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We devote part of this issue of *Critical Review* to Lotfi Zadeh.
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Neutrosophic Systems and Neutrosophic Dynamic Systems

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Abstract

In this paper, we introduce for the first time the neutrosophic system and neutrosophic dynamic system that represent new per-spectives in science. A neutrosophic system is a quasi- or (t, i, f)–classical system, in the sense that the neutrosophic system deals with quasi-terms/concepts/attributes, etc. [or (t, i, f) – terms/ concepts/attributes], which are approximations of the classical terms/concepts/attributes, i.e. they are partially true/membership/probable *(t%),* partially indeterminate *(i%),* and partially false/nonmember-ship/improbable *(f%),* where t, i, f are subsets of the unitary interval [0,1]. {We recall that 'quasi' means relative(ly), approximate(ly), almost, near, partial(ly), etc. or mathematically 'quasi' means (t, i, f) in a neutrophic way.}

Keywords

neutrosophy, neutrosophics, neutrosophic system, neutrosophic patterns, neutrosophic model, neutrosophic synergy, neutrosophic interactions, neutrosophic complexity, neutrosophic process, neutrosophic cognitive science.

1 Introduction

A system $\mathcal{S}$ in general is composed from a space $\mathcal{M}$, together with its elements (concepts) $\{e_j\}, j \in \theta$, and the relationships $\{\mathcal{R}_k\}$, $k \in \psi$, between them, where θ and ψ are countable or uncountable index sets. For a closed system, the space and its elements do not interact with the environment. For an open set, the space or its elements interact with the environment.

2 Definition of the neutrosophic system

A system is called neutrosophic system if at least one of the following occur:

a. The space contains some indeterminacy.
b. At least one of its elements x has some indeterminacy (it is not well-defined or not well-known).

c. At least one of its elements x does not 100% belong to the space; we say $x(t, i, f) \in \mathcal{M}$, with $(t, i, f) \neq (1, 0, 0)$.
d. At least one of the relationships $\mathcal{R}_o$ between the elements of $\mathcal{M}$ is not 100% well-defined (or well-known); we say $\mathcal{R}_o(t, i, f) \in \mathcal{S}$, with $(t, i, f) \neq (1, 0, 0)$.
e. For an open system, at least one $[\mathcal{R}_E(t, i, f)]$ of the system's interactions relationships with the environment has some indeterminacy, or it is not well-defined, or not well-known, with $(t, i, f) \neq (1, 0, 0)$.

2.1 Classical system as particular case of neutrosophic system

By language abuse, a classical system is a neutrosophic system with indeterminacy zero (no indeterminacy) at all system's levels.

2.2 World systems are mostly neutrosophic

In our opinion, most of our world systems are neutrosophic systems, not classical systems, and the dynamicity of the systems is neutrosophic, not classical.

Maybe the mechanical and electronical systems could have a better chance to be classical systems.

3 A simple example of neutrosophic system

Let's consider a university campus Coronado as a whole neutrosophic system $\mathcal{S}$, whose space is a prism having a base the campus land and the altitude such that the prism encloses all campus' buildings, towers, observatories, etc.

The elements of the space are people (administration, faculty, staff, and students) and objects (buildings, vehicles, computers, boards, tables, chairs, etc.).

A part of the campus land is unused. The campus administration has not decided yet what to do with it: either to build a laboratory on it, or to sell it. This is an indeterminate part of the space.

Suppose that a staff (John, from the office of Human Resources) has been fired by the campus director for misconduct. But, according to his co-workers, John was not guilty for anything wrong doing. So, John sues the campus. At this point, we do not know if John belongs to the campus, or not. John's appurtenance to the campus is indeterminate.

Assume the faculty norm of teaching is four courses per semester. But some faculty are part-timers, therefore they teach less number of courses. If an instructor teaches only one class per semester, he belongs to the campus only partially (25%), if he teaches two classes he belongs to the campus 50%, and if he teaches three courses he belongs to the campus 75%.

We may write:

Joe $(0.25, 0, 0.75) \in \mathcal{S}$
George $(0.50, 0, 0.50) \in \mathcal{S}$
and Thom $(0.75, 0.10, 0.25) \in \mathcal{S}$.

Thom has some indeterminacy (0.10) with respect to his work in the campus: it is possible that he might do some administrative work for the campus (but we don't know).

The faculty that are full-time (teaching four courses per semester) may also do overload. Suppose that Laura teaches five courses per semester, therefore Laura $(1.25, 0, 0) \in \mathcal{S}$.

In neutrosophic logic/set/probability it's possible to have the sum of components (t, i, f) different from 1:

$t + i + f > 1$, for paraconsistent (conflicting) information;
$t + i + f = 1$, for complete information;
$t + i + f < 1$, for incomplete information.

Also, there are staff that work only ½ norm for the campus, and many students take fewer classes or more classes than the required full-time norm. Therefore, they belong to the campus Coronado in a percentage different from 100%.

About the objects, suppose that 50 calculators were brought from IBM for one semester only as part of IBM's promotion of their new products. Therefore, these calculators only partially and temporarily belong to the campus.

Thus, not all elements (people or objects) entirely belong to this system, there exist many $e_j(t, i, f) \in \mathcal{S}$, with $(t, i, f) \neq (1, 0, 0)$.

Now, let's take into consideration the relationships. A professor, Frank, may agree with the campus dean with respect to a dean's decision, may disagree with respect to the dean's other decision, or may be ignorant with respect to the dean's various decisions. So, the relationship between Frank and the dean may be, for example:

$$\text{Frank} \xrightarrow{\text{agreement } (0.5,0.2,0.3)} \text{dean, i. e. not } (1, 0, 0) \text{ agreement.}$$

This campus, as an open system, cooperates with one Research Laboratory from Nevada, pending some funds allocated by the government to the campus.

Therefore, the relationship (research cooperation) between campus Coronado and the Nevada Research Laboratory is indeterminate at this moment.

4 Neutrosophic patterns

In a neutrosophic system, we may study or discover, in general, neutrosophic patterns, i.e. quasi-patterns, approximated patterns, not totally working; we say: $(t,i,f)-$patterns, i.e. t% true, i% indeterminate, and f% false, and elucidate (t,i,f) −principles.

The neutrosophic system, through feedback or partial feedback, is (t,i,f) −self-correcting, and (t,i,f) −self-organizing.

5 Neutrosophic holism

From a holistic point of view, the sum of parts of a system may be:

1. Smaller than the whole (when the interactions between parts are unsatisfactory);
2. Equals to the whole (when the interactions between parts are satisfactory);
3. Greater than the whole (when the interactions between parts are super-satisfactory).

The more interactions (interdependance, transdependance, hyper-dependance) between parts, the more complex a system is.

We have positive, neutral, and negative interactions between parts. Actually, an interaction between the parts has a degree of positiveness, degree of neutrality, and degree of negativeness. And these interactions are dynamic, meaning that their degrees of positiveness/neutrality/negativity change in time. They may be partially absolute and partially relative.

6 Neutrosophic model

In order to model such systems, we need a neutrosophic (approximate, partial, incomplete, imperfect) model that would discover the approximate system properties.

7 Neutrosophic successful system

A neutrosophic successful system is a system that is successful with respect to some goals, and partially successful or failing with respect to other goals.

The adaptivity, self-organization, self-reproducing, self-learning, reiteration, recursivity, relationism, complexity and other attributes of a classical system are extended to (t, i, f) −attributes in the neutrosophic system.

8 (t, i, f) −attribute

A (t, i, f) − attribute means an attribute that is t% true (or probable), i% indeterminate (with respect to the true/probable and false/improbable), and f% false/improbable - where t,i,f are subsets of the unitary interval [0,1].

For example, considering the subsets reduced to single numbers, if a neutrosophic system is (0.7, 0.2, 0.3)-adaptable, it means that the system is 70% adaptable, 20% indeterminate regarding adaptability, and 30% inadaptable; we may receive the informations for each attribute phase from different independent sources, that's why the sum of the neutrosophic components is not necessarily 1.

9 Neutrosophic dynamics

While classical dynamics was beset by dialectics, which brought together an entity ⟨A⟩ and its opposite ⟨antiA⟩, the neutrosophic dynamics is beset by tri-alectics, which brings together an entity ⟨A⟩ with its opposite ⟨antiA⟩ and their neutrality ⟨neutA⟩. Instead of duality as in dialectics, we have tri-alities in our world.

Dialectics failed to take into consideration the neutrality between opposites, since the neutrality partially influences both opposites.

Instead of unifying the opposites, the neutrosophic dynamics unifies the triad ⟨A⟩, ⟨antiA⟩, ⟨neutA⟩.

Instead of coupling with continuity as the classical dynamics promise, one has "tripling" with continuity and discontinuity altogether.

All neutrosophic dynamic system's components are interacted in a certain degree, repelling in another degree, and neutral (no interaction) in a different degree.

They comprise the systems whose equilibrium is the disechilibrium - systems that are continuously changing.

The internal structure of the neutrosophic system may increase in complexity and interconnections, or may degrade during the time.

A neutrosophic system is characterized by potential, impotential, and indeterminate developmental outcome, each one of these three in a specific degree.

10 Neutrosophic behavior gradient

In a neutrosophic system, we talk also about neutrosophic structure, which is actually a quasi-structure or structure which manifests into a certain degree; which influences the neutrosophic behavior gradient, that similarly is a behavior quasi-gradient - partially determined by quasi-stimulative effects; one has: discrete systems, continuous systems, hybrid (discrete and continuous) systems.

11 Neutrosophic interactions

Neutrosophic interactions in the system have the form:

$$A \;\blacksquare \xleftrightarrow[(t,\iota,f)]{\qquad\qquad} \blacksquare\; B$$

Neutrosophic self-organization is a quasi-self-organization. The system's neutrosophic intelligence sets into the neutrosophic patterns formed within the system's elements.

We have a neutrosophic causality between event E_1, that triggers event E_2, and so on. And similarly, neutrosophic structure S_1 (which is an approximate, not clearly know structure) causes the system to turn on neutrosophic structure S_2, and so on. A neutrosophic system has different levels of self-organizations.

12 Potentiality/impotentiality/indeterminacy

Each neutrosophic system has a potentiality/impotentiality/indeterminacy to attain a certain state/stage; we mostly mention herein about the transition from a quasi-pattern to another quasi-pattern. A neutrosophic open system is always transacting with the environment; since always the change is needed.

A neutrosophic system is always oscilating between stability, instability, and ambiguity (indeterminacy). Analysis, synthesis, and neutrosynthesis of existing data are done by the neutrosophic system. They are based on system's principles, antiprinciples, and nonprinciples.

13 Neutrosophic synergy

The Neutrosophic Synergy is referred to partially joined work or partially combined forces, since the participating forces may cooperate in a degree (t), may be antagonist in another degree (f), and may have a neutral interest in joint work in a different degree (i).

14 Neutrosophic complexity

The neutrosophic complex systems produce neutrosophic complex patterns. These patterns result according to the neutrosophic relationships among system's parts. They are well described by the neutrosophic cognitive maps (NCM), neutrosophic relational maps (NRM), and neutrosophic relational equations (NRE), all introduced by W. B. Vasanttha Kandasamy and F. Smarandache in 2003-2004.

The neutrosophic systems represent a new perspective in science. They deal with quasi-terms [or (t, i, f) −terms], quasi-concepts [or (t, i, f) −concepts], and quasi-attributes [or (t, i, f) −attributes], which are approximations of the terms, concepts, attributes, etc., i.e. they are partially true (t%), partially indeterminate (i%), and partially false (f%).

Alike in neutrosophy where there are interactions between ⟨A⟩, ⟨neutA⟩, and ⟨antiA⟩, where ⟨A⟩ is an entity, a system is frequently in one of these general states: equilibrium, indeterminacy (neither equilibrium, nor disequilibrium), and disequilibrium.

They form a neutrosophic complexity with neutrosophically ordered patterns. A neutrosophic order is a quasi or approximate order, which is described by a neutrosophic formalism.

The parts all together are partially homogeneous, partially heterogeneous, and they may combine in finitely and infinitely ways.

15 Neutrosophic processes

The neutrosophic patterns formed are also dynamic, changing in time and space. They are similar, dissimilar, and indeterminate (unknown, hidden, vague, incomplete) processes among the parts.

They are called neutrosophic processes.

16 Neutrosophic system behavior

The neutrosophic system's functionality and behavior are, therefore, coherent, incoherent, and imprevisible (indeterminate). It moves, at a given level, from a neutrosophic simplicity to a neutrosophic complexity, which becomes neutrosophic simplicity at the next level. And so on.

Ambiguity (indeterminacy) at a level propagates at the next level.

17 Classical systems

Although the biologist Bertalanffy is considered the father of general system theory since 1940, it has been found out that the conceptual portion of the system theory was published by Alexander Bogdanov between 1912-1917 in his three volumes of *Tectology*.

18 Classical open systems

A classical open system, in general, cannot be totally deterministic, if the environment is not totally deterministic itself.

Change in energy or in momentum makes a classical system to move from thermodynamic equilibrium to nonequilibrium or reciprocally.

Open classical systems, by infusion of outside energy, may get an unexpected spontaneous structure.

19 Deneutrosophication

In a neutrosophic system, besides the degrees of freedom, one also talk about the degree (grade) of indeterminacy. Indeterminacy can be described by a variable.

Surely, the degrees of freedom should be condensed, and the indetermination reduced (the last action is called "deneutrosophication").

The neutrosophic system has a multi-indeterminate behavior. A neutrosophic operator of many variables, including the variable representing indeterminacy, can approximate and semi-predict the system's behavior.

20 From classical to neutrosophic systems

Of course, in a bigger or more degree, one can consider the neutrosophic cybernetic system (quasi or approximate control mechanism, quasi information processing, and quasi information reaction), and similarly the neutrosophic chaos theory, neutrosophic catastrophe theory, or neutrosophic complexity theory.

In general, when passing from a classical system $\mathcal{S}_c$ in a given field of knowledge $\mathcal{F}$ to a corresponding neutrosophic system $\mathcal{S}_N$ in the same field of knowledge $\mathcal{F}$, one relaxes the restrictions about the system's space, elements, and relationships, i.e. these components of the system (space, elements, relationships) may contain indeterminacy, may be partially (or totally) unknown (or vague, incomplete, contradictory), may only partially belong to the system; they are approximate, quasi.

Scientifically, we write:

$$\mathcal{S}_N = (t, i, f) - \mathcal{S}_c,$$

and we read: *a neutrosophic system is a* (t, i, f)*–classical system*. As mapping, between the neutrosophic algebraic structure systems, we have defined neutrosophic isomorphism.

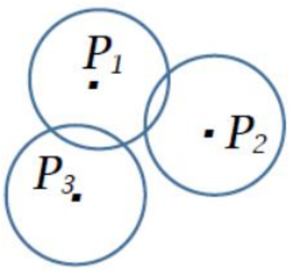
21 Neutrosophic dynamic system

The behavior of a neutrosophic dynamic system is chaotic from a classical point of view. Instead of fixed points, as in classical dynamic systems, one deals with fixed regions (i.e. neighbourhoods of fixed points), as approximate values of the neutrosophic variables [we recall that a neutrosophic variable is, in general, represented by a thick curve – alike a neutrosophic (thick) function].

There may be several fixed regions that are attractive regions in the sense that the neutrosophic system converges towards these regions if it starts out in a nearby neutrosophic state.

And similarly, instead of periodic points, as in classical dynamic systems, one has periodic regions, which are neutrosophic states where the neutrosophic system repeats from time to time.

If two or more periodic regions are non-disjoint (as in a classical dynamic system, where the fixed points lie in the system space too close to each other, such that their corresponding neighbourhoods intersect), one gets double periodic region, triple periodic region:



and so on: n −uple periodic region, for $n \geq 2$.

In a simple/double/triple/…/ $n-$ uple periodic region the neutrosophic system is fluctuating/oscilating from a point to another point.

The smaller is a fixed region, the better is the accuracy.

22 Neutrosophic cognitive science

In the Neutrosophic Cognitive Science, the Indeterminacy "I" led to the definition of the Neutrosophic Graphs (graphs which have: either at least one indeterminate edge, or at least one indeterminate vertex, or both some indeterminate edge and some indeterminate vertex), and Neutrosophic Trees (trees which have: either at least one indeterminate edge, or at least one indeterminate vertex, or both some indeterminate edge and some indeterminate vertex), that have many applications in social sciences.

Another type of neutrosophic graph is when at least one edge has a neutrosophic (t, i, f) truth-value.

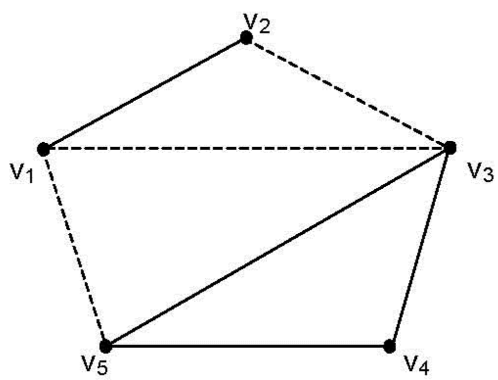
As a consequence, the Neutrosophic Cognitive Maps (Vasantha & Smarandache, 2003) and Neutrosophic Relational Maps (Vasantha & Smarandache, 2004) are generalizations of fuzzy cognitive maps and respectively fuzzy relational maps, Neutrosophic Relational Equations (Vasantha & Smarandache, 2004), Neutrosophic Relational Data (Wang, Smarandache, Sunderraman, Rogatko - 2008), etc.

A Neutrosophic Cognitive Map is a neutrosophic directed graph with concepts like policies, events etc. as vertices, and causalities or indeterminates as edges. It represents the causal relationship between concepts.

An edge is said indeterminate if we don't know if it is any relationship between the vertices it connects, or for a directed graph we don't know if it is a directly or inversely proportional relationship. We may write for such edge that (t, i, f) = (0,1,0).

A vertex is indeterminate if we don't know what kind of vertex it is since we have incomplete information. We may write for such vertex that (t, i, f) = (0,1,0).

Example of Neutrosophic Graph (edges V_1V_3, V_1V_5, V_2V_3 are indeterminate and they are drawn as dotted):



and its neutrosophic adjacency matrix is:

$$\begin{bmatrix} 0 & 1 & I & 0 & I \\ 1 & 0 & I & 0 & 0 \\ I & I & 0 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ I & 0 & 1 & 1 & 0 \end{bmatrix}$$

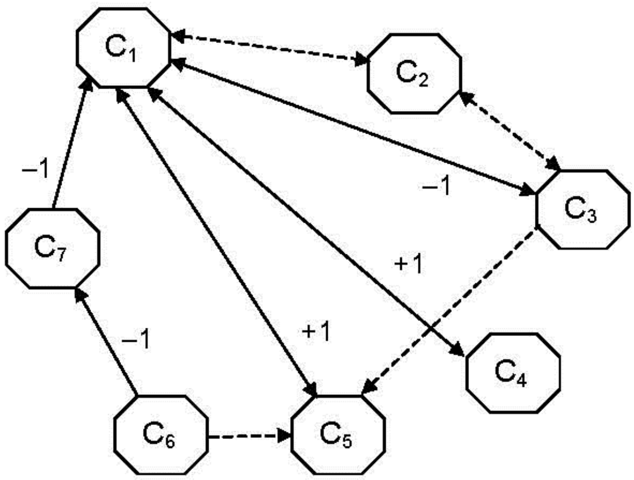
The edges mean: 0 = no connection between vertices, 1 = connection between vertices, I = indeterminate connection (not known if it is, or if it is not).

Such notions are not used in the fuzzy theory.

Let's give an example of Neutrosophic Cognitive Map (NCM), which is a generalization of the Fuzzy Cognitive Maps.

We take the following vertices:

C1 - Child Labor
C2 - Political Leaders
C3 - Good Teachers
C4 - Poverty
C5 - Industrialists
C6 - Public practicing/encouraging Child Labor
C7 - Good Non-Governmental Organizations (NGOs)



The corresponding neutrosophic adjacency matrix related to this neutrosophic cognitive map is:

$$\begin{bmatrix} 0 & I & -1 & 1 & 1 & 0 & 0 \\ I & 0 & I & 0 & 0 & 0 & 0 \\ -1 & I & 0 & 0 & I & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & I & 0 & -1 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

The edges mean: 0 = no connection between vertices, 1 = directly proportional connection, -1 = inversely proportionally connection, and I = indeterminate connection (not knowing what kind of relationship is between the vertices that the edge connects).

Now, we give another type of neutrosophic graphs (and trees): An edge of a graph, let's say from A to B (i.e. how A influences B),

may have a neutrosophic value (t, i, f), where t means the positive influence of A on B, i means the indeterminate/neutral influence of A on B, and f means the negative influence of A on B.

Then, if we have, let's say: $A -> B -> C$ such that $A -> B$ has the neutrosophic value *(t_1, i_1, f_1)* and $B -> C$ has the neutrosophic value *(t_2, i_2, f_2)*, then $A -> C$ has the neutrosophic value *(t_1, i_1, f_1)/\(t_2, i_2. f_2)*, where /\ is the AND_N neutrosophic operator.

Also, again a different type of graph: we can consider a vertex A as: $t\%$ belonging/membership to the graph, $i\%$ indeterminate membership to the graph, and $f\%$ nonmembership to the graph.

Finally, one may consider any of the previous types of graphs (or trees) put together.

23 (t, i, f) −qualitative behavior

We normally study in a neutrosophic dynamic system its long-term (t, i, f) −qualitative behavior, i.e. degree of behavior's good quality *(t),* degree of behavior's indeterminate (unclear) quality *(i),* and degree of behavior's bad quality *(f).*

The questions arise: will the neutrosophic system fluctuate in a fixed region (considered as a neutrosophic steady state of the system)? Will the fluctuation be smooth or sharp? Will the fixed region be large (hence less accuracy) or small (hence bigger accuracy)? How many periodic regions does the neutrosophic system has? Do any of them intersect [i.e. does the neutrosophic system has some n −uple periodic regions (for $n \geq 2$), and for how many]?

24 Neutrosophic state

The more indeterminacy a neutrosophic system has, the more chaotic it is from the classical point of view. A neutrosophic lineal dynamic system still has a degree of chaotic behavior. A collection of numerical sets determines a neutrosophic state, while a classical state is determined by a collection of numbers.

25 Neutrosophic evolution rule

The neutrosophic evolution rule decribes the set of neutrosophic states where the future state (that follows from a given current state) belongs to. If the set of neutrosophic states, that the next neutrosophic state will be in, is known, we have a quasi-deterministic neutrosophic evolution rule, otherwise the neutrosophic evolution rule is called quasi-stochastic.

26 Neutrosophic chaos

As an alternative to the classical Chaos Theory, we have the Neutrosophic Chaos Theory, which is highly sensitive to indeterminacy; we mean that small change in the neutrosophic system's initial indeterminacy produces huge perturbations of the neutrosophic system's behavior.

27 Time quasi-delays and quasi-feedback thick-loops

Similarly, the difficulties in modelling and simulating a Neutrosophic Complex System (also called Science of Neutrosophic Complexity) reside in its degree of indeterminacy at each system's level.

In order to understand the Neutrosophic System Dynamics, one studies the system's time quasi-delays and internal quasi-feedback thick-loops (that are similar to thick functions ad thick curves defined in the neutrosophic precalculus and neutrosophic calculus).

The system may oscillate from linearity to nonlinearity, depending on the neutrosophic time function.

28 Semi-open semi-closed system

Almost all systems are open (exchanging energy with the environment). But, in theory and in laboratory, one may consider closed systems (completely isolated from the environment); such systems can oscillate between closed and open (when they are cut from the environment, or put back in contact with the environment respectively). Therefore, between open systems and closed systems, there also is a semi-open semi-closed system.

29 Neutrosophic system's development

The system's self-learning, self-adapting, self-conscienting, self-developing are parts of the system's dynamicity and the way it moves from a state to another state – as a response to the system internal or external conditions. They are constituents of system's behavior.

The more developed is a neutrosophic system, the more complex it becomes. System's development depends on the internal and external interactions (relationships) as well.

Alike classical systems, the neutrosophic system shifts from a quasi-developmental level to another. Inherent fluctuations are characteristic to neutrosophic complex systems. Around the quasi-steady states, the fluctuations in a neutrosophic system becomes its sources of new quasi-development and quasi-behavior.

In general, a neutrosophic system shows a nonlinear response to its initial conditions. The environment of a neutrosophic system may also be neutrosophic (i.e. having some indeterminacy).

30 Dynamic dimensions of neutrosophic systems

There may be neutrosophic systems whose spaces have dynamic dimensions, i.e. their dimensions change upon the time variable.

Neutrosophic Dimension of a space has the form (t, i, f), where we are $t\%$ sure about the real dimension of the space, $i\%$ indeterminate about the real dimension of the space, and $f\%$ unsure about the real dimension of the space.

31 Noise in a neutrosophic system

A neutrosophic system's noise is part of the system's indeterminacy. A system's pattern may evolve or dissolve over time, as in a classical system.

32 Quasi-stability

A neutrosophic system has a degree of stability, degree of indeterminacy referring to its stability, and degree of instability. Similarly, it has a degree of change, degree of indeterminate change, and degree of non-change at any point in time.

Quasi-stability of a neutrosophic system is its partial resistance to change.

33 (t, i, f) −attractors

Neutrosophic system's quasi-stability is also dependant on the (t, i, f) −attractor, which $t\%$ attracts, $i\%$ its attraction is indeterminate, and $f\%$ rejects. Or we may say that the neutrosophic system $(t\%, i\%, f\%)$ −prefers to reside in a such neutrosophic attractor.

Quasi-stability in a neutrosophic system responds to quasi-perturbations.

When $(t, i, f) \to (1,0,0)$ the quasi-attractors tend to become stable, but if $(t, i, f) \to (0, i, f)$, they tend to become unstable.

Most neutrosophic system are very chaotic and possess many quasi-attractors and anomalous quasi-patterns. The degree of freedom in a neutrosophic complex system increase and get more intricate due to the type of indeterminacies that are specific to that system. For example, the classical system's noise is a sort of indeterminacy.

Various neutrosophic subsystems are assembled into a neutrosophic complex system.

34 (t, i, f) − repellors

Besides attractors, there are systems that have repellors, i.e. states where the system avoids residing. The neutrosophic systems have quasi-repellors, or (t, i, f) −repellors, i.e. states where the neutrosophic system partialy avoid residing.

35 Neutrosophic probability of the system's states

In any (classical or neutrosophic) system, at a given time ρ, for each system state τ one can associate a neutrosophic probability,

$$\mathcal{NP}(\tau) = (\mathrm{t}, \mathrm{i}, \mathrm{f}),$$

where *t, i, f* are subsets of the unit interval [0, 1] such that:

t = the probability that the system resides in τ;
i = the indeterminate probability/improbability about the system residing in τ;
f = the improbability that the system resides in τ;

For a (classical or neutrosophic) dynamic system, the neutrosophic probability of a system's state changes in the time, upon the previous states the system was in, and upon the internal or external conditions.

36 (t, i, f) −reiterative

In Neutrosophic Reiterative System, each state is partially dependent on the previous state. We call this process quasi-reiteration or (t, i, f) −reiteration.

In a more general case, each state is partially dependent on the previous n states, for $n \geq 1$. This is called n-quasi-reiteration, or $n - (t, i, f)$ −reiteration.

Therefore, the previous neutrosophic system history partialy influences the future neutrosophic system's states, which may be different even if the neutrosophic system started under the same initial conditions.

37 Finite and infinite system

A system is finite if its space, the number of its elements, and the number of its relationships are all finite.

If at least one of these three is infinite, the system is considered infinite. An infinite system may be countable (if both the number of its elements and the number of its relationships are countable), or, otherwise, uncountable.

38 Thermodynamic (t, i, f) −equilibrium

The potential energy (the work done for changing the system to its present state from its standard configuration) of the classical system is a minimum if the equilibrium is stable, zero if the equilibrium is neutral, or a maximum if the equilibrium is unstable.

A classical system may be in stable, neutral, or unstable equilibrium. A neutrosophic system may be in quasi-stable, quasi-neutral or quasi-unstable equilibrium, and its potential energy respectively quasi-minimum, quasi-null (i.e. close to zero), or quasi-maximum. {We recall that 'quasi' means relative(ly), approximate(ly), almost, near, partial(ly), etc. or mathematically 'quasi' means (t, i, f) in a neutrophic way.}

In general, we say that a neutrosophic system is in $(t, i, f) -$ equilibrium, or $t\%$ in stable equilibrium, $i\%$ in neutral equilibrium, and $f\%$ in unstable equilibrium (non-equilibrium).

When $f \gg t$ (f is much greater than t), the neutroophic system gets into deep non-equilibrium and the perturbations overtake the system's organization to a new organization.

Thus, similarly to the second law of thermodynamics, the neutrosophic system runs down to a (t, i, f) −equilibrium state.

A neutrosophic system is considered at a thermodynamic (t, i, f) −equilibrium state when there is not (or insignificant) flow from a region to another region, and the momentum and energy are uninformally at (t, i, f) −level.

39 The (t_1, i_1, f_1) −cause produces a (t_2, i_2, f_2) −effect

The potential energy (the work done for changing the system to its present state from its standard configuration) of the classical system is a minimum if the equilibrium is stable, zero if the equilibrium is neutral, or a maximum if the equilibrium is unstable.

In a neutrosophic system, a (t_1, i_1, f_1)-cause produces a (t_2, i_2, f_2)-effect. We also have cascading (t, i, f)-effects from a given cause, and we have permanent change into the system.

(t, i, f)-principles and (t, i, f)-laws function in a neutrosophic dynamic system. It is endowed with (t, i, f)-invariants and with parameters of (t, i, f)-potential (potentiality, neutrality, impotentiality) control.

40 (t, i, f) −holism

A neutrosophic system is a (t, i, f) −holism, in the sense that it has a degree of independent entity (t) with respect to its parts, a degree of indeterminate (i) independent-dependent entity with respect to its parts, and a degree of dependent entity (f) with respect to its parts.

41 Neutrosophic soft assembly

Only several ways of assembling (combining and arranging) the neutrosophic system's parts are quasi-stable. The others assemble ways are quasi-transitional.

The neutrosophic system development is viewed as a neutrosophic soft assembly. It is alike an amoeba that changes its shape. In a neutrosophic dynamic system, the space, the elements, the relationships are all flexible, changing, restructuring, reordering, reconnecting and so on, due to heterogeneity, multimodal processes, multi-causalities, multidimensionality, auto-stabilization, auto-hierarchization, auto-embodiement and especially due to synergetism (the neutrosophic system parts cooperating in a (t, i, f) −degree).

42 Neutrosophic collective variable

The neutrosophic system is partially incoherent (because of the indeterminacy), and partially coherent. Its quasi-behavior is given by the neutrosophic collective variable that embeds all neutrosophic variables acting into the (t, i, f) −holism.

43 Conclusion

We have introduced for the first time notions of neutrosophic system and neutrosophic dynamic system. Of course, these proposals and studies are not exhaustive.

Future investigations have to be done about the neutrosophic (dynamic or not) system, regarding: the neutrosophic descriptive methods and neutrosophic experimental methods, developmental and study the neutrosophic differential equations and neutrosophic difference equations, neutrosophic simulations, the extension of the classical A-Not-B Error to the neutrosophic form, the neutrosophic putative control parameters, neutrosophic loops or neutrosophic cyclic alternations within the system, neutrosophic degenerating (dynamic or not) systems, possible programs within the neutrosophic system, from neutrosophic antecedent conditions how to predict the outcome, also how to find the boundary of neutrosophic conditions, when the neutrosophic invariants are innate/genetic, what are the relationships between the neutrosophic attractors and the neutrosophic repellors, etc.

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Tri-complex Rough Neutrosophic Similarity Measure and its Application in Multi-Attribute Decision Making

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Abstract

This paper presents multi-attribute decision making based on tri-complex rough neutrosophic similarity measure with rough neutrosophic attribute values. The concept of rough neutrosophic set is a powerful mathematical tool to deal with incomplete, indeterminate and inconsistent information. The ratings of all alternatives are expressed in terms of the upper and lower approximation operators and the pair of neutrosophic sets which are characterized by truth-membership degree, indeterminacy-membership degree, and falsity-membership degree. We define a function based on tri-complex number system to determine the degree of similarity between rough neutrosophic sets. The approach of using tri-complex number system in formulating the similarity measure in rough neutrosophic environment is new. Finally, a numerical example demonstrates the applicability of the proposed approach.

Keyword

tri-complex rough neutrosophic similarity measure, rough neutrosophic set, MCDM problem, approximation operator.

1 Introduction

The concept of rough neutrosophic set is grounded by Broumi et al. [1], [2] in 2014. It is derived by hybridizing the concepts of rough set proposed by Pawlak [3] and neutrosophic set originated by Smarandache [4, 5]. Neutrosophic sets and rough sets are both capable of dealing with uncertainty and partial information. Wang et al. [6] introduced single valued neutrosophic set (SVNS) in 2010 to deal with real world problems.

Rough neutrosophic set is the generalization of rough fuzzy sets [7], [8] and rough intuitionistic fuzzy sets [9]. Mondal and Pramanik [10] applied the concept of rough neutrosophic set in multi-attribute decision making based on grey relational analysis in 2015. Mondal and Pramanik [11] also studied cosine similarity measure of rough neutrosophic sets and its application in medical diagnosis in 2015. The same authors [12] proposed multi attribute decision making using rough accuracy score function, and also proposed cotangent similarity measure under rough neutrosophic environment [13]. The same authors [14] further proposed some similarity measures namely Dice and Jaccard similarity measures in rough neutrosophic environment. Olariu [15] introduced the concept of hypercomplex numbers and studied some of its properties in 2002, then studied exponential and trigonometric form, the concept of analyticity, contour integration and residue. Mandal and Basu [16] studied hyper-complex similarity measure for SVNS and presented application in decision making. No studies have been made on multi-attribute decision making using tri-complex rough neutrosophic environment.

In this paper, we develop rough tri-complex neutrosophic multi-attribute decision making based on rough tri-complex neutrosophic similarity function (RTNSF). RNSs are represented as a tri-complex number. The distance measured between so transformed tri-complex numbers produce the similarity value. Section 2 presents preliminaries of neutrosophic sets and rough neutrosophic sets. Section 3 describes some basic ideas of tri-complex number. Section 4 presents tri-complex similarity measures in rough neutrosophic environment. Section 5 is devoted to present multi attribute decision-making method based on rough tri-complex neutrosophic similarity function. Section 6 presents a numerical example of the proposed approach. Section 7 presents comparison with existing rough neutrosophic similarity measures. Finally, section 8 presents concluding remarks and scope of future research.

2 Neutrosophic Preliminaries

Definition 2.1 [4, 5]

Let U be an universe of discourse. Then the neutrosophic set A can be presented in the form:

$$A = \{< x: T_A(x), I_A(x), F_A(x)>, x \in U\},$$

where the functions T, I, F: $U \rightarrow]^-0,1^+[$ represent respectively the degree of membership, the degree of indeterminacy, and the degree of non-membership of the element $x \in U$ to the set P satisfying the following the condition:

$$^{-}0 \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^{+}$$

Wang et al. [6] mentioned that the neutrosophic set assumes the value from real standard or non-standard subsets of $]^{-}0, 1^{+}[$ based on philosophical point of view. So instead of $]^{-}0, 1^{+}[$ Wang et al. [6] consider the interval [0, 1] for technical applications, because $]^{-}0, 1^{+}[$ is difficult to apply in the real applications such as scientific and engineering problems. For two neutrosophic sets (NSs), $A_{NS} = \{<x: T_A(x), I_A(x), F_A(x)> \mid x \in X\}$ and $B_{NS} = \{<x, T_B(x), I_B(x), F_B(x)> \mid x \in X\}$ the two relations are defined as follows:

(1) $A_{NS} \subseteq B_{NS}$ if and only if $T_A(x) \leq T_B(x), I_A(x) \geq I_B(x), F_A(x) \geq F_B(x)$

(2) $A_{NS} = B_{NS}$ if and only if $T_A(x) = T_B(x), I_A(x) = I_B(x), F_A(x) = F_B(x)$

2.2 Single valued neutrosophic sets

Definition 2.2 [6]

Assume that X be a space of points (objects) with generic elements in X denoted by x. A SVNS A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity membership function $F_A(x)$, for each point x in X, $T_A(x)$, $I_A(x)$, $F_A(x) \in [0, 1]$. When X is continuous, a SVNS A can be written as follows:

$$A = \int_x \frac{<T_A(x), I_A(x), F_A(x)>}{x} : x \in X$$

When X is discrete, a SVNS A can be written as follows:

$$A = \sum_{i=1}^{n} \frac{<T_A(x_i), I_A(x_i), F_A(x_i)>}{x_i} : x_i \in X$$

For two SVNSs, $A_{SVNS} = \{<x: T_A(x), I_A(x), F_A(x)> \mid x \in X\}$ and $B_{SVNS} = \{<x, T_B(x), I_B(x), F_B(x)> \mid x \in X\}$ the two relations are defined as follows:

(1) $A_{SVNS} \subseteq B_{SVNS}$
if and only if $T_A(x) \leq T_B(x), I_A(x) \geq I_B(x), F_A(x) \geq F_B(x)$

(2) $A_{SVNS} = B_{SVNS}$
if and only if $T_A(x) = T_Q(x), I_A(x) = I_B(x), F_A(x) = F_B(x)$ for any $x \in X$.

