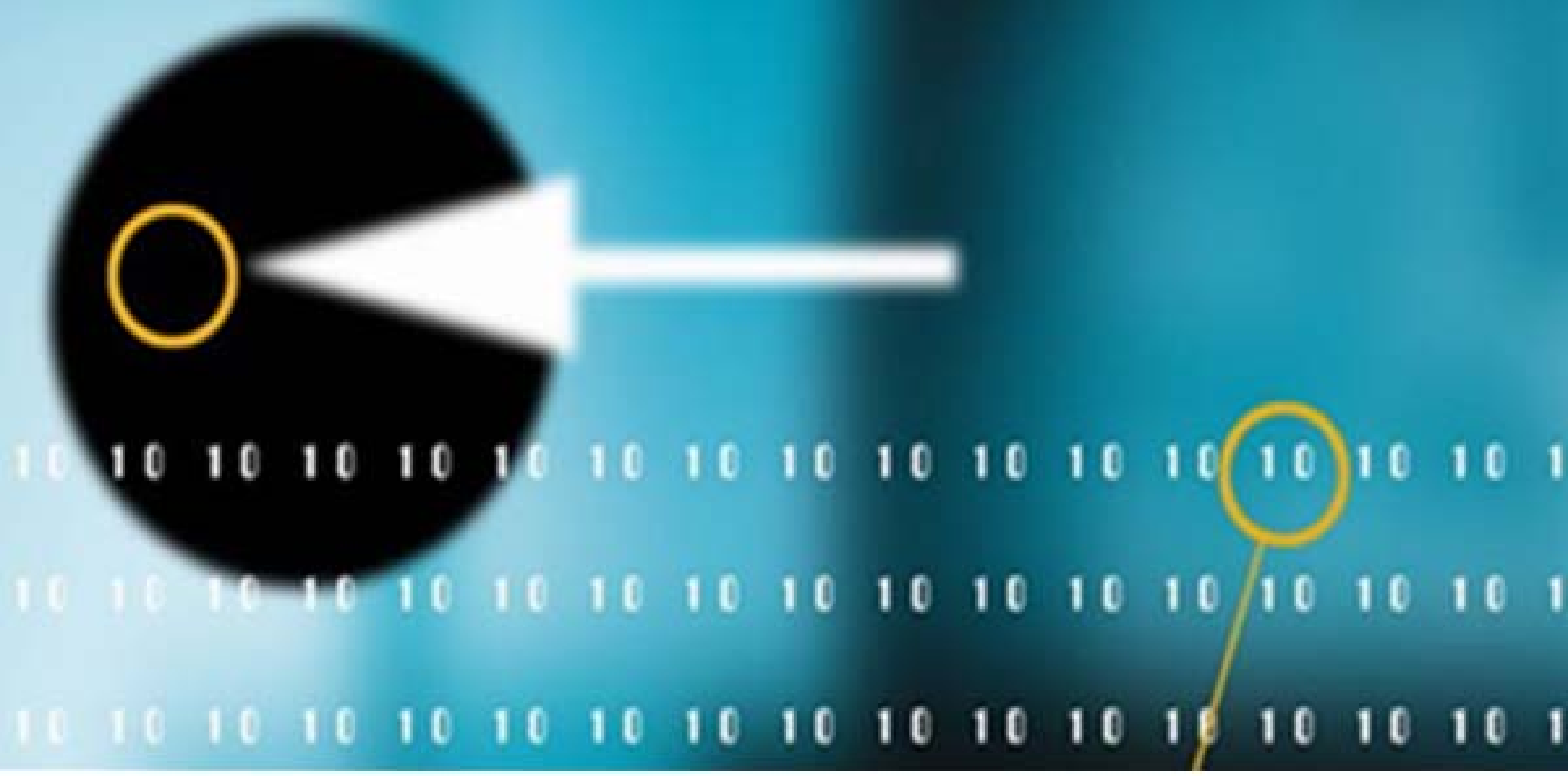

RESEARCH ON NUMBER THEORY AND SMARANDACHE NOTIONS

PROCEEDINGS OF THE SIXTH INTERNATIONAL CONFERENCE ON NUMBER THEORY AND SMARANDACHE NOTIONS

Edited by
ZHANG WENPENG
Department of Mathematics
Northwest University
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Preface

This Book is devoted to the proceedings of the Sixth International Conference on Number Theory and Smarandache Notions held in Tianshui during April 24-25, 2010. The organizers were myself and Professor Wangsheng He from Tianshui Normal University. The conference was supported by Tianshui Normal University and there were more than 100 participants. We had one foreign guest, Professor K.Chakraborty from India. The conference was a great success and will give a strong impact on the development of number theory in general and Smarandache Notions in particular. We hope this will become a tradition in our country and will continue to grow. And indeed we are planning to organize the seventh conference in coming March which will be held in Weinan, a beautiful city of shaanxi.

In the volume we assemble not only those papers which were presented at the conference but also those papers which were submitted later and are concerned with the Smarandache type problems or other mathematical problems.

There are a few papers which are not directly related to but should fall within the scope of Smarandache type problems. They are 1. A. K. S. Chandra Sekhar Rao, On Smarandache Semigroups; 2. X. Pan and Y. Shao, A Note on Smarandache non-associative rings; 3. Jiangmin Gu, A arithmetical function mean value of binary; etc.

Other papers are concerned with the number-theoretic Smarandache problems and will enrich the already rich stock of results on them.

Readers can learn various techniques used in number theory and will get familiar with the beautiful identities and sharp asymptotic formulas obtained in the volume.

Researchers can download books on the Smarandache notions from the following open source Digital Library of Science:

www.gallup.unm.edu/~smarandache/eBooks-otherformats.htm.

Wenpeng Zhang

Contents

The Sixth International Conference on Number Theory

and Smarandache Notions

The opening ceremony of the conference is occurred in Tianshui Normal University (http://www.tsnc.edu.cn).

Professor Xinke Yang

Professor Wenpeng Zhang

Professor Kalyan Chakraborty

Professor Hailong Li

Professor Wansheng He


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The Smarandache sums of products for $E(n, r)$ and $O(n, r)$

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Abstract The main purpose of this paper is using the elementary methods to study the properties of the function $E(n, r)$ and $O(n, r)$, and get two calculating formulaes for them.

Keywords The Smarandache sums of products, binomial theorem.

§1. Introduction

This paper deals with the sums of products of first n even and odd natural numbers, taken r at a time. Many interesting results about these two functions are obtained. For example, Mr. Ramasubramanian [1] and Anant W. Vyawahare [3] have already made some work in this direction. This paper is an extension of their work.

Definition. For any positive integer n and r, $E(n, r)$ are the sums of products of first n even natural numbers, taken r at a time, $r \leq n$. $O(n, r)$ are the sums of products of first n odd natural numbers, taken r at a time, $r \leq n$, they are also without repeatition.

For example:

$E(4, 1) = 2 + 4 + 6 + 8 = 20,$

$E(4, 2) = 2 \cdot 4 + 2 \cdot 6 + 2 \cdot 8 + 4 \cdot 6 + 4 \cdot 8 + 6 \cdot 8 = 140,$

$E(4, 3) = 2 \cdot 4 \cdot 6 + 2 \cdot 4 \cdot 8 + 4 \cdot 6 \cdot 8 + 2 \cdot 6 \cdot 8 = 400,$

$E(4, 4) = 2 \cdot 4 \cdot 6 \cdot 8 = 384,$

$O(4, 1) = 3 + 5 + 7 + 9 = 24,$

$O(4, 2) = 3 \cdot 5 + 3 \cdot 7 + 3 \cdot 9 + 5 \cdot 7 + 5 \cdot 9 + 7 \cdot 9 = 206,$

$O(4, 3) = 3 \cdot 5 \cdot 7 + 3 \cdot 5 \cdot 9 + 5 \cdot 7 \cdot 9 + 3 \cdot 7 \cdot 9 = 744,$

$O(4, 4) = 3 \cdot 5 \cdot 7 \cdot 9 = 945.$

We assume that $E(n, 0) = O(n, 0) = 1$.

About the properties of functions $E(n, r)$ and $O(n, r)$, we can obtain some interesting conclusions from their definitions. Following are some elementary properties of $E(n, r)$ and $O(n, r)$:

1. $E(n, n) = 2nE(n - 1, n - 1)$,
2. $O(n, n) = (2n + 1)O(n - 1, n - 1)$,
3. $E(n, 1) = n(n + 1)$,
4. $O(n, 1) = n(n + 2)$,
5. $(p+2)(p+4)(p+6) \cdots (p+2n) = E(n, 0)p^n + E(n, 1)p^{n-1} + E(n, 2)p^{n-2} + E(n, 3)p^{n-3} + \cdots + E(n, n - 1)p + E(n, n)$,

6. $(p+3)(p+5)(p+7)\cdots(p+2n+1) = O(n,0)p^n + O(n,1)p^{n-1} + O(n,2)p^{n-2} + O(n,3)p^{n-3} + \cdots + O(n,n-1)p + O(n,n)$,

7. $E(n,0)+E(n,1)+E(n,2)+\cdots+E(n,n) = O(n+1,n+1)$, that is $\sum_{r=0}^{n} E(n,r) = O(n+1,n+1)$,

8. $O(n,0)+O(n,1)+O(n,2)+\cdots+O(n,n) = E(n+1,n+1)/2$, that is $\sum_{r=0}^{n} O(n,r) = E(n+1,n+1)/2$.

The 7th and 8th properties can be obtained by putting $p=1$ in the 5th and 6th properties.

From these properties, we can use the elementary methods to study the expanding expressions for $E(n,r)$ and $O(n,r)$. The main purpose of this paper is using the elementary method to study this problem, and prove the following conclusions:

Theorem 1. For any positive integer n and r, we have the following formulas:

$$\begin{aligned} E(n,r) &= E(r,r) + 2nE(n-1,r-1) + 2(n-1)E(n-2,r-1) + \\ &\quad 2(n-2)E(n-3,r-1) + \cdots + 2(r+1)E(r,r-1)) \\ &= \left[2^r C_{n+1}^{r+1} E(n,0) + 2^{r-1} C_n^r E(n,1) + 2^{r-2} C_{n-1}^{r-1} E(n,2) + \right. \\ &\quad \left. \cdots + 2C_{n-r+2}^2 E(n,r-1)\right]/r. \end{aligned}$$

Theorem 2. For any positive integer n and r, we have the following formulas:

$$\begin{aligned} O(n,r) &= O(r,r) + (2n+1)O(n-1,r-1) + (2n-1)O(n-2,r-1) + \\ &\quad (2n-3)O(n-3,r-1) + \cdots + (2r+3)O(r,r-1)) \\ &= \left[O(n,0)\left(2^r C_n^{r+1} + 3\cdot 2^{r-1} C_n^r\right) + O(n,1)\left(2^{r-1} C_{n-1}^r + 3\cdot 2^{r-2} C_{n-1}^{r-1}\right) + \right. \\ &\quad \left. O(n,2)\left(2^{r-2} C_{n-2}^r + 3\cdot 2^{r-3} C_{n-2}^{r-2}\right) + \cdots + O(n,r-1)\left(2C_{n-r+1}^2 + 3C_{n-r+1}^1\right)\right]/r. \end{aligned}$$

§2. Proof of the theorems

In this section, we shall complete the proof of our Theorems. First we give two simple Lemmas (see [1]) which are necessary in the proof of our theorems.

Lemma 1. For any positive number n and r, we have the identity

$$E(n,r) = E(n-1,r) + 2nE(n-1,r-1), \quad r < n.$$

Lemma 2. For any positive number n and r, we have the identity

$$O(n,r) = O(n-1,r) + 2(n+1)O(n-1,r-1), \quad r < n.$$

Now we use these two Lemmas to prove our conclusions. First we use the elementary method to obtain a formula.

From Lemma 1 we know that $E(n,r) = E(n-1,r) + 2nE(n-1,r-1), \quad r < n$. Using this result repeatedly, we have:

$$E(n,r) = E(n-1,r) + 2nE(n-1,r-1),$$

$$E(n-1,r) = E(n-2,r) + 2(n-1)E(n-2,r-1),$$

$$E(n-2,r)=E(n-3,r)+2(n-2)E(n-3,r-1),$$
$$E(n-3,r)=E(n-4,r)+2(n-3)E(n-4,r-1),$$
$$\cdots$$
$$E(r+1,r)=E(r,r)+2(r+1)E(r,r-1).$$

Adding the above formulas we get:

$$\begin{aligned} E(n,r) \quad = \quad & E(r,r)+2nE(n-1,r-1)+2(n-1)E(n-2,r-1)+ \\ & 2(n-2)E(n-3,r-1)+\cdots+2(r+1)E(r,r-1)). \end{aligned} \tag{1}$$

The equation (1) is the first part of Theorem 1, now we prove the second part. Also, from the 5th property of $E(n,r)$, we have:

$$\begin{aligned} & (p+2)(p+4)(p+6)\cdots(p+2n)(p+2n+2) \\ = \quad & E(n+1,0)p^{n+1}+E(n+1,1)p^n+E(n+1,2)p^{n-1}+ \\ & E(n+1,3)p^{n-2}+\cdots+E(n+1,r+1)p^{n-r}\cdots+E(n+1,n+1). \end{aligned} \tag{2}$$

Left hand side of (2) is

$$\begin{aligned} & (p+2)(p+2+2)(p+2+4)(p+2+6)\cdots(p+2+2n-2)(p+2+2n) \\ = \quad & (p+2)\left\{E(n,0)(p+2)^n+E(n,1)(p+2)^{n-1}+\cdots+E(n,r)(p+1)^{n-r}+\cdots+E(n,n)\right\} \\ = \quad & E(n,0)(p+2)^{n+1}+E(n,1)(p+2)^n+\cdots+E(n,r)(p+2)^{n-r+1}+ \\ & \cdots+E(n,n)(p+2). \end{aligned} \tag{3}$$

Expanding each of $(p+2)^{n+1}$, $(p+2)^n$, $(p+2)^{n-1}$, $\cdots$, $(p+2)^{n-r+1}$, by binomial theorem, we get the right hand side of (3) is:

$$\begin{aligned} & E(n,0)\left[C_{n+1}^0p^{n+1}+2C_{n+1}^1p^n+\cdots+2^{r+1}C_{n+1}^{r+1}p^{n-r}+\cdots+2^{n+1}C_{n+1}^{n+1}\right] \\ + \quad & E(n,1)\left[C_n^0p^n+2C_n^1p^{n-1}+\cdots+2^rC_n^rp^{n-r}+\cdots+2^nC_n^n\right] \\ + \quad & E(n,2)\left[C_{n-1}^0p^{n-1}+2C_{n-1}^1p^{n-2}+\cdots+2^{r-1}C_{n-1}^{r-1}p^{n-r}+\cdots+2^{n-1}C_{n-1}^{n-1}\right] \\ + \quad & \cdots \\ + \quad & E(n,r)\left[C_{n-r+1}^0p^{n-r+1}+2C_{n-r+1}^1p^{n-r}+\cdots+2^{n-r+1}C_{n-r+1}^{n-r+1}\right] \\ + \quad & E(n,r+1)\left[C_{n-r}^0p^{n-r}+2C_{n-r}^1p^{n-r-1}+\cdots+2^{n-r}C_{n-r}^{n-r}\right] \\ + \quad & \cdots+E(n,n)(p+2). \end{aligned} \tag{4}$$

Comparing the coefficients of p^{n-r} from right hand side of (2) and (4), we get

$$\begin{aligned} & E(n+1,r+1) \\ = \quad & 2^{r+1}C_{n+1}^{r+1}E(n,0)+2^rC_n^rE(n,1)+2^{r-1}C_{n-1}^{r-1}E(n,2)+ \\ & \cdots+2C_{n-r+1}^1E(n,r)+C_{n-r}^0E(n,r+1). \end{aligned}$$

Simultaneously, we have the following formula from Lemma 1:

$$E(n+1,r+1)=E(n,r+1)+2(n+1)E(n,r).$$

Then we know that

$$\begin{aligned} & E(n,r+1)+2(n+1)E(n,r) \\ = \;& 2^{r+1}C_{n+1}^{r+1}E(n,0)+2^{r}C_{n}^{r}E(n,1)+2^{r-1}C_{n-1}^{r-1}E(n,2)+ \\ & \cdots+2C_{n-r+1}^{1}E(n,r)+C_{n-r}^{0}E(n,r+1) \end{aligned}$$

or

$$\begin{aligned} & 2(n+1)E(n,r)-2C_{n-r+1}^{1}E(n,r) \\ = \;& 2^{r+1}C_{n+1}^{r+1}E(n,0)+2^{r}C_{n}^{r}E(n,1)+2^{r-1}C_{n-1}^{r-1}E(n,2)+ \\ & \cdots+2^{2}C_{n-r+2}^{2}E(n,r-1). \end{aligned}$$

Because $C_{n-r+1}^{1}=n-r+1$, we have

$$\begin{aligned} & 2(n+1)E(n,r)-2C_{n-r+1}^{1}E(n,r) \\ = \;& \left[2(n+1)-2C_{n-r+1}^{1}\right]E(n,r) \\ = \;& 2\left[n+1-(n-r+1)\right]E(n,r) \\ = \;& 2rE(n,r) \\ = \;& 2^{r+1}C_{n+1}^{r+1}E(n,0)+2^{r}C_{n}^{r}E(n,1)+2^{r-1}C_{n-1}^{r-1}E(n,2)+ \\ & \cdots+2^{2}C_{n-r+2}^{2}E(n,r-1). \end{aligned}$$

Hence,

$$\begin{aligned} E(n,r) \;=\; & \left\{2^{r}C_{n+1}^{r+1}E(n,0)+2^{r-1}C_{n}^{r}E(n,1)+2^{r-2}C_{n-1}^{r-1}E(n,2)+\right. \\ & \left.\cdots+2C_{n-r+2}^{2}E(n,r-1)\right\}/r. \end{aligned} \tag{5}$$

Combining (1) and (5) we may immediately deduce Theorem 1.

Corollary 1. For any positive integer n, we have

$$E(n,2)=n(n+1)(n-1)(2+3n)/6.$$

In fact, if taking $r=2$ in (5), then

$$E(n,2)=2C_{n+1}^{3}E(n,0)+C_{n}^{2}E(n,1)=n(n+1)(n-1)(2+3n)/6.$$

If taking $r=2$ in (1), then

$$\begin{aligned} E(n,2) \;=\; & E(2,2)+2nE(n-1,1)+2(n-1)E(n-2,1)+ \\ & 2(n-2)E(n-3,1)+\cdots+6E(2,1) \\ =\; & 8+2\sum_{i=3}^{n}i^{2}(i-1) \\ =\; & 8+2\left[\sum_{i=1}^{n}i^{2}(i-1)-(1^{3}+2^{3})-(1^{2}+2^{2})\right] \\ =\; & 2\sum_{i=1}^{n}i^{2}(i-1) \\ =\; & n^{2}(n+1)^{2}/2-n(n+1)(2n+1)/3 \\ =\; & n(n+1)(n-1)(2+3n)/6. \end{aligned}$$

This completes the proof of Corollary 1.

Similarly, we can use the same method to prove Theorem 2. From Lemma 2 we know that $O(n,r) = O(n-1,r) + (2n+1)O(n-1,r-1), r < n$. Using this result repeatedly, the following formulas can be obtained:

$$O(n,r) = O(n-1,r) + (2n+1)O(n-1,r-1),$$

$$O(n-1,r) = O(n-2,r) + (2n-1)O(n-2,r-1),$$

$$O(n-2,r) = O(n-3,r) + (2n-3)O(n-3,r-1),$$

$$O(n-3,r) = O(n-4,r) + (2n-5)O(n-4,r-1),$$

$$\cdots,$$

$$O(r+1,r) = O(r,r) + (2r+3)O(r,r-1).$$

Adding the above formulas we get:

$$\begin{aligned} O(n,r) &= O(r,r) + (2n+1)O(n-1,r-1) + (2n-1)O(n-2,r-1) + \\ & \quad (2n-3)O(n-3,r-1) + \cdots + (2r+3)O(r,r-1)). \end{aligned} \tag{6}$$

This is the first part of Theorem 2. Now we prove the second part of Theorem 2. Because

$$\begin{aligned} & (p+3)(p+5)(p+7)\cdots(p+2n+1)(p+2n+3) \\ = \; & O(n+1,0)p^{n+1} + O(n+1,1)p^{n} + O(n+1,2)p^{n-1} + \\ & O(n+1,3)p^{n-2} + \cdots + O(n+1,r+1)p^{n-r} + \cdots + O(n+1,n+1). \end{aligned} \tag{7}$$

Left hand side of (7) is

$$\begin{aligned} & (p+3)(p+2+3)(p+2+5)(p+2+7)\cdots(p+2+2n-3)(p+2+2n+1) + \\ = \; & (p+3)\left\{O(n,0)(p+2)^{n} + O(n,1)(p+2)^{n-1} + \cdots \right. \\ & \left. O(n,r)(p+2)^{n-r} + \cdots + O(n,n)\right\}. \end{aligned} \tag{8}$$

Expanding each of $(p+2)^n, (p+2)^{n-1}, (p+2)^{n-3}, \cdots, (p+2)^{n-r+1}$, by binomial theorem, the right hand side of (8) is:

$$\begin{aligned} & (p+3)\left\{O(n,0)\left[C_n^0 p^n + 2C_n^1 p^{n-1} + \cdots + 2^{r+1}C_n^{r+1}p^{n-r-1} + \cdots + 2^n C_n^n\right]\right. \\ + \; & O(n,1)\left[C_{n-1}^0 p^{n-1} + 3C_{n-1}^1 p^{n-2} + \cdots + 2^r C_{n-1}^r p^{n-r-1} + \cdots + 2^{n-1}C_{n-1}^{n-1}\right] \\ + \; & O(n,2)\left[C_{n-2}^0 p^{n-2} + 2C_{n-2}^1 p^{n-3} + \cdots + 2^{r-1}C_{n-2}^{r-1}p^{n-r-1} + \cdots + 2^{n-2}C_{n-2}^{n-2}\right] \\ + \; & \cdots \\ + \; & O(n,r)\left[C_{n-r}^0 p^{n-r} + 2C_{n-r}^1 p^{n-r-1} + \cdots + 2^{n-r}C_{n-r}^{n-r}\right] \\ + \; & O(n,r+1)\left[C_{n-r-1}^0 p^{n-r-1} + 2C_{n-r-1}^1 p^{n-r-2} + \cdots + 2^{n-r-1}C_{n-r-1}^{n-r-1}\right] \\ + \; & \left.\cdots + O(n,n)\right\}. \end{aligned} \tag{9}$$

Now, comparing the coefficients of p^{n-r} from right side of (7) and (9), we get

$$\begin{aligned}
& O(n+1,r+1) \\
= \ & 2^{r+1}C_n^{r+1}o(n,0)+2^rC_{n-1}^{r}O(n,1)+2^{r-1}C_{n-2}^{r-1}O(n,2)+ \\
& \cdots+2C_{n-r}^{1}O(n,r)+C_{n-r-1}^{0}O(n,r+1)+3\left[2^rC_n^rO(n,0)\right. + \\
& \left. 2^{r-1}C_{n-1}^{r-1}+2^{r-2}C_{n-2}^{r-2}O(n,2)+\cdots+2C_{n-r+1}^{1}O(n,r-1)+C_{n-r}^{0}O(n,r)\right].
\end{aligned}$$

Simultaneously, from Lemma 2, we have:

$$O(n+1,r+1)=O(n,r+1)+(2n+3)O(n,r),$$

and

$$\begin{aligned}
& O(n,r+1)+(2n+3)O(n,r) \\
= \ & 2^{r+1}C_n^{r+1}o(n,0)+2^rC_{n-1}^{r}O(n,1)+2^{r-1}C_{n-2}^{r-1}O(n,2)+ \\
& \cdots+2C_{n-r}^{1}O(n,r)+C_{n-r-1}^{0}O(n,r+1)+3\left[2^rC_n^rO(n,0)\right. + \\
& 2^{r-1}C_{n-1}^{r-1}O(n,1)+2^{r-2}C_{n-2}^{r-2}O(n,2)+\cdots + \\
& \left. 2C_{n-r+1}^{1}O(n,r-1)+C_{n-r}^{0}O(n,r)\right],
\end{aligned}$$

or

$$\begin{aligned}
& (2n+3)O(n,r)-2C_{n-r}^{1}O(n,r)-3C_{n-r}^{0}O(n,r) \\
= \ & 2^{r+1}C_n^{r+1}O(n,0)+2^rC_{n-1}^{r}O(n,1)+ \\
& 2^{r-1}C_{n-2}^{r-1}O(n,2)+\cdots+2^2C_{n-r+1}^{2}O(n,r-1)+3\left[2^rC_n^rO(n,0)\right.+ \\
& \left. 2^{r-1}C_{n-1}^{r-1}O(n,r)+2^{r-2}C_{n-2}^{r-2}O(n,2)+\cdots+2C_{n-r+1}^{1}O(n,r-1)\right].
\end{aligned}$$

Since $C_{n-r}^{1}=n-r$ and $C_{n-r}^{0}=1$, we have

$$\begin{aligned}
& (2n+3)O(n,r)-2C_{n-r}^{1}O(n,r)-3C_{n-r}^{0}O(n,r) \\
= \ & 2rO(n,r) \\
= \ & 2^{r+1}C_n^{r+1}O(n,0)+2^rC_{n-1}^{r}O(n,1)+2^{r-1}C_{n-2}^{r-1}O(n,2)+ \\
& \cdots+2^2C_{n-r+1}^{2}O(n,r-1)+3\left[2^rC_n^rO(n,0)+2^{r-1}C_{n-1}^{r-1}O(n,1)\right. + \\
& \left. 2^{r-2}C_{n-2}^{r-2}O(n,2)+\cdots+2C_{n-r+1}^{1}O(n,r-1)\right].
\end{aligned}$$

Therefore

$$\begin{aligned}
O(n,r) \ = \ & \left[O(n,0)\left(2^rC_n^{r+1}+3\cdot 2^{r-1}C_n^r\right)\right. + \\
& O(n,1)\left(2^{r-1}C_{n-1}^{r}+3\cdot 2^{r-2}C_{n-1}^{r-1}\right) + \\
& O(n,2)\left(2^{r-2}C_{n-2}^{r}+3\cdot 2^{r-3}C_{n-2}^{r-2}\right) + \\
& \cdots + \\
& \left. O(n,r-1)\left(2C_{n-r+1}^{2}+3C_{n-r+1}^{1}\right)\right]/r. \qquad (10)
\end{aligned}$$

Now our Theorem 2 follows from (6) and (10).

Corollary 2. For any positive integer n, we have

$$O(n,2) = n(n-1)(3n^2+11n+11)/6.$$

Taking $r=2$ in (10), we have

$$\begin{aligned} O(n,2) &= \left[O(n,0)(4C_n^3+6C_n^2)+O(n,1)(2C_{n-1}^2+3C_{n-1}^2)\right]/2 \\ &= n(n-1)(3n^2+11n+11)/6. \end{aligned}$$

If taking $r=2$ in (6), we can also get

$$\begin{aligned} O(n,2) &= O(2,2)+(2n+1)O(n-1,1)+(2n-1)O(n-2,1)+ \\ &\quad (2n-3)O(n-3,1)+\cdots+7O(2,1) \\ &= 15+\sum_{p=3}^{n}(2p+1)(p-1)(p+1) \\ &= \sum_{p=1}^{n}(2p+1)(p-1)(p+1) \\ &= \sum_{p=1}^{n}(2p^3-2p+p^2-1) \\ &= n^2(n+1)^2/2-n(n+1)+n(n+1)(2n+1)/6-n \\ &= n(n-1)(3n^2+11n+11)/6. \end{aligned}$$

This completes the proof of Corollary 2.

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Smarandache isotopy of second Smarandache Bol loops

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Abstract The pair $(G_H, \cdot)$ is called a special loop if $(G, \cdot)$ is a loop with an arbitrary subloop $(H, \cdot)$ called its special subloop. A special loop $(G_H, \cdot)$ is called a second Smarandache Bol loop ($S_{2^{nd}}$BL) if and only if it obeys the second Smarandache Bol identity $(xs \cdot z)s = x(sz \cdot s)$ for all x, z in G and s in H. The popularly known and well studied class of loops called Bol loops fall into this class and so $S_{2^{nd}}$BLs generalize Bol loops. The Smarandache isotopy of $S_{2^{nd}}$BLs is introduced and studied for the first time. It is shown that every Smarandache isotope (S-isotope) of a special loop is Smarandache isomorphic (S-isomorphic) to a S-principal isotope of the special loop. It is established that every special loop that is S-isotopic to a $S_{2^{nd}}$BL is itself a $S_{2^{nd}}$BL. A special loop is called a Smarandache G-special loop (SGS-loop) if and only if every special loop that is S-isotopic to it is S-isomorphic to it. A $S_{2^{nd}}$BL is shown to be a SGS-loop if and only if each element of its special subloop is a $S_{1^{st}}$ companion for a $S_{1^{st}}$ pseudo-automorphism of the $S_{2^{nd}}$BL. The results in this work generalize the results on the isotopy of Bol loops as can be found in the Ph. D. thesis of D. A. Robinson.

Keywords Irr$G$, RC-graphs, Core$_G H$.

§1. Introduction

The study of the Smarandache concept in groupoids was initiated by W. B. Vasantha Kandasamy in [24]. In her book [22] and first paper [23] on Smarandache concept in loops, she defined a Smarandache loop (S-loop) as a loop with at least a subloop which forms a subgroup under the binary operation of the loop. The present author has contributed to the study of S-quasigroups and S-loops in [5]–[12] by introducing some new concepts immediately after the works of Muktibodh [15]–[16]. His recent monograph [14] gives inter-relationships and connections between and among the various Smarandache concepts and notions that have been developed in the aforementioned papers.

But in the quest of developing the concept of Smarandache quasigroups and loops into a theory of its own just as in quasigroups and loop theory (see [1]–[4], [17], [22]), there is the need to introduce identities for types and varieties of Smarandache quasigroups and loops. This led Jaíyéọlá [13] to the introduction of second Smarandache Bol loop ($S_{2^{nd}}$BL) described by the second Smarandache Bol identity $(xs \cdot z)s = x(sz \cdot s)$ for all x, z in G and s in H where the pair $(G_H, \cdot)$ is called a special loop if $(G, \cdot)$ is a loop with an arbitrary subloop $(H, \cdot)$. For now, a Smarandache loop or Smarandache quasigroup will be called a first Smarandache loop ($S_{1^{st}}$-loop) or first Smarandache quasigroup ($S_{1^{st}}$-quasigroup).

Let L be a non-empty set. Define a binary operation $(\cdot)$ on L : if $x \cdot y \in L$ for all $x, y \in L$, $(L, \cdot)$ is called a groupoid. If the equations: $a \cdot x = b$ and $y \cdot a = b$ have unique solutions for x and y respectively, then $(L, \cdot)$ is called a quasigroup. For each $x \in L$, the elements $x^\rho = xJ_\rho, x^\lambda = xJ_\lambda \in L$ such that $xx^\rho = e^\rho$ and $x^\lambda x = e^\lambda$ are called the right, left inverses of x respectively. Furthermore, if there exists a unique element $e = e_\rho = e_\lambda$ in L called the identity element such that for all x in L, $x \cdot e = e \cdot x = x$, $(L, \cdot)$ is called a loop. We write xy instead of $x \cdot y$, and stipulate that $\cdot$ has lower priority than juxtaposition among factors to be multiplied. For instance, $x \cdot yz$ stands for $x(yz)$. A loop is called a right Bol loop (Bol loop in short) if and only if it obeys the identity

$$(xy \cdot z)y = x(yz \cdot y).$$

This class of loops was the first to catch the attention of loop theorists and the first comprehensive study of this class of loops was carried out by Robinson [19].

The popularly known and well studied class of loops called Bol loops fall into the class of $S_{2^{nd}}$BLs and so $S_{2^{nd}}$BLs generalize Bol loops. The aim of this work is to introduce and study for the first time, the Smarandache isotopy of $S_{2^{nd}}$BLs. It is shown that every Smarandache isotope (S-isotope) of a special loop is Smarandache isomorphic (S-isomorphic) to a S-principal isotope of the special loop. It is established that every special loop that is S-isotopic to a $S_{2^{nd}}$BL is itself a $S_{2^{nd}}$BL. A $S_{2^{nd}}$BL is shown to be a Smarandache G-special loop if and only if each element of its special subloop is a $S_{1^{st}}$ companion for a $S_{1^{st}}$ pseudo-automorphism of the $S_{2^{nd}}$BL. The results in this work generalize the results on the isotopy of Bol loops as can be found in the Ph. D. thesis of D. A. Robinson.

§2. Preliminaries

Definition 1. Let $(G, \cdot)$ be a quasigroup with an arbitrary non-trivial subquasigroup $(H, \cdot)$. Then, $(G_H, \cdot)$ is called a special quasigroup with special subquasigroup $(H, \cdot)$. If $(G, \cdot)$ is a loop with an arbitrary non-trivial subloop $(H, \cdot)$. Then, $(G_H, \cdot)$ is called a special loop with special subloop $(H, \cdot)$. If $(H, \cdot)$ is of exponent 2, then $(G_H, \cdot)$ is called a special loop of Smarandache exponent 2.

A special quasigroup $(G_H, \cdot)$ is called a second Smarandache right Bol quasigroup ($S_{2^{nd}}$-right Bol quasigroup) or simply a second Smarandache Bol quasigroup ($S_{2^{nd}}$-Bol quasigroup) and abbreviated $S_{2^{nd}}$RBQ or $S_{2^{nd}}$BQ if and only if it obeys the second Smarandache Bol identity ($S_{2^{nd}}$-Bol identity) i.e $S_{2^{nd}}$BI

$$(xs \cdot z)s = x(sz \cdot s) \text{ for all } x, z \in G \text{ and } s \in H. \tag{1}$$

Hence, if $(G_H, \cdot)$ is a special loop, and it obeys the $S_{2^{nd}}$BI, it is called a second Smarandache Bol loop ($S_{2^{nd}}$-Bol loop) and abbreviated $S_{2^{nd}}$BL.

Remark 1. A Smarandache Bol loop (i.e a loop with at least a non-trivial subloop that is a Bol loop) will now be called a first Smarandache Bol loop ($S_{1^{st}}$-Bol loop). It is easy to see that a $S_{2^{nd}}$BL is a $S_{1^{st}}$BL. But the converse is not generally true. So $S_{2^{nd}}$BLs are particular types of $S_{1^{st}}$BL. Their study can be used to generalise existing results in the theory of Bol loops by simply forcing H to be equal to G.

Definition 2. Let $(G, \cdot)$ be a quasigroup (loop). It is called a right inverse property quasigroup (loop) [RIPQ(RIPL)] if and only if it obeys the right inverse property (RIP) $yx \cdot x^\rho = y$ for all $x, y \in G$. Similarly, it is called a left inverse property quasigroup (loop) [LIPQ(LIPL)] if and only if it obeys the left inverse property (LIP) $x^\lambda \cdot xy = y$ for all $x, y \in G$. Hence, it is called an inverse property quasigroup (loop) [IPQ(IPL)] if and only if it obeys both the RIP and LIP.

$(G, \cdot)$ is called a right alternative property quasigroup (loop) [RAPQ(RAPL)] if and only if it obeys the right alternative property(RAP) $y \cdot xx = yx \cdot x$ for all $x, y \in G$. Similarly, it is called a left alternative property quasigroup (loop) [LAPQ(LAPL)] if and only if it obeys the left alternative property (LAP) $xx \cdot y = x \cdot xy$ for all $x, y \in G$. Hence, it is called an alternative property quasigroup (loop) [APQ(APL)] if and only if it obeys both the RAP and LAP.

The bijection $L_x : G \to G$ defined as $yL_x = x \cdot y$ for all $x, y \in G$ is called a left translation (multiplication) of G while the bijection $R_x : G \to G$ defined as $yR_x = y \cdot x$ for all $x, y \in G$ is called a right translation (multiplication) of G. Let

$$x\backslash y = yL_x^{-1} = y\mathbb{L}_x \qquad \text{and} \qquad x/y = xR_y^{-1} = x\mathbb{R}_y$$

and note that

$$x\backslash y = z \Longleftrightarrow x \cdot z = y \qquad \text{and} \qquad x/y = z \Longleftrightarrow z \cdot y = x.$$

The operations $\backslash$ and $/$ are called the left and right divisions respectively. We stipulate that $/$ and $\backslash$ have higher priority than $\cdot$ among factors to be multiplied. For instance, $x \cdot y/z$ and $x \cdot y\backslash z$ stand for $x(y/z)$ and $x \cdot (y\backslash z)$ respectively.

$(G, \cdot)$ is said to be a right power alternative property loop (RPAPL) if and only if it obeys the right power alternative property (RPAP)

$$xy^n = \underbrace{(((xy)y)y)y \cdots y}_{n\text{-times}} \text{ i.e. } R_{y^n} = R_y^n \text{ for all } x, y \in G \text{ and } n \in \mathbb{Z}.$$

The right nucleus of G denoted by $N_\rho(G, \cdot) = N_\rho(G) = \{a \in G : y \cdot xa = yx \cdot a \ \forall\ x, y \in G\}$.

Let $(G_H, \cdot)$ be a special quasigroup (loop). It is called a second Smarandache right inverse property quasigroup (loop) [$S_{2^{nd}}$RIPQ($S_{2^{nd}}$RIPL)] if and only if it obeys the second Smarandache right inverse property ($S_{2^{nd}}$RIP) $ys \cdot s^\rho = y$ for all $y \in G$ and $s \in H$. Similarly, it is called a second Smarandache left inverse property quasigroup (loop) [$S_{2^{nd}}$LIPQ($S_{2^{nd}}$LIPL)] if and only if it obeys the second Smarandache left inverse property($S_{2^{nd}}$LIP) $s^\lambda \cdot sy = y$ for all $y \in G$ and $s \in H$. Hence, it is called a second Smarandache inverse property quasigroup (loop) [$S_{2^{nd}}$IPQ($S_{2^{nd}}$IPL)] if and only if it obeys both the $S_{2^{nd}}$RIP and $S_{2^{nd}}$LIP.

$(G_H, \cdot)$ is called a third Smarandache right inverse property quasigroup (loop) [$S_{3^{rd}}$RIPQ ($S_{3^{rd}}$RIPL)] if and only if it obeys the third Smarandache right inverse property ($S_{3^{rd}}$RIP) $sy \cdot y^\rho = s$ for all $y \in G$ and $s \in H$.

$(G_H, \cdot)$ is called a second Smarandache right alternative property quasigroup (loop) [$S_{2^{nd}}$ RAPQ($S_{2^{nd}}$RAPL)] if and only if it obeys the second Smarandache right alternative property ($S_{2^{nd}}$RAP) $y \cdot ss = ys \cdot s$ for all $y \in G$ and $s \in H$. Similarly, it is called a second Smarandache left alternative property quasigroup (loop) [$S_{2^{nd}}$LAPQ($S_{2^{nd}}$LAPL)] if and only if it obeys the second Smarandache left alternative property ($S_{2^{nd}}$LAP) $ss \cdot y = s \cdot sy$ for all $y \in G$ and $s \in H$. Hence, it is called an second Smarandache alternative property quasigroup (loop) [$S_{2^{nd}}$APQ($S_{2^{nd}}$APL)] if and only if it obeys both the $S_{2^{nd}}$RAP and $S_{2^{nd}}$LAP.

$(G_H, \cdot)$ is said to be a Smarandache right power alternative property loop (SRPAPL) if and only if it obeys the Smarandache right power alternative property (SRPAP)

$$xs^n = \underbrace{(((xs)s)s \cdots s}_{n\text{-times}} \text{ i.e. } R_{s^n} = R_s^n \text{ for all } x \in G,\ s \in H \text{ and } n \in \mathbb{Z}.$$

The Smarandache right nucleus of G_H denoted by $SN_\rho(G_H, \cdot) = SN_\rho(G_H) = N_\rho(G) \cap H$. G_H is called a Smarandache right nuclear square special loop if and only if $s^2 \in SN_\rho(G_H)$ for all $s \in H$.

Remark 2. A Smarandache; RIPQ or LIPQ or IPQ (i.e a loop with at least a non-trivial subquasigroup that is a RIPQ or LIPQ or IPQ) will now be called a first Smarandache; RIPQ or LIPQ or IPQ ($S_{1^{st}}$RIPQ or $S_{1^{st}}$LIPQ or $S_{1^{st}}$IPQ). It is easy to see that a $S_{2^{nd}}$RIPQ or $S_{2^{nd}}$LIPQ or $S_{2^{nd}}$IPQ is a $S_{1^{st}}$RIPQ or $S_{1^{st}}$LIPQ or $S_{1^{st}}$IPQ respectively. But the converse is not generally true.

Definition 3. Let $(G, \cdot)$ be a quasigroup (loop). The set $SYM(G, \cdot) = SYM(G)$ of all bijections in G forms a group called the permutation (symmetric) group of G. The triple (U, V, W) such that $U, V, W \in SYM(G, \cdot)$ is called an autotopism of G if and only if

$$xU \cdot yV = (x \cdot y)W\ \forall\ x, y \in G.$$

The group of autotopisms of G is denoted by $AUT(G, \cdot) = AUT(G)$.

Let $(G_H, \cdot)$ be a special quasigroup (loop). The set $SSYM(G_H, \cdot) = SSYM(G_H)$ of all Smarandache bijections (S-bijections) in G_H i.e $A \in SYM(G_H)$ such that $A : H \to H$ forms a group called the Smarandache permutation (symmetric) group [S-permutation group] of G_H. The triple (U, V, W) such that $U, V, W \in SSYM(G_H, \cdot)$ is called a first Smarandache autotopism ($S_{1^{st}}$ autotopism) of G_H if and only if

$$xU \cdot yV = (x \cdot y)W\ \forall\ x, y \in G_H.$$

If their set forms a group under componentwise multiplication, it is called the first Smarandache autotopism group ($S_{1^{st}}$ autotopism group) of G_H and is denoted by $S_{1^{st}}AUT(G_H, \cdot) = S_{1^{st}}AUT(G_H)$.

The triple (U, V, W) such that $U, W \in SYM(G, \cdot)$ and $V \in SSYM(G_H, \cdot)$ is called a second right Smarandache autotopism ($S_{2^{nd}}$ right autotopism) of G_H if and only if

$$xU \cdot sV = (x \cdot s)W\ \forall\ x \in G \text{ and } s \in H.$$

If their set forms a group under componentwise multiplication, it is called the second right Smarandache autotopism group ($S_{2^{nd}}$ right autotopism group) of G_H and is denoted by $S_{2^{nd}}RAU$ $T(G_H, \cdot) = S_{2^{nd}}RAUT(G_H)$.

The triple (U, V, W) such that $V, W \in SYM(G, \cdot)$ and $U \in SSYM(G_H, \cdot)$ is called a second left Smarandache autotopism ($S_{2^{nd}}$ left autotopism) of G_H if and only if

$$sU \cdot yV = (s \cdot y)W\ \forall\ y \in G \text{ and } s \in H.$$

If their set forms a group under componentwise multiplication, it is called the second left Smarandache autotopism group ($S_{2^{nd}}$ left autotopism group) of G_H and is denoted by $S_{2^{nd}}LAUT$ $(G_H,\cdot)=S_{2^{nd}}LAUT(G_H)$.

Let $(G_H,\cdot)$ be a special quasigroup (loop) with identity element e. A mapping $T\in SSYM(G_H)$ is called a first Smarandache semi-automorphism ($S_{1^{st}}$ semi-automorphism) if and only if $eT=e$ and

$$(xy\cdot x)T=(xT\cdot yT)xT \text{ for all } x,y\in G.$$

A mapping $T\in SSYM(G_H)$ is called a second Smarandache semi-automorphism ($S_{2^{nd}}$ semi-automorphism) if and only if $eT=e$ and

$$(sy\cdot s)T=(sT\cdot yT)sT \text{ for all } y\in G \text{ and all } s\in H.$$

A special loop $(G_H,\cdot)$ is called a first Smarandache semi-automorphic inverse property loop ($S_{1^{st}}$SAIPL) if and only if J_ρ is a $S_{1^{st}}$ semi-automorphism.

A special loop $(G_H,\cdot)$ is called a second Smarandache semi-automorphic inverse property loop ($S_{2^{nd}}$SAIPL) if and only if J_ρ is a $S_{2^{nd}}$ semi-automorphism.

Let $(G_H,\cdot)$ be a special quasigroup(loop). A mapping $A\in SSYM(G_H)$ is a

1. First Smarandache pseudo-automorphism ($S_{1^{st}}$ pseudo-automorphism) of G_H if and only if there exists a $c\in H$ such that $(A,AR_c,AR_c)\in S_{1^{st}}AUT(G_H)$. c is reffered to as the first Smarandache companion ($S_{1^{st}}$ companion) of A. The set of such A's is denoted by $S_{1^{st}}PAUT(G_H,\cdot)=S_{1^{st}}PAUT(G_H)$.

2. Second right Smarandache pseudo-automorphism ($S_{2^{nd}}$ right pseudo-automorphism) of G_H if and only if there exists a $c\in H$ such that $(A,AR_c,AR_c)\in S_{2^{nd}}RAUT(G_H)$. c is reffered to as the second right Smarandache companion ($S_{2^{nd}}$ right companion) of A. The set of such A's is denoted by $S_{2^{nd}}RPAUT(G_H,\cdot)=S_{2^{nd}}RPAUT(G_H)$.

3. Second left Smarandache pseudo-automorphism ($S_{2^{nd}}$ left pseudo-automorphism) of G_H if and only if there exists a $c\in H$ such that $(A,AR_c,AR_c)\in S_{2^{nd}}LAUT(G_H)$. c is reffered to as the second left Smarandache companion ($S_{2^{nd}}$ left companion) of A. The set of such A's is denoted by $S_{2^{nd}}LPAUT(G_H,\cdot)=S_{2^{nd}}LPAUT(G_H)$.

Let $(G_H,\cdot)$ be a special loop. A mapping $A\in SSYM(G_H)$ is a

1. First Smarandache automorphism ($S_{1^{st}}$ automorphism) of G_H if and only if $A\in S_{1^{st}}PAUT(G_H)$ such that $c=e$. Their set is denoted by $S_{1^{st}}AUM(G_H,\cdot)=S_{1^{st}}AUM(G_H)$.

2. Second right Smarandache automorphism ($S_{2^{nd}}$ right automorphism) of G_H if and only if $A\in S_{2^{nd}}RPAUT(G_H)$ such that $c=e$. Their set is denoted by $S_{2^{nd}}RAUM(G_H,\cdot)=S_{2^{nd}}RAUM(G_H)$.

3. Second left Smarandache automorphism ($S_{2^{nd}}$ left automorphism) of G_H if and only if $A\in S_{2^{nd}}LPAUT(G_H)$ such that $c=e$. Their set is denoted by $S_{2^{nd}}LAUM(G_H,\cdot)=S_{2^{nd}}LAUM(G_H)$.

A special loop $(G_H, \cdot)$ is called a first Smarandache automorphism inverse property loop ($S_{1^{st}}$AIPL) if and only if $(J_\rho, J_\rho, J_\rho) \in AUT(H, \cdot)$.

A special loop $(G_H, \cdot)$ is called a second Smarandache right automorphic inverse property loop ($S_{2^{nd}}$RAIPL) if and only if J_ρ is a $S_{2^{nd}}$ right automorphism.

A special loop $(G_H, \cdot)$ is called a second Smarandache left automorphic inverse property loop ($S_{2^{nd}}$LAIPL) if and only if J_ρ is a $S_{2^{nd}}$ left automorphism.

Definition 4. Let $(G, \cdot)$ and $(L, \circ)$ be quasigroups (loops). The triple (U, V, W) such that $U, V, W\ : G \to L$ are bijections is called an isotopism of G onto L if and only if

$$xU \circ yV = (x \cdot y)W\ \forall\ x, y \in G. \tag{2}$$

Let $(G_H, \cdot)$ and $(L_M, \circ)$ be special groupoids. G_H and L_M are Smarandache isotopic (S-isotopic) [and we say $(L_M, \circ)$ is a Smarandache isotope of $(G_H, \cdot)$] if and only if there exist bijections $U, V, W\ : H \to M$ such that the triple $(U, V, W)\ :\ (G_H, \cdot) \to (L_M, \circ)$ is an isotopism. In addition, if $U = V = W$, then $(G_H, \cdot)$ and $(L_M, \circ)$ are said to be Smarandache isomorphic (S-isomorphic) [and we say $(L_M, \circ)$ is a Smarandache isomorph of $(G_H, \cdot)$ and thus write $(G_H, \cdot) \succsim (L_M, \circ)$.].

$(G_H, \cdot)$ is called a Smarandache G-special loop(SGS-loop) if and only if every special loop that is S-isotopic to $(G_H, \cdot)$ is S-isomorphic to $(G_H, \cdot)$.

Theorem 1. (Jaíyéọlá [13]) Let the special loop $(G_H, \cdot)$ be a $S_{2^{nd}}$BL. Then it is both a $S_{2^{nd}}$RIPL and a $S_{2^{nd}}$RAPL.

Theorem 2. (Jaíyéọlá [13]) Let $(G_H, \cdot)$ be a special loop. $(G_H, \cdot)$ is a $S_{2^{nd}}$BL if and only if $(R_s^{-1}, L_sR_s, R_s) \in S_{1^{st}}AUT(G_H, \cdot)$.

§3. Main results

Lemma 1. Let $(G_H, \cdot)$ be a special quasigroup and let $s, t \in H$. For all $x, y \in G$, let

$$x \circ y = xR_t^{-1} \cdot yL_s^{-1}. \tag{3}$$

Then, $(G_H, \circ)$ is a special loop and so $(G_H, \cdot)$ and $(G_H, \circ)$ are S-isotopic.

Proof. It is easy to show that $(G_H, \circ)$ is a quasigroup with a subquasigroup $(H, \circ)$ since $(G_H, \cdot)$ is a special quasigroup. So, $(G_H, \circ)$ is a special quasigroup. It is also easy to see that $s \cdot t \in H$ is the identity element of $(G_H, \circ)$. Thus, $(G_H, \circ)$ is a special loop. With $U = R_t$, $V = L_s$ and $W = I$, the triple $(U, V, W)\ :\ (G_H, \cdot) \to (G_H, \circ)$ is an S-isotopism.

Remark 3. $(G_H, \circ)$ will be called a Smarandache principal isotopism (S-principal isotopism) of $(G_H, \cdot)$.

Theorem 3. If the special quasigroup $(G_H, \cdot)$ and special loop $(L_M, \circ)$ are S-isotopic, then $(L_M, \circ)$ is S-isomorphic to a S-principal isotope of $(G_H, \cdot)$.

Proof. Let e be the identity element of the special loop $(L_M, \circ)$. Let U, V and W be 1-1 S-mappings of G_H onto L_M such that

$$xU \circ yV = (x \cdot y)W\ \forall\ x, y \in G_H.$$

Let $t = eV^{-1}$ and $s = eU^{-1}$. Define $x * y$ for all $x, y \in G_H$ by

$$x * y = (xW \circ yW)W^{-1}. \tag{4}$$

From (2), with x and y replaced by xWU^{-1} and yWV^{-1} respectively, we get

$$(xW \circ yW)W^{-1} = xWU^{-1} \cdot yWV^{-1} \ \forall\ x, y \in G_H. \tag{5}$$

In (5), with $x = eW^{-1}$, we get $WV^{-1} = L_s^{-1}$ and with $y = eW^{-1}$, we get $WU^{-1} = R_t^{-1}$. Hence, from (4) and (5),

$$x * y = xR_t^{-1} \cdot yL_s^{-1} \text{ and } (x * y)W = xW \circ yW \ \forall\ x, y \in G_H.$$

That is, $(G_H, *)$ is a S-principal isotope of $(G_H, \cdot)$ and is S-isomorphic to $(L_M, \circ)$.

Theorem 4. Let $(G_H, \cdot)$ be a $\mathrm{S}_{2^{\mathrm{nd}}}$RIPL. Let $f, g \in H$ and let $(G_H, \circ)$ be a S-principal isotope of $(G_H, \cdot)$. $(G_H, \circ)$ is a $\mathrm{S}_{2^{\mathrm{nd}}}$RIPL if and only if $\alpha(f, g) = (R_g, L_f R_g^{-1} L_{f\cdot g}^{-1}, R_g^{-1}) \in \mathrm{S}_{2^{\mathrm{nd}}} RAUT(G_H, \cdot)$ for all $f, g \in H$.

