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一个包含Smarandache LCM对偶函数的方程

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摘要:在初等数论和分类讨论方法的基础上使用java程序研究函数方程 $\sum_{d|n}\frac{1}{SL*(d)}=3\Omega(n)$ 的可解性,并给出这个方程的所有正整数解的具体形式.

关键词: Smarandache LCM对偶函数;Ω 函数;分类讨论;正整数解

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An equation involving the Smarandache LCM dual function

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Abstract: Based on the elementary number theory and classification discussion methods, we studied the solvability of the equation $\sum_{d|n}\frac{1}{SL*(d)}=3\Omega(n)$ using the java program, and its all specific forms of positive integer solution were given.

Keywords: Smarandache dual LCM function; Ω function; classification discussion; positive integer solution

1 主要结论

对任意的正整数 n,著名的Smarandache LCM函数的对偶函数定义为[1]:

$$SL(n)=\min\{k|n|[1,2,\cdots,k],k\in N_+\},$$

其对偶函数定义为[2-3]:

$$SL*(n)=\max\{k|[1,2,\cdots,k]|n,k\in N_+\}.$$

许多学者对 $SL*(n)$ 的算术性质进行了研究,获得了不少有趣的结果.例如,田呈亮[4]得到当 n 为奇数时,$SL*(n)=1$;当 n 为偶数时,$SL*(n)\geqslant 2$.王妤[5]得到 $\sum_{d|n}SL*(d)=\sum_{d|n}S*(d)$ 的正整数解.陈斌[6]得到了 $\sum_{d|n}\frac{1}{S*(d)}=3\Omega(n)$ 的正整数解.赵娜娜[7-8]得到了 $\sum_{d|n}\frac{1}{SL*(d)}=\Omega(n)$ 和 $\sum_{d|n}\frac{1}{SL*(d)}=2\Omega(n)$ 的正整数解.

本文中利用初等数论和分类讨论的方法研究方程

$$\sum_{d|n}\frac{1}{SL*(d)}=3\Omega(n) \tag{1}$$

的正整数解,并得到其所有正整数解.

定理 方程(1)的奇数解为 $p_1^3p_2^5$;所有偶数解为 $2p_1^2p_2^3$;$2p_1^3p_2^2$;$2p_1p_2p_3$;$2^{10}\cdot 3^{98}$;$2^{11}\cdot 3^{54}$;$2^{13}\cdot 3^{32}$;$2^{17}\cdot 3^{21}$;$2^{20}\cdot 3^{18}$;$2^{31}\cdot 3^{14}$;$2^{53}\cdot 3^{12}$;$2^{97}\cdot 3^{11}$;$2^3 3^2 p_2^6$;$2^3 3^4 p_2^3$;$2^4 3^6 p_2^2$;$2^4 3^{40} p_2^1$;$2^5 3^3 p_2^3$;$2^5 3^{18} p_2^1$;$2^{10} 3^8 p_2^1$;$2^{13} 3^2 p_2^3$;$2^{13} 3^7 p_2^1$;

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$2^{21}3^{6}p_2^{1}$; $2^{25}3^{3}p_2^{2}$; $2^{109}3^{5}p_2^{1}$; $2^{2}3^{2}p_2^{8}$; $2^{2}3^{10}p_2^{2}$; $2^{3}3^{1}p_2^{7}$; $2^{3}3^{2}p_2^{5}$; $2^{3}3^{6}p_2^{2}$; $2^{4}3^{2}p_2^{4}$; $2^{4}3^{20}p_2^{1}$; $2^{5}3^{4}p_2^{2}$; $2^{5}3^{12}p_2^{1}$; $2^{7}3^{1}p_2^{5}$; $2^{7}3^{2}p_2^{3}$; $2^{7}3^{8}p_2^{1}$; $2^{9}3^{3}p_2^{2}$; $2^{11}3^{6}p_2^{1}$; $2^{19}3^{5}p_2^{1}$; $2^{6}p_1^{29}$; $2^{7}p_1^{17}$; $2^{8}p_1^{13}$; $2^{9}p_1^{11}$; $2^{11}p_1^{9}$; $2^{13}p_1^{8}$; $2^{17}p_1^{7}$; $2^{29}p_1^{6}$; $2^{2}\cdot p_1^{2}\cdot p_2^{5}$; $2^{2}\cdot p_1^{3}\cdot p_2^{3}$; $2^{2}\cdot p_1^{5}\cdot p_2^{2}$; $2^{3}\cdot p_1^{1}\cdot p_2^{8}$; $2^{3}\cdot p_1^{2}\cdot p_2^{3}$; $2^{3}\cdot p_1^{3}\cdot p_2^{2}$; $2^{3}\cdot p_1^{8}\cdot p_2^{1}$; $2^{4}\cdot p_1^{1}\cdot p_2^{5}$; $2^{4}\cdot p_1^{5}\cdot p_2^{1}$; $2^{5}\cdot p_1^{1}\cdot p_2^{4}$; $2^{5}\cdot p_1^{2}\cdot p_2^{2}$; $2^{5}\cdot p_1^{4}\cdot p_2^{1}$; $2^{8}\cdot p_1^{1}\cdot p_2^{3}$; $2^{8}\cdot p_1^{3}\cdot p_2^{1}$; $2^{2}\cdot p_1^{2}\cdot p_2\cdot p_3$; $2^{2}\cdot p_1\cdot p_2^{2}\cdot p_3$; $2^{2}\cdot p_1\cdot p_2\cdot p_3^{2}$.