2.3 Rough neutrosophic set

Definition 2.2.1 [1], [2]

Let Z be a non-null set and R be an equivalence relation on Z. Let A be neutrosophic set in Z with the membership function T_A, indeterminacy function I_A and non-membership function F_A. The lower and the upper approximations of A in the approximation (*Z, R*) denoted by $\underline{N}(A)$ and $\overline{N}(A)$ are respectively defined as follows:

$$\underline{N}(A)=\left\langle < x, T_{\underline{N}(A)}(x), I_{\underline{N}(A)}(x), F_{\underline{N}(A)}(x) > / z\in[x]_R, x\in Z\right\rangle \quad (1)$$

$$\overline{N}(A)=\left\langle < x, T_{\overline{N}(A)}(x), I_{\overline{N}(A)}(x), F_{\overline{N}(A)}(x) > / z\in[x]_R, x\in Z\right\rangle \quad (2)$$

where, $T_{\underline{N}(A)}(x) = \wedge_z \in [x]_R\, T_A(z)$,

$I_{\underline{N}(A)}(x) = \wedge_z \in [x]_R\, I_A(z)$, $F_{\underline{N}(A)}(x) = \wedge_z \in [x]_R\, F_A(z)$,

$T_{\overline{N}(A)}(x) = \vee_z \in [x]_R\, T_A(z)$, $I_{\overline{N}(A)}(x) = \vee_z \in [x]_R\, T_A(z)$,

$F_{\overline{N}(A)}(x) = \vee_z \in [x]_R\, I_A(z)$

So, $0 \le T_{\underline{N}(A)}(x) + I_{\underline{N}(A)}(x) + F_{\underline{N}(A)}(x) \le 3$ and $0 \le T_{\overline{N}(A)}(x) + I_{\overline{N}(A)}(x) + F_{\overline{N}(A)}(x) \le 3$ hold good. Here $\vee$ and $\wedge$ denote "max" and "min'' operators respectively. $T_A(z)$, $I_A(z)$ and $F_A(z)$ are the membership, indeterminacy and non-membership of *z* with respect to A. $\underline{N}(A)$ and $\overline{N}(A)$ are two neutrosophic sets in *Z*.

Thus, NS mappings $\underline{N}$, $\overline{N}$: $N(Z) \rightarrow N(Z)$ are respectively referred to as the lower and upper rough NS approximation operators, and the pair $(\underline{N}(A), \overline{N}(A))$ is called the rough neutrosophic set in (*Z*, R).

Based on the above mentioned definition, it is observed that $\underline{N}(A)$ and $\overline{N}(A)$ have constant membership on the equivalence classes of *R*, if $\underline{N}(A) = \overline{N}(A)$; i.e. $T_{\underline{N}(A)}(x) = T_{\overline{N}(A)}(x)$, $I_{\underline{N}(A)}(x) = I_{\overline{N}(A)}(x)$, $F_{\underline{N}(A)}(x) = F_{\overline{N}(A)}(x)$.

For any x belongs to *Z,* P is said to be a definable neutrosophic set in the approximation (*Z, R*). Obviously, zero neutrosophic set (0_N) and unit neutrosophic sets (1_N) are definable neutrosophic sets.

Definition 2.2.2 [1], [2]

Let N(A) = ($\underline{N}(A), \overline{N}(A)$) is a rough neutrosophic set in (*Z,* R). The rough complement of *N*(A) is denoted by $\sim N(A) = (\underline{N}(A)^c, \overline{N}(A)^c)$, where $\underline{N}(A)^c$, $\overline{N}(A)^c$ are the complements of neutrosophic sets of $\underline{N}(A), \overline{N}(A)$ respectively.

$$\underline{N}(A)^c = \left\langle < x, F_{\underline{N}(A)}(x), 1 - I_{\underline{N}(A)}(x), T_{\underline{N}(A)}(x) > /, x\in Z\right\rangle, \text{ and}$$

$$\overline{N}(A)^c = \left\langle < x, F_{\underline{N}(A)}(x), 1 - I_{\overline{N}(A)}(x), T_{\overline{N}(A)}(x) > /, x\in Z\right\rangle \quad (3)$$

Definition 2.2.3 [1], [2]

Let $N(A)$ and $N(B)$ are two rough neutrosophic sets respectively in *Z*, then the following definitions hold good:

$$N(A)=N(B)\Leftrightarrow \underline{N}(A)=\underline{N}(B)\wedge \overline{N}(A)=\overline{N}(B)$$

$$N(A)\subseteq N(B)\Leftrightarrow \underline{N}(A)\subseteq \underline{N}(B)\wedge \overline{N}(A)\subseteq \overline{N}(B)$$

$$N(A)\bigcup N(B)=<\underline{N}(A)\bigcup \underline{N}(B),\ \overline{N}(A)\bigcup \overline{N}(B)>$$

$$N(A)\bigcap N(B)=<\underline{N}(A)\bigcap \underline{N}(B),\ \overline{N}(A)\bigcap \overline{N}(B)>$$

$$N(A)+N(B)=<\underline{N}(A)+\underline{N}(B),\ \overline{N}(A)+\overline{N}(B)>$$

$$N(A).N(B)=<\underline{N}(A).\underline{N}(B),\ \overline{N}(A).\overline{N}(B)>$$

If A, B, C are the rough neutrosophic sets in (*Z*, R), then the following propositions are stated from definitions

Proposition 1 [1], [2]

1. $\sim A(\sim A)=A$
2. $A\bigcup B=B\bigcup A,\ A\cap B=B\cap A$
3. $(A\bigcup B)\bigcup C=A\bigcup(B\bigcup C),\ (A\bigcap B)\bigcap C=A\bigcap(B\bigcap C)$
4. $(A\bigcup B)\bigcap C=(A\bigcup B)\bigcap(A\bigcup C),\ (A\bigcap B)\bigcup C=(A\bigcap B)\bigcup(A\bigcap C)$

Proposition 2 [1], [2]

De Morgan's Laws are satisfied for rough neutrosophic sets N(A) and N(B)

1. $\sim(N(A)\bigcup N(B))=(\sim N(A))\bigcap(\sim N(B))$
2. $\sim(N(A)\bigcap N(B))=(\sim N(A))\bigcup(\sim N(B))$

For proof of the proposition, see [1], [2].

Proposition 3 [1], [2]:

If A and B are two neutrosophic sets in *U* such that $A\subseteq B$, then $N(A)\subseteq N(B)$

1. $N(A\bigcap B)\subseteq N(A)\bigcap N(B)$
2. $N(A\bigcup B)\supseteq N(A)\bigcup N(B)$

For proof of the proposition, see [1], [2].

Proposition 4 [1], [2]:

1. $\underline{N}(A) = \sim \overline{N}(\sim A)$
2. $\overline{N}(A) = \sim \underline{N}(\sim A)$
3. $\underline{N}(A) \subseteq \overline{N}(A)$

For proof of the proposition, see [1], [2].

3 Basic concept of Tri-complex number in three dimension

Olariu [15] described a system of hypercomplex numbers in three dimensions, where multiplication is associative and commutative. Hypercomplex numbers can be expressed in exponential and trigonometric forms and for which the concepts of analytic tri-complex function, contour integration and residue are well defined. Olariu [15] introduced the concept of tri-complex numbers which is expressed in the form $u = x + h_1 y + h_2 z$, the variables x, y, and z being real numbers. The multiplication rules [15] for the complex units h_1, h_2 are given by $h_1^2 = h_2$, $h_2^2 = h_1$, $1.\, h_1 = h_1$, $1.\, h_2 = h_2$, $h_1.\, h_2 = 1$. Geometrically, tri-complex number u is expressed by the point D(x, y, z). Assume that O be the origin of the x, y, z axes, T be the trisector line $x = y = z$ of the positive octant. Also, let L be the plane $x + y + z = 0$ passing through the origin O and perpendicular to T. The tricomplex number u can be expressed as the projection p of the segment OD along the line T, by the distance δ from D to the line T, and by the azimuthal angle ϕ in the plane L (see *Fig. 1* below).

Here, ϕ is the angle between the projection of D on the plane L and the straight line which is the intersection of the plane L and the plane determined by line T and x axis. ϕ satisfied the relation $0 \le \phi \le 2\pi$. The amplitude ψ of a tri-complex number is defined as $\psi = (x^3+y^3+z^3 - 3xyz)^{1/3}$. The polar angle θ of OD with respect to the tri-sector line T is presented as $\tan\theta = \delta/p$. θ satisfies the inequality $0 \le \theta \le 2\pi$. The distance d from D to the origin is obtained as $d^2 = x^2 + y^2 + z^2$. The division $1/(x + h_1 y + h_2 z)$ is possible if $\psi \neq 0$.

The product of two tri-complex numbers is equal to zero if both numbers are equal to zero, or if one of the tri-complex numbers lies in the plane L and the other on the T line. The tri-complex number $u = x + h_1 y + h_2 z$ can be represented by the point D having coordinates *(x, y, z)*. The projection $p = OQ$ of the line OD on the tri-sector line $x = y = z$, which has the unit tangent $\left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$, *is* $p = \frac{1}{\sqrt{3}}(x+y+z)$.

The distance $\delta = DQ$ from D to the tri-sector line $x = y = z$, measured as the distance from the point $D(x, y, z)$ to the point Q of coordinates $\left(\frac{x+y+z}{3}, \frac{x+y+z}{3}, \frac{x+y+z}{3}\right)$, is $\delta^2 = \frac{2}{3}\left(x^2+y^2+z^2-xy-yz-zx\right)$.

The plane through the *point* D and perpendicular to the tri-sector line T intersects the x-axis at point A of coordinates $(x + y + z, 0, 0)$, the y-axis at point B of coordinates $(0, x + y + z, 0)$, and the z-axis at point C of coordinates $(0, 0, x + y + z)$. The expression of ϕ in terms of *x, y, z* can be obtained in a system of coordinates defined by the unit vectors as follows:

$$\zeta_1 = \frac{1}{\sqrt{6}}(2, -1, -1),\ \zeta_2 = \frac{1}{\sqrt{2}}(0, -1, -1),\ \zeta_3 = \frac{1}{\sqrt{3}}(1, 1, 1).$$

The relation between the coordinates of D in the systems (ζ_1, ζ_2, ζ_3) and *x, y, z* can be presented as follows:

$$\begin{bmatrix}\zeta_1\\ \zeta_2\\ \zeta_3\end{bmatrix} = \begin{bmatrix}\frac{2}{\sqrt{6}} & \frac{-1}{\sqrt{6}} & \frac{-1}{\sqrt{6}}\\ 0 & -\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}\end{bmatrix}\begin{bmatrix}x\\ y\\ z\end{bmatrix} \tag{4}$$

$$[\zeta_1, \zeta_2, \zeta_3] = \left(\frac{1}{\sqrt{6}}(2x-y-z), -\frac{1}{\sqrt{2}}(y+z), \frac{1}{\sqrt{3}}(x+y+z)\right) \tag{5}$$

Also, $\cos\varphi = \dfrac{2x-y-z}{2(x^2+y^2+z^2-xy-yz-zx)}$ (6)

$\sin\varphi = \dfrac{\sqrt{3}(y-z)}{2(x^2+y^2+z^2-xy-yz-zx)}$ (7)

The angle θ between the line OD and the tri-sector line T is given by $tan\theta = \frac{\delta}{p}$.

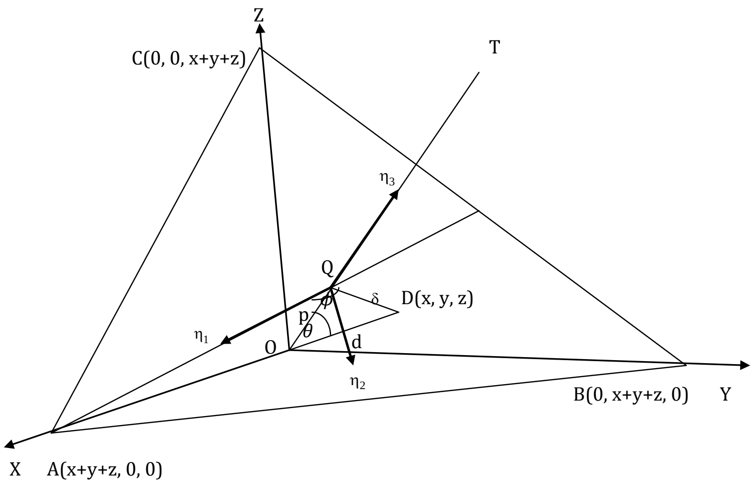

Figure 1. Tri-complex number.

Tri-complex variables p, d, θ, and ϕ for the tri-complex number $x+h_1y+h_2z$, represented by the point $D(x,y,z)$. The angle ϕ is shown in the plane parallel to L, passing through D, which intersects the tri-sector line T at Q. The orthogonal axes: η_1, η_2, η_3 intersect at the origin Q. The axis $Q\eta_1$ is parallel to the axis $O\eta_1$, the axis $Q\eta_2$ is parallel to the axis $O\eta_2$ and the axis $Q\eta_3$ is parallel to the axis $O\eta_3$, so that, in the plane ABC, the angle ϕ is measured from the line QA.

4 Tri-complex similarity measure in RNS

From the basic concept of Tri-complex number we have the following relations.

$$tan\theta = \frac{\delta}{p} = \frac{\sqrt{(x-y)^2+(y-z)^2+(z-x)^2}}{x+y+z} \tag{8}$$

where, $\delta^2 = \frac{2}{3}\left(x^2+y^2+z^2-xy-yz-zx\right)$ and $p = \frac{1}{\sqrt{3}}(x+y+z)$.

$$\cos\varphi = \frac{2x-y-z}{2(x^2+y^2+z^2-xy-yz-zx)}$$

$$\sin\varphi = \frac{\sqrt{3}(y-z)}{2(x^2+y^2+z^2-xy-yz-zx)}$$

This implies, $\tan\varphi = \frac{\sqrt{3}(y-z)}{(2x-y-z)}$ (9)

We now define a function for similarity measure between rough neutrosophic set (RNSs). The function satisfies the basic properties of similarity measure method in tri-complex system. The rough tri-complex similarity function is defined as follows (see definition 1).

Definition 1:

Let A=< $\left\langle \left(\underline{T}_A(x_i), \underline{I}_A(x_i), \underline{F}_A(x_i)\right), \left(\overline{T}_A(x_i), \overline{I}_A(x_i), \overline{F}_A(x_i)\right)\right\rangle$ > and

B = < $\left\langle \left(\underline{T}_B(x_i), \underline{I}_B(x_i), \underline{F}_B(x_i)\right), \left(\overline{T}_B(x_i), \overline{I}_B(x_i), \overline{F}_B(x_i)\right)\right\rangle$ > be two rough neutrosophic numbers in $X = \{x_i$: i = 1, 2, ..., n}.

Also let, $tan\theta_1 = \frac{\sqrt{\left(\delta T_A(x_i)-\delta I_A(x_i)\right)^2+\left(\delta I_A(x_i)-\delta F_A(x_i)\right)^2+\left(\delta F_A(x_i)-\delta T_A(x_i)\right)^2}}{\delta T_A(x_i)+\delta I_A(x_i)+\delta F_A(x_i)}$

$$tan\theta_2 = \frac{\sqrt{\left(\delta T_B(x_i)-\delta I_B(x_i)\right)^2+\left(\delta I_B(x_i)-\delta F_B(x_i)\right)^2+\left(\delta F_B(x_i)-\delta T_B(x_i)\right)^2}}{\delta T_B(x_i)+\delta I_B(x_i)+\delta F_B(x_i)}$$

$$\tan\varphi_1 = \frac{\sqrt{3}\left[\delta I_A(x_i) - \delta F_A(x_i)\right]}{2\delta T_A(x_i) - \delta I_A(x_i) - \delta F_A(x_i)}$$

$$\tan\varphi_2 = \frac{\sqrt{3}\left[\delta I_B(x_i) - \delta F_B(x_i)\right]}{2\delta T_B(x_i) - \delta I_B(x_i) - \delta F_B(x_i)}.$$

Taking, $tan\theta_1 = \Delta_{\theta_1}$, $tan\theta_2 = \Delta_{\theta_2}$, $\tan\varphi_1 = \Delta_{\phi_1}$, $\tan\varphi_2 = \Delta_{\phi_2}$, the rough tri-complex neutrosophic similarity function (RTNSF) between two neutrosophic sets A and B is defined as follows:

$S_{RTNSF}(A, B) =$

$$\frac{1}{2}\left[\frac{\left(1+\nabla_{\theta_1}\nabla_{\theta_2}\right)^2}{1+\nabla_{\theta_1}^{\ 2}+\nabla_{\theta_2}^{\ 2}+\nabla_{\theta_1}^{\ 2}\nabla_{\theta_2}^{\ 2}}+\frac{\left(1+\nabla_{\phi_1}\nabla_{\phi_2}\right)^2}{1+\nabla_{\phi_1}^{\ 2}+\nabla_{\phi_2}^{\ 2}+\nabla_{\phi_1}^{\ 2}\nabla_{\phi_2}^{\ 2}}\right] \qquad (10)$$

where,

$$\delta T_A(x_i) = \left(\frac{\underline{T}_A(x_i)+\overline{T}_A(x_i)}{2}\right),$$

$$\delta T_B(x_i) = \left(\frac{\underline{T}_B(x_i)+\overline{T}_B(x_i)}{2}\right),$$

$$\delta I_A(x_i) = \left(\frac{\underline{I}_A(x_i)+\overline{I}_A(x_i)}{2}\right),$$

$$\delta I_B(x_i) = \left(\frac{\underline{I}_B(x_i)+\overline{I}_B(x_i)}{2}\right),$$

$$\delta F_A(x_i) = \left(\frac{\underline{F}_A(x_i)+\overline{F}_A(x_i)}{2}\right),$$

$$\delta F_B(x_i) = \left(\frac{\underline{F}_B(x_i)+\overline{F}_B(x_i)}{2}\right).$$

Also, $[\delta T_A(x), \delta I_A(x), \delta F_A(x)] \neq [0, 0, 0]$ and $[\delta T_B(x), \delta I_B(x), \delta F_B(x)] \neq [0, 0, 0]$, $i = 1, 2, \ldots, n$.

The proposed rough neutrosophic operator satisfies the following conditions of similarity measures.

P1. $0 \leq S_{RTNSF}(A, B) \leq 1$

P2. $S_{RTNSF}(A, B) = S_{RTNSF}(B, A)$

P3. $S_{RTNSF}(A, B) = 1$ if $A = B$

Proof:

P1. Since $2\nabla_{\theta_1}\nabla_{\theta_2} \le \nabla_{\theta_1}{}^2+\nabla_{\theta_2}{}^2$ and $2\nabla_{\varphi_1}\nabla_{\varphi_2} \le \nabla_{\varphi_1}{}^2+\nabla_{\varphi_2}{}^2$ so it is obvious that $0 \le S_{RTNSF}(A, B) \le 1$

P2. Obviously, $S_{RTNSF}(A, B) = S_{RTNSF}(B, A)$

P3. When A = B then, $\nabla_{\theta_1}=\nabla_{\theta_2}$ and $\nabla_{\varphi_1}=\nabla_{\varphi_2}$ so, $S_{RTNSF}(A, B) = (1/2) \times (1+1) = 1$.

When, $S_{RTNSF}(A, B) = 1$ then, $2\nabla_{\theta_1}\nabla_{\theta_2} = \nabla_{\theta_1}{}^2+\nabla_{\theta_2}{}^2$ and $2\nabla_{\varphi_1}\nabla_{\varphi_2} = \nabla_{\varphi_1}{}^2+\nabla_{\varphi_2}{}^2$. It is possible when $\nabla_{\theta_1}=\nabla_{\theta_2}$ and $\nabla_{\varphi_1}=\nabla_{\varphi_2}$. This implies that A = B.

Alternative proof:

Assume that

$$H(A, B) = \frac{1}{2}\left[\frac{1}{1+\tan^2(\alpha_1-\alpha_2)}+\frac{1}{1+\tan^2(\beta_1-\beta_2)}\right]$$

$$=\frac{1}{2}\left[\frac{(1+\tan\alpha_1\tan\alpha_2)^2}{1+\tan^2\alpha_1+\tan^2\alpha_2+\tan^2\alpha_1\tan^2\alpha_2}+\frac{(1+\tan\beta_1\tan\beta_2)^2}{1+\tan^2\beta_1+\tan^2\beta_2+\tan^2\beta_1\tan^2\beta_2}\right]$$

Taking, $\tan\alpha_1=\nabla_{\theta_1}$, $\tan\alpha_2=\nabla_{\theta_2}$, $\tan\beta_1=\nabla_{\varphi_1}$, $\tan\beta_2=\nabla_{\varphi_2}$, then,

$$H(A, B) = S_{RTNSF}(A, B).$$

The function H(A, B) obviously satisfies the following conditions.

P1. $0 \le H(A, B) \le 1$ (obvious)

P2. H(A, B) = H(B, A) (obvious)

P3. When A = B then $\alpha_1=\alpha_2$ and $\beta_1=\beta_2$ then, H(A, B) = 1.

Conversely, if H(A, B) = 1 then obviously, $\alpha_1=\alpha_2$ and $\beta_1=\beta_2$.

This implies that A = B.

5 Decision making procedure under rough tri-complex neutrosophic similarity measure

In this section, we apply rough tri-complex similarity measures between RNSs to the multi-criteria decision making problem. Let $A = A_1, A_2, \dots, A_m$ be a set of alternatives and $C = C_1, C_2, \dots, C_n$ be a set of attributes.

The proposed decision making method is described using the following steps.

Step 1: Construction of the decision matrix with rough neutrosophic number

The decision maker considers a decision matrix with respect to *m* alternatives and *n* attributes in terms of rough neutrosophic numbers as follows.

$$D = \langle \underline{d}_{ij}, \overline{d}_{ij} \rangle_{m\times n} =$$

$$\begin{array}{c|cccc} & C_1 & C_2 & \cdots & C_n \\ \hline A_1 & \langle \underline{d}_{11}, \overline{d}_{11} \rangle & \langle \underline{d}_{12}, \overline{d}_{12} \rangle & \cdots & \langle \underline{d}_{1n}, \overline{d}_{1n} \rangle \\ A_2 & \langle \underline{d}_{21}, \overline{d}_{21} \rangle & \langle \underline{d}_{22}, \overline{d}_{22} \rangle & \cdots & \langle \underline{d}_{2n}, \overline{d}_{2n} \rangle \\ . & \cdots & \cdots & \cdots & \cdots \\ . & \cdots & \cdots & \cdots & \cdots \\ A_m & \langle \underline{d}_{m1}, \overline{d}_{m1} \rangle & \langle \underline{d}_{m2}, \overline{d}_{m2} \rangle & \cdots & \langle \underline{d}_{mn}, \overline{d}_{mn} \rangle \end{array} \quad (11)$$

Table1. Rough neutrosophic decision matrix.

Here $\langle \underline{d}_{ij}, \overline{d}_{ij} \rangle$ is the rough neutrosophic number according to the i-th alternative and the j-th attribute.

Step 2: Determination of the weights of attribute

Assume that the weight of the attributes C (j = 1, 2, … , n) considered by the decision-maker be w_j ((j = 1, 2, … , n)) such that $\forall w_j \in [0, 1]$ (j = 1, 2, …, n) and $\sum_{j=1}^{n} w_j = 1$.

Step 3: Determination of the benefit type attribute and cost type attribute

Generally, the evaluation attribute can be categorized into two types: benefit attribute and cost attribute. Let K be a set of benefit attribute and M be a set of cost attribute. In the proposed decision-making method, an ideal alternative can be identified by using a maximum operator for the benefit attribute and a minimum operator for the cost attribute to determine the best value of each criterion among all alternatives. Therefore, we define an ideal alternative as follows:

$$A^* = \{C_1^*, C_2^*, \ldots, C_m^*\},$$

where benefit attribute

$$C_j^* = \left[\max_i T_{C_j}^{(A_i)}, \min_i I_{C_j}^{(A_i)}, \min_i F_{C_j}^{(A_i)} \right]$$

and the cost attribute

$$C_j^* = \left[\min_i T_{C_j}^{(A_i)}, \max_i I_{C_j}^{(A_i)}, \max_i F_{C_j}^{(A_i)} \right]$$

Step 4: Determination of the overall weighted rough tri-complex neutrosophic similarity function (WRTNSF) of the alternatives

We define weighted rough tri-complex neutrosophic similarity function as follows.

$$S_{WRTNSF}(A, B) = \sum_{j=1}^{n} w_j S_{WRTNSF}(A, B) \quad (12)$$

Properties:

This weighted rough tri-complex neutrosophic operator satisfies the following conditions of similarity measures.

P1. $0 \leq S_{WRTNSF}(A, B) \leq 1$

P2. $S_{WRTNSF}(A, B) = S_{WRTNSF}(B, A)$

P3. $S_{WRTNSF}(A, B) = 1$ if $A = B$

Proofs:

P1. Since $2D_{\theta_1}D_{\theta_2} \leq D_{\theta_1}^2 + D_{\theta_2}^2$ and $2D_{\phi_1}D_{\phi_2} \leq D_{\phi_1}^2 + D_{\phi_2}^2$ and $\sum_{j=1}^{n} w_j = 1$, so it is obvious that $0 \leq S_{WRTNSF}(A, B) \leq 1$

P2. Obviously, $S_{WRTNSF}(A, B) = S_{WRTNSF}(B, A)$

P3. When A = B then, $D_{\theta_1} = D_{\theta_2}$ and $D_{\phi_1} = D_{\phi_2}$ so, $S_{WRTNSF}(A, B) = \sum_{j=1}^{n} w_j = 1$.

When, $S_{WRTNSF}(A, B) = 1$ then, $2D_{\theta_1}D_{\theta_2} = D_{\theta_1}^2 + D_{\theta_2}^2$ and $2D_{\phi_1}D_{\phi_2} = D_{\phi_1}^2 + D_{\phi_2}^2$. It is possible when $D_{\theta_1} = D_{\theta_2}$ and $D_{\phi_1} = D_{\phi_2}$. Again, $\sum_{j=1}^{n} w_j = 1$. This implies that A = B.

Step 5: Ranking the alternatives

Using the weighted rough tri-complex neutrosophic similarity measure between each alternative and the ideal alternative, the ranking order of all alternatives can be determined and the best alternative can be easily selected with the highest similarity value.

Step 6: End.

6 Numerical Example

Let us assume that a decision maker intends to select the most suitable smartphone for rough use from the four initially chosen smartphones (S_1, S_2, S_3) by considering four attributes namely: features C_1, reasonable price C_2, customer care C_3, risk factor C_4. Based on the proposed approach discussed in section 5, the considered problem is solved using the following steps:

Step 1: Construction of decision matrix with rough neutrosophic numbers

The decision maker considers a decision matrix with respect to three alternatives and four attributes in terms of rough neutrosophic numbers as follows (see the *Table 2*).

$$d_S = \left\langle \underline{N}(P), \overline{N}(P) \right\rangle_{3\times 4} =$$

	C_1	C_2	C_3	C_4
A_1	$\left\langle \begin{matrix}(0.6,0.3,0.3),\\(0.8,0.1,0.1)\end{matrix} \right\rangle$	$\left\langle \begin{matrix}(0.6,0.4,0.4),\\(0.8,0.2,0.2)\end{matrix} \right\rangle$	$\left\langle \begin{matrix}(0.6,0.4,0.4),\\(0.8,0.2,0.2)\end{matrix} \right\rangle$	$\left\langle \begin{matrix}(0.7,0.4,0.4),\\(0.9,0.2,0.2)\end{matrix} \right\rangle$
A_2	$\left\langle \begin{matrix}(0.7,0.3,0.3),\\(0.9,0.1,0.3)\end{matrix} \right\rangle$	$\left\langle \begin{matrix}(0.6,0.3,0.3),\\(0.8,0.3,0.3)\end{matrix} \right\rangle$	$\left\langle \begin{matrix}(0.6,0.2,0.2),\\(0.8,0.4,0.2)\end{matrix} \right\rangle$	$\left\langle \begin{matrix}(0.7,0.3,0.3),\\(0.9,0.3,0.3)\end{matrix} \right\rangle$
A_3	$\left\langle \begin{matrix}(0.6,0.2,0.2),\\(0.8,0.0,0.2)\end{matrix} \right\rangle$	$\left\langle \begin{matrix}(0.7,0.3,0.3),\\(0.9,0.1,0.1)\end{matrix} \right\rangle$	$\left\langle \begin{matrix}(0.7,0.4,0.6),\\(0.9,0.2,0.4)\end{matrix} \right\rangle$	$\left\langle \begin{matrix}(0.6,0.3,0.2),\\(0.8,0.1,0.2)\end{matrix} \right\rangle$

Table 2. Decision matrix with rough neutrosophic number.

Step 2: Determination of the weights of the attributes

The weight vectors considered by the decision maker are 0.30, 0.30, 0.30 and 0.10 respectively.

Step 3: Determination of the benefit attribute and cost attribute

Here three benefit types attributes C_1, C_2, C_3 and one cost type attribute C_4.

$$A^* = [(0.8, 0.1, 0.2), (0.8, 0.2, 0.2), (0.8, 0.3, 0.3), (0.0.7, 0.3,0.3)]$$

Step 4: Determination of the overall weighted rough tri-complex neutrosophic similarity function (WRHNSF)of the alternatives

We calculate weighted rough tri-complex neutrosophic similarity values as follows.

$$S_{WRTNSF}(A_1, A^*) = 0.99554$$

$$S_{WRTNSF}(A_2, A^*) = 0.99253$$

$$S_{WRTNSF}(A_3, A^*) = 0.99799$$

Step 5: Ranking the alternatives

Ranking the alternatives is prepared based on the descending order of similarity measures. Highest value reflects the best alternative. Here,

$$S_{WRTNSF}(A_3, A^*) \prec S_{WRTNSF}(A_1, A^*) \prec S_{WRTNSF}(A_2, A^*).$$

Hence, the Smartphone A_3 is the best alternative for rough use.

Step 6: End.

7 Comparison with other similarity measures

We compare our result to other existing rough neutrosophic similarity measures as follows.

Rough neutrosophic similarity measure	*Measure value*	*Ranking order*
Weighted rough Cosine similarity measure	$C_{WRNS}(A_1, A^*) = 0.99260$ $C_{WRNS}(A_2, A^*) = 0.99083$ $C_{WRNS}(A_3, A^*) = 0.99482$	$A_3 \succ A_1 \succ A_2$
Weighted rough Dice similarity measure	$D_{WRNS}(A_1, A^*) = 0.98606$ $D_{WRNS}(A_2, A^*) = 0.98559$ $D_{WRNS}(A_3, A^*) = 0.98926$	$A_3 \succ A_1 \succ A_2$
Weighted rough Jaccard similarity measure	$J_{WRNS}(A_1, A^*) = 0.97856$ $J_{WRNS}(A_2, A^*) = 0.97772$ $J_{WRNS}(A_3, A^*) = 0.97891$	$A_3 \succ A_1 \succ A_2$
Weighted rough Tri-complex similarity measure	$S_{WRTNSF}(A_1, A^*) = 0.99554$ $S_{WRTNSF}(A_2, A^*) = 0.99253$ $S_{WRTNSF}(A_3, A^*) = 0.99799$	$A_3 \succ A_1 \succ A_2$

Table 3. Comparison with other existing rough neutrosophic similarity measures.

8 Conclusion

In this paper, we have proposed rough tri-complex similarity measure based multi-attribute decision making of rough neutrosophic environment and proved some of its basic properties. We have presented an application, namely selection of best smart-phone for rough use. We have also presented comparison with other existing rough neutrosophic similarity measures. In this paper, predefined weights of the decision makers have been considered. The proposed approach can be extended for generalized hypercomplex system with weighting scheme.

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Generalized Neutrosophic Soft Multi-Attribute Group Decision Making Based on TOPSIS

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Abstract

In this study, we present Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) method for solving generalized neutrosophic soft multi-attribute group decision making problem. The concept of generalized neutrosophic soft set is the hybridization of the two concepts namely generalized neutrosophic sets and soft sets. In the decision making process, the ratings of alternatives with respect to the parameters are expressed in terms of generalized neutrosophic sets. The evaluator selects the choice parameters and AND operator of generalized neutrosophic soft sets. Generalized neutrosophic soft set is used to aggregate the individual decision maker's opinion into a single opinion based on the performance values of the choice parameters. The weights of the choice parameters are derived from information entropy method. Then, the preference of alternatives is ranked by using TOPSIS method. Finally, a numerical example is solved to show the potential applicability and effectiveness of the proposed method.

Keyword

neutrosophic set, soft set, generalized neutrosophic soft set, TOPSIS, information entropy method, multi-attribute group decision making.

1 Introduction

Multi-attribute group decision making (MAGDM) is the process of determining the best option from a list of feasible alternatives with respect to several predefined attributes offered by the multiple decision makers (DMs).

However, the rating and the weights of the attributes cannot always be preciously assessed in terms of crisp numbers due to the ambiguity of human decision and the complexity of the attributes. In order to overcome the abovementioned difficulties, Zadeh [37] proposed fuzzy set theory by introducing membership function $\mathrm{T_A}(x)$ to deal with uncertainty and partial information. Atanassov [3] incorporated the degree of non-membership as independent component and defined intuitionistic fuzzy. Smarandache [28, 29, 30, 31] proposed neutrosophic sets (NSs) by introducing degree of indeterminacy $\mathrm{I_A}(x)$ as independent element in intuitionistc fuzzy set for handling incomplete, imprecise, inconsistent information. Later, Salama and Alblowi [27] defined generalized neutrosophic sets (GNSs), where the triplet functions satisfy the condition $\mathrm{T_A}(x) \wedge \mathrm{F_A}(x) \wedge \mathrm{I_A}(x) \leq 0.5$.

In 1999, Molodtsov [23] introduced the notion of soft set theory for dealing with uncertainty and vagueness and the concept has been applied diverse practical fields such as decision making [16, 17, 18, 24], data analysis [38], forecasting [33], optimization [14], etc. Several researchers have incorporated different mathematical hybrid structures such as fuzzy soft sets [10, 11, 19], intuitionistic fuzzy soft set theory [8, 9, 20], possibility fuzzy soft set [2], generalized fuzzy soft sets [22, 35], generalized intuitionistic fuzzy soft [4], possibility intuitionistic fuzzy soft set [5], vague soft set [34], possibility vague soft set [1], neutrosophic soft sets [17], weighted neutrosophic soft sets [16], etc by generalizing and extending classical soft set theory of Molodtsov [23]. Recently, Broumi [7] studied generalized neutrosophic soft sets (GNSSs) and provided some definitions and operations of the concept. He also provided an application of GNSSs in decision making problem. Şahin, and Küçük [25] discussed a method to find out similarity measures of two GNSSs and provided an application of GNSS in decision making problem.

Hwang and Yoon [13] developed Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) method for solving classical multi-attribute decision making (MADM) problems. Liu et al. [15] proposed a new method based on generalized neutrosophic number Hamacher aggregation operators for MAGDM with single valued neutrosophic numbers. Ye [36] investigated an extended TOPSIS method for solving a MADM problems based on the single valued neutrosophic linguistic numbers under single valued neutrosophic linguistic assessment. Biswas et al. [6] extended the notion of TOPSIS method for MAGDM problems under single valued neutrosophic environment. In the paper, we have demonstrated a new mathematical model for solving generalized neutrosophic soft MAGDM problem based on TOPSIS method.

The content of the paper is structured as follows. Section 2 presents some basic definitions regarding NSs, soft sets, GNSs and GNSSs which will be useful for the construction of the paper. Section 3 is devoted to describe TOPSIS method for solving MAGDM problems under generalized neutrosophic soft environment. Section 4 is devoted to present the algorithm of the proposed TOPSIS method. A numerical problem regarding flat selection is presented to show the applicability of the proposed method in Section 5. Section 6 presents the concluding remarks and future scope of research.

2 Preliminaries

In this section, we present basic definitions regarding NSs, soft sets, GNSs and GNSSs.

2.1 Neutrosophic Set [28, 29, 30, 31]

Consider U be a space of objects with a generic element of U represented by x. Then, a neutrosophic set N on U is represented as follows:

$$N = \{x, \langle \mathrm{T}_N(x), \mathrm{I}_N(x), \mathrm{F}_N(x) \rangle \mid x \in U\}$$

where, $\mathrm{T}_N(x), \mathrm{I}_N(x), \mathrm{F}_N(x) : U \to]^-0, 1^+[$ present respectively the degrees of truth-membership, indeterminacy-membership, and falsity-membership of a point $x \in U$ to the set N with the condition $^-0 \le \mathrm{T}_N(x) + \mathrm{I}_N(x) + \mathrm{F}_N(x) \le 3^+$.

2.2 Generalized Neutrosophic Set [27]

Let U be a universe of discourse, with a generic element in U denoted by x. Then, a generalized neutrosophic set $G \subset U$ is represented as follows:

$$G = \{x, \langle \mathrm{T}_G(x), \mathrm{I}_G(x), \mathrm{F}_G(x) \rangle \mid x \in U\}$$

where, $\mathrm{T}_G(x)$, $\mathrm{I}_G(x)$, $\mathrm{F}_G(x)$ denote respectively the truth-membership function, indeterminacy-membership function, and falsity-membership function of a point $x \in U$ to the set G where the functions satisfy the condition $\mathrm{T}_G(x) \wedge \mathrm{I}_G(x) \wedge \mathrm{F}_G(x) \le 0.5$.

Definition 2.2.1 [21]

The Euclidean distance between two GNSs $S_1 = \{x_i, \langle \mathrm{T}_{S_1}(x_i), \mathrm{I}_{S_1}(x_i), \mathrm{F}_{S_1}(x_i) \rangle \mid x_i \in U\}$ and $S_2 = \{x_i, \langle \mathrm{T}_{S_2}(x_i), \mathrm{I}_{S_2}(x_i), \mathrm{F}_{S_2}(x_i) \rangle \mid x_i \in U\}$ is defined as follows:

$$\mathrm{D_{Euc}}(S_1, S_2) = \sqrt{\sum_{j=1}^{n}\left\{(\mathrm{T}_{S_1}(x_j) - \mathrm{T}_{S_2}(x_j))^2 + (\mathrm{I}_{S_1}(x_j) - \mathrm{I}_{S_2}(x_j))^2 + (\mathrm{F}_{S_1}(x_j) - \mathrm{F}_{S_2}(x_j))^2\right\}} \quad (1)$$

and the normalized Euclidean distance between two GNSs S_1 and S_2 can be defined as follows:

$$D_{Euc}^{N}(S_1,S_2$$
$$\sqrt{\frac{1}{3n}\sum_{j=1}^{n}\left\{(T_{S_1}(x_j)-T_{S_2}(x_j))^2+(I_{S_1}(x_j)-I_{S_2}(x_j))^2+(F_{S_1}(x_j)-F_{S_2}(x_j))^2\right\}} \quad (2)$$

2.3 Soft set [23]

Let X be a universal set and E be a set of parameters. Consider P (X) represents a power set of X. Also, let F be a non-empty set, where F $\subset$ E. Then, a pair (Θ, F) is called a soft set over U, where Θ is a mapping given by Θ : F $\rightarrow$ P (X).

2.4 Generalized neutrosophic soft sets [7]

Suppose X is a universal set and E is a set of parameters. Let A be a non-empty subset of E and GNS (X) denotes the set of all generalized neutrosophic sets of X. Then, the pair (Θ, A) is termed to be a GNSS over X, where Θ is a mapping given by Θ : A$\rightarrow$ GNS (X).