Proof. Let $(G_H, \cdot)$ be a special loop that has the $\mathrm{S}_{2^{\mathrm{nd}}}$RIP and let $f, g \in H$. For all $x, y \in G$, define $x \circ y = xR_g^{-1} \cdot yL_f^{-1}$ as in (3). Recall that $f \cdot g$ is the identity in $(G_H, \circ)$, so $x \circ x^{\rho'} = f \cdot g$ where $xJ'_\rho = x^{\rho'}$ i.e the right identity element of x in $(G_H, \circ)$. Then, for all $x \in G$, $x \circ x^{\rho'} = xR_g^{-1} \cdot xJ'_\rho L_f^{-1} = f \cdot g$ and by the $\mathrm{S}_{2^{\mathrm{nd}}}$RIP of $(G_H, \cdot)$, since $sR_g^{-1} \cdot sJ'_\rho L_f^{-1} = f \cdot g$ for all $s \in H$, then $sR_g^{-1} = (f \cdot g) \cdot (sJ'_\rho L_f^{-1})J_\rho$ because $(H, \cdot)$ has the RIP. Thus,

$$sR_g^{-1} = sJ'_\rho L_f^{-1} J_\rho L_{f\cdot g} \Rightarrow sJ'_\rho = sR_g^{-1} L_{f\cdot g}^{-1} J_\lambda L_f. \tag{6}$$

$(G_H, \circ)$ has the $\mathrm{S}_{2^{\mathrm{nd}}}$RIP iff $(x \circ s) \circ sJ'_\rho = s$ for all $s \in H$, $x \in G_H$ iff $(xR_g^{-1} \cdot sL_f^{-1})R_g^{-1} \cdot sJ'_\rho L_f^{-1} = x$, for all $s \in H$, $x \in G_H$. Replace x by $x \cdot g$ and s by $f \cdot s$, then $(x \cdot s)R_g^{-1} \cdot (f \cdot s)J'_\rho L_f^{-1} = x \cdot g$ iff $(x \cdot s)R_g^{-1} = (x \cdot g) \cdot (f \cdot s)J'_\rho L_f^{-1} J_\rho$ for all $s \in H$, $x \in G_H$ since $(G_H, \cdot)$ has the $\mathrm{S}_{2^{\mathrm{nd}}}$RIP. Using (6),

$$(x \cdot s)R_g^{-1} = xR_g \cdot (f \cdot s)R_g^{-1} L_{f\cdot g}^{-1} \Leftrightarrow (x \cdot s)R_g^{-1} = xR_g \cdot sL_f R_g^{-1} L_{f\cdot g}^{-1} \Leftrightarrow$$

$$\alpha(f, g) = (R_g, L_f R_g^{-1} L_{f\cdot g}^{-1}, R_g^{-1}) \in \mathrm{S}_{2^{\mathrm{nd}}} RAUT(G_H, \cdot) \text{ for all } f, g \in H.$$

Theorem 5. If a special loop $(G_H, \cdot)$ is a $\mathrm{S}_{2^{\mathrm{nd}}}$BL, then any of its S-isotopes is a $\mathrm{S}_{2^{\mathrm{nd}}}$RIPL.

Proof. By virtue of Theorem 3, we need only to concern ourselves with the S-principal isotopes of $(G_H, \cdot)$. $(G_H, \cdot)$ is a $\mathrm{S}_{2^{\mathrm{nd}}}$BL iff it obeys the $\mathrm{S}_{2^{\mathrm{nd}}}$BI iff $(xs \cdot z)s = x(sz \cdot s)$ for all $x, z \in G$ and $s \in H$ iff $L_{xs}R_s = L_s R_s L_x$ for all $x \in G$ and $s \in H$ iff $R_s^{-1} L_{xs}^{-1} = L_x^{-1} R_s^{-1} L_s^{-1}$ for all $x \in G$ and $s \in H$ iff

$$R_s^{-1} L_s^{-1} = L_x R_s^{-1} L_{xs}^{-1} \text{ for all } x \in G \text{ and } s \in H. \tag{7}$$

Assume that $(G_H, \cdot)$ is a $\mathrm{S}_{2^{\mathrm{nd}}}$BL. Then, by Theorem 2,

$$(R_s^{-1}, L_s R_s, R_s) \in \mathrm{S}_{1^{\mathrm{st}}} AUT(G_H, \cdot) \Rightarrow (R_s^{-1}, L_s R_s, R_s) \in \mathrm{S}_{2^{\mathrm{nd}}} RAUT(G_H, \cdot) \Rightarrow$$

$$(R_s^{-1}, L_s R_s, R_s)^{-1} = (R_s, R_s^{-1} L_s^{-1}, R_s^{-1}) \in \mathrm{S}_{2^{\mathrm{nd}}} RAUT(G_H, \cdot).$$

By (7), $\alpha(x,s)=(R_s, L_xR_s^{-1}L_{xs}^{-1}, R_s^{-1})\in S_{2^{\text{nd}}}RAUT(G_H,\cdot)$ for all $f,g\in H$. But $(G_H,\cdot)$ has the $S_{2^{\text{nd}}}$RIP by Theorem 1. So, following Theorem 4, all special loops that are S-isotopic to $(G_H,\cdot)$ are $S_{2^{\text{nd}}}$RIPLs.

Theorem 6. Suppose that each special loop that is S-isotopic to $(G_H,\cdot)$ is a $S_{2^{\text{nd}}}$RIPL, then the identities:

1. $(fg)\backslash f=(xg)\backslash x$,
2. $g\backslash(sg^{-1})=(fg)\backslash[(fs)g^{-1}]$

are satisfied for all $f,g,s\in H$ and $x\in G$.

Proof. In particular, $(G_H,\cdot)$ has the $S_{2^{\text{nd}}}$RIP. Then by Theorem 3, $\alpha(f,g)=(R_g, L_fR_g^{-1}L_{f\cdot g}^{-1}, R_g^{-1})\in S_{2^{\text{nd}}}RAUT(G_H,\cdot)$ for all $f,g\in H$. Let

$$Y=L_fR_g^{-1}L_{f\cdot g}^{-1}. \tag{8}$$

Then,

$$xg\cdot sY=(xs)R_g^{-1}. \tag{9}$$

Put $s=g$ in (9), then $xg\cdot gY=(xg)R_g^{-1}=x$. But, $gY=gL_fR_g^{-1}L_{f\cdot g}^{-1}=(fg)\backslash[(fg)g^{-1}]=(fg)\backslash f$. So, $xg\cdot(fg)\backslash f=x\Rightarrow(fg)\backslash f=(xg)\backslash x$.

Put $x=e$ in (9), then $sYL_g=sR_g^{-1}\Rightarrow sY=sR_g^{-1}L_g^{-1}$. So, combining this with (8), $sR_g^{-1}L_g^{-1}=sL_fR_g^{-1}L_{f\cdot g}^{-1}\Rightarrow g\backslash(sg^{-1})=(fg)\backslash[(fs)g^{-1}]$.

Theorem 7. Every special loop that is S-isotopic to a $S_{2^{\text{nd}}}$BL is itself a $S_{2^{\text{nd}}}$BL.

Proof. Let $(G_H,\circ)$ be a special loop that is S-isotopic to an $S_{2^{\text{nd}}}$BL $(G_H,\cdot)$. Assume that $x\cdot y=x\alpha\circ y\beta$ where $\alpha,\beta\,:\,H\to H$. Then the $S_{2^{\text{nd}}}$BI can be written in terms of $(\circ)$ as follows. $(xs\cdot z)s=x(sz\cdot s)$ for all $x,z\in G$ and $s\in H$.

$$[(x\alpha\circ s\beta)\alpha\circ z\beta]\alpha\circ s\beta=x\alpha\circ[(s\alpha\circ z\beta)\alpha\circ s\beta]\beta. \tag{10}$$

Replace $x\alpha$ by $\overline{x}$, $s\beta$ by $\overline{s}$ and $z\beta$ by $\overline{z}$, then

$$[(\overline{x}\circ\overline{s})\alpha\circ\overline{z}]\alpha\circ\overline{s}=\overline{x}\circ[(\overline{s}\beta^{-1}\alpha\circ\overline{z})\alpha\circ\overline{s}]\beta. \tag{11}$$

If $\overline{x}=e$, then

$$(\overline{s}\alpha\circ\overline{z})\alpha\circ\overline{s}=[(\overline{s}\beta^{-1}\alpha\circ\overline{z})\alpha\circ\overline{s}]\beta. \tag{12}$$

Substituting (12) into the RHS of (11) and replacing $\overline{x}$, $\overline{s}$ and $\overline{z}$ by x, s and z respectively, we have

$$[(x\circ s)\alpha\circ z]\alpha\circ s=x\circ[(s\alpha\circ z)\alpha\circ s]. \tag{13}$$

With $s=e$, $(x\alpha\circ z)\alpha=x\circ(e\alpha\circ z)\alpha$. Let $(e\alpha\circ z)\alpha=z\delta$, where $\delta\in SSYM(G_H)$. Then,

$$(x\alpha\circ z)\alpha=x\circ z\delta. \tag{14}$$

Applying (14), then (13) to the expression $[(x\circ s)\circ z\delta]\circ s$, that is

$$[(x\circ s)\circ z\delta]\circ s=[(x\circ s)\alpha\circ z]\alpha\circ s=x\circ[(s\alpha\circ z)\alpha\circ s]=x\circ[(s\circ z\delta)\circ s].$$

implies

$$[(x \circ s) \circ z\delta] \circ s = x \circ [(s \circ z\delta) \circ s].$$

Replace $z\delta$ by z, then

$$[(x \circ s) \circ z] \circ s = x \circ [(s \circ z) \circ s].$$

Theorem 8. Let $(G_H, \cdot)$ be a $\mathrm{S_{2^{nd}}BL}$. Each special loop that is S-isotopic to $(G_H, \cdot)$ is S-isomorphic to a S-principal isotope $(G_H, \circ)$ where $x \circ y = xR_f \cdot yL_f^{-1}$ for all $x, y \in G$ and some $f \in H$.

Proof. Let e be the identity element of $(G_H, \cdot)$. Let $(G_H, *)$ be any S-principal isotope of $(G_H, \cdot)$ say $x * y = xR_v^{-1} \cdot yL_u^{-1}$ for all $x, y \in G$ and some $u, v \in H$. Let e' be the identity element of $(G_H, *)$. That is, $e' = u \cdot v$. Now, define $x * y$ by

$$x \circ y = [(xe') * (ye')]e'^{-1} \text{ for all } x, y \in G.$$

Then $R_{e'}$ is an S-isomorphism of $(G_H, \circ)$ onto $(G_H, *)$. Observe that e is also the identity element for $(G_H, \circ)$ and since $(G_H, \cdot)$ is a $\mathrm{S_{2^{nd}}BL}$,

$$(pe')(e'^{-1}q \cdot e'^{-1}) = pq \cdot e'^{-1} \text{ for all } p, q \in G. \tag{15}$$

So, using (15),

$$x \circ y = [(xe') * (ye')]e'^{-1} = [xR_{e'}R_v^{-1} \cdot yR_{e'}L_u^{-1}]e'^{-1} = xR_{e'}R_v^{-1}R_{e'} \cdot yR_{e'}L_u^{-1}L_{e'^{-1}}R_{e'^{-1}}$$

implies that

$$x \circ y = xA \cdot yB, \ A = R_{e'}R_v^{-1}R_{e'} \text{ and } B = R_{e'}L_u^{-1}L_{e'^{-1}}R_{e'^{-1}}. \tag{16}$$

Let $f = eA$. then, $y = e \circ y = eA \cdot yB = f \cdot yB$ for all $y \in G$. So, $B = L_f^{-1}$. In fact, $eB = f^\rho = f^{-1}$. Then, $x = x \circ e = xA \cdot eB = xA \cdot f^{-1}$ for all $x \in G$ implies $xf = (xA \cdot f^{-1})f$ implies $xf = xA$ ($\mathrm{S_{2^{nd}}RIP}$) implies $A = R_f$. Now, (16) becomes $x \circ y = xR_f \cdot yL_f^{-1}$.

Theorem 9. Let $(G_H, \cdot)$ be a $\mathrm{S_{2^{nd}}BL}$ with the $\mathrm{S_{2^{nd}}RAIP}$ or $\mathrm{S_{2^{nd}}LAIP}$, let $f \in H$ and let $x \circ y = xR_f \cdot yL_f^{-1}$ for all $x, y \in G$. Then $(G_H, \circ)$ is a $\mathrm{S_{1^{st}}AIPL}$ if and only if $f \in N_\lambda(H, \cdot)$.

Proof. Since $(G_H, \cdot)$ is a $\mathrm{S_{2^{nd}}BL}$, $J = J_\lambda = J_\rho$ in $(H, \cdot)$. Using (6) with $g = f^{-1}$,

$$sJ'_\rho = sR_fJL_f. \tag{17}$$

$(G_H, \circ)$ is a $\mathrm{S_{1^{st}}AIPL}$ iff $(x \circ y)J'_\rho = xJ'_\rho \circ yJ'_\rho$ for all $x, y \in H$ iff

$$(xR_f \cdot yL_f^{-1})J'_\rho = xJ'_\rho R_f \cdot yJ'_\rho L_f^{-1}. \tag{18}$$

Let $x = uR_f^{-1}$ and $y = vL_f$ and use (16), then (18) becomes $(uv)R_fJL_f = uJL_fR_f \cdot vL_fR_fJ$ iff $\alpha = (JL_fR_f, L_fR_fJ, R_fJL_f) \in AUT(H, \cdot)$. Since $(G_H, \cdot)$ is a $\mathrm{S_{1^{st}}AIPL}$, so $(J, J, J) \in AUT(H, \cdot)$. So, $\alpha \in AUT(H, \cdot) \Leftrightarrow \beta = \alpha(J, J, J)(R_{f^{-1}}^{-1}, L_{f^{-1}}R_{f^{-1}}, R_{f^{-1}}) \in AUT(H, \cdot)$. Since $(G_H, \cdot)$ is a $\mathrm{S_{2^{nd}}BL}$,

$xL_fR_fL_{f^{-1}}R_{f^{-1}} = [f^{-1}(fx \cdot f)]f^{-1} = [(f^{-1}f \cdot x)f]f^{-1} = x$ for all $x \in G$. That is, $L_fR_fL_{f^{-1}}R_{f^{-1}} = I$ in $(G_H, \cdot)$. Also, since $J \in AUM(H, \cdot)$, then $R_fJ = JR_{f^{-1}}$ and $L_fJ = JL_{f^{-1}}$ in $(H, \cdot)$. So,

$$\beta = (JL_fR_fJR_{f^{-1}}^{-1}, L_fR_fJ^2L_{f^{-1}}R_{f^{-1}}, R_fJL_fJR_{f^{-1}}) =$$
$$(JL_fJR_{f^{-1}}R_{f^{-1}}^{-1}, L_fR_fL_{f^{-1}}R_{f^{-1}}, R_fL_{f^{-1}}R_{f^{-1}}) = (L_{f^{-1}}, I, R_fL_{f^{-1}}R_{f^{-1}}).$$

Hence, $(G_H, \circ)$ is a $S_{1^{st}}$AIPL iff $\beta \in AUT(H, \cdot)$.

Now, assume that $\beta \in AUT(H, \cdot)$. Then, $xL_{f^{-1}} \cdot y = (xy)R_fL_{f^{-1}}R_{f^{-1}}$ for all $x, y \in H$. For $y = e$, $L_{f^{-1}} = R_fL_{f^{-1}}R_{f^{-1}}$ in $(H, \cdot)$. So, $\beta = (L_{f^{-1}}, I, L_{f^{-1}}) \in AUT(H, \cdot) \Rightarrow f^{-1} \in N_\lambda(H, \cdot) \Rightarrow f \in N_\lambda(H, \cdot)$.

On the other hand, if $f \in N_\lambda(H, \cdot)$, then, $\gamma = (L_f, I, L_f) \in AUT(H, \cdot)$. But $f \in N_\lambda(H, \cdot) \Rightarrow L_f^{-1} = L_{f^{-1}} = R_fL_{f^{-1}}R_{f^{-1}}$ in $(H, \cdot)$. Hence, $\beta = \gamma^{-1}$ and $\beta \in AUT(H, \cdot)$.

Corollary 1. Let $(G_H, \cdot)$ be a $S_{2^{nd}}$BL and a $S_{1^{st}}$AIPL. Then, for any special loop $(G_H, \circ)$ that is S-isotopic to $(G_H, \cdot)$, $(G_H, \circ)$ is a $S_{1^{st}}$AIPL iff $(G_H, \cdot)$ is a $S_{1^{st}}$-loop and a $S_{1^{st}}$ commutative loop.

Proof. Suppose every special loop that is S-isotopic to $(G_H, \cdot)$ is a $S_{1^{st}}$AIPL. Then, $f \in N_\lambda(H, \cdot)$ for all $f \in H$ by Theorem 9. So, $(G_H, \cdot)$ is a $S_{1^{st}}$-loop. Then, $y^{-1}x^{-1} = (xy)^{-1} = x^{-1}y^{-1}$ for all $x, y \in H$. So, $(G_H, \cdot)$ is a $S_{1^{st}}$ commutative loop.

The proof of the converse is as follows. If $(G_H, \cdot)$ is a $S_{1^{st}}$-loop and a $S_{1^{st}}$ commutative loop, then for all $x, y \in H$ such that $x \circ y = xR_f \cdot yL_f^{-1}$,

$$(x \circ y) \circ z = (xR_f \cdot yL_f^{-1})R_f \cdot zL_f^{-1} = (xf \cdot f^{-1}y)f \cdot f^{-1}z.$$
$$x \circ (y \circ z) = xR_f \cdot (yR_f \cdot zL_f^{-1})L_f^{-1} = xf \cdot f^{-1}(yf \cdot f^{-1}z).$$

So, $(x \circ y) \circ z = x \circ (y \circ z)$. Thus, $(H, \circ)$ is a group. Furthermore,

$$x \circ y = xR_f \cdot yL_f^{-1} = xf \cdot f^{-1}y = x \cdot y = y \cdot x = yf \cdot f^{-1}x = y \circ x.$$

So, $(H, \circ)$ is commutative and so has the AIP. Therefore, $(G_H, \circ)$ is a $S_{1^{st}}$AIPL.

Lemma 2. Let $(G_H, \cdot)$ be a $S_{2^{nd}}$BL. Then, every special loop that is S-isotopic to $(G_H, \cdot)$ is S-isomorphic to $(G_H, \cdot)$ if and only if $(G_H, \cdot)$ obeys the identity $(x \cdot fg)g^{-1} \cdot f\backslash(y \cdot fg) = (xy) \cdot (fg)$ for all $x, y \in G_H$ and $f, g \in H$.

Proof. Let $(G_H, \circ)$ be an arbitrary S-principal isotope of $(G_H, \cdot)$. It is claimed that $(G_H, \cdot) \overset{R_{fg}}{\succsim} (G_H, \circ)$ iff $xR_{fg} \circ yR_{fg} = (x \cdot y)R_{fg}$ iff $(x \cdot fg)R_g^{-1} \cdot (y \cdot fg)L_f^{-1} = (x \cdot y)R_{fg}$ iff $(x \cdot fg)g^{-1} \cdot f\backslash(y \cdot fg) = (xy) \cdot (fg)$ for all $x, y \in G_H$ and $f, g \in H$.

Theorem 10. Let $(G_H, \cdot)$ be a $S_{2^{nd}}$BL, let $f \in H$, and let $x \circ y = xR_f \cdot yL_f^{-1}$ for all $x, y \in G$. Then, $(G_H, \cdot) \succsim (G_H, \circ)$ if and only if there exists a $S_{1^{st}}$ pseudo-automorphism of $(G_H, \cdot)$ with $S_{1^{st}}$ companion f.

Proof. $(G_H, \cdot) \succsim (G_H, \circ)$ if and only if there exists $T \in SSYM(G_H, \cdot)$ such that $xT \circ yT = (x \cdot y)T$ for all $x, y \in G$ iff $xTR_f \cdot yTL_f^{-1} = (x \cdot y)T$ for all $x, y \in G$ iff $\alpha = (TR_f, TL_f^{-1}, T) \in S_{1^{st}}AUT(G_H)$.

Recall that by Theorem 2, $(G_H, \cdot)$ is a $S_{2^{nd}}$BL iff $(R_f^{-1}, L_fR_f, R_f) \in S_{1^{st}}AUT(G_H, \cdot)$ for each $f \in H$. So,

$$\alpha \in S_{1^{st}}AUT(G_H) \Leftrightarrow \beta = \alpha(R_f^{-1}, L_fR_f, R_f) =$$
$$(T, TR_f, TR_f) \in S_{1^{st}}AUT(G_H, \cdot) \Leftrightarrow T \in S_{1^{st}}PAUT(G_H)$$

with $S_{1^{st}}$ companion f.

Corollary 2. Let $(G_H,\cdot)$ be a $S_{2^{nd}}$BL, let $f\in H$ and let $x\circ y = xR_f\cdot yL_f^{-1}$ for all $x,y\in G_H$. If $f\in N_\rho(H,\cdot)$, then, $(G_H,\cdot)\gtrsim(G_H,\circ)$.

Proof. Following Theorem 10, $f\in N_\rho(H,\cdot)\Rightarrow TS_{1^{st}}PAUT(G_H)$ with $S_{1^{st}}$ companion f.

Corollary 3. Let $(G_H,\cdot)$ be a $S_{2^{nd}}$BL. Then, every special loop that is S-isotopic to $(G_H,\cdot)$ is S-isomorphic to $(G_H,\cdot)$ if and only if each element of H is a $S_{1^{st}}$ companion for a $S_{1^{st}}$ pseudo-automorphism of $(G_H,\cdot)$.

Proof. This follows from Theorem 8 and Theorem 10.

Corollary 4. Let $(G_H,\cdot)$ be a $S_{2^{nd}}$BL. Then, $(G_H,\cdot)$ is a SGS-loop if and only if each element of H is a $S_{1^{st}}$ companion for a $S_{1^{st}}$ pseudo-automorphism of $(G_H,\cdot)$.

Proof. This is an immediate consequence of Corollary 4.

Remark 4. Every Bol loop is a $S_{2^{nd}}$BL. Most of the results on isotopy of Bol loops in chapter 3 of [19] can easily be deduced from the results in this paper by simply forcing H to be equal to G.

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A arithmetical function mean value of binary

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Abstract Binary polynomial and Binary pure even polynomial through the introduction of recursive method, using the two polynomial solve the problem of the average value formula with the sum of numbers of binary.

Keywords Binary polynomial, binary pure even polynomial, pure even partition of set number, characteristic function.

§1. Introduction

In the paper [1], U. S. experts in number theory Florentin Smarandache put forward function of the digital sum of the number of columns so that people study it. Paper [2-5] mainly to binary numbers and function of some low times have been studied. In this paper, using a recursive method provides a binary number and function of the mean value formula $A_p\left(2^k\right)$ and $A_p\left(N\right)$, and give evidence. with the paper [2-5] compared to the structure of the conclusions clear, convenient and practical, conclusions more comprehensive, easy to theoretical study of the characteristics of. To study the convenience of, the paper k and p both are non-negative integer, we give

Definition 1.[3] Set $m = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_s}$ $(k_1 > k_2 > \cdots > k_s \geq 0)$, call $a(m) = \sum\limits_{i=1}^{s} 1$ as the sum of numbers of binary, call $A_p(N) = \sum\limits_{m<N} a^p(m)$ as the average value of function $a(m)$.

Definition 2. Defined polynomial $t_p\left(k\right)$ as Binary polynomial: satisfy the recurrence relation $t_0\left(k\right) = 1, t_{p+1}\left(k\right) = 2kt_p\left(k\right) - kt_p\left(k-1\right)$.

Definition 3. Defined polynomial $g_p\left(k\right)$ as Binary pure even polynomial: satisfy the recurrence relation $g_0\left(k\right) = 1, g_1\left(k\right) = 0, g_{p+2}\left(k\right) = k^2 g_p\left(k\right) - k\left(k-1\right) g_p\left(k-2\right)$.

§2. The main conclusions

Theorem 1. Given $t_p\left(k\right)$ is Binary polynomial，then $A_p(2^k) = t_p\left(k\right) 2^{k-p}$.

Theorem 2. Set integer $N = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_s}$ $(k_1 > k_2 > \cdots > k_s)$, $g_h\left(k\right)$ is Binary pure even polynomial, $r_i = k_i + 2i - 2\,(i = 1, 2, \cdots, s)$, then

$$A_p(N) = \sum_{i=1}^{s}\left[\sum_{h=0}^{p}\begin{pmatrix} p \\ h \end{pmatrix} g_h\left(k_i\right) {r_i}^{p-h}\right] 2^{k_i - p}.$$

Corollary. Suppose $r_i = k_i + 2i - 2$, $g_{i4} = k_i\,(3k_i - 2)$, $g_{i6} = k_i\left(15{k_i}^2 - 30k_i + 16\right)$, then
(1) $A_1\,(N) = \sum\limits_{i=1}^{s} (k_i + 2i - 2)\,2^{k_i - 1}$, (2) $A_2\,(N) = \sum\limits_{i=1}^{s} \left({r_i}^2 + k_i\right)2^{k_i-2}$,
(3) $A_3\,(N) = \sum\limits_{i=1}^{s} \left({r_i}^3 + 3k_i r_i\right)2^{k_i-3}$, (4) $A_4\,(N) = \sum\limits_{i=1}^{s} \left({r_i}^4 + 6k_i {r_i}^2 + g_{i4}\right)2^{k_i-4}$,
(5) $A_5\,(N) = \sum\limits_{i=1}^{s} \left({r_i}^5 + 10k_i {r_i}^3 + 5g_{i4} r_i\right)2^{k_i-5}$,
(6) $A_6\,(N) = \sum\limits_{i=1}^{s} \left({r_i}^6 + 15k_i {r_i}^4 + 15g_{i4} {r_i}^2 + k_i g_{i6}\right)2^{k_i-6}$,
(7) $A_7\,(N) = \sum\limits_{i=1}^{s} \left({r_i}^7 + 21k_i {r_i}^5 + 35g_{i4} {r_i}^3 + 7g_{i6} r_i\right)2^{k_i-7}$.

§3. Proof of Theorem 1

3.1. Preparation

Lemma 1. Verify $(1+e^x)^k = \sum\limits_{p=0}^{\infty} A_p(2^k)\frac{x^p}{p!}$.

Proof. A positive integer in the binary of less than 2^k, each can take a digital "0" or "1", and affect each other, k-digit number in a j-bit integer to take a total of $\begin{pmatrix} k \\ j \end{pmatrix}$, $(j = 0, 1, 2, \cdots, k)$, this $\begin{pmatrix} k \\ j \end{pmatrix}$ integer numbers that the sum of numbers are j, their sum of p-times is $\begin{pmatrix} k \\ j \end{pmatrix} j^p$, then

$$A_p(2^k) = \sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} j^p. \tag{1}$$

According to the binomial theorem to know function

$$v_k(x) = (1+e^x)^k = \sum_{j=0}^{k} \begin{pmatrix} k \\ j \end{pmatrix} e^{jx},$$

p-derivative of function is

$$v_k{}^{(p)}(x) = \sum_{j=1}^{k} j^p \begin{pmatrix} k \\ j \end{pmatrix} e^{jx},$$

then

$$v_k{}^{(p)}(0) = \sum_{j=1}^{k} \begin{pmatrix} k \\ j \end{pmatrix} j^p. \tag{2}$$

Combining (1) and (2), we have

$$A_p(2^k) = v_k{}^{(p)}(0). \tag{3}$$

Then

$$(1+e^x)^k = v_k\left(x\right) = \sum_{p=0}^{\infty} {v_k}^{(p)}(0)\frac{x^p}{p!} = \sum_{p=0}^{\infty} A_p(2^k)\frac{x^p}{p!}.$$

Definition 4. Called function $v_k(x) = (1+e^x)^k$ is a characteristic function of $A_p(2^k)$.

3.2. Proof of Theorem 1

Derivative of function $v_k(x) = (1+e^x)^k$ is ${v_k}'(x) = k(1+e^x)^{k-1}e^x = k(1+e^x)^k - k(1+e^x)^{k-1}$. Then ${v_k}'(x) = kv_k(x) - kv_{k-1}(x)$. So ${v_k}^{(p+1)}(x) = k{v_k}^{(p)}(x) - k{v_{k-1}}^{(p)}(x)$. Then ${v_k}^{(p+1)}(0) = k{v_k}^{(p)}(0) - k{v_{k-1}}^{(p)}(0)$. The use of (3) available

$$A_{p+1}\left(2^k\right) = kA_p\left(2^k\right) - kA_p\left(2^{k-1}\right). \tag{4}$$

Suppose

$$A_p(2^k) = f_p\left(k\right)2^{k-p}, (p = 0,1,2,\cdots). \tag{5}$$

Then $A_{p+1}(2^k) = f_{p+1}\left(k\right)2^{k-p-1}$, $A_p(2^{k-1}) = f_p\left(k-1\right)2^{k-p-1}$.

All of the above three type substituted into (4) may

$$f_{p+1}\left(k\right)2^{k-p-1} = k\cdot f_p\left(k\right)2^{k-p} - k\cdot f_p\left(k-1\right)2^{k-p-1}.$$

Simplification may

$$f_{p+1}\left(k\right) = 2k\cdot f_p\left(k\right) - k\cdot f_p\left(k-1\right), (p = 0,1,2,\cdots). \tag{6}$$

By (3) available $A_0(2^k) = v\left(0\right) = 2^k$, by (5) available $A_0(2^k) = f_0\left(k\right)2^k$,

So $f_0\left(k\right)2^k = 2^k$, that

$$f_0\left(k\right) = 1. \tag{7}$$

By (6) and (7) available polynomial $f_p\left(k\right)$ to meet the definition of two terms in $t_p\left(k\right)$ recursive, so $f_p\left(k\right) = t_p\left(k\right)$.

Substituted into (5) available $A_p(2^k) = t_p\left(k\right)2^{k-p}$.

This proves the theorem.

§4. Proof of Theorem 2

4.1. Two lemma

Lemma 2.[7] $\left(\sum\limits_{p=0}^{\infty} a_p\frac{x^p}{p!}\right)\left(\sum\limits_{p=0}^{\infty} b_p\frac{x^p}{p!}\right) = \sum\limits_{p=0}^{\infty}\left[\sum\limits_{h=0}^{p}\begin{pmatrix} p \\ h \end{pmatrix} a_{p-h}b_h\right]\frac{x^p}{p!}.$

Lemma 3. $chx = \dfrac{e^x+e^{-x}}{2}$ is Hyperbolic cosine function, $g_h\left(k\right)$ is Binary pure even polynomial, verify $(chx)^k = \sum\limits_{p=0}^{\infty} g_p\left(k\right)\frac{x^p}{p!}$.

Proof. Set

$$w_k(x) = (chx)^k. \tag{8}$$

Then

$$w_k{}'(x) = k(chx)^{k-1}shx,$$

$$\begin{aligned} w_k{}''(x) &= k(k-1)(chx)^{k-2}sh^2x + k(chx)^{k-1}chx \\ &= k(k-1)(chx)^{k-2}\left(ch^2x-1\right) + k(chx)^k \\ &= k^2(chx)^k - k(k-1)(chx)^{k-2}. \end{aligned}$$

The use of (8) available $w_k{}''(x) = k^2 w_k(x) - k(k-1)w_{k-2}(x)$. So $w_k{}^{(p+2)}(x) = k^2 w_k{}^{(p)}(x) - k(k-1)w_{k-2}{}^{(p)}(x)$. That

$$w_k{}^{(p+2)}(0) = k^2 w_k{}^{(p)}(0) - k(k-1)w_{k-2}{}^{(p)}(0). \tag{9}$$

Suppose

$$w_k{}^{(p)}(0) = f_p(k), (p = 0, 1, 2, \cdots). \tag{10}$$

Then $w_k{}^{(p+2)}(0) = f_{p+2}(k)$, $w_{k-2}{}^{(p)}(0) = f_p(k-2)$.
All of the above type substituted into (9)

$$f_{p+2}(k) = k^2 f_p(k) - k(k-1)f_p(k-2). \tag{11}$$

Combining (8) and (10), we have

$$f_0(k) = w_k(0) = 1, f_1(k) = w_k{}'(0) = 0. \tag{12}$$

By (11) and (12) available polynomial $f_p(k)$ to meet the definition of three terms in $g_h(k)$ recursive, So $f_p(k) = g_p(k)$. Combined (10) available $w_k{}^{(p)}(0) = g_p(k)$. So

$$(chx)^k = w_k(x) = \sum_{p=0}^{\infty} w_k{}^{(p)}(0)\frac{x^p}{p!} = \sum_{p=0}^{\infty} g_p(k)\frac{x^p}{p!}.$$

4.2. The transformation of the problem

Given an integer $N = 2^{k_1} + 2^{k_2} + \cdots + 2^{k_s}(k_1 > k_2 > \cdots > k_s \geqslant 0)$, interval $[0, N)$ divided into the interval between s plots $N_i = \left[2^{k_1} + \cdots + 2^{k_{i-1}}, 2^{k_1} + \cdots + 2^{k_{i-1}} + 2^{k_i}\right), (i = 1, 2, \cdots, s)$, Set $A_p(N_i) = \sum\limits_{m \in N_i} a^p(m)$, then

$$A_p(N) = \sum_{i=1}^{s} A_p(N_i). \tag{13}$$

In the range of 2^{k_i} integers in interval N_i, k_i -digit number in a j-bit integer to take a total of $\begin{pmatrix} k_i \\ j \end{pmatrix}$, $(j = 0, 1, 2, \cdots, k_i)$, from the (k_i+1)-bit to the k_1 -bit numbers the sum of numbers are $(i-1)$, this $\begin{pmatrix} k_i \\ j \end{pmatrix}$ integer numbers the sum of numbers are $(i-1+j)$, their sum of p-times is $\begin{pmatrix} k_i \\ j \end{pmatrix}(i-1+j)^p$, then

$$A_p\left(N_i\right)=\sum_{j=0}^{k_i}\begin{pmatrix} k_i \\ j \end{pmatrix}(i-1+j)^p. \tag{14}$$

The use of the binomial theorem to get function

$$u_i\left(x\right)=e^{(i-1)x}(1+e^x)^{k_i}=e^{(i-1)x}\sum_{j=0}^{k_i}\begin{pmatrix} k_i \\ j \end{pmatrix}e^{jx}=\sum_{j=0}^{k_i}\begin{pmatrix} k_i \\ j \end{pmatrix}e^{(i-1+j)x},$$

p-derivative of function is

$$u_i{}^{(p)}\left(x\right)=\sum_{j=0}^{k_i}\begin{pmatrix} k_i \\ j \end{pmatrix}(i-1+j)^p e^{(i-1+j)x}. \tag{15}$$

Combining (14) and (15), we have

$$A_p\left(N_i\right)=u_i{}^{(p)}\left(0\right). \tag{16}$$

So characteristic function of $A_p\left(N_i\right)$ is $u_i\left(x\right)=e^{(i-1)x}(1+e^x)^{k_i}$.

4.3. Proof of Theorem 2

Because

$$u_i\left(x\right)=e^{(i-1)x}(1+e^x)^{k_i}=e^{\left(\frac{k_i}{2}+i-1\right)x}\left(e^{\frac{x}{2}}+e^{-\frac{x}{2}}\right)^{k_i}=2^{k_i}e^{\left(\frac{k_i}{2}+i-1\right)x}\left(ch\frac{x}{2}\right)^{k_i},$$

Set

$$r_i=k_i+2i-2,\left(i=1,2,\cdots,s\right),$$

so

$$e^{\left(\frac{k_i}{2}+i-1\right)x}=e^{\frac{r_i}{2}x}=\sum_{p=0}^{\infty}\frac{r_i{}^p}{2^p}\frac{x^p}{p!}. \tag{17}$$

Also according to Lemma 3

$$\left(ch\frac{x}{2}\right)^{k_i}=\sum_{p=0}^{\infty}g_p\left(k_i\right)\left(\frac{x}{2}\right)^p\frac{1}{p!}=\sum_{p=0}^{\infty}2^{-p}g_p\left(k_i\right)\frac{x^p}{p!}. \tag{18}$$

Combining (17), (18) and Lemma 2, we have

$$\begin{aligned} u_i\left(x\right) &= 2^{k_i}e^{\left(\frac{k_i}{2}+i-1\right)x}\left(ch\frac{x}{2}\right)^{k_i} \\ &= 2^{k_i}\left(\sum_{p=0}^{\infty}\frac{r_i{}^p}{2^p}\cdot\frac{x^p}{p!}\right)\left(\sum_{p=0}^{\infty}2^{-p}g_p\left(k_i\right)\frac{x^p}{p!}\right) \\ &= 2^{k_i}\sum_{p=0}^{\infty}\left[\sum_{h=0}^{p}\begin{pmatrix} p \\ h \end{pmatrix}\frac{r_i{}^{p-h}}{2^{p-h}}2^{-h}g_h\left(k_i\right)\right]\frac{x^p}{p!} \\ &= \sum_{p=0}^{\infty}2^{k_i-p}\left[\sum_{h=0}^{p}\begin{pmatrix} p \\ h \end{pmatrix}g_h\left(k_i\right)r_i{}^{p-h}\right]\frac{x^p}{p!}. \end{aligned} \tag{19}$$

Combining (16) and (19), we have

$$A_p\left(N_i\right) = {u_i}^{(p)}\left(0\right) = 2^{k_i - p}\left[\sum_{h=0}^{p}\begin{pmatrix} p \\ h \end{pmatrix} g_h\left(k_i\right){r_i}^{p-h}\right]. \tag{20}$$

By (13) and (20) available $A_p(N) = \sum\limits_{i=1}^{s} A_p\left(N_i\right) = \sum\limits_{i=1}^{s}\left[\sum\limits_{h=0}^{p}\begin{pmatrix} p \\ h \end{pmatrix} g_h\left(k_i\right){r_i}^{p-h}\right]2^{k_i - p}$.

This shows that the Theorem 2 is true.

§5. The relationship between Binary pure even polynomial and Pure even partition of set number

Can be seen from Theorem 2, Binary pure even polynomial in the expression is essential, I have studied a class of several, as pure even partition of set number, following pointed out that the Binary pure even polynomial and Pure even partition of set number relationships.

Definition 5.[6] j is non-negative integer, power Series $\dfrac{1}{j!}(chx-1)^j = \sum\limits_{p=0}^{\infty} S_4\left(p,j\right)\dfrac{x^p}{p!}$ the coefficient $S_4(p,j)$ is called the definition of pure even partition of set number.

Theorem 3. Factorial function $[k]_j = k\left(k-1\right)\cdots\left(k-j+1\right)$, then

Binary pure even polynomial $g_p\left(k\right) = \sum\limits_{j=0}^{\infty} S_4\left(p,j\right)[k]_j$.

Proof. Because

$$(chx)^k = \left[(chx-1)+1\right]^k = \sum_{j=0}^{k}\begin{pmatrix} k \\ j \end{pmatrix}(chx-1)^j = \sum_{j=0}^{k}[k]_j\frac{1}{j!}(chx-1)^j. \tag{21}$$

By the definition of 5 available

$$\frac{1}{j!}(chx-1)^j = \sum_{p=0}^{\infty} S_4\left(p,j\right)\frac{x^p}{p!}. \tag{22}$$

Combining (21) and (22), we have

$$(chx)^k = \sum_{j=0}^{k}[k]_j\sum_{p=0}^{\infty} S_4\left(p,j\right)\frac{x^p}{p!} = \sum_{p=0}^{\infty}\left[\sum_{j=0}^{\infty} S_4\left(p,j\right)[k]_j\right]\frac{x^p}{p!}.$$

Combination of Lemma 3 available

$$\sum_{p=0}^{\infty} g_p\left(k\right)\frac{x^p}{p!} = \sum_{p=0}^{\infty}\left[\sum_{j=0}^{\infty} S_4\left(p,j\right)[k]_j\right]\frac{x^p}{p!}.$$

So $g_p\left(k\right) = \sum\limits_{j=0}^{\infty} S_4\left(p,j\right)[k]_j$, $(p = 0,1,2,\cdots)$, prove complete.

Theorem 4. Pure even partition of set number $S_4(p,k)$ there is a recursive relationship: $S_4(0,0) = 1; S_4(1,j) = 0; S_4(p,0) = S_4(0,j) = 0$, $p \geqslant 1, j \geqslant 1$, and $S_4(p+2,j) = (2j-1)\, S_4(p,j-1) + j^2 S_4(p,j)$, $j \geqslant 1$.

Proof. Know from the Definition 5

$$\sum_{p=0}^{\infty} S_4\left(p,j\right)\frac{x^p}{p!}=\frac{1}{j!}(chx-1)^j=\frac{1}{j!}\left(\frac{x^2}{2!}+\frac{x^4}{4!}+\frac{x^6}{6!}+\cdots\right)^j.$$

Because the expansion does not contain x, so $S_4(1,j)=0$;
When $j=0$, only the constant term $S_4\left(0,0\right)=1, S_4\left(p,0\right)=0,(p\geqslant 1)$;
When $j\geqslant 1$, expansion does not contain the constant term, so $S_4(0,j)=0$;
Set $\varphi\left(x\right)=\sum\limits_{p=0}^{\infty} S_4\left(p,j\right)\frac{x^p}{p!}$.
So

$$\varphi''\left(x\right)=\sum_{p=2}^{\infty} S_4\left(p,j\right)\frac{x^{p-2}}{(p-2)!}=\sum_{p=0}^{\infty} S_4\left(p+2,j\right)\frac{x^p}{p!}. \tag{23}$$

Also, according to the Definition 5 known $\varphi\left(x\right)=\sum\limits_{p=0}^{\infty} S_4\left(p,j\right)\frac{x^p}{p!}=\frac{1}{j!}(chx-1)^j$. So $\varphi'\left(x\right)=\frac{1}{(j-1)!}(chx-1)^{j-1}shx$.
Then

$$\begin{aligned}
\varphi''\left(x\right) &= \frac{1}{(j-2)!}(chx-1)^{j-2}sh^2x+\frac{1}{(j-1)!}(chx-1)^{j-1}chx\\
&= \frac{j-1}{(j-1)!}(chx-1)^{j-2}\left(ch^2x-1\right)+\frac{1}{(j-1)!}(chx-1)^{j-1}chx\\
&= \frac{1}{(j-1)!}(chx-1)^{j-1}\left[(j-1)\left(chx+1\right)+chx\right]\\
&= \frac{1}{(j-1)!}(chx-1)^{j-1}\left[(2j-1)+j\left(chx-1\right)\right]\\
&= (2j-1)\cdot\frac{1}{(j-1)!}(chx-1)^{j-1}+j^2\cdot\frac{1}{j!}(chx-1)^j\\
&= (2j-1)\sum_{p=0}^{\infty} S_4\left(p,j-1\right)\frac{x^p}{p!}+j^2\sum_{p=0}^{\infty} S_4\left(p,j\right)\frac{x^p}{p!}.
\end{aligned}$$

So

$$\varphi''\left(x\right)=\sum_{p=0}^{\infty}\left[(2j-1)\,S_4\left(p,j-1\right)+j^2S_4\left(p,j\right)\right]\frac{x^p}{p!}. \tag{24}$$

Combining (23) and (24), we have

$$\sum_{p=0}^{\infty} S_4\left(p+2,j\right)\frac{x^p}{p!}=\sum_{p=0}^{\infty}\left[(2j-1)\,S_4\left(p,j-1\right)+j^2S_4\left(p,j\right)\right]\frac{x^p}{p!}.$$

So $S_4(p+2,j)=(2j-1)\,S_4(p,j-1)+j^2S_4(p,j)$. This proves the theorem.

Conclusion

The use of Theorem 3 and Theorem 4 expression available to Binary pure even polynomial.When the k value is not significant,applications (1) or (14) is also a convenient，when the k value is significant, application of theorem 1 or theorem 2 on the much simpler.

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The adjoint semiring part of IS-algebras [1]

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Abstract Let X be a IS-algebra, $\mathrm{AS(X)} = \{x \in \mathrm{X} | 0 * (0 * x) = 0 * x\}$ be called the adjoint semiring part of X. It be proved that AS(X) is subalgebra and ideal of X, and it is semiring about operation " + " by which $x + y = 0 * (x * y)$ and " $\cdot$ " on IS-algebra X, and its some other properties are given.

Keywords IS-algebra, ideal, subalgebra, adjoint semiring part.

§1. Introduction

The notion of BCI–algebra was formulated first in 1966 by K. Iseki. The IS-algebra (BCI-semigroup) was introduced in 1993 by Y. B. Jun. The author gave the ring part and the adjoint ring part of IS-algebra and discussed their properties in [4]. In this paper, we will give the new concept of adjoint semiring part on IS-algebras, and discuss its good properties, in order to explain its significance.

We stated the some relational definitions and conclusions for convenience of discussion.

Definition 1.[1] An algebra $(\mathrm{X}, *, 0)$ of type $(2, 0)$ is said to be a BCI-algebra if it satisfies:

(1) $((x * y) * (x * z)) * (z * y) = 0$.

(2) $(x * (x * y)) * y = 0$.

(3) $x * x = 0$.

(4) $x * y = 0$ and $y * x = 0$ imply $x = y$.

In a BCI–algebra X, define a binary relation $\leqslant$ by which $x \leqslant y$ if and only if $x * y = 0$ for any $x, y \in \mathrm{X}$, then $\leqslant$ is a partially ordered on X.

Lemma 1.[1] Let $(\mathrm{X}, *, 0)$ a BCI-algebra, for all $x, y, z \in X$, we have (1) $(x*y)*z = (x*z)*y$.

(2) $x * 0 = x$.

(3) $0 * (x * y) = (0 * x) * (0 * y)$.

(4) $0 * (0 * (0 * x)) = 0 * x$.

(5) $x * 0 = 0$ imply $x = 0$.

Definition 2.[2,3] A IS-algebra (BCI-semigroup) X is a non-empty set X with two operations "$*$" and "$\cdot$", and with a constant element 0 such that following axioms are satisfied:

(1) $(\mathrm{X}, *, 0)$ is BCI-algebra.

(2) $(\mathrm{X}, \cdot)$ is semigroup.

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(3) Distributive law: $x\cdot(y*z)=(x\cdot y)*(x\cdot z)$, $(x*y)\cdot z=(x\cdot z)*(y\cdot z)$, for any $x,y,z\in \mathrm{X}$. $x\cdot y$ is usual to be written xy and IS-algebra $(\mathrm{X},*,\cdot,0)$ is usual to be written X for short.

Lemma 2.[2] In a IS-algebra X, we have $0x=x0=0$.

Let Y be the non-empty subset of IS-algebra X, if operations " $*$ " and " $\cdot$ " are closed in Y, then $(\mathrm{Y},*,\cdot,0)$ is IS-algebra too, we call it is a subalgebra of X[2].

Definition 3.[3] Let I is a non-empty subset of IS-algebra X, It is said to be ideal of X, if

(1) For any $x\in \mathrm{X}$, for any $a\in \mathrm{I}$, then $xa, ax\in \mathrm{I}$.

(2) $x*y\in \mathrm{I}$ and $y\in \mathrm{I}$ imply $x\in \mathrm{I}$.

Definition 4.[4] In a IS-algebra X, the set

$$R(\mathrm{X})=\{x\in \mathrm{X}|0*x=x\}$$

is said to be ring part of X.

$$AR(\mathrm{X})=\{x\in \mathrm{X}|0*(0*x)=x\}$$

is said to be adjoint ring part of X.

Lemma 4.[4] In a IS-algebra X, Ring part $R(\mathrm{X})$ is a subalgebra of X and a ring that character is 2. $AR(\mathrm{X})$ is subalgebra of IS-algebra X, and adjoint ring part $AR(\mathrm{X})$ is a ring about operation " $+$ " by which $x+y=x*(0*y)$ and operation " $\cdot$ " on IS-algebra X.

§2. New concept

We first prove a theorem for introduction a new concept.

Theorem 1. In a IS-algebra X, Let $\mathrm{AS(X)}=\{x\in \mathrm{X}|0*(0*x)=0*x\}$, then

(1) AS(X) is subalgebra of IS-algebra X.

(2) Let $x+y=0*(x*y)$, then $(\mathrm{AS(X)},+,\cdot)$ is a semiring, and have

$$x+y=y+x, (x+y)+z=x+(y+z).$$

Proof. (1) Obviously, $0\in AS(\mathrm{X})$, so $AS(\mathrm{X})\neq\varnothing$.

For any $x,y\in AS(\mathrm{X})$, We have $0*(0*(x*y))=(0*(0*x))*(0*(0*y))=(0*x)*(0*y)=0*(x*y)$. that is $x*y\in \mathrm{AS(X)}$.

In addition, since $0*(0*(xy))=(0y)*((0y))*(xy))=(0\star(0*(x))y=(0*x)y=(0y)*(xy)=0*(xy)$. That is $xy\in AS(\mathrm{X})$, hence $AS(\mathrm{X})$ is a subalgebra of X.

(2) For any $x,y,z\in AS(\mathrm{X})$, we have $0*(0*(x+y))=0*(0*(0*(x*y)))=0*((0*(0*x))*(0*(0*y)))=0*((0*x)*(0*y))=0*(0*(x*y))=0*(x+y)$. then $x+y\in AS(\mathrm{X})$.

In addition, since $x+y=0*(x*y)=(0*x)*(0*y)=(0*(0*x))*(0*y)=(0*(0*y))*(0*x)=(0*y)*(0*x)=0*(y*x)=y+x$, that is $x+y=y+x$,$(x+y)+z=0*((0*(x*y))*z)=(0*(0*(x*y)))*(0*z)=(0*(x*y))*(0*z)=0*((x*y)*z)$. Hence $(x+z)+y=0*((x*z)*y)=0*((x*y)*z)=(x+y)+z$. Therefore, $AS(\mathrm{X})$ is a semigroup about above operation " $+$ ". Also, since AS(X) is closed about operation " $\cdot$ " on IS-algebra X, we have $x(y+z)=x(0*(y*z))=(x0*(xy*xz))=(0*(xy*xz))=xy+xz$.

In same reason, $(x+y)z=xz+yz$, so $(AS(\mathrm{X}),+,\cdot)$ is a semiring.

Definition 4. In a IS-algebra X, the set

$$\mathrm{AS(X)} = \{x \in \mathrm{X}|0 * (0 * x) = 0 * x\}$$

is said to be adjoint semiring part of X.

Clearly, the adjoint semiring part of X is semiring on above " $+$ " and " $\cdot$ " by Theorem 1.

Example 2. LetX $= \{0, a, b, c\}$, operation " $\cdot$ " by $xy = 0$, and operation " $*$ " is following:

$*$	0	a	b	c
0	0	c	0	a
a	a	0	a	c
b	b	c	0	a
c	c	0	c	a

Then $(\mathrm{X}, *, \cdot, 0)$ is IS-algebra, and $\mathrm{R(X)} = \{0\}$, $\mathrm{AR(X)} = \{0, a, c\}$, $\mathrm{AS(X)} = \{0, b\}$.

§3. Property

Theorem 2. $x \in \mathrm{AS(X)}$ if and only if $0 * x \in \mathrm{AS(X)}$.

Proof. If $x \in \mathrm{AS(X)}$, that is $0 * (0 * x) = 0 * x$, then $0 * (0 * (0 * x)) = 0 * (0 * x)$, therefore $0 * x \in \mathrm{AS(X)}$.

Conversely, if $0 * x \in \mathrm{AS(X)}$, that is $0 * (0 * (0 * x)) = 0 * (0 * x)$, then $0 * (0 * x) = 0 * x$, so $x \in \mathrm{AS(X)}$.

Theorem 3. $x \in \mathrm{AS(X)}$ if and only if $(0 * x) * x = 0$.

Proof. If $x \in \mathrm{AS(X)}$, then $(0 * (0 * x)) * x = (0 * x) * (0 * x) = (0 * x) * x = 0$.

Conversely, if $(0*x)*x = 0$, then $0*((0*x)*x) = (0*(0*x))*(0*x) = 0$, $(0*x)*(0*(0*x)) = (0 * (0 * (0 * x))) * x = (0 * x) * x = 0$, therefore $0 * (0 * x) = 0 * x$, that is $x \in \mathrm{AS(X)}$.

Theorem 4. Suppose $x * y \in \mathrm{AS(X)}$, we have

(1) if $x \in \mathrm{AS(X)}$, then $y \in \mathrm{AS(X)}$.

(2) if $y \in \mathrm{AS(X)}$, then $x \in \mathrm{AS(X)}$.

Proof. (1) Let $x * y \in \mathrm{AS(X)}$, $x \in \mathrm{AS(X)}$, by Theorem 2, AS(X) is close, we have $(x * y) * x = (x * x) * y = 0 * y$. Therefore, $y \in \mathrm{AS(X)}$.

(2) Let $y \in \mathrm{AS(X)}$, by Theorem 2, $0*y \in \mathrm{AS(X)}$, we have $(0*y)*(x*y) = (0*(x*y))*y = (0 * x) * (0 * y)) * y = ((0 * (0 * y)) * x) * y = ((0 * (0 * y)) * y) * x = 0 * x$. Hence $x \in \mathrm{AS(X)}$.

Theorem 5. In IS-algebra X, AS(X) is ideal of X.

Proof. In the first place, by Theorem 4 (2), $x*y \in \mathrm{AS(X)}$ and $y \in \mathrm{AS(X)}$ imply $x \in \mathrm{AS(X)}$.

In the second place, for any $x \in X, a \in \mathrm{AS(X)}$, we obtain $0*(0*(xa)) = (x0)*((x0)*(xa)) = x(0 * (0 * a)) = x(0 * a) = (x0) * (xa) = 0 * (xa)$. So $xa \in \mathrm{AS(X)}$.

In same reason, $xa \in \mathrm{AS(X)}$, hence, AS(X) is ideal of X.

Theorem 6. $\mathrm{AS(X)} \cap \mathrm{AR(X)} = \mathrm{R(X)}$.

Proof. For any $x \in \mathrm{R(X)}$, that is $0 * x = x$, then $0 * (0 * x) = 0 * x = x$, hence $x \in \mathrm{AR(X)}$ and $x \in \mathrm{AS(X)}$, that is, $x \in \mathrm{AS(X)} \cap \mathrm{AR(X)}$.

