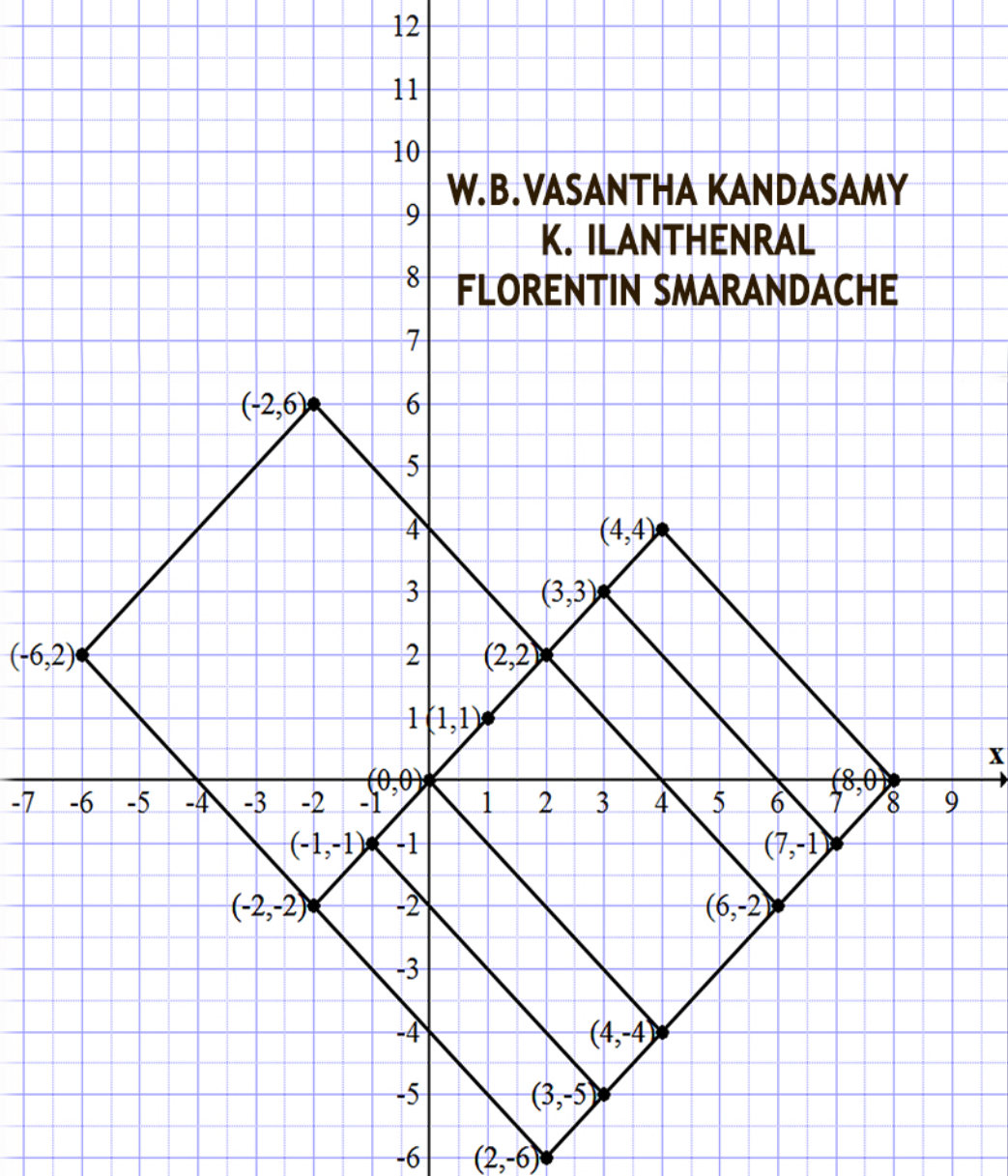# EUCLID SQUARES ON INFINITE PLANES

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Euclid Squares on Infinite Planes

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EuropaNova

2015

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CONTENTS

PREFACE

In this book for the first time the authors study the new type of Euclid squares in various planes like real plane, complex plane, dual number plane, special dual like number plane and special quasi dual number plane.

There are six such planes and they behave distinctly. From the study it is revealed that each type of squares behave in a different way depending on the plane.

We define several types of algebraic structures on them. Such study is new, innovative and interesting. However for some types of squares; one is not in a position to define product. Further under addition these squares from a group.

One of the benefits is addition of point squares to any square of type I and II is an easy translation of the square without affecting the area or length of the squares. There are over 150 graphs which makes the book more understandable.

This book is the blend of geometry, algebra and analysis.

This study is of fresh approach and in due course of time it will find lots of implications on other structures and applications to different fields.

We wish to acknowledge Dr. K Kandasamy for his sustained support and encouragement in the writing of this book.

W.B.VASANTHA KANDASAMY
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Chapter One

SPECIAL TYPES OF SQUARES IN REAL PLANE

In this chapter for the first time we define, develop and describe the concept of squares in a real plane and describe algebraic operations on them. We take different types of squares however for the first type we take the sides of the squares parallel to the x-axis and y-axis.

Now the squares can be over lapping or disjoint however we define operations on them.

In the first place we call square whose sides are parallel to the x-axis and y-axis as the real or Euclid squares of type I.

We perform algebraic operations on them. We can also study set of type I squares in the quadrant I or II or III or IV or in the total plane.

We will define operations on the set of real Euclid type I squares in the first quadrant.

We will describe the real or Euclid type I square by 4-tuple given by

A = {(a₁, b₁), (a₂, b₁), (a₂, b₂), (a₁, b₂)} this order is also preserved.

Example 1.1: Let A = {(0, 0), (1, 0), (1, 1), (0, 1)} be a real type I square.

Example 1.2: Let B = {(5, 3), (10, 3), (10, 8), (5, 8)} be a real type I square.

Example 1.3: Let A = {(0, 0), (0, 7), (–7, 7), (–7, 0)} be again a real type I square.

Figure 1.1
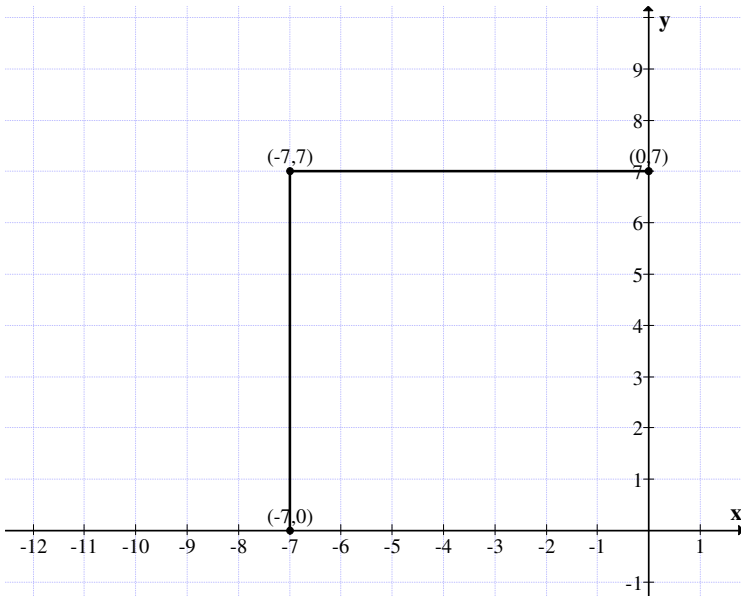
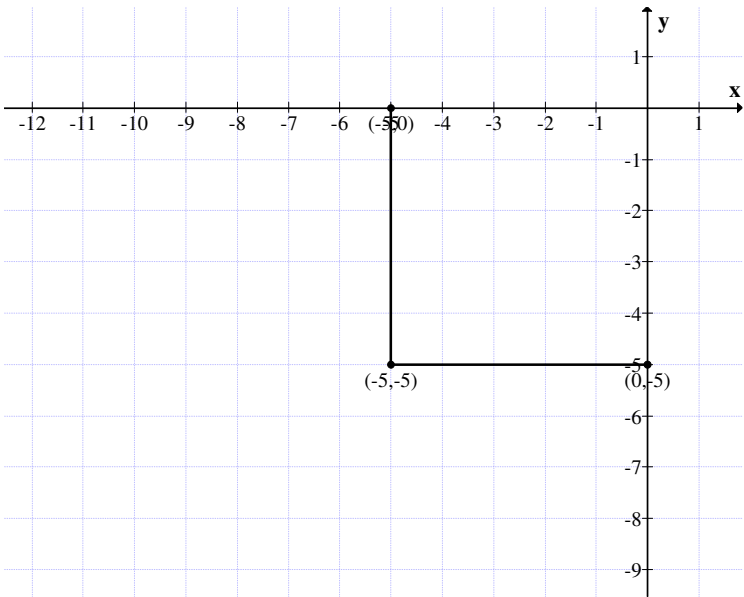*Example 1.4:* Let M = {(0, 0), (–5, 0), (–5, –5), (0, –5)} be a Euclid or real type I square.

Figure 1.2

Example 1.5: Let T = {(0, 0), (3, 0), (3, 3), (0, 3)} be the real type I square given by the following figure.

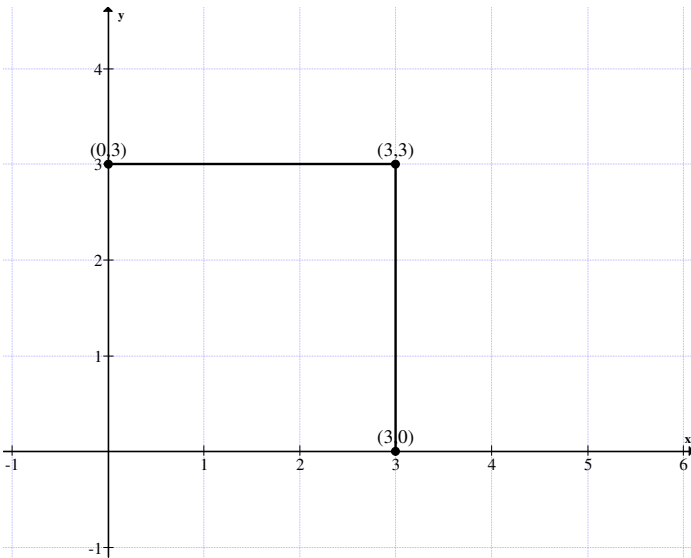*Figure 1.3*

Example 1.6: Let

$$M = \{(6, 3), (9, 3), (9, 6), (6, 6)\}$$

be the real type I square or Euclid type I square.
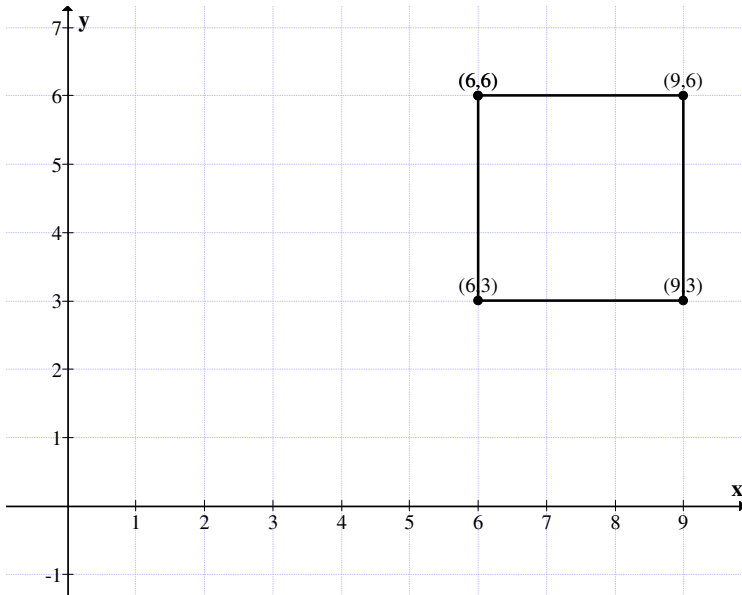

Figure 1.4

Example 1.7: Let

$$S = \{(3.2, 1.4), (6, 1.4), (3.2, 4.2), (6, 4.2)\}$$

be the Euclid or real type I square.

Example 1.8: Let

$$S = \{(0, 0), (-3, 0), (-3, -3), (0, -3)\}$$

be the Euclid type I square.
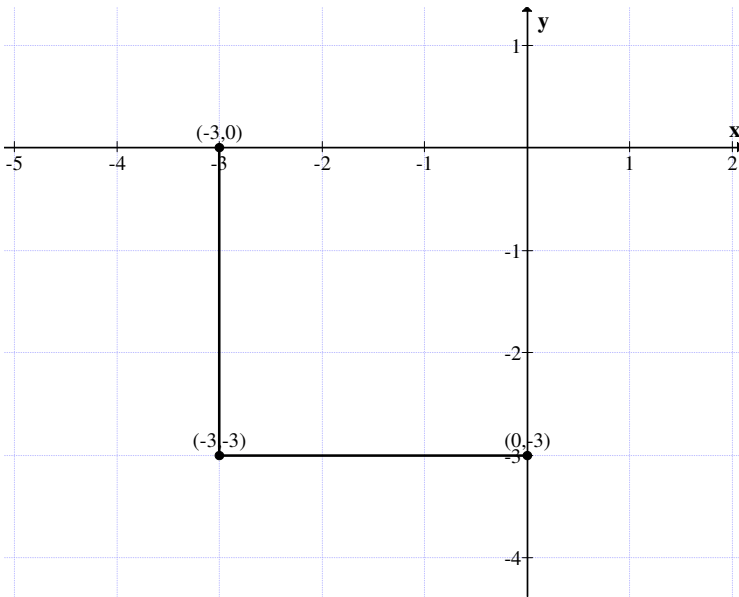
Figure 1.5

Now S_I = {Collection of all Euclid (or real) type I squares}.

We proceed on to define operations on S_I.

Let
$$A = \{(5, 3), (7, 3), (7, 5), (5, 5)\}$$
be a Euclid (or real) type I square in S_I.

$$B = \{(0, 0), (6, 0), (6, 6), (0, 6)\}$$
be another Euclid type I square in S_I.

We define

$$A + B = \{(5, 3), (7, 3), (7, 5), (5, 5)\} + \{(0, 0), (6, 0), (6, 6), (0. 6)\}$$

$$= \{(5, 3), (5, 11), (13, 3), (13, 11)\}$$

Figure 1.6

is again a Euclid square of type I.
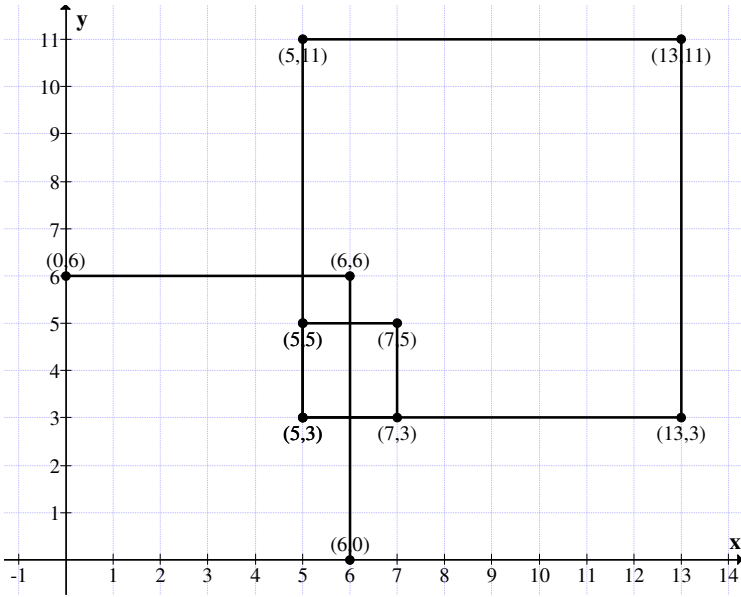
This is the way addition is made in squares of type I.

We see A + B is again a square of type I.

Let

$$A = \{(-5, 0), (0, 0), (0, 5), (-5, 5)\}$$

and

$$B = \{(0, 0), (3, 0), (3, 3), (0, 3)\} \in S_I.$$

$$\begin{aligned} A + B \ = \ \ &\{(-5, 0), (0, 0), (0, 5), (-5, 5)\} \ + \\ &\{(0, 0), (3, 0), (3, 3), (0, 3)\} \end{aligned}$$

$$= \ \ \{(-5, 0), (3, 0), (3, 8), (-5, 8)$$
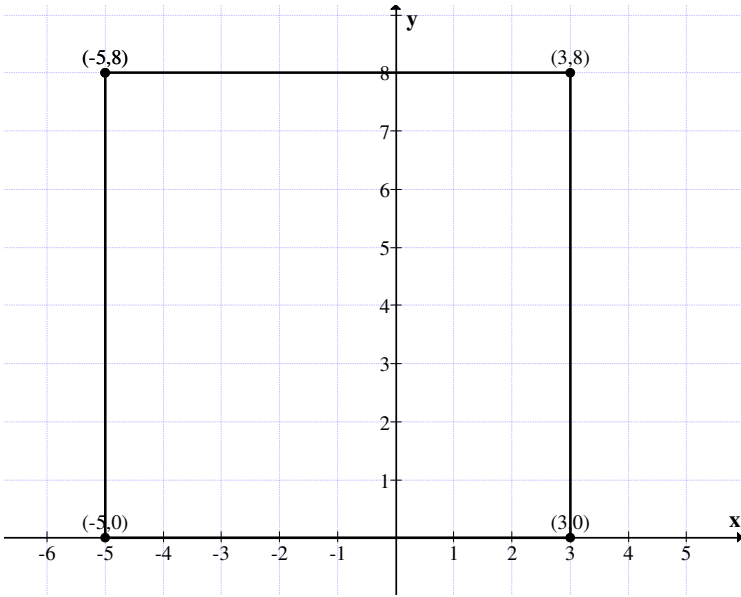
Figure 1.7

is again a square of type I.

Let

$$A = \{(-3, 0), (3, 0), (3, 6), (-3, 6)\}$$

and

$$B = \{(5, 0), (7, 0), (7, 2), (5, 2)\} \in S_I,$$

$$A + B \quad = \quad \{(-3, 0), (3, 0), (3, 6), (-3, 6)\} + $$
$$\{(5, 0), (7, 0), (7, 2), (5, 2)\}$$

$$= \quad \{(2, 0), (10, 0), (10, 8), (2, 8)\} \in S_I.$$

In view of all these we have the following theorem.

THEOREM 1.1: *Let*
S_I = *{Collection of type I Euclid (real) squares}. S_I is closed under the operation ' + '.*

Proof: Let

$$P = \{(a, b), (a + c, b), (a + c, b + c)\ (a, b + c)\}$$

and

$$M = \{(x, y), (x + d, y), (x + d, y + d), (x, y + d)\} \in S_I.$$

$$
\begin{aligned}
P + M \;=\; & \{(a + x, b + y), (a + x + (c + d), b + y), \\
& (a + x + (c + d), b + y + (c + d)), \\
& (a + x, b + y + c + d)\} \in S_I
\end{aligned}
$$

that is $P + M$ is again a Euclid square of type I hence the theorem.

Now we see if

$$A = \{((-3, 0), (0, 0), (0, 3), (-3, 3)\},$$
$$B = \{(5, 2), (10, 2), (10, 7), (5, 7)\}$$

and

$$C = \{((4, -2), (8, -2), (8, 2), (4, 2)\} \in S_I$$

be any three Euclid type I squares.

Consider

$$
\begin{aligned}
(A + B) + C \;=\; & \{((-3, 0), (0, 0), (0, 3), (-3, 3) + \{(5, 2), \\
& (10, 2), (10, 7), (5, 7))\} + \{((4, -2), \\
& (8, -2), (8, 2), (4, 2)\}
\end{aligned}
$$

$$
\begin{aligned}
\;=\; & ((2, 2), (10, 2), (10, 10), (2, 10)) + \\
& ((4, -2), (8, -2), (8, 2), (4, 2))
\end{aligned}
$$

$$
\;=\; ((6, 0), (18, 0), (18, 12), (6, 12)) \in S_I.
$$

Consider

$$
\begin{aligned}
A + (B + C) \;=\; & \{((-3, 0), (0, 0), (0, 3), (-3, 3))\} + \\
& \{((5, 2), (10, 2), (10, 7), (5, 7)) + \\
& ((4, -2), (8, -2), (8, 2), (4, 2))\}
\end{aligned}
$$

$$
\begin{aligned}
\;=\; & \{((-3, 0), (0, 0), (0, 3), (-3, 3))\} + \\
& \{((9, 0), (18, 0), (18, 9), (9, 9)\}
\end{aligned}
$$

$$= \{(6, 0), (18, 0), (18, 12), (6, 12)\} \in S_I.$$

We see $(A + B) + C = A + (B + C)$.

THEOREM 1.2: *Let S_I be the collection of all Euclid type I squares S_I is an associative set under the operation ' + '.*

Proof: Let

$$A = ((a, b), (a + x, b), (a + x, b + x), (a, b + x)),$$
$$B = ((c, d), (c + y, d), (c + y, d + y), (c, d + y))$$

and

$$C = ((e, f), (e + z, f), (e + z, f + z), (e, f + z)) \in S_I.$$

To show

$$(A + B) + C = A + (B + C)$$

Consider

$(A + B) + C \quad = \quad \{((a, b), (a + x, b), (a + x, b + x),$
$\qquad (a, b + x)) + ((c, d), (c + y, d),$
$\qquad (c + y, d + y), (c, d + y))\} + ((e, f),$
$\qquad (e + z, f), (e + z, f + z), (e, f + z))$

$\qquad = \quad \{(a + c, b + d), (a + c + x + y, b + d), (a + c + x + y,$
$\qquad b + d + x + y), (a + c, b + d + x + y)) + ((e, f), (e + z,$
$\qquad f), (e + z, f + z), (e, f + z)\}$

$\qquad = \quad ((a + c + e, b + d + f), (a + c + e + x + y + z, b + d +$
$\qquad f), (a + c + e + x + y + z, b + d + f + x + y + z),$
$\qquad (a + c + e, b + d + f + x + y + z))$

$$\text{--- I}$$

are in S_I.

$A + (B + C) \quad = \quad \{((a, b), (a + x, b), (a + x, b + x),$
$\qquad (a, b + x))\} + \{((c, d), (c + y, d),$
$\qquad (c + y, d + y), (c, d + y)) + ((e, f),$
$\qquad (e + z, f), (e + z, f + z), (e, f + z))\}$

$=$ ((a, b), (a + x, b), (a + x, b + x), (a, b + x),) + ((c + e, d + f), (c + y + e + z, d + f), (c + y + e + z, d + y + f + z), (e + c, d + f + y + z))

$=$ ((a + c + e, b + d + f), (a + c + e + x + y + z, b + d + f), (a + c + e + x + y + z, b + d + f + x + y + z), (a + e + c, b + d + f + x + y + z))

--- II

is in S_I.

Clearly I and II are the same; hence

$$(A + B) + C = A + (B + C)$$

for any A, B, C $\in$ S_I.

Thus '+' operation on S_I is associative. Hence the theorem.

Let

$$A = \{((0, 3), (6, 3), (6, 9), (0, 9))$$

and

$$B = \{((2, 5), (10, 5), (10, 13), (2, 13)\} \in S_I$$

A + B $=$ $\{((2, 8), (16, 8), (16, 22), (2, 22)\}$ I $\in$ S_I.

B + A $=$ ((2, 5), (10, 5), (10, 13), (2, 13)) +
              ((0, 3), (6, 3), (6, 9), (0, 9))

        $=$ ((2, 8), (16, 8), (16, 22), (2, 22))

I and II are identical hence A + B = B + A, so we see '+' operation on S_I is commutative.

THEOREM 1.3: *Let S_I be the collection of all Euclid squares of type I. The operation ' + ' on S_I is commutative.*

Proof: Consider any

$$A = ((a, b), (a + x, b), (a + x, b + y), (a, b + y))$$

and

$$B = ((c, d), (c + z, d), (c + z, d + t), (c, d + t)) \in S_I.$$

Now

$A + B$ = $\{((a, b), (a + x, b), (a + x, b + y), (a, b + y))\}$ +
$\{((c, d), (c + z, d), (c + z, d + t), (c, d + t))\}$

= $\{(a + c, b + d), (a + c + x + z, b + d),$
$(a + c + x + z, b + d + y + t), (a + c, b + d + y + t)\}$

--- I

$B + A$ = $\{(c, d), (c + z, d), (c + z, d + t), (c, d + t)\}$ +
$\{(a, b), (a + x, b), (a + x, b + y), (a, b + y)\}$

= $\{(a + c, b + d), (c + z + a + x, b + d),$
$(a + c + x + z, b + d + y + t), (a + c, b + d + y + t)\}$

--- II

is in S_I.

Clearly $A + B = B + A$.

Hence $\{S_I, +\}$ is commutative under the operation '+'. Hence the result.

Now we consider

$A = \{(3, 2), (7, 2), (7, 6), (3, 6)\} \in S_I$.

We see

$-A = \{(-3, -2), (-7, -2), (-7, -6), (-3, -6)\} \in S_I$

and

$A + (-A) = \{(0, 0), (0, 0), (0, 0), (0, 0)\}$.

We see $\{(0, 0), (0, 0), (0, 0), (0, 0)\}$ is the zero Euclid square of type I and for every A in S_I we have a unique $-A$ in S_I such that

$A + (-A) = \{(0, 0), (0, 0), (0, 0), (0, 0)\}$

called the zero Euclid square of type I.

Let

$C = \{(-3, 6), (0, 6), (0, 9), (-3, 9)\} \in S_I$.

Then

$-C = \{(3, -6), (0, -6), (0, -9), (3, -9)\} \in S_I$

is such that

$C + (-C) = \{(0, 0), (0, 0), (0, 0), (0, 0)\} \in S_I$.

In view of this we have the following theorem.

THEOREM 1.4: *Let $(S_I, +)$ be the set with a closed binary operation ' + '.*
$$((0, 0), (0, 0), (0, 0), (0, 0))$$
is the Euclid zero square of I.
    For every $A \in S_I$ there exists a unique $-A \in S_I$ such that
$$A + (-A) = ((0, 0), (0, 0), (0, 0), (0, 0)).$$

Proof: Let
$$A = ((a, b), (a + x, b), (a + x, b + y), (a, b + y)) \in S_I$$
be any real square of type I.
$$-A = ((-a, -b), (-a-x, -b), (-a-x, -b-y), (-a, -b-y)) \in S_I$$
is unique Euclid square of type I and is such that
$$A + (-A) = ((0, 0), (0, 0), (0, 0), (0, 0)).$$

Now we have the following theorem.

THEOREM 1.5: *Let*

$$S_I = \{Collection\ of\ all\ Euclid\ square\ of\ type\ I\}.$$

(i) S_I is an abelian group under '+'.
(ii) S_I is of infinite order.

Proof is direct and hence left as an exercise to the reader.

Suppose we take only the first quadrant and take all Euclid square of type I.

We see at the outset $(S_I, +)$ is not a group as no element in the positive quadrant has additive inverse.

Now consider $S_I^{++} = \{$All Euclid squares of type I from the first quadrant$\}$.

$\{S_I^{++}, +\}$ is only a monoid for $\{(0, 0), (0, 0), (0, 0), (0, 0)\}$ is the additive identity.

This Euclid square of type I monoid will also be known as first quadrant monoid.

Now consider the third quadrant Euclid squares of type I.

S_I^{--} = {All real squares of type I}. S_I^{--} is a semigroup under ' + '.

$$((0, 0), (0, 0), (0, 0), (0, 0))$$

acts as the identity with respect to addition.
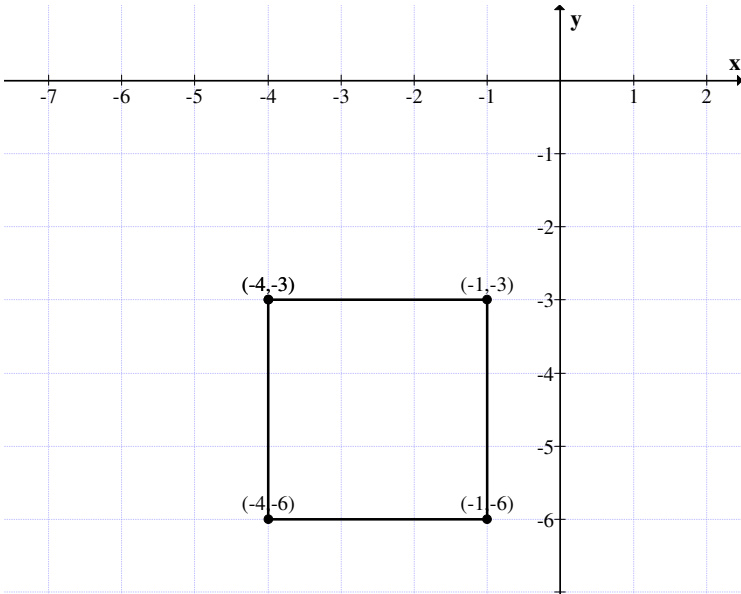
$$A = \{(-1, -3), (-4, -3), (-4, -6), (-1, -6)\}$$

Figure 1.8

and

$$B = \{(-5, -7), (-7, -7), (-7, -9), (-5, -9)\} \in S_I.$$

Clearly

$$A + B = \{(-6, -10), (-11, -10), (-11, -15), (-6, -15)\} \in S_I^{--}.$$

Thus $\{ S_I^{--}, + \}$ is a monoid.

Thus we see we can have semigroup of Euclid squares of type I under addition.

Another interesting feature about semigroup squares of type I in the first quadrant is that we see if the length of a side of one real square of type I is 5 and that of another square is 3 then their sum yield the length of the side to be 8 and the subtraction yields a square of side's length 2.

This is evident from the two real square of type I.

$$A = \{(7, 3), (9, 3), (9, 5), (7, 5)\}$$

and

$$B = \{(-2, -1), (5, -1), (5, 6), (-2, 6)\}$$

be two real squares of type I whose side length is 2 and 7 respectively.

$$A + B = \{(5, 2), (14, 2), (14, 11), (5, 11)\}.$$

The side length of A + B is 9 which is sum of the sides of the real squares A and B.

Let

$$D = \{(-3, 2), (2, 2), (2, 7), (-3, 7)\}$$

and

$$E = \{(10, -3), (14, -3), (14, 1), (10, 1)\}$$

be

$$D - E = \{(-13, 5), (-12, 5), (-12, 6), (-13, 6)\}.$$

Consider

$$E - D = \{(13, -5), (12, -5), (12, -6), (13, -6)\}.$$

We see E – D has side to be 1 unit real square of type I.

$$M = \{(9, -8), (10, -8), (10, -7), (9, -7)\}$$

and

$$N = \{(-5, 4), (-4, 4), (-4, 5), (-5, 5)\}$$

M – N = {(14, –12), (14, –12), (14, –12), (14, –12)}

a single point as both M and N are real squares of unit side length.

Now
M + N = {(4, –4), (6, –4), (6, –2), (4, –2)}.

Clearly M + N is a real square of side length 2.

Thus real squares of type I of the form
A = {(a, b), (a, b), (a, b), (a, b)}
will be known as the point square of type I.

So if
A = {(3, 2), (3, 2), (3, 2), (3, 2)}
be the point square of type I.

B = {(0, 7), (5, 7), (5, 12), (0, 12)}
be a real square of type I.

B + A = {(3, 9), (8, 9), (8, 14), (3, 14)}.

Thus we see addition of a point square of type I to a real square of type I we get a shift.

This is shown by an example.

Let
P = {(1, 3), (1, 3), (1, 3), (1, 3)}
be the point real square of type I.

Let
A = {(1, 2), (2, 2), (2, 3), (1, 3)}
be a real square of type I.

Clearly
P + A = {(2, 5), (3, 5), (3, 6), (2, 6)}.
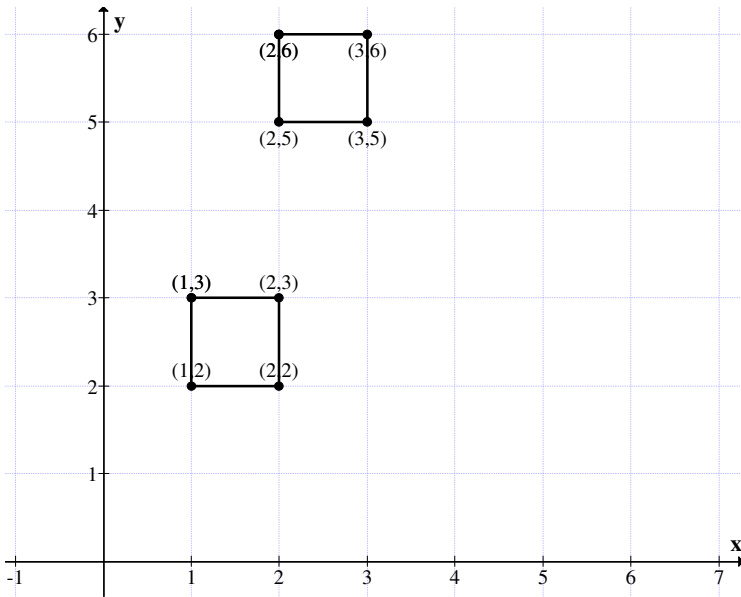This is represented by the following graph.

Figure 1.9

Thus we have shown $(S_I^{--}, +)$ is a monoid and $\{S_I^{++}, +\}$ is also a monoid.

Let

$$A = \{(5, 0), (8, 0), (8, 3), (5, 3)\}$$

and

$$B = \{(-7, 4), (-4, 4), (-4, 7), (-7, 7)\}$$

be two real squares of type I.

$$A - B = \{(12, -4), (12, -4), (12, -4), (12, -4)\}$$

is a point square of type I.

We see both A and B are real squares of type I of same side length 3 units.

Let

$$X = \{(2, 5), (7, 5), (7, 10), (2, 10)\}$$

and

$$Y = \{(-10, 1), (-5, 1), (-5, 6), (-10, 6)\}$$

be any two real square of type I.

Consider
$$X - Y = \{(12, 4), (12, 4), (12, 4), (12, 4)\}$$
is the point square of type I.

In view of all these we arrive at the following result.

THEOREM 1.6: *Let $\{S_I, +\}$ be a Euclid real square of type I. Let A and B any two real squares of type I with the length of its side being equal. Then A – B is a real point square of type I.*

Proof is direct and hence left as an exercise to the reader.

THEOREM 1.7: *Let $(S_I, +)$ be a real square of type I.*

If $A = \{(a, b), (a, b), (a, b), (a, b)\}$ be a real point square of type I. Then addition of A only shifts the position of every real point square of type I.

Proof follows from simple calculations.

Now having seen properties about real (Euclid) squares of type I under addition we now proceed onto study S_I under product.

What is the algebraic structure enjoyed by S_I under product?
Let
$$A = \{(5, 3), (7, 3), (7, 5), (5, 5)\}$$
and
$$B = \{(4, 3), (8, 3), (8, 7), (4, 7)\}$$
be two real square of type I.

$$A \times B = \{(5, 3), (7, 3), (7, 5), (5, 5)\}$$
$$\times \{(4, 3), (8, 3), (8, 7), (4, 7)\}$$

$$= \{(20, 9), (56, 9), (56, 35), (20, 35)\}.$$
The side of square A is 2 and that of B is 4. Clearly $A \times B$ is not a real square of type I.

Let
$$A = \{(3, 4), (4, 4), (4, 5), (3, 5)\}$$
and
$$B = \{(-2, 1), (-1, 1), (-1, 2), (-2, 2)\}$$
be two real squares of type I.

$$A \times B = \{(-6, 4), (4, 4), (4, 10), (-6, 10)\}.$$
Clearly $A \times B$ is not a square of type I.

DEFINITION 1.1: *Let R_I = {Collection of all rectangles such that the sides of them are parallel to the x-axis and y-axis respectively}.*

We call R_I the real or Euclid rectangle of type I. We see R_I $\cap S_I = \phi$.

Now we first define some operations on R_I and illustrate them by some examples.
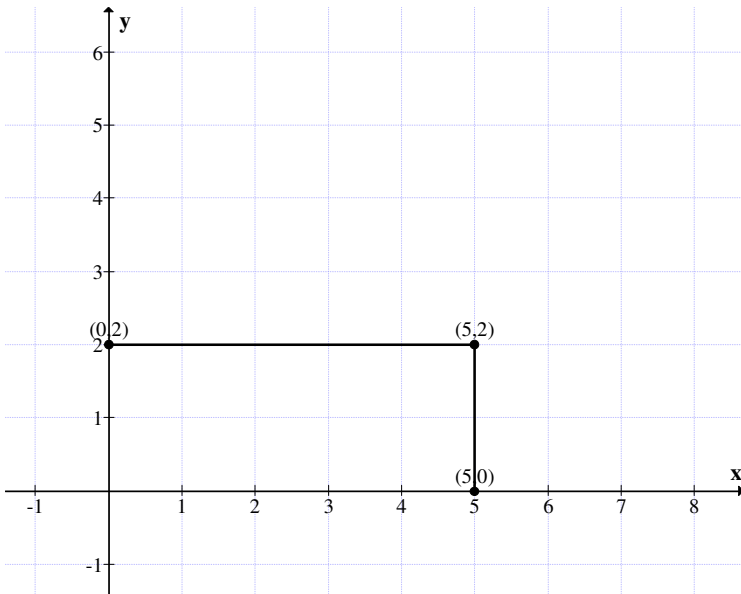Let
$$A = \{(0, 0), (5, 0), (5, 2), (0, 2)\}$$

Figure 1.10

be a Euclid rectangle of type I.
    Let

$$B = \{(1, 1), (6, 1), (6, 5), (1, 5)\}$$

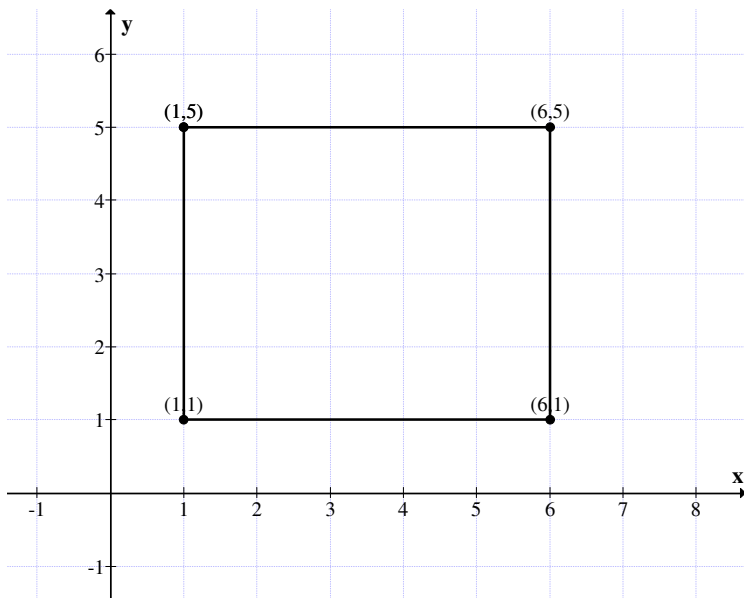be the Euclid rectangle of type I.

Figure 1.11

Let

$$A = \{(0, 6), (5, 6), (5, 9), (0, 9)\}$$

and

$$B = \{(2, -3), (5, -3), (5, -2), (2, -2)\} \in R_I;$$

$$A \times B = \{(0, -18), (25, -18), (25, -18), (0, -18)\}$$

is not a rectangle of type I.
    Let

$$A = \{(0, 6), (5, 6), (5, 9), (0, 9)\} \in R_I;$$

$$A^2 = \{(0, 36), (25, 36), (25, 81), (0, 81)\}$$

$$B^2 = \{(4, 9), (25, 9), (25, 4), (4, 4)\}$$

both are real rectangles of type I.

Thus we if
$$A = \{(0, 2), (4, 2), (4, 6), (0, 6)\}$$
is a real square of type I, then

$$A^2 = \{(0, 4), (16, 4), (16, 36), (0, 36)\}$$
is only a real Euclid rectangle of type I.

Thus the collection of Euclid square of type I is not closed under product.

THEOREM 1.8: *Let S_I = {Collection of all real or Euclid square of type I}. S_I under product $\times$ is not closed.*

Proof: Follows from the simple fact the product of any two squares of type I in general is not a Euclid square of type I.

Take
$$A = \{(0, 2), (6, 2), (6, 8), (0, 8)\} \in S_I.$$

$$A \times A = \{(0, 4), (36, 4), (36, 64), (0, 64)\}$$

is not in S_I it is only a real rectangle as per definition hence the claim.

Now we denote R_I = {Collection of all real rectangles of type I} to define '+' operation on R_I.

Example 1.9: Let $\{S_I, \times\}$ be the real or Euclid squares of type I.
Let
$$A = \{(1, 1), (2, 1), (2, 2), (1, 2)\}$$
be a unit square of type I of side 1.

$$A^2 = \{(1, 1), (4, 1), (4, 4), (1, 4)\}$$
is again square of type I of side 3.

$$A^3 = \{(1, 1), (8, 1), (8, 8), (1, 8)\}$$
is again a real or Euclid square of type I and is of side of length 7.

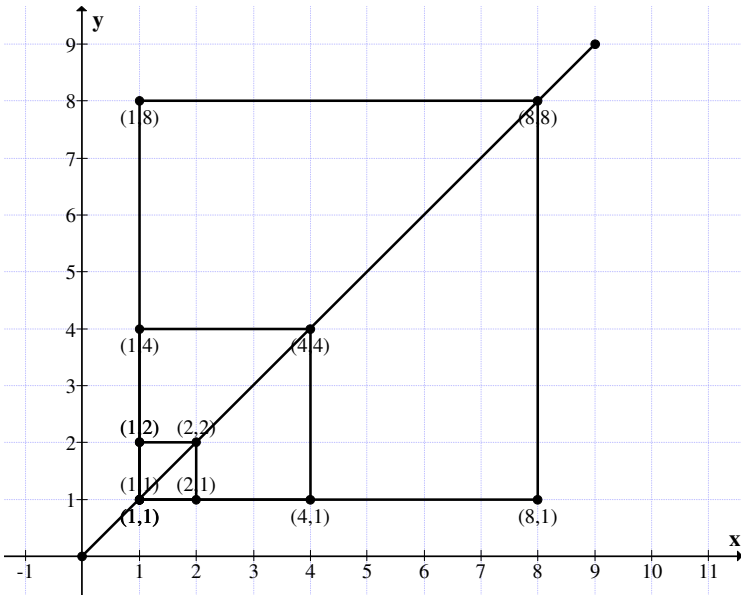We first describe this by the following diagram.

Figure 1.12

In view of this we have the following theorem.

THEOREM 1.9: *Let $(S_I, +)$ be a real or Euclid square of type I. The real square generated by $\langle\{(1, 1), (2, 1), (2, 2), (1, 2)\}\rangle$ under product is a closed set and is a semigroup and every square has the diagonal the bisector of the first quadrant.*

Proof : Now if

$$A = \{(1, 1), (2, 1), (2, 1), (1, 2)\} \in S_I$$

then

$$A^2 = A \times A = \{(1, 1), (4, 1), (4, 4), (1, 4)\}$$

$$A^3 = A^2 \times A = \{(1, 1), (8, 1), (8, 8), (1, 8)\}$$

$$A^4 = A^2 \times A^2 = \{(1, 1), (16, 1), (16, 16), (1, 16)\}$$

$$A^5 = \{(1, 1), (32, 1), (32, 32), (1, 32)\}$$
$$= \{(1, 1), (2^5, 1), (2^5, 2^5), (1, 2^5)\}$$

$$A^6 = \{(1, 1), (2^6, 1), (2^6, 2^6), (1, 2^6)\}$$

and so on.

$$A^n = \{(1, 1), (2^n, 1), (2^n, 2^n), (1, 2^n)\}$$

and $n \to \infty$.

The following observations are made

(i) Every real or Euclid square has (1, 1) to be one of the coordinate and the bisector of the angle y o x happens to be the diagonal of the square.

(ii) The length of the side is $(2-1)$, (2^2-1), (2^3-1), …, (2^n-1) and so on.

Consider the real square of type I,
$$B = \{(1, 1), (3, 1), (3, 3), (1, 3)\} \in S_I.$$
$$B^2 = \{(1, 1), (9, 1), (9, 9), (1, 9)\}.$$

Length of the side of the square
$$B = 2 = 3 - 1.$$

Length of the side of the square
$$B^2 = 8 = 3^2 - 1.$$

Length of the side of the square
$$B^3 \text{ is } 26 = 3^3 - 1$$
where
$$B^3 = \{(1, 1), (27, 1), (27, 27), (1, 27)\}$$
and so on.

Thus $\{(1, 1), (3^n, 1), (3^n, 3^n), (1, 3^n)\}$ and the length of the side is $3^n - 1$.

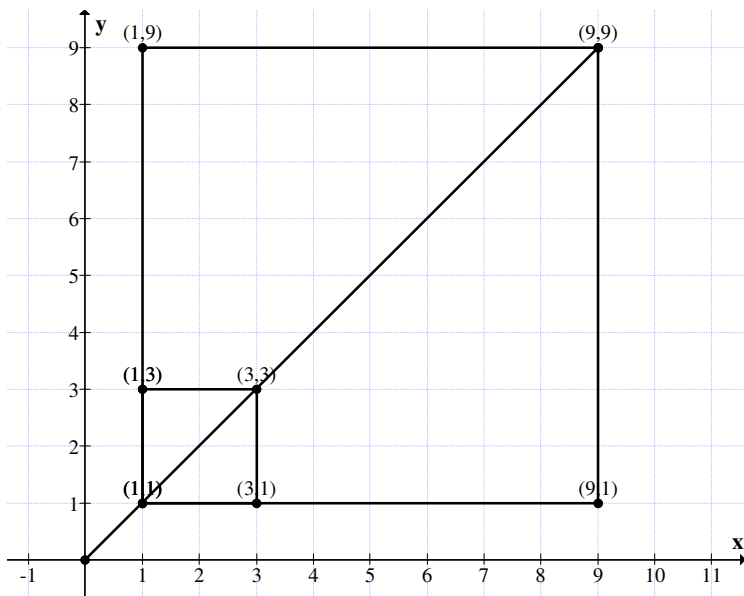The diagram associated with the subsemigroup is as follows:

Figure 1.13

Now if we take C = {(1, 1), (4, 1), (1, 4), (4, 4)} and generate a semigroup under product.
We see

$$C^2 = \{(1, 1), (16, 1), (16, 16), (1, 16)\}$$

$$C^3 = \{(1, 1), (64, 1), (64, 64), (1, 64)\}$$

and so on; thus

$$C^n = \{(1, 1), (4^n, 1), (4^n, 4^n), (1, 4^n)\}$$

and so on and this would be a subsemigroup of the semigroup generated by

$$A = \{(1, 1), (2, 1), (2, 2), (1, 4)\}.$$

That is

$$\langle C \rangle \subseteq \langle A \rangle;$$

$$\langle\{(1, 1), (4, 1), (4, 4), (1, 4)\}\rangle \subseteq \langle\{(1, 1), (2, 1), (2, 2), (1, 2)\}\rangle.$$

Consider

$$D = \{(1, 1), (5, 1), (5, 5), (1, 5)\} \in S_I.$$

$$D^2 = \{(1, 1), (25, 1), (25, 25), (1, 25)\}$$

$$D^3 = \{(1, 1), (125, 1), (125, 125), (1, 125)\}$$

and so on.

$$D^n = \{(1, 1), (5^n, 1), (5^n, 5^n), (1, 5^n)\}.$$

The length of the side of D is 4, D^2 is 24, D^3 is 124 and so on and that of D^n is $5^n - 1$.

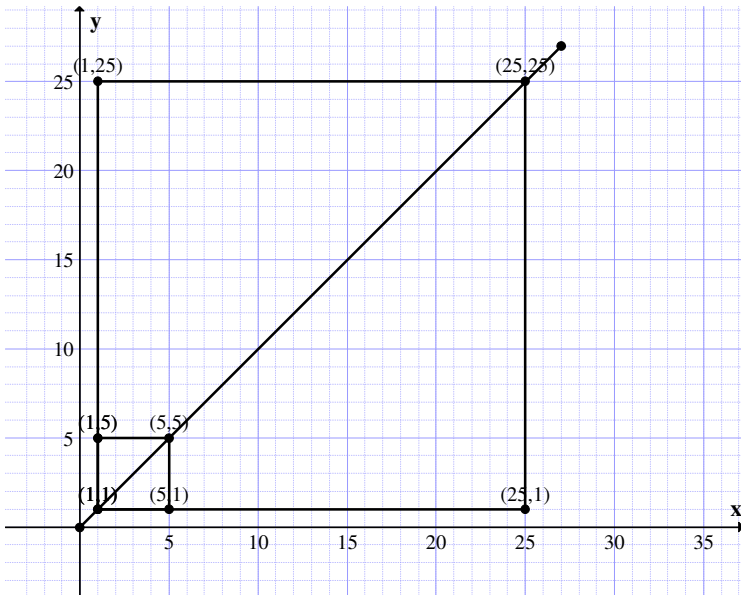We see the diagram for D, D^2, D^3 and so on is

Figure 1.14

Let E = {(1, 1), (6, 1), (6, 6), (1, 6)} be the semigroup generated by the Euclid square of type I under product.

$$E^2 = \{(1, 1), (36, 1), (36, 36), (1, 36)\}$$

$$E^3 = \{(1, 1), (6^3, 1), (6^3, 6^3), (1, 6^3)\} \dots.$$

$$E^n = \{(1, 1), (6^n, 1), (6^n, 6^n), (1, 6^n)\}$$

and so on.

Thus the length of the sides of the Euclid squares of type I generated by E are $6 - 1$, $(6^2 - 1)$, $(6^3 - 1)$, …, $(6^n - 1)$, ….

Consider F = {(1, 1), (7, 1), (7, 7), (1, 7)} be the Euclid square of type I.

F generates a semigroup under $\times$.

$$F^2 = \{(1, 1), (49, 1), (49, 49), (1, 49)\},$$

$$F^3 = \{(1, 1), (7^3, 1), (7^3, 7^3), (1, 7^3)\}$$

and so on;

$$F^n = \{(1, 1), (7^n, 1), (7^n, 7^n), (1, 7^n)\}$$

and so on.

The side length of F is 6, F^2 is $7^2 - 1$ that of F^n is $7^n - 1$ and so on.

$$G = \{(1, 1), (8, 1), (8, 8), (1, 8)\}$$

be the Euclid square of type I.

$$G^2 = \{(1, 1), (8^2, 1), (64, 64), (1, 64)\}$$

and so on.

$$G^n = \{(1, 1), (8^n, 1), (8^n, 8^n), (1, 8^n)\}$$

be the semigroups, and so on.

The length of the side of the Euclid square of type I G is 7, that of $G^2 = 64-1$, and so on and that of G^n is $8^n - 1$.

We see $\langle G \rangle$ is again a subsemigroup of $\langle A \rangle$.

Let
$$H = \{(1, 1), (9, 1), (9, 9), (1, 9)\} \in S_I$$
be the semigroup generated by H.

$$H^2 = \{(1, 1), (81, 1), (81, 81), (1, 81)\}$$
and so on;

$$H^n = \{(1, 1), (9^n, 1), (9^n, 9^n), (1, 9^n)\}$$
and so on.

Thus H generates a subsemigroup of side length 8, $9^2 - 1$, $9^3 - 1$, and so on $9^n - 1$.

Infact $\langle H \rangle \subseteq \langle B \rangle$; H is a subsemigroup of the semigroup generated by B.

Let $I = \{(1, 1), (10, 1), (10, 10), (1, 10)\}$ be the Euclid square groups of type I generates a semigroup generated by $\langle I \rangle$.

$$I^2 = \{(1, 1), (10^2, 1), (10^2, 10^2), (1, 10^2)\},$$

$$I^3 = \{(1, 1), (10^3, 1), (10^3, 10^3), (1, 10^3)\}$$
and so on;

$$I^n = \{(1, 1), (10^n, 1), (10^n, 10^n), (1, 10^n)\}$$
and so on.

The length of the side of the square I type I is 9, that of I^2 is $10^2 - 1$ and so on and that of I^n is $10^n - 1$ and we see this semigroup under product I is as follows:
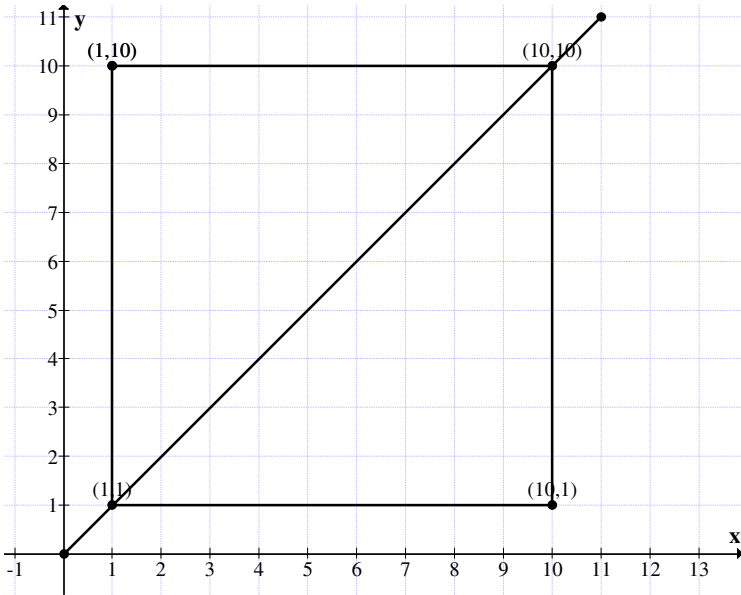
Figure 1.15

$$I^2 = \{(1, 1), (100, 1), (100, 100), (1, 100)\}$$

and so on.

