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We devote part of this issue of *Critical Review* to Lotfi Zadeh.
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Neutrosophic Axiomatic System

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Abstract

In this paper, we introduce for the first time the notions of Neutrosophic Axiom, Neutrosophic Axiomatic System, Neutrosophic Deducibility and Neutrosophic Inference, Neutrosophic Proof, Neutrosophic Tautologies, Neutrosophic Quantifiers, Neutrosophic Propositional Logic, Neutrosophic Axiomatic Space, Degree of Contradiction (Dissimilarity) of Two Neutrosophic Axioms, and Neutrosophic Model. A class of neutrosophic implications is also introduced. A comparison between these innovatory neutrosophic notions and their corresponding classical notions is made. Then, three concrete examples of neutrosophic axiomatic systems, describing the same neutrosophic geometrical model, are presented at the end of the paper.

Keywords

Neutrosophic logic, Neutrosophic Axiom, Neutrosophic Deducibility, Neutrosophic Inference, Neutrosophic Proof, Neutrosophic Tautologies, Neutrosophic Quantifiers, Neutrosophic Propositional Logic, Neutrosophic Axiomatic Space.

1 Neutrosophic Axiom

A *neutrosophic axiom* or *neutrosophic postulate* (α) is a partial premise, which is t% true (degree of truth), i% indeterminate (degree of indeterminacy), and f% false (degree of falsehood), where <t, i, f> are standard or nonstandard subsets included in the non-standard unit interval]-0, 1+[.

The non-standard subsets and non-standard unit interval are mostly used in philosophy in cases where one needs to make distinction between “absolute truth” (which is a truth in all possible worlds) and “relative truth” (which is a truth in at least one world, but not in all possible worlds), and similarly for distinction between "absolute indeterminacy" and "relative indeterminacy", and respectively distinction between "absolute falsehood" and "relative falsehood".

But for other scientific and technical applications one uses standard subsets, and the standard classical unit interval [0, 1].

As a particular case of neutrosophic axiom is the classical axiom. In the classical mathematics an axiom is supposed 100% true, 0% indeterminate, and 0% false. But this thing occurs in idealistic systems, in perfectly closed systems, not in many of the real world situations.

Unlike the classical axiom which is a total premise of reasoning and without any controversy, the neutrosophic axiom is a partial premise of reasoning with a partial controversy.

The neutrosophic axioms serve in approximate reasoning.

The partial truth of a neutrosophic axiom is similarly taken for granting.

The neutrosophic axioms, and in general the neutrosophic propositions, deal with approximate ideas or with probable ideas, and in general with ideas we are not able to measure exactly. That's why one cannot get 100% true statements (propositions).

In our life we deal with approximations. An axiom is approximately true, and the inference is approximately true either.

A neutrosophic axiom is a self-evident assumption in some degrees of truth, indeterminacy, and falsehood respectively.

2 Neutrosophic Deducing and Neutrosophic Inference

The neutrosophic axioms are employed in *neutrosophic deducing* and *neutrosophic inference* rules, which are sort of neutrosophic implications, and similarly they have degrees of truth, indeterminacy, and respectively falsehood.

3 Neutrosophic Proof

Consequently, a *neutrosophic proof* has also a degree of validity, degree of indeterminacy, and degree of invalidity. And this is when we work with not-well determinate elements in the space or not not-well determinate inference rules.

The neutrosophic axioms are at the foundation of various *neutrosophic sciences*.

The approximate, indeterminate, incomplete, partially unknown, ambiguous, vagueness, imprecision, contradictory, etc. knowledge can be neutrosophically axiomized.

4 Neutrosophic Axiomatic System

A set of neutrosophic axioms Ω is called *neutrosophic axiomatic system*, where the neutrosophic deducing and the neutrosophic inference (neutrosophic implication) are used.

The neutrosophic axioms are defined on a given space S. The space can be classical (space without indeterminacy), or neutrosophic space (space which has some indeterminacy with respect to its elements).

A neutrosophic space may be, for example, a space that has at least one element which only partially belongs to the space. Let us say the element x <0.5, 0.2, 0.3> that belongs only 50% to the space, while 20% its appurtenance is indeterminate, and 30% it does not belong to the space.

Therefore, we have three types of neutrosophic axiomatic systems:

[1] Neutrosophic axioms defined on classical space;

[2] Classical axioms defined on neutrosophic space;

[3] Neutrosophic axioms defined on neutrosophic space.

Remark:

The neutrosophic axiomatic system is not unique, in the sense that several different axiomatic systems may describe the same neutrosophic model. This happens because one deals with approximations, and because the neutrosophic axioms represent partial (not total) truths.

5 Classification of the Neutrosophic Axioms

[1] *Neutrosophic Logical Axioms*, which are neutrosophic statements whose truth-value is <t, i, f> within the system of neutrosophic logic. For example: (α or β) neutrosophically implies β.

[2] *Neutrosophic Non-Logical Axioms*, which are neutrosophic properties of the elements of the space. For example: the neutrosophic associativity $a(bc) = (ab)c$, which occurs for some elements, it is unknown (indeterminate) for others, and does not occur for others.

In general, a neutrosophic non-logical axiom is a classical non-logical axiom that works for certain space elements, is indeterminate for others, and does not work for others.

6 Neutrosophic Tautologies

A classical tautology is a statement that is universally true [regarded in a larger way, or lato sensu], i.e. true in all possible worlds (according to Leibniz's definition of "world"). For example, "M = M" in all possible worlds.

A *neutrosophic tautology* is a statement that is true in a narrow way [i.e. regarded in stricto sensu], or it is <1, 0, 0> true for a class of certain parameters and conditions, and <t, i, f> true for another class of certain parameters and conditions, where <t, i, f> ≠ <1, 0, 0>. I.e. a neutrosophic tautology is true in some worlds, and partially true in other worlds. For example, the previous assertation: "M = M".

If "M" is a number [i.e. the parameter = number], then a number is always equal to itself in any numeration base.

But if "M" is a person [i.e. the parameter = person], call him Martin, then Martin at time t1 is the same as Martin at time t1 [i.e. it has been considered another parameter = time], but Martin at time t1 is different from Martin at time t2 (meaning for example 20 years ago: hence Martin younger is different from Martin older). Therefore, from the point of view of parameters 'person' and 'time', "M = M" is not a classical tautology.

Similarly, we may have a proposition P which is true locally, but it is untrue non-locally.

A neutrosophic logical system is an approximate minimal set of partially true/indeterminate/false propositions.

While the classical axioms cannot be deduced from other axioms, there are neutrosophic axioms that can be partially deduced from other neutrosophic axioms.

7 Notations regarding the Classical Logic and Set, Fuzzy Logic and Set, Intuitionistic Fuzzy Logic and Set, and Neutrosophic Logic and Set

In order to make distinction between classical (Boolean) logic/set, fuzzy logic/set, intuitionistic fuzzy logic/set, and neutrosophic logic/set, we denote their corresponding operators (negation/complement, conjunction/ intersection, disjunction/union, implication/inclusion, and equivalence/equality), as it follows:

[1] For classical (Boolean) logic and set:

$$\neg \quad \wedge \quad \vee \quad \rightarrow \quad \leftrightarrow \tag{1}$$

[2] For fuzzy logic and set:

$$\underset{F}{\neg} \quad \underset{F}{\wedge} \quad \underset{F}{\vee} \quad \underset{F}{\rightarrow} \quad \underset{F}{\leftrightarrow} \tag{2}$$

[3] For intuitionistic fuzzy logic and set:

$$\underset{IF}{\neg} \quad \underset{IF}{\wedge} \quad \underset{IF}{\vee} \quad \underset{IF}{\rightarrow} \quad \underset{IF}{\leftrightarrow} \tag{3}$$

[4] For neutrosophic logic and set:

$$\underset{N}{\neg} \quad \underset{N}{\wedge} \quad \underset{N}{\vee} \quad \underset{N}{\rightarrow} \quad \underset{N}{\leftrightarrow} \tag{4}$$

8 The Classical Quantifiers

The classical *Existential Quantifier* is the following way:

$$\exists x \in A, P(x). \tag{5}$$

In a neutrosophic way we can write it as:

There exist $x<1, 0, 0>$ in *A* such that $P(x)<1, 0, 0>$, or:

$$\exists x < 1,0,0 > \in A, P(x) < 1,0,0 >. \tag{6}$$

The classical *Universal Quantifier* is the following way:

$$\forall x \in A, P(x). \tag{7}$$

In a neutrosophic way we can write it as:

For any $x<1, 0, 0>$ in *A* one has $P(x)<1, 0, 0>$, or:

$$\forall x < 1,0,0 > \in A, P(x) < 1,0,0 >. \tag{8}$$

9 The Neutrosophic Quantifiers

The *Neutrosophic Existential Quantifier* is in the following way:

There exist $x<t_x, i_x, f_x>$ in A such that $P(x)<t_P, i_P, f_P>$, or:

$$\exists x < t_x, i_x, f_x > \in A, P(x) < t_P, i_P, f_P >, \tag{9}$$

which means that: there exists an element x which belongs to A in a neutrosophic degree $<t_x, i_x, f_x>$, such that the proposition P has the neutrosophic degree of truth $<t_P, i_P, f_P>$.

The *Neutrosophic Universal Quantifier* is the following way:

For any $x<t_x, i_x, f_x>$ in A one has $P(x)<t_P, i_P, f_P>$, or:

$$\forall x < t_x, i_x, f_x > \in A, P(x) < t_P, i_P, f_P >, \tag{10}$$

which means that: for any element x that belongs to A in a neutrosophic degree $<t_x, i_x, f_x>$, one has the proposition P with the neutrosophic degree of truth $<t_P, i_P, f_P>$.

10 Neutrosophic Axiom Schema

A *neutrosophic axiom schema* is a neutrosophic rule for generating infinitely many neutrosophic axioms.

Examples of neutrosophic axiom schema:

[1] *Neutrosophic Axiom Scheme for Universal Instantiation.*

Let Φ(x) be a formula, depending on variable x defined on a domain D, in the first-order language L, and let's substitute x for $a \in$ D. Then the new formula:

$$\forall x \Phi(x) \rightarrow_N \Phi(a) \tag{11}$$

is $< t_{\rightarrow_N}, i_{\rightarrow_N}, f_{\rightarrow_N} >$-neutrosophically [universally] valid.

This means the following: if one knows that a formula Φ(x) holds $<t_x, i_x, f_x>$-neutrosophically for every x in the domain D, and for $x = a$ the formula $\Phi(a)$ holds $<t_a, i_a, f_a>$-neutrosophically, then the whole new formula (a) holds $< t_{\rightarrow_N}, i_{\rightarrow_N}, f_{\rightarrow_N} >$-neutrosophically, where $t_{\rightarrow_N}$ means the truth degree, $i_{\rightarrow_N}$ the indeterminacy degree, and $f_{\rightarrow_N}$ the falsehood degree –- all resulted from the neutrosophic implication $\rightarrow_N$.

[2] *Neutrosophic Axiom Scheme for Existential Generalization.*

Let Φ*(x)* be a formula, depending on variable *x* defined on a domain *D*, in the first-order language *L*, and let's substitute x for $a \in$ D. Then the new formula:

$$\Phi(a) \rightarrow_N \exists x \Phi(x) \tag{12}$$

is $< t_{\rightarrow_N}, i_{\rightarrow_N}, f_{\rightarrow_N} >$-neutrosophically [universally] valid.

This means the following: if one knows that a formula Φ*(a)* holds $<t_a, i_a, f_a>$-neutrosophically for a given x = a in the domain *D*, and for every *x* in the domain formula Φ(x) holds $<t_x, i_x, f_x>$-neutrosophically, then the whole new formula (b) holds $< t_{\rightarrow_N}, i_{\rightarrow_N}, f_{\rightarrow_N} >$-neutrosophically, where $t_{\rightarrow_N}$ means the truth degree, $i_{\rightarrow_N}$ the indeterminacy degree, and $f_{\rightarrow_N}$ the falsehood degree –- all resulted from the neutrosophic implication $\rightarrow_N$.

These are *neutrosophic metatheorems* of the mathematical neutrosophic theory where they are employed.

11 Neutrosophic Propositional Logic

We have many neutrosophic formulas that one takes as neutrosophic axioms. For example, as extension from the classical logic, one has the following.

Let $P<t_P, i_P, f_P>$, $Q<t_Q, i_Q, f_Q>$, $R<t_R, i_R, f_R>$, $S<t_S, i_S, f_S>$ be neutrosophic propositions, where $<t_P, i_P, f_P>$ is the neutrosophic-truth value of *P*, and similarly for *Q, R,* and *S*. Then:

a) *Neutrosophic modus ponens* (neutrosophic implication elimination):

$$P \rightarrow_N (Q \rightarrow_N P) \tag{13}$$

b) *Neutrosophic modus tollens* (neutrosophic law of contrapositive):

$$((P \rightarrow_N Q) \wedge_N \neg_N Q) \rightarrow_N \neg_N P \tag{14}$$

c) Neutrosophic *disjunctive syllogism* (neutrosophic disjunction elimination):

$$((P \vee_N Q) \wedge_N \neg_N P) \rightarrow_N Q \tag{15}$$

d) Neutrosophic *hypothetical syllogism* (neutrosophic chain argument):

$$((P \rightarrow_N Q) \wedge_N (Q \rightarrow_N R)) \rightarrow_N (P \rightarrow_N R) \tag{16}$$

e) ***Neutrosophic constructive dilemma*** (neutrosophic *disjunctive* version of *modus ponens*):

$$(((P \to_N Q) \wedge_N (R \to_N S)) \wedge_N (P \vee_N R)) \to_N (Q \vee_N S) \tag{17}$$

f) ***Neutrosophic distructive dilemma*** (neutrosophic *disjunctive* version of *modus tollens*):

$$\begin{gathered}(((P \to_N Q) \wedge_N (R \to_N S)) \wedge_N \\ (\neg_N Q \vee_N \neg_N S)) \to_N (\neg_N P \vee_N \neg_N R)\end{gathered} \tag{18}$$

All these neutrosophic formulae also run as *neutrosophic rules of inference*.

These neutrosophic formulas or neutrosophic derivation rules only *partially* preserve the truth, and depending on the neutrosophic implication operator that is employed the indeterminacy may increase or decrease.

This happens for one working with approximations.

While the above classical formulas in classical proportional logic are classical tautologies (i.e. from a neutrosophical point of view they are *100%* true, *0%* indeterminate, and *0%* false), their corresponding neutrosophic formulas are neither classical tautologies nor neutrosophical tautologies, but ordinary neutrosophic propositions whose $< t, i, f >$ – neutrosophic truth-value is resulted from the $\underset{N}{\to}$ neutrosophic implication

$$A < t_A, i_A, f_A > \underset{N}{\to} B < (t_B, i_B, f_B) >. \tag{19}$$

12 Classes of Neutrosophic Negation Operators

There are defined in neutrosophic literature classes of neutrosophic negation operators as follows: if $A(t_A, i_A, f_A)$, then its negation is:

$$\underset{N}{\neg} A(f_A, i_A, t_A), \tag{20}$$

or $\underset{N}{\neg} A(f_A, 1 - i_A, t_A)$, (21)

or $\underset{N}{\neg} A(1 - t_A, 1 - i_A, 1 - f_A)$, (22)

or $\underset{N}{\neg} A(1 - t_A, i_A, 1 - f_A)$, etc. (23)

13 Classes of Neutrosophic Conjunctive Operators.

Similarly: if $A(t_A, i_A, f_A)$ and $B(t_B, i_B, f_B)$, then

$$A \overset{\wedge}{_N} B = \langle t_A \overset{\wedge}{_F} t_B, i_A \overset{\vee}{_F} i_B, f_A \overset{\vee}{_F} f_B \rangle, \tag{24}$$

$$\text{or } A \overset{\wedge}{_N} B = \langle t_A \overset{\wedge}{_F} t_B, i_A \overset{\wedge}{_F} i_B, f_A \overset{\vee}{_F} f_B \rangle, \tag{25}$$

$$\text{or } A \overset{\wedge}{_N} B = \langle t_A \overset{\wedge}{_F} t_B, i_A \overset{\wedge}{_F} i_B, f_A \overset{\wedge}{_F} f_B \rangle \tag{26}$$

$$\text{or } A \overset{\wedge}{_N} B = \langle t_A \overset{\wedge}{_F} t_B, \frac{i_A + i_B}{2}, f_A \overset{\vee}{_F} f_B \rangle, \tag{27}$$

$$\text{or } A \overset{\wedge}{_N} B = \langle t_A \overset{\wedge}{_F} t_B, 1 - \frac{i_A + i_B}{2}, f_A \overset{\vee}{_F} f_B \rangle, \tag{28}$$

$$\text{or } A \overset{\wedge}{_N} B = \langle t_A \overset{\wedge}{_F} t_B, |i_A - i_B|, f_A \overset{\vee}{_F} f_B \rangle, \text{ etc.} \tag{29}$$

14 Classes of Neutrosophic Disjunctive Operators

And analogously, there were defined:

$$A \overset{\vee}{_N} B = \langle t_A \overset{\vee}{_F} t_B, i_A \overset{\wedge}{_F} i_B, f_A \overset{\wedge}{_F} f_B \rangle, \tag{30}$$

$$\text{or } A \overset{\vee}{_N} B = \langle t_A \overset{\vee}{_F} t_B, i_A \overset{\vee}{_F} i_B, f_A \overset{\wedge}{_F} f_B \rangle, \tag{31}$$

$$\text{or } A \overset{\vee}{_N} B = \langle t_A \overset{\vee}{_F} t_B, i_A \overset{\vee}{_F} i_B, f_A \overset{\vee}{_F} f_B \rangle, \tag{32}$$

$$\text{or } A \overset{\vee}{_N} B = \langle t_A \overset{\vee}{_F} t_B, \frac{i_A + i_B}{2}, f_A \overset{\wedge}{_F} f_B \rangle, \tag{33}$$

$$\text{or } A \overset{\vee}{_N} B = \langle t_A \overset{\vee}{_F} t_B, 1 - \frac{i_A + i_B}{2}, f_A \overset{\wedge}{_F} f_B \rangle, \tag{34}$$

$$\text{or } A \overset{\vee}{_N} B = \langle t_A \overset{\vee}{_F} t_B, |i_A - i_B|, f_A \overset{\vee}{_F} f_B \rangle, \text{ etc.} \tag{35}$$

15 Fuzzy Operators

Let $\alpha, \beta \in [0, 1]$.

15.1. The *Fuzzy Negation* has been defined as $\neg_F \alpha = 1 - \alpha$. (36)

15.2. While the class of *Fuzzy Conjunctions* (or *t-norm*) may be:

$$\alpha \overset{\wedge}{_F} \beta = \min\{\alpha, \beta\}, \tag{37}$$

$$\text{or } \alpha \overset{\wedge}{_F} \beta = \alpha \cdot \beta, \tag{38}$$

$$\text{or } \alpha \overset{\wedge}{_F} \beta = \max\{0, \alpha + \beta - 1\}, \text{ etc.} \tag{39}$$

15.3. And the class of *Fuzzy Disjunctions* (or *t-conorm*) may be:

$$\alpha \overset{\vee}{_F} \beta = \max\{\alpha, \beta\}, \tag{40}$$

$$\text{or } \alpha \overset{\vee}{_F} \beta = \alpha + \beta - \alpha\beta, \tag{41}$$

$$\text{or } \alpha \overset{\vee}{_F} \beta = \min\{1, \alpha + \beta\}, \text{ etc.} \tag{42}$$

15.4. Examples of *Fuzzy Implications* $x \underset{F}{\rightarrow} y$, for $x, y \in [0, 1]$, defined below:

- Fodor (1993): $I_{FD}(x,y) = \begin{cases} 1, \text{ if } x \le y \\ \max(1-x, y), \text{ if } x > y \end{cases}$ (43)
- Weber (1983): $I_{WB}(x,y) = \begin{cases} 1, \text{if } x < y \\ y, \text{if } x = 1 \end{cases}$ (44)
- Yager (1980): $I_{YG}(x,y) = \begin{cases} 1, \text{if } x = 0 \text{ and } y = 0 \\ y^x, \text{if } x > 0 \text{ or } y > 0 \end{cases}$ (45)
- Goguen (1969): $I_{GG}(x,y) = \begin{cases} 1, \text{if } x \le y \\ \frac{y}{x}, \text{if } x > y \end{cases}$ (46)
- Rescher (1969): $I_{RS}(x,y) = \begin{cases} 1, \text{if } x \le y \\ 0, \text{if } x > y \end{cases}$ (47)
- Kleene-Dienes (1938): $I_{KD}(x,y) = \max(1-x, y)$ (48)
- Reichenbach (1935): $I_{RC}(x,y) = 1 - x + xy$ (49)
- Gödel (1932): $I_{GD}(x,y) = \begin{cases} 1, \text{if } x \le y \\ y, \text{if } x > y \end{cases}$ (50)
- Lukasiewicz (1923): $I_{LK}(x,y) = \min(1, 1-x+y)$, (51)

according to the list made by Michal Baczyński and Balasubramaniam Jayaram (2008).

16 Example of Intuitionistic Fuzzy Implication

Example of *Intuitionistic Fuzzy Implication* $A(t_A, f_A) \underset{IF}{\rightarrow} B(t_B, f_B)$ is:

$$I_{IF} = \left(\left[(1-t_A) \overset{\vee}{_F} t_B \right] \overset{\wedge}{_F} \left[(1-f_B) \overset{\vee}{_F} f_A \right], f_B \overset{\wedge}{_F} (1-t_A) \right), \tag{52}$$

according to Yunhua Xiao, Tianyu Xue, Zhan'ao Xue, and Huiru Cheng (2011).

17 Classes of Neutrosophic Implication Operators

We now propose for the first time *eight new classes of neutrosophic implications* and extend a ninth one defined previously:

$$A(t_A, i_A, f_A) \underset{N}{\rightarrow} B(t_B, i_B, f_B),$$

in the following ways:

17.1-17.2. $I_{N1}\left(t_A \underset{F/IF}{\longrightarrow} t_B, i_A \underset{F}{\wedge} i_B, f_A \underset{F}{\wedge} f_B\right)$, (53)

where $t_A \underset{F/IF}{\longrightarrow} t_B$ is any fuzzy implication (from above or others) or any intuitionistic fuzzy implication (from above or others), while $\underset{F}{\wedge}$ is any fuzzy conjunction (from above or others);

17.3-17.4. $I_{N2}\left(t_A \underset{F/IF}{\longrightarrow} t_B, i_A \underset{F}{\vee} i_B, f_A \underset{F}{\wedge} f_B\right)$, (54)

where $\underset{F}{\vee}$ is any fuzzy disjunction (from above or others);

17.5-17.6. $I_{N3}\left(t_A \underset{F/IF}{\longrightarrow} t_B, \frac{i_A+i_B}{2}, f_A \underset{F}{\wedge} f_B\right)$; (55)

17.7-17.8. $I_{N4}\left(t_A \underset{F/IF}{\longrightarrow} t_B, \frac{i_A+i_B}{2}, \frac{f_A+f_B}{2}\right)$. (56)

17.9. Now we extend another neutrosophic implication that has been defined by S. Broumi & F. Smarandache (2014) and it was based on the classical logical equivalence:

$$(A \rightarrow B) \leftrightarrow (\neg A \vee B). \quad (57)$$

Whence, since the corresponding neutrosophic logic equivalence:

$$\left(A \underset{N}{\rightarrow} B\right) \underset{N}{\leftrightarrow} (\underset{N}{\neg} A \underset{N}{\vee} B) \quad (58)$$

holds, one obtains another *Class of Neutrosophic Implication Operators* as:

$$(\underset{N}{\neg} A \underset{N}{\vee} B) \quad (59)$$

where one may use any neutrosophic negation $\underset{N}{\neg}$ (from above or others), and any neutrosophic disjunction $\underset{N}{\vee}$ (from above or others).

18 Example of Neutrosophic Implication

Let's see an *Example of Neutrosophic Implication*.

Let's have two neutrosophic propositions $A\langle 0.3, 0.4, 0.2\rangle$ and $B\langle 0.7, 0.1, 0.4\rangle$. Then $A \underset{N}{\rightarrow} B$ has the neutrosophic truth value of $\underset{N}{\neg} A \underset{N}{\vee} B$, i.e.:

$$\langle 0.2, 0.4, 0.3\rangle \underset{N}{\vee} \langle 0.7, 0.1, 0.4\rangle,$$

$$\text{or } \langle \max\{0.2, 0.7\}, \min\{0.4, 0.1\}, \min\{0.3, 0.4\}\rangle,$$

$$\text{or } \langle 0.7, 0.1, 0.3\rangle,$$

where we used the neutrosophic operators defined above: $\underset{N}{\neg}\langle t, i, f\rangle = \langle f, i, t\rangle$ for neutrosophic negation, and $\langle t_1, i_1, f_1\rangle \underset{N}{\vee} \langle t_2, i_2, f_2\rangle = \langle \max\{t_1, t_2\}, \min\{i_1, i_2\}, \min\{f_1, f_2\}\rangle$ for the neutrosophic disjunction.

Using different versions of the neutrosophic negation operators and/or different versions of the neutrosophic disjunction operators, one obtains, in general, different results. Similarly as in fuzzy logic.

18.1. Another Example of Neutrosophic Implication.

Let A have the neutrosophic truth-value (t_A, i_A, f_A) , and B have the neutrosophic truth-value (t_B, i_B, f_B), then:

$$\left[A \underset{N}{\rightarrow} B\right] \underset{N}{\leftrightarrow} [(\underset{N}{\neg} A) \underset{N}{\vee} B], \tag{60}$$

where $\underset{N}{\neg}$ is any of the above neutrosophic negations, while $\underset{N}{\vee}$ is any of the above neutrosophic disjunctions.

19 General Definition of Neutrosophic Operators

We consider that the most general definition of neutrosophic operators shall be the followings:

$$A(t_A, i_A, f_A) \underset{N}{\oplus} B(t_B, i_B, f_B) = A \underset{N}{\oplus} B\langle u(t_A, i_A, f_A, t_B, i_B, f_B), \\ v(t_A, i_A, f_A, t_B, i_B, f_B),\ w(t_A, i_A, f_A, t_B, i_B, f_B)\rangle \tag{61}$$

where $\underset{N}{\oplus}$ is any binary neutrosophic operator, and

$$u(x_1, x_2, x_3, x_4, x_5, x_6), v(x_1, x_2, x_3, x_4, x_5, x_6),$$

$$w(x_1, x_2, x_3, x_4, x_5, x_6): [0,1]^6 \rightarrow [0,1].$$

Even more, the neutrosophic component functions u, v, w may depend, on the top of these six variables, on hidden parameters as well, such as: $h_1, h_2, \dots, h_n$.

