*?*" If the other one answers "*Not*" (negative, mark by 0), then the champion team is in teams $9-16$, asks again: "*Is the champion in teams* $12-16$*?*" So, this person needs to ask five times at most to find out the winner of world cup, i.e., only cost 5 instead of 32 dollars if each team has an equal chance of winning the world cup because $32 = 2^5$. According to this idea, C.E.Shannon, the founder of information theory calculated the logarithm to base 2 and proposed the conception of *information entropy* in 1948, namely the information entropy $H(\xi)$ of any random variable $\xi$ is

**Figure** 9.3. **Love at first glance**

$$H(\xi) = -\sum_x P(x)\log_2 P(x) \tag{9.2}$$

with the unit *bit* or a binary number, where $P(x)$ is the probability when the random variable $\xi = x$. For example, if all teams have the same probability of winning the champion in the game $\xi$ of guessing the winner of world cup, i.e., 1/32, its entropy is

$H(\xi) = -32 \times (1/32)\log_2(1/32) = 5$ bits by the formula (9.2). Now, "*is the information transmission uniform which can be characterized by a real number?*" The answer is certainly Not! For examples of the radio and television, the announcer needs "*cadence*" to broadcast, the actor needs "*vivid and absolutely lifelike*" and "*exciting*" on the stage performance. All the characters imply that the transmission can not be uniform passingly on the channel. Otherwise, the information received by the receiver must be dull and boring, must lose its intrinsic value. Therefore, using network flow to characterize the state of information transmission is only an approximate state and what obtained is a large-scale average state rather than the real face of information transmission. In the category of ③, all models have been established on assumption that the flow of particles or macromolecules composed of particles such as atoms or molecules passes through the channel formed naturally. Certainly, such a kind of models can characterize the flow of these particles or biological macromolecules such as blood, carbon dioxide to some extent. However, they are not uniform flow and can not be characterized by real numbers. Surely, in some artificial designing and constructing flow system of materials such as the urban water, power, heat, gas supply systems, the rainwater, sewage collecting with discharge system as well as human flow and logistics in traffic system, the uniform flow can be achieved through by control of human, and the network flow is used to characterize and simulate in case. Among them, the mileage of road network, carrying capacity and other parameters are often used as the labels on the road network, which results in the "*operation on graph*" to achieve the minimum price per kg.km in transportation to reduce the cost and shorten the time.

Conclusively, although the networks can theoretically characterize all things of categories ①-③ but the network flows can only model mainly on the artificial systems that built and controlled by humans ourselves such as the water, electricity, heat, gas, the rainwater sewage collecting with discharge and the logistics system in traffic network. Indeed, the simulation results of network flow are consistent with the material flows in such artificial systems under control which present the true face of local characters of things. However, if we enlarge the applying fields of network flow for simulating other things in categories ①-③, it is purely an assumption or approximate simulation and can not completely reflects the true face of things. This is the limitation of recognizing things by network flow and implies that it needs to be further extended for human recognition.

**1.2.Operation on Graph.** In order to solve some optimal problems in practice such as the selection of logistics center, searching the shortest transportation road or the maximum traffic volume between two cities, it is more convenient to simulate the logistics system by network flow, which transforms these problems to find respectively the minimum support tree, the shortest distance between two points or the maximum flow on the network flow model. Notice that the traffic volume on any edge $(v, u), v, u \in V(G)$ of network flow $G^L$ in real life is constraint, which must satisfy the conditions of $0 \leq L(v, u) \leq c_{vu}$, called the *permissible capacity* on edge $(v, u)$ usually. Meanwhile, a labeling mapping $L : V(G) \bigcup E(G) \to \mathbb{R}^+$ or flow $L$ is feasible if for any edge $(v, u) \in E(G)$, $0 \leq L(v, u) \leq c_{vu}$. Whence, the essence of optimization on network flow is to obtain the optimization in all feasible flows $L$.

**Figure** 9.4. **Permissible capacity**

**(1) Minimum spanning tree.** In a road network $G^L$, *how do we choose a city as the logistics center for minimizing the transportation distance to each city?* Dr.Ouyang tells Huizi that the essence of this problem is to find a spanning tree $T$ in the network $G^L$ under the condition $0 \leq L(v, u) \leq c_{vu}$ of permissible capacity for $(v, u) \in E(G)$ to minimize the sum $s(T)$ of flows on edges in $E(T)$ and ignore the size of vertex labeling, namely such a combinatorial optimization problem

$$\omega(T) = \min \left\{ s(T) = \sum_{(v,u) \in E(T)} L(v, u) \,\middle|\, T^L \prec G^L \right\} \tag{9.3}$$

that the transportation distance from the logistics center to each city is minimum on network $G^L$ and accordingly, it has an economic significance for the smallest cost and shortest time in transportation.

So, *how do we solve the problem of choosing logistics center?* Dr.Ouyang explains that it is a linear programming problem if combines the (9.3) with the permissible capacities $0 \leq L(v, u) \leq c_{vu}, (v, u) \in E(G)$. Here, the objective function is $\min \omega(T)$ under the constraint $0 \leq L(v, u) \leq c_{vu}, (v, u) \in E(G)$ and $T^L \prec G^L$ is a spanning tree of $G^L$. First of all, we can find out all the spanning trees $T$ of network $G^L$, calculate the sum $s(T)$ of tree and then, find the smallest traffic tree with $\omega(T)$ by 9.3) to get the solution of the problem, which is only feasible for those networks $G^L$ of small scale. So, *is there an easy way to find the minimum spanning tree directly on $G^L$?* The answer is Yes! Dr.Ouyang explains that a spanning tree in a graph is such a tree that contains all vertices of graph but without a circuit by definition. We can directly operate on the graph, namely remove the edge with maximum flow in a circuit of $G^L$ one by one and then obtain the spanning tree with the minimum transportation distance. Usually, this method is called the *breaking-circuits* in operation on graph, namely breaks a circuit if it exists in network until $G^L$ no longer contains a circuit and it become a tree. The breaking-circuits process is as follows:

**Step 1.** *For a connected network $G_0^L = G^L$, go to Step 2 if it is not a tree. Otherwise, go directly to Step 3;*

**Step 2.** *Choosing a circle $C^L$ with the maximum flow of an edge $(v,u)$ in network $G_0^L$, i.e, $L(v,u) = \max\{L(x,y)|(x,y) \in E(C^L)\}$, define $G_1^L = G_0^L \setminus \{(v,u)\}$. Replace $G_0^L$ by $G_1^L$ and back to Step 1;*

**Step 3.** *Program terminates if there are no circuits in $G_s^L$ after breaking-circuits in step $s$, $s \geq 1$.*

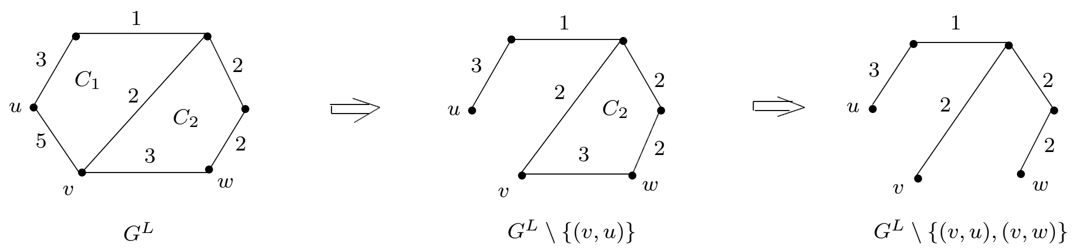



**Figure** 9.5. **Algorithm of minimum spanning tree**

For example, the network $G^L$ in Figure 9.1 contains two circles $C_1, C_2$, as shown in Figure 9.5. For finding the minimum spanning tree on network $G^L$, break the circuit $C_1$ firstly, which removes the maximum flow 5 of $(v,u)$ and gets the network $G^L \setminus \{(v,u)\}$. And then, break the circuit $C_2$ which removes the maximum flow 3 of $(v,w)$ and gets the network $G^L \setminus \{(v,u),(v,w)\}$. Now that the network $G^L \setminus \{(v,u),(v,w)\}$ is already a tree after removing edges $(v,u)$ and $(v,w)$, which is the minimum spanning tree of $G^L$ by the algorithm. This broking-circuit process is shown in Figure 9.5.

**(2) Shortest path problem.** Usually, the shortest path problem is often encountered in the practical work such as the traveling, pipeline laying, line installation, etc., namely in a network $G^L$, *how do we choose the shortest path from one location to another?* Similarly, there are also the shortest time, the greatest benefit or the least investment, etc., which can be all abstracted into the optimal problem

$$\begin{aligned}
\min_{P \in \mathcal{P}(v,u)} \omega(P) &= \min_{P \in \mathcal{P}(v,u)} \sum_{(x,y)\in E(P)} L(x,y), \\
\max_{P \in \mathcal{P}(v,u)} \omega(P) &= \max_{P \in \mathcal{P}(v,u)} \sum_{(x,y)\in E(P)} L(x,y)
\end{aligned} \tag{9.4}$$

for any two vertices $v, u \in V\left(G^L\right)$ in a connected network $G^L$ under the permissible capacity $0 \leq L(x,y) \leq c_{xy}$ for edges $(x,y) \in E(G)$, where $\mathcal{P}(v,u)$ denotes the set of paths between vertices $v, u \in V(G)$. Notice that the minimum or maximum problem in (9.4) is both a linear programming problem combining with the permissible capacities.

Dr.Ouyang asks Huizi: "*What is the minimum or maximum range for the objective function in* (9.4) *here?*" Huizi replies: "*The minimum or maximum range for the objective function in* (9.4) *should be all paths* $\mathcal{P}(v,u)$ *from* $v$ *to* $u$." Dr.Ouyang confirms her answer and says: "*Right! We can get the minimum or maximum* $\omega(P)$ *if we know the set* $\mathcal{P}(v,u)$ *for networks of smaller scale. But in general, determining* $\mathcal{P}(v,u)$ *is not an easy work. For example, how do you find the shortest path from vertex* $v$ *to* $u$ *from step by step in Figure* 9.6*?*" Huizi thinks for a while and answers: "*is it possible to first select a vertex* $v_1$ *with the smallest edge flow in the neighborhood of* $v$ *and then select a vertex* $v_2$ *with the smallest edge flow in the neighborhood of* $v_1, \cdots,$ *so choose step by step until the vertex* $u$ *is selected. Now each step in this selection is a vertex with the smallest edge flow, which should add up to a shortest path from* $v$ *to* $u$."

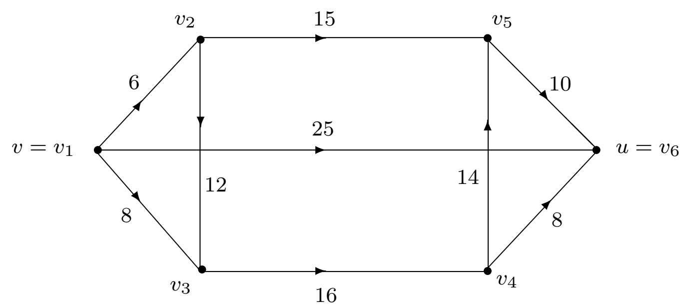



**Figure** 9.6. **Finding the shortest path**

Dr.Ouyang asks Huizi: "*How are you sure that the path chosen is a shortest path from* $v$ *to* $u$*?*" Huizi explains: "*Every time I choose a vertex with the smallest edge flow in neighborhood of the last vertex. Is the sum not the minimum distance with the shortest corresponding path from* $v$ *to* $u$*?*" Dr.Ouyang asks her to determine the shortest path from vertex $v$ to $u$ on the network shown in Figure 9.6. Huizi determines the vertices $v_1, v_2, v_3, u$ step by step by her own idea to get a path $vv_1v_2v_3u$ from $v$ to $u$ with the sum flow 44. Dr.Ouyang then asks her: "*Is you so determined the path from* $v$ *to* $u$ *the shortest path?*" Huizi looks at the network in Figure 9.6 and replies: "*Does it not seem to be the shortest path from* $v$ *to* $u$ *because there is flow* 25 *on an edge connecting vertices* $v$ *and* $u$ *in network, which is smaller than the flow* 44 *on path* $vv_1v_2v_3u$ *but I don't find where it's wrong, Dad!*" Dr.Ouyang tells Huizi that choosing a vertex with the smallest edge flow in its neighborhood as the vertex on the path from $v$ to $u$ only compares the flow size at the incident edges of vertex, not the size of flow sum on all paths from $v$ to $u$. For example, after selecting vertex $v_1$ in the network shown in Figure 9.6, $v_2$ is selected in the neighborhood of $v_1$, $v_3$ is selected in the neighborhood of $v_2$ and $u$ is finally selected. She has no comparison of the flow with other paths from $v$ to $u$ such as $vu$, which naturally results in the obtained path is not necessarily the shortest path from $v$ to $u$. After listening to Dr.Ouyang's explanation, Huizi mutters a little: "*It's me thinking the problem easily myself. It seems that we should calculate it step by step.*" Dr.Ouyang encourages her and says: "*Your idea isn't all wrong. It's correct to choose the minimum edge flow in the adjacent vertices to determine the shortest distance! The only thing that should be added is the comparison of flow sums with all paths from v to u.*"

So, *how do we determine the minimum distance and shortest path from vertices v to u?* Dr.Ouyang tells Huizi that in order to realize the comparison of the flow sum on all paths from $v$ to $u$, there is a operation on graph method, called the *Dijkstra algorithm* or *greedy algorithm* on the vertex with the smallest edge flow among the adjacent vertices which calculates the minimum distance from $v$ to any other vertex $u$ step by step, constructs the vertex set $O$ and $P = V(G) \setminus O$ until $P = \emptyset$. After the termination of this algorithm, the minimum distance $d_{vu}$ with a shortest path from vertex $v$ to each vertex $u$ can be obtained in $G^L$. Steps of this algorithm are listed as follows:

Assume that $G^L$ is a connected network. Label the vertices of $G^L$ by integers 1, 2, $\cdots$, $p = |G|$ and the labels on the starting vertex $v$, the end vertex $u$ are respectively 1 and $p$, the flow on edge $(i, j)$ is $L(i, j)$.

**Step 1.** *Let $O = \{1\}$, $P = \{1, 2, \cdots, p\} \setminus O = \{2, 3, \cdots, p\}$ initially and the initial values $u_{11} = 0$, $d_{1j} = L(1, j)$ for any vertex $j \in N_G(1)$. Otherwise, $d_{1j} = \infty$;*

**Step 2.** *Find a vertex $k$ in vertex set $P$ such that $d_{1k} = \min\{d_{1j}, j \in P\}$, replace set $O$ with set $O \bigcup \{k\}$ and set $P$ with set $P \setminus \{k\}$. If $P \neq \emptyset$ then go to Step 3. Otherwise, if $P = \emptyset$ then go Step 4;*

**Step 3.** *For any vertex $j \in P$ modify its number $d_{1j}$ with a rule that if $d_{1j} > d_{1k} + L(k, j)$, then use $d_{1k} + L(k, j)$ replacing $d_{1j}$ and if $d_{1j} \leq d_{1k} + L(k, j)$, then $d_{1j}$ and $R$ do not change. Return to Step 2 for selecting a vertex in set $P$;*

**Step 4.** *Repeat Steps 2-3 until it terminates at $P = \emptyset$.*

In this way, the vertex $k$ determined at each step in the greedy algorithm, including node $p$ is the vertex with the smallest distance from vertex 1 and the shortest distance from all vertices to vertex 1 and the shortest path from vertex 1 to $p$ are obtained when the algorithm terminates. For example, a shortest distance from the node $v$ to $u$ in Figure 9.6 is calculated step by step as follows:

①$P = \{v_2, v_3, v_4, v_5, v_6\}, d_{12} = 6, d_{13} = 8, d_{14} = d_{15} = \infty, d_{16} = 25 \to$ ②$P = \{v_3, v_4, v_5, v_6\}, d_{13} = 8, d_{14} = \infty, d_{15} = 21, d_{16} = 25 \to$ ③$P = \{v_4, v_5, v_6\}, d_{14} = 24, d_{15} = 21, d_{16} = 25 \to$ ④$P = \{v_4, v_6\}, d_{14} = 24, d_{16} = 25 \to$ ⑤$P = \{v_6\}, d_{16} = 25$, namely the shortest distance from $v$ and $u$ is 25 and meanwhile, the shortest path from $v$ to $u$ is $vu$

by reversing steps.

Now, removing the edge $(v,u)$ from the network in Figure 9.6 to get a new network $G^L \setminus \{(v,u)\}$. In this case, according to the greedy algorithm, the shortest distance between node $v$ and $u$ in $G^L \setminus \{(v,u)\}$ can be determined step by step as follows:

①$P = \{v_2, v_3, v_4, v_5, v_6\}$, $d_{12} = 6$, $d_{13} = 8$, $d_{14} = d_{15} = d_{16} = \infty \to$ ②$P = \{v_3, v_4, v_5, v_6\}, d_{13} = 8, d_{15} = 21, d_{14} = d_{16} = \infty \to$ ③$P = \{v_4, v_5, v_6\}, d_{14} = 24, d_{15} = 21, d_{16} = \infty \to$ ④$P = \{v_4, v_6\}, d_{14} = 24, d_{16} = 31 \to$ ⑤$P = \{v_6\}, d_{16} = 31$, namely the shortest distance between $v$ and $u$ is 31. Notice that $d_{16} = 31$ in ④ is obtained by $d_{16} = d_{15} + L(v_5, v_6)$ and $d_{15} = 21$ is obtained by $d_{15} = d_{12} + L(v_2, v_5)$. So, we know that $vv_2v_5u$ is a shortest path from $v$ to $u$ in $G^L \setminus \{(v,u)\}$.

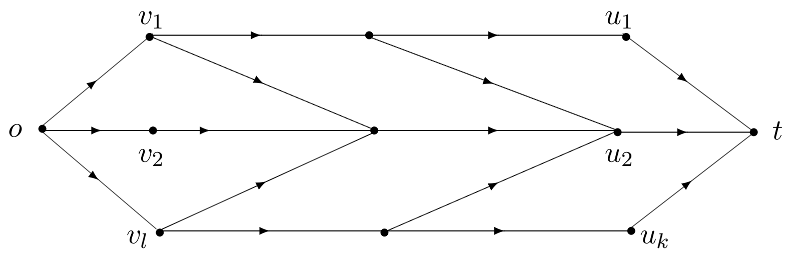



**Figure** 9.7. **Maximum flow problem**

(3) **Maximum flow problem.** Usually, the capacity limitation of road network, pipeline, carrier and the condition of flow conservation on vertex show that it is a valuable problem to discuss how to maximize the carrying capacity from one vertex $o$ to another $t$ on the network $G^L$ constraint with the device capacity. In this case, if $f$ is a feasible flow on the network $G^L$ and all inventories of intermediate vertices are 0, namely the inflow is equal to the outflow at an intermediate vertex, then the outflow $v_o^+(f)$ of vertex $o$ is the same as the inflow $v_t^-(f)$ of vertex $t$, i.e.,

$$v_o^+(f) = \sum_{v \in N_G^+(o)} L(o,n) = \sum_{u \in N_G^-(t)} L(u,t) = v_t^-(f), \qquad 9.5)$$

where the vertex $o$ is called the *source* and $t$ is called the *sink* of network, as shown in Figure 9.7. Thus, the maximum flow problem on network $G^L$ is equivalent to a linear programming problem

$$\begin{cases} \max v_o^+(f) = \max v_t^-(f), \\ v \neq s,t \ \Rightarrow \ \sum\limits_{u \in N_G^-(v)} f(u,v) = \sum\limits_{w \in N_G^+(v)} f(v,w), \\ (x,y) \in E\left(G^L\right) \ \Rightarrow \ 0 \leq f(x,y) \leq c_{xy}, \end{cases} \qquad (9.6)$$

where $f(x,y)$ and $c_{x,y}$ are respectively the feasible flow $f$ on edge $(x,y)$ and the upper permission of flow capacity.

So, *how do we solve the dynamic programming problem* (9.6)*?* Dr.Ouyang tells Huizi that it is possible to directly solve the linear programming problem (9.6) in theory but if the scale of network $G^L$ is large, directly solving (9.6) involves a large amount of calculation which is a very tedious thing and needs to be simplified according to the characters of maximum flow problem.

First of all, divide all vertices in $G^L$ into $V_1$ and $V_2$ so that $V\left(G^L\right) = V_1 \bigcup V_2,\ V_1 \bigcap V_2 = \emptyset$ with the source $o \in V_1$ and sink $t \in V_2$. Denote by $E_G\left(V_1, V_2\right)$ the set of all edges with one vertex in $V_1$ and another in $V_2$. It is easy to know that $G^L \backslash E_G\left(V_1, V_2\right)$ is a disconnected network. Generally, the edge set $E_G\left(V_1, V_2\right)$ is called a *cut set* on $G^L$, as shown in Figure 9.8; Secondly, for a feasible flow $f$ on network $G^L$, denote by $E_G^+\left(V_1, V_2\right)$ the edges from a vertex $v_1$ of $V_1$ to a vertex $u_2$ of $V_2$ in $E_G\left(V_1, V_2\right)$ and let the flow on such an edge $(v_1, u_2)$ be $f^+(v_1, u_2)$, denote by $E_G^-\left(V_1, V_2\right)$ the edges from one vertex $v_2$ of $V_2$ to a vertex $u_1$ of $V_1$ and let the flow on edge $v_2, u_1$ be $f^-(v_2, u_1)$. Then, there must be

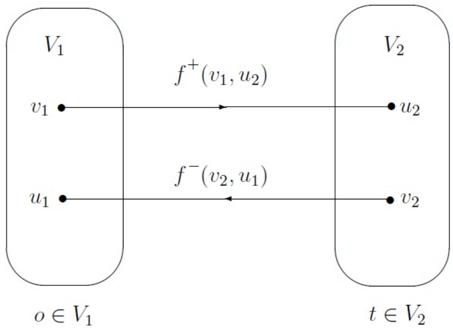



**Figure** 9.8. **Cut set**

$$E_G\left(V_1, V_2\right) = E_G^+\left(V_1, V_2\right) \bigcup E_G^-\left(V_1, V_2\right), \quad E_G^+\left(V_1, V_2\right) \bigcap E_G^-\left(V_1, V_2\right) = \emptyset$$

Notice that for any feasible flow $f$ on $G^L$, the flow on an edge in cut set $E_G\left(V_1, V_2\right)$ may go from a vertex in $V_1$ to a vertex in $V_2$ with the same direction as the source $o$ to the sink $t$, but may also be from a vertex in $V_2$ to a vertex in $V_1$ with the opposite direction from the source $o$ to the sink $t$. Their flow difference between is properly the actual flow from the source $o$ to the sink $t$, called the *capacity of cut set* $E_G\left(V_1, V_2\right)$ $\operatorname{Cap}\left(V_1, V_2\right)$, defined by

$$\operatorname{Cap}\left(V_1, V_2\right) = \sum_{(v_1, u_2) \in E_G^-(V_1, V_2)} f^-(v_1, u_2) - \sum_{(v_2, u_1) \in E_G^+(V_1, V_2)} f^+(v_2, u_1). \tag{9.7}$$

For example, the edge set $\{(v_1, v_6), (v_2, v_5), (v_3, v_4)\}$ is a cut set for vertex set $V_1 = \{v_1, v_2, v_3\}$, $V_2 = \{v_4, v_5, v_6\}$ in Figure 9.6. Its capacity $\operatorname{Cap}\left(V_1, V_2\right) = 15 + 25 + 16 = 56$.

Then, *what is the relation of maximum flow* $\max v_o^+(f)$ *of network with the capacity of cut set?* Dr.Ouyang explains that for a feasible flow $f$ on network flow $G^L$, it is easy to see that removes any cut set of network $G^L$ will result in the disconnection of source $o$ with the sink $t$, i.e., the source $o$ and the sink $t$ are no longer connected by definition. Notice that any feasible flow $f$ is through from $o$ to $t$, namely its flow will not exceed the capacity of any cut set $E_G(V_1, V_2)$ between the source $o$ and sink $t$, i.e., $v_o^+(f) \leq \mathrm{Cap}\,(V_1, V_2)$. In 1956, L.R.Ford and D.R.Fulkerson, the founders of network flow theory further proved the *maximum flow and minimum cut theorem*, i.e., *the maximum flow* $\max v_o^+(f)$ *is equal to the minimum capacity* $\min \mathrm{Cap}\,(V_1, V_2)$ *on cut set of network* $G^L$. According to this conclusion, solving the linear programming problem (9.6) is transformed to calculating the minimum capacity of cut set of network $G^L$, namely starting from a feasible flow $f$ seeks the augmenting path of this feasible flow, i.e., increasing $f^-(v_1, u_2)$ or decreasing $f^+(v_2, u_1)$ under the permission capacity to increase the capacity of cut set $E_G(V_1, V_2)$ by adjusting $f$ to a feasible flow $f'$ with a larger flow than the original feasible flow and the adjusting feasible flow still satisfies the conservation condition at each intermediate vertices. In this way, it iterates step by step until no longer exists an augmenting path from $o$ to $t$ in $G^L$ and then, the maximum flow of network $G^L$ is obtained.

**1.3.Dynamic Network Flow.** A network flow $G^L$ characterizes the behavior of a thing that actions in elements are all constants along a given path. Certainly, finding the minimum spanning tree, the shortest distance between two vertices and the maximum flow on network $G^L$ are all characterizing on local characters of network at a microcosmic level. Similarly, the complex networks characterize only the random changes of edges or connecting vertices and do not involve the quantitative characterizing of interaction in elements. So, *how do we extend the network flow to characterize the varying of flows or actions between elements?* Dr.Ouyang tells Huizi that a natural idea is extending the constant flow between vertices to a dynamic flow similar to the method that used in the maximum flow and minimum cut theorem, i.e., a *dynamic network flow* that the network $G^L$ varies on time $t$ and other parameters, and also complies with the law of flow conservation at each vertex of $G^L$, namely

$$V\left(G^L[t;\mathbf{x}]\right) = V\left(G^L\right), \quad E\left(G^L[t;\mathbf{x}]\right) = E\left(G^L\right),$$
$$L : V(G^L) \to \mathscr{F}\left(\mathbb{R}^+ \times \mathbb{R}^n,\ \mathbb{R}\right), \quad L : E(G^L) \to \mathscr{F}\left(\mathbb{R}^+ \times \mathbb{R}^n,\ \mathbb{R}\right)$$

and for $v \in V\left(G^L(t;\mathbf{x})\right)$ follows the vertex conservation law (9.1), i.e

$$\sum_{u \in N_G^-(v)} L(u,v)(t;\mathbf{x}) - \sum_{u \in N_G^+(v)} L(v,u)(t;\mathbf{x}) = L(v)(t;\mathbf{x}), \tag{9.8}$$

where $n \geq 1$ is an integer, $t \in \mathbb{R}^+, \mathbf{x} \in \mathbb{R}^n$, $\mathscr{F}\left(\mathbb{R} \times \mathbb{R}^n, \mathbb{R}\right)$ is the set of mappings $f$ from $\mathbb{R} \times \mathbb{R}^n$ to $\mathbb{R}$.

Generally, for any vertex $v \in V\left(G^L(t;\mathbf{x})\right)$, we approve the labeling $L(v)$ of vertex $v$ is determined by (9.8) and then, a dynamic network flow $G^L[t;\mathbf{x}]$ can be represented only by edge labeling of $G^L$, as shown in Figure 9.9.

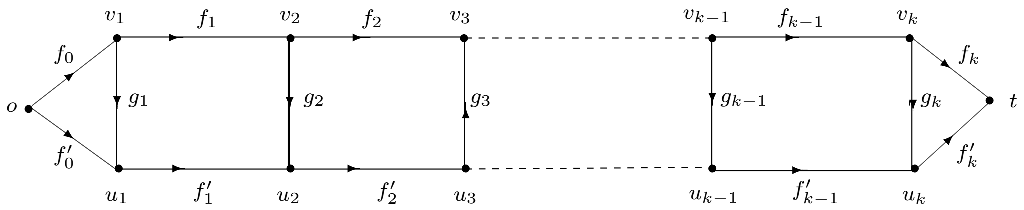



**Figure** 9.9. **Dynamic network flow**

According to the labeling $L : E(G^L) \to \mathscr{F}(\mathbb{R}^+ \times \mathbb{R}^n, \mathbb{R})$ in a dynamic network flow $G^L$ is discrete or continuous, the dynamic network flow can be divided into two classes, i.e., discrete and continuous following.

**Discrete Network Flow** *For any edge $(v, u) \in E\left(G^L\right)$, if the flow $L(v, u)$ has only a finite or countable value $L_1(v, u), L_2(v, u), \cdots, L_n(v, u), \cdots$, such a dynamic network flow is called the discrete network flow, denoted by $\left\{G^L[n]\right\}_1^\infty$, i.e.,*

| $n$ | 1 | 2 | $\cdots$ | $k$ | $\cdots$ |
|---|---|---|---|---|---|
| $G^L[n]$ | $G^{L_1}$ | $G^{L_2}$ | $\cdots$ | $G^{L_k}$ | $\cdots$ |

Notice that for an integer $n \geq 1$ and edges $(v_1, u_1), (v_2, u_2) \in E\left(G^L\right)$ in this case, the $\{L_n(v_1, u_1)\}_1^\infty$ and $\{L_n(v_2, u_2)\}_1^\infty$ are two sequences, not necessarily the same one. Particularly, if $L_n(v, u) = c_{vu}$ is a constant for any edge $(v, u) \in E\left(G^L\right)$ for integer $n \geq 1$, then $\left\{G^{L_n}\right\}_1^\infty$ degenerates to a sequence $\left\{G^L\right\}_1^\infty$ or network flow $G^L$, namely the discrete network flow $\left\{G^L[n]\right\}_1^\infty = \left\{G^L\right\}_1^\infty$, i.e., each term of the discrete network flow is the $G^L$ network flow itself.

For example, there are two discrete network flows shown in Figure 9.10, in which $(a)$ is a network flow with constants $c_i, 1 \leq i \leq 5$ but $(b)$ is a dynamic network flow.

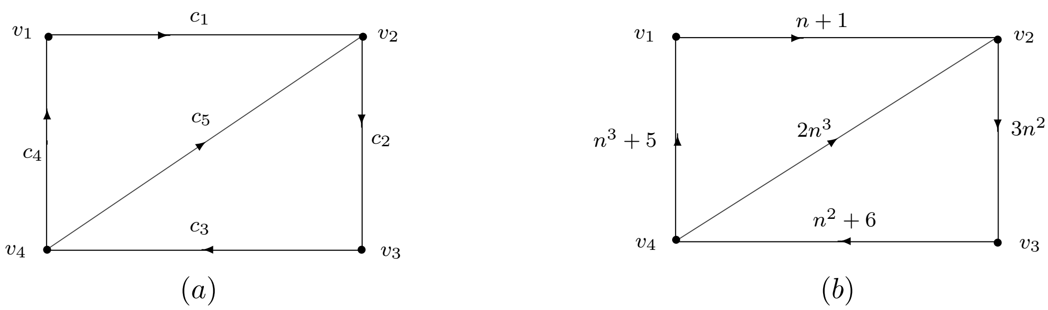



**Figure** 9.10. **Discrete dynamic network flow**

**Continuous Network Flow** *For any edge $(v, u) \in E\left(G^L\right)$, if the flow $L(v, u)$ on edge $(v, u)$ is a continuous function $f_{vu}(t, \mathbf{x})$, such a dynamic network flow is called the continuously network flow, denoted by $G^L[t, \mathbf{x}]$, where $t \in \mathbb{R}^+$ denotes the time, $\mathbf{x} \in \mathbb{R}^n$ is an variable that causes the change of network, $n \geq 1$ is an integer.*

Similarly, for any two edges $(v_1, u_1), (v_2, u_2) \in E\left(G^L\right)$, the functions $f_{v_1u_1}(t, \mathbf{x})$ and $f_{v_2u_2}(t, \mathbf{x})$ are not necessarily the same in general. So, the image of mapping $L$ on $E\left(G^L\right)$ is a family $\left\{f_{vu}(t, \mathbf{x}) : (v, u) \in E\left(G^L\right)\right\}$ of functions on graph $G$. Particularly, if $f_{vu}(t, \mathbf{x}) = f_{vu}[t]$ just is a function of time $t$ for any edge $(v, u) \in E\left(G^L\right)$, such a continuously network flow $G^L[t, \mathbf{x}]$ is varying only with the time $t$, call the *network time-flow*. Furthermore, if $f_{vu}(t, \mathbf{x}) = c_{vu}$ is a constant for any edge $(v, u) \in E\left(G^L\right)$, the network time-flow $G^L[t]$ degenerates to a sequence $\left\{G^L\right\}_1^\infty$ or network flow $G^L$. For example, there are two continuously network flows shown in Figure 9.11, in which $(a)$ is a network time-flow but $(b)$ is a continuous network flow on variables $t$ and $x$.

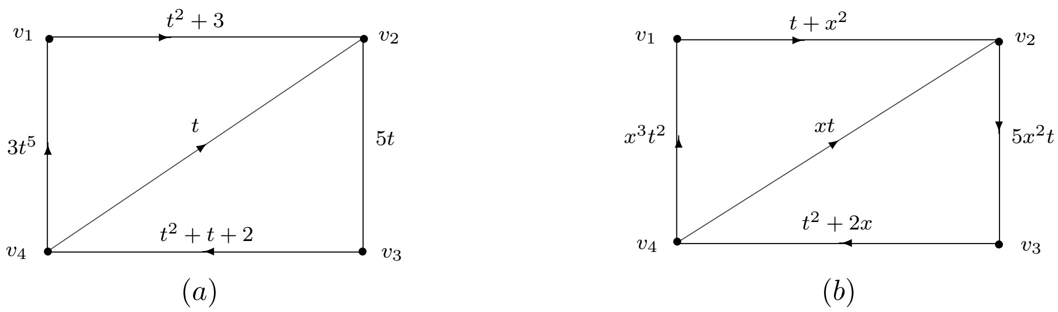



**Figure** 9.11. **Continuous network flow**

## §2. Network Flow Sequence

Certainly, everything is in the process of developing or changing states, namely the invariable or static only occurs to the observer relatively at a certain moment. In this view, the network flow is the flow of materials on a path and it is the local moving and changing of material relative to the path. And the dynamic network flow is the situation that both the path and material flow vary simultaneously in space. Huizi still has some doubts on the necessity of dynamic network in recognizing the reality of things and asks Dr.Ouyang: "*Dad, the particle mechanics, rigid body mechanics of Newton et al and the field theory of Einstein, Maxwell et al have been sufficient to explain most of the natural phenomena in objective world, why does it establish the dynamic network flow? It doesn't seem so necessary for knowing things in the universe!*" Dr.Ouyang tells her: "*You should know that the particles, rigid bodies and fields are all regular shapes in geometry but things in eyes of human, especially those with living signs such as the biological individuals or groups are irregular. Generally, it is difficult to characterize them with such models as the particles, rigid bodies or fields, which needs to establish new recognitive models, namely the dynamic network flow is proposed in such a case.*" Seeing that Huizi still does not understand this view point, Dr.Ouyang asks her a simple question: "*how do you record the moving and changing of an object in modern life?*" Huizi replies: "*I can continuously record the moving and changing states by pictures of a camera or video camera on this object at every moment and then combine them together to form a continuous process, similar to the photo record of the little girl running from states* (1) *to* (4) *in Figure* 9.12." Dr.Ouyang explains that either a camera or a video camera takes pictures on a moving object one by one. Dr.Ouyang asks Huizi: "*Why is that a camera shows a picture one by one while a video camera shows a continuous motion of an object?*" Huizi replies: "*You have told me that an image has a visual temporary retention in about 0.1-0.4 seconds on the retina of a person. As long as the camera speed is so fast that the interval between each image is less than the visual temporary retention time of a person, we will not be able to distinguish one picture from another while watching but just feel the scenery in continuous moving or changing.*"

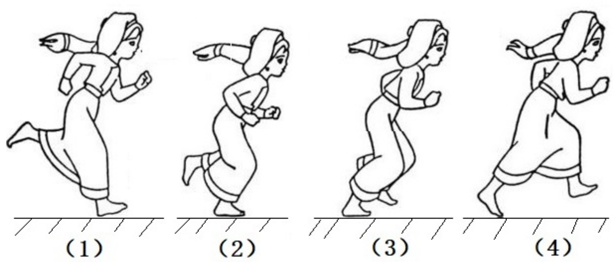


**Figure** 9.12. **A little girl running**

Dr.Ouyang nods and then asks her: "*Can the little girl's running motion be abstracted to a mass, a rigid body or a field?*" Huizi answers with a little confusion: "*It seems not to be able to because the little girl's running is flexible and can not be abstracted to a particle or rigid body, nor can she be regarded as a field.*" Dr.Ouyang further advises her that Figure 9.12 is a sketch of the little girl running and is only her appearance. Furthermore, we can add some vertices on the corners or ends of lines that represent the little girl's shape. In this way, the little girl's running can be regarded as a network flow in moving and the flow in the network is the moving character of energy in her body between vertices through the running of her legs and the swinging of her hands. Dr.Ouyang asks Huizi: "*The little girl's running motions* (1) - (4) *reflect in the leg transmission force is always the same?*" Huizi replies: "*By my own experience of running, the force transmitted through the legs is gradually stronger or weaker when accelerating or decelerating but it is always the same when running at a constant speed in which the runner is also relaxed.*" Dr.Ouyang affirms and then says: "*Good! You can see that characterizing the little girl running motions* (1) - (4) *by network flow is exactly a dynamic network flow!*" Listening to Dr.Ouyang's so explaining, Huizi seems has a little bit on dynamic network flows: "*Yes, the little girl running motion characterizing by network flow is nothing else but a dynamic network flow!*" Dr.Ouyang tells her that liking the pictures of little girl running, a dynamic network flow is an abstracted sequence of network flows of the internal and external actions of things, countable or uncountable. Among them, each is a record of state of the thing at a time $t$. Particularly, if it is countable, such a dynamic network flow is usually characterized by the network flow sequence.

**2.1.Network Flow Sequence.** Generally, for a given graph $G$, a *network flow sequence* is a list of network flows arranged in a certain order $G^{L_1}, G^{L_2}, \cdots, G^{L_n}, \cdots$, i.e., a discrete network flow $\left\{G^L[n]\right\}_1^\infty$ with operations of addition "+" and multiplication "·" as follows:

$$G^L + G^{L'} = G^{L+L'}, \quad G^L \cdot G^{L'} = G^{L \cdot L'}, \tag{9.9}$$

where the mappings $L + L'$ and $L \cdot L'$ are respectively defined by

$$L + L'(v,u) = L(v,u) + L'(v,u), \quad L' \cdot L(v,u) = L'(v,u)L(v,u) \tag{9.10}$$

for any edge $(v,u) \in E(G)$, as shown in Figure 9.13.

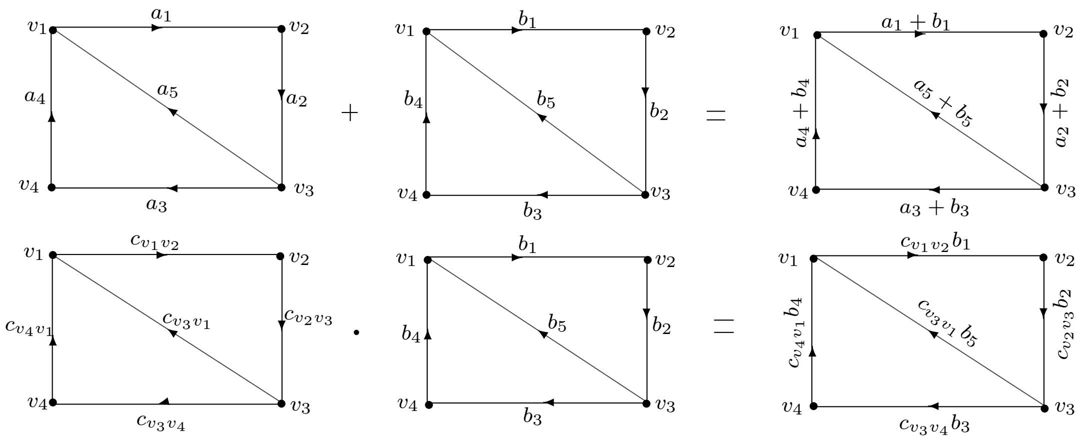



**Figure** 9.13. **Operations on network flows**

Meanwhile, define the subtraction "−", i.e., the inverse operation of "+" and the division or the inverse operation of "·" on either edge of $(v,u) \in E(G)$ with $L'(v,u) \neq 0$ respectively by

$$G^L - G^{L'} = G^L + G^{-L'} = G^{L-L'}, \quad \frac{G^L}{G^{L'}} = G^L \cdot G^{\frac{1}{L'}} = G^{\frac{L}{L'}}, \tag{9.11}$$

i.e., similar to the real numbers $\mathbb{R}$, we can define the elementary arithmetic on network flows in $\left\{G^L[n]\right\}_1^\infty$. Particularly, if $L(v,u) = c \in \mathbb{R}$ for any edge $(v,u) \in E(G)$, such a $L$ can be abbreviated as $c$, which turns the (9.9) to a special circumstance, i.e., $c \cdot G^L = G^{cL}$ and $G^L - G^{L'} = G^L + (-G^{L'})$.

Notice that the addition "+", scalar multiplication "·" on vectors $\mathbf{a}, \mathbf{b}$ and matrixes $(a_{ij})_{m\times n}, (b_{ij})_{m\times n}$ can be all viewed as the operations on a table or a particular network flow. For example, regard the entries of vector $\mathbf{a} = (a_1, a_2, \cdots, a_n) \in \mathbb{R}^n$ or matrix $M = (a_{ij})_{m\times n}$ respectively as flows on the networks $P_{n+1}$ or $mP_{n+1}$, as shown in Figures 9.14($a$) and ($b$). Then, the addition "+" and scalar multiplication "·" on vector $\mathbf{a}, \mathbf{b}$ or matrix $(a_{ij})_{m\times n}, (b_{ij})_{m\times n}$ are special cases of network flow addition "+" and multiplication "·" defined by (9.9) with $G^L = P^L_{n+1}$ or $mP^L_{n+1}$. So, the network flow addition "+", scalar multiplication "·" both are generalization of the operations on the vectors or matrixes.

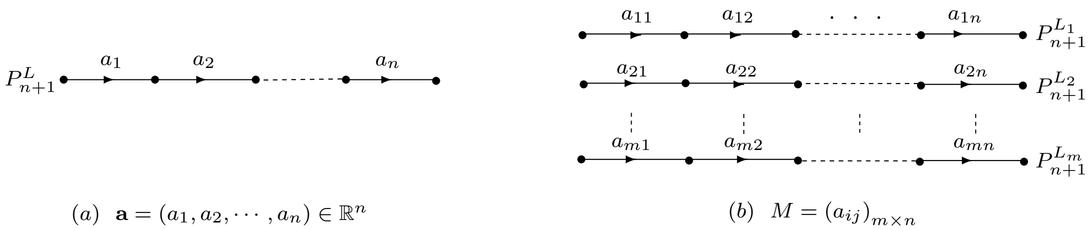



**Figure** 9.14. **Network flows with vector and matrix**

A network flow sequence $\left\{G^L[n]\right\}_1^\infty$ is usually determined by its $n$th term or the *general term* $G^{L_n}$, namely the labeling $L_n(v,u)$ or its determining rule is known on each edge $(v,u) \in \left(E(G^L)\right.$. Particularly, if $L_n(v,u)$ can be generally expressed as a known function $f_{vu}(n)$ of $n$, i.e., $L_n : (v,u) \to f_{vu}(n)$ for the general term $G^{L_n}$ in $\left\{G^L[n]\right\}_1^\infty$, we can determine the network flow sequence $\left\{G^L[n]\right\}_1^\infty$. But, *how do we get the general term* $G^{L_n}$ *of a network flow sequence* $\left\{G^L[n]\right\}_1^\infty$*?* Dr.Ouyang explains that determining the general term $G^{L_n}$ needs to establish a recursive relation of the $n$th term $G^{L_n}$ with the former $n-1$ terms $G^{L_1}, G^{L_2}, \cdots, G^{L_{n-1}}$ by applying (9.9) and (9.10) in general. In theory, if the network flow sequence $\left\{G^L[n]\right\}_1^\infty$ really characterizes the continuously changing states of a thing, Then, as the external expression or observing "*effect*" of a thing, $G^{L_n}$ should be produced by the "*cause*", including the internal and external causes. Generally, it is concluded that the variation of $G^{L_n}$ is caused by the parameter $n$ in the evolving process of network flow sequence $\left\{G^L[n]\right\}_1^\infty$, namely there are no external variables $\mathbf{x} \in \mathbb{R}^n$. So, the state $G^{L_n}$ is determined by there states $G^{L_1}, G^{L_2}, \cdots, G^{L_{n-1}}$, which implies that the there is a recursive relation of $G^{L_n}$ with $G^{L_1}, G^{L_2}, \cdots, G^{L_{n-1}}$.

**(1) Recursive formula.** A recursive formula can directly determine the general term $G^{L_n}$ in network flow sequence $\left\{G^L[n]\right\}_1^\infty$, which is a common method to establish a network flow sequence. In this case, the network flow $G^{L_n}$ is increased with a known network flow $G^{\Delta L_n} = G^{L_{f(n)}}$ on $G^{L_{n-1}}$ in evolving, namely

$$G^{L_n} = G^{L_{n-1}} + G^{\Delta L_n} = G^{L_{n-1}} + G^{L_{f(n)}}, \tag{9.12}$$

where $L_{f(n)}$ is a known function of $n$ in $G^{L_{f(n)}}$, i.e., $L_{f(n)} : (v,u) \to f_{vu}(n)$ is known for any edge $(v,u)\in E(G)$. Thus, we know that

$$G^{L_n}=G^{L_{n-1}+L_{f(n)}}=\cdots=G^{L_1+\sum\limits_{i=2}^{n}L_{f(i)}} \tag{9.13}$$

from (9.9), i.e., the labeling mapping

$$L_n=L_1+\sum_{i=2}^{n}L_{f(i)}. \tag{9.14}$$

For example, assume that the weight on edge $(v,u)\in E(G)$ is $w_{vu}$ and $L_1(v,u)=w_{vu}, f_{vu}(n)=w_{vu}n$. Then, there is

$$\begin{aligned}L_n(v,u) &= L_1(v,u)+\sum_{i=2}^{n}L_{f(i)}(v,u)\\ &= w_{vu}+\sum_{i=2}^{n}w_{vu}i=w_{vu}\sum_{i=1}^{n}i=\frac{n(n+1)}{2}w_{vu}.\end{aligned}$$

Particularly, if $f_{vu}(n)=c_{vu}$ is a constant for edge $(v,u)\in E(G)$, namely $G^{\Delta L_{f(n)}}=G^{L_c}$, then by (9.13), the general term of network flow sequence $\left\{G^L[n]\right\}_1^\infty$ is

$$G^{L_n}=G^{L_1+\sum\limits_{i=2}^{n}L_c}=G^{L_1+(n-1)L_c}\ \Rightarrow\ L_n(v,u)=L_1(v,u)+(n-1)c_{vu}. \tag{9.15}$$

(2) **Recursive equation.** A recursive equation is an implicit equation of $G^{L_n}$ with $G^{L_1},G^{L_2},\cdots,G^{L_{n-1}}$, namely

$$\mathscr{F}\left(G^{L_1},G^{L_2},\cdots,G^{L_n}\right)=G^{\mathscr{F}(L_1,L_2,\cdots,L_n)}=\mathbf{O}, \tag{9.16}$$

where the symbol $\mathbf{O}$ denotes such a network flow $G^{L_0}$ that for any edge $(v,u)\in E(G)$. $L_0:(v,u)\to 0$, namely

$$\mathscr{F}\left(L_1(v,u),L_2(v,u),\cdots,L_n(v,u)\right)=0. \tag{9.17}$$

Although the general term $G^{L_n}$ can be determined by equation (9.17) or (9.16) under the condition of known $G^{L_1},G^{L_2},\cdots,G^{L_{n-1}}$ in theory but it is difficult to get the general term $G^{L_n}$ by applying an elementary method to solve the equation (9.16) or (9.17) in general. Among them, a relatively simple case is the linear case of (9.16) or (9.17), namely there are the network flows $G^{L_{c1}},G^{L_{c2}},\cdots,G^{L_{cn}}$ holding with

$$G^{L_{c1}}\cdot G^{L_1}+G^{L_{c2}}\cdot G^{L_2}+\cdots+G^{L_{cn}}\cdot G^{L_n}=\mathbf{O}, \tag{9.18}$$

called the *n-order linear recursive equation* of network flows. In this case, for any edge $(v,u)\in E(G)$ and an integer $1\leq i\leq n$, there is such a constant $c_{vu}^{[n]}\neq 0$ that $L_i:(v,u)\to c_{vu}^{[i]}\in\mathbb{R}$. Thus, by definition

$$G^{L_n} = \sum_{i=1}^{n-1}\frac{G^{L_{c_i}}}{G^{L_{c_n}}}\cdot G^{L_i} = G^{\sum\limits_{i=1}^{n-1}\frac{L_{c_i}}{L_{c_n}}L_i} \tag{9.19}$$

we can determine $G^{L_n}$ step by step of (9.19) for integers $n=2,3,\cdots$. Particularly, if $n=2$, i.e, the 2-order linear recursive equation

$$G^{L_{c_{n-2}}}\cdot G^{L_{n-2}}+G^{L_{c_{n-1}}}\cdot G^{L_{n-1}}+G^{L_{c_n}}\cdot G^{L_n}=\mathbf{O} \tag{9.20}$$

gets the more attention because it is a generalization of the 2-order recursive number sequence. For example, assume that $L_{c_{n-2}}=a_{n-2}$, $L_{c_{n-1}}=a_{n-1}$ and $L_{c_n}=a_n$ are constants, independent on edge $(v,u)\in E(G)$. Then, the equation (9.20) can be further simplified to

$$a_{n-2}G^{L_{n-2}}+a_{n-1}G^{L_{n-1}}+a_nG^{L_n}=\mathbf{O}, \tag{9.21}$$

i.e., for any edge $(v,u)\in E(G)$, $a_{n-2}L_{n-2}+a_{n-1}L_{n-1}+a_nL_n=0$ and the recursive equation is $a_{n-2}x_{n-2}+a_{n-1}x_{n-1}+a_nx_n=0$ of a number sequence $\{x_n\}_1^\infty$. As an example, Dr.Ouyang tells Huizi that let the Fibonacci sequence $\{F_n\}_1^\infty$ with $F_n=F_{n-2}+F_{n-1}$ be the network flow equation in (9.21), i.e., $a_0=0,\ a_1=a_2=1$ and $L_n:(v,u)\to F_n$, i.e., the flow on any edge $(v,u)$ is $F_n$. Then, the network flow sequence $\left\{G^L[n]\right\}_1^\infty$ corresponds to the Fibonacci sequence $\{F_n\}_1^\infty$ in this case.

Generally, Dr.ouyang explains that all labels $L_{c_i}, 1\leq i\leq n$ in network flow equation (9.18) are different for different edges $(v_1,u_1)$ and $(v_2,u_2)$ of $G$, i.e., maybe $L_{c_i}(v_1,u_1)\neq L_{c_i}(v_2,u_2)$ in general. In this case, the network flow equation (9.18) can be regarded as $m$ number sequences $\{L_n(v,u)\}_1^\infty$ determined by linear recursive equations

$$c_{vu}^{[1]}L_1(v,u)+c_{vu}^{[2]}L_2(v,u)+\cdots+c_{vu}^{[n]}L_n(v,u)=0 \tag{9.22}$$

for any edge $(v,u)\in E(G)$, i.e., $\{\{L_n(v,u)\}_1^\infty,\ (v,u)\in E(G)\}$, where $m=|E(G)|$.

**2.2.Network Flow Distance.** Usually, a moving showing a tiger running continuously needs a high-speed camera to ensure that the two shots remain within 0.1-0.4 seconds of human vision. Generally, there are 4 parameters for characterizing the moving of a thing, namely the distance, direction, velocity and time. Thus, it is necessary to introduce the conception of *distance of network flows* when we characterize the continuous change of a thing with network flow sequence $\left\{G^L[n]\right\}_1^\infty$. So, *how do we measure the distance* $\rho\left(G^L,G^{L'}\right)$ *between network flows* $G^L$ *and* $G^{L'}$*?* Dr.Ouyang explains that if a network flow $G^L$ is viewed as a geometrical point in a Euclidean space with coordinates $(L(v,u);(v,u)\in E(G))$, then a most natural way for measuring the distance between network flows $G^L$ and $G^{L'}$ is by the distance of points $G^L$ and $G^{L'}$ in Euclidean space as that of network flows $G^L$ and $G^{L'}$, i.e

$$\rho\left(G^L, G^{L'}\right) = \sqrt{\sum_{(v,u)\in E(G)} (L(v,u) - L'(v,u))^2} = \rho\left(\mathbf{O}, G^{L-L'}\right), \qquad (9.23)$$

and particularly, the *norm* $\left|G^L\right|$ *of network* $G^L$ in $G^{L'} = \mathbf{O}$, i.e.,

$$\left|G^L\right| = \rho\left(\mathbf{O}, G^L\right) = \sqrt{\sum_{(v,u)\in E(G)} L^2(v,u)}. \qquad (9.24)$$

Now that let $G^L, G^{L'}$ and $G^{L^*}$ be three network flows. Applying the formulas (4.33) and (4.35) on vectors of Euclidean space, there must be:

(1) (Non-Negativity) $\rho\left(G^L, G^{L'}\right) \geq 0$ *with equality holds if and only if* $G^L = G^{L'}$, *i.e.,* $G^{L-L'} = \mathbf{O}$*;*

(2) (Symmetry) $\rho\left(G^L, G^{L'}\right) = \rho\left(G^{L'}, G^L\right)$*;*

(3) (Triangle Inequality) $\rho\left(G^L, G^{L'}\right) \leq \rho\left(G^L, G^{L^*}\right) + \rho\left(G^{L^*}, G^{L'}\right)$.

Now, let $m = |E(G)|$, $n \in \mathbb{R}^+$. We construct a network flow sequence $\left\{G^L[n]\right\}_1^\infty$ with norm $\left|G^{L_n}\right| = n$ by applying the definition (9.24) as follows:

First of all, let $\theta_1, \theta_2, \cdots, \theta_m$ be $m$ angles with $x$-axis in Euclidean plane $\mathbb{R}^2 = \{(x,y)| -\infty < x, y < +\infty\}$, See Figure 9.15. Meanwhile, denote all edges $(v,u) \in E(G)$ in symmetry, namely if $m$ is an even number $2k$, let $(v,u)_i^+, (v,u)_i^-$, $1 \leq i \leq k$ be all edges of $G$; if $m$ is an odd number $2k+1$, let $(v,u)_0$, $(v,u)_i^+$, $(v,u)_i^-$, $1 \leq i \leq k$ be that all edges of $G$. And then, define the flow on edge $(v,u) \in E(G)$ according to the even or odd of number $m$ as: ①If $m$ is an even number $2k$, $1 \leq i \leq k$, define the flow

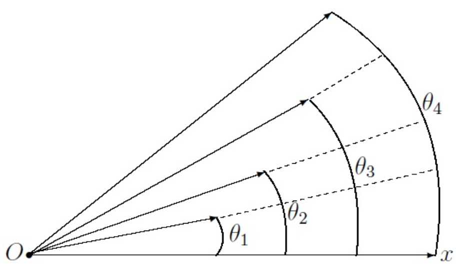



**Figure** 9.15. **Angles with $x$-axis**

$$L_n(v,u)_i^+ = \frac{n}{\sqrt{k}}\sin\theta_i, \quad L_n(v,u)_i^- = \frac{n}{\sqrt{k}}\cos\theta_i$$

on edges $(v,u)_i^+, (v,u)_i^- \in E(G)$ for an integer $i, 1 \leq i \leq k$, then the norm of network flow $G^{L_n}$ is

$$\left|G^{L_n}\right| = \sqrt{\sum_{i=1}^{k}\left({L_n}^2(v,u)_i^+ + {L_n}^2(v,u)_i^-\right)} = \sqrt{\sum_{i=1}^{k}\frac{n^2}{k}\left(\sin\theta_i^2 + \cos\theta_i^2\right)} \;=\; n;$$

②If $m$ is an odd number $2k+1$, define the flows on edges $(v,u)_0,(v,u)_i^+,(v,u)_i^- \in E(G), 1\le i\le k$ to be

$$L_n(v,u)_0 = 2n-1, \quad L_n(v,u)_i^+ = \frac{n-1}{\sqrt{k}}\sin\theta_i, \quad L_n(v,u)_i^- = \frac{n-1}{\sqrt{k}}\cos\theta_i,$$

then the norm of network flow $G^{L_n}$ is

$$\begin{aligned}\left|G^{L_n}\right| &= \sqrt{{L_n}^2(v,u)_0 + \sum_{i=1}^{k}\left({L_n}^2(v,u)_i^+ + {L_n}^2(v,u)_i^-\right)}\\ &= \sqrt{2n-1+\sum_{i=1}^{k}\frac{(n-1)^2}{k}\left(\sin\theta_i^2+\cos\theta_i^2\right)}\\ &= \sqrt{2n-1+(n-1)^2} \ = \ n,\end{aligned}$$

i.e., there is a labeling mapping $L_n$ on network flow $G^{L_n}$ with $\left|G^{L_n}\right| = n$.

**2.3.Network Flow Limitation.** Usually, the distance of two consecutive network flows in a network flow sequence $\left\{G^L[n]\right\}_1^\infty$ is an important parameter for quantitatively characterizing the changing of dynamic network flow so as to determine whether the dynamic network flow infinitely tends to a network flow or diverge. For holding on the evolving law of things, it should be noted that every thing varies always from the unstable to stable, return to an equilibrium state in the nature. In the contrast, the network flow sequence simulates this varying of things with a limiting state of sequence $\left\{G^L[n]\right\}_1^\infty$ as $n\to\infty$.

So, *what is the limitation of network flow sequence* $\left\{G^L[n]\right\}_1^\infty$*?* Dr.Ouyang tells Huizi that the limitation of network flow sequence is similar to that of the number sequence $\{x_n\}_1^\infty$, i.e., let $\left\{G^L[n]\right\}_1^\infty$ be a network flow sequence and let $G^L$ be a network flow. If there is always an integer $N$ for a given small number $\varepsilon>0$ such that if $n>N$,

$$\rho\left(G^{L_n}, G^L\right) \ < \ \varepsilon, \tag{9.25}$$

then it is said that the network flow sequence $\left\{G^L[n]\right\}_1^\infty$ is *convergent* to $G^L$, denoted by

$$\lim_{n\to\infty} G^{L_n} \ = \ G^L \tag{9.26}$$

and $G^L$ is the *limitation of network flow sequence* $\left\{G^L[n]\right\}_1^\infty$. Otherwise, if a network flow sequence $\left\{G^L[n]\right\}_1^\infty$ has no limitation, it is said to be *divergent*.

Notice that a network flow sequence is convergent or divergent dependent on the existence of limitation $G^L$ of the network flow sequence $\left\{G^L[n]\right\}_1^\infty$ by definition. For example, there are two network flow sequences shown in Figure 9.16. Among them, the network flow sequence $(a)$ is divergent but $(b)$ is convergent.

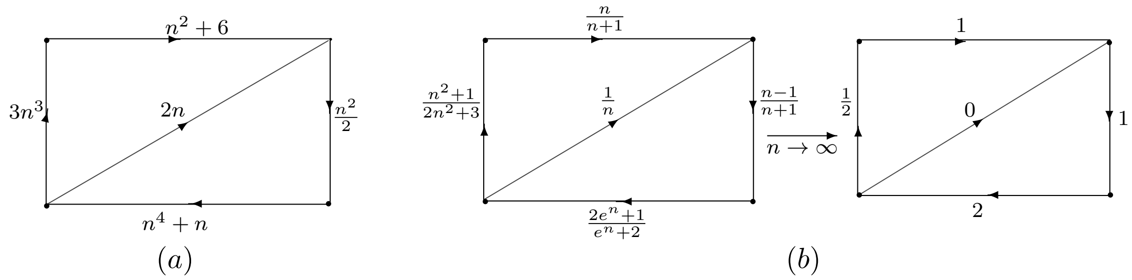



**Figure** 9.16. **Divergent and convergent network flow sequences**

By the definition (9.23) of distance of network flows, if a network flow sequence $\left\{G^L[n]\right\}_1^\infty$ converges to $G^L$, there exists an integer $N$ for any positive number $\varepsilon$ such that if $n > N$ then $\rho\left(G^{L_n} - G^L\right) < \varepsilon$. Thus, for any edge $(v,u) \in E(G)$, there must be $|L_n(v,u) - L(v,u)| \leq \rho\left(G^L - G^{L'}\right) < \varepsilon$ if $n > N$, namely the number sequence $\{L_n(v,u)\}_1^\infty$ converges to $L(v,u)$. Conversely, if for any edge $(v,u) \in E(G)$ the number sequence $\{L_n(v,u)\}_1^\infty$ converges to $L(v,u)$, by definition there is an integer $N_{vu}$ for a positive number $\varepsilon/\sqrt{|E(G)|}$ such that $|L_n(v,u) - L(v,u)| < \varepsilon/\sqrt{|E(G)|}$ if $n > N_{vu}$. Now, let $N = \max\{N_{vu}, (v,u) \in E(G)\}$. Then, if $n > N$ there must be

$$\rho\left(G^{L_n}, G^L\right) = \sqrt{\sum_{(v,u)\in E(G)} (L(v,u) - L'(v,u))^2} < \sqrt{\sum_{(v,u)\in E(G)} \frac{\varepsilon^2}{|E(G)|}} = \varepsilon,$$

by (9.23), i.e., the network flow sequence $\left\{G^L[n]\right\}_1^\infty$ converges to $G^L$. Dr.Ouyang then concludes a criterion for determining a network flow sequence $\left\{G^L[n]\right\}_1^\infty$ is convergent or not, i.e., *a network flow sequence* $\left\{G^L[n]\right\}_1^\infty$ *converges to* $G^L$ *if and only if the number sequence* $\{L_n(v,u)\}_1^\infty$ *converges to* $L(v,u)$ *for any edge* $(v,u) \in E(G)$.

According to this criterion, some important properties of convergence network flow sequence $\left\{G^L[n]\right\}_1^\infty$ are esily obtained in the following, which are similar to the number sequence $\{x_n\}_1^\infty$:

(1) **Limitation uniqueness.** Now if a network flow sequence $\left\{G^L[n]\right\}_1^\infty$ converges, the limitation $\lim\limits_{n\to\infty} G^{L_n} = G^L$ is unique. Otherwise, if $G^{L'}$ be another limitation of the sequence $\left\{G^L[n]\right\}_1^\infty$ with $\rho\left(G^L - G^{L'}\right) > 0$, let $\varepsilon = \rho\left(G^L - G^{L'}\right)/2$. Then, by the definition (9.25) of network sequence, there is such an integer $N$ that holds with

$$\rho\left(G^{L_n} - G^L\right) < \varepsilon, \quad \rho\left(G^{L'} - G^{L_n}\right) < \varepsilon$$

if $n > N$. Applying the triangle inequality and symmetry of network flow norm, there is

$$\begin{aligned} 2\varepsilon &= \rho\left(G^L - G^{L'}\right) = \rho\left(\left(G^L - G^{L_n}\right) + \left(G^{L_n} - G^{L'}\right)\right) \\ &\leq \rho\left(G^L - G^{L_n}\right) + \rho\left(G^{L_n} - G^{L'}\right) = \rho\left(G^{L_n} - G^L\right) + \rho\left(G^{L_n} - G^{L'}\right) < 2\varepsilon, \end{aligned}$$

which contradicts to the assumption that $\rho\left(G^L - G^{L'}\right) > 0$, i.e., the only possibility is $\rho\left(G^L - G^{L'}\right) = 0$ or $G^L = G^{L'}$. Consequently, the limitation of network flow sequence $\left\{G^L[n]\right\}_1^{\infty}$ must be unique if it is convergent.

(2) **Norm boundedness.** If a network flow sequence $\left\{G^L[n]\right\}_1^{\infty}$ is convergent, there must be a positive number $M$ such that $\left|G^{L_n}\right| \leq M$ for any integer $n$. *Why is it so for a convergent network flow sequence* $\left\{G^L[n]\right\}_1^{\infty}$*?* Notice that$\rho\left(G^{L_n}, \mathbf{O}\right) \leq \rho\left(G^{L_n}, G^L\right) + \rho\left(G^L, \mathbf{O}\right)$ by the triangle inequality, i.e., $\rho\left(G^{L_n}, G^L\right) \geq \left|G^{L_n}\right| - \left(\left|G^L\right|\right.$. Now, assume $\lim\limits_{n\to\infty} G^{L_n} = G^L$. Then, for a positive number $\varepsilon = 1$ there must be an integer $N$ such that for any integer $n > N$ there is $\rho\left(G^{L_n}, G^L\right) < 1$, i.e., $\left|G^{L_n}\right| \leq \left|G^L\right| + 1$. Generally, let $M = \max\left\{\left|G^L\right| + 1, \left|G^{L_n}\right|, 1 \leq n \leq N\right\}$. Then, there is $\left|G^{L_n}\right| \leq M$ for any integer $n$.

**3) Order convergence.** Certainly, a partial order "$\preceq$" on network flows can be introduced, namely for two given network flows $G^{L_1}, G^{L_2}$, if for any edge $(v,u) \in E(G)$ holds with $L_1(v,u) \leq L_2(v,u)$, then the order of network flow $G^{L_1}$ is *weak* than $G^{L_2}$ and denoted by $G^{L_1} \preceq G^{L_2}$. Now, assume that the network flow sequences $\left\{G_1^L[n]\right\}_1^{\infty}, \left\{G_2^L[n]\right\}_1^{\infty}$ are both convergent, i.e., $\lim\limits_{n\to\infty} G^{L_{1n}} = G^{L_1}$ and $\lim\limits_{n\to\infty} G^{L_{2n}} = G^{L_2}$. If there is a positive number $N_0$ such that partial order

$$G^{L_{1n}} \preceq G^{L_n} \preceq G^{L_{2n}}$$

holds if $n > N_0$, then $\left\{G^L[n]\right\}_1^{\infty}$ is also convergent with $\lim\limits_{n\to\infty} G^{L_n} = G^L$. *Why is it so?* Dr.Ouyang explains that because for any edge $(v,u) \in E(G)$, by definition the network flow sequence $\{L_{1n}(v,u)\}_1^{\infty}$ and $\{L_{2n}(v,u)\}_1^{\infty}$ are both convergent. In this case, there is $\lim\limits_{n\to\infty} L_{1n}(v,u) = \lim\limits_{n\to\infty} L_{2n}(v,u) = L(v,u)$ and $L_{1n}(v,u) \leq L_n(v,u) \leq L_{2n}(v,u)$. By the pinch theorem on two sides in number sequences, we therefore know that $\lim\limits_{n\to\infty} L_n(v,u) = L(v,u)$, i.e., $\lim\limits_{n\to\infty} G^{L_n} = G^L$.

(4) **Limitation operations.** Notice that the limitation operations (9.9) on network flow sequences $\left\{G^L[n]\right\}_1^{\infty}$ are similar to those of operations on number sequence $\{x_n\}_1^{\infty}$, namely for the convergent network flow sequences $\left\{G^L[n]\right\}_1^{\infty}$ and $\left\{G^{L'}[n]\right\}_1^{\infty}$, there are

$$\begin{aligned} \lim_{n\to\infty}\left(G^{L_n} + G^{L'n}\right) &= \lim_{n\to\infty}\left(G^{L_n}\right) + \lim_{n\to\infty}\left(G^{L'n}\right), \\ \lim_{n\to\infty}\left(G^{L_n} \cdot G^{L'n}\right) &= \lim_{n\to\infty}\left(G^{L_n}\right) \cdot \lim_{n\to\infty}\left(G^{L'n}\right). \end{aligned} \tag{9.27}$$

and furthermore, if for any edge $(v,u) \in E(G)$, $\lim\limits_{n\to\infty} L_n(v,u) \neq 0$, then $\left\{G^{L_n}/G^{L'_n}\right\}$ is a convergent network flow sequence with

$$\lim_{n\to\infty} \frac{G^{L_n}}{G^{L'_n}} = \frac{\lim\limits_{n\to\infty} G^{L_n}}{\lim\limits_{n\to\infty} G^{L'_n}}. \tag{9.28}$$

## §3. Network Flow Evolving

A network flow sequence is using finite or countable network flows to simulate the moving or changing state of a thing by showing the network flow states of a thing one by one. However, there is an intermittence or blank between one state with another, which naturally has influence on holding on the dynamic behavior of thing. For example, if the projection speed of a moving is less than 24 grids per second, for instance it is slower than 12 grids per second, the animal moving in the film will be felt to be mechanically jumping instead of the animal scene in the eyes of human because there are missing or jumping in the changes of animal shown on the screen, which is inconsistent with human's normal perception. This is also the defect of using limited or count network flows to simulate the moving or changing of a thing. So, *how do we correct the weakness of network flow sequence in characterizing the moving or changing of a thing?* Dr.Ouyang tells Huizi that a natural way is to fully show the evolving of a thing, i.e., adding all the missing states of a thing in successive network flows for giving a continuously evolving process or causality of a thing from one state to another, i.e., the continuous evolving of network flows.

So, *how do we characterize the evolving of a network flow?* Dr.Ouyang explains that the evolving of a network flow $G^L$ is from one state of $G^L$ to another while keeping the graph structure $G$ unchanged. A natural simulation way is to view the network flow $G^L$ as evolving with time $t$ and a space variable $\mathbf{x} \in \mathbb{R}^n$ and denote it by $G^L[t, \mathbf{x}]$. For simplicity, let time $t = x_{n+1}$ be the $n+1^{th}$ coordinate and denote $G^L[t, \mathbf{x}]$ by $G^L[\mathbf{x}]$, $\mathbf{x} \in \mathbb{R}^{n+1}$. Thus, the network flow $G^L : \mathbf{x} \in \mathbb{R}^{n+1} \to G^L[\mathbf{x}]$ characterizes an acting function between things, namely the changing of functions $\{L(v,u)[\mathbf{x}], (v,u) \in E(G)\}$ on time and space variable $\mathbf{x}$. So, *how do we define the continuity in the evolving of network flow $G^L[\mathbf{x}]$?* Dr.Ouyang tells Huizi that for defining the continuous evolving of network flows, it needs first to introduce the conception of $\delta$-neighborhood of a network flow $G^L[\mathbf{x}]$.

Generally, a *$\delta$-neighborhood of network flow $G^{L_0}$* is similar to the $\delta$-neighborhood of a point in Euclidean space, defined to be a set $\left\{G^L[\mathbf{x}] | \rho\left(G^L[\mathbf{x}], G^{L_0}\right) < \delta\right\}$ of network flows. Then, *what is the limitation of a dynamic network flow?* Dr.Ouyang tells his daughter that for a network flow $G^L[\mathbf{x}]$, if there exists a network flow $G^{L_0}$ such that for any given number $\varepsilon > 0$ there is always a $\delta$-neighborhood of $G^{L_0}$ holding with $\rho\left(G^L[\mathbf{x}], G^{L_0}\right) = \rho\left(G^{L[\mathbf{x}]-L_0}, \mathbf{O}\right) < \varepsilon$, then the network flow $G^{L_0}$ is called the *limitation of network flow* $G^L[\mathbf{x}]$ when $L[\mathbf{x}] \to L_0$, denoted by $\lim\limits_{L[\mathbf{x}] \to L_0} G^L[\mathbf{x}] = G^{L_0}$. Notice that there maybe not a variable $\mathbf{x}_0$ such that $G^L[\mathbf{x}_0] = G^{L_0}$ in general but for a *continuous evolving* of network flow, there must be a variable $\mathbf{x}_0$ such that $G^L[\mathbf{x_0}] = G^{L_0}$, where the continuous evolving characterizes the continuous changing of network flow $G^L[\mathbf{x}]$ along with $\mathbf{x}$, without any discontinuity or jump in its evolving process, namely for any given positive number $\varepsilon$, if there is a positive $\delta$ such that

$$|\mathbf{x}-\mathbf{x}_0|<\delta \quad\Rightarrow\quad \rho\left(G^L[\mathbf{x}], G^L[\mathbf{x}_0]\right)<\varepsilon, \tag{9.29}$$

then the network flow $G^L[\mathbf{x}]$ is said to be continuous at network flow $G^L[\mathbf{x}_0]$ and denoted by $\lim\limits_{\mathbf{x}\to\mathbf{x}_0} G^L[\mathbf{x}] = G^L[\mathbf{x}]_0$.

Generally, the difference of variables is denoted by $\Delta\mathbf{x}=\mathbf{x}-\mathbf{x}_0$ and the deference of network flows by $\Delta G^L[\mathbf{x}] = G^L[\mathbf{x}] - G^L[\mathbf{x}_0]$. Then, by the subtraction (9.11) of network flows we know that $\Delta G^L[\mathbf{x}] = G^{L[\mathbf{x}]-L[\mathbf{x}_0]} = G^{\Delta L[\mathbf{x}]}$, see Figure 9.17. In this way, the continuous evolving of network flow can be defined as $\Delta G^L[\mathbf{x}]\to\mathbf{O}$ if $\Delta\mathbf{x}\to\mathbf{0}$ or $\lim\limits_{\Delta\mathbf{x}\to\mathbf{0}} \Delta G^L[\mathbf{x}] = \mathbf{O}$. Conversely, a network flow $G^L[\mathbf{x}]$ is said to be discontinuous at a point $\mathbf{x}_0\in\mathbb{R}^{n+1}$ if there are no definition of $G^L[\mathbf{x}]$ at $\mathbf{x}_0$, not existing the limitation of $\lim\limits_{\mathbf{x}\to\mathbf{x}_0} G^L[\mathbf{x}]$ or $\lim\limits_{\mathbf{x}\to\mathbf{x}_0} G^L[\mathbf{x}]\neq G^L[\mathbf{x}_0]$. All of them are discontinuous cases of $G^L[\mathbf{x}]$ evolving at $\mathbf{x}_0$.

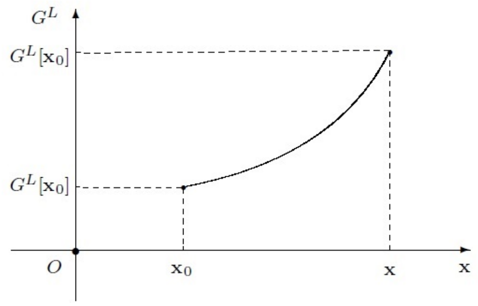



**Figure** 9.17. **Network difference**

**3.1.Equivalent Condition.** For an edge $(v,u)\in E(G)$, the evolving of network flow $G^L[\mathbf{x}]$ constraint on the edge $(v,u)$ is a function of $\mathbf{x}$, namely $L:(v,u)\to L(v,u)[\mathbf{x}]\in\mathbb{R}$. Thus, if $G^L[\mathbf{x}]$ is continuous at $G^L[\mathbf{x}_0]$, the for any given number $\varepsilon>0$ there exists a positive number $\delta$ such that $\rho\left(G^L[\mathbf{x}], G^L[\mathbf{x}_0]\right)<\varepsilon$ if $|\mathbf{x}-\mathbf{x}_0|<\delta$. Whence, there is

$$\begin{aligned}|L(v,u)[\mathbf{x}]-L(v,u)[\mathbf{x}_0]| &\le \sqrt{\sum_{(v,u)\in E(G)}\left(L(v,u)[\mathbf{x}]-L(v,u)[\mathbf{x}_0]\right)^2}\\ &= \rho\left(G^L[\mathbf{x}], G^L[\mathbf{x}_0]\right) < \varepsilon\end{aligned}$$

for any $(v,u)\in E(G)$, namely the network flow $G^L[\mathbf{x}]$ constraint on an edge $(v,u)\in E(G)$ is continuous at point $\mathbf{x}_0$. Conversely, if the network flow $G^L[\mathbf{x}]$ constraint on every edge $(v,u)\in E(G)$ is continuous at the point $\mathbf{x}_0$, namely for any positive number $\varepsilon/\sqrt{|E(G)|}$ there is a positive number $\delta_{vu}$ such that

$$|\mathbf{x}-\mathbf{x}_0|<\delta_{vu} \quad\Rightarrow\quad |L(v,u)[\mathbf{x}]-L(v,u)[\mathbf{x}_0]|<\varepsilon/\sqrt{|E(G)|},$$

let $\delta=\min\{\delta_{vu},(v,u)\in E(G)\}$. Then, there are $|L(v,u)[\mathbf{x}]-L(v,u)[\mathbf{x}_0]|<\varepsilon/\sqrt{|E(G)|}$ for any edge $(v,u)\in E(G)$ if $|\mathbf{x}-\mathbf{x}_0|<\delta$. In this case, there must be

$$\rho\left(G^L[\mathbf{x}], G^L[\mathbf{x}_0]\right) = \sqrt{\sum_{(v,u)\in E(G)}\left(L(v,u)[\mathbf{x}]-L(v,u)[\mathbf{x}_0]\right)^2} < \sqrt{|E(G)|\,\frac{\varepsilon^2}{|E(G)|}} = \varepsilon,$$

namely the network flow $G^L[\mathbf{x}]$ is continuous at $\mathbf{x}_0$. Dr.Ouyang tells Huizi that the previous discussion leads to an equivalent condition for determining that a network flow $G^L[\mathbf{x}]$ is continuous at network flow $G^L[\mathbf{x}_0]$, namely *the evolving of network flow $G^L[\mathbf{x}]$ is continuous at network flow $G^L[\mathbf{x}_0]$ if and only if for any edge $(v,u)\in E(G)$, the constraint function $L(v,u)[\mathbf{x}]$ of $G^L[\mathbf{x}]$ on edge $(v,u)\in E(G)$ is continuous at point* $\mathbf{x}_0$. Now, let $G^L[\mathbf{x}]$ and $G^{L'}[\mathbf{x}]$ be continuous evolving of network flows at point $\mathbf{x}_0$. Applying this equivalent condition with properties of the continuous function, there are

$$\begin{aligned}
\lim_{\mathbf{x}\to\mathbf{x}_0}\left(G^L[\mathbf{x}]+G^{L'}[\mathbf{x}]\right) &= \lim_{\mathbf{x}\to\mathbf{x}_0}G^L[\mathbf{x}]+\lim_{\mathbf{x}\to\mathbf{x}_0}G^{L'}[\mathbf{x}],\\
\lim_{\mathbf{x}\to\mathbf{x}_0}\left(G^L[\mathbf{x}]-G^{L'}[\mathbf{x}]\right) &= \lim_{\mathbf{x}\to\mathbf{x}_0}G^L[\mathbf{x}]-\lim_{\mathbf{x}\to\mathbf{x}_0}G^{L'}[\mathbf{x}],\\
\lim_{\mathbf{x}\to\mathbf{x}_0}\left(G^L[\mathbf{x}]\cdot G^{L'}[\mathbf{x}]\right) &= \lim_{\mathbf{x}\to\mathbf{x}_0}G^L[\mathbf{x}]\cdot\lim_{\mathbf{x}\to\mathbf{x}_0}G^{L'}[\mathbf{x}]
\end{aligned}\tag{9.30}$$

and for any edge $(v,u)\in E(G)$, if $L'(v,u)[\mathbf{x}_0]\neq\mathbf{0}$ then there is

$$\lim_{\mathbf{x}\to\mathbf{x}_0}\frac{G^L[\mathbf{x}]}{G^{L'}[\mathbf{x}_0]} = \frac{\lim\limits_{\mathbf{x}\to\mathbf{x}_0}G^L[\mathbf{x}]}{\lim\limits_{\mathbf{x}\to\mathbf{x}_0}G^{L'}[\mathbf{x}_0]}.\tag{9.31}$$

Now, let $G^L:\mathbf{x}\to G^L[\mathbf{x}]$ and $G^{L'}:G^L[\mathbf{x}]\to G^{L'}[\mathbf{x}]$ be continuous network flows. Clearly, the composition $G^{L'(L)}:\mathbf{x}\to G^{L'}\left(G^L[\mathbf{x}]\right)=G^{L'(L)[\mathbf{x}]}$ is also a network evolving, called the *composite evolving* of network flows $G^L[\mathbf{x}]$ and $G^{L'}[\mathbf{x}]$. Then, a natural question is *under what conditions the composite evolving of $G^{L'(L)[\mathbf{x}]}$ is continuous?* Dr.Ouyang tells Huizi that similar to the continuity of composite function, if $G^L[\mathbf{x}]$ at point $\mathbf{x}_0$ is continuous and $G^{L'}[\mathbf{x}]$ is continuous at network flow $G^L[\mathbf{x}_0]$, then the composite evolving of $G^{L'(L)[\mathbf{x}]}$ is also continuous at point $\mathbf{x}_0$. *Why is the composite evolving $G^{L'(L)[\mathbf{x}]}$ continuous so?* It is because that by the equivalent condition of a continuous evolving of network flow $G^L[\mathbf{x}]$ with its constraint $L(v,u)[\mathbf{x}]$ on edge $(v,u)\in E(G)$ at a point $\mathbf{x}_0$, we know that the constraint function $L'(v,u)[\mathbf{x}]$ of $G^{L'}[\mathbf{x}]$ on edge $(v,u)$ is continuous at $L'(v,u)[\mathbf{x}_0]$. And so, by applying a well-known conclusion on composite function, i.e., the composite function of continuous functions is also a continuous function, we easily know that the constraint function $L'(L)(v,u)[\mathbf{x}]$ of $G^{L'(L)[\mathbf{x}]}$ on edge $(v,u)\in E(G)$ is continuous at point $\mathbf{x}_0$. Then, applying the equivalent condition of continuous evolving of network flow again, we obtain the conclusion that the composite evolving $G^{L'(L)[\mathbf{x}]}$ is continuous at point $\mathbf{x}_0$.

Notice that the operations of continuous evolving of network flows and the continuity of composite evolving provide a convenient condition for calculating the limitation of continuous evolving of network flow, namely for each edge $(v,u)\in E(G)$ calculate the limitation $\lim\limits_{\mathbf{x}\to\mathbf{x}_0}L(v,u)[\mathbf{x}]$ and then, take the limitation as the label on edge $(v,u)\in$

$E(G)$, we get immediately the limitation $G^L[\mathbf{x}_0]$ of network flow $G^L[\mathbf{x}]$ at point $\mathbf{x}_0$. For example, as shown in Figure 9.18, $(a)$ is continuous evolving of network flow $G^L[\mathbf{x}]$, $(b)$ is its limitation of $G^L[\mathbf{x}]$ with $(x,t) \to (2,1)$, i.e., at point $(2,1)$.

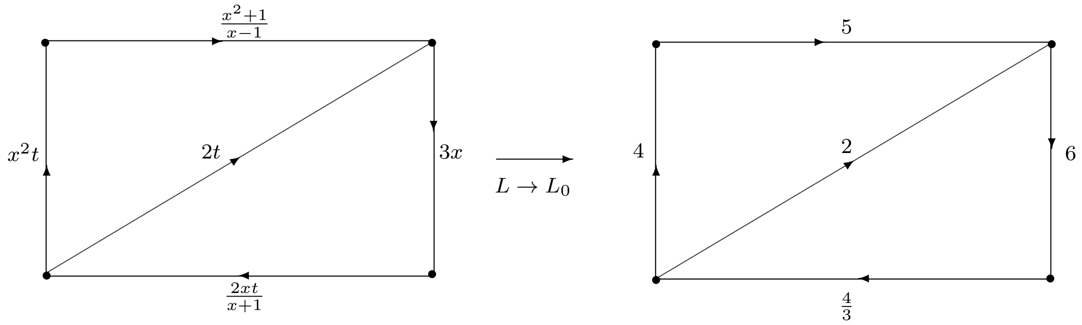



**Figure** 9.18. **Limitation of continuous evolving**

**3.2.Partial Order on Network Evolving.** For two network flows $G^{L_1}$ and $G^{L_2}$, if $L_1(v,u) < L_2(v,u)$ for any edge $(v,u) \in E(G)$, it is said that the order of $G^{L_1}$ is strictly weaker than that of $G^{L_2}$, denoted by $G^{L_1} \prec G^{L_2}$. Notice that the order relation "$\prec$" in network $G^L[\mathbf{x}]$ evolving forms a partial order relation and the partial order chain in the evolving of $G^L[\mathbf{x}]$. For the simplicity, the following discussion assumes that $\mathbf{x} = t$, namely the evolving of network flow $G^L[\mathbf{x}]$ only with one parameter $t$, i.e., $G^L[t]$.