Example:

Let X be the set of citizens under consideration and E = {very rich, rich, upper-middle-income, middle-income, lower-middle-income, poor, below-poverty-line} be the set of parameters (or qualities). Each parameter is a generalized neutrosophic word or sentence regarding generalized neutrosophic word. Here, to describe GNSS means to indicate very rich citizens, rich citizens, citizens of lower-middle-income, poor citizens, etc. Consider four citizens in the universe X given by X = (x_1, x_2, x_3, x_4) and A = {a_1, a_2, a_3, a_4} be a set of parameters, where a_1, a_2, a_3, a_4 stand for the parameters 'rich', 'middle-income', 'poor', 'below-poverty-line' respectively. Suppose that

Θ (rich) = {< a_1, 0.8, 0.3, 0.2>, < a_2, 0.6, 0.3, 0.3>, < a_3, 0.7, 0.4, 0.2>, < a_4, 0.6, 0.1, 0.2>},
Θ (middle-income) = {< a_1, 0.6, 0.1, 0.1>, < a_2, 0.5, 0.3, 0.4>, < a_3, 0.8, 0.4, 0.3>, < a_4, 0.5, 0.2, 0.2>},
Θ (poor) = {< a_1, 0.8, 0.4, 0.3>, < a_2, 0.6, 0.4, 0.1>, < a_3, 0.7, 0.3, 0.5>, < a_4, 0.7, 0.2, 0.2>},
Θ (below-poverty-line) = {< a_1, 0.8, 0.4, 0.4>, < a_2, 0.6, 0.2, 0.5>, < a_3, 0.5, 0.2, 0.2>, < a_4, 0.7, 0.4, 0.5>}.

Consequently, Θ (rich) represents rich citizens, Θ (middle-income) represents citizens of middle-income, Θ (poor) represents poor citizens and Θ (below-poverty-line) represents citizens of below-poverty-line. Therefore, the tabular representation of GNSS (Θ, A) is given below (see *Table1*).

X	a_1 = rich	a_2 = middle-income	a_3 = poor	a_4 = below-poverty-line
x_1	(0.8, 0.3, 0.2)	(0.6, 0.1, 0.1)	(0.8, 0.4, 0.3)	(0.8, 0.4, 0.4)
x_2	(0.6, 0.3, 0.3)	(0.5, 0.3, 0.4)	(0.6, 0.4, 0.1)	(0.6, 0.2, 0.5)
x_3	(0.7, 0.4, 0.2)	(0.8, 0.4, 0.3)	(0.7, 0.3, 0.5)	(0.5, 0.2, 0.2)
x_4	(0.6, 0.1, 0.2)	(0.5, 0.2, 0.2)	(0.7, 0.2, 0.2)	(0.7, 0.4, 0.5)

Table 1. Tabular representation of GNSS (Θ , A)

Definition 2.4.1 [7]

Consider (Θ_1, A) and (Θ_2, B) be two GNSSs over a common universe U. The union (Θ_1, A) and (Θ_2, B) is defined by (Θ_1, A) $\cup$ (Θ_2, B) = (Θ_3, C), where C = A $\cup$ B. The truth-membership, indeterminacy-membership and falsity-membership functions of (Θ_3, C) are presented as follows:

$$T_{\Theta_3(e)}(m) = T_{\Theta_1(e)}(m), \text{ if } e \in \Theta_1 - \Theta_2,$$
$$= T_{\Theta_2(e)}(m), \text{ if } e \in \Theta_2 - \Theta_1,$$
$$= \text{Max}\,(T_{\Theta_1(e)}(m), T_{\Theta_2(e)}(m)), \text{ if } e \in \Theta_1 \cap \Theta_2.$$

$$I_{\Theta_3(e)}(m) = I_{\Theta_1(e)}(m), \text{ if } e \in \Theta_1 - \Theta_2,$$
$$= I_{\Theta_2(e)}(m), \text{ if } e \in \Theta_2 - \Theta_1,$$
$$= \text{Min}\,(I_{\Theta_1(e)}(m), I_{\Theta_2(e)}(m)), \text{ if } e \in \Theta_1 \cap \Theta_2.$$

$$F_{\Theta_3(e)}(x) = F_{\Theta_1(e)}(m), \text{ if } e \in \Theta_1 - \Theta_2,$$
$$= F_{\Theta_2(e)}(m), \text{ if } e \in \Theta_2 - \Theta_1,$$
$$= \text{Min}\,(F_{\Theta_1(e)}(m), F_{\Theta_2(e)}(m)), \text{ if } e \in \Theta_1 \cap \Theta_2.$$

Definition 2.4.2 [7]

Suppose (Θ_1, A) and (Θ_2, B) are two GNSSs over the same universe X The intersection (Θ_1, A) and (Θ_2, B) is defined by (Θ_1, A) $\cap$ (Θ_2, B) = (Θ_4, D), where D = A $\cap$ B ($\neq \varphi$) and the truth-membership, indeterminacy-membership and falsity-membership functions of (Θ_4, D) are defined as follows:

$$T_{\Theta_4(e)}(x) = \text{Min}\,(T_{\Theta_1(e)}(m), T_{\Theta_2(e)}(m)), \; I_{\Phi_4(e)}(m) = \text{Min}\,(I_{\Theta_1(e)}(m), I_{\Theta_2(e)}(m)), F_{\Phi_4(e)}(m) = \text{Max}\,(F_{\Theta_1(e)}(m), F_{\Theta_2(e)}(m)), \; \forall e \in D.$$

Definition 2.4.3 [7]

Let (Θ_1, A) and (Θ_2, B) be two GNSSs over the identical universe U. Then 'AND' operation on (Θ_1, A) and (Θ_2, B) is defined by (Θ_1, A) $\wedge$ (Θ_2, B) = (Θ_5, K), where K = $A \times B$ and the truth-membership, indeterminacy-membership and falsity-membership functions of (Φ_5, $A \times B$) are defined as follows:

$T_{\Theta_5(\gamma,\delta)}(m)$ = Min ($T_{\Theta_1(\gamma)}(m)$, $T_{\Theta_2(\delta)}(m)$), $I_{\Theta_5(\gamma,\delta)}(m)$ = Min ($I_{\Theta_1(\gamma)}(m)$, $I_{\Theta_2(\delta)}(m)$), $F_{\Theta_5(\gamma,\delta)}(m)$ = Max ($F_{\Theta_1(\gamma)}(m)$, $F_{\Theta_2(\delta)}(m)$), $\forall\ \gamma \in A$, $\forall\ \delta \in B$, $m \in X$.

3 A generalized neutrosophic soft MAGDM based on TOPSIS method

Let C = {C_1, C_2, ..., C_n}, ($n \geq 2$) be a discrete set of alternatives in a MAGDM problem with p DMs. Let q be the total number of parameters involved in the problem, where q_i be number of parameters under the assessment of DM_i (i = 1, 2, ..., p) such that q = $\sum_{i=1}^{p} q_i$. The rating of performance value of alternative C_i, (i = 1, 2, ..., n) with respect to the choice parameters is provided by the DMs and they can be expressed in terms of GNSs. The procedure for solving neutrosophic soft MAGDM problem based on TOPSIS method is described as follows:

Step 1. Formulation of criterion matrix with SVNSs

Suppose that the rating of alternative C_i (i = 1, 2, ..., n) with respect to the choice parameter provided by the s-th (s = 1, 2, ..., p) DM is represented by GNSS (Θ_s, H_s), (s = 1, 2, ..., p) and they can be presented in matrix form $d^s_{G_{ij}}$ (i = 1, 2, ..., n, j = 1, 2, ..., q_s; s = 1, 2, ..., p). Therefore, criterion matrix for s-th DM can be explicitly formulated as follows:

$$D_G^s = \left\langle d^s_{ij} \right\rangle_{n \times q_s} = \begin{bmatrix} d^s_{11} & d^s_{12} & \dots & d^s_{1q_s} \\ d^s_{21} & d^s_{22} & \dots & d^s_{2q_s} \\ . & . & \dots & . \\ . & . & \dots & . \\ d^s_{n1} & d^s_{n2} & \dots & d^s_{nq_s} \end{bmatrix}$$

Here, d^s_{ij} = (T^s_{ij}, I^s_{ij}, F^s_{ij}) where $T^s_{ij}, I^s_{ij}, F^s_{ij} \in [0, 1]$ and $0 \leq T^s_{ij} + I^s_{ij} + F^s_{ij} \leq 3$, i = 1, 2, ..., n; j = 1, 2, ..., q_s; s =1, 2, ..., p.

Step 2. Formulation of combined criterion matrix with GNSs

In the group decision making problem, DMs assessments need to be fused into a group opinion based on the choice parameters of the evaluator. Suppose the evaluator considers *r* number of choice parameters in the decision making situation. Using 'AND' operator of GNSSs proposed by Broumi [7], the resultant GNSSs is placed in the decision matrix D_G as follows:

$$D_G = \langle d_{ij} \rangle_{p\times r} = \begin{bmatrix} d_{11} & d_{12} & \dots & d_{1r} \\ d_{21} & d_{22} & \dots & d_{2r} \\ . & . & \dots & . \\ . & . & \dots & . \\ d_{n1} & d_{n2} & \dots & d_{nr} \end{bmatrix}$$

Here, $d_{ij} = \langle T'_{ij}, I'_{ij}, F'_{ij} \rangle$ where $T'_{ij}, I'_{ij}, F'_{ij} \in [0, 1]$ and $0 \leq T'_{ij} + I'_{ij} + F'_{ij} \leq 3$, i = 1, 2, ..., n; j = 1, 2, ..., r.

Step 3. Determination of weights of the choice parameters

The evaluator selects the choice parameters in the decision making situation. In general, the weights of the choice parameters are dissimilar and completely unknown to the evaluator. In this paper, we use information entropy method in order to achieve the weights of the choice parameters. The entropy value H_j of the j-th attribute can be defined as follows:

$$H_j = 1 - \frac{1}{r} \sum_{i=1}^{p} (T_{ij}(x_i) + F_{ij}(x_i)) \left| I_{ij}(x_i) - I_{ij}^C(x_i) \right|, \ j = 1, 2, \dots, r \qquad (3)$$

Here, $0 \leq H_j \leq 1$ and the entropy weight [12, 32] of the j-th attribute is obtained from the Eq. as given below.

$$w_j = \frac{1 - H_j}{\sum_{j=1}^{r} (1 - H_j)}, \text{ with } 0 \leq w_j \leq 1 \text{ and } \sum_{j-1}^{r} w_j = 1. \qquad (4)$$

Step 4. Construction of weighted decision matrix

We obtain aggregated weighted decision matrix by multiplying weights (w_j) [26] of the choice parameters and aggregated decision matrix $\langle d_{ij} \rangle_{p\times r}$ as follows:

$$D_G^w = D_G \otimes w = \langle d_{ij} \rangle_{n\times r} \otimes w_j = \langle d_{ij}^{w_j} \rangle_{n\times r} = \begin{bmatrix} d_{11}^{w_1} & d_{12}^{w_2} & \dots & d_{1r}^{w_r} \\ d_{21}^{w_1} & d_{22}^{w_2} & \dots & d_{2r}^{w_r} \\ . & . & \dots & . \\ . & . & \dots & . \\ d_{n1}^{w_1} & d_{n1}^{w_2} & \dots & d_{nr}^{w_r} \end{bmatrix}$$

Here, $d_{ij}^{w_j} = \left\langle T_{ij}^{w_j}, I_{ij}^{w_j}, F_{ij}^{w_j} \right\rangle$ where $T_{ij}^{w_j}, I_{ij}^{w_j}, F_{ij}^{w_j} \in [0, 1]$ and $0 \le T_{ij}^{w_j} + I_{ij}^{w_j} + F_{ij}^{w_j} \le 3$, i = 1, 2, ..., n; j = 1, 2, ..., r.

Step 5. Determination of relative positive ideal solution (RPIS) and relative negative ideal solution (RNIS)

In practical decision making, the attributes are classified into two categories namely benefit type attributes (J_1) and cost type attributes (J_2). Let, R_G^{w+} and R_G^{w-} be the relative positive ideal solution (RPIS) and relative negative ideal solution (RNIS). Then, R_G^{w+} and R_G^{w-} can be defined as follows:

$$R_G^{w+} = (\left\langle T_1^{w_1+}, I_1^{w_1+}, F_1^{w_1+} \right\rangle, \left\langle T_2^{w_2+}, I_2^{w_2+}, F_2^{w_2+} \right\rangle, \ldots, \left\langle T_r^{w_r+}, I_r^{w_r+}, F_r^{w_r+} \right\rangle)$$

$$R_G^{w-} = (\left\langle T_1^{w_1-}, I_1^{w_1-}, F_1^{w_1-} \right\rangle, \left\langle T_2^{w_2-}, I_2^{w_2-}, F_2^{w_2-} \right\rangle, \ldots, \left\langle T_r^{w_r-}, I_r^{w_r-}, F_r^{w_r-} \right\rangle)$$

where

$\left\langle T_j^{w_j+}, I_j^{w_j+}, F_j^{w_j+} \right\rangle$ = < [{ $\underset{i}{\mathrm{Max}}(T_{ij}^{w_j})$ | j∈ J_1}; { $\underset{i}{\mathrm{Min}}(T_{ij}^{w_j})$ | j∈J_2}],
[{ $\underset{i}{\mathrm{Min}}(I_{ij}^{w_j})$ | j∈ J_1}; { $\underset{i}{\mathrm{Max}}(I_{ij}^{w_j})$ | j∈ J_2}], [{ $\underset{i}{\mathrm{Min}}(F_{ij}^{w_j})$ | j∈ J_1};
{ $\underset{i}{\mathrm{Max}}(F_{ij}^{w_j})$ | j∈ J_2}] >, j = 1, 2, ..., r,

$\left\langle T_j^{w_j-}, I_j^{w_j-}, F_j^{w_j-} \right\rangle$ = < [{ $\underset{i}{\mathrm{Min}}(T_{ij}^{w_j})$ | j∈ J_1}; { $\underset{i}{\mathrm{Max}}(T_{ij}^{w_j})$ | j∈J_2}],
[{ $\underset{i}{\mathrm{Max}}(I_{ij}^{w_j})$ | j∈ J_1}; { $\underset{i}{\mathrm{Min}}(I_{ij}^{w_j})$ | j∈ J_2}], [{ $\underset{i}{\mathrm{Max}}(F_{ij}^{w_j})$ | j∈ J_1};
{ $\underset{i}{\mathrm{Min}}(F_{ij}^{w_j})$ | j∈ J_2}] >, j = 1, 2, ..., r.

Step 6. Calculation of distance measure of each alternative from RPIS and RNIS

The normalized Euclidean distance of each alternative $\left\langle T_{ij}^{w_j}, I_{ij}^{w_j}, F_{ij}^{w_j} \right\rangle$ from the RPIS $\left\langle T_j^{w_j+}, I_j^{w_j+}, F_j^{w_j+} \right\rangle$ for i = 1, 2, ..., n; j = 1, 2, ..., r can be defined as follows:

$$D_{Euc}^{i+}(d_{ij}^{w_j}, d_j^{w_j+})$$

$$\sqrt{\frac{1}{3r}\sum_{j=1}^{r}\left\{(T_{ij}^{w_j}(x_j) - T_j^{w_j+}(x_j))^2 + (I_{ij}^{w_j}(x_j) - I_j^{w_j+}(x_j))^2 + (F_{ij}^{w_j}(x_j) - F_j^{w_j+}(x_j))^2\right\}} \qquad (5)$$

Similarly, normalized Euclidean distance of each alternative $\left\langle T_{ij}^{w_j}, I_{ij}^{w_j}, F_{ij}^{w_j} \right\rangle$ from the RNIS $\left\langle T_j^{w_j-}, I_j^{w_j-}, F_j^{w_j-} \right\rangle$ for i = 1, 2, ..., n; j = 1, 2, ..., r can be written as follows:

$$D^{i-}_{Euc}(d^{w_j}_{ij}, d^{w_j-}_{j})$$

$$\sqrt{\frac{1}{3r}\sum_{j=1}^{r}\left\{(T^{w_j}_{ij}(x_j)-T^{w_j-}_{j}(x_j))^2+(I^{w_j}_{ij}(x_j)-I^{w_j-}_{j}(x_j))^2+(F^{w_j}_{ij}(x_j)-F^{w_j-}_{j}(x_j))^2\right\}} \quad (6)$$

Step 7. Computation of the relative closeness co-efficient to the neutrosophic ideal solution

The relative closeness co-efficient of each alternative C_i, (i = 1, 2, …, n) with respect to the RPIS is defined as follows:

$$\rho^*_i = \frac{D^{i-}_{Euc}(d^{w_j}_{ij}, d^{w_j-}_{j})}{D^{i+}_{Euc}(d^{w_j}_{ij}, d^{w_j+}_{j}) + D^{i-}_{Euc}(d^{w_j}_{ij}, d^{w_j-}_{j})} \quad (7)$$

where, $0 \le \rho^*_i \le 1$.

Step 8. Rank the alternatives

We rank the alternatives according to the values of ρ^*_i, i = 1, 2, …, n and bigger value of ρ^*_i, i = 1, 2, …, p reflects the better alternative.

4 Proposed TOPSIS algorithm for MAGDM problems

In sum, TOPSIS algorithm for generalized neutrosophic soft MAGDM problems is designed using the following steps:

Step 1. Formulate the criterion matrix D^s_G of the s-th decision maker, s = 1, 2, …, p.

Step 2. Establish the aggregated decision matrix D_G using AND operator GNSSs the based on the choice parameters of the evaluator.

Step 3. Determine the weight (w_j) of the choice parameters using Eq. (4).

Step 4. Construct the weighted aggregated decision matrix $D^w_G = \left\langle d^{w_j}_{ij} \right\rangle_{n\times r}$.

Step 5. Identify the relative positive ideal solution (R^{w+}_G) and relative negative ideal solution (R^{w-}_G).

Step 6. Compute the normalized Euclidean distance of each alternative from relative positive ideal solution (R^{w+}_G) and relative negative ideal solution (R^{w-}_G) by Eqs. (5) and (6) respectively.

Step 7. Calculate the relative closeness co-efficient ρ_i^* using Eq. (7) of each alternative C_i.

Step 8. Rank the preference order of alternatives according to the order of their relative closeness.

5 A numerical example

Let $F = \{f_1, f_2, f_3, f_4\}$ be the set of flats characterized by different locations, prices and constructions and E = {very good, good, average good, below average, bad, very costly, costly, moderate, cheap, new-construction, not so new-constructions, old-constructions, very old-constructions} be the set of parameters. Assume that E_1 = {very good, good}, E_2 = {very costly, costly, moderate}, E_3 = {new-construction, not so new-construction} are three subsets of E. Let the GNSSs (Θ_1, E_1), (Θ_2, E_2), (Θ_3, E_3) stand for the flats 'having diverse locations', 'having diverse prices', 'having diverse constructions' respectively and they are computed by the three DMs namely DM_1, DM_2 and DM_3 respectively. The criterion decision matrices for DM_1, DM_2 and DM_3 are presented (see *Table 2, Table 3, Table 4*) respectively as follows:

U	α_1 = very good	α_2 = good
f_1	(0.9, 0.3, 0.5)	(0.5, 0.3, 0.4)
f_2	(0.6, 0.4, 0.3)	(0.5, 0.2, 0.4)
f_3	(0.8, 0.2, 0.3)	(0.7, 0.5, 0.4)
f_4	(0.7, 0.2, 0.1)	(0.7, 0.5, 0.4)

Table 2: Tabular form of GNSS (Θ_1, E_1)

U	β_1 = very costly	β_2 = costly	β_3 = moderate
f_1	(0.9, 0.3, 0.1)	(0.7, 0.3, 0.4)	(0.6, 0.2, 0.4)
f_2	(0.8, 0.3, 0.2)	(0.6, 0.5, 0.4)	(0.5, 0.4, 0.3)
f_3	(0.8, 0.5, 0.4)	(0.7, 0.2, 0.3)	(0.8, 0.3, 0.2)
f_4	(0.7, 0.2, 0.4)	(0.8, 0.4, 0.5)	(0.6, 0.5, 0.3)

Table 3: Tabular form of GNSS (Θ_2, E_2)

U	λ_1 = new-construction	λ_2 = not so new-construction
f_1	(0.8, 0.4, 0.2)	(0.7, 0.4, 0.3)
f_2	(0.9, 0.1, 0.1)	(0.6, 0.3, 0.1)
f_3	(0.5, 0.4, 0.4)	(0.8, 0.3, 0.4)
f_4	(0.4, 0.3, 0.4)	(0.6, 0.3, 0.4)

Table 4: Tabular form of GNSS (Θ_3, E_3)

The proposed TOPSIS method for solving generalized soft MAGDM problem is presented in the following steps.

Step 1: If the evaluator wishes to perform the operation '(Θ_1, E_1) AND (Θ_2, E_2)' then we will get 2×3 parameters of the form μ_{ij}, where $\mu_{ij} = \alpha_i \wedge \beta_j$, for i = 1, 2; j = 1, 2, 3. Let S = {$\mu_{12}, \mu_{13}, \mu_{21}, \mu_{22}, \mu_{23}$} be the set of choice parameters of the evaluator, where μ_{12} = (very good, costly), μ_{13} = (very good, moderate), μ_{21} = (good, very costly), etc. (see *Table 5*).

U	μ_{12}	μ_{13}	μ_{21}	μ_{22}	μ_{23}
f_1	(0.7, 0.3, 0.5)	(0.6, 0.2, 0.5)	(0.5, 0.3, 0.4)	(0.5, 0.3, 0.4)	(0.5, 0.2, 0.4)
f_2	(0.6, 0.4, 0.4)	(0.5, 0.4, 0.3)	(0.5, 0.2, 0.4)	(0.5, 0.2, 0.4)	(0.5, 0.2, 0.4)
f_3	(0.7, 0.2, 0.3)	(0.8, 0.2, 0.3)	(0.7, 0.5, 0.4)	(0.7, 0.2, 0.4)	(0.7, 0.3, 0.4)
f_4	(0.7, 0.2, 0.5)	(0.6, 0.2, 0.3)	(0.6, 0.2, 0.4)	(0.6, 0.3, 0.5)	(0.6, 0.3, 0.4)

Table 5: Tabular form of '(Θ_1, E_1) AND (Θ_2, E_2)'

Now the evaluator desires to compute (Θ_5, T) from (Θ_4, S) AND (Θ_3, E_3) for the specified parameters T = {$\mu_{13} \wedge \lambda_1, \mu_{22} \wedge \lambda_1, \mu_{12} \wedge \lambda_2, \mu_{21} \wedge \lambda_2$}, where $\mu_{13} \wedge \lambda_1$ denotes (very good, moderate, new-construction), $\mu_{12} \wedge \lambda_2$ represents (very good, costly, not so new construction), etc, (see *Table 6*).

U	$\mu_{13} \wedge \lambda_1$	$\mu_{22} \wedge \lambda_1$	$\mu_{12} \wedge \lambda_2$	$\mu_{21} \wedge \lambda_2$
f_1	(0.6, 0.2, 0.5)	(0.5, 0.3, 0.4)	(0.7, 0.3, 0.5)	(0.5, 0.3, 0.4)
f_2	(0.5, 0.1, 0.3)	(0.5, 0.1, 0.4)	(0.6, 0.3, 0.4)	(0.5, 0.2, 0.4)
f_3	(0.5, 0.2, 0.4)	(0.5, 0.2, 0.4)	(0.7, 0.2, 0.4)	(0.7, 0.3, 0.4)
f_4	(0.4, 0.2, 0.4)	(0.4, 0.3, 0.5)	(0.6, 0.2, 0.5)	(0.6, 0.2, 0.4)

Table 6: Tabular form of '(Θ_4, S) AND (Θ_3, E_3)'

Step 2. Computation of the weights of the parameters

Entropy value H_j (j = 1, 2, 3, 4) of the j-th choice parameter can be determined from Eq. (3) as follows:
H_1 = 0.42, H_2 = 0.505, H_3 = 0.45, H_4 = 0.515.
Then, normalized entropy weights are obtained as follows:
w_1 = 0.2712, w_2 = 0.2318, w_3 = 0.2564, w_4 = 0.2406, where $\sum_{j-1}^{4} w_j = 1$.

Step 3. Formulation of weighted decision matrix of the choice parameters

The tabular form of the weighted decision matrix is presented in the *Table 7*).

U	$w_1 \otimes (\mu_{13} \wedge \lambda_1)$	$w_2 \otimes (\mu_{22} \wedge \lambda_1)$	$w_3 \otimes (\mu_{12} \wedge \lambda_2)$	$w_4 \otimes (\mu_{21} \wedge \lambda_2)$
f_1	(0.22, 0.6463, 0.8286)	(0.1484, 0.7565, 0.8086)	(0.2656, 0.7344, 0.8372)	(0.1536, 0.7485, 0.8022)
f_2	(0.1714, 0.5356, 0.7214)	(0.1484, 0.5864, 0.8086)	(0.2094, 0.7344, 0.7906)	(0.1536, 0.6789, 0.8022)
f_3	(0.1714, 0.6463, 0.78)	(0.1484, 0.6886, 0.8086)	(0.2656, 0.6619, 0.7906)	(0.2515, 0.7485, 0.8022)
f_4	(0.1294, 0.6463, 0.78)	(0.1167, 0.7565, 0.8516)	(0.2094, 0.6619, 0.8372)	(0.1978, 0.6789, 0.8022)

Table 7: Tabular form of weighted decision matrix

Step 4. Determination of RPIS and RNIS

The RPIS (R_G^+) and RNIS (R_G^-) can be obtained from the weighted decision matrix as follows:

R_G^+ = < (0.22, 0.5356, 0.7214); (0.1484, 0.5864, 0.8086); (0.2656, 0.6619, 0.7906); (0.2515, 0.6789, 0.8022) >

R_G^- = < (0.1294, 0.6453, 0.8286); (0.1167, 0.7565, 0.8516); (0.2094, 0.7344, 0.8372); (0.1536, 0.7485, 0.8022) >

Step 5. Determine the distance measure of each alternative from the RPIS and RNIS

Using Eq. (5), the distance measures of each alternative from the RPIS are obtained as follows:

$$D_{Euc}^{1+} = 0.0788,\ D_{Euc}^{2+} = 0.0412,\ D_{Euc}^{3+} = 0.0527,\ D_{Euc}^{4+} = 0.0730.$$

Similarly, the distance measures of each alternative from the RNIS are obtained using Eq. (6) as follows:

$$D_{Euc}^{1-} = 0.0344,\ D_{Euc}^{2-} = 0.0731,\ D_{Euc}^{3-} = 0.0514,\ D_{Euc}^{4-} = 0.0346.$$

Step 6. Calculate the relative closeness coefficient

We now compute the relative closeness co-efficient ρ_i^*, i = 1, 2, 3, 4 using Eq. (7) as follows:

$$\rho_1^* = 0.3039,\ \rho_2^* = 0.6395,\ \rho_3^* = 0.4938,\ \rho_4^* = 0.3216.$$

Step 7. Rank the alternatives

The ranking order of alternatives based on the relative closeness coefficient is presented as follows:

$$C_2 \succ C_3 \succ C_4 \succ C_1.$$

Therefore, C_2 is the best alternative.

6 Conclusion

In this paper, we have proposed a TOPSIS method for solving MAGDM problem with generalized neutrosophic soft information. In the decision making context, the rating of performance values of the alternatives with respect to the parameters are presented in terms of GNSSs. We employ AND operator of GNSSs to combine opinions of the DMs based on the choice parameters of the evaluator. We construct weighted decision matrix after obtaining the weights of the choice parameters by using information entropy method. Then, we define RPIS and RNIS from the weighted decision matrix and Euclidean distance measure is used to compute distances of each alternative from RPISs as well as RNISs. Finally, relative closeness co-efficient of each alternative is calculated in order to select the best alternative. The authors expect that the proposed concept can be useful in dealing with diverse MAGDM problems such as personnel and project selections, manufacturing systems, marketing research problems and various other management decision problems.

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When Should We Switch from Interval-Valued Fuzzy to Full Type-2 Fuzzy (e.g., Gaussian)?

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Abstract

Full type-2 fuzzy techniques provide a more adequate representation of expert knowledge. However, such techniques also require additional computational efforts, so we should only use them if we expect a reasonable improvement in the result of the corresponding data processing. It is therefore important to come up with a practically useful criterion for deciding when we should stay with interval-valued fuzzy and when we should use full type-2 fuzzy techniques. Such a criterion is proposed in this paper. We also analyze how many experts we need to ask to come up with a reasonable description of expert uncertainty.

Keywords

interval-valued fuzzy, type-2 fuzzy, number of experts.

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1 Formulation of the Problem

Need for fuzzy logic. In many application areas, we have expert knowledge formulated by using imprecise ("fuzzy") words from natural language, such as "small", "weak", etc. To use this knowledge in automated systems, it is necessary to reformulate it in precise computer-understandable terms. The need for such a reformulation was one of the motivations behind *fuzzy logic* (see, e.g., [3], [11], [15]). Fuzzy logic uses the fact that in a computer, "absolutely true" is usually represented as 1, and "absolutely false" is represented as 0. Thus, to describe expert's intermediate degrees of confidence, it makes sense to use real numbers intermediate between 0 and 1.

In this case, to represent an imprecise word like "small", we describe, for each real number x, the degree $\mu_{small}(x) \in [0,1]$ to which the expert considers this value to be small. The corresponding function from the set of possible value to the interval [0, 1] is known as a *membership function.*

Need to go beyond [0, 1]**-valued fuzzy logic.** In most practical problems, we have several experts, and while their imprecise rules may coincide, their understanding of the meaning of the corresponding words may be slightly different. As a result, when we ask different experts, we get, in general, different membership functions corresponding to the same term – i.e., for each possible value x, we get, in general, different degrees $\mu(x)$ (describing the expert's opinion to what extent this value x satisfies the given property).

To adequately represent expert knowledge, it is desirable to capture this difference, i.e., to go beyond the original [0, 1]-valued fuzzy logic – which was oriented towards capturing the opinion of a single expert.

Interval-valued fuzzy techniques. If for the same property P and for same value x, two different degrees of confidence, e.g., 0.6 and 0.8, are both possible – according to two experts – then it makes sense to assume that for other experts, intermediate viewpoints will also be possible. In other words, if two real numbers from the interval [0, 1] are possible degrees, then all intermediate numbers should also be possible degrees. In this case, for each property P and for each value x, the set of all possible degree that x satisfies the property P is an interval. This interval can be denoted by $[\underline{\mu}(x), \overline{\mu}(x)]$.

Interval-valued fuzzy techniques have indeed been successfully used in many applications; see, e.g., [7], [8], [10].

General type-2 fuzzy techniques. The *interval-valued* techniques do not fully capture the uncertainty of the experts' opinion: these techniques just describe the interval, but they do not take into account that some values from this interval are shared by many experts, while other values are "outliers", opinions of a few unorthodox experts. To capture this difference, a reasonable idea is to describe, for each value μ from the corresponding interval $[\underline{\mu}, \overline{\mu}]$, a degree to which this value is common.

In other words, for each possible value x of the original quantity, instead of single numerical degree $\mu(x)$, we now have a *fuzzy set* (membership function) describing this degree. Such situation in which, for every possible value x of the original quantity, the experts' degree of confidence that x satisfies the given property P is itself a fuzzy number is known as *type-2 fuzzy set*.

Of course, each *interval-valued* fuzzy set is a trivial particular case of the general *type-2 fuzzy set*, corresponding to the case when the degree is 1 inside the interval $[\underline{\mu}, \overline{\mu}]$ and 0 outside this interval.

The most commonly used non-trivial type-2 fuzzy sets are the *Gaussian* ones, in which, for each *x*, the corresponding membership function of the set of all possible values μ is Gaussian: $d(\mu) = \exp\left(-\frac{(\mu-\mu_o)^2}{2\sigma^2}\right)$ for some values μ_0 and σ. Such Gaussian-valued fuzzy sets are also used in applications [7], [8].

Comment. In addition to empirical success, there are also theoretical reasons why namely Gaussian membership functions are successfully used; see, e.g., [4].

Formulation of the problem.

- On the one hand, the transition from *interval-valued* to general type-2 fuzzy sets leads to a more adequate representation of the experts' knowledge. From this viewpoint, it may sound as if it is always beneficial to use general *type-2 fuzzy sets*.
- However, on the other hand, this transition requires that we store and process additional information about the secondary membership functions. So, we should only perform this switch if we expect a reasonable advantage.

It is therefore desirable to come up with a criterion for deciding when we should switch from *interval-valued fuzzy* to general *type-2 fuzzy*. The main objective of this paper is to come up with such a criterion.

Comment. A similar problem occurs in describing measurement uncertainty: we can simply store and use the interval of possible values of measurement error, or we may want to supplement this interval withe the information about the probability of different values within this interval – i.e., with a probability distribution. Here also, we face a similar problem of deciding when it is beneficial to switch from a simpler interval description to a more complex (but more adequate) probabilistic description. A possible solution to this problem – based on information theory – is presented in [1].

2 Analysis of the Problem

One more reason why Gaussian membership functions provide a good description of the expert diversity. There are many different factors that influence the expert's degree of confidence. The actual degree produced by an individual expert is a result of the joint effect of all these factors.

Such situations, when a quantity is influenced by many different factors, are ubiquitous. There is a known result – the Central Limit Theorem (see, e.g., [13]) – that helps to describe such situations, by proving that; under reasonable assumptions the probability distribution of the joint effect of many independent factors is close to Gaussian. This is a well-known fact explaining the ubiquity of bell-shaped Gaussian (normal) distributions: they describe the distribution of people by height, by weight, by IQ, they describe the distribution of different animals and plants, they describe the measurement errors, etc.

It is therefore reasonable to assume that when we consider many experts providing their degrees of confidence, the resulting probability distribution of these degrees is also close to Gaussian (= normal), with some mean μ_0 and standard deviation σ.

For normally distributed expert estimates, what is the corresponding interval? Let us assume that for the same statement, different expert degrees of confidence are normally distributed with mean μ_0 and standard deviation σ. Let N denote the number of experts whose opinions we ask, and let $\mu_1,\ldots,\mu_N$ are degrees indicated by these experts.

If we use an interval approach, then, as the *interval-valued* degree of confidence $[\underline{\mu},\overline{\mu}]$, we take the interval formed by these degrees μ_i, i.e., the interval $[\min\limits_i \mu_i, \max\limits_i \mu_i]$.

On average, when we have a sample of N random values, then one of the ways to approximate the original distribution is to build a histogram, i.e., sort the observed values μ_i in increasing order into a sequence

$$\mu_{(1)} < \mu_{(2)} < \ldots < \mu_{(N)} \tag{1}$$

and then take a distribution that has each of the values $\mu_{(i)}$ with the same probability $\frac{1}{N}$. It is known that in the limit $N \to \infty$, this histogram distribution converges to the actual distribution (i.e., becomes closer and closer as N increases).

Thus, as a good approximation to the smallest possible value $\mu_{(1)} = \min\limits_i \mu_i$, it is reasonable to take the value $\underline{\upsilon}$ for which the probability $\Pr ob(\upsilon \le \underline{\upsilon}) = \frac{1}{N}$

Similarly, as a good approximation to the largest possible value $\mu_{(N)} = \max\limits_i \mu_i$, we can take the value $\bar{\upsilon}$ for which $\Pr ob(\upsilon \ge \bar{\upsilon}) = \frac{1}{N}$, i.e., for which

$$\Pr ob(\upsilon \le \bar{\upsilon}) = 1 - \frac{1}{N} \qquad (2)$$

For a normal distribution with mean μ_0 and standard deviation σ, the corresponding values $\underline{\upsilon}$ and $\bar{\upsilon}$ can be obtained as follows (see, e.g., [13]):

$$\underline{\upsilon} = \mu_0 - k(N)\cdot\sigma; \quad \bar{\upsilon} = \mu_0 + k(N)\cdot\sigma \qquad (3)$$

where

$$k(N) \stackrel{def}{=} \sqrt{2}\cdot erf^{-1}\left(1 - \frac{2}{N}\right) \qquad (4)$$

and the *error function erf* $\operatorname{erf}(x)$ is defined as

$$erf(x) \stackrel{def}{=} \int_{-x}^{x} \exp\left(-\frac{t^2}{2}\right) dt \qquad (5)$$

So, when is interval representation better? The value $k(N)$ increases with N and tends to ∞ when N increases. Thus, when the number of experts N is large, the lower endpoint of the interval

$$[\underline{\mu}, \overline{\mu}] = [\mu_0 - k(N)\cdot\sigma, \mu_0 + k(N)\cdot\sigma] \qquad (6)$$

becomes negative, while its upper bound becomes larger than 1. Since the values μ_i are always located within the interval [0, 1], in this case, the *interval-valued* description of uncertainty is useless: the smallest value is 0 (or close to 0), the largest value is 1 (or close to 1). In such situations, we cannot use the *interval-valued* approach, so we need to use a more computationally complex Gaussian approach.

On the other hand, if we have

$$0 < \underline{\mu} = \mu_0 - k(N)\cdot\sigma \text{ and } \overline{\mu} = \mu_0 + k(N)\cdot\sigma < 1 \qquad (7)$$

then, once we know the bounds $\underline{\mu}$ and $\overline{\mu}$, we can uniquely reconstruct both parameters μ_0 and σ as follows:

$$\mu_0 = \frac{\underline{\mu} + \overline{\mu}}{2}; \ \sigma = \frac{\overline{\mu} - \underline{\mu}}{2k(N)} \tag{8}$$

In this case, if we use the *interval-valued* approach, we do not lose any information in comparison with the *Gaussian-based* approach. Since the *interval-valued* approach is computationally easier than the *Gaussian-based* approach, it therefore makes sense to use the *interval-based* approach.

But are these expert estimates meaningful at all? What if the experts do not real have any knowledge and their degrees are all over the map? In this case, processing these ignorance-based degrees does not make any sense. How can we detect such a situation?

In the cases when experts have no meaningful knowledge, their degrees are simply uniformly distributed on the interval [0, 1]. In this case, the variance is equal to $\sigma^2 = \frac{1}{12}$, in which case $\sigma \approx 0.3$. So, we can conclude that if the empirical standard deviation is greater than or equal to 0.3, then we should simply ignore the experts' degrees – since the experts' opinions disagree too much to be useful.