2 定理的证明

定理的证明 当 $n=1$ 时，$\sum_{d|n}\frac{1}{SL*(d)}=1$，$3\Omega(n)=0$，显然 $n=1$ 不是方程(1)的正整数解，下设 $n>1$，具体讨论以下两种情况：

A：当 $n>1$ 且为奇数时，设 $n=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$，此时显然对 n 的每一个因子 d 必为奇数．即 $2\nmid d$，故 $SL*(d)=S*(d)=1$，而 $3\Omega(n)=3(\alpha_1+\alpha_2+\cdots+\alpha_k)$．故原方程等价于

$$\sum_{d|n}\frac{1}{SL*(d)}=\sum_{d|n}\frac{1}{S*(d)}=3\Omega(n).$$

由文献[2]可知，当 n 为奇数时，方程(1)的奇数解为 $n=p_1^3p_2^5$，其中 p_1，p_2 为奇素数．

B：当 $n>1$ 且为偶数时，设 $n=2^{\alpha}m$，$m=p_1^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}$，$\alpha\geqslant1$，$p_1<p_2<\cdots<p_k$．具体讨论以如下：

Ⅰ）当 $m=1$ 时，$n=2^{\alpha}$，$3\Omega(n)=3\alpha$，有

$$\sum_{d|n}\frac{1}{SL*(d)}=1+\sum_{d|2^{\alpha},d>1}\frac{1}{SL*(d)}=1+\frac{\alpha}{2},$$

故方程(1)等价于 $1+\frac{\alpha}{2}=3\alpha$，解得 $\alpha=\frac{2}{5}$．因此 $n=2^{\alpha}$ 不是方程(1)的正整数解．

Ⅱ）当 $m>1$ 时，分 $\alpha=1$ 和 $\alpha>1$ 两种情况，具体分析如下：

(a)当 $\alpha=1$ 时，$n=2m$，$m=p_1^{\alpha_1}\cdot p_2^{\alpha_2}\cdot\cdots\cdot p_k^{\alpha_k}$，$k\geqslant1$，具体讨论如下：

(1)当 $3|m$ 时，$n=2\cdot3^{\alpha_1}\cdot p_2^{\alpha_2}\cdot\cdots\cdot p_k^{\alpha_k}$，$5\leqslant p_2<p_3<\cdots<p_k$．

(ⅰ)当 $k=1$ 时，$n=2\cdot3^{\alpha_1}$，$3\Omega(n)=3(1+\alpha_1)$，有

$$\sum_{d|n}\frac{1}{SL*(d)}=\sum_{d|2\cdot3^{\alpha_1}}\frac{1}{SL*(d)}=1+\frac{1}{SL*(2)}+\sum_{i=1}^{\alpha_1}\frac{1}{SL*(3^i)}+\sum_{i=1}^{\alpha_1}\frac{1}{SL*(2\cdot3^i)}=\frac{3}{2}+\frac{4}{3}\alpha_1,$$

