

On the Smarandache Pseudo Number Sequence

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Abstract: The main purpose of this paper is to study the mean value properties of the second Smarandache pseudo-odd number sequence and pseudo-even number sequence, and give some interesting asymptotic formula for them.

Key words: pseudo-odd numbers; pseudo-even numbers; asymptotic formula.

2000 MR Subject Classification: 11B83

CLC number: O156.4 **Document code:** A

Article ID: 1002-0462 (2006) 04-0581-04

§1. Introduction

A number is called the second Smarandache pseudo-odd number if it is an even number, and some permutation of its digits is an odd number. For example: 10, 12, 14, 16, 18, 30, 32, 34, 36, 38, 50, 52, \dots are the second Smarandache pseudo-odd numbers. Let A denotes the set of all the second Smarandache pseudo-odd numbers. Similarly, we can define the second Smarandache pseudo-even number. That is, a number is called the second Smarandache pseudo-even number if it is an odd number, and some permutation of its digits is an even number, such as 21, 23, 25, 27, 29, 41, 43, 45, 47, 49, \dots are the second Smarandache pseudo-even numbers. Let B denotes the set of all the second Smarandache pseudo-even numbers. In problems of 85 and 89 of [2], Professor F. Smarandache asked us to study the properties of these sequences. About these problems, it seems that none had studied them before, at least we have not seen any related papers at present. In this paper, we use the elementary method to study the mean value properties of these two sequences, and obtain some interesting asymptotic formulae for them. That is, we shall prove the following:

Received date: 2004-09-20

Foundation item: Supported by the NSF of China(10271093)

Biography: LIU Yan-ni(1977-), female, native of Fuping, Shaanxi, engages in analytic number theory.

Theorem 1 For any real number $x \geq 1$, we have the asymptotic formulae

$$\sum_{\substack{n \in A \\ n \leq x}} 1 = \frac{1}{2}x + O\left(x^{\frac{\ln 5}{\ln 10}}\right) \quad \text{and} \quad \sum_{\substack{n \in B \\ n \leq x}} 1 = \frac{1}{2}x + O\left(x^{\frac{\ln 5}{\ln 10}}\right).$$

Theorem 2 For any real number $x \geq 1$, let $d(n)$ denotes the Dirichlet divisor function, then we have the asymptotic formulae

$$\sum_{\substack{n \in A \\ n \leq x}} d(n) = \frac{3}{4}x \ln x + \left(\frac{3}{2}\gamma - \frac{\ln 2}{2} - \frac{3}{4}\right)x + O\left(x^{\frac{\ln 5}{\ln 10} + \varepsilon}\right)$$

and

$$\sum_{\substack{n \in B \\ n \leq x}} d(n) = \frac{1}{4}x \ln x + \left(\frac{1}{2}\gamma + \frac{\ln 2}{2} - \frac{1}{4}\right)x + O\left(x^{\frac{\ln 5}{\ln 10} + \varepsilon}\right),$$

where γ is the Euler constant, ε is any fixed positive integer.

§2. Some Lemmas

To completes the proof of the theorems, we need the following two simple lemmas:

Lemma 1 For any real number $x \geq 1$, we have the asymptotic formula

$$\sum_{n \leq \frac{x+1}{2}} d(2n-1) = \frac{1}{4}x \ln x + \frac{1}{4}(2\gamma + 2 \ln 2 - 1)x + O\left(x^{\frac{1}{2} + \varepsilon}\right),$$

where γ is the Euler constant, ε is any fixed positive number.

Proof For any complex s with $\text{Re}(s) > 1$, we let generate function $f(s)$ as following:

$$f(s) = \sum_{n=1}^{\infty} \frac{d(2n-1)}{(2n-1)^s}.$$

Since $d(n)$ is a multiplicative function of n , and $f(s)$ is absolutely convergent if $\text{Re}(s) > 1$, so from the Euler product formula(see Theorem 11.6 of [3]) and the multiplicative property of $d(n)$ we have

$$f(s) = \prod_{p \neq 2} \left(1 + \frac{2}{p^s} + \frac{3}{p^{2s}} + \cdots\right) = \prod_{p \neq 2} \left(1 - \frac{1}{p^s}\right)^2 = \zeta^2(s) \left(1 - \frac{1}{2^s}\right)^2,$$

where $\zeta(s)$ is the Riemann zeta-function.

By Perron formula [2] with $s_0 = 0$, $T = x^{\frac{1}{2}}$ and $b = \frac{3}{2}$, we have

$$\sum_{2n-1 \leq x} d(2n-1) = \frac{1}{2\pi i} \int_{\frac{3}{2}-iT}^{\frac{3}{2}+iT} \zeta^2(s) \left(1 - \frac{1}{2^s}\right)^2 \frac{x^s}{s} ds + O\left(x^{\frac{1}{2} + \varepsilon}\right).$$