For any $y \in \mathrm{AS(X)} \cap \mathrm{AR(X)}$, that is $0 * (0 * x) = 0 * x = x$, hence $y \in \mathrm{R(X)}$, therefore $\mathrm{AS(X)} \cap \mathrm{AR(X)} = \mathrm{R(X)}$.

Theorem 7. If $x \in \mathrm{AS(X)}$, then $0 * x \in \mathrm{R(X)}$.

Proof. By Theorem 2, $0 * x \in \mathrm{RS(X)}$, but $0 * (0 * (0 * x)) = 0 * x$, that is $0 * x \in \mathrm{AR(X)}$, hence $0 * x \in \mathrm{R(X)}$ by Theorem 6.

Theorem 8. If $x \in \mathrm{R(X)}$ and $y \in \mathrm{AS(X)}$, then $x * y \in \mathrm{R(X)}$ and $y * x \in \mathrm{AS(X)}$.

Proof. Let $x \in \mathrm{R(X)}$ and $y \in \mathrm{AS(X)}$, that is $0 * x = x, 0 * (0 * y) = 0 * y$, we obtain $0 * (x * y) = (0 * x) * (0 * y) = (0 * (0 * y)) * x = (0 * y) * x = (0 * x) * y = x * y$. So $x * y \in \mathrm{R(X)}$.

$0 * (0 * (y * x)) = (0 * (0 * y)) * (0 * (0 * x)) = (0 * y) * (0 * x) = (0 * (y * x)$, therefore, $y * x \in \mathrm{AS(X)}$.

Theorem 9. If $x \in \mathrm{AR(X)}$ and $y \in \mathrm{AS(X)}$, then $x * y \in \mathrm{AR(X)}$.

Proof. Let $x \in \mathrm{AR(X)}$ and $y \in \mathrm{AS(X)}$, that is $0 * (0 * x) = x, 0 * (0 * y) = 0 * y$, we obtain $0 * (0 * (x * y)) = (0 * (0 * x)) * (0 * (0 * y)) = (0 * (0 * (0 * y))) * (0 * x) = (0 * y) * (0 * x) = (0 * (0 * x)) * y = x * y$. Therefore, $x * y \in \mathrm{AR(X)}$.

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On the Smarandache double factorial function

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Abstract For any positive integer n, the Smarandache double factorial function $Sdf(n)$ is defined as $Sdf(n) = \min\{m: \ m \in N, \ n \mid m!!\}$. Let $\varphi(n)$ be the Euler function. The main purpose of this paper is using the elementary methods to study the solvability of the equation $Sdf(n) + \varphi(n) = n$, and give its all positive integer solutions.

Keywords Smarandache double factorial function, Euler function, equation, solutions.

§1. Introduction and results

For any positive integer n, let $\varphi(n)$ denotes the Euler function. That is, $\varphi(n)$ denotes the number of all positive integers not exceeding n which are relatively prime to n.

For any positive integer n, the famous Smarandache double factorial function $Sdf(n)$ is defined as the smallest positive integer m such that $m!!$ is divisible by n, where the double factorial

$$m!! = \begin{cases} 1 \cdot 3 \cdot 5 \cdots (m-2) \cdot m, & \text{if } \ m \ \text{ is an odd number;} \\ 2 \cdot 4 \cdot 6 \cdots (m-2) \cdot m, & \text{if } \ m \text{ is an even number.} \end{cases}$$

That is, $Sdf(n) = \min\{m: \ m \in N, \ n \mid m!!\}$, where N denotes the set of all positive integers. For example, the first few values of $Sdf(n)$ are: $Sdf(1) = 1$, $Sdf(2) = 2$, $Sdf(3) = 3$, $Sdf(4) = 4$, $Sdf(5) = 5$, $Sdf(6) = 6$, $Sdf(7) = 7$, $Sdf(8) = 4$, $Sdf(9) = 9$, $Sdf(10) = 10$, $Sdf(11) = 11$, $Sdf(12) = 6$, $Sdf(13) = 13$, $Sdf(14) = 14$, $Sdf(15) = 5$, $Sdf(16) = 6$, $Sdf(17) = 17$, $Sdf(18) = 12$, $Sdf(19) = 19$, $Sdf(20) = 10$, $\cdots$. In reference [1] and [2], Professor F. Smarandache asked us to study the properties of $Sdf(n)$. About this problem, some authors had studied it, and obtained some interesting results, see references [3-7]. For example, Maohua Le [4] discussed various problems and conjectures about $Sdf(n)$, and obtained some useful results, one of them as follows: if $2|n$ and $n = 2^\alpha n_1$, where $\alpha, \ n_1$ are positive integers with $2 \nmid n_1$, then

$$Sdf(n) \leq \max\{Sdf(2^\alpha), \ 2Sdf(n_1)\}.$$

Fuling Zhang and Jianghua Li [5] proved that for any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{n \leq x} Sdf(n) = \frac{x \ln x}{\ln \ln x} + O\left(\frac{x \ln x}{(\ln \ln x)^2}\right).$$

Jianping Wang [6] proved that for any real number $x \geq 1$ and any fixed positive integer k, we have the asymptotic formula

$$\sum_{n\leq x} (Sdf(n) - S(n))^2 = \frac{\zeta(3)}{24} \frac{x^3}{\ln x} + \sum_{i=2}^{k} \frac{c_i \cdot x^3}{\ln^i x} + O\left(\frac{x^3}{\ln^{k+1} x}\right),$$

where $\zeta(s)$ is the Riemann zeta-function, and c_i are constants.

Bin Cheng [7] studied the solvability of the equation

$$Sdf(n) = \varphi(n),$$

and give its all positive integer solutions.

In this paper, we use the elementary method to study the solvability of the equation $Sdf(n) + \varphi(n) = n$, and give its all positive integer solutions. That is, we will prove the following:

Theorem. For any positive integer n, the equation

$$Sdf(n) + \varphi(n) = n$$

has and only has 4 positive integer solutions, they are $n = 8, 18, 27, 125$.

§2. Some preliminary lemmas

In this section, we shall give several simple lemmas which are necessary in the proof of our theorem. They are stated as follows:

Lemma 1. For $n \geq 1$, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ denotes the factorization of n into prime powers, where $p_1, p_2, \cdots, p_k$ are distinct primes and $\alpha_1, \alpha_2, \cdots, \alpha_k$ are positive integers, then we have

$$\varphi(n) = n \prod_{p|n} \left(1 - \frac{1}{p}\right) = p_1^{\alpha_1 - 1} p_2^{\alpha_2 - 1} \cdots p_k^{\alpha_k - 1} (p_1 - 1)(p_2 - 1) \cdots (p_k - 1).$$

Proof. See reference [8].

Lemma 2. If n is a square-free number, then we have

$$Sdf(n) = \begin{cases} \max\{p_1, p_2, \cdots, p_k\}, & \text{if } n = p_1 p_2 \cdots p_k \text{ and } 2 \nmid n. \\ 2 \cdot \max\{p_1, p_2, \cdots, p_k\}, & \text{if } n = 2p_1 p_2 \cdots p_k. \end{cases}$$

Lemma 3. For any positive integer n, $Sdf(n) \leq n$.

Proofs of Lemma 2 and Lemma 3 can be found in reference [9].

Lemma 4. If m is any positive integer and p is any odd prime, then we have

$$Sdf(p^m) = (2m - 1)p, \text{ for } p \geq (2m - 1).$$

Proof. See reference [10].

Lemma 5. If $2 \nmid n$ and $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the factorization of n into prime powers, then

$$Sdf(n) = \max\{Sdf(p_1^{\alpha_1}), Sdf(p_2^{\alpha_2}), \cdots, Sdf(p_k^{\alpha_k})\}.$$

Lemma 6. If $2 \mid n$ and $n = 2^\alpha n_1$, where α, n are positive integers with $2 \nmid n_1$, then

$$Sdf(n) \leq \max\{Sdf(2^\alpha), 2Sdf(n_1)\}.$$

Proofs of Lemma 5 and Lemma 6 can be found in reference [4].

§3. Proof of the theorem

In this section, we will complete the proof of our Theorem. In fact from the definition of the function $Sdf(n)$ we can easily deduce that $Sdf(1) = 1, Sdf(2) = 2, Sdf(3) = 3, Sdf(4) = 4, Sdf(5) = 5, Sdf(6) = 6, Sdf(7) = 7$, so for any positive integer $1 \leq n \leq 7$, the equation $Sdf(n) + \varphi(n) = n$ does not hold. Now we suppose that $n \geq 8$, we consider the following cases:

I. If n be an odd integer, let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the factorization of n into prime power, where $p_1 < p_2 < \cdots < p_k, p_i$ $(1 \leq i \leq k)$ is an odd prime, $\alpha_i \geq 0$ $(1 \leq i \leq k)$.

1. If $k = 1$, then $n = p^\alpha$.

1) If $\alpha = 1$, then $n = p$.

At this time, from the definition of Smarandache double factorial function $Sdf(n)$ we know $Sdf(p) = p$. From Lemma 1, we get $\varphi(p) = p - 1$. Then

$$Sdf(p) + \varphi(p) = 2p - 1 > p.$$

That is, $Sdf(n) + \varphi(n) > n$ for all n in this case.

2) If $\alpha = 2$, then $n = p^2$.

At this time, from Lemma 4 and Lemma 1, we have $Sdf(p^2) = 3p$ (p is an odd prime) and $\varphi(p^2) = p(p-1)$. Then

$$Sdf(p^2) + \varphi(p^2) = p^2 + 2p > p^2.$$

Hence $Sdf(n) + \varphi(n) > n$ in this case.

3) If $\alpha = 3$, then $n = p^3$.

i) If $p = 3$, then $n = 3^3$. According to the definition of Smarandache double factorial function $Sdf(n)$ we can easily deduce that

$$Sdf(3^3) + \varphi(3^3) = 9 + 3^2 \times 2 = 27 = 3^3.$$

So $n = 27$ is a positive integer solution of the equation $Sdf(n) + \varphi(n) = n$.

ii) If $p = 5$, then $n = 5^3$. Hence

$$Sdf(5^3) + \varphi(5^3) = 25 + 5^2 \times 4 = 125 = 5^3.$$

So $n = 125$ is a positive integer solution of the equation $Sdf(n) + \varphi(n) = n$.

iii) If $p \geq 7$, then $n = p^3$. From Lemma 4 and Lemma 1, we have $Sdf(p^3) = 5p$, and $\varphi(p^3) = p^2(p-1)$. Then

$$Sdf(p^3) + \varphi(p^3) = p(p^2 - p + 5).$$

In fact we know $p^2 - p + 5 < p^2$ when $p \geq 7$. so $p(p^2 - p + 5) < p^3$. That is, $Sdf(p^3) + \varphi(p^3) < p^3$.

4) If $\alpha \geq 4$, then $n = p^\alpha$.

From $n-\varphi(n)=p^{\alpha}-p^{\alpha-1}(p-1)=p^{\alpha-1}$, we know if it holds $Sdf(n)+\varphi(n)=n$, it must hold $Sdf(p^{\alpha})=p^{\alpha-1}=\min\{m:\ m\in N,\ p^{\alpha}\mid m!!\}$. Now we will prove that $Sdf(p^{\alpha})\neq p^{\alpha-1}$. It's obvious that $p,p^2,\cdots,p^{\alpha-1}$ are all included in $p^{\alpha-1}!!$. So if $p^{\alpha-1}!!=p^m\cdot n_1$, we can deduce that $m\geq\frac{\alpha(\alpha-1)}{2}$. But when $\alpha\geq 4$, we have $\alpha<\frac{\alpha(\alpha-1)}{2}$, so $Sdf(p^{\alpha})<p^{\alpha-1}$, that is, $Sdf(p^{\alpha})<p^{\alpha}-\varphi(p^{\alpha})$. We know $Sdf(n)+\varphi(n)<n$ for all n in this case.

2. If $k\geq 2$, then $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$.

On one hand, from Lemma 5, we have $Sdf(n)=Sdf(p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k})=\max\{Sdf(p_1^{\alpha_1}),$ $Sdf(p_2^{\alpha_2}),\cdots,Sdf(p_k^{\alpha_k})\}$. Without loss of generality we assume that $\max\{Sdf(p_1^{\alpha_1}),Sdf(p_2^{\alpha_2}),$ $\cdots,Sdf(p_k^{\alpha_k})\}=Sdf(p_k^{\alpha_k})$, then we have $Sdf(p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k})=Sdf(p_k^{\alpha_k})\leq p_k^{\alpha_k}$. On the other hand, we have $\varphi(n)=\varphi(p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k})=p_1^{\alpha_1-1}p_2^{\alpha_2-1}\cdots p_k^{\alpha_k-1}(p_1-1)(p_2-1)\cdots(p_k-1)$.

1) If $Sdf(n)=p_k^{\alpha_k}$, we have $n-Sdf(n)=p_k^{\alpha_k}(p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_{k-1}^{\alpha_{k-1}}-1)$.
From

$$p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_{k-1}^{\alpha_{k-1}}-1>p_1^{\alpha_1-1}p_2^{\alpha_2-1}\cdots p_{k-1}^{\alpha_{k-1}-1}(p_1-1)(p_2-1)\cdots(p_{k-1}-1)$$

and

$$p_k^{\alpha_k}>p_k^{\alpha_k-1}(p_k-1),$$

we have $n-Sdf(n)>\varphi(n)$. That is, $Sdf(n)+\varphi(n)<n$.

2) If $Sdf(n)<p_k^{\alpha_k}$, it's obvious that $Sdf(n)+\varphi(n)<n$.

II. If n be an even integer, let $n=2^{\alpha}p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$.

1. If $\alpha=1$, then $n=2p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$.

1) If $k=1$, then $n=2p^{\alpha}$.

i) If $\alpha=1$, then $n=2p$.

From the definition of $Sdf(n)$ we get $Sdf(2p)=2p$. So $Sdf(n)+\varphi(n)>n$ for all n in this case.

ii) If $\alpha=2$, then $n=2p^2$.

In fact, from Lemma 6 and Lemma 4, we get

$$Sdf(n)=Sdf(2p^2)\leq\max\{Sdf(2),2Sdf(p^2)\}=2Sdf(p^2)\leq 6p.$$

At the same time, noting that $n-\varphi(n)=2p^2-p(p-1)=p(p+1)$. We know $6p<p(p+1)$ when $p\geq 7$. So when $p\geq 7$, $Sdf(2p^2)+\varphi(2p^2)<2p^2$. For $p=3$ and 5, we have:

ii)' When $p=3$, $Sdf(2\times 3^2)=12$, $\varphi(2\times 3^2)=6$. So $Sdf(2\times 3^2)+\varphi(2\times 3^2)=18=2\times 3^2$. Hence $n=18$ is a solution of $Sdf(n)+\varphi(n)=n$.

ii)'' When $p=5$, $Sdf(2\times 5^2)=20$, $\varphi(2\times 5^2)=20$, and $Sdf(2\times 5^2)+\varphi(2\times 5^2)<2\times 5^2$. So $n=2\times 5^2$ is not a solution of the equation.

iii) If $\alpha=3$, then $n=2p^3$.

iii)' When $p=3$, We have $Sdf(2\times 3^3)=18=2\times 3^3$, so $Sdf(2\times 3^3)+\varphi(2\times 3^3)>2\times 3^3$.

iii)'' When $p\geq 5$, from Lemma 6 and Lemma 4, we have

$$Sdf(2\times p^3)\leq\max\{Sdf(2),2Sdf(p^3)\}=2Sdf(p^3)=10p.$$

Noting that, $2p^3-\varphi(2p^3)=2p^3-p^2(p-1)=p^2(p+1)$. We know $p^2(p+1)>10p$. So $Sdf(n)<n-\varphi(n)$, that is, $Sdf(n)+\varphi(n)<n$.

iv) If $\alpha \geq 4$, then $n = 2p^\alpha$.

It's easy to show that $n - \varphi(n) = 2p^\alpha - p^{\alpha-1}(p-1) = p^{\alpha-1}(p+1)$. In fact if it holds $Sdf(n)+\varphi(n) = n$, it must hold $Sdf(2p^\alpha) = p^{\alpha-1}(p+1)$, but it's not true. Because it's obvious that $p, p^2, \cdots, p^{\alpha-1}$ are all included in $p^{\alpha-1}(p+1)!!$, so if $p^{\alpha-1}(p+1)!! = p^m \cdot n_1$, we can deduce that $m \geq \frac{\alpha(\alpha-1)}{2}$. But when $\alpha \geq 4$, we have $\alpha < \frac{\alpha(\alpha-1)}{2}$, so $Sdf(2p^\alpha) < p^{\alpha-1}(p+1)$, that is, $Sdf(2p^\alpha) < 2p^\alpha - \varphi(2p^\alpha)$.

2) If $k \geq 2$, then $n = 2p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$.

Firstly, from Lemma 6 we have

$$\begin{aligned} Sdf(n) &= Sdf(2p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}) \\ &\leq \max\{Sdf(2), 2Sdf(p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k})\} \\ &= 2Sdf(p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}) \\ &= 2\max\{Sdf(p_1^{\alpha_1}), Sdf(p_2^{\alpha_2}), \cdots, Sdf(p_k^{\alpha_k})\}. \end{aligned}$$

For convenience we assume that $\max\{Sdf(p_1^{\alpha_1}), Sdf(p_2^{\alpha_2}), \cdots, Sdf(p_k^{\alpha_k})\} = Sdf(p_k^{\alpha_k})$. So

$$Sdf(n) \leq 2Sdf(p_k^{\alpha_k}) \leq 2p_k^{\alpha_k}.$$

And from Lemma 1, we have

$$\varphi(n) = p_1^{\alpha_1-1}p_2^{\alpha_2-1}\cdots p_k^{\alpha_k-1}(p_1-1)(p_2-1)\cdots(p_k-1).$$

i) If $Sdf(n) = 2p_k^{\alpha_k}$, then $n - Sdf(n) = 2p_k^{\alpha_k}(p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_{k-1}^{\alpha_{k-1}} - 1)$.

From

$$p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_{k-1}^{\alpha_{k-1}} - 1 > p_1^{\alpha_1-1}p_2^{\alpha_2-1}\cdots p_{k-1}^{\alpha_{k-1}-1}(p_1-1)(p_2-1)\cdots(p_{k-1}-1), \tag{1}$$

and

$$2p_k^{\alpha_k} > p_k^{\alpha_k-1}(p_k-1), \tag{2}$$

we get $n - Sdf(n) > \varphi(n)$. That is, $Sdf(n) + \varphi(n) < n$.

ii) If $Sdf(n) < 2p_k^{\alpha_k}$, it's obvious that $Sdf(n) + \varphi(n) < n$.

2. If $\alpha \geq 2$, then $n = 2^\alpha p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$.

1) If $\alpha_i = 0 (1 \leq i \leq k)$, then $n = 2^\alpha$.

i) When $\alpha = 3$, $Sdf(2^3) + \varphi(2^3) = 4 + 4 = 8$, so $n = 8$ is a solution of the equation.

ii) When $\alpha \geq 4$, $n - \varphi(n) = 2^{\alpha-1}$. In fact $Sdf(2^\alpha) \neq 2^{\alpha-1}$. Because $2, 2^2, \cdots, 2^{\alpha-1}$ are all included in $2^{\alpha-1}!!$, so if $2^{\alpha-1}!! = 2^m \cdot n_1$, we can deduce that $m \geq \frac{\alpha(\alpha-1)}{2}$. But when $\alpha \geq 4$, we have $\alpha < \frac{\alpha(\alpha-1)}{2}$, so $Sdf(2^\alpha) < 2^{\alpha-1}$. Hence $Sdf(2^\alpha) < 2^\alpha - \varphi(2^\alpha)$, that is, $Sdf(2^\alpha) + \varphi(2^\alpha) < 2^\alpha$.

2) If $k \geq 2$, then $n = 2^\alpha p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$.

For convenience we assume that $\max\{Sdf(p_1^{\alpha_1}), Sdf(p_2^{\alpha_2}), \cdots, Sdf(p_k^{\alpha_k})\} = Sdf(p_k^{\alpha_k})$. So from above Lemmas, we get

$$\begin{aligned} Sdf(n) &= Sdf(2^\alpha p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}) \\ &\leq \max\{Sdf(2^\alpha), 2Sdf(p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k})\} \\ &= \max\{Sdf(2^\alpha), 2Sdf(p_k^{\alpha_k})\} \\ &\leq 2^\alpha p_k^{\alpha_k}. \end{aligned}$$

Now note that $\varphi(n) = 2^{\alpha-1}p_1^{\alpha_1-1}p_2^{\alpha_2-1}\cdots p_k^{\alpha_k-1}(p_1-1)(p_2-1)\cdots(p_k-1)$ and $n - Sdf(n) = 2^{\alpha}p_k^{\alpha_k}(p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_{k-1}^{\alpha_{k-1}} - 1)$, using the same method in (1) and (2), we can easily deduce that $n - Sdf(n) > \varphi(n)$. It's clearly that there is no solutions satisfied $Sdf(n) + \varphi(n) = n$ in this case.

Now combining the above cases we may immediately get all positive solutions of the equation $Sdf(n) + \varphi(n) = n$, they are $n = 8, 18, 27, 125$.

This completes the proof of Theorem.

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On Smarandache Semigroups

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Abstract The notion of completely regular element of a semigroup is applied to characterize Smarandache Semigroups. Examples are provided for justification.

Keywords Semigroup, regular element, completely regular element, divisibility, idempotent element.

§1. Introduction

Smarandache notions on all algebraic and mathematical structures are interesting to the world of mathematics and researchers. The Smarandache notions in groups and the concept of Smaranadache Semigroups, which are a class of very innovative and conceptually a creative structure, have been introduced in the context of groups and a complete possible study has been taken in [11]. Padilla Raul intoduced the notion of Smarandache Semigroups in the year 1998 in the paper *Smarandache Algebraic Structures* [6].

In [5], the concept of regularity was first initiatied by J. V. Neumann for elements of rings. In general theory of semigroups, the regular semigroups were first studied by Thierrin [7] under the name demi-groupes inversifs. The completely regular semigroups were introduced by Clifford [2].

The notions of regular element, completely reagular element of a semigroup are very much useful to characterize Smarandache Semigroups. In this paper we present characterizations of Smarandache Semigroups. Besides, some more theorems on Smarandache Semigroups, examples are provided for justification. In Section 2 we give some basic definitions from the theory of semigroups (See [3]) and definition of Smarandache Semigroup (See [11]). In Section 3 we present our main characterization of Smarandache Semigroups and examples for justification.

§2. Preliminaries

Definition 2.1.[3] A semigroup is a nonempty set S in which for every ordered pair of elements $x, y \in S$, there is defined a new element called their product $xy \in S$, where for all $x, y, z \in S$ we have $(xy)z = x(yz)$.

Definition 2.2.[3] An element b of the semigroup S is called a right divisor of the element a of the semigroup if there exists in S an element x such that $xb = a$. b is called the left divisor of a if there exists in S an element y such that $by = a$.

If b is a right divisor of a, we say that a is divisible on the right by b. If b is a left divisor of a, we say that a is divible on the left by b.

Definition 2.3.[3] An element b of a semigroup S is called a right unit of the element a of the same semigroup, if $ab = a$.

Left unit is defined analogously. An element that is both a right and a left unit of some elemet is called two-sided unit of that element.

An element I which is its own two-sided unit is called an Idempotent : $I^2 = I$.

Definition 2.4.[3] An element a of a semigroup S is said to be regular, if we can find in S an element x such that $axa = a$.

A semigroup consisting entirely of regular elements is said to be Regular semigroup.

Definition 2.5.[3] An element a is said to be completely regular if we can find in S an element x such that $axa = a$; $ax = xa$.

A semigroup consisting entirely of completely regular elements is said to be completely regular.

Definition 2.6.[3] An element e of a semigroup S which is a left unit of the element $a \in S$ is called a Regualar left unit if it is divisible on the left by a.

e is called a regular right unit of a if it is a right unit of a and is divisible on the right by a.

e is called a regular two-sided unit of a if e is a two-sided unit of a and is divisible both on the left and on the right by a.

In [3], the following observations are known:

2.6.1. Concepts of regularity and complete regularity coincide for commutaive semigroup.

2.6.2. If $e \in S$ is a regular left unit of $a \in S$ there must exist an $x \in S$ such that $ea = a$, $ax = e$. The condition that e should be a right regular unit is $ae = a$, $xa = e$.

2.6.3. Every idempotent is completely regular. It is its own regular two-sided unit.

2.6.4. A regular left unit of an arbitrary element is always an idempotent.

2.6.5. No element in a semigroup S may have two regular two-sided units.

2.6.6. If an element has regular two-sided unit then it is completely regular.

§3. Proofs of the theorems

In this section we give characterizations of Smarandache Semigroups by proving the following theorems.

Theorem 3.1. A semigroup S is a Smarandache Semigroup if and only if S contains idempotents.

Proof. Let S be a Smarandache Semigroup then there is a proper subset $G \subset S$ such that G is a group under the operation defined on S. The identity element e of G is its own two-sided unit i.e., $e^2 = e$, in S. Hence, S contains idempotent.

Conversely, assume that the semigroup S contains idempotents. Let I be an arbitrary idempotent of the semigroup S. Write G_I for the set of all completely regular elements of S for which I is a regular two-sided unit. In view of (2.6.3), G_I is a nonempty subset of S as G_I contains I.

Now we show that G_I is a group under the operation on S. Let g_1, g_2 be any two elements in G_I. Since I is a regular two-sided unit of g_1 and g_2 we have for some u_1, u_2, v_1, v_2 in S.

$I = g_1u_1$, $I = g_2u_2$, $I = v_1g_1$, $I = v_2g_2$ from this we have

$$(g_1g_2)(u_2u_1) = g_1(g_2u_2)u_1 = g_1Iu_1 = g_1u_1 = I,$$

next,

$$(v_2v_1)(g_1g_2) = v_2(v_1g_1)g_2 = v_2Ig_2 = v_2g_2 = I.$$

Since, I is a two-sided unit of the element g_1g_2, I is a regular two-sided unit of g_1g_2. In view of (2.6.6), we have $g_1g_2 \in G_I$. Therefore G_I is a semigroup with unit I. Since I is clearly a two-sided unit for Iu_1I and

$$I = II = g_1u_1I = g_1(Iu_1I),$$

$$I = v_1g_1 = v_1IIg_1 = v_1g_1u_1Ig_1 = Iu_1Ig_1,$$

it follows that I is a regular two-sided unit of the element Iu_1I. In view of (2.6.6), $Iu_1I \in G_I$ and further, Iu_1I is a two- sided inverse of g_1 with respect to I. From this we get the fact that every element in G_I has a two-sided inverse in G_I as g_1 is an arbitrary element of G_I with unit I. So, the proper subset $G_I \subset S$ is a group and hence S is a Smarandache Semigroup.

Theorem 3.2. A semigroup S is a Smarandache semigroup if and only if S contains completely regular elements.

Proof. Suppose that the semigroup S is a Smarandache semigroup then there is a proper subset $G \subset S$ which is group under the operation defined on S. Clearly, the identity element $e \in G$, which is a regular two-sided unit of any arbitrary element of the semigroup, is completely regular.

On the other hand if the semigroup S contains a completely regular element, say a, then a has an idempotent element I as its regular two-sided unit. In view of the Theorem 3.1, the proper subset $G_I \subset S$ is a group. Hence, S is a Smarandache Semigroup.

Theorem 3.3. Let S be a Smarandache Semigroup. The set C of all completely regular elements of S can be expressed as the union of non-intersecting groups.

Proof. Let S be a Smarandache Semigroup, C be the set of all completely regular elements of S and H be the set of Idempotent elements of S.

In view of Theorem 3.1 and Theorem 3.2, $C \neq \phi$ and $H \neq \phi$. Let $c \in C$ then C has an idempotent I as its regualr two-sided unit. In view of Theorem 3.1 $c \in G_I$ which is always a group. In view of (2.6.5), no element may have two regular two-sided units. It follows that the groups G_I, $I \in H$ are all mutually disjoint. Therefore, $C = \cup_{I \in H} G_I$.

§4. Examples

In this section we give examples for justification.

Example 4.1. Let $S = \{e, a, b, c\}$ be a semigroup under the operation defined by the following table.

	e	a	b	c
e	e	a	b	c
a	a	e	b	c
b	b	b	c	b
c	c	c	b	c

Table 1

Clearly, the operation is commutative. Inview of (2.6.1), the completely regular elements of S are e, a, b, c as $eee = e$, $aaa = a$, $bbb = b$, $ccc = c$. Moreover the idempotent elements are e, c.

Now G_e= { e, a } as e is regular two-sided unit of e, a and G_c= { c, b } as c is regular two-sided uniit of c, b. Using the Table 1, we can easily see that G_e and G_c are groups. Further, $G_e \cap G_c = \phi$. Let $C = \{e, a, b, c\}$, we can easily see that $C = G_e \cup G_c$.

Example 4.2. Let $S = \{1, 2, 3, 4, 5, 6\}$ be a semigroup under the operation defined by $xy =$ the great common divisor of x, y for all $x, y \in S$. The composition table is as follows:

	1	2	3	4	5	6
1	1	1	1	1	1	1
2	1	2	1	2	1	2
3	1	1	3	1	1	3
4	1	2	1	4	1	2
5	1	1	1	1	5	1
6	1	2	3	2	1	6

Table 2

We can easily see that S is a commutative semigroup. The completely regular elements in S are $1, 2, 3, 4, 5, 6$ as $111 = 1$, $222 = 2$, $333 = 3$, $444 = 4$, $555 = 5$ and $666 = 6$. Write $C = \{1, 2, 3, 4, 5, 6\}$ for the set of all completely regular elements of S and $H = \{1, 2, 3, 4, 5, 6\}$ for the set of all idempotent elements of S. Now, $G_1 = \{1\}$ as 1 is the only regular two-sided element of 1. Obviously, we have $G_2 = \{2\}$, $G_3 = \{3\}$, $G_4 = \{4\}$, $G_5 = \{5\}$, $G_6 = \{6\}$. Further $G_1, G_2, G_3, G_4, G_5, G_6$ are groups and they are mutually disjoint also $C = G_1 \cup G_2 \cup G_3 \cup G_4 \cup G_5 \cup G_6$.

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Merrifield-Simmons index of zig-zag tree-type hexagonal systems

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Abstract In order to obtain a lower bound of Merrifield-Simmons index of the tree-type hexagonal systems, the zig-zag tree-type hexagonal systems are taken into consideration. In this paper, some results with respect to Merrifield-Simmons index of zig-zag tree-type hexagonal systems are shown. Using these results, hexagonal chains and hexagonal spiders with the lower bound of Merrifield-Simmons index are also determined.

Keywords Merrifield-Simmons index, zig-zag tree-type hexagonal system, hexagonal spider.

§1. Introduction

A hexagonal system is a 2−connected plane graph whose every interior face is bounded by a regular hexagon. Hexagonal systems are of great importance for theoretical chemistry because they are the natural graph representations of benzenoid hydrocarbons [2]. A hexagonal system is a tree-type one if it has no inner vertex. The zig-zag tree-type hexagonal systems are the graph representations of an important subclass of benzenoid molecules. A considerable amount of research in mathematical chemistry has been devoted to hexagonal systems [2-16].

In order to describe our results, we need some graph-theoretic notations and terminologies. Our standard reference for any graph theoretical terminology is [1].

Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Let e and u be an edge and a vertex of G, respectively. We will denote by $G-e$ or $G-u$ the graph obtained from G by removing e or u, respectively. Denote by N_u the set $\{v \in V(G) : uv \in E(G)\} \cup \{u\}$. Let H be a subset of $V(G)$. The subgraph of G induced by H is denoted by $G[H]$, and $G[V \setminus H]$ is denoted by $G - H$. Undefined concepts and notations of graph theory are referred to [11-16].

Two vertices of a graph G are said to be independent if they are not adjacent. A subset I of $V(G)$ is called an independent set of G if any two vertices of I are independent. Denote $i(G)$ the number of independent sets of G. In chemical terminology, $i(G)$ is called the Merrifield-Simmons index. Clearly, the Merrifield-Simmons index of a graph is larger than that of its proper subgraphs.

We denote by Ψ_n the set of the hexagonal chains with n hexagons. Let $B_n \in \Psi_n$. We denote by $V_3 = V_3(B_n)$ the set of the vertices with degree 3 in B_n. Thus, the subgraph $B_n[V_3]$

is a acyclic graph. If the subgraph $B_n[V_3]$ is a matching with $n-1$ edges, then B_n is called a linear chain and denoted by L_n. If the subgraph $B_n[V_3]$ is a path, then B_n is called a $zig-zag$ chain and denoted by Z_n. If the subgraph $B_n[V_3]$ is a comb, then B_n is called a helicene chain and denoted by H_n (see [11]).

Denote by $\mathbf{T}_n$ the tree-type hexagonal systems containing n hexagons. Let $\mathbf{T} = \bigcup_1^\infty \mathbf{T}_n$, and $T \in \mathbf{T}$. Let H be a hexagon of T. Obviously, H has at most three adjacent hexagons in T; if H has exactly three adjacent hexagons in T, then H is called a full-hexagon of T; if H has two adjacent hexagons in T, and, moreover, if its two vertices with degree two are adjacent, then call H a turn-hexagon of T; and if H has at most one adjacent hexagon in T, then H is called an end-hexagon of T. It is easy to see that the number of the end-hexagons of a tree-type hexagonal system of $n \geq 2$ hexagons is more two than the number of its full-hexagons. Let $T \in \mathbf{T}$ and let $B = H_1 H_2 \dots H_k, k \geq 2$ be a hexagonal chain of T. If the end-hexagon H_1 of B is also an end-hexagon of T, the other end-hexagon H_k is a full-hexagon of T, and for $2 \leq i \leq k-1$, H_i is not a full-hexagon of T, then B is called a branch of T (see [16]). If any branch of T is a zig-zag chain, then T is called zig-zag tree-type hexagonal system. Both a zig-zag hexagonal chain and zig-zag hexagonal spider are zig-zag tree-type hexagonal systems with no full-hexagon and only one full-hexagon, respectively.

§2. Some useful results

Among tree-type hexagonal systems with extremal properties on topological indices, L_n and Z_n play important roles. We list some of them about the Merrifield-Simmons index as follows.

Theorem 2.1.[6] For any $n \geq 1$ and any $B_n \in \Psi_n$, if B_n is neither L_n nor Z_n, then

$$i(Z_n) < i(B_n) < i(L_n).$$

Theorem 2.2.[6] For any $n \geq 1$ and any $T \in \mathbf{T}_{\mathrm{n}}$, if T is not L_n, then

$$i(T) < i(L_n).$$

Among many properties of $i(G)$, we mention the following results which will be used later.

Lemma 2.1.[1] Let G be a graph consisting of two components G_1 and G_2, then

$$i(G) = i(G_1)i(G_2).$$

Lemma 2.2.[1] Let G be a graph and any $u \in V(G)$, then

$$i(G) = i(G-u) + i(G-N_u).$$

Lemma 2.3.[1] Let G be a graph. For each $uv \in E(G)$. Then

$$i(G) - i(G-u) - i(G-u-v) \leq 0.$$

Moreover, the equality holds only if v is the unique neighbor of u.

Let A and B be any graphs and C be a hexagon. Let $G = A@_y^x C$. Let r and s be two adjacent vertices of B of at least degree two. Denote by $G_\eta B$ the graph obtained from G and B by identifying the edge ab with rs; by $G_\beta B$ the graph obtained from G and B by identifying the edge bc with rs; by $G_\zeta B$ the graph obtained from G and B by identifying the edge cd with rs (see [11]).

Lemma 2.4.[11] Let $A, B, G = A@_y^x C, G_\eta B$ and $G_\zeta B$, if $i(A - x) < i(A - y)$, then

$$i(G_\zeta B) < i(G_\eta B).$$

Lemma 2.5.[11] Let $A, B, G = A@_y^x C, G_\eta B, G_\beta B$ and $G_\zeta B$, then

(a) $i(G_\eta B) < i(G_\beta B)$,

(b) $i(G_\zeta B) < i(G_\beta B)$.

We add some notations which are convenient to express useful results. For a given zig-zag chain Z_k, denote by $x_k^{'}, x_k, y_k, y_k^{'}$ the four clockwise successful vertices with degree two in one of end-hexagons (see Fig. 2.1).

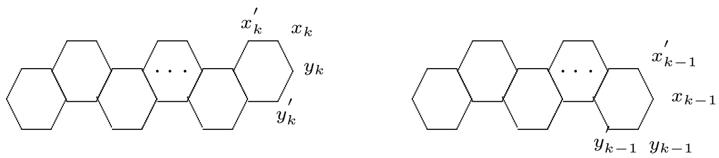


Fig. 2.1 Z_k and Z_{k-1}

Lemma 2.6. Suppose G is a zig-zag chain with k hexagons. Then

$$\begin{pmatrix} i(Z_k) \\ i(Z_k - x_k - y_k) \\ i(Z_k - x_k - x_k^{'} - y_k) \\ i(Z_k - x_k - y_k - y_k^{'}) \\ i(Z_k - y_k^{'}) \\ i(Z_k - y_k) \\ i(Z_k - y_k - y_k^{'}) \end{pmatrix} = \begin{pmatrix} 3 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 3 & 0 & 2 & 0 \\ 2 & 2 & 1 & 1 \\ 2 & 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} i(Z_{k-1}) \\ i(Z_{k-1} - y_{k-1}^{'}) \\ i(Z_{k-1} - y_{k-1}) \\ i(Z_{k-1} - y_{k-1} - y_{k-1}^{'}) \end{pmatrix}. \quad (1)$$

By applying Lemma 2.1 and Lemma 2.2, it is easy to obtain the result.

Lemma 2.7. Keep the notations as in Lemma 2.6 and suppose Z_k is a zig-zag chain with $k (k \geq 3)$ hexagons. Then

(a) $i(Z_k - x_k - y_k) > i(P_5) i(Z_{k-2}) + i(P_3) i(Z_{k-2} - y_{k-2}^{'})$,

(b) $i(Z_k - x_k - y_k - y_k^{'}) < i(P_4) i(Z_{k-2}) + i(P_3) i(Z_{k-2} - y_{k-2}^{'})$,

(c) $i(Z_k - x_k - x_k^{'} - y_k) < i(P_5) i(Z_{k-2} - y_{k-2}) + i(P_3) i(Z_{k-2} - y_{k-2} - y_{k-2}^{'})$.

Where $P_m (m = 3, 4, 5)$ is the path with m vertices.

Proof. (a) Set $f_1(k) = i(Z_k)$, $f_2(k) = i(Z_k - x_k - y_k)$, $f_3(k) = i(Z_k - x_k - y_k - y_k^{'})$, $f_4(k) = i(Z_k - x_k - x_k^{'} - y_k)$, $f_5(k) = i(Z_k - y_k^{'})$, $f_6(k) = i(Z_k - y_k)$ and $f_7(k) = i(Z_k - y_k - y_k^{'})$.

Applying Lemma 2.6 to $Z_k - x_k - y_k$, Z_{k-2} and $Z_{k-2} - y^{'}_{k-2}$, we get

$$\begin{aligned} i(Z_k - x_k - y_k) &= f_2(k) \\ &= f_1(k-1) + f_5(k-1) + f_6(k-1) + f_7(k-1) \\ &= 10f_1(k-2) + 4f_5(k-2) + 6f_6(k-2) + 2f_7(k-2) \end{aligned}$$

and

$$i(P_5)i(Z_{k-2}) + i(P_3)i(Z_{k-2} - y^{'}_{k-2}) = 13f_1(k-2) + 5f_5(k-2).$$

Since $f_1(k-2) = f_6(k-2) + f_3(k-2)$ for $k \geq 3$, then

$$\begin{aligned} \Delta_1 &= i(Z_k - x_k - y_k) - [i(P_5)i(Z_{k-2}) + i(P_3)i(Z_{k-2} - y^{'}_{k-2})] \\ &= -3f_1(k-2) - f_5(k-2) + 6f_6(k-2) + 2f_7(k-2) \\ &= 3f_6(k-2) - 3f_3(k-2) - f_5(k-2) + 2f_7(k-2). \end{aligned}$$

Since $Z_k - x_k - y_k - y^{'}_k$ is the proper subgraph of $Z_k - y_k$, then $i(Z_k - y_k) > i(Z_k - x_k - y_k - y^{'}_k)$. By Lemma 2.2, we have $2f_7(k-2) > f_5(k-2)$. Therefore $\Delta_1 > 0$.

(b) Similar to the proof of (a), by Lemma 2.6, we obtain

$$\begin{aligned} i(Z_k - x_k - y_k - y^{'}_k) &= f_3(k) \\ &= f_1(k-1) + f_6(k-1) \\ &= 5f_1(k-2) + 4f_5(k-2) + 3f_6(k-2) + 2f_7(k-2), \end{aligned}$$

and

$$i(P_4)i(Z_{k-2}) + i(P_3)i(Z_{k-2}) = 8f_1(k-2) + 5f_5(k-2).$$

Thus

$$\begin{aligned} \Delta_2 &= i(P_4)i(Z_{k-2}) + i(P_3)i(Z_{k-2}) - i(Z_k - x_k - y_k - y^{'}_k) \\ &= 3f_1(k-2) + f_5(k-2) - 3f_6(k-2) - 2f_7(k-2). \end{aligned}$$

According to Lemma 2.2, we have $f_1(k-2) = f_6(k-2) + f_3(k-2)$. So

$$\begin{aligned} \Delta_2 &= 3f_1(k-2) + f_5(k-2) - 3f_6(k-2) - 2f_7(k-2) \\ &= [2f_3(k-2) - f_7(k-2)] + [f_5(k-2) - f_7(k-2)] + f_3(k-2). \end{aligned}$$

Note that $Z_k - y_k - y^{'}_k$ is the proper subgraph of $Z_k - y^{'}_k$, then $i(Z_k - y^{'}_k) > i(Z_k - y_k - y^{'}_k)$. By Lemma 2.2, we have $2f_3(k-2) > f_7(k-2)$. Therefore $\Delta_2 > 0$.

(c) Similar to the proof of $(a), (b)$, by Lemma 2.6, we have

$$\begin{aligned} i(Z_k - x_k - y_k - x^{'}_k) &= f_4(k) \\ &= f_1(k-1) + f_5(k-1) \\ &= 6f_1(k-2) + 2f_5(k-2) + 4f_6(k-2) + f_7(k-2), \end{aligned}$$

and

$$i(P_5)i(Z_{k-2}-y_{k-2})+i(P_3)i(Z_{k-2}-y_{k-2}-y^{'}_{k-2})=13f_6(k-2)+5f_7(k-2).$$

Then

$$\begin{aligned}\Delta_3 &= i(P_5)i(Z_{k-2}-y_{k-2})+i(P_3)i(Z_{k-2}-y_{k-2}-y^{'}_{k-2})-i(Z_k-x_k-y_k-x^{'}_k)\\ &= -6f_1(k-2)-2f_5(k-2)+9f_6(k-2)+4f_7(k-2)\\ &= 3f_6(k-2)-6f_3(k-2)-2f_5(k-2)+4f_7(k-2)\\ &= 2f_1(k-3)+6f_5(k-3)-3f_6(k-3)+3f_7(k-3).\end{aligned}$$

Since $Z_{k-3}-y_{k-3}$ is the proper subgraph of Z_{k-3}, then $i(Z_{k-3})>i(Z_{k-3}-y_{k-3})$. By Lemma 2.2, we obtain $2f_5(k-3)>f_6(k-3)$. Therefore $\Delta_3>0$ and the proof of Lemma 2.7 is complete.

§3. Preliminary results

Suppose T_1, $T_2 \in \mathbf{T}$, and p_i, q_i are two adjacent vertices with degree two in $T_i, i=1,2$. Denote by $T_1(p_1,q_1)\otimes T_2(p_2,q_2)$ the tree-type hexagonal system obtained from T_1 and T_2 by identifying p_1 with p_2, and q_1 with q_2, respectively.

In the present section, for a given $T\in\mathbf{T}$, we always assume that s,t are two adjacent vertices with degree two in T. For a given linear zig-zag chain Z_k, denote by $x^{'}_k, x_k, y_k, y^{'}_k$ the four clockwise successful vertices with degree two in one of end-hexagons (see Fig. 3.1.).

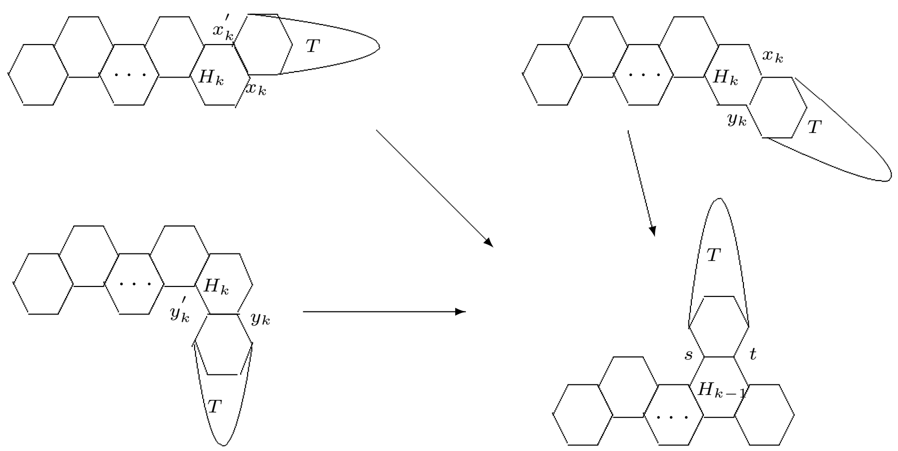


*Fig.*3.1.

Theorem 3.1. Keep the notations as Lemma 2.7. For any $T\in\mathbf{T}$ and $k\geq 3$ (see Fig. 3.1). Then

(a) $i(T(s,t)\otimes Z_k(x_k,y_k)) > i(T(s,t)\otimes Z_k(x'_{k-1},x_{k-1}))$,
(b) $i(T(s,t)\otimes Z_k(x'_k,x_k)) > i(T(s,t)\otimes Z_k(x'_{k-1},x_{k-1}))$,
(c) $i(T(s,t)\otimes Z_k(y_k,y'_k)) > i(T(s,t)\otimes Z_k(x'_{k-1},x_{k-1}))$.

Proof. (a) By Lemma 2.1 and Lemma2.2, we get

$$\begin{aligned} i(T(s,t)\otimes Z_k(x_k,y_k)) &= i(T-s-t)i(Z_k-x_k-y_k)+i(T-N_t)i(Z_k-x_k-y_k-y'_k) \\ &\quad +i(T-N_s)i(Z_k-x_k-y_k-x'_k) \\ &= i(T-s-t)f_2(k)+i(T-N_t)f_3(k)+i(T-N_s)f_4(k), \end{aligned}$$

and

$$\begin{aligned} i(T(s,t)\otimes Z_k(x'_{k-1},x_{k-1})) &= i(T-s-t)[13i(Z_{k-2})+5i(Z_{k-2}-y'_{k-2})] \\ &\quad +i(T-N_t)[8i(Z_{k-2})+5i(Z_{k-2}-y'_{k-2})] \\ &\quad +i(T-N_s)[13i(Z_{k-2}-y_{k-2})+5i(Z_{k-2}-y_{k-2}-y'_{k-2})] \\ &= i(T-s-t)[13f_1(k-2)+5f_5(k-2)] \\ &\quad +i(T-N_t)[8f_1(k-2)+5f_5(k-2)] \\ &\quad +i(T-N_s)[13f_6(k-2)+5f_7(k-2)]. \end{aligned}$$

Then

$$\begin{aligned} \Delta_4 &= i(T(s,t)\otimes Z_k(x_k,y_k))-i(T(s,t)\otimes Z_k(x'_{k-1},x_{k-1})) \\ &= i(T-s-t)\{f_2(k)-[13f_1(k-2)+5f_5(k-2)]\} \\ &\quad +i(T-N_t)\{f_3(k)-[8f_1(k-2)+5f_5(k-2)]\} \\ &\quad +i(T-N_s)\{f_4(k)-[13f_6(k-2)+5f_7(k-2)]\}. \end{aligned}$$

From Lemma 2.7, we have $f_2(k) > 13f_1(k-2)+5f_5(k-2)$, $f_3(k) < 8f_1(k-2)+5f_5(k-2)$ and $f_4(k) < 13f_6(k-2)+5f_7(k-2)$. If $i(T-N_t)\le i(T-N_s)$, then

$$\Delta_4 > i(T-N_s)[f_2(k)+f_3(k)+f_4(k)-21f_1(k-2)-10f_5(k-2)-13f_6(k-2)-5f_7(k-2)].$$

Otherwise

$$\Delta_4 > i(T-N_t)[f_2(k)+f_3(k)+f_4(k)-21f_1(k-2)-10f_5(k-2)-13f_6(k-2)-5f_7(k-2)].$$

Since $f_2(k)+f_3(k)+f_4(k)-21f_1(k-2)-10f_5(k-2)-13f_6(k-2)-5f_7(k-2)=0$, therefore $\Delta_4>0$ and similar to the proof of (a) and Lemma 2.7, we obtain

(b) $i(T(s,t)\otimes Z_k(x'_k,x_k)) > i(T(s,t)\otimes Z_k(x'_{k-1},x_{k-1}))$,

(c) $i(T(s,t)\otimes Z_k(y_k,y'_k)) > i(T(s,t)\otimes Z_k(x'_{k-1},x_{k-1}))$. The proof of Theorem 3.1 is complete.

Corollary 3.1. For any $k\ge 3$, then
(a) $i(L_n(s,t)\otimes Z_k(x_k,y_k)) > i(L_n(s,t)\otimes Z_k(x'_{k-1},x_{k-1}))$,
(b) $i(L_n(s,t)\otimes Z_k(x'_k,x_k)) > i(L_n(s,t)\otimes Z_k(x'_{k-1},x_{k-1}))$,
(c) $i(L_n(s,t)\otimes Z_k(y_k,y'_k)) > i(L_n(s,t)\otimes Z_k(x'_{k-1},x_{k-1}))$.

§4. Zig-zag tree-type hexagonal systems

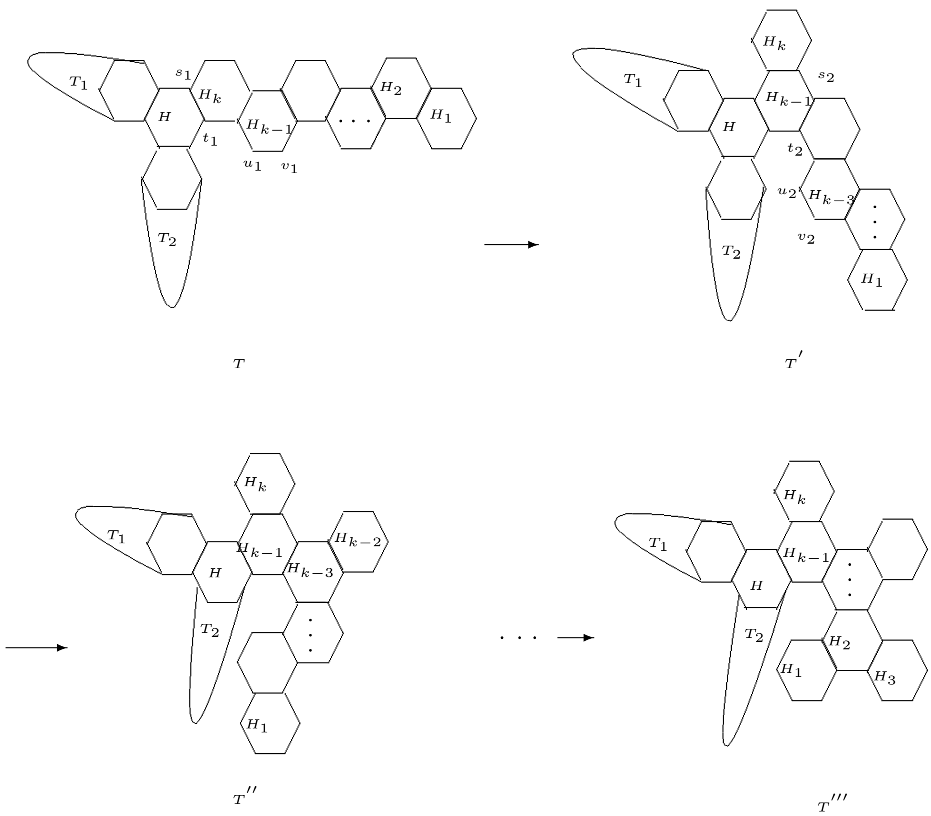
A graph G is called a zig-zag tree-type hexagonal system if it is a tree-type hexagonal system and any branch of which is zig-zag chain.

We shall use $\mathbf{Z}_{\mathrm{n}}^{*}$ to denote the set of all zig-zag tree-type hexagonal systems with n hexagons. For a given graph $Z^{*} \in \mathbf{Z}_{\mathrm{n}}^{*}$, we denote by $Z^{\perp}$ the graph obtained from Z^{*} whose every branch is transformed by transformation I (see Fig. 4.1).

A graph G is called a spider if it is a tree and contains only one vertex of degree greater than 2. For positive integer n_1, n_2, n_3, we use $S(n_1, n_2, n_3)$ to denote a hexagonal spider with three legs of lengths n_1, n_2 and n_3, respectively (see [11]).

If a hexagonal spider $S(n_1, n_2, n_3)$ whose 3 legs are linear chains, then such a graph is called a linear hexagonal spider and denoted by $L(n_1, n_2, n_3)$ (see [11]).