因此，方程(1)等价于 $\frac{3}{2}+\frac{4}{3}\alpha_1=3(1+\alpha_1)$．由于 $\frac{3}{2}+\frac{4}{3}\alpha_1<3(1+\alpha_1)$，此时方程(1)无正整数解．

(ⅱ)当 $k=2$ 时，$n=2\cdot3^{\alpha_1}\cdot p_2^{\alpha_2}$，$3\Omega(n)=3(1+\alpha_1+\alpha_2)$，有

$$\sum_{d|n}\frac{1}{SL*(d)}=\sum_{d|2\cdot3^{\alpha_1}p_2^{\alpha_2}}\frac{1}{SL*(d)}=\left(\frac{3}{2}+\frac{4}{3}\alpha_1\right)(1+\alpha_2),$$

因此，方程(1)等价于 $\left(\frac{3}{2}+\frac{4}{3}\alpha_1\right)(1+\alpha_2)=3(1+\alpha_1+\alpha_2)$．由 java 程序易知此方程无正整数解，此时方程(1)无正整数解．

(ⅲ)当 $k=3$ 时，$n=2\cdot3^{\alpha_1}\cdot p_2^{\alpha_2}p_3^{\alpha_3}$，$3\Omega(n)=3(1+\alpha_1+\alpha_2+\alpha_3)$，有

$$\sum_{d|n}\frac{1}{SL*(d)}=\sum_{d|2\cdot3^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}}\frac{1}{SL*(d)}=\left(\frac{3}{2}+\frac{4}{3}\alpha_1\right)(1+\alpha_2)(1+\alpha_3),$$

因此，方程(1)等价于 $\left(\frac{3}{2}+\frac{4}{3}\alpha_1\right)(1+\alpha_2)(1+\alpha_3)=3(1+\alpha_1+\alpha_2+\alpha_3)$，解得此方程无正整数解，此时方程(1)无正整数解．

(ⅳ)当 $k\geqslant4$ 时，$n=2\cdot3^{\alpha_1}\cdot p_2^{\alpha_2}\cdot\cdots\cdot p_k^{\alpha_k}$，$3\Omega(n)=3(1+\alpha_1+\alpha_2+\cdots+\alpha_k)$ 有

$$\sum_{d|n}\frac{1}{SL*(d)}=\sum_{d|2\cdot3^{\alpha_1}p_2^{\alpha_2}\cdots p_k^{\alpha_k}}\frac{1}{SL*(d)}=\left(\frac{3}{2}+\frac{4}{3}\alpha_1\right)(1+\alpha_2)\cdots(1+\alpha_k).$$

由数学归纳法证得,当 $k\geqslant 4$ 时,$\left(\frac{3}{2}+\frac{4}{3}\alpha_1\right)(1+\alpha_2)\cdots(1+\alpha_k)>3(1+\alpha_1+\alpha_2+\cdots+\alpha_k)$,易知此方程无正整数解,此时方程(1)无正整数解.

(2) 当 $3\nmid m$ 时,$n=2\cdot p_1^{\alpha_1}\cdot p_2^{\alpha_2}\cdot\cdots\cdot p_k^{\alpha_k}$,$5\leqslant p_1<p_2<\cdots<p_k$,此时,

$$3\Omega(n)=3(1+\alpha_1+\alpha_2+\cdots+\alpha_k),$$

$$\sum_{d|n}\frac{1}{SL*(d)}=\frac{3}{2}(1+\alpha_1)(1+\alpha_2)\cdots(1+\alpha_k).$$

因此,方程(1)等价于

$$\frac{3}{2}(1+\alpha_1)(1+\alpha_2)\cdots(1+\alpha_k)=3(1+\alpha_1+\alpha_2+\cdots+\alpha_k) \tag{2}$$

下面求解方程(2).

(i) 当 $k=1$ 时,(2)式等价于 $\frac{3}{2}(1+\alpha_1)=3(1+\alpha_1)$.解得 $\alpha_1=-1$,此时方程(1)无正整数解.

(ii) 当 $k=2$ 时,(2)式等价于 $\frac{3}{2}(1+\alpha_1)(1+\alpha_2)=3(1+\alpha_1+\alpha_2)$,即 $\alpha_1\alpha_2=1+\alpha_1+\alpha_2$,解得 $\alpha_1=2$,$\alpha_2=3$ 或者 $\alpha_1=3$,$\alpha_2=2$,即 $n=2p_1^2p_2^3$ 或者 $n=2p_1^3p_2^2$ 是方程(1)的正整数解.