To estimate the main term

$$\frac{1}{2\pi i} \int_{\frac{3}{2}-iT}^{\frac{3}{2}+iT} \zeta^2(s) \left(1 - \frac{1}{2^s}\right)^2 \frac{x^s}{s} ds,$$

we move the integral line from $s = \frac{3}{2} \pm iT$ to $s = \frac{1}{2} \pm iT$. This time, the function

$$f(s) = \zeta^2(s) \left(1 - \frac{1}{2^s}\right)^2$$

has a second order pole point at $s = 1$ with the residue

$$\lim_{s \rightarrow 1} \frac{1}{1!} \left((s-1)^2 \zeta^2(s) \left(1 - \frac{1}{2^s}\right)^2 \frac{x^s}{s} \right)' = \frac{1}{4} x \ln x + \frac{1}{4} (2\gamma + 2 \ln 2 - 1) x,$$

where γ is the Euler constant. Note that the estimate

$$\frac{1}{2i\pi} \left(\int_{\frac{3}{2}+iT}^{\frac{1}{2}+iT} + \int_{\frac{1}{2}-iT}^{\frac{3}{2}-iT} + \int_{\frac{3}{2}-iT}^{\frac{3}{2}+iT} \right) \zeta^2(s) \left(1 - \frac{1}{2^s}\right)^2 \frac{x^s}{s} ds \ll x^{\frac{1}{2}+\epsilon},$$

we may immediately get

$$\sum_{\substack{2n-1 \leq x}} d(2n-1) = \frac{1}{4} x \ln x + \frac{1}{4} (2\gamma + 2 \ln 2 - 1) x + O\left(x^{\frac{1}{2}+\epsilon}\right).$$

This completes the proof of Lemma 1.

Lemma 2 For any real number $x \geq 1$, we have the estimate

$$\sum_{\substack{2n \notin A \\ 2n \leq x}} 1 = O\left(x^{\frac{\ln 5}{\ln 10}}\right) \quad \text{and} \quad \sum_{\substack{2n-1 \notin B \\ 2n-1 \leq x}} 1 = O\left(x^{\frac{\ln 5}{\ln 10}}\right).$$

Proof First we let k be a positive integer such that $10^k \leq x < 10^{k+1}$. Then it is clear that $k \leq \log x < k + 1$. Now according to the definition of set A , we know that the largest number of the digits ($2n \leq x$) not attribute set A is 5^k . In fact, in these numbers, there are 0 one digit; There are 4×5 two digits, they are 20, 22, 24, 26, 28, \dots , 98. There are 4×5^2 three digits; The number of the k digits are $4 \times 5^{k-1}$. So the largest number of digits ($2n \leq x$) not attribute set A is $4 \times 5 + 4 \times 5^2 + \dots + 4 \times 5^{k-1} = 5^k - 5$. Note that $k \leq \log x < k + 1$, we have the estimate

$$\sum_{\substack{2n \notin A \\ 2n \leq x}} 1 \leq 5^k \leq 5^{\log x} = (5^{\log_5 x})^{\frac{1}{\log_5 10}} = x^{\frac{\ln 5}{\ln 10}}.$$

Similarly, we can also get the estimate

$$\sum_{\substack{2n-1 \notin B \\ 2n-1 \leq x}} 1 = O\left(x^{\frac{\ln 5}{\ln 10}}\right).$$

This proves the Lemma 2.

§3. Proof of the Theorems

In this section, we can easily complete the proof of the Theorems. In fact, from Lemma 2 we have

$$\sum_{\substack{n \in A \\ n \leq x}} 1 = \sum_{2n \leq x} 1 - \sum_{\substack{2n \notin A \\ 2n \leq x}} 1 = \frac{1}{2} x + O\left(x^{\frac{\ln 5}{\ln 10}}\right)$$

and

$$\sum_{\substack{n \in B \\ n \leq x}} 1 = \sum_{2n-1 \leq x} 1 - \sum_{\substack{2n-1 \notin A \\ 2n-1 \leq x}} 1 = \frac{1}{2}x + O\left(x^{\frac{\ln 5}{\ln 10}}\right).$$

This proves the Theorem 1.

Now we prove Theorem 2. From Lemma 1, Lemma 2, and note that the estimate $d(n) \ll n^\epsilon$ (see Theorem 13.12 of [3]) we have

$$\begin{aligned} & \sum_{\substack{n \in A \\ n \leq x}} d(n) \\ &= \sum_{2n \leq x} d(2n) - \sum_{\substack{2n \notin A \\ 2n \leq x}} d(2n) \\ &= \sum_{n \leq x} d(n) - \sum_{2n-1 \leq x} d(2n-1) - \sum_{\substack{2n \notin A \\ 2n \leq x}} d(2n) \\ &= x \ln x + (2\gamma - 1)x - \frac{1}{4}x \ln x - \frac{1}{4}(2\gamma + 2 \ln 2 - 1)x + O\left(x^{\frac{\ln 5}{\ln 10} + \epsilon}\right) \\ &= \frac{3}{4}x \ln x + \left(\frac{3}{2}\gamma - \frac{\ln 2}{2} - \frac{3}{4}\right)x + O\left(x^{\frac{\ln 5}{\ln 10} + \epsilon}\right). \end{aligned}$$

Similarly, we can also get the asymptotic formula

$$\begin{aligned} & \sum_{\substack{n \in B \\ n \leq x}} d(n) \\ &= \sum_{2n-1 \leq x} d(2n-1) - \sum_{\substack{2n-1 \notin B \\ 2n-1 \leq x}} d(2n-1) \\ &= \frac{1}{4}x \ln x + \left(\frac{1}{2}\gamma + \frac{\ln 2}{2} - \frac{1}{4}\right)x + O\left(x^{\frac{\ln 5}{\ln 10} + \epsilon}\right). \end{aligned}$$

This completes the proof of the Theorems.

Acknowledgments The author express her gratitude to her supervisor Professor Zhang Wen-peng for his very helpful and detailed instructions.

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