Similarly if each leg of $S(n_1, n_2, n_3)$ combining with the central hexagon is a zig-zag chain, then such graph is called a zig-zag hexagonal spider and denoted by $Z(n_1, n_2, n_3)$ (see [11]).



or

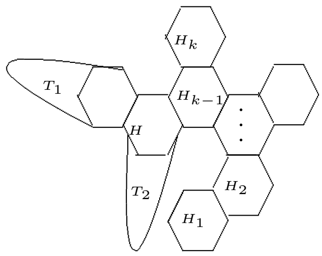


Fig. 4.1

Transformation I. Let $Z_k = H_1H_2\cdots H_k$ and $Z_k \bigotimes H$ be a branch of T (see Fig. 4.1.). Firstly, the graph $T^{'}$ can be obtained from $T - Z_k$ and Z_k by identifying the edge u_1v_1 of H_{k-1} with the edge s_1t_1 of H. Secondly, the graph $T^{''}$ can be got from $T^{'} - Z_{k-2}$ and Z_{k-2} by identifying the edge u_2v_2 of H_{k-3} with the edge s_2t_2 of H_{k-1}. Finally, by repeating this operation, the graph $T^{'''}$ can be obtained. If $T = Z_n$, only let $H = H_1$.

Theorem 4.1. For any $Z^* \in \mathbf{Z}_{\mathrm{n}}^*$ and any $n \geq 4$. Then

$$i(Z^{\perp}) \leq i(Z^*).$$

Moreover, the equality holds if and only if $Z^{\perp} \cong Z^*$.

Proof. Note that the graph $Z^{\perp}$ is obtained from Z^* whose every branch is transformed by transformation I, and by Theorem 3.1, we get $i(Z^{\perp}) \leq i(Z^*)$. Moreover, the equality holds if and only if $Z^{\perp} \cong Z^*$.

By repeating to apply transformation I on a hexagonal spider $S(n_1, n_2, n_3)$ and Z_n, and according to Theorem 3.1, we will also obtain a good lower bound of Merrifield-Simmons index of Z_n and $Z(n_1, n_2, n_3)$ as follows.

Theorem 4.2. For any $Z^*(n_1, n_2, n_3) \in Z(n_1, n_2, n_3)$ with n hexagons and any $n \geq 4$. Then

$$i(Z^{\perp}(n_1, n_2, n_3)) \leq i(Z^*(n_1, n_2, n_3)) < i(L(n_1, n_2, n_3)).$$

Moreover, the equality holds if and only if $Z^{\perp}(n_1, n_2, n_3)) \cong Z^*(n_1, n_2, n_3))$.

Theorem 4.3. For any $Z^* \in Z_n$ and $n \geq 4$. Then

$$i(Z^{\perp}) < i(Z^*) < i(L_n).$$

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An equation involving function $S_c(n)$ and $Z_*(n)$

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Abstract For any positive integer n, the Smarandache reciprocal function $S_c(n)$ is defined as the largest positive integer m such that $y \mid n!$ for all integers $1 \leqslant y \leqslant m$, and $m+1 \nmid n!$. And for any positive integer n and m, the Pseudo-Smarandache dual function $Z_*(n)$ is defined as the largest positive integer m such that $\frac{m(m+1)}{2} \mid n$. In this paper, we use the elementary methods to study the solvability of the equation $S_c(n) = Z_*(n) + n$, and give its all positive integer solutions.

Keywords Smarandache reciprocal function, Pseudo-Smarandache dual function, equation, solution.

§1. Introduction and results

In reference [1], A. Murthy introduced function $S_c(n)$, which is called the Smarandache reciprocal function. It is defined as the largest positive integer m such that $y \mid n!$ for all integers $1 \leqslant y \leqslant m$, and $m+1 \nmid n!$. That is,

$$S_c(n) = \max\left\{m: \ m \in N, \ y \mid n! \text{ for all integers } 1 \leqslant y \leqslant m, \text{ and } m+1 \nmid n!\right\}.$$

For example, the first few values of $S_c(n)$ are: $S_c(1) = 1, \ S_c(2) = 2, \ S_c(3) = 3, \ S_c(4) = 4, \ S_c(5) = 6, \ S_c(6) = 6, \ S_c(7) = 10, \ S_c(8) = 10, \ S_c(9) = 10, \ S_c(10) = 10, \ S_c(11) = 12, \ S_c(12) = 12, \ S_c(13) = 16, \ S_c(14) = 16, \ S_c(15) = 16, \ S_c(16) = 16, \ S_c(17) = 18, \ S_c(18) = 18, \cdots$.

Some authors had studied the elementary properties of $S_c(n)$, and obtained many interesting conclusions. For example:

If $S_c(n) = x$ and $n \neq 3$, then $x+1$ is the smallest prime greater than n.

On the other hand, for any positive integer n, the Pseudo-Smarandache dual function, denoted by $Z_*(n)$, is defined as the largest positive integer m such that $\frac{m(m+1)}{2} \mid n$. That is,

$$Z_*(n) = \max\left\{m: \ m \in N, \ \frac{m(m+1)}{2} \mid n\right\},$$

where N denotes the set of all positive integers.

From the definition of $Z_*(n)$, we find that the first few values of $Z_*(n)$ are:

$Z_*(1) = 1, \ Z_*(2) = 1, \ Z_*(3) = 2, \ Z_*(4) = 1, \ Z_*(5) = 1, \ Z_*(6) = 3, \ Z_*(7) = 1, \ Z_*(8) = 1, \ Z_*(9) = 2, \cdots$.

About this function, some authors had studied its properties, and obtained a series of interesting results, see references [2-5]. Such as:

For any prime $p \geq 3$, and $k \in N$,

$$Z_*(p^k) = \begin{cases} 2, & \text{if} \quad p = 3, \\ 1, & \text{if} \quad p \neq 3. \end{cases}$$

For any prime $p,\ q \geq 3$ satisfying $p = 2q - 1$, $Z_*(pq) = p$.

For all integers $a,\ b \geq 1$, $Z_*(ab) \geq \max\{Z_*(a),\ Z_*(b)\}$.

For any integer $s \geq 1$ and any prime p, $Z_*(3^s \cdot p) \geq 2$.

In reference [4], Professor Zhang Wenpeng and Li Ling proposed the equation $S_c(n) = Z_*(n) + n$, and give the following:

Conjecture. For any positive integer n, the equation

$$S_c(n) = Z_*(n) + n. \tag{1}$$

holds if and only if $n = p^{2\alpha+1}$, where p $(\geqslant 5)$ and $p^{2\alpha+1} + 2$ are primes, $\alpha \in N$.

A.A.K Majumdar studied this problem, and found several counter-examples to the conjecture. For example,

$$S_c(35) = 36 = Z_*(35) + 35,\ S_c(65) = 66 = Z_*(65) + 65,\ S_c(77) = 78 = Z_*(77) + 77.$$

The main purpose of this paper is to study the solvability of the equation (1), and find its all positive integer solutions.

That is, we shall prove the following:

Theorem. The equation (1) has infinite solutions, they are:

1. $n = p^{2\alpha+1}$, where p $(\geqslant 5)$ and $p^{2\alpha+1} + 2$ are primes, $\alpha \in N$.

2. $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} = u_i v_i t_i$, where $p_1,\ p_2, \cdots, p_r$ are distinct odd primes ≥ 5, r is a positive integer ≥ 2, and $(u_i,\ v_i) = (v_i,\ t_i) = (t_i,\ u_i) = 1$, $i \in N$ satisfying the following conditions:

(a). For any $u_i' \mid u_i,\ v_i' \mid v_i,\ v_i' \neq 2u_i' \pm 1$.

(b). For $n = \prod\limits_{i=1}^{h} (6a_i - 1)^{\alpha_i} \cdot \prod\limits_{j=h+1}^{r} (6a_j + 1)^{\alpha_j}$, if $\sum\limits_{i=1}^{h} \alpha_i = k$ is an odd integer, and $n + 2$ is a prime.

§2. Some useful lemmas

Lemma 1. For any prime $p,\ q \geq 3$ satisfying $p = 2q + 1$, then $Z_*(pq) = p - 1$.

Proof. $\frac{p(p-1)}{2} = pq$, so $Z_*(pq) = p - 1$.

Lemma 2. If $n = 3^s \cdot t$, for any positive integer s and composite integer t, $Z_*(n) \geq 2$.

Lemma 3. $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} = u_i v_i t_i$, where $p_1,\ p_2, \cdots, p_r$ are distinct odd primes ≥ 5, r is a positive integer ≥ 2, and $(u_i,\ v_i) = (v_i,\ t_i) = (t_i,\ u_i) = 1$, $i \in N$.

If $u_i' \mid u_i,\ v_i' \mid v_i,\ v_i' = 2u_i' - 1$, then $Z_*(n) = \max\{v_i'\} = v_1'$.

Proof. Let $Z_*(n) = m$, we have,

$$\frac{m(m+1)}{2} \mid u_i v_i t_i,$$

and it is clear that,

$$\frac{v_1'(v_1'+1)}{2} = u_1'v_1' \mid u_iv_it_i.$$

Then $m = v_1'$.

Lemma 4. $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r} = u_iv_it_i$, where $p_1,\ p_2,\cdots,p_r$ are distinct odd primes ≥ 5, r is a positive integer ≥ 2, and $(u_i,\ v_i) = (v_i,\ t_i) = (t_i,\ u_i) = 1,\ i \in N$.

If $u_i' \mid u_i,\ v_i' \mid v_i,\ v_i' = 2u_i'+1$, then $Z_*(n) = \max\{v_i'-1\} = v_1'-1$.

Lemma 5. If $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r} = u_iv_it_i$, where $p_1,\ p_2,\cdots,p_r$ are distinct odd primes ≥ 5, r is a positive integer ≥ 2, and $(u_i,\ v_i) = (v_i,\ t_i) = (t_i,\ u_i) = 1,\ i \in N$. If $u_i' \mid u_i,\ v_i' \mid v_i$, $v_i' \neq 2u_i' \pm 1$, then $Z_*(n) = 1$.

§3. Proof of the theorem

In this section, we shall use the elementary methods to complete the proof of our theorem.

Now we suppose that $n = 2^k \cdot s$, where s is an odd integer, we discuss the solutions in the following several cases:

(a). If n is an even integer, then $k \neq 0$.

It is clear that $n = 2,\ 4,\ 6,\ 8,\cdots$ are not the solutions of the equation (1).

(i) Let $t = 1$, then $n = 2^k$.

From the property of $Z_*(n)$, we can get $Z_*(2^k) = 1$. If $S_c(2^k) = Z_*(2^k)+2^k = 2^k+1$, then 2^k+2 must be a prime. In fact, $2 \mid (2^k+2)$. Hence, there are no solutions.

(ii) Let $t = p^\alpha$, then $n = 2^k \cdot p^\alpha$, p is an odd prime while α is a positive integer. Let $Z_*(n) = m$, we have $\frac{m(m+1)}{2} \mid 2^k \cdot p^\alpha$, that is $m(m+1) \mid 2^{k+1} \cdot p^\alpha$.

Obviously, $(m,\ m+1) = 1, (2^{k+1},\ p^\alpha) = 1$. So m must divides either 2^{k+1} or p^α, while $m+1$ must divides another.

A. If $m \mid 2^{k+1}$, $m+1 \mid p^\alpha$, then $m \geq 2$, $m+1 \geq 3$. Let $(m+1,\ p^\alpha) = d$, obviously, $d = m+1 \geq 3$. Then we can get $d \mid (m+1+n)$.

B. If $m \mid p^\alpha$, $m+1 \mid 2^{k+1}$, then $m \geq 3$, $m+1 \geq 4$. Let $(m+1,\ 2^{k+1}) = d$, obviously, $d = m+1 \geq 4$. Then we can get $2 \mid (m+1+n)$.

Therefore, there are no solutions.

(iii) Let t be a composite integer, $t = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_r^{\alpha_r}$, where $p_1, p_2,\cdots,p_r$ are distinct odd primes, and $\alpha_1,\ \alpha_2,\cdots,\alpha_r \in N$, $r \geq 2$. Let $Z_*(n) = m$, we have

$$\frac{m(m+1)}{2} \mid 2^k \cdot t,$$

that is $m(m+1) \mid 2^{k+1} \cdot t$.

It is clear that $(m+1,\ 2^k \cdot t) \geq 2$, then $(m+1+n,\ m+1) = d \geq 2$. So $m+1+n$ can not be a prime.

From the cases (i)-(iii), we know that the equation (1) has no even positive integer solutions.

(b). If n is an odd integer, we get $k = 0$ and $n = s$.

It is clear that $n = 5$ is a solution of the equation (1). We also obtained $n = 377 = 13 \cdot 29,\ n = 437 = 19 \cdot 23,\ n = 1445 = 5 \cdot 17^2,\ n = 1859 = 11 \cdot 13^2,\ n = 2387 = 7 \cdot 11 \cdot 31, \cdots$ satisfying the equation (1).

(i) Let $n = 1$, then $S_c(1) = 1,\ Z_*(1) = 1$. Obviously, $S_c(1) \neq Z_*(1) + 1$, so $n = 1$ is not a solution of the equation (1).

(ii) Let $n = 3^\alpha$, where α is a positive integer. Hence, $Z_*(3^\alpha) = 2$.

If $S_c(3^\alpha) = Z_*(3^\alpha) + 3^\alpha = 3^\alpha + 2$, then $3^\alpha + 3$ must be a prime. In fact, $3^\alpha + 3 \equiv 0(\bmod 3)$, here we obtain contradiction, so $n = 3^\alpha$ can not satisfy the equation (1).

(iii) Let $n = 3^\alpha \cdot t$, $t = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r}$, where $p_1,\ p_2,\ \cdots,\ p_r$ are distinct odd primes ≥ 5, r is a positive integer. Let $Z_*(n) = m$, we can get $\frac{m(m+1)}{2} \mid 3^\alpha \cdot t$, that is $m(m+1) \mid 2 \cdot 3^\alpha \cdot t$. From the property of $Z_*(n)$, we have $m \geq 2$, that is $m + 1 \geq 3$. So $(m + 1 + n,\ n) = d \geq 3$, and $m + 1 + n$ can not be a prime.

Therefore, there are no solutions in this case.

(iv) Let $n = p^r$, where p is an odd prime ≥ 5, r is a positive integer. Now we discuss in the following cases:

A. If $r = 1$, then $n = p$. Hence, $Z_*(n) = 1$. If $p + 2$ is a prime, then $S_c(p) = p + 1$. Both p and $p + 2$ are primes holds if and only if $4[p - 1)! + 1] + p \equiv 0(\bmod p(p + 2))$. In this case, $n = p$ satisfy the equation (1).

B. If $r = 2\alpha$, where α is a positive integer, we can get $Z_*(p^{2\alpha}) = 1$.

It is clear that $3 \nmid p^\alpha$. For $p^\alpha - 1,\ p^\alpha,\ p^\alpha + 1$ are three continuously integers, then 3 must divides one of the three forms. So $(p^\alpha - 1)(p^\alpha + 1)$ must be divided by 3. That is

$$(p^\alpha - 1)(p^\alpha + 1) \equiv 0(\bmod 3).$$

That is equivalent to $p^{2\alpha} \equiv 1(\bmod 3)$, then we can get $p^{2\alpha} + 2 \equiv 0(\bmod 3)$.

It means that if $S_c(n) = n + 1$, $n + 2$ can not be a prime.

C. If $r = 2\alpha + 1$, then $Z_*(p^{2\alpha+1}) = 1$. According to case A, we can get

$$p^{2\alpha+1} \equiv p(\bmod 3).$$

Then $p^{2\alpha+1} + 2 \equiv p + 2(\bmod 3)$.

If $p + 2 \equiv 0(\bmod 3)$, that is $p \equiv 1(\bmod 3)$, then we can get $p^{2\alpha+1} + 2 \equiv 0(\bmod 3)$.

When $p^{2\alpha+1} + 2$ is a prime, $p^{2\alpha+1}$ can be the solution.

(v) Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} = u_i v_i t_i$, where $p_1,\ p_2, \cdots, p_r$ are distinct odd primes, r is a positive integer ≥ 2, and $(u_i,\ v_i) = (v_i,\ t_i) = (t_i,\ u_i) = 1,\ i \in N$.

If there exists a equality $v_i' = 2u_i' \pm 1$, where $u_i' \mid u_i,\ v_i' \mid v_i$.

Let $Z_*(n) = m$, we have $\frac{m(m+1)}{2} \mid n$, that is $m(m + 1) \mid 2n$.

If $S_c(n) = Z_*(n) + n = m + n$, then $m + 1 + n$ must be a prime. According to Lemma 3 and 4, we can easily get that $m + 1 + n$ can not be a prime.

So in this case, the equation (1) has no solutions.

(vi) Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r} = u_i v_i t_i$, where $p_1,\ p_2, \cdots, p_r$ are distinct odd primes ≥ 5, r is a positive integer ≥ 2, and $(u_i,\ v_i) = (v_i,\ t_i) = (t_i,\ u_i) = 1,\ i \in N$. For any $u_i' \mid u_i,\ v_i' \mid v_i$, $v_i' \neq 2u_i' \pm 1$. Then we can get $Z_*(n) = 1$.

If $Z_*(n) + n + 1 = n + 2$ be a prime, n can be the solution of the equation (1).

Now we discuss the solutions in the following several parts:

A. For any prime $p_i \geq 5$, $p_i = 6a + 1$ or $p_i = 6a - 1$, every odd integers ≥ 5 can be expressed as $6a - 1$, $6a + 1$ or $6a + 3$. Obviously, $3 \mid (6a + 3)$, so $(6a + 3)$ can not be a prime.

B. Mark $n = \prod_{i=1}^{h} (6a_i - 1)^{\alpha_i} \cdot \prod_{j=h+1}^{r} (6a_j + 1)^{\alpha_j}$, and $\sum_{i=1}^{h} \alpha_i = k$, where $h \leq r$.
If k is an odd integer, we can get

$$\prod_{i=1}^{h} (6a_i - 1)^{\alpha_i} \cdot \prod_{j=h+1}^{r} (6a_j + 1)^{\alpha_j} \equiv -1(\bmod 3).$$

So $n + 2 \equiv 1(\bmod 3)$. n is the solution if $n + 2$ be a prime.

If k is an even integer, we can get $n + 2 \equiv 0(\bmod 3)$. Hence, for any integer n, $n + 2$ can not be a prime. In this case, the equation (1) has no solutions.

Thus, the theorem is established.

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The conjecture of Wenpeng Zhang with respect to the Smarandache $3n$-digital sequence

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Abstract For any positive integer n, the famous Smarandache $3n$-digital sequence $\{a(n)\}$ is defined as $a(n) = \overline{g_1 g_2}$ such that $g_1 = n$, $g_2 = 3n$. That is, the numbers that can be partitioned into two groups such that the second one is three times bigger than the first. The main purpose of this paper is using the elementary method to study the properties of the Smarandache $3n$-digital sequence, and solved a related conjecture.

Keywords Smarandache $3n$-digital sequence, elementary method, conjecture.

§1. Introduction and results

For any positive integer n, the famous Smarandache $3n$-digital sequence $a(n)$ is defined as follows: $a(n) = \overline{g_1 g_2}$, where $g_1 = n$, $g_2 = 3n$. That is, the numbers that can be partitioned into two groups such that the second one is three times bigger than the first. For example, $a(1) = 13$, $a(2) = 26$, $a(3) = 39$, $a(4) = 412$, $a(5) = 515$, $\cdots$. In reference [1], Professor F. Smarandache asked us to study the properties of the sequence $\{a(n)\}$. About this problem, professor Zhang proposed the following:

Conjecture. There does not exist any complete square number in the Smarandache $3n$-digital sequence $a(n)$. That is, the equation

$$a_n = m^2 \tag{1}$$

has no positive integer solution.

In reference [2], Jin Zhang studied this problem, and proved the following conclusions:

Proposition 1. If positive integer n is a square-free number (That is, for any prime p, if $p \mid n$, then $p^2 \dagger n$), then $a(n)$ is not a complete square number.

Proposition 2. If positive integer n is a complete square number, then $a(n)$ is not a complete square number.

In this paper, we using the elementary methods and the properties of the prime distribution to study the Smarandache $3n$-digital sequence $a(n)$, and partly solved the zhang's conjecture as following:

Theorem 1. Equation (1) has solutions, and part of the solutions can be expressed as follows:

$$n = \frac{n_1^2 \cdot (10^{p\,(p-1)\,i+k_0} + 3)}{p^2},$$

where $p^2 \mid (10^{p\,(p-1)\,i+k_0} + 3)$, $\frac{\sqrt{30}\,p}{30} < n_1 < \frac{\sqrt{3}\,p}{3}$, $i = 0,\ 1,\ 2,\ \cdots$.

Theorem 2. For any positive integer $k \geq 1$, there are infinite complete square numbers in the Smarandache kn-digital sequence $\{a_k(n)\}$. That is, the part of solutions of equation $a_k(n) = m^2$ can be expressed as

$$n = \frac{n_1^2 \cdot (10^{p\,(p-1)\,i+k_0} + k)}{p^2},$$

where $p^2 \mid (10^{p\,(p-1)\,i+k_0} + k)$, $\frac{\sqrt{10\,k}\,p}{10\,k} < n_1 < \frac{\sqrt{k}\,p}{k}$, $i = 0, 1, 2, \cdots$.

From above theorems, we can immediately obtain the following:

Corollary 1. Let b be a positive integer, if $b^2 \mid (10^{k_0}+3)$, then the solution of the equation (1) can be expressed as the following form

$$n = \frac{n_1^2 \cdot (10^{k_0} + 3)}{b^2},$$

where $\frac{\sqrt{30}\,b}{30} < n_1 < \frac{\sqrt{3}\,b}{3}$.

Corollary 2. Let b be an positive integer, if $b^2 \mid (10^{k_0} + k)$, then the solution of the equation (1) can be expressed as the following form

$$n = \frac{n_1^2 \cdot (10^{k_0} + k)}{b^2},$$

where $\frac{\sqrt{10k}\,b}{10k} < n_1 < \frac{\sqrt{k}\,b}{k}$.

§2. Some useful lemmas

To complete the proof of the theorems, we need the following several lemmas:

Lemma 1. Let p be a prime, if $p^2 \mid (10^{k_0}+3)$, then $p^2 \mid (10^{p\,(p-1)\,i+k_0}+3)$, $i = 0,\ 1,\ 2,\ \cdots$.

Proof. It is clear that if $p \mid (10^k+3)$, $(p \neq 2, 5)$, then $(10,\ p^2) = 1$. From Euler Theorem, we have $10^{\phi(p^2)} \equiv 1 (\bmod\ p^2)$. Note that $p^2 \mid (10^{k_0} + 3)$, we have

$$10^{p\,(p-1)i+k_0} \equiv -3 (\bmod\ p^2),\ \text{where}\ i = 0,\ 1,\ 2,\ \cdots,$$

so

$$p^2 \mid (10^{p\,(p-1)i+k_0} + 3),\ \text{where}\ i = 0,\ 1,\ 2,\ \cdots.$$

This completes the proof of Lemma 1.

Lemma 2. Let p be a prime, if $p^2 \nmid (10^{p-1}-1)$, then there exists a minimum positive integer $p\,\delta$ such that $10^{p\,\delta} \equiv 1(\bmod\ p^2)$.

Proof. let $\delta = \min\{\, d : 10^{\,d} \equiv 1(\bmod\ p),\ d \mid (\, p-1\,)\,\}$. Since $p^2 \nmid (10^{p-1}-1)$, then $p^2 \nmid (10^{\delta}-1)$ and

$$1 + 10^{\delta} + 10^{2\delta} + \cdots + 10^{(p-1)\delta} \equiv p \equiv 0(\bmod\ p),$$

$$(10^{\delta}-1)\,(10^{(p-1)\delta} + 10^{(p-2)\delta} + \cdots + 10^{\delta} + 1) \equiv 10^{p\,\delta} - 1 \equiv 0(\bmod\ p^2).$$

If there exists another positive integer u such that $p^2 \mid (10^{u}-1)$ and $u < p\,\delta$, then $\delta < u < p\,\delta$. It is obvious that $\delta \mid u$. Let $u = k\,\delta(1<k<p)$, since $1+10^{\delta}+10^{2\delta}+\cdots+10^{(k-1)\delta} \equiv k \not\equiv 0(\bmod\ p)$, then $p \nmid (1+10^{\delta}+10^{2\delta}+\cdots+10^{(k-1)\delta})$, and $p^2 \nmid (10^{\delta}-1)$, so we have $p^2 \nmid (10^{\delta}-1)(1+10^{\delta}+10^{2\delta}+\cdots+10^{(k-1)\delta})$, that is $p^2 \nmid (10^{k\,\delta}-1)$, which obtains a contradiction.

This completes the proof of Lemma 2.

Lemma 3. There exists a prime p and a positive integer k_0 such that $p^2 \mid (10^{k_0}+3)$.

Proof. For any positive integer k, we divided k into three sets as follows

$A = \{k \mid 10^k + 3 = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_r^{\alpha_r},$ there exist at least a $\alpha_i \ge 2, 1 \le i \le r\}$,

$B = \{k \mid 10^k + 3 = p_1 p_2 \cdots p_r,$ there exist at least a $p_i, 1 \le i \le r,$ satisfies $p_i^2 \nmid (10^{p-1}-1)\}$,

$C = \{k \mid 10^k + 3 = p_1 p_2 \cdots p_r,$ for any $p_i, 1 \le i \le r,$ satisfies $p_i^2 \mid (10^{p-1}-1)\}$.

We discuss it in following cases

Case 1. If $k \in A$, there exists a positive integer $\alpha_i \ge 2$ $(1 \le i \le r)$ then $p_i^2 \mid (10^k+3)$. This completes the Lemma 3.

Case 2. If $k \in B$, there exists at least one prime p among $p_1,\ p_2,\ \ldots,\ p_r$, which satisfies $p^2 \nmid (10^{p-1}-1)$. It is obvious that $p \mid (10^k+3)$ and $(p,\ 10) = 1$. From Lemma 2, we have $p^2 \nmid (10^{\delta}-1)$ and

$$10^{\delta\, i+k_1} \equiv -3(\bmod\ p),\ \text{where}\ i = 0,\ 1,\ 2,\ \cdots,\ k_1 \equiv k(\bmod\ \delta).$$

For any $i = 0,\ 1,\ 2,\ \cdots,\ p-1$, $\frac{10^{\delta\, i+k_1}+3}{p}$ traverse complete residue system mod p.

Otherwise, suppose that there exists $i,\ j$, such that $\frac{10^{\delta\, i+k_1}+3}{p} \equiv \frac{10^{\delta\, j+k_1}+3}{p}(\bmod\ p)$, where $0 \le i < j < p-1$, then $p^2 \mid 10^{\delta\, i+k_1}(10^{\delta(j-i)}-1)$, so we have $p^2 \mid (10^{\delta(j-i)}-1)$. That is, $10^{\delta(j-i)} \equiv 1(\bmod\ p^2)$, $1 \le j-i \le p-1$. From Lemma 2, we have $p\,\delta$ is the smallest integer such that $10^{p\,\delta} \equiv 1(\bmod\ p^2)$ and we have $p\,\delta \mid \delta(j-i)$, that is $p \mid (j-i)$, which obtains a contradiction.

So, we obtains a $i_0(0 \le i_0 < p-1)$ such that $\frac{10^{\delta\, i_0+k_1}+3}{p} \equiv 0(\bmod\ p)$, that is $p^2 \mid (10^{\delta\, i_0+k_1} + 3)$ and if $k_0 = \delta\, i_0 + k_1$, then

$$p^2 \mid (10^{k_0}+3). \tag{2}$$

Case 3. For any prime p among $p_1,\ p_2,\ \ldots,\ p_r$, if $k \in C$ and $p^2 \mid (10^{p-1}-1)$, then $10^{(p-1)j+k} + 3 \equiv 10^k + 3(\bmod\ p^2)$ $(j = 0,\ 1,\ \cdots)$. That is, $p^2 \nmid (10^{(p-1)j+k}+3), j = 0,\ 1,\ \cdots$.

Combing (1), (2) and (3), we can easily have

$$A \ne \emptyset \text{ or } B \ne \emptyset.$$

Otherwise, $k \in C$ and $10^k + 3 = p_1 p_2 \cdots p_r$. For any p_i $(1 \le i \le r), p_i^2 \mid (10^{p_i - 1} - 1)$. Which is impossible.

For example, if $k = 34$, then $49 \mid (10^{34} + 3)$, $k \in A$. If $k = 1$, then $k \in B$, and this completes the proof of Lemma 3.

§3. Proof of the theorems

In this section, we will complete the proof of the theorem. Firstly, we prove Theorem 1. Let n be k digit positive integer, from the definition of $\{a_n\}$, we have

$$a_n = \overline{g_1 g_2} = n \cdot (10^{k+1} + 3) \tag{3}$$

or

$$a_n = \overline{g_1 g_2} = n \cdot (10^{k+2} + 3). \tag{4}$$

Combining Proposition 1 and Proposition 2, we easily get the following results:

If $10^{k+1} + 3$ or $10^{k+2} + 3$ is a square-free number, then a_n is not a complete square number. Hence, if a_n is a complete square number, then $10^{k+1} + 3$ or $10^{k+2} + 3$ must be contain square element. Now, we construct the solution of equation (1) by the square number of $10^{k+1} + 3$ or $10^{k+2} + 3$.

From Lemma 1 and Lemma 3, there exists prime p and positive integer k_0 such that

$$p^2 \mid (10^{p\,(p-1)\,i + k_0} + 3), \text{ where } i = 0, 1, 2, \ldots.$$

If

$$g_1 = n = n_1^2 \cdot \frac{10^{p\,(p-1)\,i+k_0} + 3}{p^2}, \tag{5}$$

$\frac{1}{10} < \frac{3\,n_1^2}{p^2} < 1$ (That is $\frac{\sqrt{30}\,p}{30} < n_1 < \frac{\sqrt{3}\,p}{3}$), then the number $g_2 = 3n = \frac{3\,n_1^2}{p^2} \cdot (10^{p\,(p-1)\,i+k_0} + 3)$ contains $p\,(p-1)\,i + k_0$ digits, then we obtain

$$\begin{aligned} a_n &= \overline{g_1 g_2} \\ &= n_1^2 \cdot \frac{10^{p\,(p-1)\,i+k_0} + 3}{p^2} \cdot (10^{p\,(p-1)\,i+k_0} + 3) \\ &= n_1^2 \cdot p^2 \cdot (\frac{10^{p\,(p-1)\,i+k_0} + 3}{p^2})^2. \end{aligned}$$

If

$$m = n_1 \cdot \frac{10^{p\,(p-1)\,i+k_0} + 3}{p}, \text{where } i = 0, 1, 2, \ldots, \frac{\sqrt{30}\;p}{30} < n_1 < \frac{\sqrt{3}\;p}{3}, \tag{6}$$

then (6) is the solution of the formula (1), and this completes the proof of Theorem 1. we can also prove Theorem 2 using the same method of Theorem 1.

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A structure theorem of left U-*rpp* semigroups [1]

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Abstract The relation $\widetilde{\mathcal{L}}^U$ on any semigroup S provides a generalization of Green's relation $\mathcal{L}$. A semigroup S is called U-rpp semigroup if each $\widetilde{\mathcal{L}}^U$-class of S contains at lest projection e from U, where U is a non-empty subset of $E(S)$. The aim of this paper is to study a class of U-*rpp* semigroups, namely, left U-*rpp* semigroups. After giving some characterizations of left U-*rpp* semigroups, we establish a structure of this kind of semigroups.

Keywords Left U-*rpp* semigroups, right zero bands, U-left cancellative semigroups.

§1. Introduction

Suppose that S is a semigroup and $E(S)$ is the set of all idempotents of S. We now consider a non-empty subset $U \subseteq E(S)$, namely, the set of projections of S. Then a relation $\widetilde{\mathcal{L}}^U$ on S is defined as $a\widetilde{\mathcal{L}}^U b$ if and only if a and b have the same set of right identities in U, that is, for all $u \in U$, $au = a$ if and only if $bu = b$.

It can be easily verified that $\mathcal{L} \subseteq \mathcal{L}^* \subseteq \widetilde{\mathcal{L}}^U$ on a semigroup S.

We say that a semigroup S is called U-*rpp* semigroup if each $\widetilde{\mathcal{L}}^U$-class of S contains at least one projection of S and $\widetilde{\mathcal{L}}^U$ is a right congruence on S, denoted by (S, U).

Clearly, regular semigroups and *rpp* semigroups are all U-*rpp* semigroups.

We call a U-*rpp* semigroup (S, U) a left U-*rpp* semigroup if U is a subsemigroup and $xey = exy$ for any $e \in U$ and for all $x, y \in S^1$ with $y \neq 1$.

In fact, left U-*rpp* semigroups are U-*rpp* semigroups whose projections are left central. A *rpp* semigroup with left central idempotents have been studied by Ren-Shum in [2]. It was proved in [2] that the a *rpp* semigroup S with left central idempotents is isomorphic to a strong semilattice of left cancellative right stripes. In this paper, we will prove that a semigroup S is a left U-*rpp* semigroup if and only if S is a semilattice of a direct product of a U-left cancellative monoid and a right zero band; if and only if S is a strong semilattice of a direct product of a U-left cancellative monoid and a right zero band.

For any notation and terminologies not given in this paper, the reader is referred to [4], [5] and [6].

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§2. Preliminaries

Throughout this paper, (S, U) is a U-semiabundant semigroup. As usual, we denote the $\widetilde{\mathcal{L}}^U$-class of S containing the element a by $\widetilde{L}^U_a$.

The following lemmas give some basic properties of relation $\widetilde{\mathcal{L}}^U$ on (S, U).

Lemma 2.1. Let $a \in (S, U)$ and $e \in U$. Then $a\widetilde{\mathcal{L}}^U e$ if and only if $ae = a$ and for all $f \in U$, $af = a$ implies $ef = e$.

It follows immediately from definition the following results.

Lemma 2.2. If (S, U) is a U-*rpp* semigroup and e, f are elements of U, then $e\widetilde{\mathcal{L}}^U f$ if and only if $e\mathcal{L}f$.

Lemma 2.3. If (S, U) is a left U-*rpp* semigroup, then every $\widetilde{\mathcal{L}}^U$-class of S contains a unique projection.

Proof. Let (S, U) be a left U-*rpp* semigroup. Then for any $a \in (S, U)$ there exists $e \in U \cap \widetilde{L}^U_a$ such that $a = ae$. Hence, we have that $ea = eae = aee = ae = a$. If $f \in U \cap \widetilde{L}^U_a$, then it is clear that $(e, f) \in \widetilde{\mathcal{L}}^U$. By Lemma 2.2, it follows that $(e, f) \in \mathcal{L}$ so that $f = fe = ef = e$.

We now use a^* to denote the unique projection of $\widetilde{L}_a$ containing element a. It is clear that $a^*a = a = aa^*$ on a left U-*rpp* semigroup (S, U). Moreover, it can easily verified that the set of projections U of a left U-*rpp* semigroup (S, U) forms a right normal band.

Lemma 2.4. Let (S, U) be a left U-rpp semigroup. Then $\widetilde{\mathcal{L}}^U$ is a congruence on (S, U).

Proof. Since $\widetilde{\mathcal{L}}^U$ is a right congruence, we only need to show that $\widetilde{\mathcal{L}}^U$ is a left congruence. Suppose that $(a, b) \in \widetilde{\mathcal{L}}^U$ for any $a, b \in S$. Clearly, $a^* = b^*$ by Lemma 2.3. To prove that $(ca, cb) \in \widetilde{\mathcal{L}}^U$ for any $c \in (S, U)$, we suppose that $cae = ca$ for any $e \in U$.

Clearly, $(c, c^*) \in \widetilde{\mathcal{L}}^U$. Then, $(ca, c^*a) \in \widetilde{\mathcal{L}}^U$ because $\widetilde{\mathcal{L}}^U$ is a right congruence. By Definition of $\widetilde{\mathcal{L}}^U$, $c^*ae = c^*a$ and so $c^*aa^*e = c^*aa^*$. By our hypothesis, $ac^*a^*e = ac^*a^*$.

Since $(a, b) \in \widetilde{\mathcal{L}}^U$ and $\widetilde{\mathcal{L}}^U$ is a right congruence, it follows that $(ac^*a^*, bc^*a^*) \in \widetilde{\mathcal{L}}^U$ so that $bc^*a^*e = bc^*a^*$. Using the fact that $a^* = b^*$, we deduce that $bc^*b^*e = bc^*b^*$. Hence, $c^*bb^*e = c^*bb^*$, that is, $c^*be = c^*b$. It is clear from $(cb, c^*b) \in \widetilde{\mathcal{L}}^U$ that $cbe = cb$. Similarly, we can show that $cbe = cb$ implies that $cae = ca$. This shows that $(ca, cb) \in \widetilde{\mathcal{L}}^U$ so that $\widetilde{\mathcal{L}}^U$ is a left congruence on (S, U). Consequently, $\widetilde{\mathcal{L}}^U$ is a congruence on (S, U).

Lemma 2.5. If (S, U) is a left U-rpp semigroup, then $(ab)^* = a^*b^*$ for all $a, b \in (S, U)$.

Proof. Suppose that a, b are two any elements of (S, U). It is clear that $a\widetilde{\mathcal{L}}^U a^*$ and $b\widetilde{\mathcal{L}}^U b^*$. Since $\widetilde{\mathcal{L}}^U$ is a congruence on (S, U), it follows from Lemma 2.4 that $(ab, a^*b^*) \in \widetilde{\mathcal{L}}^U$. Hence, $(ab)^* = a^*b^*$ by Lemma 2.3.

Theorem 2.6. Suppose that (S, U) is a left U-*rpp* semigroup. Define a relation σ on (S, U) by $a\sigma b$ if and only if $a^*b^* = b^*$ and $b^*a^* = a^*$ for all $a, b \in (S, U)$. Then σ is a semilattice congruence on (S, U).

Proof. It is clear that σ is reflexive and symmetric.

To see that σ is transitive, we let $a\sigma b$ and $b\sigma c$. Clearly, $a^*b^* = b^*, b^*a^* = a^*$ and $b^*c^* = c^*, c^*b^* = b^*$. Thus, we have that

$$a^*c^* = a^*(b^*c^*) = (a^*b^*)c^* = b^*c^* = c^*,$$

and

$$c^*a^* = c^*(b^*a^*) = (c^*b^*)a^* = b^*a^* = a^*.$$

Hence $a\sigma c$, that is, σ is transitive. Thus σ is an equivalence relation.

Next we prove that σ is right compatible. Let $a\sigma b$ for any $a, b \in (S, U)$. Then for any $c \in (S, U)$,

$$(ac)^*(bc)^* = a^*c^*b^*c^* = c^*(a^*b^*)c^* = c^*b^*c^* = b^*c^*c^* = b^*c^* = (bc)^*,$$

and

$$(bc)^*(ac)^* = b^*c^*a^*c^* = c^*(b^*a^*)c^* = c^*a^*c^* = a^*c^*c^* = a^*c^* = (ac)^*.$$

By the definition of σ, it follows that $ac\sigma bc$. Similarly, we can prove that $ca\sigma cb$ so that σ is a congruence on (S, U). Finally, we prove that σ is a semilattice congruence on (S, U). For this purpose, we let $a, b \in (S, U)$. It follows that $(ba)^*(ab)^* = b^*a^*a^*b^* = b^*a^*b^* = a^*b^* = (ab)^*$ and $(ab)^*(ba)^* = a^*b^*b^*a^* = a^*b^*a^* = b^*a^* = (ba)^*$. Hence, $ab\sigma ba$. It is easy to see that $a^2\sigma a$. Thus, σ is indeed a semilattice congruence on (S, U).

Finally, we need the following definition in section 3.

Definition 2.7. A left U-*rpp* semigroup (S, U) is said to be U-left cancellative if for all $a, b \in (S, U)$ and for all $e \in U$, $bae = ba$ implies $ae = a$.

It is easy to see that left cancellative semigroups are U-left cancellative semigroups.

§3. Structure theorem

In this section, we will establish a structure theorem for left U-*rpp* semigroups.

Theorem 3.1. The following statements are equivalent on a semigroup S:

(i) (S, U) is a left U-*rpp* semigroup.

(ii) (S, U) is a semilattice of semigroup S_α which is a direct product of a U-left cancellative monoid M_α and a right zero band Λ_α. Moreover, $U = \bigcup_{\alpha \in Y}\{(1_\alpha, i) : 1_\alpha \text{ is the identity of } M_\alpha, i \in \Lambda_\alpha\}$.

(iii) (S, U) is a strong semilattice of semigroup S_α which is a direct product $M_\alpha \times \Lambda_\alpha$, where M_α is a U-left cancellative monoid and Λ_α is a right zero band and $U = \bigcup_{\alpha \in Y}\{(1_\alpha, i) : 1_\alpha \text{ is the identity of } M_\alpha, i \in \Lambda_\alpha\}$.

Proof. (i) $\Longrightarrow$ (ii) Let (S, U) be a left U-*rpp* semigroup. Then by Theorem 2.6 there exists a semilattice Y such that $(S, U) = \bigcup_{\alpha \in Y} S_\alpha$, where S_α is a σ-class of (S, U).

First we show that every S_α can be expressed as a direct product of a U-left cancellative monoid and a right zero band. For each $\alpha \in Y$, let $\Lambda_\alpha = S_\alpha \cap U$. Suppose that $a \in S_\alpha$. Clearly, $a\sigma a^*$ so that $a^* \in \Lambda_\alpha$. It is easy to see that $e\sigma f$ for any $e, f \in \Lambda_\alpha$. Thus $ef = f$ and $fe = e$. This implies that Λ_α is a right zero band.

Let $M_\alpha = S_\alpha e_\alpha$ for some projection $e_\alpha \in U \cap S_\alpha$ and $a, b \in M_\alpha$. Hence $a = xe_\alpha$ and $b = ye_\alpha$ for some $x, y \in S_\alpha$. Since σ is a semilattice congruence, it is clear that $xy \in S_\alpha$. Consequently, $ab = xe_\alpha ye_\alpha = xye_\alpha^2 = xye_\alpha$ which implies $ab \in S_\alpha e_\alpha = M_\alpha$. Hence, M_α is a monoid with the identity e_α.

Suppose now that $bae = ba$ for any $a, b \in M_\alpha$ and for any projection $e \in U \cap M_\alpha$. Clearly, $a = xe_\alpha$ and $b = ye_\alpha$ for some $x, y \in S_\alpha$. Hence, we have that $ye_\alpha xe_\alpha e = yxe_\alpha e = yxe = yxe_\alpha$ so that $(yxe)^* = (yxe_\alpha)^*$. It is easy to see that $x\sigma x^*$. Hence, $y^*x^*e = y^*x^*e_\alpha$ so that $x^*e = x^*e_\alpha$ since $x^*\sigma y^*$. It follows that $xx^*e = xx^*e_\alpha$, that is, $ae = ae_\alpha = a$. This shows that M_α is a U-left cancellative monoid.

Define $\varphi : M_\alpha \times \Lambda_\alpha \longrightarrow S_\alpha$ by $\varphi(x, f) = xf$ for any $x \in M_\alpha$ and any $f \in \Lambda_\alpha$. Now we can claim that φ is an isomorphism. For any $(x, f), (y, g) \in M_\alpha \times \Lambda_\alpha$, it follows that $\varphi(x, f)\varphi(y, g) = xfyg = xyfg = xyg = \varphi[(x, f)(y, g)]$ which implies that φ is a homomorphism.

Suppose that $\varphi(x, f) = \varphi(y, g)$ for $(x, f), (y, g) \in M_\alpha \times \Lambda_\alpha$.Then $xf = yg$ and so $xfe_\alpha = yge_\alpha$ where $e_\alpha \in S_\alpha$. Noticing that Λ_α is a right zero band, we immediately deduce that $xe_\alpha = ye_\alpha$, that is, $x = y$ which gives that $xf = xg$. By Lemma 2.5, we can deduce that $x^*f = x^*g$ with $x^* \in \Lambda_\alpha$. Since Λ_α is a right zero band, it then follows that $f = g$. Thus, $(x, f) = (y, g)$. This shows that φ is injective.

To see that φ is surjective, we just take any $a \in S_\alpha$. Clearly, $\varphi(ae_\alpha, a^*) = ae_\alpha a^* = aa^* = a$. This shows that φ is surjective. Hence $S_\alpha \simeq M_\alpha \times \Lambda_\alpha$.

Since (S, U) is a a left U-*rpp* semigroup, it is clear that $efg = feg$ for all $e, f, g \in U$. This shows that U is a right normal band.

(ii) $\Longrightarrow$ (iii) Suppose that (S, U) is a semilattice of $S_\alpha = M_\alpha \times \Lambda_\alpha$ such that M_α is a U-left cancellative monoid and Λ_α is a right zero band. Let $U = \bigcup_{\alpha \in Y}\{(1_\alpha, i) : 1_\alpha$ is the identity of M_α, $i \in \Lambda_\alpha\}$. Take $e_\beta = (1_\beta, j)$ such that $\alpha \geqslant \beta$. Then for each a in S_α the product $e_\beta a$ is in $S_\beta = M_\beta \times \Lambda_\beta$ and so write $e_\beta a = (x, i)$ for some $x \in M_\beta$ and some $i \in \Lambda_\beta$. Define $\varphi_{\alpha,\beta} : S_\alpha \longrightarrow S_\beta$ by $a\varphi_{\alpha,\beta} = e_\beta a$. It is clear that $\varphi_{\alpha,\alpha}$ is the identity mapping on S_α. Let $g = (1_\beta, i) \in M_\beta \times \Lambda_\beta$, where 1_β is the identity of M_β. Then we have

$$e_\beta ag = (x, i)(1_\beta, i) = (x, i) = e_\beta a,$$

and

$$g = (1_\beta, i) = (1_\beta, j)(1_\beta, i) = e_\beta g.$$

Similarly, let $b = (y, l) \in M_\alpha \times \Lambda_\alpha$ and $h = (1_\alpha, l)$. Then $hb = b$. Using the right normality of U, we obtain that

$$e_\beta ae_\beta b = e_\beta age_\beta hb = e_\beta ae_\beta ghb = e_\beta aghb = e_\beta ab,$$

which implies that

$$a\varphi_{\alpha,\beta}b\varphi_{\alpha,\beta} = (ab)\varphi_{\alpha,\beta}.$$

Hence, $\varphi_{\alpha,\beta}$ is a homomorphism.

Next, suppose that $a = (x, i) \in S_\alpha$, $h = (1_\alpha, i) \in S_\alpha$ and $\alpha \geqslant \beta \geqslant \gamma$. Then we have $ha = a$. Because U is a right normal band, it follows that

$$e_\gamma e_\beta h = e_\beta e_\gamma h = e_\beta e_\gamma \cdot e_\gamma h = e_\gamma h.$$

Thus, $a\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = e_\gamma(e_\beta a) = e_\gamma e_\beta ha = e_\gamma ha = e_\gamma a = a\varphi_{\alpha,\gamma}$.
This shows that $\varphi_{\alpha,\beta}\varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$.

Finally, noticing that, for any a in S_α and b in S_β, we deduce that $ab = e_{\alpha\beta}(ab) \in S_{\alpha\beta}$. Clearly, $e_{\alpha\beta}a \in S_{\alpha\beta}$. Then there exists $f^2 = f \in U \cap S_{\alpha\beta}$ such that $e_{\alpha\beta}af = e_{\alpha\beta}a$. Similarly, there exists $e_\beta^2 = e_\beta \in U \cap S_\beta$ such that $e_\beta b = b$ for any $b \in S_\beta$. By the right normality of U again, we have

$$e_{\alpha\beta}ae_{\alpha\beta}b = e_{\alpha\beta}afe_{\alpha\beta}e_\beta b = e_{\alpha\beta}ae_{\alpha\beta}fe_\beta b = e_{\alpha\beta}afe_\beta b = e_{\alpha\beta}ab.$$

This shows that $ab = a\varphi_{\alpha,\alpha\beta}b\varphi_{\beta,\alpha\beta}$. Hence S is indeed a strong semilattice of the semigroups $S_\alpha = M_\alpha \times \Lambda_\alpha$, denoted by $(S,U) = [Y; S_\alpha; \varphi_{\alpha,\beta}]$.

(iii) $\Longrightarrow$ (i) Let $(S,U) = [Y; S_\alpha; \varphi_{\alpha,\beta}]$ be a strong semilattice of semigroup $S_\alpha = M_\alpha \times \Lambda_\alpha$ and the set of projections $U = \bigcup_{\alpha\in Y}\{(1_\alpha, i)|\ 1_\alpha \text{ is the identity of } M_\alpha, i \in \Lambda_\alpha\}$. Let $e \in S_\alpha \cap U,\ f \in S_\beta \cap U,\ e\varphi_{\alpha,\alpha\beta} = (1_{\alpha\beta}, i),\ f\varphi_{\beta,\alpha\beta} = (1_{\alpha\beta}, j)$. Then $ef = (e\varphi_{\alpha,\alpha\beta})(f\varphi_{\beta,\alpha\beta}) = (1_{\alpha\beta}, i)(1_{\alpha\beta}, j) = (1_{\alpha\beta}, j) \in U$. It follows that U is a band.

Now we claim that for any $a, b \in S^1, b \neq 1$ and $e \in U, aeb = eab$ hold. Suppose that $a, b \in S^1$,$b \neq 1$ and $e \in U$. Then there exist $\alpha,\ \beta,\ \gamma \in Y$ such that $a \in S^1_\alpha,\ b \in S^1_\beta$ and $e \in U \cap S_\gamma$. Write $\delta = \alpha\beta\gamma,\ a\varphi_{\alpha,\delta} = (x, i),\ b\varphi_{\beta,\delta} = (y, j)$ and $e\varphi_{\gamma,\delta} = (1_\delta, k)$, we have

$$aeb = (a\varphi_{\alpha,\delta})(e\varphi_{\gamma,\delta})(b\varphi_{\beta,\delta}) = (x, i)(1_\delta, k)(y, j) = (xy, j).$$

Similarly, $eab = (xy, j)$. Thus, $eab = aeb$.

To see that each $\widetilde{\mathcal{L}}^U$-class of (S,U) contains at least one projection, we first let $a = (x, i) \in S_\alpha$ and $e = (1_\alpha, i) \in S_\alpha \cap U$. Clearly, $ea = ae = (x, i) = a$. For any $f \in S_\beta \cap U$, let $f\varphi_{\beta,\alpha\beta} = (1_{\alpha\beta}, j)$. Suppose that $af = a$. Then we have $\alpha\beta = \alpha$ and

$$af = (a\varphi_{\alpha,\alpha\beta})(f\varphi_{\beta,\alpha\beta}) = (x, i)(1_{\alpha\beta}, j) = (x, j) = a = (x, i),$$

Hence $i = j$ and $ef = (e\varphi_{\alpha,\alpha\beta})(f\varphi_{\beta,\alpha\beta}) = (1_\alpha, i)(1_{\alpha\beta}, j) = (1_\alpha, j) = (1_\alpha, i) = e$. It follows that $(a, e) \in \widetilde{\mathcal{L}}^U$.

Suppose that $(a, e) \in \widetilde{\mathcal{L}}^U, (a, f) \in \widetilde{\mathcal{L}}^U, e \in S_\alpha \cap U, f \in S_\beta \cap U$, then we have $(e, f) \in \widetilde{\mathcal{L}}^U$. Hence $e = ef = efe = fee = fe = f$. It follows that every $\widetilde{\mathcal{L}}^U$-class of (S,U) contains a unique projection. We use a^* to denote the unique projection of $\widetilde{L}_a \cap U$. It is easy to observe that $aa^* = a$.

Let $a \in S_\alpha,\ b \in S_\beta,\ c \in S_\gamma,\ e \in S_\alpha \cap U$. Write $\delta = \alpha\beta\gamma,\ a\varphi_{\alpha,\delta} = (x, i),\ b\varphi_{\beta,\delta} = (y, j),\ c\varphi_{\gamma,\delta} = (z, k),\ b^*\varphi_{\beta,\delta} = (m, n)$ and $e\varphi_{\gamma,\delta} = (1_\delta, t)$. Suppose that $(a, b) \in \widetilde{\mathcal{L}}^U$. Then $ae = a$ if and only if $be = b$. Suppose that $ace = ac$, that is, $aa^*ce = aa^*c$. Using the fact that $a^* = b^*$, we deduce that $ab^*ce = ab^*c$. Hence $(a\varphi_{\alpha,\delta})(b^*\varphi_{\beta,\delta})(c\varphi_{\gamma,\delta})(e\varphi_{\alpha,\delta}) = (a\varphi_{\alpha,\delta})(b^*\varphi_{\beta,\delta})(c\varphi_{\gamma,\delta})$, that is, $(x, i)(m, n)(z, k)(1_\delta, t) = (x, i)(m, n)(z, k)$ and $(xmz1_\delta, t) = (xmz, k)$. Then we have $mz1_\delta = mz$ by M_α is a U-left cancellative monoid and $t = k$. Hence $bce = bb^*ce = (b\varphi_{\beta,\delta})(b^*\varphi_{\beta,\delta})(c\varphi_{\gamma,\delta})(e\varphi_{\alpha,\delta}) = (ymz1_\delta, t) = (ymz, k) = (b\varphi_{\beta,\delta})(b^*\varphi_{\beta,\delta})(c\varphi_{\gamma,\delta}) = bb^*c = bc$.