Thus, we arrive at the following recommendation.

3 Recommendation: When to Use Interval-Valued Approach and When to Use Gaussian Approach

What is given. For each property and for each possible value x, we have N experts that provide us with their degrees of confidence $\mu_1, \ldots, \mu_N$ that this value x satisfies the given imprecise property (e.g., that this value x is small).

Resulting algorithm. First, we use the standard formulas to estimate the mean μ_0 and standard deviation σ of the expert's degrees μ_i:

$$\mu_0 = \frac{\mu_1 + \ldots + \mu_N}{N} \tag{9}$$

$$\sigma = \sqrt{\frac{1}{N-1} \cdot \sum_{i=1}^{N} (\mu_i - \mu_0)^2} \tag{10}$$

If $\sigma \geq 0.3$, then we conclude that the experts' opinion disagree too much to be useful.

If $\sigma < 0.3$, then, based on the number of experts N, we estimate $k(N)$ as

$$k(N) = \sqrt{2} \cdot erf^{-1}\left(1 - \frac{2}{N}\right) \tag{11}$$

Based on this value $k(N)$, we compute the values

$$\underline{\mu} = \mu_0 - k(N) \cdot \sigma \text{ and } \overline{\mu} = \mu_0 + k(N) \cdot \sigma \tag{12}$$

Then:

- if $0 < \underline{\mu}$ and $\overline{\mu} < 1$, we use *interval-valued* approach, with *interval-valued* degree $[\underline{\mu}, \overline{\mu}]$;
- otherwise, if $\underline{\mu} \le 0$ or $\overline{\mu} \ge 1$, we use a Gaussian approach, with the type-2 Gaussian degree of confidence

$$d(\mu) = \exp\left(-\frac{(\mu - \mu_o)^2}{2\sigma^2}\right) \tag{13}$$

4 Auxiliary Question: How Many Experts We Should Ask?

How many experts we should ask? For a general random variable, the larger the sample is the more accurate the estimates are. For example, if we perform measurements, then we can decrease the random component of the measurement error if we repeat the measurement many times and take the average of the measurement results. This fact follows from the Large Numbers Theorem, according to which, when the sample size increases, the sample average tends to the mean of the corresponding random variable.

This makes sense if we deal with measurements of physical quantities, where more and more accurate description of this quantity makes perfect sense – and is desirable. For degree, however, the situation is different. A person can only provide his or her degree of confidence only with a low accuracy: e.g., an expert may distinguish between marks 6 and 7 on a scale from 0 to 10, but, when describing their degree of confidence, experts cannot meaningfully distinguish between, e.g., values 61 and 62 on a scale from 0 to 100.

Comment. Issues related to decision making in fuzzy context are handled, e.g., in [2], [5], [6].

Our idea. Psychologists have found out that we usually divide each quantity into 7 plus minus 2 categories – this is the largest number of categories whose meaning we can immediately grasp; see, e.g., [9], [12] (see also [14]). For some people, this "magical number" is 7 + 2 = 9, for some it is 7 - 2 = 5. This rule is in good accordance with the fact that in fuzzy logic, to describe the expert's opinion on each quantity, we usually use 7 ± 2 different categories (such as "small", "medium", etc.).

Since on the interval [0, 1], we can only have 7 ± 2 meaningfully different degrees of confidence, the accuracy of these degrees ranges is, at best, 1/9. When we estimate the mean μ_0 based on N values, the accuracy is of order $\frac{\sigma}{\sqrt{N}}$. It does not make sense to bring this accuracy below 1/9, so it makes sense to limit the number of experts N to a value for which $\frac{\sigma}{\sqrt{N}} \approx \frac{1}{9}$, i.e., to the value $N \approx (9\cdot\sigma)^2$.

Resulting recommendation. To estimate how many experts we need to ask, we ask a small number n of experts, and, based on their degrees μ_i, estimate σ as

$$\sigma = \sqrt{\frac{1}{n-1}\cdot\sum_{i=1}^{n}(\mu_i - \mu_{av})^2} \tag{14}$$

where $\mu_{av} = \frac{1}{n}\sum_{i=1}^{n}\mu_i$.

Then, we estimate the number N of experts to ask as $N = (9\cdot\sigma)^2$.

Comment. Of course, if $N \le n$, this means that we do not have to ask any more experts, whatever information we have from n experts is enough.

Examples. If all experts perfectly agree with each other, i.e., if $\mu_i = \mu_j$ for all i and j, then $\sigma = 0$ and $N = 0$. In this case, there is no need to ask any more experts.

Similarly, if all experts more or less agree with each other and $\sigma = 0.1$, then $N < 1$, meaning also that there is no need to ask more experts.

If $\sigma = 0.2$, then $N = 3.61$, meaning that we should ask at least 4 experts to get a good estimate. For $\sigma = 0.3$, we get $N = 7.29$, meaning that we need to ask at least 7 experts.

This is about as bad as we can get: as we have mentioned, even when the expert's degrees are all over the map, i.e., uniformly distributed on the interval [0, 1], then the variance is equal to $\sigma^2 = \frac{1}{12}$, in which case $\sigma \approx 0.3$, and we get $N = 9^2\cdot\sigma^2 = \frac{81}{12} = 6.75$, meaning that we need to ask at most 7 experts.

5 Conclusion

In all cases, we need to ask at most seven experts to get a meaningful estimate (and sometimes, when the experts agree with each other, a smaller number of experts is sufficient).

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Neutrosophic Index Numbers: Neutrosophic Logic Applied In The Statistical Indicators Theory

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Abstract

Neutrosophic numbers easily allow modeling uncertainties of prices universe, thus justifying the growing interest for theoretical and practical aspects of arithmetic generated by some special numbers in our work. At the beginning of this paper, we reconsider the importance in applied research of instrumental discernment, viewed as the main support of the final measurement validity. Theoretically, the need for discernment is revealed by decision logic, and more recently by the new neutrosophic logic and by constructing neutrosophic-type index numbers, exemplified in the context and applied to the world of prices, and, from a practical standpoint, by the possibility to use index numbers in characterization of some cyclical phenomena and economic processes, e.g. inflation rate. The neutrosophic index numbers or neutrosophic indexes are the key topic of this article. The next step is an interrogative and applicative one, drawing the coordinates of an optimized discernment centered on neutrosophic-type index numbers. The inevitable conclusions are optimistic in relation to the common future of the index method and neutrosophic logic, with statistical and economic meaning and utility.

Keyword

neutrosophic-tendential fuzzy logic, neutrosophic logic, neutrosophic index, index statistical method, price index, interpreter index, neutrosophic interpreter index.

1 Introduction

Any decision, including the statistical evaluation in the economy, requires three major aspects, distinct but interdependent to a large extent, starting with *providing the needed knowledge to a certain level of credibility* (reducing uncertainty, available knowledge being incomplete and unreliable in different proportions, and the condition of certainty rarely being encountered in practice, the determinism essentially characterizing only the theory), then by the *discernment of chosing the decision option*, and, finally, by *obtaining the instrumental and quantified consensus*. In the hierarchy of measurement results qualities, the discernment of instrumental choice – by selection of the tool, of the technics, or of the method from the alternative options that characterizes all available solutions – should be declared the fundamental property of applied research. Moreover, the discernment can be placed on a scale intensity, from experimental discernment or decison discernment, selected according to the experience acquired in time, then ascending a "ladder" revealed by perpetual change of the continuous informational discernment or by the discernment obtained through knowledge from new results of research in specific activity, until the final stage of intuitional discernment (apparently rational, but mostly based on intuition), in fact the expression of a researcher's personal reasoning.

In summary, the process of making a measurement decision, based on a spontaneous and intuitive personal judgment, contains a referential system that experiences, more or less by chance, different quantifying actions satisfying to varying degrees the needs of which the system is aware in a fairly nuanced manner. The actions, the tools, the techniques and the measurement methods that are experienced as satisfactory will be accepted, resumed, fixed and amplified as accurate, and those that are experienced as unsatisfactory will be remove from the beginning. A modern discernment involves completing all the steps of the described "ladder", continuously exploiting the solutions or the alternatives enabling the best interpretation, ensuring the highest degree of differentiation, offering the best diagnostic, leading to the best treatment, with the most effective impact in real time. However, some modern measurement theories argue that human social systems, in conditions of uncertainity, resort to a simplified decision-making strategy, respectively the adoption of the first satisfactory solution, coherentely formulated, accepted by relative consensus (the Dow Jones index example is a perennial proof in this respect).

Neutrosophic logic facilitates the discernment in relation to natural language, and especially with some of its terms, often having arbitrary values. An example in this regard is the formulation of common market economies: "inflation is low and a slight increase in prices is reported," a mathematical imprecise formulation since it is not exactly known which is the percentage of price increase; still, if it comes about a short period of time and a well defined market of a product, one can make the assumption that a change to the current price is between 0% and 100% compared to the last (basic) price.

A statement like – “if one identifies a general increase in prices close to zero” (or an overall increase situated between three and five percent, or a general increase around up to five percent), “then the relevant market enjoys a low inflation” – has a corresponding degree of truth according to its interpretation in the context it was issued. However, the information must be interpreted accordingly to a certain linguistic value, because it can have different contextual meanings (for Romania, an amount of 5% may be a low value, but for the EU even an amount of five percent is certainly a very high one).

Neutrosophic logic, by employing neutrosophic-type sets and corresponding membership functions, could allow detailing the arrangement of values covering the area of representation of a neutrosophic set, as well as the correspondence between these values and their degree of belonging to the related neutrosophic set, or by employing neutrosophic numbers, especially the neutrosophic indexes explained in this paper; and could open new applied horizons, e.g. price indexes that are, in fact, nothing else than interpreters, but more special – on the strength of their special relationship with the reality of price universe.

2 Neutrosophic Logic

First of all, we should define what the Logic is in general, and then the Neutrosophic Logic in particular.

Although considered elliptical by Nae Ionescu, the most succinct and expressive metaphorical definition of the Logic remains that – Logic is “the thinking that thinks itself.”

The Logic has indisputable historical primacy as science. The science of Logic seeks a finite number of consequences, operating with sets of sentences and the relationship between them. The consequence’s or relationship’s substance is exclusively predicative, modal, or propositional, such generating Predicative Logic, Modal Logic, or Propositional Logic.

A logic calculation can be syntactic (based on evidence) and semantic (based on facts). The Classical Logic, or the Aristotelian excluded middle logic, operates only with the notions of truth and false, which makes it inappropriate to the vast majority of real situations, which are unclear or imprecise. According to the same traditional approach, an object could either belong or not belong to a set.

The essence of the new Neutrosophic Logic is based on the notion of vagueness: a neutrosophic sentence may be only true to a certain extent; the notion of belonging benefits from a more flexible interpretation, as more items may belong to a set in varying degrees. The first imprecision based logic (early neutrosophic) has existed since 1920, as proposed by the Polish mathematician and logician Jan Łukasiewicz, which expanded the truth of a proposition to all real numbers in the range [0; 1], thus generating the possibility theory, as reasoning method in conditions of inaccuracy and incompleteness [1].

In early fuzzy logic, a neutrosophic-tendential logic of Łukasiewicz type, this paradox disappears, since if φ has the value of 0.5, its own negation will have the same value, equivalent to φ. This is already a first step of a potentially approachable gradation, by denying the true statement (1) by the false statement (φ) and by the new arithmetic result of this logic, namely the new value 1- φ.

In 1965, Lotfi A. Zadeh extended the possibility theory in a formal system of fuzzy mathematical logic, focused on methods of working using nuanced terms of natural language. Zadeh introduced the degree of membership/truth (t) in 1965 and defined the *fuzzy set*.

Atanassov introduced the degree of nonmembership/falsehood (f) in 1986 and defined the *intuitionistic fuzzy set*.

Smarandache introduced the degree of indeterminacy/neutrality (i) as independent component in 1995 (published in 1998), and defined the neutrosophic set. In 2013, he refined the neutrosophic set to *n* components:

$$t_1, t_2, \dots t_j;\ i_1, i_2, \dots, i_k;\ f_1, f_2, \dots, f_2,$$

where $j + k + l = n > 3$.

The words “neutrosophy” and “neutrosophic” were coined/invented by F. Smarandache in his 1998 book. Etymologically, “neutro-sophy” (noun) [French *neutre* <Latin *neuter*, *neutral*, and Greek *sophia*, skill/wisdom] means “knowledge of neutral thought”, while “neutrosophic” (adjective), means “having the nature of, or having the characteristic of Neutrosophy”.

Going over, in fuzzy set, there is only a degree (percentage) of belonging of an element to a set (Zadeh, 1965). Atanassov introduced in 1986 the degree (percentage) of non-belonging of an element to a set, and developed the intuitionistic fuzzy set. Smarandache introduced in 1995 the degree (percentage) of indeterminacy of belonging, that is: we do not know if an element belongs, or does not belong to a set), defining the neutrosophic set.

The neutrosophy, in general, is based on the neutral part, neither membership nor non-membership, and in neutrosophic logic, in particular: neither true, nor false, but in between them. Therefore, an element $x(t, i, f)$ belongs to a neutrosophic set M in the following way: x is $t\%$ in M, $i\%$ indeterminate belonging, and $f\%$ does not belong. Or we can look at this issue in probabilistic terms as such: the chance for the element x to belong to the set M is $t\%$, the indeterminate chance to belong is $i\%$, and the chance not to belong is $f\%$.

In normalized cases, $t + i + f = 1$ (100%), but in general, if the information about the possibility of membership of the element x in the set M is independently sourced (not communicating with one another, so not influencing each other), then it may be that $0 \leq t + i + f \leq 3$.

In more general or approximated cases, t, i, f can be included intervals in [0, 1], or even certain subsets included in [0, 1], i.e. when working with inaccurate, wrong, contradictory, vague data.

In 1972, S.S.L. Chang and L. A. Zadeh sketched the use of fuzzy logic (also of tendential-neutrosophic logic) in conducting technological processes by introducing the concept of linguistic variables defined not by numbers, but as a variable in linguistic terms, clearly structured by letters of words. The linguistic variables can be decomposed into a multitude of terms, covering the full range of the considered parameter.

On the other hand, unlike the classical logic (Aristotelian, mathematic and boolean), which work exclusively with two exact numerical values (0 for false and 1 for true), the fuzzy early-neutrosophic logic was able to use a wide continuous spectrum of logical values in the range [0, 1], where 0 indicates complete falsity, and 1 indicates complete truth. However, if an object, in classical logic, could belong to a set (1) or not belong to a set (0), the neutrosophic logic redefines the object's degree of membership to the set, taking any value between 0 and 1. The linguistic refinement could be fuzzy tendential-neutrosophically redefined, both logically and mathematically, by inaccuracy, by indistinctness, by vagueness. The mathematical clarification of imprecision and vagueness, the more elastic formal interpretation of membership, the representation and the manipulation of nuanced terms of natural language, all these characterize today, after almost half a century, the neutrosophic logic.

The first major application of the neutrosophic logical system has been carried out by L.P. Holmblad and J.J. Ostergaard on a cement kiln automation [2], in 1982, followed by more practical various uses, as in high traffic intersections or water treatment plants. The first chip capable of performing the inference in a decision based on neutrosophic logic was conducted in 1986 by Masaki Togai and Hiroyuki Watanabe at AT&T Bell Laboratories, using the digital implementation of *min-max* type logics, expressing elementary union and intersection operations [3].

A neutrosophic-tendential fuzzy set, e.g. denoted by F, defined in a field of existence U, is characterized by a membership function $\mu F(x)$ which has values in the range [0, 1] and is a generalization of the concise set [4], where the belonging function takes only one of two values, zero and one. The membership function provides a measure of the degree of similarity of an element U of neutrosophic-tendential fuzzy subset F. Unlike the concise sets and subsets, characterized by net frontiers, the frontiers of the neutrosophic-tendential fuzzy sets and subsets are made from regions where membership function values gradually fade out until they disappear, and the areas of frontiers of these nuanced subsets may overlap, meaning that the elements from these areas may belong to two neighboring subsets at the same time.

As a result of the neutrosophic-tendential fuzzy subset being characterized by frontiers, which are not net, the classic inference reasoning, expressed by a *Modus Ponens* in the traditional logic, of form:

$(p\to(p\to q))\to q$, i.e.: *premise:* if p, then q
fact: p
consequence: q,

becomes a *generalized Modus Ponens,* according to the neutrosophic-tendential fuzzy logic and under the new rules of inference suggested from the very beginning by Lotfi A. Zadeh [5], respectively in the following expression:

premise: if x is A, then y is B
fact: x is A'
consequence: y is B', where $B' = A' \circ (A\to B)$.

(*Modus ponens* from classical logic could have the rule *max-min* as correspondent in neutrosophic-tendential fuzzy logic).

This inference reasoning, which is essentially the basis of the neutrosophic-tendential fuzzy logic, generated the use of expression "approximate reasoning", with a nuanced meaning. Neutrosophic-tendential fuzzy logic can be considered a first extension of meanings of the incompleteness theory to date, offering the possibility of representing and reasoning with common knowledge, ordinary formulated, therefore having found applicability in many areas.

The advantage of the neutrosophic-tendential fuzzy logic was the existence of a huge number of possibilities that must be validated at first. It could use linguistic modifiers of the language to appropriate the degree of imprecision represented by a neutrosophic-tendential fuzzy set, just having the natural language as example, where people alter the degree of ambiguity of a sentence using adverbs as *incredibly, extremely, very,* etc. An adverb can modify a verb, an adjective, another adverb, or the entire sentence.

After designing and analyzing a logic system with neutrosophic-tendential fuzzy sets [6], one develops its algorithm and, finally, its program incorporating specific applications, denoted as neutrosophic-tendential fuzzy controller. Any neutrosophic-tendential fuzzy logic consists of four blocks: the fuzzyfication (transcribing by the membership functions in neutrosophic input sets), the basic rules block (which contains rules, mostly described in a conditional manner, drawn from concise numerical data in a single collection of specific judgments, expressed in linguistic terms, having neutrosophic sets associated in the process of inference or decision), the inference block (transposing by neutrosophic inferential procedures nuanced input sets into nuanced output sets), and the defuzzyfication (transposing nuanced output sets in the form of concise numbers).

The last few decades are increasingly dominated by artificial intelligence, especially by the computerized intelligence of experts and the expert-systems; alongside, the tendential-neutrosophic fuzzy logic has gradually imposed itself, being more and more commonly used in tendential-neutrosophic fuzzy control of subways and elevators systems, in tendential-neutrosophic fuzzy-controlled household appliances (washing machines, microwave ovens, air conditioning, so on), in voice commands of tendential-neutrosophic fuzzy types, like *up*, *land*, *hover*, used to drive helicopters without men onboard, in tendential-neutrosophic fuzzy cameras that maps imaging data in medical lens settings etc.

In that respect, a bibliography of theoretical and applied works related to the tendential-neutrosophic fuzzy logic, certainly counting thousands of articles and books, and increasing at a fast pace, proves the importance of the discipline.

3 Construction of Sets and Numbers of Neutrosophic Type in the Universe of Prices heading

As it can be seen from almost all fields of science and human communication, natural language is structured and prioritized through logical nuances of terms. Valorisation of linguistic nuances through neutrosophic-tendential fuzzy logic, contrary to traditional logic, after which an object may belong to a set or may not belong to a set, allow the use with a wide flexibility of the concept of belonging [7].

Neutrosophic-tendential fuzzy numbers are used in practice to represent more precisely defined approximate values. For example, creating a budget of a business focused on selling a new technology, characterized by uncertainty in relation to the number of firms that have the opportunity to purchase it for a prices ensuring a certain profit of the producer, a price situated between 50 and 100 million lei, with the highest possible range in the interval situated somewhere between 70 and 75, provides, among other things, a variant to define concretely a neutrosophic-tendential fuzzy number *Z*, using the set of pairs (offered contractual price, possibility, or real degree of membership), that may lead to a steady price: Z = [(50, 0), (60, 0.5), (70, 1), (75, 1), (85; 0.5) (100, 0)].

Given that *X* represents a universe of discourse, with a linguistic variable referring to the typical inflation or to a slight *normal-upward* shift of the price of a product, in a short period of time and in a well-defined market, specified by the elements *x*, it can be noted $\triangle p$, where $\triangle p=(p_1-p_0)/p_0$. In the following exemplification, the values of $\triangle p$ are simultaneously considered positive for the beginning and also below 1 (it is not hypothetically allowed, in a short period of time, a price increase more than double the original price, respectively the values of $\triangle p$ are situated in the interval between 0 and 1). A neutrosophic-tendential fuzzy set *A* of a universe of discourse *X* is defined or it is characterized by a function of belonging $\mu A(x)$ or $\mu A(\triangle p)$, associating to each item *x* or $\triangle p$ a degree of membership in the set *A,* as described by the equation:

$$\mu A(x): X \to [0,1] \text{ or } \mu A(\triangle p): X \to [0,1]. \qquad (1)$$

To graphically represent a neutrosophic-tendential fuzzy set, we must first define the function of belonging, and thus the solution of spacial unambiguous definition is conferred by the coordinates *x* and $\mu\, A\,(x)$ or $\triangle p$ and $\mu\, A\,(\triangle p)$:

$$A = \{[x, \mu\, A\,(x)] \mid x \in [0,1]\}$$

or

$$A = \{[\triangle p, \mu\, A\,(\triangle p)] \mid \triangle p \in [0,1]\} \qquad (2)$$

A finite universe of discourse $X=\{x_1,x_2,...,x_n\}$ or $X=\{\triangle p_1,\triangle p_2,...,\triangle p_n\}$ can redeem, for simplicity, a notation of type:

$$A = \{\mu_1/x_1 + \mu_2/x_2 + ... + \mu_n/x_n\},$$

respectively $A = \{\mu_1/\triangle p_1 + \mu_1/\triangle p_2 + ... + \mu_n/\triangle p_n\}$.

For example, in the situation of linguistic variable “a slight increase in price,” one can detail multiple universes of discourse, be it a summary one *X* = {0, 10, 20, 100}, be it an excessive one *X* = {0, 10, 20, 30, 40, 50, 60, 70, 80, 90, 100}, the breakdowns being completed by membership functions for percentage values of variable $\triangle p$, resorting either to a reduced notation:

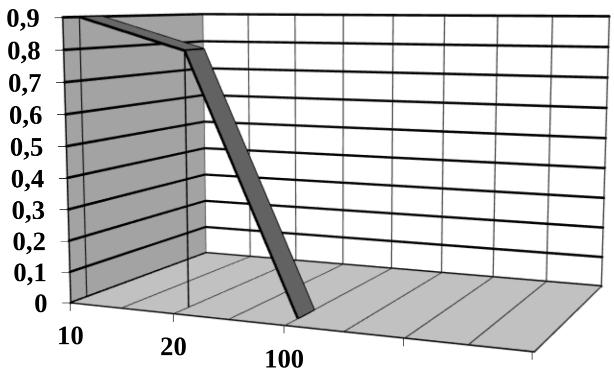
$A = [0/1 + 10 / 0{,}9 + 20 / 0{,}8 + 100 / 0]$,

or to an extended one:

$A = [0/1 + 10/0{,}9 + 20/0{,}8 + 30/0{,}7 + 40/0{,}6 + 50/0{,}5 + 60/0{,}4 + 70/0{,}3 + 80/0{,}2 + 90/0{,}1 + 100/0]$.

The meaning of this notations starts with inclusion in the slight increase of a both unchanged price, where the difference between the old price of 20 lei and the new price of a certain product is nil, thus the unchanged price belonging 100% to the set of "slight increase of price", and of a changed price of 22 lei, where $\triangle p = (p_1 - p_0) / p_0 = 0{,}1$ or 10%, therefore belonging 90% to the set of "slight increase of price", … , and, finally, even the price of 40 lei, in proportion of 0% (its degree of belonging to the analyzed set being 0).

Let us represent graphically, in a situation of a summary inflationary discourse:



Graphic 2. Neutrosophic-tendential excesively described fuzzy set.

To define a neutrosophic number, some other important concepts are required from the theory of neutrosophic set:

- the *support* of A or the strict subset of X, whose elements have nonzero degrees of belonging in A:

  $\mathrm{supp}(A) = \{x \in X \mid \mu A(x) > 0\}$

  or $\mathrm{supp}(A) = \{\triangle p \in X \mid \mu A(\triangle p) > 0\}$, (3)

- the *height* of A or the highest value of membership function [8]:

  $h(A) = \sup \mu A(x)$, where $x \in X$

  or $h(A) = \sup \mu A(\triangle p)$, where $\triangle p \in X$, (4)

- the *nucleus* of A or the *strict subset* of X, whose elements have unitary degrees of belonging in A:

  $n(A) = \{x \in X \mid \mu A(x) = 1\}$

or $n(A) = \{\triangle p \in X \mid \mu A(\triangle p) = 1\}$, (5)

- the *subset A of subset B* of neutrosophic-tendential fuzzy type: for *A* and *B* neutrosophic subsets of *X*, A becomes a *subset of B* if $\mu A(X) \leq \mu B(X)$, in the general case of any $x \in X$, (6)
- *neutrosophic-tendential fuzzy subsets equal* to *X* or $A = B \Leftrightarrow \mu A(X) = \mu B(X)$, if $A \subset B$ și $B \subset A$. (7)

The first three operations with neutrosophic-tendential fuzzy set according to their importance are broadly the same as those of classical logic (reunion, intersection, complementarity etc.), being defined in the neutrosophic-tendential fuzzy logic by characteristic membership functions. If A and B are two fuzzy or nuanced neutrosophic-tendential subsets, described by their membership functions $\mu A(x)$ or $\mu B(x)$, one gets the following results:

a. The neutrosophic-tendential fuzzy *reunion* is defined by the membership function: $\mu A \cup B(x) = \max[\mu A(x), \mu B(x)]$;

b. The neutrosophic-tendential fuzzy *intersection* is rendered by the expression: $\mu A \cap B(x) = \min[\mu A(x), \mu B(x)]$;

c. The neutrosophic-tendential fuzzy *complementarity* is theor-etically identic with the belonging function: $\mu B(x) = 1 - \mu B(x)$.

The neutrosophic-tendential fuzzy logic does not respect the classical principles of excluded middle and noncontradiction. For the topic of this article, a greater importance presents the arithmetic of neutrosophic-tendential fuzzy numbers useful in building the neutrosophic indexes and mostly the interpret indexes.

The neutrosophic-tendential fuzzy numbers, by their nuanced logic, allow a more rigorous approach of indexes in general and, especially, of interpreter indexes and price indexes, mathematically solving a relatively arbitrary linguistic approach of inflation level.

The arguments leading to the neutrosophic-type indexes solution are:

1. The inflation can be corectly defined as the rate of price growth ($\triangle p$), in relation to either the past price, when $\triangle p = (p_1 - p_0) / p_0$, or an average price, and then $\triangle p = (p_1 - p_m) / p_m$ (the index from which this rate will be extracted, just as inflation is extracted from IPCG as soon as it was quantified, will be a neutrosophic-type index number purely expressing a mathematical coefficient).

2. The denominator or the *reference base* of statistical index, from which the rate defining the inflation is extracted, is the most important value; the optimal choice acquires a special significance, while the numerator *reported level* of statistical index is the signal of variation or stationarity of the studied phenomenon. Similarly, in the nuanced logic of neutrosophic-tendential fuzzy numbers, the denominator value of $\triangle p$ (either p_0, or p_m) still remains essential, keeping the validity of the index paradox, as a sign of evolution or variation, to be fundamentally dependent on denominator, although apparently it seems to be signified by the nominator.
3. The prices of any economy can be represented as a universe of discourse *X*, with a linguistic variable related to typical inflation or to a slight *normal-upward* shift of a product price, in a short period of time and in a well-defined market, specified by the elements *x*, the variable being denoted by $\triangle p$, where $\triangle p = (p_1\text{-}p_0) / p_0$ or $\triangle p = (p_1\text{-}p_m) / p_m$.
4. The values of $\triangle p$ can be initially considered both positive and negative, but still smaller than 1. This is normal and in fact a price increase more than double the original price can not even be admitted in a short interval of time (usually a decade or a month), respectively the values of $\triangle p$ are initially placed in the interval between -1 and 1, so that in the end $\Sigma\triangle p/n$, where *n* represents the number of registered prices, the overwhelming majority of real cases to belong to the interval [0;1].
5. All operations generated by the specific arithmetic of constructing a neutrosophic number or a neutrosophic-type index are possible in the nuanced logic of neutrosophic numbers, finally being accepted even negative values or deflation processes (examples 1 and 2).
6. The *equations* with neutrosophic-tendential fuzzy numbers and the *functions* specified by neutrosophic-tendential fuzzy numbers offer a much better use in constructing the hedonic functions – that were the relative computing solution of price dynamics of new products replacing in the market the technologically obsolete products, a solution often challenged in contemporary statistics of inflation. Example 3 resolves more clearly the problem of products substitution due to new technologies, but placing the divergences in the plane of correctness of the functions specified by neutrosophic numbers,

regarding the measurement of price increases or of inflationary developments.

7. Some current calculation procedures capitalize the simplified notation

   $A= \{\mu_1/x_1+\mu_2/x_2+...+\mu_n/x_n\}$,

   respectively $A=\{\mu_1/\triangle p_1+\mu_1/\triangle p_2+...+\mu_n/\triangle p_n\}$.

   Even the calculation formula of IPCG of Laspeyres type constitutes a way to build an anticipation method of constructing neutrosophic-tendential fuzzy numbers. Thus IPCG = $\frac{\Sigma I^p (p_0 q_0)}{\Sigma (p_0 q_0)}$, where

   $\frac{(p_0 q_0)}{\Sigma (p_0 q_0)} = C_p$ where I^p = the index of month *t* compared to the average price and C_p = weighting coefficient, finally becomes IPCG = $\Sigma I^p \times C_p$, for each item or group of expenditures being required the values $\triangle p$ and C_p.

4 Index Numbers or Statistical Indexes

In Greek, *deixis* means “to indicate”, which makes the indicator to be that *which indicates* (etymologically). An indicator linguistically defines the situation, the time and the subject of an assertion. The concept of linguistic indicator becomes indicial exclusively in practical terms, respectively the pragmatism turns an indicator into an index as soon as the addressee and the recipient are clarified. The indicial character is conferred by specifying the addressee, but especially the recipient, and by determining the goals that created the indicator. The indicial is somehow similar to the symptom or to the syndrome in an illness metaphor of a process, phenomenon or system, be it political, economic or social.

The symptom or the factor analysis of illness coincides with its explanatory fundamental factor, and a preventive approach of the health of a process, a phenomenon or a system obliges to the preliminary construction of indexes. The index is also a specific and graphic sign which reveals its character as iconic or reflected sign. The iconicity degree or the coverage depth in specific signs increases in figures, tables, or charts, and reaches a statistical peak with indexes. The statistical index reflects more promptly the information needed for a correct diagnosis, in relation to the flow chart and the table. The systemic approach becomes salutary. The indexes, gathered in systems, generates the systematic indicial significance, characterized by:

- in-depth approach of complex phenomena,
- temporal and spatial ongoing investigation,
- diversification of recipients,
- extending intension (of sense) and increasing the extension coverage (of described reality),
- gradual appreciation of development,
- motivating the liasons with described reality,
- ensuring practical conditions that are necessary for clustering of temporal primary indexes or globalization of regional indexes,
- diversification of addressees (sources) and recipients (beneficiaries),
- limiting restrictions of processing,
- continued expansion of the range of phenomena and processes etc.

The complexity and the promptness of the indicial overpass any other type of complexity and even promptness.

After three centuries of existing, the index method is still the method providing the best statistical information, and the advanced importance of indexes is becoming more evident in the expediency of statistical information. The assessments made by means of indexes offer qualitatively the pattern elements defining national economies, regional or community and, ultimately, international aggregates. Thinking and practice of the statistical work emphasize the relevance of factorial analysis by the method of index, embodied in the interpreter (price) indexes of inflation, in the efficient use of labor indexes etc. Because the favorite field of indexes is the economic field, they gradually became key economic indicators. The indexes are used in most comparisons, confrontations, territorial and temporal analysis – as measuring instruments. [9]

Originating etymologically in Greek *deixis,* which became in latin *index,* the index concept has multiple meanings, e.g. *index, indicator, title, list, inscription*. These meanings have maintained and even have enriched with new one, like *hint, indication, sign*. The statistical index is accepted as method, system, report or reference, size or relative indicator, average value of relative sizes or relative average change, instrument or measurement of relative change, pure number or adimensional numerical expression, simplified representation by substituting raw data, mathematical function or distinctive value of the axiomatic index theory etc.

Defined as *pure number* or adimensional numerical expression, the index is a particular form of "numerical purity", namely of independence in relation to the measurement unit of comparable size. The term "index" was first applied to dynamic data series and is expressed as a relative number. Even today, it is considered statistically an adimensional number, achieved in relation either to two values of the same simple variable corresponding to two different periods of time or space, or to two sizes of a complex indicator, whose simple sizes are heterogeneous and can not be directly added together. The first category is that of individual (particular or elementary) indexes, and the second, known as synthetic or group indexes category, which is indeed the most important. Considered as a variation scheme of a single or of multiple sizes or phenomena, the index is a simplified representation by substituting raw data by their report, aimed at rebuilding the evolution of temporal and spatial observed quantities. Whenever a variable changes its level in time or space, a statistical index is born (Henri Guitton). Approached as statistical and mathematical function, the index generated a whole axiomatic theory which defines it as an economic measure, a function F: $D \rightarrow \mathbb{R}$, which projects a set or a set D of economic interest goals (information and data) into a set or a set of real numbers $\mathbb{R}$, which satisfies a system of relevant economic conditions – for example, the properties of monotony, homogeneity or homothety or relative identity (Wolfgang Eichhorn).

Thus, the concept of "index" is shown by a general method of decomposition and factorial analysis; it is used in practice mainly as system. The index is defined either as a report or a reference which provides a characteristic number, or as synthetic relative size, either as relative indicator (numerical adimensional indicator), or as pure number, either in the condensed version as the weighted average of relative sizes or the measure of the average relative change of variables at their different time moments, different spaces or different categories, and, last but not least, as a simplified mathematical representation, by substituting the raw data by their report through a function with the same name – index function - respectively F: $D \rightarrow \mathbb{R}$, where $F(z_1, z_2, \ldots, z_k) = z_1/z_2$, with z representing a specific variable and D the set of goals, information and data of (economic) interest, and $\mathbb{R}$ is the set of real numbers. [10; 11; 12; 13; 14]

The above mentioned properties means the following:

→ MONOTONY (A)

An index is greater than the index of whose variables resultative vector is less than the initial index vector, all other conditions being constant:

$z_1 / z_2 > x_1 / x_2 \Rightarrow F(\underline{z}) > F(\underline{x})$

or:

$z_1 \nearrow$ → $F(\underline{z})$ is strictly increasing

$z_2 \searrow$ → $F(\underline{z})$ is strictly decreasing

$z_i = ct$ → $F(\underline{z})$ is constant, where $i = \overline{3k}$

(where z is the vector of objective economic phenomenon, and $\underline{z}$ a correspondent real number).

→ HOMOGENEITY (A)

If all variables "z" have a common factor λ, the resulting index $F(\lambda \underline{z})$ is equal to the product of the common factor λ and the calculated index, if a multiplication factor λ is absent.

- *of 1st degree* (cu referire la z_1)

$F(\lambda z_1, z_2, \ldots, z_k) = \lambda F(\underline{z})$ for any $z > 0$ and $\lambda > 0$

- *of "zero" degree*

$F(\lambda \underline{z}) = F(\underline{z})$ for any $\lambda > 0$.

→ IDENTITY (A) ("STATIONARY")

If there is no change of variables ($z_1 = z_2$), the index is uniform or stationary regardless of other conditions.

$F(1, 1, z_3, \ldots, z_k) = 1$ for any $z_3, \ldots, z_k$ (for description simplification of F, we considered $z_1 = z_2 = 1$).

→ ADDITIVITY (T)

If the variable z is expressed in terms of its original value through an algebraic sum ($z_1 = z_2 + \bar{z}$), the new index $F(z_2 + \bar{z})$ is equal to the algebraic sum of generated indexes $F(z_2) + F(\bar{z})$

$F(z_2 + \bar{z}) = F(z_2) + F(\bar{z})$.

→ MULTIPLICATION (T)

If the variable z is multiplied by the values $(\lambda_1, \ldots, \lambda_k) \in \mathbb{R}_+$, than the resulting index $F(\lambda_1 z_1, \lambda_2 z_2, \ldots, \lambda_k z_k)$ is equal to the product between the differentially multiplied variable z $(\lambda_1, \ldots, \lambda_k)$ and the initial index $F(\underline{z})$

$F(\lambda_1 z_1, \ldots, \lambda_k z_k) = z(\lambda_1, \ldots, \lambda_k)\, F(\underline{z})$,

where $\lambda_i \in \mathbb{R}_+$ and $i = \overline{1\,k}$.

→ QUASILINEARITY (T)

If $a_1, a_2, \ldots, a_k$ and b are real constant, and $a_1, a_2, \ldots, a_k \neq 0$, and given the continuous and strictly monotone function f: $\mathbb{R}_+ \rightarrow \mathbb{R}$, having the inverse function f^{-1}, it verifies the relation:

$F(\underline{z}) = f^{-1}[a_1 f(z_1) + a_2 f(z_2) + \ldots + a_k f(z_k) + b]$.

→ DIMENSIONALITY (A)

If all variables z_1 and z_2 are multiplied by a certain factor λ, the resulting index is equal to the initial index, as the case of the multiplying by λ would not have been existed.

$F(\lambda z_1, \lambda z_2, \ldots, \lambda z_k) = \frac{\lambda}{\lambda} F(\underline{z}) = F(\underline{z})$, for any $z > 0$ and $\lambda > 0$.