Now

$$J = \{(1, 1), (11, 1), (11, 11), (1, 11)\}$$

be the Euclid or real square of type I.

$$J^2 = \{(1, 1), (121, 1), (121, 121), (1, 121)\}$$

and so on.

$$J^n = \{(1, 1), (11^n, 1), (11^n, 11^n), (1, 11^n)\}$$

and so on.

Thus J generates a semigroup different from the semigroups generates by A, B, C …   ---   I

Let K = {(1, 1), (12, 1), (12, 12), (1, 12)} be the Euclid or real square of type I.

$$K^2 = \{(1, 1), (144, 1), (144, 144), (1, 144)\}$$

be the real square of type I.

The side of K is 11 and that of $K^2 = 12^2 - 1 = 143$ and so on.

Thus K^n is $\{(1, 1), (12^n, 1), (12^n, 12^n), (1, 12^n)\}$ and so on. Thus K generates a semigroup.

$L = \{(1, 1), (13, 1), (13, 13), (1, 13)\}$ be the real square of type I. L under product generates a semigroup of infinite order which has diagonal passing through $(1, 1)$ and length of the sides of the squares being 12, $13^2 - 1$, $13^3 - 1$, …, $13^n - 1$.

Let $M = \{(1, 1), (14, 1), (14, 14), (1, 14)\}$ be the Euclid (real) square of type I and length of the side of the square is 13 and M generates a semigroup under product of infinite order.

Now let us consider a Euclid square of type I given by $x_1 = \{(0, 0), (1/2, 0), (1/2, \frac{1}{2}), (0, \frac{1}{2})\}$ be a Euclid square of type I. The length of the square is $\frac{1}{2}$.

The diagram is as follows.
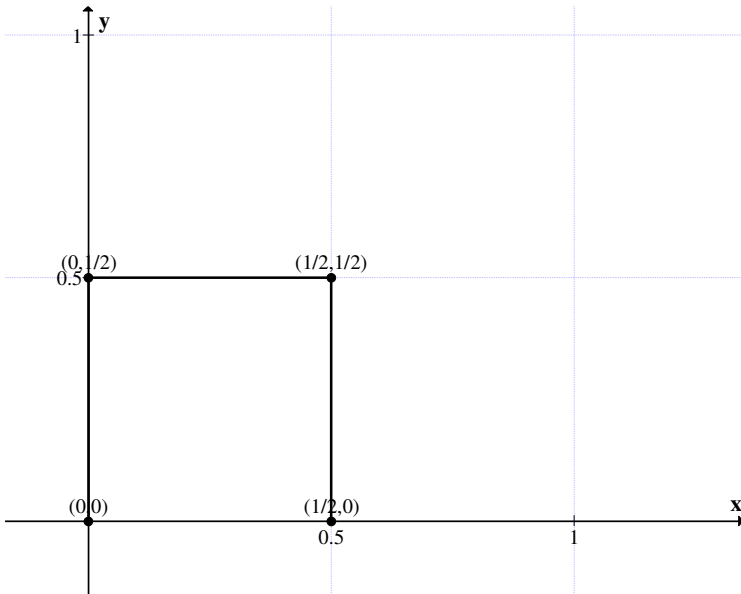
Figure 1.16

$$x_1^2 = \{(0, 0), (\tfrac{1}{4}, 0), (\tfrac{1}{4}, \tfrac{1}{4}), (0, \tfrac{1}{4})\},$$
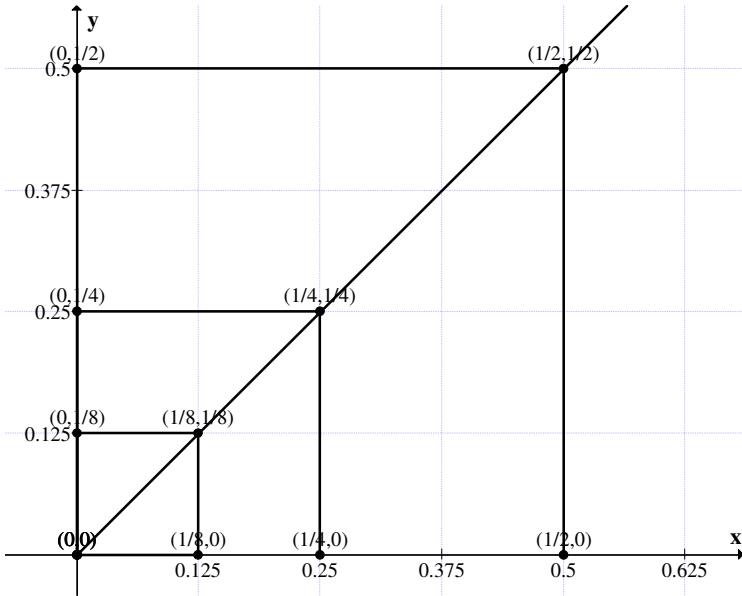

Figure 1.17

$$x_1^3 = \{(0, 0), (1/8, 0), (1/8, 1/8), (0, 1/8)\}$$

and so on.

All square of type I generated by x_1 converges from x_1 into smaller and smaller square of type I with length of the sides being ½, ¼, 1/8, 1/16, 1/32, …, $\dfrac{1}{2^n}$ and so on.

Now we consider type I squares which has the diagonal to be on the line $(1, 1)$ to (n, n) and $n \to \infty$; let this collection be denoted by M; $M = \{(1, 1), (n, 1), (n, n), (1, n) \mid n \in Z^+ \setminus \{1\}\}$ be the Euclid or real squares of type I.

Clearly M is a semigroup under $\times$ and M is of infinite order.

Now if we shift from (1, 1) any number of the form (n, n) in the square of type I then also we see under product the squares of type I form a semigroup.

Consider
$$X_1 = \{(2, 2), (3, 2), (3, 3), (2, 3)\} \in S_I.$$

$$X_1^2 = \{(4, 4), (9, 4), (9, 9), (4, 9)\}$$

and

$$X_1^3 = \{(8, 8), (27, 8), (27, 27), (8, 27)\}.$$

Length of the side of the square X_1 of type I is one, that of X_1^2 is 5 and of X_1^3 is 19.

$$X_1^4 = \{(16, 16), (81, 16), (81, 81), (16, 81)\}$$

and so on.

The length of the side of the square X_1^4 is 65.

$$X_1^5 = \{(32, 32), (243, 32), (243, 243), (32, 243)\}$$

be the square of type I.

The length of the side X_1^5 is 211 and so on.

One see the length of the sides of these squares are 1, 5, 19, 65, 211 and so on.

$$1 = 3 - 2$$
$$5 = 3^2 - 2^2$$
$$19 = 3^3 - 2^3$$
$$65 = 3^4 - 2^4$$

$211 = 3^5 - 2^5$ and so on for the real square of type I, X^n, the length of the side is $3^n - 2^n$; and so on.

However for all these the diagonal is the angle bisector of the first quadrant.

Let
$$y = \{(2, 2), (4, 2), (4, 4), (2, 4)\}$$
be a real square of type I.

$$y^2 = \{(4, 4), (16, 4), (16, 16), (4, 16)\}$$

is again a real square of type I and length of the side is 12.

The length of the side of the real square is 2.

$$y^3 = \{(8, 8), (32, 8), (32, 32), (8, 32)\}$$

is again a real unit square of type I of side length 24 and so on.

Thus

$$y^n = \{(2^n, 2^n), (4^n, 2^n), (4^n, 4^n), (2^n, 4^n)\}$$

is a Euclid square of type I of the side length $4^n - 2^n$.

Now

$$Z = x_1 \times y$$

$$= \{(2, 2), (3, 2), (3, 3), (2, 3)\}$$
$$\times \{(2, 2), (4, 2), (4, 4), (2, 4)\}$$

$$= \{(4, 4), (12, 4), (12, 12), (4, 12)\}$$

is again a real square of type I and its side length is 8.

$$Z^2 = \{(16, 16), (144, 16), (144, 144), (16, 144)\}$$

is a real square of type I and its side length is $144 - 16 = 128$ and so on.

Thus we can get infinite classes of real squares of type I which are semigroups under product; infact cyclic semigroups. These also contain subsemigroups.

Hence any type I real square of the form

$$S = \{(x, x), (x + k, x), (x + k, x + k), (x, x + k)\}$$

for all $x, k \in Z^+ \setminus \{0\}$ generates a cyclic semigroup.

Inview of all these we have the following theorem.

THEOREM 1.10: *Let S_I be the Euclid squares of type I.*

Let P = {(x, x), (x + k, x), (x + k, x + k), (x, x + k) | x, k ∈ Z⁺\ {0}} ⊆ S_I.

(i)   *P is a semigroup under product of infinite order.*

(ii)   *Every A ∈ P generates a subsemigroup of P of infinite order; infact a cyclic subsemigroup of P under product.*

(iii)   *Thus P has infinite number of cyclic subsemigroups.*

Proof: Let
$$P = \{(x, x), (x + k, x), (x + k, x + k), (x, x + k)\}$$
$$| x, k \in Z^+ \setminus \{0\}\}$$
be the collection of real squares of type I.

To show P is a semigroup of squares of type I under product.

$$A = \{\{(t, t), (t + k, t), (t + k, t + k), (t, t + k)\}$$
$$| t, k \in Z^+ \setminus \{0\}\}$$
are fixed numbers for this A and

$$B = \{\{(s, s), (s + p, s), (s + p, s + p), (s, s + p)\}$$
$$| s, p \in Z^+ \setminus \{0\}\}$$
and are fixed numbers for this B be any two real squares of type I.

$$
\begin{aligned}
A \times B \; = \; & \{(t, t), (t + k, t), (t + k, t + k), (t, t + k)\} \\
& \times \{(s, s), (s + p, s), (s + p, s + p), (s, s + p)\} \\[6pt]
= \; & \{(ts, ts), ((t + k)\,(s + p), ts), ((t + k)\,(s + p), \\
& (t + k)\,(s + p)), (ts, (t + k)\,(s + p))\}.
\end{aligned}
$$

Clearly A × B ∈ P; hence {P, ×} is closed under product so is a semigroup of real squares of type I.

Hence the proof of (1).

Consider

A ={(x, x), (x + k, x), (x + k, x + k), (x, x + k)} ∈ P.
A generates a cyclic subsemigroup of P given by

$A^2 = \{(x^2, x^2), ((x + k)^2, x^2), ((x + k)^2, (x + k)^2), (x^2, (x + k)^2)\}$
and so on.

$A^n = \{(x^n, x^n), ((x + k)^n, x^n), ((x + k)^n, (x + k)^n), (x^n, (x + k)^n)\}$
and so on are in P and is generated by A.

Thus A generates a real square of type I subsemigroup of P which is cyclic and of infinite order.

Since P has infinite cardinality. P has infinite number of cyclic subsemigroups of infinite order. Hence the theorem.

It is important to note that S_I under product is not even closed.

However S_I has subsets which are closed under product and form a semigroup.

Infact S_I has infinite number of cyclic subsemigroups of infinite order.

Now we see if
A = {(−1, −1), (−5, −1), (−5, −5), (−1, −5)}
be a square of type I, S_I.

$$A^2 = \{(1, 1), (25, 1), (25, 25), (1, 25)\}$$

$A^3 = \{(−1, −1), (−125, −1), (−125, −125), (−1, −125)\}$
We see
$A^4 = \{(1, 1), (625, 1), (625, 625), (1, 625)\}.$

If $A \in S_I^{--}$ then A^2, $A^4 \notin S_I^{--}$ but $A^3 \in S_I^{--}$ and so on.
    Let
P   = $P_I^{--} \cup P_I^{++}$
    =   {(−x, −x) (−x −k, −x), (−x − k, −x − k), (−x, −x–k)} ∪
        {(x, x) (x + k, x), (x + k, x + k), (x, x + k)}

where $x, k \in Z^+ \setminus \{0\}$.

$$P_I^{--} \cap P_I^{++} = \phi.$$

We see every $x \in P$ is such that $x^n \in P_I^{++}$ if $x \in P_I^{++}$ and if $x \in P_I^{--}$ then $x^m \in P_I^{--}$ if m is odd and if m is even $x^m \in P_I^{++}$.
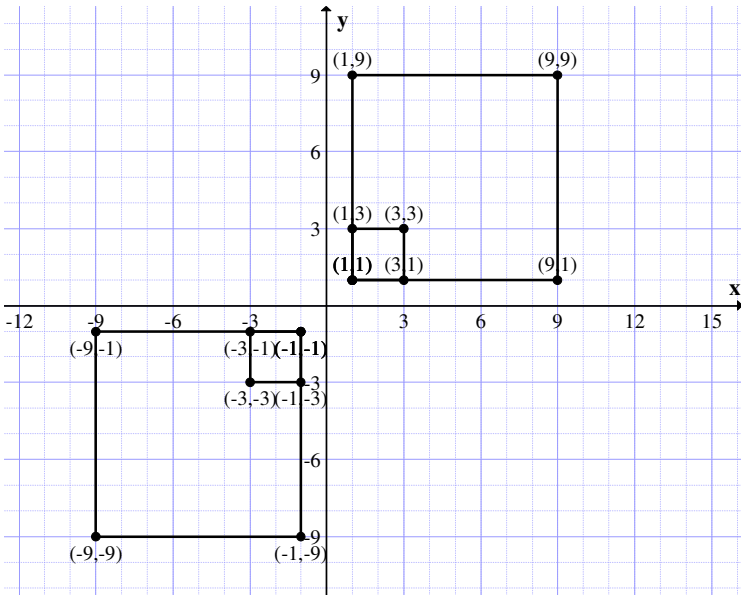
We will first illustrate this by some examples.

Let
$$x = \{(1, 1), (3, 1), (3, 3), (1, 3)\}$$
and
$$y = \{(-1, -1), (-9, -1), (-9, -9), (-1, -9)\} \in P.$$

$x \times y \quad = \quad \{(-1, -1), (-27, -1), (-27, -27), (-1, -27)\}$

$x^2 \quad = \quad \{(1, 1), (9, 1), (9, 9), (1, 9)\}$

and
$y^2 \quad = \quad \{(1, 1), (81, 1), (81, 81), (1, 81)\}$

$x \times y, x^2, y^2 \in P$ we see $x^2, y^2 \in P_I^{++}$ and $x \times y \in P_I^{--}$.

$x^2 \times y \quad = \quad \{(-1, -1), (-81, -1), (-81, -81), (-1, -81)\} \in P_I^{--}$.

$y^3 \quad = \quad \{(-1, -1), (-729, -1), (-729, -729), (-1, -729)\}$
$$\in P_I^{--}.$$

This is the way product is carried out on P. $\{P, \times\}$ is a semigroup S_I.

Clearly $\{P_I^{--}, \times\}$ is not even closed under product.

$A \in P_I^{--}$ then $A^2 \notin P_I^{--}$ but $A^3 \in P_I^{--}$.

Further if $A \in P_I^{++}$ and $B \in P_I^{--}$ then $AB \in P_I^{--}$ and $AB \notin P_I^{++}$.

We know $P_I^{++} \cap P_I^{--} = \phi$.

Now this looks like

Figure 1.18

Let

$$A = \{(1, 1), (3, 1), (3, 3), (1, 3)\}$$

and

$$B = \{(-1, -1), (-3, -1), (-3, -3), (-1, -3)\} \in P$$

$$A \times B \quad = \quad \{(-1, -1), (-9, -1), (-9, -9), (-1, -9)\} \in P_I^{--}.$$

Both

$$A^2 = B^2 \in P_I^{++}.$$

If $B \in P_I^{--}$ then $B^3, B^5, B^7, B^9, B^{11}, \ldots, B^{2n+1} \in P_I^{--}$.

However P_I^{++} is a semigroup under product and P_I^{--} is not a semigroup under product.

We cannot have real square of type I on the line bisecting the 2nd and 4th quadrant.

For

$$A = \{(-1, 1), (1, 1), (1, 3), (-1, 3)\}$$
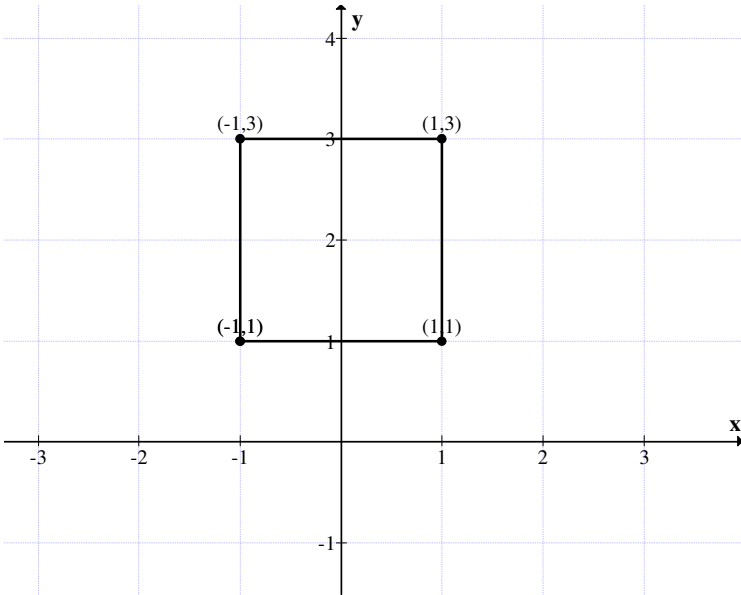
Figure 1.19
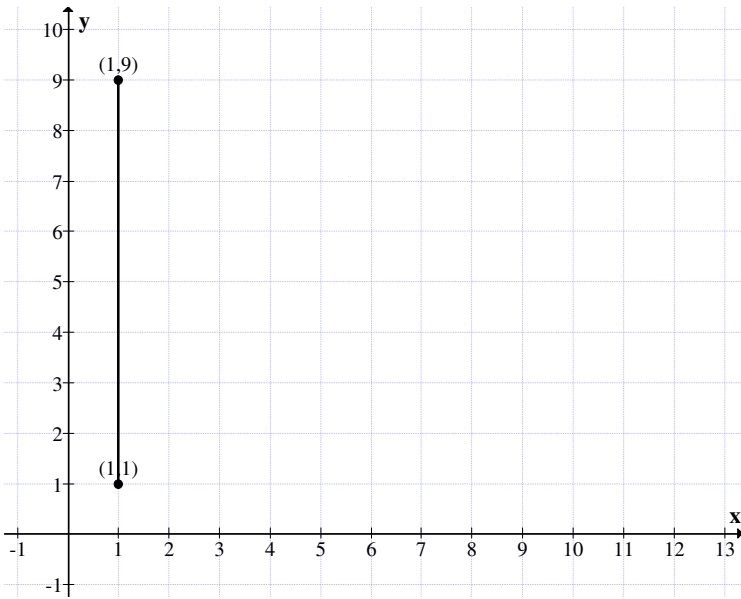
Figure 1.20

$$A^2 = \{(1, 1), (1, 1), (1, 9), (1, 9)\}$$

which is not a real square of type I given in Figure 1.20 is just a line.

Thus we cannot find any such property with these types of real squares of type I.

Now we can also have real or Euclid squares of type I using

$A = \{(-x, -x), (-x - k, -x), (-x - k, -x - k), (-x, -x - k)\}$ in S^I; when in A, x = 3 and k = 5 we get

$$A = \{(-3, -3), (-8, -3), (-8, -8), (-3, -8)\}$$

$$A^2 = \{(9, 9), (64, 9), (64, 64), (9, 64)\}$$
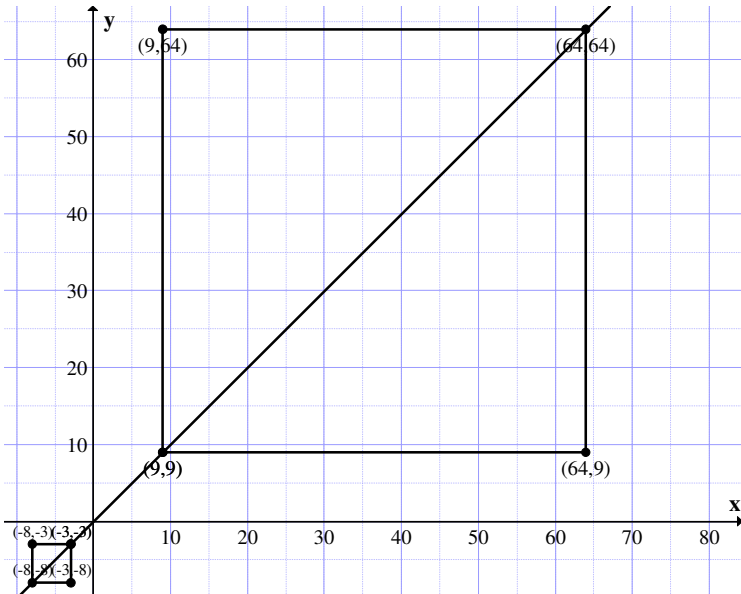
Figure 1.21

A is the 3rd quadrant and A^2 is in the first quadrant.

For if

$$B = \{(-2, -2), (-3, -2), (-3, -3), (-2, -3)\}$$

Then

$$B^2 = \{(4, 4), (9, 4), (9, 9) (4, 9)\}$$
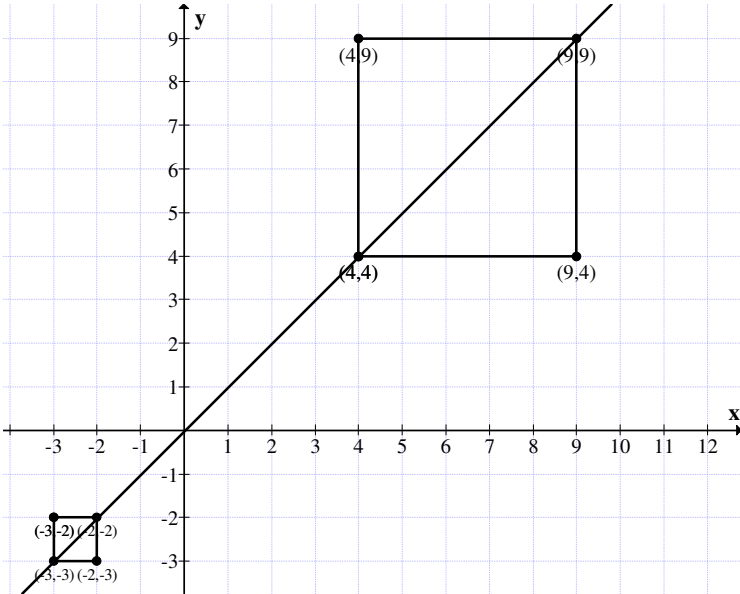
Figure 1.22

B^2 is in first quadrant where as B is in the 3rd quadrant.

Thus we have the following type of real squares of type I.
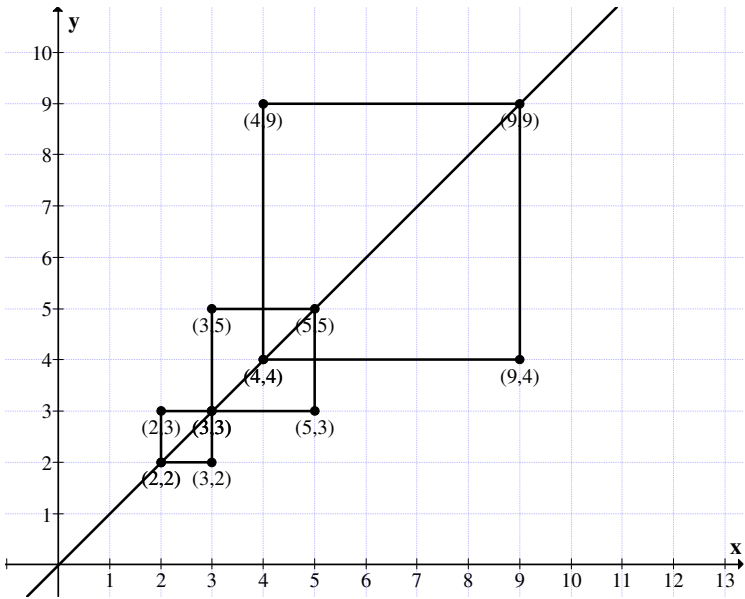
Figure 1.23

We get if in

$$A = \{(x, x), (x + k, x), (x + k, x + k), (x, x + k)\} \in S_I^{++}$$

with $x \neq 1$ then we get the diagram of the form where we get over lapping real squares of type I.

The same is true of

$$A = [(-x^1, -x), (-x-k, -x), (-x-k, -x-k), (-x, -x-k)\}$$
with $x \neq -1 \in S_I^{--}$.
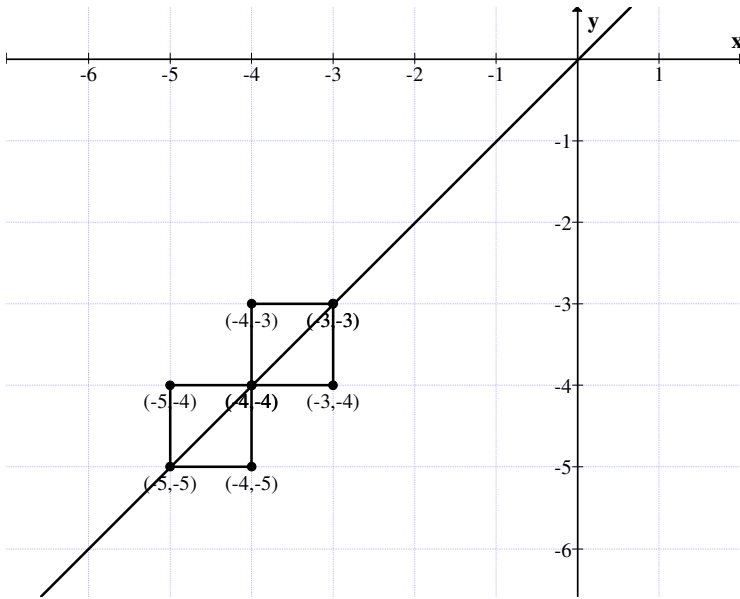
Figure 1.24

Now we want to study this type of type I Euclid square where the coordinates of the square are fractions.

Let

A = {(1/2, 1/2), (3/2, 1/2), (3/2, 3/2), (1/2, 3/2)} ∈ S_I.

A^2 = {(1/4, 1/4), (9/4, 1/4), (9/4, 9/4), (1/4, 9/4)} ∈ S_I.

A^3 = ((1/8, 1/8), (27/8, 1/8), (27/8, 27/8), (1/8, 27/8)}.

We see the behaviour of this A is very different. For the square A is contained in A^2, A^3 contains A^2 and so on.
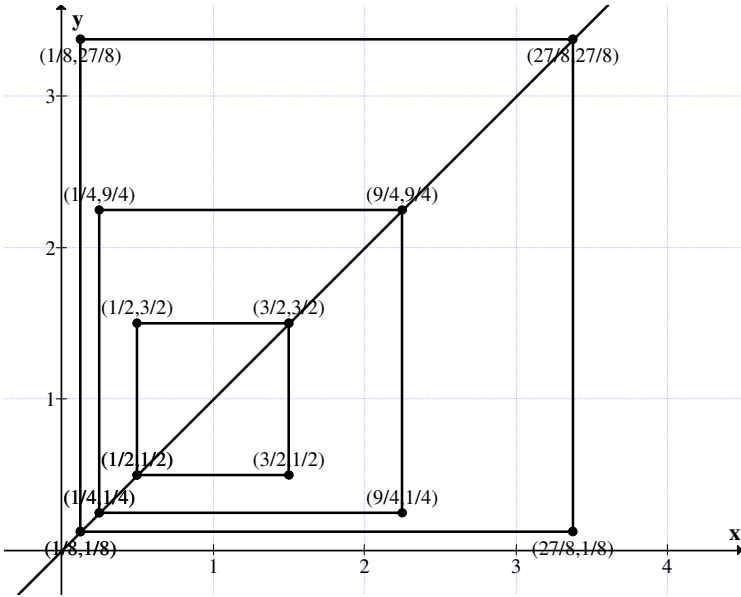
Figure 1.25

Let

$$B = \{(1/3, 1/3), (4/3, 1/3), (4/3, 4/3), (1/3, 4/3)\} \in S_I,$$

$$B^2 = \{(1/9, 1/9), (16/9, 1/9), (16/9, 16/9), (1/9, 16/9)\}$$

$$B^3 = \{(1/27, 1/27), (64/27, 1/27), (64/27, 64/27), (1/27, 64/27)\}$$

and

$$B^4 = \{(1/81, 1/81), (256/81, 1/81), (256/81, 256/81), (1/81, 256/81)\}$$

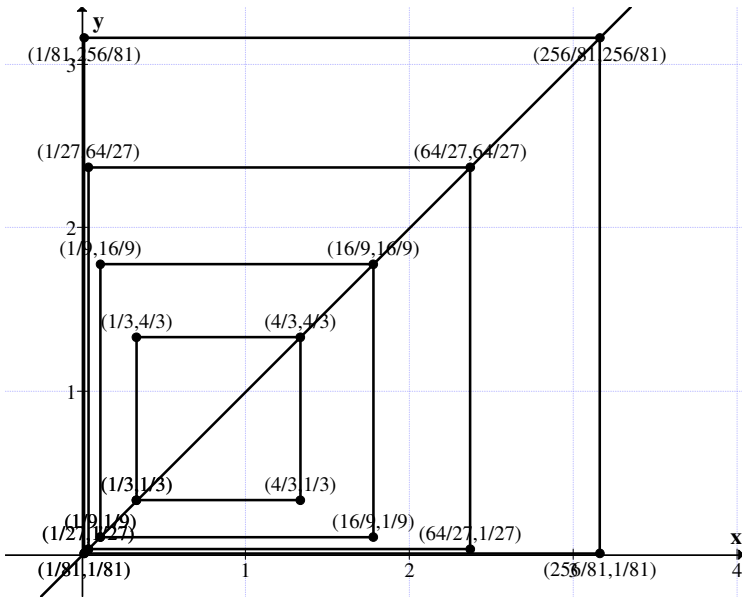The diagram of B, B^2, B^3 and B^4 is as follows:

Figure 1.26

$$B \subseteq B^2 \subseteq B^3 \subseteq B^4 \subseteq \ldots.$$

This is a very different behaviour of real squares of type I.

Let
$$C = \{(3/2, 3/2), (5/2, 3/2), (5/2, 5/2), (3/2, 5/2)\}$$
be a Euclid square of type I.

$$C^2 = \{(9/4, 9/4), (25/4, 9/4), (25/4, (25/4), (9/4, 25/4)$$
and
$$C^3 = \{(27/8, 27/8), (125/8, 27/8),$$
$$(125/8, 125/8), (27/8, 125/8)\}.$$

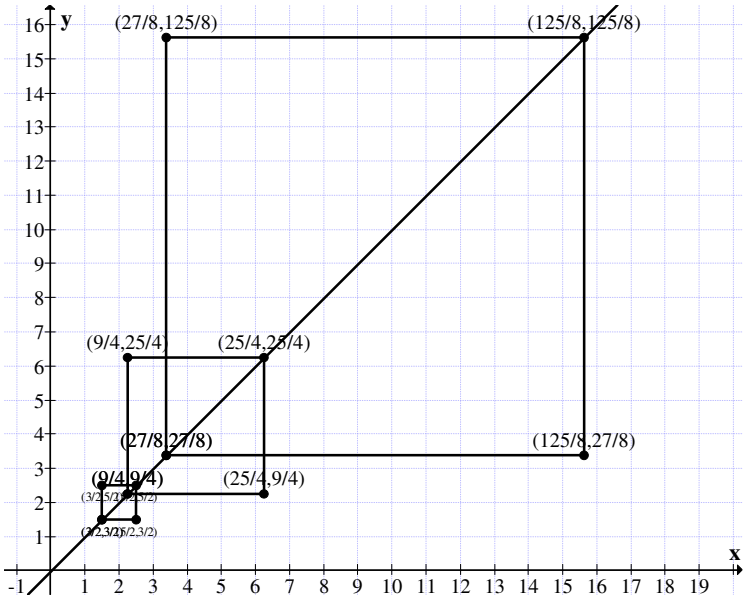The diagrammatic representation of C, C^2, C^3 is as follows:

Figure 1.27

and so on.

We see each real square is such that
$$C \cap C^2 \neq \phi, C^2 \cap C^3 \neq \phi$$
and so on.

Let
$$M = \{(6/5, 6/5), (16/5, 6/5), (16/5, 16/5), (6/5, 16/5)\} \in S_I.$$

$$M^2 = \{(36/25, 36/25), (256/25, 36/25),$$
$$(256/25, 256/25), (36/25, 265/25)\}.$$

We see $M \cap M^2 = \phi$ and so on.

Let
$$N = \{(8/7, 8/7), (15/7, 8/7), (15/7, 15/7), (8/7, 15/7)\}$$

$$N^2 = \{(64/49, 64/49), (225/49, 64/49),$$
$$(225/49, 225/49), (64/49, 225/49)\}$$

$$N \cap N^2 = \phi.$$

In view of all these we have the following theorem.

THEOREM 1.11: *Let A= {{(x, x) (x + k, x), (x + k, x + k), (x, x + k)} / x ∈ (0, 1) and k ∈ Z⁺\ {0}} ⊆ S_I be the collection of all real squares of type I.*

We have $A \subseteq A^2 \subseteq A^3 \subseteq A^4 \dots \subseteq A^n \subseteq \dots$

Proof follows from the simple fact $x^2 < x$ and $(x + k)^2 > x^2$ for all $x \in (0, 1)$ and $k \in Z^+\setminus \{0\}$.

THEOREM 1.12: *Let S_I be the collection of all real squares of type I. A = {{(x, x), (x + k, x), (x + k, x + k), (x, x + k)} | x ∈ (0, 1) and k ∈ Z⁺\ {0}} is a semigroup of real squares of type I under product and if X ∈ A then X generates a subsemigroup of A which is cyclic and*

$$X \subseteq X^2 \subseteq X^3 \subseteq X^4 \subseteq \dots$$

Proof follows from the simple fact x used the constructing lies in $0 < x < 1$. Hence the claim.

THEOREM 1.13: *Let A = {{(x, x), (x + k, x), (x + k, x + k), (x, x + k)} | x ∈ Q⁺\ (Z⁺ ∪{0}); k ∈ Z⁺\ {0}} ⊆ S_I*

(1) A is a semigroup under product.
(2) If X ∈ A then X generates a cyclic subsemigroup of A.
(3) $X^i \cap X^j \neq \{\phi\}$ if $i \neq j$ however $X^i \not\subseteq X^j$ if $i \neq j$.

The proof of this theorem is direct hence left an as exercise to the reader.

Now we proceed on to study type II real or Euclid squares.

DEFINITION 1.2 : *Let S_{II} = {Collection of all real or Euclid squares whose diagonals are parallel to x axis and y axis in the real plane}; we define S_{II} as the collection of real or Euclid squares of type II.*

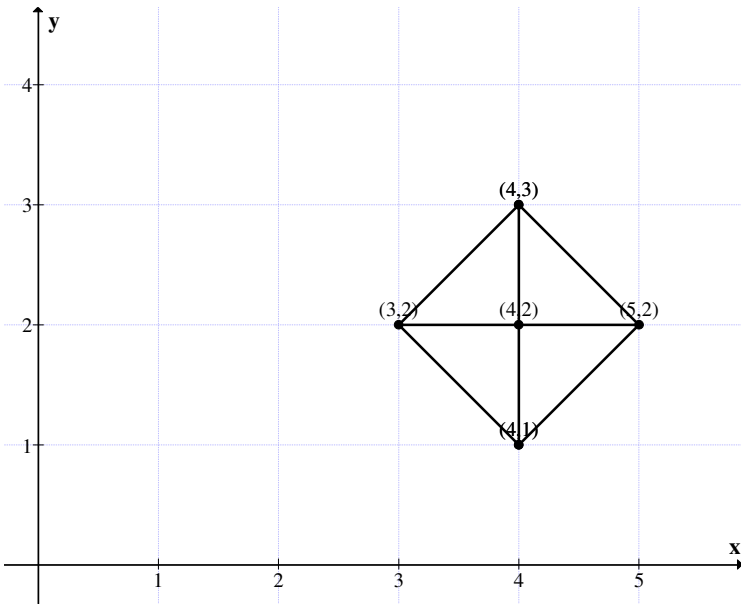We will first illustrate this situation by some examples.

Figure 1.28

The length of the side is $\sqrt{2}$.

$$A = \{(4, 1), (5, 2), (4, 3), (3, 2)\}$$

$$B = \{(5, 1), (8, 4), (5, 7), (2, 4)\}$$

be two Euclid squares of type II.

$$A + B = \{(9, 2), (13, 6), (9, 10), (5, 6)\}$$

the length of the side of

$$A + B \text{ is } \sqrt{32} \ .$$
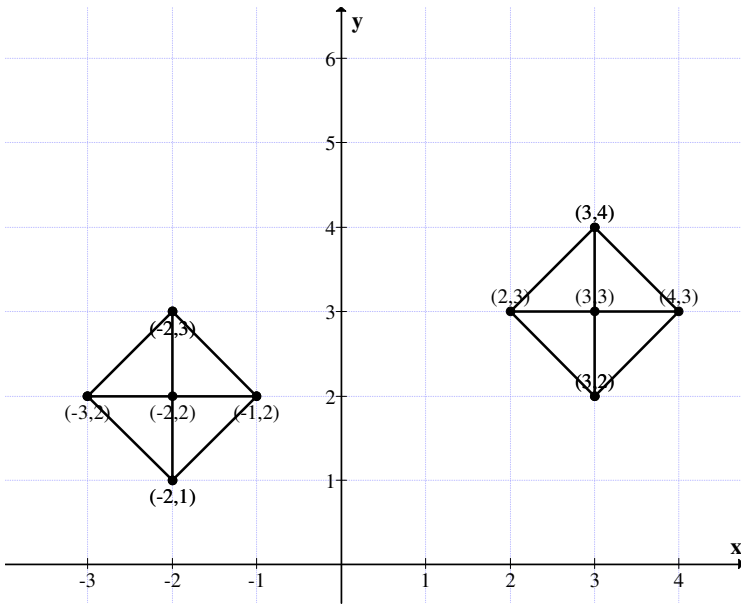
A + B is again a Euclid square of type II.

Figure 1.29

The side of the square is $\sqrt{2}$ and the type II.

$$A = \{(-2, 1), (-1, 2), (-2, 3), (-3, 2)\}$$

and

$$B = \{(3, 2), (4, 3), (3, 4), (2, 3)\}$$

be any two type II squares.

$$A + B = \{(1, 3), (3, 5), (1, 7), (-1, 5)\}$$

and the side of

$$A + B \text{ is } \sqrt{8}.$$

$$A - B = \{(-5, -1), (-5, -1), (-5, -1), (-5, -1)\}$$

is a single point square group of type II.

We assume every point of the form

$$\{(x, y), (x, y), (x, y), (x, y)\} = (x, y)$$

as a single point square of type II.

All single point real squares of type II have length of the side to be zero.

We give some more illustrations in this regard.

Let
$$A = \{(2, -5), (4, -3), (2, -10), (0, -3)\}$$
be a real square of type I. The length of the side is $\sqrt{8}$.

Now
$$B = \{(1, 3), (1, 3), (1, 3), (1, 3)\}$$
be the point square of type II.

$$A + B = \{(3, -2), (5, 0), (3, 2), (1, 0)\}.$$

This is again a real square of type II with length of the side of the square to be $\sqrt{8}$.

$$A - B = \{(1, -8), (3, -6), (1, -4), (-1, -6)\}.$$

$A - B$ is again a real square of type II and the length of the side is $\sqrt{8}$.

This is the way operation of addition or subtraction of a point square of type II with a square of type II does not change the length of the side only the position of the real square of type II is shifted to another place on the real plane.

Let
$$A = \{(8, -2), (12, 2), (8, 6), (4, 2)\}$$
be a Euclid square of type II.

$$A + A = \{(16, -4), (24, 4), (16, 12), (8, 4)\}$$
is again a Euclid square of type II the side of A is $\sqrt{32}$ where as side of 2A is $\sqrt{128}$.

Let
$$P = ((5, 3), (5, 3), (5, 3), (5, 3))$$

be a point real square of type II.

$$A - P = \{(3, -5), (7, -1), (3, 3), (-1, -1)\}.$$

The length of the side of the square $A - P$ is $\sqrt{32}$ same as that of A only the position is changed.

Now
$$A + P = \{(13, 1), (17, 5), (13, 9), (9, 5)\}$$
is again a real square of type II, length of the side of
$$A + P \text{ is } \sqrt{32}.$$

Thus the length of the side of the square does not change only the position changes.

We will illustrate this change of positions for a real square of type II.
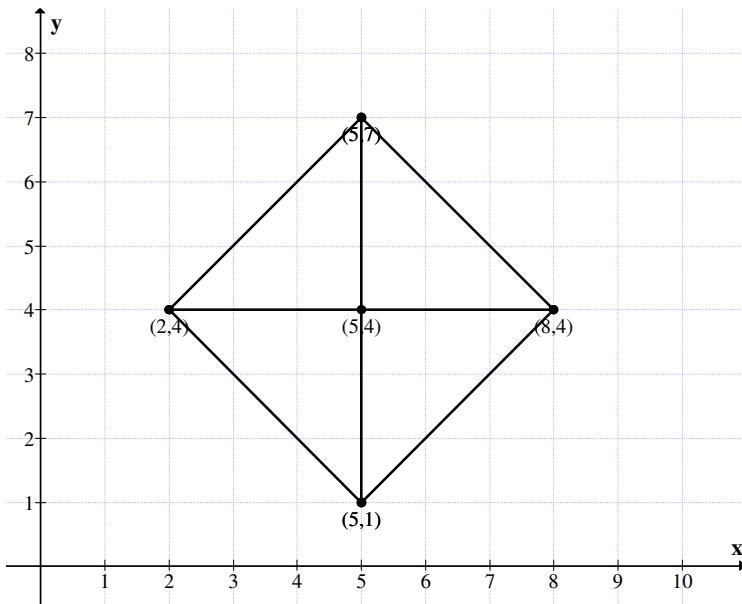

Figure 1.30

$$A = \{(5, 1), (8, 4), (5, 7), (2, 4)\}$$

a real square of type II. The length of the side of A is $\sqrt{18}$.

Let P = {(1, 1)} be a point square of type II.

$$A + P = \{(6, 2), (9, 5), (6, 8), (3, 5)\}$$

is a real square of type II.

Length of the side of A + P is also $\sqrt{18}$.

Now consider
$$A - P = \{(4, 0), (7, 3), (4, 6), (1, 3)\}$$
is real square of type II and the length of the side of
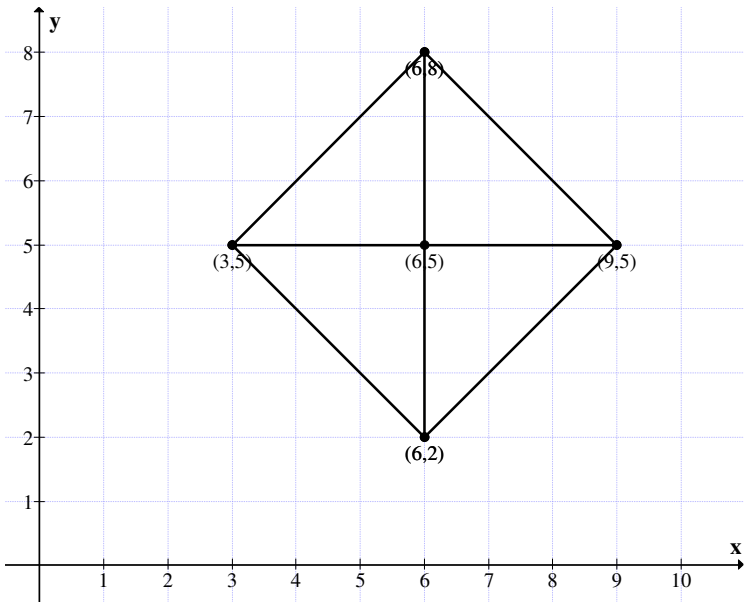$$A - P \text{ is } \sqrt{18} .$$

The diagram of A + P is

Figure 1.31

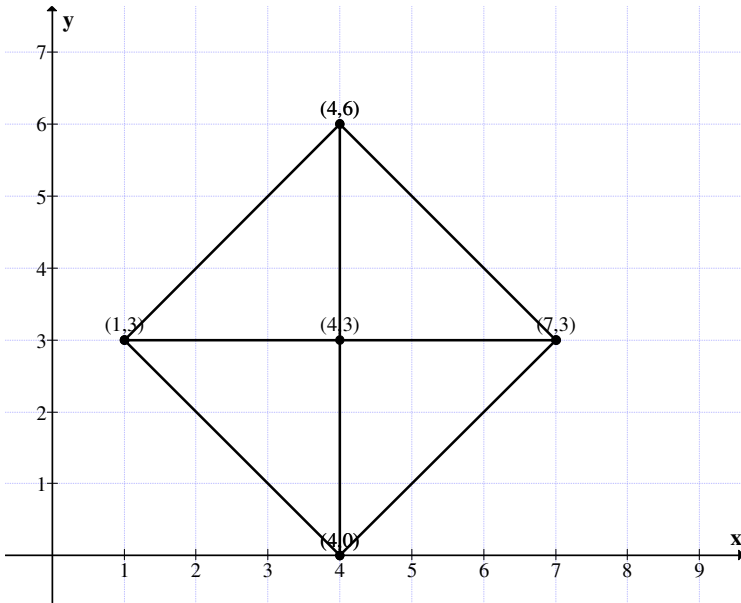The diagram for A – P is as follows, where P = (2, 2).

Figure 1.32

Now let

$$P_1 = \{(-2, -1)\}$$

be the point square of type II and length of the side of

$$A + P_1 = \{(3, 0), (6, 3), (3, 6), (0, 3)\}$$

is a real square of type II and the length of the side of

$$A + P_1 \text{ is } \sqrt{18} \,.$$

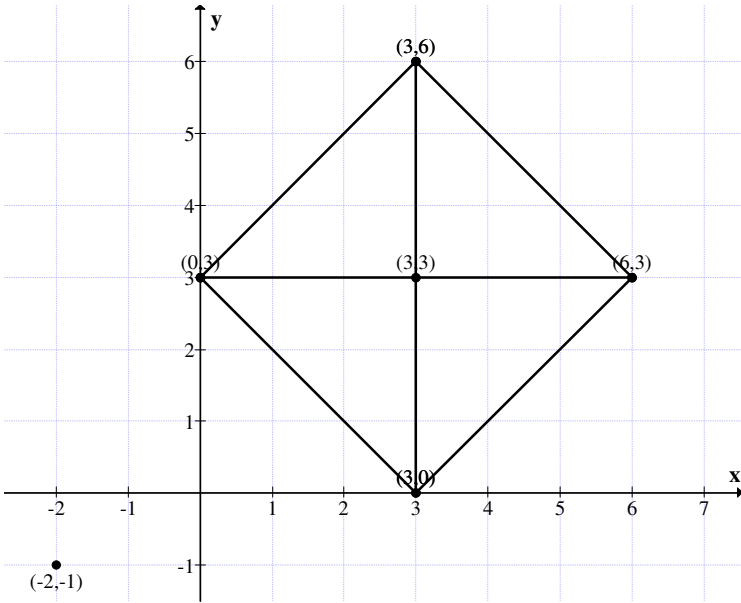The diagram for A + P_1 is as follows

Figure 1.33

We find
$$A - P_1 = \{(7, 2), (10, 5), (7, 8), (4, 5)\}.$$

The real square of type II is also has the length of its to be $\sqrt{18}$.

The diagram associated with the real or Euclid type II square $A - P_1$ is given in Figure 1.34.

The length of the side of $A - P_1$ is also $\sqrt{18}$.

Let $T = (2, 4)$ be the point square of type II.

$$A + T = \{(7, 5), (10, 8), (7, 11), (4, 8)\}$$
is a real square of type II of length of the side is $\sqrt{18}$.

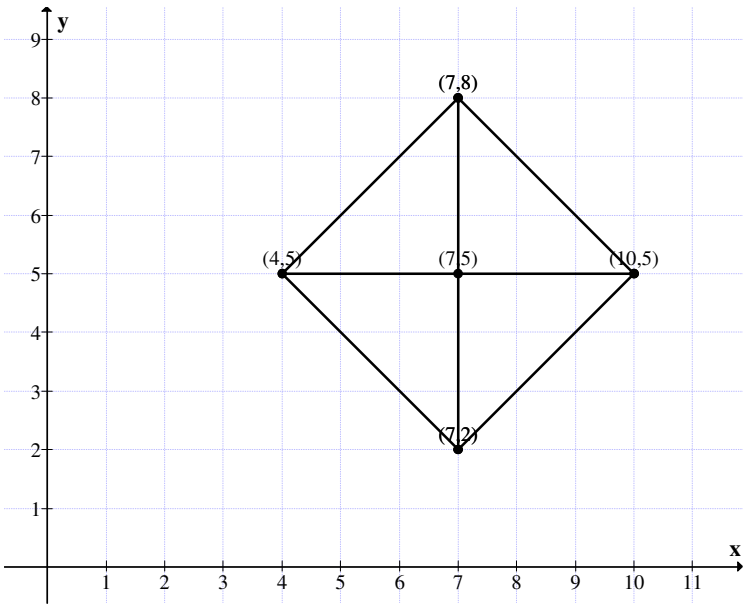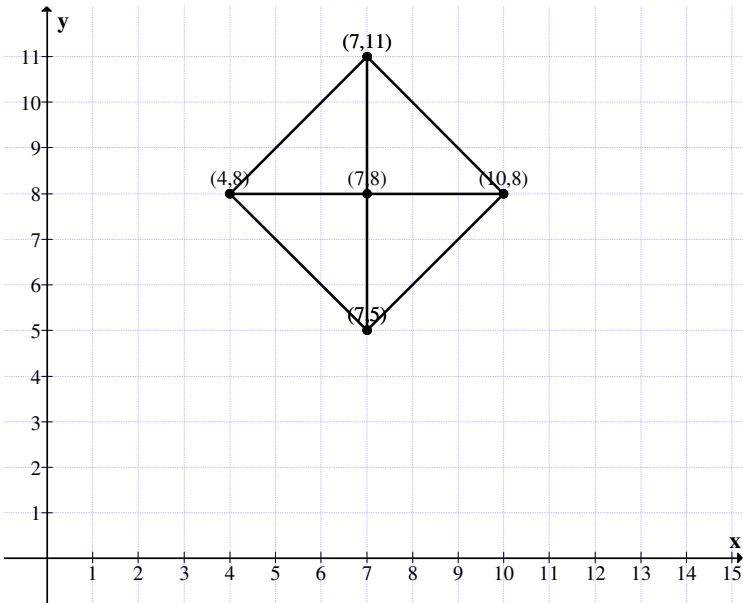The diagram of the real square of type II is given in Figure 1.35.

Figure 1.34

Figure 1.35

$$A - T = \{(3, -3), (6, 0), (3, 3), (0, 0)\}$$

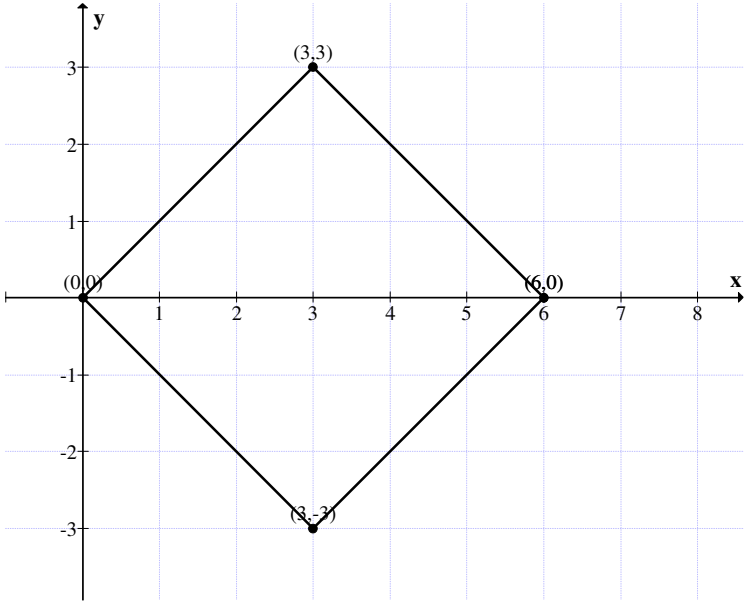is again a real square of type II with length of the side $\sqrt{18}$.