Similarly, $bce = bc$ implies that $ace = ac$. Thus, we have already proved that $(ac, bc) \in \widetilde{\mathcal{L}}^U$. This shows that $\widetilde{\mathcal{L}}^U$ is a right congruence on S. In fact we have proved that (S,U) is a left U-*rpp* semigroup.

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On the Smarandache $5n$-digital sequence

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Abstract For any positive integer n, the Smarandache $5n$-digital sequence is defined as $\{a_n\} = \{15, 210, 315, 420, 525, 630, 735, 840, 945, 1050, \cdots\}$. That is, for any element a_n in $\{a_n\}$, it can be partitioned into two groups such that the second is five times bigger than the first. The main purpose of this paper is using the elementary method to study the properties of the Smarandache $5n$-digital sequence, and obtained some usefull conclusions.

Keywords The Smarandache $5n$-digital sequence, elementary method, conjecture infinite series, convergence.

§1. Introduction and results

For any positive integer n, the Smarandache $5n$-digital sequence is defined as $\{a_n\} = \{15, 210, 315, 420, 525, 630, 735, 840, 945, 1050, \cdots\}$. That is, for any element a_n in $\{a_n\}$, it can be partitioned into two parts such that the second is five times bigger than the first. This sequence was first proposed by professor F. Smarandache, he also asked us to study the properties of $5n$-digital sequence. About this problem, it seems that none had studied it yet, at least we have not seen any related papers before. Recently, Professor Zhang Wenpeng proposed the following:

Conjecture. There does not exist any complete square number in the Smarandache $5n$-digital sequence $\{a_n\}$. That is, the equation $a_n = m^2$ has no positive integer solution.

I think that this conjecture is interesting, because if it is true, then we shall obtain a deeply properties of the Smarandache $5n$-digital sequence. In this paper, we are using the elementary method to prove that the Zhang's conjecture is correct for some special positive integers. At the same time, we also study the convergent properties of one kind infinite series involving the Smarandache $5n$-digital sequence, and give a sharper asymptotic formula for $\sum_{n\leq N} \frac{n}{a_n}$. That is, we shall prove the following conclusions:

Theorem 1. If positive integer n is a square-free number (That is, for any prime p, if $p \mid n$, then $p^2 \nmid n$), then a_n is not a complete square number.

Theorem 2. If positive integer n is a complete square number, then a_n is not a complete square number.

Theorem 3. Let z be a real number. If $z > \frac{1}{2}$, then the infinite series

$$f(z) = \sum_{n=1}^{+\infty} \frac{1}{a_n^z} \tag{1}$$

is convergent; If $z \le \frac{1}{2}$, then the infinite series (1) is divergent.

Theorem 4. For any real number $N > 1$, we have the asymptotic formula

$$\sum_{n \le N} \frac{n}{a_n} = \frac{9}{50} \cdot \frac{\ln N}{\ln 10} + O(1).$$

§2. Proof of the theorems

In this section, we shall use the elementary method to complete the proof of our theorems. First we prove Theorem 1. For any square-free number n, let $5n = b_{k(n)} b_{k(n)-1} \cdots b_2 b_1$, where $1 \le b_{k(n)} \le 9$, $0 \le b_i \le 9$, $i = 1, 2, \cdots, k(n) - 1$. Then from the definition of a_n we know that $a_n = n \cdot (10^{k(n)} + 5)$. If n is a square-free number, and there exists a positive integer m such that

$$a_n = n \cdot (10^{k(n)} + 5) = m^2. \tag{2}$$

Then from (2) and the definition of square-free number we know that $n \mid m$. Let $m = u \cdot n$, then (2) become

$$10^{k(n)} + 5 = u^2 \cdot n. \tag{3}$$

In formula (3), we know that:

(a). If $u = 1$, then (3) is impossible. Since $10^{k(n)} + 5 > \underbrace{99 \cdots \cdots 9}_{k(n)} \ge b_{k(n)} b_{k(n)-1} \cdots b_2 b_1 = 5n > n$.

(b). If $u = 2$, then (3) does not hold. In fact, if (2) holds, then $10^{k(n)} + 5 = 4 \cdot n$, since $10^{k(n)} + 5$ is an odd number, but $4 \cdot n$ is an even number, this contradicts with $10^{k(n)} + 5 = 4 \cdot n$.

(c). If $u = 3$, then (3) is impossible. In this case, we have the congruence $10^{k(n)} + 5 \equiv 6(\mod 9)$, but $u^2 \cdot n = 3^2 \cdot n \equiv 0(\mod 9)$, so (3) is not possible.

(d). If $u = 4$, then $10^{k(n)} + 5$ is an odd number, but $u^2 \cdot n = 4^2 \cdot n$ is an even number, so (3) does not hold.

(e). If $u = 5$, then we have the congruence $10^{k(n)} + 5 \equiv 5(\mod 25)$, but $u^2 \cdot n = 5^2 \cdot n \equiv 0(\mod 25)$, so (3) is impossible.

(f). If $u = 6$, then $10^{k(n)} + 5$ is an odd number, and $u^2 \cdot n = 6^2 \cdot n$ is an even number, so (3) is not correct.

(g). If $u = 7$, then we have:

(i) If $3 \dagger$n, then we have $10^{k(n)} + 5 \equiv 0(\mod 3)$, but $u^2 \cdot n = 7^2 \cdot n \equiv 0(\mod 3)$ doesn't hold, so (3) is not correct.

(ii) If $3|n$, let's $n = 3a$, where a is an even integer. It's clear that (3) doesn't hold, since $10^{k(n)} + 5$ is an odd number, while $u^2 \cdot n = 7^2 \cdot n = 49 \cdot n = 49 \cdot 3a$ is an even number.

(iii) If $3|n$, let's $n = 3a$, where a is an odd integer. From the definition of square-free number, we know that $3^2 \dagger n$ and $(3, a) = 1$, then (3) become $u^2 \cdot n = 7^2 \cdot n = 7^2 \cdot 3a = 10^{k(n)} + 5 = 9 \cdot \underbrace{111 \cdots 1}_{k(n)} + 6$, that is $7^2 \cdot a - 2 = 3 \cdot \underbrace{111 \cdots 1}_{k(n)}$.

If $a = 3b + 1$, then $7^2 \cdot a - 2 = 7^2 \cdot (3b + 1) - 2 = 7^2 \cdot 3b + 47 \ne 3 \cdot \underbrace{111 \cdots 1}_{k(n)}$.

If $a = 3b + 2$, then $7^2 \cdot a - 2 = 7^2 \cdot (3b + 2) - 2 = 7^2 \cdot 3b + 96 = 3 \cdot (7^2 b + 32) \neq 3 \cdot \underbrace{111 \cdots 1}_{k(n)}$,

so formula (3) is impossible.

(h). If $u \geq 8$, then note that $5n = b_{k(n)} b_{k(n)-1} \cdots b_2 b_1 \geq 10^{k(n)-1}$, we have the inequality

$$u^2 \cdot n \geq 8^2 \cdot n = 64n = 10 \cdot 6n + 4n \geq 10 \cdot 6n + 5 > 10 \cdot 5n + 5 > 10^{k(n)} + 5,$$

so formula (3) does not hold.

From above discussion, we know that there does not exist any positive integer u such that formula (3) hold. This proves Theorem 1.

Now we prove Theorem 2. Let $n = u^2$ be a complete square number, if there exists a positive integer m such that

$$n \cdot (10^{k(n)} + 5) = u^2 \cdot (10^{k(n)} + 5) = m^2, \tag{4}$$

then from (4) we deduce that $u \mid m$, let $m = u \cdot r$, then formula (4) become

$$10^{k(n)} + 5 = r^2. \tag{5}$$

It is clear that (5) is not possible, since $10^{k(n)} + 5 = 5 \cdot (2 \cdot 10^{k(n)-1} + 1)$ and $5 \dagger 2 \cdot 10^{k(n)-1} + 1$, this contradicts with $10^{k(n)} + 5 = r^2$. This proves Theorem 2.

Now we prove Theorem 3. For any element a_n in $\{a_n\}$, let $5n = b_{k(n)} b_{k(n)-1} \cdots b_2 b_1$, where $1 \leq b_{k(n)} \leq 9$, $0 \leq b_i \leq 9$, $i = 1,\ 2,\ \cdots,\ k(n) - 1$. Then from the definition of the Smarandache $5n$-digital sequence we have:

$$a_n = n \cdot 10^{k(n)} + 5 \cdot n = n \cdot \left(10^{k(n)} + 5\right). \tag{6}$$

On the other hand, note that for any positive integer n, if

$$\underbrace{200 \cdots 00}_{u} \leq n \leq \underbrace{199 \cdots 99}_{u+1},$$

then $5n = b_{u+1} b_u \cdots b_2 b_1$, where $1 \leq b_{u+1} \leq 9$, $0 \leq b_i \leq 9$, $i = 1,\ 2,\ \cdots,\ u$, so $k(n) = u + 1$.

Therefore we have:

$$\begin{aligned}
f(z) &= \sum_{n=1}^{+\infty} \frac{1}{a_n^z} = \sum_{n=1}^{+\infty} \frac{1}{n^z \cdot \left(10^{k(n)}+5\right)} \\
&= \sum_{i=1} \frac{1}{1^z \cdot (10+5)^z} + \sum_{2\le i\le 19} \frac{1}{i^z \cdot (10^2+5)^z} \\
&\quad + \sum_{20\le i\le 199} \frac{1}{i^z \cdot (10^3+5)^z} + \sum_{200\le i\le 1999} \frac{1}{i^z \cdot (10^4+5)^z} + \cdots \\
&\le \sum_{k=1}^{+\infty} \frac{18\cdot 10^{k-2}}{10^{z\cdot(k-2)} \cdot 10^{zk}} \\
&= 18\cdot \sum_{k=0}^{+\infty} \frac{10^{k-1}}{10^{z\cdot(k-1)} \cdot 10^{z(k+1)}} \\
&\le 18\cdot \sum_{k=0}^{+\infty} \frac{10^{k}}{10^{z\cdot(k-1)} \cdot 10^{z(k+1)}} \\
&= 18\cdot \sum_{k=0}^{+\infty} \frac{1}{10^{k\cdot(2z-1)}}. \qquad (7)
\end{aligned}$$

Now if $z > \frac{1}{2}$, then from (7) and the properties of the geometric progression we know that $f(z)$ is convergent. If $z \le \frac{1}{2}$, then from (7) we also have:

$$\begin{aligned}
f(z) &= \sum_{n=1}^{+\infty} \frac{1}{a_n^z} = \sum_{n=1}^{+\infty} \frac{1}{n^z \cdot \left(10^{k(n)}+5\right)} \\
&= \sum_{i=1} \frac{1}{1^z \cdot (10+5)^z} + \sum_{2\le i\le 19} \frac{1}{i^z \cdot (10^2+5)^z} \\
&\quad + \sum_{20\le i\le 199} \frac{1}{i^z \cdot (10^3+5)^z} + \sum_{200\le i\le 1999} \frac{1}{i^z \cdot (10^4+5)^z} + \cdots \\
&\ge \sum_{k=1}^{+\infty} \frac{18\cdot 10^{k-2}}{10^{z\cdot(k-1)} \cdot 10^{z(k+1)}} \\
&= 18\cdot \sum_{k=0}^{+\infty} \frac{10^{k-1}}{10^{zk} \cdot 10^{z(k+2)}} \\
&= 18\cdot \sum_{k=0}^{+\infty} \frac{1}{10^{2zk+2z-k+1}}. \qquad (8)
\end{aligned}$$

Then from the properties of the geometric progression and (8) we know that the series $f(z)$ is divergent if $z \le \frac{1}{2}$. This proves Theorem 3.

Now we prove Theorem 4. For any positive integer N, there exist a positive integer M such that

$$\underbrace{200\cdots 00}_{M} \le N \le \underbrace{199\cdots 99}_{M+1}.$$

Note that for any positive integer n, if

$$\underbrace{200\cdots 00}_{u} \le n \le \underbrace{199\cdots 99}_{u+1},$$

then $5n = b_{u+1}b_u\cdots b_2b_1$, where $1 \le b_{u+1} \le 9$, $0 \le b_i \le 9, i = 1,\ 2,\ \cdots,\ u$, so $k(n) = u+1$. Therefore we have

$$\begin{aligned}
\sum_{n\le N}\frac{n}{a_n} &= \sum_{n\le N}\frac{1}{10^{k(n)}+5}\\
&= \frac{1}{10+5} + \sum_{2\le n\le 19}\frac{1}{10^2+5} + \sum_{20\le n\le 199}\frac{1}{10^3+5} + \cdots\\
&\quad + \sum_{\underbrace{20\cdots 00}_{M-1}\le n\le \underbrace{19\cdots 99}_{M}}\frac{1}{10^M+5} + \sum_{\underbrace{20\cdots 00}_{M}\le n\le N}\frac{1}{10^{M+1}+5}\\
&= \frac{1}{10+5} + \frac{18}{10^2+5} + \frac{180}{10^3+5} + \cdots + \frac{18\cdot 10^{M-2}}{10^M+5} + \frac{N-\frac{10^M}{5}+1}{10^{M+1}+5}\\
&= \frac{9}{50}\left(\frac{10+5-5}{10+5} + \frac{10^2+5-5}{10^2+5} + \frac{10^3+5-5}{10^3+5} + \cdots + \frac{10^M+5-5}{10^M+5}\right)\\
&\quad + \frac{N-\frac{10^M}{5}+1}{10^{M+1}+5} - \frac{4}{75}\\
&= \frac{9}{50}\left[M - \left(\frac{5}{10+5} + \frac{5}{10^2+5} + \frac{5}{10^3+5} + \cdots + \frac{5}{10^M+5}\right)\right]\\
&\quad + \frac{N-\frac{10^M}{5}+1}{10^{M+1}+5} - \frac{4}{75}\\
&= \frac{9}{50}\cdot M - \frac{9}{10}\cdot\sum_{i=1}^{M}\frac{1}{10^i+5} + \frac{N-\frac{10^M}{5}+1}{10^{M+1}+5} - \frac{4}{75}.
\end{aligned}$$

Considering M, by the inequality

$$\underbrace{200\cdots 00}_{M} \le N \le \underbrace{199\cdots 99}_{M+1},$$

we have

$$10^M < 5N \le 10^{M+1} - 5,$$

$$M\ln 10 \le \ln 5N \le (M+1)\ln 10 + \ln\left(1 - \frac{5}{10^{M+1}}\right),$$

$$\frac{\ln 5N}{\ln 10} - \frac{\ln(1-\frac{5}{10^{M+1}})}{\ln 10} - 1 \le M \le \frac{\ln 5N}{\ln 10}.$$

Note that as $N \to +\infty$, $\ln(1 - \dfrac{5}{10^{M+1}}) = O\left(\dfrac{1}{10^M}\right)$, then

$$\frac{\ln 5N}{\ln 10} - 1 - O\left(\frac{1}{10^M}\right) \le M \le \frac{\ln 5N}{\ln 10}.$$

Combining this we may immediately deduce the congruence

$$\sum_{n\le N}\frac{n}{a_n}=\frac{9}{50}\cdot\frac{\ln N}{\ln 10}+O\left(1\right).$$

This completes the proof of Theorem 4.

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Amenable partial orders on a locally inverse semigroup [1]

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Abstract Suppose that S is a locally inverse semigroup with an inverse transversal S°. We can construct an amenable partial order on S by a McAlister cone of S°. Conversely, every amenable partial order on S can be constructed in this way. We show that the amenable partial order constructed by the set $E(S^\circ)$ of all idempotents of S° is the natural partial order on S.

Keywords Locally inverse semigroup, amenable partial order, inverse transversal.

§1. Introduction and preliminary

A semigroup S is said to be a partially ordered semigroup, or to be partially ordered, if it admits a compatible ordering $\leq$; That is, $\leq$ is a partial order on S such that

$$(\forall\, a,\, b \in S,\, x \in S^1)\ a \leq b \Longrightarrow xa \leq xb \text{ and } ax \leq bx.$$

Let S be a regular semigroup with set $E(S)$ of idempotent elements. As usual, $\preceq$ denotes the natural partial order on S. That is, for any $a,\, b \in S$,

$$a \preceq b \quad \text{if and only if} \quad a = eb = bf \quad \text{for some} \quad e,\, f \in E(S).$$

By Corollary II. 4.2 in [1], the natural partial order $\preceq$ on S is compatible with the multiplication if and only if S is a locally inverse semigroup. Thus, a locally inverse semigroup equipped with the natural partial order is a partially ordered semigroup. Particularly, an inverse semigroup is a partially ordered semigroup under the natural partial order.

McAlister introduced and studied amenable partially ordered inverse semigroup in [3].

Definition 1.1.[3] Let $(S,\, \cdot,\, \leq)$ be a partially ordered inverse semigroup. The partial order $\leq$ is said to be a left(right) amenable partial order if it coincides with $\preceq$ on idempotents and for each $a,\, b \in S$, $a \leq b$ implies $a^{-1}a \preceq b^{-1}b\, (aa^{-1} \preceq bb^{-1})$. If $\leq$ is both a left amenable partial order and a right amenable partial order on S, then $\leq$ is called an amenable partial order and S is called an amenable partially ordered inverse semigroup.

Blyth and Almeida Santos generalized (left) amenable partial orders on inverse semigroup to regular semigroup with an inverse transversal in [4]. Let S be a regular semigroup, for any

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$a \in S$, $V(a)$ denotes the all inverses of a. An *inverse transversal* of a regular semigroup S is an inverse subsemigroup S° with the property that $|S^\circ \bigcap V(a)| = 1$ for every a in S. The unique inverse of a in $S^\circ \bigcap V(a)$ is written as a° and $(a^\circ)^\circ$ as $a^{\circ\circ}$. The set of idempotents in S° is denoted by $E(S^\circ)$. We recall the following definition.

Definition 1.2.[4] Let $(S, \cdot, \leq)$ be a partially ordered regular semigroup with an inverse transversal S°. If $\leq$ coincides with $\preceq$ on idempotents and the partial order $\leq$ has the following property

$$(\forall a, b \in S)\ \ a \leq b \Longrightarrow a^\circ a \preceq b^\circ b,$$

then $\leq$ is said to be a *left amenable partial order* on S. Dually, if $a \leq b$ implies $aa^\circ \preceq bb^\circ$, then $\leq$ is called a *right amenable partial order* on S. If $\leq$ is both a left amenable partial order and a right amenable partial order on S, then $\leq$ is called an *amenable partial order* and S is called an amenable partially ordered regular semigroup with inverse transversal S°.

Suppose that S is a locally inverse semigroup with an inverse transversal S°. Blyth and Almeida Santos gave a complete description of all amenable partial orders on S and showed the natural partial order on S is the smallest amenable partial order in [5]. They also proved that every amenable partial orders on S° extends to a unique amenable partial order on S. In this paper, we will give a new characterization of the amenable partial orders on S. We can construct an amenable partial order on S by a McAlister cone of S°. Conversely, every amenable partial order on S can be constructed in this way, which simplify Blyth and Almeida Santos's work in [5]. It is easily seen that the set $E(S^\circ)$ of all idempotent elements of S° is the smallest McAlister cone of S°. We will show that the amenable partial order constructed by $E(S^\circ)$ is equal to the natural partial order on S and so the natural partial order on S is the smallest amenable partial order.

§2. Constructing amenable partial orders

Suppose that $(S, \cdot)$ is a regular semigroup with an inverse transversal S°. For any $a, b \in S$, Blyth and Almeida Santos say in [5] that S satisfies the following formulars

$$(ab)^\circ = (a^\circ ab)a^\circ = b^\circ(ab^\circ b)^\circ = b^\circ(a^\circ abb^\circ)^\circ,\ (a^\circ b)^\circ = b^\circ a^{\circ\circ},\ (ab^\circ)^\circ = b^{\circ\circ}a^\circ. \tag{1}$$

According to Blyth and Almeida Santos in [5], if S is locally inverse, then

$$(\forall a, b, c \in S) \quad a^\circ bc^\circ = a^\circ b^{\circ\circ}c^\circ. \tag{2}$$

Suppose that S is a regular semigroup with an inverse transversal S°. Blyth and Almeida Santos stated in [4] and [5] that the two subsets of $E(S)$

$$\Lambda = \{x^\circ x |\, x \in S\} \text{ and } I = \{xx^\circ |\, x \in S\}$$

are respectively right regular subband and left regular subband of $E(S)$. Hence, we immediately have the following lemma.

Lemma 2.1. Let S be a locally inverse semigroup with an inverse transversal S°. Then Λ is a right normal subband of $E(S)$ and I is a left normal subband of $E(S)$.

Let S be a locally inverse semigroup with an inverse transversal S°. The two subsets of S are

$$L = \{xx^\circ x^{\circ\circ} | x \in S\},\ R = \{x^{\circ\circ}x^\circ x | x \in S\},$$

Blyth and Almeida Santos in [5] established the following fundamental statements:

(α) L is a left normal orthodox subsemigroup of S and $I = \{xx^\circ |\, x \in S\}$ is the set of all idempotents of L;

(β) R is a right normal orthodox subsemigroup of S and $\Lambda = \{x^\circ x | x \in S\}$ is the set of all idempotents of R;

(γ) $L \bigcap R = S^\circ$, $\Lambda \bigcap I = E(S^\circ)$.

Consider the following two sets

$$\Lambda^* = \{x \in S | (\forall l \in \Lambda)\, lxl = xl\},\ I^* = \{x \in S | (\forall r \in I)\, rxr = rx\},$$

Blyth and Almeida Santos proved that Λ^* is a subsemigrup of S containing Λ and I^* is a subsemigroup of S containing I (see Theorem 3 in [5]), and after that they introduced the concepts of R-cone and L-cone of S, which generalized the notion of McAlister cone in an inverse semigroup. If a full subsemigroup Q of Λ^* with the properties that $Q \bigcap Q^\circ = E(S^\circ)$ and $xQx^\circ \subseteq Q$ for all $x \in R$, then Q is called a R-cone of S. Dually, they considered the subsemigroup of I^* and gave the notion of L-cone. They proved in [5] that an amenable partial order on S can be constructed by a R-cone P and a L-cone Q, conversely, every amenable partial order on S can be obtained in this way (see Theorems 8 and 10 in [5]). If P is a R-cone and Q is a L-cone, they also proved that $P \bigcap Q$ is a McAlister cone of S° (see Theorem 17 in [5]). We will recall the concept of McAlister cone of S° in the following.

Suppose that S is a locally inverse semigroup with an inverse transversal S°. Consider the set

$$E(S^\circ)\zeta = \{x \in S | (\forall e \in E(S^\circ))\ ex = xe\},$$

which is the centralizer of $E(S^\circ)$ in S. Blyth and Almeida Santos showed that $E(S^\circ)\zeta$ is a subsemigroup of S° (see Theorem 16 in [5]). Now, we have

Definition 2.2.[5] Suppose that S is a locally inverse semigroup with an inverse transversal S°. A subset Q of S° is said to be a McAlister cone of S° if

(i) Q is a subsemigroup of $E(S^\circ)\zeta$;

(ii) $Q \bigcap Q^\circ = E(S^\circ)$ $(Q^\circ = \{a^\circ |\ a \in Q\})$;

(iii) $(\forall x \in S)\ x^\circ Q x^{\circ\circ} \subseteq Q$.

If S is a locally inverse semigroup with an inverse transversal S°, then it is easy to see that $E(S^\circ)$ is a McAlister cone of S°. The following result will show that an amenable partial order on S also can be constructed by a McAlister cone of S°.

Theorem 2.3. Suppose that S is a locally inverse semigroup with an inverse transversal S°. Let C be a McAlister cone of S°. Then the relation $\leq_C$ defined on S by

$$x \leq_C y \Longleftrightarrow xx^\circ \preceq yy^\circ,\ x^\circ x \preceq y^\circ y,\ x^\circ y^{\circ\circ},\ y^{\circ\circ}x^\circ \in C \qquad (*)$$

is an amenable partial order on S.

Proof. It is easily seen that $\leq_C$ is reflexive. If $x \leq_C y$ and $y \leq_C x$, then $xx^\circ = yy^\circ$, $x^\circ x = y^\circ y$, $x^\circ y^{\circ\circ}$, $y^\circ x^{\circ\circ} \in C$. Thus $y^\circ x^{\circ\circ} = (x^\circ y^{\circ\circ})^\circ \in C \bigcap C^\circ = E(S^\circ)$ since C is a McAlister cone. It follows from $xx^\circ = yy^\circ$ and (1) that $x^{\circ\circ}x^\circ = (xx^\circ)^\circ = (yy^\circ)^\circ = y^{\circ\circ}y^\circ$ and so $y^\circ = y^\circ y^{\circ\circ} y^\circ = y^\circ x^{\circ\circ} x^\circ$, which gives $y^\circ \preceq x^\circ$. Likewise, $x^\circ \preceq y^\circ$ and so $x^\circ = y^\circ$, furthermore, we have $x^{\circ\circ} = y^{\circ\circ}$. Hence, $x = xx^\circ \cdot x^{\circ\circ} \cdot x^\circ x = yy^\circ \cdot y^{\circ\circ} \cdot y^\circ y = y$, thus $\leq_C$ is anti-symmetric. If $x \leq_C y$ and $y \leq_C z$, then $x^\circ x \preceq y^\circ y \preceq z^\circ z$, $xx^\circ \preceq yy^\circ \preceq zz^\circ$ and $x^\circ y^{\circ\circ}$, $y^\circ z^{\circ\circ} \in C$. We obtain from $xx^\circ \preceq yy^\circ$ that $xx^\circ yy^\circ = xx^\circ$. It follows from Definition 2.2 that $x^\circ y^{\circ\circ} y^\circ z^{\circ\circ} \in C$. We thus have

$$\begin{aligned} x^\circ y^{\circ\circ} y^\circ z^{\circ\circ} &= x^\circ x x^\circ y^{\circ\circ} y^\circ z^{\circ\circ} \\ &= x^\circ (xx^\circ yy^\circ) z^{\circ\circ} \qquad \text{(by (2))} \\ &= x^\circ z^{\circ\circ}. \end{aligned}$$

Consequently $x^\circ z^{\circ\circ} \in C$, similarly, we have $z^{\circ\circ} x^\circ \in C$ and so $x \leq_C z$. Thereby, $\leq_C$ is transitive and $\leq_C$ is a partial order on S.

Suppose that $x \leq_C y$. For any $z \in S$, we have

$$\begin{aligned} (zx)^\circ (zy)^{\circ\circ} &= x^\circ (zxx^\circ)^\circ (zy)^{\circ\circ} && \text{(by, (1))} \\ &= x^\circ (zyy^\circ xx^\circ)^\circ (zy)^{\circ\circ} \\ &= x^\circ x^{\circ\circ} x^\circ y^{\circ\circ} (zy)^\circ (zy)^{\circ\circ} \\ &= x^\circ y^{\circ\circ} (zy)^\circ (zy)^{\circ\circ} \\ &\in CE(S^\circ) \\ &\subseteq C && (E(S^\circ) \subseteq C) \end{aligned}$$

and

$$\begin{aligned} (zy)^{\circ\circ} (zx)^\circ &= (zy)^{\circ\circ} x^\circ (zxx^\circ)^\circ && \text{(by, (1))} \\ &= (zyy^\circ y)^{\circ\circ} x^\circ (zxx^\circ yy^\circ)^\circ \\ &= (zyy^\circ)^{\circ\circ} y^{\circ\circ} x^\circ (zyy^\circ xx^\circ)^\circ \\ &= (zyy^\circ)^{\circ\circ} y^{\circ\circ} x^\circ (zyy^\circ x^{\circ\circ} x^\circ)^\circ && \text{(by (2))} \\ &= (zyy^\circ)^{\circ\circ} y^{\circ\circ} x^\circ x^{\circ\circ} x^\circ (zyy^\circ)^\circ && \text{(by (1))} \\ &= (zyy^\circ)^{\circ\circ} y^{\circ\circ} x^\circ (zyy^\circ)^\circ \\ &\in C, && (y^{\circ\circ} x^\circ \in C) \end{aligned}$$

i.e., $(zx)^\circ (zy)^{\circ\circ}$, $(zy)^{\circ\circ} (zx)^\circ \in C$. It follows by Theorem 8 in [5] that $zx(zx)^\circ \preceq zy(zy)^\circ$ and $(zx)^\circ zx \preceq (zy)^\circ zy$. Thus we have $zx \leq_C zy$, therefore $\leq_C$ is compatible on the left. Dually, we have that $\leq_C$ is also compatible on the right and so $(S, \cdot, \leq_C)$ is a partially ordered semigroup.

In the following, we will show that the partial order $\leq_C$ coincides with the natural partial order on $E(S)$. Suppose that $e, f \in E(S)$ and $e \leq_C f$. Then $e^\circ e \preceq f^\circ f$, $ee^\circ \preceq ff^\circ$. It follows from Theorem 2 in [5] that $e \preceq f$. Conversely, if $e \preceq f$, then $e^\circ e \preceq f^\circ f$, $ee^\circ \preceq ff^\circ$ and $e^\circ f \in \Lambda$, $fe^\circ \in I$. Furthermore, we have $(e^\circ f)^{\circ\circ} = f^{\circ\circ} e^\circ \in E(S^\circ) \subseteq C$, likewise, $e^\circ f^{\circ\circ} \in C$. Thus $e \leq_C f$, and consequently $\leq_C$ coincides with $\preceq$ on idempotents. This shows that $\leq_C$ is an amenable partial order on S.

Assume that S is a locally inverse semigroup with an inverse transversal S° and $\leq$ is a partial order on S. We denote by $\leq^{S^\circ}$ the restriction of $\leq$ on S°. Then the following lemma is clear.

Lemma 2.4. Suppose that S is a locally inverse semigroup with an inverse transversal S°. If $\leq$ is an amenable partial order on S, then $\leq^{S^\circ}$ is an amenable partial order on S°.

Lemma 2.5. Suppose that S is a locally inverse semigroup with an inverse transversal S°. If the partial order $\leq$ is an amenable partial order on S, then

$$(\forall a,\, b \in S)\ a \leq b \Longrightarrow a^{\circ\circ} \leq^{S^\circ} b^{\circ\circ}.$$

Proof. Suppose that $a \leq b$, then $aa^\circ \preceq bb^\circ$ and $a^\circ a \preceq b^\circ b$, From (1) and (2) we have $(aa^\circ bb^\circ)^\circ = (aa^\circ b^{\circ\circ} b^\circ)^\circ = b^{\circ\circ} b^\circ a^{\circ\circ} a^\circ = (aa^\circ)^\circ = a^{\circ\circ} a^\circ$. This shows that $a^{\circ\circ} a^\circ \preceq b^{\circ\circ} b^\circ$. Likewise, $a^\circ a^{\circ\circ} \preceq b^\circ b^{\circ\circ}$. Since $\leq$ is an amenable partial order, $a^{\circ\circ} a^\circ \leq b^{\circ\circ} b^\circ$ and $a^\circ a^{\circ\circ} \leq b^\circ b^{\circ\circ}$, and consequently $a^{\circ\circ} = a^{\circ\circ} a^\circ a a^\circ a^{\circ\circ} \leq b^{\circ\circ} b^\circ b b^\circ b^{\circ\circ} = b^{\circ\circ}$. It follows from $a^{\circ\circ},\, b^{\circ\circ} \in S^\circ$ that $a^{\circ\circ} \leq^{S^\circ} b^{\circ\circ}$, as required.

Proposition 2.6. Suppose that S is a locally inverse semigroup with an inverse transversal S°. If the partial order $\leq$ is an amenable partial order on S, then there exists a McAlister cone C of S° such that $\leq_C = \leq$.

Proof. Assume that $\leq$ is an amenable partial order on S, we denote by $\leq^{S^\circ}$ the restriction of $\leq$ on S°. It follows by Lemma 2.5 that $\leq^{S^\circ}$ is an amenable partial order on S°. Let

$$C = \{x | x \in S^\circ,\, x^\circ x \leq^{S^\circ} x,\, xx^\circ \leq^{S^\circ} x\},$$

it easy to see that $E(S^\circ) \subseteq C$. By Lemma 2.1 (iii) in [3] and its dual, we have C is the subset of $E(S^\circ)\zeta$. Now let $x,\, y \in C$. Then

$$\begin{aligned}
(xy)^\circ xy &= y^\circ x^\circ xy && (x,\, y \in S^\circ)\\
&= y^\circ y y^\circ x^\circ xy &&\\
&\leq^{S^\circ} yy^\circ x^\circ xy &&\\
&= x^\circ x y y^\circ y && (E(S^\circ) \text{ is a semilattice})\\
&= x^\circ xy &&\\
&\leq^{S^\circ} xy. &&
\end{aligned}$$

Likewise, $xy(xy)^\circ \leq^{S^\circ} xy$ and so $xy \in C$. This shows that C is a subsemigroup of $E(S^\circ)\zeta$.

Suppose that $x,\, x^\circ \in C$. Then $x^\circ x \leq^{S^\circ} x$, $x^{\circ\circ} x^\circ \leq^{S^\circ} x^\circ$. From $x \in S^\circ$ we obtain $x^{\circ\circ} = x$, thus $xx^\circ \leq^{S^\circ} x^\circ$, post-multiplying this by x, we have $x \leq^{S^\circ} x^\circ x$ whence $x = x^\circ x \in E(S^\circ)$, hence, $C \bigcap C^\circ \subseteq E(S^\circ)$. On the other hand, it is clear that $E(S^\circ) \subseteq C \bigcap C^\circ$. Consequently $E(S^\circ) = C \bigcap C^\circ$.

For any $x \in S$, $a \in C$, we have

$$\begin{aligned}
(x^\circ a x^{\circ\circ})^\circ (x^\circ a x^{\circ\circ}) &= x^\circ a^\circ x^{\circ\circ} x^\circ a x^{\circ\circ} && (\text{by } (1))\\
&= x^\circ a^\circ x^{\circ\circ} x^\circ a x^{\circ\circ} x^\circ \cdot x^{\circ\circ} &&\\
&= x^\circ a^\circ a x^{\circ\circ} x^\circ \cdot x^{\circ\circ} && (a \in C \subseteq E(S^\circ)\zeta)\\
&= x^\circ a^\circ a x^{\circ\circ} &&\\
&\leq^{S^\circ} x^\circ a x^{\circ\circ}. && (a^\circ a \leq^{S^\circ} a)
\end{aligned}$$

Dually, we obtain $(x^\circ a x^{\circ\circ})(x^\circ a x^{\circ\circ})^\circ \leq^{S^\circ} x^\circ a x^{\circ\circ}$. Thus $x^\circ a x^{\circ\circ} \in C$ and so $x^\circ C x^{\circ\circ} \subseteq C$. It follows from Definition 2.2 that C is a McAlister cone of S°.

Consider the corresponding partial order $\leq_C$ given by

$$x \leq_C y \Longleftrightarrow xx^\circ \preceq yy^\circ,\ x^\circ x \preceq y^\circ y, x^\circ y^{\circ\circ},\ y^{\circ\circ}x^\circ \in C.$$

We can obtain from Theorem 2.3 that $\leq_C$ is an amenable partial order on S.

In the following, we will show that $\leq_C = \leq$.

Suppose that $x \leq_C y$. Then $xx^\circ \preceq yy^\circ$ and $x^\circ y^{\circ\circ} \in C$, hence,

$$\begin{aligned}
x^{\circ\circ}x^\circ &= (xx^\circ)^{\circ\circ} && \text{(by (1))}\\
&= (xx^\circ yy^\circ)^{\circ\circ}\\
&= (xx^\circ y^{\circ\circ}y^\circ)^{\circ\circ} && \text{(by (2))}\\
&= x^{\circ\circ}x^\circ y^{\circ\circ}y^\circ && \text{(by (1))}\\
&= y^{\circ\circ}y^\circ x^{\circ\circ}x^\circ y^{\circ\circ}y^\circ\\
&= y^{\circ\circ}(x^\circ y^{\circ\circ})^\circ(x^\circ y^{\circ\circ})y^\circ\\
&\leq^{S^\circ} y^{\circ\circ}(x^\circ y^{\circ\circ})y^\circ\\
&= y^{\circ\circ}x^\circ y^{\circ\circ}y^\circ\\
&= y^{\circ\circ}x^\circ x^{\circ\circ}x^\circ y^{\circ\circ}y^\circ\\
&= y^{\circ\circ}x^\circ(x^{\circ\circ}x^\circ y^{\circ\circ}y^\circ)\\
&= y^{\circ\circ}x^\circ x^{\circ\circ}x^\circ\\
&= y^{\circ\circ}x^\circ.
\end{aligned}$$

Since $\leq$ is an amenable partial order, $x = xx^\circ x^{\circ\circ}x^\circ x \leq xx^\circ y^{\circ\circ}x^\circ x \leq yy^\circ y^{\circ\circ}y^\circ y = y$. Thus $\leq_C \subseteq \leq$.

Suppose that $a, b \in S$ and $a \leq b$. It follows from $a \leq b$ that $aa^\circ \preceq bb^\circ$ and $a^\circ a \preceq b^\circ b$, furthermore, $b^\circ b^{\circ\circ}a^\circ = a^\circ$. By Lemma 2.5, we have $a^{\circ\circ} \leq^{S^\circ} b^{\circ\circ}$. Hence, $(a^\circ b)^\circ(a^\circ b)^{\circ\circ} = b^\circ a^{\circ\circ}a^\circ b^{\circ\circ} \leq^{S^\circ} b^\circ b^{\circ\circ}a^\circ b^{\circ\circ} = a^\circ b^{\circ\circ} = (a^\circ b)^{\circ\circ}$, i.e., $(a^\circ b)^\circ(a^\circ b)^{\circ\circ} \leq^{S^\circ} (a^\circ b)^{\circ\circ}$. Similarly, we have $(a^\circ b)^{\circ\circ}(a^\circ b)^\circ \leq^{S^\circ} (a^\circ b)^{\circ\circ}$. Thus $a^\circ b^{\circ\circ} = (a^\circ b)^{\circ\circ} \in C$. Dually, we can get $b^{\circ\circ}a^\circ \in C$. It follows from definition of $\leq_C$ that $a \leq_C b$, which implies $\leq \subseteq \leq_C$ and so $\leq_C = \leq$.

It is easy to see that $E(S^\circ)$ is the smallest McAlister cone of S°, by Proposition 2.6 and Theorem 11 in [5], we have the following result.

Theorem 2.7. Suppose that S is a locally inverse semigroup with an inverse transversal S°. Then $\leq_{E(S^\circ)}$ defined by $(*)$ is the smallest amenable partial order on S and $\leq_{E(S^\circ)} = \preceq$.

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On the generalization of the Smarandache's Cevians Theorem (II)

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Abstract In this paper, we use the Ceva's Theorem and Menelaus' Theorem to study Smarandache's Cevians Theorem (II), and give the generalization of Smarandanche's Cevians Theorem (II) in quadrilateral and pentagon.

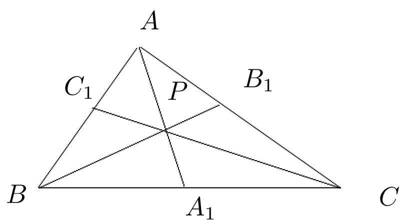
Keywords Ceva's Theorem, Menelaus' Theorem, quadrilateral, pentagon.

§1. Introduction and results

Dr. M. Khoshnevisan presented the Smarandache's Cevians Theorem (II) in the geometry of triangle, which stated as follows:

In a triangle $\triangle ABC$ (see Fig. 1) we draw the Cevians AA_1, BB_1, CC_1 that intersect in P. Then

$$\frac{PA}{PA_1} \times \frac{PB}{PB_1} \times \frac{PC}{PC_1} = \frac{AB}{A_1B} \times \frac{BC}{B_1C} \times \frac{CA}{C_1A},$$



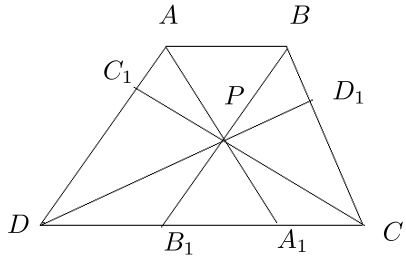
(Fig. 1)

Where the lines and following are all directive.

In this paper, we shall generalize this theorem for quadrilateral and pentagon. That is, we shall prove the following:

Theorem 1. Taking a point of P optional in quadrilateral $ABCD$ (see Fig. 2), draw the $AP, BP, CP,$ DP that intersect the opposite sides with A_1, B_1, C_1, D_1. Then

$$\frac{PA}{PA_1} \times \frac{PB}{PB_1} \times \frac{PC}{PC_1} \times \frac{PD}{PD_1} = \frac{AD}{A_1D} \times \frac{DC}{D_1C} \times \frac{CD}{C_1D} \times \frac{BC}{B_1C}.$$



(Fig. 2)

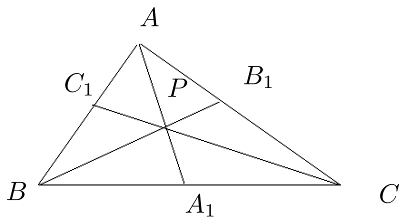
Theorem 2. The Smarandache's Cevians Theorem (II) can't be generalized to pentagon.

§2. Some lemmas

To complete the proof of the theorems, we need the following several lemmas.

Lemma 1. (Ceva's Theorem) In the triangle $\triangle ABC$ (see Fig. 3), we draw the AA_1, BB_1, CC_1 that intersect in P, then

$$\frac{AC_1}{C_1B} \times \frac{BA_1}{A_1C} \times \frac{CB_1}{B_1A} = 1.$$



(Fig. 3)

Proof. In the triangle $\triangle ABA_1$, cut by transversal CPC_1, we apply the Menelaus' Theorem

$$\frac{BC}{CA_1} \times \frac{A_1P}{PA} \times \frac{AC_1}{C_1B} = 1.$$

In the triangle $\triangle AA_1C$, cut by transversal BPB_1, we apply again the Menelaus' Theorem

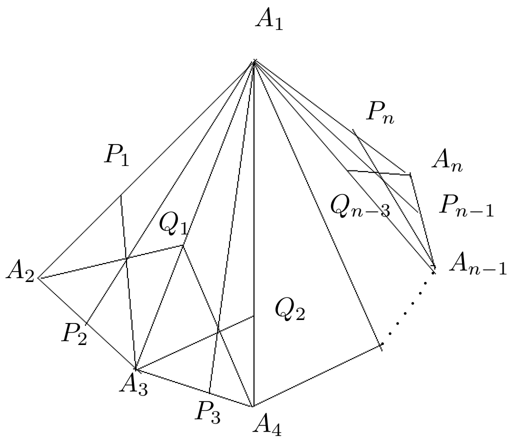
$$\frac{CB}{BA_1} \times \frac{A_1P}{PA} \times \frac{AB_1}{B_1C} = 1.$$

Divide the two above-mentioned formulas and we obtain

$$\frac{AC_1}{C_1B} \times \frac{BA_1}{A_1C} \times \frac{CB_1}{B_1A} = 1.$$

Lemma 2. (The generalization of Ceva's Theorem) In any polygon $A_1A_2\cdots A_n$ (see Fig. 4), if we have Ceva's point on $n-1$ edge, then we can determine one and only one Ceva's point on the edge of n. Thus,

$$\frac{A_1P_1}{P_1A_2} \times \frac{A_2P_2}{P_2A_3} \times \ldots \times \frac{A_{n-1}P_{n-1}}{P_{n-1}A_n} \times \frac{A_nP_n}{P_nA_1} = (-1)^n.$$



(Fig. 4)

Proof. In the triangle $\triangle A_1A_2A_3$, we apply the Ceva's Theorem

$$\frac{A_1P_1}{P_1A_2} \times \frac{A_2P_2}{P_2A_3} \times \frac{A_3Q_1}{Q_1A_1} = 1.$$

In the triangle $\triangle A_1A_3A_4$, we apply the Ceva's Theorem

$$\frac{A_1Q_1}{Q_1A_3} \times \frac{A_3P_3}{P_3A_4} \times \frac{A_4Q_2}{Q_2A_1} = 1.$$

$$\cdots$$

In the triangle $\triangle A_1A_{n-1}A_n$, we apply again the Ceva's Theorem

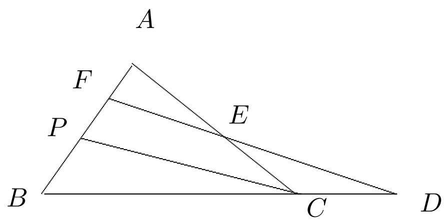
$$\frac{A_1Q_{n-3}}{Q_{n-3}A_{n-1}} \times \frac{A_{n-1}P_{n-1}}{P_{n-1}A_n} \times \frac{A_nP_n}{P_nA_1} = 1.$$

Multiplying the above-mentioned formulas we have

$$\frac{A_1P_1}{P_1A_2} \times \frac{A_2P_2}{P_2A_3} \times \cdots \times \frac{A_{n-1}P_{n-1}}{P_{n-1}A_n} \times \frac{A_nP_n}{P_nA_1} = (-1)^n.$$

Lemma 3. (Menelaus' Theorem) In the triangle $\triangle ABC$ (see Fig. 5), if a straight line intersect with AB, BC, CA or their extension at F, D, E, then

$$\frac{AF}{FB} \times \frac{BD}{DC} \times \frac{CE}{EA} = 1.$$



(Fig. 5)

Proof. Taking $CP//DF$, intersect AB at P, $\frac{BD}{DC}=\frac{FB}{PF}$, $\frac{CE}{EA}=\frac{PF}{AF}$, then

$$\frac{AF}{FB}\times\frac{BD}{DC}\times\frac{CE}{EA}=\frac{AF}{FB}\times\frac{FB}{PF}\times\frac{PF}{AF}=1.$$

Lemma 4. (The generalization of Menelaus' Theorem) In a polygon $A_1A_2\cdots A_n$, the linear L intersect with $A_1A_2, A_2A_3, \cdots, A_nA_1$ at $P_1, P_2, \cdots, P_n$, then

$$\frac{A_1P_1}{P_1A_2}\times\frac{A_2P_2}{P_2A_3}\times\ldots\times\frac{A_{n-1}P_{n-1}}{P_{n-1}A_n}\times\frac{A_nP_n}{P_nA_1}=1.$$

Proof. Apply the mathematical induction. If $n=3$, then it is the Menelaus' Theorem. Suppose the theorem holds for $n=k$. That is,

$$\frac{A_1P_1}{P_1A_2}\times\frac{A_2P_2}{P_2A_3}\times\ldots\times\frac{A_{k-1}P_{k-1}}{P_{k-1}A_k}\times\frac{A_kP_k}{P_kA_1}=1.$$

Then for $n=k+1$, apply the Menelaus' Theorem in the triangle $\triangle A_1A_kA_{k+1}$ we have

$$\frac{A_1P_k}{P_kA_k}\times\frac{A_kP_k}{P_kA_{k+1}}\times\frac{A_{k+1}P_{k+1}}{P_{k+1}A_1}=1.$$

Multiplying the above formulas and we obtain

$$\frac{A_1P_1}{P_1A_2}\times\frac{A_2P_2}{P_2A_3}\times\ldots\times\frac{A_{k-1}P_{k-1}}{P_{k-1}A_k}\times\frac{A_kP_k}{P_kA_{k+1}}\times\frac{A_{k+1}P_{k+1}}{P_{k+1}A_1}=1.$$

This shows that the theorem holds for $n=k+1$. Now the Lemma 4 follows from the induction.

§3. Proof of the theorems

In this section, we prove Theorem 1 and Theorem 2. First we prove Theorem 1. Apply the Menelaus' Theorem in the triangle $\triangle ADA_1$, we have

$$\frac{DC}{CA_1}\times\frac{A_1P}{PA}\times\frac{AC_1}{C_1D}=1.$$

In the triangle $\triangle BCB_1$,

$$\frac{CD}{DB_1}\times\frac{B_1P}{PB}\times\frac{BD_1}{D_1C}=1.$$

In the triangle $\triangle CDC_1$,

$$\frac{DA}{AC_1}\times\frac{C_1P}{PC}\times\frac{CA_1}{A_1D}=1.$$

In the triangle $\triangle CDD_1$,

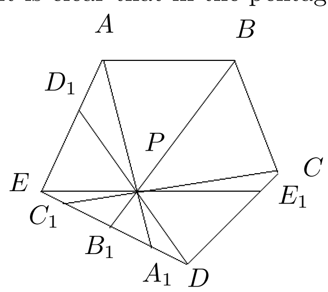
$$\frac{CB}{BD_1}\times\frac{D_1P}{PD}\times\frac{DB_1}{B_1C}=1.$$

Multiplying the above formulas we have

$$\frac{PA}{PA_1}\times\frac{PB}{PB_1}\times\frac{PC}{PC_1}\times\frac{PD}{PD_1}=\frac{AD}{A_1D}\times\frac{DC}{D_1C}\times\frac{CD}{C_1D}\times\frac{BC}{B_1C}.$$

This proves Theorem 1.

Now we prove Theorem 2. It is clear that in the pentagon (see Fig. 6)



(Fig. 6)

the conclusions are not correct. Otherwise, we have

$$\frac{PA}{PA_1} \times \frac{PB}{PB_1} \times \frac{PC}{PC_1} \times \frac{PD}{PD_1} \times \frac{PE}{PE_1} = \frac{AE}{A_1E} \times \frac{ED}{E_1D} \times \frac{DE}{D_1E} \times \frac{ED}{E_1D} \times \frac{CD}{C_1D}.$$

Connect AC, according to the conclusion in the quadrilateral, we have

$$\frac{PA}{PA_1} \times \frac{PC}{PC_1} \times \frac{PD}{PD_1} \times \frac{PE}{PE_1} = \frac{AE}{A_1E} \times \frac{DE}{D_1E} \times \frac{ED}{E_1D} \times \frac{CD}{C_1D}.$$

Because $\frac{ED}{E_1D}$ is fixed, but as the movement of B, $\frac{PB}{PB_1}$ is changing, so the conclusion is not correct. So the Smarandache's Cevians Theorem (II) can't be generalized to pentagon.

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The translational hull of strongly right Ehresmann semigroups [1]

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Abstract The concept of translational hull of semigroups was first introduced by Petrich in [8]. The translational hull of an inverse semigroup was studied by Ault in [4]. Fountain and Lawson studied the translational hull of adequate semigroups. And later on, the translational hull of strongly right or left adequate semigroups were further investigated by Ren and Shum. In this paper, we concentrate on the translational hull of strongly right Ehresmann semigroups. It is proved that the translational hull of a strongly right Ehresmann semigroup is still of the same type. Our results extends the previous results of strongly right adequate semigroups.

Keywords Translational hulls, strongly right U-ample semigroups, strongly right Ehresmann semigroups.

§1. Introduction

Recall that a mapping λ from a semigroup S into itself is called a left translation of S if $\lambda(ab) = (\lambda a)b$ for all a, b in S. Similarly, a mapping ρ from S into itself is a right translation of S if $(ab)\rho = a(b\rho)$ for all a, b in S. A left translation λ and a right translation ρ of S are said to be linked if $a(\lambda b) = (a\rho)b$ for all a, b in S. In this case, we call the pair (λ, ρ) a bitranslation of S. The set $\Lambda(S)$ of all left translations and the set $P(S)$ of all right translations of S form semigroups under the composition of mappings. The translational hull of S is the subsemigroup $\Omega(S)$ of $\Lambda(S) \times P(S)$ which consist of all bitranslations (λ, ρ). The concept of translational hull of semigroups and rings was first introduced in 1970 by Petrich in [8]. The translational hull of an inverse semigroup was first studied by Ault in [4], and the translational hull of an adequate semigroup was further studied by Fountain and Lawson in [1]. And later on, Ren and Shum investigated in 2006 the translational hull of a strongly right or left adequate semigroup in [11]. The translational hull of semigroup plays an important role in the theory of semigroups.

Let $E(S)$ be the set of all idempotents of a semigroup S and $U \subseteq E(S)$ be a non-empty subset, namely, the set of projections of S. The generalized Green relation $\widetilde{\mathcal{L}}^U$ was first defined by Lawson in [9]. For any elements a, b in S, $(a, b) \in \widetilde{\mathcal{L}}^U$ is defined if and only if a and b have

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the same set of right identities in U, that is to say, $U_a^r = U_b^r$,where

$$U_a^r = \{\, u \in U \mid au = a \,\}.$$

It is easy to check that $\mathcal{L} \subseteq \mathcal{L}^* \subseteq \widetilde{\mathcal{L}}^U$.

We now call a semigroup S a U-rpp semigroup if every $\widetilde{\mathcal{L}}^U$-class of S contains a projection of S and $\widetilde{\mathcal{L}}^U$ is a right congruence, denoted by (S,U). A U-rpp semigroup (S,U) is called a right Ehresmann semigroup if the projections of (S,U) commute. A U-rpp semigroup (S,U) is called strongly U-rpp semigroup if for any $a \in (S,U)$, there is a unique projection $e \in U$ such that $a\widetilde{\mathcal{L}}^U e$ and $a = ea$. Thus, we naturally call a right Ehresmann semigroup (S,U) a strongly right Ehresmann semigroup if (S,U) is a strongly U-rpp semigroup.

In this paper, we will show that the translational hull of an strongly right (left) Ehresmann semigroup is still of the same type.

For any notation and terminologies not given in this paper, the reader is referred to [4], [5] and [6].

§2. Preliminaries

We first give some basic results and notation from [7].