(1) **Boundedness.** For two real numbers $a, b \in \mathbb{R}, a \le b$, let $G^L[t]$ be a continuous evolving of network flow on the closed interval $[a,b]$, namely for any edge $(v,u) \in E(G)$, $L(v,u)[t]$ is a continuous function on the closed interval $[a,b]$. Then, by the definition of network flow norm (9.24) and the property of continuous function, the norm

$$\left|G^L[t]\right| = \sqrt{\sum_{(v,u)\in E(G)} L^2(v,u)[t]} \tag{9.32}$$

is a continuous function on the closed interval $[a,b]$ of $t$. By applying the boundedness of continuous function on a closed interval, we are easily know that there is a positive number $M$ holding with $\left|G^L[t]\right| \le M$, i.e., the norm of continuous evolving of network flow must be bounded on a closed interval. Furthermore, applying the property that the norm $\left|G^L[t]\right|$ is a continuous function and the existence of maximum and minimum values for a continuous function on a closed interval $[a,b]$ in calculus, we know that the norm $\left|G^L[t]\right|$ of continuous evolving of network flow will be able to reach the maximum and minimum values.

(2) **Intermediate network flow.** Certainly, there are two kinds of intermediate network flows for the continuous evolving $G^L[t]$ of network flow in a closed interval $[a,b], a, b \in \mathbb{R}$ should be discussed, i.e., the intermediate network flow under the norm or partial order condition, respectively.

The intermediate network flow $G^L[t]$ on the norm of network flow is an immediate deduction of the intermediate value theorem of continuous functions on closed interval because by (9.32) the norm $\left|G^L[t]\right|$ is a continuous function of $t$ on closed interval $[a,b]$. In this case, for any real number $C$ between $\left|G^L[a]\right|$ and $\left|G^L[b]\right|$, i.e., $\left|G^L[a]\right| < C < \left|G^L[b]\right|$ there must be a real $c \in (a,b)$ such that $\left|G^L[c]\right| = C$ by the intermediate value theorem of continuous function on a closed interval $[a,b]$.

Certainly, the intermediate network flow on partial order in continuous evolving of network flow $G^L[t]$ is affected by the partial order "$\prec$", which is not simple as the intermediate value theorem of continuous function but there exists a construction rule for determining the intermediate network flow. In this case, assume that the network evolving $G^L[t]$ is continuous, $G^L[a] = G^{L_1}, G^L[b] = G^{L_2}$ and $G^{L_1} \prec G^{L_0} \prec G^{L_2}$. Dr.Ouyang explains that the equivalent condition of continuous evolving $G^L[t]$ of network flow shows that for any edge $(v,u) \in E(G)$, $L(v,u)[t]$ is a continuous function of $t$ and by the intermediate value theorem of continuous function, if $L(v,u)[a] < L_0(v,u) < L(v,u)[b]$, there is a real number $c_{vu}$ on interval $(a,b)$ such that $L_0(v,u) = L(v,u)[c_{vu}]$, namely the network flow $G^{L_0}$ is equivalent to a mapping $L_0 : (v,u) \to L(v,u)[c_{vu}]$ on any edge $(v,u) \in E(G)$. Of course, for different edges $(v_1,u_1)$, $(v_2,u_2) \in E(G)$ the real numbers $c_{v_1u_1}, c_{v_2u_2} \in (a,b)$ is not necessarily the same. Particularly, if all numbers $c_{vu}$ are the same constant $c \in (a,b)$ for all edges $(v,u) \in E(G)$ , the network flow $G^{L_0}$ is nothing else but the network $G^L[t]$ when $t = c$ in evolving, namely $G^{L_0} = G^L[c]$.

**3.3.Network Evolving Algebra.** For any integer $m \geq 1$, Let $G^{L_1}[\mathbf{x}], G^{L_2}[\mathbf{x}], \cdots, G^{L_m}[\mathbf{x}]$ be $m$ states of continuous evolving of network flow. According to the network flow operation $(9.9)-(9.11)$ and the definition of composite evolving, the network flow has algebraic properties following.

**(1) Constant flow.** For a constant $c \in \mathbb{R}$, define a network flow $G^{L_c}$ by labeling mapping $L_c : (v,u) \to c$ for any edge $(v,u) \in E(G)$. Particularly, there is $\mathbf{O} = G^{L_0}$, $\mathbf{I} = G^{L_1}$, where $\mathbf{O}$ and $\mathbf{I}$ are respectively the identity elements of addition "$+$" and multiplication "$\cdot$" on network flows, namely

$$\mathbf{O} + G^L[\mathbf{x}] = G^L[\mathbf{x}] + \mathbf{O} = G^L[\mathbf{x}],$$
$$\mathbf{I} \cdot G^L[\mathbf{x}] = G^L[\mathbf{x}] \cdot \mathbf{I} = G^L[\mathbf{x}],$$

where a table of addition and multiplication on identity elements $\mathbf{O}$ and $\mathbf{I}$ is as follows:

$$\mathbf{O} + \mathbf{O} = \mathbf{O}, \quad \mathbf{O} + \mathbf{I} = \mathbf{I} + \mathbf{O} = \mathbf{I},$$
$$\mathbf{O} \cdot \mathbf{O} = \mathbf{O}, \quad \mathbf{I} \cdot \mathbf{O} = \mathbf{O} \cdot \mathbf{I} = \mathbf{O}, \quad \mathbf{I} \cdot \mathbf{I} = \mathbf{I}.$$

(2) **Group and bigroup.** For a network flow $G^L[\mathbf{x}]$, if $X + G^L[\mathbf{x}] = \mathbf{O}$ then $X$ is

defined to be the inverse network flow of $G^L[\mathbf{x}]$ in addition "+". Similarly, if $L(v,u)[\mathbf{x}] \neq \mathbf{0}$ for all edges $(v,u) \in E(G)$ in network flow , a network flow $Y$ that holds with $Y \cdot G^L[\mathbf{x}] = \mathbf{I}$ is defined to be the inverse network flow of $G^L[\mathbf{x}]$ in multiplication "$\cdot$". It is easy to know

$$X = -G^L[\mathbf{x}] = G^{-L}[\mathbf{x}], \qquad Y = \frac{1}{G^L[\mathbf{x}]} = G^{\frac{1}{L}}[\mathbf{x}] = G^{L^{-1}}[\mathbf{x}] \tag{9.33}$$

by the operation (9.9)$-$(9.11) of network flows and we further have the following equations

$$-G^L[\mathbf{x}] = G^{-L}[\mathbf{x}], \qquad \frac{\mathbf{I}}{G^L[\mathbf{x}]} = G^{\frac{1}{L}}[\mathbf{x}]. \tag{9.34}$$

In this way, by (9.9) – (9.11) and equation (9.33) – (9.34), all network flows $G^L[\mathbf{x}]$ on a graph $G$ form an Abelian group $\left(\left\{G^L[\mathbf{x}]\right\};+\right)$ under the addition "+" and all network flows $\left[G^L\mathbf{x}]\}\right]$ form a group of $\left(\left\{G^L[\ bfx]\right\};\cdot\right)$ under the multiplication "$\cdot$" if $L(v,u)[\mathbf{x}] \neq \mathbf{0}$ for all edges $(v,u) \in E(G)$ but do not necessarily have the commutativity, i.e., maybe not $G^L[\mathbf{x}] \cdot G^{L'}[\mathbf{x}] = G^{L'}[\mathbf{x}] \cdot GL[\mathbf{x}]$. Particularly, if $L(v,u)[\mathbf{x}] \neq \mathbf{0}$ for any edge $(v,u) \in E(G)$, all network flows $\left\{G^L[\mathbf{x}]\right\}$ form a *bigroup* under double operations, i.e., the addition "+" and the multiplication "$\cdot$", namely all network flows $\left\{G^L[\mathbf{x}]\right\}$ form a group respectively under addition "+" and multiplication "$\cdot$" in this case.

**(3) Algebraic expressions.** By the operations (9.9) – (9.11) on network flows and (9.33) – (9.34), the linear combination, polynomial and fractional expressions of network flows $G^{L_1}[\mathbf{x}]$, $G^{L_2}[\mathbf{x}], \cdots, G^{L_m}[\mathbf{x}]$ are respectively as follows:

① For $m$ real numbers $a_1, a_2, \cdots, a_m \in \mathbb{R}$, the linear combination expressions of network flows $G^{L_1}[\mathbf{x}], G^{L_2}[\mathbf{x}], \cdots, G^{L_m}[\mathbf{x}]$ are respectively

$$\begin{aligned}\left(a_1G^{L_1}[\mathbf{x}]\right) + \left(a_2G^{L_2}[\mathbf{x}]\right) + \cdots + \left(a_mG^{L_m}[\mathbf{x}]\right) &= G^{(a_1L_1)+(a_2L_2)+\cdots+(a_mL_m)}[\mathbf{x}],\\ \left(a_1G^{L_1}[\mathbf{x}]\right) \cdot \left(a_2G^{L_2}[\mathbf{x}]\right) \cdots \left(a_mG^{L_m}[\mathbf{x}]\right) &= G^{(a_1L_1)\cdot(a_2L_2)\cdot\cdots\cdot(a_mL_m)}[\mathbf{x}].\end{aligned}$$

② For $m$ real numbers $a_1, a_2, \cdots, a_m \in \mathbb{R}$, the polynomial expression of network flows $G^L[\mathbf{x}]$ is

$$(a_0\mathbf{I}) + \left(a_1G^L[\mathbf{x}]\right) + \cdots + \left(a_mG^{L^m}[\mathbf{x}]\right) = G^{a_0+(a_1L)+\cdots+(a_mL^m)}[\mathbf{x}].$$

③ For $2m$ real numbers $a_1, a_2, \cdots, a_m, b_1, b_2, \cdots, b_m \in \mathbb{R}$, the fractional expression of network flows $G^{L_1}[\mathbf{x}], G^{L_2}[\mathbf{x}], \cdots, G^{L_m}[\mathbf{x}]$ is

$$\begin{aligned}&\frac{\left(a_1G^{L_1}[\mathbf{x}]\right) + \left(a_2G^{L_2}[\mathbf{x}]\right) + \cdots + \left(a_mG^{L_m}[\mathbf{x}]\right)}{\left(b_1G^{L'_1}[\mathbf{x}]\right) + \left(b_2G^{L'_2}[\mathbf{x}]\right) + \cdots + \left(b_mG^{L'_m}[\mathbf{x}]\right)} = \frac{G^{(a_1L_1)+(a_2L_2)+\cdots+(a_mL_m)}[\mathbf{x}]}{G^{\left(b_1L'_1\right)+\left(b_2L'_2\right)+\cdots+\left(b_mL'_m\right)}[\mathbf{x}]}\\ &= G^{(a_1L_1)+(a_2L_2)+\cdots+(a_mL_m)}[\mathbf{x}] \cdot \frac{\mathbf{I}}{G^{\left(b_1L'_1\right)+\left(b_2L'_2\right)+\cdots+\left(a_mL'_m\right)}[\mathbf{x}]}\\ &= G^{\frac{(a_1L_1)+(a_2L_2)+\cdots+(a_mL_m)}{\left(b_1L'_1\right)+\left(b_2L'_2\right)+\cdots+\left(b_mL'_m\right)}}[\mathbf{x}].\end{aligned}$$

Dr.Ouyang tells Huizi that the algebra of network flow is the calculating result of network flows by arithmetic operations. Notice that the graph $G$ is always invariant in calculation. In this case, the algebraic calculation of network flows is certainly equivalent to that the calculation of mapping $L(v,u)$ on any edge $(v,u) \in E(G)$ by the general operation rules and the calculating result is used to label the edge, which is the significance of introducing the algebraic expression of network flow for characterizing the evolving of network flow.

## §4. Network Flow Calculus

Everything in the universe is in constant moving and changing. Using the network flow $G^L[\mathbf{x}]$ to simulate the moving or changing state of a thing $T$, it is necessary to have the local recognition of elements $v \in T$ with interactions $L(v,u)[\mathbf{x}], v, u \in T$ and then to get the systematic state of thing $T$ at the same time. Dr.Ouyang tells Huizi that the developing and changing of thing $T$ can be divided into two parts. One is the change of the state of $T$ as a whole or an abstract point, namely the synchronous changing among elements; another is the local action between elements $v, u \in T$ of thing $T$. For example, a gymnast to do a back flip $360^o$ can be broken down into the steps of bend the knee ready $\rightarrow$ jump $\rightarrow$ flip back $\rightarrow$ tuck the body $\rightarrow$ landing and so on. Correspondingly, one is the athlete's body rotates $360^o$ in the air to show the athlete's backflip action, as shown in Figure 9.19($a$); the other is the athlete's moving from point A through the arc to point B, as shown in Figure 9.19($b$) to show the synchronized running track of each part of the athlete's body. At this time, if let the head, arms, torso and legs of the athlete's body be elements, the network $G^L$ corresponding to the athlete is a tree $T$ with 7 vertices, as shown in Figure 9.19($c$). In this way, the moving process of an athlete's $360^o$ back flip can be simulated by the evolving of the continuous evolving of network flow $T$.

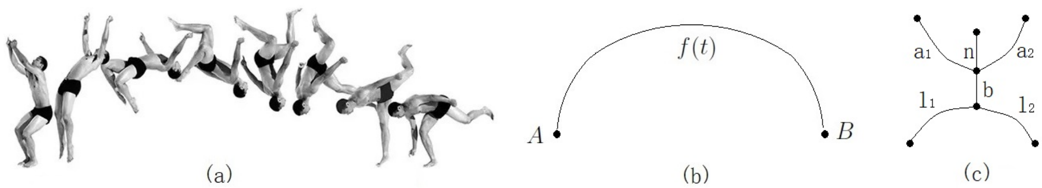



**Figure** 9.19. **Action break down of the back flip** $360^o$

So, *how do we use the evolving of network flow to simulate this synchronized changing of a thing?* Dr.Ouyang explains that the synchronized moving at this time corresponds to the athlete's moving trajectory during a back flip, namely the synchronized motion $f(t)$ of each part of the network $T$ in Figure 9.18($c$). After listening to Dr.Ouyang's explanation, Huizi says with some insight: "*Dad, I think this recognitive way is a bit like relativity! Firstly, imagine the athlete as a particle in the earth reference system of humans. And then, choose a point on the athlete's body as the relative origin to construct a coordinate system for observing the athlete's back flip action and then, combine the two together. This is the coordinate transformation method in special relativity theory, isn't it?*" Dr.Ouyang nods and explains that this approach is the recognitive way guided by Einstein in his relativity theory. After a short pause, Dr.Ouyang asks Huizi: "*An athlete performing a back flip* $360^o$ *is a visible action in human eyes or the macroscopic world. Now replacing the athlete with a biological macromolecule, which can be observed under a microscope but still can not be observed the atoms within the macromolecule. How do you simulate the state of this macromolecule?*" Huizi replies: "*Can we abstract it into a point and use particle mechanics to characterize the state of macromolecule.*" Dr.Ouyang explains that the particle mechanics can only be used to characterize the external behavior of biological molecules but not the internal structure or evolving of the atomic states. For characterizing the state of macromolecule and its internal atoms at the same time, a natural way is to simulate the state of macromolecule, including its whole external appearing and the local state of internal atoms by an evolving of network flow. A generalization of this way is to dynamically recognize the developing and changing of a thing through the combination structure inherited in a thing.

Generally, characterizing a synchronized moving by network flow $G^L[\mathbf{x}]$ needs us to introduce a mapping $f : G^L[\mathbf{x}] \to G^{f(L)}[\mathbf{x}]$ on the state of network flow, i.e.,

$$f\left(G^L[\mathbf{x}]\right) = G^{f(L)[\mathbf{x}]} \tag{9.35}$$

for dynamic simulating on a thing moving and changing.

**4.1.Continuous Mapping on Network.** A continuous mapping on network flow is designed to characterize the changing caused by a slight changing of the network flow. For two continuous evolving $G^L[\mathbf{x}], G^{L'}[\mathbf{x}]$ of network flow, let $f$ be a mapping of $G^L[\mathbf{x}]$. If for any positive number $\varepsilon$ there is a real number $\delta$ such that

$$\rho\left(G^L[\mathbf{x}], G^{L'}[\mathbf{x}_0]\right) < \delta \quad \Rightarrow \quad \rho\left(f\left(G^L[\mathbf{x}]\right), G^{L'}[\mathbf{x}_0]\right) < \varepsilon, \tag{9.36}$$

then $f$ is said to be *continuous* at $G^{L'}[\mathbf{x}_0]$, denoted by $\lim\limits_{G^L \to G^{L'}} f\left(G^L[\mathbf{x}]\right) = G^{L'}[\mathbf{x}_0]$.

Notice that $G^L[\mathbf{x}], G^{L'}[\mathbf{x}_0]$ are two continuous evolving. By definition there are

$$\begin{aligned}
\rho\left(G^L[\mathbf{x}], G^{L'}[\mathbf{x}_0]\right) < \delta \quad &\Leftrightarrow \quad \left|L(v,u)[\mathbf{x}] - L'(v,u)[\mathbf{x}_0]\right| < \frac{\delta}{\sqrt{|E(G)|}},\\
\rho\left(f\left(G^L[\mathbf{x}]\right), G^{L'}[\mathbf{x}_0]\right) < \varepsilon \quad &\Leftrightarrow \quad \left|f(L)(v,u)[\mathbf{x}] - L'(v,u)[\mathbf{x}_0]\right| < \frac{\varepsilon}{\sqrt{|E(G)|}}
\end{aligned}$$

for any edge $(v, u) \in E(G)$. So, the definition (9.35) is equivalent to

$$\left|L(v,u)[\mathbf{x}] - L'(v,u)[\mathbf{x}_0]\right| < \delta \quad \Leftrightarrow \quad \left|f(L)(v,u)[\mathbf{x}] - L'(v,u)[\mathbf{x}_0]\right| < \varepsilon \tag{9.37}$$

for any edge $(v, u) \in E(G)$, denoted by $\lim\limits_{L \to L'} f\left(G^L[\mathbf{x}]\right) = G^{L'}[\mathbf{x}_0]$.

Particularly, if the labeling $L' = f(L)$, namely the mapping $f$ is a continuous evolving of $G^L[\mathbf{x}]$ at a point $G^L[\mathbf{x}_0]$, by the equivalent condition of continuous evolving there are

$$\rho\left(G^L[\mathbf{x}], G^{L'}[\mathbf{x}_0]\right) < \delta \qquad \Leftrightarrow \quad |L(v,u)[\mathbf{x}] - L(v,u)[\mathbf{x}_0]| < \frac{\delta}{\sqrt{|E(G)|}},$$

$$\rho\left(f\left(G^L[\mathbf{x}]\right), G^L[\mathbf{x}_0]\right) < \varepsilon \quad \Leftrightarrow \quad |f(L)(v,u)[\mathbf{x}] - f(L)(v,u)[\mathbf{x}_0]| < \frac{\varepsilon}{\sqrt{|E(G)|}}$$

for any edge $(v, u) \in E(G)$ but $L(v, u)[\mathbf{x}]$ is a continuous function in this case, there must be a positive number $\delta$ such that

$$|\mathbf{x} - \mathbf{x}_0| < \delta \quad \Leftrightarrow \quad |L(v,u)[\mathbf{x}] - L(v,u)[\mathbf{x}_0]| < \frac{\delta_1}{\sqrt{|E(G)|}}.$$

And so, we get an equivalent condition for a continuous mapping $f$ on the evolving of $G^L[\mathbf{x}]$ a point, i.e.:

*A mapping $f$ on the evolving $G^L[\mathbf{x}]$ is continuous at a point $G^L[\mathbf{x}_0]$ if and only if for any positive number $\varepsilon$ and edge $(v, u) \in E(G)$, there exists a real number $\delta$ such that*

$$|\mathbf{x} - \mathbf{x}_0| < \delta \quad \Leftrightarrow \quad |f(L)(v,u)[\mathbf{x}] - f(L)(v,u)[\mathbf{x}_0]| < \varepsilon, \tag{9.38}$$

*namely for any edge $(v, u) \in E(G)$, the constraint $f(L)(v, u)[\mathbf{x}]$ of composition function $f(L)[\mathbf{x}]$ on edge $(v, u)$ is continuous at the point $\mathbf{x}_0$, denoted by $\lim\limits_{\mathbf{x} \to \mathbf{x}_0} f\left(G^L[\mathbf{x}]\right) = f\left(G^L[\mathbf{x}_0]\right)$.*

In this way, Dr.Ouyang tells Huizi that it is easy to extend the elementary functions to the continuous mapping on evolving of network flows by applying this equivalent condition.

**(1) Power mapping.** For any real number $a \in \mathbb{R}$ and an integer $n \in \mathbb{Z}^+$, an *n-power mapping* is determined by

$$f\left(G^L[\mathbf{x}]\right) = a\left(G^L[\mathbf{x}]\right)^n = G^{aL^n[\mathbf{x}]};$$

**(2) Exponential mapping.** For a real number $0 \neq a \in \mathbb{R}$, an exponential mapping is determined by

$$f\left(G^L[\mathbf{x}]\right) = a^{G^L}[\mathbf{x}] = G^{a^L}[\mathbf{x}], \quad \log G^L[\mathbf{x}] = \overrightarrow{G}^{\log L[\mathbf{x}]}.$$

Particularly, if $a = e$, there are identities of network flows, i.e.,

$$e^{G^L}[\mathbf{x}] = G^{e^{L[\mathbf{x}]}}, \quad G^L[\mathbf{x}] = G^{\ln L[\mathbf{x}]};$$

**(3) Trigonometric mapping.** A trigonometric mapping is an extension of trigonometric functions on network flow, i.e.,

$$\begin{array}{rclrcl}
\sin G^L[\mathbf{x}] &=& G^{\sin L[\mathbf{x}]}, & \cos G^L[\mathbf{x}] &=& G^{\cos L[\mathbf{x}]},\\
\tan G^L[\mathbf{x}] &=& G^{\tan L[\mathbf{x}]}, & \cot G^L[\mathbf{x}] &=& G^{\cot L[\mathbf{x}]},\\
\sinh G^L[\mathbf{x}] &=& G^{\sinh L[\mathbf{x}]}, & \cosh G^L[\mathbf{x}] &=& G^{\cosh L[\mathbf{x}]},\\
\coth G^L[\mathbf{x}] &=& G^{\coth L[\mathbf{x}]}, & \tanh G^L[\mathbf{x}] &=& G^{\tanh L[\mathbf{x}]}
\end{array}$$

and other continuous mappings on a network flow. Meanwhile, there are identities following

$$\begin{array}{rclrcl}
\left(\mathbf{I}+aG^L[\mathbf{x}]\right)^n &=& G^{(1+aL[\mathbf{x}])^n}, & \left(\mathbf{I}+\dfrac{a\mathbf{I}}{G^L[\mathbf{x}]}\right)^n &=& G^{\left(1+\frac{a}{L[\mathbf{x}]}\right)^n},\\
\dfrac{\mathbf{I}-G^{nL}[\mathbf{x}]}{\mathbf{I}-G^L[\mathbf{x}]} &=& \mathbf{I}+G^{L[\mathbf{x}]}+G^{2L[\mathbf{x}]}+\cdots+G^{(n-1)L[\mathbf{x}]} & & &
\end{array}$$

of network flows, which can be easily found by the algebraic expression of network flow, continuous mapping and (9.35) for any integer $n\in\mathbb{Z}^+$ and number $a\in\mathbb{R}$. Furthermore, there is an interesting identity of exponential mapping, i.e.,

$$e^{G^L[\mathbf{x}]}=\mathbf{I}+\frac{G^L[\mathbf{x}]}{1!}+\frac{G^{2L}[\mathbf{x}]}{2!}+\cdots+\frac{G^{nL}[\mathbf{x}]}{n!}+\cdots. \tag{9.39}$$

So, *how is the identity* (9.39) *obtained?* Dr.Ouyang explains that a simple calculation of network flow by definition implies that

$$\begin{aligned}
&\mathbf{I}+\frac{G^L[\mathbf{x}]}{1!}+\frac{G^{2L}[\mathbf{x}]}{2!}+\cdots+\frac{G^{nL}[\mathbf{x}]}{n!}+\cdots\\
&\quad= \mathbf{I}+G^{\frac{L[\mathbf{x}]}{1!}}+G^{\frac{2L[\mathbf{x}]}{2!}}+\cdots+G^{\frac{nL[\mathbf{x}]}{n!}}+\cdots\\
&\quad= G^{1+\frac{L[\mathbf{x}]}{1!}+\frac{2L[\mathbf{x}]}{2!}+\cdots+\frac{nL[\mathbf{x}]}{n!}+\cdots} = G^{e^{L[\mathbf{x}]}}
\end{aligned}$$

and so, we know $e^{G^L[\mathbf{x}]}=G^{e^{L[\mathbf{x}]}}$ by (9.35) and get the identity (9.39) following

$$e^{G^L[\mathbf{x}]}=\mathbf{I}+\frac{G^L[\mathbf{x}]}{1!}+\frac{G^{2L}[\mathbf{x}]}{2!}+\cdots+\frac{G^{nL}[\mathbf{x}]}{n!}+\cdots.$$

At the same time, it is easy to know the operation rule

$$\begin{aligned}
e^{G^L[\mathbf{x}]}\cdot e^{G^{L'}[\mathbf{x}]} &= G^{e^{L[\mathbf{x}]}}\cdot G^{e^{L'}[\mathbf{x}]}\\
&= G^{e^{L[\mathbf{x}]}\cdot e^{L'[\mathbf{x}]}}=G^{e^{L[\mathbf{x}]+L'[\mathbf{x}]}}=e^{G^{L[\mathbf{x}]+L'[\mathbf{x}]}}
\end{aligned}$$

by (9.35) and (9.39), which is the same as the well-known exponential function $e^x\cdot e^y=e^{x+y}$. At this point, Huizi find that if replace $G^L[\mathbf{x}]$ with $x$ in (9.39), then

$$e^x=1+\frac{x}{1!}+\frac{x^2}{2!}+\cdots+\frac{x^n}{n!}+\cdots$$

is just the Taylor series of exponential function $e^x$. He tells Dr.Ouyang excitedly on her discovery and says: "*Dad, replacing* $G^L[\mathbf{x}]$ *with* $x$ *in* (9.39) *obtains the Taylor series of* $e^x$*, implies that* (9.39) *is a generalization of the Taylor series of exponential function* $e^x$ *over a network flow. So what's the significance of this kind of generalization?*" Dr.Ouyang smilingly asks her if she still remember the solution of linear differential equation of $\dot{\mathbf{x}} = A\mathbf{x}$ with initial value $\mathbf{x}(t_0) = \mathbf{x}_0 \in \mathbb{R}^n$. Huizi answers the solution of this equation is $\mathbf{x}(t) = e^{tA}\mathbf{x}_0$ and $A$ is an $n \times n$-matrix. Dr.Ouyang then asks her: "*What does the* $e^{tA}$ *mean in the solution* $\mathbf{x}(t) = e^{tA}\mathbf{x}_0$*?*" Huizi replies that it is a simplified notation on the sum of matrixes, i.e.,

$$e^{tA} = I_n + A + \frac{A^2}{2!} + \cdots + \frac{A^k}{k!} + \cdots \tag{9.40}$$

Dr.Ouyang tells her: "$e^{tA}$ *is not only a simplified notation on the sum of matrixes but also a real matrix called exponential matrix. Accordingly,* (9.40) *is a matrix identity.*" Seeing Huizi is still in puzzling, Dr.Ouyang tells her: "*You can try to replace* $tA$ *with* $x$ *in the identity* (9.40)*. Then, what do you get?*" Huizi replies that she gets the Taylor series of $e^x$ again. Dr.Ouyang tells her that the formula (9.40) is a generalization of Taylor series of $e^x$ on exponential matrix $e^{tA}$. From the view point of network flow, a $m\times n$-matrix is nothing else but a special network flow, namely the union $mP_{n+1}$ network flow of $m$ paths $P_{n+1}$ network flows. Thus, the equation (9.39) can be viewed not only as a generalization of the Taylor series of exponential function $e^x$ on the network flow $G^L[\mathbf{x}]$ but also a generalization of the matrix identity (9.40), namely the Taylor series of the exponential function $e^x$ is not only an identity as a function but also as an exponential mapping on network flow, including the identity of exponential matrix (9.40), which reflects the developing way of mathematics from concretion to the abstraction.

**4.2.Network Flow Calculus.** Assumes that $G^L[t]$ is a continuous evolving of network flow on $t$ with a continuous mapping $f$ and $f\left(G^L[t+\Delta t]\right) \to f\left(\overrightarrow{G}^L[t]\right)$ if $\Delta t \to 0$. In this case, if the limitation

$$\lim_{\Delta t\to 0} \frac{f\left(G^L[t+\Delta t]\right) - f\left(G^L[t]\right)}{G^{\Delta t}} \;=\; G^{\lim\limits_{\Delta t\to 0}\frac{f(L)[t+\Delta t]-f(L)[t]}{\Delta t}}$$

exists, it is said that $f$ is *differential* at network flow $G^L[t]$, denoted by

$$\frac{df}{dt} \;=\; \lim_{\Delta t\to 0} \frac{f\left(G^L[t+\Delta t]\right) - f\left(G^L[t]\right)}{G^{\Delta t}}, \tag{9.41}$$

where $\Delta : (v,u) \in E(G) \to \Delta t$ in $G^{\Delta t}$. Now that for any edge $(v,u) \in E(G)$, if the mapping $C : (v,u) \to c_{vu}$ is a constant, $G^C$ is called a *definitive constant flow* and abbreviated $G^C$ to $C$. Notice that a definitive constant flow $C$ is not equivalent to the constant flow $G^{L_c}$.Generally, only if on any edge $(v,u) \in E(G)$ there is $c_{vu} = c$ for a constant $c$, $C$ is a constant flow of $G^{L_c}$, namely the definitive constant flow is $C = G^{L_c}$.

So, *how do we define the integral on the evolving of a network flow?* Dr.Ouyang explains that the integral of a function is the inverse of its differential and so, the integral on network flow is no exception. By the definition of $f$ is differentiable on a network flow, there is

$$\frac{d}{dt} f\left(G^L\right)[t] = F\left(G^L[t]\right) \quad \Rightarrow \quad \frac{d}{dt} f\left(G^L[t] + C\right) = F\left(G^L[t]\right).$$

Thus, we define generally the *indefinite integral* of $f$ on network flow $G^L[t]$ to be

$$\int F\left(G^L[t]\right) dt \;=\; f\left(G^L[t]\right) + C = G^{f(L)+C}. \tag{9.42}$$

So, by the (9.41) and (9.42) there is a relation of differential with the integral as follows:

$$\begin{aligned}\int \left(\frac{df}{dt}\left(\overrightarrow{G}^L[t]\right)\right) dt \;&=\; f\left(\overrightarrow{G}^L[t]\right) + C,\\ \frac{df}{dt}\left(\int \left(f\left(\overrightarrow{G}^L[t]\right)\right) dt\right) \;&=\; f\left(\overrightarrow{G}^L[t]\right).\end{aligned} \tag{9.43}$$

And meanwhile, by the definition of differential (9.41) and integral (9.42) on mapping of evolving of network flow, it is easy to know that the differential $d/dt$ and integral $\int$ are both linear mappings on network flow, namely for any two mappings $f$, $g$ on network flow $G^L[t]$ and definitive constant flows $C_1, C_2$, there are equalities

$$\begin{aligned}\frac{d}{dt}\left(C_1 \cdot f\left(G^L[t]\right) + C_2 \cdot g\left(G^L[t]\right)\right) \;&=\; C_1 \cdot \frac{d}{dt} f\left(G^L[t]\right) + C_2 \cdot \frac{d}{dt} g\left(G^L[t]\right),\\ \int \left(C_1 \cdot f\left(G^L[t]\right) + C_2 \cdot g\left(G^L[t]\right)\right) \;&=\; C_1 \cdot \int f\left(G^L[t]\right) + C_2 \cdot \int g\left(G^L[t]\right).\end{aligned}$$

Dr.Ouyang tells Huizi that if the continuous mapping $f$ is differentiable at $G^L[t]$, by the continuity of network flow $G^L[t]$ and (9.41) there is

$$\frac{df}{dt} \;=\; \lim_{\Delta t \to 0} \frac{f\left(G^L[t+\Delta t]\right) - f\left(G^L[t]\right)}{G^{\Delta t}} = G^{\lim\limits_{\Delta t \to 0} \frac{f(L)[t+\Delta t] - f(L)[t]}{\Delta t}} = G^{\frac{df(L)}{dt}},$$

namely for any edge $(v,u) \in E(G)$, the composite function $f(L)(v,u)[t]$ is differential on $t$. Conversely, if the constraint function $f(L)(v,u)[t]$ of continuous mapping $f$ with labeling $L$ on edge $(v,u)$ is differential to $t$, there must be

$$G^{\frac{df(L)}{dL}} \;=\; G^{\lim\limits_{\Delta t \to 0} \frac{f(L)[t+\Delta t] - f(L)[t]}{\Delta t}} = \lim_{\Delta t \to 0} \frac{f\left(G^L[t+\Delta t]\right) - f\left(G^L[t]\right)}{G^{\Delta t}} = \frac{df}{dt},$$

namely a continuous mapping $f$ is differentiable at the network flow $G^L[t]$. In this way, an equivalent condition is obtained for a mapping $f$ to be differentiable at network flow

$G^L[t]$, i.e. *a continuous mapping $f$ is differentiable at network flow $G^L[t]$ if and only if the constraint function $f(L)(v,u)[t]$ of composite function $f(L)$ on edge $(v,u)$ is differentiable to $t$ for any edge $(v,u)\in E(G)$.* Dr.Ouyang tells Huizi that by this equivalent condition, a computational result of differential and indefinite integral on functions could be naturally generalized to network flows. For example, the computational results listed in the following are all generalization of well-known results in calculus.

(1) **Definitive constant flow.** $\dfrac{dC}{dt} = \mathbf{O}, \quad \displaystyle\int \mathbf{O}dt = \mathbf{C}.$

(2) **Linear flow.** $\dfrac{d}{dt}\left(C\cdot G^t\right)=C, \quad \displaystyle\int Cdt = C\cdot G^t + C'$, where $C, C'$ are definitive constant flows and mapping $t:(v,u)\to t$ for any edge $(v,u)\in E(G)$.

(3) **Power flow.** $\dfrac{d}{dt}\left(G^{nL}[t]\right)=nG^{(n-1)L}[t], \quad \displaystyle\int G^{(n-1)L}[t]dt=\frac{1}{n}G^{nL}[t]$, $n\in\mathbb{Z}^+$.

(4) **Exponential flow.** $\dfrac{d}{dt}\left(e^{G^t}\right)=G^{\frac{de^t}{dt}}=G^{e^t}=e^{G^t}, \quad \displaystyle\int e^{G^t}dt=e^{G^t}.$

(5) **Logarithmic flow.** If $L\neq\mathbf{0}$ for any edge $\forall(v,u)\in E(G)$, then

$$\frac{d}{dt}\left(\ln\left|G^L[t]\right|\right)=G^{\frac{d\ln|L[t]|}{dt}}=G^{\frac{1}{L[t]}}=\frac{1}{G^L[t]}, \qquad \int\frac{dt}{G^L[t]}=\ln\left|G^L[t]\right|.$$

Similarly, the differential, integral on trigonometric and hyperbolic flows can be also obtained. For example,

$$\frac{d}{dt}\left(\sin\left(G^L\right)\right)=\cos\left(G^L\right), \qquad \int\sin\left(G^L\right)dt=-\cos\left(G^L\right),$$
$$\frac{d}{dt}\left(\cos\left(G^L\right)\right)=-\sin\left(G^L\right), \qquad \int\cos\left(G^L\right)dt=\sin\left(G^L\right),\ \cdots.$$

**4.3.Network Calculus Theorem.** As an inverse operation of network differential, the computational results by (9.42) are a family of network flows. So, the integral introduced does not a $1-1$ correspond to network flows in this way but is a $1\leftrightarrow\infty$. Then, *can we introduce a $1-1$ integral on the network flows?* The answer is Yes! Dr.Ouyang tells Huizi that it is nothing else but the *definite integral* of $f$ on network flow, which can be introduced similar to that of the area enclosed by boundary lines $x=a, x=b, y=0$ and $y=f(t)$ in the definite integral of function, the only difference is that $f(t)$ here is replaced by $G^{f(L)[t]}$, as shown in Figure 9.20, namely for two given real numbers $a,b\in\mathbb{R}, a\leq b$, let $a=x_0<t_1<t_2<\cdots<t_n=b$ be a subdivision of the closed interval $[a,b]$ with $\Delta t_i=t_k-t_{k-1}$, $\mu=\max\limits_{1\leq k\leq n}\Delta t_k$ and the constraint function $f(L)(v,u)[t]$ of mapping $f$ on the continuous evolving $G^L[t]$ of network flow for any edge $(v,u)\in E(G)$ only has finite

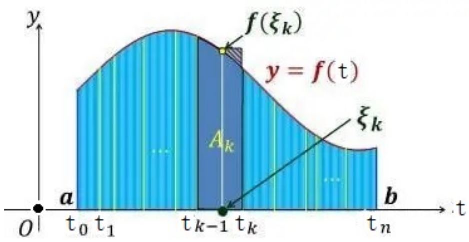



**Figure** 9.20. **Definite integral**

discontinuous points in interval $[a,b]$. In this case, if the length of the subinterval $\mu \to 0$, i.e., the two endpoints of interval gradually approach to one and the limitation of sum $\sum\limits_{k=1}^{n} f\left(G^L[\xi_k]\right) \cdot G^{\Delta t_k}$ exists, then the *definite integral of mapping* $f$ on continuous evolving $G^L[t]$ of network flow in the interval $[a,b]$ is defined to be

$$\int\limits_a^b f\left(G^L[t]\right) dt \; = \; \lim_{\mu \to 0} \sum_{k=1}^{n} f\left(G^L[\xi_i]\right) \cdot G^{\Delta t_k},$$

where for any edge $(v,u) \in E(G)$, the mapping $\Delta t_k : \; (v,u) \to \Delta_{(v,u)} t_k$ dependent on edge $(v,u)$ in network flow $G^{\Delta t_i}$ and $\xi_i \in [t_{k-1}, t_k]$ is a point in interval $[t_{k-1}, t_k]$. Notice that the function $y = f(t)$ from a curve tends to a straight line $y = f(t_k)$ or $f(t_{k-1})$ if $\mu \to 0$, i.e., the two endpoints of $f(t_k)$ and $f(t_{k-1})$ tend to one. In this case, the area $A_k$ of rectangle tends to $A_k \to f\left(G^L\right)[\xi_k] \cdot \Delta_k$, where $1 \leq i \leq n$.

By definition, it is easy to know that

$$\int\limits_a^a f\left(G^L[t]\right) dt = \mathbf{O},$$

$$\int\limits_a^b f\left(G^L[t]\right) dt + \int\limits_b^c f\left(G^L[t]\right) dt = \int\limits_a^c f\left(G^L[t]\right) dt$$

and meanwhile, by the operations of network flows there is

$$\begin{aligned}
\lim_{\mu \to 0} \sum_{i=1}^{n} f\left(G^L[\xi_i]\right) \cdot G^{\Delta t_i} &= \lim_{\mu \to 0} \sum_{i=1}^{n} G^{f(L)[\xi_i]} \cdot G^{\Delta t_i} \\
&= \lim_{\mu \to 0} \sum_{i=1}^{n} G^{f(L)[\xi_i]\Delta t_i} \\
&= \lim_{\mu \to 0} G^{\sum\limits_{i=1}^{n} f(L)[\xi_i]\Delta t_i} = G^{\lim\limits_{\mu \to 0} \sum\limits_{i=1}^{n} f(L)[\xi_i]\Delta t_i},
\end{aligned}$$

namely if $\mu \to 0$ then

$$\lim_{\mu \to 0} \sum_{i=1}^{n} f\left(G^L[\xi_i]\right) \cdot G^{\Delta t_i} \text{ exists} \;\Leftrightarrow\; \lim_{\mu \to 0} \sum_{i=1}^{n} f(L)[\xi_i]\Delta t_i \text{ exists}$$

for any edge $\forall (v,u) \in E(G)$, i.e., the composite function $f(L)(v,u)[t]$ is integral.

Notice that by the definition of differential on network flow, the formula

$$\frac{d}{dt} F\left(G^L[t]\right) = f\left(G^L[t]\right)$$

implies that for any edge $(v,u) \in E(G)$ there is $\frac{d}{dt}F(L)(v,u)[t] = f(L)(v,u)[t]$, i.e.,

$$\begin{aligned}\int_a^b f\left(G^L[t]\right)dt &= \lim_{\mu\to 0}\sum_{i=1}^{n} f\left(G^L[\xi_i]\right)\cdot G^{\Delta t_i} = G^{\lim\limits_{\mu\to 0}\sum\limits_{i=1}^{n} f(L)[\xi_i]\Delta t_i} = G^{\int_a^b f(L)[t]dt} \\ &= G^{F(L)[b]-F(L)[a]} = F\left(G^L[t]\right)\big|_{t=b} - F\left(G^L[t]\right)\big|_{t=a}\end{aligned}$$

and so, the *fundamental theorem of network calculus* is obtained, namely *if $f$ is a mapping on continuous evolving $G^L[t], t \in [a,b]$ of network flow and there are at most finite discontinuous points on the interval $[a,b]$,*

$$\frac{d}{dt}F\left(G^L[t]\right) = f\left(G^L[t]\right),$$

*then there must be*

$$\int_a^b f\left(G^L[t]\right)dt = F\left(G^L[t]\right)\big|_{t=b} - F\left(G^L[t]\right)\big|_{t=a}. \tag{9.44}$$

Obviously, (9.44) is a generalization of the fundamental theorem found by Newton-Leibniz in calculus of functions to network flows, which greatly simplifies the calculation of definite integrals on network flows. For example, $(a)$ is an evolving of network flow $G^L[t]$ and $(b)$ is its integral of network flow $G^L[t]$ on $t$ in interval $[2,4]$ in Figure 9.21.

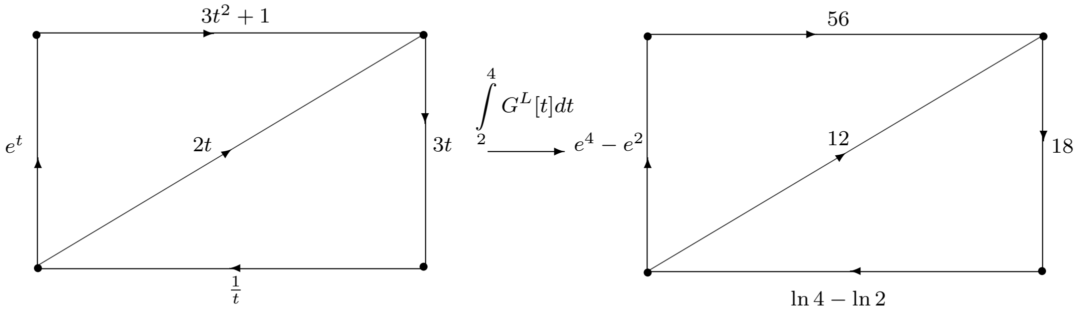



**Figure** 9.21. **An example of network flow integral**

## §5. Network Flow Equation

Generally, a *network flow equation* is an equation or system of equations whose unknown terms are network flows. Accordingly, the solution of network flow equation is the solution of this system of equations, namely the network flow is to simulate the behavior of a thing $T$ to get different expressions with unknown terms $f\left(G^L(T)\right), g\left(G^L(T)\right)$ and an objective thing $T$ does not change in the different form of humans. In this case, there must be the equation $f\left(G^L(T)\right) = g\left(G^L(T)\right)$, including the algebraic equation, functional equation and the differential equation on the network flow. Dr.Ouyang tells Huizi that whether an equation is solvable or not depends on the assumption of existence domain of the solution, namely in what domain to solve an equation. For example, the solution of algebraic equation depends on the number field in which the operation rules of solution are predetermined, the functional equation and differential equation depend on the function field in which the function characters of solution are predetermined. Similarly, the solution of the network flow equation depends on the number fields or function fields on that of the network flow $G^L$. For example, the solution to the algebraic equation $2x-4=0$ in the real number field $\mathbb{R}$ is $x=2$. Now, assume that $G=K_3$ for a complete graph of order 3 or 3-circuit, the solution of number domain is still $\mathbb{R}$. Then, the constants in this equation are constant flows of $2=K_3^{L_2}$ and $4=K_3^{L_4}$. Accordingly, the equation of $2-4X=0$ is a network flow equation


**Figure** 9.22. **I'm who I am**

$$K_3^{L_2}\cdot X - K_3^{L_4} = \mathbf{O}.$$

In this way, by the operations of network flow there is

$$K_3^{L_2}\cdot X = K_3^{L_4} + \mathbf{O} = K_3^{L_4} \quad \Rightarrow \quad X = K_3^{L_4}\cdot 1/K_3^{L_2} = K_3^{L_4/L_2} = K_3^{L_2},$$

namely the network flow solution of equations is $X=K_3^{L_2}$, which is still a constant flow.

**5.1.Linear Equation of Network Flow.** For an integer $n\geq 1$, let $\left\{G^L[t]\right\}$ be a dynamic network flow on graph $G$ and let $C_{ij}, 1\leq i\leq m, 1\leq j\leq n$ be $mn$ definitive constant flows. Then, *are there $n$ network flows $X_1, X_2,\cdots,X_n$ in $\left\{G^L[t]\right\}$ holding with linear equations*

$$\left\{\begin{array}{l} C_{11}\cdot X_1 + C_{12}\cdot X_2 + \cdots + C_{1n}\cdot X_n = G^{L_1}\\ C_{21}\cdot X_1 + C_{22}\cdot X_2 + \cdots + C_{2n}\cdot X_n = G^{L_2}\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ C_{m1}\cdot X_1 + C_{m2}\cdot X_2 + \cdots + C_{mn}\cdot X_n = G^{L_m} \end{array}\right. \tag{9.45}$$

*of network flow?* Here, $X_j = G^{L_i}$ is a network flow in $\left\{G^L[t]\right\}$ for an arbitrary integer $1\leq j\leq n$. Dr.Ouyang explains that assumes that the unknown flow on $(v,u)$ in network flow $X_j$ is $x_j^{vu}$ and the flow on edge $(v,u)$ in definitive constant flows $C_{ij}$ is $c_{ij}^{vu}$, $1\leq i\leq$

$m, 1 \leq j \leq n$. Then, the linear equation on edge $(v,u)$ of linear network flow equation (9.45) is

$$\left\{\begin{array}{l} c_{11}^{vu}x_1^{vu} + c_{12}^{vu}x_2^{vu} + \cdots + c_{1n}^{vu}x_n^{vu} = L_1(v,u) \\ c_{21}^{vu}x_1^{vu} + c_{22}^{vu}x_2^{vu} + \cdots + c_{2n}^{vu}x_n^{vu} = L_2(v,u) \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ c_{m1}^{vu}x_1^{vu} + c_{m2}^{vu}x_2^{vu} + \cdots + c_{mn}^{vu}x_n^{vu} = L_1(v,u) \end{array}\right. \tag{9.46}$$

which is the matrix equation (5.34), i.e. $\mathbf{Ax^t} = \mathbf{B}$, where

$$\mathbf{A} = \begin{pmatrix} c_{11}^{vu} & c_{12}^{vu} & \cdots & c_{1n}^{vu} \\ c_{21}^{vu} & c_{22}^{vu} & \cdots & c_{2n}^{vu} \\ \vdots & \vdots & \vdots & \vdots \\ c_{m1}^{vu} & c_{m2}^{vu} & \cdots & c_{mn}^{vu} \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} L_1(v,u) \\ L_2(v,u) \\ \vdots \\ L_m(v,u) \end{pmatrix}, \quad \mathbf{x}^t = \begin{pmatrix} x_1^{vu} \\ x_2^{vu} \\ \vdots \\ x_n^{vu} \end{pmatrix}.$$

Consequently, by the solvable condition of linear system (9.46) it is easy to known that *the linear network flow equation* (9.45) *is solvable if and only if the linear system* (9.46) *of equations is solvable on any edge* $(v,u) \in E(G)$, *i.e.,*

$$\text{rank}\left[c_{ij}^{vu}\right]_{m\times n} = \text{rank}\left[c_{ij}^{vu}\right]_{m\times(n+1)}^{+}$$

*for any edge* $(v,u) \in E(G)$, where rank$\mathbf{A}$ is the *rank* of matrix $\mathbf{A}$, namely the integer $l$ obtained by the elimination method on the matrix equation (5.34) and

$$\left[c_{ij}^{vu}\right]_{m\times(n+1)}^{+} = \begin{bmatrix} c_{11}^{vu} & c_{12}^{vu} & \cdots & c_{1n}^{vu} & L_1(v,u) \\ c_{21}^{vu} & c_{22}^{vu} & \cdots & c_{2n}^{vu} & L_2(v,u) \\ \cdots & \cdots & \cdots & \cdots & \cdots\cdots \\ c_{m1}^{vu} & c_{m2}^{vu} & \cdots & c_{mn}^{vu} & L_m(v,u) \end{bmatrix}.$$

Particularly, if $G^{L_{c_i}}$ is a definitive constant flow $G^{L_i} = G^{L_{x_i}}, 1 \leq i \leq m, 1 \leq j \leq n$, the linear network equations (9.45) is nothing else but the linear equations (5.34). In this case, the solution $X_j = G^{L_{x_j}}$ is a constant flow for integers $1 \leq j \leq n$.

**5.2.Higher Order Equation of Network Flow.** For any integer $n \geq 1$, consider the network flow equation

$$C_n \cdot X^n + C_{n-1} \cdot X^{n-1} + \cdots + C_1 \cdot X + C_0 = \mathbf{O} \tag{9.47}$$

of order $n$ in dynamic network flow $\left\{G^L[t]\right\}$, where for any edge $(v,u) \in E(G)$ and integer $1 \leq i \leq n$, the flow on edge $(v,u)$ of definitive constant flow $C_i$ is $c_i^{vu}$ with $c_n^{vu}(v,u) \neq 0$. Now that assume $X = G^L$ is a solution of network flow equation (9.47). Then, the left side of (9.47) is known as

$$C_n \cdot X^n + C_{n-1} \cdot X^{n-1} + \cdots + C_1 \cdot X + C_0$$
$$= G^{C_nL^n + C_{n-1}L^{n-1} + \cdots + C_1L + C_0} \;=\; G^{p(L)},$$

namely the network flow equation (9.47) is equivalent to $G^{p(L)} = \mathbf{O}$ by definition, where the polynomial $p\,(L) = C_nL^n + C_{n-1}L^{n-1} + \cdots + C_1L + C_0$ and meanwhile, there is

$$c_n^{vu}L^n(v,u) + c_{n-1}^{vu}L^{n-1}(v,u) + \cdots + c_1^{vu}L(v,u) + c_0^{vu} \;=\; 0 \tag{9.48}$$

for any edge $(v,u) \in E\,(G)$. Thus, by the fundamental theorem in classical algebra, the algebraic equation (9.48) has $n$ roots in the complex number field $\mathbb{C}$, i.e., $\lambda_1^{vu}, \lambda_2^{vu}, \cdots, \lambda_n^{vu}$, namely the mapping $L$ in a solution $G^L$ of network flow equation (9.47) is

$$L: \;(v,u) \to \;\{\lambda_1^{vu}, \lambda_2^{vu}, \cdots, \lambda_n^{vu}\}\,, \quad (v,u) \in E\,(G)$$

and then, a solution $X = G^L$ of network equation (9.47) is obtained.

Conversely, for any edge $(v,u) \in E(G)$, define a labeling mapping $L : (v,u) \to \{\lambda_1^{vu}, \lambda_2^{vu}, \cdots, \lambda_n^{vu}\}$ which results in a network flow $G^L$ on $G$. By assumption,

$$c_n^{vu}L^n(v,u) + c_{n-1}^{vu}L^{n-1}(v,u) + \cdots + c_1^{vu}L(v,u) + c_0^{vu} = 0,$$

it is easy to know that $G^{p(L)} = \mathbf{O}$, i.e., for such a network flow $X = G^L$ there must be

$$C_n \cdot X^n + C_{n-1} \cdot X^{n-1} + \cdots + C_1 \cdot X + C_0 = \mathbf{O}$$

and so, we get a necessary and sufficient condition for the solvability of equation (9.47) in dynamic network flow $\left\{G^L[t]\right\}$, namely *a network flow $G^L$ is a solution of* (9.47) *if and only if for any edge $(v,u) \in E\,(G)$, the labeling mapping $L : (v,u) \to \;\{\lambda_1^{vu}, \lambda_2^{vu}, \cdots, \lambda_n^{vu}\}$, where for any integer $1 \le i \le n$, $L_{c_i}(v,u) \in \mathbb{C}$, $L_{c_n}(v,u) \neq 0$ and $\lambda_1^{vu}, \lambda_2^{vu}, \cdots, \lambda_n^{vu}$ is the $n$ roots of polynomial $p(L)(v,u)[t]$ in complex number field $\mathbb{C}$.*

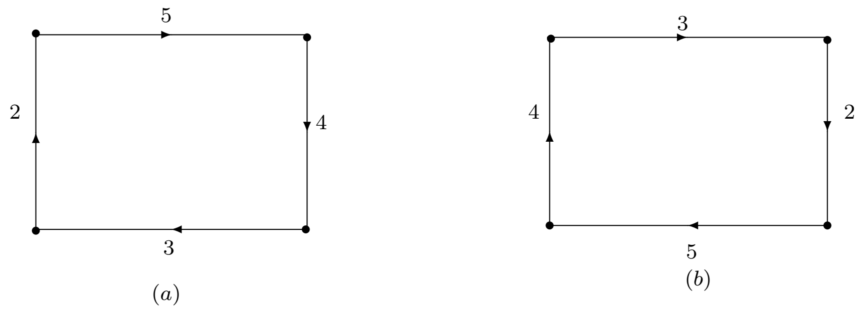



**Figure** 9.23. **Isomorphic network flows**

Notice that some of the solutions of network flow equation (9.47) constructed in the previous way are essentially the same or *isomorphism* in terminology. They are looking different only because the view point standing on the plane is different. For example, the two network flows ($a$) and ($b$) in Figure 9.23 are essentially the same and can be obtained from the rotation of drawing board, namely the network flow ($b$) can be obtained from network flow ($a$) by rotating $180^o$ around the center of graph $C_4$, which is not intrinsically different from network flow ($a$) in combination structure and edge flow. So, *how can we characterize that two network flows are essentially consistent and then, calculate how many essentially different solutions $G^L$ for network equation* (9.47)*?* Dr.Ouyang explains that the rotation of $180^o$ on $C_4^L$ in Figure 9.23 is a symmetric transformation on the image, characterized by changing vertices and edges in the network flow ($a$) to that of vertices and edges in ($b$) such that the flow of each edge is invariant. General, there are 4 symmetric transformations on a square that hold the image invariant, namely the rotations of $0^o$, $90^o$, $180^o$ and $270^o$ around the center of square. Notice that folding along the center line of opposite side of a square also keeps the square unchanged. But it is equivalent to rotating $180^o$ around the center of square. However, the four edges of $C_4^L$ in ($a$) and ($b$) all have directions. Therefore, the folding ($a$) is not equal to ($b$) because the direction of network flow obtained by folding ($a$) is different from the network flow ($b$). Similarly, there are 24 symmetric transformations on a cube to keep the cube unchanged, which are respectively 1 transformations to keep all points stationary, 9 transformations around the 3 face-face center axes, namely the cube around the center axes of faces ABCD-EFGH, ABFE-CDHG or ADHE-BCGF rotates $90^o, 180^o$ and $270^o$, 6 transformations of the cube around the edge-edge center, namely the cube around 6 axes of edge centers, i.e., AB-GH, CD-EF, AE-CG, BF-DH, AD-FG, BC-EH rotates $180^o$ and 8 transformations of the cube around 4 diagonal lines AG, CE, BH and DF rotates $120^o$, $240^o$, as shown in Figure 9.24. For simplicity, the six faces of cube in Figure 9.24 are respectively marked with the numbers $1, 2, 3, 4, 5$ and 6, i.e. $ABCD = 1, EFGH = 2, ABFE = 3, CDHG = 4, ADHE = 5, BCGF = 6$. In this way, the symmetric transformation on a cube can be simplified to the change of numbers $1, 2, 3, 4, 5, 6$, called *permutation.* For example, a rotation of $180^o$ about the $l$ axis corresponds to a transformation of $1 \leftrightarrow 1,\ 2 \leftrightarrow 2,\ 3 \leftrightarrow 4,\ 5 \leftrightarrow 6$.

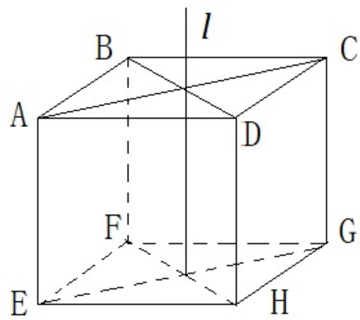



**Figure** 9.24. **Cubic symmetry**

Generally, let $\Omega = \{a_1, a_2, \cdots, a_n\}$ be a set of $n$ elements. Then, a permutation on

$\Omega$, i.e., a transformation $\sigma : \Omega \to \Omega$ is a $1-1$ mapping between its elements. Usually, denote the image of element $a_i$ under that action of $\sigma$ by $a_i^\sigma$ for integers $1 \leq i \leq n$. Then, the permutation $\sigma$ can be generally expressed as

$$\sigma = \begin{pmatrix} a_1 & a_2 & \cdots & a_n \\ a_1^\sigma & a_2^\sigma & \cdots & a_n^\sigma \end{pmatrix}.$$

Notice that the elements $a_i, 1 \leq i \leq n$ in $\Omega$ are symbols of elements. So, we can further let $\Omega = \{1, 2, \cdots, n\}$ for simplicity. In this case, the permutation $\sigma$ can be expressed as

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ 1^\sigma & 2^\sigma & \cdots & n^\sigma \end{pmatrix}.$$

Now, let $\sigma, \tau$ be two permutations on $\Omega$. The product $\sigma\tau$ is defined to be a consecutive action of $\tau$ and $\sigma$ on elements $i = 1, 2, \cdots, n$, i.e., $i^{\sigma\tau} = (^\sigma)^\tau$. For example, if

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}, \quad \tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix},$$

then their product is

$$\sigma\tau = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}.$$

Particularly, if the action of $\sigma$ on an element $a_1 \in \Omega$ is $a_1^\sigma = a_2, a_2^\sigma = a_3, \cdots, a_{m-1}^\sigma = a_m, a_m^\sigma = a_1$, namely $a_1$ comes back $a_1$ after $m$ actions of $\sigma$, called an $m$-cycle of $a_1$ under the action of $\sigma$ and denoted by $(a_1, a_2, \cdots, a_m)$ in general. And then, choose an element $b_1$ in $\Omega \backslash \{a_1, a_2, \cdots a_m\}$ under the repeated action of $\sigma$ until back to $b_1$ and get the second cycle $(b_1 b_2, \cdots, b_m)$. This process continues until each element in $\Omega$ is in a cycle. We get a conclusion that *any permutation $\sigma$ can be uniquely expressed as a disjoint cyclic product.* For example, we have $\sigma = (1243), \tau = (12)(34)$.

Generally, all permutations on set $\Omega$ with the consecutive action are called the *symmetric group $S_n$* on $\Omega$ because they really form a group satisfying the axioms of group in algebra, i.e., ①Closure, namely the consecutive action of permutations is still a permutation; ②Associativity, namely for 3 permutations $\sigma, \tau, \rho \in S_n$ and $a \in \Omega$, there is $a^{\rho(\sigma\tau)} = a^{\sigma(\tau\rho)}$; ③Existence a unit element uniquely, namely $1_{id} = (1)(2)\cdots(n)$; ④Existence a unique inverse of element, namely for any permutation $\sigma \in S_n$ exists an inverse permutation $\sigma^{-1}$ holding with $\sigma\sigma^{-1} = \sigma^{-1}\sigma = 1_{id}$. Generally, a subset $H \subset S_n$ of the symmetric group $S_n$ is called a *permutation group* if it satisfies the 4 axioms of group. For an element $a \in \Omega$, the action $\alpha^g, g \in H$ of permutation group $H$ on element $a$ is called the orbit of element

$a$ under the action of $H$, denoted by $a^H$. Notice that for $h_k, h_l \in H$, if $a^{h_k} = a^{h_l}$ then there must be $a^{h_k h_l^1} = 1_{id}$ or keep the element $a$ still under the action of $h_k h_l^{-1}$. Generally, all elements in $H$ that hold $a$ still also form a subgroup of $H$, called the *stabilizer* of $a$, denoted by $H_a$. Now that if the orbit $a^H = \{a, a^{h_1}, \cdots, a^{h_{s-1}}\}$, then the intersection of any two orbit of elements $a, a^{h_1}, \cdots, a^{h_{s-1}}$ under the action of $H$ must be an empty set. Whence, the length of all orbits are the same, i.e., a number $s$ and $s$ equals to the number of elements in $H$ that keep the element $a \in \Omega$ still or $|H_a|$. We therefore get the Burnside lemma

$$\left|a^H\right| |H_a| \ = \ |H|. \tag{9.49}$$

Particularly, let $\Omega$ be the vertex set $V(G)$ of a graph $G$. In this case, the permutation group consisted of all permutations on $V(G)$ that maintain the adjacency relation of vertices is called the *automorphism group* of $G$, denoted by $\mathrm{Aut}G$, namely for any element $g \in \mathrm{Aut}G$ and edge $(v,u) \in E(G)$, $g(v,u) = (g(v), g(u))$.

According to the isomorphism of graph, *how do we determine two network flows $G^{L_1}, G^{L_2}$ isomorphism?* Dr.Ouyang explains that a network flow isomorphism is a graph isomorphism constrained on $1-1$ corresponding of edge flows, namely if there is an automorphism $g \in \mathrm{Aut}G$ of graph $G$ such that $gL_1(v,u) = L_2 g(v,u) = L_2(g(v), g(u))$ for any edge $(v,u) \in E(G)$, then $g$ is an *isomorphism* between network flows $G^{L_1}$ and $G^{L_2}$. Particularly, $g$ is said to be an *automorphism of network flow $G^L$* if $L_1 = L_2$. Similarly, all automorphisms on $G^L$ form a permutation group, denoted by $\mathrm{Aut}G^L$. Thus, by the definition of automorphism, there is a natural inclusion between automorphism groups of graph $G$ and network flow $G^L$, i.e., $\mathrm{Aut}G^L \preceq \mathrm{Aut}G$ and also $\left|\mathrm{Aut}G^L\right| \leq |\mathrm{Aur}G|$.

Now, we can determine how many network flow solutions of equation (9.47) are essentially different in $\left\{G^L[t]\right\}$, i.e. the number $n\left(G^L\right)$ of non-isomorphic solutions $G^L$. First of all, by the solution condition of equation (9.47, there are totally $n^{E(G)}$ methods for choosing different labeling mapping $L$ on edge $(v,u) \in E(G)$ and two network flows $G^{L_1}$, $G^{L_2}$ are isomorphic if and only if there exists an automorphism $\varphi : G \to G$ of graph $G$ such that $L_2(v,u) = L_1\varphi(v,u)$ for any edge $(v,u) \in E(G)$. Secondly, let $\Omega$ be all network flows with labeling mapping

$$L: \ (v,u) \to \ \{\lambda_1^{vu}, \lambda_2^{vu}, \cdots, \lambda_n^{vu}\}, \quad (v,u) \in E(G)$$

and consider the action of automorphism group $\mathrm{Aut}G^L$ on $\Omega$. Notice that for a solution $G^{L_0}$ of network flow equation (9.47), the number of isomorphic solution of network flow of equation (9.47) is $\left|\left(G^{L_0}\right)^{\mathrm{Aut}G^L}\right|$, namely the length of orbit that $G^{L_0}$ under the action of automorphism group $\mathrm{Aut}G^L$. Notice that only the unit element $1_{id}$ of automorphism group $\mathrm{Aut}G^L$ that keeps the network flow $G^{L_0}$ stationary, i.e., $\left(\mathrm{Aut}G^L\right)_{G^{L_0}} = \{1_{id}\}$ which concludes that $\left|\left(\mathrm{Aut}G^L\right)_{G^{L_0}}\right| = 1$.

According to Burnside's lemma, $\left|\mathrm{Aut}G^L\right| = \left|(\mathrm{Aut}G^L)_{G^{L_0}}\right| \left|\left(G^{L_0}\right)^{\mathrm{Aut}G^L}\right|$. We therefore know that $\left|\left(G^{L_0}\right)^{\mathrm{Aut}G^L}\right| = \left|\mathrm{Aut}G^L\right|$, namely the orbit length of $G^{L_0}$ under the action of automorphism group $\mathrm{Aut}G^L$ is a constant, i.e., $\left|\mathrm{Aut}G^L\right|$. Consequently, the number of different isomorphic solutions of network flow of equation (9.47) in $\left\{G^L\right\}$ is

$$n\left(G^L\right) = \frac{n^{|E(G)|}}{|\mathrm{Aut}G^L|}. \tag{9.50}$$

For example, let $G = C_m$, $K_m$, or $B_m$ for any integer $m \geq 3$. Then,

$$n\left(G^L\right) = \frac{n^m}{2m}, \quad \frac{n^m}{m!} \quad \text{or} \quad \frac{n^{\frac{m(m-1)}{2}}}{m!}.$$

**5.3.Differential Equation of Network Flow.** Generally, let $G^{L_1}[t], G^{L_0}[t]$ be two known network flows in $\left\{G^L[t]\right\}$. A first order ordinary differential equation of network flow is

$$\frac{dX}{dt} = G^{L_1}[t] \cdot X + G^{L_0}[t]. \tag{9.51}$$

As is known to all, there is a closed formula for the general solution of first order ordinary differential equation. Similarly, we can also get a closed formula for the first order ordinary differential equation of network flow.

First of all, assume that $G^{L_0}[t] = \mathbf{O}$. Then, there is

$$\frac{dX}{dt} = G^{L_1}[t] \cdot X \quad \Rightarrow \quad \frac{dX}{X} = G^{L_1}[t]dt. \tag{9.52}$$

In this case, the integral of (9.52) is

$$\ln|X| = \int G^{L_1}[t]dt + C' \quad \Rightarrow \quad X[t] = C' \cdot e^{\int G^{L_1}[t]dt}, \tag{9.53}$$

where $C'$ is a definitive constant flow. Next, let $C'$ varies on $t$, namely $C' = G^L[t]$ and substitute it into equation (9.51), we know that

$$\begin{aligned}\left(\frac{d}{dt}\left(G^L[t]\right)\right) e^{\int G^{L_1}[t]dt} &+ G^L[t] \cdot e^{\int G^{L_1}[t]dt} \cdot G^{L_1}[t] \\ &= G^{L_1}[t] \cdot G^L[t] \cdot e^{\int G^{L_1}[t]dt} + G^{L_0}[t].\end{aligned}$$

Combining the liking terms in previous formula and further simplifying, there is

$$\frac{d}{dt}\left(G^L[t]\right) = G^{L_0} \cdot e^{-\int G^{L_1}dt} \quad \Rightarrow \quad G^L[t] = \int G^{L_0} \cdot e^{-\int G^{L_1}dt}dt + C,$$

where $C$ is a definitive constant flow. Thus, the general solution of network equation (9.51) can be expressed as

$$X[t] = e^{\int G^{L_1}dt} \cdot \left( \int G^{L_0} \cdot e^{-\int G^{L_1}dt} dt + C \right). \tag{9.54}$$

Similarly, the initial value problem

$$\begin{cases} \dfrac{dX}{dt} = G^{L_1}[t] \cdot X + G^{L_0}[t] \\ X|_{t=t_0} = G^{L(0)} \end{cases} \tag{9.55}$$

on the first order ordinary differential equation (9.51) of network flow can be solved by determining the definitive constant flow $C$ in (9.54). Generally, assume that the solution to the initial value problems (9.55) is $X = G^L[t] \cdot e^{\int_{t_0}^t G[L_1]^d x}$. Substituting it into equation (9.55) and combining the liking terms, we have

$$\frac{d}{dt}\left(G^L[t]\right) = G^{L_0}[t] \cdot e^{-\int_{t_0}^t G^{L_1}[x]dx} \ \Rightarrow \ G^L[t] = \int\limits_{t_0}^{t} G^{L_0}[x] \cdot e^{-\int_{t_0}^x G^{L_1}[s]ds} dx + C.$$

Consequently,

$$X[t] = \left( \int\limits_{t_0}^{t} G^{L_0}[x] \cdot e^{-\int_{t_0}^x G^{L_1}[s]ds} dx + C \right) \cdot e^{\int_{t_0}^t G^{L_1}[x]dx}$$

and so, if $t = t_0$ there must be

$$X(t_0) = \left( \int\limits_{t_0}^{t_0} G^{L_0}[x] \cdot e^{-\int_{t_0}^{t_0} G^{L_1}[s]ds} dx + C \right) \cdot e^{\int_{t_0}^{t_0} G^{L_1}[x]dx}$$

By the assumption and properties of network flow integrals, there are

$$X|_{t=t_0} = G^{L(0)}, \qquad \int_{t_0}^{t_0} G^{L_0}[x]dx = \int_{t_0}^{t_0} G^{L_1}[x]dx = \mathbf{O}.$$

Now that $X(t_0) = (\mathbf{O} + C) \cdot e^{\mathbf{O}} = C \cdot I = C$, i.e., $C = X(t_0) = G^{L(0)}$. So, the solution to the initial value problem (9.55) is

$$X[t] = \left( \int\limits_{t_0}^{t} G^{L_0}[x] \cdot e^{-\int_{t_0}^x G^{L_1}[s]ds} dx + G^{L(0)} \right) \cdot e^{\int_{t_0}^t G^{L_1}[x]dx} \tag{9.56}$$

For example, let $G^{L_1}[t]$ and $G^{L_0}[t]$ be two network flows shown respectively in $(a)$ and $(b)$ of Figure 9.25. Then, the general solution of equation (9.51) is the network flow shown in Figure 9.25($c$) in which $\alpha$ is a constant. Particularly, let $L(0) : (v,u) \to te^t$ for any edge $(v,u) \in E(G)$, i.e., the network flow shown in Figure 9.25($c$) with $\alpha = 1/2$. We get the solution of initial value problem (9.56) in this case.

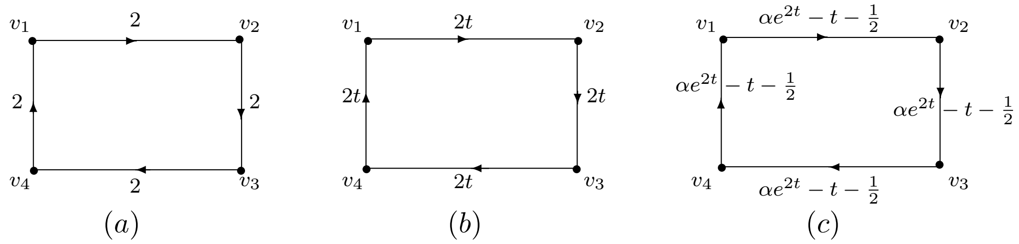



**Figure** 9.25. **Example of network flow solution of differential equation**

For an integer $n \geq 1$ and definitive constant flows $C_0, C_1, \cdots, C_{n-1}$, the $n$-order ordinary differential equation

$$\frac{d^n X}{dt^n} + C_{n-1} \cdot \frac{d^{n-1}X}{dt^{n-1}} + C_{n-1} \cdot \frac{d^{n-2}X}{dt^{n-2}} + \cdots + C_1 \cdot \frac{dX}{dt} + C_0 = \mathbf{O} \qquad (9.57)$$

of network flows $\{G^L[t]\}$ on graph $G$ is called a *constant coefficient n-order ordinary differential equation of network flow* and its corresponding characteristic equation is algebraic network flow equation

$$\Lambda^n + C_{n-1} \cdot \Lambda^{n-1} + C_{n-2} \cdot \Lambda^{n-2} + \cdots + C_1 \cdot \Lambda + C_0 = \mathbf{O}. \qquad (9.58)$$

Particularly, for edge $\forall (v,u) \in E(G)$, the characteristic equation (9.58) constraint on $(v,u)$ is an algebraic equation

$$\lambda^n + c_{n-1}^{vu}\lambda^{n-1} + c_{n-2}^{vu}\lambda^{n-2} + \cdots + c_1^{vu}\lambda + c_0^{vu} \ = \ 0. \qquad (9.59)$$

Now, assume that the $n$ roots of algebraic equation (9.59) are respectively

$$\begin{array}{l}
\lambda_1^{vu} = r_1^{vu}, \lambda_2^{vu} = r_1^{vu}, \cdots, \lambda_{m_{r_1}}^{vu} = r_1^{vu}, \\
\lambda_{m_{r_1}+1}^{vu} = r_2^{vu}, \lambda_{m_{r_1}+2}^{vu} = r_2^{vu}, \cdots, \lambda_{m_{r_1}+m_{r_2}}^{vu} = r_2^{vu}, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots , \\
\lambda_{m_{r_1}+m_{r_2}+\cdots+m_{r_{s-1}}+1}^{vu} = r_s^{vu}, \lambda_{m_{r_1}+m_{r_2}+\cdots+m_{r_{s-1}}+2}^{vu} = r_s^{vu}, \cdots, \lambda_n^{vu} = r_s^{vu}
\end{array}$$

where $m_{r_1} + m_{r_2} + \cdots + m_{r_s} = n$, then the general solution of network flow of $n$-order algebraic equation (9.58) can be expressed as $\Lambda = G^{L_\Lambda}$. In this case, the labeling mapping $L_\lambda : (v,u) \to \lambda_i^{vu}$, $1 \leq i \leq n$. Now, for edge $\forall (v,u) \in E(G)$, define a mapping $\tau : G^{L_\lambda} \to G^{\tau(L_\lambda)}$ by

$$\tau : \ L_\lambda(v,u) = \lambda_i^{vu} \ \to \left\{ \begin{array}{ll}
t^{i-1}e^{r_1^{vu}t}, & 1 \leq i \leq m_{r_1}, \\
t^{i-1}e^{r_2^{vu}t}, & m_{r_1} + 1 \leq i \leq m_{r_1} + m_{r_2}, \\
\cdots\cdots & \cdots\cdots\cdots\cdots\cdots\cdots\cdots , \\
t^{i-1}e^{r_s^{vu}t}, & m_{r_1} + m_{r_2} + \cdots + m_{r_{s-1}} + 1 \leq i \leq n,
\end{array} \right.$$

then, $\left\{G^{\tau\left(L_\lambda^1\right)}, G^{\tau\left(L_\lambda^2\right)}, \cdots, G^{\tau\left(L_\lambda^m\right)}\right\}$ constitutes a spanning basis of solution space $\mathscr{S}$ with dimension

$$\dim\mathscr{S} = \frac{n^{|E(G)|}}{|\mathrm{Aut}G^L|} \tag{9.60}$$

of network flow equation (9.57). But, *why does the solution space of network flow equation* (9.57) *look like this?* Dr.Ouyang explains that first of all, the solution of network flow of network flow equation (9.57) constraint on any edge $(v,u)\in E(G)$ is a linear differential equation

$$\frac{d^n x}{dt^n} + c_{n-1}^{vu}\frac{d^{n-1}x}{dt^{n-1}} + c_{n-2}^{vu}\frac{d^{n-2}x}{dt^{n-2}} + \cdots + c_1^{vu}\frac{dx}{dt} + c_0^{vu} = 0 \tag{9.61}$$

with constant coefficients. Thus, by the construction of solution space of linear ordinary differential equation with constant coefficients, assume the general solution of (9.61) is $x = C_1'\tau\left(L_\lambda^1\right) + C_2'\tau\left(L_\lambda^2\right) + \cdots + C_m'\tau\left(L_\lambda^m\right)$. Then, the linear combination

$$\begin{aligned} X &= C_1'\cdot G^{\tau\left(L_\lambda^1\right)} + C_2'\cdot G^{\tau\left(L_\lambda^2\right)} + \cdots + C_m'\cdot G^{\tau\left(L_\lambda^m\right)} \\ &= G^{C_1'\tau\left(L_\lambda^1\right)+C_2'\tau\left(L_\lambda^2\right)+\cdots+C_m'\tau\left(L_\lambda^m\right)} = \mathbf{O} \end{aligned}$$

by operation of network flows, namely $X$ is a network flow solution of equation (9.57). And meanwhile, the network flows $G^{\tau\left(L_\lambda^1\right)}, G^{\tau\left(L_\lambda^2\right)}, \cdots, G^{\tau\left(L_\lambda^m\right)}$ are linearly independent, which implies that there are no definitive constant flows $C_1, C_2, \cdots, C_m$ such that

$$C_1\cdot G^{\tau\left(L_\lambda^1\right)} + C_2\cdot G^{\tau\left(L_\lambda^2\right)} + \cdots + C_m\cdot G^{\tau\left(L_\lambda^m\right)} = \mathbf{O}. \tag{9.62}$$

Otherwise, there must exist an edge $(v,u)\in E(G)$ such that

$$c_1^{vu}\tau\left(L_\lambda^1(v,u)\right) + c_2^{vu}\tau\left(L_\lambda^2(v,u)\right) + \cdots + c_m^{vu}\tau\left(L_\lambda^m(v,u)\right) = 0,$$

which contradicts to the classical theory of linear differential equation that the solution set $\{\tau(\lambda_i^{vu}), 1\le i\le n\}$ is a spanning basis because the constraint of network flow equation (9.61) on edge $(v,u)$ is an $n$-order ordinary differential equation with constant coefficient. Thus, $\left\{G^{\tau\left(L_\lambda^1\right)}, G^{\tau\left(L_\lambda^2\right)}, \cdots, G^{\tau\left(L_\lambda^m\right)}\right\}$ form a spanning basis of the solution space $\mathscr{S}$ of network flow equation (9.57) in case. Notice that the dimension $\dim\mathscr{S}$ of solution space $\mathscr{S}$ is determined by the number of different isomorphic network flow solutions of algebraic network equation (9.58), which implies that the dimension of solution space $\mathscr{S}$ can be easily calculated by (9.60), i.e., the number (9.50) of different isomorphic network flow solutions of algebraic equation (9.58).

Dr.Ouyang asks Huizi whether she knows the solution of differential equation $dx/dt = x$ with initial value $x|_t = 0 = 1$. Huizi answers that its solution is $x = e^t$. Dr.Ouyang then asking her tries to solve the differential equation $dX/dt = X$ in network flows $\left\{G^L[t]\right\}$ on graph $G$ with the initial value $X|_t = 0 = \mathbf{I}$. Huizi shakes her head and then, says that she does not know how to solve such a network flow equation. Dr.Ouyang tells her that there are two methods for solving a network equation. One is assuming that its solution is $X[t] = G^L[t]$ which turns the network differential equation into $dX/dt = G^{dL/dt}[t] = G^{L[t]}$ and then, solves its constraint differential equation $dL(v,u)[t]/dt = L(v,u)[t]$ of function on any edge $(v,u) \in E(G)$. Generally, by separating variables we know that

$$\frac{dL(v,u)[t]}{L(v,u)[t]} = dt \quad \Rightarrow \quad L(v,u)[t] = c_{vu}e^t.$$

Notice that $X|_{t=0} = \mathbf{I}$, $L(v,u)[0] = c_{vu} = 1$, i.e., $L(v,u)[t] = e^t$. We get that the solution of $dX/dt = X$ in network flows $\left\{G^L\right\}$ is $X = G^{e^t} = e^{G^t}$. Another is calculating directly by network flows, i.e., separate the variables of network flow to get $dX/X = dt$ and integrate it on both sides to get

$$\ln|X| = G^t + C' = tG + C' \quad \Rightarrow \quad X = C \cdot e^{tG},$$

where $C = e^{C'}$. Next, by the initial condition $X|_{t=0} = \mathbf{I}$, $X|_{t=0} = C \cdot \mathbf{I} = \mathbf{I}$, we get the network flow solution $X = e^{tG} = G^{e^t}$.

Hearing her father's explaining, Huizi understands that why introduces the exponential mapping $e^{G^L[\mathbf{x}]}$ in network flow calculus and the significance of identity (9.39) at this time. She asks Dr.Ouyang perceptively: "*Dad, the network flow calculus is so powerful! Can it simulate everything in the universe?*" Dr.Ouyang tells her that classical calculus could be applied to characterize the state or behavior of an element of system and the network flow calculus is its extension on topological structure or graph $G$ inherited in a thing $T$. Certainly, the network flow calculus can characterize a wider range of things than that of classical calculus but it is also far from enough because it does not take the spontaneous behavior of elements into account. And meanwhile, the assumption that the flow on any edge $(v,u) \in E(G)$ follows the function $L(v,u)[\mathbf{x}]$ is too ideal also. So, it needs further to be extended for systematic recognizing things in the universe.

## §6. Comments

**6.1.** As an example, the network calculus is a generalization of calculus by combinatorial notion. Certainly, the range of things that can be characterized by network flow are very limited because the network flow is the abstraction of road or pipe nets in practice. It is a special graph labeled with real numbers. Generally, it is assumed that the observer is standing at a point in the network for local observing and then analyzing flow data on the network of parts, which are extensively applied in the transport optimization in practice [ChL]. However, *what can an observer see if this one stands outside the network flow?* In this case, the observer sees the whole evolving of network flow, including local changes of flows on edges and the whole evolving of network such as the reduction or increase of vertices, the connection or cancellation of edges between vertices and other changes. In this way, the network flow can also be applied to characterize microcosmic things theoretically. For example, the quarks in theoretical physics are regarded as the elementary particles and protons, neutrons and mesons are all regarded as compositions respectively consisting of three or two quarks with an inherited topological structure $K_3$ and $K_2$ in situation. However, there are few humans discuss the integrity of network flow and those who study network flow are keen on solving the problem of network optimization but ignore its general value in human recognition which is caused by the recognitive limitation of human.

**6.2.** Different from the usual view of network flow only as a net, the network flow calculus is regarding a network flow as a mathematical element and then establishes its mathematical system, including its operation system of network flow in case of that the vertex and edge labels are real numbers. In fact, similar to existing mathematical elements such as numbers, mappings, functions $f(x), x \in \mathbb{R}$, vectors, matrices and points, lines, open sets, etc., the whole evolving of a network flow can be really regarded as a *mathematical element*. So, *how do we characterize the mathematical element of network flow and its evolving?* There are two methods, ie., regarding the evolving of network flow as a sequence or a continuous changing process. All of them provide us an effective ways for systemic characterizing the behavior of a thing.

**6.3.** Notice that network flow evolving follows a partial order rather than a full order and a vector in $n$-Euclidean space $\mathbb{R}^n$ corresponds to path flow $P_n^L$, which enables one generalizing the vector algebra to network flows. In this way, many conceptions such as the distance of network flows, limitation of network flow sequence and the continuous mapping can be defined with various algebraic expressions on the analogy of Euclidean space $\mathbb{R}^n$. Particularly, a lots of well-know conclusions in classical mathematics such as the exponential mapping identities and the fundamental theorems can be generalized to network flows, see [Mao51]. However, different from the general algebraic equation and differential equation, the solution of network flow equation is not unique. It is necessary to classify the solution or determine the number of non-isomorphic solutions under the action of automorphism group of graphs by using permutation groups, see [BiM], [BiW], [Mao14], [Mao21] and [Xum]. This is actually to make up the difference of expression forms in recognition brought by one standing on different viewpoints.

Chapter 10

# Combinatorial Reality

*********************************************************************

**Her clothes is Floating Clouds With Face a Rose,**

**Breeze Pets the Rails and Belle in Repose;**

**If not a Fairy Queen from Heaven on High,**

**She's Goddess of Moon That Makes Flowers Shy.**

– *Pure Happiness* By Li Bai, a poet in Tang dynasty of China

## §1. Thing's Nature

Certainly, the recognition of humans of a thing $T$ is on the nature or characters shown by that of $T$. Here, a character refers to the nature of $T$ that is displayed in front of humans and can be recognized by humans. On the nature of a thing, a fundamental question is whether the character of a thing is single or multiple in the eyes of humans. Dr.Ouyang asks Huizi: "*Do you think the character of a thing is a single or multiple one?*" Huizi replies: "*I have never thought about this question but if it is single, the characterizing of state of a thing only needs one parameter, which is relatively simple and easy to hold on the state of thing.*" Dr.Ouyang tells Huizi that a single character of a thing is usually quantized to a real number $a \in \mathbb{R}$ by humans. However, everything in the universe shows on a variety of shapes, even the front, side and back of the girl are different in shape, as shown in Figure 10.1. At the same time, the characters shown by things are very different. Thus, the appearance of a thing is very different from that of different viewing. We have to hold on the characters of a thing from all sides implied in the fable of blind men with an elephant. For example, the physical characters such as the color, odor, density, temperature, stillness or motion displayed by a thing $T$; the chemical characters such as the acidity, alkalinity, oxidation, reducibility and the thermal stability of a thing and the biological characters such as the death and living, including its growth, reproduction and custom, all indicate that the character of a thing is generally diverse, namely a Smarandache multispace or multisystem

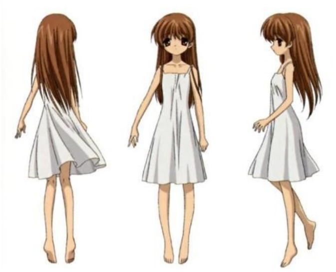


**Figure** 10.1. **A girl's shape**

$$T = \left(\bigcup_{i=1}^{n}\{\mu_i\}\right)\bigcup\left(\bigcup_{k\geq 1}\{\nu_k\}\right)$$

on a thing $T$ at time $t$ in Section 5 of Chapter 2 or Section 2 of Chapter 7, where $\mu_1, \mu_2, \cdots, \mu_n$ are all characters of thing $T$ known and $\nu_k, k \geq 1$ are unknown characters at time $t$.