Figure 1.36

We see if

$$A = \{(7, 5), (10, 8), (7, 11), (4, 8)\}$$

and

$$B = \{(4, 0), (7, 3), (4, 6), (1, 3)\}$$

be any two real square of type II in S_{II}.

$A + B = \{(11, 5), (17, 11), (11, 17), (5, 11)\}$.
The length of the side is $\sqrt{72}$.

$$A - B = \{(3, 5), (3, 5), (3, 5), (3, 5)\}$$

is a real point square of type II.

Let

$$A = \{(8, -2), (12, 2), (8, 6), (4, 2)\}$$

$$B = \{(4, 0), (7, 3), (4, 6), (1, 3)\}$$

be the real square of type II.

$$A \times B = \{(32, 0), (84, 6), (32, 36), (4, 6)\}.$$

Clearly $A \times B$ is not a Euclid square of type II.

So product is not defined in case of type II real squares. We see $\{S_{II}, + \}$ is a group.

Let

$$A = \{(8, -2), (12, 2), (8, 6), (4, 2)\}$$

and

$$B = \{(4, 0), (7, 3), (4, 6), (1, 3)\} \in S_{II}.$$

$$A + B = \{(12, -2), (19, 5), (12, 12), (5, 5)\}$$

the length of the side is $\sqrt{98}$.

$$A - B = \{(4, -2), (5, -1), (4, 0), (3, -1)\}.$$

$A - B$ is the case of type II square of side length $\sqrt{2}$.

Now given

$$A = \{(8, -2), (12, 2), (8, 6), (4, 2)\} \in S_{II},$$

$$-A = \{(-8, 2), (-12, -2), (-8, -6), (-4, -2)\}$$

is the real square of type II.

We see

$$A + (-A) = \{(0, 0), (0, 0), (0, 0), (0, 0)\}.$$

If

$$B = \{(4, 0), (7, 3), (4, 6), (1, 3)\} \in S_{II}.$$

$$-B = \{(-4, 0), (-7, -3), (-4, -6), (-1, -3)\} \in S_{II}.$$

$$B + (-B) = \{(0, 0), (0, 0), (0, 0), (0, 0)\}$$

is the point zero square of type II.

Let
$$A = \{(4, 5), (7, 8), (7, 2), (10, 5)\}$$
be a type II real square

$$A + A = \{(8, 10), (14, 16), (14, 4), (20, 10)\}$$

$$A + A + A = \{(12, 15), (21, 24), (21, 6), (30, 15)\}$$
and so on.

We call the point of intersection of the diagonals as the center of the real square.

For we see knowing the centre of the type II real squares we can find several properties about them.

Let
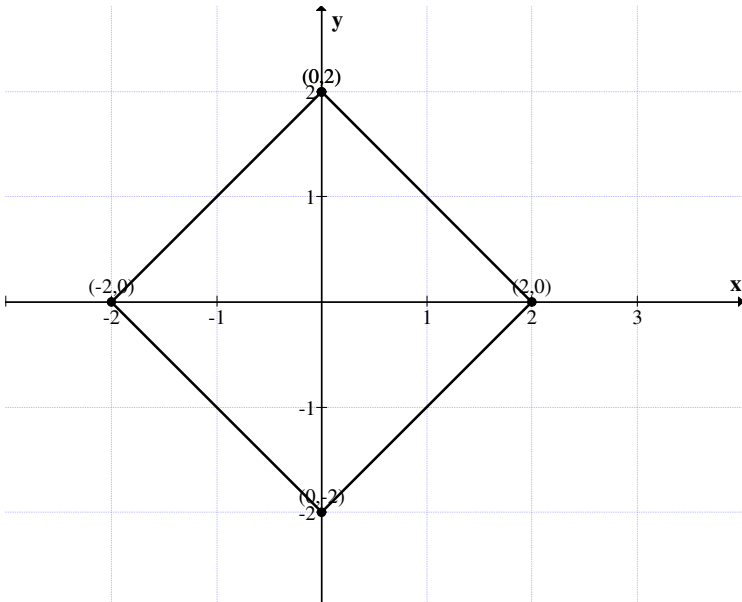$$A = \{(0, -2), (2, 0), (0, 2), (-2, 0)\}$$
be the real square of type II.

Figure 1.37
$$A + A = \{(0, -4), (4, 0), (0, 4), (-4, 0)\}$$

$$A + A + A = \{(0, -6), (6, 0), (0, 6), (-6, 0)\}$$

$$A + A + A + A = \{(0, -8), (8, 0), (0, 8), (-8, 0)\}$$

and so on.

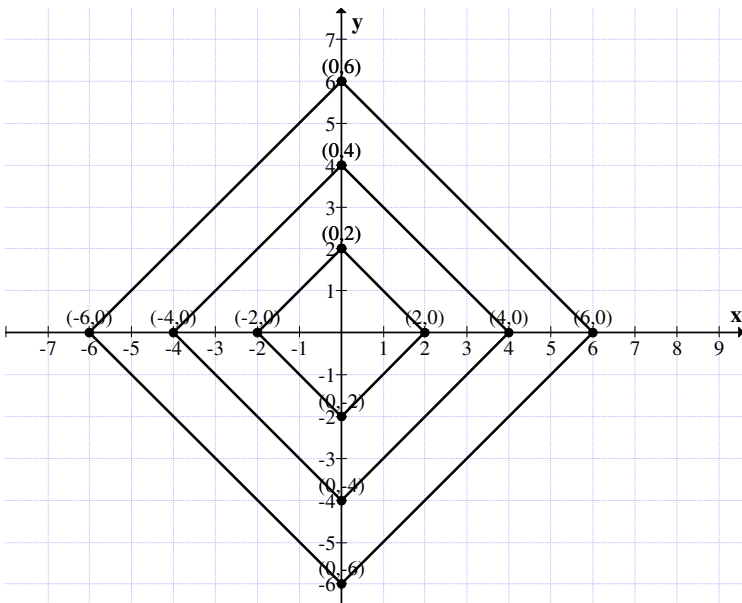We have the following diagram for A, 2A, 3A, 4A and so on.

Figure 1.38

In view of this if A is a real type II square with centre $(0, 0)$ then we see A generates a semigroup under + and

$$A \subseteq A + A \subseteq A + A + A \subseteq A + A + A + A \subseteq \ldots$$

Thus $A \subseteq 2A \subseteq 3A \subseteq 4A \subseteq 5A \subseteq \ldots$

Hence the claim.

THEOREM 1.14: *Let A = {(0, –t), (t, 0), (0, t), (–t, 0)} ∈ S$_{II}$ be a real square of type II with center (0, 0).*

> *(i)   Then A generates a semigroup under + .*
> *(ii)   A ⊆ 2A ⊆ 3A ⊆ 4A ⊆ ....*

Proof : We see A generates a semigroup under + for A + A has (0, 0) as its center.

For if

$$A = \{(0, –t), (t, 0), (0, t), (–t, 0)\}$$

then

$$nA = \underbrace{A + ... + A}_{n-times} = \{(0, –nt), (nt, 0), (0, nt), (–nt, 0)\}.$$

Clearly of nA has its center to be (0, 0).

Further we see

$$A \subseteq A + A, \; A + A \subseteq 3A, \; ... \; nA \subseteq (n + 1)A$$

and so on.

Hence the proof of the theorem.

Consider

$$A = \{(0, –5), (5, 0), (0, 5), (–5, 0)\} \in S_{II}.$$

Clearly center of A is (0, 0).

Now

$$–A = \{(0, 5), (–5, 0), (0, –5), (5, 0)\} \in S_{II}.$$

–A also has (0, 0) as its centre.

$$A + (–A) = \{(0, 0)\}$$

the point real square of type II.

Infact (0, 0) is also the additive identity point zero square of S$_{II}$.

For (0, 0) + X = X + (0, 0)   = X for all X ∈ S$_{II}$.

Thus we can say (0, 0) the real zero point square is the additive identity of S_I.

Let
$$B = \{(0, -3), (3, 0), (0, 3), (-3, 0)\} \in S_{II}$$
be the Euclid square of type II. B has (0, 0) as its center.

Now
$$-B = \{(0, 3), (-3, 0), (0, -3), (3, 0)\}.$$

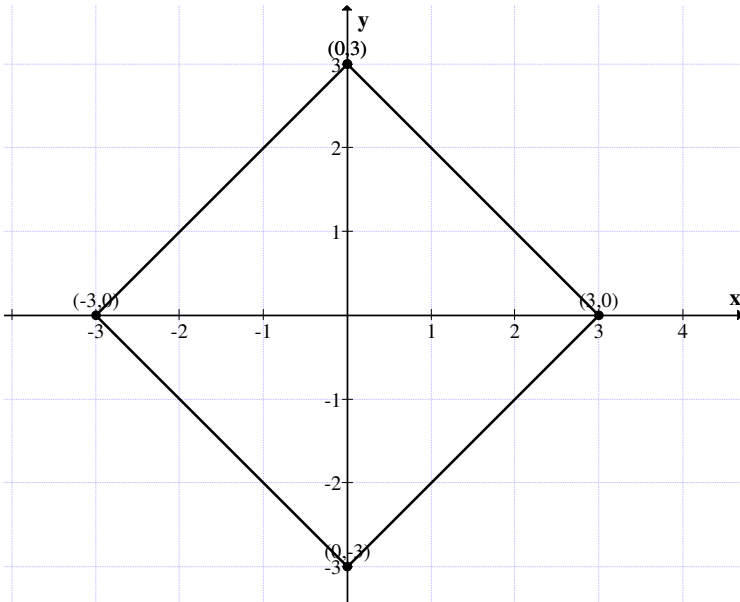If we try to give the figure of B and −B we have the following problem.

Figure 1.39

So infact both B and −B have the same figure.

Thus we can say for those real squares of type II with centre (0, 0) we see given B, −B can be got from B as follows:

We say if

$$B = \{(0, -a), (a, 0), (0, a), (-a, 0)\}$$

then the two shifts of B is as follows:

$$\{(0, a), (-a, 0), (0, -a), (a, 0)\} = -B$$

thus we see a double shift gives the additive inverse of a real square of type II which has (0, 0) as its center.

However for any real square of type II two shift or double shift will not yield to its inverse.

For if

$$A = \{(16, -4), ((24, 4), (16, 12), (8, 4)\} \in S_{II}$$

the double shift of A is

$$A' = \{(16, 12), (8, 4), (16, -4), (24, 4)\}$$

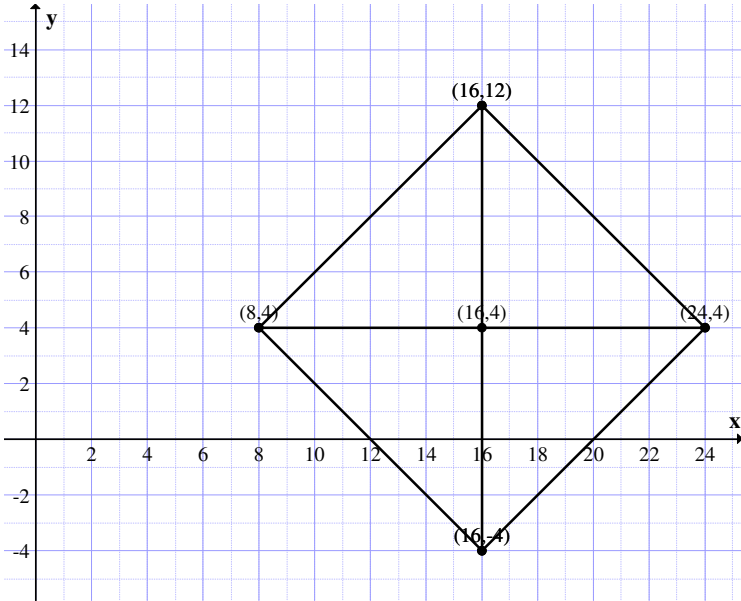the figure for A and A′ is as follows.

Figure 1.40: *Figure of A'*

Clearly A and A′ are not inverse of each other for they are identical. One can get A′ from A by rotating the line parallel to x axis through 180°.

Now we see if A is any type II real square then by rotating its first vertex 180° degrees in the anti clock wise direction we get another real square of type II. By this the position of the diagonals are changed.

Hence if A = {(3, 2), (4, 3), (3, 4), (2, 3)} be the real square of type II with the following diagram.
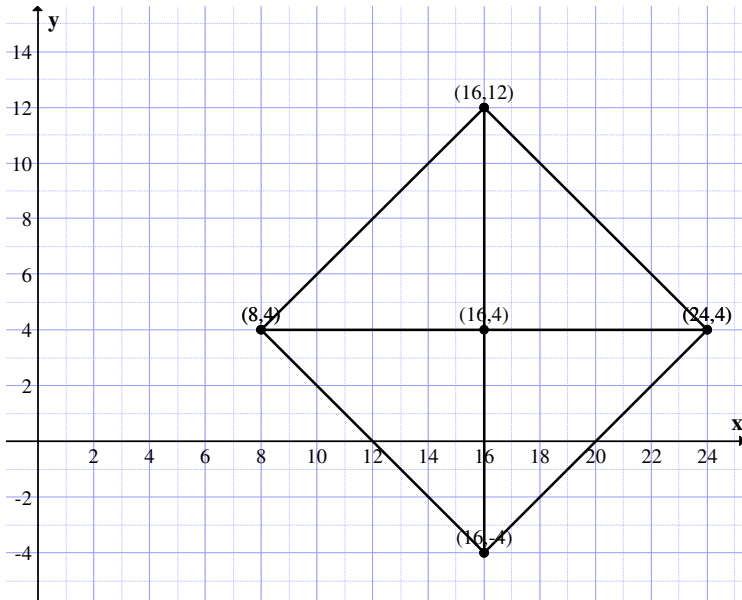

Figure 1.41

By rotating A through 180° degree in the clockwise direction we get the same figure.

By rotating A through 270°degrees we get the same figure.

By rotating A through 90° degrees we get the same figure again.

Now we will study the properties of each of these real squares of type II.

Let us consider the real square
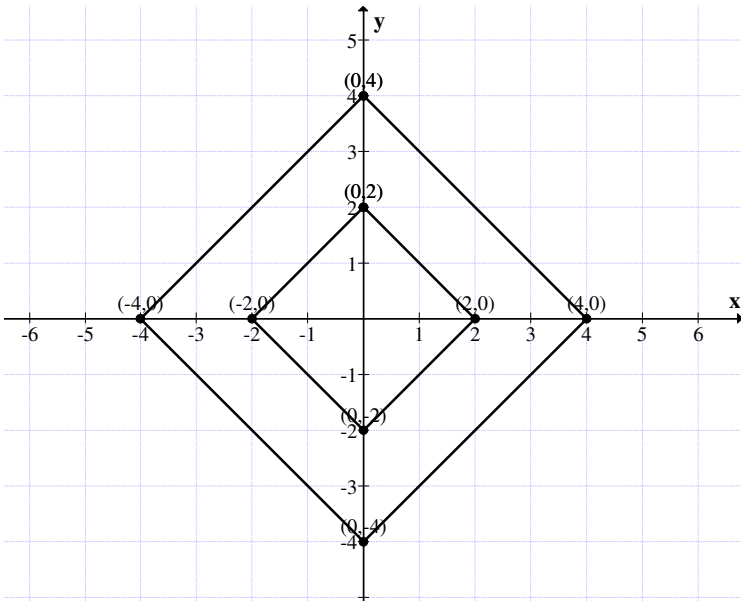$$A = \{(0, -2), (2, 0), (0, 2), (-2, 0)\} \in S_{II}.$$

Figure 1.42

Clearly A + A is again a real square of type II and $A \subseteq A + A$.

Similarly $A + A \subseteq A + A + A$ and so on.

Thus A generates a semigroup of real square with
$$A \subseteq 2A \subseteq 3A \subseteq 4A \subseteq \ldots$$

Suppose we try to find the product of A with itself.

$$A^2 = \{(0, 4), (4, 0), (0, 4), (4, 0)\}$$

$$A^3 = \{(0, -8), (8, 0), (0, 8), (-8, 0)\}$$

$$A^4 = \{(0, 16), (16, 0), (0, 16), (16, 0)\}$$

and so on.

All odd powers of A lead to real square of type II and all even powers lead to a straight line. This is described by the following figure.
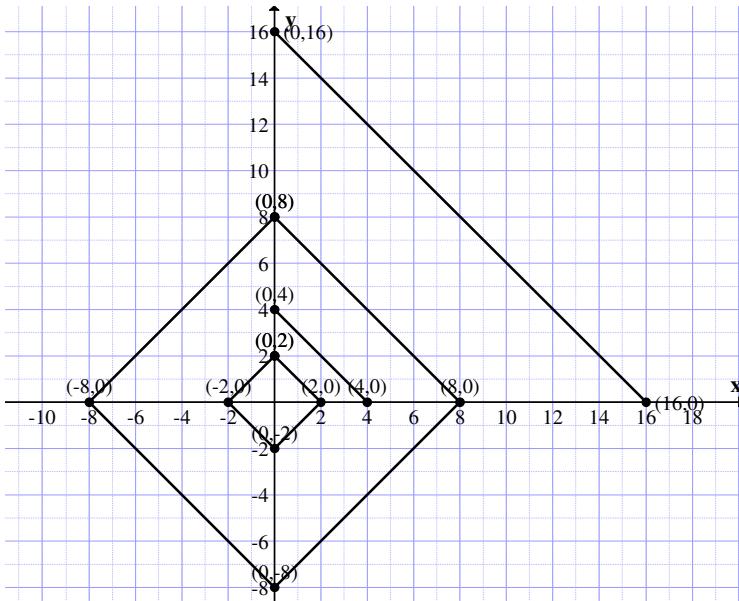

Figure 1.43

and so on.

Thus we see if

$$A = \{(0, -p), (p, 0), (0, p), (-p, 0)\}; p \in Z^+ \setminus \{0\}.$$
A^2 is a line of the form $\{(0, p), (p, 0)\}$.
$$A^3 = \{(0, -p^3), (p^3, 0), (0, p^3), (-p^3, 0)\}$$
is again a real square of type II. Thus we can claim if A is a real square of type II given in that form all even powers are

intercepts of the axes in the first quadrant and the odd powers are real squares of type II.

Inview of this we have the following theorem.

THEOREM 1.15: *Let A = {(0, –n), (n, 0), (0, n), (–n, 0)} $\in S_{II}$ where n $\in Z^+ \setminus \{0\}$ be the real square of type II.*

(1) *All odd powers of A are again real square of type II.*
(2) *All even powers of A are just the intercept line of the form {(0, n^{2t}), (n^{2t}, 0)}*

The proof is direct hence left as an exercise to the reader.

Let
$$A = \{(4, 2), (6, 4), (4, 6), (2, 4)\}$$
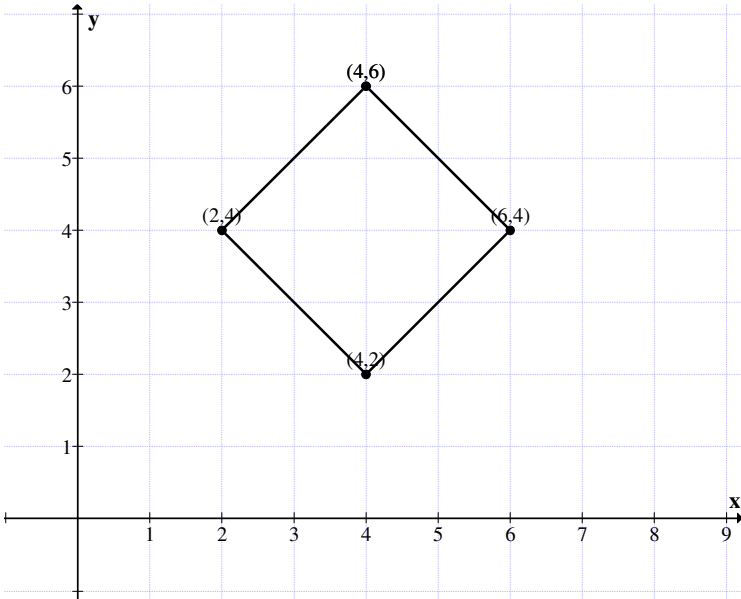be the real square of type I.

Figure 1.44

Suppose we try to find
$$A^2 = \{(16, 4), (36, 16), (16, 36), (4, 16)\}.$$

A^2 is not a real square of type II.

We can say maximum A^2 can be a quadrilateral in this case.

Let

$$A = \{(6, 5), (7, 6), (6, 7), (5, 6)\}$$

be a square of type II.

Figure 1.45
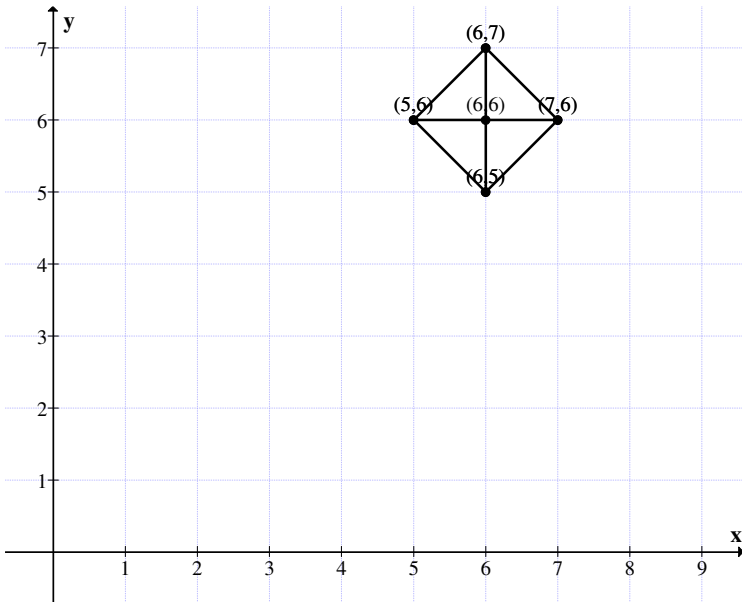
$A^2 = \{(36, 25), (49, 36), (36, 49), (25, 36)$ is not a real square type II.

It is not easy to find product of two real squares of type II. A^2 at times is a line, at times a point, at times a rectangle and at other times a quadrilateral.

So if we take

$$A = \{(0, -4), (4, 0), (0, 4), (-4, 0)\} \in S_{II}.$$

We see

$$A^2 = \{(0, 16), (16, 0), (0, 16), (16, 0)\}$$

is only a line.

However
$$A^3 = \{(0, -64), (64, 0), (0, 64), (-64, 0)\}$$
is a real square of type II.

Thus if
$$S = \{(0, -2), (2, 0), (0, 2), (-2, 0)\} \in S_{II}.$$

$$S^2 = \{(0, 4), (4, 0), (0, 4), (4, 0)\}$$
is a line.
$$S^3 = \{(0, -8), (8, 0), (0, 8), (-8, 0)\}$$

$$S^4 = \{(0, 16), (16, 0), (0, 16), (16, 0)\}$$
is a line.

S^5 is a square and so on.

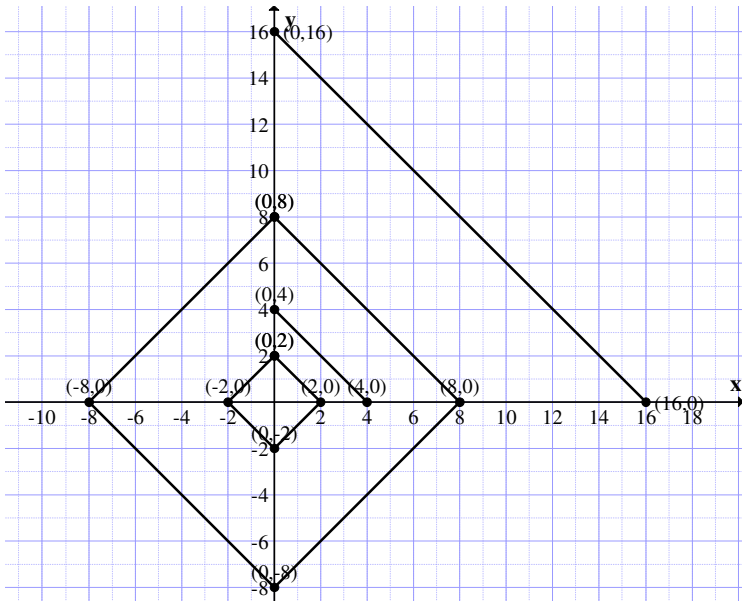The diagrammatic representation is as follows:

Figure 1.46

A square, a finite line again a square an intercept of equal length in the first quadrant and so on.

Thus we call these square which are lines as degenerate line squares.

These line squares, point squares and squares of type II can have interesting properties associated with them.

It is interesting and is kept on record that several types of translation or more non mathematical shifts in position of these squares can be easily carried out by adding or subtracting it to a described point square.

Thus instead of complicating by using functions of transformation one can easily obtain the same effect by adding to the square the point square. This sort of learning can help researchers in other fields who do not possesses a deep knowledge of mathematics.

However we are not going to deal with properties of rectangles of type I elaborately in this book.

A set of problems are suggested for the reader.

Problems :

1. Find all new and special features enjoyed by the group S_I of real or Euclid square of type I.

2. Find all subgroups of S_1 of type I Euclid squares.

3. Let $\{ S_I^{++} , + \} \subseteq S_I$ be a real square semigroup of type I.

    (i) Prove $\{ S_I^{++} , + \}$ is a monoid.
    (ii) Find all real square subsemigroups of type I.
    (iii) Is $\{ S_I^{++} , + \}$ S–subsemigroup?

4. Let $\{ S_I^{--} , + \} \subseteq \{S_I, + \}$ be a real semigroup of real squares of type I.

Study questions (i) to (iii) of problem 3 for this $\{ S_I^{--}, + \}$.

5. Is $\{ S_I^{+-}, + \}$ a real square semigroup of type I?

6. Is $\{ S_I^{-+}, + \}$ a real square semigroup of type I?

7. Compare $\{ S_I^{--}, + \}$ and $\{ S_I^{++}, + \}$, real square semigroups of type I.
   $\{ S_I^{+-}, + \}$ and $\{ S_I^{-+}, + \}$ with each other.

   Bring in the similarities and dissimilarities between them.

8. Let $\{S_I, + \}$ be the Euclid or real square of type I.

   Can the concept of quotient group of type I be defined for $\{S_I, + \}$?

9. Can $\{S_I, + \}$ have subgroups of finite order?

10. Prove S_I under $\times$ is not even a closed set.

11. Find semigroups which are subsets of S_I under product.

12. Can S_I have infinite number of subsets which are subsemigroups of S_I under $\times$?

13. Obtain some special features enjoyed by S_I under $\times$.

14. Let $A = \{(1, 1), (3, 1), (3, 3), (1, 3)\}$ be real square of type I.

    (i) Prove A generates a cyclic semigroup of infinite order under product.

(ii) Can ⟨A⟩ have subsemigroups of finite order?

(iii) Can ⟨A⟩ have infinite number of subsemigroups under product?

15. Let X = {(7, 3), (10, 3), (10, 6), (7, 6)} ∈ S_I be the real square of type I.

(i) Does X generate a semigroup under product?

(ii) Does X generate a semigroup under ' + '?

(iii) Prove X cannot generate a subgroup under + .

16. Let A = {(0, 5), (2, 5), (0, 7), (2, 7)} be a square of type I.

(i) What is the length of the side of the square A^5?

(ii) Prove A generates an infinite cyclic semigroup.

(iii) Let P_1 = {(0, 5), (0, 5), (0, 5), (0, 5)} be the point square.

Find P_1 + A, P_1 – A and A – P_1

(iv) Will – A generate a cyclic semigroup?

17. What are the special and distinct features enjoyed by type I squares?

18. Is {S_I, ×} a semigroup?

19. Let T = {(–5, 3), (–1, 3), (–5, 7), (–1, 7)} be real square of type I.

(i) Study questions (i) to (iv) of problem 16 for this T.

(ii) If M = {(–1, 7), (–1, 7), (–1, 7), (–1, 7)} be the point square.

Find M + T, M – T, T – M and 2T + M.

20. Prove the procedure of adding or subtracting a point square to any real or Euclid square of type I leads to a shift or translation of Euclid square.

Prove this sort of transformation is easy than defining functions for these translation.

21. Let B = {(3, 4), (8, 4), (8, 9), (3, 9)} and C = {(2, −3), (7, −3), (2, 2), (7, 2)} be two real or Euclid square of type I.

    (i) Find B + C.
    (ii) Find B − C and C − B.
    (iii) What is structure enjoyed by BC?
    (iv) Find (i) B^2, 2B (ii) C^3, 4C.

22. Let M = {(2, 2), (6, 2), (2, 6), (6, 6)} and P = {(13, 2), (3, 12), (3, 2), (13, 12)} be real Euclid square of type I.

    (i) Study questions (i) to (iv) of problem 21 for this M and P.
    (ii) If S = {(0, −3), (0, −3), (0, −3), (0, −3)} is a point real square.
        Find (i) SM (ii) PS (iii) PSM
    (iv) P + S + M   (v) P − S − M
    (vi) P + S − M and   (vii) M − P + S.

23. Obtain any of the special features associated with real or Euclid squares of type II.

24. Compare type I and type II real or Euclid squares.

25. Let A be a Euclid square of type I and B be a Euclid square of type II.

    (i) Is A + B or B − A or BA defined?
    (ii) Substantiate the answer in (i)

26. Give some examples of Euclid squares of type II with length of its side 5.

    How many such squares exist?

27. Let A = {(–4, 0), (4, 0), (0, 4), (0, –4)} be a Euclid square of type I.

    (i) Find A + A.

    (ii) What is the structure of A^2, A^3 and A^4?

    (iii) If P = {(5, 3), (5, 3), (5, 3), (5, 3)} is a point square find P + A, P – A, A – P and PA.

28. Let W = {(8, 3), (8, –3), (6, 0), (10, 0)} be the Euclid square of type II.

    (i) Find W + W.

    (ii) Draw this W in the Euclid plane.

    (iii) If P = {(6, –3), (6, –3), (6, –3), (6, –3)} ∈ S_I. Find W + P, W–P and P–W.

    (iv) Trace all the 3 squares W + P, W–P and P–W.

    (v) Find PW, W^2 and P^2 and trace them.

29. Let M be the real type I Euclid square given by {(–4, –9), (–1, –9), (–4, –6), (–1, –6)}.

    (i) Find area of M.

    (ii) Find center of M.

    (iii) Find 3M = M + M + M and find the length of side of 3 M.

    (iv) Is it related with the side of M?

    (v) Can M^2 be a square or a quadrilateral?

30. Let S = {(4, 5), (7, 8), (7, 2), (10, 5)} be the real square of type II.

    (i)   Find S + T where T={(3, 1), (3, 1), (3, 1), (3, 1)}.

    (ii) Find P–S where P = {(–3, –5), (–3, –5), (–3, –5), (–3, –5)}

    (iii) Find T–S, S–T, S–P and P + S.

31. Let S = {(8, 10), (14, 16), (14, 4), (20, 10)} be the real square of type II.

    Let P = {(–4, –8), (–4, –8), (–4, –8), (–4, –8)} be the point square.

    (i) Find S + P, S–P and P–S.

    (ii) Find 3S–P and P–2S.

    (iii) What is area of S + P, S–P and P–S?

    (iv) Find the length of the of the squares P + S, S–P, P–S, 3S–P and P–2S.

    (v) What is the largest size square of the 5 squares in iv?

32. Let W = {(3, –5), (7, –1), (3, 3), (–1, –1)} be the real of Euclid square of type II.

    Let P_1 = {(–3, 5), (–3, 5), (–3, 5), (–3, 5)} be the point square.

    (i) Study questions (i) to (v) using W and P_1.

    (ii) Let P_2 = {(–7, 1), (–7, 1), (–7, 1), (–7, 1)} be the real point square.

Using P_2 and W study questions (i) to (v) of problem 31.

(iii) Let $P_3 = \{(-3, 3), (-3, 3), (-3, 3), (-3, 3)\}$ be point real square.

Study questions (i) to (v) of problem 31 using P_3 and W.

(iv) Let $P_4 = \{(1, 1), (1, 1), (1, 1), (1, 1)\}$ be the real or Euclid point square.

Study questions (i) to (v) of problem 31 using P_4 and W.

(v) Now compare and contrast the solutions of (i) to (iv).

33. Let $M = \{(-3, 0), (0, 3), (3, 0), (0, -3)\}$ be the Euclid square of type II.

(i) Show M^2, M^4, … are degenerate Euclid squares.

(ii) Prove only M^3, M^5, …, $M^{2n + 1}$ are non degenerate Euclid squares.

(iii) Is the side lengths of M^3, M^5 become larger or Smaller?

(iv) Study (iii) if $M_1 = \{(-1/2, 0), (0, -1/2), (1/2, 0), (0, 1/2)\}$.

Chapter Two

SQUARES OF NEUTROSOPHIC PLANE

In this chapter we proceed onto define the two types of squares on the neutrosophic plane. Further we define algebraic operations on them.

Recall $\langle R \cup I \rangle = \{a + bI \mid a, b \in R, I^2 = I\}$ is the real neutrosophic plane.

$\langle Q \cup I \rangle = \{a + bI \mid a, b \in Q, I^2 = I\}$ is the rational neutrosophic plane,

$\langle Z \cup I \rangle = \{a + bI \mid a, b \in Z, I^2 = I\}$ is the integer neutrosophic plane. First we will built type I neutrosophic squares.

That is all squares in these planes are such that their sides are parallel to the x and y axis is taken as the neutrosophic axis. In all graph the second coordinates in nI that is if P is a point (x,y) then by default of notation it is (x, yI).

We will first illustrate this situation by an example.

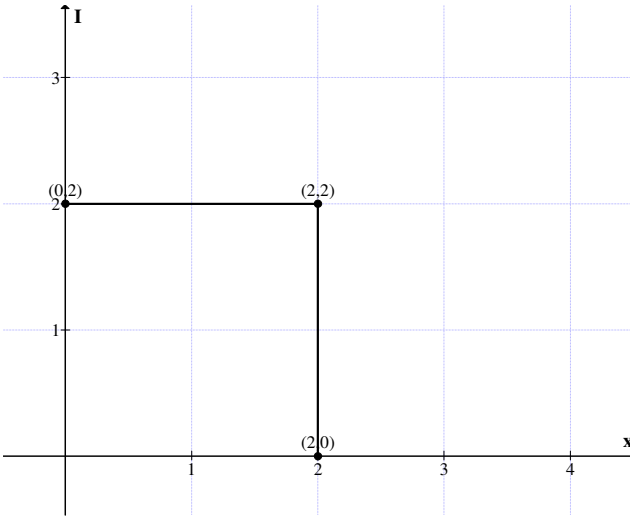***Example 2.1:*** Let N = {(0, 2I), (2,0), (0,0), (2,2I)} be the integer neutrosophic type I square.

Figure 2.1

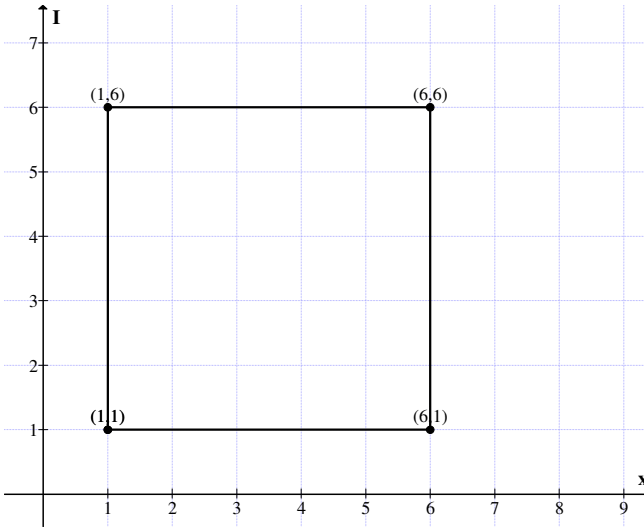***Example 2.2:*** Let B = {(1, I), (6, 6I), (1, 6I), (6, 6I)}

Figure 2.2

be the integer neutrosophic square of type I.

Example 2.3: Let S = {(0,0), (–2,0), (0,–2I), (2,2I)} be the integer neutrosophic type I square in the 3rd quadrant.
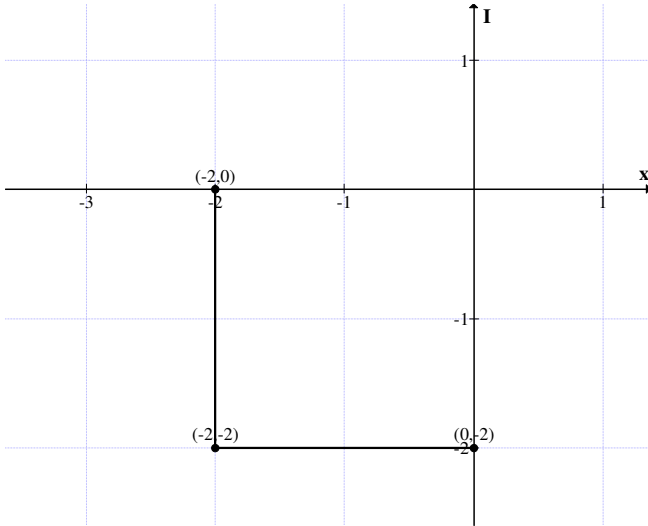

Figure 2.3

Example 2.4: Let P = {(0, 0), (3, 0), (0, –3I), (3, –3I)} be the integer neutrosophic type I square in the forth quadrant.
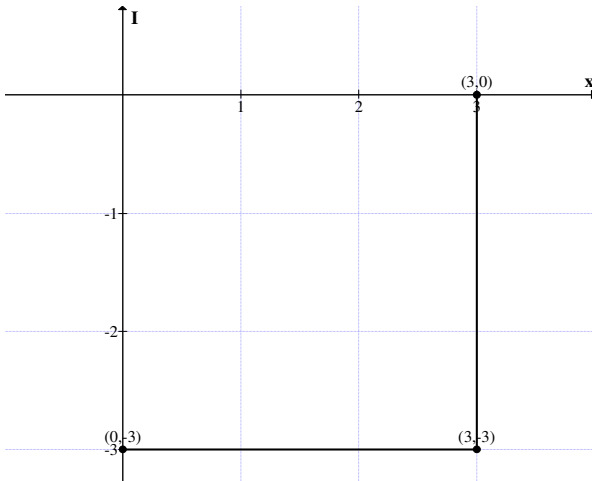

Figure 2.4

Example 2.5: Let S = {(0, 0), (0, 7I), (–7, 0), (–7, 7I)} be the integer neutrosophic square of type I in the second quadrant.

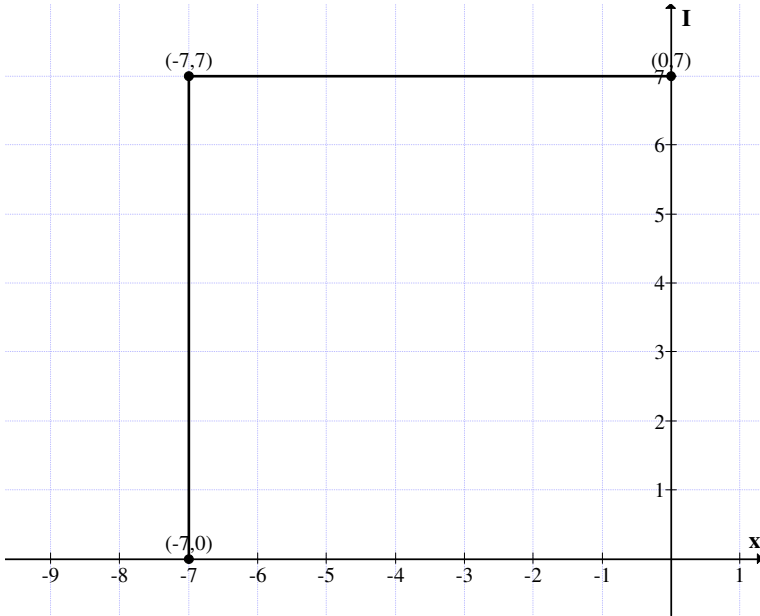***Figure 2.5***

Now

N_I = {Collection of all neutrosophic integer square of type I}.

We proceed onto built or define operations on N_I.

Let
$$A = \{(5, 3I), (7, 3I), (5, 5I), (7, 5I)\}$$
and

$$B = \{(0, 0), (6, 0), (6, 6I), (0, 6I)\}$$
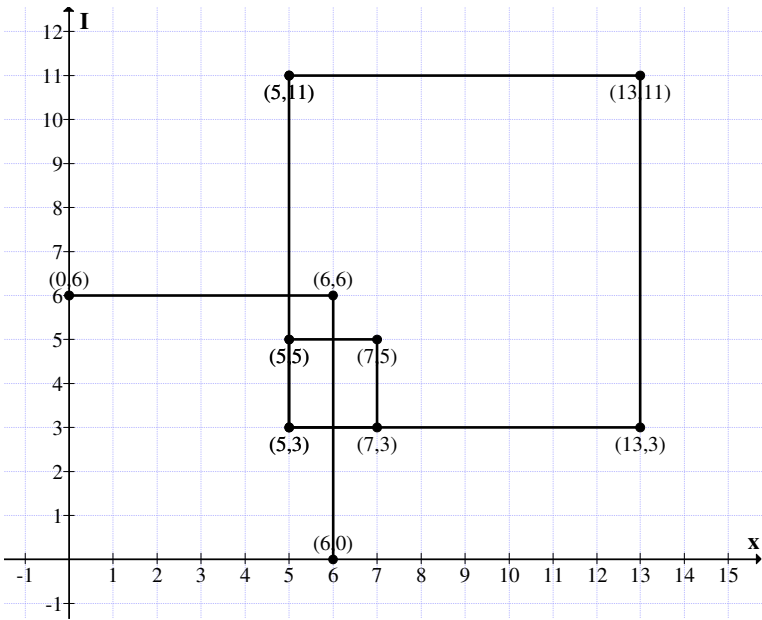
be two neutrosophic type I squares.

Figure 2.6

A + B = {(5, 3I), (5, 11I), (13, 11I), (13, 3I)}.

Thus A + B is again a neutrosophic square of type II.

Consider the zero neutrosophic square given by

0_I = {(0, 0), (0, 0), (0, 0), (0, 0)} in N_I.

Clearly for every A $\in$ N_I;

$$A + 0_I = 0_I + A = A.$$

The zero square 0_I is defined as the zero neutrosophic square of type I.

For every
    A = {(–3, 6I), (0, 6I), (0, 9I), (–3, 9I)} $\in$ N_I,
we have unique

$$- A = \{(3, -6I), (0, -6I), (0, -9I), (3, -9I)\} \in N_I$$
such that
$$A + (-A) = \{(0, 0), (0, 0), (0, 0), (0, 0)\} = 0.$$

THEOREM 2.1: *Let $\{N_I, + \}$ be the collection of all neutrosophic squares of type I in the neutrosophic plane $\langle Z \cup I\rangle$ (or $\langle R \cup I\rangle$ or $\langle Q \cup I\rangle$).*

$\{N_I, + \}$ is an abelian group of infinite order.

Proof : Given $N_I = \{$Collection of all neutrosophic squares of type I from $\langle Z \cup I\rangle$ (or $\langle R \cup I\rangle$ or $\langle Q \cup I\rangle\}$. To prove N_I is an abelian group under ' + '. Clearly N_I is of infinite order.

Any $A \in N_I$ has the following representation.

$A = \{(a, bI), (a + x, bI), (a, (b + y)I), (a + x, (b + y)I)\} \in N_I$
be any neutrosophic square of type I.

Clearly $0_I = \{(0, 0), (0, 0), (0, 0), (0, 0)\} \in N_I$.

Now
$-A = \{(-a, -bI), (-a-x, -bI), (-a-(b + y)I),$
$\qquad\qquad (-a-x, -(b + y)I)\} \in N_I$.
Further
$$A = (-A) = \{(0, 0), (0, 0), (0, 0), (0, 0)\} = 0_I.$$

Consider any

$B = \{(c, dI), (c + r, dI), (c, (d + s)I), (c + r, (d + s)I)\} \in N_I$.

$\quad = \{(a + c, (b + d)I), (a + c, (b + y + d + s)I),$
$\qquad\qquad (a + c + x + r, (b + d)I),$
$\qquad\qquad\qquad (a + c + x + r, (b + d + y + s)I) \in N_I$.

Thus for every $A, B \in N_I$. $A + B \in N_I$.

Further $A + B = B + A$ can be easily verified.

Hence $(N_I, +)$ is an abelian group of infinite order.

Consider N_I^{++} = {All neutrosophic squares of type II from the first quadrant of $\langle R \cup I \rangle$ or $\langle Q \cup I \rangle$ of $\langle Z \cup I \rangle$}.

{ N_I^{++}, + } is only a monoid.

In the first place

$$0_I = \{(0, 0), (0, 0), (0, 0), (0, 0)\} \in N_I^{++}$$

and this serves as the additive identity.

This neutrosophic monoid will also be known as the neutrosophic first quadrant monoid of type I.

Example 2.6: Let

$$A = \{(6, 3I), (10, 3I), (6, 7I), (10, 7I)\}$$

and

$$B = \{(2, 6I), (8, 6I), (2, 12I), (8, 12I)\} \in N_I^{++};$$

$$A + B = \{(8, 9I), (18, 9I), (8, 19I), (18, 19I)\} \in N_I^{++}.$$

It is easily verified N_I^{++} is a monoid of infinite order which is commutative.

Let N_I^{--} = {All neutrosophic squares of type I from the 3rd quadrant}. N_I^{--} is a monoid.

Let

$$A = \{(-1, -3I), (-5, -3I), (-5, -7I), (-1, -7I)\}$$

and

$$B = \{(-2, -1I), (-8, -I), (-2, -7I), (-8, -7I)\} \in N_I^{--}.$$

$$A + B = \{(-3, -4I), (-13, -4I), (-13, -14I), (-3, -14I)\} \in N_I^{--}.$$

Thus it is a very easy task to prove N_I^{--} is a monoid of infinite order which is commutative.

Now let

$$A = \{(-3, 2I), (2, 2I), (2, 7I), (-3, 7I)\}$$

and
$$B = \{(-10, -3I), (14, -3I), (14, I), (10, I)\}$$
be two neutrosophic squares of type I in N_I.

$$A - B = \{(-13, 5I), (-12, 5I), (-12, 6I), (-13, 6I)\}$$
and
$$B - A = \{(13, -5I), (12, -5I), (12, -6I), (13, -6I)\}$$
are unit neutrosophic squares.

Let
$$M = \{(9, -8I), (10, -8I), (10, -7I), (9, -7I)\}$$
and
$$N = \{(-5, 4I), (-4, 4I), (-4, 5I), (-5, + 5I)\} \in N_I.$$

$$M - N = \{(14, -12I), (14, -12I), (14, -12I), (14, -12I)\}$$
is a single point in the neutrosophic plane.

$$M + N = \{(4, -4I), (6, -4I), (6, -2I), (4, -2I)\}.$$

Clearly M + N is a neutrosophic square of length II.

Thus neutrosophic squares of type I can be just point squares of the form

$$S = \{(a, bI), (a, bI), (a, bI), (a, bI)\};$$

these type I squares will be known as neutrosophic point squares of type I.

Let
$$A = \{(3, 2I), (3, 2I), (3, 2I), (3, 2I)\}$$
and
$$B = \{(0, 7I), (5, 7I), (5, 12I), (0, 12I)\}$$
be the neutrosophic squares of type I.

$$B + A = \{(3, 9I), (8, 9I), (8, 14I), (3, 14I)\}$$

is again a neutrosophic square of type I with 5 as the length of the sides.

Clearly the neutrosophic square B of type I is shifted to another place.

We will illustrate this situation by an example.

Example 2.7: Let

$$P = \{(1, 3I), (1, 3I), (1, 3I), (1, 3I)\}$$

be the point neutrosophic square of type I.

Let

$$A = \{(1, 2I), (2, 2I), (2, 3I), (1, 3I)\} \in N_I.$$

$$A + P = \{(2, 5I), (3, 5I), (3, 6I), 2, 6I)\}.$$
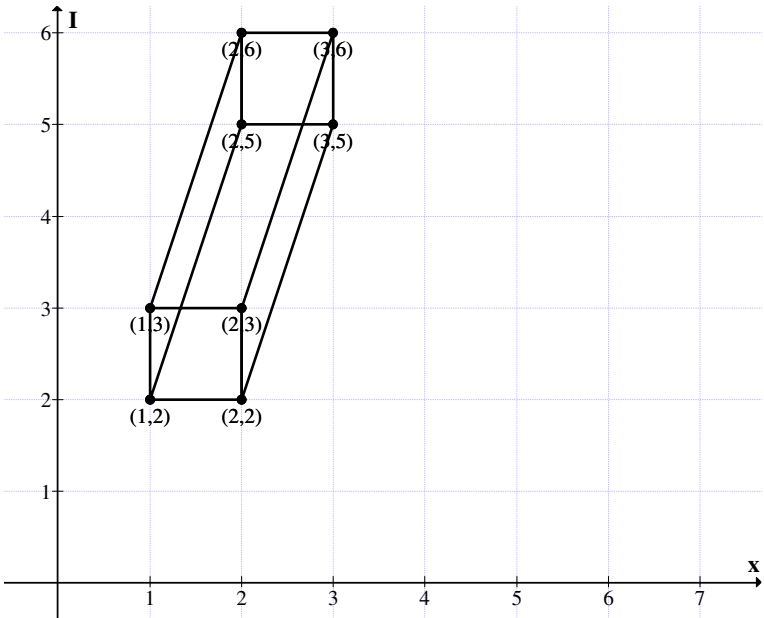
The graphical representation is as follows:

Figure 2.7

Thus the point square P shift the square A to A + P.

Example 2.8: Let

$$P = \{(2, 2I), (2, 2I), (2, 2I), (2, 2I)\}$$

be a neutrosophic point square of type I in N_I.

Let S = $\{(3, 4I), (4, 4I), (3, 5I), (4, 5I)\} \in N_I$

be a neutrosophic square of type I.

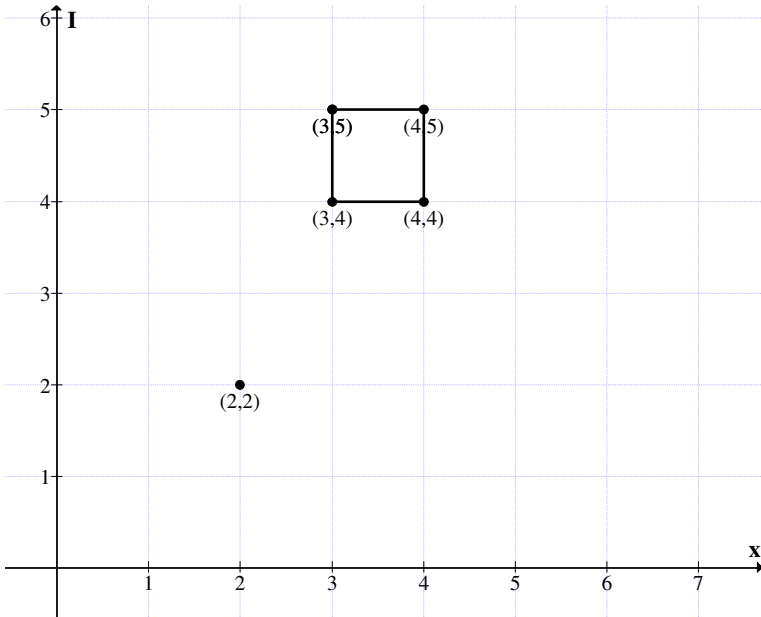The graph representation is as follows:

Figure 2.8

$$P + S = \{(5, 6I), (6, 6I), (5, 7I), (6, 7I)\}$$

is again a neutrosophic square of type I in N_I.

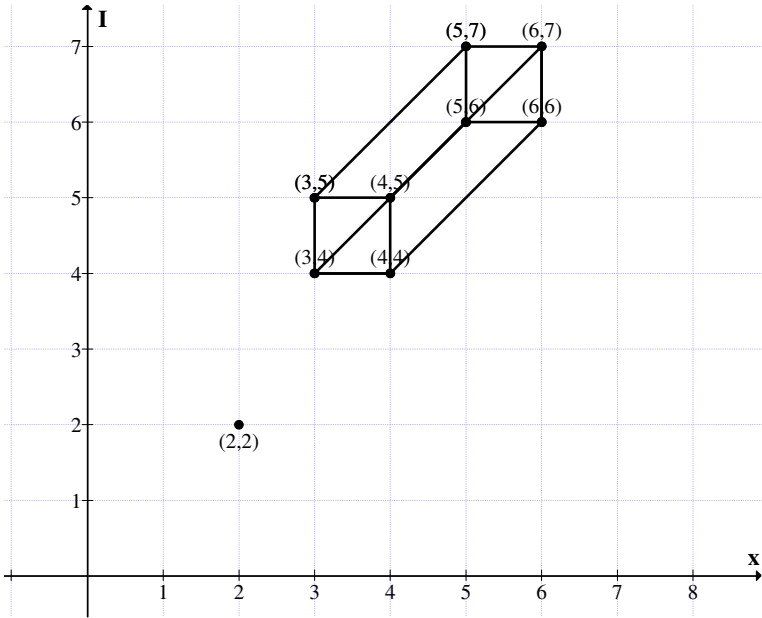The representation of P + S in the neutrosophic plane is as follows:

Figure 2.9

Thus the square is shifted to another place that is a translation takes place without altering the size the square or length of the side of the square.

