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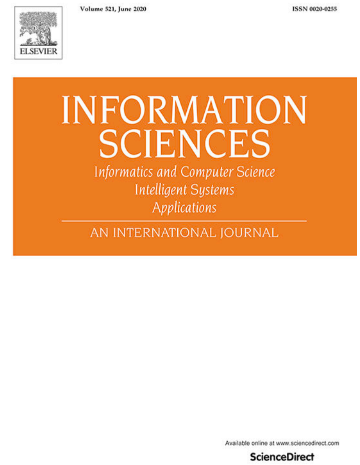

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Generalized combination rule for evidential reasoning approach and Dempster–Shafer theory of evidence

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Abstract: The Dempster–Shafer (DS) theory of evidence can combine evidence with one parameter. The evidential reasoning (ER) approach is an extension of DS theory that can combine evidence with two parameters (weights and reliabilities). However, it has three infeasible aspects: reliability dependence, unreliability effectiveness, and intergeneration inconsistency. This study aimed to establish a generalized combination (GC) rule with both weight and reliability, where ER and DS can be viewed as two particular cases, and the problems of infeasibility of the parameters can be solved. In this paper, the infeasibilities of ER are analyzed, and a generalized discounting method is introduced to reasonably discount the belief distributions of the evidence using both the weight and the reliability. A GC rule is then constructed to combine evidence by means of the orthogonal sum operation, and the corresponding theorems and corollaries are provided. Finally, the superiority of the GC rule is shown through numerical comparisons and discussion, and an illustrative example is provided to demonstrate its applicability.

Keywords: Decision analysis; generalized combination rule; evidential reasoning; Dempster-Shafer theory of evidence; weight and reliability

1. Introduction

The Dempster–Shafer (DS) theory of evidence, first introduced by Arthur P. Dempster and subsequently developed by his student Glenn Shafer [4], is a flexible and useful mathematical tool for expressing and combining information under ignorance. Decision theory, which examines the reasoning underlying the choices of decision-makers or experts [8], is employed to make an optimal selection from a finite number of alternatives. Combining DS theory with decision theory, an approach called evidential reasoning (ER) was introduced to analyze multiple-criteria decision-making (MCDM) issues in uncertain environments.

The evidence combination rule is the kernel of the ER approach. As the earliest combination rule for evidence, Dempster's rule was adopted to aggregate the probability mass assignments for criteria that are discounted by Shafer's discounting technique [36]. However, such an evidence combination rule cannot distinguish the unassigned probability masses generated by incompleteness from those generated by weight, exaggerating the final ignorance of the fusion results. Therefore, a new ER rule was established by introducing an innovative weight-normalization process (called an "ER rule with weight" in this paper), in which the residual support generalized by Shafer's discounting for the weight is allocated to a power set, and the orthogonal sum operation is employed for aggregation [37]. Since the residual support for the weight can be sufficiently separated from the global ignorance, an ER rule with weight can overcome the limitations of earlier versions of the combination rule (e.g., the inability to distinguish the unassigned probability mass generated by incompleteness from that generated by the weight). Recently, a new version of the ER rule was developed to solve the problem of evidence combination with two parameters (weight and reliability) [38]. In this recently proposed ER rule, the characteristic differences between the two parameters (weight and reliability) of the evidence are considered in the aggregation process—namely, the belief distributions are discounted by the

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weight and the reliability, and the discounted belief distributions are then combined using the orthogonal sum operation. Since the recently proposed ER rule follows Bayesian inference and can deal with the evidence-combination problem by taking the weight and reliability into account, it is considered to be more effective than the previous methods (e.g., enhanced proportional conflict redistribution rule no. 5, which can also be used to form a combination with two parameters) [27].

In this study, we aimed to establish a generalized combination (GC) rule for the ER approach and DS theory that can overcome the drawbacks of both while retaining their advantages. The specific improvement presented in this paper is the introduction of a new discounting method by thoroughly considering the characteristics of the weights and reliabilities. Based on this, a GC rule with weight and reliability is proposed, and the corresponding theorems and corollaries are provided. The reliability and weight are two distinct parameters that reflect the role of evidence from objective and subjective perspectives, respectively. The reliability of evidence is used to describe the information quality from objective and absolute perspectives[25] while the weight of evidence is used to describe information importance from subjective and relative perspectives[5]. The fusion result can only be reasonable and effective if both parameters are scientifically embodied and reflected in the process of evidence fusion. In our opinion, DS theory is capable of dealing with the problem of evidence fusion with only one parameter, and it does not distinguish the weight from the reliability. Although ER distinguishes the two parameters, the recently proposed ER rule has three drawbacks (infeasibilities) due to its inability to effectively deal with the properties of the two parameters in discounting and fusion. The first drawback involves the reliability dependence. In a situation with two parameters, it is reasonable to assume that the weight and reliability have their respective functions for evidence fusion. However, the discounting approach in the recently proposed ER rule will not work if the evidence is completely reliable. In other words, whether the weight is involved in the discounting fully depends on whether the reliability is equal to 1. The second drawback relates to the unreliability effectiveness. When all of the pieces of evidence are completely unreliable, it is logical to infer that their combined result is wholly ineffective. Unfortunately, such ineffectiveness cannot be reflected if we employ the recently proposed ER rule for the combination process. The third drawback involves intergeneration inconsistency. Two pieces of evidence that only have weights can be combined by the second-generation ER rule (ER rule with weight), while evidence with both weight and reliability can be combined by the third-generation ER rule (the recently proposed ER rule). When two combined pieces of evidence both have full or complete reliabilities, it is reasonable to ignore the effect of the reliability. Thus, such combination situations can be transferred to those with only weights. However, the third-generation ER rule is not equivalent to the second-generation rule when the two pieces of evidence are completely reliable. These three drawbacks are concretely demonstrated in Subsection 4.1.

The method proposed in this study can solve the problem of the two undistinguished parameters in DS theory as well as the aforementioned drawbacks of the ER approach. ER and DS can be seen as two particular cases of the proposed method. The rest of this paper is organized as follows. In Section 2, the literature related to DS theory and ER is reviewed, and the basic preliminary concepts are defined in Section 3. We present the generalized discounting method and the GC rule with weight and reliability in Section 4. In Section 5, the superiority of the GC rule is demonstrated through numerical comparisons and discussion. In Section 6, an illustrative example is provided to demonstrate its applicability. Section 7 concludes the paper.

2. Literature review

DS theory, as a general extension of Bayesian theory, introduces a simple method for combining evidence from multiple sources with an orthogonal sum operator. Since DS theory can not only describe uncertain information with ignorance but also combine it, it is a powerful tool for handling uncertainty in decision-making problems. Thus, it has been used in fields such as image processing [18], supply chain sustainability assessment [1], safety case confidence assessment [29], medical diagnosis [17], and wildfire risk prevention [12].

Dempster's rule, which plays a crucial role in DS theory, has been challenged for its counterintuitive combination results (also called the intuition paradox) in high-conflict situations. Specifically, the combination of all of the pieces of evidence in a lowly supportive state can produce a fully supportive state [43,42,44]. Counterintuitive combination results are still likely to arise when the conflict level (whatever it is) does not play any role. Two main types of modified approaches have been developed in recent years to solve the counterintuitive problem of DS theory.

The first type assumes that Dempster's rule is problematic and therefore develops various new combination rules. The developed rules are mainly constructed to redistribute conflict masses. For example, Yager assigned conflict masses to the frame of discernment [33-34], while Lefevre et al. distributed them over the subsets of the frame of discernment with a weighting factor [16]. Dubois et al. proposed a combination rule based on both conjunctive and disjunctive rules [7]. Smets and Kennes established a modified rule that assigned the conflict masses to an empty set from the perspective of an open-world assumption [22,28]. Dezert–Smarandache theory (DSmT) extends DS theory on the super-power set and develops a series of combination rules [27]. Although these modified combination rules are well known, some problems have been identified, such as unsatisfying associative properties and high computation complexities. New modifications of Dempster's rule are still being proposed, such as the flexible rule for evidential combination based on complete conflict and evidence weights presented by Ma et al. [20].

The other type of modification assumes that highly conflicting evidence is problematic while Dempster's rule is correct. Thus, a number of discounting methods have been developed to correct evidence prior to combination. For example, Shafer proposed a simple method to add doubt to a piece of evidence, where the belief masses of focal elements are discounted using weights, and the residual masses of the weights are assigned to the frame of discernment [26]. Elouedi et al. used a confusion matrix to discount belief functions and showed how data presented in a matrix could adjust the information [41]. Mercier et al. constructed a contextual discounting method where belief functions could be weakened or strengthened [21]. Guyard and Cherfaoui established new discounting methods with fewer computations based on canonical decomposition [24]. The aforementioned discounting methods can correct the evidence, based on which the advantage of Dempster's rule without high conflicts can be demonstrated through a combination. Nevertheless, this type of method carries the assumption that conflict evidence cannot be fully reliable.

Since DS theory cannot distinguish unassigned probability masses generated by incompleteness from those generated by weight, an ER rule with weight that mainly follows the second type of modified approach was established by introducing an innovative weight-normalization process. Over the past two decades, ER has gradually developed into a systematic approach and has been successfully applied in diverse areas. These include belief rule-based expert systems [39-40], medical quality assessment [14], smart-home subcontractor selection [23],navigational risk assessment [45], multiple-criteria R&D project selection [19], and financial investments [11]. The ER approach consists of four steps [38]: (i) a set of grades is supplied to assess attributes, (ii) a distributed framework is constructed to express assessments with belief structures, (iii) an evidence combination rule is established to aggregate the given assessments, and (iv) multi-attribute utility theory is used to rank alternatives. For steps (ii) and (iv), the ER approach has been extensively developed to handle the problem of assessment with various types of uncertainties. These include intervals or fuzziness [32,11], fuzzy linguistic assessment grades [35], interval belief degrees [30-31], coexisting uncertainties or interval uncertainties with various parameters[9-10], discrete belief structures [2], and deviated intervals [49].

The combination rule used in Step (iii), as the kernel of the ER approach, has seen three generations of development, as described in Section 1. The recently proposed ER rule follows Bayesian inference and can deal with two parameters. Thus, it has been used to construct inference models in the data-driven approximate causal field [3] to solve the expert assessment integration problem[5] and to make decisions for group MCDM [48]. Meanwhile, the ER approach has also been improved theoretically. For example, Wang et al. constructed an analytical ER methodology to solve the combination problem with interval belief degrees [30]. Zhang et al. proposed a Gini coefficient–based ER approach for making decisions in business negotiations [46]. Du and Wang presented an evidence combination rule with contrary support in the ER approach [6]. Du et al. presented a new ER combination rule that integrates subjective and objective fusions with a pair of coefficients [5].

The connections between the present and previous studies are as follows. First, the proposed method adopts the discounting idea in DS theory to process the reliabilities and extend DS theory to a more complex combination scenario with two parameters. Second, the proposed approach adopts the discounting idea in the ER to process weights and modifies the recently proposed ER rule by overcoming its three drawbacks. Third, this study adopts the orthogonal sum operation employed in both DS theory and ER to form combinations. Thus, it is a generalization of the two approaches.

3. Preliminaries

DS theory is an uncertainty reasoning approach to determine an overall belief degree by forming fusions or combinations based on different evidence. The ER approach introduces a distributed structure to address probabilistic uncertainties in MCDM problems. Several concepts of DS theory and the ER approach which are the focus of this paper, are briefly described in this section.

Definition 1 [31]**.** Suppose a possible hypothesis of a variable is $\theta_n (n = 1, \cdots, N)$, and each of the possible hypotheses is exclusive. A finite non-empty exhaustive set of all possible hypotheses $\Theta = \{\theta_1, \cdots, \theta_N\}$ is called a frame of discernment, and its power set that consists of 2^N subsets of Θ is usually expressed as follows:

$$P(\Theta) = \{\varnothing, \theta_1, \cdots, \theta_N, \{\theta_1, \theta_2\}, \cdots, \{\theta_1, \theta_N\}, \cdots, \{\theta_1, \cdots, \theta_{N-1}\}, \Theta\} \quad (1)$$

Definition 2 [4]**.** Suppose $\Theta = \{\theta_1, \cdots, \theta_N\}$ is the frame of discernment. If the mapping function $m : 2^\Theta \to [0,1]$ satisfies

$$\begin{cases} m(\theta) = 0 & \theta = \varnothing \\ m(\theta) \geqslant 0, \sum_{\theta \subseteq \Theta} m(\theta) = 1 & \theta \neq \varnothing \end{cases} \quad (2)$$

then $m(\cdot)$ is called the basic probability assignment (BPA) function of Θ. If $m(\theta)>0$, θ is named a focal element. $m(\Theta)$ reflects the degree of global ignorance, and $m(\theta)$ measures the degree of local ignorance when $\theta \subset \Theta$ and $\theta \neq \forall \theta_n$.

In Shafer's definition, the integration of the belief distribution and the weight of evidence is called the BPA function. This means that the BPA function can not only reflect the belief distribution, but it can also consider the weight of evidence. In this paper, the weight of evidence is separated from the belief distribution to facilitate further discounting, and thus, we provide the following definitions of the belief distribution and Shafer's discounting.

Definition 3 [38]**.** Suppose $(\theta, p_{\theta,i})$ shows that the evidence e_i points to proposition θ to a belief degree $p_{\theta,i}$. The profiled expression

$$b_i = \{(\theta, p_{\theta,i}), \forall \theta \subseteq \Theta, \sum\nolimits_{\theta \subseteq \Theta} p_{\theta,i} = 1\} \quad (3)$$

is called the belief distribution (BD) of e_i.

Definition 4 [26]. Suppose the BD of evidence e_i is b_i, as defined by Eq. (3), and w_i is the weight of evidence e_i, which is used to discount b_i, where $0 \le w_i \le 1$. Shafer's discounting method can be defined to generate the BPA function for the evidence e_i as follows:

$$m_i(\theta) = \begin{cases} w_i p_i(\theta) & \theta \subset \Theta \\ w_i p_i(\theta) + (1 - w_i) & \theta = \Theta \end{cases} \quad (4)$$

Definition 5 [4]**.** Suppose the BPA functions of two pieces of evidence are m_1 and m_2 on Θ and $\oplus$ is the orthogonal sum operator. The combined evidence with Dempster's rule from m_1 and m_2 for $\theta \neq \varnothing$ can be defined as follows:

$$m_{e(2)}(\theta) = [m_1 \oplus m_2](\theta) = \frac{\sum_{B \cap C = \theta, B, C \subseteq \Theta} m_1(B) m_2(C)}{1 - \sum_{B \cap C = \varnothing, B, C \subseteq \Theta} m_1(B) m_2(C)} \quad (5)$$

In Shafer's discounting method, global ignorance is produced to a certain BD, even when the evidence precisely points to a proposition without any ambiguous degree. Thus, the specificity of the original evidence cannot be well maintained. To improve Shafer's discounting, a new discounting method with a weight is defined in the ER approach in Definition 6, and the discounted result is called the weighted belief distribution (WBD). WBDs can be combined by the recursive combination rules given in Definition 7 below.

Definition 6 [38]**.** Suppose the BD of evidence e_i is b_i, as defined by Eq. (3), w_i ($0 \le w_i \le 1$) is the weight to discount b_i, and $P(\Theta)$ is the power set of Θ. The ER discounting method with a weight is defined to generate the WBD for evidence e_i as follows:

$$m_{\theta,i} = m_i(\theta) = \begin{cases} 0 & \theta = \varnothing \\ w_i p_{\theta,i} & \theta \subseteq \Theta \\ 1 - w_i & \theta = P(\Theta) \end{cases} \quad (6)$$

Definition 7 [38]**.** Suppose *I* pieces of independent evidence are each profiled by Eq. (3), and their WBDs are represented by Eq. (6). Suppose $e(i)$ denotes the combination of the first *i* pieces of evidence, and $m_{\theta,e(i)}$ is the probability mass to which θ is supported jointly by $e(i)$, with $m_{\theta,e(1)} = m_{\theta,1}$ and $m_{P(\Theta),e(1)} = m_{P(\Theta),1}$. The orthogonal sum of the first *i* WBDs is then given as follows:

$$m_{\theta,e(i)} = [m_1 \oplus \cdots \oplus m_i](\theta) = \begin{cases} 0 & \theta = \varnothing \\ \dfrac{\widehat{m}_{\theta,e(i)}}{\sum_{\vartheta \subseteq \Theta} \widehat{m}_{\vartheta,e(i)} + \widehat{m}_{P(\Theta),e(i)}} & \theta \neq \varnothing \end{cases} \quad (7a)$$

$$\widehat{m}_{\theta,e(i)} = [(1-w_i)m_{\theta,e(i-1)} + m_{P(\Theta),e(i-1)}m_{\theta,i}] + \sum_{B\cap C=\theta} m_{B,e(i-1)}m_{C,i}, \forall\theta\subseteq\Theta \quad (7b)$$

$$\widehat{m}_{P(\Theta),e(i)} = (1-w_i)m_{P(\Theta),e(i-1)} \quad (7c)$$

where w_i is the weight of e_i, which is not necessarily normalized, $0 \le m_{\theta,e(i)}, m_{P(\Theta),e(i)} \le 1$, and $\sum_{\theta\subseteq\Theta} m_{\theta,e(i)} + m_{P(\Theta),e(i)} = 1$ for $i = 1,\cdots,I$, recursively.

If there is another reliability parameter in the fusion process, the reliability can be used to discount the WBDs by Definition 8. The discounted result with both a weight and reliability is also called the weighted belief distribution with reliability (WBDR). In the ER approach, the recursive combination rule given by Definition 9 is established to form combinations for the WBDRs, and the final combined BD for each proposition is determined by Definition 10.

Definition 8 [38]**.** Suppose the BD of evidence e_i is b_i, as defined by Eq. (3), where w_i and r_i are its weight and reliability, respectively, $0 \le w_i, r_i \le 1$, and $P(\Theta)$ is the power set of Θ. The discounting method of the ER approach with both a weight and reliability is defined to generate the WBDR for evidence e_i, as follows:

$$m_{\theta,i} = m_i(\theta) = \begin{cases} 0 & \theta = \varnothing \\ \tilde{w}_i p_{\theta,i} & \theta \subseteq \Theta \\ 1-\tilde{w}_i & \theta = P(\Theta) \end{cases} \quad (8)$$

where $c_{rw,i} = 1/(1+w_i-r_i)$ is a normalization factor determined by $\sum_{\theta\subseteq\Theta} m_{\theta,i} + m_{P(\Theta),i} = 1$, and $\tilde{w}_i = c_{rw,i}w_i$ is called the new weight or the adjusted weight, $1-\tilde{w}_i = c_{rw,i}(1-r_i)$.

Definition 9 [38]**.** Suppose I pieces of independent evidence are each profiled by Eq. (3), and their WBDRs are represented by Eq. (8). Suppose $e(i)$ is defined in the same way as in Definition 7, $m_{\theta,i}$ is defined in the same as in Definition 8, and $m_{P(\Theta),e(1)} = m_{P(\Theta),1} = 1-r_1$. The combined degree of belief to which I pieces of independent evidence e_i with weight w_i and reliability r_i ($i = 1,\cdots,I$) jointly support proposition θ is given by $\widehat{m}_{\theta,e(I)}$, which is generated by recursively applying Eq. (7a) and the following two equations:

$$\widehat{m}_{\theta,e(i)} = [(1-r_i)m_{\theta,e(i-1)} + m_{P(\Theta),e(i-1)}m_{\theta,i}] + \sum_{B\cap C=\theta} m_{B,e(i-1)}m_{C,i}, \forall\theta\subseteq\Theta \quad (9a)$$

$$\widehat{m}_{P(\Theta),e(i)} = (1-r_i)m_{P(\Theta),e(i-1)} \quad (9b)$$

Definition 10 [38]**.** The combined degree of belief to which I pieces of independent evidence jointly support proposition θ is given as follows:

$$p_\theta = p_{\theta,e(I)} = \begin{cases} 0 & \theta = \varnothing \\ \dfrac{\widehat{m}_{\theta,e(I)}}{\sum_{\vartheta\subseteq\Theta} \widehat{m}_{\vartheta,e(I)}} & \theta\subseteq\Theta, \theta\ne\varnothing \end{cases} \quad (10)$$

with $\widehat{m}_{\theta,e(I)}$ given by Eq. (7b) or (9a) for $i = I$, $0 \le p_\theta \le 1$, $\forall\theta\subseteq\Theta$, and $\sum_{\theta\subseteq\Theta} p_\theta = 1$.

4. Proposed method

4.1 Infeasibility analysis of ER

As mentioned in Section 1, the ER approach has three drawbacks: reliability dependence, unreliability effectiveness, and intergeneration inconsistency. The first two arise from the reliability parameter and pertain to counterintuitive fusion results with the extreme reliability degrees. The third arises from both parameters and is related to inconsistencies between the second and third generations of the ER rule. The reason for these drawbacks is that the properties of the two parameters of the reliability and weight are not effectively reflected.

(1) Reliability dependence. The reliability dependence problem refers to the fact that the discounting in the recently proposed ER rule will lose its effectiveness when the evidence is completely reliable. Specifically, if reliability $r_i = 1$, then the weight cannot contribute to the discounting. Setting $r_i = 1$ in Eq. (8), we have $c_{rw,i} = 1/(1+w_i - r_i) = 1/w_i$, $m_{\theta,i} = c_{rw,i} w_i p_{\theta,i} = (1/w_i) w_i p_{\theta,i} = p_{\theta,i}$ for $\forall \theta \subseteq \Theta$, $m_{P(\Theta),i} = c_{rw,i}(1-r_i) = 1/w_i \times 0 = 0$, and $m_{\varnothing,i} = 0$. It is easy to find that $m_{\theta,i} = p_{\theta,i}$ for $\forall \theta \subseteq \Theta$. This means that if the evidence is completely reliable, then ER's discounted result is the same as the BD, although weight exists in the discounting. Consequently, whether the weight contributes to the discounting depends on whether the reliability is equal to 1.