Now that the characters of a thing are diverse. Dr.Ouyang then asks Huizi: "*how do we characterize the state of an element $v$ with its changes in the systemic recognition of thing?*" Huizi answers: "*For an element $v$ we can collect its all characters together to form an $n$-dimensional vector* $\mathbf{v} = (\mu_1, \mu_2, \cdots \mu_n) \in \mathbb{R}^n$ *and characterize its state change* *by vector* $\mathbf{v}$*!*" Dr.Ouyang nods and then says: "*In this case, how to characterize the change process of element* $v$ *quantitatively?*" Huizi answers: "*The change process of element* $v$ *from a state* $\mathbf{v}_1 \in \mathbb{R}^n$ *to another state* $\mathbf{v}_2 \in \mathbb{R}^n$ *can be characterized by a mapping* $f : \mathbf{v}_1 \to \mathbf{v}_2$ *on* $n$*-dimensional vectors.*" Now, assume that there are $n$ known characters $\mu_1, \mu_2, \cdots, \mu_n$ of things $T$. Dr.Ouyang then asks Huizi: "*Can we characterize state change of thing* $T$ *analogous to that of an element, i.e., by a mapping* $F : \mathbb{R}^n \to \mathbb{R}^n$ *on* $n$*-dimensional vectors?*" Huizi thinks for a while and replies: "*It should be so because* $\mu_1, \mu_2, \cdots, \mu_n$ *is known characters of thing* $T$*. Each state change of element* $v$ *can be characterized by a mapping on vector* $(\mu_1, \mu_2, \cdots, \mu_n)$*. And so, the state change of thing* $T$ *can of course be characterized by a mapping on vector* $(\mu_1, \mu_2, \cdots, \mu_n)$*.*" See her understanding is a little vague on this problem, Dr.Ouyang suggests to her: "*You can think it again. Is the state change of each element* $v$ *consistent or synchronized in evolving of thing* $T$*?*" Hearing her father so asking Huizi comes: "*Yes, Dad! The state change of* $T$ *can be described by a mapping of* $n$ *variables only if the elements* $v$ *are synchronized. But in general, the state change of elements* $v$ *are not synchronized. How should this situation be characterized?*" Dr.Ouyang tells her that the thing $T$ is composed of elements $v$, which inherits a combination structure, i.e., the labeled system graph $G^L$, as shown in Figure 10.2 and the change of thing $T$ from one state to another can be characterized by the labeled graph $G^L$, namely a mapping $F : G^{L_1} \to G^{L_2}$. In this case, the state of an element $v$ is an $n$-dimensional vector. If there are $m$ elements in the systematic recognition of $T$, then $F : G^{L_1} \to G^{L_2}$ can be viewed as a mapping on $\mathbb{R}^{nm} \to \mathbb{R}^{nm}$ that leaves the combinational structure $G$ unchanged, i.e.,

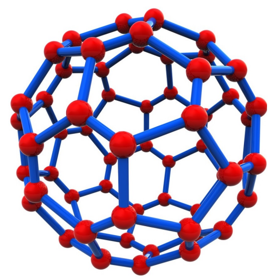


**Figure** 10.2. **Interstructure**

$$F : \left\{ \begin{array}{c} \mathbf{v}_1(\mu_1, \mu_2, \cdots, \mu_n) \\ \mathbf{v}_2(\mu_1, \mu_2, \cdots, \mu_n) \\ \cdots\cdots\cdots\cdots \\ \mathbf{v}_m(\mu_1, \mu_2, \cdots, \mu_n) \end{array} \right\} \to \left\{ \begin{array}{c} \mathbf{v}'_1(\mu_1, \mu_2, \cdots, \mu_n) \\ \mathbf{v}'_2(\mu_1, \mu_2, \cdots, \mu_n) \\ \cdots\cdots\cdots\cdots \\ \mathbf{v}'_m(\mu_1, \mu_2, \cdots, \mu_n) \end{array} \right\}, \tag{10.1}$$

Certainly, it is similar to the network flow with $L_1(v) = \mathbf{v}(\mu_1, \mu_2, \cdots, \mu_n)$, $L_2(v) = \mathbf{v}'(\mu_1, \mu_2, \cdots, \mu_n)$ for vertex $v \in V(G)$, i.e., labeled by $n$-dimensional vector. Therefore, in order to characterize the state change of thing $T$, we should first discuss vector calculus to simulate the dynamic behavior of $T$ in general.

**1.1.Vector Calculus.** Certainly, the vector calculus is a special case of network flow calculus, which is the case in Figure 9.14 $(a)$. In this case, the notations in Euclidean space can be used to represent the results of vector calculus. First of all, for any integer $n \geq 1$, let $\{\mathbf{v}_n\}_1^\infty = \{\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k, \cdots\}$ be a vector sequence. Here, $\mathbf{v}_i \in \mathbb{R}^n$ for any integer $i \geq 1$. If for a given number $\varepsilon > 0$ there is always an integer $N$ such that $|\mathbf{v}_n - \mathbf{v}| < \varepsilon$ if $n > N$, then the limitation of vector sequence $\{\mathbf{v}\}_1^\infty$ is $v$, denoted by $\lim\limits_{k\to\infty} \mathbf{v}_k = \mathbf{v}$. Now, if $\mathbf{v}_k = (v_1^k, v_2^k, \cdots, v_n^k)$ and $\mathbf{v} = (v_1, v_2, \cdots, v_n)$, then

$$|\mathbf{v}_n - \mathbf{v}| = \sqrt{\left(v_1^k - v_1\right)^2 + \left(v_2^k - v_2\right)^2 + \cdots + \left(v_n^k - v_n\right)^2} \tag{10.2}$$

and $\lim\limits_{k\to\infty} \mathbf{v}_k = \mathbf{v}$ if and only if for any integer $1 \leq i \leq n$, $\lim\limits_{k\to\infty} v_i^k = v_i$, namely a coordinate of vector $\mathbf{v}_k$ approaches to its corresponding coordinate in $\mathbf{v}$ if $k \to \infty$.

Secondly, assume that $\mathbf{v} \in \mathbb{R}^n$ and $\mathbf{v}' \in \mathbb{R}^m$ are respectively an $n$-dimensional vector and an $m$-dimensional vector. A mapping $\mathbf{f} : \mathbf{v} \to \mathbf{v}'$ or $\mathbf{f}(\mathbf{v}) = \mathbf{v}'$ implies that

$$\mathbf{f}(\mathbf{v}) = \left(f_1\left(x_1, x_2, \cdots, x_n\right), f_2\left(x_1, x_2, \cdots, x_n\right), \cdots, f_m\left(x_1, x_2, \cdots, x_n\right)\right),$$

namely $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$, where integers $n, m \geq 1$. In this case, for $\mathbf{v}, \mathbf{v}_0 \in \mathbb{R}^n$ and a given number $\varepsilon > 0$, if there exists a real number $\delta > 0$ such that $|\mathbf{f}(\mathbf{v}) - \mathbf{f}(\mathbf{v}_0)| < \varepsilon$ if $|\mathbf{v} - \mathbf{v}_0| < \delta$, then it is said that $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at point $\mathbf{v}_0$, denoted by $\lim\limits_{\mathbf{v}\to\mathbf{v}_0} \mathbf{f}(\mathbf{v}) = \mathbf{f}(\mathbf{v}_0)$. Here, $|\mathbf{v} - \mathbf{v}_0|$ and $|\mathbf{f}(\mathbf{v}) - \mathbf{f}(\mathbf{v}_0)|$ are respectively the distance of vectors $\mathbf{v}$, $\mathbf{v}_0$ in $\mathbb{R}^n$ and vectors $\mathbf{f}(\mathbf{v}), \mathbf{f}(\mathbf{v}_0)$ in $\mathbb{R}^m$, i.e., $n = n$ or $m$ in (10.2). Accordingly, a mapping $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is continuous at point $\mathbf{v}_0$ if and only if each coordinate of $\mathbf{f}$ is continuous at point $\mathbf{v}_0$.

Thirdly, assume that $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is a continuous mapping, the increment of variable $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ is $\Delta\mathbf{x} = (\Delta x_1, \Delta x_2, \cdots, \Delta x_n)$. If $\Delta\mathbf{x} \to \mathbf{0}$ the limitation

$$\lim_{\Delta\mathbf{x}\to\mathbf{0}} \frac{\mathbf{f}(\Delta\mathbf{x} + \mathbf{x}) - \mathbf{f}(\mathbf{x})}{|\Delta\mathbf{x}|}$$

exists, then it is said that $\mathbf{f}$ is differentiable at point $\mathbf{x}$, denoted by

$$\frac{\partial\mathbf{f}}{\partial\mathbf{x}} = \lim_{\Delta\mathbf{x}\to\mathbf{0}} \frac{\mathbf{f}(\Delta\mathbf{x} + \mathbf{x}) - \mathbf{f}(\mathbf{x})}{|\Delta\mathbf{x}|}. \tag{10.3}$$

Now, let $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}$ be a norm coordinate system of $\mathbb{R}^n$. Then, the differential $\dfrac{\partial\mathbf{f}}{\partial\mathbf{x}}$ of $\mathbf{f}$ at point $\mathbf{x}$ can be expressed as a Jacobian of $m \times n$ matrix, i.e

$$\frac{\partial\mathbf{f}}{\partial\mathbf{x}} = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \vdots & \vdots & \vdots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \frac{\partial f_m}{\partial x_2} & \cdots & \frac{\partial f_m}{\partial x_n} \end{pmatrix} = J_{\mathbf{f}|\mathbf{x}}.$$

In this case, the differential of $\mathbf{f}$ at point $\mathbf{x}$ can be simply expressed by Jacobian as

$$df = J_{\mathbf{f}|\mathbf{x}} dx^t, \tag{10.4}$$

i.e., the $J_{\mathbf{f}|\mathbf{x}}$ is a transformation matrix of differential function in variables. Particularly, if $m = 1$ or $n = 1$, then $J_{\mathbf{f}|\mathbf{x}}$ is respectively

$$\frac{\partial f}{\partial \mathbf{x}} = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \cdots, \frac{\partial f}{\partial x_1}\right) \quad \text{or} \quad \frac{\partial \mathbf{f}}{\partial x} = \left(\frac{\partial f_1}{\partial x}, \frac{\partial f_2}{\partial x}, \cdots, \frac{\partial f_m}{\partial x}\right)^t$$

and so, a mapping $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable at point $\mathbf{x} \in \mathbb{R}^n$ if and only if each coordinate of $\mathbf{f}$ is differentiable at point $\mathbf{x}$. By matrix multiplication, the complete differential

$$du = \sum_{i=1}^{n} \frac{\partial u}{\partial y^i} dy^i = \sum_{i=1}^{n} \frac{\partial u}{\partial y^i} \sum_{j=1}^{n} \frac{\partial f_i}{\partial x^j} dx^j = \sum_{i=1}^{n} \frac{\partial u}{\partial x^i} dx^i$$

of composite function $u = g(f)$ can be generalized to the composite mapping $\mathbf{g}(\mathbf{f})$ of $\mathbf{g} : \mathbb{R}^n \to \mathbb{R}^s$ and $\mathbf{f} : \mathbb{R}^s \to \mathbb{R}^m$ by Jacobian matrixes, i.e.,

$$J_{\mathbf{g}(\mathbf{f})|\mathbf{f}} J_{\mathbf{f}|\mathbf{x}} = J_{\mathbf{g}(\mathbf{f})|\mathbf{x}}, \quad \forall \mathbf{x} \in \mathbb{R}^n. \tag{10.5}$$

**1.2.Banach Space.** For any integer $n \geq 1$, the Euclidean space $\mathbb{R}^n$ is the most intuitively acceptable class of linear spaces. Particularly, the distance between two points is the calculation result by Pythagorean theorem, namely (10.2) in a normal coordinate system $\{\mathbf{e}_1, \mathbf{e}_2, \cdots, \mathbf{e}_n\}, |\mathbf{e}_i| = 1$ of $\mathbb{R}^n$. Then, *can we calculate generally the distance between two vectors in any linear space by introducing the metric similar to that of Euclidean space?* Dr.Ouyang tells Huizi that the metric $\rho$ can be certainly introduced for characterizing the distance between two vectors in any linear space.

**(1) Normed space.** A norm $\|\mathbf{v}\|$ of vector $\mathbf{v}$ is an extension of the length in Euclidean and a normed space is such a linear space that the norm is introduced for characterizing the length of vector. We know that only addition $+$ and scalar multiplication $\cdot$ between elements are defined on a linear space. Then, *how do we introduce a metric into a linear space?* Dr.Ouyang explains that let $(V; \mathscr{F})$ be a finite or infinite dimensional linear space over field $\mathscr{F}$. Define a function $\|\cdot\| : V \to \mathbb{R}$ on $V$ satisfying the following three conditions:

$(N_1)$ Non-negative, namely there is $\|\mathbf{v}\| \geq 0$ for any vector $\mathbf{v} \in V$ and $\|\mathbf{v}\| = 0$ if and only if $\mathbf{v} = \mathbf{0}$, call such $\mathbf{0}$ a zero element of $V$;

$(N_2)$ Homogeneity, namely $\|\alpha\mathbf{v}\| = |\alpha|\|\mathbf{v}\|$ for any vector $\mathbf{v} \in V$ and $\alpha \in \mathscr{F}$;

$(N_3)$ Triangle inequality, namely $\|\mathbf{v} + \mathbf{u}\| \leq \|\mathbf{v}\| + \|\mathbf{u}\|$ for any elements $\mathbf{v}, \mathbf{u} \in V$.

Such a function $\|\cdot\| : V \to \mathbb{R}$ if exists, it is called a *norm* on $V$. Generally, a linear space $(V; \mathscr{F})$ endowed with a norm $\|\cdot\|$ is called a *normed space*, denoted by $(V; \|\cdot\|)$.

Notice that there is a non-negative function between any two points $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ in Euclidian space $\mathbb{R}^n$, namely the distance $d : \{\mathbf{x}, \mathbf{y}\} \to d(\mathbf{x}, \mathbf{y})$ for determining the distance between two points, more points near or far to a fixed point. So, *can we introduce such a function to measure the distance between two vectors on a linear space?* Dr.Ouyang tells Huizi that the answer is Yes! In this case, for a linear space $(V; \mathscr{F})$ and vectors $\mathbf{v}, \mathbf{u}, \mathbf{w} \in V$, we define a function $\rho : V \times V \to \mathbb{R}$ satisfies the following three conditions:

$(M_1)$ Non-negative, namely $\rho(\mathbf{v}, \mathbf{u}) \geq 0$ and $\rho(\mathbf{v}, \mathbf{u}) = 0$ if and only if $\mathbf{v} = \mathbf{u}$;

$(M_2)$ Symmetry, namely $\rho(\mathbf{v}, \mathbf{u}) = \rho(\mathbf{u}, \mathbf{v})$;

$(M_3)$ Triangle inequality, namely $\rho(\mathbf{v}, \mathbf{w}) \leq \rho(\mathbf{v}, \mathbf{u}) + \rho(\mathbf{u}, \mathbf{w})$.

Such a function $\rho : V \times V \to \mathbb{R}$ is called a *metric* if it exists on $V$ and can be analogous to Euclidean space $\mathbb{R}^n$ for determining the distance between vectors. Generally, a linear space $(V; \mathscr{F})$ defined a metric is called a *metric space*, denoted as $(V; \rho)$. So, *what is the relationship between the conceptions of normed space and metric space?* Dr.Ouyang explains that for arbitrary vectors $\mathbf{v}, \mathbf{u} \in V$ of a normed space $(V; \|\cdot\|)$, by the definition of norm $\|\cdot\|$, we introduce a function $\rho : V \times V \to \mathbb{R}$ by $\rho(\mathbf{v}, \mathbf{u}) = \|\mathbf{v} - \mathbf{u}\|$. Clearly, it satisfies the conditions $(M_1) - (M_3)$, namely the linear space $V$ becomes a metric space.

It should be noted that there maybe more than one ways to introduce the norm function $\|\mathbf{V}\| : V \times V \to \mathbb{R}$ in $V$. So, *how can we determine that two norms on the same linear space $V$ are essentially consistent or equivalent?* Dr.Ouyang explains that if there are two constants $c_1, c_2$ for two norms $\|\cdot\|_1$, $\|\cdot\|_2$ on $V$ such that

$$c_1\|\cdot\|_1 \;\leq\; \|\cdot\|_2 \;\leq\; c_2\|\cdot\|_1, \tag{10.6}$$

then the norms $\|\cdot\|_1$ and $\|\cdot\|_2$ are said to be *equivalent*. In this case, the normed spaces $(V; \|\cdot\|_1)$ and $(V; \|\cdot\|_2)$ determined by two equivalent norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on the same space $V$ need not to discuss respectively because for any two vectors $\mathbf{v}, \mathbf{u} \in V$, the calculations of $\rho(\mathbf{v}, \mathbf{u})$ in $(V; \|\cdot\|_1)$ or $(V; \|\cdot\|_2)$ are only different in a constant factor irrelevant with vectors $\mathbf{v}, \mathbf{u}$.

Certainly, the equivalent norms are exist widely in the recognition of reality of things. For example, for any point $\mathbf{x} = (x_1, x_2, \cdots, x_n) \in \mathbb{R}^n$ or a vector $\overrightarrow{Ox}$ on a Euclidean space $\mathbb{R}^n$, there are three norms defined as follows:

$$\begin{aligned}
\|\mathbf{x}\|_1 &= \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}, \\
\|\mathbf{x}\|_2 &= |x_1| + |x_2| + \cdots + |x_n|, \\
\|\mathbf{x}\|_3 &= \max\{|x_1|, |x_2|, \cdots, |x_n|\}.
\end{aligned}$$

Indeed, the norms $\|\cdot\|_1$, $\|\cdot\|_2$ and $\|\cdot\|_3$ are equivalent norms on $\mathbb{R}^n$. *Why are they equivalent?* Dr.Ouyang explains that let $a_i = 1, b_i = |x_i|, 1 \leq\leq n$ in the cauchy inequality (4.34). Then, there is

$$(|x_1| + |x_2| + \cdots + |x_n|)^2 \leq n\left(x_1^2 + x_2^2 + \cdots + x_n^2\right) \leq n\left(|x_1| + |x_2| + \cdots + |x_n|\right)^2,$$

i.e.,

$$\frac{1}{\sqrt{n}}\|\mathbf{x}\|_2 \;\leq\; \|\mathbf{x}\|_1 \;\leq\; \|\mathbf{x}\|_2.$$

And meanwhile, by

$$\max\{|x_1|, |x_2|, \cdots, |x_n|\} \leq \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2} \leq \sqrt{n}\max\{|x_1|, |x_2|, \cdots, |x_n|\}$$

we know that

$$\|\mathbf{x}\|_3 \;\leq\; \|\mathbf{x}\|_1 \;\leq\; \sqrt{n}\|\mathbf{x}\|_3,$$

i.e., $\|\cdot\|_2$ and $\|\cdot\|_3$ are both equivalent to $\|\cdot\|_1$. Dr.Ouyang explains that there is a general conclusion that *any two norms on a finite dimensional linear space are equivalent* and furthermore, there is a criterion on the norm equivalent, namely *two norms $\|\cdot\|_1$ and $\|\cdot\|_2$ on a normed linear space $(V;\|\cdot\|)$ are equivalent if and only if for any point sequence $\{\mathbf{v}_n\}_1^\infty$ in $V$, $\|\mathbf{v}_n\|_1 \to 0$ if and only if $\|\mathbf{v}_n\|_2 \to 0$.*

Notice that for any number $r > 0$ and $\mathbf{x} \in \mathbb{R}^n$, there is an opened ball $\mathbb{B}(\mathbf{x}, r) = \{\mathbf{y} \in \mathbb{R}^n | d(\mathbf{x}, \mathbf{y}) < r\}$ of points $\mathbf{y}$ with distance $r$ to $\mathbf{x}$ in $\mathbb{R}^n$, where $\mathbf{x}$, $r$ are called respectively the center and radius of $\mathbb{B}(\mathbf{x}, r)$ and then, define the open set in $\mathbb{R}^n$ to construct the topological structure of $\mathbb{R}^n$ as a topological space. So, *can we define opened sets on a metric space which enables it to be a topological space?* The answer is Yes! Such a topology is called the *metric topology*. Dr.Ouyang explains that by replacing the distance $d$ in Euclidean space with the metric $\rho$ for a given metric space $(V;\rho)$, a topological structure can be introduced into the metric space $V$. Now let $U \subset V$ be a subset of $V$ and if there is a positive number $r > 0$ for any vector $\mathbf{v} \in U$ such that $\mathbb{B}_r(\mathbf{v}) \subset U$, then $U$ is defined to be an *opened set* of $V$, where $\mathbb{B}_r(\mathbf{v})$ is an opened ball with radius $r$ and center $\mathbf{v}$ in $V$, i.e., $\mathbb{B}_r(\mathbf{v}) = \{\mathbf{u} \in V | \rho(\mathbf{v}, \mathbf{u}) < r\}$. By this definition, the opened ball $\mathbb{B}_r(\mathbf{v})$ is itself an open set and all these opened sets form a topology, i.e., the metric topology on $V$. Therefore, a metric space $V$ can be also regarded as a topological space in which any vector is viewed as a point in topological space $V$.

(2) **Banach space.** Generally, the system recognition of a thing $T$ needs to extend the network flow to the continuity flow $G^L$ whose vertices and edges are labeled vectors in a Banach space. So, *what is a Banach space?* Dr.Ouyang tells Huizi that there is an

induced metric

$$\rho(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|, \quad \mathbf{x}, \mathbf{y} \in V. \tag{10.7}$$

on a normed space $(V; \|\cdot\|)$. Accordingly, a *Banach space* is a completely normed space $(V; \|\cdot\|)$. Here, a normed space is "*complete*" refers to that any of its Cauchy point sequence is converging under the induced metric $\rho$.

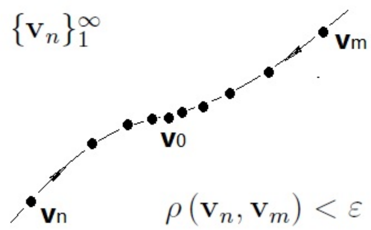



**Figure** 10.3. **Cauchy sequence**

So, *what is the Cauchy point sequence in a metric space?* Dr.Ouyang explains that let $\{\mathbf{v}_n\}_1^\infty$ be a vector or point sequence in the metric space $(V; \rho)$. For any small positive number $\varepsilon$ if there exists an integer $N > 0$ such that $\rho(\mathbf{v}_n, \mathbf{v}_m) < \varepsilon$ for integers $n,\ m > N$, then the sequence $\{\mathbf{v}_n\}_1^\infty$ is called a *Cauchy point sequence*, namely the distance between points becomes closer and closer as the ordinal number increases or if we remove the finite previous terms in sequence $\{\mathbf{v}_n\}_1^\infty$. Certainly, the distance between any two points of the remaining elements does not exceed an arbitrarily given positive number, as shown in Figure 10.3. Generally, a convergent point sequence must be a Cauchy point sequence but a Cauchy point sequence $\{\mathbf{v}_n\}_1^\infty$ is not necessarily a convergent point sequence. However, *if there is really a subsequence* $\{\mathbf{v}_{n_k}\}_1^\infty$ *of Cauchy point sequence* $\{\mathbf{v}_n\}_1^\infty$ *that is convergent to a point* $\mathbf{v}$*, then the Cauchy point sequence* $\{\mathbf{v}_n\}_1^\infty$ *must be convergent to* $\mathbf{v}$ *also.* Dr.Ouyang explains that in this case, for any given number $\varepsilon > 0$, by definition there are integers $K$ and $N$ such that: ① $\rho(\mathbf{v}_{n_k}, \mathbf{v}) < \varepsilon/2$ if $k > K$; ② $\rho(\mathbf{v}_n, \mathbf{v}_m) < \varepsilon/2$ if $n, m > N$. Now, let $k > K$ be such an integer that $n_k > N$. Then, we know that

$$\rho(\mathbf{v}_n, \mathbf{v}) \leq (\mathbf{v}_n, \mathbf{v}_{n_k}) + (\mathbf{v}_{n_k}, \mathbf{v}) < \varepsilon$$

by the triangle inequality $(M_3)$ for $n > N$, namely the Cauchy point sequence $\{\mathbf{v}_n\}_1^\infty$ is convergent to $\mathbf{v}$.

Certainly, the Cauchy point sequence characterizes a dense property of points in a complete metric space. Generally, the metric $\rho$ of a metric space $(V; \rho)$ is also a metric on its subset $U \subset V$ which satisfies the conditions $(M_1) - (M_3)$, namely $(U; \rho)$ is also a metric space called a subspace of $(V; \rho)$. Among these subspaces, there exists a kind of subspace, called the *dense subspace*. So, *what is a dense subspace?* Dr.Ouyang tells Huizi that let $U \subset V$ be a subspace of $(V; \rho)$. If each ball $\{\mathbf{u} \in V \| \rho(\mathbf{u}, \mathbf{u}_0) < r\}$ of radius $r > 0$ contains points of $U$ with $\mathbf{u}_0 \in V$, such a subset $U$ is called a *dense subspace* of $V$. Notice that $r$ can be arbitrarily close to 0 and then characterizing the density of points of $U$ in $V$. In this way, there is a general conclusion on the structure of metric spaces that *every* *metric space is a dense subspace of another complete metric space or every metric space is a dense subspace of another metric space in which the Cauchy point sequence converges.*

There are two useful classes of Banach spaces, namely for any integer $n \geq 1$, the linear spaces $(V; \mathscr{F})$ over a real number field $\mathscr{F} = \mathbb{R}$ or a complex number field $\mathscr{F} = \mathbb{C}$, called respectively the *real Banach space* $\mathbb{R}^n$ or *complex Banach space* $\mathbb{C}^n$. Particularly, the Banach spaces $\mathbb{R}^n$ and $\mathbb{C}^n$ of finitely dimension $n < \infty$.

**(3) Linear operator.** A linear operator is a generalization of linear transformation $T : \mathbb{R}^n \to \mathbb{R}^n$ on Euclidean space $\mathbb{R}^n$. Dr.Ouyang tells Huizi that let $V, U$ two normed spaces over a number field $\mathscr{F}$ and let $D \subset V$ be a linear subspace of $V$ with a mapping $T : D \to U$. For any vectors $\mathbf{v}, \mathbf{u} \in D$ and $\alpha, \beta \in \mathscr{F}$, if

$$T(\alpha\mathbf{v} + \beta\mathbf{u}) \ = \ \alpha T(\mathbf{v}) + \beta T(\mathbf{u}), \tag{10.8}$$

then $T$ is called a *linear operator* on $D$, where $D$ is called the *define domain* of $T$ in $V$, $R(T) = \{T(\mathbf{x}) | \mathbf{x} \in D\}$ is called the *image set* of $T$ in $U$. Particularly, let $\beta = 0$. Then, $T(\alpha\mathbf{v}) = \alpha T(\mathbf{v})$ by (10.8). In this case, there must be $T(\mathbf{0}) = \mathbf{0}$ if $\alpha = 0$. Generally, such a linear operator $T$ is called a *real (or complex) linear functional* if $U = \mathbb{R}$ (or $\mathbb{C}$), namely the image value of $T$ is real or complex and denoted by $\langle T, \mathbf{v}\rangle$ the action of $T$ on vector $\mathbf{v}$.

Generally, all linear operators $T : V \to U$ from a Banach space $V$ to $U$ are denoted by $L(V, U)$ and meanwhile, for any linear operator $T \in L(V, U)$,

$$\|T\| \ = \ \sup_{\mathbf{v}\in V\setminus\{\mathbf{0}\}} \frac{\|T(\mathbf{v})\|}{\|\mathbf{v}\|} \ = \ \sup_{\|\mathbf{v}\|=1} \|T(\mathbf{v})\|$$

define a function $\| \cdot \| : L(V, U) \to \mathbb{R}$, where $\sup A$ denotes the supremum of $A$. Notice that the function $\| \cdot \|$ has properties following:

$(O1)$ Non-negative, namely for any $T \in L(V, U)$,

$$\|T\| = \sup_{\mathbf{v}\in V\setminus\{\mathbf{0}\}} \frac{\|T(\mathbf{v})\|}{\|\mathbf{v}\|} \geq 0$$

and if $\|T\| = 0$ then $T(\mathbf{v}) = 0$ for any vector $\mathbf{v} \in V$, i.e., $T = \mathbf{0}$;

$(O2)$ Homogeneity, namely for any $\alpha \in \mathscr{F}$ and $T \in L(V, U)$, there is

$$\|\alpha T\| = \|\sup_{\|\mathbf{v}\|=1} \alpha T(\mathbf{v})\| = |\alpha| \|\sup_{\|\mathbf{v}\|=1} T(\mathbf{v})\| = |\alpha| \|T\|;$$

$(O3)$ Triangle inequality, namely for any operators $T_1, T_2 \in L(V, U)$, there is

$$\|T_1 + T_2\| = \sup_{\|\mathbf{v}\|=1} (T_1(\mathbf{v}) + T_2(\mathbf{v})) \leq \sup_{\|\mathbf{v}\|=1} T_1(\mathbf{v}) + \sup_{\|\mathbf{v}\|=1} T_2(\mathbf{v}) = \|T_1\| + \|T_2\|,$$

Thus, the function $\|\cdot\|$ satisfies the norm conditions $(N_1)-(N_3)$. Dr.Ouyang concludes then that $\|\cdot\|$ is a norm on the linear operators $L(V,U)$ which turns $(L(V,U);\|\cdot\|)$ into a Banach space.

For a linear operator $T: D\subset E\to U$ and point $\mathbf{v}_0\in D$, let $\{\mathbf{v}_n\}_1^\infty\subset D$ be a point sequence. If $T(\mathbf{v}_n)\to T(\mathbf{v}_0)$ when $\mathbf{v}_n\to\mathbf{v}_0$, then $T$ is *continuous* at point $\mathbf{v}_0$, i.e.,

$$\lim_{n\to\infty} T\left(\mathbf{v}_n\right)=T\left(\lim_{n\to\infty}\mathbf{v}_n\right). \tag{10.9}$$

Notice that by the properties of linear operator, $T\left(\mathbf{v}_n\right)-T\left(\mathbf{v}_0\right)=T\left(\mathbf{v}_n-\mathbf{v}_0\right)$. The condition $\mathbf{v}_n\to\mathbf{v}_0$ can be replaced by $\mathbf{v}_n\to\mathbf{0}$. In this way, a linear operator $T$ is continuous at point $\mathbf{v}_0$ of $D$ if and only if $T$ is continuous at point $\mathbf{0}$. Meanwhile, if $T$ is continuous at any point in $\mathbf{v}\in D$, by $T\left(\mathbf{v}_n-\mathbf{v}+\mathbf{v}_0\right)=T\left(\mathbf{v}_n\right)-T\left(\mathbf{v}\right)+T\left(\mathbf{v}_0\right)\to T(\mathbf{v}_0)$, namely $T$ is also continuous at the point $\mathbf{v}_0$. We get a conclusion that a linear operator $T$ is continuous everywhere on $D$ if and only if $T$ is continuous at a point on $D$ such as the point $\mathbf{0}$.

Different from the function continuous, the continuity of linear operator $T: D\to U$ on $D$ is related to whether the operator $T$ is bounded on $D$. So, *what is bounded for a linear operator $T$?* Dr.Ouyang explains that for two normed spaces $(V;\|\cdot\|_1)$, $(U;\|\cdot\|_2)$ and a linear operator $T: D\subset V\to U$, if there is a real number $C\in\mathbb{R}$ such that for any point $\mathbf{v}\in V$ there is $\|T(\mathbf{v})\|_2\leq C\|\mathbf{v}\|_1$, then $T$ is said to be a *bounded operator.* Notice that if $T$ is a bounded operator and $\{\mathbf{v}_n\}\subset D$ is a point sequence convergent to point $\mathbf{v}$, by $T(\mathbf{v}_n)-T(\mathbf{v})=T(\mathbf{v}_n-\mathbf{v})\leq C\|\mathbf{v}_n-\mathbf{v}\|$, then $T$ must be continuous operator on $D$. Conversely, if $T$ is a continuous operator but unbounded on $D$, then for any integer $n\geq 1$, there exists a non-zero point $\mathbf{v}_n$ in $D$ holding with $T(\mathbf{v}_n)\geq n\|\mathbf{v}_n\|$. Define a point sequence $\{\mathbf{u}_n\}_1^\infty$ by $\mathbf{u}_n=\mathbf{v}_n/(n\|\mathbf{v}_n\|)$. Clearly, $\|\mathbf{u}_n\|\to 0$ or $\mathbf{u}_n\to\mathbf{0}$ and $T(\mathbf{u}_n)\to T(\mathbf{0})=\mathbf{0}$ if $n\to\infty$. However, by assumption $T$ is a linear operator, there is

$$\|T\left(\mathbf{u}_n\right)\|=\left\|T\left(\frac{\mathbf{v}_n}{n\|\mathbf{v_n}\|}\right)\right\| = \frac{T\left(\mathbf{v}_n\right)}{n\|\mathbf{v}_n\|} \geq \frac{n\|\mathbf{v}_n\|}{n\|\mathbf{v}_n\|} = 1,$$

which contradicts to $T(\mathbf{u}_n)\to\mathbf{0}$, namely $T$ must be bounded on $D$. Thus, *a linear operator $T$ is continuous on $D$ if and only if $T$ is bounded on $D$.*

A linear mapping $T: V\to U$ on Banach spaces $V$, $U$ is called a *open mapping* or *closed mapping* if $T$ maps the opened (closed) set in $V$ to the opened (closed) set in $U$. Thus, *if $T$ is a continuous operator and is surjection, then $T$ must be an open mapping. Conversely, if $T$ is a closed operator, then $T$ must be a continuous operator.*

Now assume that $V$ is a real normed space. A function $p: V\to\mathbb{R}$ is called a *sublinear functional* on $V$ if: ① $p(\alpha\mathbf{v})=\alpha p(\mathbf{v})$ for any $\mathbf{v}\in V, 0<\alpha\in\mathbb{R}$; ② $p(\mathbf{v}+\mathbf{u})\leq p(\mathbf{v})+p(\mathbf{u})$

for any two vectors $\mathbf{v}, \mathbf{u} \in V$. In this case, *if there is a sublinear functional $p$ on a subspace $W \subset V$ of $V$ and there is a real linear functionals $f : W \to \mathbb{R}$ such that $f(\mathbf{w}) \leq p(\mathbf{w})$ for any $\mathbf{w} \in W$, there must be a real linear functional $\widetilde{f} : V \to \mathbb{R}$ such that $\widetilde{f}(\mathbf{v}) = f(\mathbf{v})$ for any vector $\mathbf{v} \in W$ and meanwhile, $\widetilde{f}(\mathbf{v}) \leq p(\mathbf{v})$* for any vector $\mathbf{v} \in V$, called the *Hahn-Banach theorem* in functionals.

**1.3.Hilbert Space.** Hilbert space is a space model corresponding to quantum states in quantum mechanics, which is a complete inner product space. Generally, let $\mathcal{H}$ be a linear space over complex field $\mathbb{C}$. For any $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathcal{H}$, if there exists a mapping $\langle \cdot, \cdot \rangle : (\mathbf{v}, \mathbf{u}) \in \mathcal{H} \times \mathcal{H} \to \mathbb{C}$ holding with the following conditions:

$(H_1)$ Non-negative, namely $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$ and $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$;

$(H_2)$ Conjugacy, namely $\langle \mathbf{v}, \mathbf{u} \rangle = \overline{\langle \mathbf{u}, \mathbf{v} \rangle}$, where $\overline{a + ib} = a - ib$ is the conjugate number of $a + ib \in \mathbb{C}$, $a, b \in \mathbb{R}$;

$(H_3)$ Linearity, namely $\langle \alpha\mathbf{v} + \beta\mathbf{u}, \mathbf{w} \rangle = \alpha \langle \mathbf{v}, \mathbf{w} \rangle + \beta \langle \mathbf{u}, \mathbf{w} \rangle$ for $\alpha, \beta \in \mathbb{C}$,

then the mapping $\langle \cdot, \cdot \rangle$ is called an *inner product* on $\mathcal{H}$ and $\mathcal{H}$ is an *inner product space*. In this case, let $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v} \rangle}$. Then, $\mathcal{H}$ is a normed space with norm $\| \cdot \|$. Furthermore, if the norm $\| \cdot \|$ induced by inner product is complete in $\mathcal{H}$, $\mathcal{H}$ is called a *Hilbert space*.

Then, *why does the quantum mechanics treat Hilbert space as a model of quantum states?* Dr.Ouyang tells Huizi that the states of quantum are vectors, which are divided into the *bra* $\langle \varphi|$ and *ket* $|\psi\rangle$ in quantum mechanics. Generally, the inner product $\langle |\psi\rangle, \langle \varphi| \rangle$ of a ket $|\psi\rangle$ with a bra $\langle \varphi|$ is abbreviated to $\langle \psi|\varphi\rangle$, which is a scalar with $\langle \psi|\psi\rangle \geq 0$ and the outer product $|\psi\rangle \times \langle \varphi|$ is still denoted by $|\psi\rangle \times \langle \varphi|$, which is a vector. Now, let $A$ be an operator. Formally, $A$ is always acting from left on a ket $A|\psi\rangle$ and from the right on a bra $\langle |\varphi| A$. In this way, the inner product $\langle \psi|A|\varphi\rangle$ can be viewed as the inner product of $A$ acting on a bra $\langle \psi| A$ with a ket $|\varphi\rangle$, and can be also viewed as the inner product of $A$ acting on a ket $A|\varphi\rangle$ with a bra $\langle \psi|$. Certainly, all the results are $\langle \psi|A|\varphi\rangle$. Usually, such an operator $A$ is called a *Hermitian operator*. Notice that the eigenvalues of a Hermitian operator are all real numbers, which can be measured by humans.

**Figure** 10.4. **Superposition of quantum states**

Generally, the quantum states are in superposition, as shown in Figure 10.4. Meanwhile, the linear combinations of a bra with a ket are respectively

$$\begin{aligned}\langle\psi|(c_1|\varphi_1\rangle+c_2|\varphi_2\rangle)\rangle &= c_1\langle\psi|\varphi_1\rangle+c_2\langle\psi|\varphi_2\rangle,\\ \langle(c_1|\psi_1\rangle+c_2|\psi_2\rangle)|\varphi\rangle &= c_1\langle\psi_1|\varphi\rangle+c_2\langle\psi_2|\varphi\rangle.\end{aligned}$$

Dr.Ouyang tells Huizi that a wave function is the state $|\psi\rangle$ of quantum under a coordinate system or orthonormal basis in quantum mechanics. But how to explain the physical connotation of wave function does not reach a consensus in scientific community. The famous *Copenhagen interpretation* applied three hypothesises about quantum system:

**Hypothesis 1.** A pure state in quantum mechanics is represented in terms of a normalized vector $|\psi\rangle$ in Hilbert space $\mathcal{H}$, i.e., $\langle\psi|\psi\rangle = 1$. Generally, assume that $|\psi_1\rangle, |\psi_2\rangle, \cdots, |\psi_n\rangle, \cdots$ are all possible physical states of the system. Then, the linear combination $c_1|\psi_1\rangle + c_2|\psi_2\rangle + \cdots + c_n|\psi_n\rangle + \cdots$ is also a possible state of the system. Here, $c_i, i \geq 1$ are complex numbers.

**Hypothesis 2.** For an observable physical quantity $a$ of quantum such as its position, momentum, energy and spin, there exists a corresponding Hermitian operator $A$ acting on $\mathcal{H}$, namely if measuring $a$ in the state $|\psi\rangle$, one obtains one of the eigenvalues $\lambda_j$ of the operator $A$ and the state undergoes an abrupt change to the corresponding eigenvector $|\lambda_j\rangle$. In this case, the state $|\psi\rangle$ of $a$ instantaneous collapse into the corresponding eigenstate $|\lambda_j\rangle$, i.e., $A|\lambda_j\rangle = \lambda_j|\lambda_j\rangle$. Generally, if one prepares an ensemble of many identical state $|\psi\rangle$ and measures $a$, the probability of obtaining the outcome $|\psi\rangle$ is $p_j = |\langle\lambda_j|\psi\rangle|^2$.

**Hypothesis 3.** The time evolving of state is governed by Schrödinger equation

$$i\hbar\frac{d|\psi\rangle}{dt} = H|\psi\rangle,$$

where $\hbar$ is the Planck's constant, $H$ is the Hermitian operator corresponding to the energy of system.

In this case, the quantum states form a linear space by the Hypothesis 1 and 2 with an inner product

$$\langle\psi,\varphi\rangle = \int_\Omega \overline{\psi(\mathbf{x})}\varphi(\mathbf{x})d\mathbf{x}$$

between two wave functions $\psi(\mathbf{x}), \varphi(\mathbf{x})$ and a normalized condition

$$\langle\psi,\psi\rangle = \int_\Omega \overline{\psi(\mathbf{x})}\psi(\mathbf{x})d\mathbf{x} = \int_\Omega |\psi(\mathbf{x})|^2 = 1$$

of probability on wave function $\psi(\mathbf{x})$. So, *why is this inner product space complete?* Dr.Ouyang explains that because any combination of quantum states is still a quantum state, which requires the spatial model should be complete, i.e., a *Hilbert space* and particularly, a family $\{|\lambda_i\rangle\,; i \geq 1\}$ of complete orthogonal eigenstates for a physical quantity $a$ satisfies $\langle\lambda_i|\lambda_i\rangle = 1$ and $|\lambda_i\rangle \perp |\lambda_j\rangle\,, 1 \leq i \neq j < \infty$, corresponding to the normal coordinate system in Hibert space $\mathcal{H}$. And so, a quantum state $|\psi\rangle$ of physical quantity $a$ can be expanded to the linear combination $|\psi\rangle = \sum\limits_{i\geq 1} c_i\,|\lambda_i\rangle = \sum\limits_{i\geq 1} \langle\psi|\lambda_i\rangle\,|\lambda_i\rangle$ of eigenstates. Then,

$$\langle\psi|A|\psi\rangle = \sum_{j,i} c_j^* c_i\,\langle\lambda_j|A|\lambda_i\rangle = \sum_{j,i} c_j^* c_i \lambda_i \delta_{ji} = \sum_i \lambda_i |c_i|^2 \tag{10.10}$$

is the expectation value of measuring outcome of $a$, where $|c_i|^2 = |\,\langle\lambda_i|\psi\rangle\,|^2$ is the probability of observation value $\lambda_i$, $c^*$ is the conjugate complex of $c$ and $\delta_{ij}$ is the Kronecker function, namely $\delta_{ij} = 1$ if $i = j$ and $\delta_{ij} = 0$ otherwise. Now, there is another problem should be answered in theory, i.e., *does there exist such a Hermitian operator $A$ on a Hilbert space $\mathcal{H}$?* The answer is certainly Yes, i.e., the *Riesz-Fréchet representation theorem* concludes that *if $A : \mathcal{H} \to \mathbb{C}$ is a continuous linear functional on a Hilbert space $\mathcal{H}$, then there is a vector $\mathbf{v} \in \mathcal{H}$ such that $A(\mathbf{u}) = \langle\mathbf{v},\mathbf{u}\rangle$ for any $\mathbf{u} \in \mathcal{H}$* in functionals, namely such a Hermitian operator $A$ in (10.10) must exist.

## §2. **Continuity Flow**

A systematic recognition of thing $T$ is the recognition on the behavior of system elements with interaction, namely the recognition about the state of thing $T$ on combination structure $G^L$ inherited in elements of thing $T$. This implies that it is necessary to characterize the behavior of element group under the notion that "*a thing is a combined one*" and construct a new mathematical system over the 1-dimensional topological structure $G^L$ inherited in thing $T$ so as to systematically realize things in the universe.

Huizi asks Dr.Ouyang: "*Dad, is it just the network flow calculus by constructing a mathematical system over the inherited combinatorial structure $G^L$ of a thing?*" Of course Not! Dr.Ouyang explains that the edge flow of a network flow is usually a real number, even if it is a dynamic network flow the edge flow is a function of real number field. Notice that the properties of things are diversified. It is impossible to use only one quantitative parameter, namely a real number or real function to characterize things, which is especially notable in the recognition of microscopic particles. Dr.Ouyang asks Huizi: "*Can we make a precise measuring on the behavior of microscopic particles?*" Huizi replies: "*A microscopic particle obeys the uncertainty principle, can not be accurately measured its position and momentum at the same time. And so, it can be only observed on single eigenvalue, i.e., measuring the eigenvalue of $\langle\psi|A|\psi\rangle$ of quantum state under* *the acting of Hermitian operator A.*" Dr.Ouyang then asks his daughter: "*Is the quantum state a real number or a vector?*" Huizi smilingly says: "*It is of course a vector, a vector in Hilbert space!*" Dr.Ouyang nods and then asks her: "*Right! So, can the network flow be used as a model for characterizing the behavior of a group of microscopic particles?*" Huizi immediately holds on what Dr.Ouyang has said and replies: "*Yes! In this light, the network flow can not be really used as a model for characterizing the behavior of a group of microscopic particles.*" Dr.Ouyang further explains that the theory of matter constitute concludes that every substance in the nature is made up of microscopic particles. And the diversity of particle properties indicates that network flow can not be used as a model to simulate the behavior of a thing directly but needs to be extended. And secondly, the graph structure $G$ in the network flow $G^L$ is invariable but a self-organizing system such as the biological cell is always in the "*metabolism*". Even in a human body, the study shows that there are about 3.8 million cells that born or die every second, i.e., $G$ is also in varying. Thirdly, by the 3rd law of Newtonian mechanics, the action and reaction are equal in size but opposite in direction. *Does it follows that the amount of element v acting on u is the same as the amount of element u acting on v after its self-absorption?* Of course, it is not so simple to equate because the Newtonian mechanics is only applicable in case of rigid bodies and not applicable to element groups, especially the living individuals.

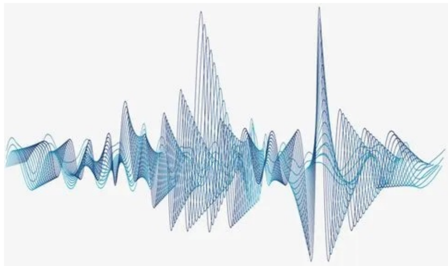


**Figure** 10.5. **Particle wave**

Generally, the vertices $v \in V(G)$ and the edges $(v,u) \in E(G)$ should be labeled by vectors correspondent to the diversity of system elements and interactions for characterizing the behavior of thing systemically, i.e., the states of elements with interaction by a labeled graph $G^L$. Then, *how can we characterize the intrinsic structural changing of the system, namely the increase, decrease of elements with the interaction changing?* Dr.Ouyang tells Huizi that a simplest way to characterize the growth or decline of individuals in population is to apply the *union* and *difference* operations of sets, i.e., $X \cup Y$ and $X\setminus Y$, $Y\setminus X$ on two sets $X, Y$. Accordingly, the graph *union* $G\bigcup G'$ and *difference* $G\setminus G'$ for two graphs $G, G'$ are defined as follows:

$$\begin{aligned} V(G\bigcup G') &= V(G)\bigcup V(G'), \quad E(G\bigcup G') &= E(G)\bigcup E(G'),\\ V(G\setminus G') &= V(G)\setminus V(G'), \quad E(G\setminus G') &= E(G)\setminus E(G'), \end{aligned}$$

namely the graph union operation $G\bigcup G'$ combines the vertices and edges of graph $G, G'$ to form a big graph and the graph difference operation $G\setminus G'$ is removing all vertices and edges of graph $G'$ from $G$ to form a small one, as shown in Figure 10.6.

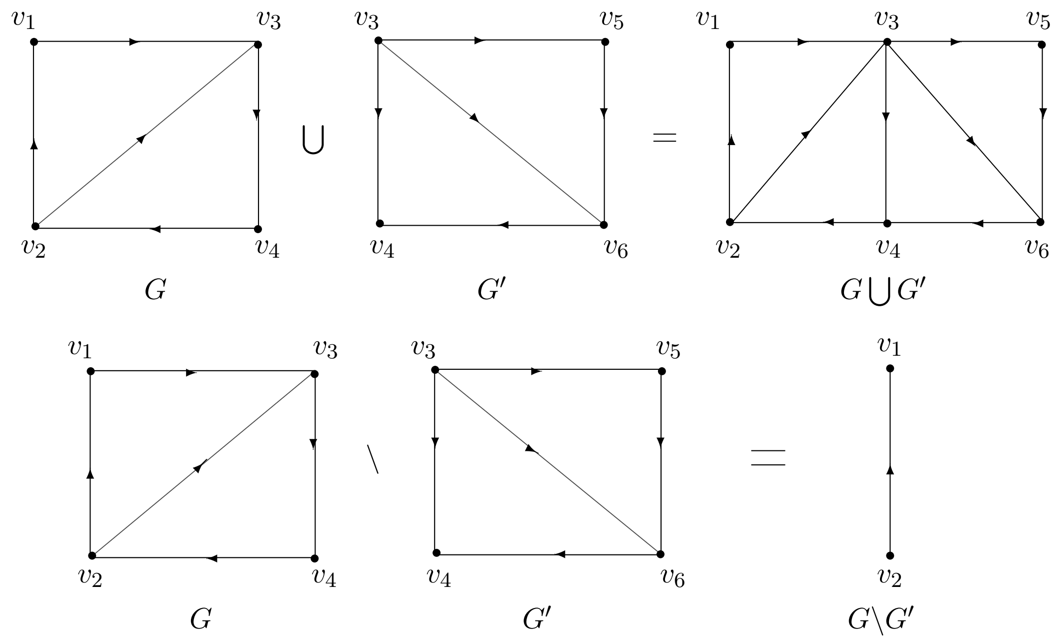



**Figure** 10.6. **The union and difference operation of graphs**

In this way, we can naturally take a graph family or sequence $\{G_n\}_1^\infty$ to discuss the graph space $\langle G_n; n \geq 1\rangle$ under the union, difference operations and then, simulate the changing of system elements. In addition, the changing of system elements can also be characterized by the labels on edges of labeled graph $G^L$. In fact, all edges $(v,u) \in E(G)$ with $L(v,u) = \mathbf{0}$ as well as those vertices $u \in N_G(v)$ associated with $L(v,u) = \mathbf{0}$ at vertex $v$ do not work in evolving of the labeled graph $G^L$, namely we agree on $G^L = G^L\setminus\{(v,u) \in E(G)|L(v,u) = \mathbf{0}\}$ and $G^L = G^L\setminus\{v \in V(G)|L(v,u) = \mathbf{0}, u \in N_G(v)\}$. And so, a complete graph $K_n$ with a sufficiently large order $n$ can be taken in advance for simulating the state of system, namely by the changing of flow on edges to simulate the state of system elements with interaction.

So, *how do we measure the reaction of element $u$ caused by the acting of element $v$ on element $u$?*

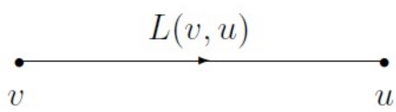



**Figure** 10.7. **Action conversion**

Dr.Ouyang tells Huizi that there is an equivalent conversion problem on the effect $L(v,u)$ of element $v$ on element $u$, as shown in Figure 10.7. Notice that the 3rd law of Newtonian mechanics may not be followed at this time. Similarly, it is assumed that the flow $L(v,u)$ from element $v$ into element $u$ is the effect on element $u$ in the network flow $G^L$ but most cases in the nature are not so. For example, the conversion of different energy forms, the equivalent carbon released by the combustion of different materials, etc. This is equivalent to the existence of an acting operator $T$ on vector $L(v,u)$, namely, the effect of $v$ on $u$ is $T : L(v,u) \to L^T(v,u)$ in mathematics. Thinking clearly these questions, Dr.Ouyang tells Huizi that a general model for perceiving a thing $T$ is established, i.e., *continuity flow* which could be used as a kind of globally mathematical element to simulate the dynamic behavior of a thing.

**2.1.Continuity Flow Model.** Certainly, the developing or evolving of a thing $T$ is the dialectical unity of internal and external causes. Among them, the internal cause is the foundation and basis, the external cause is the condition and inducement which promotes the developing or evolving of a thing through internal cause. In this way, a systematic recognition of thing $T$ can be abstracted into the input and output of its elements with interactions and then characterize the input and output states by vectors. Dr.Ouyang explains that the network flow is a simulation of substance flow in macroscopic world on the network and characterizes its input and output process. However, the labels $L(v), L(v,u)$ on vertices $v \in V(G)$, edges $(v,u) \in E(G)$ are all real numbers, i.e., $L(v), L(v,u) \in \mathbb{R}$ on network flow $G^L$, which naturally limits its simulating range. So, *how can we abstractly simulate the flow of substance in the microscopic world?* Dr.Ouyang tells Huizi that if the substance flow is observed by a microscopic person on the transporting route in the microscopic world, it can be similar to the network flow but needs to replace the image range $\mathbb{R}$ of mapping $L : V(G)\bigcup E(G) \to \mathbb{R}$ by a Banach space $(V; \|\cdot\|)$, i.e., labeling the vertices $v \in V(G)$ and edges $(v,u) \in E(G)$ by vectors in $V$, and the input, output flows also hold with the conservation of continuous flow at each vertex. Clearly, such a labeled graph $G^L$ is dynamic, can be used as an abstract model of systemically recognitive thing, called the *continuity flow* or *flow* in the context.

Generally, a continuity flow $\left(\overrightarrow{G}; L, \mathscr{A}\right)$ is an oriented embedded graph $\overrightarrow{G}$ in a topological space $\mathscr{S}$ associated with a mapping $L : v \to L(v),\ (v,u) \to L(v,u)$, 2 end-operators $A^+_{vu} : L(v,u) \to L^{A^+_{vu}}(v,u)$ and $A^+_{uv} : L(u,v) \to L^{A^+_{uv}}(u,v)$ on a Banach space $\mathscr{B}$ over a field $\mathscr{F}$ such as those shown in Figure 10.8

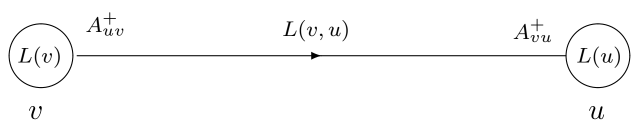



**Figure** 10.8. **Flow with end-operators**

with $L(v,u) = -L(u,v)$, $A^+_{vu}(-L(v,u)) = -L^{A^+_{vu}}(v,u)$ for $\forall (v,u) \in E(G)$ and meanwhile, holding with the continuity equation

$$\sum_{u \in N^-_G(v)} L^{A^+_{uv}}(u,v) - \sum_{u \in N^+_G(v)} L^{A^+_{uv}}(u,v) \ = \ L(v) \tag{10.11}$$

at any vertex $v \in V(G)$ of graph $G$, where $N^-_G(v), N^+_G(v)$ are respectively the in-neighborhood and out-neighborhood of vertex $v \in V(G)$, namely all vertices $N_G^-(v) \subset N_G(v)$ that flow into the vertex $v$ and all vertices $N_G^+(v) \subset N_G(v)$ that flow out of the vertex $v$ and $N_G^-(v) \cup N_G^+(v) = N_G(v)$.

Notice that the equation (10.11) is a vector equation, the operator $A_{vu}^+$, $(v,u) \in E(G)$ are all linear operators on Banach space $\mathscr{B}$, including the differential and integral operators. Thus, the continuity equations in continuity flow contain the algebraic equation, differential equation, integral equations and operator equations as special cases.

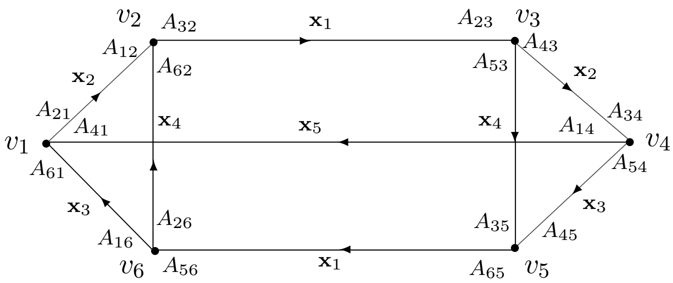



**Figure** 10.9. **An example of continuity flow**

For example, the continuity equations at vertices of the continuity flow $G^L$ given in Figure 10.9 are respectively

$$\begin{cases} (v_1): \ A_{41}(\mathbf{x}_5) + A_{61}(\mathbf{x}_3) - A_{21}(\mathbf{x}_2) = L(v_1), \\ (v_2): \ A_{12}(\mathbf{x}_2) + A_{62}(\mathbf{x}_4) - A_{32}(\mathbf{x}_1) = L(v_2), \\ (v_3): \ A_{23}(\mathbf{x}_1) - A_{43}(\mathbf{x}_2) - A_{53}(\mathbf{x}_4) = L(v_3), \\ (v_4): \ A_{34}(\mathbf{x}_2) - A_{14}(\mathbf{x}_5) - A_{54}(\mathbf{x}_3) = L(v_4), \\ (v_5): \ A_{35}(\mathbf{x}_4) + A_{45}(\mathbf{x}_3) - A_{65}(\mathbf{x}_1) = L(v_5), \\ (v_6): \ A_{56}(\mathbf{x}_1) - A_{16}(\mathbf{x}_3) - A_{26}(\mathbf{x}_4) = L(v_6). \end{cases} \tag{10.12}$$

Particularly, let

$$L(v_1) = L(v_2) = L(v_3) = L(v_4) = L(v_5) = L(v_6) = \mathbf{0},$$

$$A_{21} = A_{32} = A_{43} = A_{53} = A_{14} = A_{54} = A_{65} = A_{16} = A_{26} = 1_{\mathscr{B}},$$

$$A_{41} = A_{61} = A_{12} = A_{62} = A_{56} = \frac{d}{dt},$$

$$A_{23} = A_{34} = A_{35} = A_{45} = A_{56} = t\frac{d}{dt}.$$

Then, the operator equation (10.12) is a system of differential equations

$$\begin{cases} t\dfrac{d\mathbf{x}_1}{dt} = \mathbf{x}_2 + \mathbf{x}_4, & \dfrac{d\mathbf{x}_2}{dt} + \dfrac{d\mathbf{x}_4}{dt} = \mathbf{x}_1, \\ t\dfrac{d\mathbf{x}_2}{dt} = \mathbf{x}_5 + \mathbf{x}_3, & t\dfrac{d\mathbf{x}_4}{dt} + t\dfrac{d\mathbf{x}_3}{dt} = \mathbf{x}_1, \\ t\dfrac{d\mathbf{x}_1}{dt} = \mathbf{x}_3 + \mathbf{x}_4, & \dfrac{d\mathbf{x}_5}{dt} + \dfrac{d\mathbf{x}_3}{dt} = \mathbf{x}_2. \end{cases}$$

Similarly, take different types of operators $A_{ij}, 1 \leq i, j \leq 6$ in the operator equation (10.12) such as partial differential operator, integral operator or difference operator. Then, the different types of operator equations such as the partial differential equation, integral equation and the difference equation can be obtained by the continuity equation (10.12).

Conversely, for an operator equation, divide each equation into $m$ terms according to the number of equations $m$ contained therein, i.e

$$① \left\{ \begin{array}{l} \mathscr{F}_1(\mathbf{x}) = \mathbf{0}, \\ \mathscr{F}_2(\mathbf{x}) = \mathbf{0}, \\ \cdots\cdots\cdots\cdots, \\ \mathscr{F}_m(\mathbf{x}) = \mathbf{0}_\circ \end{array} \right. \Rightarrow ② \left\{ \begin{array}{r} A_{11}(\mathbf{x}_1) + A_{21}(\mathbf{x}_2) + \cdots + A_{m1}(\mathbf{x}_m) \ = \mathbf{0}, \\ A_{12}(\mathbf{x}_1) + A_{22}(\mathbf{x}_2) + \cdots + A_{m2}(\mathbf{x}_m) \ = \mathbf{0}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ A_{1m}(\mathbf{x}_1) + A_{2m}(\mathbf{x}_2) + \cdots + A_{mm}(\mathbf{x}_m) = \mathbf{0}. \end{array} \right.$$

Among these, if one of equations $\mathscr{F}_i(\mathbf{x}) = \mathbf{0}$ in ① does not contain variables $\mathbf{x}_k, 1 \leq k \leq m$, then let $\mathbf{x}_k = \mathbf{0}$ and still divide it into $m$ terms in ②. In this way, one can construct a continuity flow $K_m^L$ on complete graph $K_m$ of order $m$ as follows:

$$V(K_m) = \{v_1, v_2, \cdots, v_m\}, \quad E(K_m) = \{(v_i, v_j) | 1 \leq i, j \leq m\}$$

and meanwhile, for any integer $1 \leq i, j \leq m$, let the labeling mapping

$$L : v_i \to -A_{ii}(\mathbf{x}_i), \quad L : (v_i, v_j) \to \mathbf{x}_j$$

and end-operators $A^+_{v_i v_j} = A_{ij}, A^+_{v_j v_i} = A_{ji}$ on edge $(v_i, v_j)$ for integers $1 \leq i, j \leq m$. Thereafter, a continuity flow $K_m^L$ of complete graph $K_m$ corresponding to the operator equation ② is obtained.

Notice that there are many interesting cases to continuous flows. Some of them have been studied while others have not attracted the enough interest of researchers. For example, if all end-operators are $\mathbf{1}_{\mathscr{V}}$, let $L(v) = \dot{x}_v$ for any vertex $v \in V(G)$. Then, the continuity flow is a complex network discussed in Chapter 8 and if $L(v) = \mathbf{v}$ is a constant vector at any vertex $v \in V(G)$, the continuity flow is called *G-flow*. Furthermore, let $\mathscr{B} = \mathbb{R}$ in the $G$-flow. Then, a $G$-flow is nothing else but the usual network flow.

Notice that there is a good summary and refining on the evolving law of things in Chinese ancient philosophy. It concludes that there are *Yin* and *Yang* interactions or two forces complement each other inside a thing, both of them promote the evolving of thing, which is the basic law of nature of things in the universe, also the philosophical implication in Taiji diagram of Chinese, see Figure 10.10. Then, *How do*

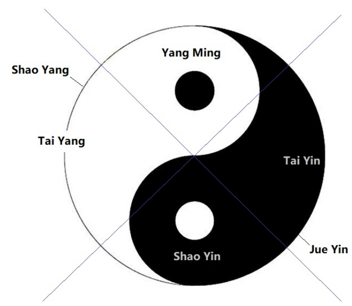



**Figure** 10.10. **Taiji philosophy**

*we hold on the philosophical implication of Taiji diagram?* Dr.Ouyang tells Huizi that Confucius wrote that "*the union of a Yin and a Yang is a kind of Tao. Its successor is good. Its carrying forward is the nature of human*" in his *Change Book – Xici Zhuan*, namely a Yin and a Yang is the way of life or law of the generation and developing of livings in universe. And so, inheriting and carrying forward this law is a good deed, also the nature of humans. Thus, by the Chinese ancient philosophy, the labels of vertices and edges on a continuity flow $G^L$ should include both the negative (Yin) $\mathbf{u}$ and positive (Yang) vectors $\mathbf{v}$ holding with $\mathbf{v}+\mathbf{u}=\mathbf{c}$, i.e., a constant vector $\mathbf{c}$ is used to realize the harmony of Yin and Yang interactions in the meaning of Taiji diagram, which is similar to the situation of the living or death of Schrodinger's cat, namely we can only observe one of the vectors $\mathbf{v}$ and $\mathbf{u}$. As a result, simply labeling by either $\mathbf{v}$ or $\mathbf{u}$ on graph $G$ can not fully reflect both positive and negative interactions, which is similar to one watching a floating iceberg that only the part above the surface of the sea can be seen, as shown in Figure 10.11. Similarly, since $\mathbf{v}+\mathbf{u}=\mathbf{c}$ is a constant vector, labeling graph $G$ with $\mathbf{v}+\mathbf{u}$ can not reflect the changing of element state with interaction also. So, *how should we integrate the Yin and Yang view of ancient Chinese philosophy into the continuity flow for characterizing a thing?* Dr.Ouyang explains that a special type of continuity flow labeled by complex vectors, called *harmonic flow* is needed in this case. Here, a *complex vector* refers to the form such as $\mathbf{x}+i\mathbf{y}\in\mathscr{B}\times\mathscr{B}$ with vectors $\mathbf{x},\mathbf{y}\in\mathscr{B}$, where $i$ is the unit of imaginary numbers, i.e., $i^2=-1$.

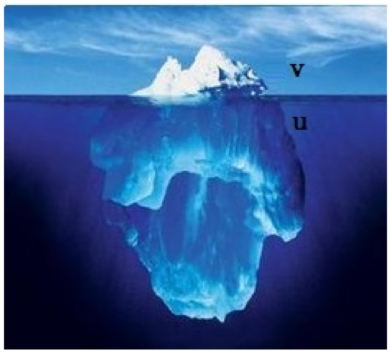



**Figure** 10.11. **Iceberg**

Generally, a harmonic flow $G^{L^2}$ is a topological graph $G$ in space with its vertices and edges labeled by vectors in a Banach space $\mathscr{B}\times\mathscr{B}$ over field $\mathscr{F}$, i.e., for any vertex $v\in V(G)$ and edge $(v,u)\in V(G)$, the labeling mapping $L^2: v\to L(v)+iL'(v)$, $L^2:(v,u)\to L(v,u)+iL'(v,u)$ holding with $L(v)+L'(v)=\mathbf{c}_v\in\mathscr{B}$, $L(v,u)+L'(v,u)=\mathbf{c}_{vu}\in\mathscr{B}$ and 2 end-operators

$$A^+_{vu}: L(v,u)+iL'(v,u)\to L^{A^+_{vu}}(v,u)+iL'^{A^+_{vu}}(v,u),$$
$$A^+_{uv}: L(v,u)+iL'(v,u)\to L^{A^+_{uv}}(u,v)+iL'^{A^+_{uv}}(u,v)$$

satisfying

$$L(v,u)=-L(u,v),\quad L'(v,u)=-L'(u,v),$$
$$A^+_{vu}(-L(v,u))=-L^{A^+_{vu}}(v,u),\quad A^+_{vu}(-L'(v,u))=-L'^{A^+_{vu}}(v,u)$$

as shown in Figure 10.12. And meanwhile, and meanwhile, holding with the continuity equation

$$\sum_{u\in N_G^-(v)} \left(L^2\left(u,v\right)\right)^{A_{uv}^+} - \sum_{u\in N_G^+(v)} \left(L^2\left(u,v\right)\right)^{A_{uv}^+} \;=\; L^2(v) \tag{10.12}$$

at any vertex $v\in V(G)$ of graph $G$.

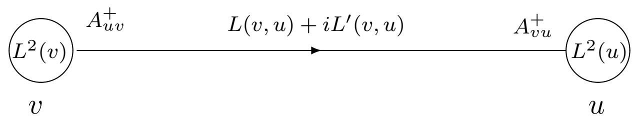



**Figure** 10.12. **Complex vector with end-operators**

Notice that for any edge $(u,v)\in E(G)$, $L(u,v)+L'(u,v)=\mathbf{c}_{uv}$, i.e., $L'(u,v)=\mathbf{c}_{uv}-L(u,v)$. Substituting it into (10.12), we know that

$$\begin{aligned}
L^2(v) &= \sum_{u\in N_G^-(v)} \left(L^2\left(u,v\right)\right)^{A_{uv}^+} - \sum_{u\in N_G^+(v)} \left(L^2\left(u,v\right)\right)^{A_{uv}^+}\\
&= \sum_{u\in N_G^-(v)} L^{A_{uv}^+}\left(u,v\right) - \sum_{u\in N_G^+(v)} L^{A_{uv}^+}\left(u,v\right) + i\sum_{u\in N_G^-(v)} \mathbf{c}_{uv}\\
&\quad -i\sum_{u\in N_G^-(v)} L^{A_{uv}^+}\left(u,v\right) - i\sum_{u\in N_G^+(v)} \mathbf{c}_{uv} + i\sum_{u\in N_G^+(v)} L^{A_{uv}^+}\left(u,v\right)\\
&= (1-i)\left(\sum_{u\in N_G^-(v)} L^{A_{uv}^+}\left(u,v\right) - \sum_{u\in N_G^-(v)} L^{A_{uv}^+}\left(u,v\right)\right)\\
&\quad +i\left(\sum_{u\in N_G^-(v)} \mathbf{c}_{uv}^{A_{uv}^+} - \sum_{u\in N_G^+(v)} \mathbf{c}_{uv}^{A_{uv}^+}\right).
\end{aligned}$$

For any vertex $v\in V(G)$ and edge $(u,v)\in E(G)$ on $G$, define

$$\begin{aligned}
\widehat{L}:\quad v \quad &\to\quad L^2(v) - i\left(\sum_{u\in N_G^-(v)} \mathbf{c}_{uv}^{A_{uv}^+} - \sum_{u\in N_G^+(v)} \mathbf{c}_{uv}^{A_{uv}^+}\right),\\
\widehat{L}:\ (u,v) \quad &\to\quad (1-i)L^2(u,v).
\end{aligned}$$

Then,

$$\sum_{u\in N_G^-(v)} \widehat{L}^{A_{uv}^+}\left(u,v\right) - \sum_{u\in N_G^+(v)} \widehat{L}^{A_{uv}^+}\left(u,v\right) \;=\; \widehat{L}(v),$$

namely the harmonic flow $G^{L^2}$ can be transformed to a continuity flow $G^{\widehat{L}}$ under the labeling mapping $\widehat{L}$, which implies that the harmonic flow $G^{L^2}$ combined with the Yin and

Yang view point of ancient Chinese philosophy is a special case of continuity flow $G^{\widehat{L}}$. On the other hand, for any vertex $v \in V(G)$ and edge $(v,u) \in E(G)$, let $L'(v) = \mathbf{0}$, $L'(v,u) = \mathbf{0}$ in harmonic flow $G^{L^2}$, then the harmonic flow $G^{L^2}$ degenerates into a continuity flow $G^L$, i.e., the continuity flow $G^L$ is a special harmonics flow. Consequently, we get a conclusion that *the difference of continuity flow $G^L$ and harmonic flow $G^{L^2}$ are only in their forms, which are essentially equivalent.* Nevertheless, the real and imaginary parts in the harmonic flow can reflect the living or death situation of a living body to some extent, which is more in line with Chinese culture, namely if the real part is $\mathbf{0}$, then the "*life*" of a living body is no longer existing in human recognition, i.e., it is in the death state. Similarly, a complex vector $A + iB, A, B \in \mathbb{R}$ can be used to characterize the living or death state of Schrodinger's cat, i.e. the cat is alive if $A \neq \mathbf{0}$. Otherwise, the cat is death if $A = \mathbf{0}$.

**2.2.Continuity Flow Expression.** Notice that the addition "+" and multiplication "·" operation (9.9) of network flows is carried out on the same graph $G$, only needs to calculate the addition or multiplication of corresponding labels on the same edge but the continuity flows $G^L, G'^{L'}$ are different, which is calculated on two possibly different graphs $G, G'$. In this case, assume that $G^L, G'^{L'}$ are two continuity flows on Banach space $\mathscr{B}$ over a field $\mathscr{F}$ and $\lambda \in \mathscr{F}$. Define the addition, multiplication and scalar multiplication as follows:

$$G^L + G'^{L'} = (G \setminus G')^L \bigcup \left(G \bigcap G'\right)^{L+L'} \bigcup (G' \setminus G)^{L'}, \tag{10.13}$$

$$G^L \cdot G'^{L'} = (G \setminus G')^L \bigcup \left(G \bigcap G'\right)^{L \cdot L'} \bigcup (G' \setminus G)^{L'}, \tag{10.14}$$

$$\lambda \cdot G^L = G^{\lambda \cdot L} \circ \tag{10.15}$$

where, for any vertex $v \in V(G)$ and edge $(v,u) \in E(G)$, $L(v), L'(v), L(v,u), L'(v,u) \in \mathscr{B}$, $L + L' : v \to L(v) + L'(v)$, $(v,u) \to L(v,u) + L'(v,u)$, $L \cdot L' : v \to L(v) \cdot L'(v)$, $(v,u) \to L(v,u) \cdot L'(v,u)$, $\lambda \cdot L : v \to \lambda \cdot L(v)$, $(v,u) \to \lambda \cdot L(v,u)$. Among them, $L(v) \cdot L'(v)$ and $L(v,u) \cdot L'(v,u)$ denotes the *Hadamard product* of vectors in Banach space $\mathscr{B}$. For example, let vectors

$$\mathbf{x} = (x_1, x_2, \cdots, x_n), \quad \mathbf{y} = (y_1, y_2, \cdots, y_n).$$

Then, the Hadamard product of $\mathbf{x}$ and $\mathbf{y}$ is

$$\mathbf{x} \cdot \mathbf{y} = (x_1 y_1, x_2 y_2, \cdots, x_n y_n). \tag{10.16}$$

Accordingly, the end-operators are replaced by $A_{vu}^{++}$, $A_{vu}^{+\otimes}$ and $A_{vu}^{+\cdot}$ with respectively actions

$$A_{vu}^{++} \quad : \quad L(v,u)+L'(v,u) \to L^{A_{vu}^{+}}(v,u)+L'^{A'^{+}_{vu}}(v,u),$$
$$A_{vu}^{+\cdot} \quad : \quad L(v,u)\cdot L'(v,u) \to L^{A_{vu}^{+}}(v,u)\cdot L'^{A'^{+}_{vu}}(v,u)$$

on edge $(v,u)\in E(G)$. Notice that $\lambda\in\mathscr{F}$ can be viewed as a vector $(\lambda,\lambda,\cdots,\lambda)$ of $\mathscr{B}$, i.e., each coordinate is $\lambda$. Then, the scalar multiplication $\lambda\cdot L$ can be seen as a special case of Hadamard product. In this case, for any vertex $v\in V(G)$ the vector on $v$ is defined as $\lambda\cdot L(v)$ after the scalar multiplication, the vector on $v$ is

$$(L+L')(v)=\begin{cases} L(v), & v\in V(G)\setminus V(G'),\\ L(v)+L'(v), & v\in V(G)\bigcap V(G'),\\ L'(v), & v\in V(G')\setminus V(G)\end{cases}$$

after addition and the vector on $v$ is

$$(L\cdot L')(v)=\begin{cases} L(v), & v\in V(G)\setminus V(G'),\\ \sum\limits_{u\in N_G(v)} L^{A_{vu}^{+}}(v,u)\cdot L'^{A'^{+}_{vu}}(v,u)), & v\in V(G)\bigcap V(G'),\\ L'(v), & v\in V(G')\setminus V(G).\end{cases}$$

after the Hadamard product. Particularly, if $G'=G$, there is

$$(L+L')(v)=L(v)+L'(v),\quad (L\cdot L')(v)=\sum_{u\in N_G(v)} L^{A_{vu}^{+}}(v,u)\cdot L'^{A'^{+}_{vu}}(v,u))$$

for any vertex $v\in V(G)$.

In this way, similar to the expression of network flow, a lots of the continuity flow expression are as follows:

①Constant continuity flow. A constant continuity flow refers to a continuity flow $G^{L_{\mathbf{C}}}$ with a labeling mapping $L_{\mathbf{C}}:(v,u)\to\mathbf{c}_{vu}$ a constant vector on any $(v,u)\in E(G)$. Particularly, for any edge $(v,u)\in E(G)$, $\mathbf{c}_{vu}=\mathbf{c}$ a same constant vector $\mathbf{c}$ is denoted by $\mathbf{c}=G^{L_{\mathbf{c}}}$. In this case, similar to the network flow, $\mathbf{O}=G^{L_{\mathbf{0}}}$ is the unit in $+$ operations, $\mathbf{I}=G^{L_{\mathbf{1}_{\mathscr{F}}}}$ is the unit in Hadamard product, namely

$$\mathbf{O}+G^L=G^L+\mathbf{O}=G^L,\qquad \mathbf{I}\cdot G^L=G^L\cdot\mathbf{I}=G^L. \tag{10.17}$$

Here, the labels on vertex of the unit $\mathbf{O}$ and $\mathbf{I}$ are all zero vector $\mathbf{0}$ or $\mathbf{1}$. Correspondingly, the unit elements $\mathbf{O}$ and $\mathbf{I}$ in network flows are special cases of continuity flow units.

②Inverse. For a continuous flow $G^L$ on Banach space $\mathscr{B}$, if $X+G^L=\mathbf{O}$ then $X$ is inverse of $G^L$ in addition, denoted by $X=-G^L$. Similarly, for a continuity flow $G^L$

without zero-vertex and zero-edge, namely for any vertex $v \in V(G)$ and edge $(v,u) \in E(G)$, $L(v) \neq \mathbf{0}$, $L(v,u) \neq \mathbf{0}$, if there is a continuity flow $Y$ such that $Y \cdot G^L = \mathbf{I}$, then $Y$ is called the inverse of $G^L$ in Hadamard product, denoted by $Y = 1/G^L$. According to the definition of continuity flow operation (10.13) – (10.15), it is easy to know

$$X = -G^L = G^{-L}, \quad Y = \frac{1}{G^L} = G^{\frac{1}{L}} = G^{L^{-1}},$$

namely the inverse of a continuity flow $G^L$ in addition and Hadamard product is respectively

$$-G^L = G^{-L}, \quad \frac{1}{G^L} = G^{\frac{1}{L}}. \tag{10.18}$$

③Algebraic expression. For any integer $n \geq 1$, let $a_1, a_2, \cdots, a_n$ be $n$ elements in field $\mathscr{F}$ and let $G_1^{L_1}, G_2^{L_2}, \cdots, G_n^{L_n}$ be $n$ continuity flows over Banach space $\mathscr{B}$. Define

$$\begin{aligned} a_1G_1^{L_1} + a_2G_2^{L_2} + \cdots + a_nG_n^{L_n} &= \left(\bigcup_{i=1}^{n} G_i\right)^{a_1L_1+a_2L_2+\cdots+a_nL_n}, \\ \left(a_1G_1^{L_1}\right) \cdot \left(a_2G_2^{L_2}\right) \cdots \left(a_nG_n^{L_n}\right) &= \left(\bigcup_{i=1}^{n} G_i\right)^{(a_1L_1)\cdot(a_2L_2)\cdots(a_nL_n)}. \end{aligned}$$

④Polynomial. For any integer $n \geq 1$, $a_1, a_2, \cdots, a_n \in \mathscr{F}$ and a continuity flow $G^L$, define the polynomials of $G^L$ by

$$\begin{aligned} a_0 + a_1G^L + a_2G^{L^2} + \cdots + a_nG^{L^n} &= G^{a_0+a_1\cdot L+a_2\cdot L^2+\cdots+a_n\cdot L^n}, \\ a_0 \cdot \left(a_1G^L\right) \cdot \left(a_2G^{L^2}\right) \cdots \left(a_nG^{L^n}\right) &= G^{a_0 \cdot(a_1L)\cdot\left(a_2L^2\right)\cdots(a_nL^n)}, \end{aligned}$$

respectively and meanwhile, if $b_1L_1' + b_2L_2' + \cdots + b_nL_n' \neq \mathbf{0}$, define

$$\begin{aligned} \frac{a_1G_1^{L_1} + a_2G_2^{L_2} + \cdots + a_nG_n^{L_n}}{b_1{G'}_1^{L_1'} + b_2{G'}_2^{L_2'} + \cdots + b_n{G'}_n^{L_n'}} &= \frac{\left(\bigcup\limits_{i=1}^{n} G_i\right)^{a_1L_1+a_2L_2+\cdots+a_nL_n}}{\left(\bigcup\limits_{i=1}^{n} G_i'\right)^{b_1L_1'+b_2L_2'+\cdots+b_nL_n'}} \\ &= \left(\left(\bigcup_{i=1}^{n} G_i\right) \bigcup \left(\bigcup_{i=1}^{n} G_i'\right)\right)^{\frac{a_1L_1+a_2L_2+\cdots+a_nL_n}{b_1L_1'+b_2L_2'+\cdots+b_nL_n'}}. \end{aligned}$$

by (10.18). Generally, let $\mathscr{G} = \{G_n | n \geq 1\}$ be a closed family under the union of graphs, namely for any graphs $G_i, G_j \in \mathscr{G}$, $G_i \cup G_j \in \mathscr{G}$ and let $\mathscr{B}$ be a Banach space over field $\mathscr{F}$, denote by $\mathscr{G}_{\mathscr{B}}$ all the continuity flows $G^L$ generated by graphs in $\mathscr{G}$ over Banach space $\mathscr{B}$, i.e., $\mathscr{G}_{\mathscr{B}} = \left\{G^L | G \in \mathscr{G}, L : V(G) \cup E(G) \to \mathscr{B}\right\}$. In this case, by definition $(\mathscr{G}_{\mathscr{B}}; +)$ and $(\mathscr{G}_{\mathscr{B}}; \cdot)$ are both Abelian groups on $\mathscr{G}_{\mathscr{B}}$. Therefore, $(\mathscr{G}_{\mathscr{B}}; +, \cdot)$ *is a bigroup of* $\mathscr{G}_{\mathscr{B}}$

*under addition "+" and multiplication "·"*. Furthermore, *if* $(\mathscr{B};+,\cdot)$ *is a field and all end-operators are* $1_{\mathscr{B}}$ *for any continuity flow* $G^L \in \mathscr{G}_{\mathscr{B}}$, *then* $(\mathscr{G}_{\mathscr{B}};+,\cdot)$ *is a field too* because for any vertex $v \in E(G)$ and edge $(v,u) \in E(G)$ there is

$$\begin{aligned}
(L_1 \cdot (L_2 + L_3))(v) &= (L_1 \cdot L_2 + L_1 \cdot L_3)(v),\\
((L_1 + L_2) \cdot L_3)(v) &= (L_1 \cdot L_3 + L_2 \cdot L_3)(v),\\
(L_1 \cdot (L_2 + L_3))(v,u) &= (L_1 \cdot L_2 + L_1 \cdot L_3)(v,u),\\
((L_1 + L_2) \cdot L_3)(v,u) &= (L_1 \cdot L_3 + L_2 \cdot L_3)(v,u)
\end{aligned}$$

and for continuity flows $G_1^{L_1}, G_2^{L_2}, G_3^{L_3} \in \mathscr{G}_{\mathscr{B}}$,

$$\begin{aligned}
G^{L_1} \cdot \left(G^{L_2} + G^{L_3}\right) &= G^{L_1} \cdot G^{L_2} + G^{L_1} \cdot G^{L_3},\\
\left(G^{L_1} + G^{L_2}\right) \cdot G^{L_3} &= G^{L_1} \cdot G^{L_3} + G^{L_2} \cdot G^{L_3}
\end{aligned}$$

in this case, namely the operations on $(\mathscr{B};+,\cdot)$ satisfies the distributive law. Here, $G = G_1 \bigcup G_2 \bigcup G_3$ and for any integer $1 \le i \le 3$, $x \in V(G) \bigcup E(G)$, $L_i$ is extended to $G$ by defining its image $L_i : x \to L_i(x)$ if $x \in V(G_i) \bigcup E(G_i)$, otherwise $L_i : x \to \mathbf{0}$. Dr.Ouyang explains that the extension labeling then holds with $G^{L_i} = G_i^{L_i}, 1 \le i \le 3$.

## §3. Banach Flow Space

Certainly, anyone is always living in a community network and the matter flow on the network can be observed by eyes of human. Meanwhile, the researcher can observes the life of microorganisms such as bacteria, viruses and fungi under the microscope. Notice that the difference between the two is that the observing on matter flow in the macroscopic world is a local behavior and we perceive things on local observing. However, the observing on microorganism is a kind of overall behavior, which is recognized by the overall appearance of a microorganism and the observing result is the state of microbial network or the cell structure of microorganism. In this way, the systematic recognition of thing in the macroscopic world or the microcosmic world is essentially the recognition of network formed by the elements of a thing, namely the state of continuous flow $G^L$.