Example 2.9: Let

$$A = \{(-1, 2I), (-1, 2I), (-1, 2I), (-1, 2I)\}$$
be the point neutrosophic square of type I.

Let
$$P = \{(3, 2I), (3, 4I), (5, 2I), (5, 4I)\} \in N_I.$$

To find P + A.

$$P + A = \{(2, 4I), (2, 6I), (4, 4I), (4, 6I)\}.$$

Suppose P_1 is the point neutrosophic square of type I.

$$P_1 = \{(-2, -2I), (-2, -2I), (-2, -2I), (-2, -2I)\} \in N_I.$$

$$P_1 + A = \{(1, 0), (1, 2I), (3, 0), (3, 2I)\}.$$

It is clearly evident both the neutrosophic squares P + A and $P_1 + A$ have the same length of the side 2 as that of A.

So the translation of A by P_1 or P_I does not alter the size of the neutrosophic square of type I.

Next consider the neutrosophic point square;

$$P_2 = \{(1, -3I), (1, -3I), (1, -3I), (1, -3I)\} \in N_I.$$
Let
$$B = \{(-1, 3I), (1, 3I), (-1, 5I), (1, 5I)\}$$
be the neutrosophic square of type I.

$$P_2 + B = \{(0, 0), (2, 0), (0, 2I), (2, 2I)\}$$
is the square after translation, now the length of the side of both the squares B and $P_2 + B$ are the same.

Thus N_I the neutrosophic squares of type I includes the zero squares and the point neutrosophic square of type I.

Example 2.10: Let

$$S = \{(-3, 0), (-3, 0), (-3, 0), (-3, 0)\}$$
be the point neutrosophic square of type I.
Let
$$A = \{(2, 4I), (5, 4I), (2, 7I), (5, 7I)\}$$
be the neutrosophic square of type I.

$$S + A = \{(-1, 4I), (2, 4I), (-1, 7I), (2, 7I)\}$$
is the neutrosophic square of type I.

Both the squares A and S + A are the same size.

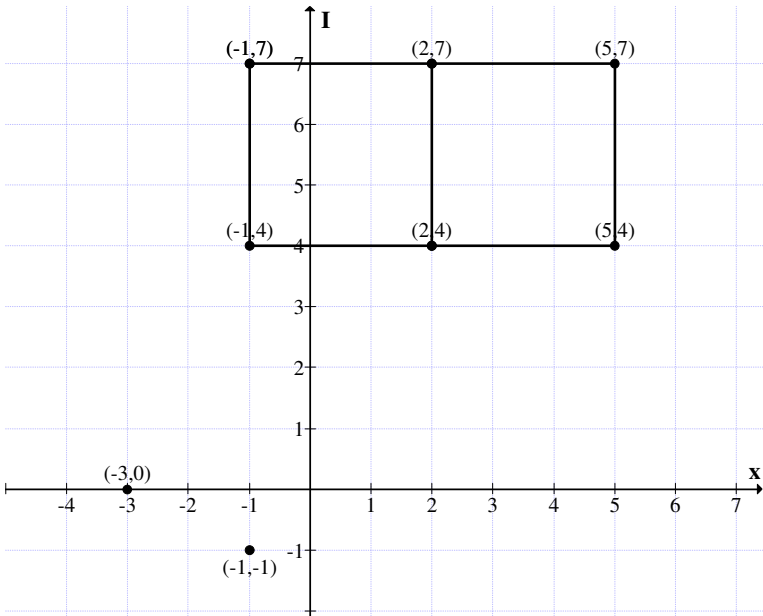They have the following graphical representation:

Figure 2.10

The addition of S with A gives a neutrosophic square of type I which has two of its coordinates to be the same.

Further one side is also common for A and S + A which has occurred due to the transformation / translation.

Example 2.11: Let
$$P = \{(0, -I), (0, -I), (0, -I), (0, -I)\}$$
be the point neutrosophic square of type I.
    Let
$$B = \{(3, 2I), (5, 2I), (3, 4I), (5, 4I)\}$$
be the neutrosophic type I square in N_I.
    To find
$$P + B = \{(3, I), (5, I), (3, 3I), (5, 3I)\}$$
be the neutrosophic square of type I.
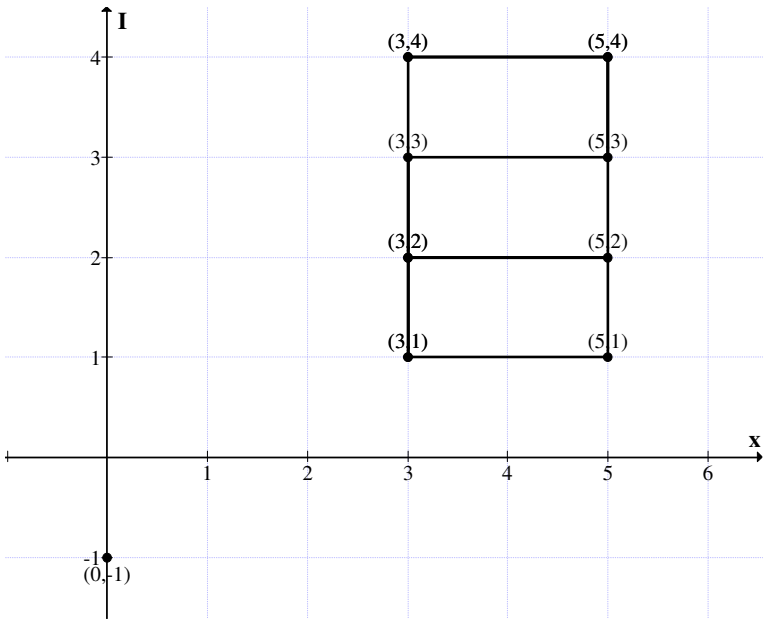
The graph of P, B and P + B is as follows:

Figure 2.11

Thus the squares B and P + B overlap which is clearly evident from the figure.

Thus the addition of a point square can make the square to have a common side of the two square can over lap or at time the square experiences a translation in the plane.

Such study in innovative and interesting.

Example 2.12: Let
$$P = \{(0, 2I), (0, 2I), (0, 2I), (0, 2I)\}$$
be the point neutrosophic square of type I.
    Let
$$S = \{(-3, -2I), (-1, -2I), (-3, 0), (-1, 0)\}$$
be the neutrosophic square of type I.

$$S + P = \{(-3, 0), (-1, 0), (-3, 2I), (-1, 2I)\}$$
be the neutrosophic square of type I.
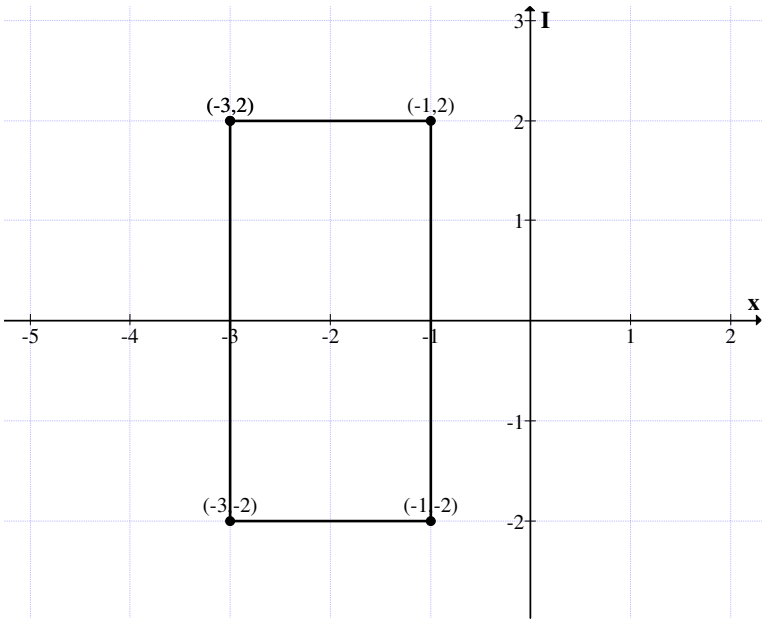
The graph of P, S and P + S is as follows:

Figure 2.12

The graph of the squares P + S and S have the line (–3, 0) to (–1, 0) to be common.

Example 2.13: Let

$$P = \{(2, 0), (2, 0), (2, 0), (2, 0)\}$$

be the neutrosophic point square of type I.

Let

$$B = \{(-4, -2I), (-3, -2I), (-4, -I), (-3, -I)\}$$

be the neutrosophic square of type I in N_I.

$$P + B = \{(-2, -2I), (-1, -2I), (-2, -I), (-1, -I)\}$$

is the neutrosophic square of type I.

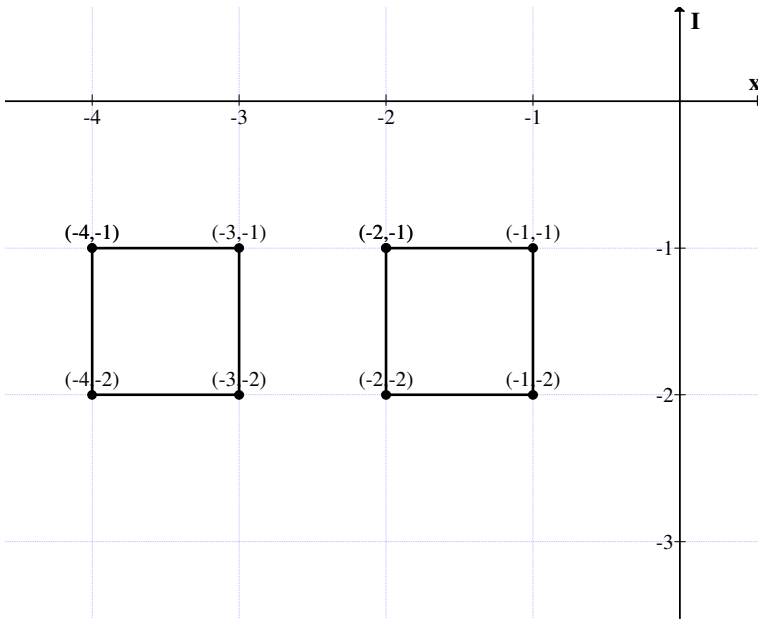Now we give the graph of P, B and P + B in the following

Figure 2.13

The two neutrosophic squares B and P + B be adjacent to each other.

Next we give examples of the differences between a point neutrosophic square P of type I and the neutrosophic square B of type I.

This will be illustrated by the following examples.

Example 2.14: Let
$$P = \{(-2, I), (-2, I), (-2, I), (-2, I)\}$$
be the point neutrosophic square of type I.

$$S = \{(4, 2I), (6, 2I), (4, 4I), (6, 4I)\}$$
be the neutrosophic square of type I.

$$P - S = \{(-6, -I), (-8, -I), (-6, -3I), (-8, -3I)\}$$
is the neutrosophic square of type I and both the square are of the same size that is side length is 2.

$$S - P = \{(6, + I), (8, + I), (6, 3I), (8, 3I)\}$$

S – P too has same size square associated with it.

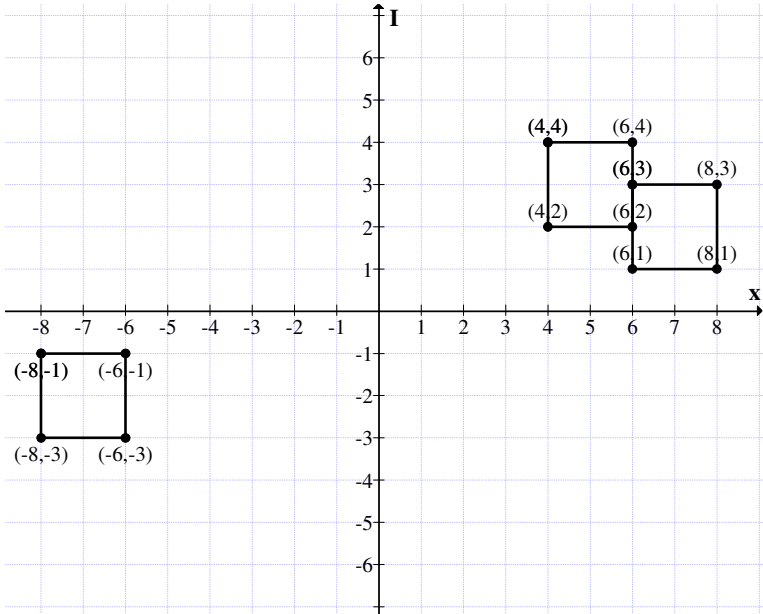We will give the graph of all the four squares in the following.

Figure 2.14

These square are of same size however half of the side line of the squares S and S–P merge as shown in the figure.

Next we give the representation of the difference between two neutrosophic squares of type I.

Example 2.15: Let
$$A = \{(5, 0), (8, 0), (8, 3I), (5, 3I)\}$$
and
$$B = \{(-7, 4I), (-4, 4I), (-4, 7I), (-7, 7I)\}$$
be two neutrosophic squares of type I of same side length.

We find A – B and B – A.

$$A - B = \{(12, -4I), (12, -4I), (12, -4I), (12, -4I)\}$$

and

$$B - A = \{(-12, 4I), (-12, 4I), (-12, 4I), (-12, 4I)\}$$

Clearly A – B ≠ B – A, but both reduce to a neutrosophic point square of type I given by (12, –4I) and (–12, 4I) respectively.
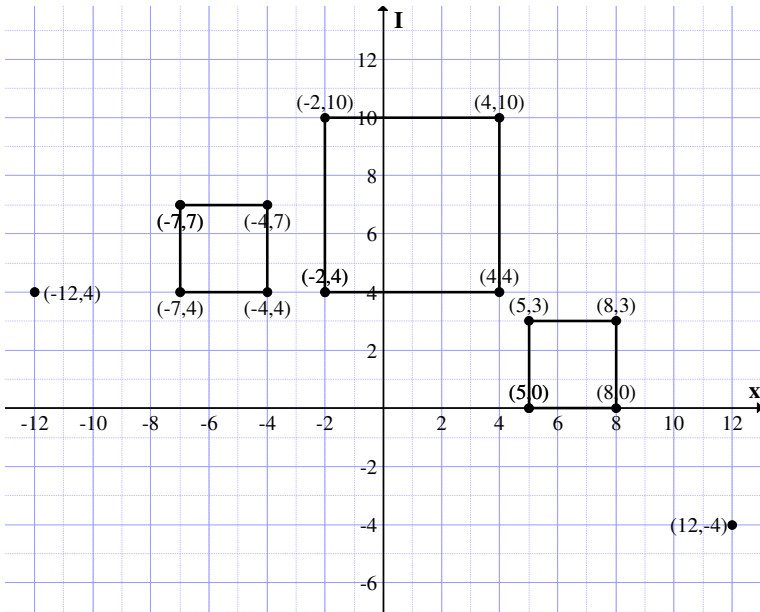
We give the graph of these four neutrosophic squares.

Figure 2.15

One can study the properties associated with the neutrosophic square of type I and their properties.

Next we find A + B. Of course A + B = B + A so we get a unique diagram.

$$A + B = \{(-2, 4I), (4, 4I), (4, 10I), (-2, 10I)\}.$$

The length of the side of this neutrosophic square is 6.

Example 2.16: Let
$$A = \{(2, 2I), (4, 2I), (4, 4I), (2, 4I)\}$$
and
$$B = \{(0, 2I), (2, 2I), (0, 4I), (2, 4I)\}$$
be two neutrosophic squares of type I.

They have 2 of the vertices to be common.

We find $A + B$, $A - B$ and $B - A$.

$$A + B = \{(2, 4I), (6, 4I), (6, 8I), (2, 8I)\}$$

$$A - B = \{(2, 0), (2, 0), (2, 0), (2, 0)\}$$
and
$$B - A = \{(-2, 0), (-2, 0), (-2, 0), (-2, 0)\}$$
are the neutrosophic square of type I.

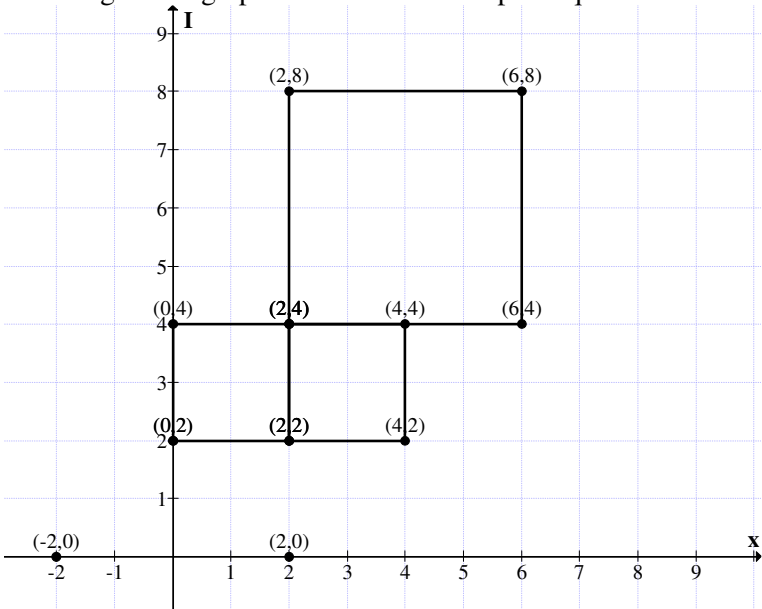We give the graph of these 5 neutrosophic squares:

Figure 2.16

This give three neutrosophic squares which touch each other in twos.

That is square A shares one edge common with B and half an edge common with A + B.

Thus we see by adding one point square to a neutrosophic square we get a shift of the neutrosophic square.

Now we see if the point square is a real point what is the type of shift and when the point is a neutrosophic point what is the shift.

Example 2.17: Let
$$A = \{(3, I), (5, I),(3, 3I), (5, 3I)\}$$

be the neutrosophic square of type I.

Let
$$P = \{(4, 0), (4, 0), (4, 0), (4, 0)\}$$

be the real point square of type I.

We find A + P, P – A and A – P.

$$A + P = \{(7, I), (9, I), (7, 3I), (9, 3I)\}$$

$$P - A = \{(1, -I), (-1, -I), (1, -3I), (-1, -3I)\}$$
and
$$A - P = \{(-1, I), (1, I), (-1, 3I), (1, 3I)\}.$$

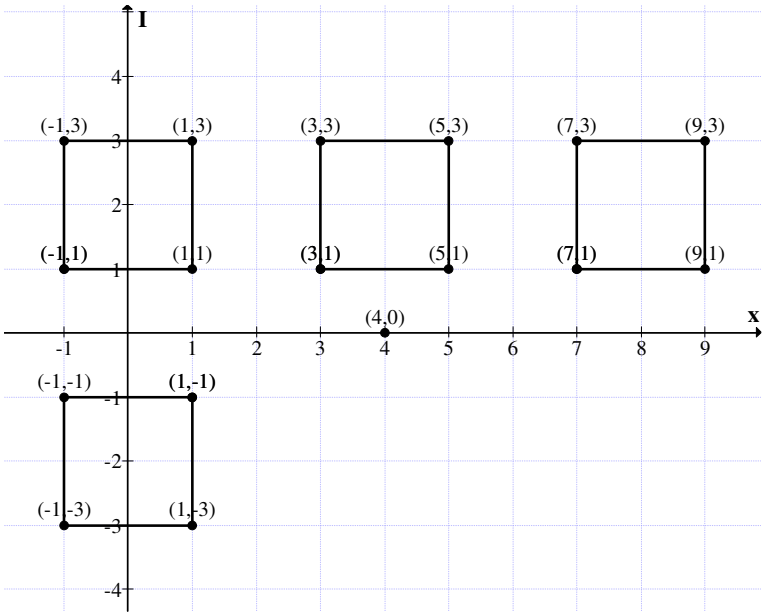We give the graph of all the four squares in the neutrosophic plane.

Figure 2.17

Thus if a real point is added the square shift parallely as shown in the figure.

Now consider for the same neutrosophic square A the neutrosophic point square N = {(0, 3I), (0, 3I), (0, 3I), (0, 3I)}.

We find A + N, A – N, N – A, N – P and P – N.

$$A + N = \{(3, 4I), (5, 4I), (3, 6I), (5, 6I)\}$$

$$A - N = \{(3, -2I), (5, -2I), (3, 0), (5, 0)\}$$

$$N - A = \{(-3, 2I), (-5, 2I), (-3, 0), (-5, 0)\}$$

$$N - P = \{(-4, 3I), (-4, 3I), (-4, 3I), (-4, 3I)\}$$

$$P - N = \{(4, -3I), (4, -3I), (4, -3I), (4, -3I)\}$$

Thus real point square difference a neutrosophic point square is again a neutrosophic point square.

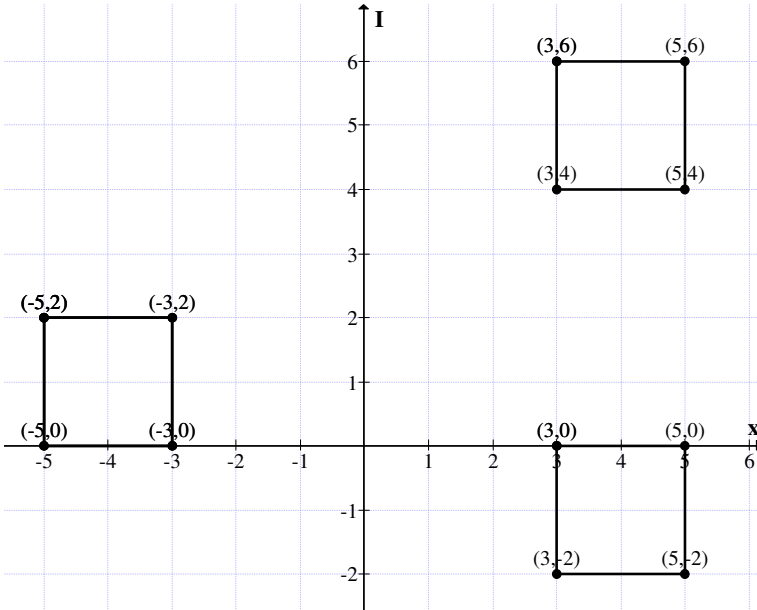Now the graph of all that 5 square is given in the following:

Figure 2.18

In view of all these following theorem are proved.

THEOREM 2.2: *Let {N_I, + } be the neutrosophic square of type I. Let A and B two neutrosophic squares with sides of same length. Then A–B and B–A are both neutrosophic point squares of type I.*

Proof is direct and hence left as an exercise to the reader.

THEOREM 2.3: *Let (N_I, +) be a neutrosophic square of type I. If A = {(a, bI), (a, bI), (a, bI), (a, bI)} where a, b $\in$ R be a point neutrosophic square. If S is any other neutrosophic square of type I with side 'm' units, then S + A or S–A or A–S gives a*

neutrosophic square of type I of side m units and shift them to different places on the neutrosophic plane.

Proof is direct and hence left as an exercise to the reader.

Next we are interested in studying what is the product of neutrosophic squares of type I with another neutrosophic square and so on.

This is first illustrated by some examples.

Example 2.18: Let

$$A = \{(5, 2I), (7, 2I), (5, 4I), (7, 4I)\}$$

and

$$B = \{(3, I), (6, I), (3, 4I), (6, 4I)\} \in N_I.$$

$A \times B = \{(15, 13I), (42, 21I), (15, 48I), (42, 68I)\}.$

Clearly $A \times B$ is not a neutrosophic square of type I.
This is clearly seen from $A \times B$.

Example 2.19: Let

$$A = \{(1, I), (2, I), (1, 2I), (2, 2I)\}$$

and

$$B = \{(-1, 0), (0, 0), (-1, I), (0, I)\} \in N_I.$$

$A \times B = \{(-1, -I), (0, 0), (-1, I), (0, 4I)\} \notin N_I.$

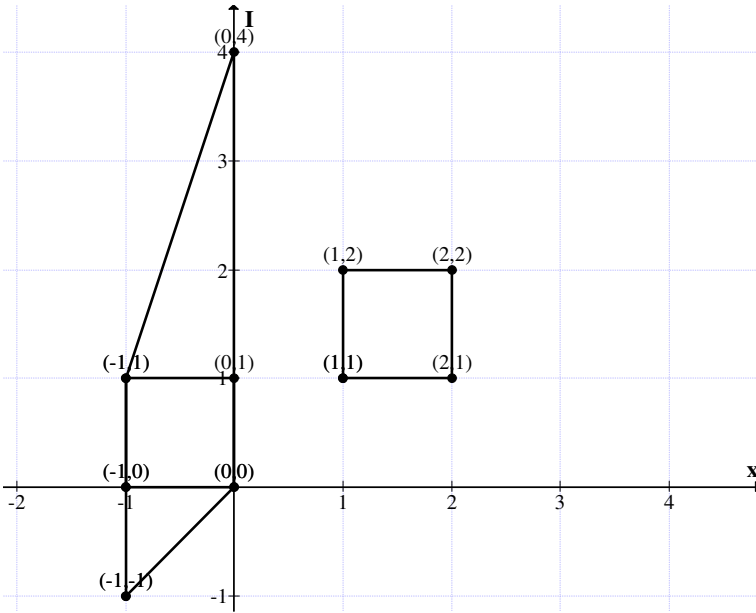We will represent A, B and A × B by the graph in the following.

Figure 2.19

$$A \times B \notin N_I.$$

Thus N_I is not compatible with product operations.

Example 2.20: Let

$$A = \{(2, I), (3, I), (2, 2I), (3, 2I)\}$$

and

$$B = \{(-1, 2I), (0, 2I), (-1, 3I), (0, 3I)\} \in N_I.$$

Thus N_I is not compatible with product operation.

Example 2.21: Let

$$A = \{(2, I), (3, I), (2, 2I), (3, 2I)\}$$

and

$$B = \{(-1, 2I), (0, 2I), (-1, 3I), (0, 3I)\} \in N_I.$$

$$A \times B = \{(-2, 5I), (0, 8I), (-2, 10I), (0, 15I)\} \notin N_I.$$

Inview of all these we have the following theorem.

THEOREM 2.4: *Let N_I be the collection of all neutrosophic square of type I.*

N_I is not closed under the product operation $\times$.

Proof: Follows from the simple fact $A \times B \notin N_I$ for any $A, B \in N_I$.

Example 2.22: Let

$$A = \{(2, 0), (4, 0), (2, 2I), (4, 2I)\}$$

and

$$P = \{(2, I), (2, I), (2, I), (2, I)\}$$

be two neutrosophic squares of type I.

$$A \times P = \{(4, 2I), (8, 4I), (4, 8I), (8, 10I)\} \notin N_I.$$

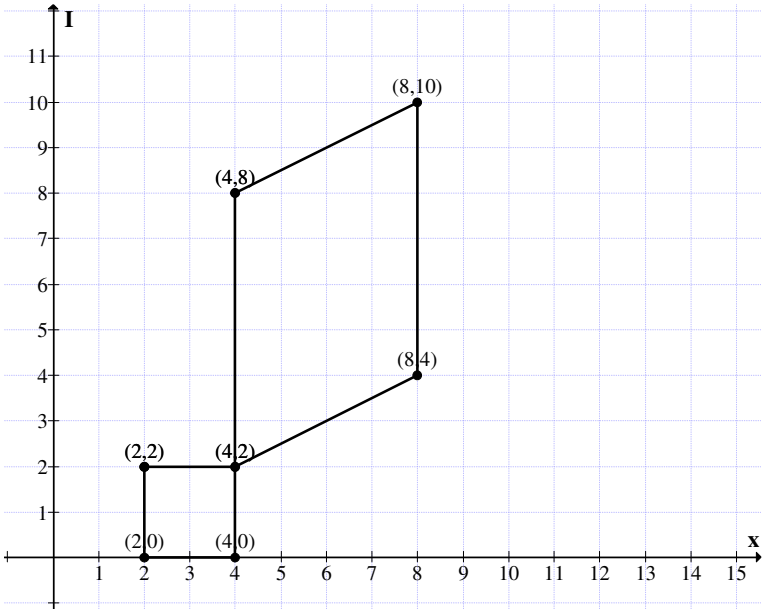We will describe this by a graph in the following.

Figure 2.20

We see A × B in two of the examples have 2 sides parallel to the neutrosophic axis where as the other sides are of different length of the neutrosophic sides.

However one gets a parallelogram.

So A × B is not a neutrosophic square of type I.

Since closure axiom is not true for N_I under product nothing can be done.

Next we proceed onto define the notion of neutrosophic rectangle of type I.

The sides are always assumed to be of different lengths and the sides are parallel to x and y axis.

We will illustrate this situation by some examples.

N_I^R = {Collection of all neutrosophic intervals of type I}.

Example 2.23: Let

$$P = \{(3, 4I), (5, 4I), (3, 8I), (5, 8I)\}$$

be the neutrosophic rectangle of type I.

The figure of P in the neutrosophic plane is given in Figure 2.21.

Example 2.24: Let

$$W = \{(-8, 4I), (-9, 4I), (-8, 6I), (-9, 6I)\}$$

be the neutrosophic rectangle of type I.

The graph of W in the neutrosophic of W in the neutrosophic plane is given in Figure 2.22.
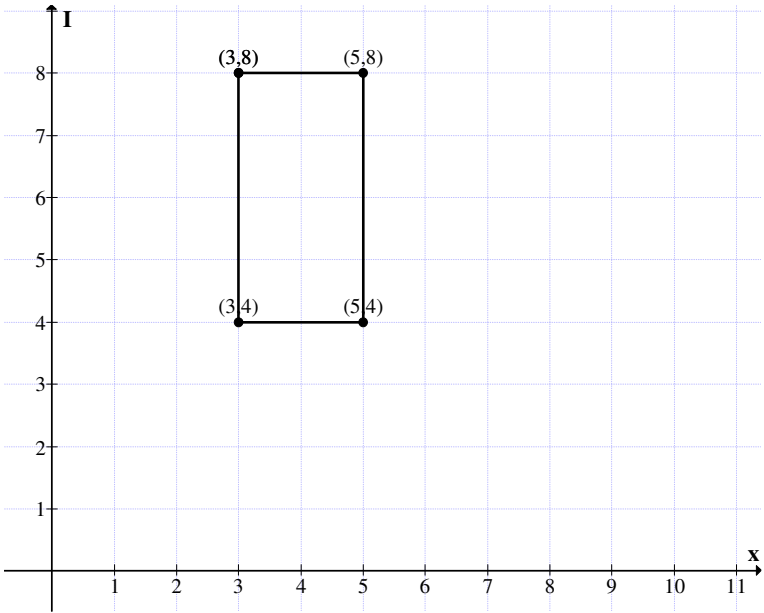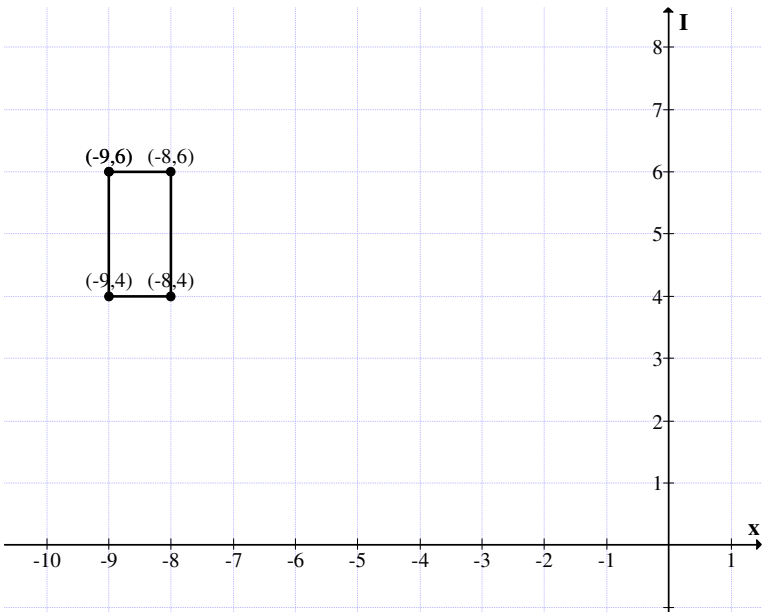
Figure 2.21

Figure 2.22

Example 2.25: Let

$$S = \{(3, 2I), (-3, 2I), (3, -2I), (-3, -2I)\} \in N_I^R$$

be the neutrosophic rectangle of type I.

The graph of S in the neutrosophic plane is as follows:
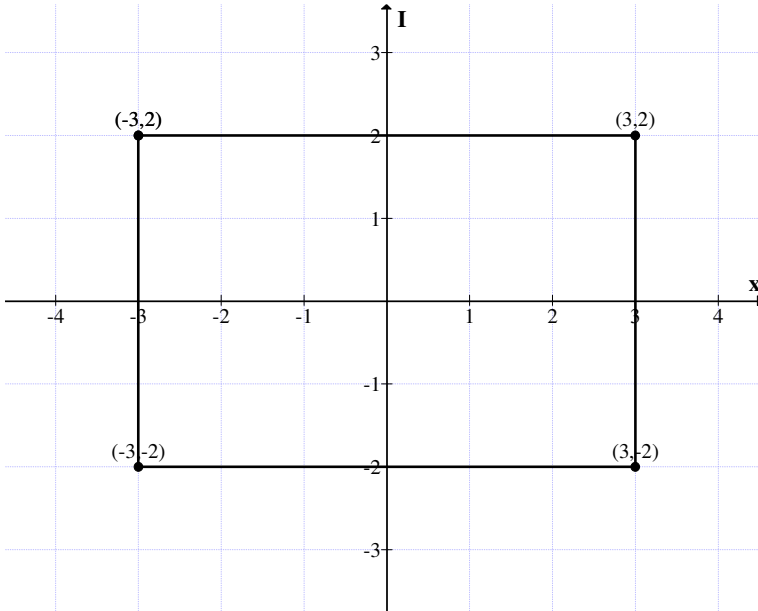

Figure 2.23

Now let
$$B = \{(1, 2I), (2, 2I), (1, 3I), (2, 3I)\}$$
be the neutrosophic square of type I.

$$B^2 = \{(1, 8I), (4, 12I), (1, 15I), (4, 21I)\}$$

is not a square only a quadrilateral having only two sides parallel to neutrosophic axis.

This is represented by the following figure in the neutrosophic plane.

Now
$$B^3 = \{(1, 26I), (8, 56I), (1, 63I), (8, 117I)\}.$$

Once again B^3 is also a quadrilateral with two sides parallel and infact parallel to the neutrosophic axis.

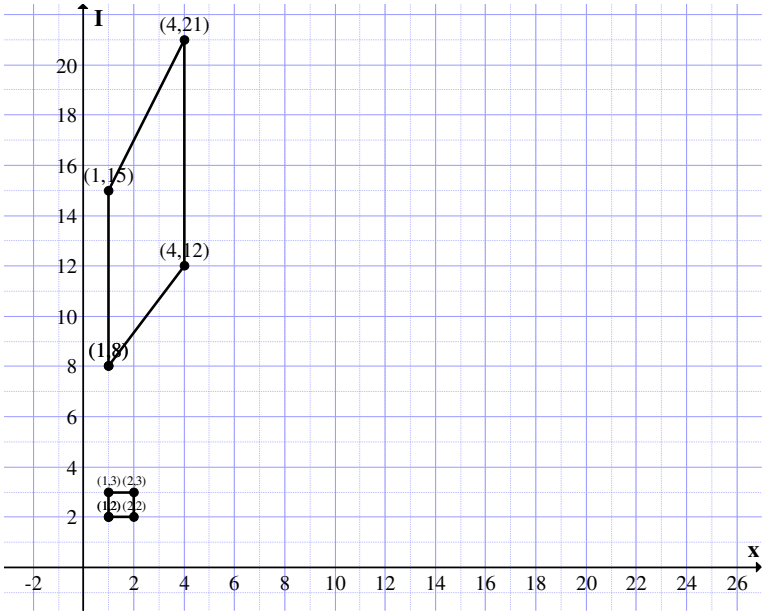Similarly B^4 is also a quadrilateral with two sides parallel to the neutrosophic axis.

Figure 2.24

Clearly B_2 is not even a parallelogram only a quadrilateral with two sides parallel.

Inview of this we have the following theorem.

THEOREM 2.5: *Let B = {a, bI), (a + r, bI), (a, bI + sI), (a + r, bI + sI)} ∈ N_I (a, a + r, b, b + s > 0), B_2, B_3, …, B_n, … are all not neutrosophic squares of type I but they are quadrilaterals such that two of its sides are parallel to the neutrosophic axis.*

Now we give examples in which we take negative values for a or b or for both.

Example 2.26: Let

$$A = \{(-1, 2I), (-3, 2I), (-1, 0), (-3, 0)\} \in N_I$$

be a neutrosophic square of type I.

$$A^2 = \{(1, 0), (9, -8I), (1, 0), (9, 0)\}.$$

Clearly A^2 is not a neutrosophic square of type I.

$$A^3 = \{(-1, 2I), (-27, -16I), (-1, 0), (27, 0)\}$$

is not a neutrosophic square of type I.

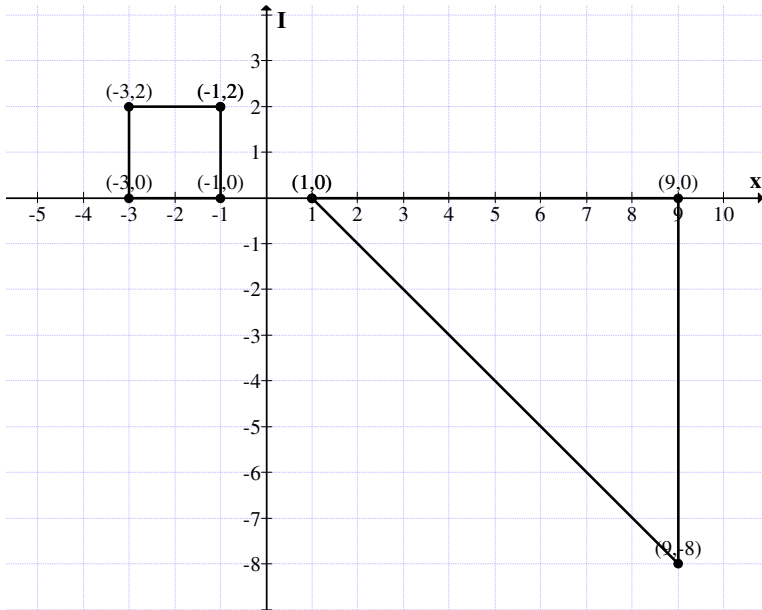We give the graph or figure associated with A, A^2, A^3 in the following.

Figure 2.25

A^2 is not even a neutrosophic quadrilateral only a neutrosophic right angled triangle.

However A^3 is a neutrosophic quadrilateral. Thus if a, b, r, s are all positive only we can say the product of the neutrosophic squares of type I is a neutrosophic quadrilateral if a, b, r, s are such that some of them are negative and some are positive we see we may not be even in a position to get a neutrosophic quadrilateral, we in this case get a neutrosophic right angled triangle.

Study in this direction is left as a open conjecture!

However we supply some more examples.

Example 2.27: Let
$$A = \{(-1, -I), (-3, -3I), (-1, -3I), (-3, -I)\}$$
be the neutrosophic square of type I.

$$A^2 = \{(1, 3I), (9, 7I), (1, 7I), (9, 27I)\} \notin N_I.$$
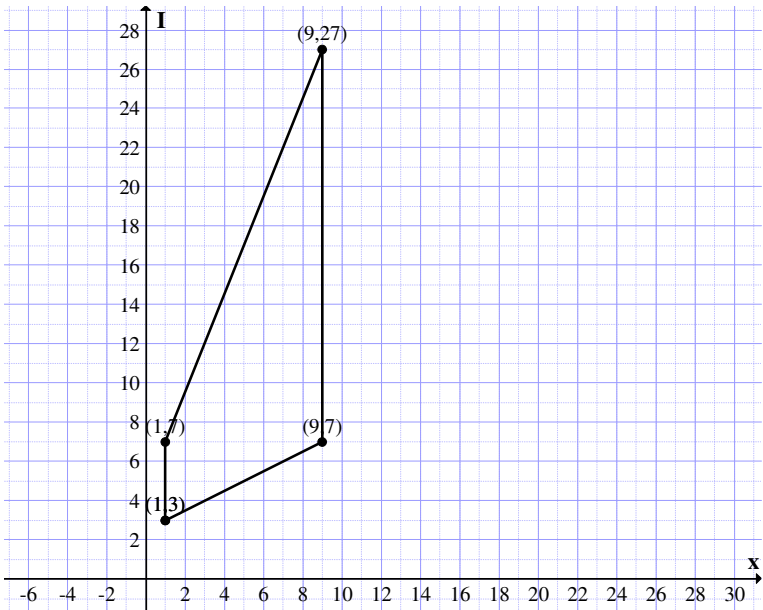Clearly A^2 is not a neutrosophic square. The graph of A^2 is as follows:

Figure 2.26

Clearly A^2 is a neutrosophic quadrilateral such that two of its sides are parallel to the neutrosophic axis.

$$A^3 = \{(-1, -7I), (-27, -69I), (-1, -3I), (-27, -17I)\}$$

is again a neutrosophic quadrilateral with two of its sides parallel to the neutrosophic axis.

Thus if a, b, r, s as in theorem 2.5 all are negative then also we get the product of the neutrosophic square of type I with itself is a neutrosophic quadrilateral with two of its sides parallel to the neutrosophic axis.

In view of this we have the following theorem.

THEOREM 2.6: *If*
A = {(a, bI), (a + r, bI), (a, bI + sI), (a + r, bI + sI)} (a, b, s, r lesser than 0) is a neutrosophic square of type I in N_I then A^2, A^3, A^4 … are all neutrosophic quadrilaterals such that two its sides are parallel to the neutrosophic axis.

Proof is direct and hence left as an exercise to the reader.

Example 2.28: Let
$$A = \{(-1, -I), (3, 3I), (3, -I), (-1, 3I)\}$$

be a neutrosophic square of type I.

$$A^2 = \{(1, 3I), (1, 3I), (1, 3I), (9, -5I), (9, 27I)\} \notin N_I.$$

A^2 is not a neutrosophic quadrilateral but is only a neutrosophic right angled triangle.

It is left as a open conjecture if a, b, r, s are such that some are positive and some are negative in the neutrosophic square A of type I, can we say A^2 will be always a neutrosophic right angled triangle?

However several of the properties associated with real squares of type I are not in general true in case of neutrosophic squares of type I.

In view of all these we have the following theorem.

THEOREM 2.7: *Let N_I be the neutrosophic square of type I.*

Let S = {(x, xI), (x + k, xI), (x, xI + kI)   ,(x + k, (x + k)I))
where x, k $\in Z^+ \setminus \{0\}\} \subseteq N_I$.

S is not even a semigroup under $\times$.

Proof is direct and hence left as an exercise to the reader.

This is the marked difference between the Euclid or real squares of type I and that of neutrosophic squares of type I.

Next we proceed onto define neutrosophic squares of type II.

DEFINITION 2.1: *Let N_{II} = {Collection of all neutrosophic squares whose diagonals are parallel to the axis in the neutrosophic plane}.*

We define N_{II} as the neutrosophic squares of type II.

We will first illustrate this situation by some examples.

Example 2.29: Let

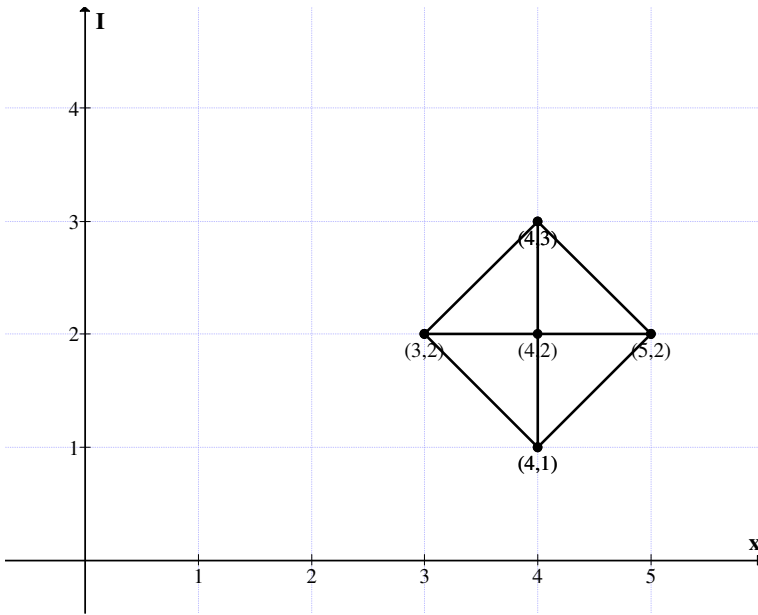$$A = \{(4, 3I), (3, 2I), (5, 2I), (4, I)\} \in N_{II}.$$

Figure 2.27

The above figure shows that the diagonals intersect at (4, 2) and they are parallel to the x and y axis respectively.

Example 2.30: Let

$$B = \{(-2, 3I), (-3, 2I), (-1, 2I), (-2, I)\}$$

be the neutrosophic square of type II.

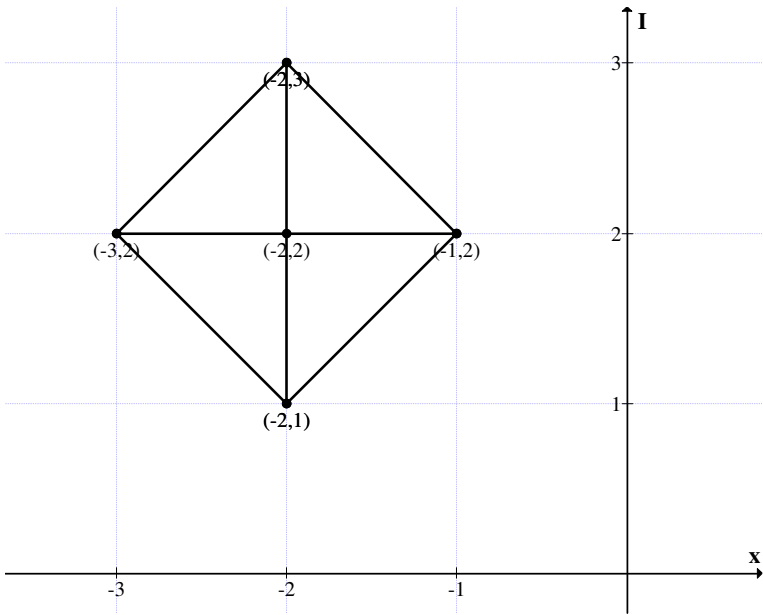This has the following representation in the neutrosophic plane.

Figure 2.28

Now having seen examples of neutrosophic squares of type II we now proceed onto define the sum or addition operation + on N_{II}.

Example 2.31: Let

$$A = \{(3, 2I), (4, 3I), (3, 4I), (2, 3I)\}$$

and

$$B = \{(-2, I), (-1, 2I), (-2, 3I), (-3, 2I)\}$$

be two neutrosophic squares of type II.

To find A + B;

$$A + B = \{(1, 7I), (3, 5I), (1, 3I), (-1, 5I)\}.$$

The diagrammatic representation of A + B is as follows:
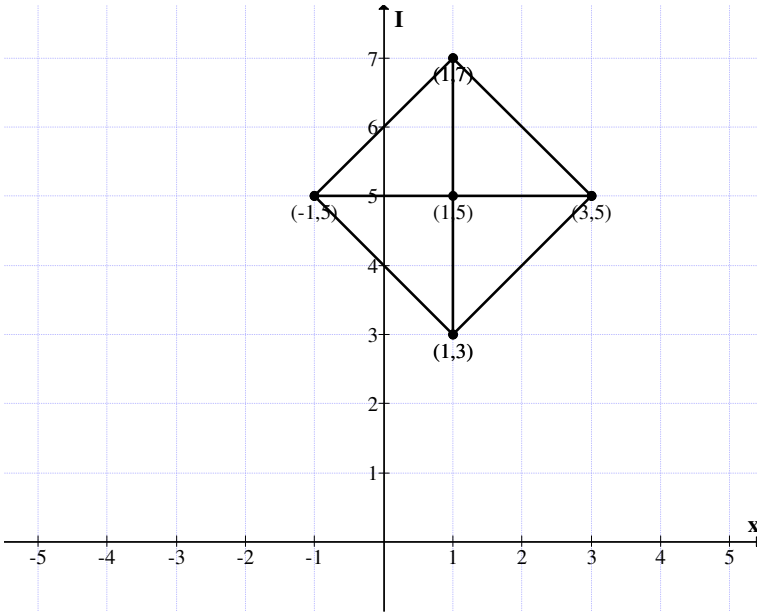
Figure 2.29

Clearly A + B is again a neutrosophic square of type II.

However se see $A + B \in N_{II}$.

Let us consider two neutrosophic square of type II.

$$A = \{(a, bI), (c, dI), (a, gI), (e, bI) \mid \frac{c + e}{2} = a$$

and

$$b = \frac{d + g}{2}, a, b, c, d, e, g \in R\}.$$

and

$$B = \{(a_1, b_1I), (c_1, d_1I), (a_1, g_1I), (e_1, b_1I) \mid \frac{c_1 + e_1}{2} = a_1$$

and

$$b_1 = \frac{d_1 + g_1}{2}, \ a_1, b_1, c_1, d_1, e_1, g_1 \in R\}.$$

Consider
$$A + B = \{(a + a_1, (b + b_1)I), (c + c_1, (d + d_1)I),$$
$$((a + a_1, g + g_1)I), (e + e_1, (b + b_1)I)\}$$

is

$$a + a_1 = \frac{c + c_1 + e + e_1}{2}$$

and

$$b + b_1 = \frac{d + d_1 + g + g_1}{2}$$

Clearly A + B is again a neutrosophic square of type II.

In view of this we assume all singletons of the form

$$P = \{(a, bI), (a, bI), (a, bI), (a, bI)\}$$

is a point neutrosophic square of type II.

The following result is true.

THEOREM 2.8: *Let N_{II} be the collection of all neutrosophic squares of type II.*

$\{N_{II}, + \}$ is a group of infinite order which is commutative.

Proof follows from the simple fact
$$0_I = \{(0, 0), (0, 0), (0, 0), (0, 0)\} \in N_{II}$$
is such that $0_I + A = A + 0_I = A$ for all $A \in N_{II}$.

Further for every $A \in N_{II}$ there exists a unique $-A \in N_{II}$ such that $A + (-A) = 0_I$.

Hence the claim.

We will illustrate this situation by an examples or two.

Example 2.32: Let

$$A = \{(6, 8I), 99, 5I), (6, 2I), (3, 5I)\} \in N_{II}.$$

Clearly
$$-A = \{(-6, -8I), (-9, -5I), (-6, -2I), (-3, -5I)\} \in N_{II}$$
and
$$A + (-A) = 0_I.$$

Example 2.33: Let

$$P = \{(8, -2I), (12, 2I), (8, 6I), (4, 2I)\} \in N_{II}.$$

$$-P = \{(-8, 2I), (-12, -2I), (-8, -6I), (-4, -2I)\} \in N_{II}$$
is such that
$$P + (-P) = 0_I.$$

Now we will show how addition of a point neutrosophic square with a square in N_{II} is performed by the following examples.

Example 2.34: Let

$$P = \{(2, I), (2, I), (2, I), (2, I)\}$$

be the neutrosophic point square.
Let
$$A = \{(8, -2I), (12, 2I), (8, 6I), (4, 2I)\}$$

be the neutrosophic square of type II.

$$A + P = \{(10, -I), (14, 3I), (10, 7I), (6, 3I)\}$$

is again a neutrosophic square of type II with point of intersection of diagonals to be (10, 3I).

Consider A–P;

$$A - P = \{(6, -3I), (10, I), (6, 5I), (2, I)\}$$

be the neutrosophic square of type II. Clearly the point of intersection of diagonals is (6, I).

Consider
$$P - A = \{(-8, 2I), (-12, -2I), (-8, -6I), (-4, -2I)\} \in N_{II}.$$

The mid point on the diagonals is (–8, –2I).

The addition or subtraction of a point square to a neutrosophic square of type II shifts the position of the neutrosophic square.

We will represent this situation by an example or two.

Example 2.35: Let

$$P = \{(1, -2I), (1, -2I), (1, -2I), (1, -2I)\}$$

be the point neutrosophic square.

$$A = \{(-2, 0), (2, 0), (0, 2I), (0, -2I)\}$$

be the neutrosophic square of type II.

$$A + P = \{(-1, -2I), (3, -2I), (1, 0), (1, -4I)\}$$

be the neutrosophic square of type II.