For a unary neutrosophic operator (for example, the neutrosophic negation), similarly:

$$\underset{N}{\neg} A(t_A, i_A, f_A) = \langle u'(t_A, i_A, f_A), v'(t_A, i_A, f_A), w'(t_A, i_A, f_A) \rangle, \quad (62)$$

where $u'(t_A, i_A, f_A), v'(t_A, i_A, f_A), w'(t_A, i_A, f_A)$: $[0, 1]^3 \to [0,1]$,

and even more u', v', w' may depend, on the top of these three variables, of hidden parameters as well, such as: $h_1, h_2, \dots, h_n$.

{Similarly there should be for a *general definition of fuzzy operators* and *general definition of intuitionistic fuzzy operators*.}

As an example, we have defined [6]: (63)

$$A(t_A, i_A, f_A) \underset{N}{\wedge} B(t_B, i_B, f_B) \\ = \langle t_A t_B, i_A i_B + t_A i_B + t_B i_A, t_A f_B + t_B f_A + i_A f_B + i_B f_A \rangle$$

these result from multiplying

$$(t_A + i_A + f_A) \cdot (t_B + i_B + f_B) \quad (64)$$

and ordering upon the below pessimistic order:

truth $\prec$ indeterminacy $\prec$ falsity,

meaning that to the *truth* only the terms of t's goes, i.e. $t_A t_B$,

to *indeterminacy* only the terms of t's and i's go, i.e. $i_A i_B + t_A i_B + t_B i_A$,

and to *falsity* the other terms left, i.e. $t_A f_B + t_B f_A + i_A f_B + i_B f_A + f_A f_B$.

20 Neutrosophic Deductive System

A *Neutrosophic Deductive System* consists of a set $\mathcal{L}_1$ of neutrosophic logical axioms, and a set $\mathcal{L}_2$ of neutrosophic non-logical axioms, and a set $\mathcal{R}$ of neutrosophic rules of inference – all defined on a neutrosophic space $\mathcal{S}$ that is composed of many elements.

A neutrosophic deductive system is said to be neutrosophically complete, if for any neutrosophic formula φ that is a neutrosophic logical consequence of $\mathcal{L}_1$, i.e. $\mathcal{L}_1 \underset{N}{\vDash} \varphi$, there exists a neutrosophic deduction of φ from $\mathcal{L}_1$, i.e. $\mathcal{L}_1 \underset{N}{\vdash} \varphi$, where $\underset{N}{\vDash}$ denotes neutrosophic logical consequence, and $\underset{N}{\vdash}$ denotes neutrosophic deduction.

Actually, everything that is neutrosophically (partially) true [i.e. made neutrosophically (partially) true by the set $\mathcal{L}_1$ of neutrosophic axioms] is neutrosophically (partially) provable.

The neutrosophic completeness of set $\mathcal{L}_2$ of neutrosophic non-logical axioms is not the same as the neutrosophic completeness of set $\mathcal{L}_1$ of neutrosophic logical axioms.

21 Neutrosophic Axiomatic Space

The space $\mathcal{S}$ is called *neutrosophic space* if it has some indeterminacy with respect to one or more of the following:

a. Its *elements*;

1. At least one element x partially belongs to the set $\mathcal{S}$, or $x(t_x, i_x, f_x)$ with $(t_X, i_X, f_X) \neq (1, 0, 0)$;

2. There is at least an element y in $\mathcal{S}$ whose appurtenance to $\mathcal{S}$ is unknown.

b. Its *logical axioms*;

1. At least a logical axiom $\mathcal{A}$ is partially true, or $\mathcal{A}(t_A, i_A, f_A)$, where similary $(t_A, i_A, f_A) \neq (1, 0, 0)$;

2. There is at least an axiom $\mathcal{B}$ whose truth-value is unknown.

c. Its *non-logical axioms*;

1. At least a non-logical axiom $\mathcal{C}$ is true for some elements, and indeterminate or false or other elements;

2. There is at least a non-logical axiom whose truth-value is unknown for some elements in the space.

d. There exist at least two neutrosophic logical axioms that have some degree of contradiction (strictly greater than zero).

e. There exist at least two neutrosophic non-logical axioms that have some degree of contradiction (strictly greater than zero).

22 Degree of Contradiction (Dissimilarity) of Two Neutrosophic Axioms

Two neutrosophic logical axioms $\mathcal{A}_1$ and $\mathcal{A}_2$ are contradictory (dissimilar) if their semantics (meanings) are contradictory in some degree d_1, while their neutrosophic truth values $<t_1, i_1, f_1>$ and $<t_2, i_2, f_2>$ are contradictory in a different degree d_2 [in other words $d_1 \neq d_2$].

As a particular case, if two neutrosophic logical axioms $\mathcal{A}_1$ and $\mathcal{A}_2$ have the same semantic (meaning) [in other words $d_1 = 0$], but their neutrosophic truth-values are different [in other words $d_2 > 0$], they are contradictory.

Another particular case, if two neutrosophic axioms $\mathcal{A}_1$ and $\mathcal{A}_2$ have different semantics (meanings) [in other words $d_1 > 0$], but their neutrosophic truth values are the same $<t_1, i_1, f_1> = <t_2, i_2, f_2>$ [in other words $d_2 = 0$], they are contradictory.

If two neutrosophic axioms $\mathcal{A}_1$ and $\mathcal{A}_2$ have the semantic degree of contradiction d_1, and the neutrosophic truth value degree of contradiction d_2, then the total degree of contradiction of the two neutrosophic axioms is $d = |d_1 - d_2|$, where $|\ \ |$ mean the absolute value.

We did not manage to design a formula in order to compute the semantic degree of contradiction d_1 of two neutrosophic axioms. The reader is invited to explore such metric.

But we can compute the neutrosophic truth value degree of contradiction d_2. If $\langle t_1, i_1, f_1\rangle$ is the neutrosophic truth-value of $\mathcal{A}_1$ and $\langle t_2, i_2, f_2\rangle$ the neutrosophic truth-value of $\mathcal{A}_2$, where $t_1, i_1, f_1, t_2, i_2, f_2$ are single values in $[0,1]$, then the neutrosophic truth value degree of contradiction d_2 of the neutrosophic axioms $\mathcal{A}_1$ and $\mathcal{A}_2$ is:

$$d_2 = \frac{1}{3}(|t_1 - t_2| + |i_1 - i_2| + |f_1 - f_2|), \tag{65}$$

whence $d_2 \in [0,1]$.

We get $d_2 = 0$, when $\mathcal{A}_1$ is identical with $\mathcal{A}_2$ from the point of view of neutrosophical truth values, i.e. when $t_1 = t_2, i_1 = i_2, f_1 = f_2$. And we get $d_2 = 1$, when $\langle t_1, i_1, f_1\rangle$ and $\langle t_2, i_2, f_2\rangle$ are respectively equal to:

$\langle 1,0,0\rangle$, $\langle 0,1,1\rangle$;

or $\langle 0,1,0\rangle$, $\langle 1,0,1\rangle$;

or $\langle 0,0,1\rangle$, $\langle 1,1,0\rangle$;

or $\langle 0,0,0\rangle$, $\langle 1,1,1\rangle$.

23 Neutrosophic Axiomatic System

The neutrosophic axioms are used, in neutrosophic conjunction, in order to derive neutrosophic theorems.

A *neutrosophic mathematical theory* may consist of a neutrosophic space where a neutrosophic axiomatic system acts and produces all neutrosophic theorems within the theory.

Yet, in a *neutrosophic formal system*, in general, the more recurrences are done the more is increased the indeterminacy and decreased the accuracy.

24 Properties of the Neutrosophic Axiomatic System

[1] While in classical mathematics an axiomatic system is consistent, in a neutrosophic axiomatic system it happens to have *partially inconsistent (contradictory) axioms*.

[2] Similarly, while in classical mathematics the axioms are independent, in a neutrosophic axiomatic system they may be *dependent in certain degree*.

[3] In classical mathematics if an axiom is dependent from other axioms, it can be removed, without affecting the axiomatic system.

[4] However, if a neutrosophic axiom is partially dependent from other neutrosophic axioms, by removing it the neutrosophic axiomatic system is affected.

[5] While, again, in classical mathematics an axiomatic system has to be complete (meaning that each statement or its negation is derivable), a neutrosophic axiomatic system is *partially complete and partially incomplete*. It is partially incomplete because one can add extra partially independent neutrosophic axioms.

[6] The *neutrosophic relative consistency* of an axiomatic system is referred to the neutrosophically (partially) undefined terms of a first neutrosophic axiomatic system that are assigned neutrosophic definitions from another neutrosophic axiomatic system in a way that, with respect to both neutrosophic axiomatic systems, is neutrosophically consistent.

25 Neutrosophic Model

A *Neutrosophic Model* is a model that assigns neutrosophic meaning to the neutrosophically (un)defined terms of a neutrosophic axiomatic system.

Similarly to the classical model, we have the following classification:

[1] *Neutrosophic Abstract Model*, which is a neutrosophic model based on another neutrosophic axiomatic system.

[2] *Neutrosophic Concrete Model*, which is a neutrosophic model based on real world, i.e. using real objects and real relations between the objects.

In general, a neutrosophic model is a *<t, i, f>-approximation*, i.e. *T%* of accuracy, *I%* indeterminacy, and *F%* inaccuracy, of a neutrosophic axiomatic system.

26 Neutrosophically Isomorphic Models

Further, *two neutrosophic models are neutrosophically isomorphic* if there is a neutrosophic one-to-one correspondence between their neutrosophic elements such that their neutrosophic relationships hold.

A neutrosophic axiomatic system is called *neutrosophically categorial* (or *categorical*) is any two of its neutrosophic models are neutrosophically isomorphic.

27 Neutrosophic Infinite Regressions

There may be situations of neutrosophic axiomatic systems that generate neutrosophic infinite regressions, unlike the classical axiomatic systems.

28 Neutrosophic Axiomatization

A *Neutrosophic Axiomatization* is referred to an approximate formulation of a set of neutrosophic statements, about a number of neutrosophic primitive terms, such that by the neutrosophic deduction one obtains various neutrosophic propositions (theorems).

29 Example of Neutrosophic Axiomatic System

Let's consider two neighboring countries M and N that have a disputed frontier zone Z:

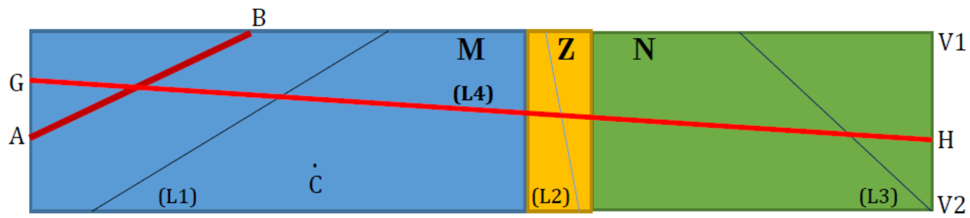

Figure 1: A Neutrosophic Model.

Let's consider the universe of discourse $U = M \cup Z \cup N$; this is a *neutrosophic space* since it has an indeterminate part (the disputed frontier).

The *neutrosophic primitive notions* in this example are: neutrosophic point, neutrosophic line, and neutrosophic plane (space).

And the *neutrosophic primitive relations* are: neutrosophic incidence, and neutrosophic parallel.

The four boundary edges of rectangle Z belong to Z (or Z is a closed set). While only three boundary edges of M (except the fourth one which is common with Z) belong to M, and similarly only three boundaries of N (except the fourth one which is common with Z) belong to N. Therefore M and N are neither closed nor open sets.

Taking a classical point P in U, one has three possibilities:

[1] $P \in M$ (membership with respect to country M);
[2] $P \in Z$ (indeterminate membership with respect to both countries);
[3] or $P \in N$ (nonmembership with respect to country M).

Such points, that can be indeterminate as well, are called *neutrosophic points*.

A *neutrosophic line* is a classical segment of line that unites two neutrosophic points lying on opposite edges of the universe of discourse U. We may have:

[1] determinate line (with respect to country M), that is completely into the determinate part M {for example ($L1$)};
[2] indeterminate line, that is completely into the frontier zone *{for example (L2)}*;
[3] determinate line (with respect to country N), that is completely into the determinate part N *{for example (L3)}*;

[4] or mixed, i.e. either two or three of the following: partially determinate with respect to M, partially indeterminate with respect to both countries, and partially determinate with respect to N {for example the red line *(L4)}*.

Through two neutrosophic points there may be passing:

[1] only one neutrosophic line {for example, through G and H passes only one neutrosophic line (*L4*)};
[2] no neutrosophic line {for example, through A and B passes no neutrosophic line, since the classical segment of line AB does not unite points of opposite edges of the universe of discourse U}.

Two *neutrosophic lines* are *parallel* is they have no common neutrosophic points.

Through a neutrosophic point outside of a neutrosophic line, one can draw:

[1] infinitely many neutrosophic parallels {for example, through the neutrosophic point C one can draw infinitely many neutrosophic parallels to the neutrosophic line *(L1)*};
[2] only one neutrosophic parallel {for example, through the neutrosophic point H that belongs to the edge *(V1V2)* one can draw only one neutrosophic parallel (i.e. V1V2) to the neutrosophic line *(L1)*};
[3] no neutrosophic parallel {for example, through the neutrosophic point H there is no neutrosophic parallel to the neutrosophic line *(L3)*}.

For example, the neutrosophic lines (*L1), (L2)* and (*L3)* are parallel. But the neutrosophic line *(L4)* is not parallel with *(L1),* nor with *(L2)* or *(L3)*.

A *neutrosophic polygon* is a classical polygon which has one or more of the following indeterminacies:

[1] indeterminate vertex;
[2] partially or totally indeterminate edge;
[3] partially or totally indeterminate region in the interior of the polygon.

We may construct several neutrosophic axiomatic systems, for this example, referring to incidence and parallel.

a) First neutrosophic axiomatic system

α1) Through two distinct neutrosophic points there is passing a single neutrosophic line.

> {According to several experts, the neutrosophic truth-value of this axiom is *<0.6, 0.1, 0.2>,* meaning that having two given neutrosophic points, the chance that only one line (that do not intersect the indeterminate zone *Z*) passes through them is *0.6,* the chance that line that passes through them intersects the indeterminate zone *Z*) is *0.1*, and the chance that no line (that does not intersect the indeterminate zone *Z*) passes through them is *0.2*.}

α2) Through a neutrosophic point exterior to a neutrosophic line there is passing either one neutrosophic parallel or infinitely many neutrosophic parallels.

> {According to several experts, the neutrosophic truth-value of this axiom is *<0.7, 0.2, 0.3>,* meaning that having a given neutrosophic line and a given exterior neutrosophic point, the chance that infinitely many parallels pass through this exterior point is *0.7*, the chance that the parallels passing through this exterior point intersect the indeterminate zone *Z* is *0.2*, and the chance that no parallel passes through this point is *0.3.*}

Now, let's apply a first neutrosophic deducibility.

Suppose one has three non-collinear neutrosophic (distinct) points *P, Q,* and *R* (meaning points not on the same line, alike in classical geometry). According to the neutrosophic axiom *(α1),* through *P, Q* passes only one neutrosophic line {let's call it *(PQ)*}, with a neutrosophic truth value *(0.6, 0.1, 0.2).* Now, according to axiom *(α2)*, through the neutrosophic point *R,* which does not lie on *(PQ),* there is passing either only one neutrosophic parallel or infinitely many neutrosophic parallels to the neutrosophic line (PQ), with a neutrosophic truth value *(0.7, 0.2, 0.3).*

Therefore,

$$(\alpha 1) \underset{N}{\wedge} (\alpha 2) = <0.6,\ 0.1,\ 0.2> \underset{N}{\wedge} <0.7,\ 0.2,\ 0.3> = <\min\{0.6,\ 0.7\}, \max\{0.1,\ 0.2\}, \max\{0.2,\ 0.3\}>= <0.6, 0.2, 0.3>, \quad (66)$$

which means the following: the chance that through the two distinct given neutrosophic points *P* and *Q* passes only one neutrosophic line, and through the exterior neutrosophic point *R* passese either only one neutrosophic parallel or infinitely many parallels to *(PQ)* is *(0.6, 0.2, 0.3),* i.e. *60%* true, *20%* indeterminate, and *30%* false.

Herein we have used the simplest neutrosophic conjunction operator $\underset{N}{\wedge}$ of the form <*min, max, max*>, but other neutrosophic conjunction operator can be used as well.

A second neutrosophic deducibility:

Again, suppose one has three non-collinear neutrosophic (distinct) points *P, Q,* and *R* (meaning points not on the same line, as in classical geometry).

Now, let's compute the neutrosophic truth value that through *P* and *Q* is passing one neutrosophic line, but through *Q* there is no neutrosophic parallel to *(PQ)*.

$$\alpha 1 \underset{N}{\wedge} (\underset{N}{\neg} \alpha 2) = <0.6, 0.1, 0.2> \underset{N}{\wedge} (\underset{N}{\neg} <0.7, 0.2, 0.3>) = <0.6, 0.1, 0.2> \underset{N}{\wedge} <0.3, 0.2, 0.7> = <0.3, 0.2, 0.7>. \quad (67)$$

b) Second neutrosophic axiomatic system

β1) Through two distinct neutrosophic points there is passing either a single neutrosophic line or no neutrosophic line. {With the neutrosophic truth-value *<0.8, 0.1, 0.0>*}.

β2) Through a neutrosophic point exterior to a neutrosophic line there is passing either one neutrosophic parallel, or infinitely many neutrosophic parallels, or no neutrosophic parallel. {With the neutrosophic truth-value *<1.0, 0.2, 0.0>*}.

In this neutrosophic axiomatic system the above propositions W1 and W2:

W1: Through two given neutrosophic points there is passing only one neutrosophic line, and through a neutrosophic point exterior to this neutrosophic line there is passing either one neutrosophic parallel or infinitely many neutrosophic parallels to the given neutrosophic line; and W2: Through two given neutrosophic points there is passing only one neutrosophic line, and through a neutrosophic point exterior to this neutrosophic line there is passing no neutrosophic parallel to the line; are not deducible.

c) Third neutrosophic axiomatic system

γ1) Through two distinct neutrosophic points there is passing a single neutrosophic line.

{With the neutrosophic truth-value *<0.6, 0.1, 0.2>*}.

γ2) Through two distinct neutrosophic points there is passing no neutrosophic line.

{With the neutrosophic truth-value *<0.2, 0.1, 0.6>*}.

δ1) Through a neutrosophic point exterior to a neutrosophic line there is passing only one neutrosophic parallel.

{With the neutrosophic truth-value *<0.1, 0.2, 0.9>*}.

δ2) Through a neutrosophic point exterior to a neutrosophic line there are passing infinitely many neutrosophic parallels.

{With the neutrosophic truth-value *<0.6, 0.2, 0.4>*}.

δ3) Through a neutrosophic point exterior to a neutrosophic line there is passing no neutrosophic parallel.

{With the neutrosophic truth-value *<0.3, 0.2, 0.7>*}.

In this neutrosophic axiomatic system we have contradictory axioms:

- *(γ1)* is in *100%* degree of contradiction with *(γ2);*
- and similarly *(δ3)* is in *100%* degree of contradiction with *[(δ1)* together with *(δ2)].*

Totally or partially contradictory axioms are allowed in a neutrosophic axiomatic systems, since they are part of our imperfect world and since they approximately describe models that are - in general - partially true.

Regarding the previous two neutrosophic deducibilities one has: (68)

$$\gamma 1 \underset{N}{\wedge} (\delta 1 \underset{N}{\vee} \delta 2) = <0.6, 0.1, 0.2> \underset{N}{\wedge} \left(<0.1, 0.2, 0.9> \underset{N}{\vee} <0.6, 0.2, 0.4>\right) = <0.6, 0.1, 0.2> \underset{N}{\wedge} <\max\{0.1, 0.6\}, \min\{0.2, 0.2\}, \min\{0.9, 0.4\}> = <0.6, 0.1, 0.2> \underset{N}{\wedge} <0.6, 0.2, 0.4> = <0.6, 0.2, 0.4>,$$

which is slightly different from the result we got using the first neutrosophic axiomatic system *<0.6, 0.2, 0.3>*, and respectively*:*

$$\gamma 1 \underset{N}{\wedge} \delta 3 = <0.6, 0.1, 0.2> \underset{N}{\wedge} <0.3, 0.2, 0.7> = <0.3, 0.2, 0.7>, \quad (69)$$

which is the same as the result we got using the first neutrosophic axiomatic system.

The third neutrosophic axiomatic system is a refinement of the first and second neutrosophic axiomatic systems. From a deducibility point of view it is better and easier to work with a refined system than with a rough system.

30 Conclusion

This paper proposes a new framework to model interdependencies in project portfolio. NCM representation model is used for modeling relation among risks.

In many real world situations, the spaces and laws are not exact, not perfect. They are inter-dependent. This means that in most cases they are not 100% true, i.e. not universal. For example, many physical laws are valid in ideal and perfectly closed systems. However, perfectly closed systems do not exist in our heterogeneous world where we mostly deal with approximations. Also, since in the real world there is not a single homogenous space, we have to use the multispace for any attempt to unify various theories.

We do not have perfect spaces and perfect systems in reality. Therefore, many physical laws function approximatively (see [5]). The physical constants are not universal too; variations of their values depend from a space to another, from a system to another. A physical constant is t% true, i% indeterminate, and f% false in a given space with a certain composition, and it has a different neutrosophical truth value <t', i', f'> in another space with another composition.

A neutrosophic axiomatic system may be dynamic: new axioms can be added and others excluded.

The neutrosophic axiomatic systems are formed by axioms than can be partially dependent (redundant), partially contradictory (inconsistent), partially incomplete, and reflecting a partial truth (and consequently a partial indeterminacy and a partial falsehood) - since they deal with approximations of reality.

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Neutrosophic Vague Set Theory

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Abstract

In 1993, Gau and Buehrer proposed the theory of vague sets as an extension of fuzzy set theory. Vague sets are regarded as a special case of context-dependent fuzzy sets. In 1995, Smarandache talked for the first time about neutrosophy, and he defined the neutrosophic set theory as a new mathematical tool for handling problems involving imprecise, indeterminacy, and inconsistent data. In this paper, we define the concept of a neutrosophic vague set as a combination of neutrosophic set and vague set. We also define and study the operations and properties of neutrosophic vague set and give some examples.

Keywords

Vague set, Neutrosophy, Neutrosophic set, Neutrosophic vague set.

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1 Introduction

Many scientists wish to find appropriate solutions to some mathematical problems that cannot be solved by traditional methods. These problems lie in the fact that traditional methods cannot solve the problems of uncertainty in economy, engineering, medicine, problems of decision-making, and others. There have been a great amount of research and applications in the literature concerning some special tools like probability theory, fuzzy set theory [13], rough set theory [19], vague set theory [18], intuitionistic fuzzy set theory [10, 12] and interval mathematics [11, 14].

Since Zadeh published his classical paper almost fifty years ago, fuzzy set theory has received more and more attention from researchers in a wide range of scientific areas, especially in the past few years.

The difference between a binary set and a fuzzy set is that in a “normal” set every element is either a member or a non-member of the set; it either has to be A or not A.

In a fuzzy set, an element can be a member of a set to some degree and at the same time a non-member of the same set to some degree. In classical set theory, the membership of elements in a set is assessed in binary terms: according to a bivalent condition, an element either belongs or does not belong to the set.

By contrast, fuzzy set theory permits the gradual assessment of the membership of elements in a set; this is described with the aid of a membership function valued in the closed unit interval [0, 1].

Fuzzy sets generalise classical sets, since the indicator functions of classical sets are special cases of the membership functions of fuzzy sets, if the later only take values 0 or 1. Therefore, a fuzzy set A in an universe of discourse X is a function $A: X \to [0, 1]$, and usually this function is referred to as the membership function and denoted by $\mu_{A(x)}$.

The theory of vague sets was first proposed by Gau and Buehrer [18] as an extension of fuzzy set theory and vague sets are regarded as a special case of context-dependent fuzzy sets.

A vague set is defined by a truth-membership function t_v and a false-membership function f_v, where $t_v(x)$ is a lower bound on the grade of membership of x derived from the evidence for x, and $f_v(x)$ is a lower bound on the negation of x derived from the evidence against x. The values of $t_v(x)$ and $f_v(x)$ are both defined on the closed interval $[0, 1]$with each point in a basic set X , where $t_v(x) + f_v(x) \leq 1$.

For more information, see [1, 2, 3, 7, 15, 16, 19].

In 1995, Smarandache talked for the first time about neutrosophy, and in 1999 and 2005 [4, 6] defined the neutrosophic set theory, one of the most important new mathematical tools for handling problems involving imprecise, indeterminacy, and inconsistent data.

In this paper, we define the concept of a neutrosophic vague set as a combination of neutrosophic set and vague set. We also define and study the operations and properties of neutrosophic vague set and give examples.

2 Preliminaries

In this section, we recall some basic notions in vague set theory and neutrosophic set theory. Gau and Buehrer have introduced the following definitions concerning its operations, which will be useful to understand the subsequent discussion.

Definition 2.1 ([18]). Let x be a vague value, $x = [t_x, 1-f_x]$, where $t_x \in [0,1]$, $f_x \in [0,1]$, and $0 \leq t_x \leq 1-f_x \leq 1$. If $t_x = 1$ and $f_x = 0$ (i.e., $x = [1,1]$), then x is called a unit vague value. If $t_x = 0$ and $f_x = 1$ (i.e., $x = [0,0]$), then x is called a zero vague value.

Definition 2.2 ([18]). Let x and y be two vague values, where $x = [t_x, 1-f_x]$ and $y = [t_y, 1-f_y]$. If $t_x = t_y$ and $f_x = f_y$, then vague values x and y are called equal (i.e. $[t_x, 1-f_x] = [t_y, 1-f_y]$).

Definition 2.3 ([18]). Let A be a vague set of the universe U. If $\forall u_i \in U$, $t_A(u_i) = 1$ and $f_A(u_i) = 0$, then A is called a unit vague set, where $1 \leq i \leq n$. If $\forall u_i \in U$, $t_A(u_i) = 0$ and $f_A(u_i) = 1$, then A is called a zero vague set, where $1 \leq i \leq n$.

Definition 2.4 ([18]). The complement of a vague set A is denoted by A^c and is defined by

$$\begin{aligned} t_{A^c} &= f_A, \\ 1-f_{A^c} &= 1-t_A. \end{aligned}$$

Definition 2.5 ([18]). Let A and B be two vague sets of the universe U. If $\forall u_i \in U$, $[t_A(u_i), 1-f_A(u_i)] = [t_B(u_i), 1-f_B(u_i)]$, then the vague set A and B are called equal, where $1 \leq i \leq n$.

Definition 2.6 ([18]). Let A and B be two vague sets of the universe U. If $\forall u_i \in U$, $t_A(u_i) \leq t_B(u_i)$ and $1-f_A(u_i) \leq 1-f_B(u_i)$, then the vague set A are included by B, denoted by $A \subseteq B$, where $1 \leq i \leq n$.