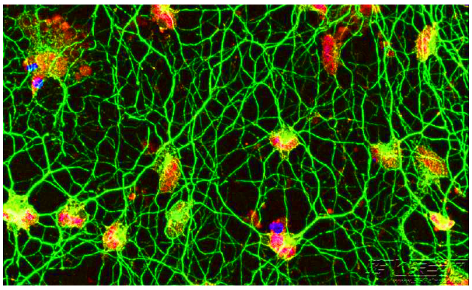


**Figure** 10.13. **Cortical cells**

Then, *can we use the continuity flow* $G^L$ *as a mathematical element to construct a mathematical space simulating the state of things in both the macroscopic or microcosmic* *world?* The answer is Yes! Dr.Ouyang explains that it is the Banach flow space $\mathscr{G}_{\mathscr{B}}$ of continuity flows $G^L$ of graphs in a family $\mathscr{G}$ on Banach space $\mathscr{B}$ by operations (10.13) and (10.15) with a norm $\|\cdot\|$, namely regard a continuity flow $G^L$ as a vector under a topological structure $G$ and then establish a mathematical system on continuity flows $G^L,\ G\in\mathscr{G}$, called the *mathematical combinatorics*.

**3.1.Linear Flow Space.** Let $\mathscr{B}$ be a Banach space over field $\mathscr{F}$. Dr.Ouyang tells Huizi that for any continuity flows $G_k, G_l, G_s \in \mathscr{G}_{\mathscr{B}}$, it is easy to verify $(\mathscr{G}_{\mathscr{B}};\mathscr{F})$ satisfies all conditions of linear space by addition + (10.13) and scalar multiplication $\cdot$ (10.15) of continuity flows.

(1) Commutative laws, namely $G_k^{L_k}+G_l^{L_l}=G_l^{L_l}+G_k^{L_k}$ because

$$\begin{aligned}
G_k^{L_k}+G_l^{L_l} &= (G_k\setminus G_l)^{L_k}\bigcup\left(G_k\bigcap G_l\right)^{L_k2+L_l2}\bigcup(wG_l\setminus G_k)^{L_l}\\
&= (G_l\setminus G_k)^{L_l}\bigcup\left(G_l\bigcap G_k\right)^{L_l2+L_k}\bigcup(G_k\setminus G_l)^{L_l}\\
&= G_l^{L_l}+G_k^{L_k}.
\end{aligned}$$

(2) Associative laws, namely $\left(G_k^{L_k}+G_l^{L_l}\right)+G_s^{L_s}=G_k^{L_k}+\left(G_l^{L_l}+G_s^{L_s}\right)$ because

$$\begin{aligned}
\left(G_k^{L_k}+G_l^{L_l}\right)+G_s^{L_s} &= \left(G_k\bigcup G_l\right)^{L_{kl}^+}+G_s^{L_s}=\left(G_k\bigcup G_l\bigcup G_s\right)^{L_{kls}^+}\\
&= G_k^{L_k}+\left(G_l\bigcup G_s\right)^{L_{ls}^+}=G_k^{L_k}+\left(G_l^{L_l}+G_s^{L_s}\right).
\end{aligned}$$

Among them, the mapping $L_{kl}^+$ is the labeling $L_k+L_l$ on graph $G_k\cup G_l$, $L_{kls}^+$ is the labeling on graph $G_k\cup G_l\cup G_s$ as follows:

$$L_{kls}^+(e)=\begin{cases}
L_k(v,u), & (v,u)\in E\left(G_k\setminus(G_l\cup G_s)\right),\\
L_l(v,u), & (v,u)\in E\left(G_l\setminus(G_k\cup G_s)\right),\\
L_s(v,u), & (v,u)\in E\left(G_s\setminus(G_k\cup G_l)\right),\\
L_{kl}^+(v,u), & (v,u)\in E\left((G_k\cap G_l)\setminus G_s\right),\\
L_{ks}^+(v,u), & (v,u)\in E\left((G_k\cap G_s)\setminus G_l\right),\\
L_{ls}^+(v,u), & (v,u)\in E\left((G_l\cap G_s)\setminus G_k\right),\\
\left(L_k+L_l+L_s\right)(v,u), & (v,u)\in E\left(G_k\bigcap G_l\bigcap G_s\right).
\end{cases}$$

(3) Unit. By definition, the zero-flow $\mathbf{O}$ is a uniquely identity on $(\mathscr{G}_{\mathscr{B}};\mathscr{F})$, namely $\mathbf{O}+G^L=G^L+\mathbf{O}=G^L$ for any continuity flow $G^L\in\mathscr{G}_{\mathscr{B}}$.

(4) Inverse, namely for any continuity flow $G^L\in\mathscr{G}_{\mathscr{B}}$, $G^{-L}$ is the uniquely inverse of $G^L$, i.e., $G^L+G^{-L}=\mathbf{O}$.

(5) Scalar multiplication, namely for any continuity flow $G^L\in\mathscr{G}_{\mathscr{B}}$ and $\alpha,\alpha_1,\alpha_2\in\mathscr{F}$, by (10.15) there are respectively

($a$) $1_{\mathscr{F}} \cdot G^L = G^L$;

($b$) $(\alpha_1\alpha_2) \cdot G^L = \alpha_1(\alpha_2 \cdot G^L)$;

($c$) $\alpha \cdot (G_k^{L_k} + G_l^{L_l}) = \alpha \cdot G_k^{L_k} + \alpha \cdot G_l^{L_l}$;

($d$) $(\alpha_1 + \alpha_2) \cdot G^L = \alpha_1 \cdot G^L + \alpha_2 \cdot G^L$.

Thus, the continuity flow space $(\mathscr{G}_{\mathscr{B}}; \mathscr{F})$ holds with all conditions of a linear space and so, it is a linear space.

**3.2.Banach Flow Space.** Generally, there are two conditions for a linear space $(\mathscr{G}_{\mathscr{B}}; \mathscr{F})$ to be a Banach space. Dr.Ouyang tells Huiz that one is to introduce a norm $\|\cdot\|$ on $(\mathscr{G}_{\mathscr{B}}; \mathscr{F})$ and another is its complete on this norm, namely each Cauchy point sequence in $(\mathscr{G}_{\mathscr{B}}; \mathscr{F})$ on the norm is convergent.

Now let $G^L \in \mathscr{G}_{\mathscr{B}}$ be a continuity flow with continuous linear end-operators on $\mathscr{B}$. Define

$$\left\|G^L\right\| = \sum_{(v,u)\in E(G)} \left\|L^{A_{vu}^+}(v,u)\right\|, \tag{10.19}$$

where $\|\cdot\|$ is the norm on Banach space $\mathscr{B}$. Then, for three continuous flows $G^L, G_k^{L_k}, G_l^{L_l} \in \mathscr{G}_{\mathscr{B}}$ there are:

(1) Non-negative, namely $\left\|G^L\right\| \geq 0$ and $\left\|G^L\right\| = 0$ if and only if $G^L = \mathbf{O}$ by definition.

(2) Homogeneity, namely $\left\|G^{\xi L}\right\| = |\xi| \left\|G^L\right\|$ for any scalar $\xi \in \mathscr{F}$.

(3) Triangle inequality, namely $\left\|G_k^{L_k} + G_l^{L_l}\right\| \leq \left\|G_k^{L_k}\right\| + \left\|G_l^{L_l}\right\|$ because $\|\mathbf{v}_1 + \mathbf{v}_2\| \leq \|\mathbf{v}_1\| + \|\mathbf{v}_2\|$ for two vectors $\mathbf{v}_1, \mathbf{v}_2 \in \mathscr{B}$ and by the definition of norm of continuity flow, we know that

$$\begin{aligned}
\left\|G_k^{L_k} + G_l^{L_l}\right\| &= \sum_{(v,u)\in E(G_k\setminus G_l)} \left\|L_k{}^{A_{vu}^+}(v,u)\right\| + \sum_{(v,u)\in E(G_l\setminus G_k)} \left\|L_l{}^{A_{vu}^+}(v,u)\right\| \\
&\quad + \sum_{(v,u)\in E(G_k\cap G_l)} \left\|L_k{}^{A_{vu}^+}(v,u) + L_l{}^{A_{vu}^+}(v,u)\right\| \\
&\leq \sum_{(v,u)\in E(G_k\setminus G_l)} \left\|L_k{}^{A_{vu}^+}(v,u)\right\| + \sum_{(v,u)\in E(G_k\cap G_l)} \left\|L_l{}^{A_{vu}^+}(v,u)\right\| \\
&\quad + \sum_{(v,u)\in E(G_l\setminus G_k)} \left\|L_k{}^{A_{vu}^+}(v,u)\right\| + \sum_{(v,u)\in E(G_k\cap G_l)} \left\|L_l{}^{A_{vu}^+}(v,u)\right\| \\
&= \left\|G_k^{L_k}\right\| + \left\|G_l^{L_l}\right\|,
\end{aligned}$$

i.e., holding with the triangle inequality $\left\|G_k^{L_k} + G_l^{L_l}\right\| \leq \left\|G_k^{L_k}\right\| + \left\|G_l^{L_l}\right\|$.

Now that the mapping $\|\cdot\| : \mathscr{G}_{\mathscr{B}} \to \mathbb{R}$ satisfies norm conditions $(N_1) - (N_3)$, i.e., it is a norm on the continuity flow space $\mathscr{G}_{\mathscr{B}}$. Then, *what is a Cauchy sequence on the* *continuous flow space* $\mathscr{G}_{\mathscr{B}}$? Dr.Ouyang explains that a continuity flow sequence $\left\{G_n^{L_n}\right\}_1^{\infty}$ on $\mathscr{G}_{\mathscr{B}}$ is a *Cauchy sequence* if for any number $\varepsilon > 0$ there is always an integer $N(\varepsilon)$ such that $\left\|G_n^{L_k} - G_l^{L_l}\right\| < \varepsilon$ when $k, l \geq N(\varepsilon)$.

Now, let $\left\{G_n^{L_n}\right\}_1^{\infty}$ be a Cauchy sequence in $\mathscr{G}_{\mathscr{B}}$ and let $\coprod = \bigcup\limits_{G\in\mathscr{G}} G$ be the union of graphs in $\mathscr{G}$. Notice that the graph family $\mathscr{G}$ is closed for the union operation. There must be $\coprod \in \mathscr{G}$. We define a labeling mapping

$$\widehat{L}_n(x) = \begin{cases} L_n(x), & x \in V\left(G_n\right) \cup E\left(G_n\right), \\ \mathbf{0}, & x \in V\left(\coprod \setminus G_n\right) \cup E\left(\coprod \setminus G_n\right) \end{cases}$$

with the same end-operators as the continuity flow $G_n^{L_n}$ but others $\mathbf{0}$ on graph $\coprod$. Then, by convention there is $G_n^{L_n} = \coprod^{\widehat{L}_n} \in \mathscr{G}_{\mathscr{B}}$. In this way, it is obvious that

$$\begin{aligned} \left\|G^{L_k} - G^{L_l}\right\| &= \left\|\coprod^{\widehat{L}_k} - \coprod^{\widehat{L}_l}\right\| = \left\|\coprod^{\widehat{L}_k - \widehat{L}_l}\right\| \\ &= \sum_{(v,u)\in E(\coprod)} \left(\left\|\widehat{L}_k^{A_{vu}^+}(v,u) - \widehat{L}_l^{A_{vu}^+}(v,u)\right\|\right). \end{aligned}$$

It should be noted that for any number $\varepsilon > 0$, if $\left\|G^{L_k} - G^{L_l}\right\| \leq \varepsilon$ for any integers $k, l \geq N(\varepsilon)$, then there must be $\left\|\widehat{L}_k^{A_{vu}^+}(v,u) - \widehat{L}_l^{A_{vu}^+}(v,u)\right\| \leq \varepsilon$, namely the sequence $\left\{\widehat{L}_n^{A_{vu}^+}(v,u)\right\}_1^{\infty}$ is a convergent Cauchy sequence in Banach space $\mathscr{B}$. Without loss of generality, let $\lim\limits_{n\to\infty} \widehat{L}_n^{A_{vu}^+}(v,u) = L_0^{A_{vu}^+}(v,u)$ for an edge $(v,u) \in E(\coprod)$. Since the operator $A_{vu}^+$ is a linear continuous operator, there is $\lim\limits_{n\to\infty} \widehat{L}_n(v,u) = L_0(v,u)$. Thus, $\lim\limits_{n\to\infty} G_n^{L_n} = \lim\limits_{n\to\infty} \coprod^{\widehat{L}_n} = \coprod^{\lim\limits_{n\to\infty}\widehat{L}_n} = \coprod^{L_0}$, namely the Cauchy sequence $\left\{G_n^{L_n}\right\}_1^{\infty}$ is convergent in $\mathscr{G}_{\mathscr{B}}$. Therefore, *the continuity flow space $\mathscr{G}_{\mathscr{B}}$ is a Banach space* by definition.

Furthermore, if $\mathscr{B}$ is a Hilbert space with inner product $\langle\cdot,\cdot\rangle$, we can define the inner product by

$$\left\langle G^L, G'^{L'}\right\rangle = \sum_{(v,u)\in E(G\cap G')} \left\langle L^{A_{vu}^+}(v,u), L'^{A_{vu}^+}(v,u)\right\rangle \tag{10.20}$$

for two continuity flows $G^L, G'^{L'} \in \mathscr{G}_{\mathscr{B}}$ and know that

(1) Non-negative, namely $\left\langle G^L, G^L\right\rangle \geq 0$ and $\left\langle G^L, G^L\right\rangle = 0$ if and only if the continuity flow $G^L = \mathbf{O}$ for any continuity flow $G^L \in \mathscr{G}_{\mathscr{B}}$.

(2) Conjugacy, namely

$$\left\langle G_k^{L_k}, G_l^{L_l}\right\rangle = \overline{\left\langle G_l^{L_l}, G_k^{L_k}\right\rangle}$$

for two continuity flows $G_k^{L_k}, G_l^{L_l} \in \mathscr{G}_{\mathscr{B}}$.

(3) Linearity, namely for continuity flows $G^L, G_k^{L_k}, G_l^{L_l} \in \mathscr{G}_{\mathscr{B}}$ and complex numbers $\alpha, \beta \in \mathbb{C}$, there is

$$\left\langle \alpha G_k^{L_k} + \beta G_l^{L_l}, G^L \right\rangle = \alpha \left\langle G_k^{L_k}, G^L \right\rangle + \beta \left\langle G_l^{L_l}, G^L \right\rangle.$$

Thus, we get a conclusion that *if $\mathscr{B}$ is a Hilbert space, the continuity flow space $\mathscr{G}_{\mathscr{B}}$ holds with all conditions $(H_1) - (H_3)$ of Hilbert space, i.e., it is a Hilbert space* by definition.

Notice that two vectors $\mathbf{v}_1, \mathbf{v_2}$ are *orthogonal* in an inner product space $V$ if $\langle \mathbf{v}_1, \mathbf{v_2} \rangle = 0$. Geometrically, two vectors $\mathbf{a}, \mathbf{b}$ are orthogonal, namely $\mathbf{a} \perp \mathbf{b}$ if and only if the angle between vectors $\mathbf{a}, \mathbf{b}$ is $90^o$ in a Euclidean space $\mathbb{R}^3$. Generally, let $G^L, G'^{L'}$ be two continuity flows. So, *are they orthogonal in what case?* Dr.Ouyang explains that by (10.20) two continuity flows $G^L$ and $G^{L'}$ are orthogonal if their inner product is 0, i.e.,

$$\left\langle G^L, G'^{L'} \right\rangle = \sum_{(v,u) \in E(G \cap G')} \left\langle L^{A_{vu}^+}(v,u), L'^{A_{vu}^+}(v,u) \right\rangle = 0,$$

namely the sum of the inner product of vectors on the same edge under the action of end-operators in intersection of graphs $G$ and $G'$ is 0.

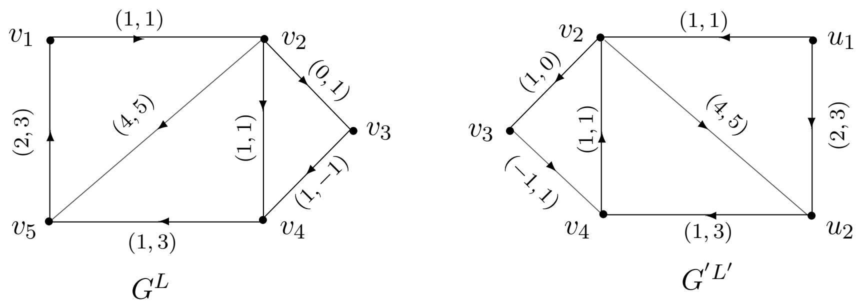



**Figure 10.14 Orthogonal continuity flows**

For example, two continuity flows $G^L$, $G^{L'}$ with all end-operators $1_{\mathscr{B}}$ and the vertex, edge sets of intersection $G \bigcap G'$ are respectively $V(G \bigcap G') = \{v_2, v_3, v_4\}$, $E(G' \bigcap G) = \{v_2v_3v_3v_4, v_4v_2\}$, i.e., a complete graph $K_3$ as shown in Figure 10.14. Then, the inner product of $G^L$ and $G'^{L'}$ is

$$\left\langle G^L, G'^{L'} \right\rangle = \langle (0,1), (1,0) \rangle + \langle (1,-1), (-1,1) \rangle + \langle (1,1), (1,1) \rangle = 0 - 2 + 2 = 0,$$

namely $G^L$ is orthogonal with $G'^{L'}$. Particularly, if $G \bigcap G' = \emptyset$, then for any labeling $L : V(G) \bigcup E(G) \to \mathscr{B}$ and $L' : V(G') \bigcup E(G') \to \mathscr{B}$ there must be $\left\langle G^L, G'^{L'} \right\rangle = 0$, which is similar to the case that

$$\langle (\alpha_1, \alpha_2, \cdots, \alpha_k, \underbrace{0, \cdots, 0}_{n-k}), (\underbrace{0, \cdots, 0}_{k}, \beta_{k+1}, \beta_{k+2}, \cdots, \beta_n) \rangle = 0$$

in Euclidean space, where $\alpha_i, \beta_j \in \mathbb{R}$ for integers $1 \leq i \leq k, k+1 \leq j \leq n$.

**3.3.$G$-Isomorphic Operator.** Generally, an operator acting on a Banach space $\mathscr{B}$ maps one vector to another. Dr.Ouyang tells Huizi that since the continuous flow space $\mathscr{G}_{\mathscr{B}}$ is a Banach space, a natural idea is to analyze those operators on $\mathscr{G}_{\mathscr{B}}$ that map one continuity flow $G^L$ to another continuity flow $G'^{L'}$ preserving the adjacency of vertices in the two continuity flows and then, extend the operator theory of Banach space.

Now, for two continuity flows $G_1^{L_1}, G_2^{L_2} \in \mathscr{G}_{\mathscr{B}}$, an operator $f : G_1^{L_1} \to G_2^{L_2}$ if satisfies conditions following:

$(C1)$ *$G_1, G_2$ are isomorphic in graphs, namely there is an isomorphism $\varphi : G_1 \to G_2$ of graph;*

$(C2)$ *$L_2 = f \circ \varphi \circ L_1$ for any edge $(v, u) \in E(G_1)$,*

then $f$ is said to be a *$G$-isomorphic operator* between continuity flows $G_1^{L_1}$ and $G_2^{L_2}$, and meanwhile, the continuity flows $G_1^{L_1}$ and $G_2^{L_2}$ are said to be *$G$-isomorphic*. Here, the operation "$\circ$" denotes the composition of mappings and a $G$-isomorphic operator $f$ is simply denoted by $f : G^{L_1} \to G^{L_2}$. Particularly, let $\varphi = \mathrm{id}_G$, i.e., the unit automorphism of graph $G$. Then, a $G$-isomorphic operator $f$ is determined by equation

$$L_2(v, u) \ = \ f \circ L_1(v, u), \quad \forall (v, u) \in E(G). \tag{10.21}$$

In other words, a $G$-isomorphic operator is such a kind of mapping on Banach space $\mathscr{B}$ that preserves the structure of graph $G$. Furthermore, if $\mathscr{B}$ is a function field on variable $\mathbf{x}$, then a $G$-isomorphic operator $f$ satisfies the equation

$$f\left(G^L[\mathbf{x}]\right) \ = \ G^{f(L[\mathbf{x}])}. \tag{10.22}$$

For example, let $\mathscr{B}$ be the real number field $\mathbb{R}$. Then, the power mapping, logarithm mapping, exponential mapping and the trigonometric mapping in network calculus are all true with (10.22). Particularly, we have the exponential identity

$$e^{G^L[\mathbf{x}]} = \mathbf{I} + \frac{G^L[\mathbf{x}]}{1!} + \frac{G^{2L}[\mathbf{x}]}{2!} + \cdots + \frac{G^{nL}[\mathbf{x}]}{n!} + \cdots . \tag{10.23}$$

Notice that that (10.23) is formally identical to (9.39) but they are essentially different because $G^L[\mathbf{x}]$ is a continuity flow and the multiplication is the Hadamard product here, which is a generalization of the network flow identity (9.39).

Dr.Ouyang tells Huizi that a $G$-isomorphism on a continuous flow $G^L$ is an automorphism of graph $G$ by definition, which is not the case in characterizing the evolving state of continuity flow but $f : G_1^{L_1} \to G_2^{L_2}$ for two non-isomorphic graphs $G_1, G_2$ in general. In this case, we need to extend the $G$-isomorphic operators. Certainly, there is a fundamental question should be answered, i.e., *the graph $G$ of a $G$-isomorphic operator $f : G_1^{L_1} \to G_2^{L_2}$*

*must be the graph $G_1$ in continuity flow $G_1^{L_1}$?* The answer is not necessarily! Dr.Ouyang explains that by the convention of continuous flow, for any graph $\widehat{G} \supset G$, if we define

$$\widehat{L}(x) = \begin{cases} L(x), & x \in V(G) \cup E(G), \\ \mathbf{0}, & x \notin V(G) \cup E(G \end{cases}$$

then $\widehat{G}^{\widehat{L}} = G^L$. Thus, we can discuss the $G$-isomorphic operator on a supergraph of $G_1, G_2$ and then, the $G$-isomorphic operator between the continuity flows $G_1^{L_1}, G_2^{L_2}$ on any two graphs $G_1, G_2$ can be characterized.

Generally, if an operator $f : {G_1}^{L_1} \to {G_2}^{L_2}$ satisfies conditions following:

$(C1')$ *There is an isomorphism $\varphi : G \to G$ with $G \supset G_1, G_2$;*

$(C2')$ *For any edge $(v, u) \in E(G_1)$ there is $L_2 = f \circ \varphi \circ L_1$ but for edge $(v, u) \in E(G_2 \setminus G_1)$, $f : \mathbf{0} \to L_2(v, u)$ and for edge $(v, u) \in E(G_1 \backslash G_2)$, $f : L(v, u) \to \mathbf{0}$, then $f$ is said to be an extended $G$-isomorphic operator.*

Now, let $G_1^{L_1}, G_2^{L_2} \in \mathscr{G}_{\mathscr{B}}$ be two continuity flows. Then, *is there extended $G$-isomorphic operator $f : G_1^{L_1} \to G_2^{L_2}$ on Banach flow space $\mathscr{G}_{\mathscr{B}}$?* The answer is affirmative. Dr.Ouyang explains that we can choose $\widehat{G} = G_1 \bigcup G_2$, define the labeling $\widehat{L}_1, \widehat{L}_2$ on $\widehat{G}$ by

$$\widehat{L}_1 = \begin{cases} L_1(v, u), & (v, u) \in E(G_1), \\ \mathbf{0}, & (v, u) \in E(\widehat{G} \backslash G_1); \end{cases} \quad \widehat{L}_2 = \begin{cases} L_2(v, u), & (v, u) \in E(G_2), \\ \mathbf{0}, & (v, u) \in E(\widehat{G} \backslash G_2) \end{cases}$$

and let the end-operators on $\widehat{G}$ the same as the end-operators on $G_1$ if the labeling mapping is $\widehat{L}_1$ or the same as the end-operators on $G_2$ if the labeling mapping is $\widehat{L}_2$. Then, $\widehat{G}^{\widehat{L}_1} = G_1^{L_1}$ and $\widehat{G}^{\widehat{L}_2} = G_2^{L_2}$. In this case, the operator $f : G_1^{L_1} \to G_2^{L_2}$ is equivalent to the operator $f : \widehat{G}^{\widehat{L}_1} \to \widehat{G}^{\widehat{L}_2}$ on graph $\widehat{G}$. Now, assume that $\varphi$ is an automorphism of graph $\widehat{G}$. Then, $f$ is a $G$-isomorphism on $\widehat{G}$. Accordingly, $f : G_1^{L_1} \to G_2^{L_2}$ is an extended $G$-isomorphism. And so, similar to the network flow calculus, we can define the limitation of continuity flow sequence, continuous operators and other conceptions and then establish the calculus theory on continuity flows. For example, *a $G$-isomorphic operator $f : \mathscr{G}_{\mathscr{B}} \to \mathscr{G}_{\mathscr{B}}$ is differentiable to $L$ if and only if $f(L)$ is differentiable to $L$.* Then, there are the relations

$$\begin{aligned} \int \left( \frac{d}{dt} f\left(G^L\right)[x] \right) dx &= f\left(G^L[x]\right) + C, \\ \frac{df}{dt} \left( \int \left( f\left(G^L[x]\right) \right) dx \right) &= f\left(G^L[x]\right) \end{aligned} \tag{10.24}$$

of differential with integral and the fundamental theorem of calculus

$$\int_a^b f\left(\overrightarrow{G}^L[t]\right) dt = F\left(\overrightarrow{G}^L[t]\right)\Big|_{t=b} - F\left(\overrightarrow{G}^L[t]\right)\Big|_{t=a} \tag{10.25}$$

on continuity flows. Here, we show how to generalize the linear operators and functionals of Banach space to the continuity flow space $\mathscr{G}_{\mathscr{B}}$.

Assume that a mapping $f:\mathscr{G}_{\mathscr{B}}\to\mathscr{G}_{\mathscr{B}'}$ is a $G$-isomorphic operator on Banach flow spaces over field $\mathscr{F}$. For two continuity flows $G_1^{L_1}$, $G_2^{L_2}\in\mathscr{G}_{\mathscr{B}}$ and scalars $\lambda,\mu\in\mathscr{F}$ if

$$f\left(\lambda G_1^{L_1}+\mu G_2^{L_2}\right)=\lambda f\left(G_1^{L_1}\right)+\mu f\left(G_2^{L_2}\right), \tag{10.26}$$

then $f$ is called a *linear operator*; if for any number $\varepsilon>0$ there always exists a real number $\delta(\varepsilon)$ such that

$$\left\|{G_1}^{L_1}-G_0^{L_0}\right\|<\delta(\varepsilon)\ \Rightarrow\ \left\|f\left(G^L\right)-f\left(G_0^{L_0}\right)\right\|<\varepsilon, \tag{10.27}$$

then $f$ is said to be *continuous* at continuity flow $G_0^{L_0}$ and if there exists a constant $\xi\in[0,\infty)$ such that $\|f(G^L)\|\leq\xi\|G^L\|$ for any continuity flow $G^L\in\mathscr{G}_{\mathscr{B}}$, then $f$ is said to be *bounded*. Furthermore, if

$$\left\|f\left(G_1^{L_1}\right)-f\left(G_2^{L_2}\right)\right\|\leq\xi\left\|G_1^{L_1}-G_2^{L_2}\right\|,\quad \xi\in[0,1) \tag{10.28}$$

for two continuity flows $G_1^{L_1},G_2^{L_2}\in\mathscr{G}_{\mathscr{B}}$, then $f$ is said to be a *contraction* on continuity flow space $\mathscr{G}_{\mathscr{B}}$. In this way, Dr.Ouyang tells Huizi that a few important conclusions on linear operators of Banach spaces can be extended to Banach flow space $\mathscr{G}_{\mathscr{B}}$ by $G$-isomorphic operators. For example, the fixed point theorem on a Banach space can be extended to Banach flow spaces $\mathscr{G}_{\mathscr{B}}$ on $G$-isomorphic contractors, namely *for any continuous $G$-isomorphic contractor $f:\mathscr{G}_{\mathscr{B}}\to\mathscr{G}_{\mathscr{B}'}$ there is only one continuity flow $G^L\in\mathscr{G}_{\mathscr{B}}$ such that $f\left(G^L\right)=G^L$.*

So, *how can we determine the continuous of a $G$-isomorphic linear operator?* It is similar to the continuous of linear operators on a Banach space. Dr.Ouyang explains that *a $G$-isomorphic linear operator $f:\mathscr{G}_{\mathscr{B}}\to\mathscr{G}_{\mathscr{B}}$ is continuous if and only if it is bounded.* Furthermore, *if $f$ is $1-1$ then the inverse operator $f^{-1}$ of $f$ is also a $G$-isomorphic continuous operator.* Meanwhile, the image Grap$f$ of a $G$-isomorphic operator $f:\mathscr{G}_{\mathscr{B}}\to\mathscr{G}_{\mathscr{B}}$ is defined by

$$\mathrm{Grap}f\ =\ \left\{\left(G^L,f\left(G^L\right)\right)\middle|\,G^L\in\mathscr{G}_{\mathscr{B}}\right\}. \tag{10.29}$$

Then, $f$ is said to be a *closed operator* if the image Grap$f$ of $f$ is closed. And so, analogous to closed operators on Banach spaces, there is a conclusion that *if $f:\mathscr{G}_{\mathscr{B}}\to\mathscr{G}_{\mathscr{B}}$ is a closed $G$-isomorphic linear operator, then $f$ is continuous.*

Usually, a functional can be regarded as a function on functions, which is a quantitative analysis method in the recognition of objective things. So, *how do we extend linear* *functionals on Banach space to Banach flow space $\mathscr{G}_{\mathscr{B}}$?* Dr.Ouyang tells Huizi that if a $G$-isomorphic linear operator $f : \mathscr{G}_{\mathscr{B}} \to \mathbb{R}$ or $\mathbb{C}$, then $f$ is said to be a *flow functional.* A fundamental question is *whether such a flow functional exists on the Banach flow space $\mathscr{G}_{\mathscr{B}}$?* The answer is Yes! Dr.Ouyang explains that it is possible to extend the functional extension theorem on Banach space, namely the Hahn-Banach theorem to Banach flow space $\mathscr{G}_{\mathscr{B}}$ to get the existence of flow functional. In this case, assume that $\mathscr{H}_{\mathscr{B}}$ is a subspace of Banach flow space $\mathscr{G}_{\mathscr{B}}$, $F : \mathscr{H}_{\mathscr{B}} \to \mathbb{C}$ is a continuous linear flow functional on $\mathscr{H}_{\mathscr{B}}$. Then, *there is a continuous linear flow functional $\widetilde{F} : \mathscr{G}_{\mathscr{B}} \to \mathbb{C}$ satisfies the conditions that if $G^L \in \mathscr{H}_{\mathscr{B}}$ then $\widetilde{F}\left(G^L\right) = F\left(G^L\right)$ and $\|\widetilde{F}\| = \|F\|$.* Particularly, *if $\mathbf{O} \neq G_0^{L_0} \in \mathscr{G}_{\mathscr{B}}$, there is a continuous linear flow functional $F$ such that $\|F\| = 1$ and $\|F(G_0^{L_0})\| = \|G_0^{L_0}\|$.* Applying this extension theorem on Banach flow space, we conclude that *for a continuity flow $G^L \in \mathscr{G}_{\mathscr{B}}$ if $F\left(G^L\right) = 0$ holds with all linear functionals $F$, there must be $\overrightarrow{G}^L = \mathbf{O}$.*

Dr.Ouyang tells Huizi that if the Banach space $\mathscr{B}$ is a complete inner product space, namely the Hilbert space $\mathcal{H}$, the Banach flow space $\mathscr{G}_{\mathcal{H}}$ is also a Hilbert space, called the *Hilbert flow space.* Notice that the Fréchet and Riesz representation theorem on Hilbert space solves the existence problem of Hermitian operator on measuring the eigenvalue of quantum state. Similarly, if the quantum state is characterized by a continuous flow $G^L$, *does this theorem hold also on the flow space $\mathscr{G}_{\mathscr{B}}$?* The answer is Yes! Dr.Ouyang explains that in this case, *let $f : \mathscr{G}_{\mathscr{B}} \to \mathbb{C}$ be a continuous linear flow functional. Then, for any continuous flow $G^L \in \mathscr{G}_{\mathscr{B}}$, there is uniquely a continuous flow of $\widehat{G}^{\widehat{L}} \in \mathscr{G}_{\mathscr{B}}$ such that $f\left(G^L\right) = \left\langle G^L, \widehat{G}^{\widehat{L}} \right\rangle$.* Thus, even if the quantum has an intrinsic combinatorial structure $G$, the Hypothesises $1-3$ for quantum system is still valid.

## §4. Continuity Flow Stability

There are two methods for characterizing the state or behavior of a thing $T$ in the system recognition. One is the method by continuity flow, namely labeling the vertices and edges by vectors on the topological structure $G$ inherited in thing $T$, holding with the continuity equation under the action of end-operator at any vertex, i.e., the continuity flow $G^L$ is constructed as a mathematical element for characterizing thing $T$ on Banach space. Another is the method by operator equation, namely the continuity equation (10.11) on the topological structure $G^L$ implied by thing $T$, i.e., applying the continuity operator equation

$$\sum_{u \in N_G^-(v)} L^{A_{uv}^+}(u,v) - \sum_{u \in N_G^+(v)} L^{A_{uv}^+}(u,v) \ = \ L(v), \quad v \in V(G)$$

characterizes thing $T$. So, *are they the same recognition on the state and behavior of thing $T$?* The answer is Yes in logic! Dr.Ouyang explains that the method of continuity flow is a systematic recognition of $T$, which simulates the state or behavior of $T$ on labeling vectors on the vertices and edges of the combination structure $G$ inherited in thing $T$. Accordingly, the operator equation is the reflection of interaction conservation on each element or vertex in thing $T$. They characterize the state or behavior of the same thing $T$. They are only different in expressing forms, with no substantial difference in logic. So, *can we know the state or behavior of a thing by an operator equation?* The answer is Yes as long as this operator equation corresponds to the continuity equation at vertices on continuity flow $G^L$ of a thing. Dr.Ouyang tells Huizi that although the two methods are logically consistent and both characterize the state of thing $T$, there are still some differences. For example, the continuity flow method is based on the intrinsic structure of thing $T$, which is intuitively more obvious and the reason that we construct the Banach flow space on continuous flow, i.e., view it as a mathematical element. Certainly, the interaction between elements in the quantitative characterizing of operator equation is an intuitive symbolic abstraction, which is related to the accuracy of variable selection and quantitative characterizing, which is a common technique in the recognition of things. However, if its corresponding with the continuity equation at vertices on continuity flow $G^L$ is removed, the operator equation can not accurately simulate the state or behavior of a thing. Particularly, whether the connotation and denotation of a variable are the same in continuity equations at different vertices as the equations (7.7) in lucky number game, whether the operator equation obtained from the continuity equation at vertices is solvable and what does it imply if it is non-solvable, etc. All of these questions determine the validity of method selected. Among these, the key is the consistency of characterizing in element state or behavior of a thing by the selected variables which is the problem should be solved in the introducing variables on observing a thing because we can accurately hold on the reality of thing only if we can correctly distinguish different influencing factors in the evolving of thing and then, correctly introduce variables to simulate the state of thing. And meanwhile, this is one of the most difficult things to recognize. For example, the different

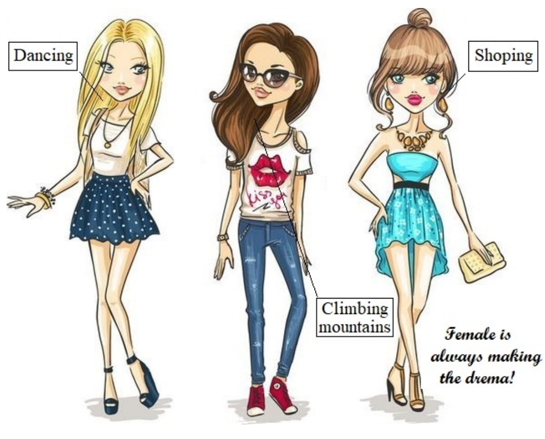



**Figure** 10.15. **Three girls in a play**

travel demands of three girls in Figure 10.15 and the reason that Bohm explained the quantum entanglement as an "*action at a distance*" with hidden variables, etc. All of these indicate the uncertainty of one's holding on the behavior of organisms or biota, especially in simulation of characterizing quantum entanglement by variables, which can only be referred to a word "*hidden*" to some extent. This implies the limitation of human recognition of things and their behavior can be not simulated or hold with a unified rule.

Theoretically, if the variables accurately correspond to elements with interactions of thing $T$, the state or behavior of $T$ can be characterized no matter $G^L$ is used by the method of continuity flow or operator equation. Now from the philosophical meaning of words that "*the heaven and earth are not kind, taking all things as humble dogs*" in *Tao Te Ching*, the solution of operator equation is to restore the true nature of a thing and there must be a solution $G^L$ of continuity flow that corresponds to the state with interactions in elements of thing $T$. Then, *why do the contradictions appear in some operator equations that simulate the state or behavior of a thing, i.e., they are non-solvable?* Dr.Ouyang tells Huizi that this situation happens in the inappropriate abstraction of elements or interactions in elements, namely two elements or interactions with different connotations are simulated by the same variable. In this case, it is necessary to introduce new variables for simulating elements, interaction states or behaviors of thing.

Huizi asks Dr.Ouyang: "*Dad! Since both of the two methods are systematic ways to recognize a thing, which method is better for holding on the truth of thing?*" Dr.Ouyang tells her that there is no a rule on which method is better to know the reality of a thing. Generally, the continuity flow is convenient for the systematic recognition on the state or behavior of a thing and the operator equation simulates the state of a thing with the solution of equation, which is beneficial to those situations that the exact solution can be obtained. However, both of them for holding on the reality of a thing have difficulties or defects. In this process, the continuity flow needs to accurately characterize the interactions in elements with interactions. For those things that are beyond human's accurate observation ability such as the distant galaxies that humans can not observe in the macroscopic world or the particles that form a binary system interacting with human observing in the microcosmic world, the reality of things can only be inferred by scientific hypothesis. Similarly, the operator equation is on the systemic recognition of a thing and it is subject also to the corresponding of variables introduced in operator equation with the element state or behavior of a thing as well as the limitations of solvability of operator equation, including the different operators such as the algebraic equation, differential equation, integral equation and the functional equation, ect. In addition, even if the solution of operator equation exists, how to solve it and whether the solution can really determine the state or behavior of thing is still unknown. This goes back to the problem implied in the Pythagoras' famous words that "*all things are numbers*", namely *can we really characterize the state or behavior of everything in the universe by numbers or an equation?* Dr.Ouyang tells Huizi: "*The idea that thinks everything in the universe can be characterized by numbers or an equation comes from an idealist belief of a few humans, i.e., the human is the master of universe and everything works according to human's rule. In fact, such a kind of view was not acceptable even in the ancient China because the ancient Chinese recognized the union of the heaven and humans.*" Then, *how should we hold on the truth of things?* Dr.Ouyang explains that since the human recognition of things is local, the reality of a thing should be colorful in the eyes of human, namely a variety of appearances or combinations of reality presented in front of the human. In this case, the stability or synchronizing of the solution of operator equation corresponding to a thing $T$ should be equivalent to the stability or synchronizing of the continuity flow because both of them can characterize the stability or synchronizing of thing $T$.

**4.1.Dynamic Equation of Element.** In an $n$-dimensional Euclidean space $\mathbb{R}^n$, assume that an element state with interaction of thing $T$ can be characterized by a vector $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ and the evolving of elements follow synchronously a time parameter $t$. In this case, the label mapping $L : v \to \dot{\mathbf{x}}_v \in \mathbb{R}^n$, $(v,u) \to \mathbf{x}_{vu} \in \mathbb{R}^n$ for vertex $v \in V(G)$ and edge $(v,u) \in E(G)$. Dr.Ouyang tells Huizi that by applying the end-operators $A^+_{vu}$ and $A^+_{uv}$, we can further assume that the actions of element $v$ on $u$, element $u$ on $v$ both are $\mathbf{x}$, i.e., $\mathbf{x}_{vu} = \mathbf{x}_{uv} = \mathbf{x}$. And so, the action of element $v$ on $u$ is $A^+_{vu}(\mathbf{x})$ on vertex $u$ and the action of element $u$ on $v$ is $A^+_{uv}(\mathbf{x})$ on vertex $v$, as shown in Figure 10.16.

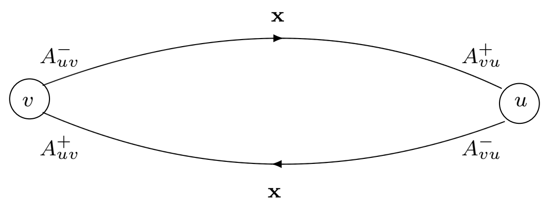



**Figure** 10.16. **Element action in Euclidean space**

In this case, the operator equation corresponding to the continuity equation at vertices on the continuity flow $G^L$ is

$$\dot{\mathbf{x}}_v = \sum_{u \in N^-_G(v)} A^+_{uv}(\mathbf{x}) - \sum_{u \in N^+_G(v)} A^+_{uv}(\mathbf{x}), \quad v \in V(G) \tag{10.30}$$

and there are totally $|G|$ operator equations. For example, there are 2 operator equations in the 2-element system composed of $v$ and $u$, as shown in Figure 10.16, i.e

$$\left\{\begin{array}{lcl} \dot{\mathbf{x}}_v &=& A^+_{uv}(\mathbf{x}) - A^-_{uv}(\mathbf{x}), \\ \dot{\mathbf{x}}_u &=& A^+_{vu}(\mathbf{x}) - A^-_{vu}(\mathbf{x}), \end{array}\right. \tag{10.31}$$

where, the variable $\mathbf{x}$ includes $\mathbf{x}$ and its derivative on time $t$, i.e., $\dot{\mathbf{x}}, \ddot{\mathbf{x}}, \cdots$, etc.

Generally, assuming the linear operator $A^+_{vu}, A^+_{uv}$ on an edge $(v,u) \in E(G)$ is continuous and $A^+_{vu}(\mathbf{0}) = A^+_{uv}(\mathbf{0}) = \mathbf{0}$, Then, the equations (10.30) and (10.31) are both autonomous equations. Thus, by expanding $A^+_{vu}(\mathbf{x})$ near the point $\mathbf{0}$ using Taylor series, there is

$$A^+_{vu}(\mathbf{x}) = \left.\frac{\partial A^+_{vu}}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{0}} \mathbf{x}^t + \alpha_{vu}(\mathbf{x}), \tag{10.32}$$

where $\alpha(\mathbf{x})$ is the remainder of $\mathbf{x}$ with order$\geq 2$ in $A^+_{vu}(\mathbf{x})$. Particularly, let the end-operator

$$A^+_{vu}(\mathbf{x}) = \left(A^{+1}_{vu}(\mathbf{x}), A^{+2}_{vu}(\mathbf{x}), \cdots, A^{+n}_{vu}(\mathbf{x})\right),$$

then

$$\left.\frac{\partial A^+_{vu}}{\partial \mathbf{x}}\right|_{\mathbf{x}=0} = \left.\begin{pmatrix} \frac{\partial A^{+1}_{vu}}{\partial x_1} & \frac{\partial A^{+1}_{vu}}{\partial x_2} & \cdots & \frac{\partial A^{+1}_{vu}}{\partial x_n} \\ \frac{\partial A^{+2}_{vu}}{\partial x_1} & \frac{\partial A^{+2}_{vu}}{\partial x_2} & \cdots & \frac{\partial A^{+2}_{vu}}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial A^{+n}_{vu}}{\partial x_1} & \frac{\partial A^{+n}_{vu}}{\partial x_2} & \cdots & \frac{\partial A^{+n}_{vu}}{\partial x_n} \end{pmatrix}\right|_{\mathbf{x}=0} = \left(a^{ij}_{vu}\right)_{n\times n} = \mathbf{A}_{vu}.$$

Thus, if let $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ and ignore the terms including $\dot{\mathbf{x}}, \ddot{\mathbf{x}}, \cdots$, there is

$$\left.\frac{\partial A^+_{vu}}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{0}} \mathbf{x}^t = \mathbf{A}_{vu}\mathbf{x}^t = \begin{pmatrix} a^{11}_{vu}x_1 + a^{12}_{vu}x_2 + \cdots + a^{1n}_{vu}x_n \\ a^{21}_{vu}x_1 + a^{22}_{vu}x_2 + \cdots + a^{2n}_{vu}x_n \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ a^{n1}_{vu}x_1 + a^{n2}_{vu}x_2 + \cdots + a^{nn}_{vu}x_n \end{pmatrix},$$

i.e.,

$$A^+_{vu}(\mathbf{x}) = \mathbf{A}_{vu}\mathbf{x}^t + \alpha_{vu}(\mathbf{x}). \tag{10.33}$$

Substituting the expansion $A^+_{vu}(\mathbf{x})$ in (10.33) into equation (10.30), then for any vertex $v \in V(G)$ there is

$$\begin{aligned} \dot{\mathbf{x}}_v &= \sum_{u\in N^-_G(v)} A^+_{uv}(\mathbf{x}) - \sum_{u\in N^+_G(v)} A^+_{uv}(\mathbf{x}) \\ &= \sum_{u\in N^-_G(v)} \mathbf{A}_{vu}\mathbf{x}^t - \sum_{u\in N^+_G(v)} \mathbf{A}_{vu}\mathbf{x}^t + \sum_{u\in N^-_G(v)} \alpha_{vu}(\mathbf{x}) - \sum_{u\in N^+_G(v)} \alpha_{vu}(\mathbf{x}). \end{aligned}$$

Denote the coefficient matrix and the remainder in previous formula respectively by

$$\begin{aligned}\mathbf{B}_{uv} &= \sum_{u\in N_G^-(v)} \mathbf{A}_{vu} - \sum_{u\in N_G^+(v)} \mathbf{A}_{vu} = \left(b_{vu}^{ij}\right)_{n\times n}, \\ \mathbf{R}(\mathbf{x}) &= \sum_{u\in N_G^-(v)} \alpha_{vu}(\mathbf{x}) - \sum_{u\in N_G^+(v)} \alpha_{vu}(\mathbf{x}) = \left(r_1(\mathbf{x}), r_2(\mathbf{x}), \cdots, r_n(\mathbf{x})\right),\end{aligned}$$

then for any element $v \in V(G)$, its dynamic equation is

$$\dot{\mathbf{x}}_v = \mathbf{B}_{vu}\mathbf{x}^t + \mathbf{R}(\mathbf{x}). \qquad (10.34)$$

Assume the state function of element $v$ to be $\mathbf{x}_v = (x_{v1}(\mathbf{x}), x_{v2}(\mathbf{x}), \cdots, x_{vn}(\mathbf{x}))$. Then, the dynamic equation (10.34) of element $v$ in coordinates is

$$\begin{cases} \dot{x}_{v1}(\mathbf{x}) = b_{vu}^{11}x_1 + b_{vu}^{12}x_2 + \cdots + b_{vu}^{1n}x_n + r_1(\mathbf{x}), \\ \dot{x}_{v2}(\mathbf{x}) = b_{vu}^{21}x_1 + b_{vu}^{22}x_2 + \cdots + b_{vu}^{2n}x_n + r_2(\mathbf{x}), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ \dot{x}_{vn}(\mathbf{x}) = b_{vu}^{n1}x_1 + b_{vu}^{n2}x_2 + \cdots + b_{vu}^{nn}x_n + r_n(\mathbf{x}). \end{cases} \qquad (10.35)$$

Particularly, if $x_{v1}(\mathbf{x}), x_{v2}(\mathbf{x}), \cdots, x_{vn}(\mathbf{x})$ are all linear functions such as $x_{v1}(\mathbf{x}) = x_1, x_{v2}(\mathbf{x}) = x_2, \cdots, x_{vn}(\mathbf{x}) = x_n$, the differential equation (10.35) then is

$$\begin{cases} \dot{x}_1 = b_{vu}^{11}x_1 + b_{vu}^{12}x_2 + \cdots + b_{vu}^{1n}x_n + r_1(\mathbf{x}), \\ \dot{x}_2 = b_{vu}^{21}x_1 + b_{vu}^{22}x_2 + \cdots + b_{vu}^{2n}x_n + r_2(\mathbf{x}), \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ \dot{x}_n = b_{vu}^{n1}x_1 + b_{vu}^{n2}x_2 + \cdots + b_{vu}^{nn}x_n + r_n(\mathbf{x}). \end{cases}$$

Furthermore, if we ignore the remainders $r_1(\mathbf{x}), r_2(\mathbf{x}), \cdots, r_n(\mathbf{x})$ of $\mathbf{x}$ with order$\geq 2$ in (10.35), then we obtain a constant coefficient linear differential equation

$$\begin{cases} \dot{x}_1 = b_{vu}^{11}x_1 + b_{vu}^{12}x_2 + \cdots + b_{vu}^{1n}x_n, \\ \dot{x}_2 = b_{vu}^{21}x_1 + b_{vu}^{22}x_2 + \cdots + b_{vu}^{2n}x_n, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots, \\ \dot{x}_n = b_{vu}^{n1}x_1 + b_{vu}^{n2}x_2 + \cdots + b_{vu}^{nn}x_n. \end{cases} \qquad (10.36)$$

by the coordinate equation (10.35).

Generally, if variables $x_{v1}(\mathbf{x}), x_{v2}(\mathbf{x}), \cdots, x_{vn}\mathbf{x}$ are nonlinear functions, we can expand them by using Taylor expansion, only take the linear part as an approximation and get a constant coefficient linear system of (10.36). In this case, the stability of solution of equation (10.36) are determined by the roots of characteristic equation of

$|\mathbf{B}_{vu} - \lambda I_n| = 0$ at the equilibrium point $\mathbf{0}$. For example, if all the real parts of roots of equation $|\mathbf{B}_{vu} - \lambda I_n| = 0$ are negative, then the solution is asymptotically stable at the equilibrium point $\mathbf{0}$ and if there exists a positive real part in roots of equation $|\mathbf{B}_{vu} - \lambda I_n| = 0$, then the solution is not stable at the equilibrium $\mathbf{0}$, etc. Certainly, there is a question should be answered in this way, i.e., *under what conditions that the stability of nonlinear system* (10.34) *is consistent with that of linear system* (10.36)*?* This question is partially answered. Dr.Ouyang explains that there is a conclusion in system stable theory, i.e., if all $A^+_{vu}(\mathbf{x})$ are hyperbolic, namely none of the real part of roots of equation $|\mathbf{B}_{vu} - \lambda I_n| = 0$ is 0, then the stability of nonlinear system (10.34) is consistent with that of the linear system (10.36) at equilibrium point $\mathbf{0}$, i.e., we can obtain the stability of nonlinear system (10.34) by linear system (10.36) in this case.

**4.2.Flow Stable State.** When the state or behavior of thing $T$ is characterized by an operator equation, the stability of $G^L$ determined by the solution of equation (10.34) greatly depends on the observing data with assuming on the state of elements $v$. And meanwhile, we can not determine the stability for all states of an element $v$, especially when the interaction in elements is nonlinear. Then, *is there a way to determine whether the state of thing $T$ is stable or not directly by continuity flow $G^L$ which does not depend on the solution of equation* (10.34)*?* The answer is Yes! Dr.Ouyang explains that this is the significance of constructing a Banach flow space by viewing a continuity flow $G^L$ as a mathematical element.

Then, *what is the meaning that a continuity flow $G^L[t]$ of thing $T$ is stable or asymptotically stable?* Dr.Ouyang explains that let $T$ be a thing with continuity flow $G^L[t]$ evolving on time $t$, $G^{L_0}$ an initial value of $G^L[t]$ and let $L : v \to \dot{\mathbf{x}}_v(t)$ be a labeling mapping. For a given positive number $\varepsilon > 0$ and an initial flow $G'^{L'_0}$ of thing $T$, if there always exists a real number $\delta > 0$ such that

$$\left\|G^L[t] - G'^{L'}[t]\right\| < \varepsilon \quad \text{or} \quad \lim_{t\to 0}\left\|G^L[t] - G'^{L'}[t]\right\| = 0, \tag{10.37}$$

if the initial flows $\left\|G^{L_0} - G^{L'_0}\right\| < \delta$, then the continuity flow $G^L[t]$ is said to be *stable* or *asymptotically stable*.

By definition, the end-operators on continuity flow $G^L$ are all linear operators acting on Banach space $\mathscr{B}$. Dr.Ouyang tells Huizi that if all edge-operator are continuous, there are constants $c_{vu}$ or $c'_{vu}$ for any edge $(v, u) \in E(G)$ or $(v, u) \in E(G')$ such that for any vector $\mathbf{x} \in \mathscr{B}$, $\|A^+_{vu}(\mathbf{X})| \leq c^+_{vu}\|\mathbf{x}\|$ and $\left\|A'^+_{vu}(\mathbf{x})\right| \leq c'^+_{vu}\|\mathbf{x}\|$ by the equivalence of a continuous operator with that of bounded operator in Banach space. And so, by the definition of norm on continuity flows, there is

$$\begin{aligned}\left\|G^L - G'^{L'}\right\| &= \left\|(G\setminus G')^L\right\| + \left\|\left(G\bigcap G'\right)^{L-L'}\right\| + \left\|(G'\setminus G)^{-L'}\right\| \\ &= \sum_{(v,u)\in E(G\setminus G')}\left\|L^{A^+_{vu}}(v,u)\right\| + \sum_{(v,u)\in E(G\bigcap G')}\left\|\left(L^{A^+_{vu}} - L'^{A'^+_{vu}}_{vu}\right)(v,u)\right\| \\ &\quad + \sum_{(v,u)\in E(G'\setminus G)}\left\|-L'^{A'^+_{vu}}(v,u)\right\|.\end{aligned}$$

Notice that the norm $\|\cdot\| \geq 0$. By the initial condition $\left\|G^L - G'^{L'}\right\| < \varepsilon$, there are respectively $\|L^{A^+_{vu}}(v,u)\| < \varepsilon$, $\|(L-L')^{A^+_{vu}}(v,u)\| < \varepsilon$ and $\|L^{A^+_{vu}}(v,u)\| < \varepsilon$ for any edge $(v,u)$ in graphs $G\setminus G'$, $G\bigcap G'$ or $G'\setminus G$. Now, the end-operator $A^+_{vu}$ is continuous. So, there are respectively $\|L(v,u)\| < \varepsilon'$, $\|(L-L')(v,u)\| < \varepsilon'$ and $\|L(v,u)\| < \varepsilon'$. Conversely, for a given positive number $\varepsilon' > 0$ and any edge $(v,u)$ in graphs $G\setminus G'$, $G\bigcap G'$ or $G'\setminus G$, if there are respectively $\|L_1(v,u)\| < \varepsilon'$, $\|(L-L')(v,u)\| < \varepsilon'$ and $\|L(v,u)\| < \varepsilon'$, define a constant

$$c^{\max}_{GG'} = \left\{\max_{(v,u)\in E(G)} c^+_{vu}, \max_{(v,u)\in E(G')} c^+_{vu}\right\}.$$

Then, there must be

$$\begin{aligned}\left\|G^L - G'^{L'}\right\| &= \sum_{(v,u)\in E(G\setminus G)}\left\|L^{A^+_{vu}}(v,u)\right\| + \sum_{(v,u)\in E(G\bigcap G')}\left\|\left(L^{A^+_{vu}} - L^{A'^+_{vu}}_{vu}\right)(v,u)\right\| \\ &\quad + \sum_{(v,u)\in E(G'\setminus G)}\left\|-L^{A'^+_{vu}}(v,u)\right\| \\ &\leq \sum_{(v,u)\in E(G\setminus G')} c^+_{vu}\|L(v,u)\| + \sum_{(v,u)\in E(G\bigcap G')} c^+_{vu}\left\|\left(L-L'\right)(v,u)\right\| \\ &\quad + \sum_{(v,u)\in E(G'\setminus G)} c^+_{vu}\|-L(v,u)\| < \left|G\bigcup G'\right| c^{\max}_{GG'}\varepsilon'.\end{aligned}$$

Particularly, let $\varepsilon' = \varepsilon/\left(|G\bigcup G'|\, c^{\max}_{GG'}\right)$. Then, $\left\|G^L - G'^{L'}\right\| < \varepsilon$, i.e., the inequality $\left\|G^L - G'^{L'}\right\| < \varepsilon$ is equivalent to that $\|L(v,u)\| < \varepsilon$, $\|(L-L')(v,u)\| < \varepsilon$ and $\|L(v,u)\| < \varepsilon$ for any edge $(v,u)$ in graphs $G\setminus G'$, $G\bigcap G'$ or $G'\setminus G$, respectively. And so, we get a local condition on the stability of continuity flow $G^L[t]$, namely *a continuity flow $G^L[t]$ is stable or asymptotically stable at the initial flow $G^L[t_0]$ if and only if $L(v,u)[t]$ is stable or asymptotically stable at the initial value $L(v,u)[t_0]$ for any edge $(v,u)\in E(G)$.*

**4.3.Flow Synchronized State.** For a labeling $L: v \to \dot{\mathbf{x}}_v$ on vertices in continuity flow $G^L[t]$, if there is a continuity flow $G_0^{L_0}[t] \neq G^L[t]$ such that

$$\lim_{t\to\infty}\left\|G^L[t] - G_0^{L_0}[t]\right\| = 0, \tag{10.38}$$

then the continuity flow $G^L[t]$ is said to be *$G$-synchronized*. At this time, a few special cases of $G^L[t]$ synchronization can be determined by known conclusions on the synchronization of solution of operator equation (10.34). Dr.Ouyang tells Huizi that there is a more convenient criterion for the case that all end-operators of $G^L[t]$ are linear continuous.

By definition, if a continuity flow $G^L[t]$ of thing $T$ is $G$-synchronized, there must be a continuity flow ${G_0}^{L_0}$ and a positive number $N(\varepsilon)$ such that $\left\|G^L - {G_0}^{L_0}\right\| < \varepsilon$ if $t \geq N(\varepsilon)$, namely for any edge $(v,u)$ in graphs $G \setminus G_0$, $G \bigcap G_0$ or $G_0 \setminus G$, there are respectively $\|L^{A^+_{vu}}(v,u)\| < \varepsilon$, $\|(L - L_0)^{A^+_{vu}}(v,u)\| < \varepsilon$ and $\|L_0^{A^+_{vu}}(v,u)\| < \varepsilon$. Thus, for any vertex $\forall v \in V(G)$, there is

$$\|L(v)\| = \left\|\sum_{u\in N_G(v)} L^{A^+_{vu}}(v,u)\right\| \leq \sum_{u\in N_G(v)} \left\|L^{A^+_{vu}}(v,u)\right\| \leq |N_G(v)|\varepsilon,$$

namely if $t \geq N\left(\dfrac{\varepsilon}{|N_G(v)| + |N_G(u)|}\right)$, then there is

$$\begin{aligned}\|L(v) - L(u)\| &\leq \|L(v)\| + \|L(u)\| \\ &< (|N_G(v)| + |N_G(u)|) \times \frac{\varepsilon}{|N_G(v)| + |N_G(u)|} = \varepsilon\end{aligned}$$

for vertices $v,u \in V(G)$, i.e., all variables $\mathbf{x}_v$, $v \in V(G)$ in equation (10.34) are synchronized. Conversely, if all variables $\mathbf{x}_v$, $v \in V(G)$ in operator equation (10.34) are synchronized, namely for any given positive number $\varepsilon > 0$ there exists a positive number $N(\varepsilon)$ such that $\|L(v) - L(u)\| < \varepsilon$ if $t \geq N(\varepsilon)$ for vertices $v,u \in V(G)$, i.e., $\lim\limits_{t\to\infty} L(v) = \lim\limits_{t\to\infty} L(u) = \mathbf{v}_0$. Without loss of generality, let $\lim\limits_{t\to\infty} G^L[t] = G^{L_0}$. Then,

$$\begin{aligned}\left\|G^L - G^{L_0}\right\| &= \sum_{(v,u)\in E(G)} \left\|\left(L^{A^+_{vu}} - L_0^{A'^+_{vu}}\right)(v,u)\right\| \\ &= \frac{1}{2}\sum_{v\in V(G)} \|(L - L_0)(v))\| \leq \frac{|G|}{2}\varepsilon,\end{aligned}$$

i.e., the continuity flow $G^L[t]$ is $G$-synchronized. Therefore, we get a conclusion that *a continuity flow $G^L[t]$ is $G$-synchronized if and only if all variables $\mathbf{x}_v$, $v \in V(G)$ in operator equation* (10.34) *are synchronized.*

Let $\Delta = G^L[t] - G_0^{L_0}$. If a continuity flow $G^L[t]$ is $G$-synchronized to a continuity flow $G_0^{L_0}$, then $\|\Delta\| = \left\|G^L - G^{L_0}\right\| \to 0$ if $t \to \infty$. Denoted $\Delta$ by $\mathbf{O}(t^{-1}G)$, i.e., an infinitesimal on $t$ of higher order. Then, there is

$$G^L[t] = G_0^{L_0} + \mathbf{O}\left(t^{-1}G\right), \tag{10.39}$$

which is similar to the synchronized track (6.1) of system element and can be applied to construct $G$-synchronized continuity flows $G^L[t]$.

For example, let the end-operators $A^+_{v_iv_{i+1}} = 2$, $A^+_{v_iv_{i-1}} = 1$ on edges in circuit $C_n$ and for any integer $1 \leq i \leq n$, define the labeling $L$ by

$$L: \ (v_i, v_{i+1}) \to \frac{\left(2^{i-1}-1\right)F(t)}{2^{i-1}} + \frac{n!}{2^{i-1}t^2}, \quad F: t \to \mathbb{R}$$

in Figure 10.17. Denote by $G^L[t]$ such a continuity flow and correspondingly, the edge flow in $G^{L_0}$ is defined by

$$f_i \ = \ \frac{\left(2^{i-1}-1\right)F(t)}{2^{i-1}}.$$

Then, $\Delta = G^L[t] - G^{L_0} = G^{L_\Delta}$, where for any integer $1 \leq i \leq n$, $L_\Delta(v_i, v_{i+1}) = n!/2^{i-1}t^2 \to 0$ if $t \to \infty$, i.e., $\Delta = \mathbf{O}(t^{-1}C_n)$. By the formula (10.39), it is easy to verify that $G^L[t]$ is $G$-synchronized with all vertex flows $F(t)$, namely all variables at vertices in $C_n$ are synchronized. Notice that this conclusion is a little different from the conclusion obtained by the method of master stability in which the variable synchronization is determined under conditions ①- ① and so, the variables $\mathbf{x}_v$ on $C_n$ are unlikely to be synchronized in Section 3 of Chapter 6.

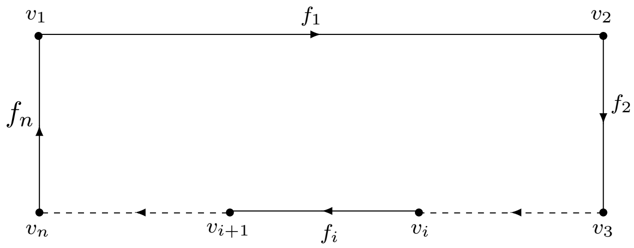



**Figure** 10.17. ***G*-synchronized on $C_n$**

§5. **System Dynamics**

The evolving of a thing $T$ can be regarded as a mechanical behavior. Certainly, we have mechanical science on this kind of mechanical behavior under conditions. For example, the particle mechanics abstracts an object $M$ to a geometrical point or particle $M$ with mass but no volume or shape, which assumes that the algebraic sum of the work of external forces and the non-conservative forces acting on the particle system is equal to the increment of mechanical energy of particle system, and then characterizes the changing laws of particle systems. Similarly, the rigid body mechanics assumes that an object $T$ is a rigid body whose size and shape remain unchanged during its moving or after force action, and then characterizes the changing laws of the object. Both of them are only ideal models under conditions for the recognition of macroscopic objects. In modern science, the field theory abstracts a thing $T$ into physical quantities such as the scalar or vector and assumes its distribution in the whole space, called the *field* so as to understand the changing behaviors and laws of things $T$. Different from the macroscopic objects, the field is an invisible or untouchable objective reality. For example, the gravitational field, electromagnetic field and the geomagnetic field as shown in Figure 10.18 are actually the external manifestation of thing action. Among them, the particle mechanics and rigid body mechanics on the certainty and knowability of things, are helpless for understanding the heterogeneous systems in the universe such as the biological groups or cell tissues. The field is a kind of systematic recognition on the interaction effect between elements of things which shows a powerful function in the recognition of those things beyond the human observing ability in the universe such as the universe noumenon, microscopic particles and other things. Similarly, the ubiquity of effect of a physical quantity in space $\mathbb{R}^3 \times \mathbb{R}$ is only an ideal assumption that does not match the idea of elementary particles as the constitute units of matter.

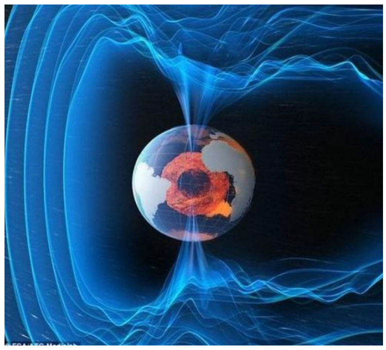


**Figure** 10.18. **Geomagnetic field**

Thus, a natural idea is to establish a *system dynamics* on the intrinsic structure of things or characterizing the dynamic behavior of continuity flow $G^L$ to simulate the state and changing of a thing, i.e., an applying of the combinatorial notion. In this regard, Dr.Ouyang explains that the state changing of a thing is an objective reality, corresponding to the changing of its elements with interaction on its intrinsic structure $G^L$. Certainly, it is a dynamic process and can be simulated by dynamics. However, the particle mechanics, rigid body mechanics or field theory in classical mechanics are all characterizing on certain kind or local characters of things but the human recognition on the state or behavior of a thing really needs such a systemic theory on continuity flows $G^L$. Meanwhile, as an element in Banach flow space $\mathscr{G}_{\mathscr{B}}$, the continuity flow $G^L$ can systematically simulate the dynamic behavior of elements with interactions, without abstracting $T$ as a particle, assuming it is a rigid body or distinguishing whether it is macroscopic or microscopic, which is essentially a systematic theory to characterize the state changing of thing $T$ by dynamic equation on topological structure $G^L$ inherited in thing $T$.

So, *how should we establish a dynamical theory on the intrinsic structure $G^L$ of thing?* Dr.Ouyang explains that on the Newton's 2nd law, there is a dynamic equation for characterizing particles moving, called the *Lagrange equation* in classical mechanics, which has been generalized to the field theory for deducing the Einstein's gravitational equation and other quantum equations. Today, it has become a general dynamic equation. Notice that a continuity flow $G^L$ characterizes the reality of the manifestation of a thing $T$ on its intrinsic structure. Theoretically, no matter the thing $T$ is regarded as a system of particles or a field, the mechanical laws followed by the continuity flow $G^L$ can be deduced by Lagrange equation and then, establish the system dynamics on $G^L$.

**5.1.Lagrange Equations.** Generally, the Lagrange equations are characterizing on the field action by Newton's 2nd law. We know that $\mathbf{F} = m\ddot{\mathbf{x}}$ is the dynamic equation of a particle $A$ with mass $m$ under the action of a conservative force field $\mathbb{R}^n$ with potential energy $\mathcal{V}(\mathbf{x})$, which can be written in coordinate form as

$$\left(-\frac{\partial \mathcal{V}}{\partial x_1}, -\frac{\partial \mathcal{V}}{\partial x_2}, \cdots, -\frac{\partial \mathcal{V}}{\partial x_n}\right) = (m\ddot{x}_1, m\ddot{x}_2, \cdots, m\ddot{x}_n), \tag{10.40}$$

where $\mathbf{x} = (x_1, x_2, \cdots, x_n)$, the $i$th coordinate of force $\mathbf{F}$ in $\mathbb{R}^n$ is $\frac{\partial \mathcal{V}}{\partial x_i}$, $1 \leq i \leq n$. Multiply both ends of (10.40) by differential vector $d\mathbf{x} = (dx_1 dx_2, \cdots, dx_n)$ to do the inner product, we know that

$$-\sum_{i=1}^{n} \frac{\partial \mathcal{V}}{\partial x_i} dx_i = \sum_{i=1}^{n} m\ddot{x}_i dx_i. \tag{10.41}$$

Now, let $\mathbf{q} = (q_1, q_2, \cdots, q_n)$ be the general coordinates of particle $A$ at time $t$, namely for any integer $1 \leq i \leq n$, $x_i = x_i(q_1, q_2, \cdots, q_n)$ and

$$dx_i = \sum_{k=1}^{n} \frac{\partial x_i}{\partial q_k} dq_k. \tag{10.42}$$

Then, there must be

$$\sum_{i=1}^{n} m\ddot{x}_i dx_i = \sum_{i=1}^{n}\sum_{k=1}^{n} m\ddot{x}_i \frac{\partial x_i}{\partial q_k} dq_k = \sum_{k=1}^{n}\sum_{i=1}^{n} m\ddot{x}_i \frac{\partial x_i}{\partial q_k} dq_k. \tag{10.43}$$

On the other hand, by (10.42) we know that

$$d\mathcal{V} = \sum_{i=1}^{n} \frac{\partial \mathcal{V}}{\partial x_i} dx_i = \sum_{i=1}^{n}\sum_{k=1}^{n} \frac{\partial \mathcal{V}}{\partial x_i}\frac{\partial x_i}{\partial q_k} dq_k = \sum_{k=1}^{n} \frac{\partial \mathcal{V}}{\partial q_k} dq_k. \tag{10.44}$$

Substitute (10.43) and (10.44) into (10.41), there is

$$\sum_{k=1}^{n}\left(\sum_{i=1}^{n} m\ddot{x}_i \frac{\partial x_i}{\partial q_k}\right) dq_k = -\sum_{i=1}^{n} \frac{\partial \mathcal{V}}{\partial q_k} dq_k. \tag{10.45}$$

Notice that $dq_k,\ k = 1, 2, \cdots, n$ are independent of each other. So, there must be

$$\sum_{i=1}^{n} m\ddot{x}_i \frac{\partial x_i}{\partial q_k} = -\frac{\partial \mathcal{V}}{\partial q_k}, \quad k = 1, 2, \cdots, n \tag{10.46}$$

in the equation (10.45). By the derivative rule on function product, we know that

$$\sum_{i=1}^{n} m\ddot{x}_i \frac{\partial x_i}{\partial q_k} = \frac{d}{dt}\left(\sum_{i=1}^{n} m\dot{x}_i \frac{\partial x_i}{\partial q_k}\right) - \sum_{i=1}^{n} m\dot{x}_i \frac{d}{dt}\frac{\partial x_i}{\partial q_k}. \tag{10.47}$$

Substituting (10.47) into (10.46), then

$$\frac{d}{dt}\left(\sum_{i=1}^{n} m\dot{x}_i \frac{\partial x_i}{\partial q_k}\right) - \sum_{i=1}^{n} m\dot{x}_i \frac{d}{dt}\frac{\partial x_i}{\partial q_k} = -\sum_{i=1}^{n} \frac{\partial \mathcal{V}}{\partial q_k} dq_k \tag{10.48}$$

for any integer $k = 1, 2, \cdots, n$. Notice that all $\partial x_i/\partial q_k$ and $\dot{q}_k$ are independent of each other. Thus,

$$\begin{aligned} \dot{x}_i &= \frac{dx_i}{dt} = \sum_{k=1}^{n} \frac{\partial x_i}{\partial q_k}\dot{q}_k, \quad \frac{\partial \dot{x}_i}{\partial \dot{q}_k} = \frac{\partial x_i}{\partial q_k}, \\ \frac{d}{dt}\frac{\partial x_i}{\partial q_k} &= \sum_{l=1}^{n} \frac{\partial^2 x_i}{\partial q_k \partial q_l}\dot{q}_l = \frac{\partial}{\partial q_l}\sum_{l=1}^{n} \frac{\partial x_i}{\partial q_l}\dot{q}_l = \frac{\partial}{\partial q_k}\dot{x}_i. \end{aligned}$$

Substituting them into (10.48), we get that

$$\frac{d}{dt}\left(\sum_{i=1}^{n} m\dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_k}\right) - \sum_{i=1}^{n} m\dot{x}_i \frac{\partial \dot{x}_i}{\partial q_k} = -\sum_{i=1}^{n} \frac{\partial \mathcal{V}}{\partial q_k} dq_k. \tag{10.49}$$

Additionally, we have known the kinetic energy $\mathcal{T}$ of particle $A$ is

$$\mathcal{T} = \frac{1}{2}m\mathbf{v}^2 = \sum_{i=1}^{n} \frac{1}{2}m\dot{x}_i^2.$$

So,

$$\frac{\partial \mathcal{T}}{\partial q_k} = \sum_{i=1}^{n} m\dot{x}_i \frac{\partial \dot{x}_i}{\partial q_k}, \quad \frac{\partial \mathcal{T}}{\partial \dot{q}_k} = \sum_{i=1}^{n} m\dot{x}_i \frac{\partial \dot{x}_i}{\partial \dot{q}_k} \circ \tag{10.50}$$

Now, comparing (10.49) with (10.50), we immediately know that

$$\frac{d}{dt}\frac{\partial \mathcal{T}}{\partial \dot{q}_k} - \frac{\partial \mathcal{T}}{\partial q_k} = -\frac{\partial \mathcal{V}}{\partial q_k}, \quad k = 1, 2, \cdots, n. \tag{10.51}$$

Notice that the particle $A$ is moving in a conservative field $\mathcal{V}(\mathbf{x})$ and the potential energy $\mathcal{V}(\mathbf{x})$ is not relevant with $\dot{q}_k$. Thus, $\partial\mathcal{V}/\partial\dot{q}_k = 0$, $k = 1, 2, \cdots, n$. In this case, if define the *Lagrangian* of particle $A$ by $\mathcal{L} = \mathcal{T} - \mathcal{V}$, then

$$\frac{\partial \mathcal{L}}{\partial q_k} = \frac{\partial \mathcal{T}}{\partial q_k} - \frac{\partial \mathcal{V}}{\partial q_k}, \quad \frac{\partial \mathcal{L}}{\partial \dot{q}_k} = \frac{\partial \mathcal{T}}{\partial \dot{q}_k} - \frac{\partial \mathcal{V}}{\partial \dot{q}_k}, \quad k = 1, 2, \cdots, n.$$

Substituting them into (10.51), we get the Lagrange equations

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_k} - \frac{\partial \mathcal{L}}{\partial q_k} = 0, \quad k = 1, 2, \cdots, n. \tag{10.52}$$

So, *how do we know the relation of Lagrangian with the field theory?* Dr.Ouyang explains that the Lagrangian $\mathcal{L}$ is a function that characterizes the state or field of particle $A$, i.e. the difference between the kinetic energy and potential energy at a position of particle $A$. Among them, the parameters in kinetic energy $\mathcal{T}$ are the derivative of $\dot{\mathbf{q}} = (\dot{q}_1, \dot{q}_2, \cdots, \dot{q}_n)$ on generalized coordinates and the parameters in potential energy $\mathcal{V}$ are the generalized coordinates $\mathbf{q} = (q_1, q_2, \cdots, q_n)$ and time $t$. Consequently, the distribution of Lagrangian $\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$ constitutes a *Lagrange field* over $\mathbb{R}^{2n+1}$. And so, the Lagrange equations (10.52) are the law that a moving particle follows in a conservative field.

**5.2.Least Action Principle.** The least action principle reflects the rule followed in the evolving of natural things, namely to maintain their own survival, changing and developing with minimum energy consumption. Usually, its expression is in the form of functional variation. Then, *what is the functional variation?* Dr.Ouyang tells Huizi that the variation was an extension of differentiation on functionals. Generally, let $K$ be a closed set in a normed space $\mathscr{B}$ and let $C(K)$ be all functions $F : K \to \mathbb{R}$. Notice that a functional $J : C(K) \to \mathbb{R}$ is a mapping of a function on closed set $K$ to a real number in $\mathbb{R}$, i.e. the *function on function*, usually denoted $J$ by $J[F]$. And so, for any function $F_0 \in C(K)$, the difference $F(K) - F_0(K)$ is said to be the *variation* of $F(K)$ at $F_0(K)$, denoted by

$$\delta F(K) = F(K) - F_0(K). \tag{10.53}$$

For example, let $K = [a, b]$ be a closed interval, the function $F, F_0 \in C[a, b], x \in [a, b]$. In this case, the variation of $F$ is $\delta F = F(x) - F_0(x)$ and $\delta F(a) = \delta F(b) = 0$. Notice that the differential operator $d/dt$ is linear. We know that

$$\delta \frac{df}{dx} = \frac{df}{dx} - \frac{df_0}{dx} = \frac{d}{dx}\delta f,$$

namely the variation $\delta$ and the derivative $d/dt$ can be commutative without affecting the calculating result.

Generally, according to the observing data on thing $T$, one usually adopts the variable $x$, dependent variable $y(x)$ and the derivative $y'(x)$, namely the function $F(x, y(x), y'(x))$ to characterize the state and behavior of $T$. In this case, for a linear functional

$$J[y(x)] \ = \ \int_{x_0}^{x_1} F(x, y(x), y'(x))dx, \quad y' = \frac{dy}{dx}$$

calculate the functional difference $\Delta J = J[y(x) + \delta y] - J[y(x)]$ and take only the linear terms in its Taylor expansion, i.e

$$\delta J = \int_{x_0}^{x_1} (\frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y')dx, \tag{10.54}$$

called the *variation* $\delta J$ *of* $J[y(x)]$. Particularly, if $F(x, y(x), y'(x))$ is an infinitely differentiable function, then

$$\begin{aligned}\Delta F &= F(x, y(x)+\delta y(x), y'(x)+\delta y'(x)) - F(x, y(x), y'(x)) \\ &= \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y' + \cdots,\end{aligned}$$

i.e., the variation of $F(x, y(x), y'(x))$ is

$$\delta F = \frac{\partial F}{\partial y}\delta y + \frac{\partial F}{\partial y'}\delta y'.$$

In this case, the formula (10.54) can be written as

$$\delta J = \delta \int_{x_0}^{x_1} F(x, y(x), y'(x))dx = \int_{x_0}^{x_1} \delta F(x, y(x), y'(x))dx. \tag{10.55}$$

Generally, if the functions $F, y_i, y_i', 1 \le i \le n$ are all differentiable, then the variation of functional

$$J[y_1, y_2, \cdots, y_n] = \int_{x_0}^{x_1} F(x, y_1, y_2, \cdots, y_n, y_1', y_2', \cdots, y_n')dx$$

is

$$\delta J = \int_{x_0}^{x_1} \delta F dx = \int_{x_0}^{x_1} \left( \sum_{i=1}^{n} \frac{\partial F}{\partial y_i}\delta y_i + \sum_{i=1}^{n} \frac{\partial F}{\partial y_i'}\delta y_i' \right) dx. \tag{10.56}$$

Notice that the variational operation is similar to that of the differential operation, except that we need to replace the differential $d$ with the variation $\delta$ in variational operation. For example, for the given functionals $F, F_1, F_2$, there are respectively

$$\delta(F_1 + F_2) = \delta F_1 + \delta F_2, \quad \delta(F_1 F_2) = F_1 \delta F_2 + F_2 \delta F_1 \quad \text{and} \quad \delta(F^n) = nF^{n-1}\delta F.$$

So, *under what conditions can a linear functional* $J[y(x)]$ *reaches its maximum or minimum?* Dr.Ouyang tells Huizi that the condition for a functional arriving at its extreme value is very similar to that of the condition for a function to arrive at its extreme value but it needs to replace the differential $d/dt$ with variation $\delta$, namely *if a linear functional* $J[y(x)]$ *arrives its extreme value at* $y(x)$*, then the variation* $\delta J[y(x)] = 0$. Notice that this is a general condition for linear functional at extremal point and it can be applied to any field characterized by $F(x, y(x), y'(x))$. Particularly, if the field $F(x, y(x), y'(x))$ is characterized by Lagrangian $\mathcal{L}(t, \mathbf{q}(t), \dot{\mathbf{q}}(t)$, then the functional $J[\mathcal{L}(t)]$ is defined to be the action $\mathcal{S}$ of Lagrange field, i.e.,

$$\mathcal{S} = \int_{t_1}^{t_2} \mathcal{L}(t, \mathbf{q}(t), \dot{\mathbf{q}}(t))dt. \tag{10.57}$$

Then, *under what condition is the action of Lagrange field minimum?* Dr.Ouyang explains that *a general condition for the linear functional $\mathcal{S}$ arriving at its minimum is* $\delta\mathcal{S} = 0$, called the *least action principle.* In this case, by derivation the condition satisfied by Lagrangian is exactly the Lagrange equations (10.52), namely

$$\frac{d}{dt}\frac{\partial\mathcal{L}}{\partial\dot{q}_k} - \frac{\partial\mathcal{L}}{\partial q_k} = 0, \quad k = 1, 2, \cdots, n.$$

**5.3.Lagrange Flow Equations.** If we regard the element state $\mathbf{x}_v[t]$ of a systemic simulation on thing $T$ as a field of element $v$, then the action between elements $v$ and $u$ can be characterized by the field action between $\mathbf{x}_u[t]$ and $\mathbf{x}_v[t]$, namely a continuity flow $G^L[t]$. On the other hand, as a vector in Banach space, the continuity flow $G^L[t]$ characterizes exactly the state and change of thing $T$ at time $t$, which should comply with the least action principle. And so, the dynamic behavior of thing $T$ can be systematically characterized by continuity flow $G^L$. For example, for a self-organizing system consisting of humans or animal bodies, the state of element $v$ is characterized by a Lagrangian $\mathcal{L}(v)[t]$ and the action $\mathcal{L}(v,u)[t]$ between elements $v$, $u$ for an edge $(v,u) \in E(G)$ is the action between fields $\mathbf{x}_v[t]$ and $\mathbf{x}_u[t]$. In this case, the action on any vertex $v$ follows the conservation law of energy, i.e., the continuity equation and the result is exactly a continuity flow $G^{\mathcal{L}}[t]$.

So, *how can we characterize the dynamic behavior of continuity flow?* Dr.Ouyang explains that assumes the mapping $\mathscr{L} : (v,u) \in E(G) \to \mathscr{L}[\mathcal{L}(t, \mathbf{x}(t), \dot{\mathbf{x}}(t))(v,u)]$ is differentiable and commutative with all end-operators $A^+_{vu}$. Notice that $G^{\mathcal{L}}[t]$ is a continuity flow, $G^{\mathscr{L}} \in \mathscr{G}_{\mathscr{B}}$ is also a continuity flow. In this case, the action and variation of continuity flow $G^{\mathscr{L}}[t]$ are defined respectively by

$$J\left[G^{\mathscr{L}}[t]\right] = \left|\int_{t_1}^{t_2} G^{\mathscr{L}[\mathcal{L}(t,\mathbf{x}(t),\dot{\mathbf{x}}(t))]}dt\right|, \quad \delta J\left[G^{\mathscr{L}}[t]\right] = \left|\delta\int_{t_1}^{t_2} G^{\mathscr{L}[\mathcal{L}(t,\mathbf{q}(t),\dot{\mathbf{q}}(t))]}dt\right|. \tag{10.58}$$

It should be noted that the variation $\delta : \mathscr{G}_{\mathscr{B}} \to \mathscr{G}_{\mathscr{B}}$ is a $G$-isomorphic operator in (10.58). By the variational rules we know that

$$\begin{aligned}\delta J\left[G^{\mathscr{L}}[t]\right] &= \left|\delta\int_{t_1}^{t_2} G^{\mathscr{L}[\mathcal{L}(t,\mathbf{q}(t),\dot{\mathbf{q}}(t))]}dt\right| \\ &= \left|G^{\delta\int_{t_1}^{t_2}\mathscr{L}[\mathcal{L}(t,\mathbf{q}(t),\dot{\mathbf{q}}(t))]dt}\right| = \left|G^{\int_{t_1}^{t_2}\sum\limits_{i=1}^{n}\left(\frac{\partial\mathscr{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial\mathscr{L}}{\partial\dot{q}_i}\right)\delta q_i dt}\right|.\end{aligned} \tag{10.59}$$

Applying the condition $\delta J\left[G^{\mathscr{L}}[t]\right](v,u) = 0$ of a functional arriving at its minimum and the property of norm on Banach flow space, we know that

$$\int_{t_1}^{t_2}\sum_{i=1}^{n}\left(\left(\frac{\partial \mathscr{L}}{\partial q_i}-\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}_i}\right)\bigg|_{(v,u)}\right)\delta q_i dt=0 \qquad (10.60)$$

for $\forall(v,u)\in E(G)$ by the least action principle. Notice that all $\delta q_i, 1\leq i\leq n$ are independent each other. Thus, the formula (10.60) is valid only if the coefficients of $\delta q_i$ are all 0, namely for any edge $(v,u)\in E(G)$,

$$\left(\frac{\partial \mathscr{L}}{\partial q_i}-\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}_i}\right)\bigg|_{(v,u)} = 0, \quad 1\leq i\leq n.$$

And so, Dr.Ouyang tells Huizi that we get the Euler-Lagrange equations

$$\frac{\partial G^{\mathscr{L}}}{\partial q_i}-\frac{d}{dt}\frac{\partial G^{\mathscr{L}}}{\partial \dot{q}_i} = \mathbf{O}, \quad 1\leq i\leq n \qquad (10.61)$$

on continuity flow $G^{\mathscr{L}}[t]$.