The graph associated with P, A is given in Figure 2.30.

$$A - P = \{(-3, 2I), (1, 2I), (-1, 4I), (-1, 0)\},$$

$$P - A = \{(3, -2I), (-1, -2I), (1, -4I), (1, 0)\}.$$
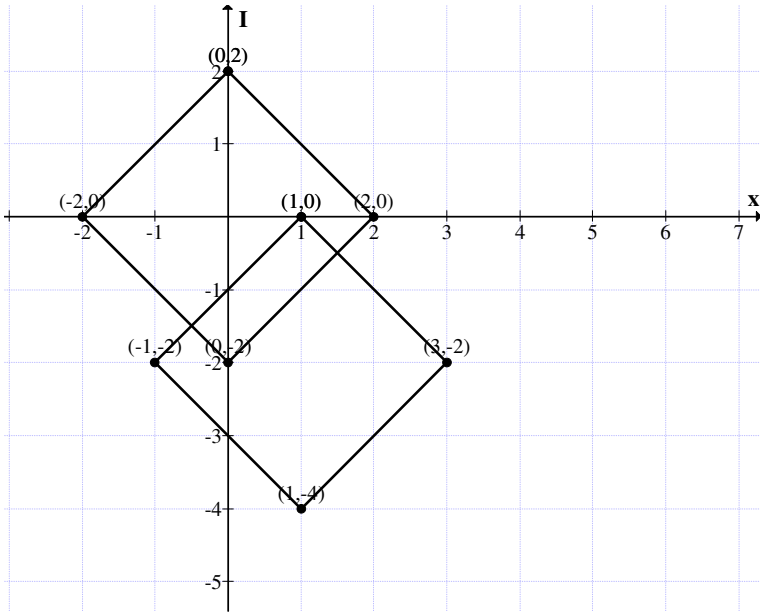
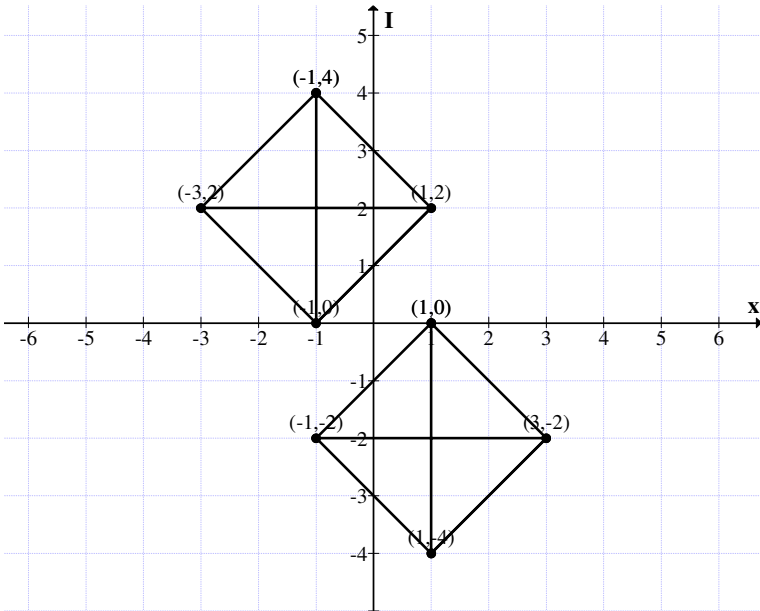The graphs of P – A and A – P is given in Figure 2.31.

Figure 2.30

Figure 2.31

These squares do not overlap.

Suppose we have two neutrosophic squares of type II. Is product defined?

Let
$$A = \{(0, 2I), (0, -2I), (2, 0), (-2, 0)\}$$
and
$$B = \{(6, 8I), (6, 2I), (9, 5I), ((3, 5I)\} \in N_{II}.$$

$$A \times B = \{(0, 28I), (0, -16I), (18, 10I) (-6, -10I)\}$$

is not a neutrosophic square of type II. Thus in general product operations is not defined on N_{II}.

Let
$$P = \{(2, 3I), (2, 3I), (2, 3I), (2, 3I)\}$$
be the point neutrosophic square.

$$A = \{(4, 5I), (7, 8I), (7, 2I), (10, 5I)\}$$
be the neutrosophic of type II.

$$A \times P = \{(8, 37I), (14, 61I), (14, 31I), (20, 55I)\}$$

is only a quadrilateral with two sides parallel to the neutrosophic axis.

Example 2.36: Let
$$A = \{(8, -2I), (12, 2I), (8, 6I), (4, 2I)\}$$
be the neutrosophic square of type II.
Let
$$P = \{(1, -2I), (1, -2I), (1, -2I), (1, -2I)\}$$
be the point neutrosophic square.

$$P \times A = \{(8, -14I), (12, -26I), (8, -22I), (4, -10I)\}$$
is not a neutrosophic square of type II.

P × A is only a quadrilateral such that two of its sides are equal and is parallel to the neutrosophic axis.

Inview of all these we have the following theorem.

THEOREM 2.9: *Let P = {(a, bI), (a, bI), (a, bI), (a, bI)} be a point neutrosophic square. M = {Any neutrosophic square of type II} ∈N_{II}. P ×M is a not a neutrosophic square of type II but is only neutrosophic quadrilateral such that two sides are equal and parallel to the neutrosophic axis.*

Proof follows from the simple working.

Thus clearly product of two neutrosophic square of type II. Hence N_{II} is not closed under product ×.

However (N_{II}, +) is a group.

Infact most of the properties enjoyed by real squares of type II are not satisfied by neutrosophic squares of type II.

Next study of interest would be will the sum or product of a neutrosophic square of type I with that of type II yield any interesting result.

We first study this situation using some examples.

Example 2.37: Let A = {(2, 2I), (4, 2I), (4, 4I), (2, 4I)} be a neutrosophic square of type I and

M = {(0, 2I), (0, –2I), (2, 0), (–2, 0)} be a neutrosophic square of type II.

However A + M is not a neutrosophic square of type I or type II. So at the outset one faces problem while adding them.

Thus study in this direction is left as a open conjecture.

When sum itself happens to be problem certainly product may or may not be a well settled problem.

In view of this situation in this book the two types of squares are dealt separately.

Next we suggest a few problems for this chapter.

Problems:

1. What are the distinct features enjoyed by neutrosophic squares of type I?

2. Compare real Euclid square of type I with neutrosophic square of type I.

3. Is $S_I \cap N_I = \phi$? (We include all point squares).

4. Let M = {(7, 8I), (8, 8I), (8, 9I), (7, 9I)} be a neutrosophic square?

    Is M a type I neutrosophic square or a type II neutrosophic square or neither?

5. Give an example of a neutrosophic square of type I of side length 7.

6. Let M = {(5, 3I), (8, 3I), (5, 5I), (8, 6I)} be the neutrosophic square of type I.

    Let P = {(7, –3I), (7, –3I), (7, –3I), (7, –3I)} be the point neutrosophic square.

    (i) Find M + M.

    (ii) Find P + M, P – M and M – P.

    (iii) Does these neutrosophic squares P + M, P–M and M–P

intersect or are disjoint?

7. Let $M = \{(-5, 2I), (-2, 2I), (-5, 5I), (-2, 5)\}$ be the neutrosophic square of type I.

Let $P = \{(-5, 2I), (-5, 2I), (-5, 2I), (-5, 2I)\}$ be a point neutrosophic square.

(i) Find $P + M$, $P–M$ and $M–P$.

(ii) If $P_1 = \{(-2, 2I), (-2, 2I), (-2, 2I), (-2, 2I)\}$ be the point neutrosophic square find $P_1 – M$, $P_1 + M$ and $M – P_1$.

(iii) If $P_2 = \{(-2, 5I), (-2, 5I), (-2, 5I), (-2, 5I)\}$ be another point neutrosophic square find $P_2 – M$, $M + P_2$ and $M – P_2$.

8. Let $S = \{(4, 2I), (6, 2I), (4, 4I), (6, 4I)\}$ be a neutrosophic square of type I.

(i) Let $P_1 = \{(3, I), (3, I), (3, I), (3, I)\}$ be a point neutrosophic square. Find $P–S$, $S + P$ and $S–P$.

(ii) Let $P_2 = \{(-3, I), (-3, I), (-3, I), (-3, I)\}$ be a point neutrosophic square. Find $P_2 + S$, $P_2–S$ and $S–P_2$.

(iii) Let $P_3 = \{\{(-3, –I), (-3, –I), (-3, –I), (-3, –I)\}$ be a point neutrosophic square.

Find $P_3 + S$, $P_3 – S$ and $S – P_3$.

(iv) Let $P_4 = \{(3, –I), (3, –I), (3, –I), (3, –I)\}$ be a point neutrosophic square. Find $P_4 + S$, $P_4 – S$ and $S – P_4$.

(v) Compare the three squares in problems (i) to (iv).

9.   Let $M = \{(-3, 4I), (0, 4I), (-3, 7I), (0, 7I)\}$ be the neutrosophic square of type I.

Let $P = \{(5, 3I), (5, 3I), (5, 3I), (5, 3I)\}$ be the point neutrosophic square.

Find $P + M$, $P–M$ and $M–P$.

10. Let $N = \{(10, 2I), (6, 2I), (10, –2I), (6, –2I)\}$ be a neutrosophic square of type I.

Let $P_1 = \{(2, 0), (2, 0), (2, 0), (2, 0)\}$ be a point square and

$P_2 = \{(0, 3I), (0, 3I), (0, 3I), (0, 3I)\}$ be a point neutrosophic square.

(i) Find $P_1 + P_2$, what is the structure of $P_1 + P_2$.

(ii) What is the structure of $P_1 – P_2$?

(iii) Find the structure of $P_2 – P_1$.

(iv) Find $N + P_1$ and $N + P_2$ and compare them.

(v) Find $P_1 – N$ and $P_2 – N$ and compare them.

(vi) Find $P_1 + P_2 + N$ and $N – (P_1 + P_2)$ are compare them.

11. Let $W = \{(4, –5I), (–3, –5I), (4, –12I), (–3, –12I)\}$ be a neutrosophic square of type I.

Let $P_1 = \{(4, 7I), (4, 7I), (4, 7I), (4, 7I)\}$ and

$P_2 = \{(–3, 4I), (–3, 4I), (–3, 4I), (–3, 4I)\}$ be neutrosophic point squares.

(i) Find $P_1 + P_2$ and $P_1 – P_2$.

(ii) Find $W + P_1$, $W + P_2$ and compare them.

(iii) Find $W - P_1$ and $W - P_2$ and compare them.

(iv) Find $P_1 - W$ and $P_2 - W$ and compare them.

(v) Find $(P_1 + P_2 - W)$ and $(P_1 + P_2 + W)$ and compare them.

(vi) Find $(P_1 + P_2 + W)$ and $(W-(P_1 + P_2))$ and compare them.

(vii) Which is largest neutrosophic square of type I?

12. Let $W_1 = \{(3, 2I), (6, 2I), (3, 5I), (6, 5I)\}$ and

$W_2 = \{(-5, 4I), (-2, 4I), (-5, 7I), (-2, 7I)\}$ be any two neutrosophic squares of type I?

(i) Find $W_1 - W_2$ and $W_2 - W_1$ and compare them.

(ii) Find $W_1 + 2W_2$ and $W_2 + 2W_1$ and compare them.

(iii) What is the geometric structure enjoyed by $W_1 \, W_2$?

13. Let $W = \{(2, -7I), (5, -7I), (2, -4I), (5, -4I)\}$ be the neutrosophic square of type I.

$M = \{(-7, 2I), (-7, 5I), (-4, 2I), (-4, 5I)\}$ be the neutrosophic square of type I.

(i) Find $M + W$.

(ii) Find $M - W$ and $W - M$.

(iii) Find $3M - W$ and $3W - M$.

(iv) Compare these neutrosophic squares.

(v) Obtain any other interesting feature you know about neutrosophic squares of type I.

14. Obtain any other interesting feature you know about neutrosophic squares of type I.

15. Enumerate all the special features enjoyed by neutrosophic squares of type II.

16. What are the main differences between neutrosophic squares of type I and type II?

17. Show $(N_{II}, +)$ is a commutative group of infinite order.

18. Can $(N_{II}, +)$ have finite order subgroups?

19. Let M = {(3, 2I), (4, 3I), (3, 4I), (2, 3I)} be a neutrosophic square of type II.

    Let A = {(–3, –4I), (–3, –4I), (–3, –4I), (–3, –4I)} be a point neutrosophic square .

    (i) Find M + A, M–A and A–M.

    (ii) Compare all the three squares with M.

    (iii) What is the structures enjoyed by MA?

    (iv) Is M × M a neutrosophic square of type II?

20. Let M = {(–1, –2I), (3, –2I), (1, 0), (1, –4I)} and

    N = {(0, 2I), (0, –2I), (2, 0), (–2, 0)} be two neutrosophic squares of type II.

    (i) Find M + N, M–N and N–M.

    (ii) Prove all the three are distinct neutrosophic squares of

type II.

(iii) Find MN, is MN a neutrosophic square?

(iv) Let P = {(2, 0), (2, 0), (2, 0), (2, 0)} be the point square.

Find M + P, N + P, P–M, P–N and M + N–P.

(v) Let R = {(0, 3I), (0, 3I), (0, 3I), (0, 3I)} be a neutrosophic point square.

(vi) Find R + N, R–N, N–R, M–R, M + R, R–M, R + N–M, R + M–N and M + N–R.

(vii)Study all the seven squares by comparison and contrast.

21. Prove (N_{II}, +) has subgroups of infinite order.

22. Let

$$M = \{(10, -I), (14, 3I), (10, 7I), (6, 3I)\}$$

and

$$N = \{(-8, 2I), (-12, -2I), (-8, -6I) (-4, -2I)\}$$

be two distinct neutrosophic squares of type II.

(i) Find M + N, M–N and N–M.

(ii) Does these exists any similarities enjoyed by these neutrosophic squares?

(iii) Let P = {(–5, –7I), (–5, –7I), (–5, –7I), (–5, –7I)} be the neutrosophic point square.

23. Let M = {(–1, –2I), (–3, –2I), (1, 0), (1, –4I)} and

N = {(6, 8I), (6, 2I), (9, 5I), (3, 5I)} be two neutrosophic square of type II. Using the same P as given in problem 22.

Study questions (i) to (iii) of problem 22 for this M and N.

24. Prove addition of a point neutrosophic square to a neutrosophic square of type II leads to the translation.

    Thus prove the work of translation is very simplified by this method.

25. Obtain any other interesting feature that can be derived on N_{II}.

26. Does there exist a subset P in N_{II} so that {P, ×} is a semigroup?

27. Translate or shift the neutrosophic square of type II say

    $$M = \{(-3, 2I), (1, 2I), (-1, 4I), (-1, 0)\}$$

    using the following set of point neutrosophic squares.

    $$P_1 = \{(3, 0), (3, 0), (3, 0), (3, 0)\}.$$

    $$P_2 = \{(0, 4I), (0, 4I), (0, 4I), (0, 4I)\}.$$

    $$P_3 = \{(-7, 0), (-7, 0), (-7, 0), (-7, 0)\}.$$

    $$P_4 = \{(0, -6I), (0, -6I), (0, -6I), (0, -6I)\}.$$

    $$P_5 = \{(2, I), (2, I), (2, I), (2, I)\}.$$

    $$P_6 = \{(3, -5I), (3, -5I), (3, -5I), (3, -5I)\}.$$

    $$P_7 = \{(-8, 7I), (-8, 7I), (-8, 7I), (-8, 7I)\}$$

and

    $$P_8 = \{(-4, -3I), (-4, -3I), (-4, -3I), (-4, -3I)\}$$

That is find $P_i + M$, $P_i - M$, $M - P_i$, $i = 1, 2, 3$, compare each of these 24 squares.

Find how many squares are disjoint.

Find those squares which has a common side.

Find those square which have common area.

Chapter Three

SQUARES OF TYPE I AND TYPE II IN THE COMPLEX PLANE AND OTHER PLANES

In this chapter first we study the properties of type I and type II squares using the complex plane

$$C = \{a + ib \mid a, b \in R, i^2 = -1\}.$$

First some examples are provided before definition are made.

Example 3.1: Let

$$P = \{(5, 3i), (7, 3i), (5, 5i), (7, 5i)\}$$

be the complex square of type I.

The representation of P in the argand plane is as follows:
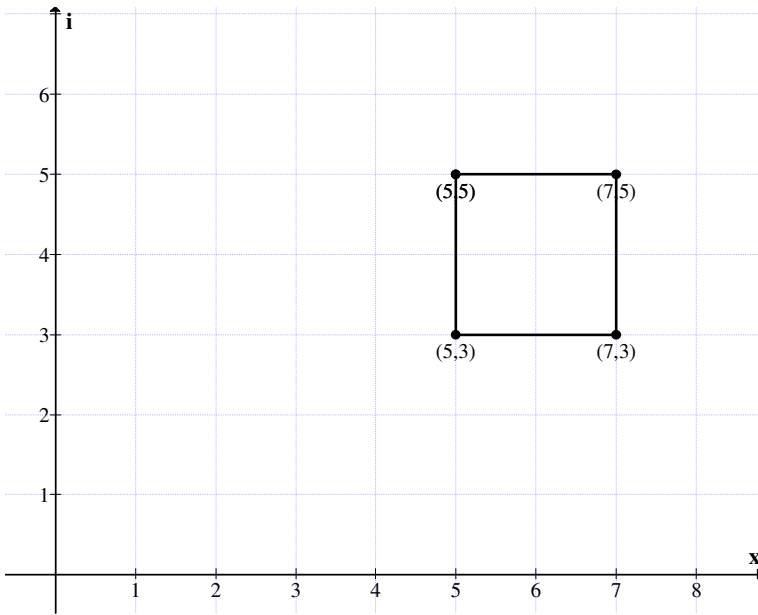
Figure 3.1

Clearly P is a square such that the sides are parallel to the axis of the plane.

Thus P is the complex square of type I.

In view of this we make the following definition.

DEFINITION 3.1: *Let C_I = {Collection of complex number squares in the complex plane such that the sides of the squares are always parallel to the x axis and imaginary axis of the plane}; Elements of C_I are defined as complex squares of type I.*

We will first give examples of them.

Example 3.2: Let

$$M = \{(3, 2i), (8, 2i), (8, 7i), (3, 7i)\}$$

be the complex square of type I.

The diagrammatic representation for M is as follows:
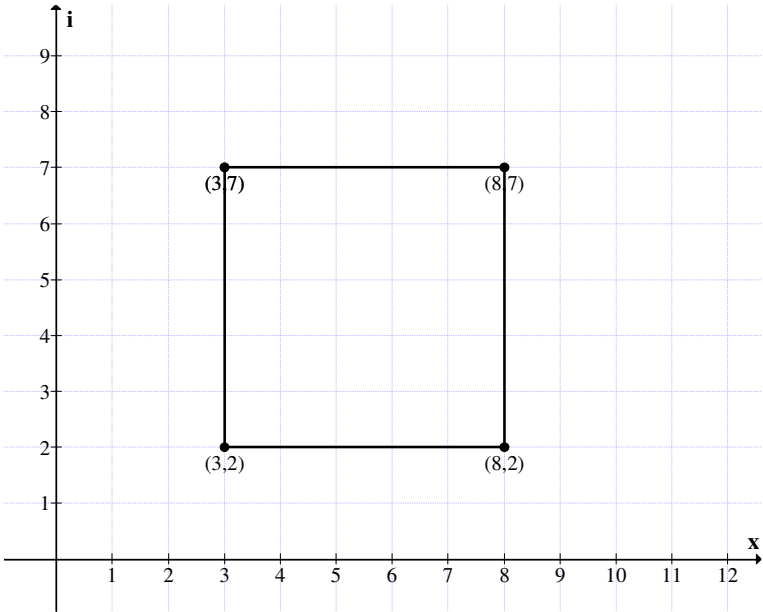
Figure 3.2

Example 3.3: Let

$$W = \{(3, -2i), (5, -2i), (3, 0), (5, 0)\}$$

be the complex square of type I.

This W has the following representation in the complex plane as given in Figure 3.3:

Example 3.4: Let

$$S = \{(-2, -4i), (-6, -4i), (-2, -8i), (-6, -8i)\}$$

be the complex square of type I.

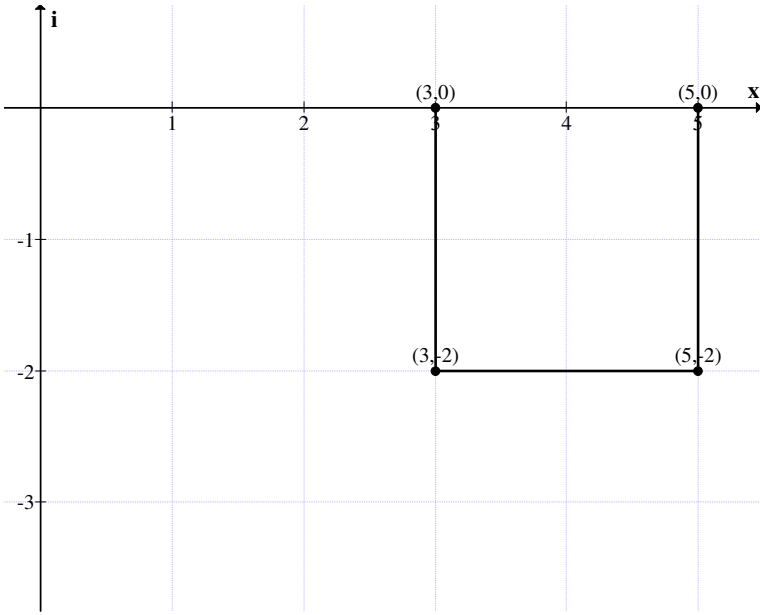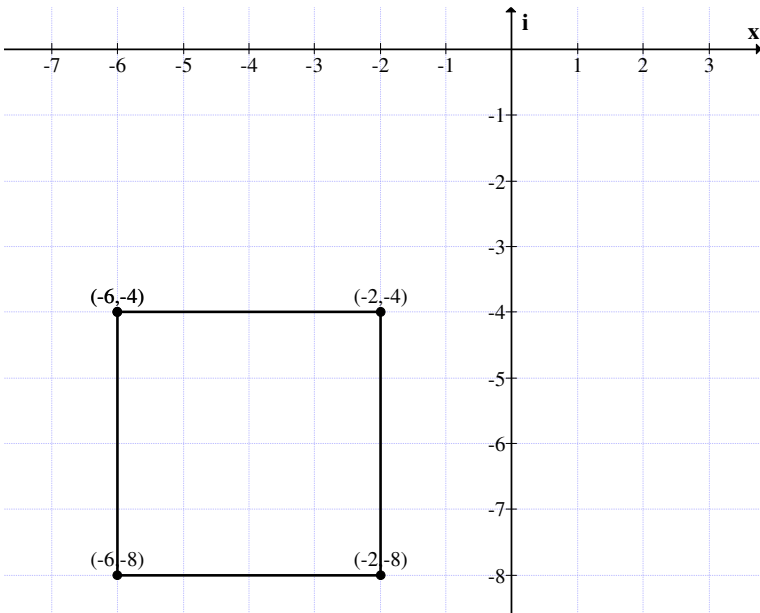This S has the following representation in the complex plane as given in Figure 3.4.

Figure 3.3

Figure 3.4

Example 3.5: Let

$$S = \{(-12, -4i), (-2, -4i), (-12, 6i), (-2, 6i)\}$$
be a complex square of type I.

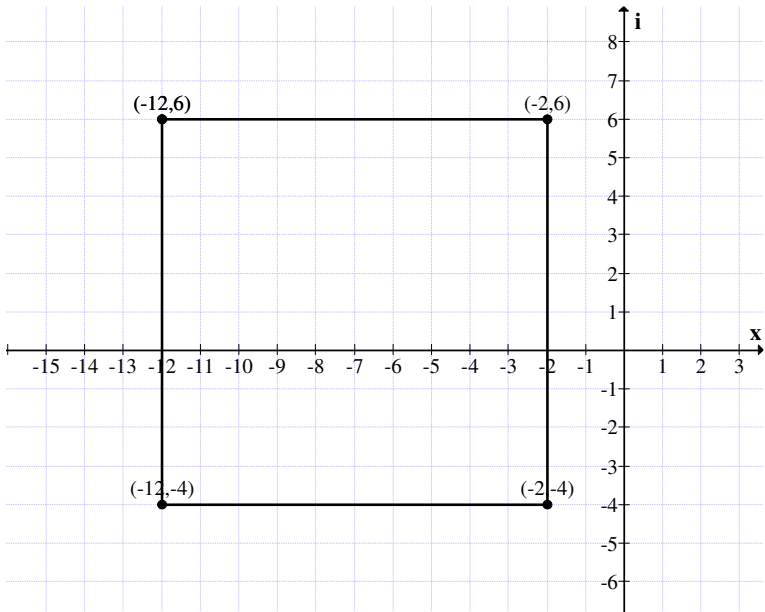S has the following diagrammatic representation in the complex plane.

Figure 3.5

Now having seen various representation of complex squares of type I, we proceed to define operations on them.

We will first illustrate the sum of two complex squares in C_I by an example.

Example 3.6: Let
$$A = \{(2, 3i), (4, 3i), (2, 5i), (4, 5i)\}$$
and

$$B = \{(-3, -6i), (0, -6i), (-3, -3i), (0, -3i)\}$$

be any two complex number squares.

$$A + B = \{(-1, -3i), (4, -3i), (-1, 2i), (4, 2i)\} \in C_I.$$

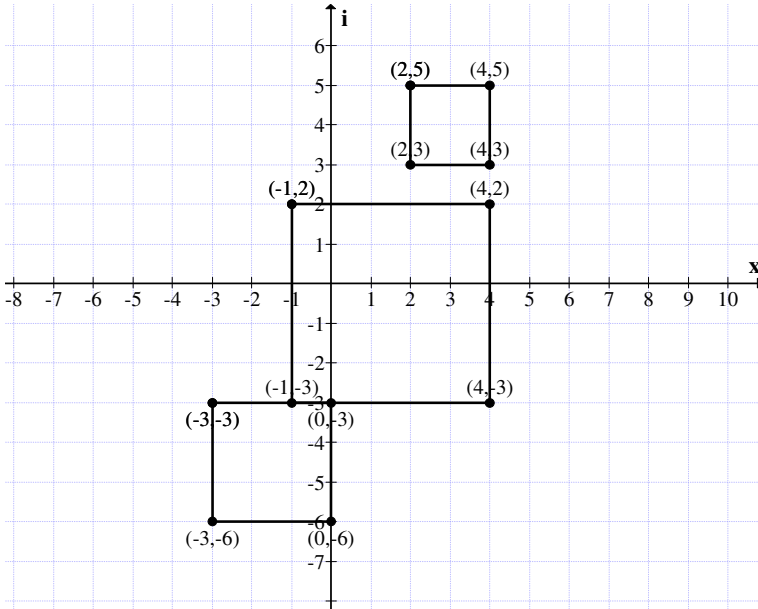We give the diagrammatic representation of A, B and A + B in the complex plane.

Figure 3.6

Clearly A + B is disjoint with A; however a small overlap of one side between A and A + B has occurred.

We find next A – B and B – A.

$$A - B = \{(5, 9i), (4, 9i), (5, 8i), (4, 8i)\}$$

and

$$B - A = \{(-5, -9i), (-4, -9i) \ (-5, -8i), (-4, -8i)\} \in C_I.$$

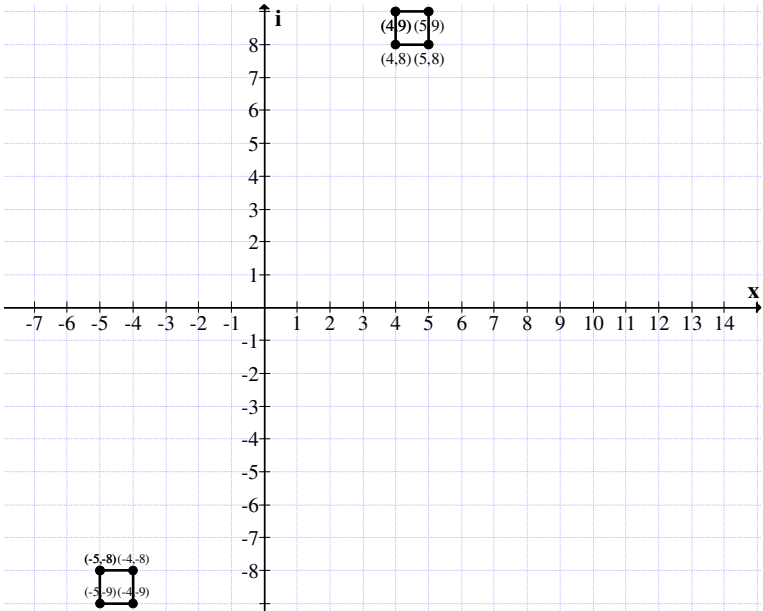In the following the graph of both A – B and B – A in the complex plane is given.

Figure 3.7

Clearly A + B, A and B have no common part with both A − B and B − A. Infact A − B and B − A happens to be diametrically opposite quadrants at the same distance from (0, 0).

Example 3.7: Let
$$P = \{(4, -2i), (7, -2i), (4, i), (7, i)\}$$
and
$$Q = \{(-2, 5i), (1, 5i), (-2, 8i), (1, 8i)\}$$
be two complex square of type I.

We find P + Q, P − Q and Q − P in the following.

$$P + Q = \{(2, 3i), (8, 3i), (2, 9i), (9, 9i)\}$$
$$P - Q = \{(6, -7i), (6, -7i), (6, -7i), (6, -7i)\}$$
$$Q - P = \{(-6, 7i), (-6, 7i), (-6, 7i), (-6, 7i)\} \in C_I.$$
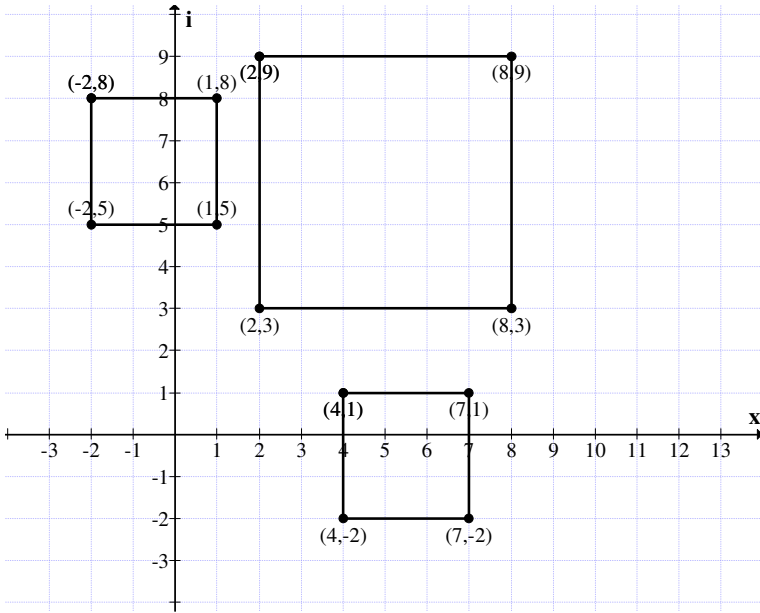The graphical representation is as follows.

Figure 3.8

All the three complex squares P, Q and P + Q are disjoint. However P – Q and Q – P are just point complex squares.

Next we conclude from this example that if two complex squares A and B of type I of same side length are subtracted that A – B and B – A are obtained; they are just point complex squares such that A – B ≠ B – A.

Next we proceed onto study the addition of a complex point square P to a complex square A of type I.

Further we also find P – A and A – P.

This is illustrated by an example or two.

Example 3.8: Let A = {(–3, 2i), (0, 2i), (–3, 5i), (0, 5i)} be a complex square of type I.
Let P = {(2, –i), (2, –i), (2, –i), (2, –i)} be a complex point square. We find P + A, A – P and P – A in the following.

$$P + A = \{(-1, i), (2, i), (-1, 4i), (2, 4i)\}$$
$$A - P = \{(-5, 3i), (-2, 3i), (-5, 6i), (-2, 6i)\}$$
$$P - A = \{(5, -3i), (2, -3i), (5, -6i), (2, -6i)\}.$$

Clearly all the three complex squares of type II of side length three.

So addition of a point square does not alter the length of the side only shift or translates the square to a different place in the complex plane.

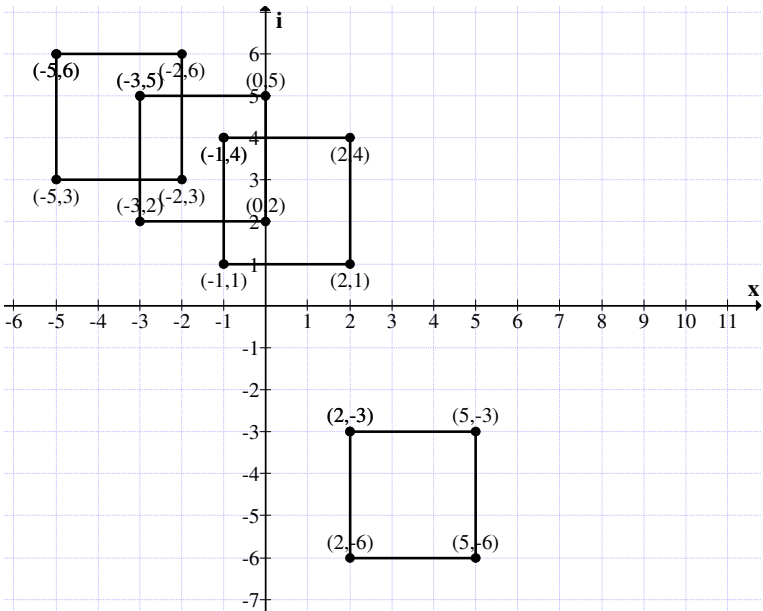We give the graph of them in the complex plane in the following.

Figure 3.9

It is clear that the complex squares P + A and A – P intersect A.

However A – P and P + A does not intersect each other.

Further the square P – A does not intersect any of the complex squares.

However all the four complex squares A + P, P – A, A and A – P have the size of the side to be the same.

Example 3.9: Let
$$B = \{(0, -3i), (2, -3i), (0, -i), (2, -i)\}$$
be the complex square of type I.
Let
$$P = \{(2, 2i), (2, 2i), (2, 2i), (2, 2i)\}$$
be the complex point square.
$$P + B = \{(2, -i), (4, -i), (2, i), (4, i)\}$$
$$P - B = \{(2, 5i), (0, 5i), (2, 3i), (0, 3i)\}$$
and $\quad B - P = \{(-2, -5i), (0, -5i), (0, -3i), (-2, -3i)\}$
are three distinct complex squares of same size.

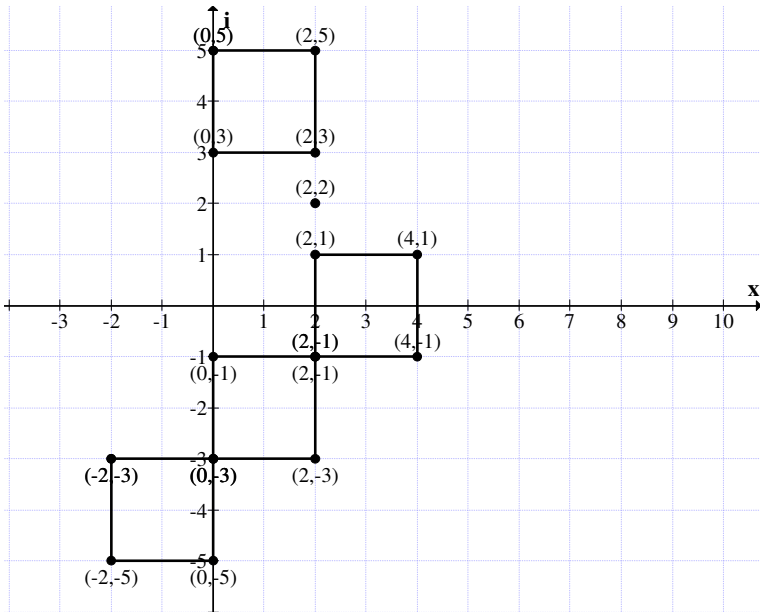The graph of them in the complex plane is as follows:

Figure 3.10

From this graph it is clear B – P and B has a common vertex of the complex square.

Further the complex squares B and P + B have a common vertex. However P – B is disconnected from all the other three complex squares.

This situation is clearly described in the diagram on the complex plane.

Thus in view of all these we include all point complex squares in the class of complex squares of type I denoted by C_I.

We prove the following theorem.

THEOREM 3.1: *Let {C_I, +} be the collection of all complex square of type I including the complex point squares.*

{C_I, +} is an abelian group of infinite order.

Proof is direct hence left as an exercise to the reader.

Next we proceed onto study whether the product operation can be defined on C_I so that C_I under X enjoys some algebraic structure.

First we will illustrate this situation by an example or two.

Example 3.10: Let

$\qquad$ A = {(2, 3i), (4, 3i), (2, 5i), (4, 5i)}

and $\qquad$ B = {(–1, 2i), (4, 2i), (–1, 7i), (4, 7i)}

be two complex squares of type I.

$\qquad$ A × B = {(–8, –7i), (10, 20i), (–37, 9i), (–19, 48i)}.

Clearly A × B is not a complex square of type I.

$\qquad$ It is infact a quadrilateral.

Example 3.11: Let

$\qquad$ A = {(0, 2i), (1, 2i), (0, 3i), (1, 3i)}

and $\qquad$ B = {(2, 0), (3, 0), (2, i), (3, i)}

be two complex squares of type I.

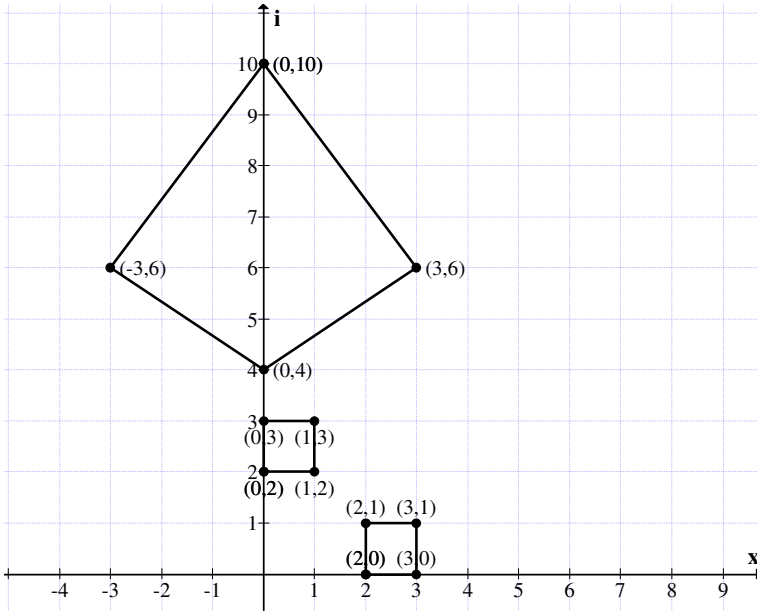$A \times B = \{(0, 4i), (3, 6i), (-3, 6i), (0, 10i)\}$ is not a complex square of type II.

Figure 3.11

Clearly the complex square product is not a complex square of type I.

Infact in this case one gets a nice quadrilateral.

Example 3.12: Let
$$A = \{(2, 2i), (3, 2i), (2, 3i), (3, 3i)\}$$
and
$$B = \{(2, 3i), (3, 3i), (2, 4i), (3, 4i)\}$$
be any two complex squares of type I.

$$A \times B = \{(-2, 10i), (3, 15i), (-10, 20i), (-3, 21i)\}$$
is not a complex square of type I.

We give the graph of all the three squares in the complex plane in the following:
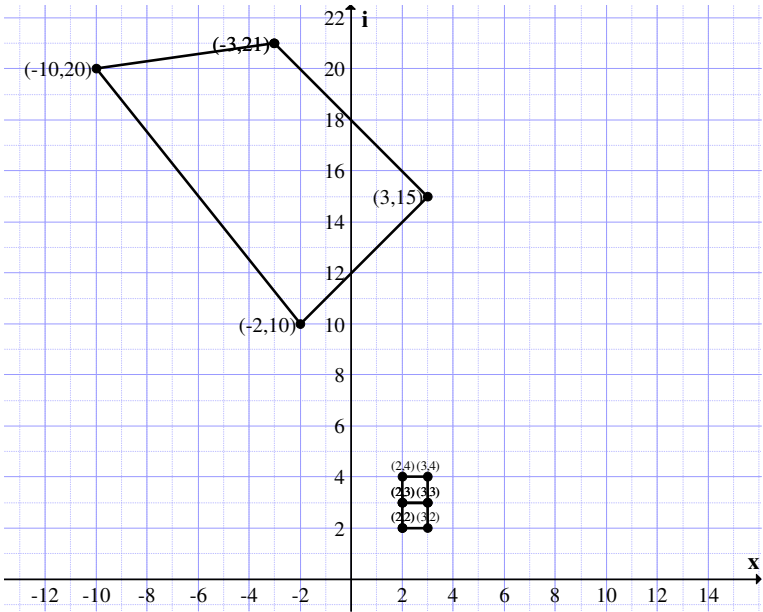
Figure 3.12

Clearly $A \times B$ is not a complex square of type I.

Thus we have the following theorem.

THEOREM 3.2: *Let*

    C_I = {Collection of all complex squares of type I}.

C_I under product is not closed.

Proof : Follows from the simple fact that for any A, B $\in$ C_I.

$$A \times B \notin C_I.$$

Thus $(C_I, +)$ is only a group under +.

Infact even if A, B are squares of type I in the first quadrant then $A \times B$ is not defined or not in C_I.

The simple procedure of translation is carried out by addition of appropriate point in C_I to any square of type in C_I.

Thus addition of a point complex square P of type I to any complex square A of type I only translates the position of A to some other position in the complex plane.

Example 3.13: Let

$$A = \{(0, 2i), (3, 2i), (0, 5i), (3, 5i)\}$$

and

$$P = \{(2, 4i), (2, 4i), (2, 4i), (2, 4i)\}$$

be the complex square in C_I.

$$P + A = \{(2, 6i), (5, 6i), (2, 9i), (5, 9i)\}$$
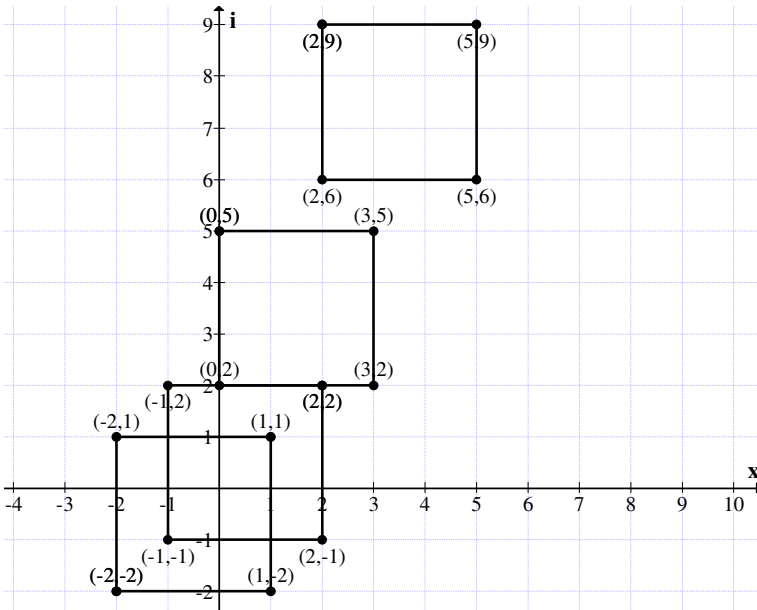
is again a complex square of type I.

Figure 3.13

$$A - P = \{(-2, -2i), (1, -2i), (-2, i), (1, i)\}$$

$$P - A = \{(2, 2i), (-1, +2i), (2, -i), (-1, -i)\}$$

is again a complex square of type I.

Clearly P – A and A – P have non trivial intersection.

However they have a common edge between the squares P – A and A.

P + A is not connected with all the three squares P, P–A and A – P.

Example 3.14: Let
$$A = \{(1, i), (3, i), (1, 3i), (3, 3i)\}$$
and
$$P = \{(0, 2i), (0, 2i), (0, 2i), (0, 2i)\}$$

be a point complex number.

We find A – P, P + A and P – A and draw the graph associated with them in the complex plane.

$$P + A = \{(1, 3i), (3, 3i), (1, 5i), (3, 5i)\}$$

$$P - A = \{(-1, i), (-3, i), (-1, -i), (-3, -i)\}$$
and
$$A - P = \{(1, -i), (3, -i), (1, i), (3, i)\}$$
are complex squares of type I.

The associated graph is given in Figure 3.14.

These squares behave in a very different way. The square P + A, A and A–P have a common edge.

Clearly the squares P–A have no common line or point or area. Thus the translation takes place in a very different way from the usual functions.
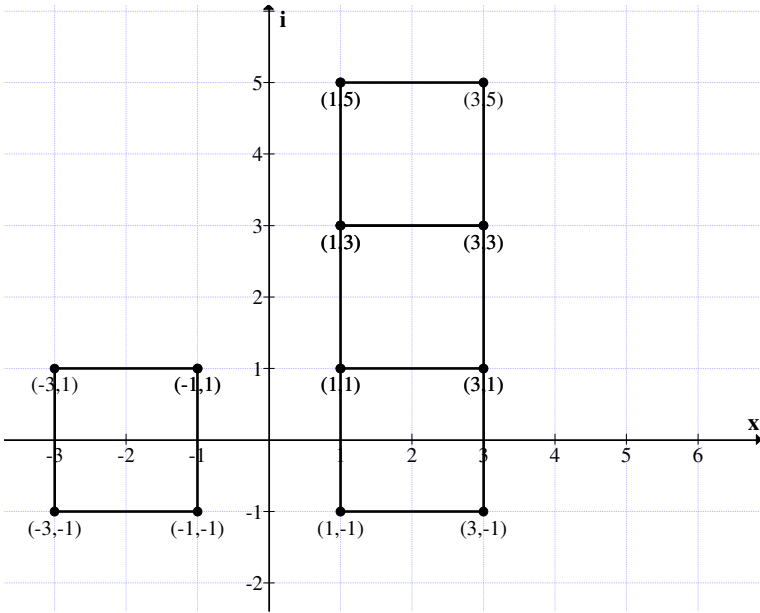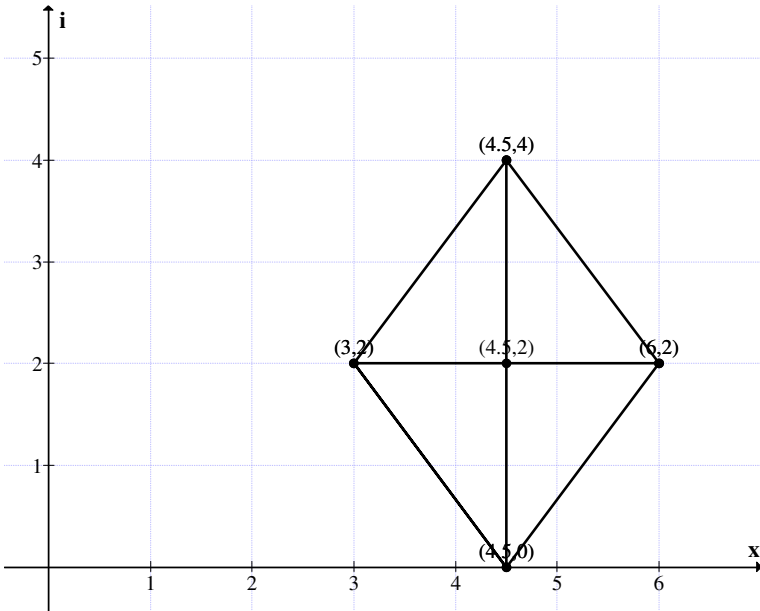
Figure 3.14

Figure 3.15

Infact beginners can use this technique for shift or translations of a complex square of type I in the complex plane.

Next we proceed onto give examples of complex squares of type II by same examples.

Example 3.15: Let

A = {(3, 2i), (6, 2i), (4.5, 4i), (4.5, 0)}
be the complex square of type II.

The graph associated with A is given in Figure 3.15.

Example 3.16: Let
M = {(–3, 0), (3, 0), (0, 3i), (0, 3i)}

be the complex square of type II.
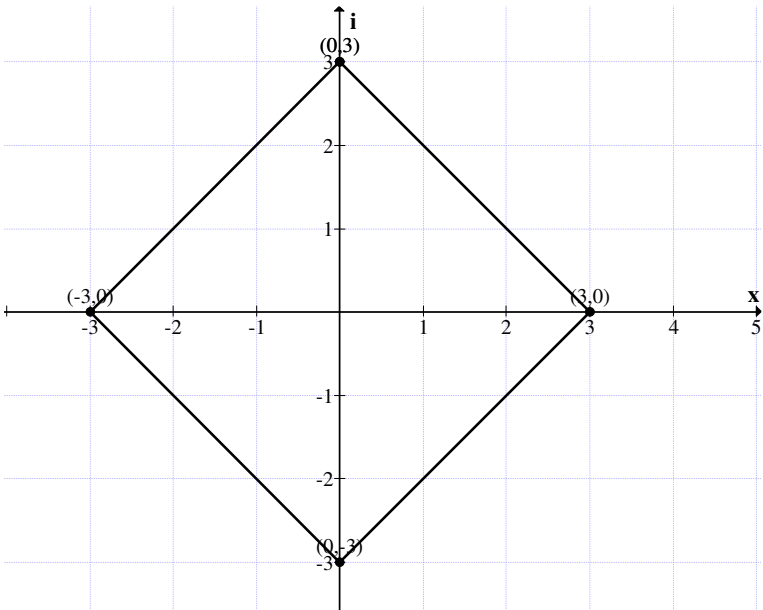
The graph of M in the complex plane is as follows:

Figure 3.16

Example 3.17: Let
$$P = \{(0, 3i), (0, 3i), (0, 3i), (0, 3i)\}$$
be the complex point square.

    Let
$$A = \{(-4, 0), (4, 0), (0, 4i), (0, -4i)\}$$
be the complex square of type II.

We find $A + P$, $A - P$ and $P - A$ in the following.
$$A + P = \{(-4, 3i), (4, 3i), (0, 7i), (0, -i)\}$$
is the complex squares of type II.

$$A - P = \{(-4, -3i), (4, -3i), (0, i), (0, -7i)\}$$
is again a complex square of type II.

$$P - A = \{(4, 3i), (-4, 3i), (0, -i), (0, 7i)\}$$
is again a complex square of type II.

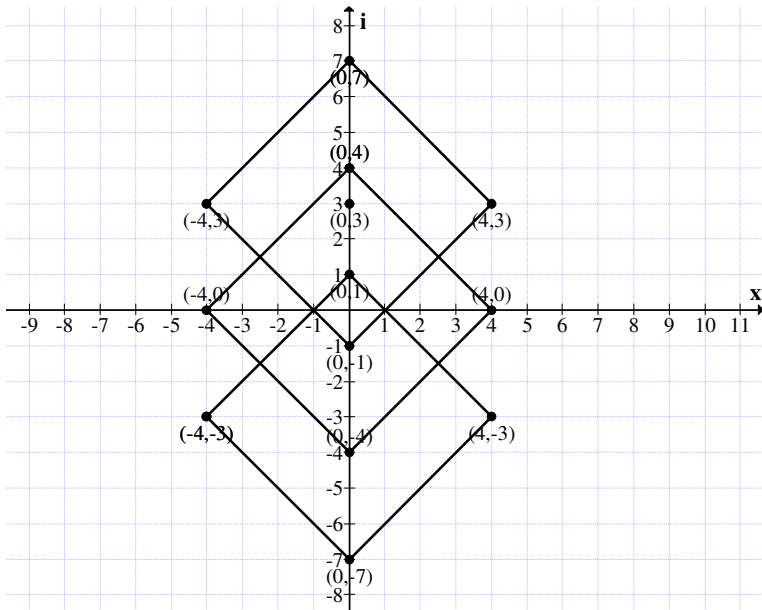   The graph of all the 5 squares in the complex plane is as follows:

Figure 3.17

Clearly P + A = P – A. All the three squares overlap in a nice way.

Further the point (0, 3i) lies on P – A = A + P.

Example 3.18: Let

$$A = \{(3, -5i), (7, -i), (3, 3i), (-1, -i)\}$$

be a complex square of type II.

Let
$$M = \{(1, i), (1, i), (1, i), (1, i)\}$$

be the complex point square.

$$A + M = \{(4, -4i), (8, 0), (4, 4i), (0, 0)\}$$

is again a complex square of type II.

$$A - M = \{(2, -6i), (6, -2i), (2, 2i), (-2, -2i)\}$$

is again a complex square of type II.

$$M - A = \{(-2, 6i), (-6, 2i), (-2, -2i), (2, 2i)\}$$

is again a complex square of type II.

In this case all the three squares distinct complex of type II.