→ INTERIORITY (T) ("AVERAGE VALUE")

The index $F(\underline{z})$ should behave as an average value of individual indicices, being inside the interval of minimum and maximum value

$$\min\left\{\frac{z_{1i}}{z_{2i}}\right\} \leq F(\underline{z}) \leq \max\left\{\frac{z_{1i}}{z_{2i}}\right\}.$$

→ MEASURABILITY (A)

The index $F(\underline{z})$ is independent, respectively it is unaffected by the measurement units in which the variables are denominated

$$F\left(\frac{z_1}{\lambda_1}, \ldots, \frac{z_K}{\lambda_K}; \lambda_1 z_1, \ldots, \lambda_k z_k\right) = F(z_1, z_2, \ldots, z_k) = F(\underline{z}).$$

→ PROPORTIONALITY (T)

(Homogeneity of 1st degree of a stationary initial index)

If an index is in the state of identity, respectively $F(1, 1, z_3, \ldots, z_k) = 1$ for any $z_3, \ldots, z_k$, the proportional increase of variable z_1 by turning it from to λ lead to a similar valure of the obtained index $F(\lambda, 1, z_3, \ldots, z_k) = \lambda$ (where $\lambda \in \mathbb{R}_+$)

→ REVERSIBILITY (T) (ANTISYMMETRY AND SYMMETRY)

Considered as an axiom, the reversibility implies a double interpretation:

- the reversibility temporal or territorial approach generates an antisymmetry of Fisher type, respectively the index calculated as a report between the current period level or the compared space and the period level or reference space must be an inverse amount of the calculated index as report between the period level or reference space and the current period level or the compared space:

$$F(z_1, z_2, \ldots, z_k) \cdot \frac{1}{F(z_2, z_1, \ldots, z_k)} = 1$$

- the factorial approach generates a symmetry of Fisher type, respectively, if the phenomenon was split into qualitative and quantitative factors ($z_1 = \Sigma n_1\theta_1$ and $z_2 = \Sigma n_0\theta_0$), changing index factors does not modify the product of new indexes (symmetry of "crossed" indexes)

$$\left(\frac{\sum n_1\theta_1}{\sum n_1\theta_0} \cdot \frac{\sum n_1\theta_0}{\sum n_0\theta_0} = \frac{\sum n_1\theta_1}{\sum n_0\theta_0}\right).$$

→ CIRCULARITY (T) (TRANZITIVITY OR CONCATENATION)

The product of successive indexes represents a closed circle, respectively an index of the first level reported to the top level of the variable.

$$F(z_1, z_2, \ldots, z_k) \cdot F(z_2, z_3, \ldots, z_k) \cdot F(z_{i-1}, z_i, \ldots, z_k) = F(z_1, z_i, \ldots, z_k).$$

→ DETERMINATION (T) (CONTINUITY)

If any scalar argument in $F(z_1, z_2, \ldots, z_k)$ tends to zero, then $F(\underline{z})$ tends as well to a unique positive value of a real number (all other variable-dependent values).

→ AGGREGATION (A) (INDEX OF INDEXES)

The index of a set of variables is equal to an aggregated index when it is derived from indexes of each group sizes. Let all sizes: $z_n = F(z_1,z_2,\ldots z_n)$ be partial indexes; the index F is aggregative if $F_n(\underline{z}) = F[F(z_1,z_2,\ldots,z_n)]$.

→ EXPANSIBILITY (A) (- specific to aggregate indexes)

$$F_n(\underline{z}) < F_{n+1}(\underline{z},0).$$

→ PRESERVING THE VALUE INDEX *(Theorem)*

The aggregated index, written in the form of average index, which corresponds to a value index equal to the real value index, preserves the value index.

→ UNICITY *(Theorem)*

An index F is not accepted as unique index if there exists two indices $F_1 \neq F_2$ such that:

$$F(z_1, z_2, \ldots, z_k) = \begin{cases} F_1 & for\ \ k \in K_1, where\ \ K_1, K_2 \in N\ \ \grave{\imath}\ \ k = \overline{1K} \\ F_2 & for\ k \in K_2 \end{cases}$$

where z is the variable, k the variables set, K_1 and K_2 are two subsets of the set N, such that $K_1 \cup K_2 = N$ and $K_1 \cap K_2 = \varnothing$. This property requires the index calculation algorithm to be the same for all analyzed variables.

→ The USE OF INDEXES

This is a property resulting from data promptitude and data availability, easiness and rapidity of calculation, from simplicity of formula and of weighting system, from truthfulness of base and practical construction of indexes.

As shown, the axiomatic theory of economy is in fact a sum of properties-conditions mostly expressed by *axioms* (**A**) defining indexes, and by *theorems* and corollaries thereof derived from axioms and from tests (**T**) whose role is also important in the construction of indexes. Depending on the system of indexes they belong to, and on the specific use, the required properties are layered by Helmut Diehl in:

→ *basic requirements* – imposed by specific circumstances of the project;
→ *required properties* – ensuring fundamental qualities and operational consistency;
→ *desirable properties* – providing some technical facilities and even some theoretical elegance;
→ *special properties* – generated by construction and method.

Gathering specific characteristics in a definition as general as possible, the index is considered an indicator, *a statistical category*, expressed through a synthetic size that renders the relative variation between two states – one "actual" (or territoriality of interest), another "baseline" – of a phenomenon, or *a relative number* resulted by the comparison of a statistical indicator values, *a measure* of the relative change of variables at different time points and in different spaces, or in different categories, set in relation to a certain characteristic feature.

The evolution in time of indexes required for over three centuries solving all sort of theoretical and methodological problems regarding the method calculation, including formula, the base choice, the weighting system and, especially, the practical construction. [10; 11; 12; 13; 14]

Process optimization of this issue is not definitively over even though its history is quite eventful, as summarized in Box No. 1, below. Moreover, even this paper is only trying to propose a new type of neutrosophic index or a neutrosophic-type index number.

Box No. 1

The index – appeared, as the modern statistics, in the school of political aritmetics – has as father an Anglican Bishop, named William Fleetwood. The birth year of the first interpreter index is 1707; it was recorded by studying the evolution of prices in England between 1440 and 1707, a work known under the title "Chicon Preciosum". The value of this first index was 30/5, respectively 600,0%, and it was built on the simple arithmetical mean of eight products: wheat, oats, beans, clothing, beer, beef, sheepmeat and ham. Moreover, the world prices – a world hardly approachable because of specific amplitude, sui generis heterogeneity and apparently infinite trend – was transformed into a homogeneous population through interpreter indexes. In 1738, Dutot C. examines the declining purchasing power of the French currency between 1515 and 1735, through a broader interpreter index, using the following formula:

(1.1) *Dutot* Index: $\frac{p_1 + p_2 + \dots + p_n}{\mathrm{P}_1 + \mathrm{P}_2 + \dots + \mathrm{P}_n} = \frac{\sum_{i=1}^{n} p_i}{\sum_{i=1}^{n} \mathrm{P}_i}$, where: p_i and P_i

= prices of current period vs. basic period.

If you multiply the numerator and denominator index by (1/n), the calculation formula of *Dutot* index becomes a mean report, respectively:

$(\sum_{i=1}^{n} p_i /\mathrm{n}):(\sum_{i=1}^{n} \mathrm{P}_i /\mathrm{n})$.

To quantify the effect of the flow of precious metals in Europe after the discovery of the Americas, the Italian historian, astronomer and economist Gian Rinaldo Carli, in 1764, used the simple arithmetic mean for three products, i.e. wheat, wine and oil, in constructing the interpreter index determined for 1500 and 1750:

(1.2) *Carli* Index: $\frac{1}{n}\left(\frac{p_1}{\mathrm{P}_1} + \frac{p_2}{\mathrm{P}_2} + \dots + \frac{p_n}{\mathrm{P}_n}\right) = \frac{1}{n}\sum_{i=1}^{n}\frac{p_i}{\mathrm{P}_i}$

As William Fleetwood has the merit of being the first to homogenize the heterogeneous variables through their ratio, using the results to ensure the necessary comparisons, the same way Dutot and Carli are praiseworthy for generating the "adimensionality" issue, namely the transformation of absolute values into relative values, generally incomparable or not reducible to a central (essential or typical) value (a value possessing an admissible coefficient of variation in statistical terms). But the most important improvement in index construction, streamlining its processing, belongs to Englishman Arthur Young, by introducing the *weight* (ponderation), i.e. coefficients meant to point the *relative importance* of the various items that are part of the index.

Young employed two weighting formulas, having as a starting point either *Dutot*:

(1.3) *Young* Index (1):

$$\frac{p_1k_1 + p_2k_2 + ... + p_nk_n}{\mathrm{P_1K_1 + P_2K_2 + ... + P_nK_n}} = \frac{\sum_{i=1}^{n} p_i k_i}{\sum_{i=1}^{n} \mathrm{P_i K_i}},$$

where k_i= coefficient of importance of product *i*,

or *Carli*:

(1.4) *Young* Index (2):

$$\frac{1}{\sum_{i=1}^{n} \mathrm{C_i}}\left(\frac{p_1}{\mathrm{P_1}}\mathrm{C_1} + \frac{p_2}{\mathrm{P_2}} + \mathrm{C_2} ... + \frac{p_n}{\mathrm{Pn}}\mathrm{C_n}\right) = \frac{1}{\sum_{i=1}^{n} \mathrm{C_i}} \times \sum_{i=1}^{n} \frac{p_i}{\mathrm{P_i}} \times \mathrm{C_i} = \sum_{i=1}^{n} \frac{p_i}{\mathrm{P_i}} \times \frac{\mathrm{C_i}}{\sum_{i=1}^{n} \mathrm{C_i}},$$

where $\frac{\mathrm{C_i}}{\sum_{i=1}^{n} \mathrm{C_i}} =$ weighting coefficient and $\sum_{i=1}^{n} (c.p.)_i = 1$.

After *Young solution* from 1812, the new problem of designing indexes has become the effect of weight variations. Sir George Shuckburgh Evelyn introduced, in 1798, the concept of "basic year", thus anticipating *the dilemma of base selection and of construction of the weighting system.* In 1863, by the index calculated as geometric mean of individual indexes, Stanley Jevons extended the issue to the formula:

(1.5) *Jevons* Index:

$$\sqrt[n]{\prod_{i=1}^{n} \frac{p_i}{\mathrm{P}_i}}$$

Jevons does not distinguish between individual indexes, giving them the same importance.

Two indexes imposed by the German school of statistics remain today, like the two terrestrial poles, structural limits of *weighting systems*. The first is the index of Etienne Laspeyres, produced in 1864, using basic period weighting, and the second is the index of Hermann Paasche, drafted in 1874, using the current period as weighting criterion:

(1.6) *Laspeyres* Index: $\frac{\sum p_{i1} q_{i0}}{\sum p_{i0} q_{i0}}$ or $\frac{\sum p_{i0} q_{i1}}{\sum p_{i0} q_{i0}}$ and

(1.7) *Paasche* Index: $\frac{\sum p_{i1} q_{i1}}{\sum p_{i0} q_{i1}}$ or $\frac{\sum p_{i1} q_{i1}}{\sum p_{i1} q_{i0}}$, where:

p_{i0}, p_{i1} = basic period prices (0) and current period prices (1)

q_{i0}, q_{i1} = basic period quantities (0) and current period quantities (1).

Although the provided indexes only checks the identity condition ($I^{X}_{1/1}$ $=X_1/X_1=1$) from Fischer's tests for elementary indexes, however they are the most commonly used in practice due to the economic content of each construction. Several "theoretical" indexes were placed close to the Laspeyres and Paasche indexes, but with the loss of specific business content, and different of Ladislaus von Bortkiewicz relationship. They can be called unreservedly indexes of "*mesonic*"-type, based on authors' wishes to situate the values within the difference (P – L), to provide a solution of equilibrium between the two limit values in terms of choosing of base. Along with the two weighting systems, other issues are born, like weighting constancy and inconsistency, or connecting the bases on the extent of aging or disuse. Of the most popular "*mesonic*"-type index formulas [5], there are the constructions using common, ordinary statistics. The simple arithmetic mean of Laspeyres and Paasche indexes is known as *Sidgwik – Drobisch* index.

(1.8) *Sidgwig -Drobisch* Index: $\frac{L+P}{2}$

The arithmetic mean of the quantities of the two periods (thus becoming weight) generates the *Marshall - Edgeworth* index or *Bowley - Edgeworth* index (1885 - 1887).

(1.9) *Marshall – Edgeworth* Index: $\frac{\sum p_{i1}(q_{i0}+q_{i1})}{\sum p_{i0}(q_{i0}+q_{i1})}$

The geometric mean of quantities in the two periods converted in weights fully describes the *Walsh* index (1901).

(1.10) *Walsh* Index: $\frac{\sum p_{i1}\sqrt{(q_{i1}\times q_{i0})}}{\sum p_{i0}\sqrt{(q_{i1}\times q_{i0})}}$

The simple geometric mean of Laspeyres and Paasche indexes is none other than the well-known *Fisher* index (1922).

(1.11) *Fisher* Index: $\sqrt{(L\times P)}$

The index checks three of the four tests of its author, Irving Fisher: the identity test, the symmetry test, or the reversibility-in-time test and the completeness test, or the factors reversibility test. The only test that is not entirely satisfied is the chaining (circularity) test. The advantage obtained by the reversibility of Fisher index:

(1.12) $F_{0/1}=\sqrt{(L_{1/0}\times P_{1/0})}=\frac{1}{\sqrt{(L_{1/0}\times P_{1/0})}}=\frac{1}{F_{1/0}}$,

is unfortunately offset by the disadvantage caused by the lack of real economic content. A construction with real practical valences is that of *R.H.I. Palgrave* (1886), which proposed a calculation formula of an arithmetic average index weighted by the total value of goods for the current period ($v_{1i}=p_{1i}\cdot q_{1i}$) :

(1.13) *Palgrave* Index: $\frac{\sum i_{1/0}\times(p_{1i}q_{i1})}{\sum p_{i1}q_{i1}}=\frac{\sum i_{1/0}\times(v_{1i})}{\sum v_{1i}}$.

The series of purely theoretical or generalized indexes is unpredictable and full of originality.

Cobb - Douglas solution (1928) is a generalization of Jevons index, using unequal weights and fulfilling three of Fisher's tests (less the completeness or the reversibility of factors):

(1.14) *Cobb – Douglas* Index: $\prod_{i=1}^{n}\left(\frac{p_i}{P_i}\right)^{\alpha_i}$, where $\alpha_i > 0$ and $\sum_{i=1}^{n}\alpha_i = 1$.

Stuvel version, an index combining the Laspeyres index „of price factor" (L^p) and the Laspeyres index „of quantity factor" (L^q), proposed in 1957, exclusively satisfies the condition of identity as its source:

(1.15) *Stuvel* Index: $\frac{L^p\text{-}P^q}{2}+\sqrt{\frac{\left(L^p\text{-}P^q\right)^2}{4}+I^{(p\times q)}}$

(where $I^{(pxq)}$ = total variation index)

Another construction, inspired this time from the „experimental" design method, based on the factorial conception, but economically ineffective, lacking such a meaning, is R.S. Banerjee index (1961), a combination of indexes as well, but of Laspeyres type and Paasche type:

(1.16) *Banerjee* Index: $\frac{L+1}{\frac{1}{P}+1}=\frac{P(L+1)}{(P+1)}$

A true turning point of classical theorizing in index theory is the *autoregressive* index.

(1.17) Autoregressive Index: $\frac{\sum\left(p_i P_i a_i^2\right)}{\sum\left(P_i\right)^2\times a_i^2}$,

Therefore, a_i means the quantities of products or weights (importance) coefficients. This only verifies the provided identity, although conditionally constructed, respectively:

$$\sum\left[p_i - P_i \times I_{\text{AUTOREGRESSIVE}}\right]^2 = \text{ minimum.}$$

Torngvist (1936) and *Divisia* (1925) indexes are results of generalizations of mathematical type, defining the following relationships:

(1.18) ln (Torngvist Index) = $\sum\frac{1}{2}\left[\frac{p_i q_i}{\sum p_i q_i}+\frac{P_i Q_i}{\sum P_i Q_i}\right]\times\ln\frac{p_i}{P_i}$,

where: $\frac{p_i q_i}{\sum p_i q_i}$ and $\frac{P_i Q_i}{\sum P_i Q_i}$ are weights of specific transactions values $p_i q_i$ and $P_i Q_i$.

The usual shape under which one meets the *Divisia* index is:

(1.19) $P_{0t}\ Q_{0t} = \frac{\sum p_{it} q_{it}}{\sum p_{i0} q_{i0}}$ as averaged value in a relationship determined by individual prices indexes, respectively:

$$P\left(i_{p1}+i_{p2}+\ldots+i_{pn}\right)=i_{pi}.$$

Contemporary multiplication processes of indexes calculation formulas have two trends, *one already visible of extrem axiomatization and mathematization,* based on *Torngvist* and *Divisia* indexes models, which culminated with the school of axiomatic indexes, and another, of resumption of the logic stream of economic significance of index construction, specific for the latest international constructions at the end of the twentieth century, respectively the integration variants of additive construction patterns or additive-multiplicative mixed models, close to the significance of real phenomena. In this regard, one could summary present the comparative advantage index or David Neven index (1895).

(1.20) *David Neven* Index:

$\left(\frac{x_k}{\sum x_k} - \frac{m_k}{\sum m_k}\right) \times 100$, where *x* and *m* are values of exports and imports in the industry *k*. The index belongs to the range of values (-100%; 100%), but rarely achieves in practice higher values than 10% or lower than – 10%. etc.

In the theory and practice of index numbers construction, to quantify and interpret the degree and the direction of the weights influence, use is made of Bortkiewicz relationship [15]. This specific relationship is based on factorial indexes and yields to the following equality:

$$I_{1/0}^{x(f_1)} : I_{1/0}^{x(f_0)} = 1 + r_{x_i f_i} \bullet Cv_{x_i} \bullet Cv_{f_i} \tag{1.21}$$

where:

$r_{x_i f_i}$ is a simple linear correlation coefficient between individual indexes of the qualitative factor x_i and individual indexes of the weights (respectively, individual indexes of the qualitative factor f_i),

Cv_{x_i} is the coefficient of variation of individual indexes of variable x to their environmental index,

Cv_{f_i} is the coefficient of variation of individual indexes of weights towards their environment index,

while $I_{1/0}^{x(f_1)} = \frac{\sum x_1 f_1}{\sum x_0 f_1}$ and $I_{1/0}^{x(f_0)} = \frac{\sum x_1 f_0}{\sum x_0 f_0}$.

The interpretation of that relationship shows that *the weighting system does not influence the index of a numerically expressed group variable, if the product of the three factors is null*, respectively $r_{x_i f_i} \times Cv_{x_i} \times Cv_{f_i} = 0$.

This is possible in three distinct situations:

a) $r_{x_i f_i} = 0 \Rightarrow x_i$ and f_i are independent to each other (there is no connection between individual indexes i^x and i^f),

b) $Cv_{x_i} = 0$ $\Rightarrow$ the absence of any variation on the part of *x* or *f*

c) $Cv_{f_i} = 0$ (individual indexes are equal to the average index).

Product sign of factors $r_{x_i f_i} \times Cv_{x_i} \times Cv_{f_i}$ *is positive or negative depending on* $r_{x_i f_i}$*, the sign of the latter being decisive.*

The interpretation of the influence of the weighting systems on the value of a synthetic index is based on the following three cases:

- The synthetic index calculated using current period weights is equal with the same index calculated with the weights of the basic period when at least one of the factors is equal to „0”.
- The synthetic index calculated using current period weights is bigger in value than the index calculated with the weights of the basic period when the three factors are different from „0” and the simple linear correlation coefficient is positive.
- The synthetic index calculated using current period weights is lower in value than the index calculated with the weights of the basic period when the three factors are different from „0” and the simple linear correlation coefficient is negative.

Applying Bortkiewicz's relationship to the interpretation of statistical indexes offers the opportunity to check the extent and direction to which the weighting system that is employed influence the value of the indexes.

The conclusive instauration of a sign in language, be it gradually, is a lengthy process, where the sign (the representative or the signifier) replaces at a certain moment the representative (the signifier). The sign substitutes an object and can express either a quality (*qualisign*), or a current existence (*synsign*), or a general law (*legisign*). Thus, the index appears as sign together with an icon (e.g.: a chart, a graphic), a symbol (e.g.: currency), a rhema (e.g.: the mere posibility), a dicent (e.g.: a fact), an argument (e.g.: a syllogism) etc. The semiotic *index* can be defined as a sign that loses its sign once the object disappears or it is destroyed, but it does not lose this status if there is no interpreter. The index can therefore easily become its own interpreter sign. Currency as sign takes nearly all detailed semiotic forms, e.g. qualisign or hard currency, symbol of a broad range of sciences, or legisign specific to monetary and banking world. As the world's history is marked by inflation, and currency implicitly, as briefly described in Box nr. 2 below, likewise the favorite index of the inflationary phenomenon remains the interpreter index.

Box No. 2

The inflation – an evolution perceived as diminishing the value or purchasing power of the domestic currency, defined either as an imbalance between a stronger domestic price growth and an international price growth, or as a major macroeconomic imbalance of material-monetary kind and practically grasped as a general and steady increase in prices – appeared long before economics. Inflationary peak periods or “critical moments” occurred in the third century, at the beginning of sixteenth century, during the entire eighteenth and the twentieth centuries. The end of the third century is marked by inflation through currency, namely excessive uncovered currency issuance in the Roman Empire, unduely and in vain approached by the Emperor Diocletian in 301 by a “famous” edict of maximum prices which sanctioned the “crime” of price increase by death penalty. The Western Roman Empire collapsed and the reformer of the Eastern Roman Empire, Constantine the Great, imposed an imperial currency, called “solidus” or "nomisma", after 306, for almost 1000 years. The beginning of the sixteenth century, due to the great geographical discoveries, brings, together with gold and silver from the "new world", over four times price increases, creating problems throughout Europe by precious metal excess of Spain and Portugal, reducing the purchasing power of their currencies and, finally, of all European money. If the seventeenth century is a century of inflationary "princes", which were maintaining wars by issuing calp fluctuating currency, the twentieth century distinguishes itself by waves of inflation, e.g. the inflation named "Great Depression began in Black Thursday", or the economic crisis in 1930, the inflation hidden in controlled and artificial imposed prices of "The Great Planning", the inflation caused by price evolutions of oil barrel, or sometimes galloping inflation of Eastern European countries' transition to market economy. Neither the "edicts" or the "assignats" of Catherine II, as financial guarantees of currency, nor the imposed or controlled prices were perennial solutions against inflation.

Inflation is driven, par excellence, by the term "excess": excessive monetary emission or inflation through currency, excessive solvable demand or inflation by demand, excessive nominal demand, respectively by loan or loan inflation, excessive cost or cost-push inflation; but rarely by the term "insufficiency", e.g. insufficient production, or supply inflation. Measurement of overall and sustained price growth – operation initiated by Bishop William Fleetwood in 1707 by estimating at about 500% the inflation present in the English economy between 1440 and 1707 – lies on the statistical science and it materializes into multiple specific assessment tools, all bearing the name of price indexes, which originated in interpreter indexes. Modern issues impose new techniques, e.g. econophysics modeling, or modeling based on neutrosophic numbers resulted from nuanced logic.

5 Neutrosofic Index Numbers or Neutrosophic-Type Interpreter Indexes

Created in the full-of-diversity world of prices, the first index was one of *interpreter* type. The term "interpreter" must be understood here by the originary meaning of its Latin component, respectively *inter* = between (middle, implicit mediation) and *pretium* = price. [16]

The distinct national or communautaire definitions, assigned to various types of price indexes, validate, by synthesizing, the statement that the interpreter index has, as constant identical components, the following features:

- measuring tool that provides an *estimate* of price trends (consumer goods in PCI, industrial goods în IPPI or import/export, rent prices, building cost etc.);
- *alienation* of goods and services (respectively, actual charged prices and tariffs);
- *price change between a fixed period* (called basic period or reference period) and *a variable period* (called current period).

The most used interpreter indexes are the following:

- PCI – Prices of Consumer (goods and services) Index measures the overall evolution of prices for bought goods and tariffs of services, considered the main tool for assessing inflation;
- IPPI – Industrial Producers' Price Index of summarizes developments and changes in average prices of products manufactured and supplied by domestic producers, actually charged in the first stage of commercialization, used both for deflating industrial production valued at current prices, and for determining inflation within "producer prices". This index is one of the few indexes endowed with power of "premonition", a true Cassandra of instruments in the so-populated world of instruments measuring inflation. Thus, IIPP anticipates the developments of IPCG. The analysis of the last 17 years shows a parallel dynamics of evolution of the two statistical tools for assessing inflation, revealing the predictive ability of IPCG dynamics, starting from the development of IIPP;
- UVI – Unit Value Index of export / import contracts characterizes the price dynamics of export / import, expanding representative goods price changes ultimately providing for products a coverage rate of maximum 92%, allowing deflation through indicators characterizing the foreign trade, and even calculating the exchange ratio;
- CLI – Cost of Living Index shows which is the cost at market prices in the current period, in order to maintain the standard of living achieved in the basic period, being calculated as a ratio between this hypothetical cost and the actual cost (consumption) of the basic period; the need for this type of interpreter index is obvious above all in the determination of real wages and real income;
- IRP – Index of Retail Price sets the price change for all goods sold through the retail network, its importance as a tool to measure inflation within "retail prices" being easily noticed;
- BCI – Building Cost Index assesses price changes in housing construction, serving for numerous rental indexation, being used independently or within IPCG, regardless of the chosen calculation method;
- FPPI-F – food products price index measures changes in prices of food products on the *farm market* (individual or associated

farmers market), providing important information about inflation on this special market;

- GDP deflator index or *the implicit deflator of GDP* – GDP price index that is not calculated directly by measuring price changes, but as a result of the ratio between nominal GDP or in current prices and GDP expressed in comparable prices (after separately deflating the individual components of this macroeconomic indicator); GDP deflator has a larger coverage as all other price indexes.

The main elements of the construction of an interpreter index refer to official name, construction aims, official computing base, weighting coefficients, sources, structure, their coverage and limits, choosing of the weighting system, of the calculation formula, method of collection, price type and description of varieties, product quality, seasonality and specific adjustments, processing and analysis of comparable sources, presentation, representation and publication. The instrumental and applied description of consumer goods price index has as guidelines: definition, the use advantages and the use disadvantages, the scope, data sources, samples used in construction, the weighting system, the actual calculation, the inflation calculated as the rate of IPCG, specific indicators of inflation, uses of IPCG and index of purchasing power of the national currency.

As there seems natural, there is a statistical correlation and a gap between two typical constructions of price index and interpreter index, IPPI and PCI. Any chart, a chronogram or a historiogram, shows the evolution of both the prices of goods purchased, and of paid services that benefited common people (according to the consumer goods price index), and of the industrial goods prices that went out of the enterprises' gate (according to the producers' price index) and are temporarily at intermediaries, following to reach the consumers in a time period from two weeks to six months, depending on the length of "commercial channel".

The interpreter indexes are statistical tools – absolutely necessary in market economies – allowing substitution of adjectival-type characterizations of inflation within an ordinal scale. As the variable measured on an ordinal scale is equipped with a relationship of order, the following ordering becomes possible:

- the level of subnormal inflation (between 0 and 3%);
- the level of (infra)normal inflation (Friedman model with yearly inflation between 3 and 5%);
- the level of moderate inflation (between 5 and 10% yearly);
- the level of maintained inflation (between 10 and 20% yearly);

- the level of persistent inflation (between 20 and 100% yearly);
- the level of enforced inflation (between 100 and 200% yearly);
- the level of accelerated inflation (between 200 and 300% yearly);
- the level of excessive inflation (over 300% yearly).

Knowing the correct level of inflation, the dynamics and the estimates of short-term price increase allow development appreciation of value indicators in real terms. The consumer goods price index, an interpreter index that can inflate or deflate all nominal value indicators, remains a prompt measurement tool of inflation at the micro and macroeconomic level.

Any of the formulas or of the classical and modern weighting systems used in price indexes' construction can be achieved by neutrosophic-tendential fuzzy numbers following operations that can be performed in neutrosophic arithmetics. A random example [17, 18, 19] relative to historical formulas and classical computing systems (maintaining the traditional name of "Index Number") is detailed for the main indexes used to measure inflation, according to the data summarized in Table 1.

The statistical data about the price trends and the quantities of milk and cheese group are presented below for two separate periods:

Table 1.

Product	Basic price p_o	Current price p_t	Total expenses in the basic period ($p_o q_o$)	Total expenses in the basic period with current prices ($p_t q_o$)	Total expenses in current period with basic prices ($p_0 q_t$)	Total expenses in the current period ($p_t q_t$)	Weighting coefficients (Cp_0) $p_o q_o / \Sigma p_o q_o$	Quantities of products bought in the basic period (q_o)	Quantities of products bought in the current period (q_t)
Milk	1,20	1,70	12,0	17,0	10,8	15,3	15,6	10	9
Butter	1,90	1,70	15,2	13,6	13,3	11,9	19,8	8	7
Yogurt	0,85	0,90	3,4	3,6	5,1	5,4	4,4	4	6
Sour cream	1,25	3,00	1,25	3,0	1,25	3,0	1,6	1	1
Cheese	7,50	8,00	45,0	48,0	52,5	56,0	58,6	6	7
Total group	12,70	15,30	76,85	85,2	82,95	91,6	100,0	-	-

A. Historical solutions (unorthodox) focused on calculating formula for the simple aggregate and unweighted index (quantities are not taken into account, although there have been changes as a result of price developments)

$$\text{I. Index Number} = \frac{\sum p_t}{\sum po} = \frac{15,30}{12,70} = 1,205.$$

The inflation rate extracted from index = 0,205 or 20,5%.

B. Contemporary solutions focused on formula for calculating the aggregate weighted index in classical system

I. Index Number by classical Laspeyres formula =

$$\frac{\sum(ptqo)}{\sum(poqo)} = \frac{85,2}{76,85} = 1{,}109.$$

The inflation rate extracted from index = 0,109 or 10,9%.

II. Index Number expressed by relative prices or individual prices indexes by Laspeyres formula =

$$\frac{\sum \frac{pt}{po}(poqo)}{\sum(poqo)} = \frac{85,2}{76,85} = 1{,}109 \text{ or } \sum \frac{pt}{po} \times Cp_0 = 1{,}109.$$

The inflation rate extracted from index = 0,109 or 10,9%.

III. Index Number by Paasche formula =

$$\frac{\sum(ptqt)}{\sum(poqt)} = \frac{91,6}{82,95} = 1{,}104.$$

The inflation rate extracted from index = 0,104 or 10,4%.

IV. Index Number by Fisher formula =

$$\sqrt{\text{Laspeyres Index Number} \times \text{ Paasche Index Number}} =$$

$$= \sqrt{1{,}109 \times 1{,}104} = 1{,}106.$$

The inflation rate extracted from index = 0,106 or 10,6%.

V. Index Number by Marshall-Edgeworth formula =

$$\frac{\sum\left[pt(qo+qt)\right]}{\sum\left[po(qo+qt)\right]} = \frac{176,8}{159,8} = 1{,}106.$$

The inflation rate extracted from index = 0,106 or 10,6%.

VI. Index Number by Tornqvist formula =

$$\Pi\left(\frac{pt}{po}\right)^{w} \text{ where } \quad w = \frac{p_o q_o}{2\sum p_o q_o} + \frac{p_t q_t}{2\sum p_t q_t} =$$

$$= \left(\frac{1,7}{1,2}\right)^{0,1616} \times \left(\frac{1,7}{1,9}\right)^{0,1639} \times \left(\frac{0,9}{0,85}\right)^{0,0516} \times \left(\frac{3,0}{1,25}\right)^{0,0245} \times \left(\frac{8,0}{7,5}\right)^{0,5985} = 1{,}106.$$

The inflation rate extracted from index = 0,106 or 10,6%.

As one can see, three Index Numbers or price indexes in Fisher, Marshall-Edgeworth and Tornqvist formulas lead to the same result of inflation of 10.6%, which is placed in median position in relation to the Laspeyres and Paasche indexes.

However, the practice imposed Laspeyres index because of obtaining a high costs and a relatively greater difficulty of weighting coefficients in the current period (t). [17]

C. Neutrosophic index-based computing solutions

Starting from the definition of "a slight increase in price" variable, denoted by $\triangle p$, where $\triangle p=(p_1-p_0)/p_0$, data from Table 1 are recalculated in Table 2, below and defines the same *unorthodox* but classic solutions (especially in the last two columns).

Table 2.

Product	Basic price p_o	Current price p_t	Quantities of products bought in the basic period (q_o)	Quantities of products bought in the current period (q_t)	Total expenses in the basic period (p_oq_o)	Total expenses in the current period ($p_t q_t$)	Classic Index Number (p_t / p_0)	$\triangle p=(p_1-p_0)/p_0$	$\triangle pq=(p_1q_1-p_0q_0)/p_0q_0$
Milk	1.20	1.70	10	9	12.0	15.3	1.4167	0.4167	0.2750
Butter	1.90	1.70	8	7	15.2	11.9	0.8947	-0.1053	-0.2171
Yogurt	0.85	0.90	4	6	3.4	5.4	1.0588	0.0588	0.5882
Sour cream	1.25	3.00	1	1	1.25	3.0	2.4000	1.4000	1.4000
Cheese	7.50	8.00	6	7	45.0	56.0	1.0667	0.0667	0.2444
Total group	12.70	15.30	-	-	76.85	91.6	1.2047	0.2047	0.1919

The identical values of Fisher, Marshall-Edgeworth and Tornqvist indices offer a hypothesis similar with the neutrosophic statistics and especially with neutrosophic frequencies. The highest similarity with the idea of neutrosophic statistics consists of the Tornqvist formula's solution. The calculus of the absolute and relative values for necessary neutrosophic frequencies is described in the Table 3.

Table 3.

Product	Total expenses in the basic period (p_0q_0)	Weighting coefficients (Cp_0) = $p_0q_0/\Sigma p_0q_0$	Total expenses in the basic period with current prices ($p_t q_o$)	Weighting coefficients (Cp_{t0}) $p_tq_0/\Sigma p_tq_0$	Total expenses in current period with basic prices (p_0q_t)	Weighting coefficients (Cp_{0t}) = $p_0q_t/\Sigma p_0q_t$	Total expenses in the current period ($p_t q_t$)	Weighting coefficients (Cp_t) = $p_tq_t/\Sigma p_tq_t$
Milk	12.0	**15.62**	17.0	*20.0*	10.8	*13.0*	15.3	**16.70**
Butter	15.2	**19.78**	13.6	*16.0*	13.3	*16.0*	11.9	**12.99**
Yogurt	3.4	**4.42**	3.6	*4.2*	5.1	*6.2*	5.4	**5.90**
Sour cream	1.25	**1.63**	3.0	*3.5*	1.25	*1.5*	3.0	**3.28**
Cheese	45.0	**58.55**	48.0	*56.3*	52.5	*63.3*	56.0	**61.13**
Total group	76.85	**100.00**	85.2	*100.0*	82.95	*100.0*	91.6	**100.00**

In this situation, the construction of major modern indexes is the same as the practical application of statistical frequencies of neutrosophic type generating neutrosophic indexes in the seemingly infinite universe of prices specific to inflation phenomena, as a necessary combination between classical indexes and thinking and logic of frequencial neutrosophic statistics [20; 21; 22; 2. 3].

Table 4.

Product	Classic Index Number (p_t / p_0) (unorthodox)	Relative Neutrosophic Frequency **RNF(0)** Weighting coefficients (Cp_0) = $p_0q_0/\Sigma p_0q_0$	Relative Neutrosophic Frequency RNF(t.0) Weighting coefficients (Cp_{t0}) $p_tq_0/\Sigma p_tq_0$	Relative Neutrosophic Frequency RNF(0.t) Weighting coefficients (Cp_{0t}) = $p_0q_t/\Sigma p_0q_t$	Relative Neutrosophic Frequency **RNF(t)** Weighting coefficients (Cp_t) = $p_tq_t/\Sigma p_tq_t$	w = [$RNF_{(0)}$ + $RNF_{(t)}$] : 2
Milk	1.4167	**15.62**	*20.0*	*13.0*	**16.70**	16.16 % or 0.1616
Butter	0.8947	**19.78**	*16.0*	*16.0*	**12.99**	16.39 % or 0.1639
Yogurt	1.0588	**4.42**	*4.2*	*6.2*	**5.90**	5.16 % or 0.0516
Sour cream	2.4000	**1.63**	*3.5*	*1.5*	**3.28**	2.45 % or 0.0245
Cheese	1.0667	**58.55**	*56.3*	*63.3*	**61.13**	59.84 % or 0.5984
Total group	1.2047	**100.0**	*100.0*	*100.0*	**100.00**	100.00 or 1.0000

In this case, index of Tornqvist type is determined exploiting the relative statistical frequencies of neutrosophic type consisting of column values (Cp_0) *and* (Cp_t) *according to the new relations:*

$$\prod\left(\frac{pt}{po}\right)^{w} \quad \text{where } w = \frac{p_o q_o}{2\sum p_o q_o} + \frac{p_t q_t}{2\sum p_t q_t} = [RNF_{(0)} + RNF_{(t)}] : 2$$

Finally, applying the values in Table 4 shows that the result $\prod\left(\frac{pt}{po}\right)^{w}$ is identical.

$$\prod\left(\frac{pt}{po}\right)^{w} = 1.4167^{\,0.1616} \times 0.8947^{\,0.1639} \times 1.0588^{0.0516} \times 2.4^{0.0245} \times 1.0667^{\,0.5985} = 1.106$$

5 Conclusion

Over time, the index became potentially-neutrosophic, through the weighting systems of the classical indexes, especially after Laspeyres and Paasche. This journey into the world of indexes method merely proves that, with Tornqvist, we are witnessing the birth of neutrosophic index, resulting from applying predictive statistical neutrosophic frequencies, still theoretically not exposed by the author of this kind of thinking, actually the first author of the present article.

Future intention of the authors is to exceed, by neutrosophic indexes, the level of convergence or even emergence of unorthodox classical indexes, delineating excessive prices (high or low) by transforming into probabilities the classical interval [0 ; 1], either by the limiting values of Paasche and Laspeyres indexes, redefined as reporting base, or by detailed application of the neutrosophic thinking into statistical space of effective prices, covered by the standard interpreter index calculation (the example of PCI index is eloquent through its dual reference to time and space as determination of ten-years average index type, by arithmetic mean, and as determination of local average index type, by geometric mean of a large number of territories according to EU methodology, EUROSTAT).

6 Notes and Bibliography

[1] An information can be considered incomplete in relation to two scaled qualitative variables. The first variable is trust given to the information by the source. by the measuring instrument or by the degree of professionalism of the expert who analyze it. the final scale of uncertainty having as lower bound the completely uncertain information and as upper bound the completely definite information. The second variable is accuracy of the information content. the information benefiting of a sure content on the scale of imprecision only when the set of specified values is single-tone. i.e. holding a unique value.