(iii) 当 $k=3$ 时,(2)式等价于 $\frac{3}{2}(1+\alpha_1)(1+\alpha_2)(1+\alpha_3)=3(1+\alpha_1+\alpha_2+\alpha_3)$,解得 $\alpha_1=\alpha_2=\alpha_3=1$,即 $n=2p_1p_2p_3$ 是方程(1)的正整数解.

(iv) 当 $k\geqslant 4$ 时,由数学归纳法证得,

$$\frac{3}{2}(1+\alpha_1)(1+\alpha_2)\cdots(1+\alpha_k)>3(1+\alpha_1+\alpha_2+\cdots+\alpha_k),$$

易知此方程无正整数解,此时方程(1)无正整数解.

(b) 当 $\alpha\geqslant 2$ 时,$n=2^\alpha m=2^\alpha\cdot p_1^{\alpha_1}\cdot p_2^{\alpha_2}\cdot\cdots\cdot p_k^{\alpha_k}$,$3\leqslant p_1<p_2<\cdots<p_k$.

(I) 当 $3|m$ 时,具体讨论如下:

(i) 当 $k=1$ 时,$n=2^\alpha 3^{\alpha_1}$,$3\Omega(n)=3(\alpha+\alpha_1)$,有

$$\sum_{d|n}\frac{1}{SL*(d)}=\sum_{d|2^\alpha\cdot 3^{\alpha_1}}\frac{1}{SL*(d)}=1+\frac{\alpha-1}{2}+\alpha_1+\frac{\alpha_1(\alpha-1)}{4}.$$

因此,方程(1)等价于 $1+\frac{\alpha-1}{2}+\alpha_1+\frac{\alpha_1(\alpha-1)}{4}=3(\alpha+\alpha_1)$,即 $2+\alpha\alpha_1=10\alpha+9\alpha_1$.由 java 程序解得 $\alpha=10$,$\alpha_1=98$;$\alpha=11$,$\alpha_1=54$;$\alpha=13$,$\alpha_1=32$;$\alpha=17$,$\alpha_1=21$;$\alpha=20$,$\alpha_1=18$;$\alpha=31$,$\alpha_1=14$;$\alpha=53$,$\alpha_1=12$;$\alpha=97$,$\alpha_1=11$.即

$$n=2^{10}\cdot 3^{98},\ n=2^{11}\cdot 3^{54},\ n=2^{13}\cdot 3^{32},\ n=2^{17}\cdot 3^{21},\ n=2^{20}\cdot 3^{18},\ n=2^{31}\cdot 3^{14},\ n=2^{53}\cdot 3^{12},\ n=2^{97}\cdot 3^{11}$$

为方程(1)的解.

(ii) 当 $k=2$ 时,$n=2^\alpha 3^{\alpha_1}p_2^{\alpha_2}$,$3\Omega(n)=3(\alpha+\alpha_1+\alpha_2)$,具体讨论如下:

①当 $p_2=5$ 时,即 $n=2^\alpha 3^{\alpha_1}5^{\alpha_2}$,有

$$\sum_{d|n}\frac{1}{SL*(d)}=\sum_{d|2^\alpha\cdot 3^{\alpha_1}5^{\alpha_2}}\frac{1}{SL*(d)}=1+\frac{\alpha-1}{2}+\alpha_1+\alpha_2+\frac{\alpha_1(\alpha-1)}{4}+\frac{\alpha_2(\alpha-1)}{2}+\alpha_1\alpha_2+\frac{\alpha_1\alpha_2(\alpha-1)}{6}.$$

因此,方程(1)等价于

$$1+\frac{\alpha-1}{2}+\alpha_1+\alpha_2+\frac{\alpha_1(\alpha-1)}{4}+\frac{\alpha_2(\alpha-1)}{2}+\alpha_1\alpha_2+\frac{\alpha_1\alpha_2(\alpha-1)}{6}=3(\alpha+\alpha_1+\alpha_2),$$