We will illustrate the graph of all the 5 squares.

The graph is given in Figure 3.18.

$$M, A + M, A - M \text{ and } M - A.$$

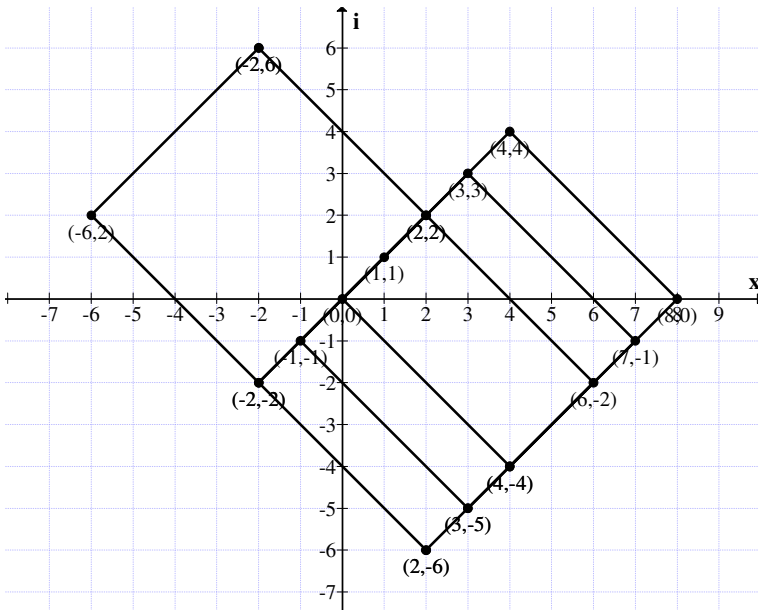Clearly all the four square intersect or have a point in common.

Figure 3.18

Example 3.19: Let

$$S = \{(3, 2i), (4, 3i), (3, 4i), (2, 3i)\}$$

be a complex square of type II.

Let

$$P = \{(2, -4i), (2, -4i), (2, -4i), (2, -4i)\}$$

be the point square.

Consider

$$S + P = \{(5, -2i), (6, -i), (5, 0), (4, -i)\}$$

is again a complex square of type II.

Now $S - P = \{(1, 6i), (2, 7i), (1, 8i), (0, 7i)\}$

$P - S = \{(-1, -6i), (-2, -7i), (-1, -8i), (0, -7i)\}$ is again a complex square of type II.
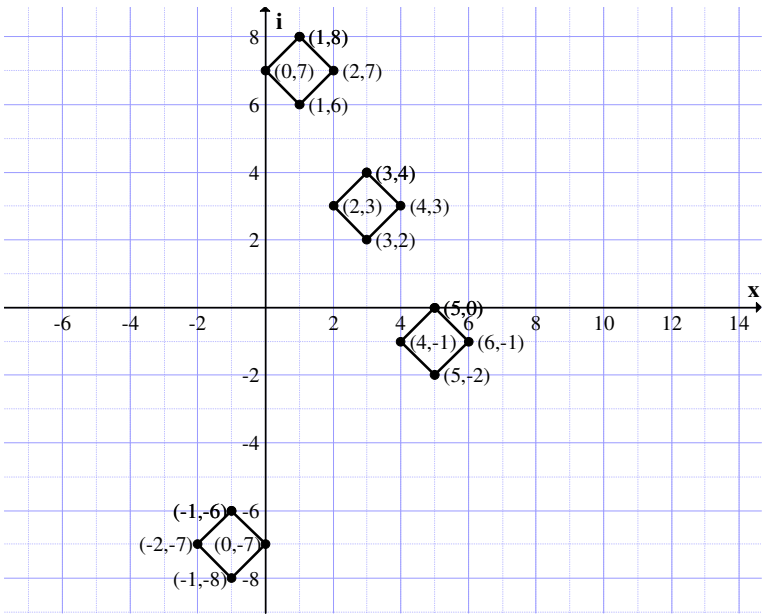
The graph of all the 5 squares are as follows:

Figure 3.19

All the 5 squares are disjoint.

How one can see how addition / subtraction of S with P gives us the same size square but shifted to 4th and 3rd quadrant.

Example 3.20: Let

$$B = \{(4, 0), (7, 3i), (4, 6i), (1, 3i)\}$$

and

$$A = \{(7, 5i), (10, 8i), (7, 11i), (4, 8i)\}$$

be two complex squares of type II.

We find A + B, A – B and B – A.

$$A + B = \{(11, 5i), (17, 11i), (11, 17i), (5, 11i)\}$$

$$A - B = \{(3, 5i),\ (3, 5i), (3, 5i), (3, 5i)\}$$

and

$$B - A = \{(-3, -5i), (-3, -5i), (-3, -5i), (-3, -5i)\}$$

are three complex squares two of them are just complex point squares but distinct or diametrically opposite to each other.

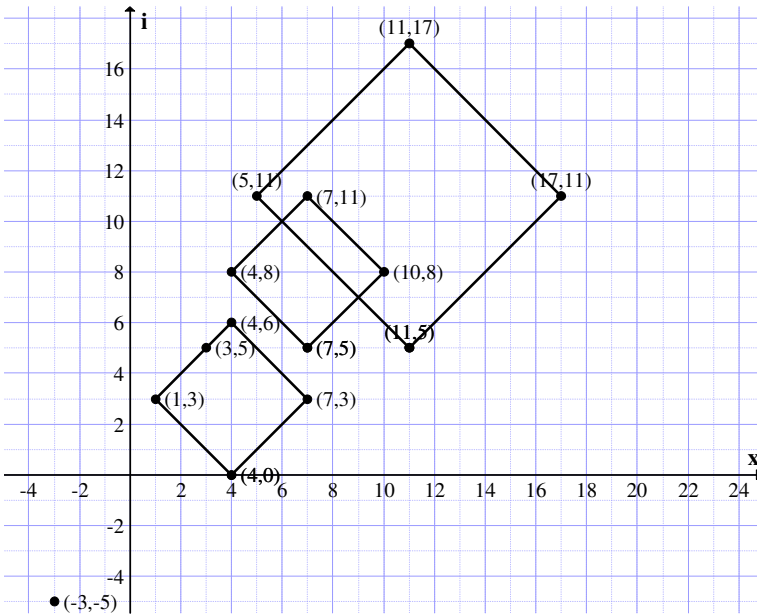The graph of these 5 complex squares is as follows:

Figure 3.20

Clearly A and A + B intersect.

Example 3.21: Let

$$A = \{(2, 4i), (4, 6i), (6, 4i), (4, 2i)\}$$

and

$$B = \{(5, 6i), (6, 7i), (7, 6i), (6, 5i)\}$$

be two complex squares of type II.

$$A + B = \{(7, 10i), (10, 13i), (13, 10i), (10, 7i)\}$$

$$A - B = \{(-3, -2i), (-2, -i), (-1, -2i), (-2, -3i)\}$$

$$B - A = \{(3, 2i), (2, i), (1, 2i), (2, 3i)\}$$

are three complex square of type II.
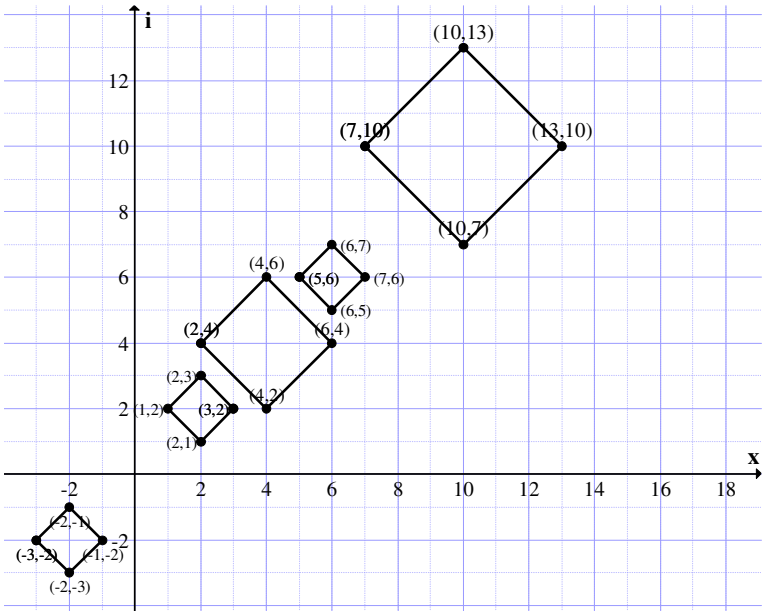
We give the graph of them.

Figure 3.21

We the complex square B – A and A – B are of same size.

However A and B are of different size and A + B is the biggest square.

Further all the 5 squares are disjoint.

Example 3.22: Let

$$A = \{(8, -2i), (12, 2i), (8, 6i), (4, 2i)\}$$

be the complex square of type II.

$$P_1 = \{(2, i), (2, i), (2, i), (2, i)\},$$

$$P_2 = \{(-3, 4i), \ (-3, 4i), (-3, 4i), (-3, 4i)\}$$

$$P_3 = \{(1, -3i), (1, -3i), (1, -3i), (1, -3i)\}$$

and

$$P_4 = \{(-3, -2i), (-3, -2i), (-3, -2i), (-3, -2i)\}$$

be four point squares.

We find $A \pm P_i$, $P_i - A$ for i = 1, 2, 3, 4.

$$A + P_1 = \{(10, -i), (14, 3i), (10, 7i), (6, 3i)\}$$

$$A + P_2 = \{(5, 2i), (9, 6i), (5, 10i), (1, 6i)\}$$

$$A + P_3 = \{(9, -5i), (13, -i), (9, 3i), (5, -i)\}$$

and

$$A + P_4 = \{(5, -4i), (9, 0), (5, 4i), (1, 0)\}$$

are the four complex squares whose graph is given in Figure 3.22.

All the four squares intersect each other. Next we find $A -$ P_i, i = 1, 2, 3, 4.

Consider

$$A - P_1 = \{(6, -3i), (10, i), (6, 5i), (2, i)\}$$

$$A - P_2 = \{(11, -6i), (15, -2i), (11, 2i), (7, -2i)\}$$

$$A - P_3 = \{(7, i), (11, 5i), (7, 9i), (3, 5i)\}$$

and

$$A - P_4 = \{(11, 0), (15, 4i), (11, 8i), (7, 4i)\}$$

are the four distinct complex squares of type II. We give the graph of all the 9 squares in the complex plane. The associated graph is given in Figure 3.23.
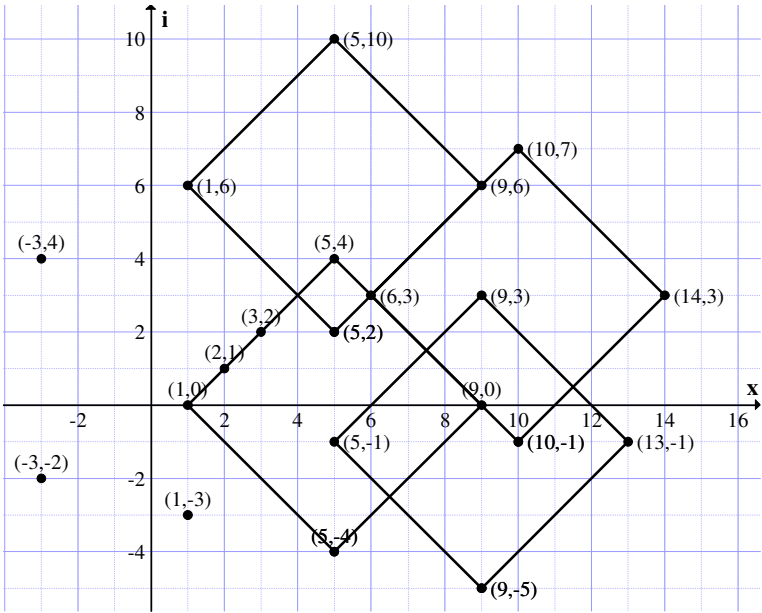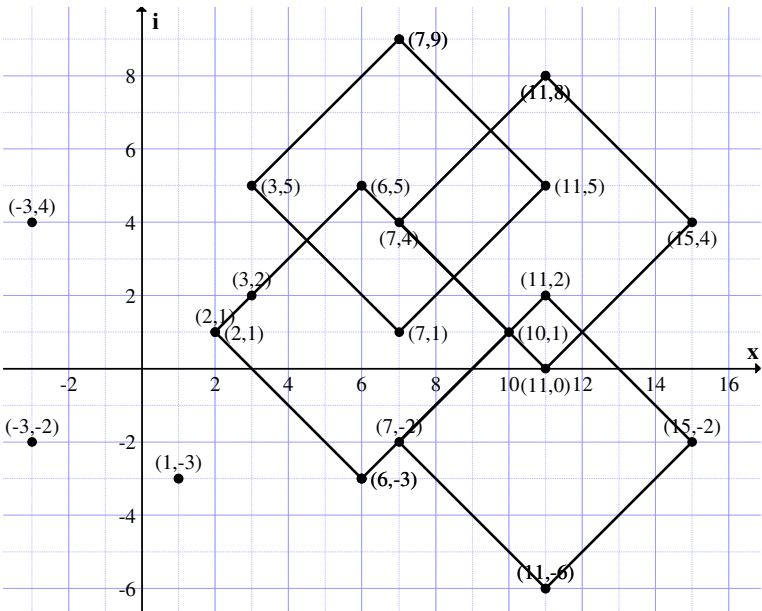
Figure 3.22

Figure 3.23

The complex square intersect each other.

Similarly one can find $P_i - A$ for $i = 1, 2, 3, 4$.

Now
$$P_1 - A = \{(-6, 3i), (-10, -i), (-6, -5i), (-2, -i)\}$$

$$P_2 - A = \{(-11, 6i), (-15, 2i), (-11, -2i), (-7, 2i)\}$$

$$P_3 - A = \{(-7, -i), (-11, -5i), (-7, -9i), (-3, -5i)\}$$
and
$$P_4 - A = \{(-11, 0), (-15, -4i), (-11, -8i), (-7, -4i)\}$$

are four distinct complex squares lying in the 3rd quadrant and they have non empty intersection.

We have seen for every $A \in C_{II}$ we have $A + 0 = 0 + A = A$ where $0 = \{(0, 0), (0, 0), (0, 0), (0, 0)\}$ zero point square which serves as the additive identity in C_{II}.

Further for every $A \in C_{II}$, $- A \in C_{II}$ such that $A + (-A) = (-A) + A = 0$ call the inverse of A and it is unique for each $A \in C_{II}$.

Thus $(C_{II}, +)$ is a group $C_{II} = \{$Collection of all complex squares whose diagonals are parallel to the x–axis and the imaginary axis$\}$, these squares in the complex plane are known as complex squares of type II.

Further type I squares and type II squares do not overlap unless they are point complex squares.

THEOREM 3.3: *Let C_{II} = {Collection of all complex squares of type II including point square}; {C_{II}, +} is an infinite abelian group.*

Proof is direct and hence left as exercise to the reader.

Now having seen the properties of these two types of complex squares we now proceed onto show the product in the squares of type II is not closed in general.

First this situation is described some examples.

Example 3.23: Let

$$A = \{(-6, 3i), (-10, -i), (-6, -5i), (2, -i)\}$$

and

$$B = \{(8, -2i), (12, 2i), (8, 6i), (4, 2i)\} \in C_{II}.$$

Consider

$$A \times B = \{(-42, 36i), (-118, -32i), (-18, -76i), (6, 0)\}.$$

Clearly $A \times B$ is not a complex square of type I or type II. This product is only a quadrilateral.

Example 3.24: Let

$$A = \{(3, 0), (-3, 0), (0, 3i), (0, -3i)\}$$

be a complex square of type II.
Let

$$B = \{(2, 0), (-2, 0), (0, 2i), (0, -2i)\} \in C_{II}.$$

$$A \times B = \{(6, 0), (6, 0), (-6, 0), (-6, 0)\}.$$

The graph of there is as follows:

That is $A \times B$ is a line on the x–axis given by $(-6, 0)$ to $(6, 0)$.

Consider
$$A = \{(-3, 0), (0, 3i), (3, 0), (0, -3i)\}$$

$$A \times A = \{(9, 0), (9, 0), (-9, 0), (-9, 0)\}$$
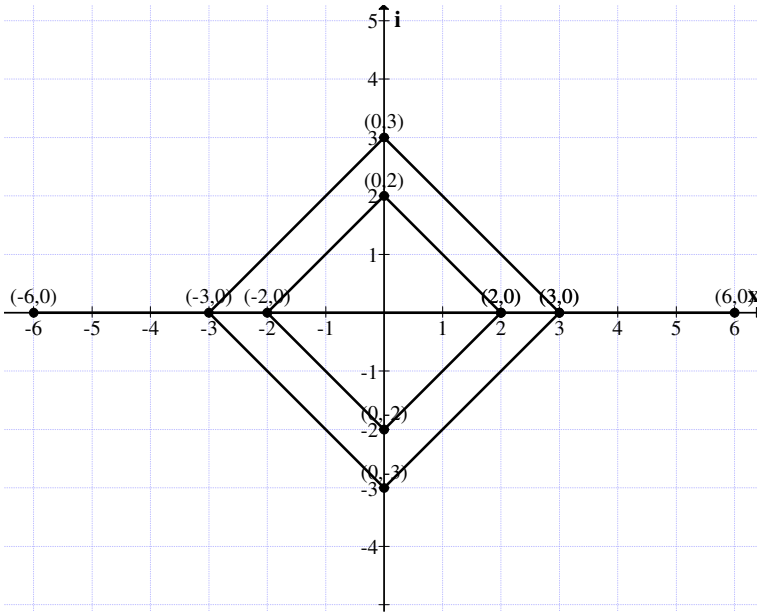is only a line on the x-axis.

Figure 3.24

Thus if $S(C_{II})$ = {Collection of all squares of the form {(a, 0), (0, ai), (–a, 0), (0, –ai)} where $a \in R$} then product happens to be line on the x–axis that is a segment of the x–axis if the squares in $S(C_{II})$ are distinct and only a point if A = B.

This is first illustrated by an example or two.

Example 3.25: Let

$$M = \{(1, 0), (0, i), (–1, 0), (0, –i)\}$$

be the complex square in $S(C_{II})$.
$$M^2 = \{(1, 0), (–1, 0)\}$$
is only a line.

$$M^3 = \{(1, 0), (–1, 0), (0, i), (0, –i)\} = M.$$

Thus in $S(C_{II})$ we have one element.

M such that $M^3 = M$, $M^4 = M^2$ and so on.

Example 3.26: Let

$$M = \{(5, 0), (-5, 0), (0, 5i), (0, -5i)\} \in S(C_{II}).$$

$$M^2 = \{(25, 0), (0, -25)\}$$

is a line.

$$M^3 = \{(125, 0), (-125, 0), (0, 125i), (0, -125i)\} \in S(C_{II}).$$

$$M^4 = \{(625, 0), (0, -625)\}$$

but $M^5 \in S(C_{II})$ and so on. Thus for every $M \in S(C_{II})$.

M generates again an element together with a collection of lines on the x–axis.

In view of this we define the following.

Let
$$A = \{(a, 0), (-a, 0), (0, ai), (0, -ai)\} \in S(C_{II}).$$

$$A^2 = \{(a^2, 0), (0, -a^2)\}$$

is a line.

$$A^4 = \{(a^4, 0), (0, a^4)\}$$

and so on.

$$A^{2n} = \{(a^{2n}, 0), (0, -a^{2n})\} \text{ and so on and } n \to \infty.$$

Thus if $LS(C_{II}) = \{$Collection of all complex squares in $S(C_{II})$ together with $\{(a^n, 0), (0, -a^n)\}$ $\{(a^n, 0), (0, a^n)\}$ where $a \in R$ (R–reals)$\}$.

Clearly point squares already are in C_{II}. The following theorem is given.

THEOREM 3.4: *$\{LS(C_{II})), \times\}$ is a semigroup under $\times$ of infinite order which is commutative.*

The proof of the theorem is direct hence left as an exercise to the reader.

We can have get another property enjoyed by $LS(C_{II})$ which is given in the following.

THEOREM 3.5: *Let $\{LS(C_{II}), \times\}$ be the semigroup. Every element generates a subsemigroup of $\{LS(C_{II}), \times\}$.*

Proof: Let

$$A = \{(a, 0), (0, -ai), (-a, 0), (0, ai)\} \in LS(C_{II}).$$

Clearly

$$A^2 = \{(a^2, 0), (0, -a^2)\}$$

and

$$A^3 = \{(a^3, 0), (0, -a^3 i), (-a^3, 0), (0, a^3 i)\} \in LS(C_{II}).$$

Thus A generates subsemigroups.

Clearly can $LS(C_{II})$ get the group structure?

Let $\{(1, 0), (0, -1)\}$ is the identity or unit element of the group.

Now if

$$A = \{(4, 0), (-4, 0), (0, 4i), (0, -4i)\} \in SL(C_{II})$$

then

$$B = \{(1/4, 0), (-1/4, 0), (0, -i/4), (0, i/4)\}$$

is such that

$$A \times B = \{(1, 0), (0, -1)\}.$$

Thus A is the inverse of B and B is the inverse of A under the product operation.

Let

$$A = \{(2, 0), (-2, 0), (0, 2i), (0, -2i)\} \in S(L(C_{II})).$$

$$A^2 = \{(4, 0), (0, -4)\}$$

$$A^4 = \{(16, 0), (0, 16)\}$$

as in the product of a line with the other line only corresponding coordinates are () multiplied.

$$A^5 = \{(32, 0), (-32, 0), (0, 32i), (0, -32i)\}$$

$$A^6 = \{(64, 0), (64, 0), (-64, 0), (0, -64)\} = \{(64, 0), (0, -64)\}$$

So inverse of any line$\{(a^2, 0), (0, a^2)\}$ is $\{((1/a^2), 0), (0, -1/a^2)\}$ only which gives $\{(1, 0), (-1, 0)\}$ the unit.

Thus for ever $A \in S(L(C_{II}))$ there is a unique B such that $A \times B = \{(1, 0), (0, -1)\}$.

If

$$A = \left\{ \left(-\left(\frac{5}{3}\right) \times 4, 0 \right), \left(0, \left(\frac{5}{3}\right) \times 4 \right) \right\}$$

then

$$A^{-1} = \left\{ \left(\frac{-3}{4 \times 5}, 0 \right), \left(0, \frac{3}{4 \times 5} \right) \right\}$$

such that

$$A \times A^{-1} = A^{-1} \times A = \{(1, 0), (0, -1)\}.$$

Thus we have group of special type of complex squares together with lines on the x-axis.

$\{(1, 0), (0, -1)\} \in S(L(C_{II})$ acts as the identity.

Now one gets a group structure for a subclass of complex squares of type II which is a group under product, unlike neutrosophic squares which are not closed under product operation.

We saw for the same type of square

$$A = \{(2, 0), (-2, 0), (0, 2I), (0, -2I)\}$$

and

$$B \in \{(3, 0), (-3, 0), (0, 3I), (0, -3I)\}$$

we have

$$A \times B = \{(6, 0), (6, 0), (0, 6I), (0, 6I)\}$$
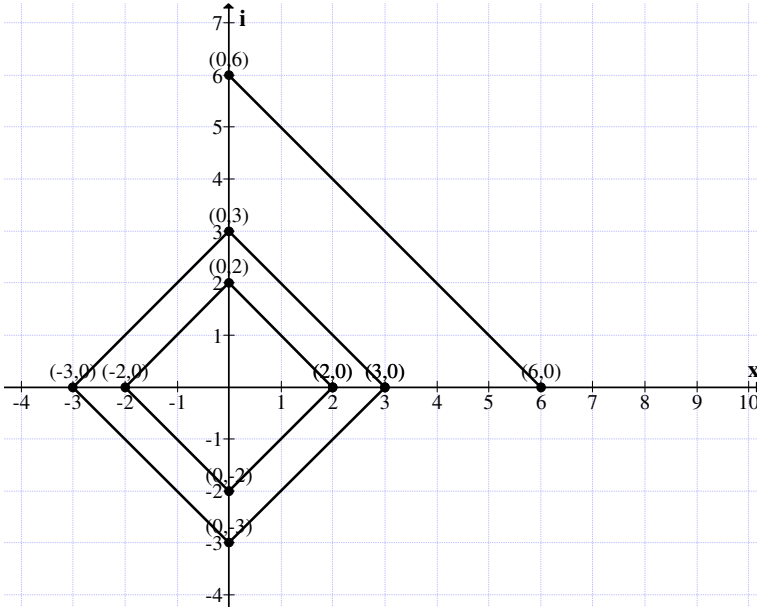
which is a line in the first quadrant.

Figure 3.25

So if

$$S(L(N_{II})) = \{\text{Collection of all neutrosophic squares}$$
$$\text{and lines of the form } \{(a, 0), (0, aI)\}.$$

Then $S(L(N_{II}))$ is only a semigroup. For I has no inverse as $I^2 = I$.

Next we proceed onto study the type of squares in the dual number line $D = \{a + bg \mid g^2 = 0, a, b \in R\}$.

We will first illustrate this situation by an example or two.

Example 3.27: Let

A = {(3, 5g), (6, 5g), (3, 8g), (6, 8g)}

be a dual number of square of type I.

Example 3.28: Let

M = {(–4, 2g), (–1, 2g), (–4, 5g), (–1, 5g)}

be a dual number of square of type I.

The representation of M in the dual plane is as follows:
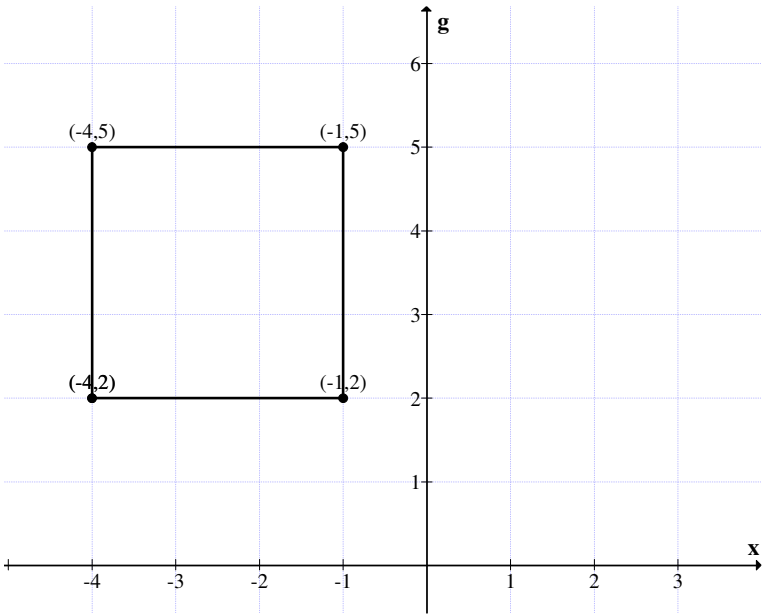

Figure 3.26

Example 3.29: Let

P = {(–3, –g), (–1, –g), (–3, g), (–1, g)}

is a dual number of square of type I.

The representation of P in the dual plane is given in Figure 3.27.

Example 3.30: Let S = {(3, –6g), (6, –6g), (3, –3g), (6,–3g)} is a dual number of square of type I.

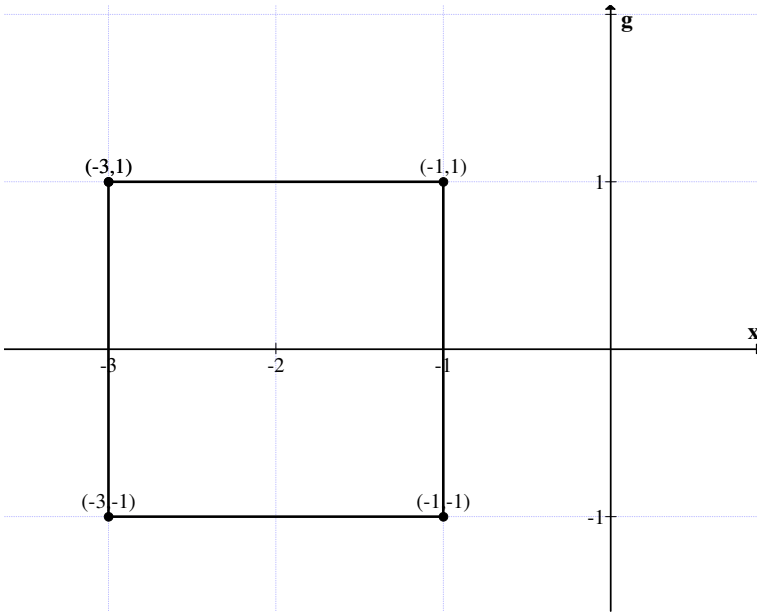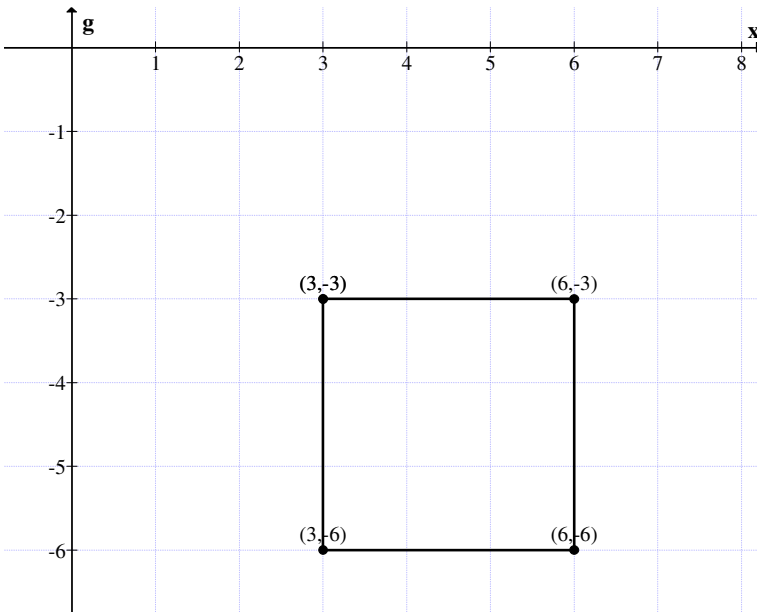The representation of P in the dual plane is given in Figure 3.28.

Figure 3.27

Figure 3.28

Example 3.31: Let
$$S = \{(7, -5g), (4, -5g), (7, -8g), (4,-8g)\}$$

is a dual number of square of type I.

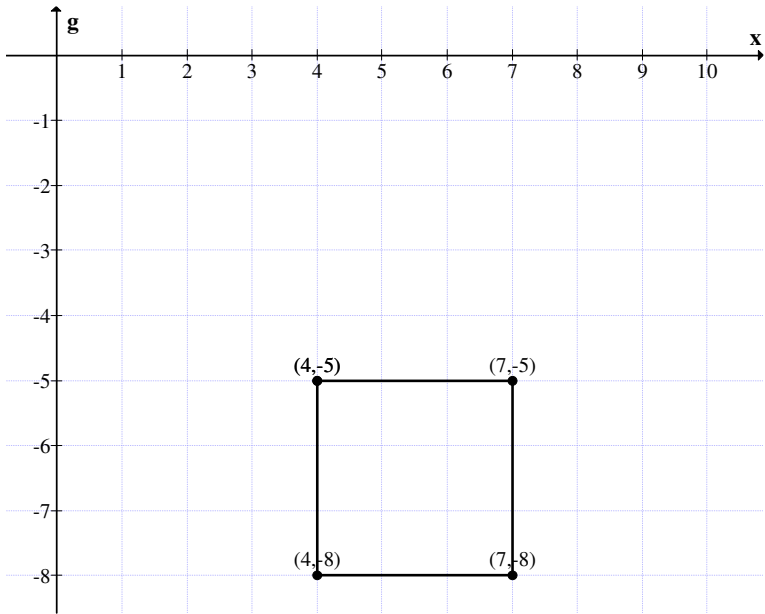The representation of P in the dual plane is given in Figure 3.29.

Figure 3.29

Example 3.32: Let

$$M = \{(3, 4g), (-3, 4g), (3, -2g), (-3,-2g)\}$$

is a dual number of square of type I.

The graph of M is given in Figure 3.27.

Now having seen different dual squares of type I we proceed onto define the operation of addition using examples.
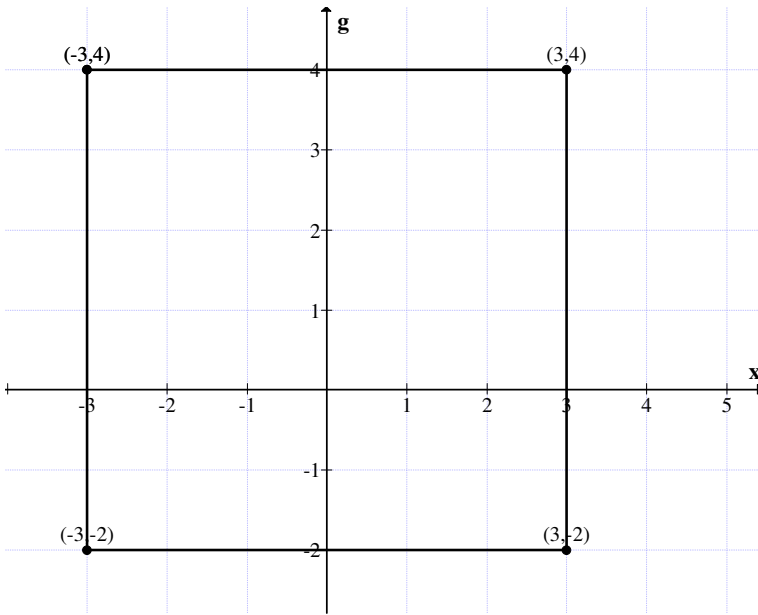
Figure 3.30

Example 3.33: Let

$$A = \{(-3, 4g), (-1, 4g), (-3, 2g), (-1, 2g)\}$$

and

$$B = \{(4, -5g), (1, -5g), (4, -2g), (1, -2g)\}$$

be any two dual squares of type I.

$$A + B = \{(1, -g), (0, -g), (1, 0), (0,0)\}.$$

The representation of these in the dual number plane is given in Figure 3.31.

Example 3.34: Let

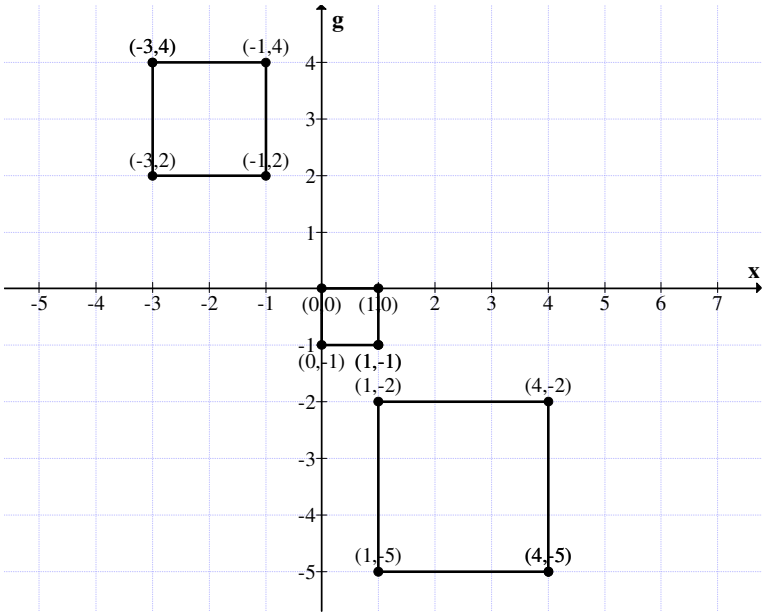$$M = \{(2, 3g), (4, 3g), (2, 5g), (4, 5g)\}$$

and

Figure 3.31

$$N = \{(3, g), (6, g), (3, 4g), (6, 4g)\}$$

be any two dual squares of type I.

We find

$$M + N, M - N \text{ and } N - M$$

$$M + N = \{(5, 4g), (10, 4g), (5, 9g), (10, 9g)\}$$

and

$$M - N = \{(-1, 2g), (-2, +2g), (-1, g), (-2, g)\}$$

and

$$N - M = \{(1, -2g), (2, -2g), (1, -g),. (2, -g)\}$$

are the dual number square.

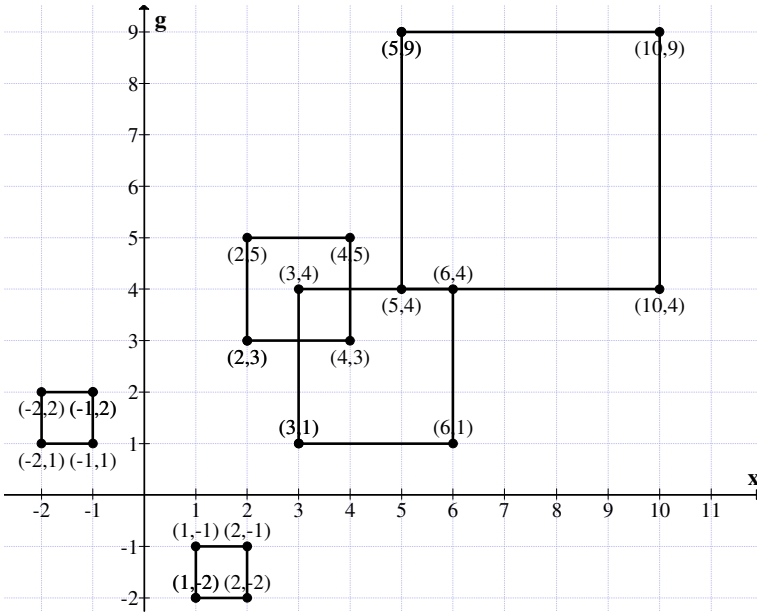The graph of all the following squares are as given in Figure 3.32.

Figure 3.32

We see the difference is two disjoint dual number square of type II.

Example 3.35: Let
$$S = \{(5, 4g),\ (3, 4g), (5, 2g), (3, 2g)\}$$
and
$$T = \{(-6, -5g), (-3, -5g), (-6, -2g), (-3, -2g)\}$$

be any two dual number squares of type I.

$$2S = \{(10, 8g), (6, 8g), (10, 4g), (6, 4g)\},$$

$$2T = \{(-12, -10g), (-6, -10g), (-12, -4g), (-6, -4g)\},$$

$$S + T = \{(-1, -g), (0, -g), (-1, 0), (0, 0)\}$$

$$S - T = \{(11, 9g), (6, 9g), (11, 4g), (6, 4g)\}$$
and
$$T - S = \{(-11, -9g), (-6, -9g), (-11, -4g), (-6, -4g)\}$$

are all distinct dual number squares of type I.

We give the graph of all these squares in the following dual number planes.

2S is a square two times of S.

Similarly 2T.

S + T is a square of length 1 where as S – T and T – S are dual number squares of whose sides are of size 5 each.
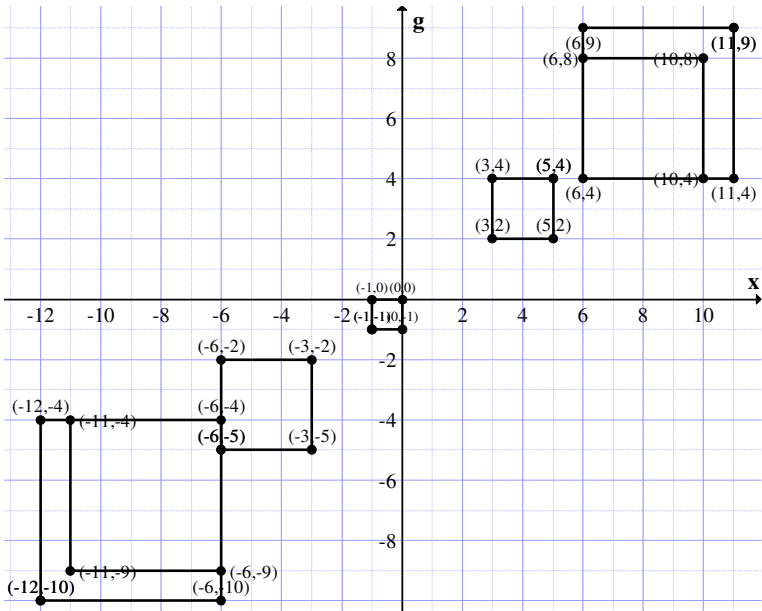

Figure 3.33

In some cases we see the squares over laps and some of them are disjoint.

Now we find how a dual number square of type gets shifted by addition to a point dual square number.

Example 3.36: Let
$$S = \{(4, 5g), (1, 5g), (4, 2g), (1, 2g)\}$$
be the dual number square of type I and

$$P = \{(3, 3g), (3, 3g), (3, 3g), (3, 3g)\}$$
be the point dual number square.

$$P + S = \{(7, 8g), (4, 8g), (7, 5g), (4, 5g)\}$$
is again a dual square of type I.

$$P - S = \{(-1, -2g), (2, -2g), (-1, g), (2, g)\}$$
is again a dual number square of type I.

$$S - P = \{(1, 2g), (-2, 2g), (1, -g), (-2, g)\}$$
is also a dual number square of type I.

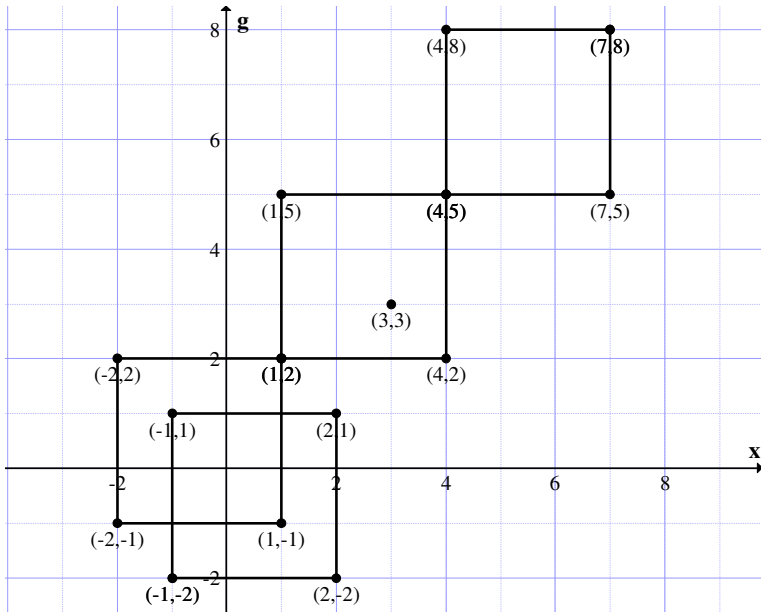We give a graphical representation of all these dual number squares in the dual number plane.

Figure 3.34

From the figure it is clear. There is each overlap or contact at a point.

Hence the claim.

Now for the same S we take the point square

$$P_1 = \{(-1, -g), (-1, -g), (-1, -g), (-1, -g)\} \in D_I.$$

We find $S + P$, $S - P_1$ and $P_1 - S$ in the following.

$$S + P_1 = \{(3, 4g), (0, 4g), (3, g), (0, g)\}$$

is a dual number square of type I.

$$S - P_1 = \{(5, 6g), (2, 6g), (5, 3g), (2, 3g)\}$$

is also a dual number square of type I.

$$P_1 - S = \{(-5, -6g), (-2, -6g), (-5, -3g), (-2, -3g)\}$$

is a dual number square of type I.

Clearly all the four dual number squares are distinct.

We give the graph of them in Figure 3.35 represented in the dual number plane.

Only three of the squares in the dual number plane intersect and the other dual number square is shifted to the 3rd quadrant of the dual number plane, the other 3 dual number squares are in the first quadrant of the dual number plane.

Thus a point outside the dual number square right in the opposite quadrant where the square S occurs gives this sort of orientation.
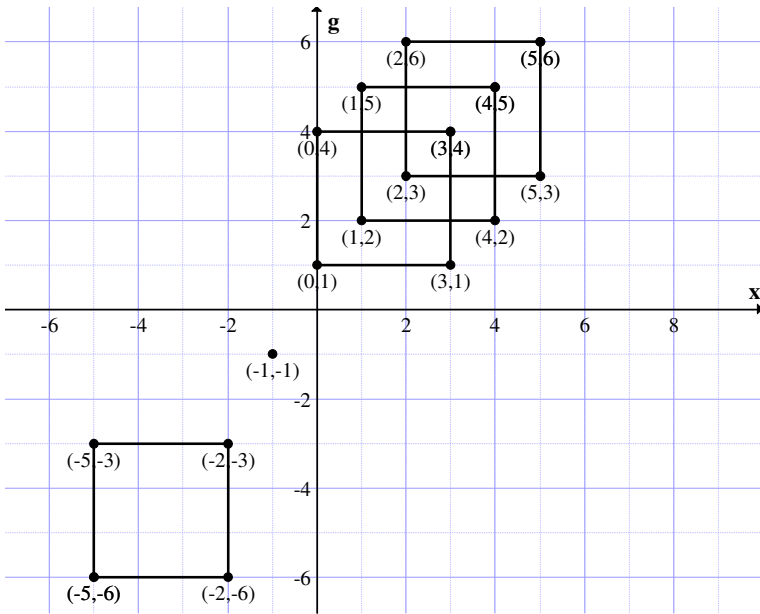
Figure 3.35

Next we proceed onto work with a point square from the second quadrant of the dual number plane.

Let

$$P_2 = \{(-4, 3g), (-4, 3g), (-4, 3g), (-4, 3g)\}$$

be the point square in the 2nd quadrant of the dual number plane.

$$S + P_2 = \{(0, 8g), (-3, 8g), (0, 5g), (-3, 5g)\}$$

$$S - P_2 = \{(8, 2g), (5, 2g), (8, -g), (5, -g)\}$$

and

$$P_2 - S = \{(-8, -2g), (-5, -2g), (-8, g), (-5, g)\}$$

are the dual number squares of type I.

We give the graph of all these three squares in the dual number plane in the following.
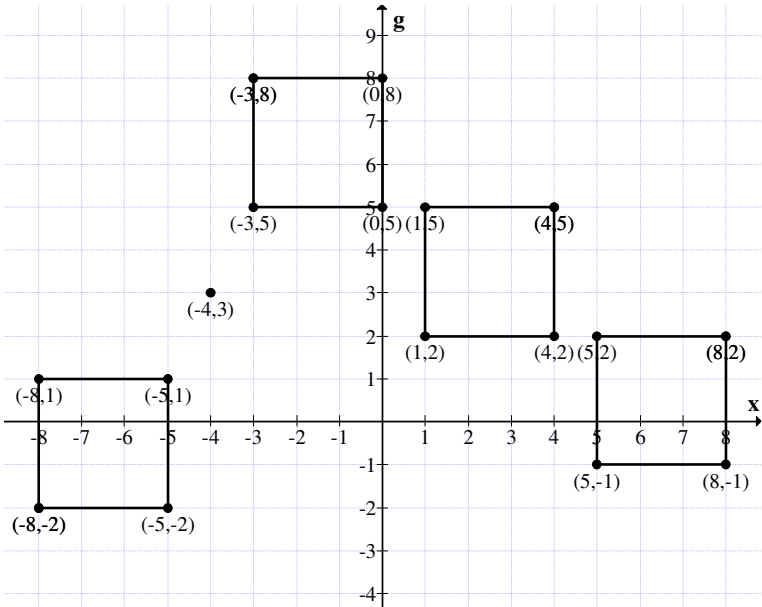
Figure 3.36

Using a point square from the 2nd quadrant we see both the dual number squares $P_2 - S$ and $S - P_2$ take place in a diametrically symmetric position in regard of both the axis.

All the squares are disconnected.

Now we take a point in the forth quadrant say

$$P_3 = \{(4, -3g), (4, -3g), (4, -3g), (4, -3g)\}$$

we find

$$S + P_3 = \{(8, 2g), (5, 2g), (8, -g), (5, -g)\},$$

$$S - P_3 = \{(0, 8g), (-3, 8g), (0, 5g), \ (-3, 5g)\}$$

and

$$P_3 - S = \{(0, -8g), (3, -8g), (0, -5g), (3, -5g)\}$$

are the three dual number squares of type I.

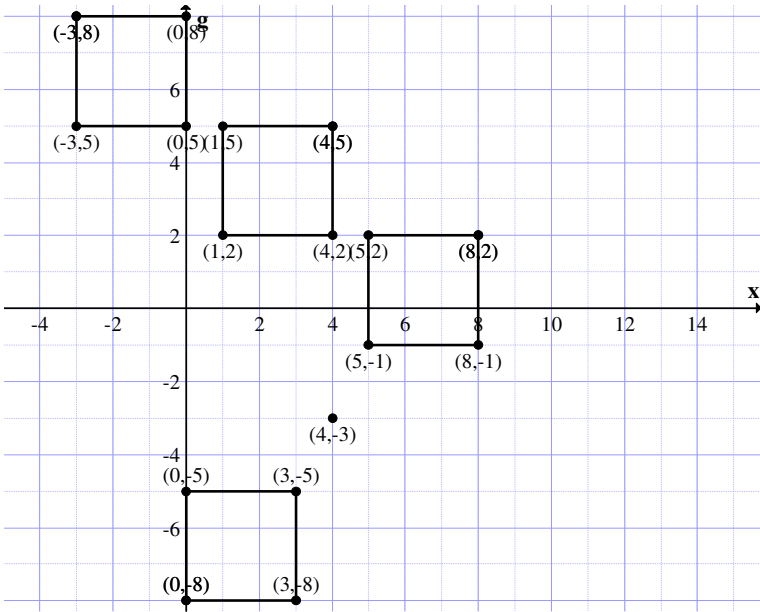We have the following figures in the dual number plane.

Figure 3.37

When the point P_3 is in the forth quadrant. The two dual number squares are symmetrically opposite.

Now having see all these we have the following result to be true.

THEOREM 3.6: *Let $\{D_I\}$ be the collection of all dual number squares of type I. D_I under addition is a group of infinite order which is commutative.*

Proof is direct and hence left as an exercise to the reader.

THEOREM 3.7: *Let D_I be the dual number squares of type I. Addition of a point square form the dual plane to any square in D_I results in a shift or translation.*

Proof : Follows from simple calculations.

We will illustrate this still by some more examples.

Example 3.37: Let

$$S = \{(5, 6g), (2, 6g), (5, 3g), (2, 3g)\}$$

be the dual number square of type I in the dual number plane.

Let

$$A = \{(-3, g), (-3, g), (-3, g), (-3, g)\}$$

be a point dual number square.

To find –S, S + A, A – S and S – A.

$$-S = \{(-5, -6g), (-2, -6g), (-5, -3g), (-2, -3g)\};$$

$$S + A = \{(2, 7g), (-1, 7g), (2, 4g), (-1, 4g)\}$$

$$S - A = \{(8, 5g), (5, 5g), (8, 2g), (5, 2g)\}$$

and

$$A - S = \{(-8, -5g), (-5, -5g), (-8, -2g), (-5, -2g)\}$$
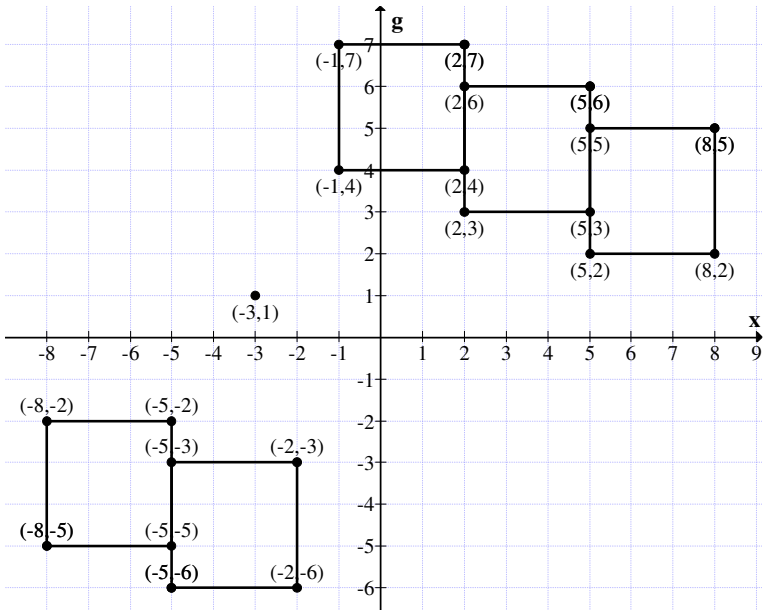
are dual number squares of type I.

Figure 3.38

They behave symmetrically as is very clear from the diagram on the dual number plane.

Example 3.38: Let
$$A = \{(-4, -2g), (-1, -2g), (-4, g), (-1, g)\}$$
and
$$B = \{(5, 3g), (2, 3g), (5, 0), (2, 0)\}$$
be any two dual number squares of type I.

$$A + B = \{(1, g), (1, g), (1, g), (1, g)\}$$
is a dual point square.

Consider
$$A - B = \{(-9, -5g), (-3, -5g), (-9, g), (-3, g)\}$$
is a square of side 6.

$$B - A = \{(9, 5g), (3, 5g), (9, -g), (3, -g)\}$$
is again a square of size 6.