Lemma 2.1. Let a, b be elements of a semigroup (S,U). Then the following statements on (S,U) are equivalent:

(i) $(a,b) \in \widetilde{\mathcal{L}}^U$.

(ii) $U_a^r = U_b^r$, where $U_a^r = \{\, u \in U \mid au = a \,\}$.

Lemma 2.2. If $a \in (S,U)$ and $e \in U$, then the following statements hold on (S,U):

(i) $(e,a) \in \widetilde{\mathcal{L}}^U$.

(ii) $ae = a$ and for all $f \in U, af = a$ implies $ef = e$.

Lemma 2.3. If (S,U) is a strongly right Ehresmann semigroup, then each $\widetilde{\mathcal{L}}^U$-class of (S,U) contains a unique projection in U.

Proof. Suppose that $a \in (S,U)$ and $e \in \widetilde{\mathcal{L}}_a^U \cap U$. Let $f \in \widetilde{\mathcal{L}}_a^U \cap U$, since (S,U) is a strongly right Ehresmann semigroup, we have $e = ef = fe = f$. This show that each $\widetilde{\mathcal{L}}^U$-class of (S,U) contains a unique projection in U.

Suppose that (S,U) is a strongly right Ehresmann semigroup and $a \in (S,U)$. Then, by Lemma 2.3, we denote the unique projection in $\widetilde{\mathcal{L}}^U$-class containing a of (S,U) by a^*.

Now we have directly from definition the following lemma.

Lemma 2.4. Let a, b be elements of a strongly right Ehresmann semigroup (S,U). Then the following conditions hold in (S,U) :

(i) $a\widetilde{\mathcal{L}}^U b$ if and only if $a^* = b^*$.

(ii) $(ab)^* = (a^*b)^*$.

(iii) $aa^* = a = a^*a$.

(iv) $(ae)^* = a^*e$.

Lemma 2.5. Let (S,U) be a strongly right Ehresmann semigroup, the following statements are equivalent:

(i) (S,U) is strongly right U-ample.

(ii) $ea = a(ea)^*$, for every $a \in (S,U)$ and every projection $e \in U$.

We call a U-rpp semigroup (S,U) a projection balanced semigroup if for any $a \in (S,U)$, there exist projections e and f in (S,U) such that $a = ea = af$. It is clear that from Lemma 2.4, a strongly right Ehresmann semigroup is a projection balanced semigroup.

Lemma 2.6. Suppose that (S,U) is a projection balanced semigroup (S,U).

(i) If λ and λ' are two left translations of (S,U), then $\lambda = \lambda'$ if and only if $\lambda e = \lambda' e$ for all $e \in U$.

(ii) If ρ and ρ' are two right translations of (S,U), then $\rho = \rho'$ if and only if $e\rho = e\rho'$ for all $e \in U$.

Proof. We only need to prove (i) since the proof of (ii) can be obtained similarly. The necessity part of (i) is clear. For the sufficiency part of (i), let a be an element of (S,U) and e be a projection such that $ea = a$. Then

$$\lambda a = \lambda(ea) = (\lambda e)a = (\lambda' e)a = \lambda'(ea) = \lambda' a.$$

Hence $\lambda = \lambda'$. Thus the proof is completed.

Lemma 2.7. Let (S,U) be a strongly right Ehresmann semigroup. If $(\lambda,\rho),(\lambda',\rho') \in \Omega(S,U)$, then the following are equivalent:

(i) $(\lambda,\rho) = (\lambda',\rho')$.

(ii) $\lambda = \lambda'$.

(iii) $\rho = \rho'$.

Proof. It is clear that (i) implies (ii) and (i) implies (iii). In fact, we only need to show that (iii) implies (i). Suppose that $\rho = \rho'$. Then by our hypothesis, for any $e \in U$ there exists a projection f such that

$$\lambda e = f(\lambda e) = (f\rho)e = (f\rho')e = f(\lambda' e).$$

And there exists a projection f' such that

$$\lambda' e = f'(\lambda' e) = (f'\rho')e = (f'\rho)e = f'(\lambda e).$$

Thus, we have that $\lambda e \mathcal{L} \lambda' e$. Since (S,U) is a projection balanced semigroup, we can easily obtain that there exists $g \in U$ such that $f(\lambda' e) = (\lambda' e)g$. Hence, $\lambda e = (\lambda' e)g$. Thus, $\lambda e = (\lambda' e)g \cdot g = (\lambda e)\cdot g$. From $\mathcal{L} \subseteq \widetilde{\mathcal{L}}^U$, it follows that $\lambda' e = (\lambda' e)\cdot g$ so that $\lambda e = \lambda' e$. By Lemma 2.6, we obtain that $\lambda = \lambda'$ and of course, $(\lambda,\rho) = (\lambda',\rho')$.

§3. Strongly right Ehresmann semigroups

In this section, we assume always that (S,U) is a strongly right Ehresmann semigroup with the set of projections U.

Let $(\lambda, \rho) \in \Omega(S, U)$ which is the translational hull of (S, U). First we define two mappings λ^*, ρ^* from (S, U) into itself by the rule that for any $a \in (S, U)$,

$$\lambda^* a = (\lambda a^*)^* a,$$

$$a\rho^* = a(\lambda a^*)^*.$$

Lemma 3.1. Let (S, U) be a strongly right Ehresmann semigroup with semilattice U of projections. Then for any $e \in U$,

(i) $\lambda^* e = e\rho^*$;

(ii) $\lambda^* e = (\lambda e)^*$;

(iii) $\lambda^* a \widetilde{\mathcal{L}}^U \lambda a$.

Proof. (i) Since all projections commute, it follows immediately that for any $e \in U$,

$$\lambda^* e = (\lambda e)^* e = e(\lambda e)^* = e\rho^*.$$

(ii) By the definition of λ^*, we have that $\lambda^* e = (\lambda e)^* e$. Since $\lambda e \widetilde{\mathcal{L}}^U (\lambda e)^*$ and $\widetilde{\mathcal{L}}^U$ is a right congruence on (S, U), it follows that $\lambda e \cdot e \widetilde{\mathcal{L}}^U (\lambda e)^* e$, that is , $\lambda e \widetilde{\mathcal{L}}^U (\lambda e)^* e$. By Lemma 2.3, each $\widetilde{\mathcal{L}}^U$-class contains a unique projection so that $(\lambda e)^* = (\lambda e)^* e$. Hence, $\lambda^* e = (\lambda e)^* e = (\lambda e)^*$.

(iii) Clearly, $\lambda a^* \widetilde{\mathcal{L}}^U (\lambda a^*)^*$. Since $\widetilde{\mathcal{L}}^U$ is a right congruence, we immediately obtain that

$$\lambda a = \lambda a^* a \widetilde{\mathcal{L}}^U (\lambda a^*)^* a = \lambda^* a.$$

Lemma 3.2. The pair (λ^*, ρ^*) is a member of the translational hull $\Omega(S, U)$ of (S, U).

Proof. First we show that λ^* is a left translation. Let a, b be elements of (S, U). Then by Lemma 3.1,

$$\begin{aligned}
\lambda^*(ab) &= (\lambda(ab)^*)^* ab = \lambda^*(ab)^* \cdot (ab) \\
&= \lambda^*(ab)^* \cdot a^* \cdot ab = a^* \cdot \lambda^*(ab)^* \cdot ab \\
&= a^* \cdot (ab)^* \rho^* \cdot ab = (a^* \cdot (ab)^*)\rho^* \cdot ab \\
&= ((ab)^* a^*)\rho^* \cdot ab = (ab)^*(a^* \rho^*) \cdot ab \\
&= (a^* \rho^*)(ab)^* \cdot (ab) = \lambda^* a^* \cdot ab \\
&= (\lambda a^*)^* a \cdot b = (\lambda^* a)b.
\end{aligned}$$

We next show that ρ^* is a right translation. Noting that $(ab)b^* = ab$, we have that $(ab)^* b^* = (ab)^*$ so that $b^*(ab)^* = (ab)^*$. Then by Lemma 3.1 and Lemma 2.4, we have

$$\begin{aligned}
(ab)\rho^* &= ab(\lambda(ab)^*)^* = ab(\lambda b^*(ab)^*)^* \\
&= (ab)((\lambda b^*)^*(ab)^*) = ab(ab)^*(\lambda b^*)^* \\
&= ab(\lambda b^*)^* = a(b(\lambda b^*)^*) \\
&= a(b\rho^*).
\end{aligned}$$

This shows that ρ^* is a right translation.

Finally, we prove that (λ^*, ρ^*) is a linked pair. It is clear that

$$\begin{aligned} a(\lambda^*b) &= a(\lambda^*b^*)b = aa^*(\lambda^*b^*)\cdot b \\ &= aa^*(b^*\rho^*)b = a(a^*b^*)\rho^*)\cdot b \\ &= a(b^*a^*)\rho^*\cdot b = a\cdot b^*(a^*\rho^*)b \\ &= a\cdot(a^*\rho^*)b^*\cdot b = (aa^*)\rho^*b \\ &= (a\rho^*)b. \end{aligned}$$

This shows that the pair (λ^*, ρ^*) is linked and so is in $\Omega(S, U)$.

Now we take the set $\Psi(S, U)$ as follows:

$$\Psi(S,U) = \{\,(\lambda,\rho) \in E(\Omega(S,U)) : \lambda U \cup U\rho \subseteq U\,\}.$$

Then, we have the following result.

Lemma 3.3. The elements of $\Psi(S, U)$ are all idempotent.

Proof. Let $(\lambda, \rho) \in \Psi(S, U)$ and $e \in U$. Then

$$\lambda^2 e = \lambda((\lambda e)e) = \lambda(e(\lambda e)) = (\lambda e)(\lambda e) = \lambda e.$$

It follows by Lemma 2.6 that $\lambda^2 = \lambda$. Again by Lemma 2.7, we have that $(\lambda, \rho)^2 = (\lambda^2, \rho^2) = (\lambda, \rho)$.

Lemma 3.4. The elements of $\Psi(S, U)$ commute with each other.

Proof. Let $(\lambda, \rho), (\lambda', \rho') \in \Psi(S, U)$. Then, by the definition of $\Psi(S, U)$, we have that $\lambda U \cup U\rho \subseteq U$ and $\lambda' U \cup U\rho' \subseteq U$. Thus, for any projection $e \in U$, we have

$$\lambda\lambda' e = \lambda\lambda'(ee) = \lambda((\lambda' e)e) = \lambda(e(\lambda' e)) = (\lambda e)(\lambda' e) = (\lambda' e)(\lambda e) = \lambda'\lambda e.$$

By Lemma 2.6, it is clear that $\lambda\lambda' = \lambda'\lambda$. Similarly, we have that $\rho\rho' = \rho'\rho$. Hence, $(\lambda, \rho)(\lambda', \rho') = (\lambda', \rho')(\lambda, \rho)$, as required.

Lemma 3.5. (λ^*, ρ^*) is an elements in $\Psi(S, U)$.

Proof. Suppose that $e \in U$. By Lemma 3.1, it is obvious that $\lambda^* e$ and $e\rho^*$ are all elements of U. Thus, $(\lambda^*, \rho^*) \in \Psi(S, U)$.

It is natural to take $\bar{U} = \Psi(S, U)$ as the set of projections of the translational hull $\Omega(S, U)$. Thus, we will prove that $(\Omega(S, U), \bar{U})$ is a strongly right Ehresmann semigroup with the set of projections $\bar{U}$.

To do this, we need the following crucial Lemma.

Lemma 3.6. Any element (λ, ρ) of $\Omega(S, U)$ is $\widetilde{\mathcal{L}}^{\bar{U}}$-related to (λ^*, ρ^*).

Proof. Firstly we show that $(\lambda, \rho)(\lambda^*, \rho^*) = (\lambda, \rho)$. Assume that e is any projection from U. Then, by Lemma 3.1, we have that

$$\lambda\lambda^* e = \lambda(e\rho^*) = \lambda(e^2\rho^*) = \lambda(e\cdot(e\rho^*)) = \lambda e(e\rho^*) = \lambda e(\lambda e)^* = \lambda e.$$

Using Lemma 2.6, we obtain $\lambda\lambda^* = \lambda$. Thus, it follows by Lemma 2.7 that $(\lambda, \rho)(\lambda^*, \rho^*) = (\lambda, \rho)$. To prove that $(\lambda, \rho)\widetilde{\mathcal{L}}^{\bar{U}}(\lambda^*, \rho^*)$, we still need show from Lemma 2.2 that for any $(\lambda', \rho') \in \bar{U}$, $(\lambda, \rho)(\lambda', \rho') = (\lambda, \rho)$ implies $(\lambda^*, \rho^*)(\lambda', \rho') = (\lambda^*, \rho^*)$. Since (S, U) is a strongly right Ehresmann semigroup and by Lemma 3.1, we have that $\lambda e \widetilde{\mathcal{L}}^{U} (\lambda e)^* = \lambda^* e$ for any $e \in U$. Suppose that $(\lambda, \rho)(\lambda', \rho') = (\lambda, \rho)$ for any $(\lambda', \rho') \in \bar{U}$. Then for any projection $e \in U$, we have

$$(\lambda e)(\lambda' e) = \lambda(e(\lambda' e)) = \lambda((\lambda' e)e) = \lambda\lambda'(ee) = \lambda\lambda' e = \lambda e.$$

This implies that $\lambda^* e\lambda' e = \lambda^* e$. On the other hand, we have that

$$\lambda^* e\lambda' e = \lambda^*(e(\lambda' e)) = \lambda^*((\lambda' e)e) = \lambda^*(\lambda' e) = \lambda^*\lambda' e.$$

Hence, $\lambda^*\lambda' e = \lambda^* e$. It follows by Lemma 2.6 that $\lambda^*\lambda' = \lambda^*$. Thus, we have proved that $(\lambda^*, \rho^*)(\lambda', \rho') = (\lambda^*, \rho^*)$. This completes the proof that (λ, ρ) and (λ^*, ρ^*) are $\widetilde{\mathcal{L}}^{\bar{U}}$- related.

Lemma 3.7. $\widetilde{\mathcal{L}}^{\bar{U}}$ on $\Omega(S, U)$ is a right congruence.

Proof. It is clear that the relation $\widetilde{\mathcal{L}}^{\bar{U}}$ on $\Omega(S, U)$ is an equivalence. We next show that it is right compatible. Suppose that $(\lambda_1, \rho_1), (\lambda_2, \rho_2), (\lambda_3, \rho_3) \in \Omega(S, U)$ and $(\lambda_1, \rho_1)\widetilde{\mathcal{L}}^{\bar{U}}(\lambda_2, \rho_2)$. Then by Lemma 3.6, we have $(\lambda_1^*, \rho_1^*)\widetilde{\mathcal{L}}^{\bar{U}}(\lambda_2^*, \rho_2^*)$. This leads to $(\lambda_1^*, \rho_1^*)\mathcal{L}(\lambda_2^*, \rho_2^*)$ on $\Omega(S, U)$. Since $\mathcal{L}$ is a right congruence on $\Omega(S, U)$, it follows that $(\lambda_1^*, \rho_1^*)(\lambda_3, \rho_3)\mathcal{L}(\lambda_2^*, \rho_2^*)(\lambda_3, \rho_3)$, that is, $(\lambda_1^*\lambda_3, \rho_1^*\rho_3)\widetilde{\mathcal{L}}^{\bar{U}}(\lambda_2^*\lambda_3, \rho_2^*\rho_3)$. Hence for any $e \in U$, by Lemma 3.1 (iii), we have that $(\lambda_1\lambda_3)^* e\widetilde{\mathcal{L}}^U \lambda_1\lambda_3 e\widetilde{\mathcal{L}}^U \lambda_1^*\lambda_3 e\widetilde{\mathcal{L}}^U(\lambda_1^*\lambda_3)^* e$, that is, $(\lambda_1\lambda_3)^* e\widetilde{\mathcal{L}}^U(\lambda_1^*\lambda_3)^* e$. Since (S, U) is a strongly right Ehresmann semigroup, it follows from Lemma 2.3 and Lemma 3.1 that $(\lambda_1\lambda_3)^* e = (\lambda_1^*\lambda_3)^* e$. By Lemma 2.6, we have $(\lambda_1\lambda_3)^* = (\lambda_1^*\lambda_3)^*$. Notice that

$$\lambda_1\lambda_3\widetilde{\mathcal{L}}^{\bar{U}}(\lambda_1\lambda_3)^* = (\lambda_1^*\lambda_3)^*\widetilde{\mathcal{L}}^{\bar{U}}\lambda_1^*\lambda_3.$$

By Lemma 2.7 and the fact that $(\lambda_1\lambda_3)^* = (\lambda_1^*\lambda_3)^*$, we have $(\rho_1\rho_3)^* = (\rho_1^*\rho_3)^*$. Dually, $\rho_1\rho_3\widetilde{\mathcal{L}}^{\bar{U}}\rho_1^*\rho_3$. Hence, $(\lambda_1, \rho_1)(\lambda_3, \rho_3)\widetilde{\mathcal{L}}^{\bar{U}}(\lambda_1^*, \rho_1^*)(\lambda_3, \rho_3)$.

Similarly, we can easily obtain that $(\lambda_2, \rho_2)(\lambda_3, \rho_3)\widetilde{\mathcal{L}}^{\bar{U}}(\lambda_2^*, \rho_2^*)(\lambda_3, \rho_3)$. Thus we deduce that $(\lambda_1, \rho_1)(\lambda_3, \rho_3)\widetilde{\mathcal{L}}^{\bar{U}}(\lambda_2, \rho_2)(\lambda_3, \rho_3)$ on $\Omega(S, U)$ so that $\widetilde{\mathcal{L}}^{\bar{U}}$ is indeed a right congruence.

Summarizing above these observations, we can prove the following main theorem.

Theorem 3.8. The translational hull of a strongly right Ehresmann semigroup is still a strongly right Ehresmann semigroup.

Proof. By using the above Lemmas, we can easily verify that $\Omega(S, U)$ is a right Ehresmann semigroup. To prove this theorem, we only need to prove that for any $(\lambda, \rho) \in \Omega(S, U)$ there exist a unique projection (λ^*, ρ^*) such that $(\lambda, \rho)\widetilde{\mathcal{L}}^{\bar{U}}(\lambda^*, \rho^*)$ and $(\lambda^*, \rho^*)(\lambda, \rho) = (\lambda, \rho)$. Since (S, U) is a strongly right Ehresmann semigroup, we have

$$\lambda^*\lambda e = (\lambda(\lambda e)^*)^*(\lambda e) = (\lambda\lambda^* e)^*(\lambda e) = (\lambda e)^*(\lambda e) = \lambda e,$$

by Lemma 2.7 and so $(\lambda^*, \rho^*)(\lambda, \rho) = (\lambda, \rho)$. This shows that $\Omega(S, U)$ is a strongly right Ehresmann semigroup. The proof is hence completed.

§4. Strongly right U-ample semigroups

We say that a strongly right Ehresmann semigroup (S, U) in which $ea = a(ea)^*$ for every element a and every projection e of (S, U) is a strongly right U-ample semigroup.

As a direct consequence of Theorem 3.8, we deduce the following theorem.

Theorem 4.1. The translational hull of a strongly right U-ample semigroup is still a strongly right U-ample semigroup.

Proof. Let (S, U) be a strongly right U-ample semigroup. By Theorem 3.8, we know that $\Omega(S, U)$ is strongly right Ehresmann. Also from the proof of this theorem we have

$$\bar{U} = \Psi(S, U) = \{ (\lambda, \rho) \in E(\Omega(S, U)) : \lambda U \cup U\rho \subseteq U \}.$$

Now let $(\lambda_1, \rho_1) \in \bar{U}$ and let $(\lambda, \rho) \in \Omega(S, U), e \in U(S)$. By our definition, we only need to show that $(\lambda_1, \rho_1)(\lambda, \rho) = (\lambda, \rho)((\lambda_1, \rho_1)(\lambda, \rho))^*$ for every element (λ, ρ) and every projection $(\lambda_1, \rho_1) \in \bar{U}$.

We now show that $\lambda_1\lambda = \lambda(\lambda_1\lambda)^*$ and $\rho_1\rho = \rho(\rho_1\rho)^*$. In fact, $\lambda_1\lambda e = \lambda_1((\lambda e)^*(\lambda e)) = (\lambda_1(\lambda e)^*)(\lambda e)$.

Since $(\lambda_1, \rho_1) \in \bar{U}$, $\lambda_1(\lambda e)^*$ is a projection in (S, U). For (S, U) is strongly right U-ample, we have

$$\lambda_1\lambda e = (\lambda_1(\lambda e)^*)(\lambda e) = (\lambda e)((\lambda_1(\lambda e)^*)(\lambda e))^* = \lambda e(\lambda_1\lambda e)^*.$$

And we also have

$$\lambda e(\lambda_1\lambda e)^* = \lambda(e(\lambda_1\lambda e)^*) = \lambda((\lambda_1\lambda e)^* e) = \lambda(((\lambda_1\lambda e)^* e)e) = \lambda((\lambda_1\lambda)^* e) = \lambda(\lambda_1\lambda)^* e.$$

Then we obtain that $\lambda_1\lambda = \lambda(\lambda_1\lambda)^*$. Hence,

$$(\lambda_1, \rho_1)(\lambda, \rho) = (\lambda, \rho)((\lambda_1, \rho_1)(\lambda, \rho))^*.$$

This shows that $\Omega(S, U)$ is a strongly right U-ample semigroup so that we have proved the theorem.

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A Note on Smarandache non-associative rings [1]

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Abstract In this note, we give the smallest non-associative ring, which is a Smarandache non-associative ring, and prove that this ring is also the SNA Moufang ring and the SNA Bol ring.

Keywords Non-associative ring, SNA-ring, SNA Moufang ring, SNA Bol ring.

§1. Introduction

In this paper [1], Vasantha Kandasamy introduced the concept of Smarandache non-associative rings, which we shortly denote as SNA-rings. This concept derived from the general definition of a Smarandache Structure (i.e., a set A embedded with a weak structure W such that a proper subset B in A is embedded with a stronger structure S) and were firstly studied in the Smarandache algebraic literature. The only non-associative structure found in Smarandache algebraic notions are Smarandache groupiods and Smarandache loops introduced in [2] and [3], which are algebraic structures with only a single binary operation defined on them that is non-associative. But SNA-rings are non-associative structures on which are defined two binary operations one associative and other being non-associative and addition distributes over multiplication both from the right and left. By [1], it is well know that the loop ring is always a SNA-ring, and the groupiod ring is also a SAN-ring when it satisfies some conditions. Those results motivate us to find the smallest non-associative ring (By smallest we mean the number of elements in them that is order is the least that is we can not find any other non-associative ring of lesser order than that). In this note, we shall give some interesting results about the mentioned problems in [1].

§2. Preliminaries

Definition 2.1. A set S together with a (binary) operation is a groupoid. A groupoid $(S, *)$ satisfying the associative law

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$$(x*y)*z = x*(y*z) \quad (x,y,z \in S)$$

is a semigroup.

Definition 2.2. A ring $(R,+,*)$ is said to be a non-associative ring if $(R,+)$ is an additive abelian group, $(R,*)$ is a non-associative semigroup (that is the binary operation $*$ on R is non-associative)such that the distributive laws

$$(x+y)*z = x*z + y*z\ ,\ x*(y+z) = x*y + x*z,$$

for all $x,y,z \in R$.

Definition 2.3. Let $(R,+,*)$ be a non-associative ring. R is said to be a SNA-ring if R contains a proper subset P is an associative ring under the operations of R.

Definition 2.4. A non-associative ring $(R,+,*)$ is said to be a Moufang ring if the Moufang identity

$$(x*y)*(z*x) = (x*(y*z))*x$$

is satisfied for all $x,y,z \in R$.

Definition 2.5. Let $(R,+,*)$ be a non-associative ring. R is said to be a Bol ring if R satisfies the Bol identity

$$((x*y)*z)*y \ = \ x*((y*z)*y), \text{ for all } x,y,z \in R.$$

In view of these we have the following interesting results.

Theorem 2.6. If R is a Moufang ring and R is also a SNA-ring then R is a SNA Moufang ring.

Theorem 2.7. Let R be a non-associative ring, which is a Bol ring. If R is a SNA-ring, then R is a SNA Bol ring.

§3. Main results

Theorem 3.1. A non-associative ring of order 2 is not exist.

Proof. Suppose that $R = \{0,1\}$ and $(R,+,*)$ is a non-associative rings, in which $(R,+)$ is an additive abelian group given by the following table:

+	0	1
0	1	1
1	1	1

Obviously, this form is single for an additive abelian group of order 2. Then by the law of addition distributive over multiplication both from the right and left we have :

$$\left.\begin{array}{c} 1*(1+1) = 1*0 \\ 1*1+1*1 = 0 \end{array}\right\} \Rightarrow 1*0 = 0,$$

and

$$\left.\begin{array}{r}(1+1)*1=0*1\\ 1*1+1*1=0\end{array}\right\}\Rightarrow 0*1=0,$$

and

$$\left.\begin{array}{r}0*(1+1)=0*0\\ 0*1+0*1=0\end{array}\right\}\Rightarrow 0*0=0,$$

and

$$\left.\begin{array}{r}1*(1+0)=1*1\\ 1*1+1*0=1*1\end{array}\right\}\Rightarrow 1*1=0 \text{ or } 1.$$

we immediately have $(R,*)$ by the following table:

$*$	0	1
0	0	0
1	0	0

$*$	0	1
0	0	0
1	0	1

It is easy to see that $(R,*)$ is a semigroup satisfied the associative law. It is contradictive with R is a non-associative ring. Thus, we can not find a non-associative ring of order 2. This completes the proof of Theorem 3.1.

Theorem 3.2. The smallest non-associative ring is of order 3 given by the following example.

Example 1. $(A,+,*)$ be a non-associative ring of 3 given by the following table:

2

$+$	0	a	b
0	0	a	b
a	a	0	a
b	b	a	0

$*$	0	a	0
0	0	0	0
a	0	0	0
b	0	a	0

It easy to see that $(A,+)$ is an additive group, in which

$$(b*b)*a=0*a=0\neq b*(b*a)=b*a=a,$$

and so $(A,*)$ is a non-associative semigroup. From Definition 2.1 and Theorem 3.1 we have Theorem 3.2.

Example 2. $(B,+,*)$ be a non-associative ring given of 4 by the following table:

$+$	0	a	b	c
0	0	a	b	c
a	a	0	a	a
b	b	a	0	b
c	c	0	b	0

$*$	0	a	b	c
0	0	0	0	0
a	0	a	0	0
b	0	c	0	0
c	0	0	0	0

Similarly, we can obtain that $(B,+,*)$ is a non-associative ring since

$$(b * a) * a = c * a = 0 \neq b * (a * a) = b * a = c,$$

Hence, $(B, *)$ is a non-associative semigroup.

It is very natural to consider whether the smallest non-associative ring is a SNA-ring. we find a proper subset $C = \{0, a\}$ of $(A, +, *)$ in Example 1, which is a associative ring given by the following table:

$+$	0	a
0	0	a
a	a	0

$*$	0	a
0	0	0
a	0	0

Thus, it follows from Definition 2.2 that $(A, +, *)$ is a SNA-ring, furthermore, we have the following theorem.

Theorem 3.3. The smallest non-associative ring is a SNA-ring.

Corollary 3.4. The least order of SNA-ring is 3.

Theorem 3.5. $(A, +, *)$ in Example 1 is the smallest SNA Moufang ring.

Proof. It is easily seen that $(A, *)$ satisfies the Bol identity

$$((x * y) * z) * y = x * ((y * z) * y), \text{ for all } x, y, z \in R.$$

From Theorem 2.6 and Theorem 3.3 we have $(A, +, *)$ is the smallest SNA Moufang ring.

Similarly, we can obtain the following result.

Theorem 3.6. $(A, +, *)$ in Example 1 is the smallest SNA Bol ring.

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An equation involving $Z_*(n)$ and $\overline{SL}(n)$

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Abstract For any positive integer n, the Pseudo-Smarandache dual function $Z_*(n)$ denotes the maximum positive integer m such that $\frac{m(m+1)}{2}$ divide n. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ denotes the factorization of n into prime powers, the Smarandach LCM dual function $\overline{SL}(n)$ is defined as $\overline{SL}(n) = \min\{p_1^{\alpha_1}, p_2^{\alpha_2}, \cdots, p_k^{\alpha_k}\}$. The main purpose of this paper is using the elementary and analytic methods to study the solvability of the equation $Z_*(n) = \overline{SL}(n)$, and give its all positive integer solutions.

Keywords Pseudo-Smarandache dual function, Smarandach dual function, equation, positive integer solution.

§1. Introduction and results

For any positive integer n, the Pseudo-Smarandache dual function, denoted by $Z_*(n)$, is defined as the maximum positive integer m such that $\frac{m(m+1)}{2}$ divide n. That is,

$$Z_*(n) = \max\left\{m : m \in Z^+, \frac{m(m+1)}{2} \mid n\right\}.$$

This function was introduced by J.Sandor in [1], where he studied the elementary properties of $Z_*(n)$, and obtained a series of interesting results. They are stated as follows:

Lemma 1. Let q be a prime such that $p = 2q - 1$ is a prime too. Then

$$Z_*(pq) = p.$$

Lemma 2. $Z_*(\frac{k(k+1)}{2}) = k$, for any integer $k \geq 1$.

Lemma 3. For any integer $a, b \geq 1$, $Z_*(ab) \geq \max\{Z_*(a), Z_*(b)\}$.

Lemma 4. Let p be a prime, then for any integer $k \geq 1$,

$$Z_*(p^k) = \begin{cases} 2, & \text{if } p = 3; \\ 1, & \text{if } p \neq 3. \end{cases}$$

Lemma 5. Any solution of the equation $Z(n) = Z_*(n)$ is of the form $n = \frac{k(k+1)}{2}$, where $k \geq 1$ is an integer.

In reference [2], A.A.K. Majumdar studied the explicit expressions of $Z_*(2p^k)$, $Z_*(3p^k)$, $Z_*(4p^k)$ and $Z_*(5p^k)$, where p is an odd prime.

On the other hand, for any positive integer n, the Smarandache LCM dual function $\overline{SL}(n)$ is defined as follows, $\overline{SL}(1) = 1$, and if $n > 1$, $\overline{SL}(n) = \min\{p_1^{\alpha_1}, p_2^{\alpha_2}, \cdots, p_k^{\alpha_k}\}$, if $n = p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$ is the prime power factorization of n. About this function, some authors had studied the solvability of the equation $\sum\limits_{d|n}\overline{SL}(n) = n$, and founded all its positive integer solutions (See reference [3]).

In this paper, we use the elementary methods to study the solvability of the equation

$$Z_*(n) = \overline{SL}(n), \tag{1}$$

and give its all positive integer solutions. That is, we shall prove the following:

Theorem. The solutions of the equation (1) can be expressed as:

1. Let n be an even integer, then $n = 2^s p^k t$, where p ($p \geq 3$ and $(p,t) = 1$) is a prime, s, t and k are positive integers satisfying the following conditions:

(a). If $p^k > 2^s$, then $p^\alpha t = 2^s + 1$, where α is a positive integer and $1 \leq \alpha \leq k$.

(b). If $p^k < 2^s$, then $p^k = 2^{\beta+1}t - 1$, where β is a positive integer and $1 \leq \beta \leq s - 1$.

2. Let n be an odd integer, then $n = 1$ or $n = p^\alpha t$, where α and t are positive integers and $(p^\alpha + 1) \mid 2t$, $p^\alpha = 2u - 1$, where u is a positive integer and $u \mid t$, let $t = a\cdot b\cdot c$, $a \neq 2b - 1$ or $a < p^\alpha$.

§3. Proof of the theorem

In this section, we shall prove our theorems directly.

a. For any even integer n, we discuss the solutions in following several cases:

(i) Let $n = 2^s$, where s is a positive integer, then from the definition of $\overline{SL}(n)$, we have $\overline{SL}(n) = 2^s$. According to Lemma 4, we have $Z_*(2^s) = 1$. It means that the equation (1) has no positive solution when $n = 2^s$.

(ii) Let $n = 2^s p^k$, where p is an odd prime, s and k are positive integer.

(A) If $2^s > p^k$, then $\overline{SL}(n) = p^k$. Suppose p^b is a divisor of p^k, where $1 \leq b \leq k$, we discuss p^b as follows:

If $p^b = 2^{a+1} + 1$, where $1 \leq a < s - 1$, then $Z_*(n) = 2^{a+1} \neq p^k = \overline{SL}(n)$.

If $p^b = 2^{a+1} - 1$, where $1 \leq a \leq s - 1$, then $Z_*(n) = 2^{a+1} - 1 = p^b$. Let $b = k$, then $Z_*(n) = \overline{SL}(n)$. Otherwise $Z_*(n) = 1 \neq \overline{SL}(n)$.

(B) If $2^s < p^k$, then $\overline{SL}(n) = 2^s$. We discuss p^b as follows:

If $p^b = 2^{a+1} + 1$, where $1 \leq a \leq s$, then $Z_*(n) = p^b - 1 = 2^{a+1}$. Let $a + 1 = s$, then $Z_*(n) = 2^s = \overline{SL}(n)$.

If $p^b = 2^{a+1} - 1$, where $1 \leq a \leq s$, then $Z_*(n) = p^b = 2^{a+1} - 1 \neq \overline{SL}(n) = 2^s$. Otherwise $Z_*(n) = 1 \neq \overline{SL}(n)$.

(iii) Let $n = 2^s p^k t$, where p is an odd prime, s, t and k are positive integer.

(A) If $2^s > p^k$, where $1 \leq b \leq k$, then $\overline{SL}(n) = p^k$. We discuss p^b as follows:

If $p^b = 2^{a+1}t + 1$, where $1 \leq a < s - 1$, then $Z_*(n) = p^b - 1 = 2^{a+1}t \neq p^k = \overline{SL}(n)$.

If $p^b = 2^{a+1}t - 1$, where $1 \leq a \leq s - 1$, then $Z_*(n) = 2^{a+1}t - 1 = p^b = \overline{SL}(n)$ while $b = k$.

If $p^b t = 2^{a+1} + 1$, where $1 \leq a \leq s$, then $Z_*(n) = p^b t - 1 = 2^{a+1} \neq p^k = \overline{SL}(n)$ for any positive integer b and t.

If $p^b t = 2^{a+1} - 1$, where $1 \leq a \leq s$, then $Z_*(n) = 2^{a+1} - 1 = p^b t = \overline{SL}(n)$, while $b = k$ and $t = 1$. Otherwise $Z_*(n) = 1 \neq \overline{SL}(n)$.

(B) If $2^s < p^k$, then $\overline{SL}(n) = 2^s$. We discuss p^b as follows:

If $p^b = 2^{a+1}t + 1$, where $1 \leq a \leq s$, then $Z_*(n) = p^b - 1 = 2^{a+1}t = 2^s = \overline{SL}(n)$, while $a + 1 = s$ and $t = 1$.

If $p^b = 2^{a+1}t - 1$, where $1 \leq a \leq s$, then $Z_*(n) = 2^{a+1}t - 1 = p^b \neq \overline{SL}(n)$.

If $p^b t = 2^{a+1} + 1$, where $1 \leq a \leq s$, then $Z_*(n) = p^b t - 1 = 2^{a+1} = 2^s = \overline{SL}(n)$ while $a + 1 = s$.

If $p^b t = 2^{a+1} - 1$, where $1 \leq a \leq s$, then $Z_*(n) = p^b t \neq 2^s = \overline{SL}(n)$ for any positive integer b and t. Otherwise $Z_*(n) = 1 \neq \overline{SL}(n)$.

(b). For any odd integer n, let $n = p^\alpha t$, $p^\alpha < t$, $(p^\alpha,\ t) = 1$ where p is an odd prime, α and t are positive integer. Let $\overline{SL}(n) = p^\alpha$. n is a solution of the equation (1) if and only if $Z_*(n) = p^\alpha$.

From the definition of $Z_*(n)$, we have $\frac{p^\alpha(p^\alpha+1)}{2} \mid p^\alpha t$, that is $(p^\alpha + 1) \mid 2t$.

(i) If $p^\alpha = 2u - 1$, where u is a positive integer and $u \mid t$, then $(p^\alpha + 1) \mid 2t$.

Let $t = a \cdot b \cdot c$, if $a = 2b - 1$ and $a > p^\alpha$, then $Z_*(n) = a \neq \overline{SL}(n)$.

If $a \neq 2b - 1$ or $a < p^\alpha$, then $\overline{SL}(n) = p^\alpha = Z_*(n)$.

(ii) If $p^\alpha \neq 2u - 1$, then $Z_*(n) \neq p^\alpha$, therefore there is no solutions in this case.

This complete the proof of our theorem.

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On the mean value of Smarandache prime part $P_p(n)$ and $p_p(n)$

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Abstract For any positive integer n, the Smarandache Superior Prime Part $P_p(n)$ is the smallest prime number greater than or equal to n. For any positive integer $n \geq 2$, the Smarandache Inferior Prime Part $p_p(n)$ is the largest prime number less than or equal to n. The main purpose of this paper is using the elementary and analytic methods to study the asymptotic properties of $(S_n(n) - I_n(n))$, $\dfrac{K_n(n)}{L_n(n)}$, $(K_n(n) - L_n(n))$, and give several interesting asymptotic formula for them, where $S_n(n) = \dfrac{1}{n}\sum\limits_{i=1}^{n} P_p(n)$, $I_n(n) = \dfrac{1}{n}\sum\limits_{i=1}^{n} p_p(n)$ and $K_n(n) = \left(\sum\limits_{i=1}^{n} P_p(n)\right)^{\frac{1}{n}}$, $L_n(n) = \left(\sum\limits_{i=1}^{n} p_p(n)\right)^{\frac{1}{n}}$.

Keywords Smarandache superior prime part sequence, Smarandache inferior prime part sequence, mean value, asymptotic formula.

§1. Introduction and results

For any positive integer $n \geq 1$, the Smarandache Superior Prime Part $P_p(n)$ is the smallest prime number greater than or equal to n, that $P_p(n) = \min\{n | n \geq p, p \text{ is a prime}\}$. For example, the first few values of $P_p(n)$ are $P_p(1) = 2, P_p(2) = 2, P_p(3) = 3, P_p(4) = 5, P_p(5) = 5, P_p(6) = 7, P_p(7) = 7, P_p(8) = 11, P_p(9) = 11, P_p(10) = 11, P_p(11) = 11, P_p(12) = 13, P_p(13) = 13, P_p(14) = 17, P_p(15) = 17, \cdots$.

For any positive integer $n \geq 2$, the Smarandache Inferior Prime Part $p_p(n)$ is the largest prime number less than or equal to n, that $p_p(n) = \max\{n | n \leq p, p \text{ is a prime}\}$. For example, the first few values of $p_p(n)$ are $p_p(2) = 2, p_p(3) = 3, p_p(4) = 3, p_p(5) = 5, p_p(6) = 5, p_p(7) = 7, p_p(8) = 7, p_p(9) = 7, p_p(10) = 7, p_p(11) = 11, p_p(12) = 11, p_p(13) = 13, p_p(14) = 13, p_p(15) = 13, \cdots$.

By the definition of these two series known for any prime q, we have $P_p(q) = p_p(q) = q$. On the sequence $\{P_p(q)\}$ and $\{p_p(q)\}$ of the nature of the study is very significant, because the Smarandache prime series and prime number distribution issues are closely linked.

Now we define

$$S_n(n) = [P_p(1) + P_p(2) + P_p(3) + \cdots + P_p(n)]/n = \frac{1}{n}\sum_{i=1}^{n} P_p(n),$$

$$I_n(n) = [p_p(1) + p_p(2) + p_p(3) + \cdots + p_p(n)]/n = \frac{1}{n}\sum_{i=1}^{n} p_p(n),$$

$$K_n(n) = \sqrt[n]{P_p(1) + P_p(2) + P_p(3) + \cdots + P_p(n)} = \left(\sum_{i=1}^{n} P_p(n)\right)^{\frac{1}{n}},$$

$$L_n(n) = \sqrt[n]{p_p(1) + p_p(2) + p_p(3) + \cdots + p_p(n)} = \left(\sum_{i=1}^{n} p_p(n)\right)^{\frac{1}{n}}.$$

In the book "Only problems, Not solutions" (See reference [1], Problems 39), Professor F. Smarandache ask us to study the properties of the sequences $\{P_p(n)\}$ and $\{p_p(n)\}$. About these problems, Scholar Yan Xiaoxia had studied it before and obtained interesting results (see reference [5]):

$$\frac{S_n(n)}{I_n(n)} = 1 + O\left(n^{\frac{1}{3}}\right), \text{ and } \lim_{n\to\infty}\frac{S_n(n)}{I_n(n)} = 1.$$

In this paper, we use the elementary and analytic methods to study the asymptotic properties of $(S_n(n) - I_n(n))$, $\frac{K_n(n)}{L_n(n)}$, $(K_n(n) - L_n(n))$, and give a shaper asymptotic formula for it. That is, we shall prove the following conclusion:

Theorem 1. For ally positive integer $n \geq 1$, we have the asymptotic formula

$$S_n(n) - I_n(n) = O\left(n^{\frac{5}{18}}\right), \lim_{n\to\infty}\frac{S_n(n) - I_n(n)}{n^{\frac{5}{18}}} = D, \lim_{n\to\infty}\left(S_n(n) - I_n(n)\right)^{\frac{1}{n}} = 1,$$

where D is computable constant.

Theorem 2. For ally positive integer $n \geq 1$, we have the asymptotic formula

$$\frac{K_3(n)}{L_3(n)} = 1 + O\left(n^{\frac{1}{3}}\right), \lim_{n\to\infty}\frac{K_3(n)}{L_3(n)} = 1, \lim_{n\to\infty}\left(K_3(n) - L_3(n)\right) = 0.$$

§2. Some lemmas

In order to complete the proof of the theorem, we need the following several lemmas.

First we have

Lemma 1. For any real number $x > 1$, we have the asymptotic formula

$$\sum_{p_{n+1}\leq x}(p_{n+1} - p_n)^2 \ll x^{\frac{23}{18}+\varepsilon},$$

where p_n denotes the n-th prime, ε denotes any fixed positive number.

Proof. This is a famous result due to D. R. Heath Brown [3] and [4].

Lemma 2. Let x be a positive real number which is large enough, then there must exist a prime P between x and $x + x^{\frac{2}{3}}$.

Proof. For any real number x which is large enough, let P_n denotes the largest prime with $P_n \leq x$. Then from Lemma 1, we may immediately deduce that

$$(P_{n+1} - P_n)^2 \ll x^{\frac{23}{18}+\varepsilon},$$

or

$$P_{n+1} - P_n \ll x^{\frac{2}{3}}.$$

So there must exist a prime P between x and $x + x^{\frac{2}{3}}$.

This proves Lemma 2.

Lemma 3. For any real number $x > 1$, we have the asymptotic formulas

$$\sum_{n \le x} P_p(n) = \frac{x^2}{2} + O\left(x^{\frac{5}{3}}\right),$$

and

$$\sum_{n \le x} p_p(n) = \frac{x^2}{2} + O\left(x^{\frac{5}{3}}\right).$$

Proof. We only prove first asymptotic formula, similarly we can deduce the second one.

Let P_k denotes the k-th prime. Then from the definition of $P_p(n)$, we know that for any fixed prime P_r, there exist $P_{r+1} - P_r$ positive integer n such that $P_p(n) = P_r$.

So we have

$$\begin{aligned}
\sum_{n \le x} P_p(n) &= \sum_{P_{n+1} \le x} (P_{n+1} - P_n) \cdot P_n \\
&= \frac{1}{2} \sum_{P_{n+1} \le x} (P_{n+1}^2 - P_n^2) - \frac{1}{2} \sum_{P_{n+1} \le x} (P_{n+1} - P_n)^2 \\
&= \frac{1}{2} P^2(x) - 2 - \frac{1}{2} \sum_{P_{n+1} \le x} (P_{n+1} - P_n)^2, \qquad (1)
\end{aligned}$$

where $P(x)$ denotes the largest prime such that $P(x) \le x$.

From Lemma 2, we know that

$$P(x) = x + x^{\frac{2}{3}} + O(1). \qquad (2)$$

Now from (1), (2) and Lemma 1, we may immediately deduce that

$$\begin{aligned}
\sum_{n \le x} P_p(n) &= \frac{1}{2} x^2 + x^{\frac{5}{3}} + x^{\frac{4}{3}} + O(1) + O\left(x^{\frac{23}{18} + \varepsilon}\right) \\
&= \frac{1}{2} x^2 + O\left(x^{\frac{5}{3}}\right).
\end{aligned}$$

This proves the first asymptotic formula of Lemma 3.

The second asymptotic formula follows from Lemma 1, Lemma 2 and the identity

$$\begin{aligned}
\sum_{n \le x} p_p(n) &= \sum_{P_{n+1} \le x} (P_{n+1} - P_n) \cdot P_n \\
&= \frac{1}{2} \sum_{P_{n+1} \le x} (P_{n+1}^2 - P_n^2) - \frac{1}{2} \sum_{P_{n+1} \le x} (P_{n+1} - P_n)^2 \\
&= \frac{1}{2} P^2(x) - \frac{1}{2} \sum_{P_{n+1} \le x} (P_{n+1} - P_n)^2 \\
&= \frac{1}{2} x^2 + x^{\frac{5}{3}} + x^{\frac{4}{3}} + O(1) + O\left(x^{\frac{23}{18} + \varepsilon}\right) \\
&= \frac{1}{2} x^2 + O\left(x^{\frac{5}{3}}\right).
\end{aligned}$$

§3. Proofs of the theorem

In this section, we shall complete the proof of Theorem 1 and Theorem 2.

Proof of Theorem 1. In fact, for any positive integer $n > 1$, from Lemma 3 and the definition of $S_n(n)$ and $I_n(n)$, let $n = x$, we have

$$\begin{aligned} S_n(n) &= \frac{1}{n}\left(\sum_{n\le x} P_p(n)\right) \\ &= \frac{1}{n}\left(\frac{1}{2}x^2 + x^{\frac{5}{3}} + x^{\frac{4}{3}} + O(1) + O\left(x^{\frac{23}{18}+\varepsilon}\right)\right) \\ &= \frac{1}{2}n + n^{\frac{2}{3}} + n^{\frac{1}{3}} + O\left(n^{\frac{5}{18}+\varepsilon}\right), \end{aligned}$$

$$\begin{aligned} I_n(n) &= \frac{1}{n}\left(\sum_{n\le x} p_p(n)\right) \\ &= \frac{1}{n}\left(\frac{1}{2}x^2 + x^{\frac{5}{3}} + x^{\frac{4}{3}} + O(1) + O\left(x^{\frac{23}{18}+\varepsilon}\right)\right) \\ &= \frac{1}{2}n + n^{\frac{2}{3}} + n^{\frac{1}{3}} + O\left(n^{\frac{5}{18}+\varepsilon}\right), \end{aligned}$$

$$S_n(n) - I_n(n) = O\left(n^{\frac{5}{18}\varepsilon_1}\right).$$

We may immediately deduce that

$$\lim_{n\to\infty} \frac{S_n(n) - I_n(n)}{n^{\frac{5}{18}}} = D, \lim_{n\to\infty} \left(S_n(n) - I_n(n)\right)^{\frac{1}{n}} = 1,$$

where D is computable constant.

This completes the proof of Theorem 1.

Proof of Theorem 2. For any positive integer $n > 1$, from Lemma 3 and the definition of $K_n(n)$ and $L_n(n)$ we have

$$K_n(n) = \sqrt[n]{P_p(1) + P_p(2) + P_p(3) + \cdots + P_p(n)} = \left(\frac{1}{2}x^2 + O\left(x^{\frac{5}{3}}\right)\right)^{\frac{1}{n}}, \tag{3}$$

$$L_n(n) = \sqrt[n]{p_p(1) + p_p(2) + p_p(3) + \cdots + p_p(n)} = \left(\frac{1}{2}x^2 + O\left(x^{\frac{5}{3}}\right)\right)^{\frac{1}{n}}. \tag{4}$$

Combining (3) and (4), we have

$$\frac{K_n(n)}{L_n(n)} = \left(\frac{\frac{1}{2}x^2 + O\left(x^{\frac{5}{3}}\right)}{\frac{1}{2}x^2 + O\left(x^{\frac{5}{3}}\right)}\right)^{\frac{1}{n}} = \left(1 + O\left(x^{-\frac{1}{3}}\right)\right)^{\frac{1}{n}} = 1 + O\left(x^{-\frac{1}{3}}\right).$$

Therefore $\lim\limits_{n\to\infty} \frac{K_n(n)}{L_n(n)} = 1$.

In addition, now note that $\lim\limits_{n\to\infty} K_n(n) = 1$, $\lim\limits_{n\to\infty} L_n(n) = 1$, we have

$$\lim_{n\to\infty} \left(K_n(n) - L_n(n)\right) = 0.$$

This completes the proof of Theorem 2.

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A representation of cyclic commutative asynchronous automata [1]

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Abstract Ito [18] provide representations of strongly connected automata by group-matrix type automata. This shows the close connection between strongly connected automata and its automorphism groups. In this paper we study cyclic commutative asynchronous automata. Some properties on endomorphism monoids of cyclic commutative asynchronous automata are given. Also, the representations of this kind of automata are provided by S-automata.

Keywords Endomorphism monoid, commute asynchronous automata, representation, semilattice-type automata.

§1. Introduction and preliminaries

Automata considered in this paper will be always automata without outputs. That is to say, an automaton $\mathbf{A} = (A, X, \delta)$ consists of the following data:

(1) A is a finite nonempty set, called a state set;

(2) X is a finite nonempty set, called an alphabet;

(3) δ is a function, called a state transition function from $A \times X$ into A.

Let X^* denote the free monoid generated by X. An element of X^* is called a word over X and ε is called the empty word. The state transition function can be extended to the function from $A \times X^*$ to A by

(1) $\delta(a, \varepsilon) = a$ for any $a \in A$;

(2) $\delta(a, xu) = \delta(\delta(a, x), u)$ for any $a \in A, x \in X$ and $u \in X^*$.

Let $\mathbf{A} = (A, X, \delta)$ and $\mathbf{B} = (B, X, \gamma)$ be automata and let ρ be a mapping from A into B. If $\rho(\delta(a, x)) = \gamma(\rho(a), x)$ holds for any $a \in A$ and $x \in X$, then ρ is called a *homomorphism* from $\mathbf{A}$ into $\mathbf{B}$. If a homomorphism ρ is bijective, then ρ is called an *isomorphism.* If there exists an isomorphism from $\mathbf{A}$ onto $\mathbf{B}$, then $\mathbf{A}$and $\mathbf{B}$ are said to be *isomorphic* to each other and denoted by $\mathbf{A} \cong \mathbf{B}$. Moreover, a homomorphism (an isomorphism) from $\mathbf{A}$ onto $\mathbf{A}$ is called an *endomorphism* (an *automorphism*) of $\mathbf{A}$. It is clear that $E(\mathbf{A})(G(\mathbf{A}))$ of all endomorphisms (automorphisms) of $\mathbf{A}$ forms a monoid (group) on the usually composition, called the *endomorphism monoid* (a*utomorphism group*) of $\mathbf{A}$.

The study of endomorphism monoids and automorphism groups of automata was started by [7] and [23] and followed by [8], [24], [1], [22], [4], [3], [2], [21] and [20].

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An automaton $\mathbf{A} = (A, X, \delta)$ is a *strongly connected* automaton if for any pair of states $a, b \in A$ there exist an element $u \in X^*$ such that $\delta(a, u) = b$. Fleck [7] proved that if $\mathbf{A} = (A, X, \delta)$ is a strongly connected automaton, then $E(\mathbf{A}) = G(\mathbf{A})$ and $\mid G(\mathbf{A}) \mid$ divides $\mid A \mid$ (for more results on automorphism group of strongly connected automata refer to [7] and [8]). Following Fleck's work, Ito [15] introduced and studied so called group-matrix type automata of order n on a group G. It is showed that for a strongly connected automata $\mathbf{A} = (A, X, \delta)$, there exists a group-matrix type automaton $\mathbf{A'} = (\widehat{G(\mathbf{A})_n}, X, \delta_\Psi)$ of order n on automorphism group $G(A)$ such that $\mathbf{A'} \cong \mathbf{A}$ (refer to [15-18] or [19], for more details). This give representation of strongly connected automata by group-matrix type automata.

As a counterpart, we want to study the automata whose endomorphism monoid are semilattices and to give representation of this kind of automata by so-called semilattice type automata. We focus on cyclic commutative asynchronous automata and their endomorphism monoids.

An automaton $\mathbf{A} = (A, X, \delta)$ is said to be *commutative* if $\delta(a, uv) = \delta(a, vu)$ for any $a \in A$ and any $u, v \in X^*$. An automaton $\mathbf{A} = (A, X, \delta)$ is an *asynchronous automaton* if $\delta(a, xx) = \delta(a, x)$ for any $a \in A$ and any $x \in X^*$. A *commutative asynchronous automaton* means a commutative and asynchronous automaton. For more information on commutative automata and asynchronous automata, refer to [12-15].

Let $\mathbf{A} = (A, X, \delta)$ be an automaton. A state g in A is called a *generator* of $\mathbf{A}$ (see, [20]) if for any $a \in A$, there exists $x \in X^*$ such that $\delta(g, x) = a$. The set of all generators of $\mathbf{A}$ is denoted by $Gen(\mathbf{A})$. An automaton is said to be *cyclic* if $Gen(\mathbf{A}) \neq \emptyset$. A cyclic commutative asynchronous automaton means a cyclic and commutative asynchronous automaton. The class of all cyclic commutative asynchronous automata is denoted by $\mathcal{CCAA}$.