Example 1. Assume $p_{\theta,i} = \{(\theta_1, 0.4), (\theta_2, 0.6)\}$, $w_i = 0.9$, and $r_i = 1.0$. According to Eq. (8), $c_{rw,i} = 1/(1+w_i - r_i) = 1/w_i = 10/9$, $m_{\theta_1,i} = c_{rw,i} w_i p_{\theta_1,i} = 10/9 \times 0.9 \times 0.4 = 0.4$, $m_{\theta_2,i} = 10/9 \times 0.9 \times 0.6 = 0.6$, $m_{P(\Theta),i} = c_{rw,i}(1-r_i) = 5/3 \times (1-1) = 0$, and $m_{\varnothing,i} = 0$. Thus, we have $m_{\theta,i} = \{(\theta_1, 0.4), (\theta_2, 0.6)\} = p_{\theta,i}$. The weight 0.9 has an apparent meaning for a piece of evidence such that its importance degree is 0.9. However, we find that the weight 0.9 contributes nothing to the discounted result. Thus, such a result of the ER approach is counterintuitive.

(2) Unreliability effectiveness. The unreliability effectiveness problem refers to the fact that the recently proposed ER rule may produce an incorrect combination result that appears to be effective from completely unreliable evidence. For a straightforward case, suppose two pieces of completely unreliable evidence e_1 and e_2 are used for a combination. It is reasonable to expect that their fusion result will produce noneffective information such as $p_{\theta,e(2)} = 0$ and $p_{\Theta,e(2)} = 1$. This means that if two pieces of completely unreliable evidence are fused, their final fusion result cannot provide any useful information. Setting reliability $r_1 = r_2 = 0$ in Eqs. (9a) and (9b), we have $\hat{m}_{\theta,e(2)} = [\tilde{w}_1 p_{\theta,1} + (1-\tilde{w}_1)\tilde{w}_2 p_{\theta,2}] + \sum_{B \cap C = \theta} \tilde{w}_1 p_{B,1} \tilde{w}_2 p_{C,2}, \forall \theta \subseteq \Theta$ and $\hat{m}_{P(\Theta),e(2)} = (1-r_2)(1-\tilde{w}_1) = 1/(1+w_1)$, where $\tilde{w}_i = c_{rw,i} w_i = w_i/(1+w_i)$, and $i = 1,2$. Since $p_{\theta,1}, p_{\theta,2} > 0$ and $w_1, w_2 > 0$, we have $0 < \tilde{w}_1, \tilde{w}_2 < 1$ and $\hat{m}_{\theta,e(2)} > 0$ for $\theta \subseteq \Theta$. From Eq. (10), the following must be satisfied: $p_{\theta,e(2)} = \hat{m}_{\theta,e(2)} \big/ \sum_{\vartheta \subseteq \Theta} \hat{m}_{\vartheta,e(2)} > 0$ for $\theta \subseteq \Theta$. Unfortunately, the expected combination result is not obtained.

Example 2. Assume two pieces of evidence are the same, their BDs are $p_{\theta,1} = p_{\theta,2} = \{(\theta_1, 0.4), (\theta_2, 0.6)\}$, their weights are $w_1 = w_2 = 0.5$, and their reliabilities are $r_1 = r_2 = 0$. Setting $r_1 = r_2 = 0$ in Eqs. (8) and (9a), we have $\tilde{w}_1 = \tilde{w}_2 = 0.5/(1+0.5) = 1/3$, $\hat{m}_{\theta_1,e(2)} = [\tilde{w}_1 p_{\theta_1,1} + (1-\tilde{w}_1)\tilde{w}_2 p_{\theta_1,2}] + \tilde{w}_1 p_{\theta_1,1} \tilde{w}_2 p_{\theta_1,2} = [1/3 \times 0.4 + (1-1/3) \times 1/3 \times 0.4] + (1/3 \times 0.4) \times (1/3 \times 0.4) = 0.24$, and $\hat{m}_{\theta_2,e(2)} = [\tilde{w}_1 p_{\theta_2,1} + (1-\tilde{w}_1)\tilde{w}_2 p_{\theta_2,2}] + \tilde{w}_1 p_{\theta_2,1} \tilde{w}_2 p_{\theta_2,2} = [1/3 \times 0.6 + (1-1/3) \times 1/3 \times 0.6] + (1/3 \times 0.6) \times (1/3 \times 0.6) = 0.37$. Inserting $\hat{m}_{\theta_1,e(2)}, \hat{m}_{\theta_2,e(2)}$ into Eq. (10), we have $p_{\theta_1,e(2)} = \hat{m}_{\theta_1,e(2)} \big/ (\hat{m}_{\theta_1,e(2)} + \hat{m}_{\theta_2,e(2)})$ $0.24/(0.24+0.37) \approx 0.39$ and $p_{\theta_2,e(2)} = \hat{m}_{\theta_2,e(2)} \big/ (\hat{m}_{\theta_1,e(2)} + \hat{m}_{\theta_2,e(2)}) = 0.37/(0.24+0.37) \approx 0.61$. We would anticipate a result that cannot provide any valuable decision information resulting from the two pieces of completely unreliable evidence. However, the calculated result of this example revealed that it has a 61% probability of being θ_2 and 39% probability of being θ_1. Thus, such a fusion result of the ER approach is counterintuitive.

(3) Intergeneration inconsistency. The intergeneration inconsistency problem concerns the fact that the recently proposed ER rule, given by Eqs. (9a) and (9b), cannot degenerate into one with only weight if all the evidence has complete reliability. As shown in Eqs. (7a)–(7c), evidence reliabilities are not considered, and they do not participate in the discounting of the ER rule with weight. The reason the reliabilities is not considered in the discounting must be explained. It is reasonable to say that all the evidence is regarded as completely reliable ($r_i = 1.0$, $\forall i$); otherwise, the reliability should be used to discount the evidence. More specifically, if all evidence is completely reliable, the fusion result of the recently proposed ER rule with two parameters should be equal to that of the ER rule with only one parameter. However, setting $r_i = 1.0$ for $\forall i$

in Eqs. (9a) and (9b), we derive $\widehat{m}_{\theta,e(i)}=\sum_{B\cap C=\theta} m_{B,e(i-1)}m_{C,i}$, $\forall\theta\subseteq\Theta$, $\widehat{m}_{P(\Theta),e(i)}=0$, which differs from the expressions in Eqs. (7b) and (7c). The third-generation ER rule is not equivalent to the second-generation one when all evidence to be combined is completely reliable. Thus, there is an intergeneration inconsistency problem in the ER approach. In fact, this problem is also caused by reliability dependence.

Example 3. Assume $p_{\theta,1}=p_{\theta,2}=\{(\theta_1,0.4),(\theta_2,0.6)\}$, $w_1=w_2=0.5$, $r_1=r_2=1$. In the first case, we use the ER rule with weight to form the combination. Inserting $p_{\theta,1},p_{\theta,2},w_1,w_2$ into Eqs. (7b) and (7c), we have $\widehat{m}_{\theta_1,e(2)}=[(1-w_2)m_{\theta_1,1}+m_{P(\Theta),1}m_{\theta_1,2}]+\sum_{B\cap C=\theta_1} m_{B,1}m_{C,2}$=$0.5\times0.5\times0.4+0.5\times0.5\times0.4+0.5\times0.4\times0.5\times0.4$=0.24, $\widehat{m}_{\theta_2,e(2)}=[(1-w_2)m_{\theta_2,1}+m_{P(\Theta),1}m_{\theta_2,2}]+\sum_{B\cap C=\theta_2} m_{B,1}m_{C,2}$=$0.5\times0.5\times0.6+0.5\times0.5\times0.6+0.5\times0.6\times0.5\times0.6$=0.39, and $\widehat{m}_{P(\Theta),e(2)}=(1-w_1)(1-w_2)$=$0.5\times0.5$=0.25. Inserting $\widehat{m}_{\theta_1,e(2)},\widehat{m}_{\theta_2,e(2)}$, and $\widehat{m}_{P(\Theta),e(2)}$ into Eq. (7a), we have $m_{\theta_1,e(2)}=\widehat{m}_{\theta_1,e(2)}/(\widehat{m}_{\theta_1,e(2)}+\widehat{m}_{\theta_2,e(2)}+\widehat{m}_{P(\Theta),e(2)})=0.24/(0.24+0.39+0.25)\approx0.27$, $m_{\theta_2,e(2)}=\widehat{m}_{\theta_2,e(2)}/(\widehat{m}_{\theta_1,e(2)}+\widehat{m}_{\theta_2,e(2)}+\widehat{m}_{P(\Theta),e(2)})$ $=0.39/(0.24+0.39+0.25)\approx0.44$, and $m_{P(\Theta),e(2)}=\widehat{m}_{P(\Theta),e(2)}/(\widehat{m}_{\theta_1,e(2)}+\widehat{m}_{\theta_2,e(2)}+\widehat{m}_{P(\Theta),e(2)})=0.25/(0.24+0.39+0.25)$ ≈0.29. From Eq. (10), we determine the final fusion results are $p_{\theta_1,e(2)}=\widehat{m}_{\theta_1,e(2)}/(\widehat{m}_{\theta_1,e(2)}+\widehat{m}_{\theta_2,e(2)})$ $=0.24/(0.24+0.39)\approx0.38$ and $p_{\theta_2,e(2)}=\widehat{m}_{\theta_2,e(2)}/(\widehat{m}_{\theta_1,e(2)}+\widehat{m}_{\theta_2,e(2)})=0.39/(0.24+0.39)\approx0.62$. In the second case, we use the recently proposed ER rule to form the combination. Inserting $p_{\theta,1},p_{\theta,2},w_1,w_2,r_1,r_2$ into Eqs. (9a) and (9b), we have $\widehat{m}_{\theta_1,e(2)}=p_{\theta_1,1}p_{\theta_1,2}$=$0.4\times0.4$=0.16, $\widehat{m}_{\theta_2,e(2)}=p_{\theta_2,1}p_{\theta_2,2}$=$0.6\times0.6$=0.36, $\widehat{m}_{P(\Theta),e(2)}=0$. Inserting $\widehat{m}_{\theta_1,e(2)}$, $\widehat{m}_{\theta_2,e(2)}$, and $\widehat{m}_{P(\Theta),e(2)}$ into Eq. (7a), we have $m_{\theta_1,e(2)}=0.16/(0.16+0.36+0)\approx0.31$, $m_{\theta_2,e(2)}=$ $0.36/(0.16+0.36+0)\approx0.69$, and $m_{P(\Theta),e(2)}=0$. From Eq. (10), we can determine that the final fusion results are $p_{\theta_1,e(2)}=\widehat{m}_{\theta_1,e(2)}/(\widehat{m}_{\theta_1,e(2)}+\widehat{m}_{\theta_2,e(2)})=0.16/(0.16+0.36)\approx0.31$ and $p_{\theta_2,e(2)}=\widehat{m}_{\theta_2,e(2)}/(\widehat{m}_{\theta_1,e(2)}+\widehat{m}_{\theta_2,e(2)})=0.36/(0.16+0.36)$ ≈0.69. The combination results in the above two cases are different for $m_{e(2)}$ or for $p_{e(2)}$, and thus, the intergeneration inconsistency problem of the ER approach is evident.

4.2 Generalized discounting method

The weight of the evidence, which is frequently defined by decision-makers, indicates the degree of importance of an evidence source relative to others. To achieve a piece of precise and unambiguous evidence, Shafer used the weight to discount the BD for the single nonempty subset θ of Θ, and the residual support of the weight was allocated to Θ. However, this discounting approach is considered to be unable to hold the specificity of the evidence. Thus, the ER approach with a weight (Eq. (6)) allocates the residual support of the weight to the power set $P(\Theta)$. We agree with the discounting approach in the ER method since it can distinguish the residual support of the weight from global ignorance. In our opinion, the residual support of the weight $1-w_i$, plays a finite role in the combination process and is thus an extrinsic property, while the global ignorance $m_i(\Theta)$ is generated by an evidence source to describe uncertainties and is therefore an intrinsic property of the evidence.

In contrast, the reliability, which is frequently estimated using statistical data methods, is the capacity of an evidence source to generate valid information. To achieve a piece of precise and unambiguous evidence, reliability is also taken into account in the ER approach. As in Eq. (8), the ER discounts the BD $p_{\theta,i}$ with weight w_i ($w_i p_{\theta,i}$) and then allocates the residual support of the reliability $1-r_i$ to the power set $P(\Theta)$ ($m_i(P(\Theta))=1-r_i$). Finally, it uses coefficient $c_{rw,i}$ for normalization. The finite role of the combination originates from the residual support of the weight in Eq. (7b) and reliability in Eq. (9a). As a result, it is reasonable to consider that the finite roles of the combination defined in the ER approach are inconsistent.

Thus, it must be determined which is the correct approach. In our opinion, the reliability is an intrinsic property of the evidence that reflects the information quality, and it has no relevance to other evidence. Any evidence source can generate the probability or BD that proposition θ may occur, but the information quality depends on its reliability. If an evidence source is not completely reliable, the generated information should first be corrected by its reliability. In other words, only the corrected probability or the corrected degree of belief is precise and unambiguous. As a result, it can be regarded as the BPA function generated by the evidence source (e.g., Example 4). Such an argument is consistent with the meaning of the BPA function. The BPA function in Shafer's book is explained as follows: "where the evidence points precisely and unambiguously to a single non-empty subset A of Θ,…we can say that the effect of the evidence is limited to providing a certain degree of support for A of Θ." Reliability is unfortunately regarded as an extrinsic property of the evidence in Eq. (8), which plays a finite role in the combination process, as does the weight. Consequently, the three aforementioned infeasibilities will inevitably appear in the ER approach.

Example 4. Assume a group of experts is asked to vote on the performance of a project, and the grade level set is $\Theta=\{\theta_1,\theta_2\}$. Assume the group consists of ten experts, in which six experts vote for θ_1 and four experts vote for θ_2. Thus, the BD is $p_{\theta,i}=\{(\theta_1,0.6),(\theta_2,0.4)\}$. Based on experience, one of the experts frequently makes a mistake, so the reliability of this piece of evidence is set as $r=(10-1)/10=0.9$. To achieve the corresponding BPA function with precise and unambiguous information as mentioned in Shafer's book, it is reasonable to discount $p(\theta_1)$ and $p(\theta_2)$ with the reliability, i.e., $r\times p(\theta_1)=0.9\times 0.6=0.54$ and $r\times p(\theta_2)=0.9\times 0.4=0.36$. It must be determined how to deal with the residual support of the reliability. Since the expert who frequently makes a mistake may give a correct or incorrect judgment, it is reasonable to assign $1-r=0.1$ to the frame of discernment Θ with the meaning that each element in Θ may be correct. As a result, the BPA function is obtained as $m=\{(\theta_1,0.54),(\theta_2,0.36),(\Theta,0.1)\}$.

Based on the properties of the weight and reliability mentioned above, we use the reliability of the evidence to discount the degree of belief for $\theta\subset\Theta$ and allocate the residual support of the reliability to Θ. We then use the weight of evidence to discount the discounted result, as shown in Eq. (6). Discounting with the reliability is used first to correct the degree of belief in terms of the intrinsic property, while discounting with the weight is then used to give the finite roles of the combination process, either for the evidence source itself or the evidence to be combined based on the extrinsic property. Therefore, we define the generalized discounting method as in Definition 11, and both the weight and reliability can participate in discounting the BD. The problem of the reliability dependence can thus be solved (e.g., Example 5).

Definition 11. Suppose w_i is the weight of evidence e_i with $0\le w_i\le 1$, r_i is the reliability of e_i with $0\le r_i\le 1$, r_i =0 corresponds to "completely unreliable", and r_i =1 corresponds to "completely reliable". The basic probability mass for e_i, discounted by both weight and reliability, is then assigned as follows:

$$m_{\theta,i}=m_i(\theta)=\begin{cases} 0 & \theta=\varnothing \\ w_i r_i p_{\theta,i} & \theta\subset\Theta \\ w_i r_i p_{\theta,i}+w_i(1-r_i) & \theta=\Theta \\ 1-w_i & \theta=P(\Theta) \end{cases} \tag{11}$$

Example 5. As in Example 1, assume that the BD generated by evidence e_i is $p_{\theta,i}=\{(\theta_1,0.4),(\theta_2,0.6)\}$ with weight $w_i=0.6$ and reliability $r_i=1.0$. Taking $p_{\theta,i}$, w_i, and r_i into Eq. (11), we have

$m_{\theta,i}$=$\{(\theta_1, 0.24), (\theta_2, 0.36), (P(\Theta), 0.40)\}$. We find that $\hat{m}_{\theta,i} \neq p_{\theta,i}$. Moreover, the discounted result has the clear meaning that the evidence e_i is exactly restricted by its weight $w_i = 0.6$ (see $\{(\theta_1, 0.24), (\theta_2, 0.36)\}$), and the evidence to be combined will be restricted by its remaining weight $1 - w_i = 0.4$ (see $\{(P(\Theta), 0.40)\}$). Thus, we propose that whether the weight is applied in the generalized discounting method is independent of the reliability.

Theorem 1. Suppose the basic probability masses for e_i are $m_{\theta,i}$, which are discounted by Eq. (11). The following must be satisfied: $\sum_{\theta \subseteq \Theta} m_{\theta,i} + m_{P(\Theta),i} = 1$.

Proof. Appendix A.1.

Corollary 1. If the weight is $w_i = 1.0$, then the generalized discounting in Eq. (11) for evidence e_i degenerates into Shafer's discounting given by Eq. (4).

Proof. Appendix A.2.

Corollary 2. If the reliability of the evidence e_i is $r_i = 1.0$, then the generalized discounting in Eq. (11) degenerates into the ER discounting with weight given by Eq. (6).

Proof. Appendix A.3.

From Definition 11, we can obtain Corollaries 1 and 2 when one parameter is set with the largest value. Comparing Corollary 1 with Shafer's discounting, we see that the discounting parameters in the two methods are different (i.e., in this paper, reliability is used as the discounting parameter, while Shafer's discounting uses the weight). It must be determined which approach is more reasonable. We propose that Corollary 1 is better than Shafer's discounting because the reliability is an intrinsic property that can be used to generate BPA functions with precise and unambiguous information. However, the weight is an extrinsic property that can be used to determine the finite roles of combination, as in Corollary 2. In fact, the parameter in Shafer's discounting (the so-called weight) has been used to reflect either the importance degree of an evidence source [26-27,38] or the capacity of that evidence source to generate valid information [25,47]. It is reasonable to say that DS theory with Shafer's discounting can deal with the combination problem with only one discounting parameter, but the parameter does not distinguish between the weight (as an extrinsic property of evidence reflecting the degree of information importance) and the reliability (as an intrinsic property of evidence reflecting the information quality).

4.3 Basic generalized combination rule

An orthogonal sum operation that follows a conjunctive probabilistic reasoning process has been used in both the DS and ER approaches. Herein, we also employ the orthogonal sum operation to perform evidence fusion. In a fusion problem with only two pieces of evidence, each is discounted using Eq. (11), and the initial fusion results are as follows:

$$\hat{m}_{\varnothing,e(2)} = \sum_{B \cap C = \varnothing} m_{B,1} m_{C,1} \quad (12a)$$

$$\hat{m}_{\theta,e(2)} = [\sum_{B \cap C = \theta, B, C \subset \Theta} m_{B,1} m_{C,1} + m_{\theta,1} m_{\Theta,2} + m_{\theta,2} m_{\Theta,1}] + [m_{\theta,1} m_{P(\Theta),2} + m_{P(\Theta),1} m_{\theta,2}], \ \theta \subset \Theta \quad (12b)$$

$$\hat{m}_{\Theta,e(2)} = m_{\Theta,1} m_{\Theta,2} + [m_{P(\Theta),1} m_{\Theta,2} + m_{\Theta,1} m_{P(\Theta),2}] \quad (12c)$$

$$\hat{m}_{P(\Theta),e(2)} = m_{P(\Theta),1} m_{P(\Theta),2} \quad (12d)$$

From Theorem 1, we know that the basic probability masses discounted by Eq. (11) have the property $\sum_{\theta \subset \Theta} m_{\theta,i} + m_{P(\Theta),i} = 1$ for $i = 1, 2$. We combine m_1 and m_2 using the orthogonal sum operation, and the sum of the probability masses for each part must be equal to unity, which is expressed as $\widehat{m}_{\varnothing,e(2)}+\sum_{\theta\subset\Theta}\widehat{m}_{\theta,e(2)}+\widehat{m}_{\Theta,e(2)}+\widehat{m}_{P(\Theta),e(2)}=1$. $\widehat{m}_{\varnothing,e(2)}$ is a conflict factor used to describe the probability mass of the empty set, and it is usually denoted as $k=\widehat{m}_{\varnothing,e(2)}$. In most of the literature, conflict factor k should be reassigned to focal elements to satisfy the requirements of the BPA definition. Following this strategy, we let $\gamma=1/(1-k)=1/(1-\widehat{m}_{\varnothing,e(2)})$ and reassign the conflict factor as follows:

$$m_{\theta,e(2)}=\gamma\widehat{m}_{\theta,e(2)}\ ,\theta\subset\Theta \quad (13a)$$

$$m_{\Theta,e(2)}=\gamma\widehat{m}_{\Theta,e(2)} \quad (13b)$$

$$m_{P(\Theta),e(2)}=\gamma\widehat{m}_{P(\Theta),e(2)} \quad (13c)$$

Theorem 2. Suppose two pieces of independent evidence are e_1 and e_2, their probability masses that have been discounted by Eq. (11) are $m_{\theta,1}$ and $m_{\theta,2}$, and the results of the combination of e_1 and e_2 are $m_{\theta,e(2)}$ as in Eqs. (13a)–(13c). The following must be satisfied: $\sum_{\theta\subseteq\Theta}m_{\theta,e(2)}+m_{P(\Theta),e(2)}=1$.

Proof. Appendix A.4.

As shown in Eqs. (13a)–(13c), the probability masses determined by combining e_1 and e_2 can be seen as intermediate combination results, since the result consists of probability masses on power set $P(\Theta)$. From Eq. (3), we know that the BD is a distribution of probability masses on focal elements. Thus, the ER approach reassigns $m_{P(\Theta),e(2)}$ to other elements to obtain the BD of a combined result. Similar to the ER approach, we reassign $m_{P(\Theta),e(2)}$ to other elements using the following Eq. (14) to determine the final combination results:

$$p_{\theta,e(2)}=m_{\theta,e(2)}/(1-m_{P(\Theta),e(2)}),\quad \theta\subseteq\Theta. \quad (14)$$

Accordingly, Fig. 1 shows the basic GC process for two pieces of evidence with two parameters, and Theorem 3 summarizes the basic GC rule. As shown in Fig. 1, Step 1 is to perform the orthogonal sum operation for two pieces of evidence, e_1 and e_2, using Eqs. (12a)–(12d); Step 2 is to reassign the conflict factor $k=\widehat{m}_{\varnothing,e(2)}$ to focal elements using Eqs. (13a)–(13c); Step 3 is to redistribute probability masses $\widehat{m}_{P(\Theta),e(2)}$ in the power set to focal elements using Eq. (14) to determine the final combination results $p_{\theta,e(2)}$ for $\theta\subseteq\Theta$.