Particularly, let $\mathscr{L}:(v,u)\to\mathcal{L}(t,\mathbf{q}(t),\dot{\mathbf{q}}(t))$ for any edge $(v,u)\in E(G)$ in equation (10.60). We then get the dynamic flow equations

$$\frac{\partial G^{\mathcal{L}}}{\partial q_i}-\frac{d}{dt}\frac{\partial G^{\mathcal{L}}}{\partial \dot{q}_i} = \mathbf{O}, \quad 1\leq i\leq n \qquad (10.62)$$

in a Lagrange field.

Notice that the formula (10.62) is Euler-Lagrange equations of continuity flow $G^{\mathcal{L}}[t]$ in a Lagrange field. Generally, the dynamic behavior of $G^{\mathscr{L}}[t]$ follows the equation (10.61). For example, define a Lagrangian

$$\mathscr{L}\left[\mathcal{L}(t,\mathbf{q}(t),\dot{\mathbf{q}}(t))\right](v,u) = \sum_{i=1}^{n}c_i\dot{q}_i^2-\sum_{1\leq i\neq j\leq n}c_{ij}q_iq_j$$

independent on all edges $(v,u)\in E(G)$. Then,

$$\frac{\partial \mathscr{L}}{\partial q_i}-\frac{d}{dt}\frac{\partial \mathscr{L}}{\partial \dot{q}_i} = 2c_i\ddot{q}_i-\sum_{j\neq i}c_{ij}q_j,$$

we get $n$ differential equations

$$\begin{cases} 2c_1\ddot{q}_1-\sum\limits_{j\neq 1}c_{1j}q_j = 0,\\ 2c_2\ddot{q}_2-\sum\limits_{j\neq 2}c_{2j}q_j = 0,\\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots,\\ 2c_n\ddot{q}_n-\sum\limits_{j\neq n}c_{nj}q_j = 0 \end{cases}$$

by (10.61) in this case.

So, *under what condition does the Euler-Lagrange equations degenerate to Lagrange equations* (10.52)*?* Dr.Ouyang tells Huizi that in general, the Lagrangian $\mathcal{L}(t, \mathbf{q}(t), \dot{\mathbf{q}}(t))$ of a flow $G^{\mathcal{L}}[t]$ depends on edge $(v, u) \in E(G)$. But if $\mathcal{L}(t, \mathbf{q}(t), \dot{\mathbf{q}}(t))$ does not depend on edges in $G$, namely the Lagrangian on all edges $(v, u) \in E(G)$ is consistent, then the equations (10.62) are simplified to the Lagrange equations (10.52). In this case, each variable $\mathbf{x}_v$ on vertex $v$ tends to the same variable $\mathbf{x}$ if $t \to \infty$, namely the continuity flow $G^{\mathcal{L}}[t]$ is $G$-synchronized. In other words, the Lagrange equations (10.52) characterize a synchronous behavior of variables $\mathbf{x}_v$ on elements $v \in V(G)$ in dynamic systems. Particularly, the dynamical behavior of a particle $A$ with mass of $m$ in conservative field.

Now that the continuity flow $G^L[t]$ plays such an important role in the system recognition of thing $T$, Huizi asks Dr.Ouyang with a little confusion: "*Can the equation* (10.61) *or* (10.62) *characterize the dynamic behavior of anything* $T$ *and establish a unified theory on things in the universe?*" Dr.Ouyang tells Huizi that it is so only in theory because we have no a unified way to determine the Lagrangian until today unless by the definition of $\mathcal{L} = \mathcal{T} - \mathcal{V}$ on conservative field. But for other fields, the Lagrangian $\mathcal{L}(t, \mathbf{q}(t), \dot{\mathbf{q}}(t))$ is pieced together with techniques according to the nature of field in most cases. Certainly, it is still an unsolved problem whether there must really be Lagrangian $\mathcal{L}$ in a physical field or the truth of a thing must be so. Even so, the equation (10.61) or (10.62) provides a theoretical framework for the systematic recognition on the state and changing behavior of thing $T$. Philosophically, it implies that the human recognition on reality of things is endless, needs to be explored and discovered constantly in some extent.

## §6. **Comments**

**6.1.** Certainly, the network flow applies to the recognition of things needs to be further extended because the vertices and edges are all labeled by real numbers in network flow and the evolving of a thing is often multi-dimensional. It is in the spring of 2014 that after I completed the review paper [Mao31] of mathematics on non-mathematics, I tried to replace flows on network flow with vectors constraint on the conservation of incoming and outgoing at any one of its vertices, called the *G-flow* for generalizing it as a mathematical element in linear space and characterize the changing of network flow with its vertices and edges not change. In this case, the evolving of network flow is similar to that of a vector except that there is an inherited combination structure $G$ under the coordinates of vector. It can be proved that if all vectors on network flows belong to a Banach or

Hilbert space $\mathscr{B}$, such a kind of network flows constitute a Banach or Hilbert space also and we can introduce the linear operators and linear functionals on them, generalize a few famous conclusions in functional analysis such as the fixed point theorem, Fréchet-Riesz representation theorem, etc., see [Con] and [GZF]. Particularly, if all flows are elements in a Hilbert space $\mathscr{B}$, the differential, integral operations can be introduced on $G$-flow and then, discuss the $G$-flow solution of equation and get the combinatorial solution of wave equation, heat conduction equation, etc., which is the seminal paper [Mao33] cited very soon after its publication.

**6.2.** Notice that the network flow is not only limited in characterizing the reality of a thing but also in its combinatorial structure $G$ is unchange in the evolving process and for any vertex $v \in V(G)$, the input flow on edge is consistent with the observed effect on vertex $v$, which is in fact an assumption of human that the measuring results on different vertices are all the same but not completely consistent with the transferring situation of substance, especially the microscopic particles. In 2015, I was invited to give an invited talk of 45 minutes in the *National Conference on Emerging Trends in Mathematics and Mathematical Sciences* (NCETMMS-2015, December 17-19, 2015), organized by the Mathematical Society of Calcutta in India (the first mathematical society in Asia, 13th mathematical society in the world) and prepared a report on combinatorial solution of non-harmonious groups. I systemically reviewed the studied cases of non-harmonious groups before and the essence of $G$-flow, thought the results should be different if measuring at different vertices and then, introduced the end-operators into $G$-flow for recognition of thing, i.e., the flows at a vertex following the conservation law of substance should be after the action of end-operator, called the *action flow* which is exactly the state equation of element introduced in [Mao38]. And then, the continuity flows are extensively discussed later in [Mao38]-[Mao54]. Correspondingly, the combinatorial solution of a non-harmonious group is nothing but a continuity flow on graph. At this time, for simulating the changing of elements in the evolving of a thing, the addition, scalar multiplication on continuity flow were introduced by the union and difference operations of graphs, and a continuity flow is regarded as a mathematical element. In this way, the vertices of a continuity flow $G^L$ can be regarded as the elements of system, the flow on edge characterizes the interaction in elements and the end-operator is the mapping or transformation of action on the acted element. Similar to network flow, the Banach or Hilbert flow space on continuity flows is obtained and the fixed point theorem, continuous mapping theorem, closed graph theorem and the Hahn-Banach theorem, Fréchet-Riesz representation theorem in functional analysis are generalized to the Banach or Hilbert flow space. Moreover, such an intrinsic structure of quantum do not affect the behavior hypothesis on microscopic particles in quantum mechanics. For a further description of particle behavior, see [Mao34], [Mao36], [Mao45] and [Mao48]. Among them, [Mao45] is a collection of my papers on combinatorial notion and the recognition of natural reality published from the time that I published the CC conjecture in 2007 to 2017, from which the readers can see my mathematical philosophy on reality of things. My recognitive view on microscopic particles in [Mao34] and [Mao36] are included in *The Worldwide List of Alternative Theories and Critics* [Cli], a large scientific handbook edited by Jean de Climont (2018 edition).

Certainly, the harmonious flow is a special case of continuous flow, namely labeling the vertices and edges by complex vectors. Particularly, if the imaginary part of the complex vector is $\mathbf{0}$, then a harmonious flow is nothing else but a continuity flow labeled by real vectors. However, according to the ancient Chinese philosophy, the harmonious flow has its significance on life and can be applied to characterize the equilibrium of Yin and Yang states on a living body. In fact, the terminology "*harmonious flow*" was introduced in [Mao46] from my reflections on the *Emperor's Inner Canon-Plain Questions* and *Emperor's Inner Canon-Lingshu Canon* when I was so ill at the time from July to August in 2018, which happens to be a real example of continuity flow of mine appearing in traditional Chinese medicine.

**6.3.** The calculus of continuity flows on Hilbert spaces are established in [Mao42]-[Mao43], [Mao49] and [Mao51] similar to that of the network flow calculus. Among them, a fundamental question is *whether the operator acting on a continuity flow is consistent with the operator acting on an element of Banach space?* The answer is Not because it needs to maintain the combinatorial structure of continuity flows, i.e., the $G$-isomorphic operator defined in [Mao51]. Certainly, the Fréchet-Riesz representation theorem on $G$-flows in [Mao33] or [Mao38] can be generalized to continuity flows similarly by $G$-isomorphic operators, namely if $\mathbf{T} : G^{\mathcal{H}} \to \mathbb{C}$ is a linear continuous $G$-isomorphic operator on a Hilbert space $\mathcal{H}$, then there exists a uniquely continuity flow $G^{L_0} \in G^{\mathcal{H}}$ such that $\mathbf{T}\left(G^L\right) = \left\langle G^L, G^{L_0}\right\rangle$ for $\forall G^L \in G^{\mathcal{H}}$, which confirms the existence of Hermitian $G$-isomorphic operators $A$ on Hilbert flow space $G^{\mathcal{H}}$ or the three hypothesises about quantum with intrinsic structure. Similarly, the fixed point theorem, continuous mapping theorem, closed graph theorem and the Hahn-Banach theorem on harmonious flows in [Mao46] are all holding immediately with the continuity flows on $G$-isomorphic operators in [Mao51].

$C_{\mathscr{M}}$

**Combin.Notion**

## Chapter 11

# Chinese Knowledge

**********************************************************************

**Great Wind, the Spring Tide.**

**Flood, Totem with Dragon; Fire, Nirvana and Phoenix.**

**Civilization Flame, not Through the Ages, only I Peerless;**

**Coexisting with the Heaven and the Earth,**

**Shine with the Sun and the Moon.**

**Chinese Culture has a Long History,**

**A History with Extensiveness, Profound and Brilliance.**

**Walk 300 Meters of Corridor,**

**Experience the Vicissitudes of Life for Five Thousand Years.**

– *Beijing World Art Museum* by X.Y.Zhu, a contemporary scholar of China

## §1. Unity of Humans with the Heaven

Generally, the recognition of a thing is humans ourselves in recognizing on the thing, which is on the individual attributes or characters of thing that can be perceived by humans and through the brain's feeling, perception, memory, thinking, imagination and others to form the conception of "*name*" or characters of a thing to distinguish it from other things. Among them, the human is the subject of recognition and the thing is the object. Then, *what is the unity of the heaven and humans?* Dr.Ouyang explains that it is a theory in ancient Chinese philosophy on the relationship in humans and the nature. It states with certainty that humans and the nature constitute a binary ecosystem to emphasize the relative and unified relationship between the heaven and humans in ancient Chinese philosophy, reflected in the dialectical relationship between the heaven, earth and humans in the *Change Book*, a symbolic system for recognition on a pure Yang and a pure Yin hexagrams, called Qian or Kun hexagrams as shown in Figure 11.1 and then, deduced 64 hexagrams by integrating the changes of Yin and Yang on lines to simulate the evolving of a thing with changing law, which is the recognition of ancient Chinese philosophers on the changing law of state of everything in the universe. Certainly, this is a systematic recognizing on evolving of thing, i.e., a systemic theory of Chinese on things in today.

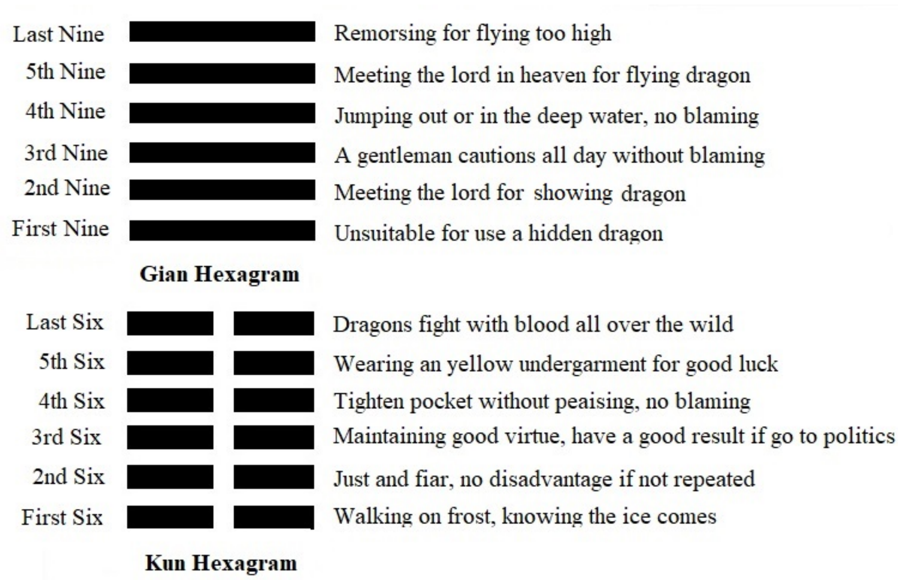



**Figure** 11.1 **Qian and Kun hexagrams**

So, *why the Qian and Kun hexagrams constitute the elements in evolving of a thing?*

Dr.Ouyang tells Huizi that the Qian hexagram constitutes all pure *Yang* symbols which corresponds to changing of the heaven or nature, symbolizing the "*going up, the heaven is full of power*", i.e., the side "*Yang*" or the vigorous, upright and unremitting self-improving in everything; the Kun hexagram constitutes all pure *Yin* symbols which corresponds to the earth, symbolizing the virtue of "*low bearing with all things*" or "*providing conditions for surviving of all things*", i.e., the side "*Yin*" or the pliability and tolerance, nourishes the growth of all things. Confucius wrote the *Hexagram's Notes* explaining the implication of words on each line in the *Change Book*, summarized that the human's imitating on the heaven should be "*vigorous as the heaven, a gentleman should strive for self-improving*" because the posture of Qian hexagram is "*vigorous, upright, pure and refining*", which has a vigorous, centered and upright energy that pushes forward the changing of thing and the human imitating on the earth should be "*low as the earth bearing with all things*" because the word "*Kun*" represents the feminine side in the changing of thing, namely its "*conformity, obeying the nature and acting according to the times*", symbolizes the conforming and obeying to the laws of universe. It is a summary of the moral character of a human who follows the way of heaven and the nature, namely the survival of humans should follows the heaven and nature. They should be surviving like the heaven, constantly pursuing progress, resolute and vigorous on one hand, and on the other hand, they should be surviving like the earth, increase their virtue and bearing with all things in the universe because only in this way, the humans could coexist and live forever with the universe.

At this time, Huizi still has some doubts in her mind and asks Dr.Ouyang: "*If humans and the nature constitute a binary ecosystem, then what position does humans occupy in this binary ecosystem? Is it equal to the nature?* " Dr.Ouyang replies that if humans and the nature are in an equal role, then they must be mutually influence, interaction and the humans can call the wind and rain, control the running of universe in this binary system. However, even if we live on the earth, we have no influence on the rotation and revolution of the earth as well as the alternation of four seasons on the earth, but an abnormal activity of nature has a great influence on humans ourselves. For example, an earthquake or tsunami. *What does this implies?* It implies that the position of humans in nature is so small or insignificant! Huizi asks Dr.Ouyang again: "*Since the influence of humans on the nature, even on the earth is very small and can be negligible, we can characterize the relationship of humans with the nature by a unary system but why*

**Figure** 11.2. **Ecosystem**

*we emphasize that humans and the nature constitute a binary system here?*" Dr.Ouyang explains that the impact of nature on humans is immediate but that of humans on the nature is a delayed effect in this case, which starts to a self-regulate after the impact of humans on the nature accumulates to a certain extent, but then it must be natural disasters. Therefore, the ancient Chinese said that humans and the nature are a union one or humans and the nature constitute a binary system as shown in Figure 11.2, emphasizes that the sustainable developing of humans needs to conform to the nature and act in accordance with the survival of all things, namely *do inaction that inconsistent with the natural law* or constantly add the *Te* that "*Tao*" embodies in the behavior of humans, advocated by Laozi in his *Tao Te Ching*. Notice that although one usually thinks that the word "*Te*" is near to "*virtue*", its meaning is different in situations. For example, Laozi's "*Te*" is the behavior of human that consistent with "*Tao*" or following the natural laws, i.e., a moral behavior of humans following the universe while Confucius's "*Te*" is the usual meaning of "*morality*" in humans ourselves. Thus, there are two topics should be discussed on the binary system of humans with the nature standing on the side of humans, i.e., the evolving laws of nature or things in the universe and the self-restraint of humans ourselves in survival and developing. Hearing her father's explaining, Huizi has a few insights on the topics and asks Dr.Ouyang: "*Dad, the first topic here is the purpose of science, isn't it?*" Dr.Ouyang tells Huizi that the first topic lies in the recognition of natural reality, which can be regarded as the purpose of science! But up to today, most of the scientific systems on natural laws established by humans are on the local recognition or conditions of humans and unconsciously ignore the factor of humans, especially the cumulative effect of humans acting on the nature as applying the achievements. For example, the frequent occurrence of haze in cities has something to do with the excessive emission of carbon dioxide in human's production and living in the past 300 years. However, the frequent occurrence of haze is essentially a process of the nature's self-regulation and returns again to the balance of carbon dioxide in the nature. Huizi puzzlingly asks Dr.Ouyang: "*What does the excessive emission mean here?*" Dr.Ouyang tells her that the "*excessive emission*" refers to the emissions that exceed the upper limitation of carbon dioxide consumed by the nature such as the photosynthesis of green plants. However, humans do not know exactly what the upper limitation is but only an estimating by the census of various resources on the earth. Huizi nods and then asks her father: "*Anyway, how can humans conduct self-restraint in harmony with the nature?*" Dr.Ouyang explains that this is exactly the problem solved by ancient Chinese philosophy, reflected in Chapter 25 of Laozi's *Tao Te Ching*, i.e., *humans follow the earth, the earth follows the heaven, the heaven follows Tao* *and Tao follows the nature*", namely the humans are restricted by the earth and needs to follow the evolving of things on the earth, the heaven, the Tao and finally all the nature's law. They must be self-restraint, i.e., constantly cultivate the morality because the nature carries the ontology of "*Tao*" and the "*Tao*" is the law or appearance of nature.

**1.1.Cosmology in Ancient Chinese.** Generally, the word of "*heaven*" in ancient Chinese philosophy means the "*universe*" or "*all things*" in the universe. Although the "*human*" is the subject of recognition, it is certainly included in "*all things*" here. In fact, the "*Tao*" in Laozi's *Tao Te Ching* is the creation and operating laws contained in the universe, nature or all things, which can not be replaced by a certain or a certain kind of operating laws. So, *how did the universe came into being and what was the pattern in the minds of ancient Chinese philosophers?* Dr.Ouyang explains that regarding the formation of universe, Laozi summarized his *creation* that "*all things are creatd of being and the being is created out of non-being*" in Chapter 40 of *Tao Te Ching*. Here, the words of "*being*" and "*non-being*" are the human perception, namely what the human perceive is "*being*" and what the human can not perceive is "*non-being*", which means that all things existing in the universe are created out of non-being dependent on the human perception. It should be noted that the way of "*being out of non-being*" is further explained in Chapter 42 of *Tao Te Ching*, i.e., "*Tao creates One, One creates Two, Two creates Three and Three creates all things*", as shown in Figure 11.3.

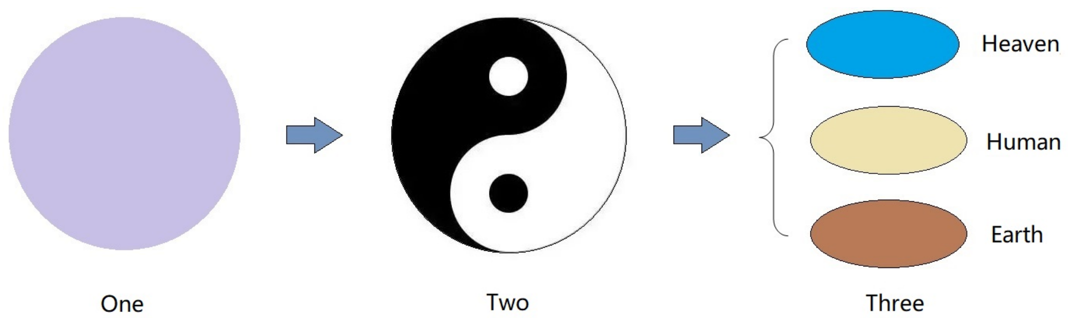



**Figure** 11.3. **All things creating process**

So, *are these One, Two, Three real numbers in* $\mathbb{R}$*?* The answer is certainly Not because they are only symbols on creating order in the process. Here, a natural question on the being of things is *why did they follow the evolving laws that Tao creates One, One creates Two, Two creates Three and Three creates all things?* Dr.Ouyang explains that this is an evolving pattern drawn philosophically by Laozi for the creation of universe because humans were created in the last step, i.e., "*Three creates all things*" and it is impossible for humans coming back to observe the creation process of things and can be only drawn a pattern that there exists congenitally a "*Tao*" that creates the "*One*", i.e., the chaotic universe. Certainly, the creation of "*Two*" from "*One*" is the chaotic universe apart for the Yin and Yang universes and then, the creation of "*Three*" from the Yin and Yang universes of "*Two*" gives the being of all things in the universe. Among them, *One, Two* and *Three* have the following meaning in Laozi's pattern on the creation of universe:

(1) **One.** "*One*" in "*Tao creates One*" belongs to the result of "*being out of non-being*" in human rationally recognizing on things in the universe, which is the starting point of human recognition on the original state of universe because "*all things are born of being*", "*One*" is the starting point of human rational recognition, i.e., "*being*", namely a chaotic state in the original universe because the "*human*" is in "*all things*". So, *what is the state of One?* Dr.Ouyang explains that the nature is the carrier of "*Tao*". Since it is "*Tao creates One*", the state of universe known to humans from non-being can be only the "*Tao*", which can be understood by Laozi's description on "*Tao*". In this regard, the Chapter 21 of *Tao Te Ching* describes "*Tao as a thing is vague, also indefinite with pattern. It is indefinite, vagues but embodying with substance. It is dark and unfathomable with essence, which is a genuine existence that can convinced of testing*", namely "*Tao*" as a natural thing is erratic and fleeting. It can be perceived as a physical object but only in a vague and indefinite pattern. Certainly, it has an essence, both pure and ethereal in the dark and unfathomable which enables humans believing in the existence of "*Tao*". At the same time, it is a depiction of humans on the creation of universe within the recognition of humans, i.e., only a drifting in a vague and indefinite pattern, not an image that a person can completely grasp or determine. Notice that "*Tao creates One*" is not in the state of nothingness but "*there are something in it*", only a little vague and indefinite, i.e., uncertain or chaotic to humans. In this respect, Tao is further described in Chapter 25 of *Tao Te Ching*, i.e., "*there is a mixed and perfect thing before the born of heaven and earth; it is limpid and silent, solitary without changing and moving around forever, namely the mother of universe; I call it the Tao for lack of a better name*", which also depicts the state of One, namely there are indeed things mixed, not nothing in "*One*" and One is created before the heaven and earth because both of them appear in "*One creates Two*". Among these, the words that "*it is limpid and silent, solitary without changing and moving around forever*" portray further the state of humans' visible universe in the period of One, namely the constantly operating of *Tao* creates all things.

(2) **Two.** "*Two*" in "*One creates Two*" refers to the "*Yin*" and "*Yang*" or Taiji diagram in ancient Chinese philosophy. It is the performance of positive and negative energy in evolving of thing. Then, *why did the universe divide from chaos into a state consisting of Yin and Yang?* The ancient Chinese philosophy holds that the evolving of

thing originates from the action of two powers, i.e., Yin and Yang or the two sides of "*contradiction*". It is the action of Yin and Yang in a thing that promotes its evolving, also known as the "*motive power*" or "*Tao*" in the evolving of thing. In this regard, Laozi pointed out this kind of cycle that "*return is the moving energy of Tao*" between Yin and Yang in Chapter 40, namely the philosophical thought contained in the Taiji diagram is the internal cause of thing evolving or Tao of the heaven that "*reducing the excess with supply the insufficient*" in Chapter 77, namely Tao of the heaven is supplying the insufficient by reducing the excess for maintaining the balance in the evolving of universe and the words "*every thing connotes Yin and Yang, achieves the balance*" following "*Tao creates One, One creates Two, Two creates Three and Three creates all things*" in Chapter 42 of *Tao Te Ching* point out that there are two sides in any thing, i.e., the positive (Yang) and negative (Yin) energies. All of these words conclude that a thing is a kind of harmonious unity in the mutual agitation and action of Yin and Yang.

**(3) Three.** "*Three*" in "*Two creates Three*" is not the number *three* in mathematics. Notice that "*Three*" is a general reference to the majority in ancient Chinese philosophy, namely the "*majority of* $\geq 3$" including the human, i.e., the human came into being not in states of "*One*" or "*Two*" but in "*Three*" and then, the creation of "*all things*". This shows that humans recognize things in the universe can be only by proposing hypothesis which can not be verified. Notice that the ancient Chinese philosophy holds with a view that "*the human is an anima gathering of the heaven and the earth*" and all things coming into being can not without the role of human. Certainly, there is a legend in ancient Chinese culture, i.e., the "*Pangu Split Open the Chaos*". It is believed that Pangu was born in the chaotic state of universe, namely "*One*". It was Pangu who pulled out a tooth in his mouth and turned it into a magic axe and swung it to cut the chaotic universe into two parts, i.e., the light rose to form the sky while the heavy fell to form the earth. Later, Nuwa created humans and formed the pattern of heaven and the earth or "*three powers*" in the universe, which corresponds to "*three geniuses*" or "*San Qing Sages*" in Chinese taoism, called the Yuanshi, Lingbao and Tao Te geniuses with deities on "*One, Two*" or "*Three*" states in the creation of universe, i.e., in charge of Tao creates One, One creates Two and Two creates Three, respectively, as shown in Figure 11.3. Notice that the order of three geniuses in Figure 11.4 is by the order rule of ancient Chinese


**Figure** 11.4. **San Qing Sages**

culture, i.e., from right to the left. Among them, the incarnation of Tao Te genius is just Laozi on earth because it is believed that Laozi passed down the taoist classic, i.e., *Tao Te Ching* is a genius but lived in the world of humans and guided the founding of taoism. And so, he is respected as the supreme leader of Chinese taoism.

So, *how to creating and through what way creating in the process of "Three creates all things"?* Dr.Ouyang tells Huizi that Laozi portrayed the creating way in Chapter 6 of Tao Te Ching, namely the words that " *the valley god is immortal, called the primitive womb of all things; it is essentially the root or mother of the heaven and earth; it seems to be existing forever, never be exhausted*". Here, the "*valley god*" refers to the character or nature of Tao embodied in things because Tao is usually illusory, like a deep valley which operates constantly and displays the evolving laws of things. So, it is called "*the valley god is immortal*". The word "*primitive*" in "*primitive womb*" is an adjective in meaning as the words of "*initial, original*" or "*first*". In this way, the process of "*Three creates all things*" is then similar to that of a biological reproduction, namely there exists a genitalia in the heaven and earth, called the "*primitive womb*" where all things in the universe are endlessly reproducing. And then, despite the variety and quantity, all things return to their initial states according to "*all things show the full of vitality, return to their own roots*" in Chapter 16 of *Tao Te Ching*, i.e., similar to the cyclic model of "*all falling leaves return to the root*" so as to maintain the continuation of successive species.

Dr.Ouyang tells Huizi that the Big Bang in theoretical physics follows Laozi's creation that of "*Tao creates One, One creates Two, Two creates Three and Three creates all things*", which is only attributing the cause of "*One creates Two*", i.e., *why does the chaotic universe creates Two* to the Big Bang in an unknown reason, a situation that ordinary humans can understand as the starting point of time $t$. Certainly, the Big Bang

**Figure** 11.5. **The Big Bang's creation**

assumes that the primitive universe inflated in all directions of space with the separation of strong, weak and electromagnetic actions, the collision of elementary particles and antiparticles, and the transformation of energy in the first few seconds after the Big Bang, i.e., the creation of "*Three*" and then, according to the embodying of"*Tao*" on different things combined into atoms, molecules and celestial bodies, gave birth to life and other things for repeating the physical combination process of "*Three creates all things*". However, Dr.Ouyang tells Huizi that there is a problem that can not explain itself by the Big Bang, namely *where did One, i.e., the chaotic universe come from on which the Big Bang happened?* The answer can only be Laozi's "*Tao creats One*" because "*One*" is the starting point of human recognition on the universe. This also shows that the Big Bang with a few derived theories are to reappear the creating process of things in the universe within the knowability of human. They are all human's understanding or recognition on the creation of universe. To some extent, this makes also humans unable to answer problems such as *what was the state of universe like before the Big Bang? whether is it a particle with an infinitesimal diameter or a space filled with unknowable matters?* Certainly, all such kinds of problems are all beyond human's knowability and maybe never have an answer unless hypothesis.

At this time, Huizi has a question on the Yin and Yang theory both in Laozi's "*Three creates all things*" and the Big Bang. She asks Dr.Ouyang: "*Dad, isn't the conception of Yin and Yang the foundation of ancient Chinese philosophy but it doesn't seem to be reflected in 'Three creates all things' or the Big Bang?*" Dr.Ouyang applauds Huizi for thinking of this question and then explains that according to the philosophy of Laozi's *Tao Te Ching*, the Yin and Yang appeared in "*One creates Two*" of the dynamic process of "*Tao creates One, One creates Two, Two creates Three and Three creates all things*". However, "*One creates Two*" is not only implying the appearing of Yin and Yang or "*Two*" creates "*Three*" but also has two implications. One is the Yin and Yang, i.e., "*Two creates Three*" or more image the "*three powers*", i.e., the "*heaven, earth and human*" and then "*Three creates all things*". Another is the Yin and Yang run through each creating process of things after "*One creates Two*" in the creation of universe. *Why do we say so?* Dr.Ouyang explains that because the original meaning of "*Two creates Three*" is the Yin and Yang reflected in the composition of each part of "*Three*", namely the heaven, earth and the humans all have two internal powers for changing, i.e., the Yin and Yang. Accordingly, there must exist two powers, i.e., the Yin and Yang in evolving of all things so as to realize the state of "*every thing connotes Yin and Yang, achieves the balance*" in Chapter 42 and "*return is the moving energy of Tao, gentleness is the function of Tao*" in Chapter 40 of *Tao Te Ching*, namely the weak in the internal positive and negative powers of thing is always following the moving of strong and gradually becoming the strong in adaptation, which is endless in cycle and then, operates according to the Tao of heaven, i.e., "*reducing the excess with supply the insufficient*" in Chapter 77 of *Tao Te Ching*.

Then, *does the Big Bang in physics incorporate completely the ancient Chinese philosophy of Yin and Yang?* The answer is Not! Dr.Ouyang tells Huizi that although the Big Bang reflects the evolving of the universe from "*One*" to "*Three*" in Laozi's creation, it does not incorporate completely the Yin and Yang view of ancient Chinese philosophy, which could not explain the original power of things in changing. Certainly, the Yin and Yang in ancient Chinese philosophy can be characterized quantitatively by vectors in mathematics. In this case, let vectors $\mathbf{v}^-$, $\mathbf{v}^+$ be respectively the Yin and Yang powers in the evolving of a thing $T$. According to ancient Chinese philosophy, the driving power to promote the changing of things $T$ should be satisfies $\mathbf{v}^- + \mathbf{v}^+ = \mathbf{c}$, namely the sum of Yin and Yang powers is a constant vector $\mathbf{c}$, where $\|\mathbf{c}\|$ is the diameter of ball $\mathbb{B}^n$, $n \geq 1$ that the Taiji diagram corresponds to and $n$ is the number of known characters $\mu_1, \mu_2, \cdots, \mu_n$ at the moment $t$. So, "*what is the expressing form of thing $T$ if it is integrated into the Yin and Yang view in ancient Chinese philosophy?*" Dr.Ouyang explains that the model of systemic recognition on thing $T$ is a continuous flow $G^L$. If it is integrated the two powers of Yin and Yang, the recognition of thing $T$ is just the harmonious flow $G^{L^2}$ in Section 2 of Chapter 10, namely for any vertex $v \in V(G)$, edge $(v,u) \in E(G)$, $L^2 : v \to L(v) + iL'(v)$, $L^2 : (v,u) \to L(v,u) + iL'(v,u)$ and $L(v) + L'(v) = \mathbf{c}_v \in \mathscr{B}$, $L(v,u) + L'(v,u) = \mathbf{c}_{vu} \in \mathscr{B}$, where $\mathbf{c}_v$ and $\mathbf{c}_{vu}$ are all constant vectors in Banach space $\mathscr{B}$.

**1.2.Unity Field of Humans with the Heaven.** Generally, the word "*heaven*" is a synonym of the word "*nature*" in Chinese culture. If the heaven and humans are regarded as the physical fields, then the unity of humans with the nature in ancient Chinese philosophy can be characterized by a labeled graph on the complete graph $K_2$ of order 2, i.e., the binary system $K_2^L$, where, the vertex set of $V(K_2) = \{v, u\}$, edge set $E(K_2) = \{(v,u), (u,v)\}$ and the labeling mapping $L$ on vertices $v$, $u$ respectively maps to fields $L(v)$ and $L(u)$ with the acting strength $L(v,u) = -L(u,v)$, namely the action between $L(v)$ and $L(u)$ is antisymmetric which is similar to equation (10.31), i.e.,

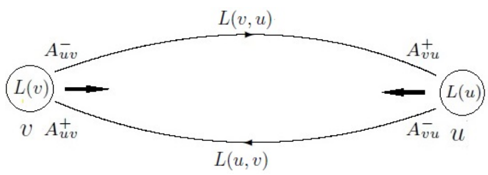



**Figure** 11.6. **Binary system**

$$\begin{cases} \dot{\mathbf{x}}_v = L^{A^+_{uv}}(u,v) - L^{A^-_{uv}}(v,u), \\ \dot{\mathbf{x}}_u = L^{A^+_{vu}}(v,u) - L^{A^-_{vu}}(u,v). \end{cases} \tag{11.1}$$

Notice that the equation (11.1) is an operator equation of system $\mathcal{F}$ with 2 interacting elements $v, u$ and observed outside system $\mathcal{F}$. In this case, the states and changing of elements $v, u$ do not depend on human's observing. For example, Let $x_v, x_u$ be respectively the population sizes in a binary system consisting of cats $u$ and mice $v$. Then, the population sizes of cats and mice will not changing due to the human's observing with a corresponding equation

$$\begin{cases} \dot{x}_v &= x_v(-\mu + cx_u), \\ \dot{x}_u &= x_u(\ \lambda \ - bx_v). \end{cases} \tag{11.2}$$

of states, namely, the labeling $L(v) = \dot{x}_v, L(u) = \dot{x}_u$ on vertices $v$ and $u$, $L(v,u) = x_v$, $L(u,v) = x_u$ on edges with end-operators $A^+_{uv} = cx_v, A^-_{uv} = \mu$, $A^+_{vu} = \lambda, A^-_{uv} = bx_u$ in Figure 11.6, where, $-\mu$ is the average growth rate of cats without mice and $\lambda$ is the average growth rate of mice without cats, $b$ and $c$ are both constants.

So, *can we apply the equation* (11.1) *to give the state equation of system of the "unity of humans with the heaven"?* The answer is Not! Dr.Ouyang explains that in this case, the humans are include in the system as the observer, as shown in Figure 11.7, i.e., observing results depend on observing of human, which is only an approximation of the reality. Strictly speaking, we can only entirely characterize the state of binary system of the "*unity of humans with the heaven*". In this case, assume that $L(v)$ and $L(u)$ are both in a Lagrange field with Lagrangian $\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$. Then, the action on the binary system from time $t_1$ to $t_2$ is


**Figure** 11.7. **Humans with the heaven**

$$J\left[K_2^L[t]\right] = \left|\int\limits_{t_1}^{t_2} K_2^{\mathcal{L}}[t]dt\right|, \qquad L(v,u) = \int\limits_{t_1}^{t_2} \mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t)dt \tag{11.3}$$

if the end-operators $A^+_{uv} = A^-_{uv} = A^+_{vu} = A^-_{vu} = 1_{\mathscr{B}}$. By the principle of least action, we know that the variation $\delta J\left[K_2^L[t]\right] = 0$ and then, the Euler-Lagrange equations of system $K_2^L[t]$ in Lagrange field $\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$ are

$$\frac{d}{dt}\frac{\partial K_2^{\mathcal{L}}}{\partial \dot{q}_k} - \frac{\partial K_2^{\mathcal{L}}}{\partial q_k} = \mathbf{O}, \quad k = 1, 2, \cdots, n. \tag{11.4}$$

Notice that if let the vertices $v$ and $u$ be respectively the humans and the nature. Then, the edge vector $L(v,u)$ is the effect of humans acting on the nature in the binary system $K_2^L$ shown in Figure 11.6. But, *can the effect of humans acting on the nature be observed by humans ourselves?* Dr.Ouyang tells Huizi that Sushi, a writer in the Song dynasty of China wrote "*why do not know the true face of Lushan mountain, only because you are in the mountain yourself*" in his one poem of *Writing Xilin Wall*, namely when a person is in the Lushan mountain himself, he can only see the local truth. In other words, this person and Lushan mountain form a binary system. What he seen is the scenery of Lushan mountain relative to what he can see, but not the whole or real face. Similarly, the observing of "*humans*" in the "*unity of humans with the heaven*" is relative in its observation of the nature. Among them, the effect of nature on humans $L(u, v)$ such as the earthquake, hurricane, tsunami and other natural disasters can immediately destroys any high-rise building that built by humans, but the effect of humans acting on the nature $L(v, u)$ is delayed because the human recognition of nature is partial or local, which result in that if $|t_2 - t_1|$ is very small, we are almost unaware the effect of humans acting on the nature. It is this unconscious behavior that causes the humans mistakenly think that the effect $L(v, u)$ of human's activities on the nature can be negligible and analysis the relationship of humans with the nature by a unary system $K_1^L$ instead of the binary system $K_2^L$, to think that the nature or the "*heaven*" can contain all human's influences on it and ignore the cumulative effect while the nature can not fully absorb them in the past for a long time. For example, the "*fire*" is the origin of human civilization and the manifestation of chemical reaction between carbon and oxygen molecules that releases the combustion energy. However, the process of carbon combustion, namely the chemical reaction $C + O_2 = CO_2$ and $2C + O_2 = 2CO$ will simultaneously produce the carbon monoxides, carbon dioxides and other wastes, which will pollute the atmospheric environment and also affect the survival of humans.

So, *what are the processes to absorb the carbon dioxides in the nature?* Dr.Ouyang tells Huizi that there are three main ways of natural absorption on the carbon dioxides, namely the photosynthesis of green plants on land and water, dissolution in water to produce carbonic acid or the reaction $2Ca + O_2 + 2CO_2 = 2CaCO_3$ with calcium ions in soil to produce calcium carbonate so as to maintain its natural balance of carbon dioxides. Notice that the humans were created in this natural balance mechanism along with "*Three creates all things*", which did not give humans special ability of survival because the words "*the heaven and earth are not kind, taking all things as humble dogs*" in Chapter 5 of *Tao Te Ching* include humans ourselves. Thus, there is a threshold $R_{human}$ for the amount of carbon dioxides in air that permits the survival of human, namely the human existing is possible only if the amount of carbon dioxides in air $R_{air} \leq R_{human}$. However, if the nature is not enough to absorb carbon dioxides emitted by human activities, the redundant carbon dioxides will float in air and it is necessary to artificially reduce carbon dioxide emissions to meet the living condition of $R_{air} \leq R_{human}$. Otherwise, if $|t_2 - t_1| \to \infty$, the cumulative amount of carbon dioxides $R_{air} \to \infty$. Once $R_{air}$ is more than $R_{human}$, i.e., $R_{air} > R_{human}$ the earth is not suitable for the survival of human and finally, the human will no longer exist on the earth.

**Figure** 11.8. **A few consequences of carbon excessive emission**

In this natural adjustment, even the accumulation $R_{air}$ of carbon dioxides in air meets the condition $R_{air} \leq R_{human}$ but $R_{air}$ has to a certain amount, will also cause the destruction of the atmospheric ozone layer, lead to the rise of earth temperature, ice sheet melting, sea level rise, extreme weather, drought, virus creation with mutation and other natural phenomena. It will result in consequences of humans, as shown in Figure 11.8. Certainly, the essence of such kinds of natural disasters is the autonomous regulation of nature to adapt to its operating law and a reaction on humans to a certain extent so as to return to the new balance of the nature. In this case, the humans need to correctly understand the relationship with the nature and actively return to the "*unity of humans with the heaven*" in ancient Chinese philosophy, namely the humans should follow the rules of binary system $K_2^L$ to realize the harmonious coexistence of humans with the nature.

## §2. **Everything's Law**

Certainly, everything existing and changing follows an inherent law in the universe. For example, "*water flows to the bottom*" is a property of water but it implies the philosophy of water such as the perfection, tenderness, humility, no competition and tolerance of all things. It lies in the fact that water has mass and moves towards the center of earth under the gravitation of earth, as shown in Figure 11.9. So, *can we regard this property of water as a law of all things?* Of course Not! Dr.Ouyang explains that this property of water is a law that water exhibits on the earth, can be not applied to other things, nor to an occasions outside the earth such as its weightlessness in the outer space. Then, *what is the law of everything?* The law of everything is the general law that all things survive and change. It is all things

**Figure** 11.9. **Water runs to the bottom**

created with "*Three creates all things*". Dr.Ouyang asks Huizi: "*Does the meaning of the law of everything here is the same as the word 'science' that we use today?*" Huizi thinks for a moment and replies: "*Dad, although the purpose of science is to reveal the law of everything, it seems that we can not regard science as the law of everything. But I can't explain why it is so.*" Dr.Ouyang agrees with Huizi's view, explains that science is only the human recognition on laws of certain things or certain kinds of things, which is within the humans' ability to understand the laws of things and can not be considered as the law of everything because the purpose of science is to reveal the natural truth of certain things or certain kinds of things, not the law that everything obeys. And meanwhile, science is the laws that human reveals on certain or certain kinds of things. It is the conclusion drawing by human's analysis and induction on observing data of things and verified also by human who repeatedly tested the conclusion, which is related to the observing conditions and the human's recognitive ability at the time of drawing the conclusion. Even if a scientific conclusion is recognized by all humans in a certain period of time, it is not necessarily the natural truth and certainly, not the law of everything. For example, on the question that *where is the center of universe?* Dr.Ouyang explains that the "*geocentrism*" of ancient Greek was not rejected until Copernicus proposed the "*heliocentric theory*" in the 15th century. However, the "*heliocentric theory*" was also found to be incorrect with the improving of spaceflight capability of humans. Certainly, we may never know exactly where the center of universe is. In fact, if the Big Bang theory is correct, the original position of the Big Bang is the center of universe but this is only a hypothesis on "*One*" of the "*Tao creates One*" in creation of the universe, whether it really is the center of universe can not be verified. After a little pause, Dr.Ouyang tells Huizi that there are famous words in the *Heart Sutra* of Buddhism classic, namely "*the color is empty, the empty is color*" refers to the limitation of human recognition on things, where the "*color*" is various appearances of things presented in front of humans. Certainly, the "*color*" is endowed by humans on things for recognition, not the nature of things. In this respect, the ancient Chinese philosophers had a deeper understanding on the law of everything than other nationalities or "*science*". They had a self-contained recognition on the creation, namely the "*Tao*" diagram in "*Tao creates One, One creates Two, Two creates Three and Three creates all things*", the "*name*" diagram of human recognition on all things and the relationship of reinforce each other in things.

**2.1.Law of Everything.** The ancient Chinese philosophers believed that the *law of everything* is "*Tao*". It is the "*Tao*" in the creation of "*Tao creates One*", i.e., the chaotic universe. It is the "*Tao*" that creates Two, creates Three and creates all things. And then in the state that "*every thing connotes Yin and Yang, achieves the balance*". So, *what is science?* Dr.Ouyang explains that science is the law of something or "*Tao*" shown on something discovered by humans, which is only the part or partial knowledge of everything's law. In other words, to find the law of everything can be seen as the ultimate goal of science. *Why do we say so?* Dr.Ouyang tells Huizi that science is to recognize the reality of things within the human ability, which is a partial recognition and its process for revealing the reality of things is an infinite recognitive process. Among them, one is "*all things*" in a large number, unknown things appear from time to time and humans just know things one by one. Its process tends to infinity; the other is the limitation of human's recognitive ability, which may not be able to perceive everything in the universe. And meanwhile, the humans' recognition is after "*Three creates all things*" in creation of all things. They did not experience "*Tao creates One, One creates Two, Two creates Three*" and deduced "*Tao, One, Two*" in the creation by hypothesis only. So, the scientific conclusions are not necessarily "*Tao*" but the local recognition of "*Tao*" by humans.

After listened to Dr.Ouyang's explanation, Huizi has some reflections on the philosophical significance of "*Tao*" in human recognition and asks Dr.Ouyang: "*Dad, what is Tao?*" Dr.Ouyang explains that it is difficult to give a precise definition of *Tao* except as the law of everything. "*Why is this so?*" Dr.Ouyang says that the humans can not hold on everything without language because language is the basis of human's expression for laws of everything. Generally, the humans need to define the connotation and extension of a thing, use language to give it an abstract "*name*" to distinguish it from other things. However, any language needs to be in a specific cultural background, a specific situation and other conditions to accurately express the meaning of words. Otherwise, it may be difficult to determine the exact meaning of words or a sentence and even cause the semantic misunderstanding to some extent. Accordingly, Laozi portrayed the recognitive process of humans in beginning of *Tao Te Ching* as "*Tao told is not the eternal Tao, Name named is not the eternal Name*", means that the "*Tao*" which can be expressed in language is definitely not the "*eternal Tao*", it is human's own recognition and a thing defined by "*name*" or language is not the thing itself but the definition of humans on this thing ourselves. And then, it explains the process of human recognition on things to be "*unnamable is the beginning of heaven and earth but naming is the mother of all particular things*". Here, the words of "*heaven, earth*" is generally refer to "*all things*" because all things are nameless in the creation of "*Tao creates One, One creates Two, Two creates Three and Three creates all things*", only according to the nature of "*Tao*" in reproduction. Certainly, it is the definition of all things by humans, i.e., to give "*names*" on things, to form the expression paradigm for recognition of things and then, know things. In this process, "*Tao*" plays the reproduction role of "*mother*".

So, *how should we know the "Tao" or law of everything?* Dr.Ouyang explains that since all things are the noumena bearing "*Tao*" and the humans can only perceive things within their ability, the recognition of "*Tao*" can be only enlightened by the characters of things. In fact, "*Tao*" is explained by Laozi himself with characters of things. For example, "*Tao*" is similized as "*Tao is likely invisible empty but use is plentiful. It is like an abyss but origins all things. Its sharpness is blunted, all tangles are united, all glares are tempered and all dust are smoothed. It is like a deep pool that never dries*" in Chapter 4 of *Tao Te Ching*, which is the enlightenment of "*Tao*" by "*Three creates all things*" in creation of the universe, means that "*Tao*" is hidden in all things. Although it is empty and disappeared but its role is as deep as the root or mother of all things; it takes in the spirit, eliminates the clutter, contains the light, mixed with dirt and is so faint as existing in trance. And then, "*Tao* " is imaged as "*water*" with words that "*the supreme good is like water which nourishes all things without trying to; it prefers the low places where no human like to stay; thus, it comes almost to Tao*" in Chapter 8 of *Tao Te Ching*, means that water nourishes all things, does not compete with other things, actively flows to and gathers in the lower place that most humans do not like, namely its nature is close to "*Tao*", etc. All are using the characters of "*things*" to similize the nature of "*Tao*".

Thus, "*Tao*" is implied in all things. However, all things are in constant moving. In such a case, the nature of "*Tao*" should includes "*moving*", namely "*moving around forever*" in words that "*it is limpid and silent, solitary without changing and moving around forever*" in Chapter 25 of *Tao Te Ching*, namely, "*Tao*" is manifested as a state of perpetual and unending moving of all things in the empty silence. Then, *what is the motivating power that pushes Tao to move around forever?* Dr.Ouyang explains that this is the words that "*return is the moving of Tao and weakness is the usefulness of Tao*" in Chapter 40 of *Tao Te Ching*", where the word "*return*" is the motivating power for "*moving around forever*", namely the Yin and Yang or the positive and negative powers included in a thing motivate its changing and developing. For example, it is the "*return*" that pushes forward the season changing of earth in the spring, summer, autumn and the winter such as shown in Figure 11.10. Certainly, it is the "*weak*" that shows the function of "*Tao*" because it is

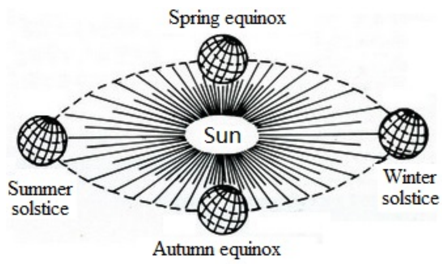



**Figure** 11.10. **Seasonal cycle**

explained that "*the weak gradually overcomes the strong*" in Chapter 36 of *Tao Te Ching*, namely "*the strong*" always wishes to maintain the status quo but "*the weak*" wants to change the status quo in thing, i.e., the motivating power that overcomes the "*strong*" to push the reformation of pattern between Yin and Yang or the weak and the strong in internal of thing. This motivating power shown in Tao of the heaven is explained by words that "*Tao of the heaven is reducing the excess with supply the insufficient*" in Chapter 77 of *Tao Te Ching*, where "*Tao of the heaven*" generally refers to the "*natural law*", namely the natural law is reducing the excess with supply the insufficient and then, to achieve the fairness of "*the heaven and earth are not kind, taking all things as humble dogs*" in order to maintain the harmony and symbiosis in things.

**2.2.Combinatorial Force Field.** Under the assumption that matter is distributed continuously, there are four fundamental forces between things known, namely the gravity, electromagnetic force, strong nuclear force and the weak nuclear force which are respectively reflected in the macrocosm world consisted of celestial bodies such as the solar system, milky way and the microcosm world consisted of elementary particles. Among them, the gravitational force discovered by Newton existing in all things, namely there is a gravitational force $\mathbf{F}$ effect that is proportional to $m_1m_2$ and inversely proportional to the square of distance $r$ between any two particles with respective masses $m_1, m_2$ but the transfer medium of gravity, namely the physical reality has not been found and have to attribute it to an action at a distance of particles. Dr.Ouyang tells Huizi that some scholars once identified it to the Aristotle's hypothetical "*ether*" by giving the "*ether*" with properties that "*massless, absolutely static*" and "*uniformly distributed*" in the universe as the transfer medium of gravity. However, there are no evidence that the scientific community agrees its existence. It was until Einstein explained the gravity as a kind of field effect of space bending that the nature of gravity and why the gravity is a kind of action at a distance in humans' eyes was answered. Then, *is the "ether" in eyes of ancient Greek philosophers the same as "thing" in words that "there is a mixed and perfect thing before the born of heaven and earth" of Laozi's creation?* The answer is Not! Dr.Ouyang explains that the "*thing*" in Laozi's creation is a kind of "*primitive matters*" that form all things in the process of "*Tao creates One, One creates Two, Two creates Three and Three creates all things*". It is a kind of physical reality in the "*One*" state of creation. However, the Aristotle's "*ether*" is an illusion that all over the space used by the ancient Greeks referring to the state of heaven or the upper atmosphere rather than "*primitive matters*" in the creation of universe. According to Laozi's creation, even if the "*ether*" exists it is included in "*all things*", which is not at the level as the "*thing*" in Laozi's "*mixed thing*".

(1) **Combined gravitational field.** Certainly, Newton's law of universal gravitation shows that there is a gravitational action between particles which is a natural law after "*Three creates all things*" in creation. In a Euclidean space $\mathbb{R}^3$, the gravitational interaction between particle $P$ with mass $m_1$ at point $\mathbf{x}$ and particle $P'$ with mass $m_2$ at point $\mathbf{y}$ is an interaction, as shown in Figure 11.11, where the distance of particles $P, P'$ is $r = |\mathbf{x}-\mathbf{y}| = |\mathbf{y}-\mathbf{x}|$. Then, the gravitational forces

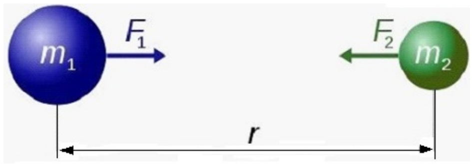


**Figure** 11.11. **Newtonian gravity**

$$F_1(\mathbf{y}) \ = \ -\frac{Gm_1m_2}{|\mathbf{y}-\mathbf{x}|^2}\mathbf{I_{xy}} \quad \text{and} \quad F_2(\mathbf{x}) \ = \ -\frac{Gm_1m_2}{|\mathbf{x}-\mathbf{y}|^2}\mathbf{I_{yx}}, \tag{11.5}$$

where $G = 6.673 \times 10^{-8} cm^3/gs^2$ is the gravitational constant, $\mathbf{I_{xy}}, \mathbf{I_{yx}}$ are respectively the unit vectors on directions $\overrightarrow{\mathbf{xy}}$ and $\overrightarrow{\mathbf{yx}}$, i.e., $\mathbf{I_{xy}} = (\mathbf{y}-\mathbf{x})/|\mathbf{y}-\mathbf{x}|$, $\mathbf{I_{yx}} = (\mathbf{x}-\mathbf{y})/|\mathbf{x}-\mathbf{y}|$.

Now that the relationship of gravitational potential $\mathcal{V}_1, \mathcal{V}_2$ with gravities $\mathbf{F}_1, \mathbf{F}_2$ are respectively

$$\mathbf{F}_1 = -\left(\frac{\partial \mathcal{V}_1(\mathbf{x})}{\partial x_1}, \frac{\partial \mathcal{V}_1(\mathbf{x})}{\partial x_2}, \frac{\partial \mathcal{V}_1(\mathbf{x})}{\partial x_3}\right), \quad \mathbf{F}_2 = -\left(\frac{\partial \mathcal{V}_2(\mathbf{x})}{\partial x_1}, \frac{\partial \mathcal{V}_2(\mathbf{x})}{\partial x_2}, \frac{\partial \mathcal{V}_2(\mathbf{x})}{\partial x_3}\right),$$

i.e., the gravitational potential

$$\mathcal{V}_1(\mathbf{x}) \ = \ -\frac{Gm_1}{|\mathbf{x}-\mathbf{y}|}, \quad \mathcal{V}_2(\mathbf{y}) \ = \ -\frac{Gm_2}{|\mathbf{x}-\mathbf{y}|}. \tag{11.6}$$

By the potentials formulae (11.6), the gravities $\mathbf{F}_1 = -\mathbf{F}_2$ between particles $P$ and $P'$ can be regarded as the action force of particle $P'$ located at point $\mathbf{y} \in \mathbb{R}^3$ in the gravitational field $\mathcal{V}_1(\mathbf{x})$ formed by particle $P$ and can be also regarded as the action force of particle $P$ located at point $\mathbf{x} \in \mathbb{R}^3$ in the gravitational field $\mathcal{V}_2(\mathbf{x})$ formed by particles $P'$. In other words, the particles $P$ and $P'$ form gravitational fields respectively with potential $\mathcal{V}_1(\mathbf{z})$ or $\mathcal{V}_2(\mathbf{z}), \mathbf{z} \in \mathbb{R}^3$ in space $\mathbb{R}^3$. Dr.Ouyang tells Huizi that the gravitational fields $\mathcal{V}_1$, $\mathcal{V}_2$ and their interactions formed respectively by particles $P$ and $P'$ can be naturally regarded as a binary system $K_2^L[\mathbf{x}]$, as shown in Figure 11.6. In this case, $L(v) = \mathcal{V}_1$, $L(u) = \mathcal{V}_2$ and $L(v,u) = \mathbf{F}_1$, $L(u,v) = \mathbf{F}_2$.

Generally, for any integer $n \geq 1$ assume that there are $n$ particles $P_1, P_2, \cdots, P_n$ with masses $m_1, m_2, \cdots, m_n$ located at $\mathbf{x}_1, \mathbf{x}_2, \cdots, \mathbf{x}_n$ in space $\mathbb{R}^3$, respectively and let $P$ be a particle with mass $M$ located at $\mathbf{x} \in \mathbb{R}^3$. So, *what is the gravities of particles $P_1, P_2, \cdots, P_n$ acting on particle $P$ by Newton's law of universal gravity?* Dr.Ouyang explains that if the Newtonian gravity is thought of as a real vector, let $\mathbf{F_{x_1x}}, \mathbf{F_{x_2x}}, \cdots, \mathbf{F_{x_nx}}$ be the gravities of particles $P_1, P_2, \cdots, P_n$ acting on particle $P$ are respectively. Then, by the linear superposition principle of vectors the gravities of particle $P_1, P_2, \cdots, P_n$ acting on particle $P$ is

$$\begin{aligned} F(\mathbf{x}) &= -\left(\mathbf{F}_{\mathbf{x_1 x}} + \mathbf{F}_{\mathbf{x_2 x}} + \cdots + \mathbf{F}_{\mathbf{x_n x}}\right) \\ &= -\sum_{i=1}^{n} \frac{GMm_i}{|\mathbf{x}_i - \mathbf{x}|^2}\mathbf{I}_{\mathbf{x_i x}} = -GM\sum_{i=1}^{n} \frac{m_i}{|\mathbf{x} - \mathbf{x}_i|^3}(\mathbf{x}_i - \mathbf{x}). \end{aligned} \tag{11.7}$$

where $\mathbf{I}_{\mathbf{x_i x}} = (\mathbf{x}_i - \mathbf{x})/|\mathbf{x}_i - \mathbf{x}|$ is the unit vector on direction $\overrightarrow{\mathbf{x}\mathbf{x_i}}$ for any integer $1 \leq i \leq n$ and it is similar to the case of two particles that the gravitational potential

$$\mathcal{V}(\mathbf{x}) = -G\sum_{i=1}^{n} \frac{m_i}{|\mathbf{x} - \mathbf{x}_i|}. \tag{11.8}$$

Notice that the particle model is an abstract model in conditions that the size or shape of an object does not matter, can be negligible or its components are all in synchronously moving. Otherwise, an object $P$ can not be abstracted to a particle. In this case, *can we establish a combined gravitational field on Newton's law of universal gravity for such an object $P$?* The answer is Yes! Dr.Ouyang explains that it is necessary to use the idea of differential and integral in calculus. Assume that the mass density $\rho(\mathbf{x}')$ of an object is continuously distributed in space. We subdivide the object $P$ with unit $U(\mathbf{x}')$ including the point $\mathbf{x}'$, i.e., $U(\mathbf{x}') \to \mathbf{x}'$ if $\mathrm{diam}U(\mathbf{x}') \to 0$. And so, we can approximately assume that $U(\mathbf{x}')$ is a particle at $\mathbf{x}'$ with mass $\rho(\mathbf{x}')^3 d\mathbf{x}'$ at a moment with $\mathrm{diam}U(\mathbf{x}') \to 0$, where $\mathrm{diam}U(\mathbf{x}')$ denotes the diameter of set $U(\mathbf{x}')$ or the maximum distance between two points in $U(\mathbf{x}')$. Then, it is similar to the derivation of formula (11.8), the continuous distribution of mass density of object $P$ forms a Newton's gravitational field with gravitational potential

$$\mathcal{V}(\mathbf{x}) = -G\int_P \frac{\rho(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|} d^3\mathbf{x}' \tag{11.9}$$

at point $\mathbf{x} \in P$.

Without loss of generality, let $P_1, P_2, \cdots, P_n$ be $n$ particles with respectively mass densities $\rho_1(\mathbf{x}), \rho_2(\mathbf{x}), \cdots, \rho_n(\mathbf{x})$ acting on a particle $P$ in general. By the formulae (11.8)−(11.9) of gravitational potential, the gravity $\mathbf{F}$ on $P$ can be regarded as a combined force of gravitational fields $\mathcal{V}_1(\mathbf{x}), \mathcal{V}_2(\mathbf{x}), \cdots, \mathcal{V}_n(\mathbf{x})$ formed respectively by objects $P_1, P_2, \cdots, P_n$ acting on $P$, i.e., a *combined gravitational field* or *Smarandache gravitational multifield* $K_n^L$. *How do the gravitational fields of $P_1, P_2, \cdots, P_n$ form a combined gravitational field $K_n^L$?* Dr.Ouyang explains that let $V(K_n) = \{v_1, v_2, \cdots, v_n\}$ be the vertex set of complete graph $K_n$ is. Define a labeling $L$ by

$$\begin{aligned} L: \quad v_i \quad &\to \quad \mathcal{V}_i(\mathbf{x}) = -G\int_{P_i} \frac{\rho_i(\mathbf{x}')}{|\mathbf{x}' - \mathbf{x}|} d^3\mathbf{x}', \\ L: \ (v_i, v_j) \quad &\to \quad \mathbf{F}_{v_i v_j} = G\int_{P_i}\int_{P_j} \frac{\rho_i(\mathbf{x}_i)\rho_i(\mathbf{x}_j)}{|\mathbf{x}_i - \mathbf{x}_j|^2} d^3\mathbf{x}_j d^3\mathbf{x}_i, \quad 1 \leq i, j \leq n \end{aligned}$$

then the labeling $L(v_i)$ on vertex $v_i$ is a gravitational field $\mathcal{V}_i(\mathbf{x})$ of object $P_i$, while the labeling $\mathbf{F}_{v_iv_j}$ on edge $(v_i, v_j)$ is gravity between objects $P_i, P_j$ or the action between gravitational fields $\mathcal{V}_i(\mathbf{x})$ and $\mathcal{V}_j(\mathbf{x})$ for integers $1 \leq i, j \leq n$. In the combined gravitational field $K_n^L$, the gravitational potential $\mathcal{V}(\mathbf{x})$ at $\mathbf{x}$ is determined by

$$\mathcal{V}(\mathbf{x}) \ = \ -G\sum_{i=1}^{n}\int_{P_i}\frac{\rho_i(\mathbf{x}')}{|\mathbf{x}'-\mathbf{x}|}d^3\mathbf{x}' \qquad (11.10)$$

for a point $\mathbf{x} \in \mathbb{R}^3$. Particularly, let $n \to \infty$. Then, the combined gravitational field $K_\infty^L$ is the gravitational field of universe determined by (11.10) or the Newtonian gravitational field. In this case, the gravity on a particle of mass $m$ in the combined gravitational field $K_\infty^L$ is $\mathbf{F}(\mathbf{x}) = m\mathcal{V}(\mathbf{x})$ and the gravity on an object $P$ with mass density $\rho(\mathbf{x})$ in the combined gravitational field $K_\infty^L$ is

$$\mathbf{F}(\mathbf{x}) = \int_P \rho(\mathbf{x})\mathcal{V}(\mathbf{x})d^3\mathbf{x} = -G\int_P \rho(\mathbf{x})\left(\lim_{n\to\infty}\sum_{i=1}^{n}\int_{P_i}\frac{\rho_i(\mathbf{x}')}{|\mathbf{x}'-\mathbf{x}|}d^3\mathbf{x}'\right)d^3\mathbf{x}. \qquad (11.11)$$

(2) **Einstein gravitational field.** Notice that Newton regarded the local action of gravity as a vector and then, we can analyze the gravity by the nature of "*force*" in universe, which is more convenient for application. However, he could not find the transfer medium finally, which was quite different from the daily experience of human on force and always felt that there was some deficiencies in Newtonian theory. However, *can we surely find the transfer medium for a force?* The answer is Not! Dr.Ouyang explains that it is easy to find a medium for common forces in life. For example, the upward force of a person carrying a burden is transferred to the rice through the pole and the rope to counteract the gravity of rice to complete the transporting in Figure 11.12. In this case, the gravity of rice is the earth's gravity to the rice but there is no intermediate transfer medium for the gravity. However, if we view the Newton's universal gravity as a field effect, namely assigning a conservative force field $\mathcal{V}(\mathbf{x})$ on every point $\mathbf{x} \in \mathbb{R}^3$ and get a weighted space $\mathbb{R}^3(\mathcal{V})$, which is a Smarandache geometry on Euclidean space $\mathbb{R}^3$, we can then explain this puzzle. Notice that the time $t$ in Newton's universal gravity exists independently of the space $\mathbb{R}^3$ but the Einstein's theory of special relativity asserts that the time $t$ can not exist independently of physical reality. We have to use a 4-dimensional Euclidean space $\mathbb{R}^4 = \{(t, x_1, x_2, x_3)|t, x_i \in \mathbb{R}, 1 \leq i \leq 3\}$ for characterizing the moving state of objects, where $t$ denotes the time and $x_1, x_2, x_3$ are the coordinates along the directions $\mathbf{e}_i, 1 \leq i \leq 3$

**Figure** 11.12. **Carrying a load**

in a coordinate system $\{O; \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$. Certainly, the essence of Einstein's gravitational field is to transform the field effect in Newton's local gravity into a geometric effect on a 4-dimensional Euclidean space $\mathbb{R}^4$ but endowed points $\mathbf{x} \in \mathbb{R}^4$ with gravitational potentials $\mathcal{V}(\mathbf{x})$ in $\mathbb{R}^4(\mathcal{V})$. In this case, since $\mathcal{V}(\mathbf{x})$ is endowed at any point $\mathbf{x}$, the geometry on $\mathbb{R}^4(\mathcal{V})$ is no longer a flat space but a curved one, i.e., a Smarandache geometry.

So, *how do we characterize the curved space $\mathbb{R}^4(\mathcal{V})$ geometrically?* Dr.Ouyang explains that first of all, the graph defined by the gravitational potential $\mathcal{V}(\mathbf{x})$ is $\Gamma[\mathbf{x}, \mathcal{V}(\mathbf{x})] = \left\{(\mathbf{x}, \mathcal{V}(\mathbf{x})|\mathbf{x} \in \mathbb{R}^4\right\}$, namely there is a neighborhood $U(\mathbf{x})$ for any point $\mathbf{x}$ homeomorphic to a 4-dimensional Euclidean space $\mathbb{R}^4$, which is a 4-dimensional manifold $M^4$ by definition. In this case, the gravity $\mathbf{F}(\mathbf{x})$ corresponds to the tangent vector at point $\mathbf{x} \in M^4$ of manifold $M^4$. By the relativity principle, namely an equation that describing the evolving law of an objective thing should have the same form in all coordinate systems, Einstein assumed that the curved space or Smarandache geometry $\mathbb{R}^4(\mathcal{V})$ is a Riemannian space with metric

$$ds^2 \ = \ g_{\mu\nu}(\mathbf{x})dx^\mu dx^\nu, \tag{11.12}$$

where $0 \leq \mu, \nu \leq 3, x^0 = t, x^1 = x_1, x^2 = x_2, x^3 = x_3$ and the Einstein's summation convention is used in (11.12), namely the summation should be proceed if the parameters $\mu, \nu$ with the same upper and lower scripts and meanwhile, $\mathcal{T}_{\mu\nu}$ is the energy momentum tensor of matter distributed in the gravitational field; secondly, the index that measures the degree of curving in geometry is called the *curvature*. Generally, the absolute ratio of tangent angle $\Delta\alpha$ of an arc $p'p$ with the corresponding arc length $\Delta s$, i.e., $|\Delta\alpha/\Delta s|$ is called the *mean curvature* $\overline{K}_{p'p}$ of arc $p'p$. In this case, if the limitation of average curvature $\lim\limits_{p' \to p} \overline{K}_{p'p}$ exists when $p' \to p$, it is called the *curvature* of curve at the point $p$, denoted by $K(p)$. Generally, the greater the curvature $K(p)$ is, the greater the curving of curve at point $p$. In this way, by the second order covariant tensor in Riemannian space, the *Ricci tensor* and *Ricci curvature scalar* are defined respectively by $R_{\mu\nu} = R^\alpha_{\mu\alpha\nu} = g^{\alpha\beta}R_{\alpha\mu\beta\nu}$ and $R = g^{\mu\nu}R_{\mu\nu}$ with the Einstein's summation convention, which can measures the deviation of space determined by Riemannian metric from that of Euclidean space and accordingly, Einstein proposed his famous equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \lambda g_{\mu\nu} = \kappa\mathcal{T}_{\mu\nu} \tag{11.13}$$

on gravitational field, where $\kappa = 8\pi G/c^4 = 2.08 \times 10^{-48}cm^{-1} \cdot g^{-1} \cdot s^2$ and $\lambda$ is called the *cosmological constant* with the term $\lambda g_{\mu\nu}$ introduced by Einstein in order to obtain a static universe. Usually, let $\lambda = 0$ and $\mathcal{T}_{\mu\nu} = \mathbf{0}$ correspond to the gravitational field in vacuum, i.e., the equation $R_{\mu\nu} - g_{\mu\nu}R/2 = 0$.

Now, let the Lagrangian of Einstein's gravitational field be $\mathcal{L} = \sqrt{-g}\,(L_G - 2\kappa L_F)$ with the action

$$J[\mathcal{L}] = \int \sqrt{-g}\,(L_G - 2\kappa L_F)\, d^4x, \tag{11.14}$$

and the energy momentum tensor

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\left\{\frac{\partial\sqrt{-g}L_F}{\partial g^{\mu\nu}} - \frac{\partial}{\partial x^\alpha}\left[\frac{\partial\sqrt{-g}L_F}{\partial g^{\mu\nu}_{,\alpha}}\right]\right\}, \tag{11.15}$$

where $L_G = R$, $L_F = L_F(g^{\mu\nu}, g^{\mu\nu}_{,\alpha})$ and $f_{,\alpha} = \partial/\partial x^\alpha$.

Notice that the variation of action $J[\mathcal{L}]$ is

$$\delta J[\mathcal{L}] = \int \sqrt{-g}\left(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R - \kappa T_{\mu\nu}\right)\delta g^{\mu\nu} d^4x$$

in this case, which implies that the principle of least action $\delta J[\mathcal{L}] = 0$ is the Einstein's gravitational equation (11.13) with the constant $\lambda = 0$.

So, *how do we solve the Einstein's gravitational equation* (11.13) *on human observing data in vacuum?* Dr.Ouyang explains that the equation (11.13) is a system of second order partial differential equations. It should be noted that the observing data on universe years show that under the condition in scale of 1 billion light years, i.e., the *cosmological hypothesis* that all points in the range of $9.46 \times 10^{20} km$ are equal to one point, different points in the universe and a point in different directions do not exist any different, namely they are uniform and isotropic at any time $t$. In this case, the spherical coordinate $(r, \theta, \phi)$ can be used to characterize the points and simplify the Riemannian metric (11.12) to be

$$d^2s = B(r)dt^2 - A(r)dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2, \tag{11.16}$$

where $A(r), B(r)$ are the undetermined functions which can be obtained by substituting them into the Einstein's gravitational equation (11.13), i.e.,

$$A(r) = \frac{1}{B(r)} = \frac{1}{1 - 2Gm/r}, \quad B(r) = 1 - \frac{2Gm}{r}. \tag{11.17}$$

Consequently, the spherically symmetric Riemannian metric (11.12) corresponding to the state of Einstein's gravitational field in vacuum is

$$ds^2 = \left(1 - \frac{2mG}{r}\right)dt^2 - \frac{dr^2}{1 - 2mG/r} - r^2d\theta^2 - r^2\sin^2\theta d\phi^2. \tag{11.18}$$

Notice that the hypothesis of uniformity and isotropy in cosmology implies that the space of matter distribution is a space with a constant curvature $K$. Accordingly, the cases of $K = 0, > 0$ or $< 0$ are respectively called the *Euclidian, ellipse* or *hyperbolic* spaces. Then, the Riemannian metric (11.12) can be expressed in case of spherical symmetry as

$$ds^2 = -c^2dt^2 + a^2(t)\left(\frac{d^2r}{1 - Kr^2} + r^2\left(d^2\theta + \sin^2\theta d\phi^2\right)\right), \tag{11.15}$$

called the *Robertson-Walker metric*, where $a(t)$ is an undetermined scale factor. Substituting $\lambda = 0$ into the Einstein's gravitational equation (11.13) in vacuum, the solution is called the *Friedmann universe* or *standard cosmology model* and the universe is in the *static, contracting* or *expanding* period for respectively $\dot{a}(t) = 0, < 0$ or $> 0$. We can then characterize and simulate the static, contracting and expanding periods in conditions.

At this moment, Huizi asks a question to Dr.Ouyang: "*Dad, now that Einstein's gravitational field is the same as Newton's gravitational field in $n \to \infty$ so, why do we study Einstein's gravitational field? Its symbols and partial differential equations seem very complicated!*" Dr.Ouyang tells Huizi that Newton's law of universal gravity is a local law on the gravity of all things. Certainly, the difference of calculated results by Newton's or Einstein's gravitational field is very small and can be negligible in the range of human activities, which is why humans move on the earth or airspace near the earth such as the launch of artificial satellites and probing into space are still applying Newton's law of universal gravity to calculate the launch speed because Newton's gravity characterizes the gravitational action between mass objects, is local but easy to calculate rather than the action case of all things. Different from the Newton's gravitational field characterizing the local behavior of gravity, Einstein's gravity characterizes the gravity of all things, i.e., "*gravitational field of all things*". Its importance lies in that under the gravity of all things, the Euclideian flat space perceived by humans on the earth or near the earth is only local, which does not exist in reality under the gravitational action of all things. Thus, everything moving is in a curved space or Smardache multispace that humans can not perceive on the earth, namely if we expect our activities beyond the earth or the milky way, we need to further hold on the law of moving in gravitational field of all things.

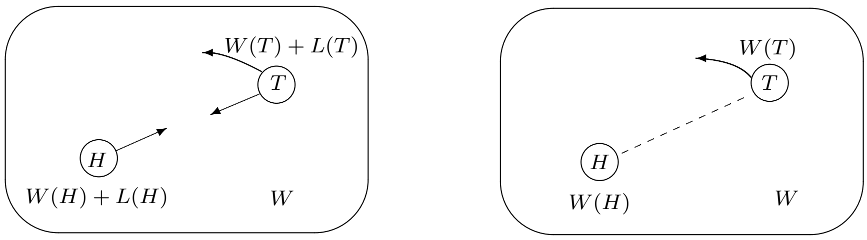
Hearing her father's explaining, Huizi has a little recognition on gravitational field and then asks Dr.Ouyang: "*Dad, can the standard universe model be used as a quantitative characterizing on the gravitational field of everything?*" Dr.Ouyang explains that Einstein's gravitational equation is Newton's gravitational field $\mathcal{V}(\mathbf{x})$ in case of $n \to \infty$, which is close to the state of "*One*" in Laozi's creation. However, the energy momentum tensor $\mathcal{T}_{\mu\nu}$ of distributed matter is an unknown quantity. So, the solution of Einstein's gravitational equation can not be obtained in the general case. And meanwhile, although the standard cosmological model can be obtained but it characterizes the creation of universe in state "*One*" under the conditions of cosmological assumptions and $\mathcal{T}_{\mu\nu} = \mathbf{0}$. In this case, the condition $\mathcal{T}_{\mu\nu} = \mathbf{0}$ actually assumes that the state of "*One*" in Laozi's creation is in vacuum without things or the energy momentum $\mathcal{T}$. Certainly, this is not consistent with Laozi's explaining that "*there is a mixed and perfect thing before the creation of heaven* *and earth*". So, *what is the energy momentum tensor $\mathcal{T}_{\mu\nu}$ in the creation of universe at the state "One'?* Dr.Ouyang explains that by the conservation law of energy, the energy momentum $\mathcal{T}$ should be a constant quantity. But, *can we measure $\mathcal{T}$ in the universe?* Surely, *how can we measure $\mathcal{T}$ of the universe in case of that we can even not determine the center or shape of universe?* This comes back to Laozi's assertion that "*Tao told is not the eternal Tao; Name named is not the eternal Name*" again, namely all talks on the universe are only the human recognition on universe, is not the true face of universe.

**(3) Human body field.** Certainly, a human body is a kind of advanced organism on the earth evolved in process that of elementary particles → atoms → biological macromolecules → human tissues and organs → human body, which is a biological complex field composed of multiple effects, including the observable and unobservable characters, where the observable characters refer to the field effects of human body field that can be detected by scientific instruments such as the electromagnetic field effect, electrostatic effect and the thermal effect generated by the biological current, electromagnetic materials and residual magnetic substances in human body, as shown in Figure 11.13; the unobservable characters refer to those of field effects that are not yet detectable by scientific instruments and partially related to the conscious guidance of brain such as the temperament and looks of a person, the sixth sense on an event and the Qigong of Chinese, i.e., an exhalation under the guidance of "*unity of humans with the heaven*" and information exchange between humans and the field of all things, a kind of mind-body practice of regulating body, breath and heart to maintain the moving of vital energy and blood in human body. Since it is human in recognizing things. The subject of recognition is human. And so, we can not avoid a fundamental question in recognition of things, namely "*can the human independent of all things in the universe?*" Dr.Ouyang explains that the human belongs to a part of all things in Laozi's "*Three creates all things*". That is, the human can not exist without all other things. Then, *why do we usually say that an objective law does not depend on the will of humans in philosophy? Do they seem contradictory?* The answer is Not! Among them, an "*objective law*" refers to the law of objective existence or Tao embodied on things that are not transferred by human's will, to emphasize that the existence of objective law is independent of the human's subjective consciousness.

**Figure** 11.13. **Body field**

Notice that the human recognition of things depends on the observing and measuring.

Different from the relationship between existence and consciousness, a more immediate question is "*can the human observes independently on a thing?*" Dr.Ouyang tells Huizi that since the human can not exist independently on all things, the observing behavior of human $H$ on a thing $T$ naturally forms a binary system $\{H, T\}$ or a binary field $K_2^L$, which is in case of existing the interactions of $H$ with $T$, as shown in Figure 11.14($a$).




($a$) Observing under the action of human field ($b$) Observing without the action of human field

**Figure** 11.14. **Observing under the action of body field**

Among them, $W(T), W(H)$ are the respective operating laws of the thing and human under the action of all things, $L(T), L(H)$ are the effect of human acting on thing $T$ or thing $T$ acting on human. Generally, the observing of human on thing $T$ is $W(T) + L(T)$ by the action of human body field. However, it is commonly believed that the observing on thing $T$ is $W(T)$ or the action of human body field on thing $T$ is $L(T) = \mathbf{0}$, namely the human observes independently on thing $T$, i.e., in situation shown in Figure 11.14($b$). So, *under what conditions hold with* $L(T) = \mathbf{0}$*?* Dr.Ouyang tells Huizi that there is a universal connection or interaction between all things by Tao, there must be $L(T) \neq \mathbf{0}$, i.e., there is no such a case that $L(T) = \mathbf{0}$ in theory. Hence, it exists only because the requirement of observing accuracy of human can let $L(T) \approx \mathbf{0}$. Generally, we use $|W(T)|/|L(T)| \geq 10^k$ to define the observing accuracy or requirement of accuracy degree, i.e., the effect of human body field on observing $T$ can be negligible if $|W(T)|/|L(T)| \geq 10^k$, where $k$ is a positive number set on observing accuracy by human beforehand. For example, let $k = 10$ and the weight or mass of a person is $80kg$ in observing of gravity. We know the earth's mass is $M = 5.975 \times 10^{24} kg$. Surely, the ratio of human gravity with that of the earth to other objects is $5.975 \times 10^{24}/80 = 7.4568 \times 10^{22}$, namely the human's gravity influence on other objects is much less than that of the earth, which can be negligible for $22 > k = 10$.

After listening to Dr.Ouyang's explanation on the interaction of binary fields, Huizi has an idea on the microscopic particles and asks Dr.Ouyang: "*Dad, are the uncertainty principle and the double slit experiment in quantum mechanics related to the fact that human observing with thing constitutes a binary system, which are affected by human body field acting on microscopic particles in observing?* Dr.Ouyang praises Huizi for her thinking of the field action and tells her that the gravity, electromagnetic force are long-range forces, the strong and weak nuclear forces are short-range forces in four fundamental forces. Under the same conditions, the strong nuclear force, electromagnetic force and the weak nuclear force are respectively $10^{38}, 10^{36}$ and $10^{32}$ times of the gravity, namely the effect of gravity on particles is far less than the strong nuclear force, electromagnetic force and the weak nuclear force, which can be ignored in microcosmic world. And meanwhile, the observing of human on particles constitutes a binary system. In this case, the effect of human body field on particles affects the observing results such as the effect of human's electromagnetic field acting on charged particles, i.e., the observing result on particles is $W(T) + L(T)$, which can not be ignored or let $L(T) \approx \mathbf{0}$. For example, the measuring of an electron velocity maybe $0.91m/s$ for the first time, $1.12m/s$ for the second time and $1.02m/s$ for the third time. This implies the principle of uncertainty in quantum mechanics that the position and momentum of a microcosmic particle can not be determined at the same time. Similarly, whether the observer is the reason that causes the interference or non-interference of light rays passing through the hole $S_1$ or $S_2$ can be explained in the double slit experiment. Of course, we still do not know what kind of the human body field is because science can not able to explain some abnormal effects in human body field until today. However, under the guidance of "*unity of humans with the heaven*" of ancient Chinese philosophy, the operation of human body is the miniature of universe. So, it is the same thing for knowing the operating of human body as that of the universe.

**2.3.Tao Following the Nature.** Then, *can the humans really know all things or science can hold on the laws or Tao of all things?* Dr.Ouyang tells Huizi that the answer is Not given by the ancient Chinese philosophy, asserted in "*Tao told is not the eternal Tao*" of Laozi because the human recognition on things or the nature is always a local recognition, i.e., what we get are all local laws shown on certain things in science and subject to the recognitive constraint of humans ourselves. Under such a circumstance, *how should the humans perceive Tao or the law of all things?* In this regard, Dr.Ouyang explains that Laozi's explanation that of "*humans follow the earth, the earth follows the heaven, the heaven follows Tao and Tao follows the nature*" in Chapter 25 of *Tao Te Ching* is an answer to this question and a guide to the developing of science, also a constraint on human's developing.

(1) **Science's developing.** Certainly, the purpose of science is to discover "*Tao*" or laws shown on things. Thus, "*humans follow the earth, the earth follows the heaven, the heaven follows Tao and Tao follows the nature*" is a realizable process of human recognition on the law of all things or "*Tao*", which should be the paradigm of science developing. But, *what are the words meaning?* Dr.Ouyang explains that first of all, we identify the

meanings of words of *earth, heaven, Tao* and *nature*. Among these, the words of "*heaven, earth*" are neither the "*Yin, Yang*" in Laozi's creation theory, nor the "*heaven, ground*" in "*the Yang was clear, lighter upward forming the heaven and the Yin was heavy, turbidness falling down to the ground*" of Pangu split open the chaos with an axe. Otherwise, the heaven and earth as the equal product of "*One creates Two*", can not logically explain why it is the "*the earth follows the heaven*" but not "*the heaven follows the earth*", namely there is no order in the heaven and the earth, not conform to the evolving of all things, i.e., "*Tao creates One, One creates Two, Two creates Three and Three creates all things*". Notice that it is emphasized that the subject of creating is Tao, i.e., it is that Tao creates One, Tao creates Two, Tao creates Three and Tao creates all things, where "*One, Two, Three*" and "*all things*" are only carriers of Tao, emphasizing that Tao can be only enlightened by laws shown on things in the universe; "*follow*" means that the imitating, also restraining behavior of humans and "*nature*" refers to the nature or the nature of all things. Thus, the words that "*humans follow the earth, the earth follows the heaven, the heaven follows Tao and Tao follows the nature*" mean that the humans should imitate "*Tao*" of things on the earth and conform to the earth; things on the earth should imitate "*Tao*" of things on the heaven and conform to the heaven; things on the heaven should imitate "*Tao*" and conform to "*Tao*", i.e., "*Tao*" is the law embodied on all things or the nature of universe.