Definition 2.7 ([18]). The union of two vague sets A and B is a vague set C, written as $C = A \cup B$, whose truth-membership and false-membership functions are related to those of A and B by

$$t_C = max(t_A, t_B),\ 1-f_C = max(1-f_A, 1-f_B) = 1-min(f_A, f_B).$$

Definition 2.8 ([18]). The intersection of two vague sets A and B is a vague set C, written as $C = A \cap B$, whose truth-membership and false-membership functions are related to those of A and B by

$$t_C = min(t_A, t_B),\ 1 - f_C = min(1 - f_A, 1 - f_B) = 1 - max(f_A, f_B).$$

In the following, we recall some definitions related to neutrosophic set given by Smarandache. Smarandache defined neutrosophic set in the following way:

Definition 2.9 [6] A neutrosophic set A on the universe of discourse X is defined as

$$A = \{< x, T_A(x), I_A(x), F_A(x) >, x \in X\}$$

where $T, I, F: X \to]^-0, 1^+[$ and $^-0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$.

Smarandache explained his concept as it follows: "For example, neutrosophic logic is a generalization of the fuzzy logic. In neutrosophic logic a proposition is $T \equiv true$, $I \equiv indeterminate$, and $F \equiv false$. For example, let's analyze the following proposition: *Pakistan will win against India in the next soccer game*. This proposition can be $(0.6, 0.3, 0.1)$, which means that there is a possibility of $60\% \equiv$ that Pakistan wins, $30\% \equiv$ that Pakistan has a tie game, and $10\% \equiv$ that Pakistan looses in the next game vs. India."

Now we give a brief overview of concepts of neutrosophic set defined in [8, 5, 17]. Let S_1 and S_2 be two real standard or non-standard subsets, then

$$S_1 \oplus S_2 = \{x \mid x = s_1 + s_2, s_1 \in S_1 \text{ and } s_2 \in S_2\},$$
$$\{1^+\} \oplus S_2 = \{x \mid x = 1^+ + s_2,\ s_2 \in S_2\},$$
$$S_1 \overline{\oplus} S_2 = \{x \mid x = s_1 - s_2,\ s_1 \in S_1 \text{ and } s_2 \in S_2\},$$
$$S_1 \odot S_2 = \{x \mid x = s_1 . s_2,\ s_1 \in S_1 \text{ and } s_2 \in S_2\},$$
$$\{1^+\} \overline{\oplus} S_2 = \{x \mid x = 1^+ - s_2,\ s_2 \in S_2\}.$$

Definition 2.10 (Containment) A neutrosophic set A is contained in the other neutrosophic set B, $A \subseteq B$, if and only if

$$\inf T_A(x) \leq \inf T_B(x),\ \sup T_A(x) \leq \sup T_B(x),$$

$$\inf I_A(x) \geq \inf I_B(x),\ \sup I_A(x) \geq \sup I_B(x),$$

$$\inf F_A(x) \geq \inf F_B(x),\ \sup F_A(x) \geq \sup F_B(x),\ \text{for all } x \in X.$$

Definition 2.11 The complement of a neutrosophic set A is denoted by $\overline{A}$ and is defined by

$$T_{\overline{A}}(x)=\{1^{+}\}\overline{\oplus}\, T_{\hat{A}}(x),$$

$$I_{\overline{A}}(x)=\{1^{+}\}\overline{\oplus}\, I_{\hat{A}}(x),$$

$$F_{\overline{A}}(x)=\{1^{+}\}\overline{\oplus}\, F_{\hat{A}}(x),\quad \text{for all } x\in X.$$

Definition 2.12 (Intersection) The intersection of two neutrosophic sets A and B is a neutrosophic set C, written as $C = A \cap B$, whose truth-membership, indeterminacy-membership and falsity-membership functions are related to those of A and B by

$$T_C(x)=T_A(x)\odot T_B(x),$$

$$I_C(x)=I_A(x)\odot I_B(x),$$

$$F_C(x)=F_A(x)\odot F_B(x),\quad \text{for all } x\in X.$$

Definition 2.11 (Union) The union of two neutrosophic sets A and B is a neutrosophic set C written as $C = A \cup B$, whose truth-membership, indeterminacy-membership and falsity-membership functions are related to those of A and B by

$$T_C(x)=T_A(x)\oplus T_B(x)\overline{\oplus}\, T_A(x)\odot T_B(x),$$

$$I_C(x)=I_A(x)\oplus I_B(x)\overline{\oplus}\, I_A(x)\odot I_B(x),$$

$$F_C(x)=F_A(x)\oplus F_B(x)\overline{\oplus}\, F_A(x)\odot F_B(x),\quad \text{for all } x\in X.$$

3 Neutrosophic Vague Set

A vague set over U is characterized by a truth-membership function t_v and a false-membership function f_v, $t_v : U \to [0,1]$ and $f_v : U \to [0,1]$ respectively where $t_v(u_i)$ is a lower bound on the grade of membership of u_i which is derived from the evidence for u_i, $f_v(u_i)$ is a lower bound on the negation of u_i derived from the evidence against u_i and $t_v(u_i)+f_v(u_i)\leq 1$. The grade of membership of u_i in the vague set is bounded to a subinterval $[t_v(u_i),1-f_v(u_i)]$ of $[0,1]$. The vague value $[t_v(u_i),1-f_v(u_i)]$ indicates that the exact grade of membership $\mu_v(u_i)$ of u_i maybe unknown, but it is bounded by $t_v(u_i) \le \mu_v(u_i) \le f_v(u_i)$ where $t_v(u_i) + f_v(u_i) \le 1$. Let U be a space of points (objects), with a generic element in U denoted by u. A neutrosophic sets (N-sets) A in U is characterized by a truth-membership function T_A , an indeterminacy-membership function I_A and a falsity-membership function F_A. $T_A(u)$; $I_A(u)$ and $F_A(u)$ are real standard or nonstandard subsets of $[0,\ 1]$. It can be written as:

$$A = \{< u,\ (T_A(u),\ I_A(u),\ F_A(u))\ >: u \in U, T_A(u),\ I_A(u),\ F_A(u) \in [0,1]\}.$$

There is no restriction on the sum of $T_A(u)$; $I_A(u)$ and $F_A(u)$, so:

$$0 \le \sup T_A(u)\ +\ \sup I_A(u)\ +\ \sup F_A(u) \le 3.$$

By using the above information and by adding the restriction of vague set to neutrosophic set, we define the concept of neutrosophic vague set as it follows.

Definition 3.1 A neutrosophic vague set A_{NV} (NVS in short) on the universe of discourse X written as

$$A_{NV} = \{< x, \hat{T}_{A_{NV}}(x), \hat{I}_{A_{NV}}(x), F_{A_{NV}}(x) >, x \in X\}$$

whose truth-membership, indeterminacy-membership and falsity-membership functions is defined as

$$\hat{T}_{A_{NV}}(x) = [T^-, T^+],\ \hat{I}_{A_{NV}}(x) = [I^-, I^+],\ \hat{F}_{A_{NV}}(x) = [F^-, F^+],$$

where

$$T^+ = 1 - F^-,\ F^+ = 1 - T^-, \text{ and}$$

$$^-0 \le T^- + I^- + F^- \le 2^+,$$

when X is continuous, a NVS A_{NV} can be written as

$$A_{NV} = \int_X < x, \hat{T}_{A_{NV}}(x), \hat{I}_{A_{NV}}(x), \hat{F}_{A_{NV}}(x) > / x,\ x \in X.$$

When X is discrete, a NVS A_{NV} can be written as

$$A_{NV} = \sum_{i=1}^{n} < x, \hat{T}_{A_{NV}}(x_i), \hat{I}_{A_{NV}}(x_i), \hat{F}_{A_{NV}}(x_i) > / x_i,\ x_i \in X.$$

In neutrosophic logic, a proposition is $T \equiv true$, $I \equiv indeterminate$, and $F \equiv false$ such that:

$$0 \le \sup T_{A_N}(u)\ +\ \sup I_{A_N}(u)\ +\ \sup F_{A_N}(u) \le 3.$$

Also, vague logic is a generalization of the fuzzy logic where a proposition is $T \equiv true$ and $F \equiv false$, such that: $t_v(u_i) + f_v(u_i) \leq 1$, he exact grade of membership $\mu_v(u_i)$ of u_i maybe unknown, but it is bounded by

$$t_v(u_i) \leq \mu_v(u_i) \leq f_v(u_i).$$

For example, let's analyze the Smarandache's proposition using our new concept: *Pakistan will win against India in the next soccer game*. This proposition can be as it follows:

$$\widehat{T}_{A_{NV}} = [0.6, 0.9],\ \widehat{I}_{A_{NV}} = [0.3, 0.4] \text{ and } \widehat{F}_{A_{NV}} = [0.4, 0.6],$$

which means that there is possibility of $60\% \ to \ 90\% \equiv$ that Pakistan wins, $30\% \ to \ 40\% \equiv$ that Pakistan has a tie game, and $40\% \ to \ 60\% \equiv$ that Pakistan looses in the next game vs. India.

Example 3.1 Let $U = \{u_1, u_2, u_3\}$ be a set of universe we define the NVS A_{NV} as follows:

$$A_{NV} = \left\{ \frac{u_1}{\langle [0.3, 0.5], [0.5, 0.5], [0.5, 0.7] \rangle}, \frac{u_2}{\langle [0.4, 0.7], [0.6, 0.6], [0.3, 0.6] \rangle}, \frac{u_3}{\langle [0.1, 0.5], [0.5, 0.5], [0.5, 0.9] \rangle} \right\}.$$

Definition 3.2 Let Ψ_{NV} be a NVS of the universe U where $\forall u_i \in U$,

$$\widehat{T}_{\Psi_{NV}}(x) = [1, 1],\ \widehat{I}_{\Psi_{NV}}(x) = [0, 0],\ \widehat{F}_{\Psi_{NV}}(x) = [0, 0],$$

then Ψ_{NV} is called a unit NVS, where $1 \leq i \leq n$.

Let Φ_{NV} be a NVS of the universe U where $\forall u_i \in U$,

$$\widehat{T}_{\Phi_{NV}}(x) = [0, 0],\ \widehat{I}_{\Phi_{NV}}(x) = [1, 1],\ \widehat{F}_{\Phi_{NV}}(x) = [1, 1],$$

then Φ_{NV} is called a zero NVS, where $1 \leq i \leq n$.

Definition 3.3 The complement of a NVS A_{NV} is denoted by A^c and is defined by

$$\widehat{T}^c_{A_{NV}}(x) = \left[1 - T^+, 1 - T^-\right],$$
$$\widehat{I}^c_{A_{NV}}(x) = \left[1 - I^+, 1 - I^-\right],$$
$$\widehat{F}^c_{A_{NV}}(x) = \left[1 - F^+, 1 - F^-\right],$$

Example 3.2 Considering *Example 3.1,* we have:

$$A^c{}_{NV} = \left\{ \frac{u_1}{\langle [0.5,0.7],[0.5,0.5],[0.3,0.5] \rangle}, \frac{u_2}{\langle [0.3,0.6],[0.4,0.4],[0.4,0.7] \rangle}, \frac{u_3}{\langle [0.5,0.9],[0.5,0.5],[0.1,0.5] \rangle} \right\}.$$

Definition 3.5 Let A_{NV} and B_{NV} be two NVSs of the universe U . If $\forall u_i \in U$,

$$\hat{T}_{A_{NV}}(u_i) = \hat{T}_{B_{NV}}(u_i),\ \hat{I}_{A_{NV}}(u_i) = \hat{I}_{B_{NV}}(u_i) \text{ and } \hat{F}_{A_{NV}}(u_i) = \hat{F}_{B_{NV}}(u_i),$$

then the NVS A_{NV} and B_{NV} are called equal, where $1 \le i \le n$.

Definition 3.6 Let A_{NV} and B_{NV} be two NVSs of the universe U. If $\forall u_i \in U$,

$$\hat{T}_{A_{NV}}(u_i) \le \hat{T}_{B_{NV}}(u_i),\ \hat{I}_{A_{NV}}(u_i) \ge \hat{I}_{B_{NV}}(u_i) \text{ and}$$
$$\hat{F}_{A_{NV}}(u_i) \ge \hat{F}_{B_{NV}}(u_i),$$

then the NVS A_{NV} are included by B_{NV}, denoted by $A_{NV} \subseteq B_{NV}$, where $1 \le i \le n$.

Definition 3.7 The union of two NVSs A_{NV} and B_{NV} is a NVS C_{NV}, written as $C_{NV} = A_{NV} \cup B_{NV}$, whose truth-membership, indeterminacy-membership and false-membership functions are related to those of A_{NV} and B_{NV} by

$$\hat{T}_{C_{NV}}(x) = \left[max\left(\hat{T}^-_{A_{NVx}}, \hat{T}^-_{B_{NVx}}\right), max\left(\hat{T}^+_{A_{NVx}}, \hat{T}^+_{B_{NVx}}\right)\right],$$
$$\hat{I}_{C_{NV}}(x) = \left[min\left(\hat{I}^-_{A_{NVx}}, \hat{I}^-_{B_{NVx}}\right), min\left(\hat{I}^+_{A_{NVx}}, \hat{I}^+_{B_{NVx}}\right)\right],$$
$$\hat{F}_{C_{NV}}(x) = \left[min\left(\hat{F}^-_{A_{NVx}}, \hat{F}^-_{B_{NVx}}\right), min\left(\hat{F}^+_{A_{NVx}}, \hat{F}^+_{B_{NVx}}\right)\right].$$

Definition 3.8 The intersection of two NVSs A_{NV} and B_{NV} is a NVS H_{NV}, written as $H_{NV} = A_{NV} \cap B_{NV}$, whose truth-membership, indeterminacy-membership and false-membership functions are related to those of A_{NV} and B_{NV} by

$$\hat{T}_{H_{NV}}(x) = \left[min\left(\hat{T}^-_{A_{NVx}}, \hat{T}^-_{B_{NVx}}\right), min\left(\hat{T}^+_{A_{NVx}}, \hat{T}^+_{B_{NVx}}\right)\right],$$
$$\hat{I}_{H_{NV}}(x) = \left[max\left(\hat{I}^-_{A_{NVx}}, \hat{I}^-_{B_{NVx}}\right), max\left(\hat{I}^+_{A_{NVx}}, \hat{I}^+_{B_{NVx}}\right)\right],$$
$$\hat{F}_{H_{NV}}(x) = \left[max\left(\hat{F}^-_{A_{NVx}}, \hat{F}^-_{B_{NVx}}\right), max\left(\hat{F}^+_{A_{NVx}}, \hat{F}^+_{B_{NVx}}\right)\right].$$

Example 3.3 Let $U = \{u_1, u_2, u_3\}$ be a set of universe and let NVS A_{NV} and B_{NV} define as follows:

$$A_{NV}=\left\{\frac{u_1}{\langle[0.3,0.5],[0.7,0.8],[0.5,0.7]\rangle},\frac{u_2}{\langle[0.4,0.7],[0.6,0.8],[0.3,0.6]\rangle},\frac{u_3}{\langle[0.1,0.5],[0.3,0.6],[0.5,0.9]\rangle}\right\}.$$

$$B_{NV}=\left\{\frac{u_1}{\langle[0.7,0.8],[0.3,0.5],[0.2,0.3]\rangle},\frac{u_2}{\langle[0.2,0.4],[0.2,0.4],[0.6,0.8]\rangle},\frac{u_3}{\langle[0.9,1],[0.6,0.7],[0,0.1]\rangle}\right\}.$$

Then we have $C_{NV} = A_{NV}\cup B_{NV}$ where

$$C_{NV}=\left\{\frac{u_1}{\langle[0.7,0.8],[0.3,0.5],[0.2,0.3]\rangle},\frac{u_2}{\langle[0.4,0.7],[0.2,0.4],[0.3,0.6]\rangle},\frac{u_3}{\langle[0.9,1],[0.3,0.6],[0,0.1]\rangle}\right\}.$$

Moreover, we have $H_{NV} = A_{NV}\cap B_{NV}$ where

$$H_{NV}=\left\{\frac{u_1}{\langle[0.3,0.5],[0.7,0.8],[0.5,0.7]\rangle},\frac{u_2}{\langle[0.2,0.4],[0.6,0.8],[0.6,0.8]\rangle},\frac{u_3}{\langle[0.1,0.5],[0.6,0.7],[0.5,0.9]\rangle}\right\}.$$

Theorem 3.1 Let P be the power set of all NVS defined in the universe X. Then $\langle P;\cup_{NV},\cap_{NV}\rangle$ is a distributive lattice.

Proof Let A, B, C be the arbitrary NVSs defined on X. It is easy to verify that

$A\cap A = A, A\cup A = A$ (idempotency),

$A\cap B = B\cap A, A\cup B = B\cup A$ (commutativity),

$(A\cap B)\cap C = A\cap(B\cap C), (A\cup B)\cup C = A\cup(B\cup C)$ (associativity), and

$A\cap(B\cup C) = (A\cap B)\cup(A\cap C), A\cup(B\cap C) = (A\cup B)\cap(A\cap C)$ (distributivity).

4 Conclusion

In this paper, we have defined and studied the concept of a neutrosophic vague set, as well as its properties, and its operations, giving some examples.

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Neutrosophic cognitive maps for modeling project portfolio interdependencies

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Abstract

Interdependency modeling and analysis have commonly been ignored in project portfolio management. In this work, we proposed a new method for modeling project portfolio interdependencies, and specially risks interdependencies, using neutrosophic logic and neutrosophic cognitive maps. This proposal has many advantages for dealing with indeterminacy making easy the elicitation of knowledge from experts. An illustrative example is presented to demonstrate the applicability of the proposed method.

Keywords

Neutrosophic logic, Neutrosophic cognitive maps, Risk interdependencies, Project portfolio interdependencies.

1 Introduction

A portfolio of project is a group of project that share resources creating relation among them of complementarity, incompatibility or synergy [1]. The interdependency modeling and analysis have commonly been ignored in project portfolio management [2].

In an international survey only 38.6 % of responders understand this element [2]. Cost increasing, the lack of benefits exploitation [3] and the incorrect selection of projects [4] are among the consequences. In this work a proposal for modeling project portfolio risk interdependencies neutrosophic cognitive maps (NCM) [11] is developed.

This paper is structured as follows: Section 2 reviews some important concepts about neutrosophic cognitive maps and risks interdependencies modeling. In Section 3, we present a framework for modeling interdependencies in project portfolio risks. Section 4 shows an illustrative example. The paper ends with conclusions and further work recommendations in Section 5.

2 Neutrosophic cognitive maps and risks interdependencies

A fuzzy cognitive maps (FCM) [5] are fuzzy graph structures for representing causal knowledge. FCM have been applied to diverse areas such as decision support and com-plex systems analysis [6]. Furthermore multiples extensions have been developed such as fuzzy grey cognitive maps [7], interval fuzzy cognitive maps [8], Intuitionistic fuzzy cognitive maps [9] and linguistic 2-tuple fuzzy cognitive maps [10].

Neutrosophic cognitive maps (NCM) were created by Vasantha & Smarandache [11] as an extension of the Fuzzy Cognitive Maps (FCMs) in which indeterminacy is included using neutrosophic logic. Neutrosophic logic is a logic in which each proposition is esti-mated to have the percentage of truth in a subset T, the percentage of indeterminacy in a subset I, and the percentage of falsity in a subset F [12].

There are five types of project portfolio interdependencies: benefit, risk, outcome, schedule and resources [13]. Risks have a positive or negative correlation with others provoking risk diversification or amplification effects. In this work project portfolio risk interdependencies are modeled using neutrosophic logic to include indeterminacy.

3 Modeling project portfolio interdependencies

Our aim is develop framework for modeling project portfolio and its interrelation based NCM. The model consists of the following phases (graphically, *Figure 1*):

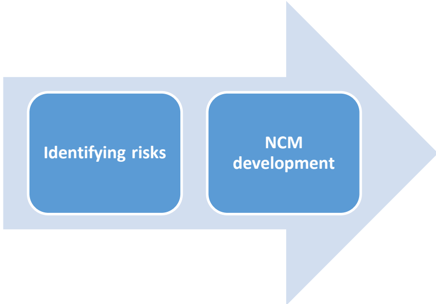

Figure 1. A Framework project portfolio risk interdependencies.

Identifying risks

The first step is the identification of risks. Risks are identified initially at project level. A portfolio risk breakdown structure with interdependencies is obtained. An example for a risk breakdown structure applicable to IT portfolios with interdependencies is shown in [13].

NCM development

The weight of the relationship among from given risk R_i to risk R_j is represented by means of neutrosophic logic. Additionally static analysis for selecting the most important risks could be developed [14].

4 Illustrative example

Five risks $\mathrm{R} = (\mathrm{r}_1, \ldots, \mathrm{r}_5)$ are identified at portfolio level (*Table 2*).

Table 1. Portfolio risks

Node	Description
$\mathbf{R_1}$	Project 1 Technical feasibility
$\mathbf{R_2}$	Project 1 Timely completion
$\mathbf{R_3}$	Project 2 Timely completion
$\mathbf{R_4}$	Project 2 Code quality
$\mathbf{R_5}$	Project 3 Timely completion
$\mathbf{R_6}$	Project 3 Cultural acceptance

Later, the expert provides the following interrelations (*Figure 2*):

$$W = \begin{matrix} 0 & 0 & 0.75 & 0 & 0 & 0 \\ 0 & 0 & 0.3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & I & 0 & 0 & 0 \\ 0 & 0 & 0.5 & I & 0 & 0 \end{matrix}$$

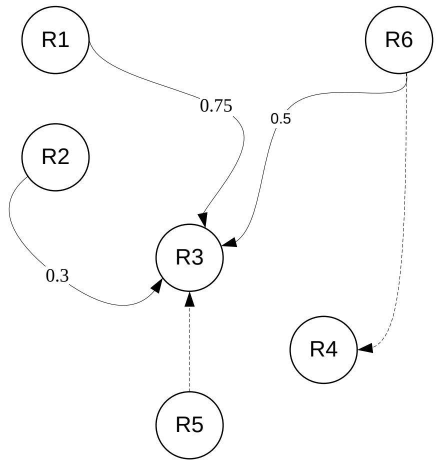

Figure 2. Neutrosophic cognitive map representing risks interdependencies.

In this example, the technical feasibility of project #1 (technical risk) could severely affect the timely completion of projects #2. Also, if no consistent tooling is used, and the agile development approach (project #3) is not culturally accepted, projects #2 are more likely to experience quality issues and time delay. Indeterminacy is introduced in project #3 risks interrelation with other risks of project #2.

5 Conclusion

This paper proposes a new framework to model interdependencies in project portfolio. NCM representation model is used for modeling relation among risks.

Building NCM follows an approach more similar to human reasoning introducing indeterminacy in relations. An illustrative example showed the applicability of the proposal. Further works will concentrate on two objectives: developing a consensus model, and extending the model to other areas of project portfolio interdependencies modeling.

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N-Valued Interval Neutrosophic Sets and Their Application in Medical Diagnosis

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Abstract

In this paper a new concept is called n-valued interval neutrosophic sets is given. The basic operations are introduced on n-valued interval neutrosophic sets such as; union, intersection, addition, multiplication, scalar multiplication, scalar division, truth-favorite and false-favorite. Then, some distances between n-valued interval neutrosophic sets (NVINS) are proposed. Also, we propose an efficient approach for group multi-criteria decision making based on n-valued interval neutrosophic sets. An application of n-valued interval neutrosophic sets in medical diagnosis problem is given.

Keywords

Neutrosophic sets, n-valued neutrosophic set, interval neutrosophic sets, n-valued interval neutrosophic sets.

1 Introduction

In 1999, Smarandache [37] proposed the concept of neutrosophic set (NS for short) by adding an independent indeterminacy-membership function which is a generalization of classic set, fuzzy set [45], intuitionistic fuzzy set [3] and so on. In NS, the indeterminacy is quantified explicitly and truth-membership (T), indeterminacy (I) membership, and false-membership (F) are completely independent and from scientific or engineering point of view, the NS operators need to be specified. Therefore, Wang et al [39] defined a single valued neutrosophic set (SVNS) and then provided the set theoretic operations and various properties of single valued neutrosophic sets and Wang et al. [40] proposed the set theoretic operations on an instance of neutrosophic set is called interval valued neutrosophic set (IVNS) which is more flexible and practical than NS. The works on single valued neutrosophic set (SVNS) and interval valued neutrosophic sets (IVNS) and their hybrid structure in theories and application have been progressing rapidly (e.g., [1,2,4-19,21,22,24-26,28-30,36,41,43]). Also, neutrosophic sets extended neutrosophic models in [13,16] both theory and application by using [27,31].

The concept of intuitionistic fuzzy multiset and some propositions with applications is originaly presented by Rajarajeswari and Uma [32-35]. After Rajarajeswari and Uma, Smarandache [38] presented n-Valued neutrosophic sets with applications. Recently, Chatterjee et al. [20], Deli et al. [18, 23], Ye et al. [42] and Ye and Ye [44] initiated definition of neutrosophic multisets with some operations. Also, the authors gave some distance and similarity measures on neutrosophic multisets. In this paper, our objective is to generalize the concept of n-valued neutrosophic sets (or neutrosophic multi sets; or neutrosophic refined sets) to the case of n-valued interval neutrosophic sets.

The paper is structured as follows; in Section 2, we first recall the necessary background on neutrosophic sets, single valued neutrosophic sets, interval valued neutrosophic sets and n-valued neutrosophic sets (or neutrosophic multi sets). Section 3 presents the concept of n-valued interval neutrosophic sets and derive their respective properties with examples. Section 4 presents the distance between two n-valued interval neutrosophic sets. Section 5 presents an application of this concept in solving a decision making problem. Section 6 concludes the paper.

2 Preliminaries

This section gives a brief overview of concepts of neutrosophic set theory [37], n-valued neutrosophic set theory [42,44] and interval valued neutrosophic set theory [40]. More detailed explanations related to this subsection may be found in [18,20,23,37,40,42,44].

Definition 2.1. [37,39] Let X be an universe of discourse, with a generic element in X denoted by x, then a neutrosophic (NS) set A is an object having the form

$$A = \{< x: T_A(x), I_A(x), F_A(x)>, x \in X\}$$

where the functions T, I, F : X→ $]^-0, 1^+[$ define respectively the degree of membership (or Truth) , the degree of indeterminacy, and the degree of non-membership (or Falsehood) of the element x ∈ X to the set A with the condition.

$$^-0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3^+$$

From philosophical point of view, the neutrosophic set takes the value from real standard or non-standard subsets of $]^-0, 1^+[$. So instead of $]-0, 1^+[$ we need to take the interval [0, 1] for technical applications, because $]^-0, 1^+[$ will be difficult to apply in the real applications such as in scientific and engineering problems.