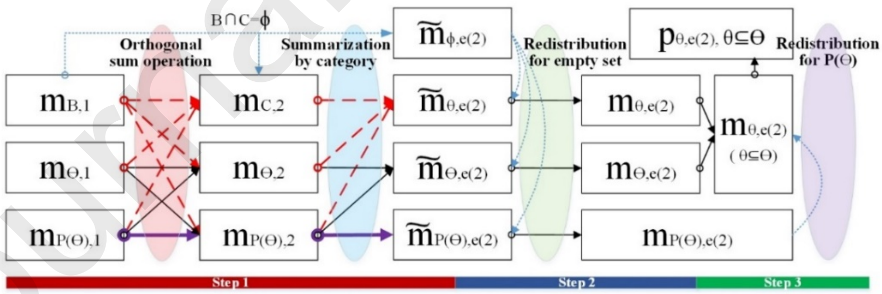

Fig. 1 The basic generalized combination process

Theorem 3. Suppose two pieces of independent evidence are e_1 and e_2 with weight w_i and reliability r_i, where $i=1,2$, and their BDs are profiled by Eq. (3) and discounted by the generalized discounting in Eq. (11). The combined BD $p_{\theta,e(2)}$ is then given as follows:

$$p_{\theta,e(2)}=\begin{cases}0 & \theta=\varnothing\\ \dfrac{\widehat{m}_{\theta,e(2)}}{\sum_{\vartheta\subseteq\Theta}\widehat{m}_{\vartheta,e(2)}} & \theta\subseteq\Theta,\theta\neq\varnothing\end{cases} \quad (15a)$$

$$\widehat{m}_{\theta,e(2)}=[\sum_{B\cap C=\theta,B,C\subset\Theta}m_{B,1}m_{C,2}+m_{\theta,1}m_{\Theta,2}+m_{\theta,2}m_{\Theta,1}]+[(1-w_2)m_{\theta,1}+(1-w_1)m_{\theta,2}],\theta\subset\Theta \quad (15b)$$

$$\widehat{m}_{\Theta,e(2)}=m_{\Theta,1}m_{\Theta,2}+[(1-w_1)m_{\Theta,2}+(1-w_2)m_{\Theta,1}] \quad (15c)$$

Proof. Appendix A.5.

Based on Eqs. (15b) and (15c), Theorem 3 reveals that the combined BD includes two parts. One part is the first square-bracketed term in Eq. (15b) and $m_{\Theta,1}m_{\Theta,2}$ in Eq. (15c); the other is the second square-bracketed term in Eq. (15b) and the square-bracketed term in Eq. (15c). We adopt the names given in the ER approach. In particular, the former is called the orthogonal sum of collective support ("orthogonal sum" for short), and the latter is called the bounded sum of individual support ("bounded sum" for short). From the bounded sum, it is easy to find that each piece of evidence has finite roles restricted by weights. From Theorem 3, we can infer Corollaries 3−5 below. Furthermore, in the evidence combination process with only one parameter, the basic GC rule is simplified into the ER rule with weight by inserting Eq. (16) into Eq. (15a), and it is simplified into Dempster's rule by inserting Eq. (17) into Eq. (15a). As a result, the ER rule with weight and Dempster's rule are two particular cases of the basic GC rule.

Corollary 3. If the reliabilities of e_1 and e_2 are both equal to 1, i.e., $r_1 = r_2 = 1$, and $m^w_{\theta,i}=w_i p_{\theta,i}$ for $\theta \subseteq \Theta$, then the combined probability masses for $\theta \subseteq \Theta$ shown in Eqs. (15b) and (15c) are calculated as follows:

$$\hat{m}_{\theta,e(2)}=[(1-w_2)m^w_{\theta,1}+(1-w_1)m^w_{\theta,2}]+\sum_{B\cap C=\theta,B,C\subseteq\Theta} m^w_{B,1}m^w_{C,2}\ ,\ \theta\subseteq\Theta \quad (16)$$

Proof. Appendix A.6.

Corollary 4. If the weights of e_1 and e_2 are both equal to 1, i.e., $w_1 = w_2 = 1$, $m^r_{\theta,i}=r_i p_{\theta,i}$ for $\theta \subset \Theta$, and $m^r_{\Theta,i}=r_i p_{\Theta,i}+1-r_i$, then the combined probability masses for $\theta \subseteq \Theta$ in Eqs. (15b) and (15c) are calculated as follows:

$$\hat{m}_{\theta,e(2)}=\sum_{B\cap C=\theta,B,C\subseteq\Theta} m^r_{B,1}m^r_{C,2}\ ,\theta\subseteq\Theta \quad (17)$$

Proof. Appendix A.7.

Corollary 5. If the reliabilities of e_1 and e_2 are both equal to 0, i.e., $r_1 = r_2 = 0$, then the BDs combined by the basic GC rule in Eq. (15a) must be $p_{\Theta,e(2)}=1$.

Proof. Appendix A.8.

4.4 Recursive generalized combination rule

Suppose there are more than two pieces of evidence to be combined, and the amount of evidence to be combined is I ($I > 2$). The combined probability masses of the first two pieces of evidence for $\varnothing$ and $\theta \subseteq \Theta$ are shown in Eqs. (12a)–(12c), and those for the power set are $\hat{m}_{P(\Theta),e(2)} = \hat{m}_{P(\Theta),1}\hat{m}_{P(\Theta),2}=(1-w_1)(1-w_2)$. Letting $\oplus$ denote the formation of a combination with the orthogonal sum operation, we have $\hat{m}_{e(2)}=m_1 \oplus m_2=(\hat{m}_{\varnothing,e(2)};\hat{m}_{\theta,e(2)},\theta\subset\Theta;\ \hat{m}_{\Theta,e(2)};\hat{m}_{P(\Theta),e(2)})$, where $\sum_{\theta\subset\Theta}\hat{m}_{\theta,e(2)}+\hat{m}_{\Theta,e(2)}+\hat{m}_{P(\Theta),e(2)}+\hat{m}_{\varnothing,e(2)}=1$. Since $\hat{m}_{\varnothing,e(2)}$ is defined as zero in generalized discounting, it should be reassigned to other elements in the combination process as expressed by Eqs. (13a)–(13c). The above combination result is simplified as $m_{e(2)}=m_1 \oplus m_2=(m_{\theta,e(2)},\theta\subset\Theta;m_{\Theta,e(2)};m_{P(\Theta),e(2)})$, where $\sum_{\theta\subset\Theta}m_{\theta,e(2)}+m_{\Theta,e(2)}+m_{P(\Theta),e(2)}=1$. The previously combined results are used to make a combination with the third piece of evidence, such as $m_{e(3)}=m_{e(2)}\oplus m_3=(m_{\theta,e(3)},\theta\subset\Theta;m_{\Theta,e(3)};m_{P(\Theta),e(3)})$. Repeating the above process, I pieces of evidence can be combined recursively. All pieces of evidence should be combined, and the final combined BD $p_{\theta,e(I)}$ is determined by reassigning $m_{P(\Theta),e(I)}$ to all of the focal elements of Θ, as in Eq. (10). Fig. 2 shows the recursive GC process for more than two pieces of evidence. As shown in Fig. 2, Step 1 is to initialize the first piece of evidence as the combined probability masses, Steps 2 to I are to form combinations for the combined probability masses of the first *i* pieces of evidence with those of the (i+1)th one using the basic GC rule repeatedly ($i=1,\cdots,I-1$), and Step I+1 is to redistribute $m_{P(\Theta),e(I)}$ and yield the final combined BD $p_{\theta,e(I)}$. Theorems 4 and 5 summarize the recursive GC rule, and both are introduced to determine the combined probability masses for the first *i* pieces of evidence and the final combined BD.

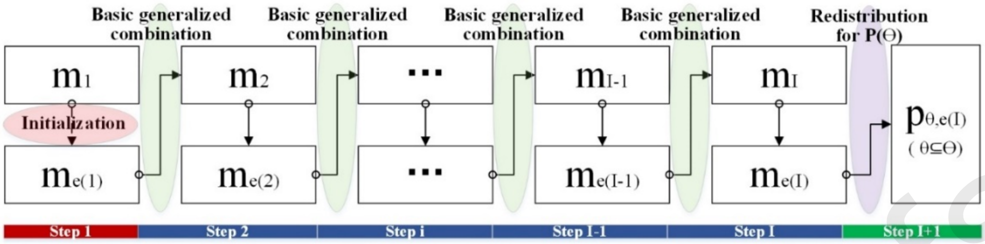

Fig. 2 The recursive generalized combination process

Theorem 4. Suppose there are *I* pieces of independent evidence to be combined, and e_i is the *i*th piece of evidence with weight w_i and reliability r_i, $i=1,\cdots,I$. The BD of e_i is profiled by Eq. (3) and discounted by Eq. (11). *e*(*i*) is the combination of the first *i* pieces of evidence, and its combined probability mass is $m_{\theta,e(i)}$, with $m_{\theta,e(1)}=m_{\theta,1}$ and $m_{P(\Theta),e(1)}=m_{P(\Theta),1}$. The orthogonal sum of the first *i* discounted probability masses are then determined by

$$m_{\theta,e(i)}=[m_1\oplus\cdots\oplus m_i](\theta)=\begin{cases}0 & \theta=\varnothing\\ \dfrac{\widehat{m}_{\theta,e(i)}}{\sum_{\vartheta\subseteq\Theta}\widehat{m}_{\vartheta,e(i)}+\widehat{m}_{P(\Theta),e(i)}} & \theta\neq\varnothing\end{cases}\quad(18a)$$

$$\widehat{m}_{\theta,e(i)}=[\textstyle\sum_{B\cap C=\theta,B,C\subset\Theta}m_{B,e(i-1)}m_{C,i}+m_{\theta,e(i-1)}m_{\Theta,i}+m_{\theta,i}m_{\Theta,e(i-1)}]+[(1-w_i)m_{\theta,e(i-1)}+m_{P(\Theta),e(i-1)}m_{\theta,i}],\theta\subset\Theta\quad(18b)$$

$$\widehat{m}_{\Theta,e(i)}=m_{\Theta,e(i-1)}m_{\Theta,i}+[m_{P(\Theta),e(i-1)}m_{\Theta,i}+(1-w_i)m_{\Theta,e(i-1)}]\quad(18c)$$

$$\widehat{m}_{P(\Theta),e(i)}=(1-w_i)m_{P(\Theta),e(i-1)}\quad(18d)$$

where $0\leq m_{\theta,e(i)}\leq 1$ for $\theta\subset\Theta$, $\theta=\Theta$, $\theta=P(\Theta)$; and $\sum_{\theta\subseteq\Theta}m_{\theta,e(i)}+m_{P(\Theta),e(i)}=1$.

Proof. Appendix A.9.

Theorem 5. The combined BDs of *I* pieces of independent evidence are determined by

$$p_\theta=p_{\theta,e(I)}=\begin{cases}0 & \theta=\varnothing\\ \dfrac{\widehat{m}_{\theta,e(I)}}{\sum_{\vartheta\subseteq\Theta}\widehat{m}_{\vartheta,e(I)}} & \theta\subseteq\Theta,\theta\neq\varnothing\end{cases}\quad(19)$$

where $\widehat{m}_{\theta,e(I)}$ is calculated by Eqs. (18b) and (18c) for $i=I$, $0\leq p_{\theta,e(I)}\leq 1$ for $\theta\subseteq\Theta$, and $\sum_{\theta\subseteq\Theta}p_{\theta,e(I)}=1$.

Proof. Appendix A.10.

Similar to Theorem 3, Theorems 4 and 5 also show that the combined BD in the recursive GC rule also includes the orthogonal sum and the bounded sum. A series of corollaries are inferred from Theorems 4 and 5.

Corollary 6. If the reliability of each piece of evidence is equal to 1, i.e., $r_i=1$ for $i=1,\cdots,I$, $m^w_{\theta,i}=w_ip_{\theta,i}$ for $\theta\subseteq\Theta$, $m^w_{\theta,e(i-1)}=\widehat{m}_{\theta,e(i-1)}/(\sum_{\vartheta\subseteq\Theta}\widehat{m}_{\vartheta,e(i-1)}+\widehat{m}_{P(\Theta),e(i-1)})$ for $\theta\subseteq\Theta$ and $\theta=P(\Theta)$, then the combined probability masses in Eqs. (18b)–(18d) can be computed as follows:

$$\widehat{m}_{\theta,e(i)}=[(1-w_i)m^w_{\theta,e(i-1)}+m^w_{P(\Theta),e(i-1)}m^w_{\theta,i}]+\textstyle\sum_{B\cap C=\theta,B,C\subseteq\Theta}m^w_{B,e(i-1)}m^w_{C,i},\ \theta\subseteq\Theta\quad(20a)$$

$$\widehat{m}_{P(\Theta),e(i)}=(1-w_i)m^w_{P(\Theta),e(i-1)}\quad(20b)$$

Proof. Appendix A.11.

Corollary 7. If the weight of each piece of evidence is equal to 1, i.e., $w_i=1$ for $i=1,\cdots,I$, $m^r_{\theta,i}=r_ip_{\theta,i}$ for $\theta\subset\Theta$, $m^r_{\Theta,i}=r_ip_{\Theta,i}+1-r_i$, $m^r_{\theta,e(i-1)}=\widehat{m}_{\theta,e(i-1)}/(\sum_{\vartheta\subseteq\Theta}\widehat{m}_{\vartheta,e(i-1)}+\widehat{m}_{P(\Theta),e(i-1)})$ for $\theta\subseteq\Theta$ and $\theta=P(\Theta)$, then the combined probability masses in Eqs. (18b)–(18d) are calculated as follows:

$$\widehat{m}_{\theta,e(i)}=\sum_{B\cap C=\theta,B,C,\subseteq\Theta} m^{r}_{B,e(i-1)}m^{r}_{C,i}\ ,\ \ \theta\subseteq\Theta \quad (21a)$$

$$\widehat{m}^{r}_{P(\Theta),e(i)}=0 \quad (21b)$$

Proof. Appendix A.12.

Corollary 8. If the reliability of each piece of evidence is equal to 0, i.e., $r_i=0$ for $i=1,\cdots,I$, then the combined BDs by the recursive GC rule in Eq. (19) must be $p_{\Theta,e(I)}=1$.

Proof. Appendix A.13.

Corollary 9. If the weight of e_i is equal to 0, i.e., $w_i=0$, its discounting result is m_i as in Eq. (11), and the combined probability masses of other $I-1$ pieces of evidence are $m_{e(I-1)}=(m_{\theta,e(I-1)},\theta\subset\Theta;m_{\Theta,e(I-1)};m_{P(\Theta),e(I-1)})$. The combined probability masses of all evidence must then be $m_{e(I)}=m_{e(I-1)}\oplus m_i=m_{e(I-1)}$.

Proof. Appendix A.14.

Corollary 10. The final combined BD of all the evidence in Corollary 9 must be $p_{e(I)}=p_{e(I-1)}$.

Proof. Appendix A.15.

Corollary 11. If the reliability of e_i is equal to 0, i.e., $r_i=0$, its discounting result is m_i in Eq. (11), and the combined probability masses of the other $I-1$ pieces of evidence are $m_{e(I-1)}=(m_{\theta,e(I-1)},\theta\subset\Theta;m_{\Theta,e(I-1)};$ $m_{P(\Theta),e(I-1)})$. The combined probability masses of all the evidence must be as follows:

$$m_{\theta,e(I)}=m_{\theta,e(I-1)}\,,\theta\subset\Theta \quad (22a)$$

$$m_{\Theta,e(I)}=m_{\Theta,e(I-1)}+w_i m_{P(\Theta),e(I-1)} \quad (22b)$$

$$m_{P(\Theta),e(I)}=m_{P(\Theta),e(I-1)}(1-w_i) \quad (22c)$$

Proof. Appendix A.16.

Corollary 12. The final combined BD of all evidence in Corollary 11 must satisfy $p_{\theta,e(I)}=\tau p_{\theta,e(I-1)}\le p_{\theta,e(I-1)}$ for $\theta\subset\Theta$ and $p_{\Theta,e(I)}=1-\tau+\tau p_{\Theta,e(I-1)}\ge p_{\Theta,e(I-1)}$, where $\tau=\sum_{\theta\subseteq\Theta}m_{\theta,e(I-1)}\Big/(\sum_{\theta\subseteq\Theta}m_{\theta,e(I-1)}+w_i m_{P(\Theta),e(I-1)})$.

Proof. Appendix A.17.

4.5 Findings and decision-making process

Based on the theoretical results of this study, a generalized discounting method is introduced to reasonably discount the BDs of the evidence using both the weight and reliability. On this basis, a GC rule is constructed to effectively combine the evidence by means of orthogonal sum operations. This GC rule, which includes basic and recursive rules, is a generalization of DS theory and ER. It can not only overcome the drawbacks of each but also inherit their advantages. When all of the pieces of evidence are completely reliable or the most important, the GC rule is simplified into the ER rule with a weight or Dempster's rule with Shafer's discounting, respectively. Evidently, when all of the pieces of evidence are both completely reliable and the most important, the GC rule is further simplified into Dempster's rule (without Shafer's discounting). Thus, the three infeasible aspects of the ER (i.e., reliability dependence, unreliability effectiveness, and intergeneration inconsistency) do not exist in the GC rule.

Fig. 3 shows the theoretical framework of the GC rule described by the relationships between the GC rule and the Theorems/Corollaries. The theoretical findings of the GC rule can be concluded from the Corollaries. As shown by ① and ② in Fig. 3, the GC rule with reliability and weight includes the basic GC rule (Theorem

3) and the recursive GC rule (Theorems 4 and 5). In ③–⑤, the GC rule with reliability and weight is simplified into the ER rule with weight, Dempster’s rule with Shafer’s discounting, and Dempster’s rule when each piece of evidence to be combined is deemed to be most important or is completely reliable. In ⑥–⑧, the GC rule with reliability and weight can obtain reasonable combination results when there exists completely unreliable or unimportant evidence to be combined.

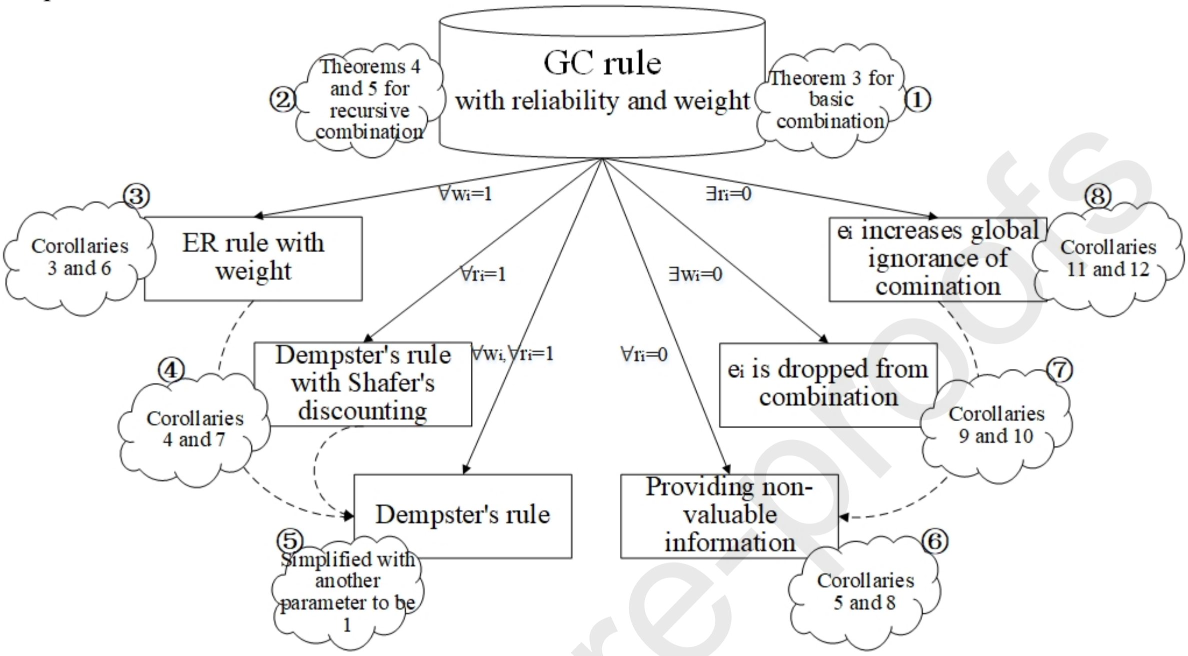

Fig. 3 Theoretical framework of the GC rule

(1) Based on Corollaries 3 and 6, if all of the evidence to be combined is completely reliable, the GC rule is simplified into the ER rule with weight (see ③ in Fig. 3). This means the intergeneration inconsistency problem can be solved by the proposed GC rule. In addition, if all the evidence to be combined is the most important (i.e., each piece of evidence does not need to be discounted by a weight), the GC rule is further simplified into Dempster’s rule without Shafer’s discounting (see ⑤ in Fig. 3).

(2) Based on Corollaries 4 and 7, if all of the evidence to be combined is the most important, the GC rule is simplified into Dempster’s rule with Shafer’s discounting (see ④ in Fig. 3). In such a situation, the combination results only consist of the orthogonal sum while the bounded sum is eliminated since the finite roles restricted by their weights are zero. Additionally, if all of the evidence to be combined is completely reliable (i.e., each piece of evidence does not need to be discounted by the reliability), the GC rule is further simplified into Dempster’s rule without Shafer’s discounting (see ⑤ in Fig. 3).

(3) Based on Corollaries 5 and 8, if all of the evidence to be combined is completely unreliable, the combined BDs cannot provide any valuable information (see ⑥ in Fig. 3). This conclusion is consistent with our intuition, and the unreliability effectiveness problem can be solved by the proposed GC rule.

