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Two equations involving the Smarandache function and its solutions

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Abstract: For any positive integer n, let $S(n)$ denote the Smarandache function. The main purpose is using the elementary methods to study the solutions of two equations involving the Smarandache function, and prove that the equations have positive integer solutions.

Key words: Smarandache function; equation; positive integer solution
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1 Introduction and results

For any positive integer n, the Smarandache function $S(n)$ is defined as the smallest positive integer m such that $n \mid m!$. From the definition and the properties of $S(n)$, see [1-2], one can easily derive that if $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ is the prime powers factorization of n, then

$$S(n) = \max_{1\le i\le k}\{S(p_i^{\alpha_i})\}.$$

About the arithmetical properties of $S(n)$, many people had studied it before, see references [3] and [4].

In this paper, we shall use the elementary methods to study the solutions of the equation $S(1^2)+S(2^2)+\cdots+S(n^2)=S\left(\frac{n(n+1)(2n+1)}{6}\right)$, and give all positive integer solutions for it. That is, we shall prove the following theorems:

Theorem 1 For any positive integer n, the equation

$$S(1^2)+S(2^2)+\cdots+S(n^2)=S\left(\frac{n(n+1)(2n+1)}{6}\right)$$

has only 2 positive integer solutions, namely $n=1, 2$.

Theorem 2 For any positive integer k, the equation

$$S(m_1)+S(m_2)+\cdots+S(m_k)=S(m_1+m_2+\cdots+m_k)$$

has at least one positive integer solution.

We conjecture that for any positive integer k, the equation

$$S(m_1)+S(m_2)+\cdots+S(m_k)=S(m_1+m_2+\cdots+m_k)$$

has infinitely many positive integer solutions. This is an open problem.

2 Proof of Theorem 1

In this section, we shall complete the proof of Theorem 1. Let $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ denote the factorization of n into prime powers, and let

$$S(n) = \max_{1\le i\le k}\{S(p_i^{\alpha_i})\} = S(p^{\alpha}),$$

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where $S(p_i^{\alpha_i}) = S(p^{\alpha})$, $1 \leqslant i \leqslant k$.

For the equation
$$S(1^2) + S(2^2) + \cdots + S(n^2) = S\left(\frac{n(n+1)(2n+1)}{6}\right),$$
it is clear that $n = 1$ is a solution of the equation. If $n > 1$, then we will discuss the problem in two cases:

(Ⅰ) If $n = 2$, then $S(1^2) + S(2^2) = S(1) + S(4) = 5$ and $S\left(\frac{n(n+1)(2n+1)}{6}\right) = S(5) = 5$, so $n = 2$ is a solution of the equation.

(Ⅱ) If $n \geqslant 3$, then $S(n^2) \geqslant 3$. From the properties of $S(n)$ we can get
$$S(1^2) + S(2^2) + \cdots + S(n^2) \geqslant 1 + 4 + 3(n-2) = 3n - 1.$$
We also know
$$(n, n+1) = 1, \ (n, 2n+1) = 1, \ (2n+1, n+1) = 1.$$
So we have
$$S\left(\frac{n(n+1)(2n+1)}{6}\right) \leqslant \max\{S(n), S(n+1), S(2n+1)\} \leqslant 2n + 1.$$
For the equation, we may get $3n - 1 \leqslant 2n + 1$. That is $n \leqslant 2$. So this time, the equation has no solution.

Now combining the above two cases, we may immediately get that the equation $S(1^2) + S(2^2) + \cdots + S(n^2) = S\left(\frac{n(n+1)(2n+1)}{6}\right)$ has only two positive integer solutions, namely $n = 1, 2$.

This completes the proof of Theorem 1.

3 Proof of Theorem 2

For any positive integer k, we shall discuss the equation
$$S(m_1) + S(m_2) + \cdots + S(m_k) = S(m_1 + m_2 + \cdots + m_k)$$
in three cases:

(Ⅰ) If $k = 1$, then it is clear that the equation has infinitely many positive integer solutions.

(Ⅱ) If $k = p$, a prime, then we taking $m_1 = m_2 = \cdots = m_k = 1$, then we have $S(1) = 1$, and $S(m_1) + S(m_2) + \cdots + S(m_k) = p = S(p) = S(m_1 + m_2 + \cdots + m_k)$, and thus the equation is satisfied.

(Ⅲ) If $k > 1$ and it is not a prime, then from the result in elementary number theory we know that there must exists a prime p between k and $2k$. Let $p - k = l$, where $1 \leqslant l \leqslant k$. Now we take $m_1 = m_2 = \cdots = m_l = 2$, $m_{l+1} = m_{l+2} = \cdots = m_k = 1$ and note that $S(1) = 1$ and $S(2) = 2$, we have
$$S(m_1) + S(m_2) + \cdots + S(m_k) = 2l + (k - l) = l + k = p$$
and
$$S(m_1 + m_2 + \cdots + m_k) = S(k + l) = S(p) = p = S(m_1) + S(m_2) + \cdots + S(m_k).$$
So $m_1 = m_2 = \cdots = m_l = 2$ and $m_{l+1} = m_{l+2} = \cdots = m_k = 1$, satisfy the equation.

Now combining the above three cases, we may immediately get that there is at least one solution satisfying the equation.

This completes the proof of Theorem 2.

Conjecture For any positive integer k, we conjecture that the equation
$$S(m_1) + S(m_2) + \cdots + S(m_k) = S(m_1 + m_2 + \cdots + m_k)$$
has infinitely many positive integer solutions.

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The derivation algebra of infinite-dimensional special modular Lie superalgebra S

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Abstract: The generators of the infinite-dimensional special modular Lie superalgebra S is first given, and then the derivation space of the infinite-dimensional special modular Lie superalgebra S to the generalized Witt modular Lie superalgebra is determined. Furthermore, the derivation algebra of S is determined.
Key words: gradation; special modular Lie superalgebra; derivation algebra

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两个包含 Smarandache函数的方程及其解

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摘 要: 利用初等方法研究两个包含 Smarandache函数的方程的解, 并且证明了这两个方程有正整数解.
关键词: Smarandache函数; 方程; 正整数解