For two NS, $A_{NS} = \{ <x , T_A(x) , I_A(x), F_A(x)> \mid x \in X \}$

and $B_{NS} = \{ <x , T_B(x) , I_B(x), F_B(x)> \mid x \in X \}$ the two relations are defined as follows:

(1) $A_{NS} \subseteq B_{NS}$ if and only if $T_A(x) \leq T_B(x)$, $I_A(x) \geq I_B(x)$, $F_A(x) \geq F_B(x)$

(2) $A_{NS} = B_{NS}$ if and only if , $T_A(x) = T_B(x)$, $I_A(x) = I_B(x)$, $F_A(x) = F_B(x)$

Definition 2.2. [40] Let X be a space of points (objects) with generic elements in X denoted by x. An interval valued neutrosophic set (for short IVNS) A in X is characterized by truth-membership function $T_A(x)$, indeteminacy-membership function $I_A(x)$ and falsity-membership function $F_A(x)$. For each point x in X, we have that $T_A(x), I_A(x), F_A(x) \subseteq [0, 1]$.

For two IVNS

$$A_{\text{IVNS}}=\{<x, [\inf T_A^1(x), \sup T_A^1(x)], [\inf I_A^1(x), \sup I_A^1(x)], [\inf F_A^1(x), \sup F_A^1(x)] >: x \in X\}$$

and

$$B_{\text{IVNS}}= \{<x, \inf T_B^1(x), \sup T_B^1(x)], [\inf I_B^1(x), \sup I_B^1(x)], [\inf F_B^1(x), \sup F_B^1(x)]> : x \in X\}$$

Then,

1. $A_{\text{IVNS}} \subseteq B_{\text{IVNS}}$ if and only if

$\inf T_A^1(x) \le \inf T_B^1(x)$, $\sup T_A^1(x) \le \sup T_B^1(x)$,
$\inf I_A^1(x) \ge \inf I_B^1(x)$, $\sup I_A^1(x) \ge \sup I_B^1(x)$,
$\inf F_A^1(x) \ge \inf F_B^1(x)$, $\sup F_A^1(x) \ge \sup F_B^1(x)$,

for all x ∈ X.

2. $A_{\mathrm{IVNS}} = B_{\mathrm{IVNS}}$ if and only if ,
$\inf T_A^1(x) = \inf T_B^1(x)$, $\sup T_A^1(x) = \sup T_B^1(x)$,
$\inf I_A^1(x) = \inf I_B^1(x)$, $\sup I_A^1(x) = \sup I_B^1(x)$,
$\inf F_A^1(x) = \inf F_B^1(x)$, $\sup F_A^1(x) = \sup F_B^1(x)$,

for any x ∈ X.

3. ${A_{\mathrm{IVNS}}}^c$= $\{x,[\inf F_A^1(x), \sup F_A^1(x)], [1- \ sup\ I_A^1(x), 1 - inf\ I_A^1(x)],$
$[inf\ T_A^1(x), sup\ T_A^1(x)]: x \in X\}$

4. $A_{\mathrm{IVNS}} \cap B_{\mathrm{IVNS}}$={<x, $[\inf T_A^1(x) \wedge \inf T_B^1(x), \sup T_A^1(x) \wedge \sup T_B^1(x)]$,
$[\inf I_A^1(x) \vee \inf I_B^1(x), \sup I_A^1(x) \vee \sup I_B^1(x)]$,
$[\inf F_A^1(x) \vee \inf F_B^1(x), \sup F_A^1(x) \vee \sup F_B^1(x)]$>: x ∈ X}

5. $A_{\mathrm{IVNS}} \cup B_{\mathrm{IVNS}}$={<x, $[\inf T_A^1(x) \vee \inf T_B^1(x), \sup T_A^1(x) \vee \sup T_B^1(x)]$,
$[\inf I_A^1(x) \wedge \inf I_B^1(x), \sup I_A^1(x) \wedge \sup I_B^1(x)]$,
$[\inf F_A^1(x) \wedge \inf F_B^1(x), \sup F_A^1(x) \wedge \sup F_B^1(x)]$>: x ∈ X}

6. $A_{\mathrm{IVNS}} \backslash$
B_{IVNS}={<x,[min{$\inf T_A^1(x), \inf F_B^1(x)$}, min{$\sup T_A^1(x), \sup F_B^1(x)$}],
[max(inf $I_A^1(x)$,1-sup $I_B^1(x)$), max(sup $I_A^1(x)$,1-inf $I_B^1(x)$)],
[max(inf$F_A^1(x)$,inf $T_B^1(x)$), max(sup $F_A^1(x)$, sup $T_B^1(x)$)]>: x ∈ X}

7. A_{IVNS}+ B_{IVNS} ={<x,[min($\inf T_A^1(x) + \inf T_B^1(x)$, 1), min($\sup T_A^1(x) +$ $\sup T_B^1(x)$, 1)],
[min($\inf I_A^1(x) + \inf I_B^1(x)$, 1), min($\sup I_A^1(x) + \sup I_B^1(x)$, 1)],
[min($\inf F_A^1(x) + \inf F_B^1(x)$, 1), min($\sup F_A^1(x) + \sup F_B^1(x)$, 1)] >
: x ∈ X},

8. A_{IVNS}.a={<x,[min($\inf T_A^1(x)$. a ,1), min($\sup T_A^1(x)$. a ,1)],
[min($\inf I_A^1(x)$. a ,1), min($\sup I_A^1(x)$. a ,1)],
[min($\inf F_A^1(x)$. a ,1), min($\sup F_A^1(x)$. a ,1)],
[min($\inf I_A^1(x)$. a ,1), min($\sup I_A^1(x)$. a ,1)] >: x ∈ X},

9. A_{IVNS}/a={<x,[min($\inf T_A^1(x)$/a ,1), min($\sup T_A^1(x)$/a ,1)],
[min($\inf I_A^1(x)$/ a ,1), min($\sup I_A^1(x)$/ a ,1)],
[min($\inf F_A^1(x)$/ a ,1), min($\sup F_A^1(x)$. a ,1)] > : x ∈ X},

10. ΔA_{IVNS} ={<x,[min($\inf T_A^1(x) + \inf I_A^1(x)$, 1), min($\sup T_A^1(x) +$ $\sup I_B^1(x)$, 1)], [0,0],
[$\inf F_A^1(x)$, $\sup F_A^1(x)$] >: x ∈ X},

11. ∇A_{IVNS} ={<x,[$\inf T_A^1(x)$, $\sup T_A^1(x)$], [0,0],

$[\min(\inf F_A^1(x) + \inf I_A^1(x), 1), \min(\sup F_A^1(x) + \sup I_B^1(x), 1)] >: x \in X\}$

Definition 2.3. [20,42] Let E be a universe. A n-valued neutrosophic sets on E can be defined as follows:

$$A = \{<x, (T_A^1(x), T_A^2(x), ..., T_A^p(x)), (I_A^1(x), I_A^2(x), ..., I_A^p(x)), (F_A^1(x), F_A^2(x), ..., F_A^p(x))>: x \in X\}$$

where

$T_A^1(x), T_A^2(x), ..., T_A^p(x), I_A^1(x), I_A^2(x), ..., I_A^p(x), F_A^1(x), F_A^2(x), ..., F_A^p(x): E \to [0, 1]$ such that

$$0 \le T_A^i(x) + I_A^i(x) + F_A^i(x) \le 3 \text{ for } i=1, 2, ..., p \text{ for any } x \in X,$$

Here, $(T_A^1(x), T_A^2(x), ..., T_A^p(x))$, $(I_A^1(x), I_A^2(x), ..., I_A^p(x))$ and $(F_A^1(x), F_A^2(x), ..., F_A^p(x))$

is the truth-membership sequence, indeterminacy-membership sequence and falsity-membership sequence of the element x, respectively. Also, P is called the dimension of n-valued neutrosophic sets (NVNS) A.

3 N-Valued Interval Neutrosophic Sets

Following the n-valued neutrosophic sets (multiset or refined set) and interval neutrosophic sets defined in [20,38,42,44] and Wang et al. in [40], respectively. In this section, we extend these sets to n-valued interval valued neutrosophic sets.

Definition 3.1. Let X be a universe, a n-valued interval neutrosophic sets (NVINS) on X can be defined as follows:

$$A = \{x, \begin{pmatrix} [\inf T_A^1(x), \sup T_A^1(x)], [\inf T_A^2(x), \sup T_A^2(x)], \dots, \\ [\inf T_A^p(x), \sup T_A^p(x)] \end{pmatrix},$$

$$\begin{pmatrix} [\inf I_A^1(x), \sup I_A^1(x)], [\inf I_A^2(x), \sup I_A^2(x)], \dots, \\ [\inf I_A^p(x), \sup I_A^q(x)] \end{pmatrix},$$

$$[\inf F_A^1(x), \sup F_A^1(x)], ([inf\ F_A^2(x), sup\ F_A^2(x)], \dots, ([inf\ F_A^p(x), sup\ F_A^r(x)]): x \in X\}$$

where

$\inf T_A^1(x), \inf T_A^2(x), \dots, \inf T_A^p(x), \inf I_A^1(x), \inf I_A^2(x), \dots, \inf I_A^p(x), \inf F_A^1(x), \inf F_A^2(x), \dots, \inf F_A^q(x), \sup T_A^1(x), \sup T_A^2(x), \dots, \sup T_A^p(x),$

$\sup I_A^1(x)$, $\sup I_A^2(x)$,…,$\sup I_A^q(x)$,
$\sup F_A^1(x)$, $\sup F_A^2(x)$,…,$\sup F_A^r(x) \in [0, 1]$

such that $0 \le \sup T_A^i(x) + \sup I_A^i(x) + \sup F_A^i(x) \le 3$, $\forall$ i=1, 2,…, p.

In our study, we focus only on the case where p=q=r is the interval truth-membership sequence, interval indeterminacy-membership sequence and interval falsity-membership sequence of the element x, respectively. Also, p is called the dimension of n-valued interval (NVINS) A. Obviously, when the upper and lower ends of the interval values of $T_A^i(x)$, $I_A^i(x)$, $F_A^i(x)$ in a NVINS are equal, the NVINS reduces to n-valued neutrosophic set (or neutrosophic multiset proposed in [17,20]).

The set of all n-valued interval neutrosophic set on X is denoted by NVINS(X).

Example 3.2. Let X={x_1, x_2} be the universe and A is an n- valued interval neutrosophic sets

A= { <x_1,{[.1, .2], [.2, .3]},{[.3, .4],[.1, .5]},{[.3, .4], [.2, .5]}>,
<x_2,{[.3, .4], [.2,.4]},{[.3, .5],[.2, .4]},{[.1, .2], [.3, .4]}>}

Definition 3.3. The complement of A is denoted by A^c and is defined by

$$A^c = \{x, ([\inf F_A^1(x), \sup F_A^1(x)] , ([\inf F_A^2(x), \sup F_A^2(x)], \dots, ([inf\ F_A^p(x), sup\ F_A^p(x)]),$$
$$\begin{pmatrix} [1 - sup\ I_A^1(x), 1 - inf\ I_A^1(x)], [1 - sup\ I_A^2(x), 1 - inf\ I_A^2(x)], \dots, \\ [1 - sup\ I_A^p(x),\ 1 - inf\ I_A^p(x)] \end{pmatrix},$$
$$([inf\ T_A^1(x), sup\ T_A^1(x)], [inf\ T_A^2(x), sup\ T_A^2(x)], \dots, [inf\ T_A^p(x), sup\ T_A^p(x)]) : x \in X\}.$$

Example 3.4. Let us consider the Example 3.5. Then we have,

A^c= { <x_1,{[.3,.4],[.2,.5]},{[.6,.7],[.5,.9]},{[.1,.2],[.2,.3]}>,

<x_2,{[.1,.2],[.3,.4]},{[.5,.7],[.6,.8]},{[.3,.4],[.2,.4]}>}

Definition 3.5. For $\forall$ i=1, 2,…,p if $\inf T_A^i(x) = \sup T_A^i(x) = 0$ and $\inf I_A^i(x) = \sup I_A^i(x) = \inf F_A^i(x) = \sup F_A^i(x) = 1$, then A is called null n-valued interval neutrosophic set denoted by Φ, for all x ∈ X.

Example 3.6. Let X={x_1, x_2} be the universe and A is an n-valued interval neutrosophic sets

Φ= { <x_1,{[0, 0], [0, 0]},{[1, 1], [1, 1] },{[1, 1], [1, 1]}>,
<x_2,{[0, 0], [0, 0]},{[1, 1], [1, 1]},{[1, 1], [1, 1]}>}.

Definition 3.7. For ∀ i=1,2,…,p if $\inf T_A^i(x)$ = $\sup T_A^i(x)$ =1 and $\inf I_A^i(x)$=$\sup I_A^i(x)$=$\inf F_A^i(x)$ =$\sup F_A^i(x)$=0, then A is called universal n-valued interval neutrosophic set denoted by E, for all x ∈ X.

Example 3.8. Let X={x_1, x_2} be the universe and A is an n-valued interval neutrosophic sets

E= { <x_1,{ [1, 1], [1, 1] },{[0, 0], [0, 0] },{[0, 0], [0, 0]}>,
<x_2{ [1, 1], [1, 1] },{[0, 0], [0, 0] },{[0, 0], [0, 0]}>}.

Definition 3.9. A n-valued interval neutrosophic set A is contained in the other n-valued interval neutrosophic set B, denoted by A⊆B, if and only if

$\inf T_A^1(x) \le \inf T_B^1(x)$, $\inf T_A^2(x) \le \inf T_B^2(x)$,..., $\inf T_A^p(x) \le \inf T_B^p(x)$,
$\sup T_A^1(x) \le \sup T_B^1(x)$, $\sup T_A^2(x) \le \sup T_B^2(x)$,..., $\sup T_A^p(x) \le \sup T_B^p(x)$,
$\inf I_A^1(x) \ge \inf I_B^1(x)$, $\inf I_A^2(x) \ge \inf I_B^2(x)$,..., $\inf I_A^p(x) \ge \inf I_B^p(x)$,
$\sup I_A^1(x) \ge \sup I_B^1(x)$, $\sup I_A^2(x) \ge \sup I_B^2(x)$,..., $\sup I_A^p(x) \ge \sup I_B^p(x)$,
$\inf F_A^1(x) \ge \inf F_B^1(x)$, $\inf F_A^2(x) \ge \inf F_B^2(x)$,..., $\inf F_A^p(x) \ge \inf F_B^p(x)$,
$\sup F_A^1(x) \ge \sup F_B^1(x)$, $\sup F_A^2(x) \ge \sup F_B^2(x)$,..., $\sup F_A^p(x) \ge \sup F_B^p(x)$

for all x ∈ X.

Example 3.10. Let X={x_1, x_2} be the universe and A and B are two n-valued interval neutrosophic sets

A= { <x_1,{[.1, .2], [.2, .3]},{[.4, .5], [.6, .7]},{[.5, .6],[.7, .8]}>,
<x_2,{[.1, .4],[.1, .3]},{[.6, .8], [.4, .6]},{[.5, .6], [.6, .7]}>}

and

B= { <x_1,{[.5, .7], [.4, .5]},{[.3,.4],[.1,.5]},{[.3,.4],[.2,.5]}>,
<x_2,{[.2, .5],[.3, .6]},{[.3, .5],[.2,.4]},{[.1, .2], [.3, .4]}>}

Then, we have A ⊆ B.

Definition 3.11. Let A and B be two n-valued interval neutrosophic sets. Then, A and B are equal, denoted by A= B if and only if A ⊆ B and B⊆ A.

Proposition 3.12. Let A, B, C ∈NVINS(X).Then,

1. ∅⊆ A
2. A ⊆ A

3. A ⊆ E
4. A ⊆ B and B ⊆ C →A ⊆ C
5. K=L and L=M↔ $K = M$
6. K⊆L and L⊆K↔ $K = L$.

Definition 3.13. Let A and B be two n-valued interval neutrosophic sets. Then, intersection of A and B, denoted by A∩B, is defined by

A∩B={<x,$([\inf T_A^1(x) \wedge \inf T_B^1(x), \sup T_A^1(x) \wedge \sup T_B^1(x)], [\inf T_A^2(x) \wedge \inf T_B^2(x), \sup T_A^2(x) \wedge \sup T_B^2(x)], \ldots, [\inf T_A^p(x) \wedge \inf T_B^p(x), \sup T_A^P(x) \wedge \sup T_B^P(x)]), ([\inf I_A^1(x) \vee \inf I_B^1(x), \sup I_A^1(x) \vee \sup I_B^1(x)], [\inf I_A^2(x) \vee \inf I_B^2(x), \sup I_A^2(x) \vee \sup I_B^2(x)], \ldots, [\inf I_A^p(x) \vee \inf I_B^p(x), \sup I_A^P(x) \vee \sup I_B^P(x)]), ([\inf F_A^1(x) \vee \inf F_B^1(x), \sup F_A^1(x) \vee \sup F_B^1(x)], [\inf F_A^2(x) \vee \inf F_B^2(x), \sup F_A^2(x) \vee \sup F_B^2(x)], \ldots, [\inf F_A^p(x) \vee \inf F_B^p(x), \sup F_A^P(x) \vee \sup F_B^P(x)])$>: x ∈ X}

Example 3.14. Let U={x_1, x_2} be the universe and A and B are two n-valued interval neutrosophic sets

A= { <x_1,{[.1, .2], [.2, .3]},{[.4, .5], [.6, .7]},{[.5, .6], [.7, .8]}>,
<x_2,{[.1, .4], [.1,.3]},{[.6, .8], [.4, .6]},{[.3, .4], [.2, .7]}>}

and

B= { <x_1,{[.3, .7], [.3, .5]},{[.2, .4], [.3, .5]},{[.3, .6], [.2, .7]}>,
<x_2,{[.3, .5], [.4,.6]},{[.3, .5], [.4, .5]},{[.3, .4], [.1, .2]}>}

Then,

A∩B={<x_1,{[.1,.2],[.2,.3]},{[.4,.5],[.6,.7]},{[.5,.6],[.7,.8]}>,
<x_2,{[.1,.4],[.1,.3]},{[.6,.8],[.4,.6]},{[.3,.4],[.2,7]}>}

Proposition 3.15. Let A, B, C∈NVINS(X). Then,

1. A∩A=A
2. A∩ $\emptyset = \emptyset$.
3. A∩E=A
4. A∩B= B∩A
5. $(A$∩B) ∩ C = A ∩ (B ∩ C).

Proof: The proof is straightforward.

Definition 3.16. Let A and B be two n-valued interval neutrosophic sets. Then, union of A and B, denoted by A∪B, is defined by

A∪B = {x,<$\big([\inf T_A^1(x) \vee \inf T_B^1(x), \sup T_A^1(x) \vee \sup T_B^1(x)], [\inf T_A^2(x) \vee \inf T_B^2(x), \sup T_A^2(x) \vee \sup T_B^2(x)], \dots, [\inf T_A^p(x) \vee \inf T_B^p(x), \sup T_A^P(x) \vee \sup T_B^P(x)]\big)$,$\big([\inf I_A^1(x) \wedge \inf I_B^1(x), \sup I_A^1(x) \wedge \sup I_B^1(x)], [\inf I_A^2(x) \wedge \inf I_B^2(x), \sup I_A^2(x) \wedge \sup I_B^2(x)], \dots, [\inf I_A^p(x) \wedge \inf I_B^p(x), \sup I_A^P(x) \wedge \sup I_B^P(x)]\big)$, $\big([\inf F_A^1(x) \wedge \inf F_B^1(x), \sup F_A^1(x) \wedge \sup F_B^1(x)], [\inf F_A^2(x) \wedge \inf F_B^2(x), \sup F_A^2(x) \wedge \sup F_B^2(x)], \dots, [\inf F_A^p(x) \wedge \inf F_B^p(x), \sup F_A^P(x) \wedge \sup F_B^P(x)]\big)$>: x ∈ X}

Proposition 3.17. Let A, B, C ∈ NVINS(X).Then,

1. A∪A=A.
2. A∪ ∅ = A.
3. A∪E=E.
4. A∪B= B∪A.
5. $(A \cup B) \cup C = A \cup (B \cup C)$.

Proof: The proof is straightforward.

Definition 3.18. Let A and B be two n-valued interval neutrosophic sets. Then, difference of A and B, denoted by A \B, is defined by

A\B={x,([min{inf $T_A^1(x)$, inf $F_B^1(x)$} , min{sup $T_A^1(x)$, sup $F_B^1(x)$}], [min{inf $T_A^2(x)$, inf $F_B^2(x)$} , min{sup $T_A^2(x)$, sup $F_B^2(x)$}],..., [min{inf $T_A^p(x)$, inf $F_B^p(x)$} , min{sup $T_A^p(x)$, sup $F_B^p(x)$}]),([max(inf $I_A^1(x)$,1-sup $I_B^1(x)$),max(sup$I_A^1(x)$,1-inf$I_B^1(x)$)],[max(inf$I_A^2(x)$,1-sup$I_B^2(x)$),max(sup$I_A^2(x)$,1-inf $I_B^2(x)$)],...,[max(inf$I_A^p(x)$,1-sup$I_B^p(x)$),max(sup$I_A^p(x)$,1-inf$I_B^p(x)$)]),([max(inf $F_A^1(x)$,inf$T_B^1(x)$), max(sup $F_A^1(x)$, sup$T_B^1(x)$)], [max(inf $F_A^2(x)$,inf$T_B^2(x)$), max(sup$F_A^2(x)$,Sup$T_B^2(x)$)],..., [max(inf $F_A^p(x)$,inf$T_B^p(x)$), max(sup $F_A^p(x)$, sup$T_B^p(x)$)]): x ∈ X}

Example 3.19. Let X= $\{x_1, x_2\}$ be the universe and A and B are two n-valued interval neutrosophic sets

A= { <x_1,{[.1, .2], [.2, .3]},{[.4, .5], [.6, .7]},{[.5, .6], [.7, .8]}>,
<x_2,{[.1, .4], [.1, .3]},{[.6,.8], [.4, .6]},{[.3, .4], [.2, .7]}>}

and

B= { <x_1,{[.3, .7], [.3, .5]},{[.2, .4], [.3, .5]},{[.3, .6], [.2, .7]}>,
<x_2,{[.3, .5], [.4, .6]},{[.3,.5], [.4, .5]},{[.3, .4], [.1, .2]}>}

Then,

A \ B={ < x_1 ,{[.1, .2], [.2, .3]},{[.6 ,.8], [.6, .7]},{[.5, .7], [.7, .8]}>,
<x_2,{[.1, .4], [.1, .2]},{[.6,.8], [.5, .6]},{[.3, .5], [.4, 7]}>}

Definition 3.20. Let A and B be two n-valued interval neutrosophic sets. Then, addition of A and B, denoted by A $\widetilde{\mp}$B, is defined by

A$\widetilde{\mp}$ = {<x,([$\min(\inf T_A^1(x) + \inf T_B^1(x), 1)$, $\min(\sup T_A^1(x) + \sup T_B^1(x), 1)$],
[$\min(\inf T_A^2(x) + \inf T_B^2(x), 1)$, $\min(\sup T_A^2(x) + \sup T_B^2(x), 1)$],...,[$\min(\inf T_A^p(x) + \inf F_B^p(x), 1)$, $\min(\sup T_A^p(x) + \sup T_B^p(x), 1)$])
([$\min(\inf I_A^1(x) + \inf I_B^1(x), 1)$, $\min(\sup I_A^1(x) + \sup I_B^1(x), 1)$], [$\min(\inf I_A^2(x) + \inf I_B^2(x), 1)$, $\min(\sup I_A^2(x) + \sup I_B^2(x), 1)$],...,[$\min(\inf T_A^p(x) + \inf T_B^p(x), 1)$, $\min(\sup T_A^p(x) + \sup T_B^p(x), 1)$]),([$\min(\inf F_A^1(x) + \inf F_B^1(x), 1)$, $\min(\sup F_A^1(x) + \sup F_B^1(x), 1)$], [$\min(\inf F_A^2(x) + \inf F_B^2(x), 1)$, $\min(\sup F_A^2(x) + \sup F_B^2(x), 1)$],...,[$\min(\inf F_A^p(x) + \inf F_B^p(x), 1)$, $\min(\sup F_A^p(x) + \sup F_B^p(x), 1)$]>: x ∈ X }.

Example 3.21. Let X=**{**x_1, x_2} be the universe and A and B are two n-valued interval neutrosophic sets

A= { <x_1,{[.1, .2], [.2, .3]},{[.4, .5], [.6, .7]},{[.5, .6], [.7, .8]}>,
<x_2,{[.1, .4], [.1, .3]},{[.6, .8], [.4, .6]},{[.3, .4], [.2, .7]}>}

and

B= { <x_1,{[.3, .7], [.3, .5]},{[.2, .4], [.3, .5]},{[.3, .6], [.2, .7]}>,
<x_2,{[.3, .5], [.4, .6]},{[.3, .5], [.4, .5]},{[.3, .4], [.1, .2]}>}

then,

A$\widetilde{\mp}$B={<x_1,{[.4,.9],[.5,.8]},{[.6,.9],[.9,1]},{[.8,1],[.9,1]}>,
<x_2,{[.4,.9],[.5,.9]},{[.9,1],[.8,1]},{[.6,.8],[.3,9]}>}.

Proposition 3.22. Let A, B, C ∈ NVINS(X).Then,

1. A$\widetilde{\mp}$B=B$\widetilde{\mp}$A.
2.(A$\widetilde{\mp}$B)$\widetilde{\mp}$C = A$\widetilde{\mp}$(B$\widetilde{\mp}$C).

Proof: The proof is straightforward.

Definition 3.23. Let A and B be two n-valued interval neutrosophic sets. Then, scalar multiplication of A, denoted by A $\tilde{.}$a, is defined by

$$A\tilde{.}a = \{x, ([\min(\inf T_A^1(x).a, 1), \min(\sup T_A^1(x).a, 1)], \begin{bmatrix}\min(\inf T_A^2(x).a, 1),\\ \min(\sup T_A^2(x).a, 1)\end{bmatrix}$$

$,...,\left[\min(\inf T_A^p(x).\mathrm{a},1), \min(\sup T_A^p(x).\mathrm{a},1)\right]),$
$([\min(\inf I_A^1(x).\mathrm{a},1), \min(\sup I_A^1(x).\mathrm{a},1)],$
$[\min(\inf I_A^2(x).\mathrm{a},1), \min(\sup I_A^2(x).\mathrm{a},1)],...,$
$\left[\min(\inf I_A^p(x).\mathrm{a},1), \min(\sup I_A^p(x).\mathrm{a},1)\right]),$
$([\min(\inf F_A^1(x).\mathrm{a},1), \min(\sup F_A^1(x).\mathrm{a},1)],$
$[\min(\inf I_A^1(x).\mathrm{a},1), \min(\sup I_A^1(x).\mathrm{a},1)],...,$
$\left[\min(\inf I_A^p(x).\mathrm{a},1), \min(\sup I_A^p(x).\mathrm{a},1)\right]): x \in X\}.$

Example 3.24. Let X={x_1, x_2} be the universe and A and B are two n-valued interval neutrosophic sets

A= { <x_1,{[.1, .2], [.2, .3]},{[.4, .5], [.6, .7]},{[.5, .6], [.7, .8]}>,
<x_2,{[.1, .4], [.1, .3]},{[.6,.8], [.4, .6]},{[.3, .4], [.2, .7]}>}

and

B= { <x_1,{[.3, .7], [.3, .5]},{[.2, .4], [.3, .5]},{[.3, .6], [.2, .7]}>,
<x_2,{[.3, .5], [.4, .6]},{[.3,.5], [.4, .5]},{[.3, .4], [.1, .2]}>},

then,

A.̃2 = { <x_1,{[.2, .4] ,[.4 ,.6]},{[.8, 1],[1, 1]},{[1, 1],[1, 1]}>,
<x_2,{[.2,.8],[.2, .6]},{[1, 1],[.8, 1]},{[.6, .8],[.4, 1]}>}

Proposition 3.25. Let A, B, C ∈ NVINS(X). Then,

1. A.̃B=B.̃A
2. $(A\tilde{.}B)\tilde{.}C = A\tilde{.}(B\tilde{.}C)$

Proof: The proof is straightforward.