(4) Based on Corollaries 9 and 10, if a piece of evidence to be combined is not important at all, it will be dropped (ignored or eliminated) and have no influence on the combination (see ⑦ in Fig. 3). This conclusion is consistent with intuition since the piece of evidence, which is completely unimportant in the problem, may have no impact on the decision-making (regardless of its reliability). For example, with a group decision-making problem with *I* experts, each expert may be seen as a piece of evidence, and the importance degree of

the suggestions given by each expert depends on his or her decision-making weight. If the weight of one expert is equal to zero, it means his or her suggestion will not be considered at all in the decision-making for various reasons (e.g., the expert has no right to participate in the decision-making), and it is reasonable to make the final decision based on the suggestions given by the experts whose weights are greater than zero. Meanwhile, it should be determined whether it is possible that the weight of each piece of evidence source could be equal to zero. Since the sum of the weights of evidence sources is usually set to 1, there must be positive weights. In an extreme case, if only one piece of evidence has a positive weight with $w_i=1$, and the remaining pieces of evidence are all weighted zero, then the combined results for all of the evidence must be determined by the evidence with $w_i=1$.

(5) Based on Corollaries 11 and 12, if a piece of evidence to be combined is completely unreliable but is important for solving the decision-making problem with weight $w_i > 0$, it will increase the global ignorance (see ⑧ in Fig. 3). It must be determined whether this conclusion is reasonable. For example, in the MCDM problem with I criteria, each criterion that may be regarded as a piece of evidence that is necessary for determining the collective evaluation value of the alternative (choice). The alternative can be well evaluated by integrating the performances of all of the criteria/evidence. However, if the performance of the alternative on the i^{th} criterion is missing, the collective evaluation value determined by the performances of the remaining criteria must have some uncertainty. The more important the i^{th} criterion is, the more uncertain the collective evaluation value becomes. The missing information can be regarded as the absence of useful information in the completely unreliable evidence. Thus, Corollaries 11 and 12 are consistent with intuition.

There are three kinds of inputs in the GC rule: the BD generated by the evidence source, the reliability of the evidence, and the weight of the evidence. If we employ the GC rule to make a decision by combining all of the pieces of evidence, the above three kinds of inputs should be determined in advance, based on which all of the pieces of evidence are combined recursively. Since global or local ignorance may exist in the combination result, the pignistic probability is frequently employed to make the final decision. The decision-making process with the GC rule is summarized as follows:

Step 1: Generate BDs by evidence sources. The frame of discernment $\Theta = \{\theta_1, \cdots, \theta_N\}$ is established first, and then BD b_i defined by Eq. (3) is generated by the i^{th} piece of evidence source e_i, $i = 1, \cdots, I$.

Step 2: Set the reliability and weight of each piece of evidence. For evidence e_i, its reliability r_i ($0 \le r_i \le 1$) is determined based on the capacity to generate valid information, and its weight w_i ($0 \le w_i \le 1$) is determined based on the importance degree of the decision problem, $i = 1, \cdots, I$.

Step 3: Combine all of the evidence with reliabilities and weights recursively. For evidence e_i, the BD b_i is discounted by the generalized discounting method given by Eq. (11) to obtain probability masses m_i, $i = 1, \cdots, I$. The probability masses of all of the pieces of evidence are combined using the recursive GC rule given by Eqs. (18a)–(18d) to determine the combined probability masses $m_{\theta,e(I)}$ for $\theta \subset \Theta$, $\theta=\Theta$, and $\theta=P(\Theta)$. Finally, probability mass $m_{P(\Theta),e(I)}$ is reassigned to all of the focal elements of Θ in Eq. (19), and the final combined BD $p_{\theta,e(I)}$ for $\theta \subseteq \Theta$ is obtained.

Step 4: Make a decision under a specified principle. Pignistic probability is popular for determining the probability corresponding to each hypothesis, and it can be computed by $BetP(\theta_n) = \sum_{\theta_n \in \theta \subset \Theta} p_{\theta,e(I)} / |\theta|$ for $n = 1, \cdots, N$, where $BetP(\theta_i)$ is the probability that hypothesis θ_n is likely to occur. Thus, the hypothesis θ_*

with the highest pignistic probability $BetP(\theta_*)=\max(BetP(\theta_1),\cdots,BetP(\theta_N))$ can be seen as the final decision.

5. Comparisons and discussion

DS theory and the ER approach are two special cases of the proposed GC rule. Therefore, each is compared with the proposed GC rule by using the same example as in the original literature for the ER rule with both weight and reliability [38]. In this example, the frame of discernment is set as $\Theta=\{\theta_1,\theta_2,\theta_3\}$ $=\{A,B,C\}$, and three pieces of evidence—e_1, e_2, and e_3—are to be combined. Table 1 lists the generated BDs. The purpose of comparing the proposed GC rule with DS theory is to prove that if the two parameters of the reliability and weight are undistinguished, then an unreasonable combination result may be generated. Meanwhile, the purpose of comparing the proposed GC rule with the ER approach is to prove that the three infeasible aspects of the ER can be overcome by the GC rule.

Table 1 The BDs given by evidence e_1-e_3

	A	B	C	{A,B}	{A,C}	{B,C}	{A,B,C}
e_1	0.8000	—	—	0.1000	0.1000	—	—
e_2	0.4000	0.3000	—	0.2000	—	0.1000	—
e_3	0.1000	0.3000	0.5000	—	—	—	0.1000

5.1 Comparison with DS theory and discussion

In DS theory, Dempster's rule is established to fuse the evidence with precise and unambiguous BDs. If the BDs corresponding to the evidence are not precise and unambiguous, Shafer's discounting method, given by Eq. (4), is employed to discount the BDs with a discounting parameter in advance. As mentioned above, DS theory deals with the combination problem with only one parameter and does not distinguish weights from reliabilities. To compare DS theory and the proposed GC rule, we introduce four cases that reflect the combination problem with only one parameter, in which the weight or reliability is set to 1. Cases 1 and 3 are used to show that the proposed GC rule is equivalent to DS theory when each piece of evidence is the most important in the combination while Cases 2 and 4 are used to show that the proposed GC rule is superior to DS theory when each piece of evidence in the combination is completely reliable. The two kinds of cases can be regarded as a combination problem with only reliabilities or only weights.

Case 1: Weights of e_1 and e_2 are $w_1 = w_2 = 1.0$, and their reliabilities are $r_1 = r_2 = 0.5$.

Because the weights of e_1 and e_2 are $w_1 = w_2 = 1.0$, they can be regarded as the most important in the combination. It is not difficult to determine that both pieces of evidence do not need to be discounted from the perspective of the weights, and thus, the combination is simplified into a problem with only one parameter. The discounting parameter in this case is the reliability. First, we insert the BDs in the second and third rows of Table 1 and the reliabilities of e_1 and e_2 into Eq. (4) to perform Shafer's discounting and obtain their BPA functions $m_{DS,1}$ and $m_{DS,2}$, as shown in the second and third rows of Table 2. Next, we take the BPA functions $m_{DS,1}$ and $m_{DS,2}$ in Eq. (5) to make a combination with Dempster's rule and obtain the fusion result of e_1 and e_2, as shown in the fourth and fifth rows of Table 2. In Table 2, $\widehat{m}_{DS,e(2)}$ is the combined result within the probability mass of the empty set, and $m_{DS,e(2)}$ is the final combination result obtained by normalizing the probability mass of the empty set into focal elements.

Table 2 Discounted and combination results of the DS theory for Case 1

	Φ	A	B	C	{A,B}	{A,C}	{B,C}	{A,B,C}
$m_{DS,1}$	—	0.4000	—	—	0.0500	0.0500	—	0.5000
$m_{DS,2}$	—	0.2000	0.1500	—	0.1000	—	0.0500	0.5000
$\widehat{m}_{DS,e(2)}$	0.0875	0.4450	0.0850	0.0025	0.0800	0.0250	0.0250	0.2500
$m_{DS,e(2)}$	—	0.4877	0.0932	0.0027	0.0877	0.0274	0.0274	0.2740

The proposed GC rule is employed to form a combination of e_1 and e_2 via the following steps. First, we insert the BDs from the second and third rows of Table 1 and the weights and reliabilities of e_1 and e_2 into Eq. (11) and obtain the discounted probability masses $m_{GC,1}$ and $m_{GC,2}$, as shown in the second and third rows of Table 3. Next, we insert $m_{GC,1}$ and $m_{GC,2}$ into Theorems 3 and 4 to obtain the fusion result of e_1 and e_2, as shown in the fourth and fifth rows of Table 3. In Table 3, $m_{GC,e(2)}$ is the joint probability mass within the probability mass of the power set, and $p_{GC,e(2)}$ is the final fusion result obtained by normalizing the probability mass of the power set back to the focal elements.

Table 3 Discounted and combination results of the GC for Case 1

	A	B	C	{A,B}	{A,C}	{B,C}	{A,B,C}	P(Θ)
$m_{GC,1}$	0.4000	—	—	0.0500	0.0500	—	0.5000	—
$m_{GC,2}$	0.2000	0.1500	—	0.1000	—	0.0500	0.5000	—
$m_{GC,e(2)}$	0.4877	0.0932	0.0027	0.0877	0.0274	0.0274	0.2740	—
$p_{GC,e(2)}$	0.4877	0.0932	0.0027	0.0877	0.0274	0.0274	0.2740	—

Comparing the second and third rows of Table 2 with those of Table 3, we find that the discounted probability masses generated by the DS theory and the proposed GC rule are the same. This means that reliability as an intrinsic property is well reflected, whether in DS theory or in the proposed GC rule. Furthermore, comparing the fifth row of Table 2 with the fifth row of Table 3, we also find that there is no difference between the fusion results of the two methods. Both comparisons show that these two methods are equivalent to each other when the evidence to be combined is the most important, and the GC can be simplified into the DS when each piece of evidence to be combined is the most important (Corollary 4).

Case 2: Weights of e_1 and e_2 are $w_1 = w_2 = 0.5$, and their reliabilities are $r_1 = r_2 = 1.0$.

Because the reliabilities of e_1 and e_2 are $r_1 = r_2 = 1.0$, they can be seen as completely reliable in the combination. Both pieces of evidence do not need to be discounted from the perspective of the reliabilities, and thus the combination is simplified into a problem with only one parameter. The discounting parameter in this case is the weight. Following a similar computing process to that in Case 1, the discounting and combination results of DS in this case are the same, as shown in Table 2. The reason these two cases have the same discounting and combination results is that the DS can deal with the combination problem with only one parameter, and weight and reliability are not distinguished.

The proposed GC rule is also employed to form a combination for e_1 and e_2 by following similar steps to those in Case 1. First, we insert the BDs from the second and third rows of Table 1 and $w_1 = w_2 = 0.5$, $r_1 = r_2 = 1.0$ into Eq. (11), and we obtain the discounted probability masses $m_{GC,1}$ and $m_{GC,2}$ as shown in the second and third rows of Table 4. Next, we insert $m_{GC,1}$ and $m_{GC,2}$ into Theorems 3 and 4 to obtain the fusion results $m_{GC,e(2)}$ and $p_{GC,e(2)}$, as shown in the fourth and fifth rows of Table 4.

Table 4 Discounted and combination results of the GC for Case 2

	A	B	C	{A,B}	{A,C}	{B,C}	{A,B,C}	P(Θ)

$m_{GC,1}$	0.4000	—	—	0.0500	0.0500	0.4000	—	0.5000
$m_{GC,2}$	0.2000	0.1500	—	0.1000	—	0.0500	—	0.5000
$m_{GC,e(2)}$	0.4877	0.0932	0.0027	0.0877	0.0274	0.0274	—	0.2500
$p_{GC,e(2)}$	0.6717	0.1283	0.0038	0.1208	0.0377	0.0377	—	—

Comparing Table 2 with Table 4, differences between the DS and GC approaches are evident in terms of not only of the discounted results but also their combination results. The DS uses Shafer's discounting to form a discount for each piece of evidence and allocates residual support of the weight to the global ignorance Θ ={A,B,C}, e.g., $m_{\Theta,1} = w_1 p_1(\Theta) + (1 - w_1) = 0 \times 0.1 + (1 - 0.5) = 0.5$, and $m_{\Theta,2} = w_2 p_2(\Theta) + (1 - w_2) = 0 \times 0.1 + (1 - 0.5) = 0.5$, while the GC makes a discount and allocates residual support of the weight to the power set $P(\Theta)$ (see Eq. (11)), e.g., $m(P(\Theta)) = 1 - w_1 = 1 - 0.5 = 0.5$ and $m(P(\Theta)) = 1 - w_2 = 1 - 0.5 = 0.5$. It must be determine which of these is correct. As mentioned in Subsection 4.2, the residual support of the weight $1 - w_i$ (i=1,2) is an extrinsic property, and it plays a finite role in the combination. Unfortunately, the DS cannot distinguish the global ignorance and the residual support of the weight. Thus, it undoubtedly disturbed the characteristics of the original evidence, i.e., there exists no global ignorance in the BDs ($p_{\Theta,i}$ =0 for i=1,2), but global ignorance appears in the discounted results ($m_{\Theta,i} > 0$ for i=1,2). In contrast, the proposed GC rule can well distinguish the residual support of the weight from the global ignorance, and thus we believe that the fusion result of the GC is more reasonable than that of the DS in this case.

Case 3: Weights and reliabilities of e_1 and e_2 are the same as in Case 1, and those of e_3 are $w_3 = 1.0$ and $r_3 \in [0,1.0]$.

In the combination made by DS theory, the BPA function $m_{DS,3}$ of e_3 is derived by Shafer's discounting with reliability r_3 as the discounting parameter, and then the final fusion result of three pieces of evidence can be determined by $m_{DS,e(3)} = m_{DS,e(2)} \oplus m_{DS,3}$, where $m_{DS,e(2)}$ is the fusion result of e_1 and e_2 as shown in Table 2, and $\oplus$ denotes the formation of a combination with Dempster's rule. In the combination made by the GC, the discounted probability mass of e_3 is derived by the generalized discounting given by Eq. (11), and then the joint probability masses of three pieces of evidence are determined as $m_{GC,e(3)} = m_{GC,e(2)} \oplus m_{GC,3}$, where $m_{GC,e(2)}$ is the fusion result of e_1 and e_2, as shown in Table 3, and $\oplus$ denotes the formation of a combination with the GC rule. The final fusion result $p_{GC,e(3)}$ is computed by inserting $m_{GC,e(3)}$ into Eq. (19).

Because the reliability r_3 is a variable that ranges from 0 to 1, we let r_3 take values from 0 to 1 with a step of 0.01. The final fusion results of the DS with different reliabilities are shown in Fig. 4, while those of the GC are shown in Fig. 5. Figs. 4 and 5 show that the final fusion results for the two methods were the same for each reliability in the range of 0 to 1. This means that the GC can be simplified to the DS in any reliability-valued situation when each piece of evidence to be combined is the most important.

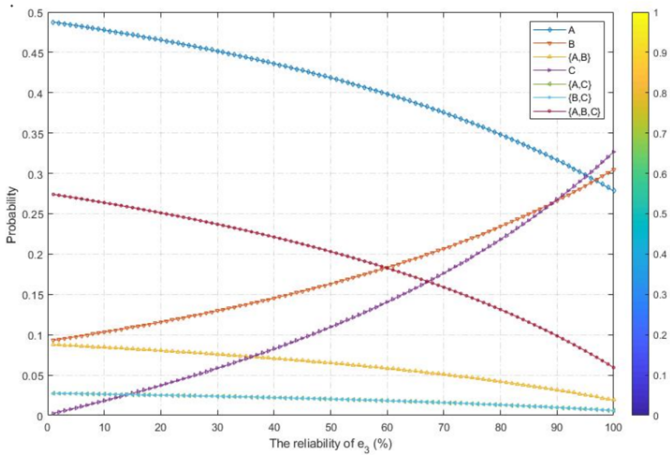

Fig.4 Final fusion results of DS for $r_3 \in [0,1]$

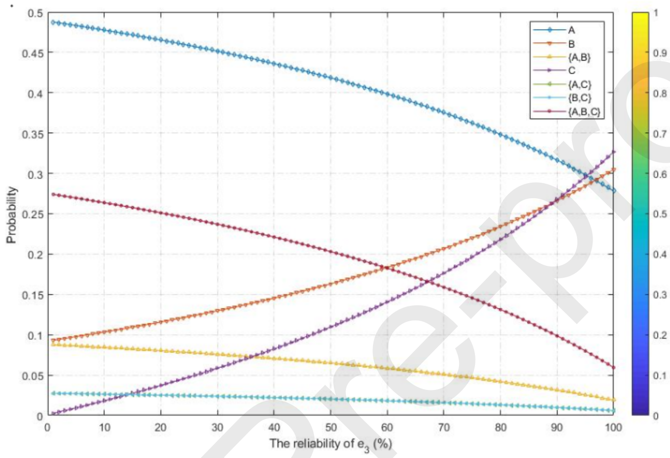

Fig.5 Final fusion results of GC for $r_3 \in [0,1]$

Case 4: Weights and reliabilities of e_1 and e_2 are the same as those in Case 2, and those of e_3 are $r_3 = 1.0$ and $w_3 \in [0,1.0]$.

Similar to the combination process in Case 3, the final fusion results of the DS and GC approaches can be determined with the defined parameters in this case. Two differences should be pointed out. The weight of e_3 is regarded as the discounting parameter in DS theory, and the other is that the fusion results $m_{DS,e(2)}$ and $m_{GC,e(2)}$ used in this case are those shown in Tables 2 and 4, respectively. Because the weight w_3 is a variable in the range of 0-1, we let w_3 take values from 0 to 1 with a step of 0.01. The final fusion results of DS approach with different weights are the same as shown in Fig. 4, while those of the GC approach with different weights are shown in Fig. 6. As shown in Figs. 4 and 6, the final fusion results between the DS and GC approaches are very different in terms of the valued weights in the range of 0-1. The DS approach cannot distinguish the global ignorance and the residual support of the weight, while the GC approach can solve this problem well. Thus, we believe that the fusion result of the GC rule proposed in this paper is more precise and reasonable than that of DS theory in any weight-valued scenario of this case.

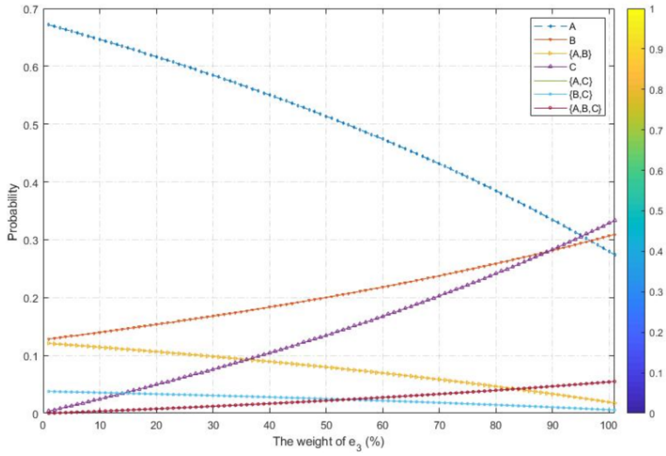

Fig.6 Final fusion results of GC for $w_3 \in [0,1]$

5.2 Comparison with ER approach and discussion

The ER approach can deal with the evidence combination problem in scenarios with existing weights and reliabilities. In the ER approach, the concept of WBDR is used to characterize evidence, and then the orthogonal sum operation is employed to combine the WBDRs. The process for the GC rule introduced in this paper is similar to that of the ER approach, but the difference lies in the discounting method (Eq. (11)) and combination rule (Eqs. (18a)–(18d)). Here, we compare the ER and GC approaches using three cases to test the three kinds of infeasibilities described in Subsection 3.1. In this subsection, Case 1 is used to show that the reliability dependence and intergeneration inconsistency problems may arise in the ER approach but not in the proposed GC rule. Cases 2 and 3 show that while the ER and GC approaches can both completely drop unimportant evidence from the combination, the unreliability effectiveness problem that may arise in the ER approach does not occur in the GC approach. In each case, the ER and GC approaches are both employed to create a combination recursively through the following three steps.

(i) Taking each BD in Table 1 and its weight and reliability as inputs, the ER approach employs Eq. (8) to generate the WBDR $m_{WBDR,i}$, and the GC approach employs Eq. (11) to generate the discounted result $m_{GC,i}$, where i=1,2,3.

(ii) The ER approach takes $m_{WBDR,i}$ as inputs and employs Eqs. (7a), (9a), and (9b) to form a combination and derive $m_{WBDR,e(2)}$ and $m_{WBDR,e(3)}$. The GC approach takes $m_{GC,i}$ as inputs and employs Eqs. (18a)–(18d) to make a combination and derive $m_{GC,\ e(2)}$ and $m_{GC,e(3)}$; e(2) and e(3) denote the combination made by the first two and first three pieces of evidence, respectively.

(iii) The ER approach employs Eq. (10) to obtain the final combined BDs $p_{WBDR,e(3)}$, and the GC approach employs Eq. (19) to obtain final combined BDs $p_{GC,e(3)}$.

Case 1: Reliabilities of e_1, e_2 and e_3 are $r_1 = r_2 = 1$, and $r_3 = 0.6$, and their weights are $w_1 = 0.7$, $w_2 = 0.4$, and $w_3 = 0.8$.

The fusion results determined by the ER and GC approaches in this case are listed in Tables 5 and 6, respectively. The comparison of the second and third rows of Tables 1 and 5 shows that the discounted results of the ER approach using both the weight and reliability are the same as those with the original BDs. It must be determined whether such discounted results are reasonable. The reliabilities of e_1 and e_2 are $r_1 = r_2 = 1$, and their weights are $w_1 = 0.7$ and $w_2 = 0.4$, resulting in a reliability dependence problem, i.e., if the reliability is completely reliable, then the ER's discounting result is the same as that of the BD, regardless of the weight values. Thus, the discounted results in the ER approach are unreasonable since the discounted result is equal to the original BD when the reliability is equal to 1.