Now we give yet another examples.

Example 3.39: Let
$$A = \{(2, 3g), (4, 3g), (2, 5g), (4, 5g)\}$$
and
$$B = \{(1, 4g), (3, 4g), (1, 6g), (3, 6g)\}$$
be any two dual number squares of type I.

We find $A + B$, $A - B$ and $B - A$.

$$A + B = \{(3, 7g), (7, 7g), (3, 11g), (7, 11g)\}$$

$$A - B = \{(1, -g), (1, -g), (1, -g), (1, -g)\}$$
and
$$B - A = \{(-1, g), (-1, g), (-1, g), (-1, g)\}$$
are three dual number squares of type I.

Of these three squares $B - A$ and $A - B$ are just dual number point squares.

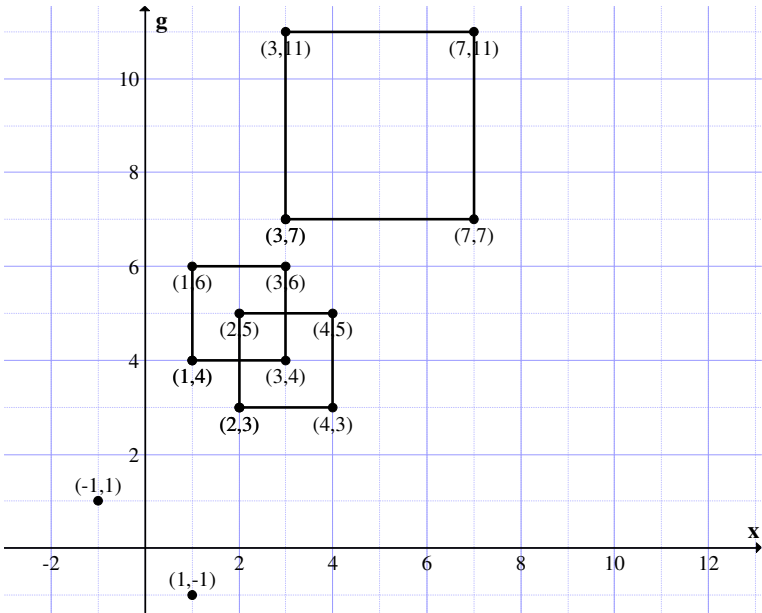We give the graph of these in the dual number plane.

Figure 3.39

We see the dual number squares A and B intersect, however A + B is disjoint with A and B.

Finally the squares A – B and B – A are just point squares.

Example 3.40: Let
$$A = \{(10, 0), (5, 0), (10, -5g), (5, -5g)\}$$
be any dual number square of type I.

We find A × A;

$$A \times A = \{(100, 0), (25, 0), (100, -100g), (25, -50g)\}$$
is not a dual number square of type I. So this A is not compatible under product.

Let

$$A = \{(3, 2g), (6, 2g), (3, 5g), (6, 5g)\}$$

and

$$B = \{(-2, -3g), (2, -3g), (-2, g), (2, g)\}$$

be any two dual number squares of type I.

We find

$$A \times B = \{(-6, -13g), (12, -14g), (-6, -7g), (12, 16g)\}$$

is not a dual number square of type I.

We see D_I under product is not even closed.

Let

$$M = \{(5, 2g), (8, 2g), (5, 5g), (8, 5g)\}$$

be a dual square of type I.

Let

$$P = \{(-2, 3g), (-2, 3g), (-2, 3g), (-2, 3g)\}$$

be a dual number point square.

$$P \times M = \{(-10, 11g), (-16, 20g), (-10, 5g), (-16, 14g)\}$$

is not a dual number square of type I.

$$P_1 = \{(1, 2g), (1, 2g), (1, +2g), (1, 2g)\}$$

be the dual number point square.

$$P_1 \times M = \{(5, 12g), (8, 18g), (5, 15g), (8, 21g)\}$$

is not a dual number square of type I.

Infact all these all quadrilaterals with two of its sides parallel to the y–axis.

Example 3.41: Let

$$M = \{(3, 2g), (5, 2g), (3, 4g), (5, 4g)\}$$

be the dual number square of type I.

Let

$$P = \{(0, -2g), (2, -2g), (0, 0), (2, 0)\}$$

be another dual number square of type I.

$$M \times P = \{(0, -6g), (10, -6g), (0, 0), (10, 8g)\}$$

is not a square of type I.

We will illustrate the situation by the graph in the dual number plane in the following:
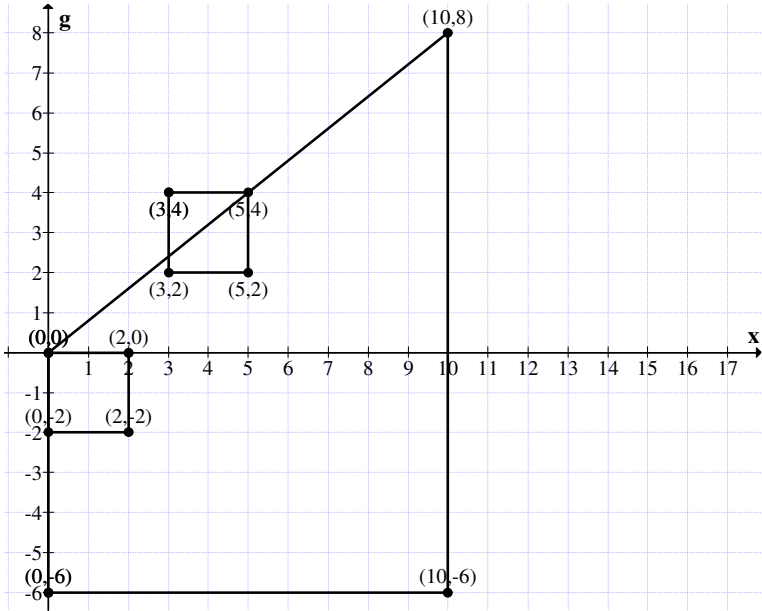

Figure 3.40

M × P is a quadrilateral with two of its sides parallel to the dual axis.

Example 3.42: Let
$$B = \{(4, 3g), (6, 3g), (4, 5g), (6, 5g)\}$$
be a dual number square of type I.

$$P = \{(1, 2g), (1, 2g), (1, 2g), (1, 2g)\}$$
be the point dual number square.

$$P \times B = \{(4, 11g), (6, 15g), (4, 13g), (6, 17g)\}$$
is not a dual number square of type I.

P × B is only a dual number quadrilateral.

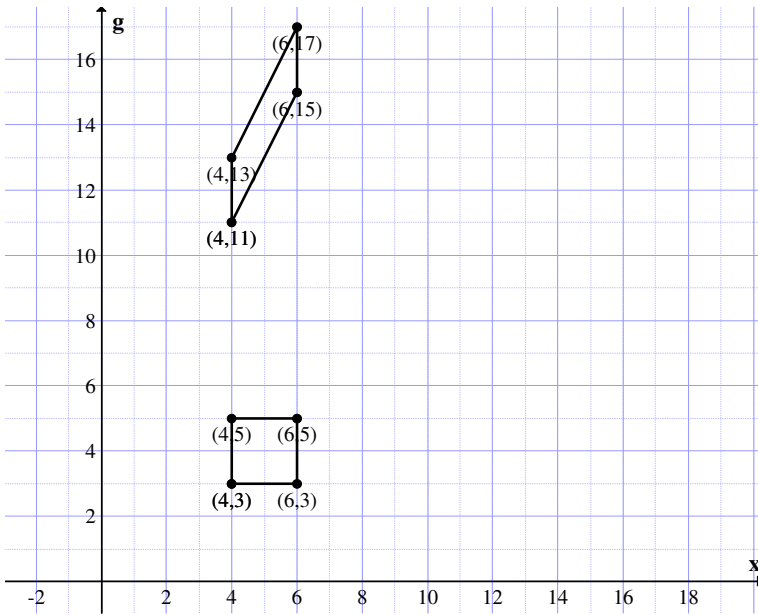We give the graph of B, P and P × B in the dual number plane.

Figure 3.40

Clearly B × P is a dual number quadrilateral such that the sides are parallel to the dual number axis as seen from the figure.

Next we proceed onto describe the notion of dual number squares of type II first by examples then by the routine definition.

Example 3.43: Let
$$B = \{(-5, 4g), (5, 4g), (0, 12g), (0, 14g)\}$$
be a dual number square of type II.

Clearly the diagonals of the square B are such that they are parallel to the axis of the dual number plane.

Example 3.44: Let
$$A = \{(2, 3g), (3, 4g), (4, 3g), (3, 2g)\}$$
be the dual number square of type II.
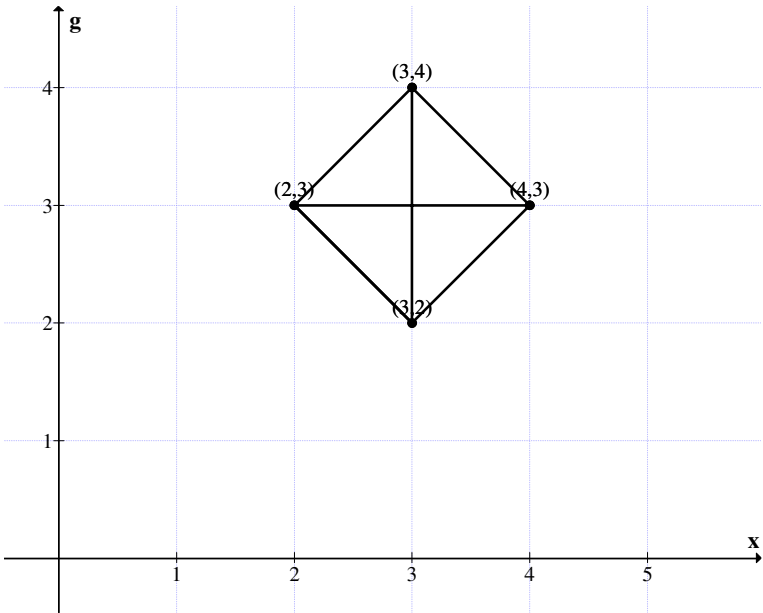
The graph of A in the dual number plane is as follows:

Figure 3.41

Example 3.45: Let

$$A = \{(2, 3i), (3, 4i), (4, 3i), (3, 2i)\}$$

be the complex square of type I.

$$A^2 = \{(-5, 12i), (-7, 24i), (7, 24i), (5, 12i)\}$$

$$A^3 = \{(-46, 9i), (-117, 44i), (-44, 117i), (-9, 46i)\}$$

and so on.

Clearly A^2, A^3 and so on are not complex squares of type I. Infact they are just complex quadrilaterals.

Suppose we take the same square in the dual number plane then

$$A_d = \{(2, 3g), (3, 4g), (4, 3g), (3, 2g)\}$$

is structure of the square A in the dual number plane.

$$A_d^2 = \{(4, 12g), (9, 24g), (16, 24g), (9, 12g)\}$$
is not a square of type I.

Clearly A_d^2 is a dual number parallelogram.

Unlike in the complex plane in the dual plane this dual square product with itself attain a nice parallelogram structure.

Thus we have also compared how by varying the plane the structure of the square also varies.

Several other properties of both types of squares can be studied by interested reader.

This work is considered as a matter of routine and hence left as an exercise to the reader.

Next we define squares of type I and type II in the special dual like number plane.

We denote in this book
$$D^1 = \{a + bg \mid g^2 = g, a, b \in R\}$$
the special dual like number plane.

Let $D_I^1 = \{$all squares in this special dual like number plane in which the sides of the squares are always parallel to the x–axis and special dual like number axis$\}$.

We will first illustrate this situation by some examples.

Example 3.46: Let
$$A = \{(5, 2g), (7, 2g), (5, 4g), (7, 4g)\}$$

be the special dual like number square.

We give the graph representation of A in the special dual like number plane.
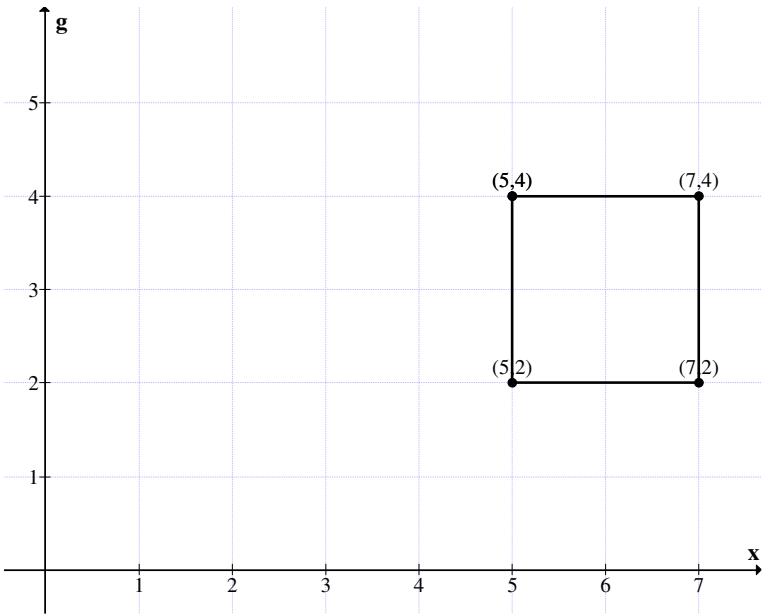
Figure 3.42

Example 3.47: Let

$$P = \{(-2, 5g), (1, 5g), (-2, 8g), (1, 8g)\}$$

be the special dual like number square of type I.

The graph representation P in the special dual like number plane is given in Figure 3.43.

Now the graph of P is a shown in the plane.

Example 3.48: Let

$$B = \{(-2, -g), (2, -g), (-2, 3g), (2, 3g)\}$$

be the special dual like number square of type I.

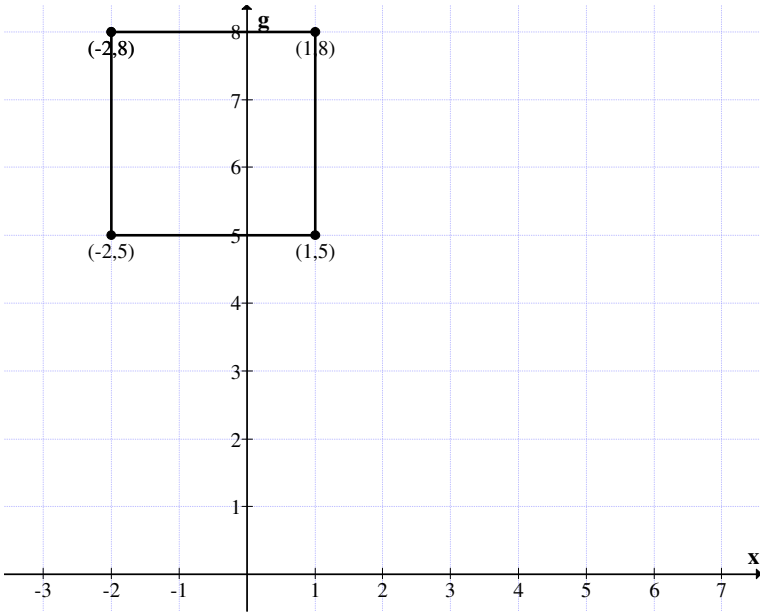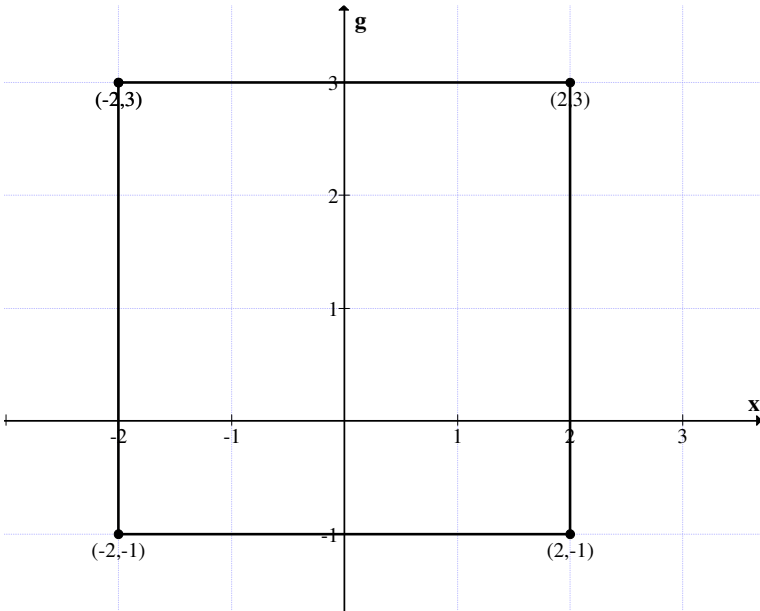The graph representation of B is given in Figure 3.44.

Figure 3.43

Figure 3.44

Next we show by examples how addition and product are performed.

Example 3.49: Let

$$M = \{(3, 4g), (5, 4g), (3, 6g), (5, 6g)\}$$

be the special dual like number squares of type I.

$$-M = \{(-3, 4g), (-5, -4g), (-3, -6g), (-5, -6g)\}$$

is also a special dual like number square of type I.

$$M + (-M) = \{(0, 0), (0, 0), (0, 0), (0, 0)\}.$$

Consider

$$M + M = \{(6, 8g),\ (10, 8g), (6, 12g), (10, 12g)\}$$

is again a special dual like number square of type I.

Let

$$P = \{(2, g), (2, g), (2, g), (2, g)\}$$

be a point square in the special dual like number plane.

We find M + P, P – M and M – P in the following.

$$M + P = \{(5, 5g), (7, 5g), (5, 7g), (7, 7g)\}$$

and

$$M - P = \{(1, 3g), (3, 3g), (1, 5g), (3, 5g)\}$$

$$P - M = \{(-1, -3g), (-3, -3g), (-1, -5g), (-3, -5g)\}$$

are all special dual like number squares of type I.

We give the graphic representation of these squares in the special dual like number plane.
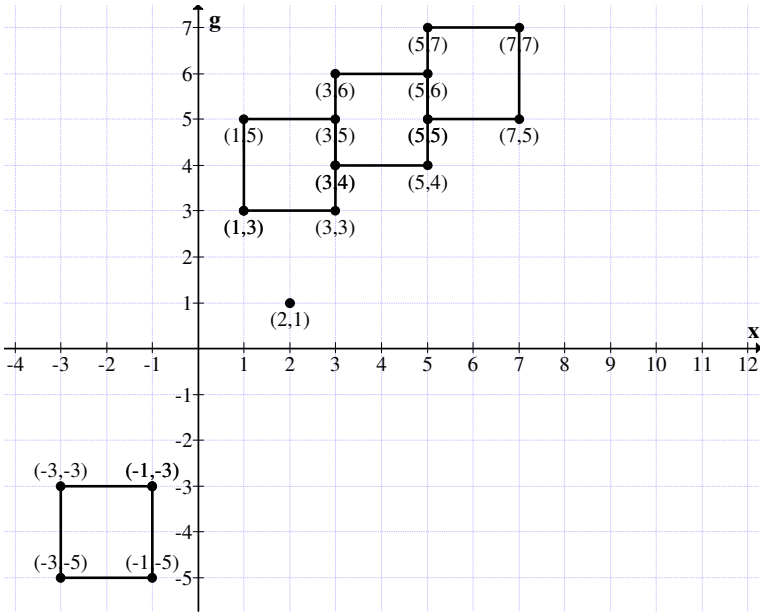
Figure 3.45

Let

$$P_1 = \{(-2, 3g), (-2, 3g), (-2, 3g), (-2, 3g)\}$$

be the point square.

We will find $M + P_1$, $M - P_1$ and $P_1 - M$.

$$M + P_1 = \{(1, 7g), (3, 7g), (1, 9g), (3, 9g),$$

$$M - P_1 = \{(5, g), (7, g), (5, 3g), (7, 3g)\}$$

and

$$P_1 - M = \{(-5, -g), (-7, -g), (-5, -3g), (-7, -3g)\}$$

be the special dual like number square of type I.

The graphs of these squares in the special dual like number plane is as follows:
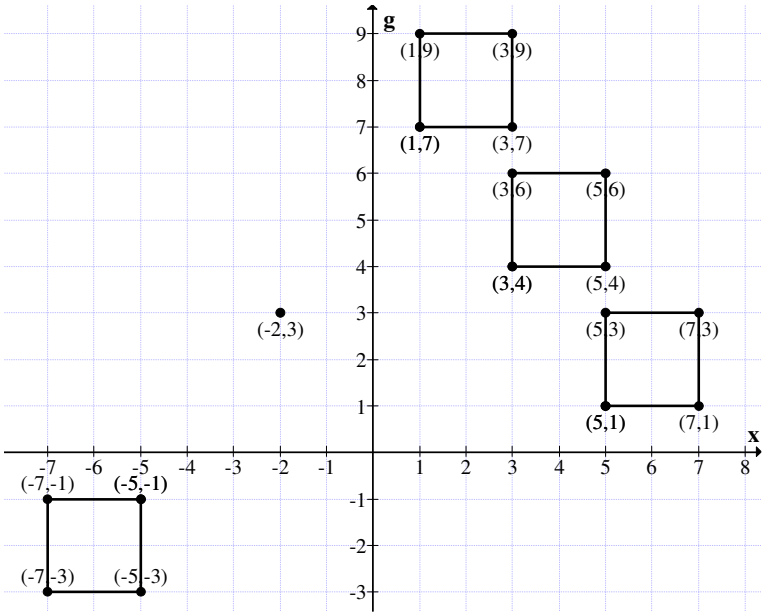
Figure 3.46

Thus from these examples we can easily have the following results.

THEOREM 3.8: *{D_1^1, +} is a group of infinite order which is commutative.*

Proof follows from simple and direct working.

We will supply some more examples before we proceed to work on other properties.

Example 3.50: Let

$$S = \{(-3, 2g), (-1, 2g), (-3, 4g), (-1, 4g)\}$$

be the special dual like number square of type I.
Let

$$P = \{(1, 0), (1, 0), (1, 0), (1, 0)\}$$

be a point square we find

$$S + P, S - P \text{ and } P - S.$$

$$S + P = \{(-2, 2g), (0, 2g), (-2, 4g), (0, 4g)\}$$

$$S - P = \{(-4, 2g), (-2, 2g), (-4, 4g), (-2, 4g)\}$$

and

$$P - S = \{(4, -2g), (2, -2g), (4, -4g), (2, -4g)\}$$

are special dual like number squares of type I.

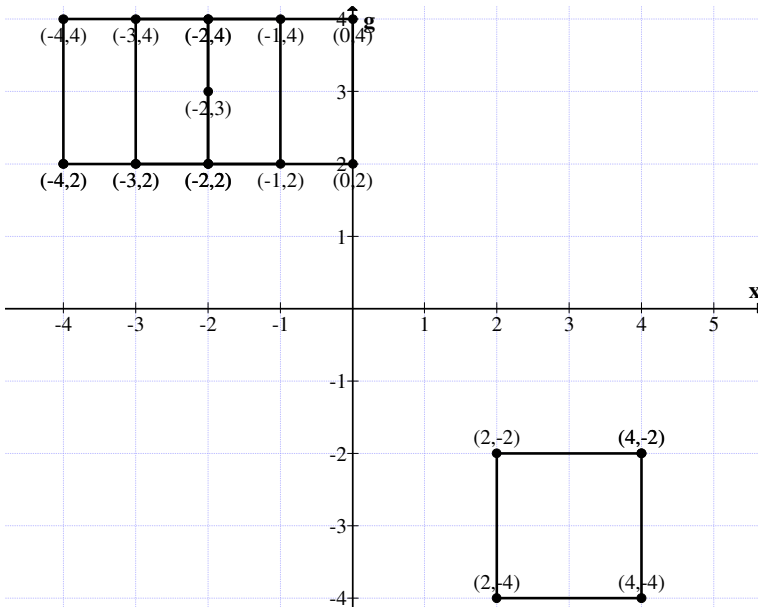We give the representation of them in the special dual like number plane in the following.

Figure 3.47

From the figure it is clear how the squares overlap over each other and only the square P – S takes a diametrically opposite place in the plane.

Example 3.51: Let

$$M = \{(2, 3g), (4, 3g), (2, 5g), (4, 5g)\}$$

and

$$N = \{(-1, g), (1, g), (-1, 3g), (1, 3g)\}$$

be two special dual like number squares of type I.

$$M \times N = \{(-2, 2g), (4, 10g), (-2, -16g), (4, 32g)\}$$

is clearly not a square of type I.

Infact $M \times N$ is a quadrilateral with two of its sides parallel to the special dual number axis.

Example 3.52: Let

$$M = \{(-3, -4g), (0, -4g), (-3, -g), (0, -g)\}$$

and

$$N = \{(2, -3g), (3, -3g), (2, -2g), (3, -2g)\}$$

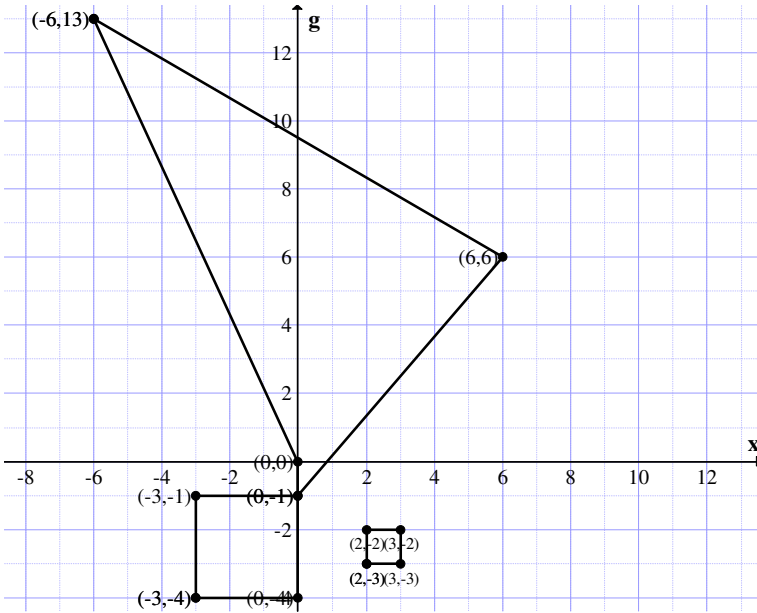be any two special dual like number squares of type I.

$$MN = \{(-6, 13g), (0, 0), (6, 6g), (0, -g)\}$$

is not a square of type I but only a quadrilateral.

We show the representation of MN in the special dual like number plane as given in Figure 3.48.

Thus MN assumes a form very different from the usual square.

Why this deviation takes place we are not in a position to explain now.

Example 3.53: Let

$$M = \{(-5, -g), (-2, -g), (-5, 2g), (-2, 2g)\}$$

and

$$N = \{(4, 3g), (3, 3g), (4, 2g), (3, 2g)\}$$

be any two special dual like number squares.

$$M \times N = \{(-20, 22g), (-6, -12g), (-20, 2g), (-6, 6g)\}$$

is a special dual like number quadrilateral in which the two sides of $M \times N$ are parallel and are parallel to the special dual like number axis.

Hence just the dual number plane the squares of type I in the special dual like number plane behave similarly except for the second coordinate becomes larger and larger or smaller as $g^2 = 0$ in dual like number planes $g^2 = g$.

Thus this study is also innovative and interesting.

Now we proceed onto define the type II special dual like number squares first by examples.

Example 3.54: Let
$$M = \{(-3, 0), (3, 0), (0, 3g), (0, -3g)\}$$
be a special dual like number squares.

Clearly M is not a type I squares. For only the diagonals of this square are parallel to the coordinate axis.

So M is a special dual like number square of type II.

The graphics representation of M is given in the special dual like number plane in the following.

Figure 3.49

Clearly the diagonals are parallel to the axis of the special dual like number plane.

Example 3.55: Let

$$M = \{(4, 6g), (2, 4g), (4, 2g), (6, 4g)\}$$

be the special dual like number square of type II.

The representation of M in the special dual like number plane is as follows:

Figure 3.50

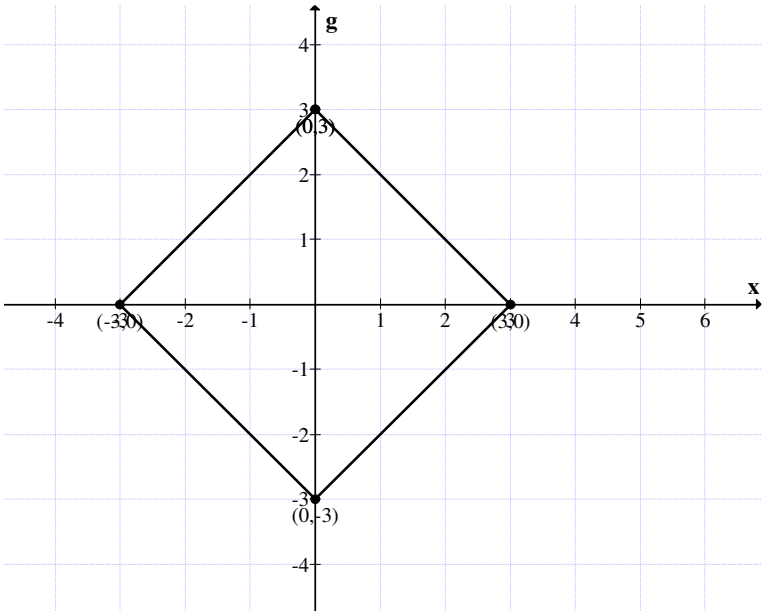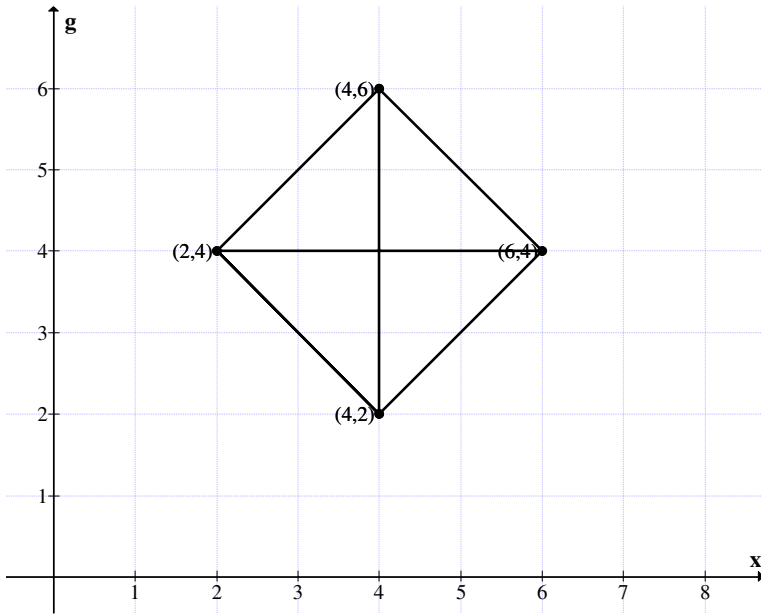Clearly the diagonals of M are parallel to the axis evident from the diagram.

Example 3.56: Let

$$W = \{(2, 4g), (4, 6g), (6, 4g), (4, 2g)\}$$

be the special dual like number square of type II.

The graph in the special dual like number plane is as follows:

Figure 3.51

Having shown examples of special dual like number squares of type II we make the definition of the same.

DEFINITION 3.2: *Let* D_{II}^l *= {Collection of all squares in the special dual like number plane such that the diagonals are parallel to the x axis and the special dual like number axis}; we define elements of* D_{II}^l *as special dual like number squares of type II.*

We will give some more examples before we proceed onto define operations on the collection D_{II}^l.

Example 3.57: Let

$$M = \{(-2, -4g), (-4, -6g), (-6, -4g), (-4, -2g)\}$$

be the special dual like number square of type II.

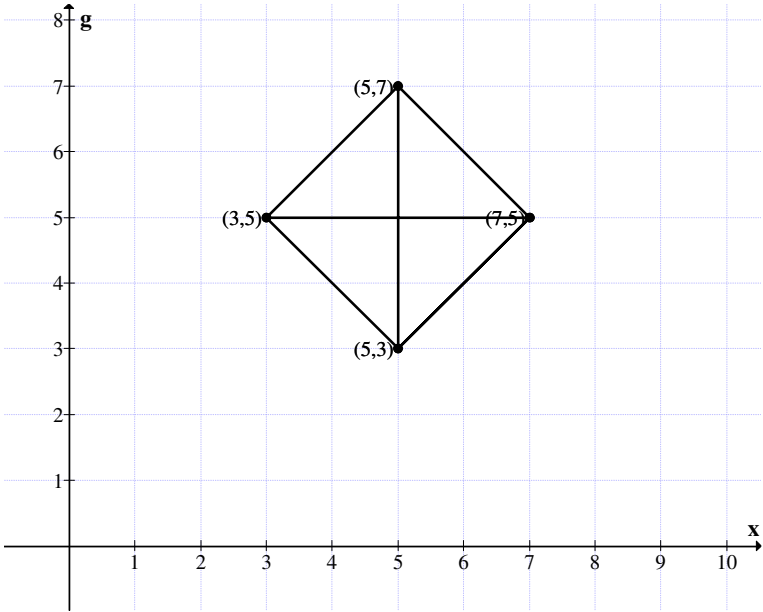The graph of M in the special dual like number plane is as follows:
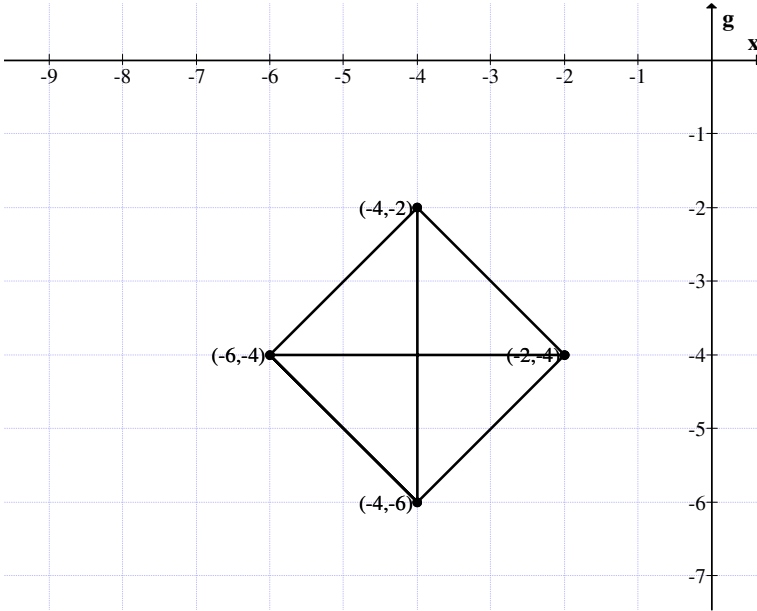
Figure 3.52

Example 3.58: Let

$$B = \{(4, 3g), (3, 2g), (2, 3g), (3, 4g)\}$$

be the special dual like number square of type II.
Let

$$P_1 = \{(0, -4g), (0, -4g), (0, -4g), (0, -4g)\}$$

be the point square.

$$B + P_1 = \{(4, -g), (3, -2g), (2, -g), (3, 0)\}$$

is a special dual number square of type II.

$$B - P_1 = \{(4, 7g), (3, 6g), (2, 7g), (3, 8g)\}$$

is again a special dual number square of type II.

$$P_1 - B = \{(-4, -7g), (-3, -6g), (-2, -7g), (-3, -8g)\}$$

is also a special dual number square of type II.

We give the diagrammatic representation of all these squares in the special dual number plane in the following.
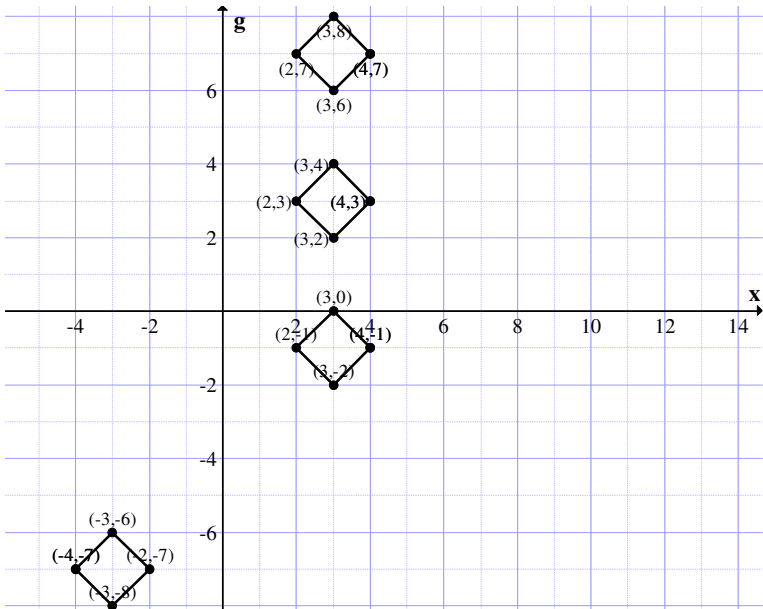
Figure 3.53

All the four square are of same size but lie in first and forth quadrant.

Now taking

$$P_2 = \{(1, 0), (1, 0), (1, 0), (1, 0)\}$$

as the point square the graph of $B + P_2$, $B - P_2$ and $P_2 - B$ is given in Figure 3.54.
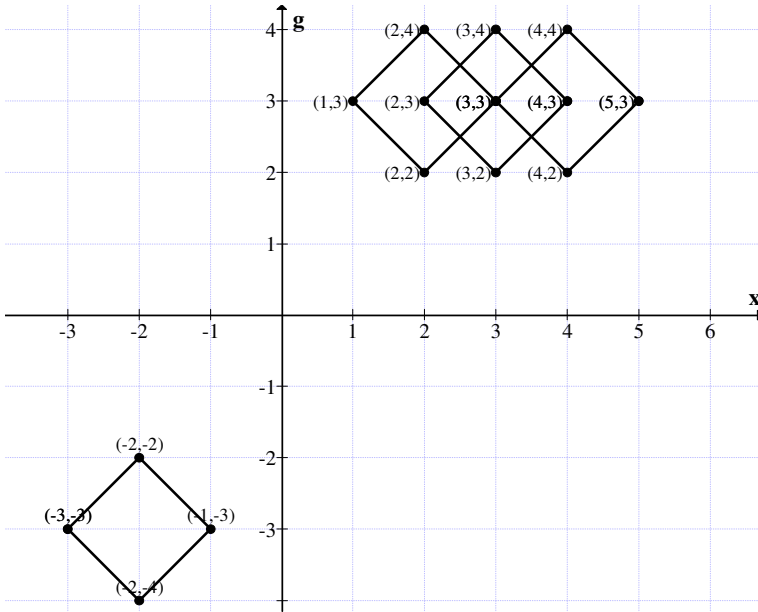
Figure 3.54

Three of the squares intersect and they occupy the first and the 3rd quadrant.

Let

$$P_3 = \{(-2, 0), (-2, 0), (-2, 0), (-2, 0)\}$$

be a square.

$$B + P_3 = \{(2, 3g), (1, 2g), (0, 3g), (1, 4g)\}$$

$$B - P_3 = \{(6, 3g), (5, 2g), (4, 3g), (5, 4g)\}$$

and

$$P_3 - B = \{(-6, -3g), (-5, -2g), (-4, -3g), (-5, -4g)\}$$

are the three special dual like number squares of type II.

We give the graph of these squares in the special dual like number plane in the following.
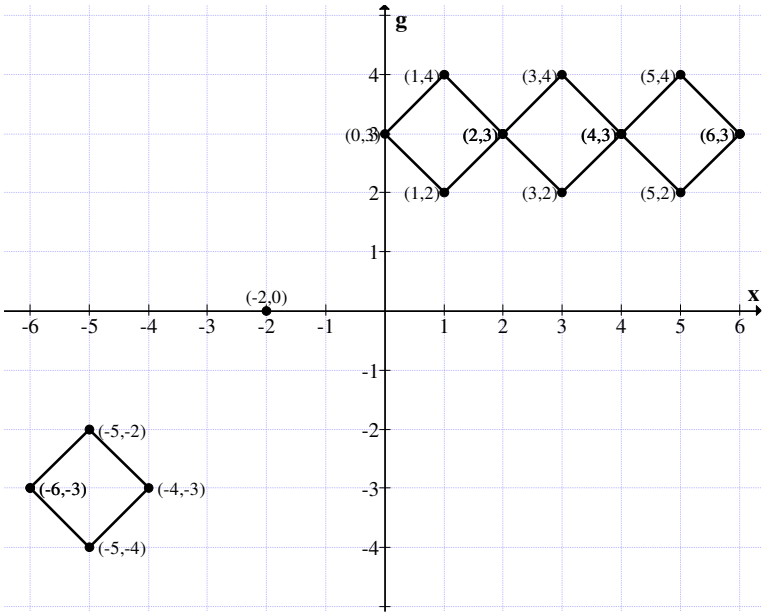
Figure 3.55

We see the three squares each shares one common vertex as shown in the figure.

However the square $P_3 - B$ is disjoint from the rest of the squares.

Now let

$$P_4 = \{(-2, g), (-2, g), (-2, g), (-2, g)\}$$

be a point square.

To find $P_4 + B$, $B - P_4$ and $P_4 - B$.

$$B + P_4 = \{(2, 4g), (1, 3g), (0, 4g), (1, 5g)\}$$

$$B - P_4 = \{(6, 2g), (5, g), (4, 2g), (5, 3g)\}$$

and

$$P_4 - B = \{(-6, -2g), (-5, -g), (-4, -2g), (-5, -3g)\}$$

are the square.

We give the diagrammatic representation of these squares in the special dual like number plane.
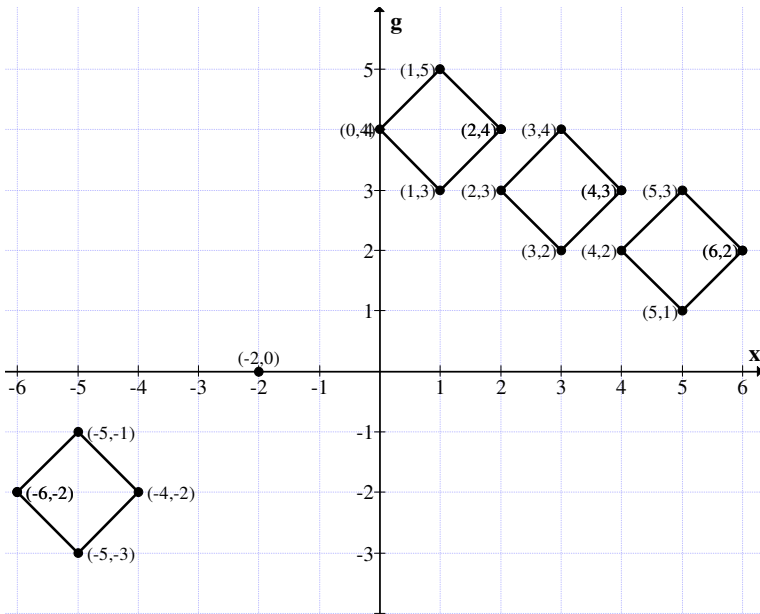
Figure 3.56

With this shift of using P_4 all the four squares are disjoint and the square $P_4 - B$ lies in the 3rd quadrant of the special dual like number plane.

Now it is easily evident by adding a point square from the special dual like number plane the special dual like number squares of type II only shifts from place to place and the size of the square remains the same in all cases.

Thus in view of all these we have the following.

THEOREM 3.9: *Let D_{II}^l be the collection of all special dual like number squares of type II.*

{ D_{II}^l , +} is an abelian group and is of infinite order.

The proof is direct and hence left as an exercise to the reader.

Thus addition of a point squares translates or shift a type II square in the special dual like number plane.

Now we proceed onto define the notion of product in the special dual like number squares of type II.

Let
$$A = \{(8, -2g), (12, 2g), (8, 6g), (4, 2g)\}$$
and
$$B = \{(4, 0), (7, 3g), (4, 6g), (1, 3g)\}$$
be any two elements of D_{II}^l .

$A \times B = \{(4, 20g), (32, 108g), (84, 56g), (32, 8g)\} \notin D_{II}^l$.

Thus $A \times B$ is not a special dual like number square of type II only a special dual like number quadrilateral.

Let us now find

$A \times A = \{(64, -28g), (144, 52g), (64, 132g), (16, 20g)\}$
is not a special dual like number square only a quadrilateral.

THEOREM 3.10: *Let D_{II}^l be the collection of special dual like number squares of type II.*

D_{II}^l in general is not even closed under the product operation.

Proof follows from the simple fact for any A, B $\in D_{II}^l$, $A \times B \notin D_{II}^l$.

Example 3.59: Let
$$A = \{(-6, 0), (0, 6g), (6, 0), (0, -6g)\}$$

be the special dual like number square of type II.

$$A^2 = \{(36, 0), (0, 36g), (36, 0), (0, 36g)\}$$

is not a square of type II.

$$A^3 = \{(-216, 0), (0, -216g), (216, 0), (0, 216g)\} \in D_{II}^1 .$$
A^2 is just a line.

$$A^4 = \{(6^4, 0), (0, 6^4 g)\}$$
again a line and so on.

Example 3.60: Let
$$V = \{(3, 0), (0, 3g), (-3, 0), (0, -3g)\}$$

be a special dual like number square of type II.

$$V^2 = \{(9, 0), (0, 9g)\}$$
and
$$V^3 = \{(27, 0), (0, 27g), (-27, 0), (0, -27g)\}$$
and
$$V^4 = \{(81, 0), (0, 81g)\}$$
and so on.

This is represented by the following graph given in Figure 3.57 in the special dual like number plane.

Next we give one more example before we make some conclusions about these new structures.

Example 3.61: Let

$$W = \{(2, 0), (-2, 0), (0, 2g), (0, -2g)\}$$

be a special dual like number square of type II.
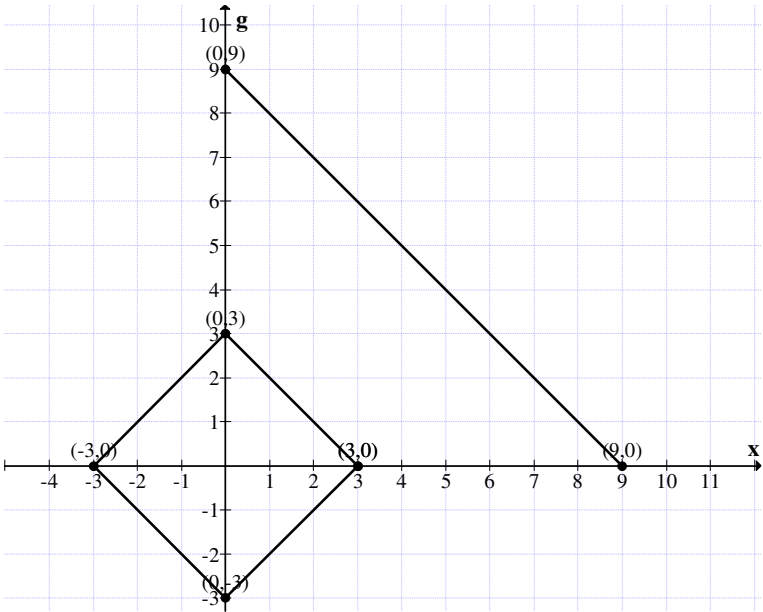
Figure 3.57

$$W^2 = \{(2^2, 0), (0, 2^2g)\},$$

$$W^3 = \{(2^3, 0), (-2^3, 0), (0, 2^3g), (0, -2^3g)\},$$

$$W^4 = \{(2^4, 0), (0, 2^4g)\}$$

and so on.

We see

$$W^{2n+1} = \{(2^{n+1}, 0), (0, 2^{n+1}g), (-2^{n+1}, 0), (0, -2^{n+1}g)\}$$

and

$$W^{2n} = \{(2^n, 0), (0, 2^ng)\}.$$

We give the graphic representation of W, W^2, W^3 so on in the special dual like number plane in Figure 3.58.

In view of all these we have the following.
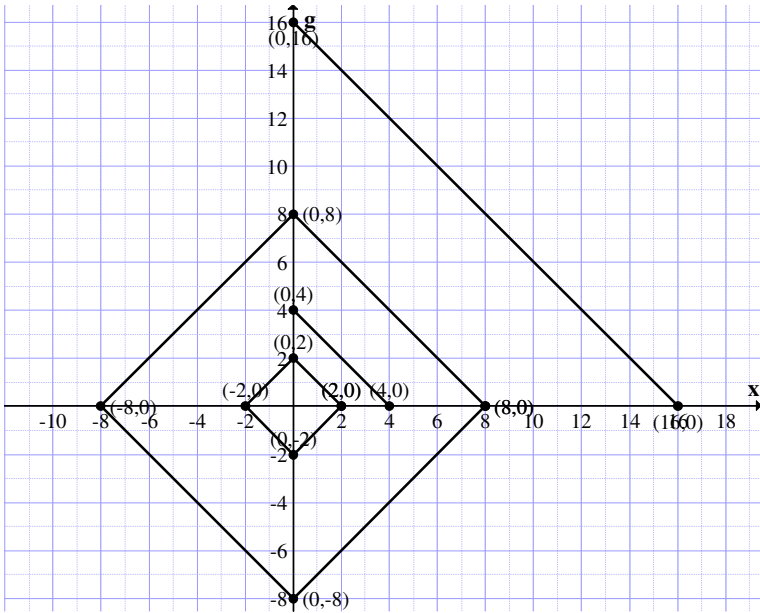
Figure 3.58

Let L $(S(D_{II}^1))$ = {Collection of all square of type II of the form {(a, 0) m (–a, 0), (0, ag), (0,–ag)} $\cup$ {All lines of the form {(a^{2t}, 0), (0, a^{2t}g)} a $\in$ R, 2 < t < ∞} it is easily verified.

$L(S(D_{II}^1))$ is closed under the operation $\times$ product.

Inview of this we have the following.

THEOREM 3.11: *Let {L(S(D_{II}^l)), $\times$} be a semigroup of infinite order which is commutative.*

Proof follows from direct working hence left as an exercise to the reader.

It is important to note D_{II}^1 is not even closed for products of every pair of elements.

This has been already proved by examples.

Now we find the product of a special dual like number squares of type II with a point square.

Let
$$A = \{(2, 3g), (4, 3g), (3, 2g), (3, 4g)\}$$
be any special dual like number square of type II.

$$P_1 = \{(3, -4g), (3, -4g), (3, -4g), (3, -4g)\}$$
be a point square.

$$A \times P_1 = \{(6, -11g), (12, -19g), (9, -14g), (9, -16g)\}$$

is a special dual like number square of type II.

Let us take
$$P_2 = \{(-2, g), (-2, g), (-2, g), (-2, g)\}$$
to be point square.

We find
$$A \times P_2 = \{(-4, -g), (-8, g), (-6, g), (-6, -g)\}$$

is again a special dual number square of type II.

Let us take
$$P_3 = \{(-1, -2g), (-1, -2g), (-1, -2g), (-1, -2g)\}$$

to be a special dual number point square.

We find
$$A \times P_3 = \{(-2, -13g), (-4, -17g), (-3, -12g), (-3, -18g)\}$$

is again a special dual number square of type II of same size.

Thus in view of all these we arrive at the following result.

THEOREM 3.12: *Let D_{II}^l be the collection of all special dual like number squares of type II.*

The product of every A in D_{II}^l by a point square in the special dual number plane shifts or translates A's position in the special dual number plane without altering the size of A.

Proof is direct and hence left as an exercise to the reader.

Now having seen the properties of special types of squares in the special dual like number planes we leave rest of the work to the reader as it is left as a matter of routine.

We now proceed onto introduce the properties of special types of squares in special quasi dual number plane we denote in this book by

$$Q^l = \{a + bg \mid a, b \in R, g^2 = -g\}.$$

Example 3.62: Let
$$A = \{(5, 2g), (3, 2g), (5, 0), (3, 0)\}$$
be a special quasi dual number square of type I.
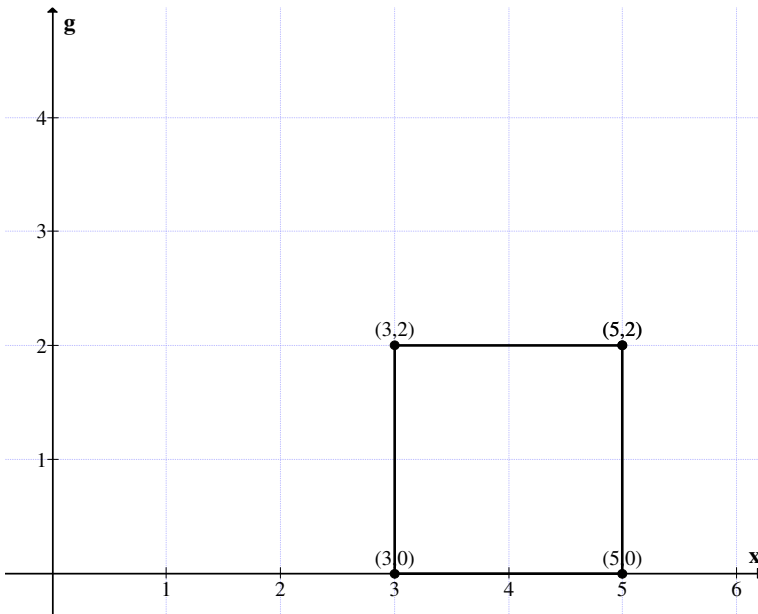
We give the graphic representation is as follows:

Figure 3.59

Example 3.63: Let $M = \{(-4, 5g), (-1, 5g), (-4, 8g), (-1, 8g)\}$ be the special quasi dual number square of type I.