可化简为 $6+3\alpha\alpha_1+6\alpha\alpha_2+10\alpha_2\alpha_1+2\alpha\alpha_1\alpha_2=30\alpha+27\alpha_1+30\alpha_2$.由 java 程序解得 $\alpha=3$,$\alpha_1=2$,$\alpha_2=6$;$\alpha=3$,$\alpha_1=4$,$\alpha_2=3$;$\alpha=4$,$\alpha_1=6$,$\alpha_2=2$;$\alpha=4$,$\alpha_1=40$,$\alpha_2=1$;$\alpha=5$,$\alpha_1=3$,$\alpha_2=3$;$\alpha=5$,$\alpha_1=18$,$\alpha_2=1$;$\alpha=10$,$\alpha_1=18$,$\alpha_2=1$;$\alpha=13$,$\alpha_1=2$,$\alpha_2=3$;$\alpha=13$,$\alpha_1=7$,$\alpha_2=1$;$\alpha=21$,$\alpha_1=6$,$\alpha_2=1$;$\alpha=25$,$\alpha_1=3$,$\alpha_2=2$;$\alpha=109$,$\alpha_1=5$,$\alpha_2=1$.

即 $n=2^3 3^2 p_2^6$;$n=2^3 3^4 p_2^3$;$n=2^4 3^6 p_2^2$;$n=2^4 3^{40} p_2^1$;$n=2^5 3^3 p_2^3$;$n=2^5 3^{18} p_2^1$;$n=2^{10} 3^8 p_2^1$;$n=2^{13} 3^2 p_2^3$;

$n=2^{13}3^7p_2^1$;$n=2^{21}3^6p_2^1$;$n=2^{25}3^3p_2^2$;$n=2^{109}3^5p_2^1$ 为方程(I)的解.

②当 $p_2>5$ 时,$n=2^{\alpha}3^{\alpha_1}p_2^{\alpha_2}$,$3\Omega(n)=3(\alpha+\alpha_1+\alpha_2)$,有

$$\sum_{d|n}\frac{1}{SL*(d)}=\sum_{d|2^{\alpha}\cdot 3^{\alpha_1}p^{\alpha_2}}\frac{1}{SL*(d)}=1+\frac{\alpha-1}{2}+\alpha_1+\alpha_2+\frac{\alpha_1(\alpha-1)}{4}+\frac{\alpha_2(\alpha-1)}{2}+\alpha_1\alpha_2+\frac{\alpha_1\alpha_2(\alpha-1)}{4}.$$

因此,方程(I)等价于

$$1+\frac{\alpha-1}{2}+\alpha_1+\alpha_2+\frac{\alpha_1(\alpha-1)}{4}+\frac{\alpha_2(\alpha-1)}{2}+\alpha_1\alpha_2+\frac{\alpha_1\alpha_2(\alpha-1)}{4}=3(\alpha+\alpha_1+\alpha_2),$$

可化简为 $2+\alpha\alpha_1+2\alpha\alpha_2+3\alpha_2\alpha_1+\alpha\alpha_1\alpha_2=10\alpha+9\alpha_1+10\alpha_2$.由 java 程序解得

$\alpha=2$,$\alpha_1=2$,$\alpha_2=8$;$\alpha=2$,$\alpha_1=10$,$\alpha_2=2$;$\alpha=3$,$\alpha_1=1$,$\alpha_2=17$;$\alpha=3$,$\alpha_1=2$,$\alpha_2=5$;$\alpha=3$,$\alpha_1=6$,$\alpha_2=2$;$\alpha=4$,$\alpha_1=2$,$\alpha_2=4$;$\alpha=4$,$\alpha_1=20$,$\alpha_2=1$;$\alpha=5$,$\alpha_1=4$,$\alpha_2=2$;$\alpha=5$,$\alpha_1=12$,$\alpha_2=1$;$\alpha=7$,$\alpha_1=1$,$\alpha_2=5$;$\alpha=7$,$\alpha_1=2$,$\alpha_2=3$;$\alpha=7$,$\alpha_1=8$,$\alpha_2=1$;$\alpha=9$,$\alpha_1=3$,$\alpha_2=2$;$\alpha=11$,$\alpha_1=6$,$\alpha_2=1$;$\alpha=19$,$\alpha_1=5$,$\alpha_2=1$.即

$n=2^23^2p_2^8$;$n=2^23^{10}p_2^2$;$n=2^33^1p_2^7$;$n=2^33^2p_2^5$;$n=2^33^6p_2^2$;$n=2^43^2p_2^4$;$n=2^43^{20}p_2^1$;$n=2^53^4p_2^2$;$n=2^53^{12}p_2^1$;$n=2^73^1p_2^5$;$n=2^73^2p_2^3$;$n=2^73^8p_2^1$;$n=2^93^3p_2^2$;$n=2^{11}3^6p_2^1$;$n=2^{19}3^5p_2^1$ 为(I)式的解.