Table 5 The ER combination results

	A	B	C	{A,B}	{A,C}	{B,C}	{A,B,C}	P(Θ)
$m_{WBDR,1}$	0.8000	—	—	0.1000	0.1000	—	—	—
$m_{WBDR,2}$	0.4000	0.3000	—	0.2000	—	0.1000	—	—
$m_{WBDR,3}$	0.0667	0.2000	0.3333	—	—	—	0.0667	0.3333
$m_{WBDR,e(2)}$	0.8923	0.0615	0.0308	0.0154	—	—	—	—
$m_{WBDR,e(3)}$	0.8626	0.0888	0.0254	0.0233	—	—	—	—
$p_{WBDR,e(3)}$	0.8626	0.0888	0.0254	0.0233	—	—	—	—

Table 6 The GC combination results

	A	B	C	{A,B}	{A,C}	{B,C}	{A,B,C}	P(Θ)
$m_{GC,1}$	0.5600	—	—	0.0700	0.0700	—	—	0.3000
$m_{GC,2}$	0.1600	0.1200	—	0.0800	—	0.0400	—	0.6000
$m_{GC,3}$	0.0480	0.1440	0.2400	—	—	—	0.3680	0.2000
$m_{GC,e(2)}$	0.6058	0.0523	0.0031	0.0794	0.0466	0.0133	—	0.1996
$m_{GC,e(3)}$	0.5360	0.1094	0.0893	0.0622	0.0365	0.0104	0.1012	0.0550
$p_{GC,e(3)}$	0.5672	0.1158	0.0945	0.0658	0.0386	0.0110	0.1071	—

Furthermore, the reliability dependence problem directly leads to the loss of focal elements in the fusion process. A comparison of the fifth through the seventh rows of Table 5 with the second through fourth rows shows that {A,C} of e_1, {B,C} of e_2, and {A,B,C} of e_3 do not exist in the ER fusion results, but they are focal elements of the evidence to be combined. It must be determined whether such combination results are reasonable.

The combined BD determined by the ER approach includes part of the bounded sum, and this part has the apparent meaning that the residual support of the weight $1-w_i$ is used to restrict the roles played by other evidence in the combination. Because the weights of all of the evidence are less than 1, the evidence to be combined is allowed to play finite roles ($1-w_i$) in the combination. Therefore, the focal elements for each piece of evidence should exist in the fusion results. Unfortunately, the mentioned focal elements are lost in the ER fusion results.

In contrast, neither the reliability dependence problem nor the loss of focal elements are incurred in the GC fusion results. A comparison of second and third rows of Tables 1 and 6 shows that discounted results of the GC approach using both weight and reliability are different from the original BDs. The discounted results of the GC approach have clear meanings, i.e., $m_{GC,i}(\theta)$ for $\theta \subseteq \Theta$ is the discounted BD, which is corrected by its reliability and restricted by its weight, and $m_{GC,i}(\theta)$ for $\theta = P(\Theta)$ is determined by $1-w_i$, which restricts the role of the combination for the evidence to be combined. In addition, the fifth to seventh rows of Table 6 show that the focal elements of all of the evidence (including the lost focal elements in the ER's fusion) are all retained in the GC fusion results. As a result, it is reasonable to believe that the GC fusion results are superior to the ER fusion results.

Note that e_1 and e_2 are both completely reliable, and their weights are less than 1. Generally, if we combine the two pieces of evidence in this situation, the weights should participate in the combination, while the reliabilities could be omitted for their full reliabilities. Thus, the combination problem of e_1 and e_2 with weight and reliability should be equivalent to that with only weight. We now adopt the ER rule with only weight to combine e_1 and e_2 and obtain the fusion results that are listed in the fourth and fifth rows of Table 7. The final fusion results of e_1 and e_2 determined by the ER approach with weights and reliabilities could be computed by inserting $m_{WBDR,e(2)}$ from Table 5 into Eq. (10) to obtain $p_{WBDR,e(2)}$ (see the sixth row of Table 7). Similarly, the final fusion results of e_1 and e_2 determined by the GC approach could be computed by inserting $m_{GC,e(2)}$ from Table 6 into Eq. (19) to obtain $p_{GC,e(2)}$ (see the seventh row of Table 7). Comparing $p_{WBD,e(2)}$ with $p_{WBDR,e(2)}$, an inter-generation inconsistency problem in the ER approach is incurred, i.e., the fusion results determined by the ER rule with two parameters are inconsistent with those determined by the ER rule with one parameter. However, comparing $p_{WBD,e(2)}$ with $p_{GC,e(2)}$, the fusion results determined by the ER rule with one parameter are consistent with those determined by the GC rule. This means that the inter-generation inconsistency problem can be well solved by the GC rule.

Table 7 Combination results of e_1 and e_2 determined by two kinds of methods

	A	B	C	{A,B}	{A,C}	{B,C}	{A,B,C}	P(Θ)
$m_{WBD,1}$	0.5600	—	—	0.0700	0.0700	—	—	0.3000
$m_{WBD,2}$	0.1600	0.1200	—	0.0800	—	0.0400	—	0.6000
$m_{WBD,e(2)}$	0.6058	0.0523	0.0031	0.0794	0.0466	0.0133	0.0196	0.1800
$p_{WBD,e(2)}$	0.7387	0.0638	0.0038	0.0968	0.0568	0.0162	0.0238	—
$p_{WBDR,e(2)}$	0.8923	0.0615	0.0308	0.0154	—	—	—	—
$p_{GC,e(2)}$	0.7387	0.0638	0.0038	0.0968	0.0568	0.0162	0.0238	—

Case 2: Weight of one piece of evidence is equal to 0, i.e., $w_i = 0$.

We assume that the weight of e_3 is 0 ($w_3 = 0$) and the other parameters are the same as they were in Case 1 of this subsection. The combination results generated by the ER and GC approaches are listed in Tables 8 and 9, respectively. As shown in Table 8, the combined results of the first two determined by the ER approach are equal to those of the first three. Similar results are shown in Table 9 for the GC approach. Although there are two aspects of problems, i.e., reliability dependence and intergeneration inconsistency, in the fusion results of the first two determined by the ER approach, these problems have no influence on the judgement of the effectiveness of the fusion results with a third piece of evidence. Based on our intuition, if the combined evidence is completely unimportant, it is dropped from the combination and has no influence on the fusion results. Consequently, we believe that the ER and GC approaches both effectively form a combination for the problem in which one piece of evidence to be combined is completely unimportant. If the weights of more than one piece of evidence are set to 0, it is inferred that such a conclusion may also be applicable for the ER and GC approaches.

Table 8 The ER combination results with w_3=0

	A	B	C	{A,B}	{A,C}	{B,C}	{A,B,C}	P(Θ)
$m_{WBDR,e(2)}$	0.8923	0.0615	0.0154	0.0308	—	—	—	—
$p_{WBDR,e(2)}$	0.8923	0.0615	0.0154	0.0308	—	—	—	—
$m_{WBDR,e(3)}$	0.8923	0.0615	0.0154	0.0308	—	—	—	—
$p_{WBDR,e(3)}$	0.8923	0.0615	0.0154	0.0308	—	—	—	—

Table 9 The GC combination results with w_3=0

	A	B	C	{A,B}	{A,C}	{B,C}	{A,B,C}	P(Θ)
$m_{GC,e(2)}$	0.6058	0.0523	0.0031	0.0794	0.0466	0.0133	—	0.1996
$p_{GC,e(2)}$	0.7568	0.0653	0.0039	0.0992	0.0582	0.0166	—	—
$m_{GC,e(3)}$	0.6058	0.0523	0.0031	0.0794	0.0466	0.0133	—	0.1996
$p_{GC,e(3)}$	0.7568	0.0653	0.0039	0.0992	0.0582	0.0166	—	—

Case 3: Reliability of one piece of evidence is equal to 0, i.e., $r_i = 0$.

We assume that the reliability of e_3 is 0 ($r_3 = 0$), and the other parameters are the same as in Case 1 of this subsection. The fusion results generated by the ER and GC approaches are listed in Tables 10 and 11, respectivley. From Table 10, the focal element {A,B,C} of e_3 is lost, and the probability masses of focal elements are all changed after making a combination with e_3 using the ER approach (see the third and fifth rows of Table 10). This means that the unreliability effectiveness problem occurs in the ER approach. As shown in Table 11, the probability masses of all the focal elements except for Θ={A,B,C} and $P(\Theta)$ are not changed, and that of Θ={A,B,C} increased after making a combination with e_3 by the GC approach (see the second and fourth rows of Table 11). The final results $p_{GC,e(3)}$ are determined by redistributing the probability masses of the power set $m_{GC,e(3)}(P(\Theta))$ on the other focal elements, with the result that $p_{GC,e(3)}(\theta)$ for $\theta \subseteq \Theta$ is decreased and that for Θ={A,B,C} is increased compared to the last round of combination results (see the third and fifth rows in Table 11). The two methods yielded different fusion results. It must be determined which results are reasonable. Based on our intuition, if one piece of evidence is completely unreliable, it can be regarded as missing information. The more important the missing information is, the more uncertain the final fusion results become. It is logical to infer that the uncertainties of the final fusion results should be enlarged after making a combination with the completely unreliable but important evidence e_3 ($r_3 = 0$ and $w_3 = 0.8$). Consequently, it is reasonable to believe that the GC fusion results are superior to the ER fusion results in this case.

Table 10 The ER combination results with r_3=0

	A	B	C	{A,B}	{A,C}	{B,C}	{A,B,C}	P(Θ)
$m_{WBDR,e(2)}$	0.8923	0.0615	0.0154	0.0308	—	—	—	—
$p_{WBDR,e(2)}$	0.8923	0.0615	0.0154	0.0308	—	—	—	—
$m_{WBDR,e(3)}$	0.8777	0.0749	0.0281	0.0193	—	—	—	—
$p_{WBDR,e(3)}$	0.8777	0.0749	0.0281	0.0193	—	—	—	—

Table 11 The GC combination results with r_3=0

	A	B	C	{A,B}	{A,C}	{B,C}	{A,B,C}	P(Θ)
$m_{GC,e(2)}$	0.6058	0.0523	0.0031	0.0794	0.0466	0.0133	—	0.1996
$p_{GC,e(2)}$	0.7568	0.0653	0.0039	0.0992	0.0582	0.0166	—	—
$m_{GC,e(3)}$	0.6058	0.0523	0.0031	0.0794	0.0466	0.0133	0.1596	0.0399
$p_{GC,e(3)}$	0.6309	0.0545	0.0032	0.0827	0.0485	0.0139	0.1663	—

To further discuss the above argument, we assume that the reliability of e_3 is 0 ($r_3 = 0$), its weight w_3 ranges from 0 to 1 with a step of 0.01, and the other parameters are the same as those in Case 1 of this subsection. The final fusion results of the ER and GC approaches with different weights are shown in Figs. 7 and 8, respectivley. Fig. 7 shows that the uncertainty (global ignorance) in the combined result of ER approach remains at zero regardless of the value set as the weight of the evidence e_3, and the probability masses of focal elements changed to minor extents. Fig. 8 shows that the fusion results of combining e_3 in the GC approach may increase the uncertainty of the first two pieces of evidence when e_3 is completely unreliable but important. The larger the weight (importance) of e_3 is, the larger the uncertainty becomes. In extreme situations, the fusion results of combining e_3 with the GC approach are the same as those of the first two pieces of evidence when e_3 is completely unreliable and completely unimportant (r_3=0 and w_3=0). The fusion results of combining e_3 increase the uncertainty of the first two pieces of evidence to the greatest extent when e_3 is completely unreliable and the most important (r_3=0 and w_3=1). Such results are consistent with intuition, i.e., if the evidence makes no contribution to the decision (w_3=0), it will be dropped from the combination regardless of its reliability. Otherwise, if the evidence can make a contribution to the decision (w_3>0) but it is completely unreliable, the uncertainty of the combined result may increase. Consequently, we believe that the GC fusion results are superior to the ER fusion results in this case.

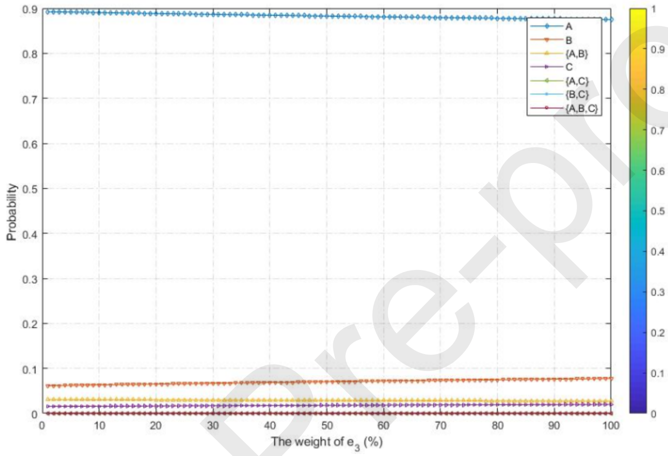

Fig.7 Final fusion results of ER for $w_3 \in [0,1]$

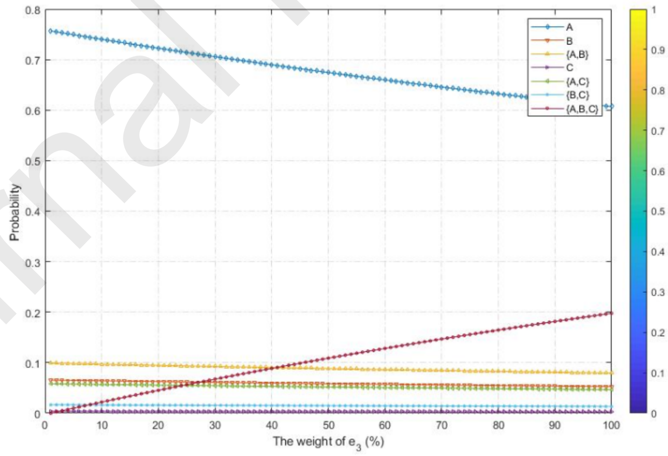

Fig.8 Final fusion results of GC for $w_3 \in [0,1]$

In a more extreme situation, each piece of evidence is completely unreliable, i.e., $r_1 = r_2 = r_3 = 0$, and the weights of all of the evidence are the same as those in Case 1 of this subsection. The combination results of the ER and GC approaches are listed in Tables 12 and 13, respectively. As shown in Table 12, the final fusion results of the ER are $p_{WBDR,e(3)}$. This means that each of the focal elements may occur with precise probabilities, and A has the largest probability (0.4679) of occurring. As shown in Table 13, the final fusion results of the GC approach are $p_{GC,e(3)}$, and this means that A, B, or C may occur, but we do not know which is correct. In other words, we obtain nothing from the fusion results of e_1, e_2, and e_3. Generally, any useful information is incapable of being obtained from completely unreliable evidence regardless of the weights. However, the ER can obtain some effective information from completely unreliable evidence. Based on our intuition, if all of the pieces of evidence are completely unreliable, then we can obtain nothing from them. In other words, if one obtains a result from all of the completely unreliable evidence, it must be incorrect. As a result, it is reasonable to believe that the GC fusion results are superior to the ER fusion results.

Table 12 The ER combination results with r_1=r_2=r_3=0

	A	B	C	{A,B}	{A,C}	{B,C}	{A,B,C}	P(Θ)
$m_{WBDR,1}$	0.3294	—	—	0.0412	0.0412	—	—	0.5882
$m_{WBDR,2}$	0.1143	0.0857	—	0.0571	—	0.0286	—	0.7143
$m_{WBDR,3}$	0.0444	0.1333	0.2222	—	—	—	0.0444	0.5556
$m_{WBDR,e(2)}$	0.3867	0.0575	0.0012	0.0682	0.0307	0.0175	—	0.4382
$m_{WBDR,e(3)}$	0.3302	0.1355	0.1319	0.0495	0.0223	0.0127	0.0236	0.2943
$p_{WBDR,e(3)}$	0.4679	0.1920	0.1869	0.0701	0.0316	0.0180	0.0334	—

Table 13 The GC combination results with r_1=r_2=r_3=0

	A	B	C	{A,B}	{A,C}	{B,C}	{A,B,C}	P(Θ)
$m_{GC,1}$	—	—	—	—	—	—	0.7000	0.3000
$m_{GC,2}$	—	—	—	—	—	—	0.4000	0.6000
$m_{GC,3}$	—	—	—	—	—	—	0.8000	0.2000
$m_{GC,e(2)}$	—	—	—	—	—	—	0.8200	0.1800
$m_{GC,e(3)}$	—	—	—	—	—	—	0.9640	0.0360
$p_{GC,e(3)}$	—	—	—	—	—	—	1.0000	0.0000

6. Illustrative example

The protection and sustainable use of China's marine biological resources have become increasingly urgent due to the decline of offshore fishery resources and the worsening of the ecological environment. Marine ranching is rapidly growing in China. This is considered to be a sustainable fishery mode that is ecofriendly for fisheries, aquaculture, and capture-based aquaculture [13,15]. By the end of 2019, China had built more than 233 marine ranches, including 110 national marine ranching demonstration zones (MRDZs), and it had released more than 60.94 million air cubic meters of reefs [50]. Marine ranching takes ecological security as the core objective in all construction, production, and recreational activities. To ensure a good ecological environment, abundant biological resources, and sustainable fishery development, such activities should not damage the integrity of the ecological environment and biological resources.

To achieve the core objective of marine ranching, it is important to evaluate the ecological security of MRDZs. We suppose that the government plans to evaluate a specific MRDZ (called MRDZ-A). Fig. 9 shows the evaluation framework and the decision information, consisting of the evaluation criteria system, evidence reliability and weight, and evidence source (experts). This relevant decision information will be described hereafter. This is the first time to evaluate the ecological security of MRDZ-A, and the collected data are inadequate. Thus, uncertainties exist in the evaluation process. The GC rule proposed in this paper is employed to evaluate the ecological security of MRDZ-A. This demonstrates the process of using the GC rule to solve a real-world problem under uncertainty. According to the steps in Subsection 4.5, the decision-making process is as follows.

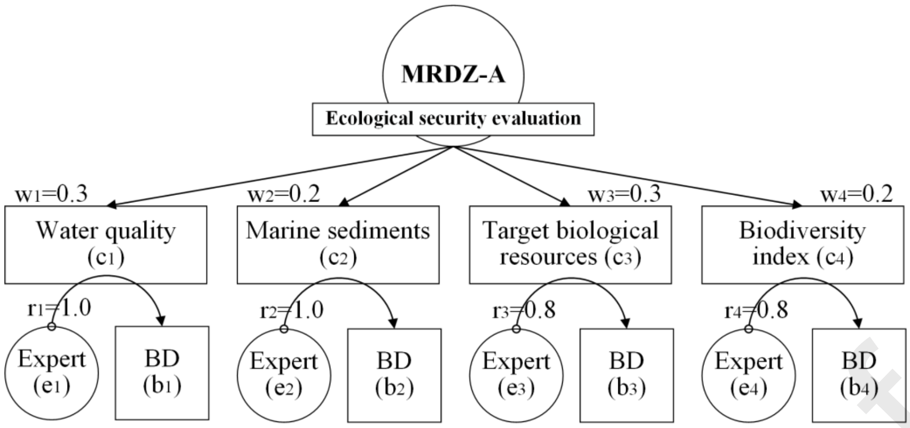

Fig.9 Evaluation framework for MRDZ-A

Step 1: Generate BDs by evidence sources. The ecological security of MRDZs is evaluated from the two aspects of marine environment and biological resources. The former is reflected by the water quality (c_1) and marine sediment (c_2) while the latter is reflected by the target biological resources (c_3) and biodiversity index (c_4). Thus, the evaluation criteria system is constructed as C={c_1,c_2,c_3,c_4}. The performance of MRDZ-A for criterion c_i is evaluated by experts e_i, $i=1,\cdots,4$. Five grades—Excellent (E), Good (G), Average (A), Poor (P), and Worst (W)—are used to express the evaluation information. Thus, the frame of discernment is constructed as $\Theta=\{\theta_1,\cdots,\theta_5\}$ ={W, P, A, G, E}. Expert e_i gives the evaluation information with the BDs as in Eq. (23), and each b_i can be considered to be a piece of evidence.

$$\begin{cases} b_1 = \{E, 0.2; A, 0.4; (P, G), 0.3; \Theta, 0.1\} \\ b_2 = \{(A, G), 0.6; (P, E), 0.4\} \\ b_3 = \{P, 0.5; (A, G), 0.5\} \\ b_4 = \{G, 0.2; (W, P), 0.3; A, 0.3; \Theta, 0.2\} \end{cases} \tag{23}$$

Step 2: Set the reliability and weight of each piece of evidence. The reliability and weight of evidence b_i are equal to the reliability of expert e_i and the weight of criterion c_i, respectively. Suppose experts e_1 and e_2 can make fully correct judgments, and e_3 and e_4 can make judgments with 80% correct information. The reliabilities of the experts can thus be obtained by their capacities to give valid information, which are set as r_1=r_2=1.0, r_3=r_4=0.8. In addition, regarding the ecological security of MRDZ-A, the water quality (c_1) and target biological resources (c_3) are considered to be slightly more important than the marine sediment (c_2) and biodiversity index (c_4). Thus, the weights of the criteria were set as w_1=w_3=0.3, and w_2=w_4=0.2.

Step 3: Combine all evidence with reliabilities and weights recursively.