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[4] A concise set *A* in existence domain *U* (providing the set of values allowed for a variable) can be objectively defined by:

a. listing all the items contained;

b. finding a definition conditions such specified: A= {x|x knowing certain conditions};

c. introduction of a zero-one membership function for A. denoted by μA(x) or characteristic (community. discriminatory or indicative) function {where: A≥μA(x)=1. if... and μA(x)=0 if...}, the subet A being thus equivalent to the function of belonging.

meaning that, mathematically, knowing μA(x) becomes equivalent to know A itself.

[5] When A' = A and B' = B. the rule identifies or becomes the particular case of a classical Modus Ponens. (Zadeh. L. A.. 1994. *Foreword*. in II. Marks J.F. ed., *The Neutrosophic Logic Technology and its Applications*. IEEE Publications).

[6] The logic system with tendential-neutrosophic fuzzy sets essentially transposes highly nuanced words in concise numbers; it was introduced in the vocabulary of contemporary logic by E. H. Mamdani in 1983. at San Francisco IEEE conference about tendential-neutrosophic fuzzy systems.

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[8] The height value mathematically described by the relationship (4) classifies the neutrosophic-tendential fuzzy subsets in *normal* or *normalized*. when h (A) = 1. i.e. $\exists x \in X$ such that μ A (x) = 1 or $\exists(\Delta p) \in X$. such that μ A (△p) =1; and in *subnormal* or *subnormalized* when it appears impossible the existence of a height equal to 1. (Gâlea Dan. Leon Florin. *Teoria mulţimilor fuzzy / Fuzzy Set Theory*).

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Neutrosophic Actions, Prevalence Order, Refinement of Neutrosophic Entities, and Neutrosophic Literal Logical Operators

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Abstract

In this chapter we define for the first time three neutrosophic actions and their properties. We then introduce the prevalence order on $\{T, I, F\}$ with respect to a given neutrosophic operator "o", which may be subjective - as defined by the neutrosophic experts. And the refinement of neutrosophic entities <A>, <neutA>, and <antiA>.

Then we extend the classical logical operators to neutrosophic literal logical operators and to refined literal logical operators, and we define the refinement neutrosophic literal space.

Keywords

neutrosophy, neutrosophics, neutrosophic actions, prevalence order, neutrosophic operator, refinement of neutrosophic entities, neutrosophic literal logical operators, refined literal logical operators, refinement neutrosophic literal space.

1 Introduction

In Boolean Logic, a proposition $\mathcal{P}$ is either true (T), or false (F). In Neutrosophic Logic, a proposition $\mathcal{P}$ is either true (T), false (F), or indeterminate (I).

For example, in Boolean Logic the proposition $\mathcal{P}_1$:

"1+1=2 (in base 10)"

is true, while the proposition $\mathcal{P}_2$:

"1+1=3 (in base 10)"

is false.

In neutrosophic logic, besides propositions $\mathcal{P}_1$ (which is true) and $\mathcal{P}_2$ (which is false), we may also have proposition $\mathcal{P}_3$:

"1+1= ?(in base 10)",

which is an incomplete/indeterminate proposition (neither true, nor false).

1.1 Remark

All conjectures in science are indeterminate at the beginning (researchers not knowing if they are true or false), and later they are proved as being either true, or false, or indeterminate in the case they were unclearly formulated.

2 Notations

In order to avoid confusions regarding the operators, we note them as:

Boolean (classical) logic:

$$\neg,\quad \wedge,\quad \vee,\quad \underline{\vee},\quad \rightarrow,\quad \leftrightarrow$$

Fuzzy logic:

$$\underset{F}{\neg},\quad \underset{F}{\wedge},\quad \underset{F}{\vee},\quad \underset{F}{\underline{\vee}},\quad \underset{F}{\rightarrow},\quad \underset{F}{\leftrightarrow}$$

Neutrosophic logic:

$$\underset{N}{\neg},\quad \underset{N}{\wedge},\quad \underset{N}{\vee},\quad \underset{N}{\underline{\vee}},\quad \underset{N}{\rightarrow},\quad \underset{N}{\leftrightarrow}$$

3 Three Neutrosophic Actions

In the frame of neutrosophy, we have considered [1995] for each entity ⟨A⟩, its opposite ⟨antiA⟩, and their neutrality ⟨neutA⟩ {i.e. neither ⟨A⟩, nor ⟨antiA⟩}. Also, by ⟨nonA⟩ we mean what is not ⟨A⟩, i.e. its opposite ⟨antiA⟩, together with its neutral(ity) ⟨neutA⟩; therefore:

$$\langle \text{non}A\rangle = \langle \text{neut}A\rangle \vee \langle \text{anti}A\rangle.$$

Based on these, we may straightforwardly introduce for the first time the following neutrosophic actions with respect to an entity <A>:

1. To neutralize (or to neuter, or simply to neut-ize) the entity <A>. [As a noun: neutralization, or neuter-ization, or simply neut-ization.] We denote it by <neutA> or neut(A).

2. To antithetic-ize (or to anti-ize) the entity <A>. [As a noun: antithetic-ization, or anti-ization.] We denote it by <antiA> ot anti(A).

This action is 100% opposition to entity <A> (strong opposition, or strong negation).

3. To non-ize the entity <A>. [As a noun: non-ization]. We denote it by <nonA> or non(A).
It is an opposition in a percentage between (0, 100]% to entity <A> (weak opposition).

Of course, not all entities <A> can be neutralized, or antithetic-ized, or non-ized.

3.1 Example

Let

⟨A⟩="Phoenix Cardinals beats Texas Cowboys".

Then,

⟨neutA⟩="\"Phoenix Cardinals has a tie game with Texas Cowboys\"";

⟨antiA⟩="\"Phoenix Cardinals is beaten by Texas Cowboys\"";

⟨nonA⟩="\"Phoenix Cardinals has a tie game with Texas Cowboys,"
"or Phoenix Cardinals is beaten by Texas Cowboys\"."

3.2 Properties of the Three Neutrosophic Actions

$$\text{neut}(\langle \text{anti}A\rangle) = \text{neut}(\langle \text{neutA}\rangle) = \text{neut}(A);$$

$$\text{anti}(\langle \text{anti}A\rangle) = A;\ \text{anti}(\langle \text{neut}A\rangle) = \langle A\rangle \text{ or } \langle \text{anti}A\rangle;$$

$$\text{non}(\langle \text{anti}A\rangle) = \langle A\rangle \text{ or } \langle \text{neut}A\rangle;\ \text{non}(\langle \text{neut}A\rangle) = \langle A\rangle \text{ or } \langle \text{anti}A\rangle.$$

4 Neutrosophic Actions' Truth-Value Tables

Let's have a logical proposition P, which may be true (T), Indeterminate (I), or false (F) as in previous example. One applies the neutrosophic actions below.

4.1 Neutralization (or Indetermination) of P

neut(P)	T	I	F
	I	I	I

4.2 Antitheticization (Neutrosophic Strong Opposition to P)

anti(P)	T	I	F
	F	$T \vee F$	T

4.3 Non-ization (Neutrosophic Weak Opposition to P):

non(P)	T	I	F
	$I \vee F$	$T \vee F$	$T \vee I$

5 Refinement of Entities in Neutrosophy

In neutrosophy, an entity ⟨A⟩ has an opposite ⟨antiA⟩ and a neutral ⟨neutA⟩. But these three categories can be refined in sub-entities $\langle A\rangle_1, \langle A\rangle_2, \dots, \langle A\rangle_m$, and respectively $\langle \text{neut}A\rangle_1, \langle \text{neut}A\rangle_2, \dots, \langle \text{neut}A\rangle_n$, and also $\langle \text{anti}A\rangle_1$, $\langle \text{anti}A\rangle_2, \dots, \langle \text{anti}A\rangle_p$, where m, n, p are integers ≥1, but $m + n + p \geq 4$ (meaning that at least one of ⟨A⟩, ⟨antiA⟩ or ⟨neutA⟩ is refined in two or more sub-entities).

For example, if ⟨A⟩=white color, then
⟨antiA⟩=black color,
while ⟨neutA⟩=colors different from white and black.

If we refine them, we get various nuances of white color: $\langle A\rangle_1, \langle A\rangle_2$, …, and various nuances of black color: $\langle \text{anti}A\rangle_1, \langle \text{anti}A\rangle_2$, …, and the colors in between them (red, green, yellow, blue, etc.): $\langle \text{neut}A\rangle_1, \langle \text{neut}A\rangle_2, \dots$.

Similarly as above, we want to point out that not all entities <A> and/or their corresponding (if any) <neutA> and <antiA> can be refined.

6 The Prevalence Order

Let's consider the classical literal (symbolic) truth (T) and falsehood (F).

In a similar way, for neutrosophic operators we may consider the literal (symbolic) truth (T), the literal (symbolic) indeterminacy (I), and the literal (symbolic) falsehood (F).

We also introduce the prevalence order on $\{T, I, F\}$ with respect to a given binary and commutative neutrosophic operator "o".

The neutrosophic operators are: neutrosophic negation, neutrosophic conjunction, neutrosophic disjunction, neutrosophic exclusive disjunction, neutrosophic Sheffer's stroke, neutrosophic implication, neutrosophic equivalence, etc.

The prevalence order is partially objective (following the classical logic for the relationship between T and F), and partially subjective (when the indeterminacy I interferes with itself or with T or F).

For its subjective part, the prevalence order is determined by the neutrosophic logic expert in terms of the application/problem to solve, and also depending on the specific conditions of the application/problem.

For $X \neq Y$, we write $X℗Y$, or $X \succ_o Y$, and we read X prevails to Y with respect to the neutrosophic binary commutative operator "o", which means that $XoY = X$.

Let's see the below examples. We mean by "o": conjunction, disjunction, exclusive disjunction, Sheffer's stroke, and equivalence.

7 Neutrosophic Literal Operators & Neutrosophic Numerical Operators

7.1 If we mean by neutrosophic literal proposition, a proposition whose truth value is a letter: either T or I or F. The operators that deal with such logical propositions are called neutrosophic literal operators.

7.2 And by neutrosophic numerical proposition, a proposition whose truth value is a triple of numbers (or in general of numerical subsets of the interval [0, 1]), for examples A(0.6, 0.1, 0.4) or B([0, 0.2], {0.3, 0.4, 0.6}, (0.7, 0.8)). The operators that deal with such logical propositions are called neutrosophic numerical operators.

8 Truth-Value Tables of Neutrosophic Literal Operators

In Boolean Logic, one has the following truth-value table for negation:

8.1 Classical Negation

$\neg$	T	F
	F	T

In Neutrosophic Logic, one has the following neutrosophic truth-value table for the neutrosophic negation:

8.2 Neutrosophic Negation

$\neg_N$	T	I	F
	F	I	T

So, we have to consider that the negation of I is I, while the negations of T and F are similar as in classical logic.

In classical logic, one has:

8.3 Classical Conjunction

$\wedge$	T	F
T	T	F
F	F	F

In neutrosophic logic, one has:

8.4 Neutrosophic Conjunction (AND_N), version 1

$\wedge_N$	T	I	F
T	T	I	F
I	I	I	I
F	F	I	F

The objective part (circled literal components in the above table) remains as in classical logic, but when indeterminacy I interferes, the neutrosophic expert may choose the most fit prevalence order.

There are also cases when the expert may choose, for various reasons, to entangle the classical logic in the objective part. In this case, the prevalence order will be totally subjective.

The prevalence order works for classical logic too. As an example, for classical conjunction, one has $F \succ_c T$, which means that $F \wedge T = F$. While the prevalence order for the neutrosophic conjunction in the above tables was:

$$I \succ_c F \succ_c T,$$

which means that $I \wedge_N F = I$, and $I \wedge_N T = I$.

Other prevalence orders can be used herein, such as:

$$F \succ_c I \succ_c T,$$

and its corresponding table would be:

8.5 Neutrosophic Conjunction (AND_N), version 2

$\wedge_N$	T	I	F
T	T	I	F
I	I	I	F
F	F	F	F

which means that $F_{\wedge_N} I = F$ and $I_{\wedge_N} I = I$; or another prevalence order:

$$F \succ_c T \succ_c I,$$

and its corresponging table would be:

8.6 Neutrosophic Conjunction (AND_N), version 3

$\wedge_N$	T	I	F
T	T	T	F
I	T	I	F
F	F	F	F

which means that $F_{\wedge_N} I = F$ and $T_{\wedge_N} I = T$.

If one compares the three versions of the neutrosophic literal conjunction, one observes that the objective part remains the same, but the subjective part changes.

The subjective of the prevalence order can be established in an optimistic way, or pessimistic way, or according to the weights assigned to the neutrosophic literal components *T, I, F* by the experts.

In a similar way, we do for disjunction. In classical logic, one has:

8.7 Classical Disjunction

$\vee$	T	F
T	T	T
F	T	F

In neutrosophic logic, one has:

8.8 Classical Disjunction (OR_N)

$\vee_N$	T	I	F
T	T	T	T
I	T	I	F
F	T	F	F

where we used the following prevalence order:

$$T \succ_d F \succ_d I,$$

but the reader is invited (as an exercise) to use another prevalence order, such as:

$$T \succ_d I \succ_d F,$$

Or

$$I \succ_d T \succ_d F, \text{ etc.,}$$

for all neutrosophic logical operators presented above and below in this paper.

In classical logic, one has:

8.9 Classical Exclusive Disjunction

$\veebar$	T	F
T	F	T
F	T	F

In neutrosophic logic, one has:

8.10 Neutrosophic Exclusive Disjunction

$\veebar_N$	T	I	F
T	F	T	T
I	T	I	F
F	T	F	F

using the prevalence order

$$T \succ_d F \succ_d I.$$

In classical logic, one has:

8.11 Classical Sheffer's Stroke

$\vert$	T	F
T	F	T
F	T	T

In neutrosophic logic, one has:

8.12 Neutrosophic Sheffer's Stroke

$\vert_N$	T	I	F
T	F	T	T
I	T	I	I
F	T	I	T

using the prevalence order

$$T \succ_d I \succ_d F.$$

In classical logic, one has:

8.13 Classical Implication

$\rightarrow$	T	F
T	T	F
F	T	T

In neutrosophic logic, one has:

8.14 Neutrosophic Implication

$\rightarrow_N$	T	I	F
T	T	I	F
I	T	T	F
F	T	T	T

using the subjective preference that $I \rightarrow_N T$ is true (because in the classical implication T is implied by anything), and $I \rightarrow_N F$ is false, while $I \rightarrow_N I$ is true because is similar to the classical implications $T \rightarrow T$ and $F \rightarrow F$, which are true.

The reader is free to check different subjective preferences.

In classical logic, one has:

8.15 Classical Equivalence

$\leftrightarrow$	T	F
T	T	F
F	F	T

In neutrosophic logic, one has:

8.15 Neutrosophic Equivalence

$\leftrightarrow_N$	T	I	F
T	T	I	F
I	I	T	I
F	F	I	T

using the subjective preference that $I \leftrightarrow_N I$ is true, because it is similar to the classical equivalences that $T \to T$ and $F \to F$ are true, and also using the prevalence:

$$I \succ_e F \succ_e T.$$

9 Refined Neutrosophic Literal Logic

Each particular case has to be treated individually.

In this paper, we present a simple example. Let's consider the following neutrosophic logical propositions:

T = Tomorrow it will rain or snow.
T is split into
→ Tomorrow it will rain.
→ Tomorrow it will snow.
F = Tomorrow it will neither rain nor snow.
F is split into
→ Tomorrow it will not rain.
→ Tomorrow it will not snow.
I = Do not know if tomorrow it will be raining, nor if it will be snowing.
I is split into
→ Do not know if tomorrow it will be raining or not.
→ Do not know if tomorrow it will be snowing or not.

Then:

$\neg_N$	T_1	T_2	I_1	I_2	F_1	F_2
	F_1	F_2	$T_1 \vee F_1$	$T_2 \vee F_2$	T_1	T_2

It is clear that the negation of T_1 (Tomorrow it will raining) is F_1 (Tomorrow it will not be raining). Similarly for the negation of T_2, which is F_2.

But, the negation of I_1 (Do not know if tomorrow it will be raining or not) is "Do know if tomorrow it will be raining or not", which is equivalent to "We know that tomorrow it will be raining" (T_1), or "We know that tomorrow it will not be raining" (F_1).

Whence, the negation of I_1 is $T_1 \vee F_1$, and similarly, the negation of I_2 is $T_2 \vee F_2$.

9.1 Refined Neutrosophic Literal Conjunction Operator

$\wedge_N$	T_1	T_2	I_1	I_2	F_1	F_2
T_1	T_1	$T_{1\,2}$	I_1	I_2	F_1	F_2
T_2	$T_{1\,2}$	T_2	I_1	I_2	F_1	F_2
I_1	I_1	I_1	I_1	I	F_1	F_2
I_2	I_2	I_2	I	I_2	F_1	F_2
F_1	F_1	F_1	F_1	F_1	F_1	F
F_2	F_2	F_2	F_2	F_2	F	F_2

where $T_{1\,2} = T_1 \wedge T_2 =$ "Tomorrow it will rain and it will snow".

Of course, other prevalence orders can be studied for this particular example.

With respect to the neutrosophic conjunction, F_l prevail in front of I_k, which prevail in front of T_j, or $F_l \succ I_k \succ T_j$, for all $l, k, j \in \{1, 2\}$.

9.2 Refined Neutrosophic Literal Disjunction Operator

$\vee_N$	T_1	T_2	I_1	I_2	F_1	F_2
T_1	T_1	T	T_1	T_1	T_1	T_1
T_2	T	T_2	T_2	T_2	T_2	T_2
I_1	T_1	T_2	I_1	I	F_1	F_2
I_2	T_1	T_2	I	I_2	F_1	F_2
F_1	T_1	T_2	F_1	F_1	F_1	$F_1 \vee F_2$
F_2	T_1	T_2	F_2	F_2	$F_1 \vee F_2$	F_2

With respect to the neutrosophic disjunction, T_j prevail in front of F_l, which prevail in front of I_k, or $T_j \succ F_l \succ I_k$, for all $j, l, k \in \{1, 2\}$.

For example, $T_1 \vee T_2 = T$, but $F_1 \vee F_2 \notin \{T, I\ F\} \cup \{T_1, \mathrm{T}_2, I_1, \mathrm{I}_2, F_1, \mathrm{F}_2\}$.

9.3 Refined Neutrosophic Literal Space

The Refinement Neutrosophic Literal Space $\{T_1, T_2, I_1, I_2, F_1, F_2\}$ is not closed under neutrosophic negation, neutrosophic conjunction, and neutrosophic disjunction. The reader can check the closeness under other neutrosophic literal operations.

A neutrosophic refined literal space

$$S_N = \{T_1, T_2, \dots, T_p;\ I_1, I_2, \dots, I_r;\ F_1, F_2, \dots, F_s\},$$

where p, r, s are integers ≥ 1, is said to be closed under a given neutrosophic operator "θ_N", if for any elements $X, Y \in S_N$ one has $X_{\theta_N} Y \in S_N$.

Let's denote the extension of S_N with respect to a single θ_N by:

$$S_{N_1}^C = (S_N, \theta_N).$$

If S_N is not closed with respect to the given neutrosophic operator θ_N, then $S_{N_1}^C \neq S_N$, and we extend S_N by adding in the new elements resulted from the operation $X\theta_N Y$, let's denote them by $A_1, A_2, \dots A_m$.

Therefore,

$$S_{N_1}^C \neq S_N \cup \{A_1, A_2, \dots A_m\}.$$

$S_{N_1}^C$ encloses S_N.

Similarly, we can define the closeness of the neutrosophic refined literal space S_N with respect to the two or more neutrosophic operators $\theta_{1_N}, \theta_{2_N}, \dots, \theta_{w_N}$, for $w \geq 2$.

S_N is closed under $\theta_{1_N}, \theta_{2_N}, \dots, \theta_{w_N}$ if for any $X, Y \in S_N$ and for any $i \in \{1, 2, \dots, w\}$ one has $X_{\theta_{i_N}} Y \in S_N$.

If S_N is not closed under these neutrosophic operators, one can extend it as previously.

Let's consider: $S_{N_w}^C = \left(S_N, \theta_{1_N}, \theta_{2_N}, \dots, \theta_{w_N}\right)$, which is S_N closed with respect to all neutrosophic operators $\theta_{1_N}, \theta_{2_N}, \dots, \theta_{w_N}$, then $S_{N_w}^C$ encloses S_N.

10 Conclusion

We have defined for the first time three neutrosophic actions and their properties. We have introduced the prevalence order on $\{T, I, F\}$ with respect to a given neutrosophic operator "o", the refinement of neutrosophic entities <A>, <neutA>, and <antiA>, and the neutrosophic literal logical operators and refined literal logical operators, and the refinement neutrosophic literal space.

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Structural Properties of Neutrosophic Abel-Grassmann's Groupoids

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Abstract .

In this paper, we have introduced the notion of neutrosophic $(2,2)$ -regular, neutrosophic strongly regular neutrosophic $\mathbf{AG}$-groupoids and investigated these structures. We have shown that neutrosophic regular, neutrosophic intra-regular and neutrosophic strongly regular $\mathbf{AG}$-groupoid are the only generalized classes of neutrosophic $\mathbf{AG}$-groupoid. Further we have shown that non-associative regular, weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, $(2,2)$ -regular and strongly regular $\mathbf{AG}^{*}$ -neutrosophic groupoids do not exist.

Keyword

A neutrosophic $\mathbf{AG}$-groupoid, left invertive law, medial law and paramedial law.
[2000]20M10 and 20N99

Introduction

We know that in every branch of science there are lots of complications and problems appear which affluence the uncertainties and impaction. Most of these problems and complications are concerning with human life. These problems also play pivotal role for being subjective and classical. Common used methods are not sufficient to apply on these problems. To solve these complications, concept of fuzzy sets was published by Lotfi A.Zadeh in 1965, which has a wide range of applications in various fields such as engineering, artificial intelligence, control engineering, operation research, management science, robotics and many more. Zadeh introduced fuzzy sets to address uncertainities. By use of fuzzy sets the manipulated data and information of uncertainties can be prossessed. The idea of fuzzy sets was particularly designed to characterize uncertainty and vagueness and to present dignified tools in order to deal with the ambiguity intrinsic to the various problems. Fuzzy logic gives a conjecture morphology that enables approximate human reasoning capabilities to be applied to knowledge-based systems. The concept of fuzzy logic gives a mathematical potency to deal with the uncertainties associated with the human intellectual processes, such as reasoning and judgment.

In literature, a lot of theories have been developed to contend with uncertainty, imprecision and vagueness. In which, theory of probability, rough set theory fuzzy set theory, intiutionistic fuzzy sets etc, have played imperative role to cope with diverse types of uncertainties and imprecision entrenched in a system. But all these above theories were not sufficient tool to deal with indeterminate and inconsistent information in believe system. F.Samrandache noticed that the law of excluded middle are presently inactive in the modern logics and also by getting inspired with sport games (winning/tie/defeating), voting system (yes/ NA/no), decision making (making a decision/hesitating/not making) etc, he developed a new concept called neutrosophic set (NS) which is basically generalization of fuzzy sets and intiutionistic fuzzy sets. NS can be described by membership degree, and indeterminate degree and non-membership degree.

The neutrosophic logic is an extended idea of neutrosophy. Fuzzy theory is used when uncertainty is modeled and when there is indeterminancy involved we use neutrosophic theory. The neutrosophic algebraic structures have defined very recently. Basically, Vasantha K andasmy and Florentin Smarandache present the concept of neutrosophic algebraic structures by using neutrosophic theory. A number of the neutrosophic algrebraic structures introduced and considered include neutrosophic fields, neutrosophic vector spaces, neutrosophic groups, neutrosophic bigroups, neutrosophic N-groups, neutrosophic bisemigroups, neutrosophic N-semigroup, neutrosophic loops, neutrosophic biloops, neutrosophic N-loop, neutrosophic groupoids, neutrosophic bigroupoids and neutrosophic AG-groupoids.

Preliminaries

Abel Grassmann's groupoid abbreviated as an AG-groupoid is a groupoid whose element satisfies the left invertive law i.e $(ab)c=(cb)a$ for all $a,b,c\in S$ .An AG-groupoid is a non associative and non-commutative algebraic structure mid way between a groupoid and commutative semigroup. AG-groupoids generalizes the concept of commutative semigroup and have an important application within the theory of flocks.

An **AG**-groupoid , is a groupoid **S** holding the left invertive law

$$(ab)c=(cb)a,\ \text{for all}\ a,b,c\in \mathbf{S}.$$

This left invertive law has been obtained by introducing braces on the left of ternary commutative law $abc=cba$.

Basic Laws of AG-groupoid

In an **AG** -groupoid, the medial law holds

$$(ab)(cd)=(ac)(bd),\ \text{for all}\ a,b,c,d\in \mathbf{S}.$$

In an **AG**-groupoid **S** with left identity, the paramedial law holds

$$(ab)(cd)=(dc)(ba),\ \text{for all}\ a,b,c,d\in \mathbf{S}.$$

Further if an **AG**-groupoid contains a left identity, the following law holds

$$a(bc)=b(ac),\ \text{for all}\ a,b,c\in \mathbf{S}.$$

Since the world is full of indeterminacy, the neutrosophics found their place into contemporary research. In 1995, Florentin Smarandache introduced the idea of neutrosophy. Neutrosophic logic is an extension of fuzzy logic. Madad Khan et al., for the first time introduced the idea of a neutrosophic LA-semigroup in [4]. Moreover $SUI = \{a + bI :$ where $a, b \in S$ and I is literal indeterminacy such that $I^2 = I$ becomes neutrosophic LA-semigroup under the operation $*$ defined as:
$(a + bI) * (c + dI) = ac + bdI$ For all $(a + bI), (c + dI) \in SUI$. That is $(SUI, *)$ becomes neutrosophic LA-semigroup. They represented it by $N(S)$.

$$[(a_1 + a_2I)(b_1 + b_2I)](c_1 + c_2I) = [(c_1 + c_2I)(b_1 + b_2I)](a_1 + a_2I), \quad (1)$$

holds for all $(a_1 + a_2I), (b_1 + b_2I), (c_1 + c_2I) \in N(S)$.
It is since than called the neutrosophic left invertive law. A neutrosophic groupoid satisfying the left invertive law is called a neutrosophic left almost semigroup and is abbreviated as neutrosophic LA-semigroup.
In a neutrosophic LA-semigroup $N(S)$ medial law holds i.e

$$\begin{aligned}&[(a_1 + a_2I)(b_1 + b_2I)][(c_1 + c_2I)(d_1 + d_2I)]\\ &= [(a_1 + a_2I)(c_1 + c_2I)][(b_1 + b_2I)(d_1 + d_2I)],\end{aligned} \quad (2)$$

holds for all $(a_1 + a_2I), (b_1 + b_2I), (c_1 + c_2I), (d_1 + d_2I) \in N(S)$.
There can be a unique left identity in a neutrosophic LA-semigroup. In a neutrosophic LA-semigroup $N(S)$ with left identity $(e + eI)$ the following laws hold for all $(a_1 + a_2I), (b_1 + b_2I), (c_1 + c_2I), (d_1 + d_2I) \in N(S)$.

$$\begin{aligned}&[(a_1 + a_2I)(b_1 + b_2I)][(c_1 + c_2I)(d_1 + d_2I)]\\ &= [(d_1 + d_2I)(b_1 + b_2I)][(c_1 + c_2I)(a_1 + a_2I)],\end{aligned} \quad (3)$$

$$\begin{aligned}&[(a_1 + a_2I)(b_1 + b_2I)][(c_1 + c_2I)(d_1 + d_2I)]\\ &= [(d_1 + d_2I)(c_1 + c_2I)][(b_1 + b_2I)(a_1 + a_2I)],\end{aligned} \quad (4)$$

and

$$(a_1 + a_2I)[(b_1 + b_2I)(c_1 + c_2I)] = (b_1 + b_2I)[(a_1 + a_2I)(c_1 + c_2I)]. \quad (5)$$

for all $(a_1 + a_2I), (b_1 + b_2I), (c_1 + c_2I) \in N(S)$.
(3) is called neutrosophic paramedial law and a neutrosophic LA semigroup satisfies (5) is called Neutrosophic AG**-groupoid.
Now, $(a + bI)^2 = a + bI$ implies $a + bI$ is idempotent and if holds for all $a + bI \in N(S)$ then $N(S)$ is called idempotent neutrosophic LA-semigroup.
This structure is closely related with a neutrosophic commutative semigroup, because if a Neutrosophic **AG**-groupoid contains a right identity, then it becomes a commutative semigroup.

A neutrosophic **AG**-groupoid N(S) with neutrosophic left identity becomes a Neutrosophic semigroup N(**S**) under new binary operation "∘" defined as

$$(x_1 + x_2 I) \circ (y_1 + y_2 I) = [(x_1 + x_2 I)(a_1 + a_2 I)](y_1 + y_2 I)$$

quasi regular if all elements of N(S)) are left quasi regular.

for all $(x_1 + x_2 I), (y_1 + y_2 I) \in N(\mathbf{S})$.

It is easy to show that ∘ is associative

$$\begin{aligned}
&[(x_1 + x_2 I) \circ (y_1 + y_2 I)] \circ (z_1 + z_2 I) \\
&= [[[(x_1 + x_2 I)(a_1 + a_2 I)](y_1 + y_2 I)](a_1 + a_2 I)](z_1 + z_2 I) \\
&= [[(z_1 + z_2 I)(a_1 + a_2 I)][[(x_1 + x_2 I)(a_1 + a_2 I)](y_1 + y_2 I)]] \\
&= [(x_1 + x_2 I)(a_1 + a_2 II)][[(z_1 + z_2 I)(a_1 + a_2 I)](y_1 + y_2 I)] \\
&= [(x_1 + x_2 I)(a_1 + a_2 I)[(y_1 + y_2 I)(a_1 + a_2 I)](z_1 + z_2 I)] \\
&= (x_1 + x_2 I) \circ [(y_1 + y_2 I) \circ (z_1 + z_2 I)].
\end{aligned}$$

Hence N(S) is a neutrosophic semigroup

Regularities in Neutrosophic AG -groupoids

An element $a + bI$ of a neutrosophic **AG** -groupoid $N(\mathbf{S})$ is called a **regular** element of $N(\mathbf{S})$ if there exists $x_1 + x_2 I \in N(\mathbf{S})$ such that $a + bI = [(a + bI) * (x_1 + x_2 I)](a + bI)$ and N(**S**) is called regular if all elements of N(**S**) are regular.

An element $a + bI$ of neutrosophic **AG**-groupoid N(**S**) is called a **weakly regular** element of N(**S**) if there exist $(x_1 + x_2 I), (y_1 + y_2 I) \in N(\mathbf{S})$ such that $a + bI = [(a + bI)(x_1 + x_2 I)][(a + bI)(y_1 + y_2 I)]$ and $N(\mathbf{S})$ is called weakly regular if all elements of N(**S**) are weakly regular.

An element $a + bI$ of a neutrosophic **AG** -groupoid $N(\mathbf{S})$ is called an **intra-regular** element of $N(\mathbf{S})$ if there exist $x_1 + x_2 I, y_1 + y_2 I \in N(\mathbf{S})$ such that $a + bI = [(x_1 + x_2 I)(a + bI)^2](y_1 + y_2 I)$ and $N(\mathbf{S})$ is called intra-regular if all elements of $N(\mathbf{S})$ are intra-regular.

An element $a + bI$ of a neutrosophic **AG** -groupoid $N(\mathbf{S})$ is called a **right regular** element of $N(\mathcal{S})$ if there exists $x_1 + x_2 I \in N(\mathbf{S})$ such that $a + bI = (a + bI)^2 (x_1 + x_2 I) = [(a + bI)(a + bI)](x_1 + x_2 I)$ and $N(\mathbf{S})$ is called right regular if all elements of $N(\mathbf{S})$ are right regular.

An element $a + bI$ of a Neutrosophic **AG** -groupoid $N(\mathbf{S})$ is called **left regular** element of **N(S)** if there exists $x_1 + x_2 I \in N(\mathcal{S})$ such that $a + bI = (x_1 + x_2 I)(a + bI)^2 = (x_1 + x_2 I)[(a + bI)(a + bI)]$ and $N(\mathbf{S})$ is called left regular if all elements of $N(\mathbf{S})$ are left regular.

An element $a + bI$ of a Neutrosophic **AG** -groupoid $N(\mathbf{S})$ is called a **left quasi regular** element of $N(\mathbf{S})$ if there exist $x_1 + x_2 I, y_1 + y_2 I \in N(\mathbf{S})$ such that $a + bI = [(x_1 + x_2 I)(a + bI)][(y_1 + y_2 I)(a + bI)]$ and $N(\mathbf{S})$ is called left quasi regular if all elements of N(S)) are left quasi regular.

An element $a+bI$ of a Neutrosophic **AG** -groupoid $N(\mathbf{S})$ is called a **completely regular** element of $N(\mathbf{S})$ if $a+bI$ is regular, left regular and right regular. $N(\mathbf{S})$ is called completely regular if it is regular, left and right regular.

An element $a+bI$ of a Neutrosopic **AG**-groupoid $N(\mathbf{S})$ is called a **(2,2)-regular** element of $N(\mathbf{S})$ if there exists $x_1+x_2I \in N(\mathbf{S})$ such that $a+bI=[(a+bI)^2(x_1+x_2I)](a+bI)^2$ and $N(\mathbf{S})$ is called $(2,2)$-regular **AG**-groupoid if all elements of $N(\mathbf{S})$ are $(2,2)$-regular.

An element $a+bI$ of a Neutrosophic **AG**-groupoid $N(\mathbf{S})$ is called a **strongly regular** element of $N(\mathbf{S})$ if there exists $x_1+x_2I \in N(\mathbf{S})$ such that $a+bI=[(a+bI)(x_1+x_2I)](a+bI)$ and $(a+bI)(x_1+x_2I)=(x_1+x_2I)(a+bI)$. $N(\mathbf{S})$ is called strongly regular Neutrosophic **AG**-groupoid if all elements of $N(\mathbf{S})$ are strongly regular.

A Neutrosophic **AG**-groupoid $N(\mathbf{S})$ is called Neutrosophic **AG***-groupoid if the following holds

$$[(a_1+a_2I)(b_1+b_2I)](c_1+c_2I)=(b_1+b_2I)[(a_1+a_2I)(c_1+c_2I)]$$

for all $a_1+a_2I, b_1+b_2I, c_1+c_2I \in N(\mathbf{S})$.

In Neutrosophic **AG*** -groupoid N(S) the following law holds

A Neutrosophic **AG**-groupoid may or may not contains a left identity. The left identity of a Neutrosophic **AG**-groupoid allow us to introduce the inverses of elements in a Neutrosophic **AG** -groupoid. If an **AG**-groupoid contains a left identity, then it is unique.

Example 1 *Let us consider a Neutrosophic* **AG**-*groupoid* **N(S)**$=\{1+1I,\ 1+2I,\ 1+3I,\ 2+1I,\ 2+2I,\ 2+3I,\ 3+1I,\ 3+2I,\ 3+3I\}$ *in the following multiplication table.*

$*$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$
$1+1I$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$
$1+2I$	$1+3I$	$1+1I$	$1+2I$	$2+2I$	$2+1I$	$2+2I$	$3+3I$	$3+1I$	$3+2I$
$1+3I$	$1+2I$	$1+3I$	$1+1I$	$2+3I$	$2+3I$	$2+1I$	$3+2I$	$3+3I$	$3+1I$
$2+1I$	$3+1I$	$3+2I$	$3+3I$	$1+1I$	$1+2I$	$1+3I$	$2+1I$	$2+2I$	$2+3I$
$2+2I$	$3+3I$	$3+1I$	$3+2I$	$1+3I$	$1+1I$	$1+2I$	$2+3I$	$2+1I$	$2+2I$
$2+3I$	$3+2I$	$3+3I$	$3+1I$	$1+2I$	$1+3I$	$1+1I$	$2+2I$	$2+3I$	$2+1I$
$3+1I$	$2+1I$	$2+2I$	$2+3I$	$3+1I$	$3+2I$	$3+3I$	$1+1I$	$1+2I$	$1+3I$
$3+2I$	$2+3I$	$2+1I$	$2+2I$	$3+3I$	$3+1I$	$3+2I$	$1+3I$	$1+1I$	$1+2I$
$3+3I$	$2+2I$	$2+3I$	$2+1I$	$3+2I$	$3+3I$	$3+1I$	$1+2I$	$1+3I$	$1+1I$

Lemma 1 *If $N(\mathbf{S})$ is a regular, weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, $(2,2)$-regular or strongly regular neutrosophic **AG**-groupoid, then $N(S) = N(\mathbf{S})^2$.*

Proof *Let $N(\mathbf{S})$ be a Neutrosophic regular **AG**-groupoid, then $N(\mathbf{S})^2 \subseteq N(\mathbf{S})$ is obvious. Let $a+bI \in N(\mathbf{S})$, then since $N(\mathbf{S})$ is regular so there exists $x+yI \in N(\mathbf{S})$ such that $a+bI = [(a+bI)(x+yI)](a+bI)$.*
Now
$a+bI = [(a+b)(x+yI)](a+bI) \in N(\mathbf{S})\mathbf{N}(\mathbf{S})$
$N(\mathbf{S}) \subseteq N(\mathbf{S})^2$
Similarly if $N(\mathbf{S})$ is weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, $(2,2)$-regular or strongly regular, then we can show that $N(\mathbf{S}) = N(\mathbf{S})^2$.

The converse is not true in general, because in Example lil, $N(\mathbf{S}) = N(\mathbf{S})^2$ holds but $N(\mathbf{S})$ is not regular, weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, $(2,2)$-regular and strongly regular, because $d_1 + d_2 I \in N(\mathbf{S})$ is not regular, weakly regular, intra-regular, right regular, left regular, left quasi regular, completely regular, $(2,2)$-regular and strongly regular.