Definition 3.26. Let A and B be two n-valued interval neutrosophic sets. Then, scalar division of A, denoted by A $\tilde{/}$ a, is defined by

$A\tilde{/}\,\mathrm{a}=\{x,[\min(\inf T_A^1(x)/\mathrm{a},1), \min(\sup T_A^1(x)/\mathrm{a},1)],$
$[\min(\inf T_A^2(x)/\mathrm{a},1), \min(\sup T_A^2(x)/\mathrm{a},1)]$
$,...,\left[\min(\inf T_A^p(x)/\mathrm{a},1), \min(\sup T_A^p(x)/\mathrm{a},1)\right]),([\min(\inf I_A^1(x)/$
$\mathrm{a},1), \min(\sup I_A^1(x)/\mathrm{a},1)], [\min(\inf I_A^2(x)/\mathrm{a},1), \min(\sup I_A^2(x)/$
$\mathrm{a},1)],...,\left[\min(\inf I_A^p(x)/\mathrm{a},1), \min(\sup I_A^p(x)/\mathrm{a},1)\right]),$
$\left[[\min(\inf F_A^1(x)/\mathrm{a},1), \min(\sup F_A^1(x).\mathrm{a},1)]\right], [\min(\inf F_A^2(x)/$
$\mathrm{a},1), \min(\sup F_A^2(x)/\mathrm{a},1)], ...,\left[\min(\inf I_A^p(x)/\right.$
$\left.\mathrm{a},1), \min(\sup I_A^p(x)/\mathrm{a},1)\right]): x \in X\}.$

Example 3.27. Let X={x_1, x_2} be the universe and A and B are two n-valued interval neutrosophic sets

A= { <x_1,{[.1, .2], [.2, .3]},{[.4, .5], [.6, .7]},{[.5, .6], [.7, .8]}>,
<x_2,{[.1, .4], [.1, .3]},{[.6, .8], [.4, .6]},{[.3, .4], [.2, .7]}>}

and

B= { <x_1,{[.3, .7], [.3, .5]},{[.2, .4], [.3, .5]},{[.3, .6], [.2, .7]}>,
<x_2,{[.3, .5], [.4, .6]},{[.3, .5], [.4, .5]},{[.3, .4], [.1, .2]}>},

then,

A/˜2= { <x_1,{[.05, .1], [.1, .15]},{[.2, .25], [.3, .35]},{[.25, .3],[.35, .4]}>,
<x_2,{[.05,.2], [.05, .15]},{[.3, .4],[.2, .3]},{[.15, .2], [.1, .35]}>}

Definition 3.28. Let A and B be two n-valued interval neutrosophic sets. Then, truth-Favorite of A, denoted by $\tilde{\Delta}$A, is defined by

$$\tilde{\Delta}A = \{x, ([\min(\inf T_A^1(x) + \inf I_A^1(x), 1), \min(\sup T_A^1(x) + \sup I_B^1(x), 1)], [\min(\inf T_A^2(x) + \inf I_A^2(x), 1), \min(\sup T_A^2(x) + \sup I_A^2(x), 1)], ..., [\min(\inf T_A^p(x) + \inf I_A^p(x), 1), \min(\sup T_A^p(x) + \sup I_A^p(x), 1)]), ([0,0], [0,0], ..., [0,0]), ([\inf F_A^1(x), \sup F_A^1(x)], [\inf F_A^2(x), \sup F_A^2(x)], ..., [\inf F_A^p(x), \sup F_A^p(x)]) : x \in X\}$$

Example 3.29. Let X=**{**x_1, x_2} be the universe and A and B are two n-valued interval neutrosophic sets

A= { <x_1,{[.1, .2], [.2, .3]},{[.4, .5], [.6, .7]},{[.5, .6], [.7, .8]}>,
<x_2,{[.1, .4], [.1, .3]},{[.6, .8], [.4, .6]},{[.3, .4], [.2, .7]}>}

and

B= { <x_1,{[.3, .7], [.3, .5]},{[.2, .4], [.3, .5]},{[.3, .6], [.2, .7]}>,
<x_2,{[.3, .5], [.4, .6]},{[.3,.5], [.4, .5]},{[.3, .4], [.1, .2]}>}

Then,

$\tilde{\Delta}$A= { <x_1,{[.5, .7],[.8, 1]},{[0, 0],[0, 0]},{[. 5, .6], [.7, .8]}>, <x_2,{[.7, 1], [. 5, .9]},{[0, 0],[0, 0]},{[. 3, .4],[.2, .7]}>}

Proposition 3.30. Let A, B, C ∈ NVINS(X).Then,

1. $\tilde{\Delta}\tilde{\Delta}$A=$\tilde{\nabla}A$.
2. $\tilde{\Delta}\,(A \cup B) \subseteq \tilde{\Delta}\,A \cup \tilde{\Delta}\,B$.
3. $\tilde{\Delta}\,(A \cap B) \subseteq \tilde{\Delta}\,A \cap \tilde{\Delta}\,B$
4. $\tilde{\Delta}\,(A \tilde{+} B) \subseteq \tilde{\Delta}\,A \tilde{+}\,\tilde{\Delta}\,B$.

Proof: The proof is straightforward.

Definition 3.31. Let A and B be two n-valued interval neutrosophic sets. Then, false-Favorite of A, denoted by $\tilde{\nabla}$A, is defined by

$\widetilde{\nabla}$A=
{x([$\inf T_A^1(x)$, $\sup T_A^1(x)$], [$\inf T_A^2(x)$, $\sup T_A^2(x)$],...,
[$\inf T_A^p(x)$, $\sup T_A^p(x)$]),([0,0], [0,0],...,[0,0]), [min($\inf F_A^1(x) +$
$\inf I_A^1(x), 1)$, min($\sup F_A^1(x) +$ $\sup I_B^1(x), 1)$], [min($\inf F_A^2(x) +$
$\inf I_A^2(x), 1)$, min($\sup F_A^2(x) +$ $\sup I_A^2(x), 1)$],...,[min($\inf F_A^p(x) +$
$\inf I_A^p(x), 1)$, min($\sup F_A^p(x) +$ $\sup I_A^p(x), 1)$]): x ∈ X }

Example 3.32. Let X={x_1, x_2} be the universe and A and B are two n-valued interval neutrosophic sets

A= { <x_1,{[.1, .2], [.2, .3]},{[.4, .5], [.6, .7]},{[.5, .6], [.7, .8]}>,
<x_2,{[.1, .4], [.1, .3]},{[.6,.8], [.4, .6]},{[.3, .4], [.2, .7]}>}

and

B= { <x_1,{[.3, .7], [.3, .5]},{[.2, .4], [.3, .5]},{[.3, .6], [.2, .7]}>,
<x_2,{[.3, .5], [.4, .6]},{[.3,.5], [.4, .5]},{[.3, .4], [.1, .2]}>}

Then,

$\widetilde{\nabla}$A = { <x_1,{[.1, .2],[.2, .3]},{[0, 0],[0, 0]},{[. 9, 1], [1, 1]}>,
<x_2,{[.1, .4], [. 1, .3]},{[0, 0], [0, 0]},{[. 9, 1], [.6, 1]}>}

Proposition 3.33. Let A, B, C ∈ NVINS(X). Then,

1. $\widetilde{\nabla}\widetilde{\nabla}$A = $\widetilde{\nabla}A$.
2. $\widetilde{\nabla}(A \cup B) \subseteq \widetilde{\nabla}A \cup \widetilde{\nabla}B$.
3. $\widetilde{\nabla}(A \cap B) \subseteq \widetilde{\nabla}A \cap \widetilde{\nabla}B$.
4. $\widetilde{\nabla}(A \widetilde{\mp} B) \subseteq \widetilde{\nabla}A \widetilde{\mp} \widetilde{\nabla}B$.

Proof: The proof is straightforward.

Here ∨, ∧, +, . , /, $\tilde{.}$, $\tilde{/}$ denotes maximum, minimum, addition, multiplication, scalar multiplication, scalar division of real numbers respectively.

Definition 3.34. Let E is a real Euclidean space E^n. Then, a NVINS A is convex if and only if

$\inf T_A^i(x)(\lambda x_1$ +(1- λ) $x_2) \geq$ min($\inf T_A^i(x_1)$, $\inf T_A^i(x_2)$),
$\sup T_A^i(x)(\lambda x_1$ +(1- λ) $x_2) \geq$ min($\sup T_A^i(x_1)$, $\sup T_A^i(x_2)$),
$\inf I_A^i(x)(\lambda x_1$ +(1- λ) $x_2) \leq$ min($\inf I_A^i(x_1)$, $\inf I_A^i(x_2)$),
$\sup I_A^i(x)(\lambda x_1$ +(1- λ) $x_2) \leq$ min($\sup i_A^i(x_1)$, $\sup I_A^i(x_2)$),
$\inf F_A^i(x)(\lambda x_1$ +(1- λ) $x_2) \leq$ min($\inf F_A^i(x_1)$, $\inf F_A^i(x_2)$),
$\sup F_A^i(x)(\lambda x_1$ +(1- λ) $x_2) \leq$ min($\sup F_A^i(x_1)$, $\sup F_A^i(x_2)$),

for allx_1, x_2 ∈ E and all λ ∈ [0 ,1] and i= 1, 2,..., p.

Theorem 3.35. If A and B are convex, so is their intersection.

Proof: Let C= A ∩ B

InfT_C^j (λx_1 +(1- λ)x_2) ≥ min (Inf T_A^j (λx_1 +(1- λ)x_2) , Inf T_B^j (λx_1 +(1- λ)x_2)) ,supT_C^j (λx_1 +(1- λ)x_2) ≥ min (sup T_A^j (λx_1 +(1- λ)x_2) , sup T_B^j (λx_1 +(1- λ)x_2)) , , Inf I_C^j (λx_1 +(1- λ)x_2)≤ max (Inf I_A^j (λx_1 +(1- λ)x_2) , Inf I_B^j (λx_1 +(1- λ)x_2)), supI_C^j (λx_1 +(1- λ)x_2) ≤ max (sup I_A^j (λx_1 +(1- λ)x_2) , sup i_B^j (λx_1 +(1- λ)x_2)) , Inf F_C^j (λx_1 +(1- λ)x_2)≤ max (Inf F_A^j (λx_1 +(1- λ)x_2) , Inf F_B^j (λx_1 +(1- λ)x_2)), supF_C^j (λx_1 +(1- λ)x_2) ≤ max (inf F_A^j (λx_1 +(1- λ)x_2) ,inf F_B^j (λx_1 +(1- λ)x_2)) since A and are convex: Inf T_A^j (λx_1 +(1- λ)x_2) ≥ min (Inf T_A^j (x_1)+ Inf T_A^j (x_2)), sup T_A^j (λx_1 +(1- λ)x_2) ≥ min (sup T_A^j (x_1)+ sup T_A^j (x_2)) , Inf I_A^j (λx_1 +(1- λ)x_2) ≤ max (Inf I_A^j (x_1)+ Inf I_A^j (x_2)) , sup I_A^j (λx_1 +(1- λ)x_2)≤max (sup I_A^j (x_1)+ sup I_A^j (x_2)) ,inf F_A^j (λx_1 +(1- λ)x_2)≤max (inf F (x_1)+ inf F_A^j (x_2)) , sup F_A^j (λx_1 +(1- λ)x_2)≤max (sup F_A^j (x_1)+ sup F_A^j (x_2)) , inf T_B^j (λx_1 +(1- λ)x_2)≥max (inf T_B^j (x_1)+ inf T_B^j (x_2)) , sup T_B^j (λx_1 +(1- λ)x_2)≥min (sup T_B^j (x_1)+ sup T_A^j (x_2)) , inf I_B^j (λx_1 +(1- λ)x_2)≤max (inf I_B^j (x_1)+ inf I_A^j (x_2)) , sup I_B^j (λx_1 +(1- λ)x_2 ≤max (sup I_B^j (x_1)+ sup I_A^j (x_2)) , inf F_B^j (λx_1 +(1- λ)x_2)≤max (inf F_B^j (x_1)+ inf F_A^j (x_2)) , sup F_B^j (λx_1+(1- λ)x_2)≤max (sup F_B^j (x_1)+ sup F_A^j (x_2)).

Hence,

infT_C^j (λx_1 +(1- λ)x_2)≥min(min (inf T_A^j (x_1)+ inf T_A^j (x_2)) , min (inf T_B^j (x_1)+ inf T_B^j (x_2)))= min (inf T_A^j (x_1)+ inf T_B^j (x_1)), min (inf T_A^j (x_2)+ inf T_B^j (x_2))= min (inf T_C^j (x_1)+ inf T_C^j (x_2)),
sup T_C^j (λx_1 +(1- λ)x_2)≥min(min (supT_A^j (x_1)+ sup T_A^j (x_2)) , min (sup T_B^j (x_1)+ sup T_B^j (x_2)))= min (supT_A^j (x_1)+sup T_B^j (x_1)), min (sup T_A^j (x_2)+ sup T_B^j (x_2))= min (sup T_C^j (x_1)+ sup T_C^j (x_2)),
infI_C^j (λx_1 +(1- λ)x_2)≤max(max (inf I_A^j (x_1)+ inf I_A^j (x_2)) , max(inf I_B^j (x_1)+ inf I_B^j (x_2)))= max(infI_A^j (x_1)+inf I_B^j (x_1)),max (inf I_A^j (x_2)+ inf I_B^j (x_2))= max (inf I_C^j (x_1)+ inf I_C^j (x_2)).

Definition 3.36. An n-valued interval neutrosophic set is strongly convex if for any two points x_1 and $x_2$and any λ in the open interval (0.1).

inf $T_A^i(x)$(λx_1 +(1- λ) x_2)> min(inf $T_A^i(x_1)$, inf $T_A^i(x_2)$),

$\sup T_A^i(x)(\lambda x_1 + (1-\lambda)\, x_2) > \min(\sup T_A^i(x_1), \sup T_A^i(x_2))$,
$\inf I_A^i(x)(\lambda x_1 + (1-\lambda)\, x_2) < \min(\inf I_A^i(x_1), \inf I_A^i(x_2))$,
$\sup I_A^i(x)(\lambda x_1 + (1-\lambda)\, x_2) < \min(\sup i_A^i(x_1), \sup I_A^i(x_2))$,
$\inf F_A^i(x)(\lambda x_1 + (1-\lambda)\, x_2) < \min(\inf F_A^i(x_1), \inf F_A^i(x_2))$,
$\sup F_A^i(x)(\lambda x_1 + (1-\lambda)\, x_2) < \min(\sup F_A^i(x_1), \sup F_A^i(x_2))$,
for allx_1 ,x_2 in X and all λ in [0 ,1] and i= 1, 2,...,p.

Theorem 3.37. If A and B are strongly convex, so is their intersection.

Proof: The proof is similar to Theorem 3.25

4 Distances between n-valued interval neutrosophic sets

In this section, we present the definitions of the Hamming, Euclidean distances between n-valued interval neutrosophic sets, generalized weighted distance and the similarity measures between n-valued interval neutrosophic sets based on the distances, which can be used in real scientific and engineering applications.

On the basis of the Hamming distance and Euclidean distance between two interval neutrosophic set defined by Ye in [43], we give the following Hamming distance and Euclidean distance between NVINSs as follows:

Definition 4.1 Let A and B two n-valued interval neutrosophic sets, Then, the Hamming distance is defined by:

1- $d_{HD} = \frac{1}{p}\sum_{j=1}^{p}\frac{1}{6}\sum_{i=1}^{n}\big[\left|\mathrm{infT}_A^j(\mathrm{x_i}) - \mathrm{infT}_B^j(\mathrm{x_i})\right| + \left|\mathrm{supT}_A^j(\mathrm{x_i}) - \mathrm{supT}_B^j(\mathrm{x_i})\right| + \left|\mathrm{infI}_A^j(\mathrm{x_i}) - \mathrm{infI}_B^j(\mathrm{x_i})\right| + \left|\mathrm{supI}_A^j(\mathrm{x_i}) - \mathrm{supI}_B^j(\mathrm{x_i})\right| + \left|inf\,\mathrm{F}_A^j(\mathrm{x_i}) - \mathrm{infF}_B^j(\mathrm{x})\right| + \left|sup\,\mathrm{F}_A^j(\mathrm{x_i}) - \mathrm{supF}_B^j(\mathrm{x})\right|\big]$

The normalized Hamming distance is defined by:

2- $d_{NHD} = \frac{1}{p}\sum_{j=1}^{p}\frac{1}{6\,n}\sum_{i=1}^{n}\big[\left|\mathrm{infT}_A^j(\mathrm{x_i}) - \mathrm{infT}_B^j(\mathrm{x_i})\right| + \left|\mathrm{supT}_A^j(\mathrm{x_i}) - \mathrm{supT}_B^j(\mathrm{x_i})\right| + \left|\mathrm{infI}_A^j(\mathrm{x_i}) - \mathrm{infI}_B^j(\mathrm{x_i})\right| + \left|\mathrm{supI}_A^j(\mathrm{x_i}) - \mathrm{supI}_B^j(\mathrm{x_i})\right| + \left|inf\,\mathrm{F}_A^j(\mathrm{x_i}) - \mathrm{infF}_B^j(\mathrm{x})\right| + \left|sup\,\mathrm{F}_A^j(\mathrm{x_i}) - \mathrm{supF}_B^j(\mathrm{x})\right|\big]$

However, the difference of importance is considered in the elements in the universe. Therefore, we need to consider the weights of the elements $\mathrm{x_i}$ (i=1, 2,..., n) into account. In the following, we defined the weighted Hamming distance with w={$\mathrm{w_1}$,$\mathrm{w_2}$,...,$\mathrm{w_n}$}

3- eighted normalized Hamming distance is defined by:

$$d_{wHD}=\frac{1}{p}\sum_{j=1}^{p}\frac{1}{6\,n}\sum_{i=1}^{n}w_i\Big[\left|\mathrm{infT}_A^j(\mathrm{x_i})-\mathrm{infT}_B^j(\mathrm{x_i})\right|+\left|\mathrm{supT}_A^j(\mathrm{x_i})-\mathrm{supT}_B^j(\mathrm{x_i})\right|+\left|\mathrm{infI}_A^j(\mathrm{x_i})-\mathrm{infI}_B^j(\mathrm{x_i})\right|+\left|\mathrm{supI}_A^j(\mathrm{x_i})-\mathrm{supI}_B^j(\mathrm{x_i})\right|+\left|inf\mathrm{F}_A^j(\mathrm{x_i})-\mathrm{infF}_B^j(\mathrm{x})\right|+\left|sup\mathrm{F}_A^j(\mathrm{x_i})-\mathrm{supF}_B^j(\mathrm{x})\right|\Big]$$

If $\mathrm{w_i}=(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n})$, then (3) reduces to the Normalized Hamming distance.

Example 4.2. Let $X=\{x_1, x_2\}$ be the universe and A and B are two n- valued interval neutrosophic sets

$A=\{<x_1,\{[.1,.2],[.2,.3]\},\{[.4,.5],[.6,.7]\},\{[.5,.6],[.7,.8]\}>, <x_2,\{[.1,.4],[.1,.3]\},\{[.6,.8],[.4,.6]\},\{[.3,.4],[.2,.7]\}>\}$

and

$B=\{<x_1,\{[.3,.7],[.3,.5]\},\{[.2,.4],[.3,.5]\},\{[.3,.6],[.2,.7]\}>, <x_2,\{[.3,.5],[.4,.6]\},\{[.3,.5],[.4,.5]\},\{[.3,.4],[.1,.2]\}>\}$

Then, we have d_{HD}= 0.4.

Definition 4.3. Let A, B two n-valued interval neutrosophic sets. Thus,

1. The Euclidean distance d_{ED} is defined by:

$$d_{ED}=\left\{\frac{1}{p}\sum_{j=1}^{p}\frac{1}{6}\sum_{i=1}^{n}\left[\left(\mathrm{infT}_A^j(\mathrm{x_i})-\mathrm{infT}_B^j(\mathrm{x_i})\right)^2+\left(\mathrm{supT}_A^j(\mathrm{x_i})-\mathrm{supT}_B^j(\mathrm{x_i})\right)^2+\left(\mathrm{infI}_A^j(\mathrm{x_i})-\mathrm{infI}_B^j(\mathrm{x_i})\right)^2+\left(\mathrm{supI}_A^j(\mathrm{x_i})-\mathrm{supI}_B^j(\mathrm{x_i})\right)^2+\left(inf\mathrm{F}_A^j(\mathrm{x_i})-\mathrm{infF}_B^j(\mathrm{x})\right)^2+\left(sup\mathrm{F}_A^j(\mathrm{x_i})-\mathrm{supF}_B^j(\mathrm{x})\right)^2\right]\right\}^{\frac{1}{2}}$$

2. The normalized Euclidean distance d_{NED} is defined by:

$$d_{NED}=\left\{\frac{1}{p}\sum_{j=1}^{p}\frac{1}{6\,n}\sum_{i=1}^{n}\left[\left(\mathrm{infT}_A^j(\mathrm{x_i})-\mathrm{infT}_B^j(\mathrm{x_i})\right)^2+\left(\mathrm{supT}_A^j(\mathrm{x_i})-\mathrm{supT}_B^j(\mathrm{x_i})\right)^2+\left(\mathrm{infI}_A^j(\mathrm{x_i})-\mathrm{infI}_B^j(\mathrm{x_i})\right)^2+\left(\mathrm{supI}_A^j(\mathrm{x_i})-\mathrm{supI}_B^j(\mathrm{x_i})\right)^2+\left(inf\mathrm{F}_A^j(\mathrm{x_i})-\mathrm{infF}_B^j(\mathrm{x_i})\right)^2+\left(sup\mathrm{F}_A^j(\mathrm{x_i})-\mathrm{supF}_B^j(\mathrm{x})\right)^2\right]\right\}^{\frac{1}{2}}$$

However, the difference of importance is considered in the elements in the universe. Therefore, we need to consider the weights of the elements x_i (i=1, 2,..., n) into account. In the following, we defined the weighted Euclidean distance with $w=\{w_1, w_2,\ldots,w_n\}$

3. The weighted Euclidean distance d_{WED} is defined by:

$$d_{WED}=\left\{\frac{1}{p}\sum_{j=1}^{p}\frac{1}{6\,n}\sum_{i=1}^{n} w_i\left[\left(\mathrm{infT}_A^j(x_i)-\mathrm{infT}_B^j(x_i)\right)^2+\left(\mathrm{supT}_A^j(x_i)-\mathrm{supT}_B^j(x_i)\right)^2+\left(\mathrm{infI}_A^j(x_i)-\mathrm{infI}_B^j(x_i)\right)^2+\left(\mathrm{supI}_A^j(x_i)-\mathrm{supI}_B^j(x_i)\right)^2+\left(inf\mathrm{F}_A^j(x_i)-\mathrm{infF}_B^j(x)\right)^2+\left(sup\mathrm{F}_A^j(x_i)-\mathrm{supF}_B^j(x)\right)^2\right]\right\}^{\frac{1}{2}}$$

If $w_i=(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n})$, then (3) reduces to the Normalized Euclidean distance.

Example 4.4. Let X={x_1, x_2} be the universe and A and B are two n- valued interval neutrosophic sets

A= { <x_1,{[.1, .2], [.2, .3]},{[.4, .5], [.6, .7]},{[.5, .6], [.7, .8]}>,
<x_2,{[.1, .4], [.1, .3]},{[.6,.8], [.4, .6]},{[.3, .4], [.2, .7]}>}

and

B= { <x_1,{[.3, .7], [.3, .5]},{[.2, .4], [.3, .5]},{[.3, .6], [.2, .7]}>,
<x_2,{[.3, .5], [.4, .6]},{[.3,.5], [.4, .5]},{[.3, .4], [.1, .2]}>},

then, we have d_{ED}= 0.125.

Definition 4.5. Let A, B two n-valued interval neutrosophic sets. Then based on Broumi et al.[11] we proposed a generalized interval valued neutrosophic weighted distance measure between A and B as follows:

$$d_\lambda(A\,,B)=\left\{\frac{1}{p}\sum_{j=1}^{p}\frac{1}{6\,n}\sum_{i=1}^{n} w_i\left[\left|\mathrm{infT}_A^j(x_i)-\mathrm{infT}_B^j(x_i)\right|^\lambda+\left|\mathrm{supT}_A^j(x_i)-\mathrm{supT}_B^j(x_i)\right|^\lambda+\left|\inf \mathrm{I}_A^j(x_i)-\inf \mathrm{I}_B^j(x_i)\right|^\lambda+\left|\sup \mathrm{I}_A^j(x_i)-\sup \mathrm{I}_B^j(x_i)\right|^\lambda+\left|\inf \mathrm{F}_A^j(x_i)-\inf \mathrm{F}_B^j(x)\right|^\lambda+\left|\sup \mathrm{F}_A^j(x_i)-\sup \mathrm{F}_B^j(x)\right|^\lambda\right]\right\}^{\frac{1}{\lambda}}$$

If λ=1 and $w_i=(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n})$, then the above equation reduces to the normalized Hamming distance.

If λ=2 and $w_i=(\frac{1}{n},\frac{1}{n},\ldots,\frac{1}{n})$,then the above equation reduces to the normalized Euclidean distance.

Theorem 4.6. The defined distance $d_K(A, B)$ between NVINSs A and B satisfies the following properties (1-4), for (k=HD, NHD, ED, NED);

1. $d_k(A, B) \geq 0$,
2. $d_k(A, B)$ =0 if and only if A = B; for all A,B ∈ NVINSs,

3. $d_k(A, B) = d_k(B, A)$,
4. If $A \subseteq B \subseteq C$, for A, B, C ∈ NVINSs, then $d_k(A, C) \geq d_k(A, B)$ and $d_k(A, C) \geq d_k(B, C)$.