First, each BD is discounted by the generalized discounting method given by Eq. (11), and we obtain the corresponding probability masses. For example, the probability masses discounted for b_1 are calculated as $m_1(E)=w_1r_1p_1(E)=0.3\times1.0\times0.2=0.06$, $m_1(A)=w_1r_1p_1(A)=0.3\times1.0\times0.4=0.12$, $m_1((P,G))=w_1r_1p_1((P,G))=0.3\times1.0\times0.3=0.09$, $m_1(\Theta)=w_1r_1p_1(\Theta)+w_1(1-r_1)=0.3\times1.0\times0.1+0.3\times(1.0-1.0)=0.03$, and $m_1(P(\Theta))=1-w_1=1-0.3=0.70$. The probability masses (m_i, $i=1,\cdots,4\}$) corresponding to all the BDs are obtained as follows:

$$\begin{cases} m_1 = \{E, 0.06; A, 0.12; (P, G), 0.09; \Theta, 0.03; P(\Theta), 0.70\} \\ m_2 = \{(A, G), 0.12; (P, E), 0.08; P(\Theta), 0.80\} \\ m_3 = \{P, 0.12; (A, G), 0.12; \Theta, 0.06; P(\Theta), 0.70\} \\ m_4 = \{G, 0.032; (W, P), 0.048; A, 0.048; \Theta, 0.072; P(\Theta), 0.80\} \end{cases} \tag{24}$$

Second, the probability masses of all of the pieces of evidence are combined by the recursive GC rule in Eqs. (18a)–(18d), and the combined probability masses are determined. The recursive combination process is as follows:

$m_{e(2)} = m_1 \oplus m_2 =$ $(E,0.0537;A,0.1123;G,0.0110;P,0.0073;(P,G),0.0732;(A,G),0.0891;(P,E),0.0594;\Theta,0.0244;P(\Theta),0.5696)$

$m_{e(3)} = m_{e(2)} \oplus m_3 =$ $(E,0.0428;A,0.1036;G,0.0194;P,0.0982;(P,G),0.0584;(A,G),0.1570;(P,E),0.0473;\Theta,0.0553;P(\Theta),0.4181)$

$m_{e(4)} = m_{e(3)} \oplus m_4 =$ $(E,0.0388;A,0.1305;G,0.0411;P,0.0991;(P,G),0.0529;(A,G),0.1422;(P,E),0.0429;(W,P),0.0236;\Theta,0.0814;P(\Theta),0.3475)$

Third, the probability mass $m_{P(\Theta),e(4)}$ is reassigned to all of the focal elements of Θ by Eq. (19), and the final combined BD is obtained as follows:

$p_{e(4)} = \{E,0.0594;A,0.2000;G,0.0630;P,0.1519;(P,G),0.0810;(A,G),0.2180;(P,E),0.0657;(W,P),0.0362;\Theta,0.1247\}$

Step 4: Make a decision using a specified principle. Inserting $p_{e(4)}$ into $BetP(\theta_n) = \sum_{\theta_n \in \theta \subset \Theta} (p_{\theta,e(l)} / |\theta|)$, we calculate the pignistic probabilities of each hypothesis as $BetP(W) = 0.0430$, $BetP(P) = 0.2684$, $BetP(A) = 0.3339$, $BetP(G) = 0.2374$, and $BetP(E) = 0.1172$. The combined result shows that the ecological security of MRDZ-A has a 4.30% probability of being Worst, a 26.84% probability of being Poor, a 33.39% probability of being Average, a 23.74% probability of being Good, and a 11.72% probability of being Excellent. Clearly, Average is the hypothesis with the highest pignistic probability. Thus, the ecological security of MRDZ-A is ultimately evaluated as Average.

Based on Steps 1–4 above, we can conclude the following. (1) The evaluation information can be described with BDs, and it can reflect the local or global ignorance that exists in the decision. (2) The quality and importance of the evaluation information, which can be described by reliability and weight, are both allowed to participate in the decision making. (3) The evaluation information with reliability and weight can be easily combined with the GC rule, and pignistic probability can be computed to make the final decision. The proposed GC rule-based decision-making process is therefore effective and practical for solving real-world problems.

7. Conclusions

DS theory is a flexible and useful tool for expressing and combining uncertain information with ignorance, but it cannot distinguish the weight and the reliability of the evidence. As an extension of DS theory, ER with weight combines evidence with the bounded sum of the individual support and the orthogonal sum of the collective support. However, the most recent version (ER with both weight and reliability) has three infeasible aspects. In this study, a GC rule with both weight and reliability was proposed that can solve the problems related to the parameters in the ER approach and DS theory. ER and DS can be seen as two particular cases of the GC rule. The present study has four main contributions, which are described below.

First, the three infeasible aspects of the ER (i.e., reliability dependence, unreliability effectiveness, and intergeneration inconsistency) were analyzed in terms of the extreme values of the two parameters. Second, a generalized discounting method with weight and reliability was introduced based on the properties of the two parameters. Third, a GC rule consisting of basic and recursive forms was established to combine evidence using reliabilities and weights, and the corresponding theorems and corollaries were provided. Finally, the proposed GC rule was compared with DS and ER to show its superiority, and it was also applied to a real-world example to demonstrate its applicability.

This study provides significant insight that evidence fusion should consider not only objective information quality but also subjective information importance. The fusion result will only be reasonable and effective if both perspectives are scientifically embodied and reflected in the process of evidence fusion. Otherwise, the intuition paradox may arise in the fusion results.

As mentioned in Section 2, Dempster's rule plays a crucial role in DS theory. However, the intuition paradox may arise in situations with high or low conflict, and the combination results may be counterintuitive. In our view, the intuition paradox arises because either the objective or subjective feature of the evidence is not well considered in the combinations. High conflict between evidence may be caused by the objective reliability of the evidence source. In this situation, the intuition paradox can be partly solved by discounting the evidence with the reliability to obtain completely reliable evidence and then make a combination using Dempster's rule. Such treatment assumes that each piece of evidence has equal importance to the most extreme degree.

Even if all of the pieces of evidence are completely reliable, the importance degree of each should also be considered in the combination to solve a specific decision-making problem. A "one-vote veto" is necessary for evidence with the most importance in the combination, but it is not necessary for evidence with the least importance. For example, the unqualified appearance that is seen as a completely reliable evidence can directly decide an actor will not pass the interview in the selection of film stars, while such a decision may not occur in other regular interviews, since this piece of evidence is very important in the former situation but is unimportant in the latter situation. Consequently, it is necessary to take both the objective quality and the subjective importance of the evidence into account in the combination. Only when both aspects are well considered in the combination can we obtain a satisfying result that conforms to intuition. After all, the so-called intuition paradox belongs to the category of subjective cognition. The proposed GC rule is an effective attempt to solve the intuition paradox.

In the era of big data, evidence with different weights and reliabilities can be easily obtained through information technology. The proposed GC rule may provide an alternative way to solve the fusion problems that arise from big data. It should be noted that the weight and the reliability of evidence in the proposed GC rule are required to be crisp values. Therefore, a valuable direction for future research would be to further study the GC rule using uncertain weights or reliabilities.

Acknowledgment

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Appendix A. Proof of theorem and corollary

Appendix A.1. Proof of Theorem 1

Proof. There is $\sum_{\theta\subseteq\Theta} m_{\theta,i} = \sum_{\theta\subset\Theta} m_{\theta,i} + m_{\Theta,i} = \sum_{\theta\subset\Theta} w_i r_i p_{\theta,i} + [w_i r_i p_{\Theta,i} + w_i(1-r_i)] = w_i r_i \sum_{\theta\subseteq\Theta} p_{\theta,i} + w_i - w_i r_i$. Because $\sum_{\theta\subseteq\Theta} p_{\theta,i} = 1$, we have $\sum_{\theta\subseteq\Theta} m_{\theta,i} = w_i r_i + w_i - w_i r_i = w_i$ and thus $\sum_{\theta\subseteq\Theta} m_{\theta,i} + m_{P(\Theta),i} = w_i + 1 - w_i = 1$.

Appendix A.2. Proof of Corollary 1

Proof. Inserting $w_i = 1.0$ into Eq. (11), we have $m_i(\theta) = w_i r_i p_{\theta,i} = r_i p_{\theta,i}$ for $\theta \subset \Theta$, $m_i(\theta) = w_i r_i p_{\theta,i} + w_i(1-r_i) = r_i p_{\theta,i} + 1 - r_i$ for $\theta = \Theta$, $m_i(\theta) = 1 - w_i = 0$ for $\theta = P(\Theta)$. Obviously, the discounted result is equivalent to the form of Shafer's discounting as in Eq. (4).

Appendix A.3. Proof of Corollary 2

Proof. Inserting $r_i = 1.0$ into Eq. (11), we have $m_i(\theta) = w_i r_i p_{\theta,i} = w_i p_{\theta,i}$ for $\theta \subset \Theta$, $m_i(\theta) = w_i r_i p_{\theta,i} + w_i(1-r_i) = w_i p_{\theta,i}$ for $\theta = \Theta$, $m_i(\theta) = 1 - w_i$ for $\theta = P(\Theta)$. The first two results could be rewritten $m_i(\theta) = w_i p_{\theta,i}$ for $\theta \subseteq \Theta$. Obviously, the discounted result is equivalent to the form of ER's discounting with weight as in Eq. (6).

Appendix A.4. Proof of Theorem 2

Proof. Inserting Eqs. (13a)-(13c) into the following expression, we have $\sum_{\theta\subseteq\Theta} m_{\theta,e(2)} + m_{P(\Theta),e(2)} = \sum_{\theta\subset\Theta} m_{\theta,e(2)} + m_{\Theta,e(2)} + m_{P(\Theta),e(2)} = \gamma(\sum_{\theta\subset\Theta} \widehat{m}_{\theta,e(2)} + \widehat{m}_{\Theta,e(2)} + \widehat{m}_{P(\Theta),e(2)})$. Because of $\gamma = 1/(1-k) = 1/(1-\widehat{m}_{\varnothing,e(2)})$, we have $\sum_{\theta\subseteq\Theta} m_{\theta,e(2)} + m_{P(\Theta),e(2)} = (\sum_{\theta\subset\Theta} \widehat{m}_{\theta,e(2)} + \widehat{m}_{\Theta,e(2)} + \widehat{m}_{P(\Theta),e(2)}) / (1-\widehat{m}_{\varnothing,e(2)})$. Because of $\widehat{m}_{\varnothing,e(2)} + \sum_{\theta\subset\Theta} \widehat{m}_{\theta,e(2)} + \widehat{m}_{\Theta,e(2)} + \widehat{m}_{P(\Theta),e(2)} = 1$, we have $\sum_{\theta\subseteq\Theta} m_{\theta,e(2)} + m_{P(\Theta),e(2)} = (1-\widehat{m}_{\varnothing,e(2)})/(1-\widehat{m}_{\varnothing,e(2)}) = 1$.

Appendix A.5. Proof of Theorem 3

Proof. From Eq. (11), the discounted results on $P(\Theta)$ of two pieces of evidence are $m_{P(\Theta),1} = 1 - w_1$ and $m_{P(\Theta),2} = 1 - w_2$. Inserting the expressions of $m_{P(\Theta),1}$ and $m_{P(\Theta),2}$ which are discounted by Eq. (11) into Eq. (12b) and Eq. (12c), we get $\widehat{m}_{\theta,e(2)} = [\sum_{B\cap C=\theta,B,C\subset\Theta} m_{B,1}m_{C,2} + m_{\theta,1}m_{\Theta,2} + m_{\theta,2}m_{\Theta,1}] + [(1-w_2)m_{\theta,1} + (1-w_1)m_{\theta,2}]$ for $\theta \subset \Theta$, $\widehat{m}_{\Theta,e(2)} = m_{\Theta,1}m_{\Theta,2} + [(1-w_1)m_{\Theta,2} + (1-w_2)m_{\Theta,1}]$. From Theorem 2, since $\sum_{\theta\subseteq\Theta} m_{\theta,e(2)} + m_{P(\Theta),e(2)} = 1$ for $i = 1,2$, we have $\sum_{\theta\subseteq\Theta} m_{\theta,e(2)} = 1 - m_{P(\Theta),e(2)}$. From Eqs. (13a)-(13b) we have $m_{\theta,e(2)} = \widehat{m}_{\theta,e(2)}/(1-\widehat{m}_{\varnothing,e(2)})$ for $\theta \subset \Theta$ and $\theta = \Theta$. Thus the Eq. (14) can be expressed as:

$$p_{\theta,e(2)} = \frac{m_{\theta,e(2)}}{1-m_{P(\Theta),e(2)}} = \frac{m_{\theta,e(2)}}{\sum_{\theta\subset\Theta} m_{\theta,e(2)} + m_{\Theta,e(2)}} = \frac{\gamma\widehat{m}_{\theta,e(2)}}{\gamma\sum_{\theta\subset\Theta}\widehat{m}_{\theta,e(2)} + \gamma\widehat{m}_{\Theta,e(2)}} = \frac{\widehat{m}_{\theta,e(2)}}{\sum_{\theta\subseteq\Theta}\widehat{m}_{\theta,e(2)}}, \quad \theta \subseteq \Theta$$

Appendix A.6. Proof of Corollary 3

Proof. Inserting $m_{\theta,i}(\theta \subset \Theta, \theta = \Theta, \theta = P(\Theta))$ as in Eq. (11) into Eqs. (15b)-(15c), we derive

$$\begin{aligned}\widehat{m}_{\theta,e(2)} = &\sum_{B\cap C=\theta,B,C\subset\Theta} w_1 r_1 p_{B,1} \times w_2 r_2 p_{C,2} + w_1 r_1 p_{\theta,1} \times [w_2 r_2 p_{\Theta,2} + w_2(1-r_2)] + w_2 r_2 p_{\theta,2} \\ &\times [w_1 r_1 p_{\Theta,1} + w_1(1-r_1)] + (1-w_2) w_1 r_1 p_{\theta,1} + (1-w_1) w_2 r_2 p_{\theta,2}, \quad \theta \subset \Theta \end{aligned} \quad (A.1)$$

$$\widehat{m}_{\Theta,e(2)} = [w_1 r_1 p_{\Theta,1} + w_1(1-r_1)] \times [w_2 r_2 p_{\Theta,2} + w_2(1-r_2)] + (1-w_1)[w_2 r_2 p_{\Theta,2} + w_2(1-r_2)] + (1-w_2)[w_1 r_1 p_{\Theta,1} + w_1(1-r_1)] \quad (A.2)$$

When $r_1 = r_2 = 1$, $\widehat{m}_{\theta,e(2)}$ and $\widehat{m}_{\Theta,e(2)}$ as in Eqs. (A.1)-(A.2) can be expressed as

$$\widehat{m}_{\theta,e(2)} = \sum_{B\cap C=\theta,B,C\subset\Theta} w_1 p_{B,1} \times w_2 p_{c,2} + w_1 p_{\theta,1} \times w_2 p_{\Theta,2} + w_2 p_{\theta,2} \times w_1 p_{\Theta,1} + (1-w_2) w_1 p_{\theta,1} + (1-w_1) w_2 p_{\theta,2}, \quad \theta \subset \Theta \quad (A.3)$$

$$\widehat{m}_{\Theta,e(2)} = w_1 p_{\Theta,1} \times w_2 p_{\Theta,2} + (1-w_1) \times w_2 p_{\Theta,2} + (1-w_2) \times w_1 p_{\Theta,1} \quad (A.4)$$

Since $m^w_{\theta,i} = w_i p_{\theta,i}$ for $\theta \subseteq \Theta$, Eqs.(A.3) and (A.4) can be expressed as

$$\widehat{m}_{\theta,e(2)} = [\sum_{B\cap C=\theta,B,C\subset\Theta} m^w_{B,1} m^w_{C,2} + m^w_{\theta,1} m^w_{\Theta,2} + m^w_{\theta,2} m^w_{\Theta,1}] + [(1-w_2) m^w_{\theta,1} + (1-w_1) m^w_{\theta,2}]$$

$$=\sum_{B\cap C=\theta,\, B,C\subseteq\Theta} m^w_{B,1}m^w_{C,2} \ +[(1-w_2)m^w_{\theta,1}+(1-w_1)m^w_{\theta,2}\,,\quad \theta\subset\Theta \quad (A.5)$$

$$\hat{m}_{\Theta,e(2)}=m^w_{\Theta,1}m^w_{\Theta,2}+[(1-w_1)m^w_{\Theta,2}+(1-w_2)m^w_{\Theta,1}]=\sum_{B,C=\Theta} m^w_{B,1}m^w_{C,2}+[(1-w_2)m^w_{\theta,1}+(1-w_1)m^w_{\theta,2}\,,\quad \theta{=}\Theta \quad (A.6)$$

Inserting both into unified forms, we have $\hat{m}_{\theta,e(2)}{=}[(1-w_2)m^w_{\theta,1}+(1-w_1)m^w_{\theta,2}]{+}\sum_{B\cap C=\theta,B,C\subseteq\Theta} m^w_{B,1}m^w_{C,2}$, $\theta\subseteq\Theta$.

Appendix A.7. Proof of Corollary 4

Proof. Inserting $m_{\theta,i}(\theta\subset\Theta,\ \theta{=}\Theta,\theta{=}P(\Theta))$ as in Eq. (11) into Eqs. (15b)-(15c), we get $\hat{m}_{\theta,e(2)}$ and $\hat{m}_{\Theta,e(2)}$ as in Eqs. (A.1)-(A.2). Inserting $w_1=w_2=1$ into Eqs. (A.1)-(A.2), we have

$$\hat{m}_{\theta,e(2)}=\sum_{B\cap C=\theta,B,C\subset\Theta} r_1p_{B,1}\times r_2p_{c,2}+r_1p_{\theta,1}\times[r_2p_{\Theta,2}+(1-r_2)]+r_2p_{\theta,2}\times[r_1p_{\Theta,1}+(1-r_1)]\,,\quad \theta\subset\Theta \quad (A.7)$$

$$\hat{m}_{\Theta,e(2)}=[r_1p_{\Theta,1}+(1-r_1)]\times[r_2p_{\Theta,2}+(1-r_2)]\,,\theta{=}\Theta \quad (A.8)$$

Since $m^r_{\theta,i}{=}r_ip_{\theta,i}$ for $\theta\subset\Theta$, $m^r_{\Theta,i}{=}r_ip_{\Theta,i}+1-r_i$, Eqs. (A.7) and (A.8) can be expressed as

$$\hat{m}_{\theta,e(2)}=\sum_{B\cap C=\theta,B,C\subset\Theta} m^r_{B,1}m^r_{C,2}+m^r_{\theta,1}m^r_{\Theta,2}+m^r_{\theta,2}m^r_{\Theta,1}=\sum_{B\cap C=\theta,B,C\subseteq\Theta} m^r_{B,1}m^r_{C,2}\,,\quad \theta\subset\Theta \quad (A.9)$$

$$\hat{m}_{\Theta,e(2)}=m^r_{\Theta,1}m^r_{\Theta,2}=\sum_{B,C=\Theta} m^w_{B,1}m^w_{C,2} \quad (A.10)$$

Inserting the both into unified forms, we determine $\hat{m}_{\theta,e(2)}{=}\sum_{B\cap C=\theta,B,C\subseteq\Theta} m^r_{B,1}m^r_{C,2}$, $\theta\subseteq\Theta$.

Appendix A.8. Proof of Corollary 5

Proof. Inserting $m_{\theta,i}(\theta\subset\Theta,\ \theta{=}\Theta,\theta{=}P(\Theta))$ as in Eq. (11) into Eqs. (15b)-(15c), we get $\hat{m}_{\theta,e(2)}$ and $\hat{m}_{\Theta,e(2)}$ as in Eqs. (A.1)-(A.2). Inserting $r_1=r_2=0$ into Eqs. (A.1)-(A.2), we get $\hat{m}_{\theta,e(2)}{=}0\,,\theta\subset\Theta$; $\hat{m}_{\Theta,e(2)}=w_1w_2+\ (1-w_1)w_2{+}(1-w_2)w_1{=}w_1{+}w_2-w_1w_2$. Inserting the above results into Eq. (15a), we determine $p_{\theta,e(2)}{=}0/(0{+}w_1{+}w_2-w_1w_2)=0\,,\theta\subset\Theta$; $p_{\Theta,e(2)}{=}(w_1{+}w_2-w_1w_2)/(0+w_1{+}w_2-w_1w_2){=}1$.

Appendix A.9. Proof of Theorem 4

Proof. For $i=2$, $m_{\theta,e(1)}=m_{\theta,1}$ for $\theta\subset\Theta$, $\theta{=}\Theta$ and $\theta{=}P(\Theta)$, and thus Eqs. (18b)-(18d) are obviously equivalent to Eqs. (12b)-(12d). Meanwhile, we also have $\hat{m}_{\varnothing,e(2)}{=}\sum_{B\bigcap C=\varnothing} m_{B,1}m_{C,1}$ as shown in Eq. (12a) and $\hat{m}_{\varnothing,e(2)}{+}\sum_{\theta\subset\Theta}\hat{m}_{\theta,e(2)}{+}\hat{m}_{\Theta,e(2)}{+}\hat{m}_{P(\Theta),e(2)}{=}1$ which can be rewritten by $\sum_{\theta\subset\Theta}\hat{m}_{\theta,e(2)}{+}\hat{m}_{\Theta,e(2)}{+}\hat{m}_{P(\Theta),e(2)}{=}1-\hat{m}_{\varnothing,e(2)}$. Because $\hat{m}_{\varnothing,e(2)}$ is defined to be zero and it should be reassigned into other parts, we have $m_{\theta,e(2)}{=}\hat{m}_{\theta,e(2)}/1-\hat{m}_{\varnothing,e(2)}=\hat{m}_{\theta,e(2)}/(\sum_{\theta\subset\Theta}\hat{m}_{\theta,e(2)}{+}\hat{m}_{\Theta,e(2)}{+}\hat{m}_{P(\Theta),e(2)})=\hat{m}_{\theta,e(2)}/(\sum_{\vartheta\subseteq\Theta}\hat{m}_{\vartheta,e(2)}+\hat{m}_{P(\Theta),e(2)})$ for $\theta\neq\varnothing$ and $m_{\varnothing,e(2)}{=}0$ which are shown as in Eq. (18a). From Theorem 2, we know $\sum_{\theta\subseteq\Theta} m_{\theta,e(2)}+m_{P(\Theta),e(2)}=1$.

For $i=i-1$, suppose that Eqs. (18a)-(18d) are true and it means that there exists $m_{\theta,e(i\text{-}1)}=[m_1\oplus\cdots\oplus m_{i\text{-}1}](\theta)$, with $0\le m_{\theta,e(i\text{-}1)}\le 1$, for $\theta\subset\Theta$, $\theta{=}\Theta$, $\theta{=}P(\Theta)$ and $\sum_{\theta\subseteq\Theta} m_{\theta,e(i-1)}{+}m_{P(\Theta),e(i-1)}{=}1$.