Secondly, for holding on law of all things the humans should establish the knowledge system of things from "*Tao*" of things on the earth to "*Tao*" on all things because the humans live on the earth, subject to the constraints of "*Tao*" of things on the earth. In other words, the human appears on the earth because there are suitable conditions or "*Tao*" for human existence, namely if we know "*Tao*" of things on the earth, we can then realize "*Tao*" related to human's survival and know the laws that humans should conform to, i.e., the "*humanity*" because "*humans follow the earth, the earth follows the heaven, the heaven follows Tao and Tao follows the nature*". Notice that "*Tao*" of things on the earth is the "*Tao*" that can be perceived within the human ability, which is the embodiment of "*Tao*" attached to things on the earth. In such a thinking, the science should be first the recognition of "*Tao*" of things on the earth, namely holding on the law of things on earth to serve humans and then, use "*Tao*" of things on the earth to understand locally the "*Tao*" outside the earth, the heaven, universe and the nature of all things, which is the "*Tao*" of science's developing, i.e., a recognitive way that of humans ourself $\rightarrow$ "*Tao*" of things on the earth $\rightarrow$ "*Tao*" of thing on the heaven $\rightarrow$ enlightening "*Tao*" of all things. Certainly, it is a gradual and long, even an infinite process because Laozi asserted that "*Tao told is not the eternal Tao*" in Chapter 1 of *Tao Te Ching*, including the science.

**(2) Human's self-discipline.** Certainly, there is a fundamental question in establishing process of science through the law of things, namely "*what is the purpose of science? is it let humans to rule all things?*" The answer is certainly Not! Dr.Ouyang explains that the purpose of science is to enable humans to live in harmony with the nature and act in accordance with the "*humanity*" or "*Te*" following Tao, i.e., "*Tao always remains in inaction, but it acts on everything*" in Chapter 37 of *Tao Te Ching*, which includes two sides shown in binary system $K_2^L$ of the "*unity of humans with the heaven*". Firstly, the purpose of establish science according to Tao on things within our ability is for humans' ourselves "*multiply in an endless succession*". Indeed, Tao gives humans the recognitive ability on Tao with restriction, i.e., science is only a local or partial "*Tao*" recognized by humans or the words that "*Tao told is not the eternal Tao*" as Laozi asserted. In this case, the humans can only conform to Tao and live harmoniously with other things rather than govern them. Such a kind of behavior is usually called "*Te*" and accordingly, the behavior of humans follows "*Tao*" and harmoniously lives with other things is called "*inaction*" by Laozi. Secondly, it is a symbiosis with other things that humans conform to Tao. Otherwise, if humans' behavior deviates from "*Tao*", all things will adjust themselves according to Tao so as to adapt to humans' deviation and return to the harmony of all things. It should be noted that the self-adjustment of other things is usually accompanied with the natural disaster in the eyes of human if the adjustment accumulates to a certain extent.

Then, *how do we discipline ourselves to live in harmony with other things?* Dr.Ouyang explains that it is necessary to further understand the meaning of "*humans follow Tao*" in words that "*humans follow the earth, the earth follows the heaven, the heaven follows Tao and Tao follows the nature*" under the notion of "*unity of humans with the heaven*". That is to say, while we establish science through "*Tao*" of things on the earth, humans need to discipline ourselves, abide, imitate and then follow Tao of things on the earth, on the heaven and all things in the universe. *How do we self-discipline?* It was answered by Laozi in his *Tao Te Ching.* For example, the words that "*the five colors blind the eye, the five tones deafen the ear, the five flavors numb the taste, the riding and hunting madden the human, the rare goods misbehave the human; that is why the sage only eats with enough food, not satisfy his eyes with a riot of colours*" in Chapter 12 of Laozi's *Tao Te Ching.* Certainly, all of these words are cautioned to humans that they should be self-disciplined on color, tone, flavor, goods, benefit and other prejudices or preferences on things, namely the behavior of humans following Tao should be self-disciplined on the "*five colors*", "*five tones*", "*five flavors*" and other prejudices or preferences to avoid them becoming blind, deaf, numbing, maddening or misbehaving on things that deviates Tao or the humanity, should give up the "*greed*" in their subjective thought and abandon in the pursuing of bright or flashy only on surface too, which are all needing the self-restraint of humans by further understanding the "*humanity*" or conforming to "*Tao*". Historically, most of natural disasters such as the earthquake, tsunami, hurricane, flood, mud slide and the infectious disease that resulted a great loss of mankind were caused by humans against "*Tao*" even though they looked like natural but were actually *man-made*. Certainly, to avoid or reduce the natural disasters is in activities of humans following "*Tao*", i.e., with "*Te*" of harmony and symbiosis with the nature, which is the essence of "*unity of humans with the heaven*" in ancient Chinese philosophy.

## §3. Meridians on Human Body

Certainly, the Yin and Yang theory, i.e., *the union of a Yin and a Yang is a kind of Tao* and *one falls but another rises*" between the Yin and Yang powers implied in Taiji diagram is the motive power from the birth to grave or a spiral evolving of thing, namely the mutual promoting and restraining of "*five elements*", i.e., "*the metal, wood, water, fire and the earth*" which correspond to five planets of solar system in ancient Chinese philosophy. In this case, the "*promoting*" means that there exists a cyclicly promoting power in the five elements, namely the metal promotes the water, the water promotes the wood, the wood promotes the fire, the fire promotes the earth and the earth promotes the metal; the "*restraining*" means that there exists a cyclicly restraining power in the five elements, namely the metal restrains the wood, the wood restrains the earth, the earth restrains the the water, the water restrains the fire and the fire restrains the metal. It should be noted that the mutual promoting and restraining of thing are the two indispensable powers in the evolving of thing. And meanwhile, it is precisely the mutual promoting and restraining in thing that maintains the harmony and coexistence of all things, i.e., "*Tao of the heaven is reducing the excess with supply the insufficient*" in Chapter 77 of Laozi's *Tao Te Ching*, which needs to be followed by humans, namely "*who can offer his excess to the insufficient? Only the human follows Tao*". Notice that the human body is a condensed version of the heaven and earth or the universe in ancient Chinese philosophy, i.e., the human body itself follows "*Tao*". It should follow "*Tao*" of the heaven to maintain the living of human, i.e., "*reducing the excess with supply the insufficient*" to maintain the balance of Yin and Yang in human body, which is the philosophical throught of traditional Chinese medicine, namely the emperor's words that "*Yin and Yang are Tao of the heaven and earth, which are the order and law, parents of thing's evolving, beginning of living and* *death, and the gods' house of all things. Thus, a treatment must find out the primitive cause of disease*" in the chapter of *Presentation of Yin and Yang* in *Emperor's Inner Canon - Plain Questions*. Here, the words that "*treatment must find out the primitive cause of disease*" implies that a treatment should return to the balance of Yin and Yang of human body, i.e., come back Tao of the heaven.

At this time, a question comes to Huizi's mind. She asks Dr.Ouyang: "*Dad, the periodic table lists* 118 *elements and maybe new elements of matter added. Is it too little for the ancient Chinese to characterize the state and behavior of things only with the interactions in elements of metal, wood, water, fire and earth? Is such a characterizing on the state or behavior of thing too general with a mysterious feeling?*" Dr.Ouyang then asks her back: "*Does the system recognize that the more elements chosen and the more finely subdivided on thing are, the better to hold on the state or behavior of thing?*" Huizi replies her father: "*Is it better to hold on the details of state and behavior of thing if it is subdivided into finer details?*" Dr.Ouyang tells her that this recognitive view is not necessarily correct in Chinese. The answer of "*Yes*" or "*Not*" to this question reflects exactly the difference of the western and Chinese cultures. Certainly, the western culture recognizes a thing with emphasis on the subtlety and precision, which brings about "*science*" in western. However, the Chinese culture believes that there should be a moderation on everything for the objective, namely the excess or deficiency will cause a thing to develop in the opposite direction, i.e., it is suitable to meet the needs, should not overdo. And so, the Chinese does not advocate the way of infinite subdivision on a thing for recognition. For example, the words that "*cutting off half on a stick of length one foot everyday, it will never be exhausted*" in Chapter *Tian Xia* of *Zhuang Zi* are explaining such a subdivision may be infinite. Clearly, the remaining lengthes of the stick form a number sequence $\left\{\frac{1}{2^n}\right\}_1^\infty$, as shown in Figure 11.15, which is certainly not the way that humans perceive things should take. Different from the western science, the ancient Chinese used a systematic way to perceive things, i.e., systematically held on the state and behavior of things by the evolving of Yin and Yang with promoting and restraining of the five elements, which is the "*system science*" by today's terminologies in western science. Meanwhile, this also indicates that the systematic perceiving of thing $T$ does not always need to decompose $T$ into elementary particles or cells because the subdivision of $T$ into elementary particles or cells does not

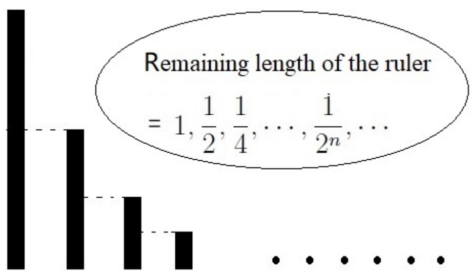



**Figure** 11.15. **Infinite subdividition**

necessarily benefit and even increases the complexity for recognition of thing $T$ in some cases. This is also the reason why the recognitive unit is called "*element*", not "*elementary particle*" or "*cell*" in systematic recognition. Such a thinking is especially prominent in the traditional Chinese medicine that simulates the human body by 12 meridians, Ren and Du meridians with the vital energy and blood running on them instead of $10^{14}$ cells of human body, applies the 362 acupoints consisting of 310 on 12 meridians, 52 on Ren and Du meridians to regulate the balance of Yin and Yang on patient in order to achieve the human rehabilitation, which is exactly the thought of "*harmony of humans with the heaven*". Among them, if we abstract the 362 acupoints to vertices, all roads of the vital energy and blood running from one acupoint to another on meridians of human body to edges, the result is nothing else but exactly a harmonic flow $G_h^{L^2}$ of vertices 362. Now, let $\widehat{G}_{cell}$ be the network of human cells. Then, $|\widehat{G}_{cell}|/|G_h^{L^2}| \sim 10^{12}$, namely $G_h^{L^2}$ is less 12 orders of magnitude than the network $\widehat{G}_{cell}$ of human cells.

**3.1.Representation on Human Body.** As one of things in "*Three creates all things*", the human body naturally follows Tao. Otherwise, Tao would not creates the human in "*Three creates all things*". In other words, the presence of human is a result of nature which includes also the "*anthropic principle*" in opposition and the science can be established by the running of human body. And meanwhile, we can recognize the "*humanity*" by Tao of the heaven or nature of all things because "*humans follow the earth, the earth follows the heaven, the heaven follows Tao and Tao follows the nature*". Dr.Ouyang tells Huizi that the traditional Chinese medicine is just on such a philosophy shown in the *Emperor's Inne Canon - Plain Questions* and *Emperor's Inner Canon - Lingshu Canon*, i.e., regulates the moving of vital energy and blood in human body according to Tao of the heavenly so as to recover the balance of Yin and Yang in body of the patient.

(1) **Presenting of body with heaven and earth.** Certainly, there is a natural corresponding between the presenting of human body and the operation of heaven and earth, which is characterized in a passage of Bo Gao's answering on a question of the emperor in Chapter *Xie Ke* of *Emperor's Inner Canon - Lingshu Canon*, i.e., the words that "*The round heaven and cubic earth correspond to the round head and cubic foot of human; the sun and moon in the heaven correspond to two eyes of human; the nine states on the earth correspond to the nine orifices of human; the wind and rain correspond to the joy and anger of human. The thunder and lightning correspond to the tone and sound of human; the four seasons of year correspond to the four limbs of human; the five tones of heaven correspond to the five internal organs of human; the six temperaments of heaven correspond to the six hollow organs of human; the winter and summer correspond to the*

*cold and heat of human; the ten stems of heaven correspond to the ten fingers of human; the twelve branches of earth correspond to the ten fingers of feet and the cock, balls of human; a women has no the cock and balls but can be pregnant for making up; the Yin and Yang of heaven correspond to the husband and wife; the 365 days of year correspond to 365 acupuncture points on human; the high mountain on earth correspond to the shoulders and knees; the deep canyons on earth correspond to the axilla and popliteal space; the twelve channels of water on earth correspond to the twelve meridians of human; the spring flow on earth corresponds to the defense energy of human; the grass on earth corresponds to the hair on head of human; the day and night of heaven correspond to the getting up and going to bed; the stars correspond to the teeth of human; the hills on earth correspond to the humping joints of human; the rocks on earth correspond to the high bones; the trees on earth correspond to the tendons of human; the thing gathering places on earth correspond to the flesh places on human body. The* 12 *months of year correspond to the* 12 *bone joints of human; the barren land without grass in four seasons on earth correspond to the childless of human. All of these are the counterparts of human body with the heaven and earth*", which explain the operating of human body corresponding to the heaven and earth.

(2) **Running of body with heaven.** All the explanations in the *Emperor's Inner Canon-Plain Questions* and *Emperor's Inner Canon-Lingshu Canon* are the corresponding of running of human body with the heaven and recuperating the balance of Yin and Yang in human body. For example, Qi Bo's interpreted on the running of vital energy in human body, i.e., the words that "*A cyclicly running in the heaven will passes through 28 mansions. Subdividing the space of two successively mansions into 36 equal parts, there are 1008 degree in the cycle. Correspondingly, a cyclicly running of vital energy on meridians of human body will need* 1008 *degrees, i.e., it will travels 28 mansions everyday. There are 28 meridians distributed in the upper and lower, left and right, front and back of human body; the length of* 28 *meridians on a whole body is* $16\times 3.3333+2\times 0.3333m$, *corresponding exactly to 28 mansions of a cyclicly running on the heaven. Let the leakage drop of water with 100 scales be standard of measuring the day and night, also the standard on running of the vital energy. A human exhales once, the pulse will beat twice, the vital energy will run* $3\times 0.0333m$ *and inhales once, the pulse with also beat twice, the vital energy runs* $3\times 0.0333m$ *and completes a breath, i.e., one exhaling and one inhaling, the vital energy runs* $6\times 0.0333m$. *Clearly, the vital energy runs* $6\times 0.333m$ *in 10 breathes passing through 2 degrees on the cycle of heaven,* $16\times 3.3333+2\times 0.3333m$ *in 270 breathes, and completes a cyclicly running on meridians of the human body with water dropping 2 scales and once again on the human body in 540 breathes with water dropping 4 scales, passing through 40*

*degrees on the cycle of heaven. Similarly, the vital energy runs 10 times on meridians of human body in 2700 breathes with water dropping 20 scales, passing through 5 mansions and 20 degrees, and 50 times on the human body in 13500 breathes with water dropping 100 scales and passing through all of 28 mansions. At this time, the water drips out, which is exactly the same as a day. Thus, a human is healthy without disease as long as the pulse and vital energy can run constantly on meridians day and night for 50 times and then, such a human will live with the heaven and earth. In such a case, the length of vital energy running is* $810 \times 3.3333m$ *in total*" are explaining the corresponding between the running of vital energy on human body with 28 mansions, day and night. Furthermore, the words that "*liver governs body in spring,* $\cdots$*; heart governs body in summer,* $\cdots$" and "*if the disease appears in liver, the patient should be recovered in summer. If not, it will become heavier, even to death in autumn. Otherwise, it will go on winter,* $\cdots$" in Chapter *Time Theory of Zang Qi* by Qi Bo are explaining the corresponding of vital energy running on the internal organs of human with the seasons of year and the evolving pattern of disease.

(3) **Body with five elements.** The ancient Chinese philosophy held on that all things are composed of "*five elements*", the mutual promoting and restraining with the "*five elements*" is a manifestation of "*Tao*". Thus, there exists a natural corresponding in the vital organs of human and the five elements shown in Chapters *Jin Gui Zhen Yan*, *Presentation of Yin and Yang* of *Emperor's Inner Canon-Plain Questions*, which is based on the inner moving, state or identity of appearance of things to identify the common characters with interactions of the same moving over a Petersen graph, as shown in Figure 11.16, namely they are classified according to the "*metal, wood, water, fire and earth*", i.e., {metal} $\leftrightarrow$ {lung, large intestine }, {wood{$\leftrightarrow$ {liver, gallbladder }, {water} $\leftrightarrow$ {kidney, bladder }, {fire} $\leftrightarrow$ {heart, small intestine} and {earth} $\leftrightarrow$ {spleen, pancreas, stomach}. For example, the promoting in five elements that the metal promotes the water, the water promotes the wood, the wood promotes the fire, the fire promotes the earth and the earth promotes the metal corresponds to the vital organs of human, i.e., the strong of lung, large intestine in function $\to$ the strong of kidney, bladder in function, the strong of kidney, bladder in function $\to$ the strong of liver, gallbladder in function, the strong of liver, gallbladder in function $\to$ the strong of heart, small intestine in function, the strong of the heart and small intestine in

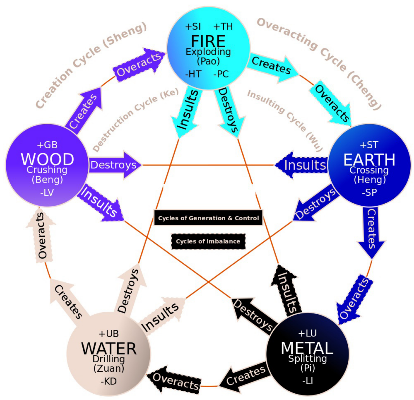
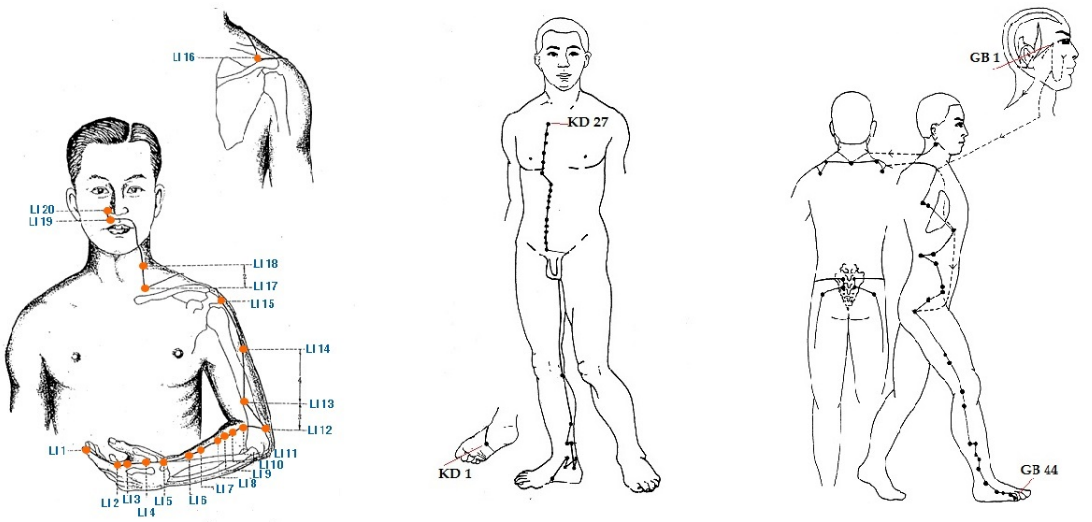



Figure 11.16. **Vital organs**

function → the strong of spleen, pancreas and stomach in function; the strong of spleen, pancreas and stomach in function → the strong of lung and large intestine in function. Similarly, the restraining in five elements that the metal restrains the wood, the wood restrains the earth, the earth restrains the the water, the water restrains the fire and the fire restrains the metal corresponds to the vital organs of human, i.e., the weak of lung, large intestine in function → the weak of liver, gallbladder in function, the weak of liver, gallbladder in function → the weak of spleen, pancreas and stomach in function, the weak of spleen, pancreas and stomach in function → the weak of kidney, bladder in function, the weak of kidney, bladder in function → the weak heart, small intestine in function, the weak of heart, small intestine in function → the weak of lung, large intestine in function.
**3.2.Twelve Meridians of Human Body.** Certainly, the Chinese medicine is guided by the philosophical thought that human body follows "*Tao*" of the heaven, which not only establishes the corresponding on presenting of human body with the operating of heaven and earth, but also discover deeply the essence of human's living is in the vital energy running on the twelve meridians of human body following Tao of the heaven and its corresponding with diseases.




Hand -Yangming Meridian (LI) Foot - Shaoyin Meridian (KI) Foot - Shaoyang Meridian (GB)

**Figure** 11.17. **Examples of meridians on human body**

(1) **Distribution of twelve meridians.** The twelve meridians of human body are distributed in the inner and outer parts of the limbs, corresponding to the vital organs of human body. Generally, the three Hand-Yin meridians distributed in the inner part of upper limb are respectively the lung meridian of Hand-Taiyin (LU), heart meridian of Hand-Shaoyin (HT) and the pericardium meridian of Hand-Jueyin (PC); the three

Foot-Yin meridians distributed in the inner part of lower limb are respectively the s-pleen meridian of Foot-Taiyin (SP), kidney Meridian of Foot-Shaoyin (KI) and the liver meridian of Foot-Jueyin (LR); the three Hand-Yang meridians distributed in the outer side of upper limb are respectively the large intestine meridian of Hand-Yangming (LI), small intestine meridian of Hand-Taiyang (SI) and the Sanjiao (triple burner) meridian of Hand-Shaoyang (SJ); the three Foot-Yang meridians distributed in the outer side of the lower limb are respectively the stomach meridian of Foot-Yangming (ST), bladder meridian of FootTaiyang (BL), gallbladder meridian of Foot-Shaoyang (GB). All of them have the functions of running the vital energy, connecting the internal and external organs, communicating the up and down, etc., as shown in Figure 11.17 of three meridians.

Notice that the vital energy on meridians is usually portrayed by terminologies of "*Qi*" and "*Xue*" in traditional Chinese medicine, which are not the usual gas coming out of mouth or blood seen by human eyes but the professional terms describing the running of vital energy in human body. Certainly, "*Qi*" is classified into "*Ying Qi*" and "*Wei Qi*" respectively running in or outside the meridians, i.e., a kind of energy that runs along meridians in the human body, namely the words that "*a human accepts the vital energy in grain, the grain is eaten into the stomach, passes in the internal organs and then, all internal organs receive the vital energy: the clearness consists of Ying Qi running in meridians, the turbidness consists of Wei Qi running outside meridians, both cyclicly run on meridians of human body without endless*" in Chapter *Generating with Converge of Ying Qi and Wei Qi* of *Emperor's Inner Canon-Lingshu Canon* and "*Xue*" is the material embodiment of Ying Qi and Wei Qi, i.e., the words that "*Ying Qi and Wei Qi constitute of the essence of Qi; Xue is the spirit of Qi. Both of them are the same name of vital energy*". Indeed, it is the constantly moving of vital energy on meridians of human body that holds on the function of human, makes it being a "*human*" and nothing else. So, "*what are an acupoint?*" Dr.Ouyang tells Huizi that an acupoint also known as the *acupuncture point*", is a very sensitive point on meridian and will change the running state of vital energy on meridians if it is stimulated. For example, there are $2 \times 20$ acupoints on the large intestine meridian of Hand-Yangming, $2 \times 27$ acupoints on the kidney meridian of Foot-Shaoyin and $2 \times 44$ acupoints on the gallbladder meridian of Foot-Shaoyang in Figure 11.17, etc.

(2) **Running of vital energy on twelve meridians.** Generally, the vital energy on each of the twelve meridians of human body has a fixed rule of running. For example, the large intestine of the Hand-Yangming in Figure 11.17 originates from the radial tip of index finger, goes up along the radial side of index finger, passes between the first and second metacarpal bones, enters between the longus and brevis extensor tendons, goes up

along the lateral front of the human upper limb to the shoulder, passes behind the shoulder to the spinous process of seventh cervical spine at the back of neck and goes down into the supraclaricular fossa, connects to the lung and goes down through the diaphragm which belongs to the large intestine.

So, *how does the vital energy work on the twelve meridians of human body as a whole?* Dr.Ouyang explains that the moving of vital energy in the twelve meridians of human body follows rule that the three Hand-Yin meridians from the chest to the hand, the three Hand-Yang meridians from the hand to the head, the three Foot-Yang meridians from the head to the foot, the three Foot-Yin meridians from the foot to the chest, i.e., initially from the lung meridian to the liver meridian, then one by one and finally, coming back the lung meridian with times, as shown in Figure 11.18, i.e., the lung meridian of Hand-Taiyin → the large intestine meridian of Hand-Yangming → the stomach meridian of Foot-Yangming → the spleen meridian of Foot-Taiyin → the heart meridian of Hand-Shaoyin → the intestine meridian of Hand-Taiyang → the bladder meridian of Foot-Taiyang → the kidney meridian of Foot-Shaoyin → the pericardium meridian of Hand-Jueyin → the triple burner meridian of Hand-Shaoyang → the gallbladder of Foot-Shaoyang → the liver meridian of Foot-Jueyin → the lung meridian of Hand-Taiyin. Whence, "*humans follow Tao*" is in form of the vital energy running on the twelve meridians of human body repeatedly according to the 24 hours formed by "*Tao*" of the haven and earth, to maintain the living of human body.

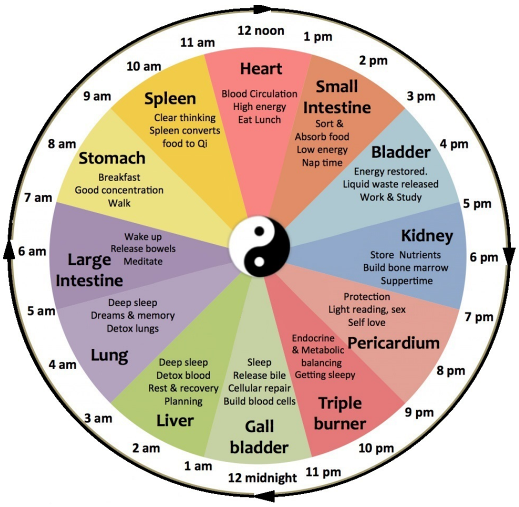



**Figure** 11.18. **Running of vital energy**

(3) **Balance of Yin and Yang on twelve meridians.** As a product of Tao, the running of vital energy on the twelve meridians of human body is in accordance with Tao of the heaven, i.e., an embodiment of "*the union of a Yin and a Yang is a kind of Tao*" or the balance of Yin and Yang in running. In this way, to maintain the balance of Yin and Yang in the moving of vital energy, i.e., the balance of body's Yang Qi and Yin essence is the basis of a health human. Conversely, a human is sicked either by the external factors of *wind, cold, heat, wet, dryness* and *fire*, called the "*six Yin*" in traditional Chinese medicine or by the seven emotions of "*joy, anger, worry, thinking, sadness, fear, shock*", which breaks the balance of Yin and Yang in human body, explained in Chapter

*Presentation of Yin and Yang* of *Emperor's Inner Canon-Plain Questions*, namely the emperor's words that "*the furious hurts Yin, the overjoyed hurts Yang, the rebellion of Qi ascending, $\cdots$. So, the overflowing of Yin will transform to Yang, the overflowing of Yang will transform to Yin*". And so, the traditional Chinese medicine adopts a method of "*syndrome differentiation and treatment*" to regulate the balance of Yin and Yang in human body, i.e., identifying the cause, nature, location, meridians, the deviation of Yin and Yang by *observing, smelling, inquiring and feeling the pulse*" and then, selecting the use of drug, acupuncture, cupping or comprehensive treatment.

Notice that the syndrome differentiation and treatment of traditional Chinese medicine is to treat the patients in order to realize the dialectical purpose of "*healing the wounded and rescuing the dying*". Then, *how does the traditional Chinese medicine treat patients?* Dr.Ouyang tells Huizi that the traditional Chinese medicine strictly follows Tao of the heaven, i.e., *reducing the excess with supply the insufficient*", namely to reduce the excess of Yang Qi and Yin essence in human body and replenish the less side so as returning to the balance of Yin and Yang of human body. For achieving this goal, the Chinese medicine applies two methods for the body back to the balance of Yin and Yang, i.e., using the Chinese medicine or acupuncture. Among them, the Chinese medicine is mainly on natural plants, supplemented by animals and minerals. Its essence lies in aiming at the disease to integrate the medicinal properties of medicinal materials to implement "*reducing Yang with supply Yin*" or "*reducing Yin with supply Yang*" on the sicked body, so that it can return to the balance of Yin and Yang; the acupuncture is a general term of the acupuncture and moxibustion, is an amazing method to return the vital energy on twelve meridians to the balance of Yin and Yang in case of the absence of applicable medicines or directly, as shown in Figure 11.19, i.e., the words of emperor asked Qi Bo in Chapter *Nine Needles and Twelve Reasons* of *Emperor's Inner Canon-Lingshu Canon*, namely the words that "*I would like to heal the sick without poisons and needle stone by using tiny needles to regulate the vital energy on meridians, catering to its direction and opposition, $\cdots$. I like to hear how to do*", imply that the emperor would like to protect his humans from the harms of medicines and stone needle, required on how to use the tiny needles on the skin of human body to dredge the meridians and regulate the harmony of Yin and Yang on meridians so that the vital energy runs smoothly in the human body. Qi Bo replied his question


**Figure** 11.19. **Acupuncture**

by words that "*Along with the past is in direction and the toward is in opposition. Thus, knowing the running direction of vital energy on meridians is necessary for using tiny needles. A needling in the opposition of vital energy is reducing the excess and along with the direction is supplying the insufficient. This is the essence of needling*", imply that the pulse on an acupoint is weak and small or peace and smooth according to the vital energy already passed or will pass. In this way, by the direction of vital energy on meridians the needle against to the direction of vital energy is contrary to its coming trend, which is in reducing its excess and it will become the balance from the excess. Conversely, along with the direction of vital energy the needle is to replenish its deficiency and it will become the balance form the insufficient, to realize "*reducing the excess with supply the insufficient*" and return to the balance of Yin and Yang on human body.

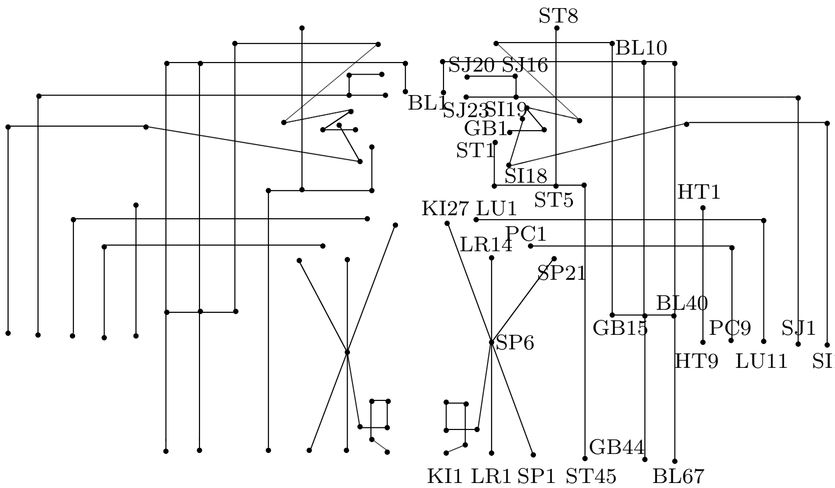



**Figure** 11.20. **Harmonic flow** $G^{L^2}$ **of twelve meridians**

**3.3.Harmonic Flow Model of Human Body.** According to the theory of traditional Chinese medicine, the moving of vital energy on the twelve meridians of human body corresponds to the living of human. Dr.Ouyang tells Huizi that the moving of vital energy on the twelve meridians of human body can be abstracted as a harmonic flow $G^{L^2}[t]$, as shown in Figure 11.20, namely let the acupoints on the twelve meridians of human body be vertices and let the sub-meridians between two successive acupoints be edges to form a graph $G$ with a labeling mapping $L^2 : v \in V(G) \to L(v) + iL'(v)$, $L : (v,u) \in E(G) \to L(p) + iL'(p)$, where $p$ is a point on a meridian from acupoints $v$ to $u$, $L(p), L'(p)$ are respectively the Yang Qi and the Yin essence at point $p$. According to the traditional Chinese medicine, the Yang qi and Yin essence naturally follow the conservation law of continuity flow at vertex $v \in V(G)$ on each of twelve meridians on human body.

**(1) Dynamic equation.** Notice that $L(v), L(v,u)$ both are functions on time $t$ for vertex $v, u \in V(G)$ and $(v, u) \in E(G)$ in the harmonic flow $G^{L^2}[t]$ of twelve meridians, which can be characterized by corresponding points $p$ on twelve meridians to coordinates $\mathbf{x}$ in a Euclidean space $\mathbb{R}^3$. Then, the state of human in gravitational field, especially the gravitational field of earth can be characterized by the harmonic flow formed by the twelve meridians of human body. In this case, if the Lagrangian of human in gravitational field is $\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t)$, then the equation can be derived from the continuity flow equations in Lagrange field, namely the dynamic behavior of twelve meridians on human body in a gravitational field can be characterized by

$$\frac{\partial G^{\mathcal{L}^2}}{\partial q_i} - \frac{d}{dt}\frac{\partial G^{\mathcal{L}^2}}{\partial \dot{q}_i} = \mathbf{O}, \quad 1 \leq i \leq 3, \tag{11.16}$$

where $G^{\mathcal{L}^2}$ is the harmonic flow that corresponds to the twelve meridians of human body. Particularly, let $P$ be the abstracting particle of human body. Then, the behavior of human body in gravitational field can be characterized by Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \mathbf{0}, \quad 1 \leq i \leq 3. \tag{11.17}$$

Generally, if "*humans follow the earth*", the space of humans' activity is approximately the gravitational field of earth. In this caes, Dr.Ouyang tells Huizi that by choosing spherical coordinates $(r, \theta, \phi)$, the equation (11.16) of the twelve meridians of human body could be simplified to a system of second order differential equations at a point $p$ on the twelve meridians. First of all, as these shown in Figure 11.21 $\mathbf{r}$ is a vector at point $P$ in coordinate system $\{O, \mathbf{e}_x, \mathbf{e}_y, \mathbf{e}_z\}$, the expansion of local coordinates $\{P; \mathbf{e}_r, \mathbf{e}_\theta, \mathbf{e}_\phi\}$ of $d\mathbf{r}$ is $d\mathbf{r} = \mathbf{e}_r dr + \mathbf{e}_\theta r d\theta + \mathbf{e}_\phi r \sin\theta d\phi$. Dividing its both ends by $dt$, then $\dot{\mathbf{r}} = \mathbf{e}_r \dot{r} + \mathbf{e}_\theta r\dot{\theta} + \mathbf{e}_\phi r \sin\theta\dot{\phi}$; Secondly, a calculation shows that the kinetic energy is $\mathcal{T}_p^2 = \mathcal{T}_p^+ + i\mathcal{T}_p^-$, the potential energy $\mathcal{V}_p^2 = \mathcal{V}_p^- + i\mathcal{V}_p^-$ and Lagrangian $\mathcal{L}_p^2 = \mathcal{L}_p^+ + i\mathcal{L}_p^-$ of Yang qi and Yin essence at points $p$ on twelve meridians of the human body, where $\mathcal{T}_p^+, \mathcal{T}_p^-, \mathcal{V}_p^+, \mathcal{V}_p^-$ and $\mathcal{L}_p^+, \mathcal{L}_p^-$ are respectively

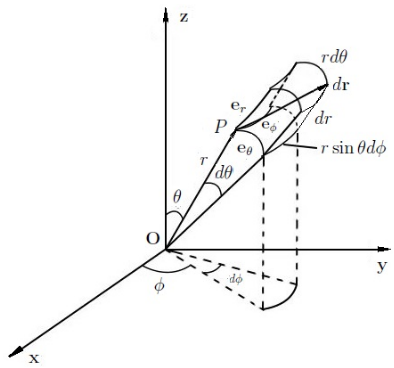



**Figure** 11.21. **Differential elements**

$$\begin{aligned}
\mathcal{T}_p^+ &= \frac{1}{2}m_p^+\dot{\mathbf{r}}^2 = \frac{1}{2}m_p^+\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right), \quad \mathcal{V}^+ = m_p^+ gr,\\
\mathcal{T}_p^- &= \frac{1}{2}m_p^-\dot{\mathbf{r}}^2 = \frac{1}{2}m_p^-\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right), \quad \mathcal{V}_p^- = m_p^- gr,\\
\mathcal{L}_p^+ &= \mathcal{T}_p^+ - \mathcal{V}_p^+ = \frac{1}{2}m_p^+\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) - m_p^+ gr,\\
\mathcal{L}_p^- &= \mathcal{T}_p^- - \mathcal{V}_p^- = \frac{1}{2}m_p^-\left(\dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2\right) - m_p^- gr.
\end{aligned}$$

where, $m_p^+, m_p^-$ are respectively the masses of Yang qi and Yin essence at point $p$. Then,

$$\begin{aligned}
\frac{\partial \mathcal{L}_p^+}{\partial \dot{r}} &= m_p^+ \dot{r}, \quad \frac{\partial \mathcal{L}_p^+}{\partial \dot{\theta}} = m_p^+ r^2 \dot{\theta}, \quad \frac{\partial \mathcal{L}_p^+}{\partial \dot{\phi}} = m_p^+ r^2 \sin^2\theta \dot{\phi}, \quad \frac{\partial \mathcal{L}_p^+}{\partial \phi} = 0,\\
\frac{\partial \mathcal{L}_p^-}{\partial \dot{r}} &= m_p^- \dot{r}, \quad \frac{\partial \mathcal{L}_p^-}{\partial \dot{\theta}} = m_p^- r^2 \dot{\theta}, \quad \frac{\partial \mathcal{L}_p^-}{\partial \dot{\phi}} = m_p^- r^2 \sin^2\theta \dot{\phi}, \quad \frac{\partial \mathcal{L}_p^-}{\partial \phi} = 0,\\
\frac{\partial \mathcal{L}_p^+}{\partial r} &= m_p^+ r\dot{\theta}^2 + m_p^+ r \sin^2\theta \dot{\phi}^2 - m_p^+ g, \quad \frac{\partial \mathcal{L}_p^+}{\partial \theta} = m_p^+ r^2 \sin\theta\cos\theta \dot{\phi}^2,\\
\frac{\partial \mathcal{L}_p^-}{\partial r} &= m_p^- r\dot{\theta}^2 + m_p^- r \sin^2\theta \dot{\phi}^2 - m_p^- g, \quad \frac{\partial \mathcal{L}_p^-}{\partial \theta} = m_p^- r^2 \sin\theta\cos\theta \dot{\phi}^2.
\end{aligned}$$

Substituting them into equations (11.16), we get the dynamic equation

$$\left\{\begin{aligned}
&\left(m_p^+ + im_p^-\right)\left(\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2 + g\right) = 0,\\
&\left(m_p^+ + im_p^-\right)\left(r^2\sin\theta\cos\theta\dot{\phi}^2 - \frac{d}{dt}\left(r^2\dot{\theta}\right)\right) = 0,\\
&\left(m_p^+ + im_p^-\right)\frac{d}{dt}\left(r^2\sin^2\theta\dot{\phi}\right) = 0,
\end{aligned}\right.$$

which can be further simplified to a dynamic equation

$$\left\{\begin{aligned}
&\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2 + g = 0,\\
&r^2\sin\theta\cos\theta\dot{\phi}^2 - \frac{d}{dt}\left(r^2\dot{\theta}\right) = 0,\\
&\frac{d}{dt}\left(r^2\sin^2\theta\dot{\phi}\right) = 0.
\end{aligned}\right. \tag{11.18}$$

of Yang qi and Yin essence at a point $p$ of twelve meridians.

Notice that equation (11.18) is the dynamic equation of any point on the twelve meridians of human body under the gravitational field of earth rather than the gravitational field of everything, i.e., a dynamic equation under the action of local gravity.

(2) **Acupuncture model of twelve meridians.** Certainly, the running of vital energy on twelve meridians of human body is an example of a harmonic flow $G^{L^2}$. Dr.Ouyang explains that the essence of human body follows Tao of the heaven is the balance of Yang qi and Yin essence at any point on the twelve meridians. In this way, the acupuncture of traditional Chinese medicine is a natural method to return the balance of Yin and Yang to the human body. So, *whether reducing the excess with supply the insufficient in traditional Chinese acupuncture can be quantitatively characterized by harmonic flow $G^{L^2}$ or simulating?* The answer is Yes! Dr.Ouyang explains that a healthy human has a balance between Yang qi and Yin essence at each point on the twelve meridians of body. In this case, if there is an imbalance point on the twelve meridians, the tiny needles can be pierced to acupoints on corresponding meridians to adjust the running of the vital energy in meridians. Generally, the tiny needle point along the direction of vital energy on meridian is

"*supply*"; the tiny needle against the direction of vital energy on meridian is "*reducing*" or "*discharging*". At the same time, the depth of needle should be moderated, namely the words that "*A disease has ups and downs, the piercing also has the shallow and deep. All of them have themselves Tao. Too much will damage the internal organs, too less will form the external obstruction by negative action, i.e., both of the shallow and deep are not correct, will affect the moving of vital energy in internal organs and cause serious disease later*" that Qi Bo replied the Emperor's question on the key points of acupuncture in *Emperor's Inner Canon-Plain Questions*", imply that a disease is maybe on the surface or inside of human body and there should be different depths for needling, namely the needling should be shallow if the disease is still on the surface of body. Otherwise, it should be deep. But, no matter in any case, the needling should be pierced to a certain depth of the disease position because the needling too deeply will damage the internal organs, too shallow will fail to reach the disease position, obstructs the running of vital energy and cause a chance for disease pathogens to invade, resulting in a greater harm to human body as the disorder of internal organs in function and following serious diseases.

In the moderating case on needling, assume that the amount of Yang insufficient (or Yang excess) is $\mathbf{A}$ at an acupoint on the twelve meridians. Then, the amount of Yin excess (or Yin insufficient) must be $-\mathbf{A}$ and the amount that needs to be supplied (or reduced) is $A - iA$ for returning to the balance of Yin and Yang at this acupoint. Such a recovering in Chinese acupuncture can be abstracted, namely define the input operation $O(v) = \mathbf{A} - i\mathbf{A}$ or the output operation $O(v) = \mathbf{A} - i\mathbf{A}$ to adjust the amount of Yang Qi and Yin essence at vertex $v \in V(G)$ in $G^L$, as shown in Figure 11.22,

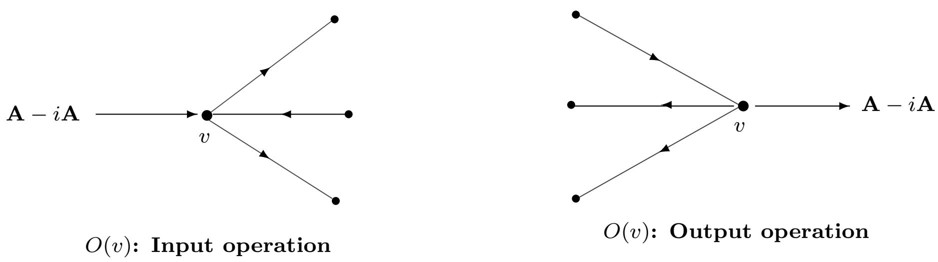



**Figure** 11.22. **Input and output operations on harmonic flow**

where, the input and output operation can be regarded as an acupuncture but can be also describe the invasion of external factors into the human body. Thus, if $G^L$ is a continuity flow with the input (output) operation $O$ on $s$ nodes $v_i \in V(G)$, $1 \leq i \leq s$ in a harmonic flow $G^{L^2}$ with respective inputs (outputs) $\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s$, then the output (input) operation $O$ only needs to apply on vertices $v_i \in V(G)$, $1 \leq i \leq s$ with the respective output (input) amounts $\mathbf{A}_1, \mathbf{A}_2, \cdots, \mathbf{A}_s$ and then $G^L$ will return to a harmonic flow $G^{L^2}$ in theory. Denote the minimum number of output (input) operations by $\varpi\left(G^L\right)$. Then, there must be $\varpi\left(G^L\right)\leq s$.

Notice that the running flow of vital energy on the twelve meridians of human body follows the rule of three meridians of Hand-Yin from the chest to hand, the three meridians of Hand-Yang from the hand to the head, the three meridians of Foot-Yang from the head to the foot and the three meridians of Foot-Yin from the foot to the chest, i.e., beginning from the lung meridian to the liver meridian, then one by one and finally, from the liver meridian to the lung meridian completing a circulation. Thus, the chest, hand, head and foot of human body are overlapping or turnover places among the running of vital energy on the twelve meridians. Generally, let the vertex $v_1$ =human chest, $v_2$ =human hand, $v_3$ =human head and $v_4$ =human foot. Then, the running flow of vital energy on twelve meridians of human body can be characterized as those shown in Figure 11.23.

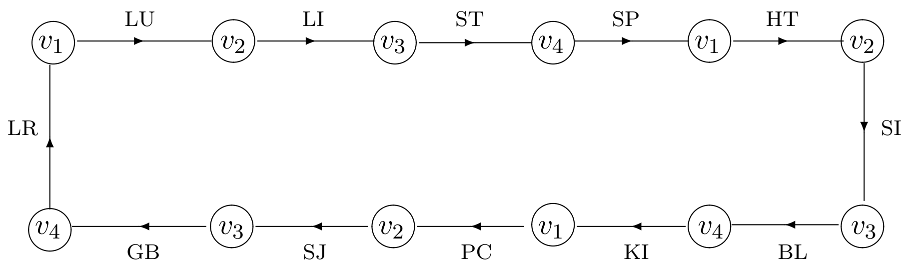



**Figure** 11.23. **Running flow of vital energy on 12e meridians**

Notice that the harmonic flow $G^{L^2}$ of twelve meridians on human body is constituted by viewing acupoints as vertices and the sub-meridians connecting two successive acupoints as edges. Furthermore, the graph $G$ of twelve meridians shown in Figure 11.20 will turn to a connected graph after introducing vertices $v_1 - v_4$ and its structure can be further determined by the distribution and structure of twelve meridians. Notice that the graphs of the lung meridian of Hand-Taiyin (LU), heart meridian of Hand-Shaoyin (HT), pericardium meridian of Hand-Jueyin (PC), spleen meridian of Foot-Taiyin (SP), kidney meridian of Foot-Shaoyin (KI), liver meridian of Foot-Jueyin (LR), large intestine meridian of Hand-Yangming (LI), small intestine meridian of Hand-Taiyang (SI), Sanjiao meridian of Hand-Shaoyang (SJ), stomach meridian of Foot-Yangming (ST) and the gallbladder meridian of Foot-Shaoyang (GB) of vital energy are all *paths*, and the graph of bladder meridian of Foot-Taiyang (BL) is a circle attached with two paths. So, the graph of vital energy moving on twelve meridians of human body is shown in Figure 11.24, where $P_k$ is a path of $k$ vertices.

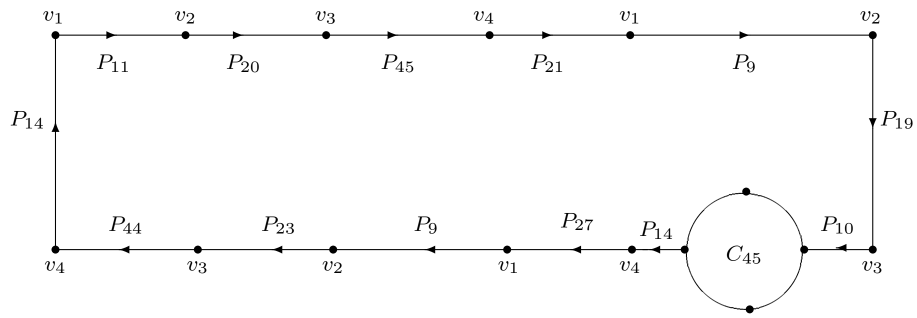



**Figure** 11.24. **Transporting graph of vital energy on twelve meridians**

Then, *can we determine the number $\varpi(G^{L^2})$ of input or output operations $O$ on the transporting graph $G$ so that $G^L$ can return to a harmonious flow?* The answer is Yes! Dr.Ouyang explains that an input (output) operation $O$ corresponds to a needle pricking and so $\varpi(G^{L^2})$ can be immediately determined by the structure of graph $G$. For example, $\varpi(P_n^{L^2})=1$, $\varpi(C_n^{L^2})=1$ and $\varpi(K_n^{L^2})=n-2$, namely a non-harmonic flow on complete graph $K_n$ can be transformed to a harmonic flow $K_n^{L^2}$ by performing $n-2$ input (output) operations $O$ at most on complete graph $K_n$, $n\geq 3$. Generally, let $G$ be a connected graph of order $\geq 3$ and for any vertex $v\in V(G)$, let $[v]_p$ be the subgraph derived by all pathes joining $v$ in $G$. Now, if $G^L$ is a continuity flow by operating $O$ on vertices $v^1,v^2,\cdots,v^{k_0}\in V(G)$ of a harmonic flow $G^{L^2}$ such that

$$G\setminus\left\{\left[v^1\right]_p,\left[v^2\right]_p,\cdots,\left[v^{k_0}\right]_p\right\}=\left(\bigcup_{i=1}^{k_1}C_{n_i}\right)\bigcup\left(\bigcup_{j=1}^{k_2}P_{n_j}\right)\bigcup\left(\bigcup_{l=1}^{k_3}K_1\right),\qquad(11.19)$$

namely if there are $k_1$ circles, $k_2$ pathes and $k_3$ isolated vertices after removal the vertices $v^1,v^2,\cdots,v^{k_0}$ with their associated pathes in graph $G$, then

$$\varpi\left(G^{L^2}\right)\leq k_0+k_1+k_2.\qquad(11.20)$$

Dr.Ouyang tells Huizi that the transporting graph $G$ of vital energy on the twelve meridians of human body is a union of two circuits coincided with an arc. Consequently, $\varpi(G^{L^2})\leq 4$ by (11.20), namely choosing 4 acupoints on twelve meridians can turn an imbalance of Yin and Yang in running of vital energy to balance in theory. However, a few of doctors prefer to follow the guidance of *Emperor's Inner Canon-Lingshu Canon* that prick on the acupoints on hands and feet of meridian where the disease is located, namely they only using the acupoints "*below the elbow on hand meridians and below the knee on feet meridians*" and the number of acupoints used may be larger than 4 but the effect is more significant in practice.

Huizi has hated the bitterness of traditional Chinese medicine since she was a child and so, she has always been a little rejecting it when she grew up. After listening to

Dr.Ouyang's explanation, she suddenly realizes the importance of harmonic flows and says: "*Dad, it is a bit magical to think of a human body as a harmonic flow* $G^{L^2}$. *Could you tell me how the ancients discovered this corresponding?*" Dr.Ouyang tells her that traditional Chinese medicine was the product of ancients following Tao under the thought of "*unity of humans with the heaven*", "*humans follow the earth*", the result of *Shennong taste herbs* and imitating Tao of other animals. Even so, how the twelve meridians on human body and other eight extraordinary meridians were discovered is still a mystery! Someone say that they were discovered by "*taking pain place as a acupoint*" by the ancient Chinese after years of accumulation and a taoist says that he observed the moving of vital energy on twelve meridians as he fell into a state of tranquility in meditation once but there is no a universally accepted answer because even under modern scientific instruments, the vital energy flows on the twelve meridians and the eight extraordinary meridians can not be observed. It seems that it is not a "*physical reality*" but it has a real impact on human's health. This goes back to a fundamental question in human recognition, namely "*is it true that only those things that can be perceived are physical reality?*" The answer seems to be very different! Otherwise, how to explain the hidden variables in quantum entanglement, how to understand the gravitational field and electromagnetic field effect! Certainly, the five thousand years civilization history of Chinese shows that the twelve meridians and eight extraordinary meridians on human body are indeed physical reality, just need to give a further scientific explanation because the running of vital energy on meridians of human body corresponds a harmonic flow $G^{L^2}$ with $|G| = 309 - 12 = 297$ shown in Figure 11.24, which can be regarded as the living model of human. After a slight pause, Dr.Ouyang asks Huizi a question: "*What do you think the relationship of Einstein's gravity with that of Newton?*" Huizi replies: "*Newtonian gravity is a real force but that of Einstein is a bending effect of space, not a real force.*" Dr.Ouyang tells her: "*This is a misunderstanding on Einstein's gravity. Is it possible that either Newton or Einstein's theory is wrong? Of course Not! Newton's gravity is introduced by local action of gravity, which is similar to the information transmission in quantum entanglement. It follows the general vector operation rules and is easily understood. Superficially, the Einstein's gravity is to explain the action at a distance in Newtonian gravity, namely under the action of Newtonian gravity, the space is not flat that humans perceive on the earth but a curved one. It is the bending effect that causes the Newtonian gravity. However, both of them are explaining the universal gravity between particles.*" Huizi is still puzzled: "*Why are they so different?*" Dr.Ouyang explains that the two theories are just different in forms, namely the Newtonian gravity is the gravity between two particles, which is only a local theory.

However, the Einstein's gravity is an integral theory on gravity in case of the number $n \to \infty$ of particles, i.e., the gravity of all things in universe. Similarly, the Chinese medicine is an integral theory but the western medicine is a local theory on human body. However, a local theory is generally accepted by the human himself because its effect can be felt in a short time but it is attend to only one thing and losing sight on others. Certainly, an integral theory is complex and the curative effect may be slow in the feeling of human but it is consistent with Tao of the heaven, i.e., the ancient Chinese recognition on "*unity of humans with the heaven*" and the environment of humans.

## §4. **Agricultural Civilization**

Notice that the purpose of human recognition of things is to live in harmony with the nature, i.e., establishes the human civilization or "*Te*" follows Tao, where the word of "*civilization*" consists of the culture and living way, namely the humans adapt to the nature by means of science and technology, adjust human's behavior by social norms in order to maximize the needs of human survival and the cultural embodiment of harmonious coexistence with the nature. So, *why did Laozi explain that humans follow the earth, the earth follows the heaven, the heaven follows Tao and Tao follows the nature in Tao Te Ching?* Dr.Ouyang explains that Laozi believes that humans follows Tao is to imitate all livings on the earth. Certainly, a main difference of the human with other livings on the earth lies in his subjective initiative in meeting the needs of living. However, the human's subjective initiative is not necessarily complied with Tao. So, it needs to be self-restraint of human. The essence of agricultural civilization is the "*working at sunrise, rest at sunset and digging wells for drinking, farming fields for eating*", as shown in Figure 11.25, which is the ancient Chinese follow Tao, conform to Tao of the heaven and earth, namely the embodiment of ancient Chinese follow the earth, the heaven and follow Tao. So, *why do we assert that the Chinese agricultural civilization is the embodiment of humans' Te in accordance with Tao?* Dr.Ouyang explains that the Chinese agricultural civilization as a human civilization which is self-sufficient and self-constrained because it follows seasons of year, i.e., Tao of the heaven and humans worked hard for following Tao of crop growth in fields. They were living on the harmony of humans with the nature at that times.

**Figure** 11.25. **Farmland**

**4.1.Tao of Livings on Earth.** There are about two million species of living things on the earth, including animals, plants and microorganisms. Among them, about 1.5 million species of animals, 350,000 species of plants are the main forms of life that human eyes can see. Certainly, the majority of livings live on the land or under the surface of oceans within a thickness of about 100 meters of earth. Generally, the range of life activity and its living environment on the earth are called "*biosphere*", which is about 10000m above the sea level to 11000m below the sea level, namely the lower part of the atmosphere, the upper part of the lithosphere, the whole soil and hydrosphere. Notice that the main forms of life seen by human eyes are animals and plants on the earth. Thus, the "*humans follow Tao*" is the living of humans following other livings on the earth because the human is the product of "*Three creates all things*" on earth by Tao, should be confined to the earth on one hand and on the other hand, all other livings on the earth have no the subjective initiative as the humans. They are living on conforming to Tao. That is why humans following Tao should imitate other living in order conforming to the nature.

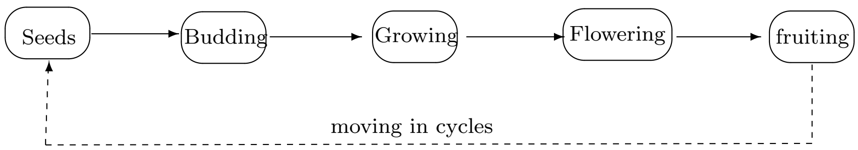



**Figure** 11.26. **Growth of plant**

(1) **Plants living.** Generally, plants can not grow without adequate sunlight, suitable temperature and soil. In this process, the photosynthesis between light and green plants produces chlorophyll, which promotes plant growth; the suitable temperature can promote the photosynthesis; the soil provides the root system for the crops, storing water, air and mineral elements needed to grow; the leaves fall to the ground after the fruit mature in "*falling leaves settle on their roots*", decay into the soil after a period of time, providing nutrients for plant growth; the seeds fall to the ground when they mature or migrate to another place to take root and then, bud for the next cycle of life, as those of shown in Figure 11.26. Notice that different plants have different environments, soil conditions and the growth cycles. The herbaceous plants change generally into three seasons, following the phenomenological law of "*budding in spring, growth in summer, fruiting in autumn, storage in winter*". For the living of human, Laozi's "*humans follow the earth*" implies that the human can simulate the growth of wild herbs, plant the grains and other herbs in a planned way under the same climate and environmental conditions, i.e., living from the collecting herbs in wild to planting and supplying the living of human in a planned way.

Certainly, there is also a nutrient transporting system in plant that supports its living. Generally, the water, minerals and nutrients needed by stems and leaves of a plant come from the soil, are absorbed and transported from the soil by its underground roots. The leaves depend on the supporting of stems and branches to extend in space for full photosynthesis; the roots depends on the photosynthesis of leaves to obtain the organic nutrients they need, which form the transporting system of nutrients similar to that of transporting of vital energy on meridians of the human body, namely a continuity flow of $T^L$ on a tree $T$, as shown in Figure 11.27. In this case, the vertex set $V(T)$ consists of its leaves, intersections of the trunk, stem and the end of roots, the edge set $E(T)$ consists of the trunk, stem and the branch of plant with a labeling mapping $L : v \to \{L_1(v), L_2(v)\}$, $L : (v, u) \to \{L_1(p), L_2(p)\}$ for vertices $v \in V(T)$ and edge $(v, u) \in E(T)$, where $p$ is a point on the trunk, stem or branch of plant between vertices $v$ and $u$; $L_1(p), L_2(p)$ denote respectively the amounts of up and down running nutrients at point $p$, which are not necessarily balanced and absorbed respectively by roots and leaves of plant. It should be noted that the leaves, branches and roots of different plants are not necessarily the same. Accordingly, the corresponding tree $T$ is not necessarily isomorphic. And meanwhile, it is different from the moving of vital energy on twelve meridians of human body, the vertices on tree $T$ are the abstractions of leaves, intersections of the trunk, stem and roots of plant, not the fixed acupoints on meridians of human body and increase with time $t$ until its end of life. Dr.Ouyang tells Huizi that according to the balance of Yin and Yang of ancient Chinese philosophy, there should also be acupoints and meridians on which run the vital energy similar to that of vital energy on meridians of human body to maintain the living of plant.

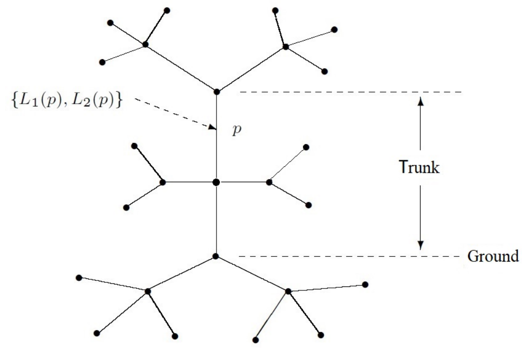



**Figure** 11.27. **Plant transporting** $T^L$

(2) **Animal living.** An animal, whether it is egg-laying or viviparous, is a product of "*Tao*" in "*Three creates all things*". Certainly, different organisms are linked together by a series of eating or being eaten, i.e., the food chain or the nutritional chain of animals such as the rabbits, cattle and sheep that feed on plants, the wolves, tigers and lions that feed on meat, the birds, fish and shrimp that feed on scraps, and also the parasites such as fleas, bacteria and viruses. All of them obtain the nutrients needed by the body in a fixed way or living follow "*Tao*". Usually, we can domesticate wild animals such as cattle, horses, sheep, pigs and other mammals, birds, fish and so on from the wild to domestic animals so that their wild gradually change and can serve the human life if not violating the way of animal survival. Among them, one is to hold on the laws of animal

survival, habit and reproduction; another is the simulating of animal's living environment and conditions. It can be classified and domesticated, raised and bred in a planned way to meet human needs. Meanwhile, some endangered animals are protected by humans for the biological diversity in nature.

No animal can survive without foraging for nutrients, including the water, sugars, proteins, lipids, vitamins and minerals. Certainly, different animals have slightly different digestive processes on different foods but essentially, they absorb nutrients through food to provide the vital energy needed by animals and living, showing the foraging, feeding, attack, defense, territory, reproduction, social, communication and other behaviors of animals. Such a kind of phenomenological behaviors of animals adapting to environment are the prerequisite for humans to master the living conditions and habits of animals and then, simulate the living environment and organize feeding animals in a planned way. Generally, the food enters the mouth of animal is ground and mixed by the teeth, swallowed into its stomach is initially dissolved under the action of stomach acid and the gastrointestinal peristalsis and then, enters the small intestine to absorb nutrients such as water, electrolytes, proteins, fats to provide the nutrients for different internal organs and finally, the waste is discharged from the body.

Then, *does an animal have also twelve meridians on body that run the vital energy?* Dr.Ouyang explains that the animal body is the same as the human body that is a product of Tao. As long as the animal and the human body have the same viscera, the needed nutrition is through the digesting and absorbing of stomach, spleen, lung and other viscera to deliver throughout the whole body. Thus, the nutrition transporting of human body in traditional Chinese medicine can apply to the animal body such as the words that "*eating and drinking get into the stomach with vital energy transported into the spleen, the temper disperses essence attributed to the lung, the water transports lower to the bladder with the essence's distributing on meridians of viscera and the nutrition transporting on viscera is estimated that often so with the balance of Yin and Yang, i.e., the same as the running of four seasons in order*" in Chapter *Other Matters on Meridians* of *Emperor's Inner Canon-Plain Questions*, namely there also exist twelve meridians and eight extraordinary meridians on animal body corresponding to the human body. It is based on this philosophical thinking that veterinarians in traditional Chinese medicine

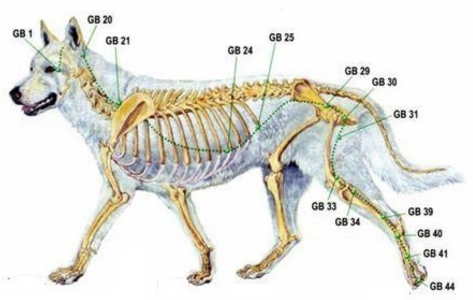



**Figure** 11.28. **GB-meridian on dog**

put forward the twelve meridians and eight odd meridians on the dog bodies of cattle, sheep, horses, dogs and other livestock. For example, the gallbladder meridian of Foot-Shaoyang (GB) on the body is shown in Figure 11.28. According to the veterinary theory of traditional Chinese medicine, the essence of a living animal body is the running of vital energy on twelve meridians similar to that of human body, i.e., corresponds to a harmonic flow $G^{L^2}$ with 362 vertices and satisfies the Euler-Lagrange equations

$$\frac{\partial G^{\mathcal{L}^2}}{\partial q_i} - \frac{d}{dt}\frac{\partial G^{\mathcal{L}^2}}{\partial \dot{q}_i} = \mathbf{O}, \quad 1 \leq i \leq 3, \tag{11.22}$$

where $\mathcal{L}$ is the Lagrangian of a conservative force field or approximately, the gravitational field of earth.

**4.2.Humans' Living Tao.** Huizi asks Dr.Ouyang: "*Dad, why do humans must follow Tao and can not do as they wish?*" Dr.Ouyang explains that humans were created by Tao. Thus, they have "*Te*", i.e., following Tao is the embodiment of Tao in "*Three creates all things*". Otherwise, there would be no human on the universe! This fact is mainly shown in two sides. The first is the living conditions of humans. Why the humans came into beings on the earth is because the earth itself has a suitable climate, water, food and other conditions for human's living, namely if there were no the conditions for living of human on the earth or the humans go on the opposition of Tao, i.e., destroying the climate, water, food and other conditions prepared for humans' living on the earth, there would be no human on the earth. Secondly, as a special kind of beings on the earth, humans follow Tao is an inevitable rule in Tao creating human or human body, namely the human body has the specific structural, functional and the nutritional needing with the specific living conditions which determine that the human should follows Tao. Otherwise, it will lead to end of the human if he or she evolves counter to Tao such as eats food or medicine that his body can not digest or extract nutrients, not only fail to provide the nutrients but also harming to the tissues or functions of body.

In such a situation, Huzzi asks Dr.Ouyang a question: "*How should humans follow Tao?*" Dr.Ouyang tells her that the essence of following Tao is having "*Te*" for humans. He explains that the Chinese agricultural civilization is a typical example of "*humans follow Tao*" by means of following livings on the earth, which is the Chinese scheme of living under the limited recognition on all things, namely the words that "*Tao told is not the eternal Tao; Name named is not the eternal Name*" in *Tao Te Ching*, i.e., do not disturb the nature or harmoniously coexistence with nature within the permissible limits, namely the words "*that is why the sage only eats with enough food, not satisfy his eyes with a riot of colours*" and "*act in inaction, deal things with nothing happened, taste in tasteless*, i.e., without against Tao in Chapter 63, it implies that humans need to follow Tao and have "*Te*", abandon the prejudice or preference of humans and then, follow Tao and act with Te, i.e., the humanity.

**(1) Living conditions.** A main characteristic of Chinese agricultural civilization is that all human activities were proceed around human's living, i.e., the "*clothes, food, residence and traveling*", where the "*clothes*" means "*dressing*", which emphasizes the protection function cold in living of humans following Tao, i.e., from the instinctive clothes of hay, leaves, bark and hides to keep out the cold to the woven fabric of hemp, organization of mulberry and silkworm breeding, drawing silk for making clothes, all are reflecting the clothing on cold, rain and shame function; the "*food*" means the satiety of "*diet*" in living of humans following Tao, i.e., from picking wild fruit, hunting to organize grain planting and raising livestock such as the horse, cow, pig, sheep, chicken, duck and goose, all are in the supplying of nutrition that the human body needs; the "*residence*" means "*dwelling*" in living of humans following Tao, i.e., from imitating other animals living in tree tops and caves to digging burrows, to building thatched cottage and house with natural materials or fired bricks, all are the dwelling in protecting humans from the wind, rain and the invasion of other animals; the "*traveling*" refers to "*traveling model*" in living of humans following Tao, i.e., from walking, riding horses and other livestock traveling to horse-drawn carriages and ox carts, which reflects the function of shortening the traveling time in space. Therefore, Chinese agricultural civilization is an typical example of harmony of humans with the nature and how to follow Tao.

**Figure** 11.29. **Farm working**

**(2) Industry.** Certainly, the developing of Chinese agricultural civilization mainly focused on the cultivation of five grains and the breeding of pigs, horses, cattle, sheep, chicken, duck and goose etc., which served the agricultural production to meet human's living or "*eats with enough food, not satisfy his eyes with a riot of colours*". And so, the industry was mainly consisted of the handicraft industry or workshop which did not intrude upon the nature or in harmony with the nature within the permissible limits in that period, including the silk and cotton weaving techniques developed in clothing, the smelting of bronze, iron, gold and other metals, forging techniques, ceramic firing techniques and construction techniques developed in eating and dwelling as well as the technology of making chariots and shipbuilding from natural wood and animal hide developed in traveling, etc.

(3) **Social undertakings.** The social undertakings in period of Chinese agricultural civilization include the medical care, humanistic education and social norms of conducting. Among them, the traditional Chinese medicine with its veterinarians plays an important role in guaranteeing the developing of agricultural society, which is established on the philosophy that the human body and other animals follow Tao. Certainly, it is a systematic medical theory that regulates human or animal diseases according to Tao; the essence of humanistic education is to educate humans following Tao, recorded in hundred schools of thought of the Chinese agricultural civilization. For example, the words that "*Tao of the heaven is reducing the excess with supply the insufficient; but the humans are different, they reduce the insufficient with supply the excess*" in Chapter 77 of *Tao Te Ching* or "*the rich are getting more richer and the poor are getting more poorer*", namely the humans are against to Tao of the heaven and depart from "*taking all things as humble dogs*". In this case, humans are really be educated to follow Tao with self-restraint, namely the words that "*keep humans from contention by disregarding the talent, keep humans from stealing by not valuing rare goods, keep humans from upset by not desirable too; that is why the sage suggest to simplify the minds, fill up the stomachs, weak the wills and strengthens the body of human*" in Chapter 3 of *Tao Te Ching*, namely without the human-made worship of *talent*, there will be no jealousy and competing, without the human-made value of rare goods, there will be no thieves, without showing what induces the desiring, the human heart will not upset against to Tao. Thus, the sage administers a society will soothe humans' minds and weaken the "*competing, stealing, disorder*" in mind and let them have the needs of living and strong physique but weakening their ambition because the human's mind against to Tao and needs to be educated following Tao. Historically, the Chinese agricultural society is such a society that consists of the scholars, peasants, artisans and merchants. One of it's important issues is regulating the behavior of humans following Tao in social governance. The words that "*the heaven and earth are not kind, taking all things as humble dogs; the sage is not kind, taking all humans as humble dogs*" in Chapter 3 of *Tao Te Ching* are following the "*unkind*" of heaven and earth because such a behavior is departing from "*taking all things as humble dogs*", i.e., Tao of the heaven and earth. And meanwhile, the words that "*the sage desires the undesirable of others, not value rare goods, learns what the other do not learn and make up faults of the others, follows the nature of all things*" in Chapter 64 of *Tao Te Ching*, implies the sage clearly knows what he wants and not wants, what he learns and not learn, never values rare goods, can remedy the mistakes in the conduct of others, including humans not following Tao and then, follows Tao himself.

Dr.Ouyang asks Huizi: "*Do you know why there is the activity of offering sacrifices to the heaven and earth in Chinese agricultural civilization?*" Huizi replies: "*don't you think the sacrifice is superstitious?*" Dr.Ouyang tells her: "*Certainly, you can't think so!*" And then, he explains that the words that "*Tao told is not the eternal Tao*" implies that a human must have awe on the nature and self-restraint, which is what missed in today's society and needs to make up. Superficially, the offering sacrifices to the heaven and earth is a superstitious activity. However, it is a kind of humanistic cultivation. It is human's inner gratitude and reverence for the heaven and earth so as to pray for favorable weather in the coming year. And meanwhile, this is also to awaken human's self-reflection, especially in facing on some great disasters, i.e., a human's inner self-reflection on the past, reviews it is in following or against to Tao of the heaven and earth, and where should be improved, etc. In fact, this is the essence of Chinese agricultural civilization.

## §5. **Chinese Science**

A lot of humans believe that any objective thing $T$ can be subdivided into hadrons, leptons, mesons, quarks and other elementary particles for accurate recognition and affirm that there is no science on the land of China by adopting the *reductionism* as the standard of science. Certainly, this is incorrect but a misunderstanding on science and the reductionism. Dr.Ouyang explains that the essence of reductionism is to look for the smallest element of changing in something $T$ but not necessarily the elementary particles that make up matter and the biology is a counterexample on such a view point because the genes, DNA and cells are all not elementary particles but biological macromolecules, which are made up of elementary particles themselves. Furthermore, even the subdivision of an organism $T$ into cells and genes does not necessarily contribute to the recognition on reality and even artificially complicate the understanding in cases. This is an important view in Chinese philosophy, i.e., the words that "*enough is as good as the feast*", namely its purpose is holding on the reality of thing. Here, the word of "*reality*" of thing $T$ refers to the state of $T$ that is real in the past, present or the future, regardless of whether it is observed or understood by humans. So, *what is science?* Dr.Ouyang tells Huizi that science is the human's knowledge on the laws or Tao on things. Certainly, it is human's local recognition of Tao on things because the

**Figure** 11.30. **Universe's ending**

recognitive ability subject to "*humans follow the earth*" is limited. However, it belongs to the category of science as long as it is the recognition on laws of things. For example, *does it belong to science to study the universe with pictures in 300,000 light years away by telescopes?* The answer is certainly Yes! Dr.Ouyang explains that this is a scientific study on the state of universe in 300,000 years ago.

Then, *what is the Chinese science?* Dr.Ouyang tells Huizi that the Chinese science came into being by the philosophy of "*humans follow the earth, the earth follows the heaven, the heaven follows Tao and Tao follows the nature*" in ancient China with a purpose of serving the agricultural civilization, including the branches of astronomy, mathematics, geography, agronomy, traditional Chinese medicine and architecture with achievements. For examples, ①Astronomy, established the 24 *solar terms* and 12 *two-hour periods* by relationship of the earth, sun and the moon corresponding to the farming season and the running of human body; ②Mathematics, found that *if the lengthes of right angled sides are 3 and 4 then the length of hypotenuse must be 5*, i.e., a special case of Pythagorean theorem in geometry with applications to measure the areas of farmland and the distance from the earth to the sun, the Chinese remainder theorem in algebra, established the "*circle cutting method*", namely to calculate the area of a circle by increasing the number of inner polygonal edges of circle with applying to the calculation of circumference ratio $\pi$, which implies the thought of limitation and definite integral; ③Physics, discovered many important laws and physical principles such as the lever principle, buoyancy, acoustics and optics including the relation of light with shadows, keyhole imaging and force analysis of wood structures; ④Agronomy, discovered the relationship between the heaven, earth, humans and the agriculture, the grain cultivation and livestock raising; ⑤Chinese medicine, discovered the corresponding of internal organs of human body with the twelve meridians and eight extraordinary meridians, Yin, Yang and the five elements, established the theory of traditional Chinese medicine and acupuncture to heal human and animal body; ⑥Geography, established the geographic information with analysing on the products, medicines of land and river flows in China for the irrigation and harvest of crops; ⑦Architecture, established by the "*corbel bracket*" as the representative of wooden architecture and the Chinese architectural forms, layouts, designs and construction norms; ⑧Other branches of science related to the *unity of humans with the heaven* such as the health preservation and nutrition in China etc. All of these are worthy of the glory of China's scientific achievements with reputation as an ancient civilization.

However, Dr.Ouyang tells Huizi that such a categorizing of the achievements is not enough to show the soul or essence of Chinese science. Huizi asks Dr.Ouyang: "*Dad, what* *is the soul or essence of Chinese science?*" Dr.Ouyang tells her that the soul or essence of Chinese science lies in the enlightening Tao on things or the purpose of science, which leads to humans following Tao, i.e., the coexistence of humans with the nature. Its essence is in the overall handing on state and changing of thing rather than the reduction of infinite subdivision. For an overall holding on thing, the ancient Chinese simulated the state and changing laws of thing by 64 hexagrams in *Change Book*, an scientific book in ancient China which is the soul or essence of Chinese science and the contribution of Chinese to science in history. Huizi takes a moment after hearing this explaining and then asks Dr.Ouyang: "*Dad, the Change Book is a fortune-telling book. How can it be science?*" Dr.Ouyang tells her that if the *Change Book* is a fortune-telling book, why did Confucius write *Ten Wings* for *Change Book* in his later years and told his adherents that "*if add me a few years or learn the Change Book before the age of fifty, I can have no great mistakes*" because Confucius believed that the "*changing*" shown in *Change Book* is "*accurate as Tao of the heaven and earth*", which implies that the *Change Book* is in simulating Tao of the heaven and earth, including the running law of heaven and earth. It is a classic physics of ancient Chinese. So, he wrote "*Ten Wings*" or put 10 wings on the *Change Book* as an education classic of humanistic education or humans how to follow Tao.

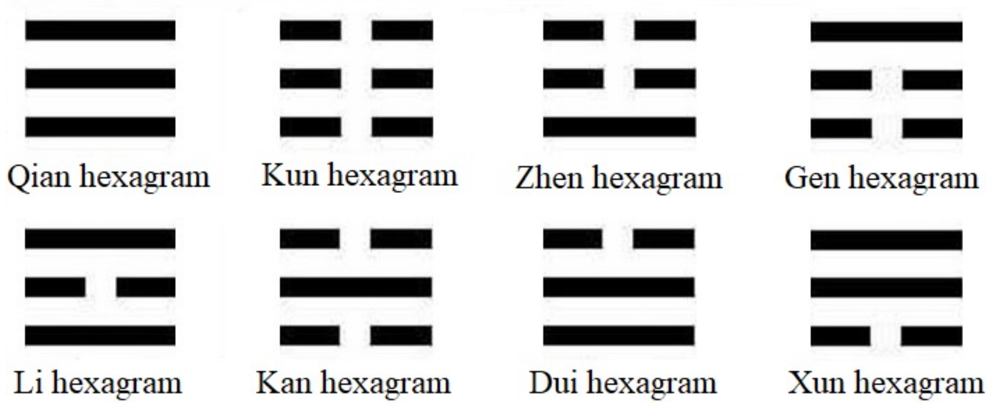



**Figure** 11.31. **Hexagrams**

**5.1.State Corresponding.** Certainly, the ancient Chinese philosophers believed that a thing is changing or evolving with mutual interaction, which can be regarded as a kind of field action in today. In this case, the creation in the process of "*Three creates all things*" is caused by the interaction of Yin and Yang states or fields. Usually, the Yang state, called the "*Yang line*" is denoted by a short line segment "-" or by $\mathbf{A}^+$; the Yin state, called the "*Yin line*" is denoted by two short line segments "--" or by $\mathbf{A}^-$. Then, the "*Three*" in "*Three creates all things*" can be respectively characterized by 3 places with one or two short lines segments, namely the cases of interactions in three fields. Notice that each line can be Yang line $\mathbf{A}^+$ or Yin line $\mathbf{A}^-$, there are a total of $2^3 = 8$ combinations of physical states as shown in Figure 11.31, which are respectively called the Qian, Kun, Zhen, Gen, Li, Kan, Dui and Xun hexagrams, i.e., the 8 different combining states with Yang and Yin in three places.

According to the ancient Chinese philosophy, the state of any thing can be formed by the combination of 8 natural phenomena, i.e., the heaven, earth, thunder, mountain, fire, water, lake and wind. Among them, the trend of heaven is vigorous and unremitting self-improvement; the trend of earth is inclusive or bearing all things; the trend of thunder is that vibrations awaken all things from hibernation; the trend of mountain is blocking, namely the wind and water turn to meet the mountain; the trend of fire is to let the combustion heat spreading around; the trend of water is to flow the low place, forming a flat surface but different depths of traps or dangerous situation; the trend of swamp is calm on the surface but as soon as we open a hole, the water will flow to other places; the trend of wind is quietly blowing. So, *how did the ancient Chinese philosophers abstract these 8 natural phenomena as the changing of Yin and Yang on lines of the eight hexagrams?* Dr.Ouyang explained that the changing direction of three lines in hexagram is moving from bottom to up, namely a cyclically moving of first line $\to$ 2nd line $\to$ 3rd line $\to$ first line, corresponding to a labeled graph $C_3^L$. By the different combinations of Yin and Yang on 3 lines, the 8 hexagrams correspond to the 8 natural phenomena are respectively: ①Qian hexagram corresponds to the heaven with 3 Yang lines, i.e., 3 fields of $(\mathbf{A}^+, \mathbf{A}^+, \mathbf{A}^+)$ and a upward trend of with the energetic, unyielding; ②Kun hexagram corresponds to the earth with 3 Yin lines, i.e., 3 fields of $(\mathbf{A}^-, \mathbf{A}^-, \mathbf{A}^-)$ and a downward trend of below other things; ③Zhen hexagram corresponds to the thunder, i.e., 3 fields of $(\mathbf{A}^+, \mathbf{A}^-, \mathbf{A}^-)$ and one Yang line is located under two Yin lines with the trend that the Yang line will break through the two Yin lines, which is extended to the thunder in beginning of spring rain and all things on the earth are vigorous; ④Gen hexagram corresponds to the mountain, i.e., 3 fields of $(\mathbf{A}^-, \mathbf{A}^-, \mathbf{A}^+)$ and one Yang line locates on two Yin lines with the trend that the Yang line prevents the upward of two Yin lines; ⑤Li hexagram corresponds to the fire, i.e., 3 fields $(\mathbf{A}^+, \mathbf{A}^-, \mathbf{A}^+)$ and one Yin line between two Yang lines with the trend of strong outside and soft inside; ⑥Kan hexagram corresponds to the water, i.e., 3 fields of $(\mathbf{A}^-, \mathbf{A}^+, \mathbf{A}^-)$ and one Yang line into the trip of two Yin lines with the trend that the water is continuously coming, i.e., there is danger in danger; ⑦Dui hexagram corresponds to the lake, i.e., 3 fields of $(\mathbf{A}^+, \mathbf{A}^+, \mathbf{A}^-)$ and one Yin line above two Yang lines with the trend that there is a mouth upper and extended to the talking about, disputing or misunderstanding and so on; ⑧Xun hexagram corresponds to the wind, i.e., 3 fields $(\mathbf{A}^-, \mathbf{A}^+, \mathbf{A}^+)$ and one Yin line under two Yang line with the trend that there is wind below and it is coming. It should be Noted that the appearance of eight natural phenomena such as heaven, earth, thunder, mountains, fire, water, lake and wind is related to the relative positions of the sun, moon and earth, as well as the effects of gravitational field and earth magnetic field, namely these 8 natural phenomena are closely related to the position, seasons and two-hours of earth. For example, the spring breeze is generally the south wind and the autumn wind is generally the northwest wind. And so, the eight hexagrams should be all matched with the azimuths, four seasons and twelve two-hours on earth as well as the mutual promoting and restraining of "*five elements*" so as to truly simulate the changing state of thing "*one side increasing with another decreasing*", as shown in Figure 11.31. Among them, the Taiji diagram in the middle of primitive eight hexagrams is the recognition of ancient Chinese following Tao of the heaven, i.e., the balance of Yin and Yang in everything. It should be noted that the direction is upward south and downward north, left east and right west in Figure 11.32, which implies that the Qian Yang is upward in the south, the Kun Yin is downward in the north, the sun rises from the east in Li and sets to the west with Kan; the northeast appears the spring thunder in Zhen, the southwest flows the autumn wind in Xun, the northwest has more high mountain in Gen, the southeast has more lakes in Xun.


**Figure** 11.32. **Primitive** 8 **hexagrams**

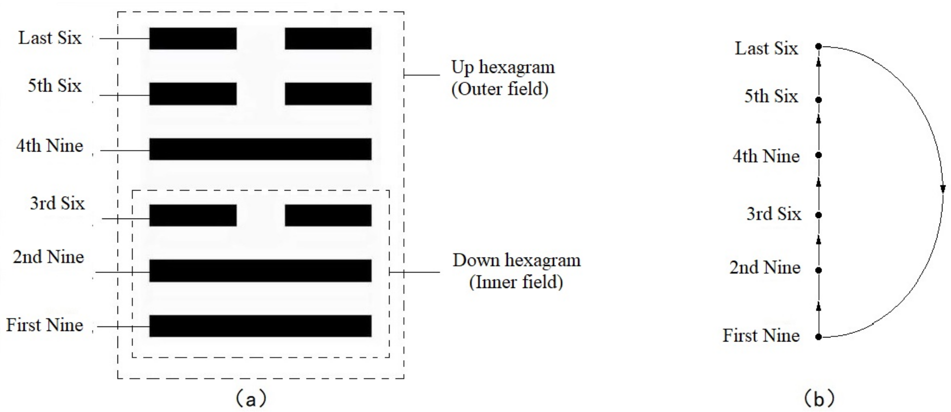



**Figure** 11.33. **Inner and outer field of hexagram**

Generally, these eight natural phenomena have also the cross-action and mutual transformation with the Yin and Yang powers. So, *how do we characterize this cross-action* *by hexagrams?* Dr.Ouyang explains that it is necessary to place one hexagram (outer) on another hexagram (inner), called the *double hexagram* to observe the influence of thing changing, as shown in Figure 11.33($a$). Certainly, there are $8 \times 8 = 64$ double hexagrams or field interactions which characterize the state changing of internal field under acting of the outer field. Here, the surname of Yang line is the number nine and the Yin line is the number six according to the position of lines from bottom to the top, as shown in Figure 11.33($a$). Similar to the eight hexagram in Figure 11.31, the Yin and Yang lines in the inner and outer hexagrams are respectively regarded as fields of $\mathbf{A}^-$ and $\mathbf{A}^+$. Then, the six lines are in a cyclically moving from the bottom to the top in a hexagram, i.e., first line $\to$ 2nd line $\to$ 3rd line $\to$ 4th line $\to$ 5th line $\to$ 6th line $\to$ first line, namely the cyclically moving of six lines corresponds to a continuity flow $C_6^L$ or the evolving of Yin and Yang fields, as shown in Figure 11.33($b$), i.e., $(\mathbf{A}^+, \mathbf{A}^+, \mathbf{A}^-, \mathbf{A}^+, \mathbf{A}^-, \mathbf{A}^-)$, denoted by $(+, +, -, +, -, -)$ simply.