(iii)当 $k=3$ 时,$n=2^{\alpha}3^{\alpha_1}p_2^{\alpha_2}p_3^{\alpha_3}$,$5\leqslant p_2<p_3$,$3\Omega(n)=3(\alpha+\alpha_1+\alpha_2+\alpha_3)$ 具体讨论如下:

①当 $p_2=5$,$p_3=7$,$\alpha_1=\alpha_2=\alpha_3=1$ 时,$n=2^{\alpha}\cdot 3\cdot 5\cdot 7$,$3\Omega(n)=3(\alpha+3)$ 有

$$\sum_{d|n}\frac{1}{SL*(d)}=6+2\alpha+\frac{2}{3}(\alpha-1)+\frac{\alpha-2}{8}+\frac{1}{7},$$

因此,方程(I)等价于 $6+2\alpha+\frac{2}{3}(\alpha-1)+\frac{\alpha-2}{8}+\frac{1}{7}=3(\alpha+\alpha_1+\alpha_2+\alpha_3)$.

由于 $6+2\alpha+\frac{2}{3}(\alpha-1)+\frac{\alpha-2}{8}+\frac{1}{7}<3(\alpha+\alpha_1+\alpha_2+\alpha_3)$,因此,方程(I)无正整数解.

②当 $p_2\neq 5$,$5<p_2<p_3$,$\alpha_1=\alpha_2=\alpha_3=1$ 时,$n=2^{\alpha}\cdot 3\cdot p_2\cdot p_3$,$3\Omega(n)=3(\alpha+3)$,

有 $\sum_{d|n}\frac{1}{SL*(d)}=3\alpha+5$,由于 $3\alpha+5<3(\alpha+3)$,因此,方程(I)无正整数解.

③当 $\alpha_1>1$,$\alpha_2>1$,$\alpha_3>1$ 时,$3\Omega(n)=3(\alpha+\alpha_1+\alpha_2+\alpha_3)$,有

$$\sum_{d|n}\frac{1}{SL*(d)}=(\alpha_1-1)+(\alpha_2-1)+(\alpha_3-1)+\frac{(\alpha-1)(\alpha_1-1)}{4}+\frac{(\alpha-1)(\alpha_2-1)}{2}+\frac{(\alpha-1)(\alpha_3-1)}{2}+$$
$$(\alpha_1-1)(\alpha_2-1)+(\alpha_1-1)(\alpha_3-1)+(\alpha_2-1)(\alpha_3-1)+\frac{(\alpha-1)(\alpha_1-1)(\alpha_2-1)}{4}+$$
$$\frac{(\alpha-1)(\alpha_1-1)(\alpha_3-1)}{4}+\frac{(\alpha-1)(\alpha_2-1)(\alpha_3-1)}{2}+(\alpha_1-1)(\alpha_2-1)(\alpha_3-1)+$$
$$\frac{(\alpha-1)(\alpha_1-1)(\alpha_2-1)(\alpha_3-1)}{4}+1+\frac{\alpha-1}{2},$$

由于 $\sum_{d|n}\frac{1}{SL*(d)}>3\Omega(n)$,因此方程(I)无正整数解.

(iv)因此,当 $k\geqslant 4$ 时,方程(I)也无正整数解.

(II)当 $3\nmid m$ 时,$n=2^{\alpha}m=2^{\alpha}\cdot p_1^{\alpha_1}\cdot p_2^{\alpha_2}\cdot\cdots\cdot p_k^{\alpha_k}$,$3\Omega(n)=3(\alpha+\alpha_1+\cdots+\alpha_k)$,有

$$\sum_{d|n}\frac{1}{SL*(d)}=\frac{1}{2}(1+\alpha)(1+\alpha_1)\cdots(1+\alpha_k).$$

(i)当 $k=1$ 时,方程(I)等价于 $1+\alpha\alpha_1=5\alpha+5\alpha_1$,由 java 程序解得 $\alpha=6$,$\alpha_1=29$;$\alpha=7$,$\alpha_1=17$;$\alpha=8$,$\alpha_1=13$;$\alpha=9$,$\alpha_1=11$;$\alpha=11$,$\alpha_1=9$;$\alpha=13$,$\alpha_1=8$;$\alpha=17$,$\alpha_1=7$;$\alpha=29$,$\alpha_1=6$.