For $i=i$, we should make combinations for the first $i-1$ pieces of evidence with the i^{th} one by $m_{\theta,e(i)}=m_{\theta,e(i-1)}\oplus m_{\theta,i}$. Similar to Eqs. (12a)-(12d), the orthogonal sum of $m_{\theta,e(i-1)}$ and $m_{\theta,i}$ without normalization can be expressed as follows.

$$\hat{m}_{\varnothing,e(i)}{=}\sum_{B\bigcap C=\varnothing} m_{B,e(i-1)}m_{C,i}$$

$$\hat{m}_{\theta,e(i)}{=}[\sum_{B\cap C=\theta,B,C\subset\Theta} m_{B,e(i-1)}m_{C,i}+m_{\theta,e(i-1)}m_{\Theta,i}+m_{\theta,i}m_{\Theta,e(i-1)}]+[(1-w_i)m_{\theta,e(i-1)}+m_{P(\Theta),e(i-1)}m_{\theta,i}]$$

$$\hat{m}_{\Theta,e(i)}=m_{\Theta,e(i-1)}m_{\Theta,i}+[m_{P(\Theta),e(i-1)}m_{\Theta,i}+(1-w_i)m_{\Theta,e(i-1)}]$$

$$\hat{m}_{P(\Theta),e(i)}{=}(1-w_i)m_{P(\Theta),e(i-1)}$$

Since $\sum_{\theta\subset\Theta}\hat{m}_{\theta,e(i)}{+}\hat{m}_{\Theta,e(i)}{+}\hat{m}_{P(\Theta),e(i)}{=}1-\hat{m}_{\varnothing,e(i)}$, we have $m_{\theta,e(i)}{=}\hat{m}_{\theta,e(i)}/1-\hat{m}_{\varnothing,e(i)}=\hat{m}_{\theta,e(i)}/(\sum_{\theta\subset\Theta}\hat{m}_{\theta,e(i)}{+}\hat{m}_{\Theta,e(i)}{+}\hat{m}_{P(\Theta),e(i)})$ $=\hat{m}_{\theta,e(i)}/(\sum_{\vartheta\subseteq\Theta}\hat{m}_{\vartheta,e(i)}+\hat{m}_{P(\Theta),e(i)})$ for $\theta\neq\varnothing$ and $m_{\varnothing,e(i)}{=}0$ which are shown as in Eq. (18a). In addition, $\sum_{\theta\subseteq\Theta} m_{\theta,e(i)}{+}m_{P(\Theta),e(i)}=(\sum_{\theta\subseteq\Theta}\hat{m}_{\theta,e(i)}+\hat{m}_{P(\Theta),e(i)})/(1-\hat{m}_{\varnothing,e(i)})=(\sum_{\theta\subseteq\Theta}\hat{m}_{\theta,e(i)}+\hat{m}_{P(\Theta),e(i)})/(\sum_{\theta\subseteq\Theta}\hat{m}_{\theta,e(i)}{+}\hat{m}_{P(\Theta),e(i)})=1.$

Appendix A.10. Proof of Theorem 5

Proof. From the proof of Theorem 4, it is known that $\sum_{\theta\subseteq\Theta} m_{\theta,e(i)} + m_{P(\Theta),e(i)} = 1$ and $m_{\theta,e(i)} = \hat{m}_{\theta,e(i)} / (1-\hat{m}_{\varnothing,e(i)}) = \hat{m}_{\theta,e(i)} / (\sum_{\vartheta\subseteq\Theta} \hat{m}_{\vartheta,e(i)} + \hat{m}_{P(\Theta),e(i)})$ for $\theta \neq \varnothing$, $i = 1,\cdots,I$. Because $\sum_{\theta\subseteq\Theta} m_{\theta,e(I)} + m_{P(\Theta),e(I)} = 1$, the final combined BD of I pieces of evidence can be determined by reassigning the probability masses of power set back to focal elements, i.e., $p_{\theta,e(I)} = m_{\theta,e(I)} / (1-m_{p(\Theta),e(I)}) = m_{\theta,e(I)} / \sum_{\theta\subseteq\Theta} m_{\theta,e(I)}$ for $\theta \subseteq \Theta, \theta \neq \varnothing$. In addition, from Eqs. (13a)-(13c) we know $m_{\theta,e(i)} = \gamma \hat{m}_{\theta,e(i)}$ for $\theta \neq \varnothing$, $\gamma = 1/(1-\hat{m}_{\varnothing,e(i)})$, $i = 1,\cdots,I$, thus we have $p_{\theta,e(I)} = m_{\theta,e(I)} / \sum_{\theta\subseteq\Theta} m_{\theta,e(I)} = \gamma\hat{m}_{\theta,e(I)} / \sum_{\theta\subseteq\Theta} \gamma\hat{m}_{\theta,e(I)} = \hat{m}_{\theta,e(I)} / \sum_{\theta\subseteq\Theta} \hat{m}_{\theta,e(I)}$ for $\theta \subseteq \Theta, \theta \neq \varnothing$. Obviously, $\sum_{\theta\subseteq\Theta} p_{\theta,e(I)} = \sum_{\theta\subseteq\Theta} (\hat{m}_{\theta,e(I)} / \sum_{\theta\subseteq\Theta} \hat{m}_{\theta,e(I)}) = \sum_{\theta\subseteq\Theta} \hat{m}_{\theta,e(I)} / \sum_{\theta\subseteq\Theta} \hat{m}_{\theta,e(I)} = 1$. Because $p_{\theta,e(I)} \geq 0$ and $\sum_{\theta\subseteq\Theta} p_{\theta,e(I)} = 1$, it is obvious to find that $0 \leq p_{\theta,e(I)} \leq 1, \forall\, \theta \subseteq \Theta$.

Appendix A.11. Proof of Corollary 6

Proof. For $i = 2$, $\hat{m}_{\theta,e(2)} = [(1-w_2)m^w_{\theta,1} + (1-w_1)m^w_{\theta,2}] + \sum_{B\cap C=\theta,B,C\subseteq\Theta} m^w_{B,1} m^w_{C,2}$ for $\theta \subseteq \Theta$ has been proved by Corollary 3. Because $m^w_{\theta,e(1)} = m^w_{\theta,1}$ and $m^w_{P(\Theta),e(1)} = m^w_{P(\Theta),1} = 1-w_1$, we take the both into the above expression of $\hat{m}_{\theta,e(2)}$ and thus Eq. (20a) can be obtained. Besides, for $i = 2$ we have $\hat{m}_{P(\Theta),e(2)} = (1-w_2)(1-w_1) = (1-w_2)m^w_{P(\Theta),1}$ $= (1-w_2)m^w_{P(\Theta),e(1)}$, and thus Eq. (20b) is obtained.

For $i = i-1$, we suppose that Eqs. (20a)-(20b) are true and it means that there exist

$$\hat{m}_{\theta,e(i-1)} = [(1-w_{i-1})m^w_{\theta,e(i-2)} + m^w_{P(\Theta),e(i-2)} m^w_{\theta,i-1}] + \sum_{B\cap C=\theta,B,C\subseteq\Theta} m^w_{B,e(i-2)} m^w_{C,i-1},\ \theta \subseteq \Theta \quad \text{(A.11)}$$

$$\hat{m}_{P(\Theta),e(i-1)} = (1-w_{i-1}) m^w_{P(\Theta),e(i-2)} \quad \text{(A.12)}$$

Note that, $m^w_{\theta,e(i-1)} = \hat{m}_{\theta,e(i-1)} / (\sum_{\vartheta\subseteq\Theta} \hat{m}_{\vartheta,e(i-1)} + \hat{m}_{P(\Theta),e(i-1)})$, for $\theta \subseteq \Theta$ and $\theta = P(\Theta)$.

For $i = i$, we combine $m_{\theta,i}$ as in Eq. (11) with $m^r_{\theta,e(i-1)}$ as above for $\theta \subset \Theta, \theta = \Theta, \theta = P(\Theta)$, and get $\hat{m}_{\theta,e(i)}$ ($\theta \subset \Theta$), $\hat{m}_{\Theta,e(i)}$ and $\hat{m}_{P(\Theta),e(i)}$ as follows.

$$\hat{m}_{\theta,e(i)} = \sum_{B\cap C=\theta,B,C\subset\Theta} m^w_{B,e(i-1)} w_i r_i p_{c,i} + m^w_{\theta,e(i-1)}[w_i r_i p_{\Theta,i} + w_i(1-r_i)] + w_i r_i p_{\theta,i} m^w_{\Theta,e(i-1)}$$
$$+[(1-w_i)m^w_{\theta,e(i-1)} + m^w_{P(\Theta),e(i-1)} w_i r_i p_{\theta,i}],\ \theta \subset \Theta \quad \text{(A.13)}$$

$$\hat{m}_{\Theta,e(i)} = m^w_{\Theta,e(i-1)}[w_i r_i p_{\Theta,i} + w_i(1-r_i)] + m^w_{P(\Theta),e(i-1)}[w_i r_i p_{\Theta,i} + w_i(1-r_i)] + (1-w_i)m^w_{\Theta,e(i-1)} \quad \text{(A.14)}$$

$$\hat{m}_{P(\Theta),e(i)} = m^w_{P(\Theta),e(i-1)}(1-w_i) \quad \text{(A.15)}$$

Since $r_i = 1$, we take it into Eqs. (A.13)-(A.15) and get Eqs. (A.16)-(A.18).

$$\hat{m}_{\theta,e(i)} = \sum_{B\cap C=\theta,B,C\subset\Theta} m^w_{B,e(i-1)} w_i p_{c,i} + m^w_{\theta,e(i-1)} w_i p_{\Theta,i} + w_i p_{\theta,i} m^w_{\Theta,e(i-1)} + [(1-w_i)m^w_{\theta,e(i-1)} + m^w_{P(\Theta),e(i-1)} w_i p_{\theta,i}],\ \theta \subset \Theta \quad \text{(A.16)}$$

$$\hat{m}_{\Theta,e(i)} = m^w_{\Theta,e(i-1)} w_i p_{\Theta,i} + m^w_{P(\Theta),e(i-1)} w_i p_{\Theta,i} + (1-w_i)m^w_{\Theta,e(i-1)} \quad \text{(A.17)}$$

$$\hat{m}_{P(\Theta),e(i)} = m^w_{P(\Theta),e(i-1)}(1-w_i) \quad \text{(A.18)}$$

Since $m^w_{\theta,i} = w_i p_{\theta,i}$ for $\theta \subseteq \Theta$, we take it into Eqs. (A.16)-(A.18) and get

$$\hat{m}_{\theta,e(i)} = \sum_{B\cap C=\theta,B,C\subset\Theta} m^w_{B,e(i-1)} m^w_{C,i} + m^w_{\theta,e(i-1)} m^w_{\Theta,i} + m^w_{\theta,i} m^w_{\Theta,e(i-1)} + [(1-w_i)m^w_{\theta,e(i-1)} + m^w_{P(\Theta),e(i-1)} m^w_{\theta,i}],\ \theta \subset \Theta \quad \text{(A.19)}$$

$$\hat{m}_{\Theta,e(i)} = m^w_{\Theta,e(i-1)} m^w_{\Theta,i} + m^w_{P(\Theta),e(i-1)} m^w_{\Theta,i} + (1-w_i)m^w_{\Theta,e(i-1)} \quad \text{(A.20)}$$

$$\hat{m}_{P(\Theta),e(i)} = m^w_{P(\Theta),e(i-1)}(1-w_i) \quad \text{(A.21)}$$

Since $\sum_{B\cap C=\theta,B,C\subset\Theta} m^w_{B,e(i-1)} m^w_{C,i} + m^w_{\theta,e(i-1)} m^w_{\Theta,i} + m^w_{\theta,i} m^w_{\Theta,e(i-1)} = \sum_{B\cap C=\theta,B,C\subseteq\Theta} m^w_{B,e(i-1)} m^w_{C,i}$ for $\theta \subset \Theta$, $m^w_{\Theta,e(i-1)} m^w_{\Theta,i} =$ $\sum_{B\cap C=\theta,B,C\subseteq\Theta} m^w_{B,e(i-1)} m^w_{C,i}$ for $\theta = \Theta$, Eqs. (A.19)-(A.21) can be simplified into Eqs. (20a)-(20b).

Appendix A.12. Proof of Corollary 7

Proof. For $i = 2$, it can be obtained from Eqs. (A.1)-(A.2) that

$$\hat{m}_{\theta,e(2)} = \sum_{B\cap C=\theta,B,C\subset\Theta} r_1 p_{B,1} \times r_2 p_{c,2} + r_1 p_{\theta,1} \times [r_2 p_{\Theta,2} + (1-r_2)] + r_2 p_{\theta,2} \times [r_1 p_{\Theta,1} + (1-r_1)],\ \theta \subset \Theta \quad \text{(A.22)}$$

$$\hat{m}_{\Theta,e(2)} = [r_1 p_{\Theta,1} + (1-r_1)] \times [r_2 p_{\Theta,2} + (1-r_2)],\ \theta = \Theta \quad \text{(A.23)}$$

Since $m^r_{\theta,i} = r_i p_{\theta,i}$ for $\theta \subset \Theta$, $m^r_{\Theta,i} = r_i p_{\Theta,i} + 1 - r_i$, $m^r_{\theta,e(i-1)} = \hat{m}_{\theta,e(i-1)} / (\sum_{\vartheta\subseteq\Theta} \hat{m}_{\vartheta,e(i-1)} + \hat{m}_{P(\Theta),e(i-1)})$ for $\theta \subseteq \Theta$

and $\theta = P(\Theta)$, Eqs. (A.22)-(A.23) can be simplified into

$$\widehat{m}_{\theta,e(2)} = \sum_{B\cap C=\theta,B,C\subset\Theta} m^r_{B,1}\times m^r_{C,2} + m^r_{\theta,1}\times m^r_{\Theta,2} + m^r_{\theta,2}\times m^r_{\Theta,1} = \sum_{B\cap C=\theta,B,C\subseteq\Theta} m^r_{B,1} m^r_{C,2}\ ,\quad \theta\subset\Theta \quad \text{(A.24)}$$

$$\widehat{m}_{\Theta,e(2)} = m^r_{\Theta,1}\times m^r_{\Theta,2}\ ,\quad \theta=\Theta \quad \text{(A.25)}$$

Obviously, Eqs. (A.24)-(A.25) can be transferred into a unified form as in Eq. (21a). Besides, there exist $\widehat{m}^r_{P(\Theta),e(2)} = (1-w_2)(1-w_1) = 0$ for the reason that $w_i = 1$ for $i = 1,2$.

For $i = i-1$, we suppose that Eqs. (21a)-(21b) are true and it means that there exist

$$\widehat{m}_{\theta,e(i-1)} = \sum_{B\cap C=\theta,B,C\subseteq\Theta} m^r_{B,e(i-2)} m^r_{C,i-1}\ ,\quad \theta\subseteq\Theta \quad \text{(A.26)}$$

$$\widehat{m}^r_{P(\Theta),e(i-1)} = 0 \quad \text{(A.27)}$$

Note that, $m^r_{\theta,e(i-1)} = \widehat{m}_{\theta,e(i-1)} \Big/ \left(\sum_{\vartheta\subseteq\Theta}\widehat{m}_{\vartheta,e(i-1)} + \widehat{m}_{P(\Theta),e(i-1)}\right) = \widehat{m}_{\theta,e(i-1)} \Big/ \sum_{\vartheta\subseteq\Theta}\widehat{m}_{\vartheta,e(i-1)}$ for $\theta\subseteq\Theta$ and $m^r_{P(\Theta),e(i-1)} = 0$.

For $i = i$, we combine $m_{\theta,i}$ as in Eq.(11) with $m^r_{\theta,e(i-1)}$ as above for $\theta\subset\Theta, \theta=\Theta, \theta=P(\Theta)$, and get $\widehat{m}_{\theta,e(i)}$ ($\theta\subset\Theta$), $\widehat{m}_{\Theta,e(i)}$ and $\widehat{m}_{P(\Theta),e(i)}$ as follows.

$$\widehat{m}_{\theta,e(i)} = \sum_{B\cap C=\theta,B,C\subset\Theta} m^r_{B,e(i-1)} w_i r_i p_{C,i} + m^r_{\theta,e(i-1)}[w_i r_i p_{\Theta,i} + w_i(1-r_i)] + w_i r_i p_{\theta,i} m^r_{\Theta,e(i-1)} + [(1-w_i)m^r_{\theta,e(i-1)} + m^r_{P(\Theta),e(i-1)} w_i r_i p_{\theta,i}],$$

$$\theta\subset\Theta \quad \text{(A.28)}$$

$$\widehat{m}_{\Theta,e(i)} = m^r_{\Theta,e(i-1)}[w_i r_i p_{\Theta,i} + w_i(1-r_i)] + m^r_{P(\Theta),e(i-1)}[w_i r_i p_{\Theta,i} + w_i(1-r_i)] + (1-w_i)m^r_{\Theta,e(i-1)} \quad \text{(A.29)}$$

$$\widehat{m}_{P(\Theta),e(i)} = m^r_{P(\Theta),e(i-1)}(1-w_i) \quad \text{(A.30)}$$

Since $w_i = 1$ and $m^r_{P(\Theta),e(i-1)} = 0$, we take both into Eqs. (A.28)-(A.30) and get Eqs. (A.31)-(A.33).

$$\widehat{m}_{\theta,e(i)} = \sum_{B\cap C=\theta,B,C\subset\Theta} m^r_{B,e(i-1)} r_i p_{c,i} + m^r_{\theta,e(i-1)}[r_i p_{\Theta,i} + (1-r_i)] + r_i p_{\theta,i} m^r_{\Theta,e(i-1)}\ ,\quad \theta\subset\Theta \quad \text{(A.31)}$$

$$\widehat{m}_{\Theta,e(i)} = m^r_{\Theta,e(i-1)}[r_i p_{\Theta,i} + (1-r_i)] + m^r_{P(\Theta),e(i-1)}[r_i p_{\Theta,i} + (1-r_i)] \quad \text{(A.32)}$$

$$\widehat{m}_{P(\Theta),e(i)} = m^r_{P(\Theta),e(i-1)}(1-w_i) = 0 \quad \text{(A.33)}$$

Since $m^r_{\theta,i} = r_i p_{\theta,i}$ for $\theta\subset\Theta$ and $m^r_{\Theta,i} = r_i p_{\Theta,i} + 1 - r_i$, Eqs. (A.31)-(A.33) can be simplified into

$$\widehat{m}_{\theta,e(i)} = \sum_{B\cap C=\theta,B,C\subset\Theta} m^r_{B,e(i-1)} m^r_{C,i} + m^r_{\theta,e(i-1)} m^r_{\Theta,i} + m^r_{\theta,i} m^r_{\Theta,e(i-1)} = \sum_{B\cap C=\theta,B,C\subseteq\Theta} m^r_{B,e(i-1)} m^r_{C,i}\ ,\quad \theta\subset\Theta \quad \text{(A.34)}$$

$$\widehat{m}_{\Theta,e(i)} = m^r_{\Theta,e(i-1)} m^r_{\Theta,i} + m^r_{P(\Theta),e(i-1)} m^r_{\Theta,i} \quad \text{(A.35)}$$

$$\widehat{m}_{P(\Theta),e(i)} = m^r_{P(\Theta),e(i-1)}(1-w_i) = 0 \quad \text{(A.36)}$$

Obviously, Eqs. (A.34)-(A.35) can be transferred into a unified form as shown in Eq. (21a), and Eq. (A.36) is equal to Eq. (21b).

Appendix A.13. Proof of Corollary 8

Proof. From Theorem 5, there are $\sum_{\theta\subseteq\Theta} p_{\theta,e(I)} = 1$ and $0 \le p_{\theta,e(I)} \le 1$ for $\theta\subseteq\Theta$. If we want to prove that $p_{\Theta,e(I)} = 1$, we just need to prove $\sum_{\theta\subset\Theta} p_{\theta,e(I)} = 1 - p_{\Theta,e(I)} = 0$. Since $0 \le p_{\theta,e(I)} \le 1$ for $\theta\subseteq\Theta$, $\sum_{\theta\subset\Theta} p_{\theta,e(I)} = 0$ requires that $p_{\theta,e(I)} = 0$ for $\forall\theta\subset\Theta$. In addition, also from Eq. (19) we know that $p_{\theta,e(I)} = \widehat{m}_{\theta,e(I)} \Big/ \sum_{\vartheta\subseteq\Theta}\widehat{m}_{\vartheta,e(I)}$ for $\theta\subseteq\Theta$, it means that $p_{\theta,e(I)} = 0$ is equivalent to $\widehat{m}_{\theta,e(I)} = 0$ for $\forall\theta\subset\Theta$ and thus we just prove $\widehat{m}_{\theta,e(I)} = 0$ for $\forall\theta\subset\Theta$ as follows.

For $i = 2$, $p_{\Theta,e(I)} = 1$ has been proved by Corollary 5 and it is equivalent to $\widehat{m}_{\theta,e(2)} = 0$ for $\forall\theta\subset\Theta$ as mentioned above.

For $i = i-1$, we suppose that $p_{\Theta,e(i-1)} = 1$ for $\forall\theta\subset\Theta$ is true and it means that there exists $\widehat{m}_{\theta,e(i-1)} = 0$ for $\forall\theta\subset\Theta$. Note that, we also have $m_{\theta,e(i-1)} = \widehat{m}_{\theta,e(i-1)} \Big/ \left(\sum_{\vartheta\subseteq\Theta}\widehat{m}_{\vartheta,e(i-1)} + \widehat{m}_{P(\Theta),e(i-1)}\right) = 0$ for $\forall\theta\subset\Theta$.

For $i = i$, from Eq.(18b) we know that

$$\widehat{m}_{\theta,e(i)} = \sum_{B\cap C=\theta,B,C\subset\Theta} m_{B,e(i-1)} w_i r_i p_{C,i} + m_{\theta,e(i-1)}[w_i r_i p_{\Theta,i} + w_i(1-r_i)] + w_i r_i p_{\theta,i} m_{\Theta,e(i-1)} + [(1-w_i)m_{\theta,e(i-1)} + m_{P(\Theta),e(i-1)} w_i r_i p_{\theta,i}],\ \theta\subset\Theta.$$

Since $m_{\theta,e(i-1)} = 0$ for $\forall\theta\subset\Theta$ and $r_i = 0$, we find that $\widehat{m}_{\theta,e(i)} = 0$ for $\forall\theta\subset\Theta$. When $i = I$, there also exists $\widehat{m}_{\theta,e(I)} = 0$ for $\forall\theta\subset\Theta$ and it is equivalent to $p_{\Theta,e(I)} = 1$.