In Section 2 we study the generator of automata in $\mathcal{CCAA}$. We conclude that an automaton in $\mathcal{CCAA}$ has unique generator. In Section 3 we study endomorphism monoids of automata in $\mathcal{CCAA}$. In Section 4 we give a representation of an automata in $\mathcal{CCAA}$ by S-automata.

For undefined notions and notations concerning automata we refer to [9] and [19].

§2. Generator of automata in $\mathcal{CCAA}$

Recall the following notations. For $w \in X^*$, write $\mid w \mid$ for the length of w and $Con(w)$ for the content of w. Also, $\mid w \mid_x$ denote the number of occurrences of x in w (refer to [9]). In this section we will study commutative asynchronous automata.

Lemma 2.1. Let $\mathbf{A} = (A, X, \delta)$ be a commutative asynchronous automaton and let a, b be a pair of states in $\mathbf{A}$. If there exist a word $w \in X^*$ such that $\delta(a, w) = b$, then $\delta(a, x) = b$ for any $x \in Con(w)$ and so $\delta(b, w) = b$.

Proof. Let $\mathbf{A} = (A, X, \delta)$ be a commutative asynchronous automaton and let a, b be a pair of states in $\mathbf{A}$. Suppose that there exists a word $w \in X^*$ such that $\delta(a, w) = b$. Without lose of generality, assume that $w = x_1x_2x_3$. Then we have

$$\begin{aligned} b &= \delta(a,w) = \delta(a, x_1x_2x_3) \\ &= \delta(a, x_1x_3x_2) && \text{(since } \mathbf{A} \text{ is commutative)} \\ &= \delta(\delta(a, x_1x_3), x_2) \\ &= \delta(\delta(a, x_1x_3), x_2^2) && \text{(since } \mathbf{A} \text{ is asynchronous)} \\ &= \delta(\delta(a, x_1x_2x_3), x_2) \\ &= \delta(b, x_2). \end{aligned}$$

This implies that $\delta(b, x_2) = b$. Similarly, we can show that $\delta(b, x_1) = \delta(b, x_3) = b$. That is to say, $\delta(a, x) = b$ for any $x \in Con(w)$. Hence, it immediately follows that $\delta(b, w) = b$.

Lemma 2.2. Let $\mathbf{A} = (A, X, \delta)$ be a commutative asynchronous automaton and let a, b be a pair of states in $\mathbf{A}$. If $\delta(a, u) = b$ and $\delta(b, v) = a$ for some $u, v \in X^*$, then $a = b$.

Proof. Let $\mathbf{A} = (A, X, \delta)$ be a commutative asynchronous automaton and let a, b be a pair of states in $\mathbf{A}$. Suppose that $\delta(a, u) = b$ and $\delta(b, v) = a$ for some $u, v \in X^*$. Then it immediately follows from Lemma 1 that $\delta(a, v) = a$. Hence, we have

$$a = \delta(b, v) = \delta(\delta(a, u), v) = \delta(a, uv) = \delta(a, vu) = \delta(\delta(a, v), u) = \delta(a, u) = b.$$

This shows that $a = b$.

The above Lemma 2 shows that a commutative asynchronous automaton (A, X, δ) must not be a strongly connected automaton, except for $|A| = 1$.

For an automaton $\mathbf{A} = (A, X, \delta)$ in $\mathcal{CCAA}$, It is true that $\mathbf{A}$ have unique generator. In fact, if g, h be generators of $\mathbf{A}$, then there exist $u, u \in X^*$ such that $\delta(g, u) = h$ and $\delta(h, v) = g$. Thus it follows from Lemma 2 that $g = h$. We have shown

Proposition 2.3. Let $\mathbf{A} = (A, X, \delta)$ be an automaton in $\mathcal{CCAA}$. Then $\mathbf{A}$ have unique generator.

§3. The endomorphism monoids of automata in $\mathcal{CCAA}$

The following give some properties of an endomorphism of automaton $\mathbf{A}$ in $\mathcal{CCAA}$.

Given $\mathbf{A} \in \mathcal{CCAA}$. In order to give the characterizations of the endomorphism monoid of automaton $\mathbf{A}$, the characteristic monoid $C(\mathbf{A})$ of automaton $\mathbf{A}$ is needed (see, [6]). Let $\bar{x}$ denote the set $\{y \in X^* | (\forall a \in A)\delta(a, x) = \delta(a, y)\}$ for any $x \in X^*$ and $C(\mathbf{A})$ the set $\{\bar{x} \mid x \in X^*\}$. Then $C(\mathbf{A})$ is a monoid under the operation defined by $\bar{x}\bar{y} = \overline{xy}$. It is called the characteristic monoid $C(\mathbf{A})$ of automaton $\mathbf{A}$.

Lemma 3.1.[6] If $\mathbf{A} = (A, X, \delta) \in \mathcal{CCAA}$, then

(i) $E(\mathbf{A}) \cong C(\mathbf{A})$;

(ii) $\mid E(\mathbf{A}) \mid = \mid A \mid$.

Let $\mathbf{A} = (A, X, \delta)$ be a commutative asynchronous automaton. For any $u \in X^*$, define mapping λ_u from A into A as follows:

$$\lambda_u(a) = \begin{cases} a & \text{if } u = \varepsilon, \\ \delta(a, x) & \text{if } u \in A^+ (= A^* \setminus \{\varepsilon\}). \end{cases}$$

By $\Lambda(X^*)$, we denote the set $\{\lambda_u \mid u \in X^*\}$.

Proposition 3.2. If $\mathbf{A} = (A, X, \delta) \in \mathcal{CCAA}$, then

(i) $\Lambda(X^*)$ is a commutative idempotent monoid under the usual composition;

(ii) $E(\mathbf{A}) = \Lambda(X^*)$;

(iii) $(E(\mathbf{A}), \preceq)$ is a complete lattice, where $\preceq$ is the natural partial order on the endomorphism monoid $E(\mathbf{A})$.

Proof. To prove part (i), notice that for any $\lambda_u, \lambda_v \in \Lambda(X^*)$ and any $a \in A$, we have

$$\lambda_u \circ \lambda_v(a) = \delta(a, vu) = \delta(a, uv) = \lambda_v \circ \lambda_u(a) = \lambda_{uv}.$$

Also, it is easy to verify that $\lambda_u(\lambda_v\lambda_w) = (\lambda_u\lambda_v)\lambda_w$ and $\lambda_u\lambda_\varepsilon = \lambda_u$ hold for any $u, v, w \in X^*$. Then $\Lambda(X^*)$ form a monoid under the usual composition and λ_ε is the identity. Now, we prove that ${\lambda_u}^2 = \lambda_u$. Since $\mathbf{A}$ is an asynchronous automaton, ${\lambda_u}^2(a) = \delta(a, uu) = \delta(a, u) = \lambda_u(a)$ hold for any $a \in A$. Then $\Lambda(X^*)$ is a commutative idempotent monoid.

To prove part (ii), notice that it is a rutin matter to verify that $C(\mathbf{A}) \cong \Lambda(X^*)$. From Lemma 3 (i) it follows $E(\mathbf{A}) \cong \Lambda(X^*)$. Now, it is enough to show that $\Lambda(X^*) \subseteq E(\mathbf{A})$. For any $\lambda_u \in \Lambda(X^*)$, $x \in X$ and any $a \in A$, we have

$$\lambda_u(\delta(a, x)) = \delta(\delta(a, x), u) = \delta(a, xu) = \delta(a, ux) = \delta(\delta(a, u), x) = \delta(\lambda_u(a)).$$

that is to say, $\lambda_u \in E(\mathbf{A})$ and hence $\Lambda(X^*) \subseteq E(\mathbf{A})$. Therefore, $E(\mathbf{A}) = \Lambda(X^*)$.

To show part (iii), we know form part (i) and (ii) that $E(\mathbf{A})$ is a commutative monoid of idempotents. Then $(E(\mathbf{A}), \preceq)$ is a meet semilattice, where $\preceq$ is the natural partial order defined in [10], as follows:

$$(\forall \rho, \sigma \in E(\mathbf{A}))\rho \preceq \sigma \Longleftrightarrow \rho \circ \sigma = \sigma \circ \rho = \rho.$$

Then, we prove that $(E(\mathbf{A}), \preceq)$ is a lattice. Since the identity mapping $\lambda_\varepsilon \in E(\mathbf{A})$ and $\lambda_\varepsilon \circ \rho = \rho \circ \lambda_\varepsilon = \rho$, then $\rho \preceq \lambda_\varepsilon$ for any $\rho \in E(\mathbf{A})$. Therefore, λ_ε be the top element [5] in the meet semilattice $(E(\mathbf{A}), \preceq)$. Also, we have following truthes: For any $\rho, \sigma \in E(\mathbf{A})$ the greatest lower bond of ρ and σ is $\rho \circ \sigma$; $E(\mathbf{A})$ is finite. By Theorem 2.16 in [5], $(E(\mathbf{A}), \preceq)$ is a complete lattice.

§4. S-automata

In order to provide a representation of automata in $\mathcal{CCAA}$, we introduce S-automaton.

Definition 4.1. Let $(S, \leq)$ be a finite meet semilattice. An automaton $\mathbf{S} = (S, X, \delta_\varphi)$ is called a S-automaton, if the following conditions are satisfied

(1) S is the set of states;

(2) X is a set of inputs;

(3) δ_φ is a state transition function which is defined by $\delta_\varphi(s, x) = s \wedge \varphi(x)$, where $s \in S$, $x \in X$ and φ is a mapping from X into S.

Since X^+ is the free semigroup on X, the mapping φ in the above definition can be extend to a homomorphism from semigroup X^+ into semigroup $(S, \wedge)$ as follows:

$$(\forall x \in X, \forall u \in X^+)\varphi(xu) = \varphi(x) \wedge \varphi(u).$$

It is easy to verify that a S-automaton is a commutative asynchronous automaton.

Let $\mathbf{A} = (A, X, \delta)$ be an automaton in $\mathcal{CCAA}$ and g be the unique generator of $\mathbf{A}$. From Proposition 2 (*ii*), $\Lambda(X^*) = E(\mathbf{A})$. Define a mapping φ from X^* into $\Lambda(X^*)$:

$$(\forall u \in X^*)\varphi(u) = \lambda_u.$$

We can easily verify that φ is a homomorphism from the free monoid X^* into $E(\mathbf{A})$. Furthermore, let $S = E(\mathbf{A})$. By Definition 1, $(E(\mathbf{A}), X, \delta_\varphi)$ is a S-automaton.

Define a mapping θ form A onto $E(\mathbf{A})$:

$$(\forall a \in A)\theta(a) = \lambda_u, \text{ where } \delta(g, u) = a.$$

From Lemma 1, we know that for any $a \in A$ there exists only one word $u \in X^*$ such that $\delta(g, u) = a$. So, θ is well defined. Also, it is easy to verify that θ is bijective.

Now, we will prove that θ is a homomorphism from $\mathbf{A}$ into $((E(\mathbf{A}), X, \delta_\varphi)$ for any $a \in A$ and any $x \in X$, i.e., $\theta(\delta(a, x)) = \delta_\varphi(\theta(a), x)$. Suppose now $a \in A$ and $\delta(g, u) = a$ for some $u \in X^*$. On one hand, for any $b \in A$ we have

$$(\theta(\delta(a, x)))(b) = (\theta(\delta(\delta(g, u), x)))(b) = (\theta(\delta(g, ux)))(b) = \lambda_{ux}(b);$$

On the other hand,

$$\begin{aligned} (\delta_\varphi(\theta(a), x))(b) &= (\delta_\varphi(\lambda_u, x))(b) = (\lambda_u \wedge \varphi(x))(b) \\ &= (\lambda_u \wedge \lambda_x)(b) = (\lambda_u \circ \lambda_x)(b) \\ &= \lambda_{ux}(b). \end{aligned}$$

This implies that $\theta(\delta(a, x)) = \delta_\varphi(\theta(a), x)$ and hence θ is a homomorphism. Therefore, $\mathbf{A} \cong (\Omega(A), X, \delta_\varphi)$.

Thus, we have proved

Theorem 4.2. Let $\mathbf{A} = (A, X, \delta)$ be an automaton in $\mathcal{CCAA}$ and let S be a semilattice such that $S = E(\mathbf{A})$. Then $\mathbf{A}$ is isomorphic to some S-automaton.

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Partial Lagrangian and conservation laws for the perturbed Boussinesq partial differential equation [1]

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Abstract In this paper, the partial Lagrangian approach is developed to construct conservation laws for some perturbed partial differential equations. Using that approach, approximate conservation laws for the perturbed Boussinesq equation.

Keywords Perturbed partial differential equation, partial Lagrangian, conservation law.

§1. Introduction

There are a number of equations with relatively small parameters or perturbed equations arising from mathematics, physics and other applied fields. To solve such problem approximately or to construct an approximation of it gave rise to the perturbation method as well as approximate symmetry method. The two methods have grown up together, whose combination greatly extends the scope and depth of both methods in themselves. This includes their effective use in constructing approximate symmetries and approximate conservation laws for perturbed partial differential equations (PDEs).

On the study of perturbed PDEs, in [1], approximate conservation laws were introduced via the approximate Noether symmetries associated with a Lagrangian of the perturbed equation. The relationship between symmetries and conservation laws was elaborated in [2]. In [3]-[4], it was shown that approximate Lie-Bäcklund symmetries and approximate conserved vectors can be utilized to construct approximate Lagrangians, and thereupon approximate Noether symmetries and new associated conservation laws for perturbed equations can be constructed by using the Lagrangians. In [5], a basis of approximate conservation laws for perturbed PDEs was discussed. In [6], how to construct conservation laws of Euler-Lagrange-type equations via Noether-type symmetry operators associated with partial Lagrangians was shown. In [7], Johnpillai *et al* found an effective way to construct approximate conservation laws of perturbed equations via approximate Nother-type symmetry operators associated with partial Lagrangians. Recently, we gave an exact definition of partial Lagrangian and partial Euler-Lagrange-type equation in [8] to clarify Definition 6 in [7] which is actually the approximate

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Lagrangian, and applied the approach of approximate Noether-type symmetry operators associated with partial Lagrangians to the nonlinear wave equation with damping and obtained its approximate conserved vectors and approximate conservation laws in general form.

In this paper, we intend to discuss the approximate conservation laws for the perturbed Boussinesq equation with weak damping in terms of our new definition of partial Lagrangian and partial Euler-Lagrange-type equation.

One form of the perturbed Boussinesq equation takes [9]

$$u_{tt}+(u^2)_{xx}+u_{xxxx}=\epsilon(\alpha(u)u_x^n+\beta(u)),$$

where $u=u(x,t)$, $\alpha(u)$ and $\beta(u)$ are arbitrary functions, n is any positive integer, ϵ is a small parameter. Specially, when $\epsilon=0$, it degenerates into the Boussinesq equation

$$u_{tt}+(u^2)_{xx}+u_{xxxx}=0.$$

§2. Approximate conversation laws for the perturbed Boussinesq equation

In the following, as applications of the theory presented in [8], we characterize approximate conserved vectors and conservation laws of the perturbed Boussinesq equation with weak damping via approximate Noether-type symmetry operators associated with partial Lagrangians.

$$u_{tt}+(u^2)_{xx}+u_{xxxx}=\epsilon(\alpha(u)u_x^n+\beta(u)). \tag{1}$$

We distinguish the following cases according to the choice of n.

Case 1. $n=1$. Eq. (1) becomes

$$u_{tt}+(u^2)_{xx}+u_{xxxx}=\epsilon(\alpha(u)u_x+\beta(u)). \tag{2}$$

Eq. (2) admits a partial Lagrangian $L=-\frac{1}{2}u_t^2-\frac{1}{2}u_x^2-uu_x^2+\frac{1}{2}u_{xx}^2$. Thus the partial Euler-Lagrange-type equation is

$$\frac{\delta L}{\delta u}=\epsilon(\alpha(u)u_x+\beta(u))-u_x^2.$$

Using Eq. in [8], for $i=1,2$, $k=1$, we have

$$\begin{aligned}&(X_0+\epsilon X_1)L+D_i(\xi_0^i+\epsilon\xi_1^i)L\\ =\;&[(\eta_0-\xi_0^ju_j)+\epsilon(\eta_1-\xi_1^ju_j)][\epsilon(\alpha(u)u_x+\beta(u))-u_x^2]+D_i(B_0^i+\epsilon B_1^i),\end{aligned} \tag{3}$$

where

$$X_0+\epsilon X_1=(\xi_0^1+\epsilon\xi_1^1)\frac{\partial}{\partial t}+(\xi_0^2+\epsilon\xi_1^2)\frac{\partial}{\partial x}+(\eta_0+\epsilon\eta_1)\frac{\partial}{\partial u}+\zeta_0\frac{\partial}{\partial u_t}+\zeta_1\frac{\partial}{\partial u_x}+\zeta_{11}\frac{\partial}{\partial u_{xx}}, \tag{4}$$

and

$$\begin{aligned}
\zeta_0 &= \eta_{0t}+\eta_{0u}u_t-(\xi^1_{0t}+\xi^1_{0u}u_t)u_t-(\xi^2_{0t}+\xi^2_{0u}u_t)u_x-\epsilon[\eta_{1t}+\eta_{1u}u_t \\
&\quad -(\xi^1_{1t}+\xi^1_{1u}u_t)u_t-(\xi^2_{1t}+\xi^2_{1u}u_x)u_x-\xi^1_1u_{tt}-\xi^2_1u_{tx}]+\epsilon(\xi^1_1u_{tt}+\xi^2_1u_{tx}), \quad (5)\\
\zeta_1 &= \eta_{0x}+\eta_{0u}u_x-(\xi^1_{0x}+\xi^1_{0u}u_x)u_t-(\xi^2_{0x}+\xi^2_{0u}u_x)u_x-\epsilon[\eta_{1x}+\eta_{1u}u_x \\
&\quad -(\xi^1_{1x}+\xi^1_{1u}u_x)u_t-(\xi^2_{1x}+\xi^2_{1u}u_x)u_x-\xi^1_1u_{tx}-\xi^2_1u_{xx}]+\epsilon(\xi^1_1u_{xt}+\xi^2_1u_{xx}), \quad (6)\\
\zeta_{11} &= \eta_{0xx}+\eta_{0xu}u_x+(\eta_{0xu}+\eta_{0uu}u_x)u_x+\eta_{0u}u_{xx}-[\xi^1_{0xx}+\xi^1_{0xu}u_x \\
&\quad +(\xi^1_{0xu}+\xi^1_{0uu}u_x)u_x+\xi^1_{0u}u_{xx}]u_t-2(\xi^1_{0x}+\xi^1_{0u}u_x)u_{xt}-[\xi^2_{0xx}+\xi^2_{0xu}u_x \\
&\quad +(\xi^2_{0xu}+\xi^2_{0uu}u_x)u_x+\xi^2_{0u}u_{xx}]u_x-2(\xi^2_{0x}+\xi^2_{0u}u_x)u_{xx}+\epsilon(\xi^1_1u_{xxt}+\xi^2_1u_{xxx}) \\
&\quad +\epsilon\{\eta_{1xx}+\eta_{1xu}u_x+(\eta_{1xu}+\eta_{1uu}u_x)u_x+\eta_{1u}u_{xx}-[\xi^1_{1xx}+\xi^1_{1xu}u_x \\
&\quad +(\xi^1_{1xu}+\xi^1_{1uu}u_x)u_x+\xi^1_{1u}u_{xx}]u_t-2(\xi^1_{1x}+\xi^1_{1u}u_x)u_{tx}-\xi^1_1u_{xxt}-[\xi^2_{1xx}+\xi^2_{1xu}u_x \\
&\quad +(\xi^2_{1xu}+\xi^2_{1uu}u_x)u_x+\xi^2_{1u}u_{xx}]u_x-2(\xi^2_{1x}+\xi^2_{1u}u_x)u_{xx}-\xi^2_1u_{xxx}\}. \quad (7)
\end{aligned}$$

Now Eq. (3) turns to be

$$\begin{aligned}
&-\eta_0u_x^2-\epsilon\eta_1u_x^2-\zeta_0u_t+(1+2u)\zeta_1u_x+\zeta_{11}u_{xx} \\
&+L[\xi^1_{0t}+\xi^1_{0u}u_t+\epsilon(\xi^1_{1t}+\xi^1_{1u}u_t)]+L[\xi^2_{0t}+\xi^2_{0u}u_x+\epsilon(\xi^2_{1x}+\xi^2_{1u}u_x)] \\
=\;&[\eta_0-\xi^1_0u_t-\xi^2_0u_x+\epsilon(\eta_1-\xi^1_1u_t-\xi^2_1u_x)][\epsilon(\alpha(u)u_x+\beta(u))-u_x^2] \\
&+\epsilon(B^1_{1t}+B^1_{1u}u_t)+\epsilon(B^2_{1x}+B^2_{1u}u_x)+B^1_{0t}+B^1_{0u}u_t+B^2_{0x}+B^2_{0u}u_x. \quad (8)
\end{aligned}$$

Substituting expressions of L, ζ_0, ζ_1 and ζ_{11} into Eq. (8) and setting $\epsilon^2=0$, then equating the coefficients of zeroth- and first-order of ϵ including the coefficients of different orders of partial derivatives of u in that to zero, after simplification, we have the following system of determining equations:

$$\begin{aligned}
&\xi^1_0=\xi^2_0=\eta_{0u}=\eta_{0xx}=0, \quad -\eta_{0t}-B^1_{0u}=0, \\
&-B^1_{0t}-B^2_{0x}=0, \quad -(1+2u)\eta_{0x}-B^2_{0u}=0, \\
&\xi^1_1=\xi^2_1=\eta_{1u}=\eta_{1xx}=0, \quad -\eta_{1t}-B^1_{1u}=0, \\
&-B^1_{1t}-B^2_{1x}-\beta(u)\eta_0=0, \quad -(1+2u)\eta_{1x}-B^2_{1u}-\alpha(u)\eta_0=0. \quad (9)
\end{aligned}$$

Solving system (9) results in:

Case 1.1. $\beta''(u)\neq 0$. We have

$$\xi^1_0=\xi^2_0=\xi^1_1=\xi^2_1=\eta_0=0,\ \eta_1=(c_1t+c_2)x+c_3t+c_4,\ B^1_0=g_3(t,x),$$

$$B^2_0=g_2(t,x),\ B^1_1=-(c_1x+c_3)u+g_1(t,x),\ B^2_1=-u(1+u)(c_1t+c_2)+g_4(t,x).$$

Thus, an approximate Noether-type symmetry operator for Eq. (2) reads:

$$\mathcal{X}=\mathcal{X}_0+\epsilon\mathcal{X}_1=\epsilon\left[(c_1t+c_2)x+c_3t+c_4\right]\frac{\partial}{\partial u}.$$

The corresponding approximate conserved vector is obtained by Eq.

$$\mathcal{T}^i=\mathcal{B}^i-L\xi^i-\mathcal{W}\frac{\delta L}{\delta u_i}-\sum_{s\geq 1}D_{i_1\cdots i_s}(\mathcal{W})\frac{\delta L}{\delta u_{ii_1\cdots i_s}}+O(\epsilon^{k+1}), \quad i=1,\ldots,n. \quad (10)$$

in [8] as

$$\begin{aligned}
\mathcal{T}^1 &= \{-(c_1x+c_3)u+[(c_1t+c_2)x+c_3t+c_4]u_t+g_1(t,x)\}\epsilon+g_3(t,x),\\
\mathcal{T}^2 &= \{(u_x+2uu_x+u_{xxx})[(c_1t+c_2)x+c_3t+c_4]-(u^2+u+u_{xx})(c_1t+c_2)\\
&\quad +g_4(t,x)\}\epsilon+g_2(t,x),
\end{aligned}$$

and the approximate conservation law for Eq. (2) is

$$(D_t\mathcal{T}^1+D_x\mathcal{T}^2)|_{\text{Eq.}(2)}=[\alpha(u)u_x+\beta(u)][(c_1t+c_2)x+c_3t+c_4]\epsilon^2=O(\epsilon^2).$$

Where c_i $(i\in Z)$ are arbitrary constants and the functions $g_i\equiv g_i(t,x)$ $(i\in Z)$ satisfy the PDEs: $g_{4,x}+g_{1,t}=0,\ g_{2,x}+g_{3,t}=0$.

Case 1.2. $\beta^{''}(u)=0$.

Case 1.2.1. $\beta^{'}(u)\neq 0$, $\alpha^{'}(u)=0$. We have

$$\xi_0^1=\xi_0^2=\xi_1^1=\xi_1^2=0,\ \eta_0=(c_6t+c_1)x+c_4t+c_5,\ \beta(u)=c_1u+c_2,\ \alpha(u)=c_3,$$

$$\eta_1=\frac{1}{6}\left[c_1(c_6x+c_4)-c_3c_6\right]t^3+\frac{1}{2}\left[c_1(c_7x+c_5)-c_3c_7\right]t^2+(c_8x+c_{10})t+c_9x+c_{11},$$

$$B_0^1=-(c_6x+c_4)u+h_3(t,x),\quad B_0^2=-u(1+u)(c_6t+c_7)+h_2(t,x),$$

$$B_1^1=-\left\{\frac{1}{2}[c_1(c_6x+c_4)-c_3c_6]t^2+c_1(c_7x+c_5)t+c_8x+c_{10}\right\}u+h_1(t,x),$$

$$B_1^2=-u(1+u)\left(\frac{1}{6}c_1c_6t^3+\frac{1}{2}c_1c_7t^2+c_8t+c_9\right)-c_3[(c_6x+c_4)t+c_7x+c_5]u+h_4(t,x).$$

Thus, we obtain the following approximate Noether-type symmetry operator for Eq. (2)

$$\begin{aligned}
\mathcal{X} &= [(c_6t+c_1)x+c_4t+c_5]\frac{\partial}{\partial u}\\
&+\epsilon\left\{\frac{1}{6}[c_1(c_6x+c_4)-c_3c_6]t^3+\frac{1}{2}[c_1(c_7x+c_5)-c_3c_7]t^2+(c_8x+c_{10})t+c_9x+c_{11}\right\}\frac{\partial}{\partial u},
\end{aligned}$$

and the following approximate conserved vector

$$\begin{aligned}
\mathcal{T}^1 &= \left\{\left[\frac{1}{2}(c_3c_6-c_1c_4)t^2+(c_3c_7-c_1c_5)t-\left(\frac{1}{2}c_1t(c_6t+c_7)+c_8\right)x-c_{10}\right]u\right.\\
&\quad+\left[\frac{1}{6}(c_1c_4-c_3c_6)t^3+\frac{1}{2}(c_1c_5-c_3c_7)t^2+\left(\frac{1}{6}c_1c_6t^3+\frac{1}{2}c_1c_7t^2+c_8t+c_9\right)x\right.\\
&\quad\left.\left.+c_{10}t+c_{11}\right]u_t+h_1(t,x)\right\}\epsilon-(c_6x+c_4)u+[(c_6t+c_7)x+c_4t+c_5]u_t+h_3(t,x),\\
\mathcal{T}^2 &= \left\{\left[\frac{1}{6}(c_6(c_1x-c_3)+c_1c_4)t^3+\frac{1}{2}(c_7(c_1x-c_3)+c_1c_5)t^2+(c_8x+c_{10})t\right.\right.\\
&\quad\left.+c_9x+c_{11}\right](u_x+2uu_x+u_{xxx})-\left(\frac{1}{6}c_1c_6t^3+\frac{1}{2}c_1c_7t^2+c_8t+c_9\right)\\
&\quad\left.\times(u^2+u+u_{xx})\ -c_3[(c_6t+c_7)x+c_4t+c_5]u+h_4(t,x)\right\}\epsilon\\
&\quad+[(c_6t+c_7)x+c_4t+c_5](u_x+2uu_x+u_{xxx})-(c_6t+c_7)(u^2+u+u_{xx})+h_2(t,x).
\end{aligned}$$

Eq. (2) is reduced to

$$u_{tt}+(u^2)_{xx}+u_{xxxx}=\epsilon(c_3u_x+c_1u+c_2), \tag{11}$$

then the approximate conservation law for Eq. (11) reads

$$
\begin{aligned}
& (D_t\mathcal{T}^1 + D_x\mathcal{T}^2)|_{\text{Eq.(11)}} \\
= & \left\{\frac{1}{6}[c_1(c_6x+c_4)-c_3c_6]t^3 + \frac{1}{2}[c_1(c_7x+c_5)-c_3c_7]t^2 + (c_8x+c_{10})t + c_9x + c_{11}\right\} \\
& \times(c_3u_x + c_1u + c_2)\epsilon^2 = O(\epsilon^2).
\end{aligned}
$$

Where c_i $(i\in Z)$ are arbitrary constants and functions $h_i \equiv h_i(t,x)$ $(i\in Z)$ satisfy the PDEs: $h_{2,x} + h_{3,t} = 0$, $h_{1,t} + h_{4,x} + c_2[(c_6t+c_7)x + c_4t + c_5] = 0$.

Case 1.2.2. $\beta^{'}(u) \neq 0$, $\alpha^{'}(u)\neq 0$. We have

$$
\begin{aligned}
\xi_0^1 = \xi_0^2 = \xi_1^1 = \xi_1^2 = 0, \quad \eta_0 = c_3t + c_4, \quad B_0^1 = -c_3u + f_3(t,x), \\
\eta_1 = \frac{1}{6}c_1c_3t^3 + \frac{1}{2}c_1c_4t^2 + (c_5x+c_7)t + c_6x + c_8, \quad B_0^2 = f_2(t,x), \\
B_1^1 = -\left(\frac{1}{2}c_1c_3t^2 + c_1c_4t + c_5x + c_7\right)u + f_1(t,x), \\
B_1^2 = -(c_5t+c_6)(u+u^2) - (c_3t+c_4)\int^u \alpha(z)dz + f_4(t,x).
\end{aligned}
$$

Thus, we obtain the following approximate Noether-type symmetry operator for Eq. (2):

$$
\mathcal{X} = (c_3t+c_4)\frac{\partial}{\partial u} + \epsilon\left[\frac{1}{6}c_1c_3t^3 + \frac{1}{2}c_1c_4t^2 + (c_5x+c_7)t + c_6x + c_8\right]\frac{\partial}{\partial u}.
$$

The approximate conserved vector corresponding to the operator $\mathcal{X}$ is

$$
\begin{aligned}
\mathcal{T}^1 = & \left[\left(\frac{1}{6}c_1c_3t^3 + (c_5x+c_7)t + c_6x + c_8\right)u_t - \left(\frac{1}{2}c_1c_3t^2 + c_1c_4t + c_7\right)u + f_1(t,x)\right]\epsilon \\
& +(c_3t+c_4)u_t - c_3u + f_3(t,x), \\
\mathcal{T}^2 = & \left[\left(\frac{1}{6}c_1c_3t^3 + \frac{1}{2}c_1c_4t^2 + (c_5x+c_7)t + c_6x + c_8\right)(u_x + 2uu_x + u_{xxx})\right. \\
& \left. -(c_5t+c_6)(u^2+u+u_{xx}) - (c_3t+c_4)\int^u \alpha(z)dz + f_4(t,x)\right]\epsilon \\
& +(c_3t+c_4)(u_x+2uu_x+u_{xxx}) + f_2(t,x).
\end{aligned}
$$

Then Eq. (2) becomes

$$
u_{tt} + (u^2)_{xx} + u_{xxxx} = \epsilon(\alpha(u)u_x + c_1u + c_2), \tag{12}
$$

and the approximate conservation law for Eq. (12) reads

$$
\begin{aligned}
& (D_t\mathcal{T}^1 + D_x\mathcal{T}^2)|_{\text{Eq.(12)}} \\
= & \left[\frac{1}{6}c_1c_3t^3 + \frac{1}{2}c_1c_4t^2 + (c_5x+c_7)t + c_6x + c_8\right](\alpha(u)u_x + c_1u + c_2)\epsilon^2 = O(\epsilon^2).
\end{aligned}
$$

Where $c_i(i\in Z)$ are arbitrary constants and $f_i \equiv f_i(t,x)$ $(i\in Z)$ are functions satisfying

$$
f_{2,x} + f_{3,t} = 0, \quad f_{1,t} + f_{4,x} + c_2(c_3t+c_4) = 0.
$$

Case 2. $n > 1$. Eq. (1) has a partial Lagrangian $L = -\frac{1}{2}u_t^2 - \frac{1}{2}u_x^2 - uu_x^2 + \frac{1}{2}u_{xx}^2$. The partial Euler-Lagrange-type equation is

$$\frac{\delta L}{\delta u} = \epsilon(\alpha(u)u_x^n + \beta(u)) - u_x^2.$$

Using Eq. in [8], for $i = 1, 2$, $k = 1$, we have

$$\begin{aligned} & (X_0 + \epsilon X_1)L + D_i(\xi_0^i + \epsilon\xi_1^i)L \\ = \ & [(\eta_0 - \xi_0^j u_j) + \epsilon(\eta_1 - \xi_1^j u_j)][\epsilon(\alpha(u)u_x^n + \beta(u)) - u_x^2] + D_i(B_0^i + \epsilon B_1^i), \end{aligned} \tag{13}$$

where $X_0 + \epsilon X_1$, ζ_0, ζ_1 and ζ_{11} are defined by formulae (4)-(7) respectively.

Substitution of the known expressions into Eq. (13) and expansion of it, then the vanishing of the coefficients of zeroth- and first-order of ϵ as well as the coefficients of different orders of partial derivatives of u in that to zero, after simplification, we arrive at the following system of determining equations:

$$\begin{aligned} \xi_0^1 = \xi_0^2 = \eta_{0u} = \eta_{0xx} = \xi_1^1 = \xi_1^2 = \eta_{1u} = \eta_{1xx} = 0, \\ -\eta_0\alpha(u) = 0, \quad -\eta_{0t} - B_{0u}^1 = 0, \quad -B_{0t}^1 - B_{0x}^2 = 0, \\ -(1+2u)\eta_{0x} - B_{0u}^2 = 0, \quad -\eta_{1t} - B_{1u}^1 = 0, \\ -(1+2u)\eta_{1x} - B_{1u}^2 = 0, \quad -B_{1t}^1 - \beta(u)\eta_0 - B_{1x}^2 = 0. \end{aligned} \tag{14}$$

To solve system (14), we distinguish the following two cases.

Case 2.1. $\alpha(u) \neq 0$. We get the following solution to system (14):

$$\xi_0^1 = \xi_0^2 = \xi_1^1 = \xi_1^2 = \eta_0 = 0, \quad \eta_1 = (c_1 t + c_2)x + c_3 t + c_4, \quad B_0^1 = g_4(t,x),$$

$$B_0^2 = g_2(t,x), \quad B_1^1 = -(c_1 x + c_3)u + g_3(t,x), \quad B_1^2 = -u(1+u)(c_1 t + c_2) + g_1(t,x).$$

Thus, we have the following approximate Noether-type symmetry operator of Eq. (1)

$$\mathcal{X} = \epsilon[(c_1 t + c_2)x + c_3 t + c_4]\frac{\partial}{\partial u}.$$

The approximate conserved vector corresponding to the operator $\mathcal{X}$ by formulae (10) is

$$\begin{aligned} \mathcal{T}^1 &= \{-(c_1 x + c_3)u + [(c_1 x + c_3)t + c_2 t + c_4]u_t + g_3(t,x)\}\epsilon + g_4(t,x), \\ \mathcal{T}^2 &= \{[(c_1 x + c_3)t + c_2 x + c_4](u_x + 2uu_x + u_{xxx}) - (c_1 t + c_2)(u^2 + u + u_{xx}) \\ & \quad + g_1(t,x)\}\epsilon + g_2(t,x), \end{aligned}$$

and the approximate conservation law for Eq. (1) reads

$$\begin{aligned} & (D_t\mathcal{T}^1 + D_x\mathcal{T}^2)|_{\text{Eq.(1)}} \\ = \ & [(c_1 x + c_3)t + c_2 x + c_4][\alpha(u)u_x^n + \beta(u)]\epsilon^2 = O(\epsilon^2). \end{aligned}$$

Where c_i $(i \in Z)$ are arbitrary constants and $g_i \equiv g_i(t,x)$ $(i \in Z)$ are functions satisfying

$$g_{2,x} + g_{4,t} = 0, \quad g_{1,x} + g_{3,t} = 0.$$

Case 2.2. $\alpha(u) = 0$. We have the following solution to system (14):

$$\beta(u) = c_1 u + c_2, \quad \alpha(u) = 0, \quad \xi_0^1 = \xi_0^2 = \xi_1^1 = \xi_1^2 = 0, \quad \eta_0 = (c_5 x + c_3)t + c_6 x + c_4,$$

$$\eta_1 = \frac{1}{6}c_1(c_5 x + c_3)t^3 + \frac{1}{2}c_1(c_6 x + c_4)t^2 + (c_9 x + c_7)t + c_{10}x + c_8,$$

$$B_0^1 = -(c_5 x + c_3)u + f_4(t,x), \quad B_0^2 = -u(1+u)(c_5 t + c_6) + f_2(t,x),$$

$$B_1^1 = -\left[\frac{1}{2}c_1(c_5 x + c_3)t^2 + c_1(c_6 x + c_4)t + c_9 x + c_7\right] u + f_3(t,x),$$

$$B_1^2 = -u(1+u)\left(\frac{1}{6}c_1 c_5 t^3 + \frac{1}{2}c_1 c_6 t^2 + c_9 t + c_{10}\right) + f_1(t,x).$$

Thus, an approximate Noether-type symmetry operator for Eq. (1) yields:

$$\begin{aligned}\mathcal{X} &= [(c_5 x + c_3)t + c_6 x + c_4]\frac{\partial}{\partial u} + \epsilon\left[\frac{1}{6}c_1(c_5 x + c_3)t^3\right.\\ &\left.+\frac{1}{2}c_1(c_6 x + c_4)t^2 + (c_9 x + c_7)t + c_{10}x + c_8\right]\frac{\partial}{\partial u}.\end{aligned}$$

The approximate conserved vector of Eq. (1) is given by

$$\begin{aligned}\mathcal{T}^1 &= \left\{\left[\frac{1}{6}c_1(c_5 x + c_3)t^3 + \frac{1}{2}c_1(c_6 x + c_4)t^2 + (c_9 x + c_7)t + c_{10}x + c_8\right] u_t\right.\\ &\left.-\left[\frac{1}{2}c_1(c_5 x + c_3)t^2 + c_1(c_6 x + c_4)t + c_9 x + c_7\right] u + f_3(t,x)\right\}\epsilon\\ &+[(c_5 x + c_3)t + c_6 x + c_4]u_t - (c_5 x + c_3)u + f_4(t,x),\\ \mathcal{T}^2 &= \left\{\left[\frac{1}{6}c_1(c_5 x + c_3)t^3 + \frac{1}{2}c_1(c_6 x + c_4)t^2 + (c_9 x + c_7)t + c_{10}x + c_8\right](u_x + 2uu_x + u_{xxx})\right.\\ &\left.-\left(\frac{1}{6}c_1 c_5 t^3 + \frac{1}{2}c_1 c_6 t^2 + c_9 t + c_{10}\right)(u^2 + u + u_{xx}) + f_1(t,x)\right\}\epsilon\\ &+[(c_5 x + c_3)t + c_6 x + c_4](u_x + 2uu_x + u_{xxx}) - (c_5 t + c_6)(u^2 + u + u_{xx}) + f_2(t,x).\end{aligned}$$

Then Eq. (1) becomes

$$u_{tt} + 2u_x^2 + 2uu_{xx} + u_{xxxx} = \epsilon(c_1 u + c_2), \tag{15}$$

and the approximate conservation law for Eq. (15) is

$$\begin{aligned}&(D_t\mathcal{T}^1 + D_x\mathcal{T}^2)|_{\text{Eq.(15)}}\\ =\ &(c_1 u + c_2)\left[\frac{1}{6}c_1(c_5 x + c_3)t^3 + \frac{1}{2}c_1(c_6 x + c_4)t^2 + (c_9 x + c_7)t + c_{10}x + c_8\right]\epsilon^2\\ =\ &O(\epsilon^2).\end{aligned}$$

Where c_i $(i \in Z)$ are arbitrary constants and $f_i \equiv f_i(t,x)$ $(i \in Z)$ are functions satisfying

$$f_{2,x} + f_{4,t} = 0, \quad f_{1,x} + f_{3,t} + c_2[(c_5 x + c_3)t + c_6 x + c_4] = 0.$$

§3. Concluding remarks

In terms of our exact definition on the concepts of partial Lagrangian and partial Euler-Lagrange-type equation, the approximate conservation laws for the perturbed Boussinesq equationwith weak damping have been derived via the partial Lagrangian approach.

It comes to light that as the order of the perturbed PDE increases its corresponding partial Lagrangian gets more complicated, so only first-order Lagrangian was involved in other literature up to date. A partial Lagrangian for one perturbed PDE may not exist or exist but is not unique. How to determine it and forge links between them? The construction of conservation laws for higher-order perturbed PDEs remains also a problem to be explored.

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Some formulate for the Fibonacci and Lucas numbers

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Abstract This paper mainly study some identities involving Fibonacci and Lucas numbers of interest. By using the properties of Chebyshev polynomials and combining the elementary and combinatorial method, we establish identities that $\sum_{j=0}^{d-1} F_{n+j}F_{m+j}x^j$, $\sum_{j=0}^{d-1} L_{n+j}L_{m+j}x^j$, $\sum_{j=0}^{d-1} F_{n_1+j}F_{n_2+j}F_{n_3+j}$, and $\sum_{j=0}^{d-1} L_{n_1+j}L_{n_2+j}L_{n_3+j}$. These identities are extensions of $\sum_{j=0}^{d-1} F_{n+j}F_{m+j}$ and $\sum_{j=0}^{d-1} L_{n+j}L_{m+j}$ which have been proved by Brian Curtin before.

Keywords Fibonacci numbers, Lucas numbers, Chebyshev polynomials.

§1. Introduction

Let $n \in N$, the Fibonacci sequence $\{F_n\}$ and the Lucas sequence $\{L_n\}$ have attracted the attention of professional as well as amateur mathematicians, and play an important role in many fields of mathematics. Also there exist many identities involving these sequences of interest. See reference [1] for a good summary. Now we turn to the Chebyshev polynomials of the first and second kind $T_n(x)$ and $U_n(x)(n = 0, 1, \cdots)$ which are given by

$$T_n(x) = \frac{1}{2}\left[(x + \sqrt{x^2-1})^n + (x - \sqrt{x^2-1})^n\right], \quad |x| < 1; \tag{1}$$

$$U_n(x) = \frac{1}{2\sqrt{x^2-1}}\left[(x + \sqrt{x^2-1})^{n+1} - (x - \sqrt{x^2-1})^{n+1}\right], \quad |x| < 1. \tag{2}$$

In 2007, Ma and Zhang [3] showed two nice connections between the Cheyshev polynomials and Fibonacci sequence and Lucas sequence, respectively. That is, let i be the square root of -1, m and n be any positive integers, then we have the identities

$$T_n\left(\frac{i}{2}\right) = \frac{i^n}{2}L_n, \text{ and } U_n\left(\frac{i}{2}\right) = i^n F_{n+1}. \tag{3}$$

Inspired by the work of professor Zhang Wenpeng, in this paper, we establish some combinational identities involving the Fibonacci and Lucas numbers, which continue the work of Brian Curtin [4] in a different way.

§2. Preparations for the proofs of the theorems

Firstly we note that there are following nice connections between Fibonacci and Lucas numbers (see [1]).

$$\begin{cases} L_{n+m} = F_{m-1}L_n + F_m L_{n+1}, \\ L_{n-m} = (-1)^m (F_{m+1}L_n - F_m L_{n+1}); \end{cases} \tag{4}$$

$$\begin{cases} F_m L_n + F_n L_m = 2F_{n+m}, \\ F_n L_m - F_m L_n = 2(-1)^m F_{n-m}; \end{cases} \tag{5}$$

$$\begin{cases} L_n L_m = L_{n+m} + (-1)^m L_{n-m}, \\ 5F_n F_m = L_{n+m} - (-1)^m L_{n-m}. \end{cases} \tag{6}$$

On the other hand, as to the two kinds of Chebyshev polynomials, we introduce a group of formulas which are useful later, when we deal with the main results in this paper.

Lemma 2.1. For all positive integers n and m,

$$T_{n+m}(x) = T_n(x)U_m(x) - T_{n-1}(x)U_{m-1}(x); \tag{7}$$

$$U_{n+m}(x) = U_n(x)T_m(x) + T_{n+1}(x)U_{m-1}(x). \tag{8}$$

Proof. To make the situation quite clear, we denote $A = x+\sqrt{x^2-1}$, and $B = x-\sqrt{x^2-1}$. Hence from (1.1) and (1.2), the terms on the right-hand side of the identity (2.4) can be written

$$\begin{aligned} & T_n(x)U_m(x) - T_{n-1}(x)U_{m-1}(x) \\ = & \frac{1}{4\sqrt{x^2-1}}\left[(A^n + B^n)(A^{m+1} - B^{m+1}) - (A^{n-1} + B^{n-1})(A^m - B^m)\right] \\ = & \frac{1}{4\sqrt{x^2-1}}\left[A^{n+m}(A - \frac{1}{A}) - B^{n+m}(B - \frac{1}{B})\right] = \frac{1}{4\sqrt{x^2-1}}(A^{n+m} + B^{n+m})(A-B) \\ = & \frac{1}{2}(A^{n+m} + B^{n+m}) = T_{n+m}(x). \end{aligned}$$

This proves identity (2.4). In fact, the equation (2.5) is also easily to be proved in the same way. Moreover, from lemma we will say more. Let $x = \frac{i}{2}$, then using identities (1.3), (2.4) and (2.5), we can have

$$\begin{cases} \frac{i^{n+m}}{2} L_{n+m} = \frac{i^{n+m}}{2} F_{m-1}L_n - \frac{i^{n+m-2}}{2} F_m L_{n+1}; \\ i^{n+m-1} F_{n+m} = \frac{i^{n+m-1}}{2} F_m L_n + \frac{i^{n+m-1}}{2} F_n L_m. \end{cases} \tag{9}$$

That is

$$\begin{cases} L_{n+m} = F_{m-1}L_n + F_m L_{n+1}; \\ 2F_{n+m} = F_m L_n + F_n L_m. \end{cases} \tag{10}$$

These formulas have been mentioned above, as a problem to be proved in reference [1], but the method used here is different and more simple.

§3. Some summations for Fibonacci and Lucas numbers

In this section, we will introduce you some summation involving Fibonacci and Lucas numbers.

Theorem 3.1. For positive integers d, m, n, and real numbers $x \neq -1$,

$$\begin{aligned}
\sum_{j=0}^{d-1} F_{n+j}F_{m+j}x^j &= \frac{x^d}{5(x^2-3x+1)}(xL_{n+m+2d-2} - L_{n+m+2d} - xL_{n+m-2} + L_{n+m}) \\
&- (-1)^m \frac{1-(-1)^d}{5(x+1)} L_{n-m}; \qquad (11) \\
\sum_{j=0}^{d-1} L_{n+j}L_{m+j}x^j &= \frac{x^d}{x^2-3x+1}(xL_{n+m+2d-2} - L_{n+m+2d} - xL_{n+m-2} + L_{n+m}) \\
&+ (-1)^m \frac{1-(-1)^d}{x+1} L_{n-m}. \qquad (12)
\end{aligned}$$

Proof. According to the identity (1.5), we set $P(x,y) = \sum_{j=0}^{d-1}(-x)^j U_{n+j-1}(y)U_{m+j-1}(y)$, for real numbers x and complex numbers y. So that if $y = \frac{i}{2}$, we can obtain

$$\sum_{j=0}^{d-1} F_{n+j}F_{m+j}x^j = \frac{1}{i^{m+n}} P\left(x, \frac{i}{2}\right). \qquad (13)$$

Now we use (2.5) to write

$$\begin{aligned}
P(x,y) &= -\sum_{j=0}^{d-1}(-x)^j \left(U_{n-1}(y)T_j(y) + T_n(y)U_{j-1}(y)\right)\left(U_{m-1}(y)T_j(y) + T_m(y)U_{j-1}(y)\right) \\
&= -U_{n-1}(y)U_{m-1}(y)M_1(x,y) - \left(U_{n-1}(y)T_m(y) + U_{m-1}(y)T_n(y)\right)M_2(x,y) \\
&\quad -T_n(y)T_m(y)M_3(x,y),
\end{aligned}$$

where $M_1(x,y) = \sum_{j=0}^{d-1}(-x)^j T_j^2(y)$, $M_2(x,y) = \sum_{j=0}^{d-1}(-x)^j T_j(y)U_{j-1}(y)$, and $M_3(x,y)$ $= \sum_{j=0}^{d-1}(-x)^j U_{j-1}^2(y)$. Because of formulas (1.3)

$$P\left(x, \frac{i}{2}\right) = i^{m+n}F_nF_mM_1\left(x, \frac{i}{2}\right) - \frac{i^{n+m}}{2}(F_nL_m + L_nF_m)M_2\left(x, \frac{i}{2}\right) - \frac{i^{n+m}}{4}L_mL_nM_3\left(x, \frac{i}{2}\right).$$

Then combining (3.3)

$$\sum_{j=0}^{d-1} F_{n+j}F_{m+j}x^j = F_nF_mM_1\left(x, \frac{i}{2}\right) - \frac{1}{2i}(F_nL_m + L_nF_m)M_2\left(x, \frac{i}{2}\right) - \frac{1}{4}L_mL_nM_3\left(x, \frac{i}{2}\right). \qquad (14)$$

From above, it suffice to compute $M_1\left(x,\frac{i}{2}\right)$, $M_2\left(x,\frac{i}{2}\right)$, and $M_3\left(x,\frac{i}{2}\right)$, respectively. However by using (1.1) and if $x \neq -1$,

$$\begin{aligned}
M_1\left(x,\frac{i}{2}\right) &= \frac{1}{4}\sum_{j=0}^{d-1}(-x)^j(A^{2j}+2+B^{2j}) \\
&= \frac{1}{4}\left[\frac{(-x)^dA^{2d}-1}{-xA^2-1}+2\frac{(-x)^d-1}{-x-1}+\frac{(-x)^dB^{2d}-1}{-xB^2-1}\right] \\
&= \frac{1}{4}\left[\frac{(-x)^{d+1}A^{2d-2}-(-x)^dA^{2d}+xB^2+(-x)^{d+1}B^{2d-2}-(-x)^dB^{2d}+xA^2+2}{(xA^2+1)(xB^2+1)}\right. \\
&\quad \left.+2\frac{1-(-x)^d}{x+1}\right] \\
&= \frac{(-x)^{d+1}T_{2d-2}(y)-(-x)^dT_{2d}(y)}{2(xA^2+1)(xB^2+1)}+\frac{x(A^2+B^2)+2}{4(xA^2+1)(xB^2+1)}+\frac{1-(-x)^d}{2(x+1)}.
\end{aligned}$$

Then let $y=\frac{i}{2}$, we obtain

$$M_1\left(x,\frac{i}{2}\right)=\frac{x^d}{4(x^2-3x+1)}(xL_{2d-2}-L_{2d})+\frac{2-3x}{4(x^2-3x+1)}+\frac{1-(-x)^d}{2(x+1)}.$$

In the same way, it is easy to derive

$$\begin{aligned}
M_2\left(x,\frac{i}{2}\right) &= \frac{-ix^d}{2(x^2-3x+1)}(xF_{2d-2}-F_{2d})-\frac{xi}{2(x^2-3x+1)}; \\
M_3\left(x,\frac{i}{2}\right) &= \frac{-x^d}{5(x^2-3x+1)}(xL_{2d-2}-L_{2d})+\frac{3x-2}{5(x^2-3x+1)}+2\frac{1-(-x)^d}{5(x+1)}.
\end{aligned}$$

Then take them into (3.4), we have

$$\begin{aligned}
&\sum_{j=0}^{d-1}F_{n+j}F_{m+j}x^j \\
&= \frac{x^{d+1}}{20(x^2-3x+1)}\left[5F_nF_mL_{2d-2}+5(F_nL_m+L_nF_m)F_{2d-2}+L_nL_mL_{2d-2}\right] \\
&-\frac{x^d}{20(x^2-3x+1)}\left[5F_nF_mL_{2d}+5(F_nL_m+L_nF_m)F_{2d}+L_nL_mL_{2d}\right] \\
&+\frac{x}{20(x^2-3x+1)}\left[-15F_nF_m+5F_nL_m+5L_nF_m-3L_nL_m\right] \\
&+\frac{1}{20(x^2-3x+1)}\left[10F_nF_m+2L_nL_m\right]+\frac{1-(-x)^d}{10(x+1)}(5F_nF_m-L_nL_m).
\end{aligned}$$

Observing the relationships between Fibonacci and Lucas numbers such as identities (2.1), (2.2), and (2.3), it follows that

$$\begin{aligned}
\sum_{j=0}^{d-1}F_{n+j}F_{m+j}x^j &= \frac{x^d}{5(x^2-3x+1)}(xL_{n+m+2d-2}-L_{n+m+2d}-xL_{n+m-2}+L_{n+m}) \\
&\quad -(-1)^m\frac{1-(-1)^d}{5(x+1)}L_{n-m}.
\end{aligned}$$

We concentrate now on the proofs of formula (3.2). At this time, it is only need to let $P(x,y)=\sum_{j=0}^{d-1}(-x)^jT_{n+j}(y)T_{m+j}(y)$, so that $\sum_{j=0}^{d-1}L_{n+j}L_{m+j}x^j=\frac{4}{i^{n+m}}P\left(x,\frac{i}{2}\right)$. Similarly, we use (2.4) to write

$$\begin{aligned}P(x,y) &= \sum_{j=0}^{d-1}(-x)^j\left(T_n(y)U_j(y)-T_{n-1}(y)U_{j-1}(y)\right)\left(T_{m-1}(y)U_j(y)-T_{m-1}(y)U_{j-1}(y)\right)\\ &= T_n(y)T_m(y)M_1(x,y)-\left(T_n(y)T_{m-1}(y)+T_{n-1}(y)T_m(y)\right)M_2(x,y)\\ &\quad -T_{n-1}(y)T_{m-1}(y)M_3(x,y),\end{aligned}$$

where $M_1(x,y)=\sum_{j=0}^{d-1}(-x)^jU_j^2(y)$, $M_2(x,y)=\sum_{j=0}^{d-1}(-x)^jU_j(y)U_{j-1}(y)$, and $M_3(x,y)$ $=\sum_{j=0}^{d-1}(-x)^jU_{j-1}^2(y)$. Hence

$$\begin{aligned}\sum_{j=0}^{d-1}L_{n+j}L_{m+j}x^j &= L_nL_mM_1\left(x,\frac{i}{2}\right)-\frac{1}{i}(L_nL_{m-1}+L_{n-1}L_m)M_2\left(x,\frac{i}{2}\right)\\ &\quad -L_{m-1}L_{n-1}M_3\left(x,\frac{i}{2}\right).\end{aligned}\tag{15}$$

After computing $M_1\left(x,\frac{i}{2}\right)$, $M_2\left(x,\frac{i}{2}\right)$, and $M_3\left(x,\frac{i}{2}\right)$, respectively, we have if $x\neq -1$

$$\begin{aligned}M_1\left(x,\frac{i}{2}\right) &= \frac{x^d}{5(x^2-3x+1)}(xL_{2d}-L_{2d+2})-\frac{2x-3}{5(x^2-3x+1)}+2\frac{1-(-x)^d}{5(x+1)};\\ M_2\left(x,\frac{i}{2}\right) &= \frac{-ix^d}{5(x^2-3x+1)}(xL_{2d-1}-L_{2d+1})-\frac{i(x+1)}{5(x^2-3x+1)}+i\frac{1-(-x)^d}{5(x+1)};\\ M_3\left(x,\frac{i}{2}\right) &= \frac{x^d}{5(x^2-3x+1)}(xL_{2d}-xL_{2d-2})+\frac{3x-2}{5(x^2-3x+1)}+2\frac{1-(-x)^d}{5(x+1)}.\end{aligned}$$

Then take them into (3.5), we derive

$$\begin{aligned}\sum_{j=0}^{d-1}L_{n+j}L_{m+j}x^j &= \frac{x^d}{x^2-3x+1}(xL_{n+m+2d-2}-L_{n+m+2d}-xL_{n+m-2}+L_{n+m})\\ &\quad +(-1)^m\frac{1-(-1)^d}{x+1}L_{n-m}.\end{aligned}$$

§4. Further study

However, we can say something more. In reference [4], Brian Curtin has given that

$$\sum_{j=0}^{d-1}F_{n+j}F_{m+j}=\begin{cases}F_dF_{n+m+d-1}, & \text{if d is even,}\\ \frac{1}{5}(L_dL_{n+m+d-1}-(-1)^nL_{m-n}), & \text{if d is odd.}\end{cases}$$

And

$$\sum_{j=0}^{d-1}L_{n+j}L_{m+j}=\begin{cases}5F_dF_{n+m+d-1}, & \text{if d is even,}\\ L_dL_{n+m+d-1}+(-1)^nL_{m-n}, & \text{if d is odd.}\end{cases}$$

By comparing these results with those in Corollary 3.2, we can easily derive an interest result:

$$L_{n+2d-1}-L_{n-1}=\begin{cases} 5F_dF_{n+d-1}, & \text{if } 2\mid \text{d},\\ L_dL_{n+d-1}, & \text{if } 2\nmid \text{d}.\end{cases} \tag{16}$$

Clearly, from these formulas many more relationships emerge by specialization. In particular, if we let $x=1$, identities (3.1) and (3.2) will produce the following identities which has been proved by Brian Curtin in reference [4], and his results are just one of the corollaries of Theorem 3.1. That is

Corollary 3.2. For positive integers d, m, and n,

$$\sum_{j=0}^{d-1}F_{n+j}F_{m+j}=\begin{cases} \frac{1}{5}(L_{n+m+2d-1}-L_{n+m-1}), & \text{if } 2\mid \text{d},\\ \frac{1}{5}(L_{n+m+2d-1}-L_{n+m-1}-(-1)^mL_{n-m}), & \text{if } 2\nmid \text{d};\end{cases} \tag{17}$$

$$\sum_{j=0}^{d-1}L_{n+j}L_{m+j}=\begin{cases} L_{n+m+2d-1}-L_{n+m-1}, & \text{if } 2\mid \text{d},\\ L_{n+m+2d-1}-L_{n+m-1}+(-1)^mL_{n-m}, & \text{if } 2\nmid \text{d}.\end{cases} \tag{18}$$

Here we should note that in the proof of Theorem 3.1, we hypothesize $x\neq -1$, and if let $x=-1$, we can derive the Corollary 3.3 similarly.