Proof: it is easy to see that $d_k(A, B)$ satisfies the properties (D1)-(D3). Therefore, we only prove (D4). Let $A \subseteq B \subseteq C$, then,

$$\inf T_A^j(x_i) \leq \inf T_B^j(x_i) \leq \inf T_C^j(x_i), \text{supf } T_A^j(x_i) \leq \sup T_B^j(x_i) \leq \sup T_C^j(x_i), \inf I_A^j(x_i) \geq \inf I_B^j(x_i) \geq \inf I_C^j(x_i), \text{supf } I_A^j(x_i) \geq \sup I_B^j(x_i) \geq \sup I_C^j(x_i),$$

and

$$\inf F_A^j(x_i) \geq \inf F_B^j(x_i) \geq \inf F_C^j(x_i), \text{supf } F_A^j(x_i) \geq \sup F_B^j(x_i) \geq \sup F_C^j(x_i),$$

for k= (HD, NHD, ED, NED), we have

$$\left|\inf T_A^j(x_i) - \inf T_B^j(x_i)\right|^k \leq \left|\inf T_A^j(x_i) - \inf T_C^j(x_i)\right|^k, \left|\sup T_A^j(x_i) - \sup T_B^j(x_i)\right|^k \leq \left|\sup T_A^j(x_i) - \sup T_C^j(x_i)\right|^k,$$

$$\left|\inf T_B^j(x_i) - \inf T_C^j(x_i)\right|^k \leq \left|\inf T_A^j(x_i) - \inf T_C^j(x_i)\right|^k, \left|\sup T_B^j(x_i) - \sup T_B^j(x_i)\right|^k \leq \left|\sup T_A^j(x_i) - \sup T_C^j(x_i)\right|^k,$$

$$\left|\inf I_A^j(x_i) - \inf I_B^j(x_i)\right|^k \leq \left|\inf I_A^j(x_i) - \inf I_C^j(x_i)\right|^k, \left|\sup I_A^j(x_i) - \sup I_B^j(x_i)\right|^k \leq \left|\sup I_A^j(x_i) - \sup I_C^j(x_i)\right|^k,$$

$$\left|\inf I_B^j(x_i) - \inf I_C^j(x_i)\right|^k \leq \left|\inf I_A^j(x_i) - \inf I_C^j(x_i)\right|^k, \left|\sup I_B^j(x_i) - \sup I_B^j(x_i)\right|^k \leq \left|\sup I_A^j(x_i) - \sup I_C^j(x_i)\right|^k$$

$$\left|\inf I_A^j(x_i) - \inf I_B^j(x_i)\right|^k \leq \left|\inf I_A^j(x_i) - \inf I_C^j(x_i)\right|^k, \left|\sup I_A^j(x_i) - \sup I_B^j(x_i)\right|^k \leq \left|\sup I_A^j(x_i) - \sup I_C^j(x_i)\right|^k,$$

$$\left|\inf I_B^j(x_i) - \inf I_C^j(x_i)\right|^k \leq \left|\inf I_A^j(x_i) - \inf I_C^j(x_i)\right|^k, \left|\sup I_B^j(x_i) - \sup I_B^j(x_i)\right|^k \leq \left|\sup I_A^j(x_i) - \sup I_C^j(x_i)\right|^k,$$

$$\left|\inf F_A^j(x_i) - \inf F_B^j(x_i)\right|^k \leq \left|\inf F_A^j(x_i) - \inf F_C^j(x_i)\right|^k, \left|\sup F_A^j(x_i) - \sup F_B^j(x_i)\right|^k \leq \left|\sup F_A^j(x_i) - \sup F_C^j(x_i)\right|^k,$$

$$\left|\inf F_B^j(x_i) - \inf F_C^j(x_i)\right|^k \leq \left|\inf F_A^j(x_i) - \inf F_C^j(x_i)\right|^k,$$

$$\left|\sup F_B^j(x_i) - \sup F_B^j(x_i)\right|^k \leq \left|\sup F_A^j(x_i) - \sup F_C^j(x_i)\right|^k$$

Hence

$$\left|\inf T_A^j(x_i) - \inf T_B^j(x_i)\right|^k + \left|\sup T_A^j(x_i) - \sup T_B^j(x_i)\right|^k + \left|\inf I_A^j(x_i) - \inf I_B^j(x_i)\right|^k + \left|\sup I_A^j(x_i) - \sup I_B^j(x_i)\right|^k + \left|\inf F_A^j(x_i) - \inf F_B^j(x_i)\right|^k + \left|\sup F_A^j(x_i) - \sup F_B^j(x_i)\right|^k \leq \left|\inf T_A^j(x_i) - \inf T_C^j(x_i)\right|^k + \left|\sup T_A^j(x_i) - \sup T_C^j(x_i)\right|^k + \left|\inf I_A^j(x_i) - \inf I_C^j(x_i)\right|^k + \left|\sup I_A^j(x_i) - \sup I_C^j(x_i)\right|^k + \left|\inf F_A^j(x_i) - \inf F_C^j(x_i)\right|^k + \left|\sup F_A^j(x_i) - \sup F_C^j(x_i)\right|^k$$

$$\begin{aligned}\frac{1}{p}\sum_{j=1}^{p}\frac{1}{6}\sum_{i=1}^{n}&\left|\inf T_A^j(x_i) - \inf T_B^j(x_i)\right|^k + \left|\sup T_A^j(x_i) - \sup T_B^j(x_i)\right|^k \\ &+ \left|\inf I_A^j(x_i) - \inf I_B^j(x_i)\right|^k \\ &+ \left|\sup I_A^j(x_i) - \sup I_B^j(x_i)\right|^k \\ &+ \left|\inf F_A^j(x_i) - \inf F_B^j(x_i)\right|^k \\ &+ \left|\sup F_A^j(x_i) - \sup F_B^j(x_i)\right|^k \\ &\leq \frac{1}{p}\sum_{j=1}^{p}\frac{1}{6}\sum_{i=1}^{n}\left|\inf T_A^j(x_i) - \inf T_C^j(x_i)\right|^k \\ &+ \left|\sup T_A^j(x_i) - \sup T_C^j(x_i)\right|^k \\ &+ \left|\inf I_A^j(x_i) - \inf I_C^j(x_i)\right|^k \\ &+ \left|\sup I_A^j(x_i) - \sup I_C^j(x_i)\right|^k \\ &+ \left|\inf F_A^j(x_i) - \inf F_C^j(x_i)\right|^k \\ &+ \left|\sup F_A^j(x_i) - \sup F_C^j(x_i)\right|^k\end{aligned}$$

Then $d_k\,(A, B) \leq d_k(A, C)$

$$\left|\inf T_B^j(x_i) - \inf T_C^j(x_i)\right|^k + \left|\sup T_B^j(x_i) - \sup T_C^j(x_i)\right|^k + \left|\inf I_B^j(x_i) - \inf I_C^j(x_i)\right|^k + \left|\sup I_B^j(x_i) - \sup I_C^j(x_i)\right|^k + \left|\inf F_B^j(x_i) - \inf F_C^j(x_i)\right|^k + \left|\sup F_B^j(x_i) - \sup F_C^j(x_i)\right|^k \leq \left|\inf T_A^j(x_i) - \inf T_C^j(x_i)\right|^k + \left|\sup T_A^j(x_i) - \sup T_C^j(x_i)\right|^k + \left|\inf I_A^j(x_i) - \inf I_C^j(x_i)\right|^k + \left|\sup I_A^j(x_i) - \sup I_C^j(x_i)\right|^k + \left|\inf F_A^j(x_i) - \inf F_C^j(x_i)\right|^k + \left|\sup F_A^j(x_i) - \sup F_C^j(x_i)\right|^k$$

$$\frac{1}{p}\sum_{j=1}^{p}\frac{1}{6}\sum_{i=1}^{n}\left|\inf T_B^j(x_i)-\inf T_C^j(x_i)\right|^k+\left|\sup T_B^j(x_i)-\sup T_C^j(x_i)\right|^k$$
$$+\left|\inf I_B^j(x_i)-\inf I_C^j(x_i)\right|^k+\left|\sup I_B^j(x_i)-\sup I_C^j(x_i)\right|^k$$
$$+\left|\inf F_B^j(x_i)-\inf F_C^j(x_i)\right|^k+\left|\sup F_B^j(x_i)-\sup F_C^j(x_i)\right|^k$$
$$\le\frac{1}{p}\sum_{j=1}^{p}\frac{1}{6}\sum_{i=1}^{n}\left|\inf T_A^j(x_i)-\inf T_C^j(x_i)\right|^k$$
$$+\left|\sup T_A^j(x_i)-\sup T_C^j(x_i)\right|^k+\left|\inf I_A^j(x_i)-\inf I_C^j(x_i)\right|^k$$
$$+\left|\sup I_A^j(x_i)-\sup I_C^j(x_i)\right|^k+\left|\inf F_A^j(x_i)-\inf F_C^j(x_i)\right|^k$$
$$+\left|\sup F_A^j(x_i)-\sup F_C^j(x_i)\right|^k$$

Then d_k (B, C) $\le d_k$(A, C).

Combining the above inequalities with the above defined distance formulas (1)-(4), we can obtain that d_k (A, B) $\le d_k$ (A, C) and d_k (B, C) $\le d_k$ (A, C) for k= (HD, NHD, ED, NED).

Thus the property (D4) is obtained.

It is well known that similarity measure can be generated from distance measure. Therefore we may use the proposed distance measure to define similarity measures.

Based on the relationship of similarity measure and distance we can define some similarity measures between NVINSs A and B as follows:

Definition 4.7. The similarity measure based on s_{NVINS}(A, B)= 1-d_k(A, B), s_{NVINS}(A,B) is said to be the similarity measure between A and B, where A, B ∈ NVINS.

Theorem 4.8. The defined similarity measure s_{NVINS}(A, B) between NVINSs A and B satisfies the following properties (1-4),

1. s_{NVINS}(A, B) = s_{NVINS}(B, A).
2. s_{NVINS}(A, B) = (1, 0, 0)=$\underline{1}$ .if A=B for all A, B ∈ NVINSs.
3. s_{NVINS}(A, B) ∈[0,1]
4. If A⊆B⊆C for all A, B, C ∈ NVNSs then s_{NVINS}(A, B)≥ s_{NVINS}(A, C) and s_{NVINS}(B, C) ≥ s_{NVINS}(A, C).

From now on, we use

$$A=\{x,\begin{pmatrix}[\inf T_A^1(x),\sup T_A^1(x)],[\inf I_A^1(x),\sup I_A^1(x)],\\ [inf\ F_A^1(x),sup\ F_A^1(x)]\end{pmatrix},\ldots,$$

$$\left(\begin{array}{c}[\inf T_A^P(x), \sup T_A^P(x)], [\inf I_A^P(x), \sup I_A^P(x)], \\ [inf\ F_A^P(x), sup\ F_A^P(x)]),\end{array}\right): \mathrm{x} \in \mathrm{E}\}$$

instead of

$$\mathrm{A}=\{x, \left(\begin{array}{c}[\inf T_A^1(x), \sup T_A^1(x)], [\inf T_A^2(x), \sup T_A^2(x)], \ldots, \\ \left[\inf T_A^p(x), \sup T_A^p(x)\right]\end{array}\right),$$

$$\left(\begin{array}{c}[\inf I_A^1(x), \sup I_A^1(x)], [\inf I_A^2(x), \sup I_A^2(x)], \ldots., \\ \left[\inf I_A^p(x), \sup I_A^q(x)\right]\end{array}\right),$$

$$([\inf F_A^1(x), \sup F_A^1(x)] , ([\inf F_A^2(x), \sup F_A^2(x)], \ldots.,$$

$$(\left[inf\ F_A^p(x), sup\ F_A^r(x)\right]): \mathrm{x} \in \mathrm{E}\}.$$

5 Medical Diagnosis using NVINS

In what follows, let us consider an illustrative example adopted from Rajarajeswari and Uma [32] with minor changes and typically considered in [17,20,37]. Obviously, the application is an extension of intuitionistic fuzzy multi sets [17,20,32,33,34].

"As Medical diagnosis contains lots of uncertainties and increased volume of information available to physicians from new medical technologies, the process of classifying different set of symptoms under a single name of disease becomes difficult. In some practical situations, there is the possibility of each element having different truth membership, indeterminate and false membership functions. The proposed similarity measure among the patients Vs symptoms and symptoms Vs diseases gives the proper medical diagnosis. The unique feature of this proposed method is that it considers multi truth membership, indeterminate and false membership. By taking one time inspection, there may be error in diagnosis. Hence, this multi time inspection, by taking the samples of the same patient at different times gives best diagnosis" [32].

Now, an example of a medical diagnosis will be presented.

Example 5.1. Let P={P_1,P_2,P_3} be a set of patients, D={Viral Fever, Tuberculosis, Typhoid, Throat disease} be a set of diseases and S={Temperature, cough, throat pain, headache, body pain} be a set of symptoms. Our solution is to examine the patient at different time intervals (three times a day), which in turn give arise to different truth membership, indeterminate and false membership function for each patient.

Table I: Q (the relation Between Patient and Symptoms)

Q	Temperature	Cough	Throat pain	Headache	Body Pain
P_1	[.3,.4],[.4,.5],[.3,.7] [0,.3],[.1,.3],[0,.5] [0,.6],[.4,.5],[.3,.4]	[.1,.2],[.3,.6],[.6,.8] [0,.5],[.4,.7],[.4,.5] [.2,.3],[0,.5],[.4,.6]	[0,.5],[.2,.6],[0,.4] [.3,.4],[.2,.3,[.3,.4] [0,.7],[.3,.7],[.3,.5]	[.2,.3],[.3,.5],[0,.7] [.4,.5],[.4,.7],[.3,.6] [.2,.6],[0,.6],[.3,.4]	[0,.4],[.6,.7],[.2,.5] [.2,.4],[.4,.5],[.1,.2] [.1,.3],[.1,.3],[.2,.3]
P2	[.2,.3],[.4,.5],[.1,.2] [.4,.5],[.2,.5],[0,.3] [.6,.7,[.4,.5],[.4,.5]	[.5,.7],[0,.4],[.7,.8] [.6,.7],[0,.5],[.4,.5] [.4,.6],[.2,.7],[0,.3]	[.5,.6],[0,.6],[.2,.3] [.4,.7],[.4,.6],[.3,.4] [.1,.3],[.2,.3],[.5,.7]	[.2,.5],[.5,.6],[.1,.5] [.2,.3],[.2,.5],[.5,.6] [.1,.3],[.3,.4],[.4,.5]	[.2,.4],[.4,.6],[.1,.4] [0,.5],[.2,.4],[.5,.6] [.5,.7],[0,.7],[.2,.4]
P3	[.1,.3],[0,.5],[.4,.6] [.1,.2],[.3,.4],[.2,.5] [.2,.4],[.4,.5],[.3,.7]	[.2,.3],[.0,.7],[.1,.4] [.5,.6],[0,.3],[.3,.5] [.3,.5],[.2,.5],[.4,.6]	[.2,.4],[.3,.6],[0,.6] [.4,.5],[0,.3],[.3,.4] [.5,.7],[.4,.6],[.3,.7]	[.2,.3],[.5,.6],[.4,.5] [.2,.4],[0,.4],[.2,.7] [.4,.5],[.2,.3],[.3,.5]	[0,.6],[.4,.7],[.2,.3] [.2,.3],[.2,.3],[.1,.2] [.0,.6],[.2,.4],[.4,.6]

Let the samples be taken at three different timings in a day (in 08:00,16:00,24:00).

Table II: R (the relation among Symptoms and Diseases)

R	Viral Fever	Tuberculosis	Typhoid	Throat diseas
Temerature	[.2,.4],[.3,.5],[.3,.7]	[.1,.4],[.2,.6],[.6,.7]	[0,.3],[.4,.6],[0,.2]	[.3,.4],[.2,.5],[0,.6]
Cough	[.2,.4],[.2,.3],[0,.5]	[.3,.4],[.2,.5],[.7,.8]	[.3,.4],[.2,.3],[.1,.2]	[.4,.5],[.1,.3],[0,.5]
Throat Pain	[0,.4],[.2,.4],[.2,.4]	[0,.2,[.3,.6],[.6,.7]	[.1,.2],[.4,.5],[.3,.4]	[.2,.4],[.2,.5],[.3,.7]
Headache	[.4,.7],[0,.3],[.3,.5]	[.1,.2],[0,.5],[0,.6]	[.3,.4],[.2,.3],[.2,.5]	[0,.3],[.3,.6],[.2,.5]
Body Pain	[.1,.4],[.2,.5],[.3,.4]	[.5,.7],[.4,.5],[.2.5]	[.2,.3],[.2,.4],[.2.3]	[0,.4],[.1,.2],[.1.3]

Table III: The Hamming distance between NVINS Q and R

Hamming Distance	Viral Fever	Tuberculosis	Typhoid	Throat diseas
P_1	**0.1511**	0.1911	0.1678	0.1689
P_2	0.1911	0.2089	0.1789	**0.1644**
P_3	**0.1433**	0.1967	0.1533	0.1456

Table IV: The similarity Measure between NVINS Q and R

Similarity measure	Viral Fever	Tuberculosis	Typhoid	Throat diseas
P_1	**0.8489**	0.8089	0.8322	0.8311
P_2	0.8089	0.7911	0.8211	**0.8356**
P_3	**0.8567**	0.8033	0.8467	0.8544

The highest similarity measure from the Table IV gives the proper medical diagnosis. Therefore, patient P_1 suffers from Viral Fever, P_2 suffers from Throat disease and P_3 suffers from Viral Fever.

6 Conclusion

In this paper, we give n-valued interval neutrosophic sets and desired operations such as; union, intersection, addition, multiplication, scalar multiplication, scalar division, truth-favorite and false-favorite. The concept of n-valued interval neutrosophic set is a generalization of interval valued neutrosophic set, single valued neutrosophic sets and single valued neutrosophic multi sets. Then, we introduce some distances between n-valued interval neutrosophic sets (NVINS) and propose an efficient approach for group multi-criteria decision making based on n-valued interval neutrosophic sets. The distances have natural applications in the field of pattern recognition, feature extraction, region extraction, image processing, coding theory etc.

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A Comparison of Combined Overlap Block Fuzzy Cognitive Maps (COBFCM) and Combined Overlap Block Neutrosophic Cognitive Map (COBNCM) in finding the hidden patterns and indeterminacies in Psychological Causal Models: Case Study of ADHD

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Abstract

In spite of researchers' concerns to find causalities, reviewing the literature of psychological studies one may argue that the classical statistical methods applied in order to find causalities are unable to find uncertainty and indeterminacies of the relationships between concepts.

In this paper, we introduce two methods to find effective solutions by identifying "hidden" patterns in the patients' cognitive maps. Combined Overlap Block Fuzzy Cognitive Map (COBFCM) and Combined Overlap Block Neutrosophic Map (COBNCM) are effective when the number of concepts can be grouped and are large in numbers. In the first section, we introduce COBFCM, COBNCM, their applications, and the advantages of COBNCM over COBFCM in some cases. In the second section, we explain eight overlapped cognitive concepts related to ADHD in children and apply COBNCM and COBFCM to analyze the modeled data, comparing their results. Conclusions, limitations, and implications for applying COBNCM in other psychological areas are also discussed.

Keywords

Fuzzy Cognitive Map, Neutrosophic Cognitive Map, Fuzzy model, Causal model, ADHD, Methodology.

1 Introduction

A portfolio of project is a group of project that share resources creating relation among them of complementarity, incompatibility or synergy [1]. The interdependency modeling and analysis have commonly been ignored in project portfolio management [2].

Identifying causalities is one of the most important concerns of researchers, one may find out reviewing the literature of psychological research. Although there are some statistical methods to investigate this issue, all, or majority, rely on quantitative data. Less attention was directed towards scientific qualitative knowledge and experience. In some methods based on theoretical basics such as structural equation modeling (SEM), there is no chance to find optimal solutions, hidden patterns and indeterminacies (possibilities) of causal relationships between variables, which are common in psychological research. Therefore, for linking quantitative and qualitative knowledge, it seems an urge to use methods as fuzzy cognitive maps or neutrosophic cognitive maps in psychological research. The two methods are rooted in cognitive map (CM). The cognitive maps for representing social scientific knowledge and describing the methods that is used for decision-making were introduced by Axelrod in 1976. The fuzzy cognitive map (FCM) was proposed by Kosko (1986) to present the causal relationship between concepts and analyze inference patterns. Kosko (1986, 1988, 1997) considered fuzzy degree of inter relationships between concepts, its nodes corresponding to a relevant node and the edges stating the relation between two nodes, denoted by a sign. A positive sign implies a positive relation; moreover, any increase in its source value leads to increase in its target value. A negative sign stages a negative relation and any increase or decrease in its source value leads to reverse effect to its target value. If there is no edge between two nodes in a cognitive map, it means that there is no relation between them (Zhang et al., 1998). In a simple fuzzy cognitive map, the relation between two nodes is determined by taking a value in interval [-1, 1].

While -1 corresponds to the strongest negative value, +1 corresponds to strongest positive value. The other values express different levels of influence (Lee, et al., 2003). Fuzzy cognitive maps are important mathematical models representing the structured causality knowledge for quantitative inferences (Carvalho & Tome, 2007). FCM is a soft computing technique that follows an approach similar to the human reasoning and decision-making process (Markinos, et al., 2004). Soft computing is an emerging field that combines and synergies advanced theories and technologies such as Fuzzy Logic, Neural Networks, Probabilistic reasoning and Genetic Algorithms. Soft computing provides a hybrid flexible computing technology that can solve real world problems. Soft computing includes not only the previously mentioned approaches, but also useful combinations of its components, e.g. Neurofuzzy systems, Fuzzy Neural systems, usage of Genetic Algorithms in Neural Networks and Fuzzy Systems, and many other hybrid methodologies (Stylios & Peter, 2000). FCM can successfully represent knowledge and human experiences, introduce concepts to represent the essential elements, cause and effect relationships among the concepts, to model the behavior of a system (Kandasamy, 1999, 2004). This method is a very simple and powerful tool that is used in numerous fields (Thiruppathi, et al. 2010). When dataset is an unsupervised one and there is uncertainty within the concepts, this method is very useful. The FCM give us the hidden patterns; this method is one effective method, providing a tool for unsupervised data. In addition, using this method, one can analyze the data by directed graphs and connection matrices where nodes represent concepts and edges - strength of relationships (Stylios & Groumpos, 2000). FCM works on the opinion of experts or another uncertainty results like the obtained results using structural equation modeling (SEM). FCM clarify optimal solution by using a simple way, while other causal models such as SEM are complicated. They do not perform well to clarify what-if scenario, for example, their results do not clarify what happens to marital satisfaction if Alexithymia is very high and Family intimacy is very low. Another advantage of FCM is its functioning on experts' opinions (Thiruppathi et al. 2010). FCM is a flexible method used in several models to display several types of problems (Vasantha Kandasamy & Devadoss, 2004; Vasantha Kandasamy & Kisho, 1999). Although by using this method we are able to study uncertainty and find hidden patterns, the FCM is unable to investigate indeterminate relationships, which is a limitation in psychological causal models. A solution to overcome this limitation is the Neutrosophic Cognitive Map (NCM).

Vasantha Kandasamy and Smarandache (2003) proposed the neutrosophic cognitive maps, making it possible to mitigate the limitation of fuzzy cognitive maps, which cannot represent the indeterminate relations between variables. The capability of neutrosophic cognitive maps to represent indetermination facilitates the apprehension of systems complexity, and thus elucidates and predicts their behaviors in the absence of complete information.

Neutrosophic Cognitive Map (NCM) relies on Neutrosophy. Neutrosophy is a new branch of philosophy introduced by Smarandache in 1995 as a generalization of dialectics, which studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. Neutrosophic Cognitive Map (NCM) is the generalization and combination of the Fuzzy Cognitive Map in which indeterminacy is included. Fuzzy theory only measures the grade of membership or the non-existence of a membership in a revolutionary way, but failing to attribute the concept when the relationship between concepts in debate are indeterminate (Vasantha Kandasamy & Smarandache, 2007). A Neutrosophic Cognitive Map is a neutrosophic directed graph with concepts like policies, events etc. as nodes and causalities, or indeterminacies as edges. It represents the causal relationship between concepts defined by Smarandache (2001) and Vasantha Kandasamy (2007). Fuzzy cognitive maps deals with the relation / non-relation between two nodes or concepts, but it declines to attribute the relation between two conceptual nodes when the relation is an indeterminate one. In Neutrosophic Logic, each proposition is estimated to have the percentage of truth in a subset T, the percentage of indeterminacy in a subset I, and the percentage of falsity in a subset F. Every logical variable x is described by an ordered triple x = (T, I, F), where T is the degree of truth, F is the degree of false and I - the level of indeterminacy. Neutrosophy means that any proposition has a percentage of truth, a percentage of indeterminacy and a percentage of falsity (some of these percentages may be zero). Neutrosophy also makes distinctions between absolute truth (a proposition true in all possible worlds), which is denoted by 1, and relative truth (a proposition which is true in at least one world, but not in all), which is denoted by I (Smarandache & Liu, 2004). Sometimes, in psychological and educational research, the causality between the two concepts, i.e. the effect of C_i on C_j is indeterminate. Chances of indeterminacy are possible and frequent in case of unsupervised data. Therefore, the NCM is a flexible and effective method based on fuzzy cognitive map for investigating the relations of psychological casual models in which indeterminate relationships are not unusual. We describe the basic components in detail to explain differences between the two methods.

2 Combined Overlap Block Fuzzy Cognitive Maps (COBFCM) and Combined Overlap Block Neutrosophic Cognitive Map (COBNCM)

We can combine arbitrarily FCM and NCM connection matrices $F_1, F_2, \ldots, F_K$ by adding augmented FCM and NCM matrices, $F_1, \ldots, F_K$. Each augmented matrix F_i has *n-rows* and *n-columns*; *n* equals the total number of distinct concepts used by the experts. We permute the rows and columns of the augmented matrices to bring them into mutual coincidence. Then we add the F_i's point wise to yield the combined FCM and NCM matrix F, $F = \Sigma F_i$. We can then use F to construct the combined FCM and NCM directed graph. The combination can be in disjoint or overlapping blocks.

Combined overlap block fuzzy cognitive maps (COBFCM) were introduced and applied in social sciences by Vasantha Kandasamy et al. (2004), and combined overlap block neutrosophic cognitive map (COBNCM) - by Vasantha Kandasamy & Smarandache (2007). In these two methods, finite number of NCM and FCM can be combined together to produce the joint effect of all NCM and FCM. In NCM method, $N(E_1)$, $N(E_2), \ldots, N(E_p)$ are considered the neutrosophic adjacency matrices, with nodes $C_1, C_2, \ldots, C_n$, and $E_1, E_2, \ldots, E_p$ are the adjacency matrices of FCM with nodes $C_1, C_2, \ldots, C_n$. The combined NCM and the combined FCM are obtained by adding all the neutrosophic adjacency matrices $N(E_1) \ldots N(E_p)$ and adjacency matrices by $E_1, \ldots, E_P$ respectively. We denote the Combined NCM adjacency neutrosophic matrix by $N(E) = N(E_1) + N(E_2) + \ldots + N(E_p)$ and the Combined FCM adjacency matrix by $E = E_1 + E_2 + \ldots + E_p$. Both models $\{C_1, C_2, C_3, \ldots C_n\}$ contain *n* concepts associated with *P* (a given problem). We divide the number of concepts $\{C_1, C_2, C_3, \ldots C_n\}$ into *K* classes $S_1, S_2, S_3, \ldots S_K$, where the classes are such that $S_i \cap S_{i+1} \neq \phi$, $\cup S_i = \{C_1, C_2, \ldots, C_n\}$ and $|S_i| \neq |S_j|$, if $i \neq j$ in general. To introduce these methods in detail, we explain their basic components below.