Appendix A.14. Proof of Corollary 9

Proof. Since the combination of evidence is made with the orthogonal sum operation, the recursive GC rule in this paper satisfies commutative law. Suppose the $I-1$ pieces of evidence (except the evidence e_i) have been combined by the recursive GC rule and their combined probability masses are $m_{e(I-1)}$=$(m_{\theta,e(I-1)},\theta\subset\Theta$; $m_{\Theta,e(I-1)};m_{P(\Theta),e(I-1)})$. Now we make combination for $m_{e(I-1)}$ and m_i by Eqs. (18b)-(18d) and we get

$$\widehat{m}_{\theta,e(I)}=\sum_{B\cap C=\theta,B,C\subset\Theta}m_{B,e(I-1)}w_ir_ip_{C,i}+m_{\theta,e(I-1)}[w_ir_ip_{\Theta,i}+w_i(1-r_i)]$$
$$+w_ir_ip_{\theta,i}m_{\Theta,e(i-1)}+[(1-w_i)m_{\theta,e(I-1)}+m_{P(\Theta),e(I-1)}w_ir_ip_{\theta,i}]\text{ for }\theta\subset\Theta\quad(A.37)$$

$$\widehat{m}_{\Theta,e(I)}=m_{\Theta,e(I-1)}[w_ir_ip_{\Theta,i}+w_i(1-r_i)]+m_{P(\Theta),e(I-1)}[w_ir_ip_{\Theta,i}+w_i(1-r_i)]+(1-w_i)m_{\Theta,e(I-1)}\quad(A.38)$$

$$\widehat{m}_{P(\Theta),e(I)}=m_{P(\Theta),e(I-1)}(1-w_i)\quad(A.39)$$

Inserting w_i=0 into the above expressions and we find that $\widehat{m}_{\theta,e(I)}$=$m_{\theta,e(I-1)}$ for $\theta\subset\Theta$, $\widehat{m}_{\Theta,e(I)}=m_{\Theta,e(I-1)}$ and $\widehat{m}_{P(\Theta),e(I)}$=$m_{P(\Theta),e(I-1)}$. Inserting them into Eq. (18a), we have $m_{\theta,e(I)}=\widehat{m}_{\theta,e(I)}\Big/(\sum_{\theta\subseteq\Theta}\widehat{m}_{\theta,e(I)}+\widehat{m}_{P(\Theta),e(I)})$ $=m_{\theta,e(I-1)}\Big/(\sum_{\theta\subseteq\Theta}m_{\theta,e(I-1)}+m_{P(\Theta),e(I-1)})$ for $\theta\subset\Theta$, θ=Θ and θ=$P(\Theta)$. From Theorem 4 we know $\sum_{\theta\subseteq\Theta}m_{\theta,e(i\text{-}1)}+$ $m_{P(\Theta),e(i-1)}$=1, so we have $m_{\theta,e(I)}$=$m_{\theta,e(I-1)}$ for $\theta\subset\Theta$, θ=Θ and θ=$P(\Theta)$. Thus there is $m_{e(I)}$=$m_{e(I-1)}\oplus m_i$=$m_{e(I-1)}$.

Appendix A.15. Proof of Corollary 10

Proof. From Eq. (18a), we have $m_{\theta,e(I)}=\widehat{m}_{\theta,e(I)}\Big/(\sum_{\theta\subseteq\Theta}\widehat{m}_{\theta,e(I)}+\widehat{m}_{P(\Theta),e(I)})$. It is equivalent to $\widehat{m}_{\theta,e(I)}$=$m_{\theta,e(I)}(\sum_{\theta\subseteq\Theta}\widehat{m}_{\theta,e(I)}+\widehat{m}_{P(\Theta),e(I)})$. From Eq. (19), the combined BD of all evidence in corollary 9 can be determined by Eq. (A.40).

$$p_{\theta,e(I)}=\frac{\widehat{m}_{\theta,e(I)}}{\sum_{\vartheta\subseteq\Theta}\widehat{m}_{\theta,e(I)}}=\frac{m_{\theta,e(I)}(\sum_{\theta\subseteq\Theta}\widehat{m}_{\theta,e(I)}+\widehat{m}_{P(\Theta),e(I)})}{\sum_{\vartheta\subseteq\Theta}[m_{\vartheta,e(I)}(\sum_{\theta\subseteq\Theta}\widehat{m}_{\theta,e(I)}+\widehat{m}_{P(\Theta),e(I)})]}=\frac{m_{\theta,e(I)}}{\sum_{\vartheta\subseteq\Theta}m_{\vartheta,e(I)}},\theta\subseteq\Theta\quad(A.40)$$

Similarly, the combined BD of *I*-1 pieces of evidence in corollary 9 can be determined by Eq. (A.41).

$$p_{\theta,e(I-1)}=\frac{\widehat{m}_{\theta,e(I-1)}}{\sum_{\vartheta\subseteq\Theta}\widehat{m}_{\theta,e(I-1)}}=\frac{m_{\theta,e(I-1)}(\sum_{\theta\subseteq\Theta}\widehat{m}_{\theta,e(I-1)}+\widehat{m}_{P(\Theta),e(I-1)})}{\sum_{\vartheta\subseteq\Theta}[m_{\vartheta,e(I-1)}(\sum_{\theta\subseteq\Theta}\widehat{m}_{\theta,e(I-1)}+\widehat{m}_{P(\Theta),e(I-1)})]}=\frac{m_{\theta,e(I-1)}}{\sum_{\vartheta\subseteq\Theta}m_{\vartheta,e(I-1)}},\theta\subseteq\Theta\quad(A.41)$$

Since $m_{e(I)}$=$m_{e(I-1)}$ which is proved by corollary 9, we have

$$p_{\theta,e(I)}=\frac{m_{\theta,e(I)}}{\sum_{\vartheta\subseteq\Theta}m_{\vartheta,e(I)}}=\frac{m_{\theta,e(I-1)}}{\sum_{\vartheta\subseteq\Theta}m_{\vartheta,e(I-1)}}=p_{\theta,e(I-1)}\quad(A.42)$$

Appendix A.16. Proof of Corollary 11

Proof. Similar to Corollary 9, we make a combination for $m_{e(I-1)}$ and m_i by Eqs. (18b)-(18d) and we get Eqs. (A.37)-(A.39). Inserting r_i=0 into Eqs. (A.37)-(A.39) and it is obvious to find that

$$\widehat{m}_{\theta,e(I)}=m_{\theta,e(I-1)}w_i+(1-w_i)m_{\theta,e(I-1)}=m_{\theta,e(I-1)}\text{ for }\theta\subset\Theta\quad(A.43)$$

$$\widehat{m}_{\Theta,e(I)}=m_{\Theta,e(I-1)}w_i+m_{P(\Theta),e(I-1)}w_i+(1-w_i)m_{\Theta,e(I-1)}=m_{\Theta,e(I-1)}+w_im_{P(\Theta),e(I-1)}\quad(A.44)$$

$$\widehat{m}_{P(\Theta),e(I)}=m_{P(\Theta),e(I-1)}(1-w_i)\quad(A.45)$$

Note that, $\sum_{\theta\subset\Theta}\widehat{m}_{\theta,e(I)}+\widehat{m}_{\Theta,e(I)}+\widehat{m}_{P(\Theta),e(I)}=\sum_{\theta\subseteq\Theta}m_{\theta,e(I-1)}+w_im_{P(\Theta),e(I-1)}+m_{P(\Theta),e(I-1)}(1-w_i)=\sum_{\theta\subseteq\Theta}m_{\theta,e(I-1)}$ $+m_{P(\Theta),e(I-1)}$=1. Inserting Eqs. (A.43)-(A.45) into Eq. (18a), we have $m_{\theta,e(I)}=\widehat{m}_{\theta,e(I)}\Big/(\sum_{\theta\subseteq\Theta}\widehat{m}_{\theta,e(I)}+\widehat{m}_{P(\Theta),e(I)})$ $=m_{\theta,e(I-1)}$ for $\theta\subset\Theta$, $m_{\Theta,e(I)}=\widehat{m}_{\Theta,e(I)}\Big/(\sum_{\theta\subseteq\Theta}\widehat{m}_{\theta,e(I)}+\widehat{m}_{P(\Theta),e(I)})=m_{\Theta,e(I-1)}+w_im_{P(\Theta),e(I-1)}$, and $\widehat{m}_{P(\Theta),e(I)}=$ $\widehat{m}_{P(\Theta),e(I)}\Big/(\sum_{\theta\subseteq\Theta}\widehat{m}_{\theta,e(I)}+\widehat{m}_{P(\Theta),e(I)})$ $=m_{P(\Theta),e(I-1)}(1-w_i)$.

Appendix A.17. Proof of Corollary 12

Proof. From Eqs. (19), (22a), (22b) and (A.40), we have

$$p_{\theta,e(I)}=\frac{m_{\theta,e(I)}}{\sum_{\theta\subseteq\Theta}m_{\theta,e(I)}}=\frac{m_{\theta,e(I)}}{\sum_{\theta\subset\Theta}m_{\theta,e(I)}+m_{\Theta,e(I)}}=\frac{m_{\theta,e(I-1)}}{\sum_{\theta\subseteq\Theta}m_{\theta,e(I-1)}+w_i m_{P(\Theta),e(I-1)}},\quad \theta\subset\Theta \quad \text{(A.46)}$$

From Eqs. (19) and (A.41), we have $p_{\theta,e(I-1)}=\widehat{m}_{\theta,e(I-1)}\Big/(\sum_{\theta\subseteq\Theta}\widehat{m}_{\theta,e(I-1)})=m_{\theta,e(I-1)}\Big/\sum_{\theta\subseteq\Theta}m_{\theta,e(I-1)}$ for $\theta\subseteq\Theta$. It is equivalent to $m_{\theta,e(I-1)}=p_{\theta,e(I-1)}\sum_{\theta\subseteq\Theta}m_{\theta,e(I-1)}$ for $\theta\subseteq\Theta$. Inserting the expression of $m_{\theta,e(I-1)}$ into Eq. (A.46) and denoting $\tau=\sum_{\theta\subseteq\Theta}m_{\theta,e(I-1)}\Big/(\sum_{\theta\subseteq\Theta}m_{\theta,e(I-1)}+w_i m_{P(\Theta),e(I-1)})$, we have

$$p_{\theta,e(I)}=\frac{m_{\theta,e(I-1)}}{\sum_{\theta\subseteq\Theta}m_{\theta,e(I-1)}+w_i m_{P(\Theta),e(I-1)}}=\frac{p_{\theta,e(I-1)}\sum_{\theta\subseteq\Theta}m_{\theta,e(I-1)}}{\sum_{\theta\subseteq\Theta}m_{\theta,e(I-1)}+w_i m_{P(\Theta),e(I-1)}}=\tau p_{\theta,e(I-1)},\quad \theta\subset\Theta \quad \text{(A.47)}$$

From theorem 5, we know $\sum_{\theta\subseteq\Theta}p_{\theta,e(I)}=1$ and $\sum_{\theta\subseteq\Theta}p_{\theta,e(I-1)}=1$. They are equivalent to $p_{\Theta,e(I)}=1-\sum_{\theta\subset\Theta}p_{\theta,e(I)}$ and $p_{\Theta,e(I-1)}=1-\sum_{\theta\subset\Theta}p_{\theta,e(I-1)}$. Thus we have

$$p_{\Theta,e(I)}=1-\sum_{\theta\subset\Theta}p_{\theta,e(I)}=1-\sum_{\theta\subset\Theta}\tau p_{\theta,e(I-1)}=1-\tau\sum_{\theta\subset\Theta}p_{\theta,e(I-1)}=1-\tau(1-p_{\Theta,e(I-1)})=1-\tau+\tau p_{\Theta,e(I-1)} \quad \text{(A.48)}$$

Since $0\le\sum_{\theta\subseteq\Theta}m_{\theta,e(I-1)}\Big/(\sum_{\theta\subseteq\Theta}m_{\theta,e(I-1)}+w_i m_{P(\Theta),e(I-1)})\le 1$, it is obvious that $0\le\tau\le 1$ and further we have $p_{\theta,e(I)}=\tau p_{\theta,e(I-1)}\le p_{\theta,e(I-1)}$. From $0\le\tau\le 1$ and $0\le p_{\Theta,e(I-1)}\le 1$, we know $(1-\tau)\ge 0$ and $(1-p_{\Theta,e(I-1)})\ge 0$. Thus

$$p_{\Theta,e(I)}-p_{\Theta,e(I-1)}=1-\tau+\tau p_{\Theta,e(I-1)}-p_{\Theta,e(I-1)}=1-\tau-(1-\tau)p_{\Theta,e(I-1)}=(1-\tau)(1-p_{\Theta,e(I-1)})\ge 0 \quad \text{(A.49)}$$

From Eq. (A.49), we know $p_{\Theta,e(I)}-p_{\Theta,e(I-1)}\ge 0$ and it is equivalent to $p_{\Theta,e(I)}\ge p_{\Theta,e(I-1)}$.

References

[1] Bappy M. M., Ali S. M., Kabir G.,Paul S. K., Supply chain sustainability assessment with Dempster-Shafer evidence theory: Implications in cleaner production, Journal of Cleaner Production. 237 (2019) 117771.

[2] Chen S., Wang Y., Shi H., Zhang M.,Lin Y., Evidential reasoning with discrete belief structures, Information Fusion. 41 (2018) 91-104.

[3] Chen Y., Chen Y., Xu X., Pan C., Yang J.,Yang G., A data-driven approximate causal inference model using the evidential reasoning rule, Knowledge-Based Systems. 88 (2015) 264-272.

[4] Dempster A. P., Upper and Lower Probabilities Induced by a Multivalued Mapping, Annals of Mathematical Statistics. 38 (2) (1967) 325-339.

[5] Du Y., Wang Y.,Qin M., New evidential reasoning rule with both weight and reliability for evidence combination, Computers & Industrial Engineering. 124 (2018) 493-508.

[6] Du Y.,Wang Y., Evidence combination rule with contrary support in the evidential reasoning approach, Expert Systems with Applications. 88 (2017) 193-204.

[7] Dubois D.,Prade H., On the unicity of dempster rule of combination, International Journal of Intelligent Systems. 1 (2) (1986) 133-142.

[8] Edwards W., The Theory of Decision Making, Psychological Bulletin. 51 (1954) 380-417.

[9] Fu C.,Chin K., Robust evidential reasoning approach with unknown attribute weights, Knowledge-Based Systems. 59 (2014) 9-20.

[10] Fu C.,Wang Y., An interval difference based evidential reasoning approach with unknown attribute weights and utilities of assessment grades, Computers & Industrial Engineering. 81 (2015) 109-117.

[11] Gao Q.,Xu D., An empirical study on the application of the Evidential Reasoning rule to decision making in financial investment, Knowledge-Based Systems. 164 (2019) 226-234.

[12] González C., Castillo M., García-Chevesich P.,Barrios J., Dempster-Shafer theory of evidence: A new approach to spatially model wildfire risk potential in central Chile, Science of The Total Environment. 613-614 (2018) 1024-1030.

[13] Kitada S., Economic, ecological and genetic impacts of marine stock enhancement and sea ranching: A systematic review, Fish and Fisheries. 19 (3) (2018) 511-532.

[14] Kong G., Xu D., Yang J.,Ma X., Combined medical quality assessment using the evidential reasoning approach, Expert Systems with Applications. 42 (13) (2015) 5522-5530.
[15] Lee S. I.,Zhang C. I., Evaluation of the Effect of Marine Ranching Activities on the Tongyeong Marine Ecosystem, Ocean Science Journal. 53 (3) (2018) 557-582.
[16] Lefevre E., Colot O.,Vannoorenberghe P., Belief function combination and conflict management, Information Fusion. 3 (2) (2002) 149-162.
[17] Li Z., Wen G.,Xie N., An approach to fuzzy soft sets in decision making based on grey relational analysis and Dempster–Shafer theory of evidence: An application in medical diagnosis, Artificial Intelligence in Medicine. 64 (3) (2015) 161-171.
[18] Lin T., Partition belief median filter based on Dempster–Shafer theory for image processing, Pattern Recognition. 41 (1) (2008) 139-151.
[19] Liu F., Chen Y., Yang J., Xu D.,Liu W., Solving multiple-criteria R&D project selection problems with a data-driven evidential reasoning rule, International Journal of Project Management. 37 (1) (2019) 87-97.
[20] Ma W., Jiang Y.,Luo X., A flexible rule for evidential combination in Dempster–Shafer theory of evidence, Applied Soft Computing. 85 (2019) 105512.
[21] Mercier D., Lefèvre É.,Delmotte F., Belief functions contextual discounting and canonical decompositions, International Journal of Approximate Reasoning. 53 (2) (2012) 146-158.
[22] P. S., The combination of evidence in the transferable belief model, IEEE Transactions on Pattern Analysis and Machine Intelligence. 12 (5) (1990) 447-458.
[23] Polat G., Cetindere F., Damci A.,Bingol B. N., Smart Home Subcontractor Selection Using the Integration of AHP and Evidential Reasoning Approaches, Procedia Engineering. 164 (2016) 347-353.
[24] R. G.,V. C., Study of Discounting Methods Applied to Canonical Decomposition of Belief Functions, in: Proc. 2018 21st International Conference on Information Fusion (FUSION),2018, pp.2505-2512.
[25] Sarabi-Jamab A.,Araabi B. N., How to decide when the sources of evidence are unreliable: A multi-criteria discounting approach in the Dempster–Shafer theory, Information Sciences. 448-449 (2018) 233-248.
[26] Shafer G., A Mathematical Theory of Evidence, Princeton University Press,1976.
[27] Smarandache F.,Dezert J. (2006). Proportional Conflict Redistribution Rules for Information Fusion. Advances and Applications of DSmT for Information Fusion. NewYork, American Research Press.
[28] Smets P.,Kennes R., The transferable belief model, Artificial Intelligence. 66 (2) (1994) 191-234.
[29] Wang R., Guiochet J., Motet G.,Schön W., Safety case confidence propagation based on Dempster–Shafer theory, International Journal of Approximate Reasoning. 107 (2019) 46-64.
[30] Wang Y., Yang J., Xu D.,Chin K., The evidential reasoning approach for multiple attribute decision analysis using interval belief degrees, European Journal of Operational Research. 175 (1) (2006) 35-66.
[31] Wang Y., Yang J., Xu D.,Chin K., On the combination and normalization of interval-valued belief structures, Information Sciences. 177 (5) (2007) 1230-1247.
[32] Xu D., Yang J.,Wang Y., The evidential reasoning approach for multi-attribute decision analysis under interval uncertainty, European Journal of Operational Research. 174 (3) (2006) 1914-1943.
[33] Yager R. R., On the dempster-shafer framework and new combination rules, Information Sciences. 41 (2) (1987) 93-137.
[34] Yager R. R., On the fusion of imprecise uncertainty measures using belief structures, Information Sciences. 181 (15) (2011) 3199-3209.
[35] Yang J. B., Wang Y. M., Xu D. L.,Chin K. S., The evidential reasoning approach for MADA under both probabilistic and fuzzy uncertainties, European Journal of Operational Research. 171 (1) (2006) 309-343.
[36] Yang J.,Singh M. G., An evidential reasoning approach for multiple-attribute decision making with uncertainty, IEEE Transactions on Systems, Man, and Cybernetics. 24 (1) (1994) 1-18.

[37] Yang J.,Xu D., On the evidential reasoning algorithm for multiple attribute decision analysis under uncertainty, IEEE Transactions on Systems, Man, and Cybernetics - Part A: Systems and Humans. 32 (3) (2002) 289-304.
[38] Yang J.,Xu D., Evidential reasoning rule for evidence combination, Artificial Intelligence. 205 (2013) 1-29.
[39] Yang L., Wang Y., Chang L.,Fu Y., A disjunctive belief rule-based expert system for bridge risk assessment with dynamic parameter optimization model, Computers & Industrial Engineering. 113 (2017) 459-474.
[40] Yang L., Wang Y., Liu J.,Martínez L., A joint optimization method on parameter and structure for belief-rule-based systems, Knowledge-Based Systems. 142 (2018) 220-240.
[41] Z. E., E. L.,D. M., Discountings of a Belief Function Using a Confusion Matrix, in: Proc. 2010 22nd IEEE International Conference on Tools with Artificial Intelligence,2010, pp.287-294.
[42] Zadeh L. A., Book Review: A Mathematical Theory of Evidence, 5 (4) (1984) 81-83.
[43] Zadeh L. A., Review of A Mathematical Theory of Evidence, Ai Magazine. 5 (3) (1984) 235--247.
[44] Zadeh L. A., A Simple View of the Dempster-Shafer Theory of Evidence and Its Implication for the Rule of Combination, Ai Magazine. 7 (2) (1986) 85-90.
[45] Zhang D., Yan X., Zhang J., Yang Z.,Wang J., Use of fuzzy rule-based evidential reasoning approach in the navigational risk assessment of inland waterway transportation systems, Safety Science. 82 (2016) 352-360.
[46] Zhang X., Wang Y., Chen S., Chu J.,Chen L., Gini coefficient-based evidential reasoning approach with unknown evidence weights, Computers & Industrial Engineering. 124 (2018) 157-166.
[47] Zhao J., Xue R., Dong Z., Tang D.,Wei W., Evaluating the reliability of sources of evidence with a two-perspective approach in classification problems based on evidence theory, Information Sciences. 507 (2020) 313-338.
[48] Zhou M., Liu X., Chen Y.,Yang J., Evidential reasoning rule for MADM with both weights and reliabilities in group decision making, Knowledge-Based Systems. 143 (2018) 142-161.
[49] Zhou M., Liu X., Yang J., Chen Y.,Wu J., Evidential reasoning approach with multiple kinds of attributes and entropy-based weight assignment, Knowledge-Based Systems. 163 (2019) 358-375.
[50] Zhou X., Zhao X., Zhang S.,Lin J., Marine Ranching Construction and Management in East China Sea: Programs for Sustainable Fishery and Aquaculture, Water. 11 (2019) 1237.

Highlights

Infeasibilities of evidential reasoning (ER) with weight and reliability are analyzed
Generalized discounting method is defined to discount evidence with two parameters
Generalized combination (GC) rule is established to make combinations for evidence
A series of theorems and corollaries of the proposed GC rule are proved
Comparison and discussion are made with ER and Dempster-Shafer theory of evidence

Credit Author Statement

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