故 $n=2^6p_1^{29}$;$n=2^7p_1^{17}$;$n=2^8p_1^{13}$;$n=2^9p_1^{11}$;$n=2^{11}p_1^9$;$n=2^{13}p_1^8$;$n=2^{17}p_1^7$;$n=2^{29}p_1^6$ 为方程(I)的解.

(ii)当 $k=2$ 时,方程(I)等价于 $\frac{1}{2}(1+\alpha)(1+\alpha_1)(1+\alpha_2)=3(\alpha+\alpha_1+\alpha_2)$,由 java 程序解得

$\alpha=2$,$\alpha_1=2$,$\alpha_2=5$;$\alpha=2$,$\alpha_1=3$,$\alpha_2=3$;$\alpha=2$,$\alpha_1=5$,$\alpha_2=2$;$\alpha=3$,$\alpha_1=1$,$\alpha_2=8$;$\alpha=3$,

$\alpha_1=2$,$\alpha_2=3$;$\alpha=3$,$\alpha_1=3$,$\alpha_2=2$;$\alpha=3$,$\alpha_1=8$,$\alpha_2=1$;$\alpha=4$,$\alpha_1=1$,$\alpha_2=5$;$\alpha=4$,$\alpha_1=5$,$\alpha_2=1$;$\alpha=5$,$\alpha_1=1$,$\alpha_2=4$;$\alpha=5$,$\alpha_1=2$,$\alpha_2=2$;$\alpha=5$,$\alpha_1=4$,$\alpha_2=1$;$\alpha=8$,$\alpha_1=1$,$\alpha_2=3$;$\alpha=8$,$\alpha_1=3$,$\alpha_2=1$.即 $n=2^2\cdot p_1^2\cdot p_2^5$;$n=2^2\cdot p_1^3\cdot p_2^3$;$n=2^2\cdot p_1^5\cdot p_2^2$;$n=2^3\cdot p_1^1\cdot p_2^8$;$n=2^3\cdot p_1^2\cdot p_2^3$;$n=2^3\cdot p_1^3\cdot p_2^2$;$n=2^3\cdot p_1^8\cdot p_2^1$;$n=2^4\cdot p_1^1\cdot p_2^5$;$n=2^4\cdot p_1^5\cdot p_2^1$;$n=2^5\cdot p_1^1\cdot p_2^4$;$n=2^5\cdot p_1^2\cdot p_2^2$;$n=2^5\cdot p_1^4\cdot p_2^1$;$n=2^8\cdot p_1^1\cdot p_2^3$;$n=2^8\cdot p_1^3\cdot p_2^1$为(1)式的解.

(iii)当 $k=3$ 时,方程(1)等价于 $\frac{1}{2}(1+\alpha)(1+\alpha_1)(1+\alpha_2)(1+\alpha_3)=3(\alpha+\alpha_1+\alpha_2+\alpha_3)$.

解得 $\alpha=2$,$\alpha_1=2$,$\alpha_2=1$,$\alpha_3=1$;$\alpha=2$,$\alpha_1=1$,$\alpha_2=2$,$\alpha_3=1$;$\alpha=2$,$\alpha_1=1$,$\alpha_2=1$,$\alpha_3=2$.即 $n=2^2\cdot p_1^2\cdot p_2\cdot p_3$;$n=2^2\cdot p_1\cdot p_2^2\cdot p_3$;$n=2^2\cdot p_1\cdot p_2\cdot p_3^2$为方程(1)的解.

(iv)当 $k\geqslant 4$ 时,用数学归纳法易证 $\frac{1}{2}(1+\alpha)(1+\alpha_1)\cdots(1+\alpha_k)>3(\alpha+\alpha_1+\cdots+\alpha_k)$.

因此,方程(1)无正整数解.

综上所述,定理得证.

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(上接第366页)

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