Theorem1 *If $N(\mathbf{S})$ is a Neutrosophic **AG**-groupoid with left identity ($\mathbf{AG}^{**}$-groupoid), then $N(\mathbf{S})$ is intra-regular if and only if for all $a+bI \in N(\mathbf{S})$, $a+bI = [(x_1+x_2I)(a+bI)][(a+bI)(z_1+z_2I)]$ holds for some $x_1+x_2I, z_1+z_2I \in N(\mathbf{S})$.*

Proof *Let $N(\mathbf{S})$ be an intra-regular Neutrosophic **AG**-groupoid with left identity ($\mathbf{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathbf{S})$ there exist $x_1+x_2I, y_1+y_2I \in N(\mathbf{S})$ such that $a+bI = [(x_1+x_2I)(a+bI)^2](y_1+y_2I)$. Now by using Lemma1, $y_1+y_2I = (u_1+u_2I)(v_1+v_2I)$ for some $u_1+u_2I, v_1+v_2I \in N(\mathbf{S})$.*

$$\begin{aligned}
& a+bI \\
&= [(x_1+x_2I)(a+bI)^2](y_1+y_2I) \\
&= [(x_1+x_2I)[(a+bI)(a+bI)]](y_1+y_2I) \\
&= [(a+bI)[(x_1+x_2I)(a+bI)]](y_1+y_2I) \\
&= [(y_1+y_2I)[(x_1+x_2I)(a+bI)]](a+bI)
\end{aligned}$$

$$
\begin{aligned}
&= [(y_1+y_2I)[(x_1+x_2I)(a+bI)]][(x_1+x_2I)(a+bI)^2](y_1+y_2I)\\
&= [[(u_1+u_2I)(v_1+v_2)][(x+yI)(a+bI)]][(x_1+x_2I)(a+bI)^2](y_1+y_2I),\\
&= [[(a+bI)(x_1+x_2I)][(v_1+v_2)(u_1+u_2I)]][(x_1+x_2I)(a+bI)^2](y_1+y_2I)\\
&= [[(a+bI)(x_1+x_2I)](t_1+t_2I)][(x_1+x_2I)(a+bI)^2](y_1+y_2I)\\
&= [[(x_1+x_2I)(a+bI)^2](y_1+y_2I)](t_1+t_2I)][(a+bI)(x_1+x_2I)]\\
&= [[(t_1+t_2I)(y_1+y_2I)][(x_1+x_2I)(a+bI)^2]][(a+bI)(x_1+x_2I)]\\
&= [[(a+bI)^2(x_1+x_2I)][(y_1+y_2I)(t_1+t_2I)]][(a+bI)(x_1+x_2I)]\\
&= [[(a+bI)^2(x_1+x_2I)][(s_1+s_2I)][(a+bI)(x_1+x_2I)],\\
&= [[(s_1+s_2I)(x_1+x_2I)](a+bI)^2][(a+bI)(x_1+x_2I)]\\
&= [[(s_1+s_2I)(x_1+x_2I)][(a+bI)(a+bI)]][(a+bI)(x_1+x_2I)]\\
&= [[(a+bI)(a+bI)][(x_1+x_2I)(s_1+s_2I)]][(a+bI)(x_1+x_2I)]\\
&= [[(a+bI)(a+bI)](w_1+w_2I)[(a+bI)(x_1+x_2I)],\\
&= [[(w_1+w_2I)(a+bI)](a+bI)][(a+bI)(x_1+x_2I)]\\
&= [(z_1+z_2I)(a+bI)][(a+bI)(x_1+x_2I)],\\
&= [(x_1+x_2I)(a+bI)][(a+bI)(z_1+z_2I)]
\end{aligned}
$$

where $(w_1+w_2I)(a+bI)=(z_1+z_2I)\in N(S)$ *where* $(x_1+x_2I)(s_1+s_2I)=(w_1+w_2I)\in N(S)$ *where* $(y_1+y_2I)(t_1+t_2I)=(s_1+s_2I)\in N(S)$ *where* $(v_1+v_2)(u_1+u_2I)=(t_1+t_2I)\in N(S)$ *where* $[(u_1+u_2I)(v_1+v_2)=(y_1+y_2I)$ $\in N(S)$

Conversely, *let for all* $a+bI\in N(\mathbf{S})$, $a+bI=[(x_1+x_2I)(a+bI)][(a+bI)(z_1+z_2I)]$ *holds for some* $x_1+x_2I, z_1+z_2I\in N(\mathbf{S})$. *Now by using* (4), (1), (2) *and* (3), *we have*

$bI+a$

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$])z_2 I+z_1 ()bI+a ([])bI+a ()x_2 I+x_1 ([=$

$])z_2 I+z_1 (])bI+a ()x_2 I+x_1 ([)bI+a ([=$

$])z_2 I+z_1 (])bI+a ()x_2 I+x_1 ([])z_2 I+z_1 ()bI+a ([])bI+a ()x_2 I+x_1 ([=])$

$z_2 I+z_1 (])bI+a ()x_2 I+x_1 ([])z_2 I+z_1 (])bI+a ()x_2 I+x_1 ([)bI+a ([=)bI$

$+a (]])z_2 I+z_1 (])bI+a ()x_2 I+x_1 ([])z_2 I+z_1 (])bI+a ()x_2 I+x_1 ([[=)bI+$

$a (^2])z_2 I+z_1 (])bI+a ()x_2 I+x_1 ([[=$

$)bI+a (]^2)z_2 I+z_1 (^2])bI+a ()x_2 I+x_1 ([[=$

$)bI+a (]])z_2 I+z_1 ()z_2 I+z_1 ([]^2)bI+a (^2)x_2 I+x_1 ([[=$

$)bI+a (]])z_2 I+z_1 (^2)bI+a ([])z_2 I+z_1 (^2)x_2 I+x_1 ([[=$

$)bI+a (])z_2 I+z_1 (])z_2 I+z_1 (^2)x_2 I+x_1 ([[^2)bI+a ([=$

$)bI+a (])z_2 I+z_1 (])z_2 I+z_1 (^2)x_2 I+x_1 ([])bI+a ()bI+a ([[=$

$)bI+a (]]^2)x_2 I+x_1 (])z_2 I+z_1 ()z_2 I+z_1 ([[])bI+a ()bI+a ([[=$

$)bI+a (]]^2)x_2 I+x_1 (^2)z_2 I+z_1 ([])bI+a ()bI+a ([[=$

$)bI+a (]])bI+a ()bI+a ([]^2)z_2 I+z_1 (^2)x_2 I+x_1 ([[=$

$)bI+a (]])bI+a ()bI+a ([)t_2 I+t_1 ([=,$

$)u_2 I+u_1 (]^2)bI+a ()t_2 I+t_1 ([=$

where $[(x_1 + x_2 I)^2 (z_1 + z_2 I)^2] = (t_1 + t_2 I) \in N(S)$ *and* $(a+bI) = (u_1 + u_2 I) \in N(S)$ *where* $(a+bI) = (u_1 + u_2 I) \in N(S)$

Thus $N(\mathbf{S})$ is intra-regular.

Theorem 2 *If $N(\mathbf{S})$ is a Neutrosophic $\mathbf{AG}$-groupoid with left identity ($\mathbf{AG}^{**}$-groupoid), then the following are equivalent.*

(i) $N(\mathbf{S})$ is weakly regular.

(ii) $N(\mathbf{S})$ is intra-regular.

Proof $(i) \Rightarrow (ii)$ Let $N(\mathbf{S})$ be a weakly regular *Neutrosophic* $\mathbf{AG}$-groupoid with left identity (Neutrosophic AG**-groupoid), then for any $a+bI \in N(\mathbf{S})$ there exist $x_1 + x_2 I, y_1 + y_2 I \in N(\mathbf{S})$ such that $a+bI = [(a+bI)(x_1 + x_2 I)][(a+bI)(y_1 + y_2 I)]$ and by Lemma1, $x_1 + x_2 I = (u_1 + u_2 I)(v_1 + v_2 I)$ for some $(u_1 + u_2 I),(v_1 + v_2 I) \in N(\mathbf{S})$ Let

$(v_1 + v_2I)(u_1 + u_2I) = (t_1 + t_2I) \in N(\mathbf{S})$. Now by using (3), (1), (4) and (2), we have

$$\begin{aligned}
a+bI &= [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)] \\
&= [(a+bI)(a+bI)][(x_1+x_2I)(y_1+y_2I)] \\
&= (x_1+x_2I)[(a+bI)^2(y_1+y_2I)] \\
&= [(u_1+u_2I)(v_1+v_2I)][(a+bI)^2(y_1+y_2I)] \\
&= [(y_1+y_2I)(a+bI)^2][(v_1+v_2I)(u_1+u_2I)] \\
&= [(y_1+y_2I)(a+bI)^2](t_1+t_2I)
\end{aligned}$$

Thus $N(\mathbf{S})$ is intra-regular.

$(ii) \Rightarrow (i)$ Let $N(S)$ be a intra regular Neutrosophic AG-groupoid with left identity (Neutrosophic $A\mathbf{G}^{**}$-groupoid), then for any $a+bI \in N(S)$

$$\begin{aligned}
a+bI &= [(y_1+y_2I)(a+bI)^2](t_1+t_2I) \\
&= [(y_1+y_2I)(a+bI)^2][(v_1+v_2I)(u_1+u_2I)] \\
&= [(u_1+u_2I)(v_1+v_2I)][(a+bI)^2(y_1+y_2I)] \\
&= (x_1+x_2I)[(a+bI)^2(y_1+y_2I)] \\
&= [(a+bI)(a+bI)][(x_1+x_2I)(y_1+y_2I)] \\
&= [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)]
\end{aligned}$$

$\Rightarrow$ Thus $N(S)$ is weakly regular $\cdot$

Theorem 3 *If $N(\mathbf{S})$ is a Neutrosophic $\mathbf{AG}$-groupoid (Neutrosophic$\mathbf{AG}^{**}$-groupoid), then the following are equivalent.*

(i) $N(\mathbf{S})$ is weakly regular.

(ii) $N(\mathbf{S})$ is right regular.

Proof $(i) \Rightarrow (ii)$ Let $N(\mathbf{S})$ be a weakly regular Neutrosophic $\mathbf{AG}$-groupoid ($\mathbf{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathbf{S})$ there exist $x_1+x_2I, y_1+y_2I \in N(\mathbf{S})$ such that $a+bI = (a+bI)(x_1+x_2I)(a+bI)(y_1+y_2I)$ and let $(x_1+x_2I)(y_1+y_2I) = (t_1+t_2I)$ for some $(t_1+t_2I) \in N(\mathbf{S})$. Now by using (2), we have

$$\begin{aligned}
a+bI &= [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)] \\
&= [(a+bI)(a+bI)][(x_1+x_2I)(y_1+y_2I)] \\
&= (a+bI)^2(t_1+t_2I)
\end{aligned}$$

Thus $N(\mathbf{S})$ is right regular.

$(ii) \Rightarrow (i)$ It follows from Lemma1 and (2) .

$$\begin{aligned}
a+bI &= (a+bI)^2(t_1+t_2I) \\
&= [(a+bI)(a+bI)](t_1+t_2I) \\
&= [(a+bI)(a+bI)][(x_1+x_2I)(y_1+y_2I)] \\
&= [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)]
\end{aligned}$$

where $(t_1+t_2I)=(x_1+x_2I)(y_1+y_2I)\in N(\mathbf{S})$. Thus $N(S)$ is weakly regular.

Theorem 4 *If $N(\mathbf{S})$ is a Neutrosophic **AG**-groupoid with left identity (Neutrosophic AG**-groupoid), then the following are equivalent.*

(i) $N(\mathbf{S})$ *is weakly regular.*

(ii) $N(\mathbf{S})$ *is left regular.*

Proof $(i)\Rightarrow(ii)$ Let $N(\mathbf{S})$ be a weakly regular Neutrosophic **AG**-groupoid with left identity (Neutrosophic AG**-groupoid), then for any $a+bI\in N(\mathbf{S})$ there exist $(x_1+x_2I),(y_1+y_2I)\in N(\mathbf{S})$ such that $a+bI=[(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)]$. Now by using (2) and (3), we have

$$\begin{aligned}a+bI &= [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)]\\ &= [(a+bI)(a+bI)][(x_1+x_2I)(y_1+y_2I)]\\ &= [(y_1+y_2I)(x_1+x_2I)][(a+bI)(a+bI)]\\ &= [(y_1+y_2I)(x_1+x_2I)](a+bI)^2\\ &= (t_1+t_2I)(a+bI)^2,\end{aligned}$$

where $[(y_1+y_2I)(x_1+x_2I)]=(t_1+t_2I)$ for some $(t_1+t_2I)\in N(\mathbf{S})$. Thus $N(\mathbf{S})$ is left regular.

$(ii)\Rightarrow(i)$ It follows from Lemma1, (3) and (2).

$$\begin{aligned}a+bI &= (t_1+t_2I)(a+bI)^2\\ &= [(y_1+y_2I)(x_1+x_2I)](a+bI)^2\\ &= [(y_1+y_2I)(x_1+x_2I)][(a+bI)(a+bI)]\\ &= [(a+bI)(a+bI)][(x_1+x_2I)(y_1+y_2I)]\\ &= [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)],\end{aligned}$$

Where $(y_1+y_2I)(x_1+x_2I)=(t_1+t_2I)$ for some $(t_1+t_2I)\in N(\mathbf{S})$. Thus $N(S)$ is

weakly regular.

Theorem 5 *If $N(\mathbf{S})$ is a Neutrosophic **AG**-groupoid with left identity (Neutrosophic AG**-groupoid), then the following are equivalent.*

(i) $N(\mathbf{S})$ *is weakly regular.*

(ii) $N(\mathbf{S})$ *is left quasi regular*

Proof $(i)\Rightarrow(ii)$ Let $N(\mathbf{S})$ be a weakly regular *Neutrosophic***AG**-groupoid with left identity, then for $a+bI\in N(\mathbf{S})$ there exists $(x_1+x_2I),(y_1+y_2I)\in N(\mathbf{S})$ such that $a+bI=[(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)]$

$$\begin{aligned}a+bI &= [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)]\\ &= [(y_1+y_2I)(a+bI)][(x_1+x_2I)(a+bI)].\end{aligned}$$

Thus $N(S)$ is left quasi regular.

$(ii) \Rightarrow (i)$ Let $N(\mathbf{S})$ be a left quasi regular Neutrosophic **AG** -groupoid with left identity then.for $a+bI \in N(\mathbf{S})$ there exists $(x_1+x_2I),(y_1+y_2I) \in N(\mathbf{S})$ such that $a+bI=[(y_1+y_2I)(a+b)][(x_1+x_2I)(a+b)]$

$$\begin{aligned} a+bI &= [(y_1+y_2I)(a+b)][(x_1+x_2I)(a+b)] \\ &= [(a+b)(x_1+x_2I)][(a+b)(y_1+y_2I)]. \end{aligned}$$

Thus $N(S)$ is weakly regular.

Theorem 6 *If $N(\mathbf{S})$ is a Neutrosophic* **AG**-*groupoid with left identity, then the following are equivalent.*

(i) $N(\mathbf{S})$ *is* $(2,2)$-*regular.*

(ii) $N(\mathbf{S})$ *is completely regular.*

Proof $(i) \Rightarrow (ii)$ Let $N(\mathbf{S})$ be a $(2,2)$-regular *Neutrosophic***AG**-groupoid with left identity, then for $a+bI \in N(\mathbf{S})$ there exists $(x_1+x_2I) \in N(\mathbf{S})$ such that $a+bI=[(a+bI)^2(x+yI)](a+bI)^2$. Now

$$\begin{aligned} a+bI &= [(a+bI)^2(x_1+x_2I)](a+bI)^2 \\ &= [(y_1+y_2I)(a+bI)^2], \end{aligned}$$

where $(a+bI)^2(x_1+x_2I)=(y_1+y_2I) \in N(\mathbf{S})$, and by using (3), we have

$$\begin{aligned} a+bI &= [(a+bI)^2(x_1+x_2I)][(a+bI)(a+bI)] \\ &= [(a+bI)(a+bI)][(x_1+x_2I)[(a+bI)^2] \\ &= (a+bI)^2(z_1+z_2I), \end{aligned}$$

where $(x_1+x_2I)(a+bI)^2=(z_1+z_2I) \in N(\mathbf{S})$. and by using (3), (1) and (4), we have

$$\begin{aligned} & a+bI \\ &= [(a+bI)^2[(x_1+x_2I)[(a+bI)(a+bI)]] \\ &= [(a+bI)(a+bI)][(x_1+x_2I)(a+bI)^2] \\ &= [(a+bI)(a+bI)][[(e_1+e_2I)(x_1+x_2I)][a+bI)(a+bI)]] \\ &= [(a+bI)(a+bI)][[(a+bI)(a+bI)][(x_1+x_2I)(e_1+e_2I)]] \\ &= [(a+bI)(a+bI)][(a+bI)^2(t_1+t_2I)] \\ &= [[(a+bI)^2(t_1+t_2I)](a+bI)](a+bI) \\ &= [[[(a+bI)(a+bI)](t_1+t_2I)](a+bI)](a+bI) \\ &= [[(t_1+t_2I)(a+bI)][(a+bI)](a+bI)]](a+bI) \\ &= [[(a+bI)(a+bI)][(t_1+t_2I)(a+bI)]](a+bI) \\ &= [[(a+bI)(t_1+t_2I)][(a+bI)(a+bI)]](a+bI) \\ &= [(a+bI)[[(a+bI)(t_1+t_2I)](a+bI)]](a+bI) \\ &= [(a+bI)(u_1+u_2I)](a+bI) \end{aligned}$$

where $(t_1+t_2I)=(x_1+x_2I)(e_1+e_2I)\in N(\mathbf{S})$ & where $(u_1+u_2I)=(a+bI)^2(t_1+t_2I).\in N(\mathbf{S})$. Thus $N(\mathbf{S})$ is neutrosophic left regular, right regular and regular, so $N(\mathbf{S})$ is completely regular.

$(ii)\Rightarrow(i)$ Assume that $N(\mathbf{S})$ is a completely regular neutrosophic **AG** -groupoid with left identity, then for any $a+bI\in N(\mathbf{S})$ there exist $(x_1+x_2I),(y_1+y_2I),(z_1+z_2I)\in N(\mathbf{S})$ such that $(a+bI)=[(a+bI)(x_1+x_2I)](a+bI)$, $(a+bI)=(a+bI)^2(y_1+y_2I)$ and $(a+bI)=(z_1+z_2I)(a+bI)^2$. Now by using (1), (4) and (3), we have

$$\begin{aligned}
&a+bI\\
&=[(a+bI)(x_1+x_2I)](a+bI)\\
&=[[(a+bI)^2(y_1+y_2I)](x_1+x_2I)][(z_1+z_2I)(a+bI)^2]\\
&=[[(x_1+x_2I)(y_1+y_2I)](a+bI)^2][(z_1+z_2I)(a+bI)^2]\\
&=[[(x_1+x_2I)(y_1+y_2I)][(a+bI)(a+bI)]][(z_1+z_2I)(a+bI)^2]\\
&=[[(a+bI)(a+bI)][(y_1+y_2I)(x_1+x_2I)]][(z_1+z_2I)(a+bI)^2]\\
&=[(a+bI)^2[(y_1+y_2I)(x_1+x_2I)]][(z_1+z_2I)(a+bI)^2]\\
&=[[(z_1+z_2I)(a+bI)^2][(y_1+y_2I)(x_1+x_2I)]](a+bI)^2\\
&=[[(z_1+z_2I)(y_1+y_2I)][(a+bI)^2(x_1+x_2I)]](a+bI)^2\\
&=[(a+bI)^2[[(z_1+z_2I)(y_1+y_2I)](x_1+x_2I)]](a+bI)^2\\
&=[(a+bI)^2(v_1+v_2I)](a+bI)^2,
\end{aligned}$$

where $[(z_1+z_2I)(y_1+y_2I)][(x_1+x_2I)]=(v_1+v_2I)\in N(\mathbf{S})$. This shows that $N(\mathbf{S})$ is $(2,2)$-regular.

Lemma2 *Every weakly regular neutrosophic* **AG** *-groupoid with left identity (Neutrosophic AG**-groupoid) is regular.*

Proof Assume that $N(\mathbf{S})$ is a weakly regular Neutrosophic **AG** -groupoid with left identity *(Neutrosophic AG**-groupoid)*, then for any $(a+bI)\in N(\mathbf{S})$ there exist $(x_1+x_2I),(y_1+y_2I)\in N(\mathbf{S})$ such that $a+bI=[(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)]$. Let $(x_1+x_2I)(y_1+y_2I)=t_1+t_2I\in N(\mathbf{S})$ and $[(t_1+t_2I)[(y_1+y_2I)(x_1+x_2I)]](a+bI)=(u_1+u_2I)\in N(\mathbf{S})$. Now by using (1) , (2), (3) and (4), we have

$$
\begin{aligned}
& a+bI \\
&= [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)] \\
&= [[(a+bI)(y_1+y_2I)](x_1+x_2I)](a+bI) \\
&= [[(x_1+x_2I)(y_1+y_2I)](a+bI)](a+bI) \\
&= [(t_1+t_2I)(a+bI)](a+bI) \\
&= [(t_1+t_2I)[[(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)]]](a+bI) \\
&= [(t_1+t_2I)[[(a+bI)(a+bI)][(x_1+x_2I)(y_1+y_2I)]]](a+bI) \\
&= [(t_1+t_2I)[[(y_1+y_2I)(x_1+x_2I)][(a+bI)(a+bI)]]](a+bI) \\
&= [(t_1+t_2I)[(a+bI)[[(y_1+y_2I)(x_1+x_2I)](a+bI)]]](a+bI) \\
&= [(a+bI)[(t_1+t_2I)[[(y_1+y_2I)(x_1+x_2I)](a+bI)]]](a+bI) \\
&= [(a+bI)(u_1+u_2I)](a+bI),
\end{aligned}
$$

where $[(t_1+t_2I)[[(y_1+y_2I)(x_1+x_2I)](a+bI)]] = u_1+u_2I \in N(\mathbf{S})$. Thus $N(\mathbf{S})$ is regular.

The converse of Lemma 2 is not true in general, as can be seen from the following example.

Example2 *[ref10] Let us consider a Neutrosophic* **AG** *-groupoid* $\mathbf{N(S)} = \left\{ \begin{array}{l} 1+1I, 1+2I, 1+3I, 1+4I, 2+1I, 2+2I, 2+3I, 2+4I, \\ 3+1I, 3+2I, 3+3I, 3+4I, 4+1I, 4+2I, 4+3I, 4+4I \end{array} \right\}$ *with left identity* 3 *in the following Cayley's table.*

$\bullet$	$1+1I$	$1+2I$	$1+3I$	$1+4I$	$2+1I$	$2+2I$	$2+3I$	$2+4I$	$3+1I$	$3+2I$	$3+3I$	$3+4I$	$4+1I$	$4+2I$	$4+3I$	$4+4I$
$1+1I$	$2+2I$	$2+2I$	$2+4I$	$2+4I$	$2+2I$	$2+2I$	$2+4I$	$2+4I$	$4+2I$	$4+2I$	$4+4I$	$4+4I$	$4+2I$	$4+2I$	$4+4I$	$4+4I$
$1+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$4+2I$	$4+2I$	$4+2I$	$4+2I$	$4+2I$	$4+2I$	$4+2I$	$4+2I$
$1+3I$	$2+1I$	$2+2I$	$2+3I$	$2+4I$	$2+1I$	$2+2I$	$2+3I$	$2+4I$	$4+1I$	$4+2I$	$4+3I$	$4+4I$	$4+1I$	$4+2I$	$4+3I$	$4+4I$
$1+4I$	$2+1I$	$2+2I$	$2+1I$	$2+2I$	$2+1I$	$2+2I$	$2+1I$	$2+2I$	$4+1I$	$4+2I$	$4+1I$	$4+2I$	$4+1I$	$4+2I$	$4+1I$	$4+2I$
$2+1I$	$2+2I$	$2+2I$	$2+4I$	$2+4I$	$2+2I$	$2+2I$	$2+4I$	$2+4I$	$2+2I$	$2+2I$	$2+4I$	$2+4I$	$2+2I$	$2+2I$	$2+4I$	$2+4I$
$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$
$2+3I$	$2+1I$	$2+2I$	$2+3I$	$2+4I$	$2+1I$	$2+2I$	$2+3I$	$2+4I$	$2+1I$	$2+2I$	$2+3I$	$2+4I$	$2+1I$	$2+2I$	$2+3I$	$2+4I$
$2+4I$	$2+1I$	$2+2I$	$2+1I$	$2+2I$	$2+1I$	$2+2I$	$2+1I$	$2+2I$	$2+1I$	$2+2I$	$2+1I$	$2+2I$	$2+1I$	$2+2I$	$2+1I$	$2+2I$
$3+1I$	$1+2I$	$1+2I$	$1+4I$	$1+4I$	$2+2I$	$2+2I$	$2+4I$	$2+4I$	$3+2I$	$3+2I$	$3+4I$	$3+4I$	$4+2I$	$4+2I$	$4+4I$	$4+4I$
$3+2I$	$1+2I$	$1+2I$	$1+2I$	$1+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$3+2I$	$3+2I$	$3+2I$	$3+2I$	$4+2I$	$4+2I$	$4+2I$	$4+2I$
$3+3I$	$1+1I$	$1+2I$	$1+3I$	$1+4I$	$2+1I$	$2+2I$	$2+3I$	$2+4I$	$3+1I$	$3+2I$	$3+3I$	$3+4I$	$4+1I$	$4+2I$	$4+3I$	$4+4I$
$3+4I$	$1+1I$	$1+2I$	$1+1I$	$1+2I$	$2+1I$	$2+2I$	$2+1I$	$2+2I$	$3+1I$	$3+2I$	$3+1I$	$3+2I$	$4+1I$	$4+2I$	$4+1I$	$4+2I$
$4+1I$	$1+2I$	$1+2I$	$1+4I$	$1+4I$	$2+2I$	$2+2I$	$2+4I$	$2+4I$	$1+1I$	$1+2I$	$1+4I$	$1+4I$	$2+2I$	$2+2I$	$2+4I$	$2+4I$
$4+2I$	$1+2I$	$1+2I$	$1+2I$	$1+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$	$1+1I$	$1+2I$	$1+2I$	$1+2I$	$2+2I$	$2+2I$	$2+2I$	$2+2I$
$4+3I$	$1+1I$	$1+2I$	$1+3I$	$1+4I$	$2+1I$	$2+2I$	$2+3I$	$2+4I$	$1+2I$	$1+2I$	$1+3I$	$1+4I$	$2+1I$	$2+2I$	$2+3I$	$2+4I$
$4+4I$	$1+1I$	$1+2I$	$1+1I$	$1+2I$	$2+1I$	$2+2I$	$2+1I$	$2+2I$	$1+2I$	$1+2I$	$1+1I$	$1+2I$	$2+1I$	$2+2I$	$2+1I$	$2+2I$

Theorem 7 *If* $N(\mathbf{S})$ *is a Neutrosophic* **AG** *-groupoid with left identity (Neutrosophic AG**-groupoid) then the following are equivalent.*

(i) $N(\mathbf{S})$ *is weakly regular.*

(ii) $N(\mathbf{S})$ *is completely regular.*

Proof $(i) \Rightarrow (ii)$

Let $N(\mathbf{S})$ be a weakly regular Neutrosophic **AG**-groupoid (**AG****-groupoid), then for any $a+bI \in N(\mathbf{S})$ there exist $(x_1+x_2I),(y_1+y_2I) \in N(\mathbf{S})$ such that $a+bI=(a+bI)(x_1+x_2I)(a+bI)(y_1+y_2I)$ and let ($x_1+x_2I)(y_1+y_2I)=(t_1+t_2I)$ for some $(t_1+t_2I) \in N(\mathbf{S})$. Now by using (2), we have

$$\begin{aligned} a+bI &= [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)] \\ &= [(a+bI)(a+bI)][(x_1+x_2I)(y_1+y_2I)] \\ &= (a+bI)^2(t_1+t_2I), \end{aligned}$$

where $(x_1+x_2I)(y_1+y_2I)=t_1+t_2I \in N(\mathbf{S})$. Thus $N(\mathbf{S})$ is right regular.

Let $N(\mathbf{S})$ be a weakly regular Neutrosophic **AG**-groupoid with left identity *(Neutrosophic AG**-groupoid*) then for any $a+bI \in N(\mathbf{S})$ there exist $(x_1+x_2I),(y_1+y_2I) \in N(\mathbf{S})$ such that $a+bI=[(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)]$. Now by using (2) and (3), we have

$$\begin{aligned} a+bI &= [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)] \\ &= [(a+bI)(a+bI)][(x_1+x_2I)(y_1+y_2I)] \\ &= [(y_1+y_2I)(x_1+x_2I)][(a+bI)(a+bI)] \\ &= [(y_1+y_2I)(x_1+x_2I)](a+bI)^2 \\ &= (t_1+t_2I)(a+bI)^2, \end{aligned}$$

Where ($y_1+y_2I)(x_1+x_2I)=(t_1+t_2I)$ for some $(t_1+t_2I) \in N(\mathbf{S})$. Thus $N(\mathbf{S})$ is left regular.

Assume that $N(\mathbf{S})$ is a weakly regular Neutrosophic **AG**-groupoid with left identity *(Neutrosophic AG**-groupoid*) , then for any $a+bI \in N(\mathbf{S})$ there exist $(x_1+x_2I),(y_1+y_2I) \in N(\mathbf{S})$ such that $a+bI=[(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)]$. Let $(x_1+x_2I)(y_1+y_2I)=(t_1+t_2I) \in N(\mathbf{S})$ and

$[(t_1+t_2I)[(y_1+y_2I)(x_1+x_2I)]](a+bI)=(u_1+u_2I) \in N(\mathbf{S})$. Now by using (1) , (2), (3) and (4), we have

$$\begin{aligned} a+bI &= [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)] \\ &= [[(a+bI)(y_1+y_2I)](x_1+x_2I)](a+bI) \\ &= [[(x_1+x_2I)(y_1+y_2I)](a+bI)](a+bI) \\ &= [(t_1+t_2I)(a+bI)](a+bI) \\ &= [(t_1+t_2I)[[(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)]]](a+bI) \\ &= [(t_1+t_2I)[[(a+bI)(a+bI)][(x_1+x_2I)(y_1+y_2I)]]](a+bI) \\ &= [(t_1+t_2I)[[(y_1+y_2I)(x_1+x_2I)][(a+bI)(a+bI)]]](a+bI) \\ &= [(t_1+t_2I)[(a+bI)[[(y_1+y_2I)(x_1+x_2I)](a+bI)]]](a+bI) \\ &= [(a+bI)[(t_1+t_2I)[[(y_1+y_2I)(x_1+x_2I)](a+bI)]]](a+bI) \\ &= [(a+bI)(u_1+u_2I)](a+bI), \end{aligned}$$

where $[(t_1+t_2I)[[(y_1+y_2I)(x_1+x_2I)](a+bI)]]=(u_1+u_2I)\in N(\mathbf{S})$. Thus $N(\mathbf{S})$ is regular. Thus $N(S)$ is completely regular.

$(ii)\Rightarrow(i)$

Assume that $N(S)$ is completely regular Neutrosophic **AG**-groupoid with left identity ($Neutrosophic$**AG****-groupoid), then for any $(a+bI)\in N(\mathbf{S})$ there exist $(t_1+t_2I)\in N(\mathbf{S})$ such that $(a+bI)=(a+bI)^2(x_1+x_2I)$, $(a+bI)=(y_1+y_2I)(a+bI)^2$, $a+bI=[(a+bI)(z_1+z_2I)](a+bI)$

$$\begin{aligned}a+bI &= (a+bI)^2(x_1+x_2I)\\ &=[(a+bI)(a+bI)][(v_1+v_2I)(u_1+u_2I)]\\ &=[(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)],\end{aligned}$$

where $(x_1+x_2I)=(v_1+v_2I)(u_1+u_2I)\in N(\mathbf{S})$. Thus $N(S)$ is Neutrosophic weakly regular

$$\begin{aligned}a+bI &= (x_1+x_2I)(a+bI)^2\\ &=[(u_1+u_2I)(v_1+v_2I)](a+bI)^2\\ &=[(u_1+u_2I)(v_1+v_2I)[(a+bI)(a+bI)]\\ &=[(a+bI)(a+bI)][(u_1+u_2I)(v_1+v_2I)]\\ &=[(a+bI)(u_1+u_2I)][(a+bI)(v_1+v_2I)],\end{aligned}$$

where $(x_1+x_2I)=(u_1+u_2I)(v_1+v_2I)$ for some $(x_1+x_2I)\in N(\mathbf{S})$. Thus $N(S)$ is weakly regular.

$$\begin{aligned}&a+bI\\ &=[(a+bI)(z_1+z_2I)](a+bI)\\ &=[[(a+bI)^2(x_1+x_2I)](z_1+z_2I)][(y_1+y_2I)(a+bI)^2]\\ &=[[(z_1+z_2I)(x_1+x_2I)][(a+bI)(a+bI)]][(y_1+y_2I)[(a+bI)(a+bI)]]\\ &=[(a+bI)[[(z_1+z_2I)(x_1+x_2I)](a+bI)]][(a+bI)[(y_1+y_2I)(a+bI)]]\\ &=[(a+bI)(t_1+t_2I)][(a+bI)(w_1+w_2I)],\end{aligned}$$

where $(t_1+t_2I)=[[(z_1+z_2I)(x_1+x_2I)](a+bI)]]\in N(\mathbf{S})$ & $(w_1+w_2I)=[(y_1+y_2I)(a+bI)]\in N(S)$. Thus $N(\mathbf{S})$ is weakly regular.

Lemma 3 *Every strongly regular Neutrosophic* **AG**-*groupoid with left identit (Neutrosophic AG**-groupoid) is completely regular.*

Proof Assume that $N(\mathbf{S})$ is a strongly regular Neutrosophic $\mathbf{AG}$-groupoid with left identity ($Neutrosophic\mathbf{AG}^{**}$-groupoid), then for any $(a+bI)\in N(\mathbf{S})$ there exists $(x_1+x_2I)\in N(\mathbf{S})$ such that $a+bI=[(a+bI)(x_1+x_2I)](a+bI)$ and $(a+bI)(x_1+x_2I)=(x_1+x_2I)(a+bI)$. Now by using (1), we have

$$\begin{aligned} a+bI &= [(a+bI)(x_1+x_2I)](a+bI) \\ &= [(x_1+x_2I)(a+bI)](a+bI) \\ &= [(a+bI)(a+bI)](x_1+x_2I) \\ &= (a+bI)^2(x_1+x_2I). \end{aligned}$$

This shows that $N(\mathbf{S})$ is right regular and by Theorems 4 and 7, it is clear to see that $N(\mathbf{S})$ is completely regular.

Theorem 8 *In a Neutrosophic* $\mathbf{AG}$*-groupoid* $\mathbf{N(S)}$ *with left identity (Neutrosophic AG**-groupoid) the following are equivalent.*

(i) $N(\mathbf{S})$ *is weakly regular.*

(ii) $N(\mathbf{S})$ *is intra-regular.*

(iii) $N(\mathbf{S})$ *is right regular.*

(iv) $N(\mathbf{S})$ *is left regular.*

(v) $N(\mathbf{S})$ *is left quasi regular.*

(vi) $N(\mathbf{S})$ *is completely regular.*

(vii) *For all* $a+bI\in N(\mathbf{S})$, *there exist* $(x_1+x_2I),(y_1+y_2I)\in N(\mathbf{S})$ *such that* $a+bI=[(x_1+x_2I)(a+bI)][(a+bI)(y_1+y_2I)]$.