3 Concepts and edges

In Combined Overlap Block Fuzzy Cognitive Maps (COBFCM) and Combined Overlap Block Neutrosophic Cognitive Map (COBNCM), the edges are qualitative concepts considered as nodes and causal influences. Concept nodes possess a numeric state, which denotes qualitative measures of the concepts present in the conceptual domain. When the nodes of FCM are a fuzzy set, they are called fuzzy nodes. Fuzzy means the concepts are not quantitative, they are uncertain, and we have to study them using linguistic variables, such as "very high", "high", "middle", etc. The nodes or concepts are presented by C_1, C_2,

C_3 ,.....,C_n. The state of concepts is portrayed as a vector. In COBNCM, we assume each node is a neutrosophic vector from neutrosophic vector space V. Let C_1, C_2, ..., C_n denote *n* nodes, So a node C_i will be represented by (x_1, ...,x_n), where x_k's - zero or one or *I* (*I* is the indeterminate) and x_k = 1 means that the node C_k is in the *ON* state, and x_k =0 means the node is in the *OFF* state, and x_k = I means the nodes state is an *indeterminate* at that time or in that situation. Let C_1, C_2... C_n be the nodes of COBNCM and let A = (a_1, a_2,..., a_n), where $a_i \in \{0, 1, I\}$. A is called the instantaneous state neutrosophic vector and it denotes the *ON – OFF – indeterminate* state position of the node at an instant:

a_i = 0 if a_i is off (no effect),

a_i = 1 if a_i is on (has effect),

a_i = I if a_i is indeterminate (effect cannot be determined),

for i = 1, 2,..., n.

In COBNCM, the nodes C_1, C_2, ..., C_n are nodes and not indeterminate nodes, because they indicate the concepts which are well known. But the edges connecting C_i and C_j may be indeterminate, i.e. an expert may not be in the position to say that C_i has some causality on C_j, either he will be in the position to state that C_i has no relation with C_j; in such cases, the relation between C_i and C_j, which is indeterminate, is denoted by I. The COBFCM with edge weights or causalities from the set {-1, 0, 1} are called simple, and COBNCM with edge weight from {-1, 0, 1, I} are called simple COBNCM. In COBFCM, the edges (e_{ij}) take values in the fuzzy causal interval [-1, 1], e_{ij} =0, e_{ij}>0 and e_{ij}<0 indicate no causality, positive and negative causality, respectively. In simple FCM, if the causality occurs, it occurs to a maximal positive or negative degree. Every edge in COBNCM is weighted with a number in the set {-1, 0, 1, I}. e_{ij} is the weight of the directed edge C_iC_j, $e_{ij} \in \{-1, 0, 1, I\}$. e_{ij} = 0 if C_i does not have any effect on C_j, e_{ij} = 1 if increase (or decrease) in C_i causes increase (or decrease) in C_j, e_{ij} = –1 if increase (or decrease) in C_i causes decrease (or increase) in C_j . e_{ij} = I if the relation or effect of C_i on C_j is an indeterminate. In such cases, it is denoted by dotted lines in the model.

4 Adjacency Matrix

In COBFCM and COBNCM, the edge weights are presented in a matrix. This matrix is defined by E= (e_{ij}), where e_{ij} indicates the weight of direct edge C_iC_j and $e_{ij} \in$ {0, 1,-1}, and by N (E) = (e_{ij}), where e_{ij} is the weight of the directed edge C_i C_j, where $e_{ij} \in$ {0, 1, -1, I}. We denote by N(E) the neutrosophic adjacency matrix of the COBNCM. It is important to note that all matrices used in these methods are always a square matrix with diagonal entries as zeros. All off-diagonal entries are edge weights that link adjacent nodes to each other. A finite number of FCM and NCM can be combined together to produce the joint effect of all FCM and NCM. Suppose $E_1, E_2, E_3 \ldots E_P$ and $N(E_1), N(E_2), N(E_3) \ldots N(E_P)$ are adjacency matrices of FCM and neutrosophic adjacency matrix of NCM, respectively, with nodes $C_1, C_2, C_3, \ldots, C_n$. Then combined FCM and NCM are obtained by adding all the adjacency matrices (Vasantha Kandasamy & Smarandache, 2003). In combined overlap FCM and NCM, all entries of all different overlapped matrices are put in a whole matrix and added to each other.

5 Inference process

The states of concepts are rendered as vectors. Therefore, the inference process of FCM and NCM can be represented by an iterative matrix calculation process. Let V_0 be the initial state vector, V_n be the state vector after *n th* iterative calculation, and W be the causal effect degree matrix; then the inference process can be defined as a repeating calculation of Equation 1 until the state vector converges to a stable value or fall in to an infinite loop. Suppose $X_1 = [1\ 0\ 0\ 0 \ldots 0]$ is the input vector and E is the associated adjacency matrix. X_1E is obtained by multiplying X_1 by the matrix E. We obtain $X_1E = [x_1, x_2, x_3, \ldots x_n]$ by replacing x_i by 1, if $x_i > c$, and x_i by 0, if $x_i < c$ (c is a suitable positive integer). After updating the thresholding concept, the concept is included in the updated vector by making the first coordinate as 1 in the resulting vector. Suppose $X_1E \rightarrow X_2$, then X_2E is considered; the same procedure is repeated until it gets limit cycle or a fixed point (Thiruppathi, et al., 2010).

$$V_{n+1} = f\,(V_n \times W + V_n), \qquad (1)$$

where the f is usually simply defined as $f(x) = f_0(x) = 1\ (x \geq 1),\ 0\ (1 > x > -1)$ and $-1\ (-1 \leq x)$.

If the equilibrium state of a dynamical system is a unique state vector, then it is called a fixed point. Consider FCM and NCM with $C_1, C_2 \ldots, C_n$ as nodes. For example, let us start the dynamical system by switching on C_1. Let us assume that NCM and FCM settle down with C_1 and C_n ON, i.e. the state vector remains as $(1, 0, \ldots, 1)$; this state vector $(1, 0, \ldots, 0, 1)$ is called the fixed point; if FCM and NCM settle down with a state vector repeating in the form $A_1 \rightarrow A_2 \rightarrow \ldots \rightarrow A_i \rightarrow A_1$, then this equilibrium is called a limit cycle of NCM and FCM (Tabar, 1991).

Let $C_1, C_2, \ldots, C_n$ be the vector of FCM and NSM. Let E be the associated adjacency matrix. Let us find the hidden pattern when x_1 is switched on when an input is given as the vector $A_1 = (1, 0, 0, \ldots, 0)$; the data should pass through the neutrosophic matrix $N_{(E)}$; this is done by multiplying A_1 by the matrix N(E). Let $A_1N_{(E)} = (a_1, a_2, \ldots, an)$ with the threshold operation, by replacing a_i by 1, if $a_i > k$, and a_i by 0, if $a_i < k$, and a_i by I, if a_i is not an integer.

$$f(k)\begin{Bmatrix} a_i < k \rightarrow a_i = 0 \\ a_i > k \rightarrow a_i = 1 \\ a_i = b + c \times I \rightarrow a_i = b \\ a_i = c \times I \rightarrow a_i = I \end{Bmatrix}$$

(k depends on researcher's opinion, for example K=1 or 0.5).

Note that $(a_1, a_2 \ldots a_n)$ and $(a'_1, a'_2, \ldots, a'_n)$ are two neutrosophic vectors. We say $(a_1, a_2, \ldots, a_n)$ is equivalent to $(a'_1, a'_2, \ldots, a'_n)$ denoted by $(a_1, a_2, \ldots, a_n) \sim (a'_1, a'_2, \ldots, a'_n)$, if we get $(a'_1, a'_2, \ldots, a'_n)$ after thresholding and updating the vector $(a_1, a_2, \ldots, a_n)$, after passing through the neutrosophic adjacency matrix N(E). The initial state vector in FCM and NCM is included 0 and 1 only (OFF and ON states, respectively). But after it passes through the adjacency matrix, the updating resultant vector may have entries from (0 and 1) in FCM and from (0, 1, I) in NCM, respectively. In this case, we cannot confirm the presence of that node (ON state), nor the absence (OFF state). Such possibilities are present only in the case of NCM.

6 Cyclic and acyclic FCM and NCM

If FCM and NCM possess a directed cycle, it is said to be cyclic (to have a feedback) and we call it a dynamical system. FCM and NCM are acyclic if they do not possess any directed cycle.

7 FCM versus NCM

Vasantha Kandasamy and Smarandache (2003) summarize the differences between FCM and NCM:

[1] FCM indicates the existence of causal relation between two concepts, and if no relation exists, it is denoted by 0.

[2] NCM does not indicate only the existence or absence of causal relation between two concepts, but also gives representation to the indeterminacy of relations between any two concepts.

[3] We cannot apply NCM for all unsupervised data. NCM will have meaning only when relation between at least two concepts C_i and C_j are indeterminate.

[4] The class of FCM is strictly contained in the class of NCM. All NCM can be made into FCM by replacing I in the connection matrix by 0.

[5] The directed graphs in case of NCM are called neutrosophic graphs. In the graphs, there are at least two edges, which are related by the dotted lines, meaning the edge between those two vertices is an indeterminate.

[6] All connection matrices of the NCM are neutrosophic matrices. They have in addition to the entries 0, 1, –1, the symbol I.

[7] The resultant vectors, i.e. the hidden pattern resulting in a fixed point or a limit cycle of a NCM, can also be a neutrosophic vector, signifying the state of certain conceptual nodes of the system to be an indeterminate; indeterminate relation is signified by I.

[8] Because NCM measures the indeterminate, the expert of the model can give careful representation while implementing the results of the model.

[9] In case of simple FCM, we have the number of instantaneous state vectors to be the same as the number of resultant vectors, but in the case of NCM the number of instantaneous state vectors is from the set {0,1}, whereas the resultant vectors are from the bigger set {0, 1, I}.

[10] Neutrosophic matrix {N (E)} converts to adjacency matrix (E) by easily recoding I to 0.

8 *Case study:* The comparison of COBFCM and COBNCM to find solution for ADHD

Attention-Deficit/Hyperactivity Disorder (ADHD) is not only the most common neuro-developmental disorder of childhood today, but also the most studied. Literature reviews report very different prevalence estimates. The DSM-IV states that the prevalence of ADHD is about 3–5% among school-age children [American Psychiatric Association, 1994]. Some of consequences of untreated ADHD children are social skills deficits, behavioral disinhibition and emotional skills deficits. Therefore, early diagnosis of ADHD is very important. The purpose of this paper is the comparison of application of COBFCM and COBNCM to identify the risk groups. When data is an unsupervised one and based on experts' opinions and there is uncertainty in the concepts, COBFCM is the best option, and when data is an unsupervised one and there is indeterminacy in the concepts, COBNCM is a preferred method. The comparison of these methods clarifies this fundamental point and the relationship of to-be-determined and not-to-be-determined between the concepts, including the effect on results in casual models in psychological research.

Based on experts' opinions (five child and developmental psychologists) and the corresponding literature, we determined eight cognitive concepts related to ADHD:

[1] C_1: Mother's harmful substance use;
[2] C_2: Mother's low physical self-efficacy;
[3] C_3: Mother's bad nutrition;
[4] C_4: Mother's depression;
[5] C_5: Family conflict;
[6] C_6: Father's addiction;
[7] C_7: Child's emotional problems;
[8] C_8: Child's hyper activity.

9 Combined Overlap Block NCM

We divide these concepts in to 3 equal length classes; each class has just four concepts in the following manner:

$S_1=\{C_1,C_2,C_3,C_4\}$, $S_2=\{C_2,C_4,C_5,C_6\}$ and $S_3=\{C_4,C_5,C_7,C_8\}$

These three classes are offered to experts in order to determine relationships and the strength. In addition, we asked them to delineate edges that have indeterminate effects by dotted lines in the figures and by I in the corresponding matrices. The directed graph and relation matrix for the S_1, S_2 and S_3 given by the expert is as follow:

Figure 1

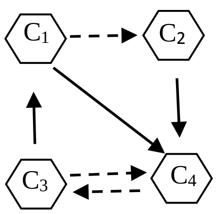


Figure 2

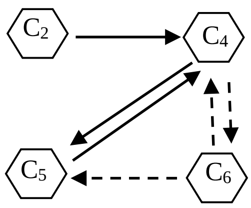
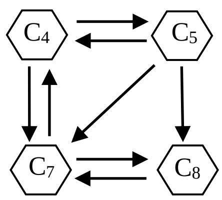


Figure 3



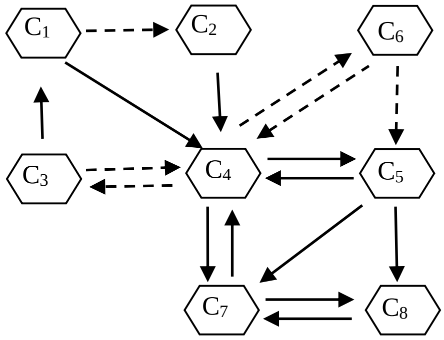
$$\begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ C_4 \end{array} \begin{array}{c} \begin{array}{cccc} C_1 & C_2 & C_3 & C_4 \end{array} \\ \begin{pmatrix} 0 & \mathbf{I} & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & \mathbf{I} \\ 0 & 0 & \mathbf{I} & 0 \end{pmatrix} \end{array}$$

$$\begin{array}{c} \\ C_2 \\ C_4 \\ C_5 \\ C_6 \end{array} \begin{array}{c} \begin{array}{cccc} C_2 & C_4 & C_5 & C_6 \end{array} \\ \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & \mathbf{I} \\ 0 & 1 & 0 & 0 \\ 0 & \mathbf{I} & \mathbf{I} & 0 \end{pmatrix} \end{array}$$

$$\begin{array}{c} \\ C_4 \\ C_5 \\ C_7 \\ C_8 \end{array} \begin{array}{c} \begin{array}{cccc} C_4 & C_5 & C_7 & C_8 \end{array} \\ \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \end{array}$$

The combined overlap block connection matrix of NCM is given by E (N).

$$E(N) = \begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \end{array} \begin{array}{c} C_1\ C_2\ C_3\ C_4\ C_5\ C_6\ C_7\ C_8 \\ \begin{pmatrix} 0, I, 0, 1, 0, 0, 0, 0 \\ 0, 0, 0, 2, 0, 0, 0, 0 \\ 1, 0, 0, I, 0, 0, 0, 0 \\ 0, 0, I, 0, 1, I, 1, 0 \\ 0, 0, 0, 2, 0, 0, 1, 1 \\ 0, 0, 0, I, I, 0, 0, 0 \\ 0, 0, 0, 1, 0, 0, 0, 1 \\ 0, 0, 0, 0, 0, 0, 1, 0 \end{pmatrix} \end{array}$$



The combined overlap block connection matrix of FCM is given by E.

$$E \quad \begin{array}{c} \\ C_1 \\ C_2 \\ C_3 \\ C_4 \\ C_5 \\ C_6 \\ C_7 \\ C_8 \end{array} \begin{array}{c} C_1\ C_2\ C_3\ C_4\ C_5\ C_6\ C_7\ C_8 \\ \begin{pmatrix} 0, 0, 0, 1, 0, 0, 0, 0 \\ 0, 0, 0, 2, 0, 0, 0, 0 \\ 1, 0, 0, 0, 0, 0, 0, 0 \\ 0, 0, 0, 0, 1, 0, 1, 0 \\ 0, 0, 0, 2, 0, 0, 1, 1 \\ 0, 0, 0, 0, 0, 0, 0, 0 \\ 0, 0, 0, 1, 0, 0, 0, 1 \\ 0, 0, 0, 0, 0, 0, 1, 0 \end{pmatrix} \end{array}$$

10 Hidden Patterns

Now, using the combined matrix E(N), we can determine any hidden patterns embedded in the matrix. Suppose the concept C_4 (Mother's depression) is in the ON state. So, initial vector for studying the effects of these concepts on the dynamical system E is A= [0 0 0 1 0 0 0 0]. Let A state vector depicting the ON state of Mother's depression passing the state vector A in to the dynamical system E (N):

A=[0 0 0 1 0 0 0 0]

AE(N) =[0, 0, I, 0, 1, I, 1, 0] $\rightarrow$ [0 0 I 1 1 I 1 0]=A_1

A_1E(N)=[I, 0, I, 2*I^2 + 3, I^2 + 1, I, 2, 2] $\rightarrow$ [I 0 I 1 1 I 1 1]=A_2

A_2E(N)=[I, I^2, I, 2*I^2 + I + 3, I^2 + 1, I, 3, 2] $\rightarrow$
[I I I 1 1 I 1 1]=A_3

A_3E(N) =[I, I^2, I, 2*I^2 + 3*I + 3, I^2 + 1, I, 3, 2] $\rightarrow$
[I I I 1 1 I 1 1]=A_4=A_3.

Since A_4=A_3 (we have reached the fixed point of the dynamical system). A_3 is determined to be a hidden pattern. Now again using the COBFCM we can determine hidden patterns embedded in the matrix (E), such as COBNCM, here initial vector considered A= [0 0 0 1 0 0 0 0], i.e. we suppose the Mother's depression is high. The results obtained are as following:

$$AE=[0\ 0\ 0\ 1\ 1\ 0\ 1\ 0] = A_1$$

$$A_1E=[\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ 1] = A_2$$

$$A_2E=[\ 0\ 0\ 0\ 1\ 1\ 0\ 1\ 1] = A_3 = A_2$$

By A_3=A_2 we have reached the fixed point of the dynamical system. A_2 is determined to be a hidden pattern using the COBFCM.

11 Weighted Method

We can use the weighted method to clarify the results, when there is a tie between the concepts inputs. Suppose the resultant vector be A= [10 0 1 1 1 0], i.e., the half of the concepts suggest that the given problem exists, but other three suggest that the problem is not justified on the basis of available concept. In this case, we can adopts a simple weighted approach where in each of the concepts can be assigned weights based on experts' opinions. For example, C_1=20%, C_2=10%, C_3=10%, C_4=60%, C_5=25%, C_6=30%, C_7=20%. The ON - OFF state for each Concept in A vector leads to a weighted average score of the corresponding concepts. Suppose the initial vector is A= [0 0 0 0 0 1 0]; based on the resultant vector and the experts' weights for the concepts, we can find a weighted average score. In this case, Geometric mean is an accurate and appropriate measure for calculating average score, because the data are expressed in percentage terms. The resulting of the example equals to 30% (which tends towards absence of the problem (since this is <50%, the point of no difference).

The results based on the COBNCM indicated when a mother suffering from depression, i.e. the C_4 is in the ON state; there will be family conflict, child's emotional problems, Child's hyper activity and also there *may be* Mother's harmful substance use, Mother's low physical self-efficacy, Mother's bad nutrition and Father's addiction. Based on the results of this study using the COBFCM, when a mother is depressed, there will be child's hyperactivity, emotional problems, and family conflict. Although, based on the results of the two models mother's depression being the main cause of ADHD, based on the COBFCM we cannot determine the occurrence of possibilities of some corresponding concepts in developing ADHD.

12 Discussion

It is important to note that in COBFCM e_{ij} measures only absence or presence of influence of the node C_i on C_j, but untill now any researcher has not contemplated the indeterminacy of any relation between two nodes C_i and C_j. When researchers deal with unsupervised data, there are situations when no relation can be determined between two nodes (Vasantha Kandasamy & Smarandache, 2005). The presence of I in any coordinate implies the expert cannot tell the presence of that node, i.e. *on state* after passing through N (E), nor can we say the absence of the node, i.e. *off state* - the effect on the node after passing through the dynamical system is indeterminate, so it is represented by *I*. Thus, only in case of NCM we can identify that the effect of any node on other nodes can also be indeterminate. Such possibilities and analysis is totally absent in the case of FCM. Therefore, the COBFCM only indicates that what happens for C_j when C_i is in an ON state, but it cannot indicate the effects of the concepts on each other in neutral states. In other words, by using COBFCM, some of the latent layers of the relationships between the concepts are not discovered. Thus, only the COBNCM helps in such conditions.

The core of psychology and education is theoretical. Theories themselves consist of constructs, concepts and variables, which are expressed by linguistic propositions - to describe, explain and predict the phenomena. For these characteristics of theory, Smarandache (2001) believes that no theory is exempted from paradoxes, because of language imprecision, metaphoric expression, various levels or meta-levels of understanding/interpretation, which might overlap. These propositions do not mean a fixed-valued components structure and it is dynamic, i.e. the truth value of a proposition may change from one place to another place and from one time to another time, and it changes with respect to the observer (subjectivity). For example, the proposition "Family conflict leads to divorce" does not mean a fixed-valued components structure; this proposition may be stated 35% true, 45% indeterminate, and 45% false at time t_1; but at time t_2 may change at 55% true, 49% indeterminate, and 32% false (according with new evidences, sources, etc.); or the proposition " Jane is depressed " can be (.76,.56, .30) according to her psychologist, but (.85, .25, .15) according to herself, or (.50, .24, .35) according to her friend, etc. Therefore, considering the indeterminacies in investigating the causal relationships in psychological and educational research is important, and it is closer to the human mind reasoning. A good method in this condition is using the NCM, as seen before, using the FCM leads to ignoring indeterminacies (by converting the e_{ij}=I to e_{ij}=0), and this ignoring itself leads to the covering the latent effects of the concepts of the causal models. It is recommended that in the conditions that indeterminacies are important, researchers use the NCM method.

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An Example of Guiding Scientific Research with Philosophical Principles Based on Uniqueness of Truth and Neutrosophy Deriving Newton's Second Law and the like

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Abstract

According to the principle of the uniqueness of truth, there should be only one truth, namely law of conservation of energy, in the area of Newton Mechanics. Through the example of free falling body, according to the neutrosophic principle considering neutralities (the small ball is falling to the middle positions), this paper derives the original Newton's second law and the original law of gravity respectively by using the law of conservation of energy.

Keywords

Uniqueness of truth, Neutrosophy, Newton Mechanics, Law of conservation of energy, Newton's second law, Law of gravity.

1 Introduction

Philosophers often say that, there should be a unique truth. According to this principle, and taking into account that the law of conservation of energy is the most important law in the natural sciences, therefore in the area of Newtonian mechanics, the law of conservation of energy should be the unique truth.

The law of conservation of energy states that the total energy of an isolated system remains constant.

As well-known, in Newton's classical mechanics, there were four main laws: the three laws of Newton and the law of gravity. If the law of conservation of energy is choosing as the unique truth, then in principle, all the Newton's four laws can be derived according to the law of conservation of energy; after studying carefully we find that this conclusion may be correct. According to the neutrosophic principle considering neutralities (the small ball is falling to the middle positions), this paper discusses how to derive the original Newton's second law and the original law of gravity respectively by using the law of conservation of energy.

2 Basic Contents of Neutrosophy

Neutrosophy is proposed by Prof. Florentin Smarandache in 1995.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

This theory considers every notion or idea <A> together with its opposite or negation <Anti-A> and the spectrum of "neutralities" <Neut-A> (i.e. notions or ideas located between the two extremes, supporting neither <A> nor <Anti-A>). The <Neut-A> and <Anti-A> ideas together are referred to as <Non-A>.

Neutrosophy is the base of neutrosophic logic, neutrosophic set, neutrosophic probability and statistics used in engineering applications (especially for software and information fusion), medicine, military, cybernetics, and physics.

Neutrosophic Logic is a general framework for unification of many existing logics, such as fuzzy logic (especially intuitionistic fuzzy logic), paraconsistent logic, intuitionistic logic, etc.

The main idea of NL is to characterize each logical statement in a 3D Neutrosophic Space, where each dimension of the space represents respectively the truth (T), the falsehood (F), and the indeterminacy (I) of the statement under consideration, where T, I, F are standard or non-standard real subsets of]-0, 1+[without necessarily connection between them.

From the basic contents of Neutrosophy we can see that, the neutralities are very important indeed.

More information about Neutrosophy can be found in references [1, 2].

3 Deriving the original Newton's second law by using the law of conservation of energy

In this section, only Newton's second law can be derived, but we have to apply the law of gravity at the same time, so we present the general forms of Newton's second law and the law of gravity with undetermined constants firstly.

Assuming that for the law of gravity, the related exponent is unknown, and we only know the form of this formula is as it follows:

$$F = -\frac{GMm}{r^D}, \tag{1}$$

where: D is an undetermined constant, in the next section we will derive that its value is equal to 2.

Similarly, assuming that for Newton's second law, the related exponent is also unknown, and we only know the form of this formula is as follows

$$F = ma^{D'}, \tag{2}$$

where: D' is an undetermined constant, in this section we will derive that its value is equal to 1.

As shown in Figure 1, supposing that circle O' denotes the Earth, M denotes its mass; m denotes the mass of the small ball (treated as a mass point P), A O' is a plumb line, and coordinate y is parallel to AO'. The length of AC is equal to H, and O'C equals the radius R of the Earth.

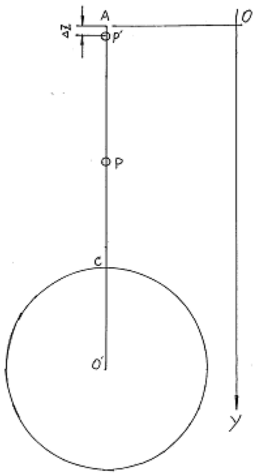

Figure 1 A small ball free falls in the gravitational field of the Earth.

We also assume that it does not take into account the motion of the Earth and only considering the free falling of the small ball in the gravitational field of the Earth (from point A to point C).

For this example, the value of v_P^2 which is the square of the velocity for the small ball located at point P (somewhere in the Middle, namely the small ball is falling to the middle position) will be investigated. To distinguish the quantities calculated by different methods, we denote the value given by the law of gravity and Newton's second law as v_P^2, while v'^2_P denotes the value given by the law of conservation of energy.

Now we calculate the related quantities according to the law of conservation of energy.

From Eq.(1), the potential energy of the small ball located at point P is as follows

$$V = -\frac{GMm}{(D-1)r_{O'P}^{D-1}} . \tag{3}$$

According to the law of conservation of energy, we can get

$$-\frac{GMm}{(D-1)r_{O'A}^{D-1}} = \frac{1}{2}mv'^2_P - \frac{GMm}{(D-1)r_{O'P}^{D-1}} , \tag{4}$$

and therefore

$$v'^2_P = \frac{2GM}{D-1}[\frac{1}{r_{O'P}^{D-1}} - \frac{1}{(R+H)^{D-1}}] . \tag{5}$$

Now we calculate the related quantities according to the law of gravity and Newton's second law.