Corollary 3.3. For positive integers d, m, and n,

$$\sum_{j=0}^{d-1}(-1)^jF_{n+j}F_{m+j}=\begin{cases} \frac{1}{5}(F_{n+m-1}-F_{n+m+2d-1}), & \text{if } 2\mid \text{d},\\ \frac{1}{5}(F_{n+m+2d-1}-F_{n+m-1}-(-1)^m dL_{n-m}), & \text{if } 2\nmid \text{d};\end{cases} \tag{19}$$

$$\sum_{j=0}^{d-1}(-1)^jL_{n+j}L_{m+j}=\begin{cases} F_{n+m-1}-F_{n+m+2d-1}, & \text{if } 2\mid \text{d},\\ F_{n+m+2d-1}-F_{n+m-1}+(-1)^m dL_{n-m}, & \text{if } 2\nmid \text{d}.\end{cases} \tag{20}$$

What's more, similar methods can be applied when $\sum_{j=0}^{d-1}F_{n+j}F_{m+j}x^j$, and $\sum_{j=0}^{d-1}L_{n+j}L_{m+j}x^j$ are replaced by $\sum_{k=0}^{d-1}F_{m+k}F_{n+k}F_{e+k}$, and $\sum_{j=0}^{d-1}L_{n+j}L_{m+j}L_{e+j}$. As I know, this problem hasn't been studied before. Next we will show the processes of the proofs in detail, and the results are given as follows:

Theorem 3.4. For positive integers n_1, n_2, n_3 and d,

$$\begin{aligned}
5\sum_{j=0}^{d-1} F_{n_1+j}F_{n_2+j}F_{n_3+j} &= \frac{1}{2}L_{n_1+n_2+n_3}F_{3d} - \frac{1}{2}F_{n_1+n_2+n_3-1} + \\
&\quad (-1)^d \sum_{i=0}^{3}(-1)^{n_i}F_{n_1+n_2+n_3+d-n_i-2} \\
&\quad - \sum_{i=0}^{3}(-1)^{n_i}F_{n_1+n_2+n_3-n_i-2}; \qquad (21)\\
\sum_{j=0}^{d-1} L_{n_1+j}L_{n_2+j}L_{n_3+j} &= \frac{1}{2}L_{n_1+n_2+n_3+3d-1} - \frac{1}{2}L_{n_1+n_2+n_3-1} \\
&\quad -(-1)^d \sum_{i=0}^{3}(-1)^{n_i}L_{n_1+n_2+n_3+d-n_i-2} \\
&\quad - \sum_{i=0}^{3}(-1)^{n_i}L_{n_1+n_2+n_3-n_i-2}; \qquad (22)
\end{aligned}$$

Proof. To make the processes more clear, we use n, m and e instead of n_1, n_2 and n_3. This time we let $P(x) = \sum_{k=0}^{d-1} i^{k+3}U_{m+k-1}(x)U_{n+k-1}(x)U_{e+k-1}(x)$. It follows from (1.3) that

$$\sum_{j=0}^{d-1} F_{n+j}F_{m+j}F_{e+j} = \frac{1}{i^{m+n+e}}P\left(\frac{i}{2}\right). \qquad (23)$$

On the other hand,

$$\begin{aligned}
&P(x) = \sum_{k=0}^{d-1} i^{k+3}U_{m+k-1}(x)U_{n+k-1}(x)U_{e+k-1}(x) \\
= &\sum_{k=0}^{d-1} i^{k+3}\left[U_{n-1}(x)T_k(x) - T_n(x)U_{k-1}(x)\right]\left[U_{m-1}(x)T_k(x) - T_m(x)U_{k-1}(x)\right] \\
&\left[U_{e-1}(x)T_k(x) - T_e(x)U_{k-1}(x)\right] \\
= &\; i^3U_{n-1}U_{m-1}U_{e-1}M_1(x) + i^3\left[U_{m-1}(x)U_{n-1}(x)T_e(x) + U_{m-1}(x)T_n(x)U_{e-1}(x)\right. \\
&\left. +T_m(x)U_{n-1}(x)U_{e-1}(x)\right]M_2(x) + i^3[T_m(x)U_{n-1}(x)T_e(x) + U_{m-1}(x)T_n(x)T_e(x) \\
&+T_m(x)T_n(x)U_{e-1}(x)]M_3(x) + i^3T_m(x)T_n(x)T_e(x)M_4(x).
\end{aligned}$$

where $M_1(x) = \sum_{k=0}^{d-1} i^kT_k^3(x)$, $M_2(x) = \sum_{k=0}^{d-1} i^kT_k^2(x)U_{k-1}(x)$, $M_3(x) = \sum_{k=0}^{d-1} i^kT_k(x)U_{k-1}^2(x)$, and

$M_4(x)=\sum_{k=0}^{d-1} i^k U_{k-1}^3(x)$. Therefore let $x=\frac{i}{2}$ and combining identity (3.12), we derive

$$\begin{aligned}\sum_{j=0}^{d-1} F_{n+j}F_{m+j}F_{e+j} &= F_nF_mF_eM_1\left(\frac{i}{2}\right)+\frac{i}{2}\left(F_nF_mL_e+F_nL_mF_e+L_nF_mF_e\right)M_2\left(\frac{i}{2}\right)\\ &\quad -\frac{1}{4}\left(F_nL_mL_e+L_nL_mF_e+L_nF_mL_e\right)M_3\left(\frac{i}{3}\right)-\frac{i}{8}L_nL_mL_eM_4\left(\frac{i}{3}\right)\\ &= aM_1\left(\frac{i}{2}\right)+\frac{ib}{2}M_2\left(\frac{i}{2}\right)-\frac{c}{4}M_3\left(\frac{i}{2}\right)-\frac{id}{8}M_4\left(\frac{i}{2}\right),\end{aligned}$$

where $a=F_nF_mF_e$, $b=F_nF_mL_e+F_nL_mF_e+L_nF_mF_e$, $c=F_nL_mL_e+L_nL_mF_e+L_nF_mL_e$, and $d=L_nL_mL_e$. By computing $M_i(x)$, $(i=1,2,3,4)$, we obtain

$$\begin{aligned}M_1(x)&=\frac{1}{16}L_{3d-1}-\frac{3}{8}i^{2d}L_{d-2}+\frac{19}{16}; \quad M_2(x)=\frac{1}{8i}F_{3d-1}-\frac{1}{4i}i^{2d}F_{d-2}-\frac{3}{8i};\\ M_3(x)&=-\frac{1}{20}L_{3d-1}-\frac{1}{10}i^{2d}L_{d-2}+\frac{1}{4}; \quad M_4(x)=-\frac{1}{10i}F_{3d-1}-\frac{3}{5i}i^{2d}F_{d-2}+\frac{i}{2}.\end{aligned}$$

Take this group of values into identity above, we have

$$\begin{aligned}\sum_{j=0}^{d-1} F_{n+j}F_{m+j}F_{e+j} &= \frac{1}{80}(5a+c)L_{3d-1}+\frac{1}{80}(5b+d)F_{3d-1}+\frac{i^{2d}}{40}(-15a+c)L_{d-2}\\ &\quad +\frac{i^{2d}}{40}(-5b+3d)F_{d-2}+\frac{1}{16}(19a-3b-c+d).\end{aligned}$$

Compute that

$$\begin{aligned}5a+c &= 5F_nF_mF_e+F_nL_mL_e+L_nL_mF_e+L_nF_mL_e\\ &= (5F_nF_m+L_nL_m)L_e+5(F_nL_m+L_nF_m)F_e\\ &= 2L_{n+m}L_e+10F_{n+m}F_e=4L_{n+m+e},\end{aligned}$$

and

$$\begin{aligned}5b+d &= 5F_nF_mL_e+5F_nL_mF_e+5L_nF_mF_e+L_nL_mL_e\\ &= (5F_nF_m+L_nL_m)L_e+5(F_nL_m+L_nF_m)F_e\\ &= 2L_{n+m}L_e+10F_{n+m}F_e=4L_{n+m+e}.\end{aligned}$$

The processes to compute $-15a+c$, $-5b+3d$, and $19a-3b-c+d$ are omitted here, and the final results are as follows:

$$\begin{aligned}-15a+c&=4(-1)^eF_{n+m-e}+4(-1)^mF_{n+e-m}+4(-1)^nF_{e+m-n};\\ -5b+3d&=4(-1)^eL_{n+m-e}+4(-1)^mL_{n+e-m}+4(-1)^nL_{e+m-n}.\end{aligned}$$

And let C be the constant numbers $C=\frac{1}{16}(19a-3b-c+d)$, we have

$$C=-\frac{1}{10}F_{n+m+e-1}-\frac{(-1)^e}{5}F_{n+m-e-2}-\frac{(-1)^n}{5}F_{e+m-n-2}-\frac{(-1)^m}{5}F_{e+n-m-2}.$$

Thus

$$\begin{aligned}
&\sum_{j=0}^{d-1} F_{n+j}F_{m+j}F_{e+j} \\
= \;& \frac{1}{20}\left(L_{n+m+e}L_{3d-1}+L_{n+m+e}F_{3d-1}\right)+\frac{i^{2d}}{10}\left((-1)^e F_{n+m-e}L_{d-2}\right. \\
& +(-1)^e L_{n+m-e}F_{d-2}+(-1)^m F_{n+e-m}L_{d-2}+(-1)^m L_{n+e-m}F_{d-2} \\
& \left.+(-1)^n F_{e+m-n}L_{d-2}+(-1)^n L_{e+m-n}F_{d-2}+C\right) \\
= \;& \frac{1}{10}L_{n+m+e}F_{3d}+\frac{i^{2d}}{5}\left((-1)^e F_{n+m+d-e-2}+(-1)^m F_{n+e+d-m-2}\right. \\
& \left.+(-1)^n F_{e+m+d-n-2}\right)+C.
\end{aligned}$$

Formula (3.11), as a matter of fact, is easily to be proved in precisely same way. So that we will not show the processes of proof any more. Hence, we have finished the proofs of the theorems in this paper by now.

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Some identities involving the classical Catalan Numbers [1]

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Abstract The classical Catalan number is an important counting function, and it has a wide application in combinational mathematics and graph theory. The main purpose of this paper is using the elementary method to study one kind summation problem involving the classical Catalan numbers, and give some interesting identities for it.

Keywords Catalan numbers, elementary method, identities.

§1. Introduction

For any positive integer n, the famous Catalan numbers b_n are defined as follows:

$$b_n = \frac{\binom{2n}{n}}{n+1}, \ n = 0, \ 1, \ 2, \ 3. \cdots .$$

For example, the first several Catalan numbers are: b_0=1, b_1=1, b_2=2, b_3=5, b_4=14, b_5=42, b_6=132,$\cdots$. This sequence has some wide applications in combinational mathematics and graph theory. It had been studied by our qing mathematician Antu Ming. There are still many people have studied its properties at present, some related papers see references [1] and [2].

In this paper, we shall study the calculating problem of the summation

$$\sum_{a_1+a_2+\cdots+a_k=n} b_{a_1} b_{a_2} \cdots b_{a_k}, \tag{1}$$

where $\sum\limits_{a_1+a_2+\cdots+a_k}$ denotes the summation over all k-tuples with no-negative integer coordinates $(a_1, a_2, \cdots, a_k)$ such that $a_1 + a_2 + \cdots + a_k = n$.

We shall use the elementary method to give an exact calculating formula for (1). That is, we shall prove the following conclusions:

Theorem 1. For any positive integers n and k with $2 \le k \le n$, we have the identity

$$\sum_{a_1+a_2+\cdots+a_k=n} b_{a_1} b_{a_2} \cdots b_{a_k} = \sum_{m=0}^{k} (-1)^{m+k+n} \binom{k}{m} \frac{2^n \cdot m!!}{(n+k)!(m-2k-2n)!!}.$$

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Theorem 2. For any positive integers n and k with $2 \leq k \leq n$, the summation

$$\sum_{a_1+a_2+\cdots+a_k=n} b_{a_1}b_{a_2}\cdots b_{a_k}$$

can be expressed by the linear combination of $b_{n+1},\ b_{n+2}\cdots$. Especially, for $k = 2,\ 3,\ 4,\ 5,\ 6,\ 7$, we have the identities

$$\sum_{a_1+a_2=n} b_{a_1}b_{a_2} = b_{n+1}.$$

$$\sum_{a_1+a_2+a_3=n} b_{a_1}b_{a_2}b_{a_3} = b_{n+2} - b_{n+1}.$$

$$\sum_{a_1+a_2+a_3+a_4=n} b_{a_1}b_{a_2}b_{a_3}b_{a_4} = b_{n+3} - 2b_{n+2}.$$

$$\sum_{a_1+a_2+a_3+a_4+a_5=n} b_{a_1}b_{a_2}b_{a_3}b_{a_4}b_{a_5} = b_{n+4} + b_{n+2} - 3b_{n+3}.$$

$$\sum_{a_1+a_2+a_3+a_4+a_5+a_6=n} b_{a_1}b_{a_2}b_{a_3}b_{a_4}b_{a_5}b_{a_6} = b_{n+5} + 3b_{n+3} - 4b_{n+4}.$$

$$\sum_{a_1+a_2+a_3+a_4+a_5+a_6+a_7=n} b_{a_1}b_{a_2}b_{a_3}b_{a_4}b_{a_5}b_{a_6}b_{a_7} = b_{n+6} + 6b_{n+4} - b_{n+3} - 5b_{n+5}.$$

§2. Proof of the theorems

In this section, we shall use the elementary methods and the properties of the Catalan numbers to prove our Theorems directly. First we prove Theorem 1. From the properties of the Catalan numbers we know that

$$2(1-\sqrt{1-x}) = x\sum_{n=0}^{\infty}\frac{b_n x^n}{4^n}. \tag{2}$$

Then from the properties of the power series we have

$$\left(2(1-\sqrt{1-x})\right)^k = x^k\left(\sum_{n=0}^{\infty}\frac{b_n x^n}{4^n}\right)^k = x^k\sum_{n=0}^{\infty}\left(\sum_{a_1+a_2+\cdots+a_k=n} b_{a_1}b_{a_2}\cdots b_{a_k}\right)\frac{x^n}{4^n}. \tag{3}$$

On the other hand, note that the power series expansion of $(1-x)^{\frac{m}{2}}$

$$(1-x)^{\frac{m}{2}} = \sum_{n=0}^{\infty}\frac{(-1)^n\cdot m!!}{2^n\cdot n!\cdot(m-2n)!!}x^n. \tag{4}$$

Applying (4) we have

$$\begin{aligned}\left(2(1-\sqrt{1-x})\right)^k &= 2^k\cdot\sum_{m=0}^{k}\tbinom{k}{m}(-1)^m(1-x)^{\frac{m}{2}}\\ &= 2^k + 2^k\cdot\sum_{m=1}^{k}\tbinom{k}{m}(-1)^m\sum_{n=0}^{\infty}\frac{(-1)^n\cdot m!!}{2^n\cdot n!\cdot(m-2n)!!}x^n\\ &= 2^k + 2^k\cdot\sum_{n=0}^{\infty}\frac{(-1)^n}{2^n\cdot n!}\left(\sum_{m=1}^{k}\binom{k}{m}(-1)^m\frac{m!!}{(m-2n)!!}\right)x^n. \end{aligned}\tag{5}$$

Then comparing the coefficients of x^{n+k} in (3) and (5) we may immediately deduce the identity

$$\sum_{a_1+a_2+\cdots+a_k=n} b_{a_1}b_{a_2}\cdots b_{a_k} = \sum_{m=1}^{k}(-1)^{m+n+k}\binom{k}{m}\frac{m!!\cdot 2^n}{(k+n)!(m-2k-2n)!!}.$$

This proves Theorem 1.

Now we prove Theorem 2. According to (2), we can deduce the identities

$$(1-x)^{\frac{1}{2}} = 1-\frac{1}{2}\sum_{n=0}^{\infty}\frac{b_n x^{n+1}}{4^n}, \tag{6}$$

$$(1-x)^{\frac{3}{2}} = 1-\frac{3}{2}x+\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{b_n}{4^n}-\frac{b_{n+1}}{4^{n+1}}\right)x^{n+2}, \tag{7}$$

$$(1-x)^{\frac{5}{2}} = 1-\frac{5}{2}x+\frac{15}{8}x^2+\frac{1}{2}\sum_{n=0}^{\infty}\left(2\frac{b_{n+1}}{4^{n+1}}-\frac{b_n}{4^n}-\frac{b_{n+2}}{4^{n+2}}\right)x^{n+3}, \tag{8}$$

and

$$(1-x)^{\frac{7}{2}} = 1-\frac{7}{2}x+\frac{35}{8}x^2-\frac{35}{16}x^3+\frac{1}{2}\sum_{n=0}^{\infty}\left(3\frac{b_{n+2}}{4^{n+2}}-3\frac{b_{n+1}}{4^{n+1}}-\frac{b_{n+3}}{4^{n+3}}+\frac{b_n}{4^n}\right)x^{n+4}. \tag{9}$$

Taking $k=2$ in (3), we have

$$4(1-\sqrt{1-x})^2 = 4\sum_{m=0}^{2}\binom{2}{m}(-\sqrt{1-x})^m = x^2\sum_{n=0}^{\infty}\frac{1}{4^n}\left(\sum_{a_1+a_2=n} b_{a_1}b_{a_2}\right)x^n. \tag{10}$$

From (6) and (10) we have

$$\sum_{a_1+a_2=n} b_{a_1}b_{a_2} = b_{n+1}.$$

Taking $k=3$ in (3), and applying (7) we have

$$\sum_{a_1+a_2+a_3=n} b_{a_1}b_{a_2}b_{a_3} = b_{n+2}-b_{n+1}.$$

Similarly, we can also deduce the identities

$$\sum_{a_1+a_2+a_3+a_4=n} b_{a_1}b_{a_2}b_{a_3}b_{a_4} = b_{n+3}-2b_{n+2},$$
$$\sum_{a_1+a_2+a_3+a_4+a_5=n} b_{a_1}b_{a_2}b_{a_3}b_{a_4}b_{a_5} = b_{n+4}+b_{n+2}-3b_{n+3},$$
$$\sum_{a_1+a_2+a_3+a_4+a_5+a_6=n} b_{a_1}b_{a_2}b_{a_3}b_{a_4}b_{a_5}b_{a_6} = b_{n+5}+3b_{n+3}-4b_{n+4},$$
$$\sum_{a_1+a_2+a_3+a_4+a_5+a_6+a_7=n} b_{a_1}b_{a_2}b_{a_3}b_{a_4}b_{a_5}b_{a_6}b_{a_7} = b_{n+6}+6b_{n+4}-b_{n+3}-5b_{n+5}.$$

This completes the proof of Theorem 2.

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Hosoya index of zig-zag tree-type hexagonal systems [1]

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Abstract In order to obtain a larger bound of Hosoya index of the tree-type hexagonal systems, the zig-zag tree-type hexagonal systems are taken into consideration. In this paper, some results with respect to Hosoya index of the zig-zag tree-type hexagonal systems are shown. Using the results, hexagonal chains and hexagonal spiders with the larger bound of Hosoya index are determined.

Keywords Hosoya index, zig-zag tree-type hexagonal system, hexagonal spider.

§1. Introduction

A hexagonal system is a 2−connected plane graph whose every interior face is bounded by a regular hexagon. Hexagonal systems are of great importance for theoretical chemistry because they are the natural graph representations of benzenoid hydrocarbons [2]. A hexagonal system is a tree-type one if it has no inner vertex. The zig-zag tree-type hexagonal systems are the graph representations of an important subclass of benzenoid molecules. A considerable amount of research in mathematical chemistry has been devoted to hexagonal systems [2−16].

In order to describe our results, we need some graph-theoretic notations and terminologies. Our standard reference for any graph theoretical terminology is [1].

Let $G = (V, E)$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Let e and u be an edge and a vertex of G, respectively. We will denote by $G-e$ or $G-u$ the graph obtained from G by removing e or u, respectively. Denote by N_u the set $\{v \in V(G) : uv \in E(G)\} \cup \{u\}$. Let H be a subset of $V(G)$. The subgraph of G induced by H is denoted by $G[H]$, and $G[V \setminus H]$ is denoted by $G-H$. Undefined concepts and notations of graph theory are referred to $[11-16]$.

Two edges of a graph G are said to be independent if they are not adjacent. A subset M of $E(G)$ is called a matching set of G if any two vertices of M are independent. Denote $m(G)$ the number of matchings sets of G. In chemical terminology, $m(G)$ is called the Hosoya index. Clearly, the Hosoya index of a graph is larger than that of its proper subgraphs.

We denote by Ψ_n the set of the hexagonal chains with n hexagons. Let $B_n \in \Psi_n$. We denote by $V_3 = V_3(B_n)$ the set of the vertices with degree 3 in B_n. Thus, the subgraph $B_n[V_3]$ is a acyclic graph. If the subgraph $B_n[V_3]$ is a matching with $n-1$ edges, then B_n is called a

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linear chain and denoted by L_n. If the subgraph $B_n[V_3]$ is a path, then B_n is called a zig-zag chain and denoted by Z_n. If the subgraph $B_n[V_3]$ is a comb, then B_n is called a helicene chain and denoted by H_n (see [11]).

Denote by $\mathbf{T}_n$ the tree-type hexagonal systems containing n hexagons. Let $\mathbf{T} = \bigcup_1^\infty \mathbf{T}_n$, and $T \in \mathbf{T}$. Let H be a hexagon of T. Obviously, H has at most three adjacent hexagons in T; if H has exactly three adjacent hexagons in T, then H is called a full-hexagon of T; if H has two adjacent hexagons in T, and, moreover, if its two vertices with degree two are adjacent, then call H a turn-hexagon of T; and if H has at most one adjacent hexagon in T, then H is called an end-hexagon of T. It is easy to see that the number of the end-hexagons of a tree-type hexagonal system of $n \geq 2$ hexagons is more two than the number of its full-hexagons. Let $T \in \mathbf{T}$ and let $B = H_1H_2 \dots H_k, k \geq 2$ be a hexagonal chain of T. If the end-hexagon H_1 of B is also an end-hexagon of T, the other end-hexagon H_k is a full-hexagon of T, and for $2 \leq i \leq k-1$, H_i is not a full-hexagon of T, then B is called a branch of T (see [16]). If any branch of T is a zig-zag chain, then T is called zig-zag tree-type hexagonal system. Both a zig-zag hexagonal chain and zig-zag hexagonal spider are zig-zag tree-type hexagonal systems with no full-hexagon and only one full-hexagon, respectively.

§2. Some useful results

Among tree-type hexagonal systems with extremal properties on topological indices, L_n and Z_n play important roles. We list some of them about the Hosoya index as follows.

Theorem 2.1.[6] For any $n \geq 1$ and any $B_n \in \Psi_n$, if B_n is neither L_n nor Z_n, then

$$m(L_n) < m(B_n) < m(Z_n).$$

Theorem 2.2.[16] For any $n \geq 1$ and any $T \in \mathbf{T}_\mathrm{n}$, if T is not L_n, then

$$m(T) > m(L_n).$$

Among many properties of $m(G)$, we mention the following results which will be used later.

Lemma 2.1.[1] Let G be a graph consisting of two components G_1 and G_2, then

$$m(G) = m(G_1)m(G_2).$$

Lemma 2.2.[1] Let G be a graph and any $uv \in E(G)$, then

$$m(G) = m(G - uv) + m(G - u - v).$$

Lemma 2.3.[1] Let G be a graph. For each $uv \in E(G)$, then

$$m(G) - m(G - u) - m(G - u - v) \geq 0.$$

Moreover, the equality holds only if v is the unique neighbor of u.

Let A and B be any graphs and C be a hexagon. Let $G = A@_y^x C$. Let r and s be two adjacent vertices of B of at least degree two. Denote by $G_\eta B$ the graph obtained from G and

B by identifying the edge ab with rs; by $G_\beta B$ the graph from G and B by identifying the edge bc with rs; by $G_\zeta B$ the graph from G and B by identifying the edge cd with rs (see [11]).

Lemma 2.4.[11] Let $A, B, G = A@_y^x C, G_\eta B$ and $G_\zeta B$, if $m(A-x) > m(A-y)$, then

$$m(G_\zeta B) > m(G_\eta B).$$

Lemma 2.5.[11] Let $A, B, G = A@_y^x C, G_\eta B, G_\beta B$ and $G_\zeta B$, then

(a) $m(G_\eta B) > m(G_\beta B)$,

(b) $m(G_\zeta B) > m(G_\beta B)$.

We add some notations which are convenient to express useful results. For a given zig-zag chain Z_k, denote by $x_k^{'}, x_k, y_k, y_k^{'}$ the four clockwise successful vertices with degree two in one of end-hexagons (see Fig. 2.1.).

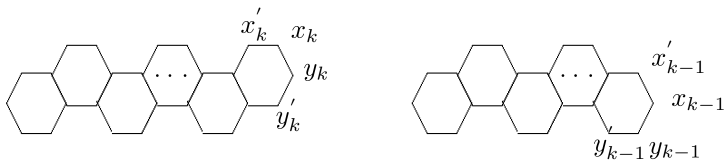


Fig. 2.1. Z_k and Z_{k-1}.

Lemma 2.6. Suppose G is a zig-zag chain with k hexagons. Then

$$\begin{pmatrix} m(Z_k) \\ m(Z_k - x_k^{'}) \\ m(Z_k - x_k) \\ m(Z_k - x_k - y_k) \\ m(Z_k - y_k - y_k^{'}) \\ m(Z_k - x_k - y_k^{'} - y_k) \\ m(Z_k - x_k - y_k - x_k^{'}) \end{pmatrix} = \begin{pmatrix} 5 & 3 & 3 & 2 \\ 3 & 2 & 0 & 0 \\ 2 & 1 & 2 & 1 \\ 2 & 1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \end{pmatrix} \begin{pmatrix} m(Z_{k-1}) \\ m(Z_{k-1} - x_{k-1}) \\ m(Z_{k-1} - x_{k-1}^{'}) \\ m(Z_{k-1} - x_{k-1} - x_{k-1}^{'}) \end{pmatrix}.$$

By applying Lemma 2.1 and Lemma 2.2, it is easy to obtain the result.

Lemma 2.7. Keep the notations as in Lemma 2.6, and suppose Z_k is a zig-zag chain with $k (k \geq 3)$ hexagons. Then

(a) $m(Z_k - x_k^{'} - x_k) < m(C_6)m(Z_{k-2}) + m(P_5)m(Z_{k-2} - x_{k-2})$,

(b) $m(Z_k - x_k^{'} - x_k - y_k) > m(P_5)m(Z_{k-2}) + m(P_4)m(Z_{k-2} - x_{k-2})$,

(c) $2m(Z_{k-1} - y_{k-1}) + m(Z_{k-1} - y_{k-1} - y_{k-1}^{'}) < m(C_6)m(Z_{k-2} - x_{k-2}^{'}) + m(P_5)m(Z_{k-2} - x_{k-2} - x_{k-2}^{'})$,

(d) $m(Z_{k-1} - y_{k-1}) + m(Z_{k-1} - y_{k-1} - y_{k-1}^{'}) + m(Z_k - x_k^{'} - x_k) < m(C_6)m(Z_{k-2}) + m(P_5)m(Z_{k-2} - x_{k-2}) + m(P_5)m(Z_{k-2} - x_{k-2}^{'}) + m(P_4)m(Z_{k-2} - x_{k-2} - x_{k-2}^{'})$.

Where C_m and P_m $(m = 3, 4, 5, 6)$ are the circle and the path with m vertices, respectively.

Proof. (a) Set $f_1(k) = m(Z_k)$, $f_2(k) = m(Z_k - x_k^{'})$, $f_3(k) = m(Z_k - x_k)$, $f_4(k) = m(Z_k - x_k - x_k^{'})$, $f_5(k) = m(Z_k - y_k^{'})$, $f_6(k) = m(Z_k - y_k)$, $f_7(k) = m(Z_k - y_k - y_k^{'})$, $f_8(k) = m(Z_k - x_k - y_k - y_k^{'})$ and $f_9(k) = m(Z_k - x_k - y_k - x_k^{'})$.

Applying Lemma 2.6 to $Z_k - x_k^{'} - x_k$, Z_{k-2} and $Z_{k-2} - x_{k-2}$, we get

$$\begin{aligned} m(Z_k - x_k^{'} - x_k) &= f_4(k) \\ &= 2f_1(k-1) + f_3(k-1) \\ &= 12f_1(k-2) + 8f_2(k-2) + 7f_3(k-2) + 5f_4(k-2), \end{aligned}$$

and

$$m(C_6)m(Z_{k-2}) + m(P_5)m(Z_{k-2} - x_{k-2}) = 18f_1(k-2) + 8f_3(k-2).$$

For $k \geq 3$, we have

$$\begin{aligned} \Delta_1 &= m(C_6)m(Z_{k-2}) + m(P_5)m(Z_{k-2} - x_{k-2}) - m(Z_k - x_k^{'} - x_k) \\ &= 6f_1(k-2) - 8f_2(k-2) + f_3(k-2) - 5f_4(k-2). \end{aligned}$$

By Lemma 2.1, we obtain $m(Z_{k-2}) = m(Z_{k-2} - x_{k-2}^{'} - x_{k-2}) + m(Z_{k-2} - x_{k-2}^{'}x_{k-2})$, and $m(Z_{k-2} - x_{k-2}^{'}x_{k-2}) = m(Z_{k-2} - x_{k-2}^{'}) + m(Z_{k-2} - x_{k-2}^{'} - y_{k-3})$. Thus

$$\begin{aligned} \Delta_1 &= 6f_1(k-2) - 8f_2(k-2) + f_3(k-2) - 5f_4(k-2) \\ &= f_3(k-2) + f_4(k-2) + 4m(Z_{k-2} - x_{k-2}^{'} - y_{k-3}) - 2m(Z_{k-2} - x_{k-2}^{'} - y_{k-3} - y_{k-3}^{'}). \end{aligned}$$

Since $Z_{k-2} - x_{k-2}^{'} - y_{k-3} - y_{k-3}^{'}$ is the proper subgraph of $Z_{k-2} - x_{k-2}^{'} - y_{k-3}$, then $m(Z_{k-2} - x_{k-2}^{'} - y_{k-3}) > m(Z_{k-2} - x_{k-2}^{'} - y_{k-3} - y_{k-3}^{'})$. Therefore $\Delta_1 > 0$.

(b) Similar to the proof of (a), by Lemma 2.6, we obtain

$$\begin{aligned} m(Z_k - x_k^{'} - x_k - y_k) &= f_9(k) \\ &= f_1(k-1) + f_3(k-1), \end{aligned}$$

and applying Lemma 2.6 to Z_{k-1} and $Z_{k-1} - x_{k-1}$, we have

$$f_1(k-1) + f_3(k-1) = 7f_1(k-2) + 5f_2(k-2) + 4f_3(k-2) + 3f_4(k-2),$$

and

$$m(P_5)m(Z_{k-2}) + m(P_4)m(Z_{k-2} - x_{k-2}) = 8f_1(k-2) + 5f_3(k-2).$$

Thus

$$\begin{aligned} \Delta_2 &= m(P_5)m(Z_{k-2}) + m(P_4)m(Z_{k-2}) - m(Z_k - x_k^{'} - x_k - y_k) \\ &= f_1(k-2) - 5f_2(k-2) + f_3(k-2) - 3f_4(k-2). \end{aligned}$$

Note that

$$m(Z_{k-2}) = m(Z_{k-2} - x_{k-2}^{'}x_{k-2}) + m(Z_{k-2} - x_{k-2}^{'} - x_{k-2}),$$

and

$$m(Z_{k-2} - x_{k-2}^{'}x_{k-2}) = m(Z_{k-2} - x_{k-2}^{'}) + m(Z_{k-2} - x_{k-2}^{'} - y_{k-3}),$$

we get

$$\begin{aligned}\Delta_2 &= f_1(k-2) - 5f_2(k-2) + f_3(k-2) - 3f_4(k-2)\\ &= m(Z_{k-2} - x'_{k-2} - y_{k-3}) + m(Z_{k-2} - x'_{k-2} - x_{k-2} - y_{k-3}) - 4f_2(k-2) - 4f_4(k-2).\end{aligned}$$

Since $Z_{k-2}-x'_{k-2}-y_{k-3}$ and $Z_{k-2}-x'_{k-2}-x_{k-2}-y_{k-3}$ are proper subgraphs of $Z_{k-2}-x'_{k-2}$, then $m(Z_{k-2}-x'_{k-2}-y_{k-3}) < m(Z_{k-2}-x'_{k-2})$, and $m(Z_{k-2}-x'_{k-2}-x_{k-2}-y_{k-3}) < m(Z_{k-2}-x'_{k-2})$. Therefore $\Delta_2 < 0$.

(c) Similar to the proof of $(a), (b)$, by Lemma 2.6, we get

$$m(C_6)m(Z_{k-2} - x'_{k-2}) + m(P_5)m(Z_{k-2} - x'_{k-2} - x_{k-2}) = 18f_2(k-2) + 8f_4(k-2),$$

and

$$\begin{aligned}2m(Z_{k-1} - y_{k-2}) + m(Z_{k-1} - y'_{k-1} - y_{k-1}) &= 6f_1(k-2) + 3f_2(k-2) + 4f_3(k-2)\\ &\quad +2f_4(k-2).\end{aligned}$$

Then

$$\begin{aligned}\Delta_3 &= m(C_6)m(Z_{k-2} - x'_{k-2}) + m(P_5)m(Z_{k-2} - x'_{k-2} - x_{k-2}) - 2m(Z_{k-1} - y_{k-2})\\ &\quad -m(Z_{k-1} - y'_{k-1} - y_{k-1})\\ &= -6f_1(k-2) + 15f_2(k-2) - 4f_3(k-2) + 6f_4(k-2).\end{aligned}$$

Note that $m(Z_{k-2}) = m(Z_{k-2} - x'_{k-2}x_{k-2}) + m(Z_{k-2} - x'_{k-2} - x_{k-2})$, and $m(Z_{k-2} - x'_{k-2}x_{k-2}) = m(Z_{k-2} - x'_{k-2}) + m(Z_{k-2} - x'_{k-2} - y_{k-3})$, thus

$$\begin{aligned}\Delta_3 &= -6f_1(k-2) + 15f_2(k-2) - 4f_3(k-2) + 6f_4(k-2)\\ &= 9m(Z_{k-2} - x'_{k-2}y_{k-3}) - m(Z_{k-2} - x'_{k-2} - x_{k-2} - y_{k-3})\\ &\quad -4m(Z_{k-2} - x'_{k-2} - x_{k-2} - x_{k-3}y_{k-3}) - 4m(Z_{k-2} - x'_{k-2} - x_{k-2} - x_{k-3} - y_{k-3}).\end{aligned}$$

Since $Z_{k-2} - x'_{k-2} - x_{k-2} - y_{k-3}$, $Z_{k-2} - x'_{k-2} - x_{k-2} - y_{k-3}$ and $Z_{k-2} - x'_{k-2} - x_{k-2} - x_{k-3}y_{k-3}$ are proper subgraphs of $Z_{k-2} - x'_{k-2}y_{k-3}$, then

$$m(Z_{k-2} - x'_{k-2} - x_{k-2} - y_{k-3}) < m(Z_{k-2} - x'_{k-2}y_{k-3}),$$

$$m(Z_{k-2} - x'_{k-2} - x_{k-2} - y_{k-3}) < m(Z_{k-2} - x'_{k-2}y_{k-3}),$$

and

$$m(Z_{k-2} - x'_{k-2} - x_{k-2} - x_{k-3}y_{k-3}) < m(Z_{k-2} - x'_{k-2}y_{k-3}).$$

Therefore $\Delta_3 > 0$.

(d) Similar to the proof of (c), we obtain

$$\begin{aligned}\Delta_4 &= m(C_6)m(Z_{k-2}) + m(P_5)m(Z_{k-2} - x_{k-2}) + m(P_5)m(Z_{k-2} - x'_{k-2})\\ &\quad +m(P_4)m(Z_{k-2} - x_{k-2} - x'_{k-2}) - m(Z_{k-1} - y_{k-1}) - m(Z_{k-1} - y_{k-1} - y'_{k-1})\\ &\quad -m(Z_k - x'_k - x_k) > 0.\end{aligned}$$

The proof of Lemma 2.7 is complete.

§3. Preliminary results and proofs

Suppose T_1, $T_2 \in \mathbf{T}$, and p_i, q_i are two adjacent vertices with degree two in $T_i, i = 1, 2$. Denote by $T_1(p_1, q_1) \otimes T_2(p_2, q_2)$ the tree-type hexagonal system obtained from T_1 and T_2 by identifying p_1 with p_2, and q_1 with q_2, respectively.

In the present section, for a given $T \in \mathbf{T}$, we always assume that s, t are two adjacent vertices with degree two in T. For a given linear zig-zag chain Z_k, denote by $x_k^{'}, x_k, y_k, y_k^{'}$ the four clockwise successful vertices with degree two in one of end-hexagons (see Fig. 3.1.).

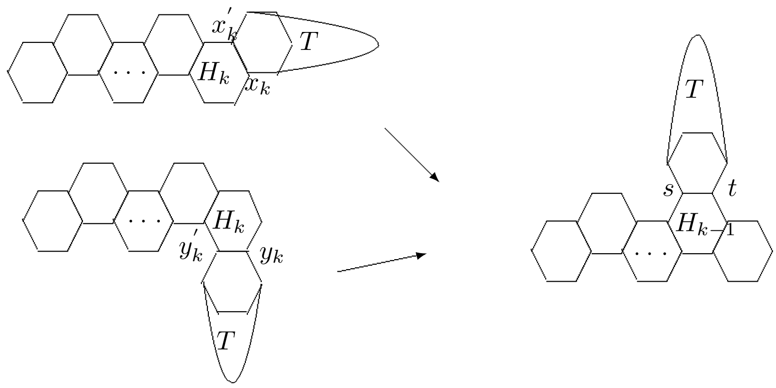


Fig. 3.1. $T(s,t) \otimes Z_k(x_k^{'}, x_k), T(s,t) \otimes Z_k(y_k, y_k^{'})$ and $T(s,t) \otimes Z_k(x_{k-1}^{'}, x_{k-1})$.

Theorem 3.1. Keep the notations as Lemma 2.7. For any $T \in \mathbf{T}$ and $k \geq 3$ (see Fig. 3.1.). Then

(a) $m(T(s,t) \otimes Z_k(x_k^{'}, x_k)) < m(T(s,t) \otimes Z_k(x_{k-1}^{'}, x_{k-1}))$,
(b) $m(T(s,t) \otimes Z_k(y_k, y_k^{'})) < m(T(s,t) \otimes Z_k(x_{k-1}^{'}, x_{k-1}))$.

Proof. (a) By Lemma 2.1 and Lemma2.2, we get

$$\begin{aligned} & m(T(s,t) \otimes Z_k(x_{k-1}^{'}, x_{k-1})) \\ = \ & m(T-st)[18f_1(k-2)+8f_3(k-2)] + m(T-t)[8f_1(k-2)+5f_3(k-2)] \\ & +m(T-s)[18f_2(k-2)+8f_4(k-2)] + m(T-t-s)[18f_1(k-2)+8f_3(k-2) \\ & +8f_2(k-2)+5f_4(k-2)], \end{aligned}$$

and

$$\begin{aligned} & m(T(s,t) \otimes Z_k(x_k^{'}, x_k)) \\ = \ & m(T-st)f_4(k-2) + m(T-t)f_9(k-2) + m(T-s)[2f_6(k-1)+f_7(k-1)] \\ & +m(T-t-s)[f_6(k-1)+f_7(k-1)+f_4(k)]. \end{aligned}$$

Thus

$$\begin{aligned}\Delta_5 &= m(T(s,t)\otimes Z_k(x'_{k-1},x_{k-1})) - m(T(s,t)\otimes Z_k(x'_k,x_k))\\ &= \Delta_1 m(T-st)+\Delta_2 m(T-t)+\Delta_3 m(T-s)+\Delta_4 m(T-s-t).\end{aligned}$$

According to Lemma 2.7 and Lemma 2.1, we obtain $\Delta_i > 0 (i = 1, 3, 4)$, $\Delta_2 < 0$ and $m(T-t) = m(T-t-s) + m(T-t-s-N_s)$. So

$$\Delta_5 > (\Delta_1 + 2\Delta_2 + \Delta_3 + \Delta_4)m(T-t-s).$$

Similar to the proof of lemma 2.7, we get $\Delta_1 + 2\Delta_2 + \Delta_3 + \Delta_4 > 0$, therefore $\Delta_5 > 0$.

Similar to the proof of Lemma 2.7 and Theorem 3.1 (a), we obtain $m(T(s,t)\otimes Z_k(y_k, y'_k)) < m(T(s,t)\otimes Z_k(x'_{k-1}, x_{k-1}))$ and the proof of Theorem 3.1 is complete.

Corollary 3.1. For any $k \geq 3$ and $n > 0$. Then

(a) $m(L_n(s,t)\otimes Z_k(x'_k, x_k)) < m(L_n(s,t)\otimes Z_k(x'_{k-1}, x_{k-1}))$,

(b) $m(L_n(s,t)\otimes Z_k(y_k, y'_k)) < m(L_n(s,t)\otimes Z_k(x'_{k-1}, x_{k-1}))$.

§4. Zig-zag tree-type hexagonal systems

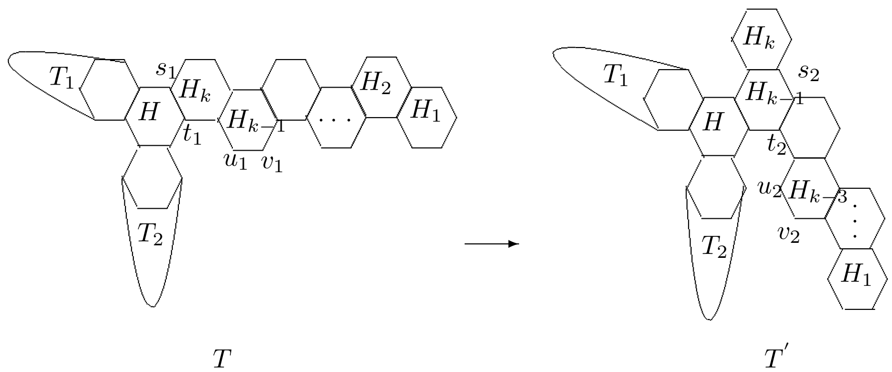
A graph G is called a zig-zag tree-type hexagonal system if it is a tree-type hexagonal system and any branch of which is zig-zag chain.

We shall use $\mathbf{Z}^*_{\mathrm{n}}$ to denote the set of all zig-zag tree-type hexagonal systems with n hexagons. For a given graph $Z^* \in \mathbf{Z}^*_{\mathrm{n}}$, we denote $Z^{\perp}$ the graph obtain from Z^* whose every branch is transformed by transformation I (see Fig. 4.1).

A graph G is called a spider if it is a tree and contains only one vertex of degree greater than 2. For positive integer n_1, n_2, n_3, we use $S(n_1, n_2, n_3)$ to denote a hexagonal spider with three legs of lengths n_1, n_2 and n_3, respectively (see [11]).

If a hexagonal spider $S(n_1, n_2, n_3)$ whose 3 legs are linear chains, then such a graph is called a linear hexagonal spider and denoted by $L(n_1, n_2, n_3)$ (see [11]).

Similarly if each leg of $S(n_1, n_2, n_3)$ combining with the central hexagon is a zig-zag chain, then such graph is called a zig-zag hexagonal spider and denoted by $Z(n_1, n_2, n_3)$ (see [11]).



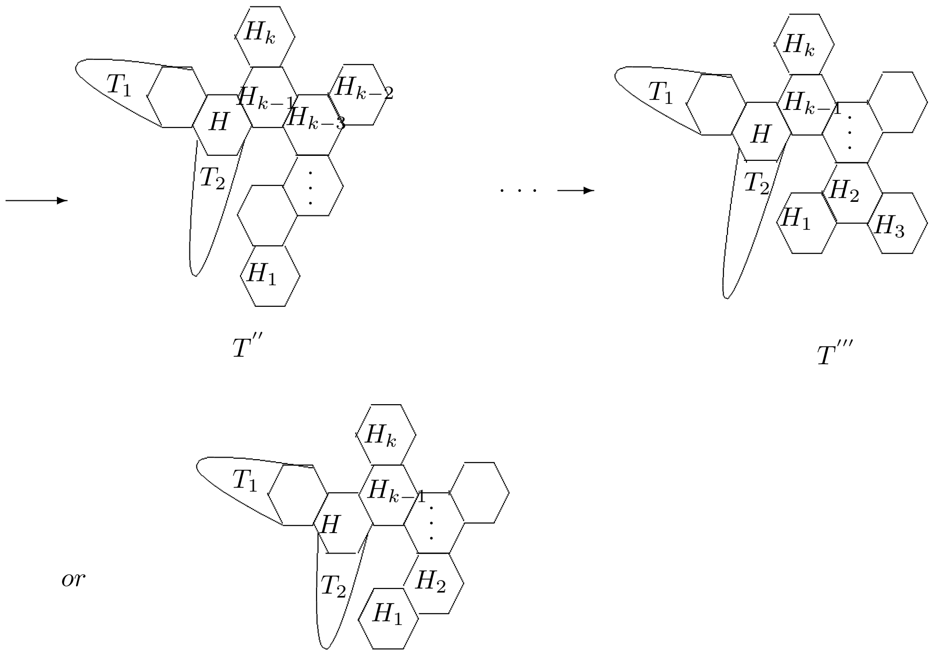


Fig. 4.1. Transformation I.

Transformation I. Let $Z_k = H_1 H_2 \cdots H_k$ and $Z_k \bigotimes H$ be a branch of T (see Fig. 4.1.). Firstly, the graph $T^{'}$ can be obtained from $T - Z_k$ and Z_k by identifying the edge $u_1 v_1$ of H_{k-1} with the edge $s_1 t_1$ of H. Secondly, the graph $T^{''}$ can be got from $T^{'} - Z_{k-2}$ and Z_{k-2} by identifying the edge $u_2 v_2$ of H_{k-3} with the edge $s_2 t_2$ of H_{k-1}. Finally, by repeating this operation, the graph $T^{'''}$ can be obtained. If $T = Z_n$, only let $H = H_1$.

Theorem 4.1. For any $Z^* \in \mathbf{Z}_{\mathrm{n}}^*$ and any $n \geq 4$. Then

$$m(Z^{\perp}) \geq m(Z^*).$$

Moreover, the equality holds if and only if $Z^{\perp} \cong Z^*$.

Proof. If $Z^{\perp}$ is not Z^*, note that the graph $Z^{\perp}$ is obtained from Z^* whose every branch is transformed by transformation I, and by Theorem 3.1, we get $m(Z^{\perp}) > m(Z^*)$. Moreover, the equality holds if and only if $Z^{\perp} \cong Z^*$, and the proof of Theorem 4.1 is complete.

By repeating to apply transformation I on a hexagonal spider $S(n_1, n_2, n_3)$ and Z_n, and according to Theorem 3.1, we also obtain a good larger bound of Hosoya index of Z_n and $Z(n_1, n_2, n_3)$ as follows.

Theorem 4.2. For any $Z^*(n_1, n_2, n_3) \in Z(n_1, n_2, n_3)$ with n hexagons and any $n \geq 4$, then

$$m(Z^{\perp}(n_1, n_2, n_3)) \geq m(Z^*(n_1, n_2, n_3)) > m(L(n_1, n_2, n_3)).$$

Moreover, the equality holds if and only if $Z^{\perp}(n_1, n_2, n_3) \cong Z^*(n_1, n_2, n_3) \cong Z(2, 2, 2)$.

Theorem 4.3. For any $Z^* \in Z_n$ and $n \geq 4$. Then

$$m(Z^{\perp}) > m(Z^*) > m(L_n).$$

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All these papers are original and have been refereed. The themes of these papers range from the mean value or hybrid mean value of Smarandache type functions, the mean value of some famous number theroretic functions acting on the Smarandache sequences, and the solvability of the Smarandache equations.

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