Proof $(i)\Rightarrow(ii)$ Let $N(\mathbf{S})$ be weakly regular *Neutrosophic* $\mathbf{AG}$-groupoid with left identity *(Neutrosophic AG**-groupoid)*, then for any $a+bI\in N(\mathbf{S})$ there exist $(x_1+x_2I),(y_1+y_2I)\in N(\mathbf{S})$ such that $a+bI=[(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)]$ and by Lemma 1, $(x_1+x_2I)=(u_1+u_2I)(v_1+v_2I)$ for some $(u_1+u_2I),(v_1+v_2I)\in N(\mathbf{S})$. Let $(v_1+v_2I)(u_1+u_2I)=(t_1+t_2I)\in N(\mathbf{S})$. Now by using (3), (1), (4) and (2), we have

$$\begin{aligned} a+bI &= [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)] \\ &= [(a+bI)(a+bI)][(x_1+x_2I)(y_1+y_2I)] \\ &= (x_1+x_2I)[(a+bI)^2(y_1+y_2I)] \\ &= [(u_1+u_2I)(v_1+v_2I)][(a+bI)^2(y_1+y_2I)] \\ &= [(y_1+y_2I)(a+bI)^2][(v_1+v_2I)(u_1+u_2I)] \\ &= [(y_1+y_2I)(a+bI)^2](t_1+t_2I) \end{aligned}$$

where $(v_1+v_2I)(u_1+u_2I)=(t_1+t_2I)\in N(\mathbf{S})$. Thus $N(\mathbf{S})$ is intra-regular

$(ii) \Rightarrow (iii)$ Let $N(\mathbf{S})$ be a weakly regular *Neutrosophic* $\mathbf{AG}$-groupoid with left identity $(Neutrosophic\mathbf{AG}^{**}$-groupoid$)$, then for any $a+bI \in N(\mathbf{S})$ there exist $(x_1+x_2I),(y_1+y_2I) \in N(\mathbf{S})$ such that $a+bI = [(x_1+x_2I)(a+bI)^2](y_1+y_2I)$

$$\begin{aligned} a+bI &= [(x_1+x_2I)(a+bI)^2](y_1+y_2I) \\ &= [[(u_1+u_2I)(v_1+v_2I)][(a+bI)(a+bI)]](y_1+y_2I) \\ &= [(a+bI)^2((v_1+v_2I)(u_1+u_2I))](y_1+y_2I) \\ &= [(y_1+y_2I)((v_1+v_2I)(u_1+u_2I))](a+bI)^2 \\ &= [(y_1+y_2I)((v_1+v_2I)(u_1+u_2I))][(a+bI)(a+bI)] \\ &= [(a+bI)(a+bI)][((v_1+v_2I)(u_1+u_2I))(y_1+y_2I)] \\ &= (a+bI)^2(s_1+s_2I), \end{aligned}$$

where $x_1+x_2I = (u_1+u_2I)(v_1+v_2I) \in N(S)$ & $s_1+s_2I = [(y_1+y_2I)[v_1+v_2I)(u_1+u_2I)]] \in N(\mathbf{S})$. Thus $N(S)$ is right regular

$(iii) \Rightarrow (iv)$ Let $N(\mathbf{S})$ be a right regular *Neutrosophic* $\mathbf{AG}$-groupoid with left identity $(Neutrosophic\mathbf{AG}^{**}$-groupoid$)$, then for any $a+bI \in N(\mathbf{S})$ there exist

$x_1+x_2I \in N(\mathbf{S})$ such that $a+bI = (a+b)^2(x_1+x_2I)$

$$\begin{aligned} a+bI &= (a+b)^2(x_1+x_2I) \\ &= [(a+bI)(a+bI)](x_1+x_2I) \\ &= [(a+bI)(a+bI)][(u_1+u_2I)(v_1+v_2I)] \\ &= [(v_1+v_2I)(u_1+u_2I)][a+bI)(a+bI)] \\ &= (y_1+y_2I)(a+bI)^2 \end{aligned}$$

Where $(y_1+y_2I) = [(v_1+v_2I)(u_1+u_2I)] \in N(\mathbf{S})$. Thus $N(S)$ is left regular

$(iv) \Rightarrow (v)$ Let $N(\mathbf{S})$ be a left regular *Neutrosophic* $\mathbf{AG}$-groupoid with left identity *(Neutrosophic AG**-groupoid*), then for any $a+bI \in N(\mathbf{S})$ there exist $(x_1+x_2I) \in N(\mathbf{S})$ such that $(a+bI) = (x_1+x_2I)(a+bI)^2$

$$\begin{aligned} a+bI &= (x_1+x_2I)(a+bI)^2 \\ &= [(u_1+u_2I)(v_1+v_2I)][(a+bI)(a+bI)] \\ &= [(u_1+u_2I)(a+bI)][(v_1+v_2I)(a+bI)] \end{aligned}$$

Thus $N(S)$ is left quasi regular

$(v) \Rightarrow (vi)$ Let $N(\mathbf{S})$ be a left quasi regular *Neutrosophic* **AG**-groupoid with left identity (*Neutrosophic*$\mathbf{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathbf{S})$ there exist $x_1+x_2I \in N(\mathbf{S})$ such that $a+bI = [(x_1+x_2I)(a+bI)][(y_1+y_2I)(a+bI)]$

$$\begin{aligned} a+bI &= [(x_1+x_2I)(a+bI)][(y_1+y_2I)(a+bI)] \\ &= [(a+bI)(a+bI)][(y_1+y_2I)(x_1+x_2I)] \\ &= (a+bI)^2(v_1+v_2I) \end{aligned}$$

where $v_1+v_2I = [(y_1+y_2I)(x_1+x_2I)] \in N(\mathbf{S})$

Thus $N(S)$ is right regular $\Rightarrow (1)$

Let $N(\mathbf{S})$ be a left quasi regular *Neutrosophic* **AG**-groupoid with left identity (*Neutrosophic*$\mathbf{AG}^{**}$ -groupoid), then for any $a+bI \in N(\mathbf{S})$ there exist $x_1+x_2I \in N(\mathbf{S})$ such that $a+bI = [(x_1+x_2I)(a+bI)][(y_1+y_2I)(a+bI)]$

$$\begin{aligned} a+bI &= [(x_1+x_2I)(a+bI)][(y_1+y_2I)(a+bI)] \\ &= [(x_1+x_2I)(y_1+y_2I)][(a+bI)(a+bI)] \\ &= (u_1+u_2I)(a+bI)^2 \end{aligned}$$

where $(u_1+u_2I) = [(x_1+x_2I)(y_1+y_2I)] \in N(\mathbf{S})$

Thus $N(S)$ is left regular $\Rightarrow (2)$

Let $N(\mathbf{S})$ be a left quasi regular *Neutrosophic* **AG**-groupoid with left identity (*Neutrosophic*$\mathbf{AG}^{**}$ -groupoid), then for any $a+bI \in N(\mathbf{S})$ there exist $(x_1+x_2I) \in N(\mathbf{S})$ such that $a+bI = [(x_1+x_2I)(a+bI)][(y_1+y_2I)(a+bI)]$

$$\begin{aligned} a+bI &= [(x_1+x_2I)(a+bI)][(y_1+y_2I)(a+bI)] \\ &= [(a+bI)(a+bI)][(y_1+y_2I)(x_1+x_2I)] \\ &= [((y_1+y_2I)(x_1+x_2I))(a+bI)](a+bI) \\ &= [(v_1+v_2I)(a+bI)](a+bI), \\ &= [(v_1+v_2I)[(x_1+x_2I)(a+bI))((y_1+y_2I)(a+bI)]](a+bI) \\ &= [(v_1+v_2I)[[(x_1+x_2I)(y_1+y_2I)][(a+bI)(a+bI)]](a+bI) \\ &= [(v_1+v_2I)[(a+bI)[[(x_1+x_2I)(y_1+y_2I)](a+bI)]]](a+bI) \\ &= [(a+bI)[(v_1+v_2I)[[(x_1+x_2I)(y_1+y_2I)](a+bI)]]](a+bI) \\ &= [(a+bI)(t_1+t_2I)](a+bI) \end{aligned}$$

where $(v_1+v_2I) = (y_1+y_2I)(x_1+x_2I) \in N(\mathbf{S})$ & where $t_1+t_2I = [(v_1+v_2I)[[(x_1+x_2I)(y_1+y_2I)](a+bI)]] \in N(\mathbf{S})$

Thus $N(S)$ is regular $\Rightarrow (3)$.

By (1).(2) & (3) $N(S)$ is completely regular.

$(vi)\Rightarrow(i)$ Let $N(\mathbf{S})$ be a complete regular *Neutrosophic* $\mathbf{AG}$-groupoid with left identity ($Neutrosophic\mathbf{AG}^{**}$-groupoid), then for any $a+bI\in N(\mathbf{S})$ there exist $(x_1+x_2I)\in N(\mathbf{S})$ such that $a+bI=(a+bI)^2(x_1+x_2I)$, $a+bI=(y_1+y_2I)(a+bI)^2$, $a+bI=[(a+bI)(z_1+z_2I)]\,(a+bI)$

$$\begin{aligned} a+bI &= (a+bI)^2(x_1+x_2I)\\ &=[(a+bI)(a+bI)][(u_1+u_2I)(v_1+v_2I)]\\ &=[(a+bI)(u_1+u_2I)][(a+bI)(v_1+v_2I)]\end{aligned}$$

where $(x_1+x_2I)=[(u_1+u_2I)(v_1+v_2I)]\in N(\mathbf{S})$. Thus $N(S)$ is weakly regular.

$$\begin{aligned} a+bI &= (y_1+y_2I)(a+bI)^2\\ &=[(v_1+v_2I)(u_1+u_2I)][(a+bI)(a+bI)]\\ &=[(a+bI)(u_1+u_2I)][(a+bI)(v_1+v_2I)]\end{aligned}$$

where $(y_1+y_2I)=[(v_1+v_2I)(u_1+u_2I)]\in N(\mathbf{S})$. Thus $N(S)$ is neutrosophic weakly regular.

$$\begin{aligned} &a+bI\\ &=[(a+bI)(z_1+z_2I)](a+bI)\\ &=[[(a+bI)^2(x_1+x_2I)](z_1+z_2I)][(y_1+y_2I)(a+bI)^2]\\ &=[[(z_1+z_2I)(x_1+x_2I)][(a+bI)(a+bI)]][(y_1+y_2I)[(a+bI)(a+bI)]]\\ &=[(a+bI)[[(z_1+z_2I)(x_1+x_2I)](a+bI)]][(a+bI)[(y_1+y_2I)(a+bI)]]\\ &=[(a+bI)(t_1+t_2I)][(a+bI)(w_1+w_2I)]\end{aligned}$$

where $t_1+t_2I=[[(z_1+z_2I)(x_1+x_2I)](a+bI)]]\in N(\mathbf{S})$ & $(w_1+w_2I)=[(y_1+y_2I)(a+bI)]\in N(\mathbf{S})$. Thus $N(\mathbf{S})$ is neutrosophic weakly regular.

$(ii)\Rightarrow(vii)$ Let $N(\mathbf{S})$ be an intra-regular *Neutrosophic* $\mathbf{AG}$-groupoid with left identity ($\mathbf{AG}^{**}$-groupoid), then for any $a+bI\in N(\mathbf{S})$ there exist $(x_1+x_2I),(y_1+y_2I)\in N(\mathbf{S})$ such that $a+bI=[(x_1+x_2I)(a+bI)^2](y_1+y_2I)$. Now by using Lemma1, $(y_1+y_2I)=(u_1+u_2I)(v_1+v_2I)$ for some $u_1+u_2I, v_1+v_2I\in N(\mathbf{S})$. Thus by using (4), (1) and (3), we have

$$\begin{aligned} &a+bI\\ &=[(x_1+x_2I)(a+bI)^2](y_1+y_2I)\\ &=[(x_1+x_2I)[(a+bI)(a+bI)]](y_1+y_2I)\\ &=[(a+bI)[(x_1+x_2I)(a+bI)]](y_1+y_2I)\\ &=[(y_1+y_2I)[(x_1+x_2I)(a+bI)]](a+bI)\end{aligned}$$

$$
\begin{aligned}
&= [(y_1 + y_2I)[(x_1 + x_2I)(a + bI)]][(x_1 + x_2I)(a + bI)^2](y_1 + y_2I) \\
&= [[(u_1 + u_2I)(v_1 + v_2)][(x + yI)(a + bI)]][(x_1 + x_2I)(a + bI)^2](y_1 + y_2I) \\
&= [[(a + bI)(x_1 + x_2I)][(v_1 + v_2)(u_1 + u_2I)]][(x_1 + x_2I)(a + bI)^2](y_1 + y_2I) \\
&= [[(a + bI)(x_1 + x_2I)](t_1 + t_2I)][(x_1 + x_2I)(a + bI)^2](y_1 + y_2I) \\
&= [[(x_1 + x_2I)(a + bI)^2](y_1 + y_2I)](t_1 + t_2I)][(a + bI)(x_1 + x_2I)] \\
&= [[(t_1 + t_2I)(y_1 + y_2I)][(x_1 + x_2I)(a + bI)^2]][(a + bI)(x_1 + x_2I)] \\
&= [[(a + bI)^2(x_1 + x_2I)][(y_1 + y_2I)(t_1 + t_2I)]][(a + bI)(x_1 + x_2I)] \\
&= [[(a + bI)^2(x_1 + x_2I)][(s_1 + s_2I)][(a + bI)(x_1 + x_2I)] \\
&= [[(s_1 + s_2I)(x_1 + x_2I)](a + bI)^2][(a + bI)(x_1 + x_2I)] \\
&= [[(s_1 + s_2I)(x_1 + x_2I)][(a + bI)(a + bI)]][(a + bI)(x_1 + x_2I)] \\
&= [[(a + bI)(a + bI)][(x_1 + x_2I)(s_1 + s_2I)]][(a + bI)(x_1 + x_2I)] \\
&= [[(a + bI)(a + bI)](w_1 + w_2I)[(a + bI)(x_1 + x_2I)] \\
&= [[(w_1 + w_2I)(a + bI)](a + bI)][(a + bI)(x_1 + x_2I)] \\
&= [(z_1 + z_2I)(a + bI)][(a + bI)(x_1 + x_2I)] \\
&= [(x_1 + x_2I)(a + bI)][(a + bI)(z_1 + z_2I)]
\end{aligned}
$$

where $[(u_1 + u_2I)(v_1 + v_2) = (y_1 + y_2I) \in N(S)$ & where $(v_1 + v_2)(u_1 + u_2I) = (t_1 + t_2I) \in N(S)$ & where $(y_1 + y_2I)(t_1 + t_2I) = (s_1 + s_2I) \in N(S)$ & where $(x_1 + x_2I)(s_1 + s_2I) = (w_1 + w_2I) \in N(S)$ & where $(w_1 + w_2I)(a + bI) = (z_1 + z_2I) \in N(S)$

$(vii) \Rightarrow (ii)$ let for all $a + bI \in N(\mathbf{S})$, $a + bI = [(x_1 + x_2I)(a + bI)][(a + bI)(z_1 + z_2I)]$ holds for some

$(x_1 + x_2I),(z_1 + z_2I) \in N(\mathbf{S})$. Now by using (4), (1), (2) and (3) we have

$$
\begin{aligned}
& a+bI \\
&= [(x_1+x_2I)(a+bI)][(a+bI)(z_1+z_2I)] \\
&= [(a+bI)[(x_1+x_2I)(a+bI)](z_1+z_2I)] \\
&= [(x_1+x_2I)(a+bI)][(a+bI)(z_1+z_2I)][(x_1+x_2I)(a+bI)](z_1+z_2I)] \\
&= [(a+bI)[(x_1+x_2I)(a+bI)](z_1+z_2I)][(x_1+x_2I)(a+bI)](z_1+z_2I)] \\
&= [[(x_1+x_2I)(a+bI)](z_1+z_2I)][(x_1+x_2I)(a+bI)](z_1+z_2I)]](a+bI) \\
&= [[(x_1+x_2I)(a+bI)](z_1+z_2I)]^2(a+bI) \\
&= [[(x_1+x_2I)(a+bI)]^2(z_1+z_2I)^2](a+bI) \\
&= [[(x_1+x_2I)^2(a+bI)^2][(z_1+z_2I)(z_1+z_2I)]](a+bI) \\
&= [[(x_1+x_2I)^2(z_1+z_2I)][(a+bI)^2(z_1+z_2I)]](a+bI) \\
&= [(a+bI)^2[[(x_1+x_2I)^2(z_1+z_2I)](z_1+z_2I)](a+bI) \\
&= [[(a+bI)(a+bI)][(x_1+x_2I)^2(z_1+z_2I)](z_1+z_2I)](a+bI) \\
&= [[(a+bI)(a+bI)][[(z_1+z_2I)(z_1+z_2I)](x_1+x_2I)^2]](a+bI) \\
&= [[(a+bI)(a+bI)][(z_1+z_2I)^2(x_1+x_2I)^2]](a+bI) \\
&= [[(x_1+x_2I)^2(z_1+z_2I)^2][(a+bI)(a+bI)]](a+bI) \\
&= [(t_1+t_2I)[(a+bI)(a+bI)]](a+bI), \\
&= [(t_1+t_2I)(a+bI)^2](u_1+u_2I)
\end{aligned}
$$

where $[(x_1+x_2I)^2(z_1+z_2I)^2]=(t_1+t_2I)\in N(S)$ and $(a+bI)=(u_1+u_2I)\in N(S)$ where $(a+bI)=(u_1+u_2I)\in N(S)$. Thus $N(S)$ is neutrosophic intra regular.

Remark *Every intra-regular, right regular, left regular, left quasi regular and completely regular* **AG** *-groupoids with left identity* (**AG**** *-groupoids*) *are regular.* The converse of above is not true in general. Indeed, from Example 1, regular **AG**-groupoid with left identity is not necessarily intra-regular.

Theorem 9 *In a Neutrosophic* **AG** *-groupoid* **N(S)** *with left identity, the following are equivalent.*

(i) N(**S**) *is weakly regular.*
(ii) N(**S**) *is intra-regular.*
(iii) N(**S**) *is right regular.*

(iv) N(**S**) *is left regular.*
(v) N(**S**) *is left quasi regular.*
(vi) N(**S**) *is completely regular.*

(vii) *For all* $a+bI \in N(\mathbf{S})$, *there exist* $(x_1+x_2I),(y_1+y_2I) \in N(\mathbf{S})$ *such that* $a+bI=[(x_1+x_2I)(a+bI)][(a+bI)(y_1+y_2I)]$.

$(viii)$ **N(S)** *is* $(2,2)$*-regular.*

Proof $(i) \Rightarrow (ii)$ Let $N(\mathbf{S})$ be a weakly regular *Neutrosophic* **AG**-groupoid with left identity (*Neutrosophic***AG****-groupoid), then for any $a+bI \in N(\mathbf{S})$ there exist $(x_1+x_2I),(y_1+y_2I) \in N(\mathbf{S})$ such that $a+bI=[(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)]$ and by Lemma1, $(x_1+x_2I)=(u_1+u_2I)(v_1+v_2I)$ for some $(u_1+u_2I),(v_1+v_2I) \in N(\mathbf{S})$. Let $(v_1+v_2I)(u_1+u_2I)=(t_1+t_2I) \in N(\mathbf{S})$. Now by using (3), (1), (4) and (2), we have

$$\begin{aligned}
a+bI &= [(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)] \\
&= [(a+bI)(a+bI)][(x_1+x_2I)(y_1+y_2I)] \\
&= (x_1+x_2I)[(a+bI)^2(y_1+y_2I)] \\
&= [(u_1+u_2I)(v_1+v_2I)][(a+bI)^2(y_1+y_2I)] \\
&= [(y_1+y_2I)(a+bI)^2][(v_1+v_2I)(u_1+u_2I)] \\
&= [(y_1+y_2I)(a+bI)^2](t_1+t_2I)
\end{aligned}$$

where $(v_1+v_2I)(u_1+u_2I)=(t_1+t_2I) \in N(\mathbf{S})$. Thus $N(\mathbf{S})$ is intra-regular.

$(ii) \Rightarrow (iii)$ Let $N(\mathbf{S})$ be a intra regular *Neutrosophic* **AG**-groupoid with left identity (*Neutrosophic***AG****-groupoid), then for any $a+bI \in N(\mathbf{S})$ there exist $x_1+x_2I, y_1+y_2I \in N(\mathbf{S})$ such that $a+bI=[(x_1+x_2I)(a+bI)^2](y_1+y_2I)$

$$\begin{aligned}
a+bI &= [(x_1+x_2I)(a+bI)^2](y_1+y_2I) \\
&= [[(u_1+u_2I)(v_1+v_2I)][(a+bI)(a+bI)]](y_1+y_2I), \\
&= [(a+bI)^2((v_1+v_2I)(u_1+u_2I))](y_1+y_2I) \\
&= [(y_1+y_2I)((v_1+v_2I)(u_1+u_2I))](a+bI)^2 \\
&= [(y_1+y_2I)((v_1+v_2I)(u_1+u_2I))][(a+bI)(a+bI)] \\
&= [(a+bI)(a+bI)][((v_1+v_2I)(u_1+u_2I))(y_1+y_2I)] \\
&= (a+bI)^2(s_1+s_2I)
\end{aligned}$$

where $(x_1+x_2I)=(u_!+u_2I)(v_1+v_2I) \in N(\mathbf{S})$ & where $(s_1+s_2I)=[(y_1+y_2I)[v_1+v_2I)(u_1+u_2I)]] \in N(\mathbf{S})$. Thus $N(S)$ is right regular

$(iii) \Rightarrow (iv)$ Let $N(\mathbf{S})$ be a right regular *Neutrosophic* **AG**-groupoid with left identity (*Neutrosophic***AG****-groupoid), then for any $a+bI \in N(\mathbf{S})$ there exist $(x_1+x_2I) \in N(\mathbf{S})$ such that $a+bI=(a+b)^2(x_1+x_2I)$

$$\begin{aligned} a+bI &= (a+b)^2(x_1+x_2I) \\ &= [(a+bI)(a+bI)](x_1+x_2I) \\ &= [(a+bI)(a+bI)][(u_1+u_2I)(v_1+v_2I)] \\ &= [(v_1+v_2I)(u_1+u_2I)][a+bI)(a+bI)] \\ &= (y_1+y_2I)(a+bI)^2 \end{aligned}$$

where $(y_1+y_2I)=[(v_1+v_2I)(u_1+u_2I)]\in N(\mathbf{S})$. Thus $N(S)$ is left regular

$(iv)\Rightarrow(v)$ Let $N(\mathbf{S})$ be a left regular *Neutrosophic* **AG**-groupoid with left identity ($Neutrosophic$**AG****-groupoid), then for any $a+bI\in N(\mathbf{S})$ there exist $(x_1+x_2I)\in N(\mathbf{S})$ such that $a+bI=(x_1+x_2I)(a+bI)^2$

$$\begin{aligned} a+bI &= (x_1+x_2I)(a+bI)^2 \\ &= [(u_1+u_2I)(v_1+v_2I)][(a+bI)(a+bI)] \\ &= [(u_1+u_2I)(a+bI)][(v_1+v_2I)(a+bI)] \end{aligned}$$

Thus $N(S)$ is left quasi regular

$(v)\Rightarrow(vi)$ Let $N(\mathbf{S})$ be a left quasi regular *Neutrosophic* **AG**-groupoid with left identity ($Neutrosophic$**AG****-groupoid), then for any $a+bI\in N(\mathbf{S})$ there exist $x_1+x_2I\in N(\mathbf{S})$ such that $a+bI=[(x_1+x_2I)(a+bI)][(y_1+y_2I)(a+bI)]$

$$\begin{aligned} a+bI &= [(x_1+x_2I)(a+bI)][(y_1+y_2I)(a+bI)] \\ &= [(a+bI)(a+bI)][(y_1+y_2I)(x_1+x_2I)] \\ &= (a+bI)^2(v_1+v_2I) \end{aligned}$$

where $(v_1+v_2I)=[(y_1+y_2I)(x_1+x_2I)]\in N(\mathbf{S})$. Thus $N(S)$ is neutrosophic right regular. Let $N(\mathbf{S})$ be a left quasi regular *Neutrosophic* **AG**-groupoid with left identity ($Neutrosophic$**AG****-groupoid), then for any $a+bI\in N(\mathbf{S})$ there exist $(x_1+x_2I)\in N(\mathbf{S})$ such that $a+bI=[(x_1+x_2I)(a+bI)][(y_1+y_2I)(a+bI)]$

$$\begin{aligned} a+bI &= [(x_1+x_2I)(a+bI)][(y_1+y_2I)(a+bI)] \\ &= [(x_1+x_2I)(y_1+y_2I)][(a+bI)(a+bI)] \\ &= (u_1+u_2I)(a+bI)^2 \end{aligned}$$

where $(u_1+u_2I)=[(x_1+x_2I)(y_1+y_2I)]\in N(\mathbf{S})$. Thus $N(S)$ is neutrosophic left regular ·

Let $N(\mathbf{S})$ be a neutrosophic left quasi regular *Neutrosophic* **AG**-groupoid with left identity ($Neutrosophic\mathbf{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathbf{S})$ there exist $x_1 + x_2I \in N(\mathbf{S})$ such that $a+bI = [(x_1+x_2I)(a+bI)][(y_1+y_2I)(a+bI)]$

$$
\begin{aligned}
& a+bI \\
&= [(x_1+x_2I)(a+bI)][(y_1+y_2I)(a+bI)] \\
&= [(a+bI)(a+bI)][(y_1+y_2I)(x_1+x_2I)] \\
&= [((y_1+y_2I)(x_1+x_2I))(a+bI)](a+bI) \\
&= [(v_1+v_2I)(a+bI)](a+bI) \\
&= [(v_1+v_2I)[(x_1+x_2I)(a+bI))((y_1+y_2I)(a+bI)]](a+bI) \\
&= [(v_1+v_2I)[[(x_1+x_2I)(y_1+y_2I)][(a+bI)(a+bI)]](a+bI) \\
&= [(v_1+v_2I)[(a+bI)[[(x_1+x_2I)(y_1+y_2I)](a+bI)]]](a+bI) \\
&= [(a+bI)[(v_1+v_2I)[[(x_1+x_2I)(y_1+y_2I)](a+bI)]]](a+bI) \\
&= [(a+bI)(t_1+t_2I)](a+bI)
\end{aligned}
$$

where $(v_1+v_2I) = (y_1+y_2I)(x_1+x_2I) \in N(\mathbf{S})$ & where $(t_1+t_2I) = [(v_1+v_2I)[[(x_1+x_2I)(y_1+y_2I)](a+bI)]] \in N(\mathbf{S})$. Thus $N(S)$ is regular $\Rightarrow$ (3)
By (1).(2) & (3) $N(S)$ is Neutrosophic completely regular.

$(vi) \Rightarrow (i)$ Assume that $N(S)$ is neutrosophic completely regular Neutrosophic **AG**-groupoid with left identity ($Neutrosophic\mathbf{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathbf{S})$ there exist $t_1+t_2I \in N(\mathbf{S})$ such that $a+bI = (a+bI)^2(x_1+x_2I)$, $a+bI = (y_1+y_2I)(a+bI)^2$, $a+bI = [(a+bI)(z_1+z_2I)](a+bI)$

$$
\begin{aligned}
& a+bI \\
&= [(a+bI)(z_1+z_2I)](a+bI) \\
&= [[(a+bI)^2(x_1+x_2I)](z_1+z_2I)][(y_1+y_2I)(a+bI)^2] \\
&= [[(z_1+z_2I)(x_1+x_2I)][(a+bI)(a+bI)]][(y_1+y_2I)[(a+bI)(a+bI)]] \\
&= [(a+bI)[[(z_1+z_2I)(x_1+x_2I)](a+bI)]][(a+bI)[(y_1+y_2I)(a+bI)]] \\
&= [(a+bI)(t_1+t_2I)][(a+bI)(w_1+w_2I)]
\end{aligned}
$$

Thus $N(S)$ is Neutrosophic weakly regular.

$(ii) \Rightarrow (vii)$ Let $N(\mathbf{S})$ be a Neutrosophic intra-regular *Neutrosophic* **AG**-groupoid with left identity ($\mathbf{AG}^{**}$-groupoid), then for any $a+bI \in N(\mathbf{S})$ there exist $(x_1+x_2I),(y_1+y_2I) \in N(\mathbf{S})$ such that $a+bI = [(x_1+x_2I)(a+bI)^2](y_1+y_2I)$. Now by using Lemma1, $(y_1+y_2I) = (u_1+u_2I)(v_1+v_2I)$ for some $(u_1+u_2I),(v_1+v_2I) \in N(\mathbf{S})$.

$$\begin{aligned}
& a+bI \\
&= [(x_1+x_2I)(a+bI)^2](y_1+y_2I) \\
&= [(x_1+x_2I)[(a+bI)(a+bI)]](y_1+y_2I) \\
&= [(a+bI)[(x_1+x_2I)(a+bI)]](y_1+y_2I) \\
&= [(y_1+y_2I)[(x_1+x_2I)(a+bI)]](a+bI) \\
&= [(y_1+y_2I)[(x_1+x_2I)(a+bI)]][(x_1+x_2I)(a+bI)^2](y_1+y_2I) \\
&= [[(u_1+u_2I)(v_1+v_2)][(x+yI)(a+bI)]][(x_1+x_2I)(a+bI)^2](y_1+y_2I), \\
&= [[(a+bI)(x_1+x_2I)][(v_1+v_2)(u_1+u_2I)]][(x_1+x_2I)(a+bI)^2](y_1+y_2I) \\
&= [[(a+bI)(x_1+x_2I)](t_1+t_2I)][(x_1+x_2I)(a+bI)^2](y_1+y_2I) \\
&= [[(x_1+x_2I)(a+bI)^2](y_1+y_2I)](t_1+t_2I)][(a+bI)(x_1+x_2I)] \\
&= [[(t_1+t_2I)(y_1+y_2I)][(x_1+x_2I)(a+bI)^2]][(a+bI)(x_1+x_2I)] \\
&= [[(a+bI)^2(x_1+x_2I)][(y_1+y_2I)(t_1+t_2I)]][(a+bI)(x_1+x_2I)] \\
&= [[(a+bI)^2(x_1+x_2I)][(s_1+s_2I)][(a+bI)(x_1+x_2I)], \\
&= [[(s_1+s_2I)(x_1+x_2I)](a+bI)^2][(a+bI)(x_1+x_2I)] \\
&= [[(s_1+s_2I)(x_1+x_2I)][(a+bI)(a+bI)]][(a+bI)(x_1+x_2I)] \\
&= [[(a+bI)(a+bI)][(x_1+x_2I)(s_1+s_2I)]][(a+bI)(x_1+x_2I)] \\
&= [[(a+bI)(a+bI)](w_1+w_2I)[(a+bI)(x_1+x_2I)], \\
&= [[(w_1+w_2I)(a+bI)](a+bI)][(a+bI)(x_1+x_2I)] \\
&= [(z_1+z_2I)(a+bI)][(a+bI)(x_1+x_2I)], \\
&= [(x_1+x_2I)(a+bI)][(a+bI)(z_1+z_2I)]
\end{aligned}$$

where $(w_1+w_2I)(a+bI)=(z_1+z_2I)\in N(S)$ where $(x_1+x_2I)(s_1+s_2I)=(w_1+w_2I)\in N(S)$ where $(y_1+y_2I)(t_1+t_2I)=(s_1+s_2I)\in N(S)$ where $(v_1+v_2)(u_1+u_2I)=(t_1+t_2I)\in N(S)$ where $(u_1+u_2I)(v_1+v_2)=(y_1+y_2I)\in N(S)$

$(vii)\Rightarrow(ii)$ let for all $a+bI\in N(\mathbf{S})$, $a+bI=[(x_1+x_2I)(a+bI)][(a+bI)(z_1+z_2I)]$ holds for some $(x_1+x_2I),(z_1+z_2I)\in N(\mathbf{S})$. Now by using (4), (1), (2) and (3), we have

$$\begin{aligned}
&a+bI\\
&=[(x_1+x_2I)(a+bI)][(a+bI)(z_1+z_2I)]\\
&=[(a+bI)[(x_1+x_2I)(a+bI)](z_1+z_2I)]\\
&=[(x_1+x_2I)(a+bI)][(a+bI)(z_1+z_2I)][(x_1+x_2I)(a+bI)](z_1+z_2I)]\\
&=[(a+bI)[(x_1+x_2I)(a+bI)](z_1+z_2I)][(x_1+x_2I)(a+bI)](z_1+z_2I)]\\
&=[[(x_1+x_2I)(a+bI)](z_1+z_2I)][(x_1+x_2I)(a+bI)](z_1+z_2I)]](a+bI)\\
&=[[(x_1+x_2I)(a+bI)](z_1+z_2I)]^2(a+bI)\\
&=[[(x_1+x_2I)(a+bI)]^2(z_1+z_2I)^2](a+bI)\\
&=[[(x_1+x_2I)^2(a+bI)^2][(z_1+z_2I)(z_1+z_2I)]](a+bI)\\
&=[[(x_1+x_2I)^2(z_1+z_2I)][(a+bI)^2(z_1+z_2I)]](a+bI)\\
&=[(a+bI)^2[[(x_1+x_2I)^2(z_1+z_2I)](z_1+z_2I)](a+bI)\\
&=[[(a+bI)(a+bI)][(x_1+x_2I)^2(z_1+z_2I)](z_1+z_2I)](a+bI)\\
&=[[(a+bI)(a+bI)][[(z_1+z_2I)(z_1+z_2I)](x_1+x_2I)^2]](a+bI)\\
&=[[(a+bI)(a+bI)][(z_1+z_2I)^2(x_1+x_2I)^2]](a+bI)\\
&=[[(x_1+x_2I)^2(z_1+z_2I)^2][(a+bI)(a+bI)]](a+bI)\\
&=[(t_1+t_2I)[(a+bI)(a+bI)]](a+bI),\\
&=[(t_1+t_2I)(a+bI)^2](u_1+u_2I)
\end{aligned}$$

where $[(x_1+x_2I)^2(z_1+z_2I)^2]=(t_1+t_2I)\in N(S)$ and $(a+bI)=(u_1+u_2I)\in N(S)$ where $(a+bI)=(u_1+u_2I)\in N(S)$. Thus $N(S)$ is intra regular.

$(vi)\Rightarrow(viii)$ Assume that $N(S)$ is completely regular Neutrosophic **AG**-groupoid with left identity ($Neutrosophic$**AG****-groupoid) ,then for any $a+bI\in N(\mathbf{S})$ there exist $(t_1+t_2I)\in N(\mathbf{S})$ such that $a+bI=(a+bI)^2(x_1+x_2I)$, $a+bI=(y_1+y_2I)(a+bI)^2$, $a+bI=[(a+bI)(z_1+z_2I)](a+bI)$

$$\begin{aligned}
a+bI&=(a+bI)^2(x_1+x_2I)\\
&=[(a+bI)(a+bI)][(v_1+v_2I)(u_1+u_2I)]\\
&=[(a+bI)(x_1+x_2I)][(a+bI)(y_1+y_2I)]
\end{aligned}$$

where $(x_1+x_2I)=(v_1+v_2I)(u_1+u_2I)\in N(\mathbf{S})$ Thus $N(S)$ is weakly regular

$$\begin{aligned}
a+bI&=(x_1+x_2I)(a+bI)^2\\
&=[(u_1+u_2I)(v_1+v_2I)](a+bI)^2\\
&=[(u_1+u_2I)(v_1+v_2I)][(a+bI)(a+bI)]\\
&=[(a+bI)(a+bI)][(u_1+u_2I)(v_1+v_2I))]\\
&=[(a+bI)(u_1+u_2I)][(a+bI)(v_1+v_2I)]
\end{aligned}$$

where $(x_1 + x_2 I) = (u_1 + u_2 I)(v_1 + v_2 I)$ for some $(x_1 + x_2 I) \in N(S)$. Thus $N(S)$ is weakly regular.

$$\begin{aligned} & a+bI \\ &= [(a+bI)(z_1+z_2I)](a+bI) \\ &= [[(a+bI)^2(x_1+x_2I)](z_1+z_2I)][(y_1+y_2I)(a+bI)^2] \\ &= [[(z_1+z_2I)(x_1+x_2I)][(a+bI)(a+bI)]][(y_1+y_2I)[(a+bI)(a+bI)]] \\ &= [(a+bI)[[(z_1+z_2I)(x_1+x_2I)](a+bI)]][(a+bI)[(y_1+y_2I)(a+bI)]] \\ &= [(a+bI)(t_1+t_2I)][(a+bI)(w_1+w_2I)] \end{aligned}$$

where $(t_1 + t_2 I) = [[(z_1 + z_2 I)(x_1 + x_2 I)](a + bI)]] \in N(\mathbf{S})$ & $(w_1 + w_2 I) = [(y_1 + y_2 I)(a + bI)] \in N(\mathbf{S})$. Thus $N(S)$ is weakly regular.

$(viii) \Rightarrow (vi)$ Let $N(\mathbf{S})$ be a $(2,2)$-regular *Neutrosophic* **AG**-groupoid with left identity, then for $a + bI \in N(\mathbf{S})$ there exists $x_1 + x_2 I \in N(\mathbf{S})$ such that $a + bI = [(a+bI)^2(x+yI)](a+bI)^2$. Now

$$\begin{aligned} a+bI &= [(a+bI)^2(x_1+x_2I)](a+bI)^2 \\ &= [(y_1+y_2I)(a+bI)^2] \end{aligned}$$

where $(a+bI)^2(x_1+x_2I) = (y_1+y_2I) \in N(\mathbf{S})$ and by using (3), we have

$$\begin{aligned} a+bI &= (a+bI)^2[(x_1+x_2I)[(a+bI)(a+bI)]] \\ &= [(a+bI)(a+bI)][(x_1+x_2I)[(a+bI)^2]] \\ &= (a+bI)^2(z_1+z_2I) \end{aligned}$$

where $(x_1+x_2I)(a+bI)^2 = (z_1+z_2I) \in N(\mathbf{S})$. and by using (3), (1) and (4), we have

$$\begin{aligned} & a+bI \\ &= [(a+bI)^2[(x_1+x_2I)[(a+bI)(a+bI)]] \\ &= [(a+bI)(a+bI)][(x_1+x_2I)(a+bI)^2] \\ &= [(a+bI)(a+bI)][[(e_1+e_2I)(x_1+x_2I)][a+bI)(a+bI)]] \\ &= [(a+bI)(a+bI)][[(a+bI)(a+bI)][(x_1+x_2I)(e_1+e_2I)]] \\ &= [(a+bI)(a+bI)][(a+bI)^2(t_1+t_2I)] \\ &= [[(a+bI)^2(t_1+t_2I)](a+bI)](a+bI) \\ &= [[[(a+bI)(a+bI)](t_1+t_2I)](a+bI)](a+bI) \\ &= [[(t_1+t_2I)(a+bI)][(a+bI)](a+bI)]](a+bI) \\ &= [[(a+bI)(a+bI)][(t_1+t_2I)(a+bI)]](a+bI) \\ &= [[(a+bI)(t_1+t_2I)][(a+bI)(a+bI)]](a+bI) \\ &= [(a+bI)[[(a+bI)(t_1+t_2I)](a+bI)]](a+bI) \\ &= [(a+bI)(u_1+u_2I)](a+bI) \end{aligned}$$

Thus N(S) is left regular, right regular and regular, so N(S) is completely regular.

$(vi) \Rightarrow (viii)$ Assume that $N(\mathbf{S})$ is a completely regular Neutrosophic **AG**-groupoid with left identity, then for any $a+bI \in N(\mathbf{S})$ there exist $(x_1+x_2I),(y_1+y_2I),(z_1+z_2I) \in N(\mathbf{S})$ such that $a+bI=[(a+bI)(x_1+x_2I)](a+bI)$, $a+bI=(a+bI)^2(y_1+y_2I)$ and $a+bI=(z_1+z_2I)(a+bI)^2$. Now by using (1), (4) and (3), we have

$$
\begin{aligned}
& a+bI \\
&= [(a+bI)(x_1+x_2I)](a+bI) \\
&= [[(a+bI)^2(y_1+y_2I)](x_1+x_2I)][(z_1+z_2I)(a+bI)^2] \\
&= [[(x_1+x_2I)(y_1+y_2I)](a+bI)^2][(z_1+z_2I)(a+bI)^2] \\
&= [[(x_1+x_2I)(y_1+y_2I)][(a+bI)(a+bI)]][(z_1+z_2I)(a+bI)^2] \\
&= [[(a+bI)(a+bI)][(y_1+y_2I)(x_1+x_2I)]][(z_1+z_2I)(a+bI)^2] \\
&= [(a+bI)^2[(y_1+y_2I)(x_1+x_2I)]][(z_1+z_2I)(a+bI)^2] \\
&= [[(z_1+z_2I)(a+bI)^2][(y_1+y_2I)(x_1+x_2I)]](a+bI)^2 \\
&= [[(z_1+z_2I)(y_1+y_2I)][(a+bI)^2(x_1+x_2I)]](a+bI)^2 \\
&= [(a+bI)^2[[(z_1+z_2I)(y_1+y_2I)](x_1+x_2I)]](a+bI)^2 \\
&= [(a+bI)^2(v_1+v_2I)](a+bI)^2
\end{aligned}
$$

where $[(z_1+z_2I)(y_1+y_2I)][(x_1+x_2I)]=(v_1+v_2I) \in N(\mathbf{S})$. Thus $N(S)$ is $(2,2)$-regular.

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