The ancient Chinese philosophers believed that the state of thing $T$ following Tao can be characterized by the 64 hexagrams or field interactions under the action of Yin and Yang powers, i.e., the continuity flows of $C_6^{L_i}, 1 \leq i \leq 64$ in Table 11.1 following. Let $p \in S_6$ be a permutation in symmetric group $S_6$, $S = \mathrm{Im}\left\{C_6^{L_i}, 1 \leq i \leq 64\right\}$ and define a transform $\pm$ by $\pm : \ - \to +, \ \pm : + \to -$. Then, it is easy to know that the set $\mathrm{Im}S$ is invariant under the action of permutation $p$ and the transformation $\pm$, i.e.,

$$\mathrm{Im}^p S = \mathrm{Im}S, \quad \mathrm{Im}^{\pm} S = \mathrm{Im}S. \tag{11.23}$$

Notice that the order $C_6^{L_1} \to C_6^{L_2} \to C_6^{L_3} \to \cdots \to C_6^{L_{63}} \to C_6^{L_{64}}$ of 64 hexagrams or field interactions is in reflecting "*Three*" how to "*creating*" all things under the action of Yin and Yang powers, explained by Confucius in his *Hexagrams' Order* of *Ten Wings*, namely the words that "*there are the heaven and earth in first and then, created all things following with the Tun hexagram, where the word "Tun" refers to the word of "filled", i.e., all things filled beween the heaven and earth, also implies the beginning of all things; the beginning of thing must be uncultured, so the next is Meng hexagram which implies the puerile thing also; a puerile thing can not be brought up without food and drink, so the next is Xu hexagram after Meng, where 'Xu' refers to the word of 'need' or Tao of eating and drinking; the eating and drinking are easily with the litigation, so the next is Song hexagram; a litigation arises from lots of humans, so the next is Shi hexagram, where 'Shi' refers to lots of humans;* $\cdots$*; the ability of human beyond the common humans is surely succeed in action, so it is Ji Ji hexagram after Xiao Guo hexagram; things have no ending in the eyes of human, so the next is Wei Ji hexagram after Ji Ji*", are explaining the law of things changing in cyclic order.

Denote the 8 natural phenomena respectively by signs, i.e., the heaven by $\Uparrow$, the earth by $\Downarrow$, the thunder by $\aleph$, the mountain by $\wedge$, the fire by $\Im$, the water by $\vee$, the lake by $\uplus$ and the wind by $\Xi$. Then, the 64 hexagrams with continuity flows $C_i^L, 1 \leq i \leq 64$ are shown in Table 11.1.

| **Order** | **Hexagram** | **Sign** | $C_6^{L_i}, 1 \leq i \leq 32$ | **Order** | **Hexagram** | **Sign** | $C_6^{L_i}, 33 \leq i \leq 64$ |
|---|---|---|---|---|---|---|---|
| 1 | Qian | $\Uparrow\Uparrow$ | $(++++++)$ | 33 | Dun | $\Uparrow\wedge$ | $(--++++)$ |
| 2 | Kun | $\Downarrow\Downarrow$ | $(------)$ | 34 | Da Zhuang | $\aleph\Uparrow$ | $(++++--)$ |
| 3 | Zhun | $\vee\aleph$ | $(+---+-)$ | 35 | Jin | $\Im\Downarrow$ | $(---+-+)$ |
| 4 | Meng | $\wedge\vee$ | $(-+---+)$ | 36 | Ming Yi | $\Downarrow\Im$ | $(+-+---)$ |
| 5 | Xu | $\vee\Uparrow$ | $(+++-+-)$ | 37 | Jia Ren | $\Xi\Im$ | $(+-+-++)$ |
| 6 | Song | $\Uparrow\vee$ | $(-+-+++)$ | 38 | Kui | $\Im\uplus$ | $(++-+-+)$ |
| 7 | Shi | $\Downarrow\vee$ | $(-+----)$ | 39 | Jian | $\vee\wedge$ | $(--+-+-)$ |
| 8 | Bi | $\vee\Downarrow$ | $(----+-)$ | 40 | Jie$^1$ | $\aleph\vee$ | $(-+-+--)$ |
| 9 | Xiao Xu | $\Xi\Uparrow$ | $(+++-++)$ | 41 | Sun | $\wedge\uplus$ | $(++---+)$ |
| 10 | Lv$^1$ | $\Uparrow\uplus$ | $(++-+++)$ | 42 | Yi$^2$ | $\Xi\aleph$ | $(+---++)$ |
| 11 | Tai | $\Downarrow\Uparrow$ | $(+++---)$ | 43 | Guai | $\uplus\Uparrow$ | $(+++++-)$ |
| 12 | Pi | $\Uparrow\Downarrow$ | $(---+++)$ | 44 | Gou | $\Uparrow\Xi$ | $(-+++++)$ |
| 13 | Tong Ren | $\Uparrow\Im$ | $(+-++++)$ | 45 | Cui | $\uplus\Downarrow$ | $(---++-)$ |
| 14 | Da You | $\Im\Uparrow$ | $(++++-+)$ | 46 | Sheng | $\Downarrow\Xi$ | $(-++---)$ |
| 15 | Qian | $\Downarrow\wedge$ | $(--+---)$ | 47 | Kun | $\uplus\vee$ | $(-+-++-)$ |
| 16 | Yu | $\aleph\Downarrow$ | $(---+--)$ | 48 | Jing | $\vee\Xi$ | $(-++-+-)$ |
| 17 | Sui | $\uplus\aleph$ | $(+--++-)$ | 49 | Ge | $\uplus\Im$ | $(+-+++-)$ |
| 18 | Gu | $\wedge\Xi$ | $(-++--+)$ | 50 | Ding | $\Im\Xi$ | $(-+++-+)$ |
| 19 | Lin | $\Downarrow\uplus$ | $(++----)$ | 51 | Zhen | $\Im\Im$ | $(+--+--)$ |
| 20 | Guan | $\Xi\Downarrow$ | $(----++)$ | 52 | Gen | $\wedge\wedge$ | $(--+--+)$ |
| 21 | Shi He | $\Im\aleph$ | $(+--+-+)$ | 53 | Jian | $\Xi\wedge$ | $(--+-++)$ |
| 22 | Bi | $\wedge\Im$ | $(+-+--+)$ | 54 | Gui Mei | $\aleph\uplus$ | $(++-+--)$ |
| 23 | Bo | $\wedge\Downarrow$ | $(-----+)$ | 55 | Feng | $\aleph\Im$ | $(+-++--)$ |
| 24 | Fu | $\Downarrow\aleph$ | $(+-----)$ | 56 | Lv$^2$ | $\Im\wedge$ | $(--++-+)$ |
| 25 | Wu Wang | $\Uparrow\aleph$ | $(+--+++)$ | 57 | Xun | $\Xi\Xi$ | $(-++-++)$ |
| 26 | Da Xu | $\Uparrow\wedge$ | $(+++--+)$ | 58 | Dui | $\uplus\uplus$ | $(++-++-)$ |
| 27 | Yi$^1$ | $\wedge\aleph$ | $(+----+)$ | 59 | Huan | $\Xi\Im$ | $(-+--++)$ |
| 28 | Da Guo | $\uplus\Xi$ | $(-++++-)$ | 60 | Jie$^2$ | $\wedge\uplus$ | $(++--+-)$ |
| 29 | Xi Kan | $\vee\vee$ | $(-+--+-)$ | 61 | Zhong Fu | $\Xi\uplus$ | $(++--++)$ |
| 30 | Li | $\Im\Im$ | $(+-++-+)$ | 62 | Xiao Guo | $\aleph\wedge$ | $(--++--)$ |
| 31 | Xian | $\uplus\wedge$ | $(--+++-)$ | 63 | Ji Ji | $\vee\Im$ | $(+-+-+-)$ |
| 32 | Heng | $\aleph\Xi v$ | $(-+++--)$ | 64 | Wei Ji | $\Im\vee$ | $(-+-+-+)$ |

**Table** 11.1. **Double** 64 **hexagrams or field interactions**

**5.2.State Changing.** Clearly, the Ancient Chinese philosophy believes that the state changing of thing is the result of interaction in Yin and Yang powers within a thing, namely the strength of Yang power in thing is from prosperity to decline, the strength of

Yin power is from the decline to prosperity and then, the strength of Yang power from decline to prosperity, the strength of Yin power from prosperity to decline, $\cdots$, i.e., a cyclic changing. Let $f^-(t)$, $f^+(t)$ be respectively the strength functions of Yin and Yang fields at time $t$. Then, such a cyclic evolving of Yin and Yang powers on $t$ can be expressed as

$$\begin{aligned}&\cdots \min f^- \to \max f^- \to \min f^- \to \max f^- \to \min f^- \cdots,\\ &\cdots \max f^+ \to \min f^+ \to \max f^+ \to \min f^+ \to \max f^+ \cdots\end{aligned} \tag{11.24}$$

and there is $f^+(t)+f^-(t)=c$, a constant at any time $t$. Thus, if the function $f^-(t)$ reaches its maximum then $f^+(t)$ is minimum at time $t^-$. Conversely, if $f^+(t)$ is maximum at time $t^+$ then $f^-(t)$ is minimum at time $t^+$. Thus, the evolving of $f^-(t)$, $f^+(t)$ characterizes the "*return is the moving of Tao and weakness is the usefulness of Tao*" in Chapter 40 of *Tao Te Ching*, namely, it's evolving can be characterized by a 2-tuple $\{f^+(t), f^-(t)\}$. Accordingly, the cyclic changing of thing $T$ can be characterized by a closed curve $C$ labeled by a function of $\{f^-(p), f^+(p)\}$ at point $p$ on $C$. In other words, the whole evolving process of thing $T$ can be characterized by a 2-tuple $\{C, L^2\}$ under the action of Yin and Yang powers. Here, $C$ is a closed simple curve or circle $C[t], t\in\mathbb{R}$ with uncountable points and a mapping $L^2: p\to\{f^-(p), f^+(p)\}$ at any point $p\in C$. Certainly, for an integer $n\in\mathbb{Z}^+$ it is also possible to choose $n$ points on curve $C$ to characterize the evolving of 2-tuple $\{C, L\}$ for simplicity. In this case, the closed curve $C$ is replaced by a circuit $C_n$ and the 2-tuple $\{C, L^2\}$ by a harmonic flow $C_n^{L^2}$, as shown in Figure 11.34 in case of $n=6$. So, *how many points on the closed curve $C$ is appropriate for characterizing the evolving of thing $T$?* Dr.Ouyang explains that the Yin and Yang powers of thing have a central axial symmetry. Generally, the larger the value of number $n$ is, the more detailed process on evolving of thing will be characterized but the more observing points and data will be needed. Generally, the points are arranged symmetrically along the central axis of Yin and Yang powers.

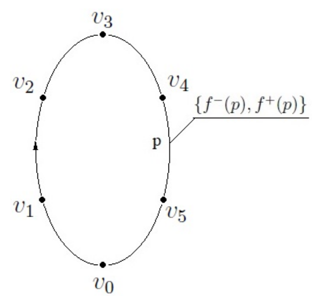



**Figure** 11.34. $C_6^L$ **evolving**

Then, *why does it use* 6 *lines in each hexagram of Change Book, not the* 5 *or* 7 *lines?* Dr.Ouyang tells Huizi that the evolving of any thing can be roughly divided into 6 stages, i.e., beginning, forming, smoothing, rising, peaking and falling. For example, the creation through by "*Tao creates One, One creates Two, Two creates Three and Three creates all things*" can be simply divided into 6 stages as the chaos, forming, inflating, smoothing, limiting and collapsing corresponding to the "*going up, the heaven is full of power*" of Qian hexagram in *Change Book*, namely the first ninth is a hidden dragon, unsuitable for using corresponding to the chaotic universe; the 2nd nine is a showing dragon corresponding to the forming universe; the 3rd nine is an alerting dragon corresponding to the inflating universe; the 4th nine is a jump dragon corresponding to the smoothing universe; the 5th nine is a flying dragon corresponding to the limiting universe; the last nine is a regret dragon for flying too high corresponding to the collapsing universe and then, begin to the next cycle, i.e., the evolving of things "*from a cradle to the grave*" follows the rule of "*six realms of reincarnation*".

Generally, the state changing $\{C, L^2\}$ of thing $T$ can be simulated by a mapping $\iota : T \to \mathrm{Im}S$, the trend of harmonic flow $C_6^{L^2}$ and the interaction between Yin and Yang powers including: ①the trend of harmonic flow $C_6^{L^2}$ is $v_0 \to v_1 \to v_2 \to v_3 \to v_4 \to v_5 \to v_0$; ②the inner field and outer field follow the mutual promoting and restraining of five elements, namely the metal promotes the water, the water promotes the wood, the wood promotes the fire, the fire promotes the earth and the earth promotes the metal, and the metal restrains the wood, the wood restrains the earth, the earth restrains the the water, the water restrains the fire and the fire restrains the metal. For example, Xu hexagram is $C_6^{L_5}$ with $(+++-+-)$, i.e., an inner field Qian hexagram in metal with an outer field Kan hexagram in water. Now that the metal promotes the water, so it is "*beneficial to the initiative attack by applying the advantage*"; Da Zhuang hexagram is $C_6^{L_{34}}$ with $(++++--)$, i.e., an inner field Qian hexagram in metal with an outer field Zhen hexagram in wood. Now that the metal restrains the wood, so it needs the integrity, i.e., "*hold on the integrity*"; ③all lines follow that "*the union of a Yin and a Yang is a kind of Tao*", a Yin field under a Yang field is smooth but restraining if over a Yang field; ④the first, 2nd, 3rd lines in inner field are matched with the 4th, 5th and the last lines in outer field, following the rule that "*the same charges repel, different charges attract each other*", namely one Yin line is smoothly matched with one Yang line, two Yin lines or two Yang lines are restrained; ⑤the first, 3rd and 5th positions in hexagram are Yang positions of lines, the 2nd, 4th and last positions are Yin positions of lines. A Yang line on Yang position and a Yin line on Yin position are "*right*". Otherwise, it is "*wrong*" on line position. After that, the 2nd and 5th lines in hexagram are the "*middle*" position of lines. A Yang line in middle is symbolizing a virtue of strong, a Yin line in middle is symbolizing a gentling and agreeable virtue. For example, $C_6^{L_{63}}$ is Ji Ji hexagram with $(-+-+-+)$, i.e., all lines are both right and middle, all up lines are matched with that of down lines; ⑥the recombination of one hexagram can presents a number of related hexagrams. Generally, a hexagram by selecting the 5th, 4th, 3rd lines to be the outer field and the 4th, 3rd, 2nd lines to be the inner field is called its *mutual hexagram*. For example, the mutual hexagram of Xian hexagram $C_6^{L_{31}}$ with $(--+++-)$ is Bo hexagram $C_6^{L_{23}}$ with $(-+++++)$; a hexagrams obtained by transforming the dynamic line in a hexagram, i.e., from Yin to Yang or from Yang to Yin is called its *changed hexagram*. For example, let the six 5th line be dynamic in Qian hexagram $C_6^{L_{15}}$ with $(--+---)$. Then, its changed hexagram is Jian hexagrams $C_6^{L_{39}}$ with $(--+-+-)$.

Dr.Ouyang tells Huizi that the ancient Chinese philosophers believe that to hold on evolving of thing should first determine which hexagram in $\mathrm{Im}S$ is suitable to characterize the behavior of thing under the action of Yin and Yang fields, determine which one of lines is the dynamic line and then, judge the trend of thing from the changing of Yin and Yang lines. Huizi asks Dr.Ouyang: "*What should I do if I can't determine the hexagram or the line fields?*" Dr.Ouyang tells Huizi that as the soul and essence of Chinese science, there are 64 hexagrams or harmonic flows $C_6^{L_i^2}, 1 \leq i \leq 64$ from the balance of Yin and Yang by Tao of the heaven in *Change Book* for simulating the overall states of thing. A key point is to determine which hexagram of the 64 hexagrams a thing follows and which line is dynamic. This is the way that the ancient Chinese philosopher systematically perceives a thing. Usually, the 64 hexagrams with lines correspond to things that can not be accurately known in situation, namely there are unknowable matters affecting the evolving of thing. In this case, it is necessary to give up the human's subjective thinking to simulate the evolving of thing following Tao. For example, the "*Great Divination*" explained by Confucius in *Xi Chi Zhuan*, namely take 49 out of the 50 Achillea Sibirca and divide them into two piles randomly for simulating Taiji, Yin and Yang, heaven, earth and humans, Taiyin, Shaoyin, Taiyang and Shaoyang, count them four by four and draw a hexagram, and have also other ways for determining the hexagrams by later generations on the great divination. Notice that its essence is to simulate Tao of the heaven for determining the hexagram and then, predict the evolving of thing according to the positions, trends, mutual and changed hexagrams of lines in the hexagram, which is not praying the heaven but abandoning the subjective consciousness of human to get the developing law of unknown things, namely following the dialectical relationship that "*objective things do not depend on human's will*" in today. Huizi asks Dr.Ouyang in puzzling: "*Dad, isn't it a bit like fortune-telling?*" Seeing his daughter in ambiguity on this issue and needing to enlighten, Dr.Ouyang asks her: "*Do you think the 'fortune-telling' is a derogatory term or a positive one?*" Huizi replies: "*It is of course a derogatory one, a word for cheating!*" Dr.Ouyang asks her again: "*if the fortune-telling is derogatory, why did Confucius say that 'one can't* *be a gentleman if he does not know his destiny'? You know, the 'gentleman' in Confucius mouth is a virtuous human rather than a common one!*" Huizi asks with more indefinitely: "*Is it science lacking and humans' ignorant in the period of Confucius?*" Dr.Ouyang explains: "*Of course Not! Confucius lived in the period of agricultural civilization of China, which is a period that all schools of thoughts are active and the thoughts deeply on the dialectical relationship of humans with the nature. Thus, it is impossible to judge in a deceptive way whether a human is virtuous or his behavior follows Tao*". Huizi asks back Dr.Ouyang still in puzzling: "*Dad, do you think the 'fortune-telling' is positive or derogatory?*" Dr.Ouyang tells her that the term "*fortune-telling*" must been positive, i.e., understanding and following Tao of the heaven rather than cheating that we understand today in the period of Confucius because Confucius regarded the "*cultivating virtue*" as the essence in education. After hearing Dr.Ouyang's explanation, Huizi has a little knowledge on *Change Book* and asks Dr.Ouyang further: "*Dad, the hexagram with lines defined like this can only give the trend or probability of thing, not a definite conclusion. Is that so?*" Seeing that his daughter is able to present such a question, Dr.Ouyang replies happily: "*Yes, it is caused by the recognitive limitation of humans following the doubt, will not use it if have no doubt on a thing, where the 'doubt' means that there are doubts on the evolving of thing and need to be relieved. So, the result can only be characterized by possibility.*" Dr.Ouyang then explains further: "*To characterize the behavior of things by probability can be also seen in modern science. For example, the modules of state vectors $|\phi\rangle$, $\langle\psi|$ in quantum mechanics are nothing else but the probability functions. The only difference is in that the quantum mechanics uses an infinite number of basic vectors whereas the Change Book uses only* 64 *states* $C_6^{L_i}, 1 \leq i \leq 64$." Huizi suddenly see light after hearing Dr.Ouyang's such an explanation: "*Surely, the quantum mechanics also uses probability to characterize the behavior of microscopic particles but in terms of the recognition of state of things, the Change Book is more than 5000 years earlier than that of quantum mechanics, which is indeed Chinese science on the foresight!*"

Dr.Ouyang tells Huizi that *Change Book* is essentially the ancient Chinese philosopher's recognition on Tao of things or science in today's terms. In this respect, different from the western science on local recognition, the *Change Book* simulates the whole evolving, i.e., "*from the birth to the grave*" or Tao in the life cycle of thing, which should be the mainstream of scientific developing. However, the *hedonism* has prevailed during the evolving of human civilization in the last hundred years, which results in that the notion of ancient Chinese philosophers is not universally accepted by the scientific community but is the local recognition or science in flood to applications because science's local recognition on Tao showing on things is easy to satisfy one's demands in present to some extent, which results in the excessive intrusion on nature and inconsistent with that the harmony of humans with the nature or following Tao for humans' living.

## §6. **Comments**

**6.1.** Generally, the evaluation on scientific achievements of a country or nation lies in the standard. Certainly, the branch developing of science is easy to achieve local achievements on scientific subjects which created really the western industries but only now, it has left the nature with many worldwide problems in the passed more than 300 years, also affecting the human survival. Different from the developing of western science, Chinese science has its own distinctive characters. It emphasizes the "*unity of humans with the nature*" and holds the law of thing evolving with the whole life cycle of thing. Certainly, it is not easy to achieve local scientific achievements and can not bring the industrial revolution into being but it will not also bring the crisis to human existence. Notice that the *Science and Civilisation in China*, a multi-volume book compiled by a British science historian Joseph Needham according to science subjects in western, systematically surveyed the developing of ancient Chinese science and its contribution to world civilization, including the philosophy, mathematics, physics, chemistry, astronomy, geography, biology, agronomy, medicine and many other fields. However, it is not enough to show the contribution of Chinese to human civilization. For example, the *Change Book* using 64 hexagrams to model the state of things, applying 6 lines of hexagram to simulate the 6 stages in its whole life cycle and other systematic sciences on recognition of the whole process of things are not include in this book, not accepted by the western science also because this life-cycle approach to the overall knowledge of things does not fit into any subject of science and some Chinese themselves studying in the west also regard it as a rule of divination rather than science, including the corresponding and operating of the twelve meridians with the internal organs of human body. However, the most of Chinese sciences are established on *Change Book* such as the theories of Chinese traditional medicine, Chinese astronomy and Chinese architecture.

**6.2.** The ancient Chinese regarded the balance of Yin and Yang as Tao of the heaven. They lived with the criterion of "*unite of humans with the nature*", the mutual promoting and restraining of "*five elements*" and recognized the evolving of unknown things by *Change Book*, i.e., a phenomenological science shown in the classics such as those of *Change Book*, *Tao Te Ching* and *Emperor's Inner Canon*, See [Fur1]-[Fur3], [Lin], [Luj1]-[Luj2], [Ren1]-

[Ren2], [Zha] and [Zhu], etc. Among these, an interpretation or "*enlightening*" on the Laozi's creation of "*Tao creates One, One creates Two, Two creates Three and Three creates all things*" and the rule of humans, i.e., "*humans follow the earth, the earth follows the heaven, the heaven follows Tao and Tao follows the nature*" can be found in J.C.Lu [Luj1] and then, an insight into One, Two and Three on the whole creation of universe shows that Laozi's creation contains the Big Bang theory as a special case.

**6.3.** The essence of twelve meridians in traditional Chinese medicine lies in that the human body corresponds to the running of vital energy on the twelve meridians and Ren, Du meridians in theory, i.e., a harmonic flow with 362 vertices and the vital energy flow running on edges, see [Mao46] and [Ren2] for details. Thus, although the continuity flow is a generalization of network flow or $G$-flow [Mao38], a real example is the abstraction of human body by the vital energy running on twelve meridians and Ren, Du meridians, i.e., a harmonic flow. Similarly, the animal bodies such as those of the horses, cattle, sheep and dog can be also characterized by harmonic flows of vital energies running on their twelve meridians and Ren, Du meridians and furthermore, we should find harmonic flows on all livings running with vital energy on meridians.

**6.4.** Usually, the researchers who study *Change Book* are divided into two groups, i.e., those on the changing principle of hexagrams and lines, and those on divining, to predict the trend of things by Tao of the heaven contained in hexagrams. Certainly, both of them are actually analyzing the relationship of hexagrams and lines with the evolving of things. In the humanity, Confucius interpreted the *Change Book* as a moral classic of Confucianism and believed that "*one can't be a gentleman if he does not know his destiny*", see [Fur1] for details and put forward the requiring of "*study the nature of things for acquiring knowledge*" for a gentleman on knowing the developing laws of things. And so, the *Change Book* is not only a book but also presents a scientific method for holding on the truth of things, i.e., a system science in ancient China.

*Progress is the activity of today and the assurance of tomorrow.*

– By R.W.Emerson, an American thinker

## Chapter 12

# Science's Philosophy

************************************************************************

**All Mountains Cover the Sun on the Sky,**

**See the Yellow River Flows Into the Sea;**

**We can Enjoy a Grander Sight,**

**If We Climb to a Greater Height.**

– *Climbing Stork Tower* by Z.Wang, a poet in Tang dynasty of China

## §1. **Conditional Truth**

Certainly, science is the human's knowledge on the verifiable basis of objective things with changing laws, and it is established on the human perception with priori assumptions on the characters of things, including: ①All objective things have characters that can be identified by humans; ②All characters of objective things are inherent to the objective things themselves that do not change by whom or when they are observed; ③All characters of objective things can be characterized in a generally accepted way. Usually, the purpose of these assumptions is to let the characters of objective things exist independently of human and can not be changed by their observed actions. Otherwise, if a character of thing is beyond human's perception ability or can not be accurately characterized, then it must not be fully recognized by humans. Accordingly, one can not perceive or accurately perceive the thing in this case. For example, Heisenberg's uncertainty principle in quantum mechanics, namely one can not accurately measure the momentum and position of a microcosmic particle at the same time, indicates that one can not accurately perceive the behavior of a microcosmic particle but can only perceive it with the probability of its occurrence in a certain place such as the superposition state of living and death of Schrodinger's cat in Figure 12.1 because science is human's recognition of laws showing on things or Tao while the sense organs of human, i.e., the eyes, ears, nose, tongue, body and mind have limitations in perceiving things, which is caused by human recognitive limitations.

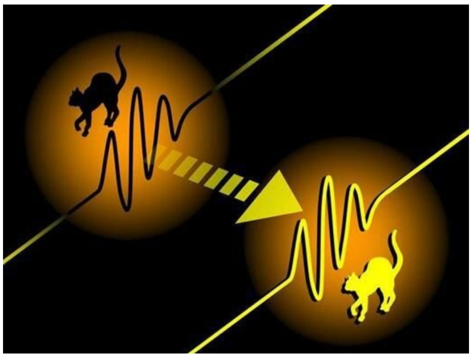


**Figure** 12.1. **Superposition**

A general paradigm of scientific recognition is shown in Figure 12.2,

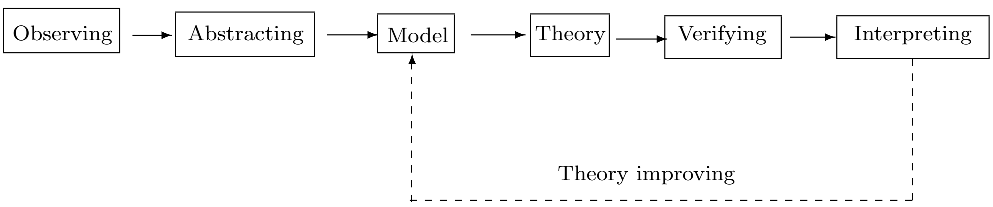



**Figure** 12.2. **Scientific recognitive paradigm**

where the observing is the basis of recognition of human on characters of thing. It is purposefully perceiving on thing to acquire the showing characters or empirical facts of thing. On this basis, the causes of thing appearing are analyzed, researching questions and analysis models are put forward, the data collecting, abstracting, analysis and verifying are carried out in a planned and purposeful way, a preliminary interpreting or answering is obtained and then, returns to the model or practice test to hold on the behavior of thing. Here, the scientific abstracting lies in holding on the essence from phenomena and forming scientific conceptions or characters; the recognitive model is a thought model for simulating the behavior of thing, including the mathematical model; the verifying is on the theory by scientific experiment; the interpreting is applying the conception, judging, reasoning and other logical thinking to restore the causality of thing characters and form the conclusion on thing.

So, *why do we say that science is a conditional truth?* Dr.Ouyang explains that science is the knowledge formed by humans on things. It is a systematic knowledge formed by forming theories from abstractions and hypothesis under certain conditions, including observing tool, time, position, environment and the characters of thing and then, verifying the behavior of thing under the same conditions. Some of these conditions are artificial and take the form of propositional statements such as "*if* $\cdots$*, then* $\cdots$" or specified conditions such as "*has* $\cdots$ *conclusion in vacuum*". Certainly, some of them are implicit in the conclusion because most of the human observation on thing is carried out on the earth, by gravity, geomagnetic or environmental action, etc. For the conditional truth, Huizi asks Dr.Ouyang in puzzle: "*Dad, it is seemingly wrong! Is* $1+1=2$ *only true in condition?*" Dr.Ouyang smilingly says: "*Surely,* $1+1=2$ *is in an implicit condition, namely it is assumed that the operation is carried out in a number field containing the symbols* 1 *and* 2 *holding with* $1+1=2$ *such as the number field* $\mathbb{R}$*. However,* $1+1=0$ *in* $\mathbb{Z}_2$ *and* $1+1=1$ *in logical truth. Both of them are not* $1+1=2$*. Such kind of implicit conditions are often overlooked but take it for granted that* $1+1=2$*, namely assuming that it is running on* $\mathbb{R}$ *and ignoring the condition.*" Huizi nods and then says: "*It's really so! But even so, what's the value of emphasizing that science is conditional?*" Dr.Ouyang explains that the human's recognition on everything's law is through by local Tao on things, i.e., the science established by humans is not the everything's law but the local recognition of local Tao or the conditional truth. Here, the purpose of pointing out the limitation or condition of science is not to detract from the function of science but to awaken humans to look at the scientific achievements in the binary system formed by humans with the nature, examining the applying conditions of science under the harmonious coexistence of humans with the nature again, which has been neglected by humans ourselves in the past.

**1.1.Conditional Science.** Generally, science is divided into 5 categories, i.e., the natural science, social science, thinking science, formal science and the interdisciplinary science.

Among them, the natural science is on the material structure, form, character and moving law of the nature; other sciences are based on or derived from the natural science. For showing the conditional conclusions, we select some achievements in the mathematics, physics, chemistry and the biology of natural sciences as examples in the following.

(1) **Mathematics.** Generally, mathematics is the abstracting product of objective things with evolving, namely a system of symbols on logical conformity and a mathematical conclusion is holding in a specific mathematical system. Dr.Ouyang explains that a set $\mathcal{A}$ with operations is called a *mathematical system* if the operating result of any two elements of $\mathcal{A}$ still belongs to $\mathcal{A}$. For example, the real numbers $\mathbb{R}$, complex numbers $\mathbb{C}$, groups, rings, fields, vector spaces with calculus, geometry, functional and operator systems established on them, etc. Generally, a *mathematical reality* is the result of logical reasoning in a given symbolic system or under given conditions. It is a kind of symbolic reality and its conditions can be expressed in two forms: one is in a self-explanatory mathematical system. For example, "*the sum of three interior angles in a triangle is* $180^o$" is holding in Euclidean geometry with operations in the real number $\mathbb{R}$, which can not in any other geometric system because the sum of three interior angles in a triangle is greater than $180^o$ in elliptic geometry and the sum of three interior angles in a triangle is less than $180^o$ in hyperbolic geometry, as shown in Figure 12.3$(a)$ and $(b)$, respectively; another is adding additional conditions in a mathematical system. For example, the Lagrange theorem that "*for any subgroup* $H \prec G$ *of finite group* $G$, $|G| = |H||G:H|$" in group theory, its additional condition is that "$G$ *is a group with* $|G| < \infty$", namely the number of its elements is finite, where $|G:H|$ denotes the coset number of subgroup $H$ in $G$, i.e., $|G|/|H|$.

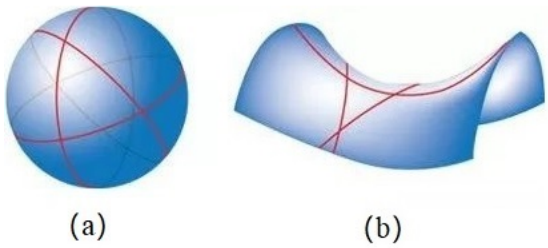



**Figure** 12.3. **Non-Euclidean triangle**

Notice that mathematics is the symbolic abstraction of special behaviors in cases or situations of objective things and the product of logical reasoning by humans, which may be not necessarily correspond to things but the real symbols. Generally, if we use a mathematical system $\mathcal{A}$ to characterize the behavior of a thing $T$, we essentially assume that the behavior of thing $T$ follows the mathematical rules in $\mathcal{A}$. This is of course a man-made ideal hypothesis similar to that of the Pythagorean assertion, i.e., "*all things are numbers*", which is identifying a thing in terms of its a particular character and leading to that mathematics can characterize all things. However, it is not the case, i.e., the scope of mathematical reality is small than the natural reality. For example, Dr.Ouyang tells Huizi that $1+1=2$ is the result of counting two disjoint objects together but can not apply to the case of counting elements in two intersecting sets $A$ and $B$ because $|A\cup B| = |A| - |B| + |A\cap B|$ in this case, and $|A\cup B| = |A| + |B|$ if and only if $|A\cap B| = 0$ or $A\cap B = \emptyset$. So, *why is* $1+1=2$ *but not* $1+1=3$*?* Dr.Ouyang explains that $1+1=2$ corresponds to counting objects one by one and so, there is the natural number set $\mathbb{Z}^+ = \{1,2,3\cdots\}$, which further extends to the integer $\mathbb{Z}$, real number $\mathbb{R}$ and the complex number $\mathbb{C}$. In the case of counting, $1+1$ is the successor of number 1, i.e., the number 2, $2+1$ is the successor of number 2, i.e., the number 3, $\cdots$. In this way, there must be $1+1=2, 2+1=3, \cdots$, which is why $1+1=2$, not $1+1=3$ in mathematics because the successor of number 1 is 2, not 3. However, there really exists the biological phenomenon of $1+1$ =3 in the nature, namely a male and a female can generate a baby, which can be regarded as a counting result with time in the natural number system $\mathbb{Z}^+$ such as the Fibonacci problem, etc.

(2) **Physics.** Physics is the science of understanding the moving laws and matter structure in the nature. It finds out patterns and theories from observing and analyzing various phenomena on matter and energy in the nature, including the physical phenomena, matter structure, action and moving laws. Notice that physics is the understanding of natural laws with interpretation of causality in the nature. For example, the classical mechanics has successfully explained many astronomical phenomena, the particle physics has determined the smallest detectable particles for explaining the origin of universe.

Certainly, the main character of physics lies in the exploration and recognition of Tao on things. Generally, all laws of physics are bound to be restricted by human's observing conditions, reference frame and the recognitive ability, which are conditional. Here we list some of them. ①The three laws of Newton are the laws of physics on macroscopic objects with slow velocity of moving and an inertial frame of reference, which can be not applied to microcosmic particles. Among them, the condition of Newton's 1st law is that there are no external force, that of the 2nd law is that there is an external force and the condition of 3rd law is that the transferring medium of force is completely elastic; ②The mechanical energy is the sum of the kinetic and potential energy of a particle and the condition of its conservation is that the particle is in a conservative force field. In this case, the potential and kinetic energy of particle can change but the total mechanical energy remains the same; the conditions of Coulomb's

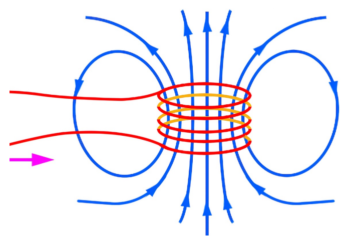


**Figure** 12.4. **Magnetic flux**

law in electromagnetic fields are electrostatic field in vacuum and the charges $q_1, q_2$ are point charge, the condition of electromagnetic induction is that the flux passes through a conductor loop as shown in Figure 12.4. Thus, the instant electromotive force $\mathbf{E}_i$ in the conductor loop is proportional to the changing rate of magnetic flux $\mathbf{\Phi}$ through the loop with respect to time, expressed as

$$\mathbf{E}_i = -\frac{\mathbf{d\Phi}}{dt} \tag{12.1}$$

in the SI system of units, called the *Faraday's Law of electromagnetic induction*; ④The three laws of thermodynamics characterize the propagation of heat energy. The condition of the 1st law of thermodynamics is the situation of heat transferring from the outside to a closed system, i.e., its part increases the energy in the system and part is used for the system to do work on the outside world, namely the law of heat transferring and conversion inside and outside of the system. The conditions of the 2nd and 3rd laws of thermodynamics are in a closed linear system, which implies that the heat propagation can only spread from the hot to cold, which is irreversible and increases the entropy, reaches its maximum value as the system is in the stable equilibrium; ⑤The condition of the law of photoelectric effect is that light illuminates a metal plate. In this case, if the frequency $\nu$ of incident light is greater than the limiting frequency $\nu_0$ of metal plate, the electron $e$ on the metal plate will escape the photon of energy $\hbar\nu$, result in the photoelectric effect and generally, the higher the frequency $\nu$ of incident light is, the greater the initial kinetic energy of electron is, and holds with Einstein's equation

$$\hbar\nu \ = \ e\mu + \frac{1}{3}m_e v_e^2 \tag{12.2}$$

on the photoelectric effect, where $\hbar = 6.63 \times 10^{-34}$ is the Planck's constant, $e\mu$ is the work of an electron escaping from the metal, $m_e = 0.511MeV$ is the electron mass, $v_e$ is the escaping velocity of electron; ⑥The condition of Einstein's mass-energy equation $E = mc^2$ is in the relativistic mechanics, namely the two distinct characters, i.e., the energy and mass of particle of classical mechanics are the different behavior of the same character in relativistic mechanics, i.e., they are only different in a squared factor $c^2$ of light speed $c$.

(3) **Chemistry.** Generally, chemistry is studying the composition, properties and structure of substances at the molecular and atomic level, as well as the changing laws of substances under conditions. Usually, the material changing is divided into physical changing and chemical changing, where the physical changing refers to a change in the form or state of a substance but no new substance is generated. As opposed to the physical changing, a chemical changing refers to the process of atomic or electron conversion, transferring or recombination between molecules under conditions to generate new molecules accompanied with energy changing.

Certainly, the main purpose of chemistry is to discover the recombination conditions and laws of matter on atomic conservation and then, understand the formation of matter at the level of molecules and atoms. Some common equations for chemical reactions are listed following: ①Combustion, such as the magnesium $2Mg + O_2 \to 2MgO$, iron $3Fe + 2O_2 \to Fe_3O_4$, hydrogen $2H_2 + O_2 \to 2H_2O$, red phosphorus $4P + 5O_2 \to 2P_2O_5$, sulfur powder $S + O_2 \to SO_2$, carbon $C + O_2 \to CO_2$, $2C + O_2 \to CO_2$, carbon monoxide $2CO + O_2 \to 2CO_2$, methane $CH_4 + 2O_2 \to CO_2 + 2H_2O$, alcohol $C_2H_5OH + 3O_2 \to 2CO_2 + 3H_2O$ etc.; ②Halogenation, such as the $2Na + Cl_2 \to 2ClNa$, $2Fe + 3Cl_2 \to 2FeCl_3$, $Cu+Cl_2 \to CuCl_2$, etc.; ③Vulcanization reactions, such as the $2Na+S \to Na_2S$, $Fe + S \to FeS$, $2Cu + S \to Cu_2S$, $O_2 + S \to SO_2$, etc.; ④Nitriding reactions, such as the $3Mg + N_2 \to Mg_3N_2$, $N_2 + O_2 \to 2NO$, $2NO + O_2 \to 2NO_2$, $3NO_2 + H_2O \to 2HNO_3 + NO$ etc.; ⑤Iron reactions, such as $3Fe + 2O_2 \to Fe_3O_4 \to Fe_3O_4 \to Fe_3O_4 \to Fe_3O_4 \to FeSO_4 + H_2$, $Fe + H_2SO_4 \to FeSO_4 + H_2$, $Fe + 2HCl \to FeCl_2 + H_2$, etc; ⑥Sodium reaction, such as the $4Na + O_2 \to Na_2O$, $2Na + O_2 \to Na_2O_2$, $2Na + Cl_2 \to 2NaCl$, $2Na + S \to Na_2S$, $Na_2O + H_2O \to 2NaOH$, $Na_2O + CO_2 \to Na_2CO_3$, $Na_2O + 2HCl \to 2NaCl + H_2O$, etc.; ⑦Methane reactions, such as the methane combustion $CH_4 + 2O_2 \to CO_2 + 2H_2O$ in Figure 12.5 and $CH_4 + Cl_2 \to CH_3Cl + HCl$, $CH_4 \to C + 2H_2$, etc.; ⑧Ethylene reactions, such as the $H_2C = CH_2 + 3O_2 \to 2CO_2 + 2H_2O$, $H_2C = CH_2 + H_2 \to CH_3CH_3$, $H_2C = CH_2 + HCl \to CH_3CH_3Cl$, $H_2C = CH_2 + H_2O \to CH_3CH_2OH$, polymerization $nH_2C \to \lbrack CH_2CH_2 \rbrack_n$, etc. ⑨Acetylene reactions, such as the $HC \equiv CH + 5O_2 \to 4CO_2 + 2H_2O$, $HC \equiv CH + H_2 \to H_2C = CH_2$, $HC \equiv CH + HCl \to CH_2 = CHCl$, polymerization reaction $nHC \equiv CH \to \lbrack CH - CH \rbrack_n$, etc.; ⑩Benzene reactions, such as $2C_6H_6 + 15O_2 \to 12CO_2 + 6H_2O$, $C_7H_8 + 9O_2 \to 7CO_2 + 4H_2O$, etc.

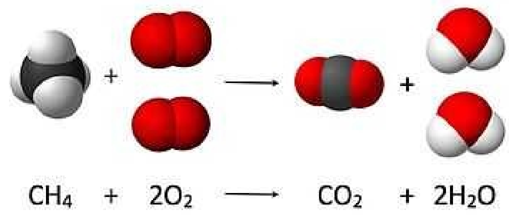



**Figure** 12.5. **Methane combustion**

Notice that substances in the nature are in a stable state and do not spontaneously react in general. Thus, the chemical reactions between substances need external conditions to occur. For example, there should be combustible substances in the combustion reaction such as the wood, gasoline, alcohol and books, paper, clothing, the combustion supporting substances such as air and the fire source such as flames, sparks, etc. In this way, the combustion is generated by molecules under the external conditions. For example, the carbon combustion $C + O_2 \to CO_2$, $2C + O_2 \to 2CO$ means that a carbon molecule and an oxygen molecule generate a carbon dioxide molecule in a full combustion or two carbon monoxide molecules are generated in an inadequate combustion. However, the actual combustion substance is not pure carbon but also contains other substances. In this case, other things besides carbon dioxide and carbon monoxide are generated also after the combustion. For example, the wood is a compound composed mainly of carbon, hydrogen, oxygen and other elements, exists usually in the form of cellulose or sugars, gums, esters, water and other molecules. Usually, its combustion mainly generates the carbon dioxide, water vapor, formic acid, acetic acid and various flammable gases at about $200^oC$, a small amount of water vapor and carbon monoxide at $200^o - 280^oC$, the flammable vapors, particulate matter at $280^o - 500^oC$ and mostly carbon at $500^oC$.

(4) **Biology.** All creatures have the ability of "*survival of the fittest*", namely the character adapted to the environment in which they are living in order to eliminate those that are not adapted to the environment. Biology is studying the structure, function, occurring and developing of livings, including the plants, animals and microorganisms. Among them, different combinations of cells are the product of organisms adapted to the environment in which they are living. The main compositions in cell are the proteins and nucleic acids. Among them, the protein is the main carrier of life activities while the nucleic acid is the carrier of genetic information and important polymer materials in life. Generally, a large number of composing elements are C, H, O, N, etc., trace elements are Fe, Mn, Zn, Cu, B, Mo and so on for livings on the earth. And the basic character of livings is metabolism, namely the constant exchange of matter and energy for livings with the outside world.

**Figure** 12.6. **Plant growth**

So, *what are the basic conditions of living?* Dr.Ouyang explained that the plants, animals and microorganisms have different living condition, even the creature of same species may not live in the same condition. For example, ①Living condition of plant includes light, water, temperature and humidity, air and soil, etc., where the light and water are the basic factors in plant growth, the appropriate temperature and humidity are the environmental condition for plant growth, the air is the intermediary of plant photosynthesis, absorption carbon dioxide and releasing oxygen, respiration and the soil is the source of nutrients for plants; ②Living condition of animal includes suitable temperatures, oxygen, water, food and a certain amount of living space. Among them, the oxygen is the condition of aerobic respiration of animals and the food is the condition of nutrients or energy intake. Thus, an adequate supply of oxygen and food is necessary to sustain the animal life; ③Living condition of microorganism includes nutrients such as water, carbon, nitrogen, inorganic salts, growth factors and oxygen. In this case, the oxygen concentration and the pH of water required by different microorganisms may not be the same.

Generally, the living condition of higher animal is more harsher than that of the lower, i.e., the lower animal can survive but the higher may not under the same condition. For example, ①Among animals such as pigs, horses, cattle and sheep, they can eat grass and live but human can not because the stomach of human can not digest most leaves and grasses; ②Surface water on the earth can be divided into five classes of Category I - Category V, among which the water of Category I is available for human consumption and drinking after simple purification to eliminate harmful bacteria, viruses and micro-organisms in water. Certainly, the water quality of Classes II and III is slightly polluted and needs to be purified by routine professional treatment before it can be used for human drinking. However, the fish and shrimp can be cultured in the water of Classes II and III. And meanwhile, the human should not drink the water of Classes IV and V but the bacteria, viruses and microorganisms can survive in water of Classes IV and V; ③There are some immune mechanisms in animals that resists to the viruses or hostile environments but the human has not. For example, there are 61 viruses carried in bats which are human diseases. Usually, a bat can reach more than $100^oC$ during flight with its heart frequency nearly 200 times every second, which is incomparable to that of human.

**1.2.Science's Rule.** Notice that science is a conditional truth on the law of things. Its applications must consider the condition of science in the binary system $K_2^L$ of humans with the nature, i.e., while it enhances the well-being of humans so as to realize humans' actions according to Tao because the intrusion of human activity into the nature is a cumulative effect of time delay and the nature's reaction to humans may occur not in the present but in the offspring of human. In this case, a fundamental rule of scientific application is that it should guide the human activities that do not violate or intrude upon the operating of nature, i.e

**Eternal Tao's Rule.** *The operating rule of nature can be not changed or adjusted due to humans' action.*

Here, the "*eternal Tao*" is the same words that "*Tao told is not the eternal Tao*" in Chapter 1 of "*Tao Te Ching*" or natural law, namely the essence of living of humans in harmony with the nature or following Tao lies in acting according to the eternal Tao rather than in conforming laws on some things. Certainly, the eternal Tao's rule contains two implications. One is the nature will not adjust its operating for adapting the need of humans because "*the heaven and earth are not kind, taking all things as humble dogs*", i.e., compared with all things, the humans is too small, too less without superiority and another, all human activities should not intrude on the nature or within its permissible limits. Otherwise, the operating or reaction of nature against human intrusion must be a disaster for humans ourselves.

This eternal Tao's rule seems simple but it is often rejected by someone standing on the unary system $K_1^L$ or the human's arrogance. However, as long as a human stands on the notion of harmonious coexistence of humans with the nature, the ancient Chinese "*unity of humans with the heaven*" or acting follows Tao, this is a basic rule that all human activities should follow. However, the human civilization, especially after it into the industrial civilization, i.e., the human activities led by science, this rule is not strictly observed by humans, even mistakenly believe that the nature can tolerate all wastes in human activities. For example, the burning of leaves and branches in fire, i.e., $C + O_2 \to CO_2$ or $2C + O_2 \to 2CO$ motivated the human civilization came into being but *are the carbon dioxide and residue in the burning of leaves and branches under the human control, not intrusive to nature?* The answer is Not! Dr.Ouyang explains that the natural absorption of carbon dioxide is through the natural mechanisms such as the photosynthesis of green plants, water and soil, and the absorption capacity is limited. In the era of agricultural civilization, the carbon dioxide mainly came from the respiration of animals and plants, the decay and deterioration of corpses, natural fires in the mountains or forests and human's cooking fires. Among them, the "*respiration of animals and plants, decay and deterioration of corpses, natural fire in mountains or forests*" belong to the natural behavior of Tao while the "*human's cooking fire*" is a subjective behavior of human, not necessarily acting in accordance with Tao. But such a amount of carbon dioxide produced is very small, still within the nature's absorption capacity $R_{air} < R_{human}$. At that time, humans did not pay attention to the eternal Tao's rule and believed that the nature could tolerate all wastes of humans and their behavior would not cause a big danger to the natural operating. However, the industrial civilization was too dependent on the burning of fossil energy such as the coal, oil, paraffin, natural gas and wood, the mining and production of chemical products such as the coal, petroleum and those of energy-dependent industrial production and transportation emit more carbon dioxide than that the nature can absorb. At this time, the excessive carbon dioxide emitted naturally floats in air and the accumulation over years may cause the amount of $R_{air}$ in air to break the field balance existed in the binary system $K_2^L$. And then, the field $L(u)$ follows Tao to adapt to the emissions of excessive carbon dioxide, along with the natural disasters in the eyes of human.

Then, *how should we look at and apply science to human activities that follow Tao?* Dr.Ouyang explains that science is a local recognition or conditional truth on everything's law. It is true that human activities led by science need to strictly follow the eternal Tao's rule because the misuse of science will also destroy the condition of human survival, must be restricted or limited.

**Science's Rule.** *All human activities led by science should not disturb the nature.*

Here, the science's rule refers to the rule that human activities should be followed by science under the thought of harmonious coexistence of humans with the nature or the ancient Chinese of "*unity of humans with the heaven*", which contains that: ①Scientific exploration and research should be free from natural disturbance, including the discharge of some toxic and harmful substances in scientific experiments, probes, satellites, space stations, rocket debris and explosive fragments in outer space exploration must be recovered and properly disposed rather than allowed them floating freely in air after being scrapped and interfering with the normal operating of universe; ②All objects produced by the scientific application whether they are in solid, liquid or gas should be intrusion-free to the nature, where the word of "*intrusion-free*" is aimed at the natural operating, including the air, soil, water and all livings. Otherwise, the product that may disturb the natural operating should be actively retained or disposed of without disturbance. Let's consider the application of pesticides, fertilizers, rubber products and fire as an example. Certainly, the fertilizers and pesticides can increase the production of agricultural products but destroy the ecological balance in the soil; the rubber products do not degrade naturally in a short time while they improve the human life; the burning of a fire produces carbon monoxide and carbon dioxide, etc. All of these are necessary for humans to take the initiative to carry out natural disposal or retention of the products without intrusion and can not be abandoned to natural absorption or degradation; ③Scientific research and application are limited to the improving of human welfare, which should not go against the way of "*living*" because in the binary system $K_2^L$ of humans with the nature, the intrusion-free on nature includes no intrusion on humans ourselves, namely science is limited only to peaceful application. For example, the developing and application of atomic energy, transgenic, gene editing, virus research and others that may intrude the nature should be strictly limited to the peaceful use of humans.

[**Einstein's Lifelong Regret**] Certainly, the peaceful application of science is the basic embodiment of science's rule, which opposes to war. So, *how should we look at war and can science be applied to war?* Dr.Ouyang explains that the essence of war is the plunder of resources between races or families, which is a serious intrusion of humans to the nature. Then, *does the application of science to war go against the science's rule?* The answer is Yes because it has intensified the intrusion of humans to the nature, which is bound to violate the science's rule. Dr.Ouyang tells Huizi that there is a story on Einstein in world war II from his first suggesting to President Franklin D.Roosevelt by applying the nuclear fission theory to create the atomic bomb and then, to be shocked by the two atomic bombs destroyed cities of Hiroshima and Nagasaki in Japan, killing tens of thousands of humans, advocating the peaceful application of science in all his life.

Here is how it happened. In August 1939, Einstein wrote a two-page letter to Roosevelt, the president of United States in order to prevent the Nazi Party in power of German from being the first using nuclear fission theory to produce atomic bombs and destroy humans, suggested that the American government should hold on the initiative of war by investing persons, material and financial resources in research and manufacture of atomic bombs. It was proposed that the president Roosevelt could delegate a trusted human with the task of communicating, coordinating and organizing the departments concerned to ensure the supply of uranium to the United States and meanwhile, raising the funds, accelerating the test and cooperating with the industrial laboratories that have the relevant equipments for carrying out. After receiving Einstein's letter, Roosevelt accepted Einstein's suggestion and formed a uranium advisory committee to serve on the developing of atomic bombs in October 1939, which later joined to the Manhattan Project of United States and developed the two atomic bombs dropped on Hiroshima and Nagasaki later, namely the *Little Boy* and *Fat Man*".

On August 6, 1945, the army of United States dropped the atomic bombs of *Little Boy* and *Fat Man* on Hiroshima and Nagasaki, instantly flattening the two cities of Japanese (see Figure 12.7). This explosion killed 30,000 people and left hundreds of thousands homeless. In the following 15 years, the death toll has increased to 220,000 due to the radiation and radioactive contamination of this explosion.

**Figure** 12.7. **Atomic bombings on Hiroshima and Nagasaki**

However, Einstein did not participate in making bombs and worked on the *Manhattan*

*Project* because he was a German, also a left-wing political activist. When he heard about the consequences of atomic bombings on Hiroshima and Nagasaki, he was deeply regretted, called his letter to Roosevelt was "*Woe to me*", said that "*had I known that the Germans would not succeed in producing an atomic bomb, I would have never lifted a finger*" and told his friend that "*I always condemn the use of atomic bombs in Japan, but nothing can be done to prevent that fateful decision*".

From the atomic bombings of Hiroshima and Nagasaki until his death, Einstein devoted himself to science's morality and the world peace. He repeatedly talked about the peaceful application of science on various occasions. Certainly, his approach showed the notion of "*unity of humans with the heaven*" of ancient Chinese, which is the harmonious coexistence of humans with the nature in today. In July 1948, Einstein's letter to the *International Intellectual Peace Congress* included the words that the tragedy of scientists is that they help to produce more terrible and powerful weapons of destruction. Thus, it is the duty of scientist to prevent scientific achievements from being used for anti-human purposes. Only two days before his death, he signed the declaration on peaceful application of scientific achievements led by B.Russell, a British mathematical philosopher and social critic, urging all world leaders to realize the dangers of nuclear war and find the peaceful means for settling all disputes between them. This is the famous *Russell-Einstein Manifesto*, a powerful plea for world peace issued in 1955.

## §2. **Theory of Everything**

Science is a knowledge system established on human's perception of things with their evolving. The perceptive subject is human and it is the human's local recognition on things in a special place and time of universe. Similar to the blind men in fable of *blind men with an elephant*, everyone wishes to take his local recognition on a character of thing as the whole one. Then, *can the local recognition of thing really be the whole one?* The answer is Not. Dr.Ouyang tells Huizi that the human recognition on reality of thing is the "*Name named*" in *Tao Te Ching*. As asserted by Laozi that "*Name named is not the eternal Name*", the recognition on reality of thing is always in "*human's perception*", as shown in Figure 12.8. And meanwhile, it can only be a gradually process of improving human's local recognition of thing. Generally,

**Figure** 12.8. **Human's perception**

the greatest risk in accepting the local truth to be the reality is that it is false and needs to be rejected if there exists a counterexample. For example, Pythagoras' assertion that "*all things are numbers*" in ancient Greece is to explain all things with the character of "*number*", which is certainly not true. Furthermore, the *number* of Pythagoras is the rational number, which let his follower Hippasus giving a counterexample, namely the diagonal length $\sqrt{2}$ of a unit square is a non-rational number which concluded that the perception of "*all things are numbers*" is a misperception.

Now that the human perception of things or science is local, *what are the disadvantages and advantages compared with the whole perception?* Generally, its disadvantage is that it is only a kind of conditional truth of things and its advantage is that it can be verified or falsified within the scope of human ability. But at the same time, the verification or falsification itself is also restricted by human's recognitive limitation. Even if it has been verified at some time, the human may find that it is not the reality of things but only the conditional truth, even a false recognition with the evolving of time and recognitive means. For example, Newton's gravity is a local recognition on the gravity among mass objects while Einstein's gravity is essentially in the gravitational field that the number of mass objects $n \to \infty$ in Newton's gravity, which is a kind of whole recognition. However, the verification condition is still a local test within human's ability. It is the rule from special to general in human recognition of things. So, *is there a theory of everything that can explain how things changing?* The answer is Not! Dr.Ouyang explains that the law of changing in everything is Tao, following that "*Tao told is not the eternal Tao*", namely it is impossible to have such a theory of everything for perception in philosophy.

Then, *why does it put forward the Theory of Everything in science?* The essence of *Theory of Everything* in science is not establishing the "*eternal Tao*" but to recognize some seemingly different natural phenomena to meet that: ①Existing laws are special cases under a broader unified theory with more succinct in form or representation; ②Unified theory can be tested experimentally; ③Unified theory can propose the prediction that can be tested experimentally, which is in fact a gradual process of human recognition on things and frequently occurred in the developing history of science. For example, Newtonian mechanics unified the moving and stillness of object, and Maxwell's electromagnetic theory unified the electricity, magnetism and light, etc. Certainly, such a kind of unified theory is still a local theory, not the theory of everything in essence.

**2.1.Unified Field Theory.** The unified field theory is an idea led by Einstein that unifies the fundamental forces on field theory. Generally, the phenomena occurring in the universe known by humans can be reduced to the action of four fundamental forces, namely the gravity, electromagnetic force, strong and weak nuclear force. Among them, the gravitational and electromagnetic forces are a kind of long-range force, i.e., they can act in distance from 0 to $+\infty$ but the acting intensity decreases with the distance increasing; the strong and weak nuclear forces are the short-range forces with a small range of acting, the acting intensity decreases rapidly with increasing the distance and automatically disappears while the distance reaches a certain value. So, *can we construct a unified field theory characterizing the acting of four forces for explaining all physical phenomena?* This is the unified field theory advocated by Einstein in assuming of "*the nature should follows the principle of simplicity*" and applying the simplicity and intuition of field to unify the *complexity* in the acting of four fundamental forces to explain the natural phenomena in the universe. But up to now, this work has only been completed the unification of gravity and electromagnetic force, the electromagnetic force with the strong nuclear force and the weak nuclear force, but not yet completed the unification of gravity with the strong and weak nuclear forces. It is still running on the process to unify the four fundamental forces.

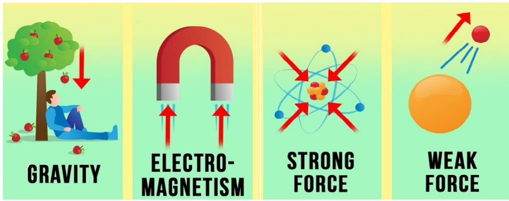



**Figure** 12.9. **4 fundamental forces**

(1) **Field geometry.** Certainly, the field geometry is an idea developed by Einstein to characterize the matter field as a curved space or Riemannian geometry under the action of forces. Notice that Newtonian gravity is the gravitational action between two mass objects. As the number $n \to \infty$ of objects, the gravitational action on a mass object at any point in space $\mathbb{R}^3 \times \mathbb{R}$ is the gravitational actions of all other objects, which is equivalent to the bending effect of space, i.e., *Einstein's gravitational field.* Similarly, *can the electromagnetic force, the strong and weak nuclear forces be translated into geometrical properties of space?* Dr.Ouyang explains that if the answer is Yes, the local action of four fundamental forces can be characterized as the gravitational action with a metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu, \quad \mu,\nu = 0,1,2,3 \tag{12.3}$$

on $\mathbb{R}^3 \times \mathbb{R}$ or a Riemannian space $M^4$. However, because of the difference in gravity, electromagnetic force, strong and weak nuclear forces, this idea has not been successful yet. Among them, the gravity and electromagnetic force are long-range forces while the strong and weak nuclear forces are short-range forces, namely the strong and weak nuclear forces do not act at a long distance and the short-range gravitational effect can be ignored. In this situation, a natural idea is to assume that the four fundamental forces act independently in their own space-time, namely the gravitational space-time $\mathbb{R}^3 \times \mathbb{R}$, electromagnetic force space-time $\mathbb{R} \times \mathbb{R}$, strong and weak nuclear forces space-time $\mathbb{R}^4 \times \mathbb{R}$, $\mathbb{R}^2 \times \mathbb{R}$ and all of them share a time dimension $t \in \mathbb{R}$, i.e., to characterize them by geometry with a metric

$$ds^2 = g_{\mu\nu}dx^\mu dx^\nu + ds_\Omega^2, \quad \mu,\nu = 0,1,2,3 \tag{12.4}$$

on Riemannian manifold, where the first term on the right side of equation (12.4) characterizes the space-time bending effect caused by the gravity and electromagnetic force, $\Omega$ is the space that can accommodate the independent action of strong and weak nuclear forces, $ds_\Omega^2$ is the space-time bending effect caused by the strong and weak nuclear forces, which satisfies $\dim\Omega \geq 4+2=6$. Usually, let $\dim\Omega = 6$, i.e., a $4+6=10$-dimensional space can accommodate the gravity, electromagnetic force, strong and weak nuclear forces independent action. Hearing her father's explanation, Huizi is a little surprised and says: "*Dad, it's what an amazing idea! But all humans live in a 4-dimensional space-time, so how do we explain those of six extra-dimensions?*" Dr.Ouyang tells Huizi that it is not known whether the spatial dimension has changed in the creation of Laozi's "*Tao creates One, One creates Two, Two creates Three and Three creates all things*" but it is certain that the spatial dimension seemly did not change in the creation that God only waved his hand and then, all things appeared. And so, applying the metric (12.4) to characterize the creation of universe must reflects the change in space dimensions after the Big Bang, i.e., the spatial dimension $4+\dim\Omega \to 4$ by a priori assumption so as not contradicting to the human perception. For the purpose, it is assumed that the dimensions in 3 directions expanded and extended dramatically after the Big Bang, forms the space that we perceive today and meanwhile, the other 7 directional dimensions dramatically curled and shrank until it is beyond the range of human perception, and the force action in the 3-dimensional macroscopic universe follows the law of gravity while the three forces in the 7-dimensional microcosmic universe follow the laws of electromagnetic force, strong and weak nuclear forces, satisfying the compactness condition on space, namely a point in the human perceived space $\mathbb{R}^3 \times \mathbb{R}$ is not a point but a compact space $\Omega$, as shown in Figure 12.10. Dr.Ouyang concludes that it is a wildly rational guess. Of course, due to the recognitive limitation of humans, it is impossible to verify whether there is really a compact space $\Omega$ at each perception point but it is possible to unite the fundamental forces into a geometric space $\mathbb{R}^4 \times \Omega$. Particularly, the following works had been accomplished.

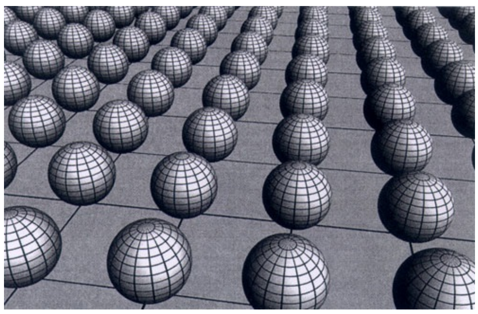


**Figure** 12.10. **Curved space model**

**Kaluza-Klein theory.** Notice that Einstein's gravitational theory is to add a time dimension $\mathbb{R}$ on $\mathbb{R}^3$ and get the gravitational geometry $M^4$ on Euclidean space $\mathbb{R}^4$, where the time dimension $\mathbb{R}$ is perpendicular to $\mathbb{R}^3$. So, *what does it contain by adding a dimension $\mathbb{R}$ on $\mathbb{R}^4$, i.e., Einstein gravitational field $M^4$ to get a $(4+1)$-geometry $M^5$?* Certainly, this is the equation of Einstein's gravitational field in $(4+1)$-dimensional space, also known as the *Kaluza-Klein theory*. Dr.Ouyang tells Huizi that the line element or metric on the gravitational geometry $M^5$ is

$$ds^2 = g_{\mu\nu}dx^{\mu}dx^{\nu}, \quad \mu,\nu = 0,1,2,3,4 \tag{12.5}$$

in this case but it is somewhat relaxed comparing with Einstein's gravitational field of $M^4$, namely it is no longer required to be invariant in all transformation of coordinates but only the Lorentz transformation. In this case, the form of Einstein's equations of gravitational field has changed to include an additional term characterizing the "*electromagnetic force*" conforming to Maxwell's equations. So, *what about this extra 5th dimension?* In Kaluza's own interpretation, "*the 5th dimension is curled into a very small circle at each point of $\mathbb{R}^3$ and no one can detect its existence*", which is the origin of curved space, namely the sphere $\Omega$ attached to each point in Figure 12.10 is now a circle $S^1$. In this case, the change in 5th dimension should be periodic, independent of the other dimensions, i.e., $x^4(t) = x^4(t+2k\pi), t\in\mathbb{R}, k\in\mathbb{Z}$, holds with $d^4g^{\mu\nu} = 0$ for integers $\mu,\nu = 0,1,2,3$ or 4 and then, the metric of $M^5$ can be expressed as

$$ds^2 = e^{2\alpha\phi}g_{\mu\nu}dx^{\mu}dx^{\nu} + e^{2\beta\phi}\left(dx^4 + \mathcal{A}_{\mu}dx^{\mu}\right)^2, \tag{12.6}$$

where $\mathcal{A}_{\mu}$ is the potential vector field satisfying $d^4\mathcal{A}_{\mu} = 0$, $\phi$ is a scalar field with $d^4\phi = 0$.

**String theory.** The success of Kalutza-Klein theory in unifying the gravity and electromagnetic force in 5-dimensional space gives humans hoping of unifying the gravity, electromagnetic force, strong and weak nuclear forces by adding more extra dimensions on the $\mathbb{R}^4$, namely let $\dim\Omega = 6$ in (12.4) and apply $(3+7)$-dimensional space to unify the action of gravity, electromagnetic force, strong and weak nuclear forces. Among them, a fundamental question on elementary particles arise, i.e., "*is an elementary particle a point with mass?*" Dr.Ouyang explains that the Bohr's model of atom indeed imagined the nucleus and electron as 3-dimensional spheres with sufficiently small diameter. But this has been broken in quantum mechanics or field theory. For example, characterizing the behavior of microcosmic particle appearing at a point in space $\mathbb{R}^3$ in terms of probability in quantum mechanics actually breaks the assumption that a particle is a 3-dimensional sphere with sufficiently small diameter because *if the elementary particles are 3-dimensional spheres with sufficiently small diameters, how can they appear at various points in space $\mathbb{R}^3$!* This is not logically consistent with the concept of elementary particles and can be only explained by the effect of elementary particles which is not a particle by definition.

Then, *how do the field effects of elementary particles come from?* Dr.Ouyang tells Huizi that unlike the matter made up of elementary particles, *string theory* holds that the basic units of natural matter are small oscillating strings and the elementary particles are only the field effects produced by the apparent behavior of different oscillations of these strings, as shown in Figure 12.11.

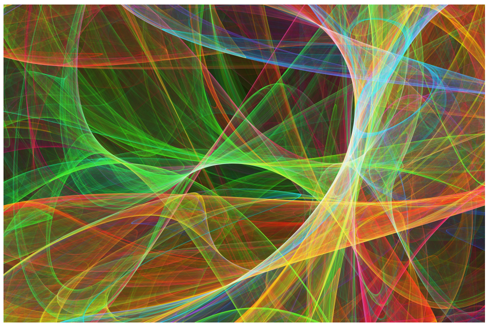


**Figure** 12.11. **String with fields**

In this case, the string can be open or closed. An open string looks like a line segment and a closed string looks like a closed curve, as shown in Figure 12.12. Two open strings joined at the end can still be an open string and an open string splits into two open strings at a point; an open string joining at an end can be a closed string, a closed string splitting into an open string at a point, etc. Particularly, the ends of open strings can be regarded as charged particles also. If one end of the open string is a positive particle and another is an antiparticle, then the two ends join together must be annihilated to form a photon. Similarly, if a closed string splits into an open string at a point, then there must be one end with a positive particle and an antiparticle at the other end. Thus, the splitting and joining of strings naturally correspond to the moving of particles.

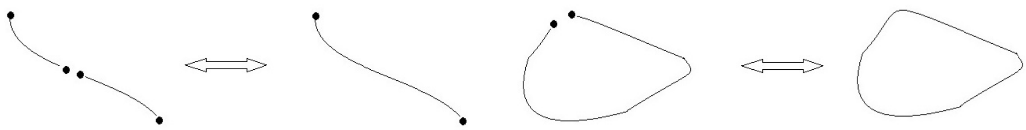


**Figure** 12.12. **Operations of joining and splitting on strings**

In this way, similar to that of Einstein's characterizing on gravitational effect with the curvature of curved space, string theory uses the high-dimensional curved space $M^{10}$ to unify the action of gravity, electromagnetic force, strong and weak nuclear forces to construct a variety of compact space $\Omega$ which is consistent with the human perception in 3-dimensional space. But different crimping modes can form countless compact spaces $\Omega$, which unconsciously increases the complexity of problem artificially. Furthermore, the geometrical space in which the four fundamental forces added should satisfy the supersymmetry by observing, namely the symmetry between bosons and fermions. And so, there are five string theories in the $(3+7)$-dimensional space, namely Type $I$, Type $IIA$, Type $IIB$, Hybrid $O(32)$ and Hybrid $E_8 \times E_8$ which give us five different models of the universe. Among them, the Type $I$ superstring theory contains open string and closed string, while the Type $IIA$ superstring theory only contains closed string and does not break the parity conservation in 9-dimensional space, the $IIB$ superstring theory only contains closed string, which breaks the parity conservation in 9-dimensional space but does not break the parity conservation after compact into 3-dimensional space. In the two hybrid string theories of $O(32)$ and $E_8 \times E_8$, the closed string vibrates respectively in left and right handed space, etc. Notice that if we add another dimension on the 10-dimensional space, i.e., to get a 11-dimensional space-time, it can be proved that these five different string theories are essentially equivalent, which is the *M-theory* proposed by E.Wtten et al.

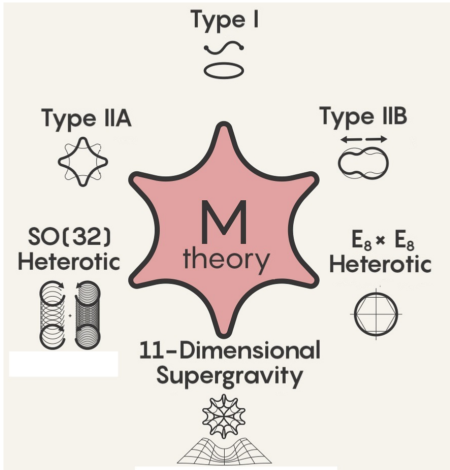



**Figure** 12.13. $M$**-Theory**

So, *does $M$-theory geometrically unify the gravity, electromagnetic force, strong and weak nuclear forces?* Dr.Ouyang explains that $M$-theory formally unites the gravity, electromagnetism, strong and weak nuclear forces but that this work is still far from completing the unification of four fundamental forces. First of all, the essence of geometrization of force field in string theory and $M$-theory is to divide the action of gravity, electromagnetic force, strong and weak nuclear forces by the distribution of dimensions in higher dimensional space, which is only a mathematical form of inclusion theory and the action of known force can be regarded as its special case in form, but it is not more concise. Secondly, both of the string theory and $M$-theory can not be verified, namely the string theory and $M$-theory can only be regarded as mathematically correct, and whether they are consistent with the real universe can not be tested, which is the fatal flaw of string theory and $M$-theory. Thirdly, the string theory and $M$-theory do not offer predictions that can not be tested even in $(3+7)$ or $(3+7+1)$-dimensional spaces. Thus, the string theory and $M$-theory can only be regarded as a formal theory or science of human recognition on unifying the four fundamental forces.

(2) **Gauge field theory.** Notice that the gauge field theory is a notion that unifies the microscopic particle fields under the action of local symmetry. The previous discussion on string theory and $M$-theory shows that although they characterize the action of electromagnetic force, strong and weak nuclear forces by adding extra dimensions and formally unify the four fundamental forces in $(3+7)$ or $(3+8)$-dimensional manifolds $M^{10}, M^{11}$,

they fail to reveal the acting essence of four fundamental forces. In the quantum field theory, particles are in excitation states at space-time points and different particles correspond to different quantum fields. Notice that this corresponding between particle and quantum field state $\phi$ requires to abandon the formal factors, namely there must be a transformation $\Lambda$ between two different quantum fields $\phi, \phi'$ on the same particle, i.e., $\Lambda : \phi \to \phi'$ because the field $\phi, \phi'$ depicts the same particle state, which are only artificially different in forms, i.e., they should be in accordance with the principle of relativity. Particularly, the transformation $\Lambda$ yields $\phi'$ only on a local region of field $\phi$ instead of the whole one. Such a transformation $\Lambda$ is called the *gauge transformation*. So, *what is the gauge field?* Dr.Ouyang tells Huizi that a *gauge field* refers to such a field of $\phi$ that $\phi^{\Lambda} = \phi$, i.e., invariable or its Lagrangian $\mathcal{L}^{\Lambda} = \mathcal{L}$ is invariable when the gauge transformation $\Lambda$ acts on every space-time point of field $\phi$ or on a certain regional space-time point. For example, there are two kinds of gauge transformations on a complex scalar field $\phi$ of mass $m$ with Lagrangian $\mathcal{L} = \partial_\mu \phi^\dagger \partial^\mu \phi - m^2 \phi^\dagger \phi$, i.e.,

①**Global transform:** $\phi \to \phi' = e^{i\gamma}\phi$ for real number $\gamma \in \mathbb{R}$, which changes the phase of field $\phi$ with a constant $\gamma$;

②**Local transformation:** $\phi \to \phi' = e^{i\gamma(\mathbf{x})}\phi$, $\mathbf{x} \in U \subset \mathbb{R}^3$.

Here, $\phi^\dagger$ is the complex conjugate of $\phi$. In this way, the first type of gauge transformation is to keep the Lagrangian constant; the second type of gauge transformation has certain restrictions on the form of Lagrangian, namely the partial differential operator operating on $\phi$ should be $\partial_\mu \to \partial_\mu + iq\mathbf{A}_\mu$, where $q$ is a properly defined constant, $\mathbf{A}_\mu$ is a field satisfying the gauge invariance to $\mathbf{A}_\mu \to \mathbf{A}_\mu - \frac{1}{q}\partial_\mu \gamma$. Now that the gauge transformations can be global or local, i.e., the gauge transformation on the global domain is the transformation on coordinate system but the local one is not necessarily the transformation on coordinate system, which is the difference between the gauge transformation and the coordinate transformation in Einstein's general relativity. Similarly, there are gauge transformations on the vector field, spinor field, etc.

So, *why do we say that gauge fields reveal the essence of force action?* Dr.Ouyang explains that after a particle equals to a quantum field state, all the acting field of microscopic particles are gauge fields. Accordingly, there should be four types of gauge bosons or mediated mesons transforming the force acting, namely the graviton $g$, photon $\gamma$, gluon $g^k, 1 \le k \le 8$ and the intermediate bosons $W^+, W^-, Z$, respectively in the gravitational field, electromagnetic field, strong and weak action fields. And so, characterizing the mediated meson is equivalent to the acting of fundamental force. Among them, the fermions, $W^+, W^-$ and $Z$ bosons have masses while the photons, gluons have zero masses. Although they are gauge fields, *why do the fermions, $W^+, W^-$ and $Z$ bosons have non-zero masses while the photons, gluons have masses of zero?* Dr.Ouyang explains that for satisfying the local gauge invariance, the gauge boson needs to be set to have mass zero. But *why do the fermions, $W^+, W^-$ and $Z$ bosons have zero masses while the photons and gluons have zero masses?* To answer this question requires a *spontaneous symmetry breaking mechanism* in gauge field, namely when the natural law observed on the physical system has a certain symmetry but the physical system itself does not have this symmetry, the system will spontaneously break symmetry, and the system originally with this symmetry will eventually become no longer exhibit this symmetry. In the spontaneous symmetry breaking, Higgs is a kind of mass particle everywhere in the universe. It participates in the spontaneous symmetry breaking of gauge boson so that the mass of gauge boson originally with zero mass is no longer zero and will obtain non-zero mass. Notice that the essence of characterizing elementary particles and force acting by gauge field lies in the equivalence relation of "*gauge field $\Leftrightarrow$ elementary particle*" and "*force action $\Leftrightarrow$gauge field*". Whence, the essence of unifying the action of gravity, electromagnetic force, strong and weak nuclear forces lies in constructing a unidied gauge field that contains the action of four fundamental forces. And meanwhile, by the appearing of unknown gauge field in the unified gauge field predicts the new elementary particle and then, discovers it by experimental exploration, which is a successful example of discovering new particles by theory in particle physics.

The work of unifying the electromagnetic force and the weak nuclear force, i.e., the *electroweak theory* was completed in the 1960s. The Lagrangian in the unified field is

$$\mathcal{L}_{IVB} = \frac{g}{\sqrt{2}}\left(J^{\mu^-}W_\mu^+ + J^{\mu^+}W_\mu^-\right)\frac{g}{\cos\theta_w}J_\mu^{NC}Z^\mu, \tag{12.7}$$

where

$$\begin{aligned}
J^{\mu^+} &= \sum_l \overline{\nu}_l\gamma^\mu\left(\frac{I-\gamma_5}{2}\right)l + \sum_q \overline{q}\gamma^\mu\left(\frac{I-\gamma_5}{2}\right)q',\\
J^{\mu^-} &= \sum_l \bar{l}\gamma^\mu\left(\frac{I-\gamma_5}{2}\right)\nu_l + \sum_q \overline{q}'\gamma^\mu\left(\frac{I-\gamma_5}{2}\right)q,\\
J_\mu^{NC} &= \sum_f g_L^f\overline{f}\gamma_\mu\left(\frac{I-\gamma_5}{2}\right)f + \sum_{f\neq\nu} g_R^f\overline{f}\gamma_\mu\left(\frac{I-\gamma_5}{2}\right)f
\end{aligned}$$

and the notations $l$ and $q$ denote the lepton and quark, respectively; $f$ is a fermion which acquires mass by interacting with the Higgs field; $\nu$ is the neutrino, $g_L^f$ and $g_R^f$ are respectively the coupling constants of left and right-hand particle state, $g$ is the coupling constant and $\theta_w$ is the weak action rotation angle.

The success of electroweak theory further advances the unifying of electromagnetic force, strong and weak nuclear forces, which is the GSW theory completed in the 1970s. Its unified field is called the *Standard Particle Model* with the Lagrangian

$$
\begin{aligned}
\mathcal{L}_{GSW} \quad = \quad & -\frac{1}{4}B_{\mu\nu}B^{\mu\nu} - \frac{1}{4}W^i_{\mu\nu}W^{i\mu\nu} + \sum_{e\to\mu,\tau} i\left(\overline{L}_e\gamma^\mu D_\mu L_e + \overline{R}_e\gamma^\mu D_\mu R_e\right) \\
& + \sum_{i=1}^{3} i\left(\overline{L}_i\gamma^\mu D_\mu L_i + \overline{R}_{ui}\gamma^\mu D_\mu R_{ui} + \overline{R}_{di}\gamma^\mu R_{di}\right) \\
& + (D_\mu\phi)^\dagger (D_\mu\phi) - m^2\phi^\dagger\phi - \lambda\left(\phi^\dagger\phi\right)^2 + \sum_{e\to\mu,\tau} g_1\left(\overline{L}_e\phi R_e + \overline{R}_e\phi^\dagger L_e\right) \\
& - \sum_{u\to c,t;d\to s,b} \left(g_d\overline{L}_u\phi R_d + g_u\overline{L}_u\phi_c R_u + h.c\right), \qquad (12.8)
\end{aligned}
$$

where the terms in first and second rows are the freedom gauge field $B_\mu$, $W^I_\mu$, leptons and quarks in order, the terms in third and fourth rows are respectively the freedom Higgs field and its coupling with leptons, quarks, the notation $h.c$ denotes the Hermitian conjugate item in front of the brackets.

Notice that the *Standard Particle Model* is still a phenomenological theory with more than a dozen adjustable parameters and randomness. However, the predicted intermediate bosons $W^+, W^-, Z$ and Higgs by this model were announced to have been found by CERN in 1983 and 2012, respectively, implying that this model is consistent with the practice. Certainly, the *Standard particle Model* does not include the gravity and is still not a unified theory on the action of four fundamental forces. Meanwhile, some predictions remain unproven until today, implies that the *Standard Particle Model* is still a local theory. For example, it predicts that the proton is not a stable particle,which will go to the decay is not consistent with the experimental observing and it can not explain why a large number of neutrons, neutrinos and photons are produced in nuclear fission or fusion, etc.

**2.2.Continuity Flow Model of Forces.** If all phenomena in the universe can be simplified into the action of gravity, electromagnetic force, strong and weak nuclear forces, the behavior and evolving of things in the universe can be simulated by the continuity flow. Among the four fundamental forces, the gravity $\mathbf{F}_g$ acts between mass objects, the electromagnetic force $\mathbf{F}_e$ acts between charge particles, the strong nuclear force $\mathbf{F}_s$ acts between hadrons (baryons, mesons) and the weak nuclear force $\mathbf{F}_w$ acts between quarks and leptons, leptons and leptons. Thus, classifying the forces by the acting range, the four fundamental forces mainly appear as ①Gravity $\mathbf{F}_g$ in range $r \geq 10^{-10}m$; ②Electromagnetic force $\mathbf{F}_e$ in range $10^{-10}m \leq r \leq 10^{-8}m$; ③Strong nuclear force $\mathbf{F}_s$ in range $10^{-18}m \leq r \leq 10^{-15}m$; ④Weak nuclear force $\mathbf{F}_w$ in rang $r \leq 10^{-18}m$. Consequently, all things in the universe naturally form a continuity flow by the constitution theory of matters as follows:

Define a graph $G[AEP]$ by its vertex set $V(G[AEP]) = \{\text{quarks, leptons}\}$, edge set $E(G[AEP]) = \{$ (quarks, quarks), (quarks, leptons), (lepton, lepton)$\}$, namely the edges in $G[AEP]$ is formed between quarks and quarks, quarks and leptons or leptons and leptons, which can have the gravitational interaction, electromagnetic interaction, strong or weak nuclear force interaction. Accordingly, the vertex labeling $L : v \in V(G[AEP]) \to v$ on $G[AEP]$ and the edge labeling is

$$L : (v,u) \in E(V(G[AEP])) \to \begin{cases} \mathbf{F}_g, & r_{vu} \geq 10^{-10}m; \\ \mathbf{F}_e, & 10^{-10}m \leq r_{vu} \leq 10^{-8}m; \\ \mathbf{F}_s, & 10^{-18}m \leq r \leq 10^{-15}m; \\ \mathbf{F}_w, & r_{vu} \leq 10^{-18}m \end{cases} \tag{12.9}$$

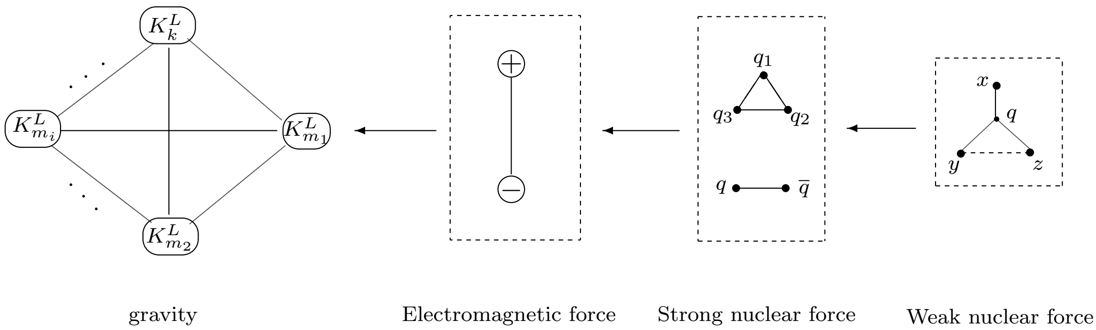



**Figure** 12.14. **Continuity flow** $G^L[AEP]$ **hierarchy**

where $r_{vu}$ is the distance between elementary particles $v$ and $u$, $q$ is a quark, $\overline{q}$ is the antiquark of $q$, $x, y, z \in \{e^{\pm}, \nu_e, \mu^{\pm}, \nu_\mu, \tau^{\pm}, \nu_\tau\}$ are all the leptons.