For the small ball located at any point P, we have

$$dv/dt = a . \tag{6}$$

We also have

$$dt = \frac{dy}{v} ,$$

therefore

$$vdv = ady . \tag{7}$$

According to Eq. (1), along the plumb direction, the force acted on the small ball is as follows

$$F_a = \frac{GMm}{r_{O'P}^D}. \tag{8}$$

From Eq. (2), it gives

$$a = (\frac{F_a}{m})^{1/D'} = (\frac{GM}{r_{O'P}^D})^{1/D'}. \tag{9}$$

According to Eq.(7), we have

$$vdv = \{\frac{GM}{(R+H-y)^D}\}^{1/D'} dy. \tag{10}$$

For the two sides of this expression, we run the integral operation from A to P; it gives:

$$v_P^2 = 2(GM)^{1/D'} \int_0^{y_p} (R+H-y)^{-D/D'} dy$$

$$v_P^2 = 2(GM)^{1/D'} \{-\frac{1}{1-D/D'}[(R+H-y)^{1-D/D'}]\Big|_0^{y_P}\}$$

$$v_P^2 = \frac{2(GM)^{1/D'}}{(D/D')-1}[\frac{1}{r_{O'P}^{(D/D')-1}} - \frac{1}{(R+H)^{(D/D')-1}}].$$

Let $v_P^2 = v_P^{'2}$, then we should have: $1 = 1/D'$, and $D-1 = (D/D')-1$; these two equations all give: $D' = 1$, this means that for free falling problem, by using the law of conservation of energy, we strictly derive the original Newton's second law $F = ma$.

Here, although the original law of gravity cannot be derived (the value of D may be any constant, certainly including the case that D=2), we already prove that the original law of gravity is not contradicted to the law of conservation of energy, or the original law of gravity is tenable accurately.

4 Deriving the original law of gravity by using the law of conservation of energy

In order to really derive the original law of gravity for the example of free falling problem, we should consider the case that a small ball free falls from point A to point P' (point P' is also shown in Figure1, it is located at the middle position closed to point A) through a very short distance ΔZ (the two endpoints of the interval ΔZ are point A and point P').

As deriving the original Newton's second law, we already reach

$$v'^2_{P'} = \frac{2GM}{D-1}[\frac{1}{(R+H-\Delta Z)^{D-1}} - \frac{1}{(R+H)^{D-1}}],$$

where: $R+H-\Delta Z = r_{O'P'}$.

For the reason that the distance of ΔZ is very short, and in this interval the gravity can be considered as a linear function, therefore the work W of gravity in this interval can be written as follows

$$W = F_{av}\Delta Z = \frac{GMm}{(R+H-\frac{1}{2}\Delta Z)^D}\Delta Z,$$

where F_{av} is the average value of gravity in this interval ΔZ, namely the value of gravity for the midpoint of interval ΔZ.

Omitting the second order term of ΔZ ($\frac{1}{4}(\Delta Z)^2$), it gives

$$W = \frac{GMm\Delta Z}{(R^2+H^2+2RH-R\Delta Z-H\Delta Z)^{D/2}}.$$

As the small ball free falls from point A to point P', its kinetic energy is as it follows:

$$\frac{1}{2}mv'^2_{P'} = \frac{GMm}{D-1}[\frac{(R+H)^{D-1}-(R+H-\Delta Z)^{D-1}}{(R^2+H^2+2RH-R\Delta Z-H\Delta Z)^{D-1}}].$$

According to the law of conservation of energy, we have

$$W = \frac{1}{2}mv'^2_{P'}.$$

Substituting the related quantities into the above expression, it gives

$$\frac{GMm}{D-1}[\frac{(R+H)^{D-1}-(R+H-\Delta Z)^{D-1}}{(R^2+H^2+2RH-R\Delta Z-H\Delta Z)^{D-1}}]$$

$$= \frac{GMm\Delta Z}{(R^2+H^2+2RH-R\Delta Z-H\Delta Z)^{D/2}}.$$

To compare the related terms, we can reach the following three equations

$$D-1=1$$

$$D/2=D-1$$

$$\Delta Z=(R+H)^{D-1}-(R+H-\Delta Z)^{D-1}.$$

All of these three equations will give the following result

$$D=2.$$

Thus, we already derive the original law of gravity by using the law of conservation of energy.

5 Conclusion and Further Topic

According to the above results it can be said that, for the free falling problem, we do not rely on any experiment, only apply law of conservation of energy to derive the original Newton's second law and the original law of gravity.

In references [3, 4], based on the equation given by Prof. Hu Ning according to general relativity and Binet's formula, we get the following improved Newton's formula of universal gravitation

$$F=-\frac{GMm}{r^2}-\frac{3G^2M^2mp}{c^2r^4}, \tag{11}$$

where: G is the gravitational constant, M and m are the masses of the two objects, r is the distance between the two objects, c is the speed of light, p is the half normal chord for the object m moving around the object M along with a curve, and the value of p is given by: $p = a(1-e^2)$ (for ellipse), $p = a(e^2-1)$ (for hyperbola), $p = y^2/2x$ (for parabola).

This improved Newton's universal gravitation formula can give the same results as given by general relativity for the problem of planetary advance of perihelion and the problem of gravitational defection of a photon orbit around the Sun.

For the problem of planetary advance of perihelion, the improved Newton's universal gravitation formula reads

$$F=-\frac{GMm}{r^2}-\frac{3G^2M^2ma(1-e^2)}{c^2r^4}. \tag{12}$$

For the problem of gravitational defection of a photon orbit around the Sun, the improved Newton's universal gravitation formula reads

$$F = -\frac{GMm}{r^2} - \frac{1.5GMmr_0^2}{r^4}, \tag{13}$$

where r_0 is the shortest distance between the light and the Sun, if the light and the Sun is tangent, it is equal to the radius of the Sun.

The funny thing is that, for this problem, the maximum gravitational force given by the improved Newton's universal gravitation formula is 2.5 times of that given by the original Newton's law of gravity.

The further topic is how to apply the law of conservation of energy to derive Eqs.(11), (12), (13), and the like.

In this regard, philosophical principles (including principles of Neutrosophy and the like), will play a major role.

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On Neutrosophic Ideals of Neutrosophic BCI-Algebras

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Abstract

The objective of this paper is to introduce and study neutrosophic ideals of neutrosophic BCI-algebras. Elementary properties of neutrosophic ideals of neutrosophic BCI-algebras are presented.

Keywords

BCI/BCK-algebra, Neutrosophic set, Neutrosophic BCI/BCK- algebra, Neutrosophic ideal.

1 Introduction

BCI/BCK-algebras are generalizations of the concepts of set-theoretic difference and propositional calculi. These two classes of logical algebras were introduced by Imai and Iséki [8, 9] in 1966. It is well known that the class of MV-algebras introduced by Chang in [4] is a proper subclass of the class of BCK- algebras which in turn is a proper subclass of the class of BCI-algebras. Since the introduction of BCI/BCK-algebras, a great deal of literature has been produced, for example see [5, 9, 10, 11, 14]. For the general develop- ment of BCI/BCK-algebras, the ideal theory plays an important role. Hence much research emphasis has been on the ideal theory of BCI/BCK-algebras, see [3, 6, 7, 15].

By a BCI-algebra we mean an algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following axioms, for all $x, y, z \in X$,

(1) $((x*y)*(x*z))*(z*y)=0$, (2) $(x*(x*y))*y=0$,

(3) $x*x=0$,

(4) $x*y=0$ and $y*x=0$ imply $x=y$.

Example 1.

(1) Every abelian group is a BCI-algebra, with group substraction and 0 the group identity.

(2) Consider $X = \{0, a, b\}$. Then, X with the following Cayley table is a BCI-algebra.

$*$	0	a	b
0	0	0	b
a	a	0	b
b	b	b	0

We can define a partial ordering $\leq$ by $x \leq y$ if and only if $x*y=0$.

If a BCI-algebra X satisfies $0*x=0$ for all $x \in X$, then we say that X is a BCK-algebra. Any BCK-algebra X satisfies the following axioms for all $x, y, z \in X$,

(1) $(x*y)*z=(x*z)*y$,

(2) $((x*z)*(y*z))*(x*y)=0$,

(3) $x*0=x$,

(4) $x*y=0 \Rightarrow (x*z)*(y*z)=0, (z*y)*(z*x)=0$.

Example 2.

(1) The subsets of a set form a BCK-algebra, where $A*B$ is the difference $A \setminus B$.

(2) A Boolean algebra is a BCK-algebra, if $A*B$ is defined to be $A \wedge \neg B$ (A does not imply B).

A subset A of a BCI/BCK-algebra $(X, *, 0)$ is called a subalgebra of X if $x*y \in A$ for all $x, y * A$.

Let $(X, *, 0)$ be a BCI-algebra. A subset A of X is called an ideal of X if the following conditions hold:

(1) $0 \in A$.

(2) For all $x, y \in X$, $x*y \in A$ and $y*A$ implies that $x \in A$.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. Neutrosophic set and neutrosophic logic were introduced in 1995 by

Smarandache as generalizations of fuzzy set and respectively intuitionistic fuzzy logic. In neutrosophic logic, each proposition has a degree of truth (T), a degree of indeterminancy (I), and a degree of falsity (F), where T, I, F are standard or non-standard subsets of $]^{-}0, 1^{+}[$, see [16, 17, 18]. Neutrosophic logic has wide applications in science, engineering, Information Technology, law, politics, economics, finance, econometrics, operations research, optimization theory, game theory and simulation etc.

The notion of neutrosophic algebraic structures was introduced by Kandasamy and Smarandache in 2006, see [12, 13]. Since then, several researchers have studied the concepts and a great deal of literature has been produced. For example, Agboola et al. in [1] continued the study of some types of neutrosophic algebraic structures. Agboola and Davvaz introduced the concept of neutrosophic BCI/BCK-algebras in [2].

Let X be a nonempty set and let I be an indeterminate.

The set $X(I) =< X, I >= \{(x, yI) : x, y \in X\}$ is called a neutrosophic set generated by X and I. If + and . are ordinary addition and multiplication, I has the following properties:

1) $I + I + \cdots + I = nI$.
2) $I + (-I) = 0$.
3) $I.I. \cdots .I = I^n = I$ for all positive integer n.
4) $0.I = 0$.
5) I^{-1} is undefined and therefore does not exist.

If $*: X(I) \times X(I) \to X(I)$ is a binary operation defined on $X(I)$, then the couple $(X(I), *)$ is called a neutrosophic algebraic structure and it is named according the axioms satisfied by $*$. If $(X(I), *)$ and $(Y(I), *')$ are two neutrosophic algebraic structures, the mapping $\varphi : (X(I), *) \to (Y(I), *')$ is called a neutrosophic homomorphism if the following conditions hold:

(1) $\varphi((w, xI) * (y, zI)) = \varphi((w, xI)) *' \varphi((y, zI))$.

(2) $\varphi(I) = I \ \forall (w, xI), (y, zI) \in X(I)$.

We recall the definition of a neutrosophic group.

Definition 1.1.

Let $(G, *)$ be a group. Then, the neutrosophic group is generated by I and G under $*$ defined by $(< G, I >, *)$. The present paper is concerned with the introduction of the concept of neutrosophic of ideals of neutrosophic BCI-algebras. Some elementary properties of neutrosophic ideals of neutrosophic BCI-algebras are presented. First, we recall some basic concepts from [2].

Definition 1.2.

Let $(X, *, 0)$ be any BCI/BCK-algebra and let $X(I) = < X, I >$ be a set generated by X and I. The triple $(X(I), *, (0, 0))$ is called a neutrosophic BCI/BCK-algebra. If (a, bI) and (c, dI) are any two elements of $X(I)$ with $a, b, c, d \in X$, we define

$$(a, bI) * (c, dI) = (a * c, (a * d \wedge b * c \wedge b * d)I) \tag{1}$$

An element $x \in X$ is represented by $(x, 0) \in X(I)$ and $(0, 0)$ represents the constant element in $X(I)$. For all $(x, 0), (y, 0) \in X$, we define

$$(x, 0) * (y, 0) = (x * y, 0) = (x \wedge \neg y, 0), \tag{2}$$

where $\neg y$ is the negation of y in X.

Definition 1.3.

Let $(X, *, 0)$ be any BCI/BCK-algebra and let $X(I) =< X, I >$ be a set generated by X and I. The triple $(X(I), *, (0,))$ is called a neutrosophic BCI/BCK-algebra. If (a, bI) and (c, dI) are any two elements of $X(I)$ with $a, b, c, d \in X$, we define

$$(a, bI) * (c, dI) = (a * c, (a * d \wedge b * c \wedge b * d)I) \tag{3}$$

An element $x \in X$ is represented by $(x, 0) \in X(I)$ and $(0, 0)$ represents the constant element in $X(I)$. For all $(x, 0), (y, 0) \in X$, we define

$$(x, 0) * (y, 0) = (x * y, 0) = (x \wedge \neg y, 0) \tag{4}$$

where $\neg y$ is the negation of y in X.

Example 3.

Let $(X(I), +)$ be any commutative neutrosophic group.

For all $(a, bI), (c, dI) \in X(I)$ define

$$(a, bI) * (c, dI) = (a, bI) - (c, dI) = (a - c, (b - d)I). \tag{5}$$

Then, $(X(I), *, (0,0))$ is a neutrosophic BCI-algebra.

Theorem 1.4.

(1) Every neutrosophic BCK-algebra $(X(I), *, (0,0))$ is a neutrosophic BCI-algebra.

(2) Every neutrosophic BCK-algebra $(X(I), *, (0,0))$ is a BCI-algebra and not the converse.

(3) Let $(X(I), *, (0,0))$ be a neutrosophic BCK-algebra. Then, $(a, bI) * (0,0) = (a, bI)$ if and only if $a = b$.

Definition 1.5.

Let $(X(I), *, (0,0))$ be a neutrosophic BCI/BCK-algebra. A non-empty subset $A(I)$ is called a neutrosophic subalgebra of $X(I)$ if the following conditions hold:

1) $(0,0) \in A(I)$.

2) $(a, bI) * (c, dI) \in A(I)$ for all $(a, bI), (c, dI) \in A(I)$.

3) $A(I)$ contains a proper subset which is a BCI/BCK-algebra.

If $A(I)$ does not contain a proper subset which is a BCI/BCK-algebra, then $A(I)$ is called a pseudo neutrosophic subalgebra of $X(I)$.

2 Main Results

Theorem 2.1.

Let $(X(I), *, (0,0))$ be a neutrosophic BCI-algebra and let $X_\omega(I)$ be a subset of $X(I)$ defined by

$$X_\omega(I) = \{(x, xI) : x \in X\}. \tag{6}$$

Then, $X_\omega(I)$ is a neutrosophic subalgebra of $X(I)$.

Proof.

Obviously, $(0,0) \in X_\omega(I)$. Let $(x, xI), (y, yI) \in X_\omega(I)$ be arbitrary. Then, we have

$$(x, xI) * (y, yI) = (x * y, (x * y)I) = (x \wedge \neg y, (x \wedge \neg y)I) \in X_\omega(I).$$

Remark 1.

Since $(X_\omega(I), *, (0,0))$ is a neutrosophic subalgebra, then $X_\omega(I)$ is a neutrosophic BCI-algebra in its own right.

Example 4.

Let $X_\omega(I) = \{(0,0), (a, aI), (b, bI), (c, cI)\}$ be a set and let $*$ be a binary operation defined on $X_\omega(I)$ as shown in the Cayley table below:

$*$	$(0,0)$	(a,aI)	(b,bI)	(c,cI)
$(0,0)$	$(0,0)$	$(0,0)$	(c,cI)	(b,bI)
(a,aI)	(a,aI)	$(0,0)$	(c,cI)	(b,bI)
(b,bI)	(b,bI)	(b,bI)	$(0,0)$	(c,cI)
(c,cI)	(c,cI)	(c,cI)	(b,bI)	$(0,0I)$

Then, $(X_\omega(I), *, (0,0))$ is a neutrosophic BCI-algebra.

Definition 2.2.

Let $(X(I), *, (0,0))$ be a neutrosophic BCI-algebra. A subset $A(I)$ is called a neutrosophic ideal of $X(I)$ if the following conditions hold:

(1) $(0,0) \in A(I)$.

(2) For all $(a, bI), (c, dI) \in X(I)$, $(a, bI)*(c, dI) \in A(I)$ and $(c, dI) \in A(I)$ implies that $(a, bI) \in A(I)$.

Definition 2.3.

Let $(X(I), *, (0,0))$ be a neutrosophic BCI-algebra and let $A(I)$ be a neutrosophic ideal of $X(I)$.

1) $A(I)$ is called a closed neutrosophic ideal of $X(I)$ if $A(I)$ is also a neutrosophic subalgebra of $X(I)$.

2) $A(I)$ is called a closed pseudo neutrosophic ideal of $X(I)$ if $A(I)$ is also a pseudo neutrosophic subalgebra of $X(I)$.

Lemma 2.4.

Let A(I) be a closed neutrosophic ideal of neutrosophic BCI-algebra $(X(I), *, (0, 0))$. Then,

1) $A(I) * A(I) = A(I)$.

2) $(a, bI) * A(I) = A(I)$ if and only if $(a, bI) \in A(I)$.

Definition 2.5.

Let $A_\omega(I)$ be a non-empty subset of $X_\omega(I)$.

1) $A_\omega(I)$ is called a neutrosophic α-ideal of $X_\omega(I)$ if the following conditions hold:

a. $(0,0) \in A_\omega(I)$.

b. For all $(x, xI), (y, yI), (z, zI) \in X_\omega(I)$, $((x, xI) * (z, zI)) * ((y, yI) * (z, zI)) \in A_\omega(I)$ and $(y, yI) \in A_\omega(I)$ imply that $(x, xI) \in A_\omega(I)$.

2) $A_\omega(I)$ is called a neutrosophic β-ideal of $X_\omega(I)$ if the following conditions hold:

a. $(0, 0) \in A_\omega(I)$.

b. For all $(x, xI), (y, yI), (z, zI) \in X_\omega(I)$, $(x, xI) * ((y, yI) * (z, zI)) \in A_\omega(I)$ and $(y, yI) \in A_\omega(I)$ imply that $(x, xI) * (z, zI) \in A_\omega(I)$.

Theorem 2.6.

Every neutrosophic α-ideal of $X_\omega(I)$ is a neutrosophic ideal of $X_\omega(I)$.

Proof.

Putting $(z, zI) = (0, 0)$ in Definition 2.5 (1-b), the result follows.

Theorem 2.7.

Every neutrosophic β-ideal of $X_\omega(I)$ is a neutrosophic ideal of $X_\omega(I)$.

Proof.

Follows easily by putting $(z, zI) = (0, 0)$ in Definition 2.5 (2-b).

Theorem 2.8.

Let $A_\omega(I)$ and $B_\omega(I)$ be neutrosophic α-ideal and neutrosophic β-ideal of $X_\omega(I)$, respectively. Then,

$$A_\omega(I) * B_\omega(I) = \{(a, aI) * (b, bI) : (a, aI) \in A_\omega(I), (b, bI) \in B_\omega(I)\} \quad (7)$$

is a neutrosophic ideal of $X_\omega(I)$.

Proof.

Follows easily from Theorems 2.6 and 2.7.

Theorem 2.9.

Let $(X(I), *, (0, 0))$ be a neutrosophic BCI-algebra and let $A(I)$ be a neutrosophic ideal of $X(I)$. For all $(a, bI), (c, dI) \in X(I)$, let τ be a relation defined on $X(I)$ by

$$(a, bI)\tau(c, dI) \iff (a, bI) * (c, dI), (c, dI) * (a, bI) \in A(I).$$

Then, τ is a congruence relation on $X(I)$.

Proof.

It is clear that τ is an equivalence relation on $X(I)$. For τ to be a congruence relation on $X(I)$, we must show that for all $(x, yI) \neq (0, 0)$ in $X(I)$, $(a, bI)\tau$

(c, dI) implies that $(a, bI) * (x, yI)\tau(c, dI) * (x, yI)$ and $(x, yI) * (a, bI)\tau(x, yI) * (c, dI)$. To this end, if $(a, bI)\tau(c, dI)$, then $(a, bI) * (c, dI), (c, dI) * (a, bI) \in A(I)$ that is $(a * c, (a * d \wedge b * c \wedge b * d)I), (c * a, (c * b \wedge d * a \wedge d * b)I) \in A(I)$ so that $(a \wedge \neg c, (a \wedge \neg d \wedge b \neg c)I), (c \wedge \neg a, (c \wedge \neg b \wedge d \wedge \neg a)I) \in A(I)$ and thus $a \wedge \neg c, a \wedge \neg d \wedge b \neg c, c \wedge \neg a, c \wedge \neg b \wedge d \wedge \neg a \in A$.

Now, let $(p, qI) = (a, bI) * (x, yI)$ and $(u, vI) = (c, dI) * (x, yI)$.

Then,

$$(p, qI) = (a * x, (a * y \wedge b * x \wedge b * y)I)$$

$$= (a \wedge \neg x, (a \wedge \neg y \wedge b \wedge \neg x)I).$$

$$(u, vI) = (c * x, (c * y \wedge d * x \wedge d * y)I)$$

$$= (c \wedge \neg x, (c \wedge \neg y \wedge d \wedge \neg x)I).$$

Now, we have

$$(p, qI) * (u, vI) = (p * u, (p * v \wedge q * u \wedge q * v)I)$$

$$= (p \wedge \neg u, (p \wedge \neg v \wedge q \wedge \neg u)I) \equiv (m, kI),$$

where

$$m = p \wedge \neg u = a \wedge \neg x \wedge (\neg c \vee x) = a \wedge \neg x \wedge \neg c \in A. \ k = p \wedge \neg v \wedge q \wedge \neg u$$

$$= a \wedge \neg x \wedge (\neg c \vee x) \wedge a \neg y \wedge b \wedge \neg x \wedge (\neg c \vee y \vee \neg d \vee x)$$

$$= a \wedge \neg x \wedge \neg c \wedge \neg y \wedge b \wedge (\neg c \vee y \vee \neg d \vee x)$$

$$= (a \wedge \neg c \wedge b \wedge \neg x \wedge \neg y) \vee (a \wedge \neg d \wedge b \wedge \neg c \wedge \neg x \wedge \neg y) \in A.$$

These show that $(m, kI) \in A(I)$ that is $((a, bI) * (x, yI)) * ((c, dI) * (x, yI)) \in A(I)$. Similarly, it can be shown that $(((x, yI) * (a, bI)) * ((x, yI) * (c, dI)) \in A(I)$. Thus, $(a, bI) * (x, yI)\tau(c, dI) * (x, yI)$ and $(x, yI) * (a, bI)\tau(x, yI) * (c, dI)$. Hence, τ is a congruence relation on $X(I)$.

For all $(a, bI) \in X(I)$, let $[(a, bI)]$ denote the congruence class containing (a, bI) and let $X(I)/A(I)$ denote the set of all congruence classes. For all $[(a, bI)], [(c, dI)] \in X(I)/A(I)$, we define

$$[(a, bI)] = (a, bI) * A(I), \tag{8}$$

$$[(0, 0)] = \{(0, 0) * (x, yI) : (x, yI) \in A(I)\}. \tag{9}$$

Theorem 2.10.

Let $A(I)$ be a closed neutrosophic ideal of neutrosophic BCI-algebra $(X(I), *, (0, 0))$. Then, $(X(I)/A(I), *, [(0, 0)])$ is a neutrosophic BCI-algebra.

Definition 2.11.

Let $(X(I), *, (0, 0))$ and $(X^{'}(I), \circ, (0^{'}, 0^{'}))$ be two neutrosophic BCI-algebras. A mapping $\varphi : X(I) \to X^{'}(I)$ is called a neutrosophic homomorphism if the following conditions hold:

(1) $\varphi((a,bI)*(c,dI))=\varphi((a,bI))\circ\varphi((c,dI)), \forall (a,bI),(c,dI) \in X(I)$.

(2) $\varphi((0,I)) = (0,I)$.

If in addition:

1) φ is injective, then φ is called a neutrosophic monomorphism.

2) φ is surjective, then φ is called a neutrosophic epimorphism.

3) φ is a bijection, then φ is called a neutrosophic isomorphism. A bijective neutrosophic homomorphism from $X(I)$ onto $X(I)$ is called a neutrosophic automorphism.

Definition 2.12.

Let $\varphi : X(I) \to Y(I)$ be a neutrosophic homomorphism of neutrosophic BCI-algebras.

(1) $Ker\varphi = \{(a,bI) \in X(I) : \varphi((a,bI)) = (0,0)\}$.

(2) $Im\varphi = \{\varphi((a,bI)) \in Y(I) : (a,bI) \in X(I)\}$.

Theorem 2.13.

Let $\varphi : X(I) \to Y(I)$ be a neutrosophic homomorphism of neutrosophic BCI-algebras. Then, $Ker\varphi$ is not a neutrosophic ideal of $X(I)$.

Proof.

The proof is straightforward since $(0,I) \in X(I)$ can not be mapped to $(0,0) \in Y(I)$.

Theorem 2.14.

Let $A(I)$ be a closed neutrosophic ideal of neutrosophic BCI-algebra $(X(I), *, (0,0))$. Then, the mapping $\varphi : X(I) \to X(I)/A(I)$ defined by

$$\varphi((x,yI)) = [(x,yI)], \quad \forall (x,yI) \in X(I)$$

is not a neutrosophic homomorphism.

Proof.

Straightforward since $\varphi((0,I)) = [(0,I)] \neq (0,I)$.

Theorem 2.15.

Let $\varphi : X_\omega(I) \rightarrow Y_\omega(I)$ be a neutrosophic homomorphism. Then, $Ker\varphi$ is a closed neutrosophic ideal of $X_\omega(I)$.

Proof.

Obvious.

Theorem 2.16.

Let $\varphi : X_\omega(I) \rightarrow Y_\omega(I)$ be a neutrosophic homomorphism and let $A[I]$ be a neutrosophic ideal of $X_\omega(I)$ such that $Ker\varphi \subseteq A[I]$. Then, $\varphi^{-1}(\varphi(A[I])) = A[I]$.

Proof.

Same as the classical case.

Theorem 2.17.

Let $A[I]$ be a neutrosophic ideal of $X_\omega(I)$. Then, the mapping $\varphi : X_\omega(I) \rightarrow X_\omega(I)/A[I]$ defined by

$$\varphi((x, xI)) = [(x, xI)], \quad \forall (x, xI) \in X_\omega(I)$$

is a neutrosophic homomorphism.

Proof.

The proof is straightforward.

Theorem 2.18.

Let $\varphi : X_\omega(I) \rightarrow Y_\omega(I)$ be a neutrosophic epimorphism. Then, $X_\omega(I)/Ker\varphi \cong Y_\omega(I)$.

Proof.

Same as the classical case.

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