Dr.Ouyang tells Huizi that the hierarchy of $G^L[AEP]$ can be generally identified by Figure 12.14 from the constitutive theory of matter, i.e., ①In the range $r \geq 10^{-10}m$ it is dominated by gravity. In this case, there is a gravity between any two mass objects, namely if remove all leptons in $G^L[AEP]$, the residue is a complete graph $K^L_{\widehat{k}}$, where $\widehat{k}$ is the number of hadrons in the universe; ②In the range $10^{-10}m \leq r \leq 10^{-8}m$, i.e., in the atomic and molecular level of gravitational field of ① it is dominated by the electromagnetic forces. At this time, the charge particles comply with principle of "*the same charges repel, different charges attract each other*" in the formation of $G^L[AEP]$, namely its local structure is a star $S^L_{1,l}$ with a nucleus as the center, all electrons as the leaf, where $l$ is the number of electrons in the atom; ③In the range $10^{-18}m \leq r \leq 10^{-15}m$, i.e., in the atomic nucleus and hadronic level of gravitational field of ①, it is locally constituted by complete subgraphs $K^L_3$ or $K^L_2$ in $G^L[AEP]$, where $K^L_3$ and $K^L_2$ are baryons or mesons formed by quarks respectively; ④In the range $r \leq 10^{-18}m$, i.e., in the quark and lepton scale of gravitational field of ①, there are the weak action of $\beta$ decay $n \to p + e^- + \overline{\nu}_e$, $p \to n + e^+ + \nu_e$, $e^- + p \to n + \nu_e$, namely the leptons are adjacent with protons and neutrons in the nucleus and locally constitute a star $S^L_{1,s}$ in $G^L[AEP]$, where $s$ is the number of leptons adjacent with a proton or neutron. The connection between leptons $x, y, z$ depends on whether they are in one decay. For example, if an electron $e^-$ and an antineutrino $\overline{\nu}_e$, a positron $e^+$ and a neutrino $\nu_e$ are respectively in a decay of neutron or proton, then they must be connected.

So, *how can we characterize the acting field of four fundamental forces?* Dr.Ouyang tells Huizi that the dominant force in the scale of $r \geq 10^{-10}m$ is gravity. In this case, the field equation is Einstein's gravitational field equation

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \lambda g_{\mu\nu} = \kappa \mathcal{T}_{\mu\nu}; \tag{12.10}$$

and in the scale of $r < 10^{-10}m$, the electromagnetic force, strong and weak nuclear forces will appear. It can characterize the fundamental force in that the external field is the gravitational field and the internal field is the electromagnetic field, strong and weak nuclear force fields, respectively. It is similar to the situation of upper and lower hexagrams in *Changing Book*. At this time, the moving of particles by the action of fundamental force is composed of two parts. One is the mass particles composed of molecules such as the stars and other celestial bodies move under the action of gravity, namely all vertices of a mass particle in $G^L[AEP]$ move in synchronization. Another is the moving of particles including the charged particle, hadron and lepton, i.e., the star graph $S^L_{1,l}$, complete subgraph $K^L_3$, $K^L_2$ and $K^L_1$ in inner field. And the particle field $\phi$ obeys the Klein-Gordon equation, i.e

$$\left(\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2\right)\phi + \left(\frac{mc}{\hbar}\right)^2\phi = 0, \tag{12.11}$$

where $\hbar, c$ and $m$ are the Planck's constant, speed of light and the mass of particle, respectively, $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$. In this case, as long as the energy tensor $\mathcal{T}_{\mu\nu}$ of particle field can be obtained, then the Einstein's gravitational field equation (12.10) can be solved to determine the particle field $\phi$ under the action of gravitational field.

For example, the moving of a charged particle is affected by electromagnetic field and meanwhile, the electromagnetic field is also affected by the gravitational field. In this case, assuming that the mass of charged particle is $m$ with charge $q$, then the electromagnetic field with $E(r) = q/r^2$. By the assumption of spherical symmetry, let the metric form be

$$d^2s = B(r)dt^2 - A(r)dr^2 - r^2d\theta^2 - r^2\sin^2\theta d\phi^2. \tag{12.12}$$

Accordingly, define the energy tensor of electromagnetic field

$$\begin{aligned}\mathcal{T}_{\mu\nu} &= -\left(g_{\sigma\nu}F_{\mu\lambda}F^{\sigma\lambda}+\frac{E^2}{2}g_{\mu\nu}\right)\\ &= \frac{E(r)}{c^2}\begin{bmatrix} B & 0 & 0 & 0\\ 0 & -A & 0 & 0\\ 0 & 0 & r^2 & 0\\ 0 & 0 & 0 & r^2\sin^2\theta\end{bmatrix}\neq 0_{4\times 0},\end{aligned}$$

where

$$F^{\mu\nu}=\frac{E(r)}{c^2}\begin{bmatrix} 0 & -1 & 0 & 0\\ 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix},\qquad F_{\mu\nu}=\frac{E(r)}{c^2}\begin{bmatrix} 0 & 1 & 0 & 0\\ -1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}.$$

Then, the gravitational field equation $R_{\mu\nu}=-8\pi GT_{\mu\nu}$ can be solved by

$$ds^2=(1+\frac{4G\pi q^2}{c^4r^2}-\frac{2Gm}{c^2r})dt^2-\frac{dr^2}{1+\frac{4G\pi q^2}{c^4r^2}-\frac{2Gm}{c^2r}}-r^2d\theta^2-r^2\sin^2\theta d\phi^2. \qquad (12.13)$$

**2.3.Mathematical Universe Hypothesis.** The mathematical universe hypothesis implies that we live in a relational reality, i.e., the structure of universe obeys the mathematical rule of humans, proposed by M.Teegmark from the United States in 2003, claiming that *the natural reality outside of human is a mathematical structure, namely the universe can not only be characterized by mathematics but the universe itself is a mathematical structure.* This hypothesis essentially excites most researchers because it allows mathematical characterizing the state and behavior of everything. Likewise, it is not so much a hypothesis as a notion of what things are, similar to the combinational conjecture for mathematical science. Dr.Ouyang asks Huizi: "*Do you remember the combinational conjecture for mathematical science?*" Huizi replies: "*Is the conjecture asserts that any mathematical science can be reconstructed from or made by combinatorialization?*" Dr.Ouyang answers her: "*Yes! This conjecture is also known as CC conjecture, advanced by the combinatorial notion on things in the universe! Here, the 'mathematical science' in this context is not only the mathematics but includes also all subjects that can be characterized quantitatively by mathematics such as the theoretical physics, theoretical chemistry, etc.*" Huizi asks Dr.Ouyang: "*Surely, the combinatorial conjecture for mathematical science is also aimed at the recognition of everything. Is it consistent with the mathematical universe hypothesis?*" Dr.Ouyang tells his daughter: "*It seems so! But due to the limitation of human* *recognition, the combinational conjecture for mathematical science is more consistent with the human recognition which is a combinatorial notion on local recognitions of thing.*" Huizi asks Dr.Ouyang: "*And then, does it implies that the behavior of everything in the universe can be characterized by the continuity flow $G^L[AEP]$? If so, it is too complicated because the continuity flow $G^L[AEP]$ is not still a lot of details unclear now!*" Dr.Ouyang smilingly tells her that the universe itself is complex on the microscopic level and explains that the essence of human recognition is to determine the continuity flow $G^L[AEP]$ if there are really four fundamental forces in the universe. Of course, the continuity flow $G^L[AEP]$ needs to be further simplified and clarified. However, due to the limitation of human recognition, including the limitation of mathematics, to determine the continuity flow $G^L[AEP]$ must be an infinite and gradual process. In other words, it is only a model but incomplete to anyone in theory.

Huizi is still somewhat vague about the role of combinatorial conjecture for mathematical science, murmuring to myself: "*If we can not determine it completely, what value does it have? The mathematical universe hypothesis is certain but it seems we can not conclude so!*" She uncertainly says to Dr.Ouyang: "*Dad, the mathematical universe hypothesis seems logically inconsistent with 'Tao told is not the eternal Tao' in Laozi's Tao Te Ching. Is it wrong!*" Dr.Ouyang asks Huizi: "*Does a thing can be characterized mathematically implies that one can hold on the behavior of thing?*" Huizi replies: "*Of course! I think the behavior of a thing that can be characterized mathematically must be holden by humans!*" Dr.Ouyang tells Huizi that this view is not correct because mathematics consists of the deterministic and non-deterministic mathematics. And meanwhile, a mathematical recognition is mostly characterizing on a character of thing's behavior, which is a formal deduction system in accordance with the principle of logical consistency, is not necessarily complete and there are also propositions that can not be proved or falsified in this formal system! What is more, there are a lot of thing's behaviors can not be simulated by mathematics. For example, *does the quantum mechanics characterizes microscopic particles with possibilities implies that humans can hold on them?* Of course Not because the behavior of microscopic particles has an uncertainty principle relative to the human observing, namely the human measuring on microscopic particles can not simultaneously determine the position and momentum of a microscopic particle. Thus, the mathematical characterizing or simulating of a particle behavior is not equivalent to the particle itself, which indicates that the mathematical universe hypothesis is not contradict to "*Tao told is not the eternal Tao*" in *Tao Te Ching* to some extent because the former corresponds to the recognitive result while the latter corresponds to the recognitive thoughts, which needs us to further extend mathematics following the principle of logical consistency such as the *mathematics combinatorics* and so on to expand the human's recognitive ability. This is the value both of the mathematical universe hypothesis and CC conjecture.

## §3. **End of Science**

The main counterpoint to the end of science is the theory of everything, namely a notion that humans can establish the theory of everything that explains all observed natural phenomena without needing to know the unknown. For examples, the ancient Chinese theory of Yin and Yang, the five elements and eight hexagrams; the ancient Greek theory of Archimedes' *unified all theories* by several axioms and Democritus reduced all natural phenomena to the collisions of atoms; Newton proposed three laws of mechanics and the law of universal gravity for explaining the moving of celestial bodies; Maxwell unified the three fields of electricity, magnetism and light by electromagnetic field; Einstein explained the gravity as the bending effect of space, unified the effect of light and electricity, mass and energy, devoted the latter half of his life to unifying the gravity with electromagnetic force, etc., none of them did not wish to establish a theory of everything to end the science and meanwhile, the human could find all "*causes*" from "*results*" of natural phenomena to dominate the operating of universe by the theory of everything, no longer worrying about the *living* and *death* of human because they could *immortality*, no longer lack of expectation for *poverty* because the human could "*liking wind gets wind, wishing rain gets rain*", namely they would control the universe to satisfy all their needs, as shown in Figure 12.13. *What a beautiful picture is on the eyes of human!* However, it is absolutely nonsense that the human could be the master of universe, which also contradicts to the knowledge of ancient Chinese on Tao of the heaven and humanity in philosophy because Tao creates humans in "*Three creates all things*" and meanwhile, it does not give the immortality to human and needs the humans ourselves to follow the earth in relationship with the nature. So, *is science really coming to an end?* Of course Not! Dr.Ouyang explains that the purpose of unifying is making the existing theories more concise, explains more natural phenomena and finds the unknown because science is a gradually established knowledge on laws or rules of things within the limits of human ability, Thus, science will never be

**Figure** 12.14. **Universe's master**

ended unless they are no longer aware of the unknown due to inertia. In such case, science is certainly not the theory of everything because there are still unknown things in the universe to humans.

In this case, *why do some scientists think that science is going to end?* Dr.Ouyang explains that science will never be ended as the whole but it can be really ended in its parts. Particularly, why they think that science is going to end is because they all stand on their own studying field and feeling, i.e., the part not the whole. Generally, the end of science's part mainly includes three cases, i.e., one theory is unified into another theory, an idea or method can no longer expand human recognitive ability on things and the part in science is contrary to the notion of "*harmony and coexistence of humans with the nature*", needs the humans initiative to give up the end.

**3.1.End of Recognitive Theory.** A theory is a logical system of recognition and holding on thing or its characters formed through by *Abstract* $\rightarrow$ *Hypothesis* $\rightarrow$ *Conception* $\rightarrow$ *Judgment* $\rightarrow$ *Reasoning* $\rightarrow$ *Theory* and other thinking processes in a certain period of time. Certainly, a theory comes from the practice which is the recognitive result of reasoning. Its purpose is to guide human understanding and holding on the behavior of thing. Generally, a theory needs to verify multiply by *Practice* $\rightarrow$ *Understand* $\rightarrow$ *Practice again* $\rightarrow$ *Understand again* $\rightarrow \cdots$, which is a circular and endless spiraling process for constantly improving and revising the existing theory. Among them, the *practice* is the source of theory, the *recognition* is the result of human's reasoning on thing, the *practice again* is verifying the theory through practice once again, the *understanding again* is the revision and sublimation of theory and then, perfecting the recognition of humans on thing.

**(1) Revising a theory** A typical characteristic of an incomplete theory is that it is not enough to fully characterize the behavior of thing, namely there exists a contradiction in theory with the practice. Usually, the reason of existing contradiction lies in the incorrect ways in establishing theory, including: ①Abstracting, such as the incorrect ways of abstracting or the abstracting of intrinsically different factors into the same one; ②Hypotheses, such as the assuming model does not fully coincide with the behaviour of thing or the characters of behaviour have been artificial added or subtracted; ③Judgment and reasoning, such as the errors of judgment on the behaviour of thing, reasoning with a false assumption or reasoning is illogical etc. Usually, the inconsistency between theory and practice can be found through the verifying of practice again. So, it is necessary to revise the existing theory and form a new one to guide the practice of humans.

Dr.Ouyang tells Huizi that although the practice can verify whether a theory is correct or not in theory, but human is the subject to judge whether a theory conforms to the practice or not and it is subject also to the constraints of verifying time, methods and means as well as the human's recognitive ability. For example, *why did the geocentric theory shown in Figure* 12.15 *existing for more than two thousand years before it was replaced by Copernicus's heliocentric theory?* The answer is because all humans commonly observe on the earth, the sun rose in the east and set in the west, the moon took over the sun's place in the evening and set in the early morning and the humans felt that all planets moved around the earth. According to the general relativity of Einstein, the human observing on the earth is to let the earth as the origin, the *geocentric theory* is an inevitable recognitive result. And meanwhile, humans are also willing to accept the *geocentric theory* cognitively because it implies that the earth is the center of universe and then, humans ourselves would be the master of universe. But this recognition of human standing on the earth is false. Once the *geocentric theory* is replaced by the *heliocentric theory*, the *geocentric theory* would be a false theory and then, the humans would lose the ranking of master in the universe. Even today, we could not determine whether there is a center in the universe. Certainly, if the Big Bang theory is true, the origin point of Big Bang must be the center of universe but it is just a hypothesis that the humans could not verify.

**Figure** 12.15. **Geocentric model**

(2) **Completing a theory.** A theory is complete if it has fully characterized the corresponding behavior of thing. In this case, the theory on the thing must be ended without the development of a new theory because humans have completely holden on such thing. But *are there such a complete theories existing in science?* Dr.Ouyang tells Huizi that there are lots of theories that end up being complete, mainly appearing in "*humans follows the earth*" or the knowledge of things on the earth within the human recognition ability. For example, the cultivation of *five grains* for human living, as the maize cultivation shown in Figure 12.16 and the crop breeding, crop cultivation, animal husbandry, genetics and veterinary medicine for animal feeding, etc. And meanwhile, constraint on "*humans follows the earth*", the theory of humans on extraterrestrial objects and whether there is extraterrestrial organisms are incomplete. It is necessary

**Figure** 12.16. **Maize cultivation**

to further expand the scope of recognition and establish a scientific system of humans.

(3) **Unifying theories.** Notice that the essence of Smarandache multispace or multisystem $S_1 \cup S_2 \cup \cdots \cup S_m$, i.e., the union of $m$ distinct spaces or systems $S_1, S_2, \cdots, S_m$ from each other becoming a system $\widetilde{S}$ or labeled graph $G^L[\widetilde{S}]$ lies in the recognition of behavior of systems $S_1, S_2, \cdots, S_m$ by a whole system $\widetilde{S}$ or labeled graph $G^L[\widetilde{S}]$, which is standing on a higher perspective to observe the behavior of $S_1, S_2, \cdots, S_m$. This is the general paradigm of science's developing, namely standing on a universal framework to unify different theories for recognizing the behavior of thing by local recognition. In this way, the old theory is replaced by the new one, which turns to be a special case or part of the new theory. Such examples are numerous in the history of science's developing. For example, ①Newton's three laws of mechanics and the law of universal gravity unified the moving of celestial bodies and moving on the ground by *forces* and explained the natural phenomena with the principles of mechanics, realizing the moving of objects in the sky or on the ground which Aristotle et al of ancient Greece believed these were followed in two different laws; ②Maxwell's equations of electromagnetic fields, i.e.,

$$\left\{\begin{array}{lcll} \mathrm{div}H & = & 0 & ⓐ \\ \mathrm{rot}E & = & -\dfrac{1}{c}\dfrac{\partial H}{\partial t} & ⓑ \\ \mathrm{div}E & = & 4\pi\rho & ⓒ \\ \mathrm{rot}H & = & \dfrac{1}{c}\dfrac{\partial E}{\partial t} + \dfrac{4\pi}{c}J & ⓓ \end{array}\right. \tag{12.14}$$

unified Gauss law of electric and magnetic fields ⓐ, Faraday Law ⓑ, Coulomb's Law ⓒ and Ampere's Law ⓓ, where $\rho$ is charge density, $J$ is current density, $E$ is the intensity of electric field and $H$ is the intensity of magnetic field. Certainly, the physical meaning of Gauss's law ⓐ is that the magnetic flux through any closed surface is 0; Faraday's law is ⓑ in equation (12.1); Coulomb's lawⓒ is similar to Newton's law of universal gravitation, the force between charges $q_1, q_2$ in vacuum is proportional to the product of charges $q_1q_2$ and inversely proportional to the distance between charges; Ampere's law is a relation between the current $I$ on the circuit and the intensity $H$ of magnetic field generated on the vertical plane of circuit under the right-hand spiral rule shown in Figure 12.17., namely the line integral $\int_l H \cdot dl = I$, where div and rot are respectively the divergence and curl in vector analysis of $\mathbb{R}^3 = \{(x, y, z), x, y, z \in \mathbb{R}\}$,

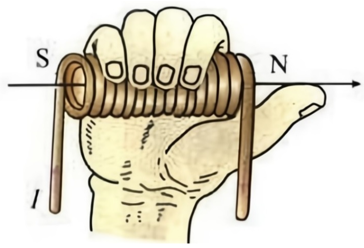



**Figure** 12.17. **Right-hand spiral rule**

defined respectively by

$$\begin{aligned}\mathrm{div}H &= \frac{\partial H_x}{\partial x}+\frac{\partial H_y}{\partial y}+\frac{\partial H_z}{\partial z},\\ \mathrm{rot}H &= (\mathrm{rot}H_x, \mathrm{rot}H_y, \mathrm{rot}H_z).\end{aligned}$$

where,

$$\mathrm{rot}H_x=\frac{\partial H_z}{\partial y}-\frac{\partial H_y}{\partial z},\quad \mathrm{rot}H_y=\frac{\partial H_x}{\partial z}-\frac{\partial H_z}{\partial x},\quad \mathrm{rot}H_x=\frac{\partial H_y}{\partial x}-\frac{\partial H_x}{\partial y};$$

③Einstein unified the independent existence of time $\mathbb{R}$ and space $\mathbb{R}^3$ in Newtonian mechanics by *space-time* $\mathbb{R}^4$, namely the time and space are not independent but relative. In addition, the mass $m$ and energy $E$ are unified by the mass-energy equation $E = mc^2$ in theory of special relativity, namely the mass is the embodiment of energy of a moving object, which ends the absolute space-time view of Newtonian mechanics and the view that mass and energy are two independent characters of matter. Notice that everything in the universe constitutes the continuity flow $G^L[AEP]$ under the condition of interaction. The purpose of unification on scientific theories is to construct $G^L[AEP]$, which of course can be only an infinite and gradual process.

**3.2.End of Recognitive Thought.** Notice that science is a local knowledge on the law of things. Then, *how does science recognize the laws of things?* Dr.Ouyang tells Huizi that it is human idea or hypothesis that guides the human recognition of things, namely through the elementary perception of thing, it is assumed that the character of thing are consistent with some abstractedly known rules, and the known rules are used to simulate and depict the character of thing. In the systematic recognition of thing, the behavior of thing $T$ is characterized by the behaviors of its elements $S_1, S_2, \cdots, S_m$, which is actually the S-marandache multispace $\widetilde{S}$ or labeled graph $G^L[S]$, i.e., recognizing the state and behavior $G^L[S]$ of thing $T$ by some characters $\chi_{i_j}(S_i)$ of element $S_i$, which is the majority of cases in which humans recognize unknown things. For example, the six blind men touched a certain part of the elephant's body, namely its legs, tail, trunk, ears, belly and teeth in the fable of blind men with an elephant, as shown in Figure 12.18. And so, the results of the six blind men's perception are also very different, namely the elephant is like a pillar, a rope, a pipe, a fan, a wall or a radish because it is very difficult to hold on the whole behavior only from the local perception on thing $T$.

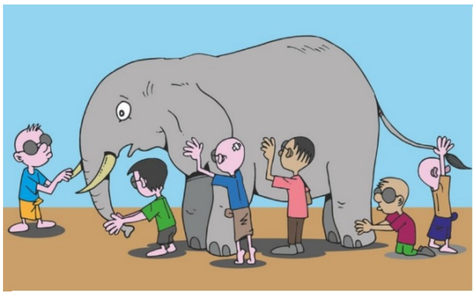


**Figure** 12.18. **Blind recognizing**

Then, *under what conditions can that a thought or hypothesis be considered to complete the recognition of thing T ?* Dr.Ouyang explains that there is a criterion for a complete thought or hypothesis, i.e., ①All behaviors of elements $S_1, S_2, \cdots, S_m$ have known; ②All characters $\chi_{i_j}(S_i), 1 \leq i_j \leq s$ of element $S_i$, $1 \leq i \leq m$ have been determined. In this case, humans have known all behaviors of elements in the composition of thing $T$, which can be regarded as the humans hold on the behavior of thing $T$. Otherwise, the existing thought or hypothesis can not guide human to hold on thing $T$, it is necessary to innovate a new thought or hypothesis to replace the old one to recognize thing $T$, which is the general process of recognizing.

It should be noted that the subject of putting forward recognitive thoughts or hypotheses is the human, who is recognizing things $T$. However, a human's life is limited, so is his recognitive and innovation abilities. What's more, most recognitive thoughts or hypotheses are put forward on the results of human's existing perception, just like the philosophical notion implied in fable of the blind men with an elephant, namely the human perceive unknown things is similar to that of the blind man in the fable, it is a gradual recognition from the local to the overall. Thus, most of the recognitive thoughts and hypotheses put forward by humans are directed at the local characters of thing $T$, which easily leads to the end of one recognitive thought such as those of: ①Existing recognitive thought verified by practice is inconsistent with the behavior of thing $T$, even incorrect and new recognitive thoughts or hypotheses have to be put forward; ②Existing thought or hypotheses fully characterizes the behavior of things $T$; ③Existing thought or hypotheses has characterized the known behavior of thing $T$ but new behavior of thing $T$ has emerged, it needs to suggest a new thought or hypotheses to characterize; ④Existing thought or hypotheses have partially characterized the behavior of thing $T$ but the humans can not suggest a new thought or hypotheses that characterizes the unknown behavior of thing $T$.

Among these, the ① is a common situation in the development of science because the recognitive thought or hypothesis is the idea put forward by humans, i.e., "*failure is the mother of success*" and a correct recognition is the result produced by the countless incorrect recognitions; the ② is the most ideal situation in recognition of things $T$ but it is not common nowadays in the development of science because those things that have not been fully recognized, can not be fully perceived by humans or the perceptual results constitute a complex system, which are all not enough to characterize them completely today; the ③ is a normal mode in the recognition of thing, namely most recognitions gradually approach to the reality but holding on their behaviors needs the continuously improving and innovating. Here, the ④ situation needs to be further analyzed, namely there are no new thoughts or hypotheses for characterizing the unknown behavior of things $T$. Certainly, this is the bottleneck in development of science and someone will mistakenly believe that science is going to end in this situation. So, *when does the human come up without new thoughts or hypothesis on unknown behavior of thing $T$?* Dr.Ouyang explains that this is bound to happen for any individual to perceive thing because life is limited, the recognition is limited but any thing is in constantly changing. And meanwhile, the evolving of any thing is from the prosperity to the decline, there are no exception on the human recognition. Generally, as a human reaches a certain age, there may emerge the inert instincts of ①Inactive thoughts and curious about the unknown; ②Rejoiced in its successes; ③Constraints with the established ways of thinking, can not come up with new thought or hypothesis which leads to the end of individual recognitive thought. Such a situation affects the innovation of his followers and also explains why the book of *The End of Science*, written by a senior writer J.Horgan of *Scientific American* is controversial in academic world because his conclusion that *science is going to end* was deduced by his interviews with some famous scholars in the scientific fields of philosophy, physics, cosmology, biology, sociology, neuroscience and so on, which is just the end of personal perception, i.e., the end of personal recognitive thought in their own fields.

**3.3.Active End by Humans.** The active end of science by humans ourselves is not the end of science but the restriction or end of certain fields or directions in science, namely science needs also to have standard in its developing, following Tao. This standard is conducive to promoting "*harmonious coexistence of humans with the nature*". Notice that the effect of nature on humans is instantaneous in the binary system $K_2^L$ of humans and the nature, namely humans can perceive the effect of nature on humans, including the "*favorable weather*" or "*natural disaster*" at the same time. However, the effect of humans acting on the nature is a cumulative one with the property of occurring over time, even after several generations. In such cases, science with its application should complies with the *Science's Rule* in Section 1 of this chapter, and adhere to the principle that all human activities guided by science should be free from intrusion upon the nature. This is particularly important in the social developing model guided by the nature of "*Matthew effect*" or "*profit by capital*", namely to limit or end them mainly in the following three categories:

**Class 1.** *Affects the operating of universe.*

According to the creation that "*Tao creates One, One creates Two, Two creates Three and Three creates all things*" of Laozi, the human was created in *Three creates all things*". Thus, exploring the origin of all things in universe by human recognition is the restoration of universe before humans and it is quite difficult to restore the past state in "*One*", "*Two*" or "*Three*" of the universe. Although there are many hypotheses and theories, so far the human recognition on the operating of universe is almost stuck in the stage of hypothesis and phenomenology. Not only that, but we still do not fully understand how the earth works. For example, the earth is rotating and also revolves around the sun with an average angular velocity of $4.167 \times 10^3 rad/s$ and a linear velocity of $465m/s$ at the equator, which is the result of earth's evolution over 2.5 billion years. So, *what are the effects of a faster or slower rotation of the earth on humans?* Theoretically, the faster the earth spins, the greater the centrifugal force and the less the gravitational pull. It is estimated that if the earth's rotation speed increases 17 times, the centrifugal force and gravity are in an equilibrium state, objects near the equator will float on the surface of earth due to the weightlessness. If the earth rotates 20 times faster, everything on the surface of earth will gradually disintegrate to pieces, fly into space and finally, the earth will gradually disintegrate and all life on the earth, including humans ourselves will cease to exist. And that is just estimated on the interaction between gravity and centrifugal force, which is far more complicated than such estimates prediction. In that case, all scientific researches and explorations that may affect the operating of universe should have to end. For example, the planning to plunder gold and other rare or precious metals on extraterrestrial planets or capture asteroids, as shown in Figure 12.19 in the one-sided pursuit of human's economic interests will eventually affect the operating of universe and change the order of universe, which may cause immeasurable damage on the earth, including humans. It may take hundreds of thousands of years at least for the universe to adjust back to the conditions on earth suitable for human's living. And meanwhile, the environment, climate changing and the collision of celestial bodies in this natural adjustment process will lead to the earth, including the space near earth no longer suitable for human survival and whether the space technology allows to migrate humans to a new planet is unknown at that time. So, *how can humans ourselves avoid such a disaster of self-destruction?* Dr.Ouyang tells Huizi that we must discipline ourselves, i.e., limit such parts of science that may affect the operating of universe from now on.

**Figure** 12.19. **Capture asteroid**

**Class 2.** *Destructing the biodiversity.*

Usually, the biodiversity refers to a variety of animals, plants and microorganisms regularly combining in a range, namely the different organisms adapt to the living environment as well as the interdependence and interaction among organisms constitute a stable ecological complex population through material circulation, energy flow and information exchange, including the diversity of animals, plants and microbial species, the diversity of species heredity, variation and the diversity of ecosystem. Notice that the existence of everything is Tao with an outstanding in its living environment, namely the *survival of the fittest*. Certainly, the destruction of biodiversity directly affects food chains, climate changing and will increase the risk of populations carrying unknown viruses with spreading and human health. Dr.Ouyang tells Huizi that humans have only a limited understanding of most other organisms and do not fully know the living conditions of most organisms until today. Thus, science is easy applying to destroy the ecological environment and affect the biological diversity in the development of profit by capital. Such kind of behaviors should be actively ended or limited. For example, the pesticide is an artificial product that can effectively control pests and diseases, eliminate weeds and improve crop yield and quality. However, the pesticide is not degraded or difficult to be degraded naturally, which leads to the environmental pollution and a small amount of pesticide residues on crops, damaging the soil structure, polluting water resources and killing organisms or microorganisms in soil and water to a certain extent. For another example, the ecosphere of a region is the evolving result of universe for billions of years. The invasion of alien species will lead to they competing with indigenous organisms for living space and food, spreading diseases and inducing genetic pollution caused by hybridization with indigenous organisms, which will reduce the survival ability of indigenous organisms, lead to the reduction or extinction of indigenous organisms and also destroy the diversity of organisms in the region.

**Figure** 12.20. **Biodiversity**

**Class 3.** *Affecting human's own survival.*

Certainly, the human is a product of the environment but *why does the human appear on the earth and not on other planets?* The answer is that the earth has the suitable conditions for human living, including the distance from the sun, earth's gravity, atmosphere that reduces the ultraviolet rays of sunlight to the suitable of human living, oxygen-rich air and suitable drinking water, food and space for human living. And meanwhile, the human himself is also suitable for living in the earth environment. So, *are there other planets in the universe suitable for human living?* Dr.Ouyang tells Huizi that although some research institutes claim that they have discovered extraterrestrial planets suitable for human living in the universe, the human spaceflight technology is not yet sufficient to send human to those planets for field verifying but analysing only on the observing data collected by human on the earth. Actually, a simplest criterion for determining whether an extraterrestrial planet is suitable for human living is to verify the existence of "*human*" and not just conditions for the existence of other creatures, animals or a few of living conditions of humans on the planet because the human is created by the environment and if it is suitable for human living, by the law of everything or by the creation of Tao, there should be "*human*" like the "*earthman*" on the planet, not just other creatures or living conditions. But so far, no extraterrestrial planet has been found creature to exist like human on the earth. Therefore, by the "*humans follow the earth*", human can only survive on the earth. Thus, to ensure the survival of human inevitably involves two aspects. First of all, is the protection of the earth's environment, ending or limiting those behaviors that destroy the earth's environment for capital profit, including the scientific researches and applications in Classes 1 and 2. Secondly, the protection of human body itself, ending or limiting those scientific researches that try to change the law of human reproduction and human body operating because no matter their impaction on the earth environment or on the human body itself, such scientific researches have a time-delay effect on human and they affect maybe not the present but the generation of human. Thus, science with its applications must be ended or limited under the notion of harmonious coexistence of humans with the nature. For example, the sheep shows in Figure 12.21 is Dolly, a cloned sheep. Because of the violation of natural law, Dolly was cloned accompanying with her mother's existing disease and similar to her mother from birth to aging. She was premature aging phenomenon, died at the age of six. Similarly, there are cloning technologies on pigs, monkeys, cows and the biotechnology that goes against the laws of nature such as the gene editing which cuts gene fragments on human embryos, must be ended or limited.

**Figure** 12.21. **Dolly**

§4. **Tao Without Intention**

Certainly, the infinity of scientific recognition and the limitation of individual scientific career show that it is impossible for anyone to know the law of everything in his lifetime and so, it is impossible to hold on all things in the universe. This fact forms a dilemma in human recognition of things to some extent. On the one hand, humans are always working on the recognition of things through science, as shown in Figure 12.22. But on the other hand, any personal recognition is only a fragment on the scientific recognition of things, namely an individual can only contribute the local laws on things. As a result, the personal research and exploration are doomed to reach the other side of recognition of everything, namely there is always a time that he or she feels powerless. In this case, a human is easily to wander for recognitive hesitation when he or she is in the dearth of innovative thought, moving to other things to replace the scientific research on things or to accept the religion as his recognition for seeking the peace of mind, which is common for a human to seek the peace of mind in fear of unknowns. In recent years, someone take Newton and Einstein as examples, claiming that Newton and Einstein turned to Christianity or Buddhism in their later years as a proof of the ultimate answer that the end of science is theology or Buddhism. Then, *how do we think about the idea that the end of science is theology or Buddhism, and whether Newton, Einstein, etc. really believed that the end of science is theology or Buddhism?* As everyone knows that science is the systematic recognition of things, which is a verifiable knowledge system that does not acknowledge the supernatural forces or the anthropomorphic "*God*" and the religion is the worship of "*God*", which is the personified of supernatural forces. They are different in essence. And so, the opinion that Newton and Einstein's belief in religion proves that "*the end of science is theology or Buddhism*" is a misreading of Newton and Einstein's scientific thoughts. Dr.Ouyang explains that the history of human civilization shows that the purpose of recognizing things is for human living. Thus, although the religion is the spiritualist worship of the personified "*God*" of supernatural forces, it also contains the element of recognizing things and can be used as a reference for science.

**Figure** 12.22. **What's the end of science**

**4.1.Science Dependent Origination out of Emptiness.** Notice the words of Laozi explaining *Tao* and *Name* in his *Tao Te Ching*, i.e., "*One realizes the subtlety of Tao without desiring, but see only an outline of Tao with desiring*", where the words of "*without desiring*" and "*with desiring*" is a state of being that comes into one's knowledge of a thing which impinges upon it, namely the recognition result without desiring is "*subtlety*" of

Tao but only an "*outline*" of Tao with desiring, which defines the different connotations of "*names*" by humans. Among them, the scientific recognition is a kind of "*with desiring*", which is similar to the word of "*origination*" in Buddhism.

Certainly, the essence of science lies in the recognition of "*origination*" of things to form a systematic knowledge of humans on things. Then, *what is the origination and what is the emptiness of thing?* Dr.Ouyang tells Huizi that the "*origination*" is a basic conception in Buddhism. It holds that everything does not exist out of thin air and can not exist alone in the world, i.e., everything can only being on the karma and its linkage. Once the karma and linkage are lost, the thing itself will come to non-being, namely the nature of thing is "*empty*". Here, the word of "*karma*" refers to the internal cause of being of thing, the word of "*linkage*" is the external cause of thing, namely the karma with linkage are the condition of a thing being, i.e., "*everything is being in karma with linkage*" in the law of causation in Buddhism. This implies that the being of all things is relative. They are mutually being with conditions. Otherwise, they can not being if losing the condition with relationship in world. Notice that the "*dependent origination out of emptiness*" is a recognition of all things in Buddhism. It holds that the nature of thing is emptiness, not reality which is only the character given by the six sense organs of human on thing because the human recognition is also a kind of origination, i.e., the profound meaning of words that "*if one can see a thing out of its image, then he or she holds on the nature of thing*" in *Diamond Sutra* or "*the color is the same as the emptiness, the emptiness is the same as the color; the color is empty, the empty is color and it is the same with the sensation, perception, decisions and consciousness*" in *Heart Sutra*, namely everything is recognized by human, it is human in recognition to give things different "*color*", i.e., the *emptiness* is the nature of thing, where the "*color*" refers to all things and phenomena produced by the karma with linkage that the six sense organs of human can perceive such as the mountain, river, landscape and fishes in the eyes of human, as shown in Figure 12.23. Certainly, the words of "*color*" and "*emptiness*" in here are similar to the words of "*being*" and "*non-being*" in words that "*all things are created from being, and the being is out of non-being*" of Laozi's *Tao Te Ching*, namely both of them are the recognition and definition of human's six sense organs on things.

**Figure** 12.23. **Fishing coming back**

And so, science as a systematic knowledge of laws on things, is a kind of "*Dependent*

*Origination out of Emptiness*" in Buddhism which is mainly shown in three aspects: ①Now that science is the human knowledge of laws on things, the laws are the "*origination*" in things and it is the human recognizing the laws, it must be also an "*origination*" of human, i.e., a "*Dependent Origination out of Emptiness*" because it must be unknown or "*empty*" before human recognizing on the thing; ②Anyone has his or her specialty in profession. Generally, what does one study or not study depends more on one's knowledge, also with one's interests, hobbies and pursuing. For example, someone are interested in discovering the law of material composition and the evolving of universe, someone are interested in discovering the chemical reactions to produce new substances or the law of biological evolution and heredity, and someone are interested in the law of social group evolving, etc. These are all the "*specialty in profession*". Here, the "*interest*" is the "*origination*" of scientific research; ③Opportunity. Usually, different recognitive thoughts will produce different results for recognizing the same thing. Why someone succeed but someone fail in the recognition of the same thing or study the same topic, it is because their thoughts, methods and means are different, where the "*thought*" is also a kind of "*origination*", including the observing, calculating, hypothesis, model, experiment and establishing theory of thing. It is not only the "*chance*" in human recognition but also the "*origination*" of individual effort and recognitive ability.

Similarly, the application of science is an "*emptiness*" also. The reason why some scientific achievements can be successfully transformed and bring benefits to human society while others remain only at the theoretical level and can not be transformed or not only the application can not increases the human welfare but also brings harm to the harmonious coexistence of humans with the nature, is also the "*origination*" of these achievements. Certainly, those achievements that can realize the transforming for benefiting humans are the "*origination*" that they can benefit humans, and those who can not realize the transforming or the transforming will bring harm to humans or the nature are also these scientific achievements "*origination*" themselves with harm to humans or the nature. This is an important issue needing to be aware for carrying out a scientific research or applying a scientific achievement, namely whether it is beneficial or harmful to the harmonious coexistence of humans with the nature.

**4.2.Science Verifying Religion.** Notice that both of the science and religion are human's recognition on things, and both of them satisfy the living needs of human to a certain extent. Certainly, the difference between them is that the science is about questioning, reasoning and evidence while the religion is unquestioning worship to God. So, *what is the relationship in science and religion?* Dr.Ouyang tells Huizi that different humans give different answers to this question standing on different perspectives. Someone stand on the side of science and hold simply that the science and religion are contradictorily irreconcilable, namely the science represents the truth but the religion represents the falsehood. Someone stand on the side of religion and hold simply that science can only satisfy the material needs of human to a certain extent but lack of the spiritual satisfaction. They take Newton and Einstein as examples, claim that "*the end of science is theology*" and hold that only by relying on God can humans prosper. What needs to be pointed out in particular is that both of these views have certain limitations or bias to human's life. First of all, both of the science and religion are human recognition of things, there are no decisive opposition between them. They satisfy respectively in the material or spiritual field of human's needs. Secondly, neither the science nor the religion can fully satisfies the needs of humans. Each of them has its own specific social value. They are needed harmoniously to solve the problems in development of humans together. Thirdly, the science and religion are influencing and promoting each other, which is a kind of unity of opposites. At this point, there is a misunderstanding on that Newton, Einstein and other scientists believed in religion. Someone consider that they believed in religion later proved that "*the end of science is theology*", which is essentially not the case. Dr.Ouyang tells Huizi that Newton, Einstein and other scientists believed in religion not because of the blind worship of "*God*" but found the scientific inspiration in religious, which confirmed or disproved the religious recognition on things. Such thought is reflected in Newton's famous book of *Philosophiae Naturalis Principia Mathematica* as well as Einstein's views on science and religion, especially his famous words that "*science without religion is lame, religion without science is blind*", namely the dialectical relationship of science with religion is influencing and promoting each other.

**Figure** 12.24. **Newton**

**Newton's Philosophy of Science.** The Anglican Church is a denomination of Protestant Christ, which is one of three main Christian Protestant denominations. When Newton was born, the Church of England was Anglican and everyone had a religious belief in that time. Thus, believing in God is a natural thing in such a society. However, Newton's belief in God is not a blind worship but regarding God as a non-anthropomorphic "*God of the nature*", which is similar to Yuanshi, Lingbao and Tao Te geniuses in Taoist of Chinese. His philosophy of science is expressed respectively in: ①Did not believe that Jesus is a God, not accepted the *trinity theory*, namely God created the universe, God saved humans through Jesus Christ, God was with Christians and the Church through the holy spirit in Christianity, did not believe in the undead souls or immortality. In his eyes, the worship of Jesus is only a form of "*idol worship*", a form of original sin. Indeed, he accepted the God's creation but he did not agree that God created a perfect universe without further managing; ②Insisted that his study purpose of *Bible* was to discover the hidden information of *Bible* and try to extract its scientific parts and to explain its recognition of things.

Guided by this philosophy, Newton himself not only believed in God but also took God as the driving power for studying science. His major scientific achievements covered mechanics, mathematics, optics, heat and philosophy. For example, he suggested that there is an absolute Cartesian coordinate system $\mathbb{R}^3$ characterizing the moving of object with an independent parameter time $t \in \mathbb{R}$. Thus, by summarizing the conclusions of his predecessors, he systemically proposed his three laws of classical mechanics, including ①Inertia law, which states that an object always moves uniformly in a straight line or at rest if it is not subject to any external force; ②Force $F$ is the cause of acceleration $a$ generated of a moving body, where $a$ is proportional to the magnitude of force $F$ and inversely proportional to the mass of object $m$, i.e. $F = ma$; ③Action & reaction law, i.e., the action force $F$ and reaction force $F'$ operate on the same straight line in equal and opposite directions, namely $F = -F'$ and the law of universal gravity which states that the gravity is proportional to the product of masses of two objects, inversely proportional to the square of distance between the centers of two bodies, etc., all of these correctly characterize the laws of macroscopic objects in low speed by mechanical moving, unify the mechanics of celestial bodies and objects on the earth, establish the theoretical system of classical mechanics. In order to solve the trajectory calculation of moving object such as the tangential problem, quadrature problem, instantaneous velocity, the maximum and minimum problem of trajectory, he unified the techniques of solving infinitesimal problems since the ancient Greece into two operations, namely the differential and integral in mathematics, established the mutual inverse relationship in the two operations and completed the critical step in invention of calculus, provided the most effective tool for modern science. Newton himself called the *fluxional calculus*. It is recorded that Newton may have studied calculus earlier than Leibnitz but Leibnitz's work on calculus was published earlier than Newton's, which is why it is called the Newton-Leibnitz calculus today.

Newton always commented his achievements were on "*following the instruction of God*", "*thinking following the thoughts of God*" in order to meet the society at that time and summed up his world view by saying that "*this being governs all things, not as the soul* *of the world, but as Lord over all ⋯*" in his *Philosophiae Naturalis Principia Mathematica* and claimed that everything he did scientifically was to verify God's existence such as those words that "*when I wrote my treatise about our system, I had an eye upon such principles as might work with considering men for the belief of a Deity and nothing can rejoice me more than to find it useful for that purpose*". So, *does Newton's self-evaluation mean that he was idolizing God?* Of course Not! Dr.Ouyang explains that although Newton was belief in Christ, his true meaning of "*God*" was "*natural God*" rather than the Christian God and his scientific research verified the truth. He believed that after the creation of universe, the various laws of moving set for all things could be understood by humans. Thus, Newton's cosmology is essentially a simple "*mechanical cosmology*", i.e., to understand the universe by the mechanical moving rather than the blindly worshiping God.

Then, *is Newton's explanation of the mechanical moving of universe acceptable to Christianity?* On May 12, 1713, C.Roger, a fellow of Trinity College of England summarized Newton's achievements as explaining the movings of celestial bodies by gravity and wrote that "*this superb work of Newton will rise up as a most well-defended stronghold against the attacks of atheists: nor could anything be happier than to draw a dart against that impious band from this quiver*" in preface to the second edition of Mathematical Principles of Natural Philosophy, is the creationist affirmation of Newton's achievements. This reflects from one side that Newton found the inspiration of scientific creation in religion and explained or confirmed the religion's recognition of universe with scientific thoughts.

**Einstein's Philosophy of Science.** Different from Newton, Einstein lived in an age of freedom in religious belief. He was like Newton in deeply religion and a firm believer in the transcendent God. One of his well-known words that "*science without religion is lame, religion without science is blind*" has led someone to believe that he was a theist. However, he rejected the anthropomorphic or personalized understanding of word "*God*" and never thought the religious belief a sign of stupidity but he believed the unity and logical consistency of all things in the universe, praised the beauty of universe and the nature. For example, he replied a question "*if scientists pray*" asked by a young girl in sixth grade with words that "*a research scientist will hardly be inclined to believe that events could be influenced by a prayer, i.e. by a wish addressed to a supernatural being*" but "*it must be admitted that our actual knowledge of these laws is only imperfect and fragmentary, so that, actually, the belief in the existence of basic*

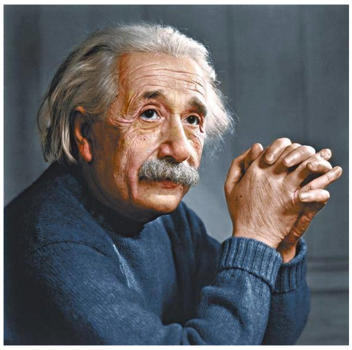


**Figure** 12.25. **Einstein**

*all-embracing laws in nature also rests on a sort of faith*". He summed up his religious beliefs in a letter to a friend: "*the word God is for me nothing but the expression and product of human weaknesses, the Bible a collection of venerable but still rather primitive legends. No interpretation, no matter how subtle, can change anything about this. ⋯. For me the Jewish religion like all other religions is an incarnation of the most childish superstition. ⋯. I can not see anything 'chosen' about them*" and so on. All of these words show that Einstein was religious but not idolatrous.

Einstein once analyzed the contribution of science and religion to living of human in one of his papers. He considered that the science is a time-old effort by systematic thinking to connect the perceived phenomena in the universe as thoroughly as possible, and to reconstruct the being through the process of conception. And meanwhile, a religiously educated human has been freed as far as he can from the selfish desires, and is preoccupied with the thoughts, feelings and aspirations which he maintains for his own sake because they are his personal worth. In this way, the religion is the long cause of humans which aims to make humans sober and fully aware of these values and objectives, and to constantly strengthen and expand their influence. If one understands the religion and science according to these definitions, there is no conflicting between them. Thus, the science can only assert *what is* and not *what ought to be* but outside its scope, all valuable judgments are still necessary; the religion is only concerned with the evaluation of human thoughts and actions, and it can not talk about various facts and their relations, namely there should be no legitimate conflicting for science with the religion.

Similar to Newton's religious view, Einstein believed that God was not a personalized "*God*" but the unity and logical consistency of all things in the universe and then, found the laws of things. Certainly, his philosophy of science led to his outstanding achievements such as those of achievements: ①Special relativity, namely the physical laws expressed in space-time in all inertial reference systems should be formally invariant under Lorenz transformations, the speed $c$ of light is a constant in all inertial reference frames, which is the maximum speed of an object's moving observed by human; ②Proposed the mass-energy equation $E = cm^2$ to unify the energy $E$ and the mass $m$ of an object; ③General relativity, namely an equation that describing the evolving law of an objective thing should have the same form in all coordinate systems; ④Proposed the equations of gravitational field, suggesting that space-time is curved by the existence of matters and the gravitational field is contrary to the human intuition, which is in fact a curved one even light; ⑤Explained the precession time difference of Mercury perihelion, predicted the gravitational redshift that the spectrum moves towards the red end in strong gravitational action and the gravitation deflects light rays, substantiated by observing later; ⑥Proposed the photon hypothesis and successfully explained the photoelectric effect and so on.

So, *what is the most remarkable of Einstein's scientific achievements?* The answer is of course the general relativity! Dr.Ouyang tells Huizi that the theory of general relativity is nearly equal to the name of Einstein in eyes of humans because Einstein's theory of general relativity completely changed the absolute space in the recognition of Newton and others, and reversed the local recognizing of space-time on the earth, namely the real material space of universe is a curved one rather than the flat one in human intuition. Then, *what is the relation of general relativity with religions?* Dr.Ouyang explains that the essence of general relativity in recognition of things is the "*origination*", i.e., the recognitive relativity in *Dependent Origination out of Emptiness* of Buddhism, and science is the human recognition of things. It is the six sense organs' perception of human in recognizing things, which must complies with "*Dependent Origination out of Emptiness*" or the relativity in recognizing, namely the observing is dependent on the observer which constitutes a binary system $K_2^L$ in Section 2.2 of Chapter 11. This is Einstein's theory of general relativity. On the other hand, Einstein's theory of general relativity is also a scientific interpretation of *Dependent Origination out of Emptiness*, which confirms the recognitive view of *Dependent Origination out of Emptiness* in Buddhism to a certain extent. This case once again shows that the science and religion can promote each other and develop together because the correct recognition of things in religion is also the precious wealth of human civilization and can certainly be applied for reference and promotes the development of science. And similarly, the science can verify the recognition in religion, promotes the recognition on things and harmonious coexistence with it.

**4.3.Science Itself no Calming Mind.** Now that the science and religion can promote each other and develop together. So, *why do some scientists abandon science later and turn to the religion, believing that the end of science is theology?* Dr.Ouyang tells Huizi that Tao has no an intention of human, everything is the result of evolving of Tao, even the heaven and earth shown in words that "*the heaven and earth are not kind, taking all things as humble dogs*", namely the heaven, earth and Tao are careless, do not intentionally tend to a thing in the universe. Accordingly, as the local recognition of Tao on things, the science can certainly not calm the mind of human because the mind is from human own. Whether it is calmness depends on human rather than science or Tao. Hence, the reason of why some scientists give up science, turn to the religion and think that the end of science is theology is not because the religion really reveals the law of everything or not but the life of human is limited, his or her ability to recognize the laws of things is limited also, namely everyone must have the end of scientific career. In this case, the science can not explain all doubts on the unknown and so, it can naturally not calm one's mind. Then, *what matter can calm the mind of human?* Dr.Ouyang explains that in the eyes of human, ①Plant living requires only materials such as the rain, air, fertilizers and the photosynthesis to sustain its reproduction from the germination, flowering to the fruiting; ②Animal unless the human is mainly living on the physical needs with reproduction, only with a little needs on the spirit. For example, the animals often fight for foods, as shown in Figure 12.26 or fight for clan leader who controls the community; ③As the spirit of all creatures, the human living has both of the physical and spiritual needs, including the usual needs of clothing, food, housing and transportation, but more is indispensable on the spiritual needs such as the independence, free and satisfies outside the physical needs. This is the biggest difference of human with an animal. For example, a few characters of human are characterized in Chapter 12 of *Tao Te Ching*, namely the words that "*the five colors blind the eye, the five tones deafen the ear, the five flavors numb the taste, the riding and hunting madden the human, the rare goods misbehave the human; that is why the sage only eats with enough food, not satisfy his eyes with a riot of colours*". Among these, five colors, five sounds, five flavors and those rare products can be met the human's spiritual needs to some extent but an excessive pursuit is against to Tao, also called the "*greedy, anger, infatuation*" in Buddhism, which can make a human blind, deaf, happy in mouth, mad in mind or go to anomaly, namely he or she must follows Tao and need to learn the sage's living "*only eats with enough food, not satisfy his eyes with a riot of colours*".

**Figure** 12.26. **Lion predation**

However, *how many humans in the world are bound by rules of the sages?* Dr.Ouyang tells Huizi that for most humans, their livings are still in both the materials and spirit. That is, when one's material needs is not enough to meet the needs of family, one must pursues the materials for calming one's mind. However, if one's material needs suffices to meet the needs of family, one will inevitably turn to the spiritual needs, including ①Addicting to entertainment of human; ②Enthusiastic with the welfare of others; ③Devoting to scientific research; ④Believing in religious and others for calming one's mind. Among these, the science is to recognize the laws of things. Thus, when a scientist's thought and innovation are exhausted or the limitation of science no longer calming one' mind, one will inevitably seek other ways to calm the mind. In the situation, the pursuit of material comforts, entertainments, seeking the well-being of others and the religious belief may be the alternatives. The religious belief, especially the theory of God creation in religion can subjectively answer the doubt of human on the nature and calm one's mind to some extent. This is why someone take Newton and Einstein as examples and believe that "*the end of science is theology or Buddhism*" and turn to religion. Naturally, the science and religion are not in the absolute opposition but they are promoting each other.

## §5. **Scientific Life**

Certainly, science is the human recognition on the law of things, technology is the application of science to serve the human society such as those of means, techniques and skills improving the human's ability to adapt to the nature and enriching the social life. Thus, the combination of science and technology is to improve the social productivity, prosperity of social life and speed up the process of human civilization. Among them, the social demand is the driving power for developing science and technology which plays a guiding, selecting and regulating role in the development of science and technology. And meanwhile, the progress of science and technology promotes the productive factors, their combination and innovation, affects the improvement of social infrastructure and promotes the development of society. For example, the steam engine, hydraulic press and the internal combustion engine, which are driven by steam power, water pressure, electricity and atomic energy have improved the development of textile industry, mining industry, metallurgy industry and transportation, and the appearing of vehicles such as automobiles, trains, ships and airplanes; the development of science and technology such as artificial materials, space technology, electronic computer, internet and artificial intelligence as well as the wide application of robots in production and life have reduced the intensity and danger of human labor and improved the convenience of life further, highlighting the role of science and technology in promoting the development of human society. The human's material and spiritual needs have been greatly satisfied. At the same time, the reaction of nature to human activities such as those of the earthquake, haze, flood, mud-rock flow, extreme weather, etc., also occurred frequently in years. So, *what is the ultimate goal of science*

**Figure** 12.27. **Science changing life**

*and technology? is it to promote humans to become the masters of universe?* Of course Not! Dr.Ouyang tells Huizi that the human's exploration of nature is always a process and science is only a partial recognition of the laws of things, liking a tourist's view. In fact, humans ourselves can never being the master of universe. We can only live in harmony with the nature confirmed by the history of human developing, which is naturally related to a fundamental question, namely *what is the ultimate goal of humans?* Dr.Ouyang explains that the *ultimate goal of humans* is twofold. One is the "*survival*" or continuity of the race and another is to realize the human's spiritual needs, namely to pursue the psychological satisfaction or happiness. By this goal, *how should science develop to serve the harmonious coexistence of humans with the nature?* The answer should be on the binary system $K_2^L$ of humans with the nature, namely the realizing the human's racial continuity and spiritual needs should be the goal but the "*harmonious coexistence with the nature*" should be the norm of humans, i.e., developing such science with technology that follows the "*harmonious coexistence with the nature*" rather than that only satisfies the needs of humans ourselves. In this situation, it is necessary to further recognizing the unknown things and strengthening human's self-restraint.

**5.1.Dilemma of Science.** Notice that science is human's local recognition of things. Its purpose is to establish the continuity flow $G^L[AEP]$ of all things, which can not calm the mind of human but the technology is not because it serves human's society and enriches human's life. Unlike the Tao without intention, the technology or science with technology must have the original intension, i.e., an intension comes from the mind of human. Thus, the appropriate development of science for serving human's material and spiritual needs lies in maintaining the conservation of vertices in binary system $K_2^L$ of humans with the nature, namely the waste caused by human activities does not invade the nature or the waste is retained by humans because in the binary system of humans with the nature, science is human's recognition of laws of things within their ability. The laws of recognition are all local laws or conditional truth and we do not know the extent of disturbance caused by the waste of human activities on the nature until today. This is the scientific dilemma, i.e., on the one hand, the application of science can enrich human life but on the other hand, especially by the influence of "*profit by capital*", it is easy to exaggerate the beneficial side of a scientific achievement while ignoring or failing to see its harmful side to the nature. Thus, if the scientific research with application is unrestrained or abused, it will break the conservation of vertices in binary system $K_2^L$ formed over billions of years of humans with the nature, and resuming the conservation of vertices in $K_2^L$ by natural regulation is inevitably accompanied with the environmental changes, which will directly affect the human living and even lead to the end of humans. Therefore, it is necessary to verify the bilateral influence of science and technology on humans and the nature under the notion of harmonious coexistence of humans with the nature so as to guide the development of science and technology.

**(1) Benefits.** Certainly, the progress in science and technology has greatly enriched the material needs of humans, including clothing, food, housing and transportation and meanwhile, brings human the spiritual enjoyment and happiness in life. Here, we take physics, chemistry and biology as examples to illustrate the benefits of scientific and technological progress to humans: ①Physics, its profound understanding on composition of matter from the macroscopic to the microscopic level has contributed to the development of human society such as the steam engine, electric current theory with applications in transport and communications, the nuclear energy, lasers, computer technology and the network technology have contributed to the industrial revolution and technological progress. It can be said that without the understanding of matter structure in physics, there would be no other sciences, no modern life of human in today; ②Chemistry is useful to human society in accompanying its development in addressing how new substances can be produced through chemical reactions to meet the needs of human life daily. Conclusively, without chemistry there would be no modern life of human. For example, the energy is the basis of large-scale industrial production, including the energy developing and production such as the coal, oil, natural gas mining, storage and the electricity production, etc., all of them are on the chemical reactions $C + O_2 \to CO_2$, $2C + 2O \to 2CO$ and subsequently, there are the industrial production to meet the social needs such as those of the chemical fertilizers used to increase the yield of crops to meet the needs of human survival, the chemical pesticides to kill the harmful insects to promote crop growth, the medicine to cure diseases to prolong the life of human, the plastics and synthetic fibers widely used in modern industry and consumer goods such as keyboards, mice, plastic cups, slippers and other daily goods, industrial instruments, electronic equipment, mobiles and other industrial products as well as the military explosives, bombs and missiles, etc. None of them is not with the chemical application; ③Biology is the basis of agronomy and medicine, including the fields of farming, animal husbandry, fisheries, medicine, pharmaceutical production and health science. For example, the genetic engineering and cell engineering have a great impact on human society such as that the plants are directed to reform and the excellent varieties with high quality, high yield and drought tolerance are cultivated; the transgenic technology can improve crop insect resistance, solve the crop pesticide residues and improve the crop quality and output; the antigenic genes can produce edible plant vaccines; some animal organs can replace the human organs to continue the life of human; the cloning can detect genetic defects in fetuses and treat neurological damage, to achieve asexual reproduction to save the endangered species; the gene editing can correct defective genes for therapeutic purposes; the environmental pollution can be carried out through genetic engineering, recycling and utilization as well as the use of animal cell large-scale culture technology to produce the vaccines, animal and crop varieties breeding, the differences of virus strains detecting, the bacterial species identifying to treat the patient and promoting the development of human society.

**(2) Harms.** Generally, humans are used to judging the benefits of a scientific and technological achievement only from the human side in the binary system $K_2^L$ formed by humans with the nature while ignoring its effect or cumulative effect on the nature, which usually results in the discovery of harm of the achievement on humans after a period of time. This is caused by the limitation of human recognition. Here, we take physics, chemistry and biology as examples for explaining the harms of science and technology: ①Physics, it caused some unnecessary damage to the human environment while it promotes the progress of human society. For example, the physics advances the development of space industry to explore the outer space. However, an increasing number of satellites, space stations, probes, rocket debris and explosive debris are floating in the sky whether they are in use or defunct, which disrupt the normal operating of universe to some extent, threat to humans further exploration of space because the probe could be destroyed if it collides with any of the uncertain floating objects. And meanwhile, a few of harms caused by the application of physical achievements can also be found even in the daily life. For example, the communication facilities and equipment such as the mobiles, radio stations, television stations and the microwave stations of communication need to send the electromagnetic signals into air from time to time while they bring convenience to humans, which will interfere with the brain wave operating to some extent as shown in Figure 12.28, causing the effects on human body; ②Chemistry, while the chemical industry is producing new substances by chemical reactions to meet the material needs of humans, the chemical residues whether they are in states of gaseous, liquid or solid are largely abandoned to the nature as long as they are no use to humans, which naturally give rise to the environmental problems. For example, the environmental pollution, resource depletion, side effects of

**Figure** 12.28. **Brain wave**

medicine, pesticide residues and so on, have led to the prevalence of some deadly diseases such as the cancer. So, *why do these things appear inconsistent with the goal of science and technology for benefit to humans?* Dr.Ouyang tells Huizi that the reason is that humans only have a superficial or partial understanding on the stability, toxicity diffusion and the natural consumption of chemicals and lack a complete conclusion of their impact on humans and the nature. This can be a problem even with chemicals that are benefits to humans. For examples, the plastic can protect one from getting wet in rainy days but it can not be degraded by the nature in a short time; the accumulation of pesticides in food chain and the mechanism of their impact on organisms are still unclear until today, which also challenges the diversity of species; ③Biology, a few of negative impacts have emerged to the living while it brings the benefits to humans. For example, the genetic engineering destroys the beneficial insects that feed the crop while it makes the insect resistance; to eat the genetically modified food may cause the modified genes to invade human cells and produce pathogenic viruses, causing harmful or lethal effects, including the cancer and other negative effects; the cloning impacts on the nature, society and the social ethics, which reduces the genetic variation but results in low immunity or other loss of function of body, may destroy a biological population and increase the risk of disease transmission to some extent; the gene editing corrects or excises the defective one but it may affect the normal operating of other genes so that the edited body is no longer suitable for existing in the living environment at the same time, etc.

So, *how can we get out of the dilemma of science?* Dr.Ouyang explains that the main step of getting out of the dilemma of science is to evaluate a scientific or technological achievement in the binary system $K_2^L$ of humans with the nature rather than simply to evaluate whether it is beneficial to humans ourselves. Notice that the effect of nature on humans is immediate while the effect of human activities on the nature is a cumulative or delayed effect. In this case, *how should we understand the conservation in field equation* (11.1) *and then get out of the dilemma of science?* Dr.Ouyang tells Huizi that the field action should be a historical sum. Without loss of generality, let $L(v,u) = (L_1(t), L_2(t), \cdots, L_n(t))$ be the acting vector of human activities on $n$ natural events at time $t$ in equation (11.1) and let $\mathcal{R}^P[t] = (\mathcal{R}_1[t], \mathcal{R}_2[t], \cdots, \mathcal{R}_n[t])$ be the natural absorption vector. Then,

$$A_{vu}^{+}: \ L(v,u) \ \rightarrow \int\limits_{t=0}^{t} L(v,u)[s]ds = \int\limits_{t=0}^{t} (L_1(s), L_2(s), \cdots, L_n(s))\, ds \tag{12.15}$$

is the intrusion of human activities on the nature constrained with the inequality

$$\int_{t=0}^{t} \left(L(v,u)[s] - \mathcal{R}^P[s]\right) ds \ \leq \ \mathcal{N}^P_{\max}, \tag{12.16}$$

which is the natural permissible condition of human activities. Among them, for any integer $1 \leq i \leq n$, $L_i(t)$ and $\mathcal{R}_i(t)$ are respectively the intrusion and natural absorption amounts of human activities on $i$th natural event in $K_2^L$, $t$ is the duration of human activities and the vector $\mathcal{N}^P_{\max} = (\mathcal{N}_1^{\max}, \mathcal{N}_2^{\max}, \cdots, \mathcal{N}_n^{\max})$ is the maximum pattern of natural events permitted for human survival. Notice that the permissible condition (12.16) is a vector inequality, which implies that

$$\int_{t=0}^{t} \left(L_i[s] - \mathcal{R}_i[s]\right) ds \ \leq \ \mathcal{N}_i^{\max} \tag{12.17}$$

for any integer $1 \leq i \leq n$.

**5.2.Science's Constraint.** Generally, it is believed that there are no forbidden fields for scientific researching and exploring by the goal of science, namely science can carry out researching and exploring on anything in the universe. However, it is not the case under the notion of harmonious coexistence of humans with the nature because it implies that there must be constraints on the scientific researching, exploring with application. Particularly, it should not be engaged in or applied to matters that may harm to humans or the nature. This should be the moral criterion in the development of science.

(1) **Natural constraint.** Certainly, the fields of science are necessarily limited by nature's constraints on human ability, i.e., the natural constraints. Dr.Ouyang tells Huizi that this is mainly reflected in the limits of recognition imposed by Tao on human ability in "*Three creates all things*". For example, if the Big Bang theory is correct, it would be impossible for any kind of artificial vehicle to travel faster than the light. Now that the Big Bang has passed over 13.82 billion light-years. It would takes 13.82 billion years for an aircraft to travel even at the speed of light. In this case, assuming that the ratio of universe expansion speed to the speed of light is $\alpha$, then the universe will expand by $138.2\alpha$ light years after 13.82 billion light years, i.e., the aircraft will need to fly another $138.2\alpha$ light years and then, it needs to fly $138.2\alpha^2$ light years again after the $138.2\alpha$ light years passed, $\cdots$, which form a geometric series $138.2\alpha, 138.2\alpha^2\cdots$. Therefore, this aircraft needs to fly

$$138.2 \times \sum_{i=1}^{\infty} \alpha^i = 138.2 \times \left(1 + \alpha + \alpha^2 + \cdots + \alpha^k + \cdots\right) \tag{12.18}$$

light years for arriving the boundary of universe. However, the formula (12.18) is an infinite series. Then, *is it a finite number?* Dr.Ouyang tells Huizi that by the convergent condition of series, if $|\alpha| < 1$, i.e., the expansion rate of universe is less than the speed of light, the series (12.18) is convergent to $138.2/(1-\alpha)$, then the aircraft flying at the speed of light can finally arrive at the boundary of universe in a limited time (assuming that the flight energy supply is sufficient and the human life to meet the exploring needs). Unfortunately, the universe expansion speed is the Hubble constant, which is calculated to be $74.20 \pm 3.6km/s$ with the minimum $70.06m/s$. Its ratio to the speed of light is $\alpha \geq 2.335 \geq 1$, namely the series (12.18) is divergent. As a result, an artificial vehicle traveling even at the speed of light would not be able to arrive the boundary of universe unless it enters a period of collapse. However, it is not known whether the humans would still exist at that time.

Dr.Ouyang tells Huizi that the natural limitation that the speed of artificial aircraft to be less than the expansion speed of universe implies that it is impossible for humans to face to face with a large number of things in the universe or to verify them in the field. This is also the reason that why Laozi explained "*humans follow the earth, the earth follows the heaven, the heaven follows Tao and Tao follows the nature*" in his *Tao Te Ching*, namely the living of humans should follow the living of all things on earth as far as possible because all humans are constrained by the earth. In fact, even on the earth, the six sense organs of human also limit the range of human's perception of things on or near the earth, namely the scope of human recognition of things or science is naturally limited by the fact that all humans are the earthmen.

**(2) Life constraint.** The scope of science is limited by the harmonious coexistence of humans with the nature, namely the human actively to limit or end some parts of science with their applied fields in order to realize its ultimate goal. Dr.Ouyang explains that the harmonious coexistence of humans with the nature requires that science with its guiding to human activities should satisfy two conditions. One is science with its application benefit to humans. Another is science with its application does not disturb the nature, namely science with its application should benefit humans without disturbing nature. These two conditions seem simple but even today, a great deal of human activities guiding by science do not satisfy the requirements, especially in synthetic materials and a large number of industrial processes. Generally, the chemical reaction of synthetic materials or industrial processes under conditions $\mathcal{C}$ is

$$A_1 + A_2 + \cdots + A_k \ \rightarrow \ B_1 + B_2 + \cdots + B_l, \tag{12.19}$$

where the symbol $A_i$ denotes the molecule involved in the chemical reaction and $B_j$ denotes the molecule produced by the chemical reaction, $1 \leq i \leq k, 1 \leq j \leq l$ for integers $k \geq 1, l \geq 1$. Dr.Ouyang tells Huizi that any naturally substance has evolved over billions of years in a stable state. Thus, the paradigm (12.19) is the chemical reaction between the molecules $A_1, A_2, \cdots, A_k$, which can also represent the chemical decomposition under certain conditions, i.e., the case of $k = 1$. Here, there are two issues of particular concern, namely ①$A_1, A_2, \cdots, A_k$ are $k$ pure molecules involved in chemical reaction that can be realized only in the laboratory conditions. In practice, the compound contains not only the molecules $A_1, A_2, \cdots, A_k$ but also other compounds or impurities. A similar situation also appears in the chemical decomposition, i.e., the case of $k = 1$; ②Outputs of $B_1, B_2, \cdots, B_l$ are not all useful to humans. Assuming that $B_1, B_2, \cdots, B_{l_0}$ are beneficial but $B_{l_0+1}, B_{l_0+2}, \cdots, B_l$ are not beneficial to humans, then the outputs $B_{l_0+1}, B_{l_0+2}, \cdots, B_l$ are the materials synthesis or industrial waste in process. Let us take the industry of textile printing and dyeing as an example. Generally, every printing and dyeing $1000kg$ of textiles will consume $100000 - 200000kg$ of water, resulting in $80\% - 90\%$ waste water, contain pulp and its decomposition, fiber scraps, enzyme pollutants, fats, nitrogen compounds, bleach, sodium thiosulfate, alkali sulfide, aniline, copper sulfate, formaldehyde, terephthalic acid, glycol and other harmful substances. So, *can we discharge the waste water directly into rivers in printing and dyeing?* Of course Not! Dr.Ouyang explains that the discharging waste water in printing and dyeing directly into rivers seems satisfying the first condition but disturbs the nature at the same time, which will cause the pollution of rivers, result in the fish, shrimp and other aquatic life gradually disappearing in the river, affect the ecological environment and endanger the human survival also. Thus, the waste water and other industrial wastes $B_{l_0+1}, B_{l_0+2}, \cdots, B_l$ produced in the industry of printing and dyeing should be properly disposed on the standard before allowed them to discharge into the nature. So, *does the synthetic or industrial processes dispose of wastes $B_{l_0+1}, B_{l_0+2}, \cdots, B_l$ on the standard necessarily imply that there are no disturbing on the nature?* The answer is Not! Dr.Ouyang explains that the wastes $B_{l_0+1}, B_{l_0+2}, \cdots, B_l$ are disposed on the standard of humans and do not back to their natural states. For example, the disposal of waste water in printing and dyeing is only to satisfy the discharge standards but its harmful substances have not completely clear yet, i.e., back to the original water quality. After a pause, Dr.Ouyang continuously explains that it is essentially an inverse process of chemical reaction (12.19) that disposes the waste to the synthetic materials or industrial process materials back to their original natural state, i.e.,

$$B_1 + B_2 + \cdots + B_l \ \rightarrow \ A_1 + A_2 + \cdots + A_k \tag{12.20}$$

in the natural mechanism. Then, what needs is its inverse condition $\mathcal{C}^-$ for human to study further because the formula (12.19) has broken the natural mechanism. In this case, if all of the wastes $B_{l_0+1}, B_{l_0+2}, \cdots, B_l$ are disposed back to their origins $A_1, A_2, \cdots, A_k$, then it needs generally $\mathcal{C}^- \supset \mathcal{C}$. For example, the disposal of wastes $B_{l_0+1}, B_{l_0+2}, \cdots, B_l$ in the decomposition of compound $A_1$, the natural mechanism is rejoining all those products $B_1, B_2, \cdots, B_{l_0}$ beneficial to humans back to $B_{l_0+1}, B_{l_0+2}, \cdots, B_l$ and then, they can be synthesized into $A_1$ and the energy consumed in this process must be greater than that of the decomposition of (12.19). However, humans will not obtain benefits of $B_1, B_2, \cdots, B_{l_0}$ in this way. In other words, obtaining benefits of $B_1, B_2, \cdots, B_{l_0}$ from material synthesis or industrial processes will inevitably produce wastes $B_{l_0+1}, B_{l_0+2}, \cdots, B_l$, which can not be restored to the original state of disposal, i.e., discharging them into the nature inevitably intruders the nature and we are faced to discover a new mechanism for disposal of wastes $B_{l_0+1}, B_{l_0+2}, \cdots, B_l$ beneficial to humans without the disturbing nature. However, such a mechanism has not been found yet. In this situation, the first is human to practice economy, against extravagance and waste in material needs, to create a cycle chain for outputs $B_1, B_2, \cdots, B_l$ in industrial production, synthetic materials or industrial process, including the waste water, waste gas and liquid such as the cyclic application of carbon dioxide. And meanwhile, the wastes not included in the cyclic chain should be properly disposed of pollution-free. If necessary, the wastes should be retained or sealed up so as not to intrude on the nature. Another is the active end or limit all scopes of science with its applications out of the harmonious coexistence of humans with the nature, including those that attempt to change the genetic structure of human body or organisms, relieve pain for certain groups of hallucinogens, medicine or drugs, the extravagant consumption of someone who pursue "*five colors, five sounds, five flavors*", the sensory stimulation that may *madden the mind* or *misbehave*, serving only the profit of capital and other scope of science with its applications that harm humans or disturb the nature such as the excessive grabbing or plundering natural resources, etc.

**5.3.Human Civilization's Constraint.** Certainly, the human civilization is the sum of humanistic spirit, invention and creation accumulated in history, which is understanding and adapting to laws in the nature, in accordance with the pursuit of humanistic spirit, recognized and accepted by the majority of humans. It emphasizes that humans must live in harmony with the nature on the survival. Accordingly, science with its applica-

**Figure** 12.29. **Great wall**

tions is the driving power for the progress of human civilization. Joseph Needham, a British historian of science, has a famous words that "*for scientists, insurmountable principle is to work for human civilization*", namely the mission of scientist is to understand the law of things and then, promotes the progress of human civilization approaching to the ultimate goal. And so, science with its applications is naturally restricted by human civilization while it promotes the progress of human civilization, i.e., not to engage in researching, exploring and development of those running against to the harmonious coexistence of humans with the nature. Then, *how does the human civilization constrains science with its applications?* Dr.Ouyang tells Huizi that the constraint of human civilization on science with its applications is reflected in the fact that science with its applying scopes is restricted by the social civilization of a certain period and can not violates the social civilization such as the "*public order and good customs*" of society. Notice that the civilization is the embodiment of social morality and the morality is the measure of social civilization. Thus, the essence of human civilization's constraint on science with its application is the constraint Te on social individuals, including the scientists.

So, *what is Te in Chinese culture, what kind of behavior is "Te being" or "Te non-being"?* Here, the word of *Te* refers to the behavior in accordance with the inherent nature or personality. Dr.Ouyang explains that "*Te*" in *Tao Te Jing* is such a behavior that follows Tao, i.e., the law of everything and believes that "*the great Te is exclusively depended on Tao in form*", namely everything is operating in accordance with Tao and then, it is said to be "*Te being*". Otherwise, its behavior against to Tao is the "*Te non-being*". Notice that everything is defined by the six sense organs of human, including all kinds of expression of things. Certainly, everything is following Tao except the human himself in the eyes of human because human has the ability of thinking, judging and selecting, which is the biggest difference between humans and other animals. But, it is exactly because of this point that human's behavior may against to Tao, i.e., appearing the behavior of "*Te non-being*". So, *what is the relationship between "Te" and all things?* The answer is explained by the words that "*Tao creates them, Te raises them, matter forms their shape and power completes them; and so, all things worship Tao and esteem Te*" in Chapter 51 of *Tao Te Ching*, namely everything is created by Tao, raised by Te, formed shape by matter and completed by the environment or nature, i.e., Te with everything is the relationship between raising and be raised. In this way, everything follows Tao with Te as the nature which needs the imitation of humans, namely the code of conduct that "*humans follow the earth, the earth follows the heaven, the heaven follows Tao and Tao follows the nature*" is in restraining the human behavior "*Te being*" or harmonious coexistence of humans with the nature to sustain the human civilization.

Notice that science is only the human's partial recognition of laws on things or local Tao, it is in revealing or recognizing laws of things and prompting human activities in accordance with Tao or Te being, namely guiding the human activities Te being lies in the fact that science's recognition of the laws of things is beneficial to the realization of human ultimate goal or civilization, which requires all individual behaviors "*Te being*" or "*cultivating morality with mind*". Then, *why does ask the human to "cultivating morality with mind"?* Dr.Ouyang explains that science is the recognition or local Tao of things, there is a dilemma, i.e., benefit or harm to humans. A nature of Tao is without intention but the human always has intention or mind. And so, if the human can not take the "*cultivating morality with mind*" as the premise of science with its application, those of "*selfishness*" existing only benefit to local regions or someone's pursuit of luxury enjoying, the plundering resource will harm to humans or the ecological environment and results in disasters, even self-destruction of humans because these activities are just showing of the "*Te non-being*", which goes against to the human civilization.

In this situation, *how should science promote human progress within the constraint of human civilization?* Dr.Ouyang tells Huizi that first of all, the harmonious coexistence of humans with the nature is the whole expression of human civilization and science is the local recognition or conditional truth of humans to the laws of things, which needs humans to confirm this fact. Secondly, under the notion of harmonious coexistence of humans with the nature, the only criterion for science to promote the human progress lies in that science with its applications does not disturb the nature while it increases the human welfare so as to achieve the harmonious coexistence of humans with the nature, forgotten in the development of past hundred years by humans. For the ultimate goal of humans, it is important to verify all of the scientific results, scientific researching and exploring, all human activities guided by science are complying with the notion of harmonious coexistence of humans with the nature or not, and ended immediately or strictly limited those of scientific projects whose researching or application is harmful to humans or excessively intruding into the nature. Meanwhile, we should correctly understand the conditions for scientific achievements, promote scientific research that is conducive to harmonious coexistence of humans with the nature and actively change the intrusion on nature in existing technologies of humans so as to realize no intrusion on the nature. This is certainly the biggest problem facing all humans in the 21st century while science promotes the progress under the constraint of human civilization. It is a challenge, also an opportunity to science with its applications and needs to be faced by all humans.

## §6. **Comments**

**6.1.** The key of science's philosophy lies in discussing the ultimate goal of science or some ultimate problems in the development of science such as *how science should develop, what problems should be studied in science*, etc. Certainly, science serves for the ultimate goal of humans. So, *what is the ultimate goal of humans? To dominate the earth, to conquer the universe?* Of course Not! Historically, as the humans are in the shortage of material, the leader leads his members to fight or war with other families for the survival and continuation of his own family. However, humans are more in the greedy of spiritual enjoying in the day of material prosperity. Therefore, the ultimate goal of humans is to satisfy the material needs and also realize the spiritual needs of humans. See [Dyq]. And meanwhile, it is impossible for any individual in the universe to dominate the earth or conquer the universe, even the humans ourselves are no exception because "*the heaven and earth are not kind, taking all things as humble dogs*", see [Fur1] and [Fur2] for details. Thus, the ultimate goal of science serving humans is to promote the progress of civilization under the "*unity of humans with the heaven*" or harmonious coexistence with the nature, see [Dan]. Now that science is human's local recognition of laws of things, the application of which benefits humans is bound to no disturbing the nature. Otherwise, its effects of reaction will inevitably turn back on humans when it accumulates to a certain extent, resulting in a dilemma of science. However, to get out of this dilemma lies in the humans ourselves, namely how to abide by the eternal Tao's rule without disturbing the nature so as to achieve a harmonious coexistence of humans with the nature while it seems benefiting to humans. This is the biggest challenge to all activities of humans guided by science in 21st century, see [Mao44] and [Mao47] for details.

**6.2.** The essence of theory of everything is to establish a Smarandache multispaces or multisystems that contains all different theories, i.e., unifying all existing local theories into a union one as described in [Mao22] and [Sma1]. The difficulty lies in how to characterize the Smarandache multispace or multisystem to accommodate the existing local theories and solves the contradiction of each other. Among them, the unified field theory advocated by Einstein is a typical example. Theoretically, there must be a unified characterizing of everything under the framework of universe. For example, the continuity flow $G^L[AEP]$ formed by the 4 fundamental forces between all things, but whether $G^L[AEP]$ can be characterized by an equation is unknown because the recognitive limitation of humans leads to the identification of vertices and the interactions between vertices is always on the process in $G^L[AEP]$, namely one can understand it only locally. In this case, it is the human's wishful thinking that the law of everything can be holden on local recognition. And meanwhile, it is an assumption that the moving of everything can be characterized only by one equation because the limitation of mathematics is even greater than the limitation of human recognition. Its answer must be Not. However, *does this imply Einstein's unified field theory meaningless?* Of course Not! The value of unified field theory is really the process of human's approach to the nature. In this situation, it is necessary to further expand and establish the mathematics on Smarandache multispace or multisystem over the classical mathematics, namely the *mathematical combinatorics*. For Einstein's theory of general relativity and the gravitational field, see [Car], [Ein1], [Thi] and [Wus], for Kaliza-Klein theory and the string theory see [Pol], [Smo] and [Wes], and for the gauge fields see [Ble], [ChL1], [Mat], [Nam] and [Wan] for details.

**6.3.** Certainly, I had not read the book of *The End of Science* by J.Horgan before I finished my paper [Mao47]. After the paper [Mao47] published on a physics journal of American, I saw the Chinese translation of *The End of Science* in a bookstore. I was excited by the explanations of some famous scholars about the end of their science. I spent a week reading through this book because I was concerning the relationship of science with the nature in 2018-2019. After reading through it, I sent J.Horgan an E-mail explaining my philosophy of science and sent him an electronic copy of [Mao47]. He graciously replied me in the following day. J.Horgan's book was first published in 1996 and once it was published, it caused a huge response, also criticism from the scientific community because if there is an end to science, it is impossible for humans to understand the universe, which is in great contrast to most scientists. But in fact, Laozi's "*Tao told is not the eternal Tao, Name named is not the eternal Name*" and Avalokitesvara Bodhisattva's "*The color is the same as the emptiness, the emptiness is the same as the color; The color is empty, the empty is color and it is the same with the sensation, perception, decisions and consciousness*", all of these implies that the human recognition of universe is the human's own recognition, which is limited and not necessarily the reality. Certainly, I do not agree with J.Horgan's opinion of the end of science according to some scholars' personal perception because any person's life is limited, the end of personal academic thought is inevitable. But it can not infer that the end of whole science. Indeed, the true worth of *The End of Science* is to awaken of human's attention to the philosophy of science, namely how science should evolve in accordance with the ultimate goals of humans.

**6.4.** Someone take Newton and Einstein as examples and hold that "*the end of science is religion or theology*" is a misreading of Newton and Einstein's philosophy of science, and a partial denying on human civilization by contrasting science with the religion. Notice that the religion also has the correct recognition of things by predecessors, which can be used for reference by the development of science. And meanwhile, the history of human civilization shows that it is a universal rule for development of science by absorbing thoughts from philosophical classics, including the religious classics. Now that science is "*being out of non-being*" of human recognition, it needs to develop on the recognition of predecessors and then, innovate the recognition of things in the universe. My personal experience bears this out. From 2003 to 2005, I was a postdoctoral researcher in *Chinese Academy of Sciences*. My main work was to further study the surface embedding of graphs and count on non-isomorphic unrooted maps, which are explained in [Liu2], [MoT], [Mul], [MrW] and [NeS] details. My research work was once fallen in thought exhaustion and unable to extricated itself. However, I was enlightened by the dialectical relationship between Tao and name, being and non-being in *Tao Te Ching* and then, understanding the mission of science lies in the recognition of non-being, lies in the recognition of Tao and put forward the expanding of human recognition of things by combinatorial notion. In fact, my approach is similar to Newton, Einstein and other scientists, namely absorbe thoughts from the philosophical classics for promoting the development of science and then, serve the progress of human civilization and realize the harmonious coexistence of humans with the nature. Personally, I think this should be the right way for the development of science.

*Science without religion is lame, religion without science is blind.*

– By A.Einstein, an American physicist.

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**ABSTRACT.** Holding on a thing is essentially to recognize the cause and effect in its evolving. The combinatorial notion implied in words of the sophist in the fable of the blind men with an elephant, is a philosophical notion for humans recognizing things with only the local recognition of things. Under this notion, this book surveys the quantitative understanding of things. It starts with some ultimate questions on universe usually raised by children, explains the systematic recognition of things including the reference frame, relativity principle, system with synchronization, non-harmonious systems with non-solvable equations, complex networks, etc., establishes the model of continuity flow, a kind of mathematics over topological graph and comments the role of science in the harmonious coexistence of humans with the nature, which is a systematic review on mathematical science and philosophy, suitable for researchers in mathematics and applied mathematics, systems science, complex systems, complex networks and philosophy of science.

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