Let
$$P = \{(2, 4g), (5, 4g), (2, 7g), (5, 7g)\}$$
be another square of type I in the same plane.

$$P + M = \{(-2, 9g), (4, 9g), (-2, 15g), (4, 15g)\}$$
is again a special quasi dual number square of type I.

$$P - M = \{(6, -g), (6, -g), (6, -g), (6, -g)\}$$
is a point square in this plane.

$$M - P = \{(-6, g), (-6, g), (-6, g), (-6, g)\}$$
is also a point square. Both the point squares M – P and P – M are diametrically opposite to each other.

Example 3.64: Let
$$M = \{(3, 2g), (5, 4g), (3, 4g), (5, 2g)\}$$
be the special quasi dual number square of type I.

$$\{(0, 0)\} = \{(0, 0), (0, 0), (0, 0), (0, 0)\}$$
is the zero point square.
Clearly
$$M + \{(0, 0)\} = \{(0, 0)\} + M = M.$$
Now
$$-M = \{(-3, -2g), (-5, -4g), (-3, -4g), (-5, -2g)\}$$

is again a special quasi dual number square of type I.

We see
$$M + (-M) = -M + M = \{(0, 0)\}.$$

Thus it is evident to every special quasi number square of type I say A we have a unique –A

$$A + (-A) = \{(0, 0), (0, 0), (0, 0), (0, 0)\}$$
$$= (-A) + A.$$

Thus we define the following.

DEFINITION 3.3: Q_I^l = *{Collection of all squares of the form {(a, bg), (a+c, bg), (a, (b+c)g), (a+c, (b+c)g)} where a, b, c $\in$ R, $g^2 = -g$} is defined as the collection of all special quasi dual number squares of type I.*

We have already given examples of them.

Now we have the following result.

THEOREM 3.13: *Let Q_I^l be the collection of all special quasi dual number squares of type I. { Q_I^l , +} is an abelian group of infinite order.*

Proof is direct and hence left as an exercise to the reader.

We give some more illustration of them in the following.

Example 3.65: Let

$$M = \{(2, 4g), (-1, 4g), (2, g), (-1, g)\}$$

and

$$N = \{(3, 5g), (1, 5g), (3, 3g), (1, 3g)\}$$

be any two special quasi dual number square of type I.

We find M + N, M – N and N – M in the following:

$$M + N = \{(5, 9g), (0, 9g), (5, 4g), (0, 4g)\}$$

$$M - N = \{(-1, -g), (-2, -g), (-1, -2g), (-2, -2g)\}$$

and

$$N - M = \{(1, g), (2, g), (1, 2g), (2, 2g)\}$$

are all special dual quasi number squares of type I.

We represent these squares graphically in the special quasi dual number plane in the following.
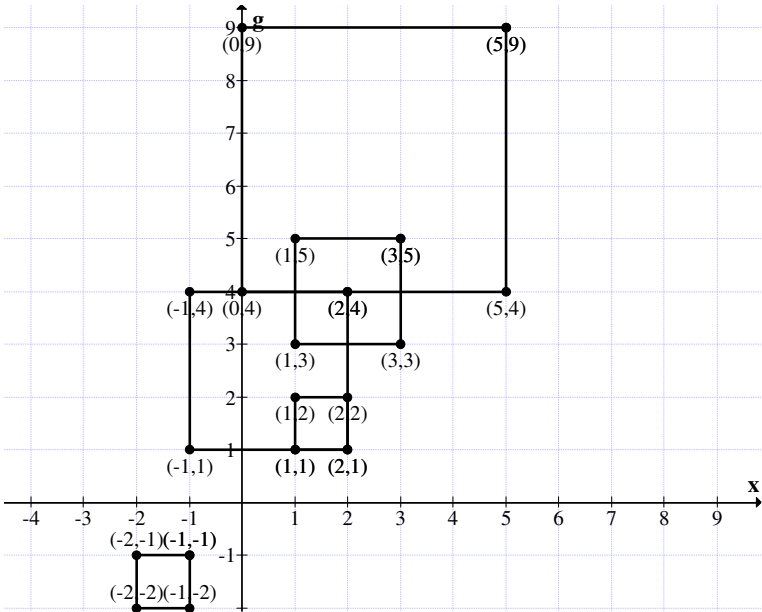
Figure 3.60

We see as in case of usual squares of type I in this case also three of the squares overlap and one square viz. N–M is disjoint with the other three and is just diametrically opposite to the square M–N.

Example 3.66: Let

$$M = \{(5, 2g), (2, 2g), (5, -g), (2, -g)\}$$

and

$$P_1 = \{(2, 0), (2, 0), (2, 0), (2, 0)\}$$

be the special dual quasi number square of type I and point square respectively.

We find $M + P_1$, $M–P_1$ and $P_1–M$ in the following.

$$M + P_1 = \{(7, 2g), (4, 2g), (7, -g), (4, -g)\}$$

$$M - P_1 = \{(3, 2g), (0, 2g), (3, -g), (0, -g)\}$$

and

$P_1 - M = \{(-3, -2g), (0, -2g), (-3, g), (0, g)\}$
are the special quasi dual number squares of type I.

We give the diagrammatic representation of them in the following in the special quasi dual number plane.
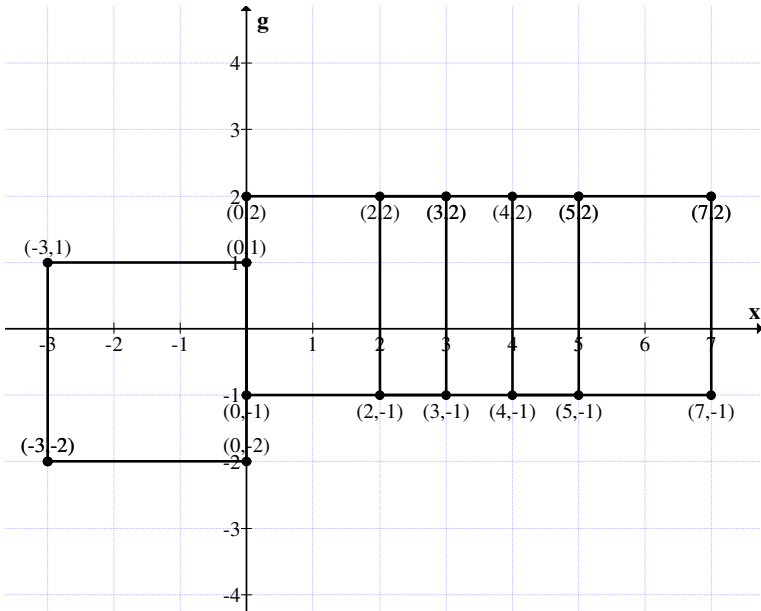

Figure 3.61

All the four squares either intersect or have a part of the side to be common.

Let us take

$P_2 = \{(0, -g), (0, -g), (0, -g), (0, -g)\}$
be the point square.

Let M be as above.

We find $M + P_2$, $M - P_2$ and $P_2 - M$ and find their values.

$M + P_2 = \{(5, g), (2, g), (5, -2g), (2, -2g)\}$

$$M - P_2 = \{(5, 3g), (2, 3g), (5, 0), (2, 0)\}$$

and

$$P_2 - M = \{(-5, -3g), (-2, -3g), (-5, 0), (-2,0)\}$$

are the three special quasi dual number squares of type I.

We give the graph of $M + P_2$, $M - P_2$ and $P_2 - M$ in the special quasi dual number plane in the following.
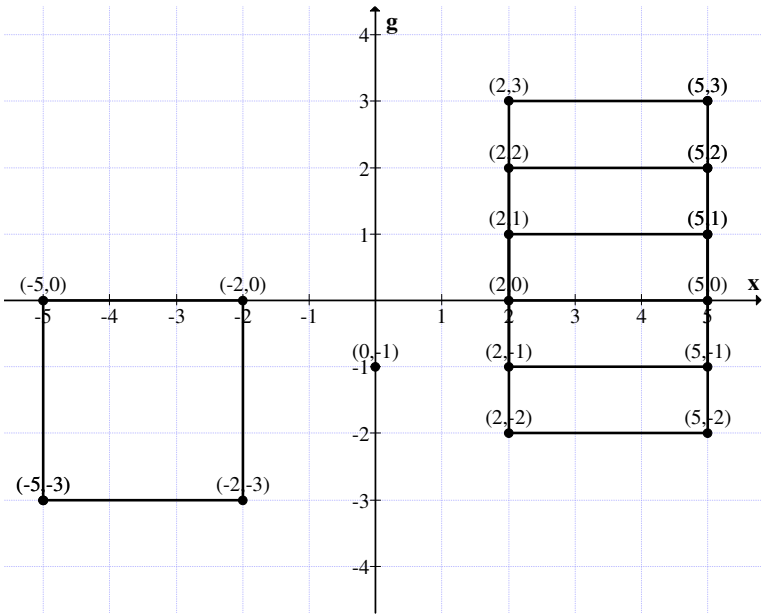

Figure 3.62

Thus three of the squares overlap or have a non empty intersection.

However the square $P_2 - M$ is disjoint from the rest of the three squares.

Let

$$P_3 = \{(2, g), (2, g), (2, g), (2, g)\}$$

be the point square.

To find

$$M + P_3 = \{(7, 3g), (4, 3g), (7, 0), (4, 0)\}$$

$$M - P_3 = \{(3, g), (0, g), (3, -2g), (0, -2g)\}$$

and

$$P_3 - M = \{(-3, -g), (0, -g), (-3, 2g), (0, 2g)\}$$

are three distinct special quasi dual number squares of type I.

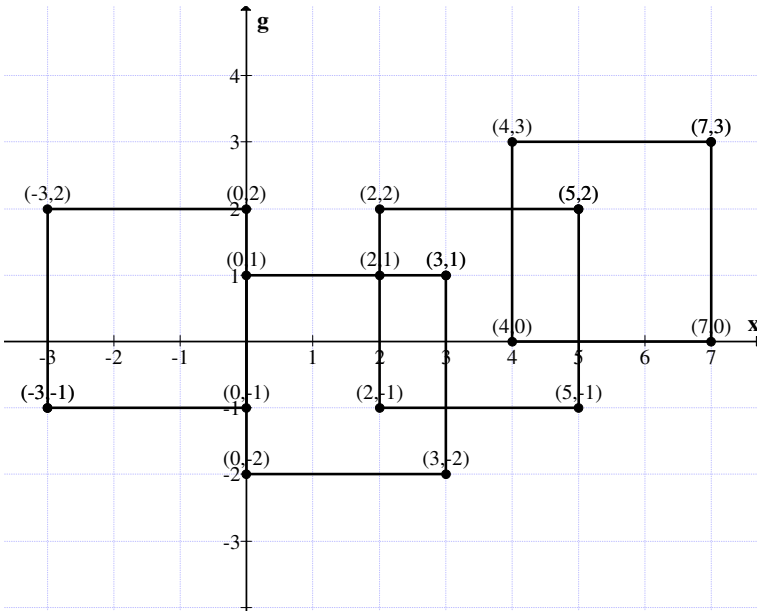The graphic representation is the special quasi dual number plane is as follows:

Figure 3.63

The three squares over lap and the square $P_3 - M$ is disjoint from other squares.

Thus the addition of a point square with the special quasi dual number square of type I shifts the square to a different place in the plane without affecting the size of the square.

Thus addition only translates the position of the squares.

Now we see weather the product of a two squares in Q_I^1 is defined.

Let

$$A = \{(3, 2g), (1, 2g), (3, 0), (1, g)\}$$

and

$$B = \{(5, -g), (2, -g), (5, -4g), (2, -4g)\}$$

be any two special quasi dual number squares of type I.

$$A \times B = \{(15, 9g), (2, 5g), (15, -20g), (2, 2g)\}$$

is not a special quasi dual square which is not type I.

It is only a quadrilateral such that the two sides are parallel and infact parallel to the special quasi dual number axis.

We will represent this by more examples.

Suppose

$$M = \{(2, g), (4, g), (2, 3g), (4, 3g)\}$$

and

$$N = \{(-1, -g), (2, -g), (-1, 2g), (2, 2g)\}$$

be two special quasi dual number squares of type I.

We find

$$M \times N = \{(-2, -2g), (8, -g), (-2, -5g), (8, 8g)\}$$

is not a square in the first place.

Infact a quadrilateral of a special type as given in Figure 3.64.

Thus we see $M \times N$ is a quadrilateral as seen from the figure two of its sides are parallel to the special quasi dual number axis.

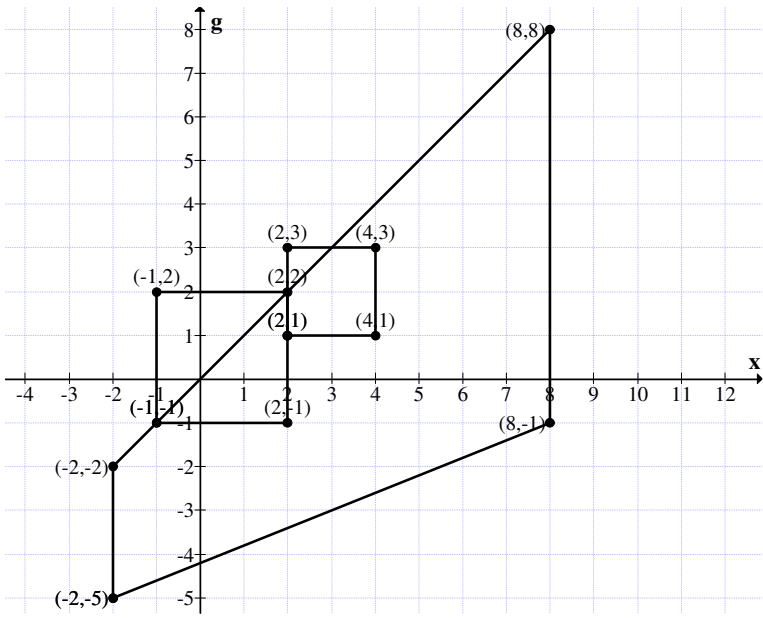**Figure 3.64**

Let

$$W = \{(0, 3g), (0, g), (-2, 3g), (-2, g)\}$$

and

$$V = \{(4, 0), (1, 0), (4, -3g), (1, -3g)\}$$

be any two special quasi dual number squares of type I.

$$W \times V = \{(0, 12g), (0, g), (-8, 27g), (-2, 10g)\}$$

is a quadrilateral and is not a square of type I.

The graph of it is given in Figure 3.65.

The quadrilateral happens to look very odd as seen from the diagram.
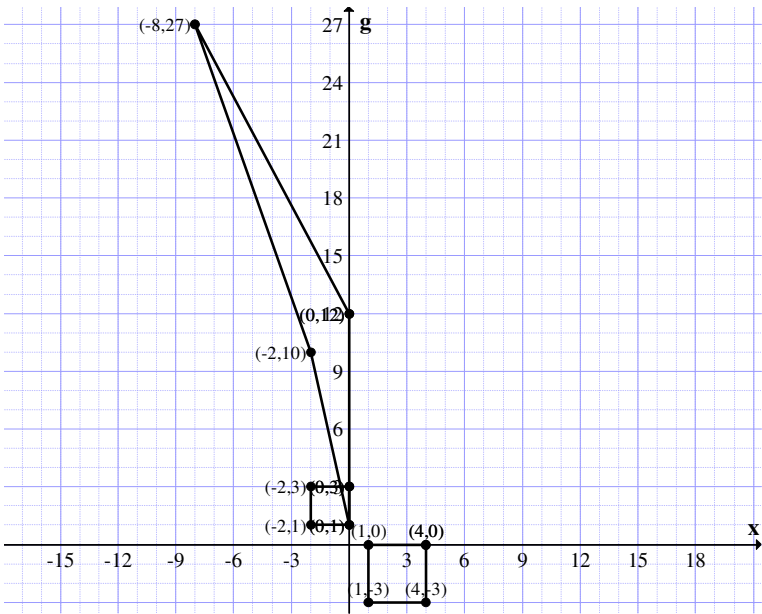
Figure 3.66

Now we find the product of a special quasi dual number square of type I with a point square from the special quasi dual number plane.

Let
$$A = \{(3, 2g), (0, 2g), (3, -g), (0, -g)\}$$

be a special quasi dual number space of type I.

Let
$$B = \{(2, 4g), (2, 4g), (2, 4g), (2, 4g)\}$$

be a point square in the same plane.

$$A \times B = \{(6, 8g), (0, -4g), (6, 14g), (0, -5g)\}.$$

The diagram in the special quasi dual number plane is as follows:
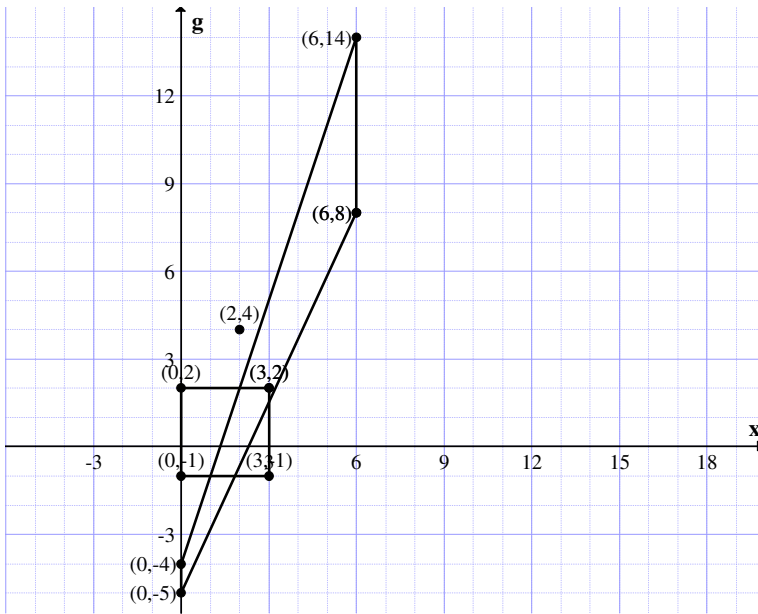
Figure 3.66

The diagram looks different from a quadrilateral. Thus we are not in a position to get the real representation of the product of a point square with a square of type I.

Example 3.67: Let
$$A = \{(3, 4g), (1, 4g), (3, 2g), (1, 2g)\}$$
be a point square

$$P = \{(-1, g), (-1, g), (-1, g), (-1, g)\}$$
in the special quasi dual number plane.

We find
$$A \times P = \{(-3, -5g), (-1, -7g), (-3, -g), (-1, -3g)\}$$
is not a special quasi dual square of type I and only a quadrilateral.

This is represented in the special quasi dual plane in the following.
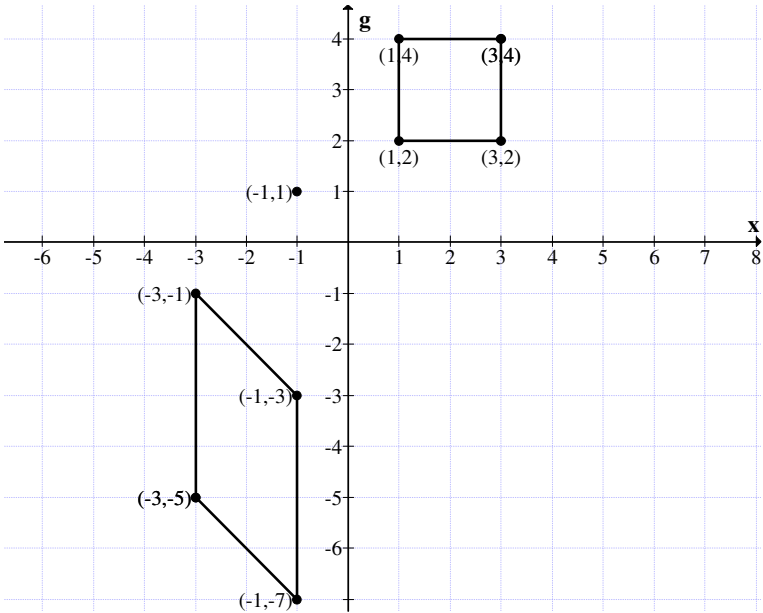
Figure 3.67

In some cases we get a quadrilateral or some times a very undefined structure, however the structure of the product of two squares of type I or a square of type I with a point square is left as an open conjecture.

Let us consider very special square of type I say

$$A = \{(0, 0),\ (2, 0), (0, 2g), (2, 2g)\} \in D_I^1 .$$

$$A \times A = \{(0, 0), (4, 0), (0, -4g), (4, -4g)\}$$

$$A \times A \times A = \{(0, 0), (8, 0), (0, 8g), (8, 8g)\}$$

and so on.

This is represented as graph in Figure 3.68.

We see A generates a square under product in the first and the forth quadrant.
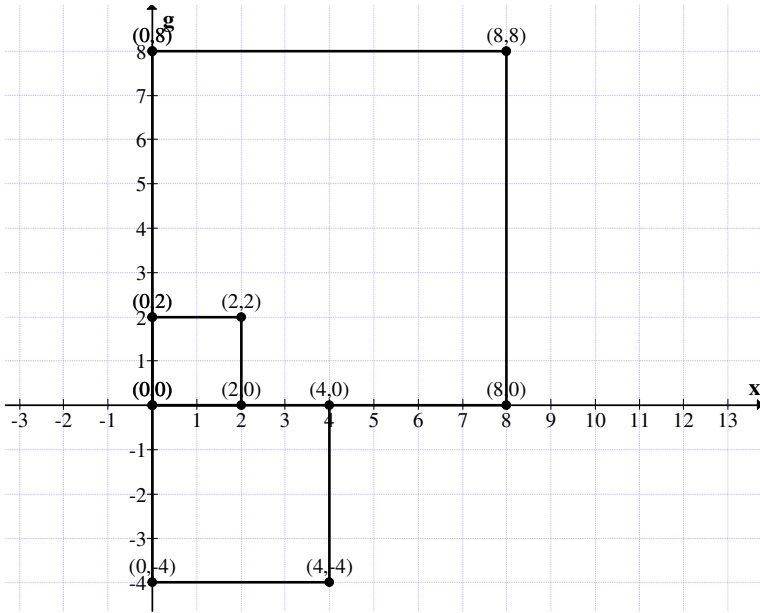
Figure 3.68

So A under product generates a semigroup of infinite order.

Let

B = {(0, 0), (–2, 0), (0, 2g), (–2, +2g)}

be the special quasi dual number square of type I.

$$B^2 = \{(0, 0), (4, 0), (0, -4g), (4, -4g)\}$$

and

$$B^3 = \{(0, 0), (-8, 0), (0, -8g), (-8, 8g)\}$$

and so on.

This is represented by the following graph in the special quasi dual number plane.
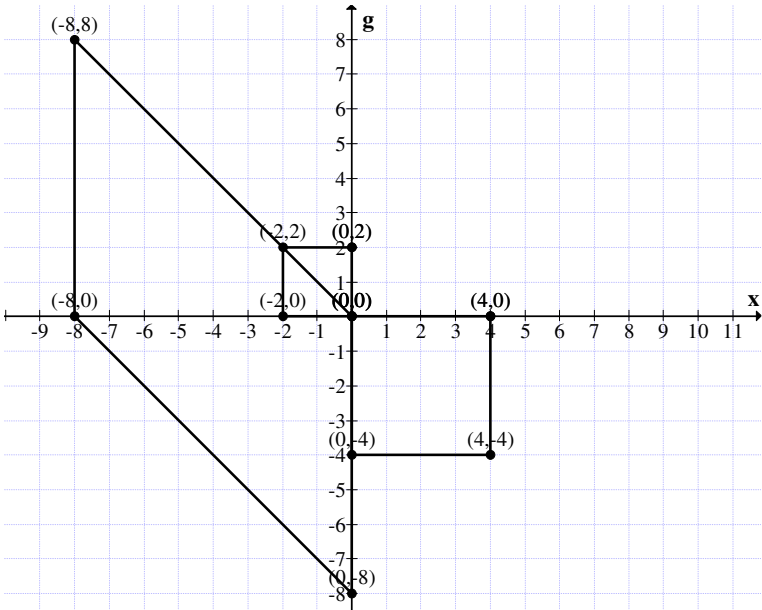
Figure 3.69

and so on.

We see if the position of the square is in the 2nd quadrant than this does not generate a semigroup under product.

Suppose the square finds its place in the 3rd quadrant say

$$A = \{(0, 0), (-2, 0), (0, -2g), (-2, -2g)\}$$

be a square of type I in the special quasi dual numbers.

$$A^2 = \{(0, 0), (4, 0), (0, -4g), (4, -4g)\}$$

and

$$A^3 = \{(0, 0), (-8, 0), (0, -8g), (-8, -8g)\}$$

and so on.

This is represented by the following graph in the special quasi dual number plane.
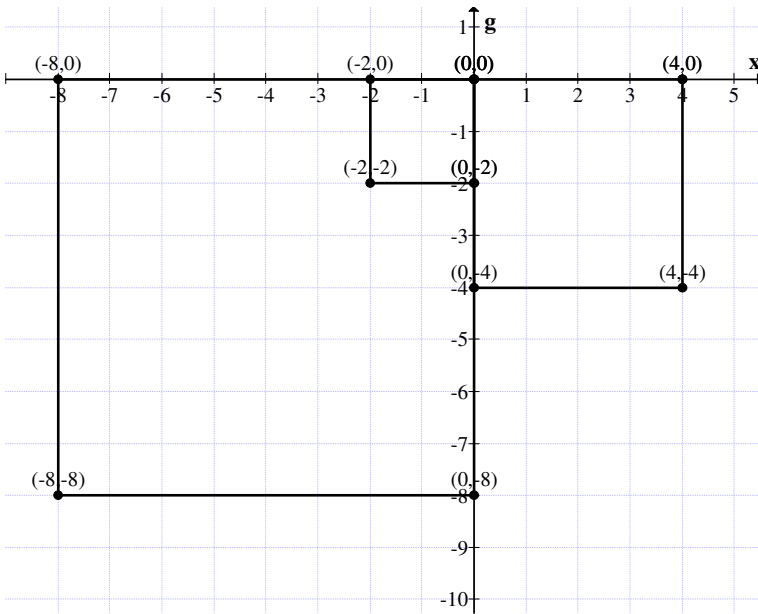
Figure 3.70

We see all the squares generates by A are in the 3rd and 4th quadrant alternatively.

Suppose

$$A = \{(0, 0), (2, 0), (0, -2g), (2, -2g)\}$$

be the special quasi dual number square of type I.

$$A^2 = \{(0, 0), (4, 0), (0, -4g), (4, -4g)\},$$

$$A^3 = \{(0, 0), (8, 0), (0, -8g), (8, -8g)\}$$

and so on.

We give the diagrammatic representation of A, A^2, A^3 and so on in the special quasi dual number plane in Figure 3.71.

We see A, A^2, A^3 and so on find its place only in the forth quadrant.
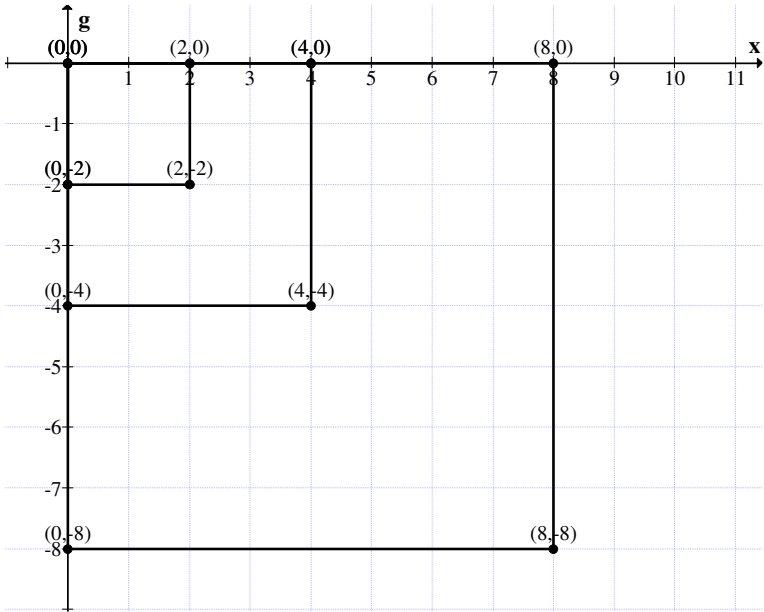
Figure 3.71

Thus A generates a semigroup under product.

Now we proceed onto define the type II squares in the special quasi dual number plane in the following.

DEFINITION 3.4: *Let* Q_{II}^l = *{Collection of all squares in the special dual quasi number plane such that the diagonals are parallel to the axis}; we define* Q_{II}^l *as the special quasi dual squares of type II.*

First we will illustrate this situation by some examples.

Example 3.68: Let

$$M = \{(4, 7g), (3, 6g), (2, 7g), (3, 8g)\}$$

be a special quasi dual number square of type II.

The graph is as follows:
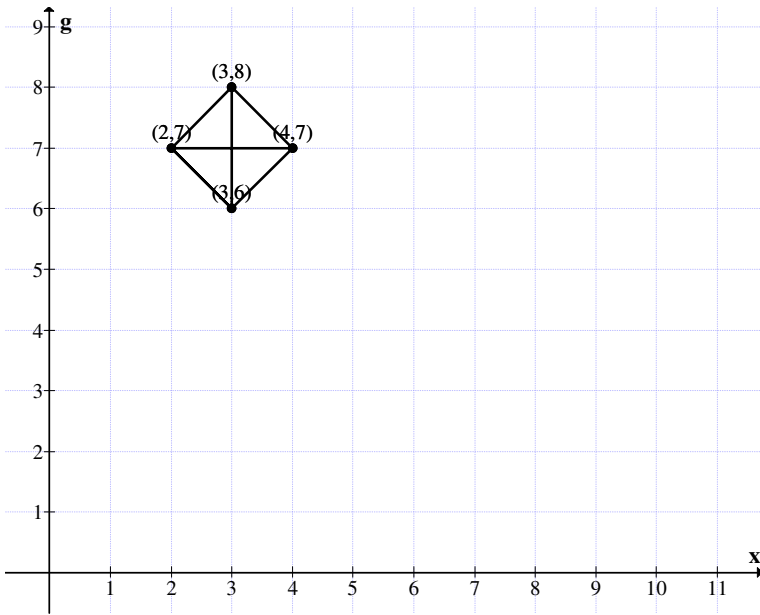
Figure 3.72

Example 3.69: Let
$$P = \{(8, -2g),\ (12, 2g), (8, 6g), (4, 2g)\}$$

be the special dual like number square of type II.

The graph associated with P is given in Figure 3.72.

Example 3.70: Let

$$W = \{(-3, 0),\ (3, 0), (0, 3g), (0, -3g)\}$$

be the special quasi dual number square of type II.

The graph of W in the special quasi dual numbers plane is given in Figure 3.43.
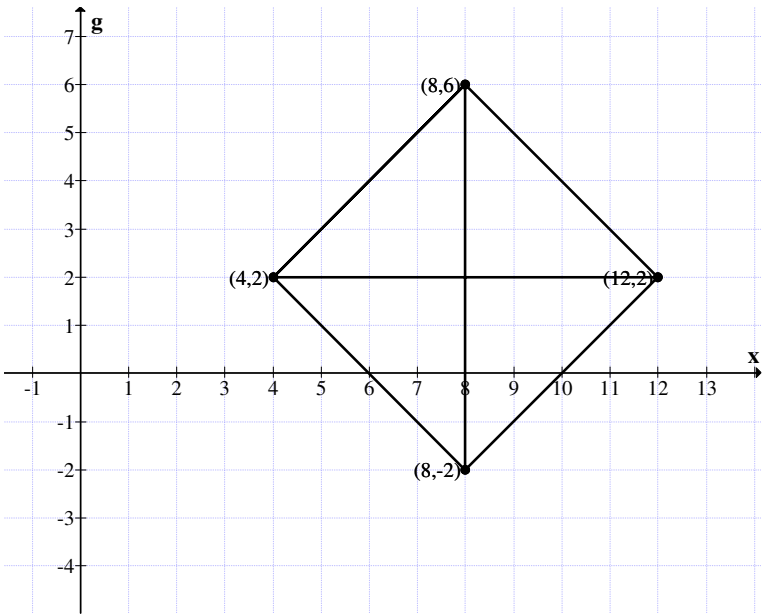
Figure 3.72
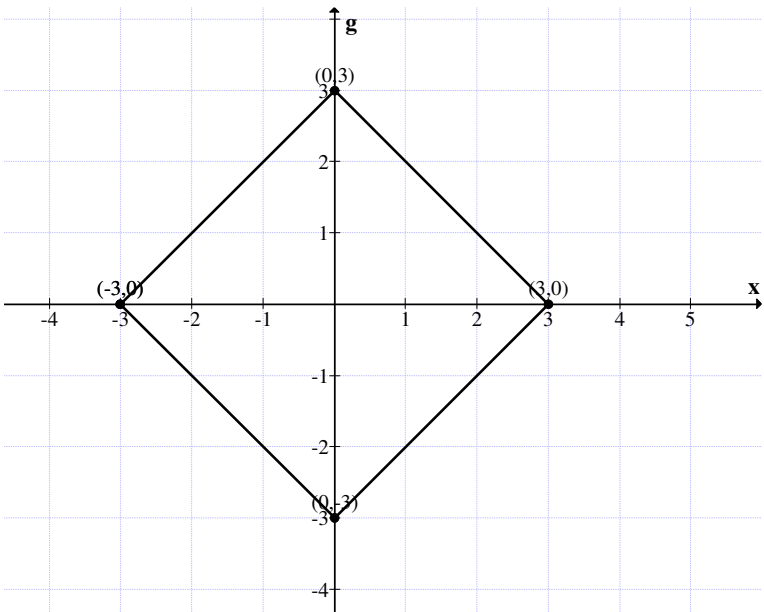
Figure 3.73

From these examples we have seen the structure of type II squares in Q_{II}^1.

We now proceed onto define operations on them.

Let
$$A = \{(8, -2g), (4, 2g), (12, 2g), (8, 6g)\}$$
and
$$B = \{(2, 3g) , (3, 4g), (4, 3g), (3, 2g)\}$$

be any two squares of type II.

$$A + B = \{(11, 0), (11, 10g), (6, 5g), (16, 5g)\}$$

is again a type II square.

Here we wish to keep on record type II squares can be parallelograms or rectangles such the diagonals are parallel to the axis of the plane.

So they need not be squares.

Now
$$A - B = \{(5, -4g), (5, 2g), (8, -g), (2, -g)\}$$

is again a type II square.

$$B - A = \{(-5, 4g), (-5, -2g), (-8, g), (-2, g)\}$$

is again a type II square.

We will provide the graph of all these squares in the special dual quasi number plane in the Figure 3.74.

We see the squares of type II, A, B, A − B and A + B have some type of over laps however B − A is disjoint with all the other squares evident from the figure.
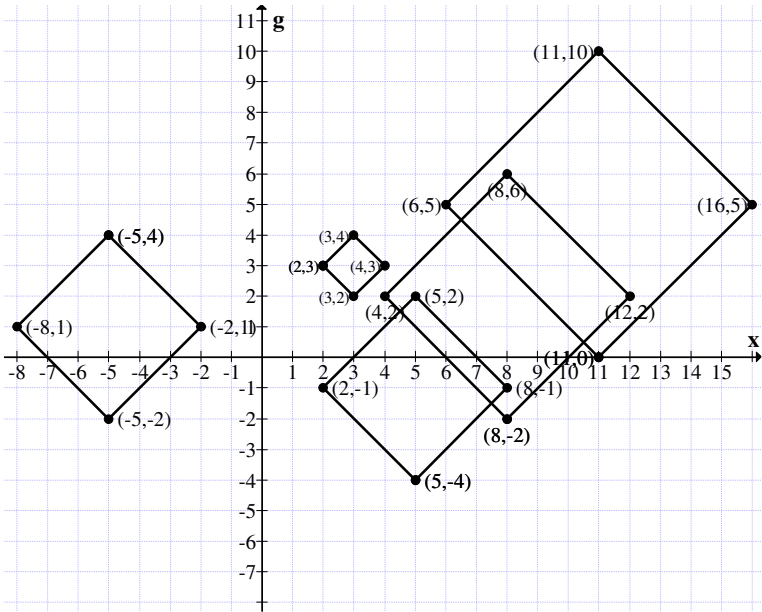
Figure 3.74

Let us consider Q_{II}^l be the collection of all squares in the special quasi dual number plane. $\{(0,0)\} = \{(0, 0), (0, 0), (0, 0), (0, 0)\}$ acts as the identity. Clearly for ever $A \in Q_{II}^l$ there exists a $-$ A in Q_{II}^l such that

$$A + (-A) = \{(0,0)\}$$

Further for every A, B in $A + B \in Q_{II}^l$.

Thus Q_{II}^l is a group. Inview of this we have the following theorem.

THEOREM 3.14: *Let* Q_{II}^l *be the collection of all type II squares of all type II squares of the special quasi dual number plane.* $\{Q_{II}^l, +\}$ *is an abelian group of infinite order which is commutative.*

The proof is direct and hence left as an exercise to the reader.

We give an example or two.

Example 3.71: Let

$$M = \{(6, 0), \ (-6, 0), (0, 6g), (0, -6g)\}$$

be a special quasi dual number square of type II.

Let
$$P = \{(3, -2g), (3, -2g), (3, -2g), (3, -2g)\}$$

be a point square.

$$M + P = \{(9, -2g), (-3, -2g), (3, 4g), (3, -8g)\}$$

is again a special quasi dual number square of type II.

$$M - P = \{(3, 2g), (-9, 2g), (-3, 8g), (-3, -4g)\}$$

is again a special quasi dual number square of type II.

$$P - M = \{(-3, -2g), (9, -2g), (3, -8g), (3, 4g)\}$$

is again a special dual quasi number square of type II.

We now give the graph of all these squares in the special quasi dual number plane in Figure 3.75.

Two of the squares in this case coincide.

We see addition of a point square has shifted the position of the square in the special quasi dual number plane.

So addition of point square to these squares shifts or translates the position of the square.
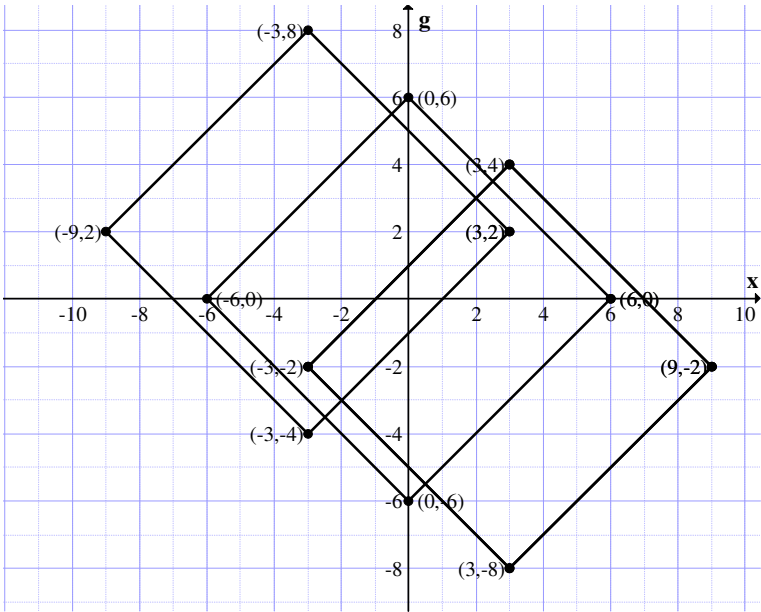
Figure 3.75

Next we try to define the product on Q_{II}^1 .

Let

$$A = \{(-3, -2g), (9, -2g), (3, -8g), (3, 4g)\}$$

and

$$B = \{(3, 2g), (-9, 2g), (-3, 8g), (-3, -4g)\} \in Q_{II}^1 .$$

$$A \times B = \{(-9, -8g), (-81, 30g), (-9, 112g), (-9, -8g)\}.$$

Clearly the product is not a square of type II, so
$$A \times B \notin Q_{II}^1 .$$

Let

$$A = \{(-3, -2g), (9 -2g), (3, -8g), (3, 4g)\}$$

and

$$P = \{(4, -g), (4, -g), (4, -g), (4, -g)\}$$

be a point square.

$$A \times P = \{(-12, -7g), (36, -9g), (12, -43g), (12, -15g)\}$$

is not a point square so the product of a type II square with a point square is not a type II square. $A \times P \notin Q_{II}^1$.

Now we see Q_{II}^1 is not closed under product.

THEOREM 3.15: $\{Q_{II}^1, \times\}$ *is not a even closed under product.*

Proof is direct and hence left as an exercise to the reader.

We will give one or two examples.

Example 3.72: Let

$$A = \{(4, 0), (7, 3g), (4, 6g), (1, 3g)\}$$
be a type II square.

$$P = \{(2, -g), (2, -g), (2, -g), (2, -g)\}$$
be the point square.

$$A \times P = \{(8, -4g), (14, 2g), (8, 14g), (2, 8g)\}$$

is again a square of type II, so it remains as an open problem to find type II squares whose product with other type III squares given a type II square in all the planes introduced in this book.

Now we consider planes of the form

$$A = \{(2, 0), (-2, 0), (0, 2g), (0 - 2g)\} \in Q_{II}^1 .$$

$$A^2 = \{(4, 0), (4, 0), (0, -4g), (0, -4g)\}$$

$$A^3 = \{(8, 0), (-8, 0), (0, 8g), (0, -8g)\} \in Q_{II}^1$$

and so on.

This attains the following graphical representation.

Figure 3.76

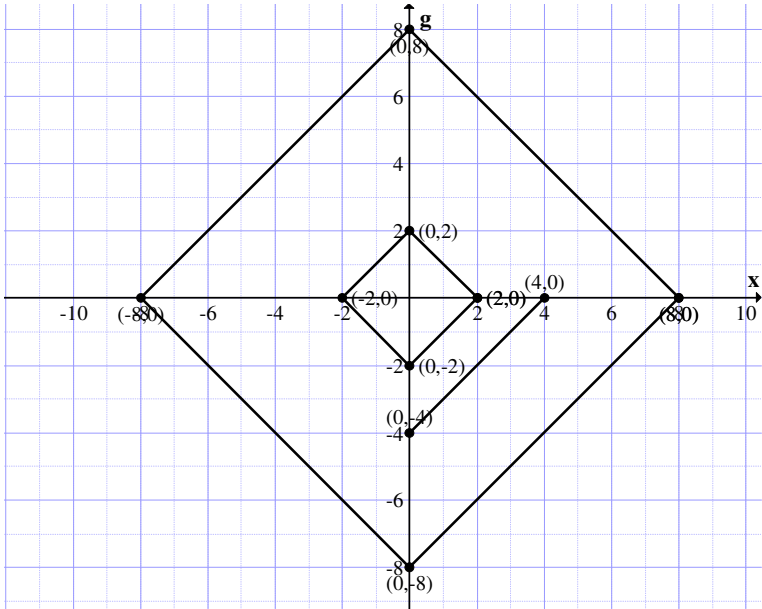So the collection with a line of the form $\{(a^n, 0),\ (0, -a^n g)\}$ generates a semigroup.

Let

$$A = \{(2, 0), (0, 2g), (-2, 0), (0, -2g)\}$$

and

$$B = \{(3, 0),\ \ (-3, 0), (0, 3g), (0, -3g)\}$$

$$A \times B = \{(6, 0), (6, 0), (0, -6g),(0, -6g)\}$$

$A \times B$ is a line.

So we denote by LS $(Q_{II}^1) = \{$Collection of all type II squares of the form $\{(a, 0), (-a, 0), (0, ag), (0, -ag)\} \cup \{(a^n, 0),$

$(0, -a^n g)$} where $a \in R$ and {$(ab, 0), (0, -abg)$} or so on say {$(m, 0), (0, -mg)$}, $m \in R$}.

Then LS (Q_{II}^1) under product is a semigroup.

Infact every square in LS (Q_{II}^1) can generate a subsemigroup of infinite order.

However if

$$B = \{(2, 0), (-2, 0), (0, 2g), (0, -2g)\}$$

and

$$P = \{(5, 2g), (5, 2g), (5, 2g), (5, 2g)\}$$

then

$$B \times P = \{(10, 4g), (-10, -4g), (0, 6g), (0, -14g)\}$$

is not a special quasi dual number square of type II only a quadrilateral.

Let

$$M = \{(1, 0), (-1, 0), (0, g), (0, -g)\}$$

be a square of type II.

Let

$$P = \{(2, -g), (2, -g), (2, -g), (2, -g)\}$$

be a point square.

$$M \times P = \{(2, -g), (-2, g), (0, 3g), (0, -3g)\}$$

we give the graph of them in the special quasi dual number plane in Figure 3.77.

This is a very special type the product square contains the square M.

Let us consider

$$W = \{(2, 0), (-2, 0), (0, 2g), (0, -2g)\} \in Q_{II}^1.$$
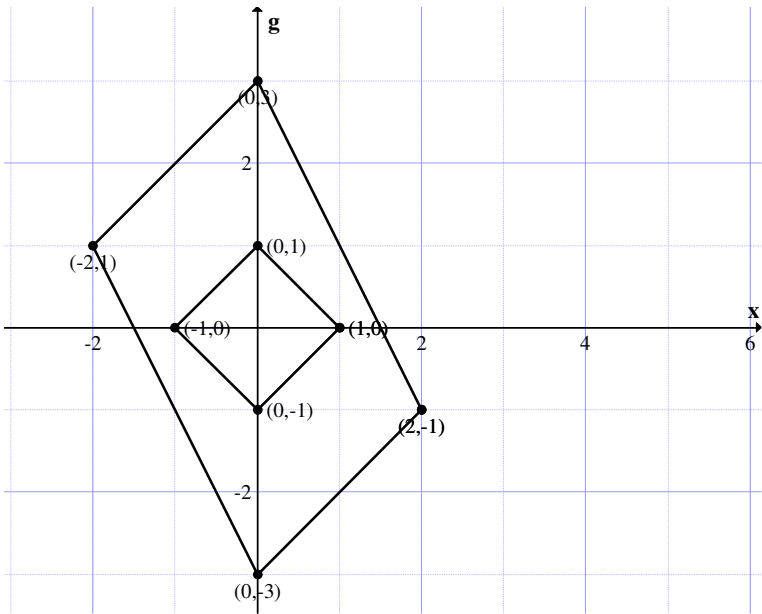
Figure 3.77

Let
$$P = \{(-2, g), (-2, g), (-2, g), (-2, g)\}$$

be a point square.

$$P \times W = \{(-4, 2g), (4, -2g), (0, -5g), (0, 5g)\}$$

is a quadrilateral.

The graph of these 3 squares is given in Figure 3.78.
Clearly $W \times P$ is not a square of type II.
Let
$$V = \{(3, 0), (0, 3g), (-3, 0), (0, -3g)\}$$

be a square of type II.

$$P = \{(1, -2g), (1, -2g), (1, -2g), (1, -2g)\}$$

be a point square.
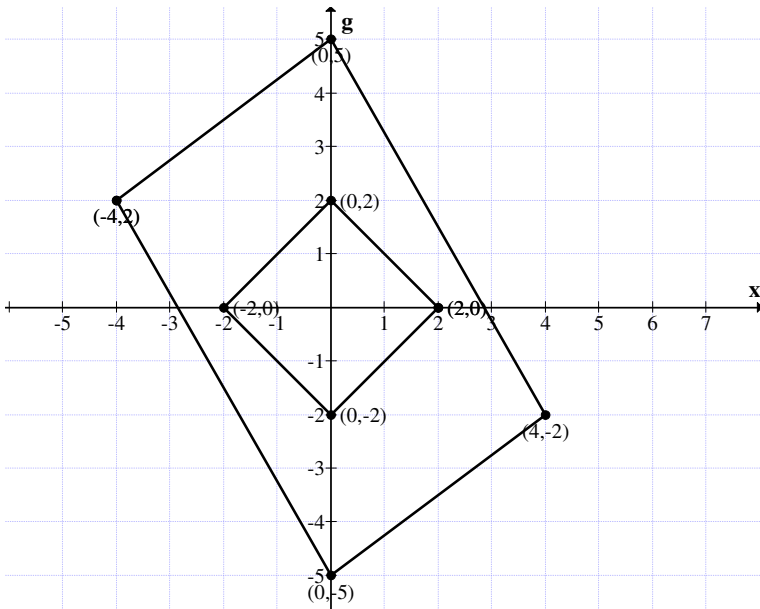
Figure 3.78

$$V \times P = \{(3, -6g), (0, 9g), (-3, 6g), (0, -9g)\}$$

is again a square of type II in the special dual like number plane.

We give the graphs of these squares in the special dual like number plane in Figure 3.79.

We see from the graph is a quadrilateral. One can define and describe this sort of properties.

Now we have seen for any $A \in LS(Q_{II}^l)$. $A \times B \notin LS(Q_{II}^l)$ even if B is a point square in the special quasi dual number plane.

Thus take $A, B \in LS(Q_{II}^l)$, let

$$A = \{(6, 0), (-6, 0), (0, -6g), (0, 6g)\}$$

and

$$B = \{(2, 0), (-2, 0), (0, -2g), (0, 2g)\}.$$
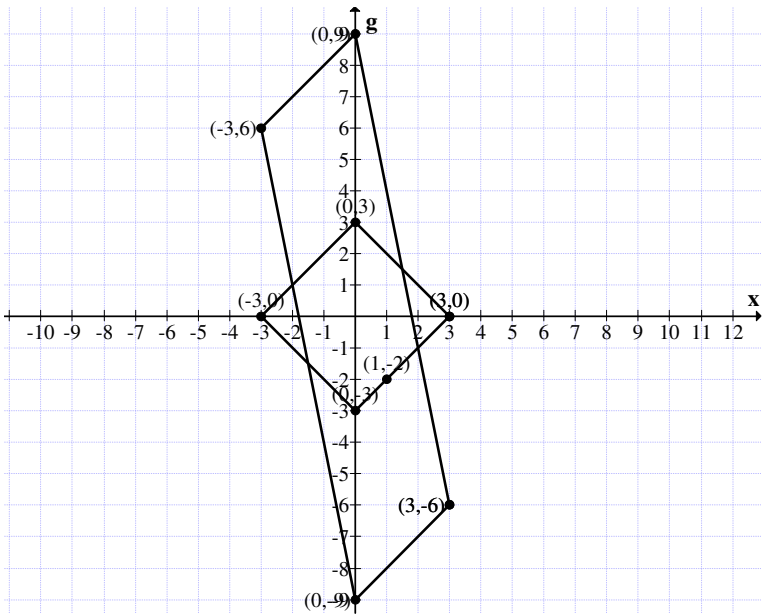
Figure3.79

We find
$$A \times B = \{(12, 0), (12, 0), (0, -12g), (0, -12g)\}$$

is a straight line which is in LS(Q_{II}^1) as desired.

Thus product of all squares of the form

A = {(a, 0), (–a, 0), (0, ag), (0, –ag)} degenerates to a line {(a^2, 0), (0, $-a^2$g)} in the forth quadrant.

However for any point square P in the special quasi dual number plane the product of A with P for any P ∈ LS(Q_{II}^1) in general is not in LS(Q_{II}^1).

Thus having worked about of all properties in different planes.

We suggest some problems for the reader.

Some of these problems are at research level and some at the moderate level.

Some of these problems can be treated as conjectures.

Overall these types of squares in these planes behave in a very different way.

Problems

1. Find some special and distinct features enjoyed by complex square of type I.

2. Prove $\{C_I, +\}$ is a group of infinite order.

3. Can $\{C_I, +\}$ have substructures which are subsemigroups?

4. Show $\{C_I, \times\}$ is not even closed.

5. Prove the sum of a point square with a complex square of type I shifts or translates the square without affecting its size.

6. Let A = $\{(3, -4i), (6, -4i), (3, -i), (6, -i)\}$ be a complex square of type I.

   P_1 = $\{(0, 2i), \ (0, 2i), (0, 2i), (0, 2i), (0, 2i)\}$ be the point square.

   (i) Find $P_1 + A$, $P_1 - A$, $A - P_1$ and draw the graph of all the three squares in the complex plane.

   (ii) If P_1 is replaced by P_2 = $\{(2, 0), (2, 0), (2, 0), (2, 0)\}$ study question (i).

(iii) If $P_3 = \{(2, 2i), (2, 2i), (2, 2i), (2, 2i)\}$ be the complex square find $P_3 + A$, $P_3 - A$ and $A - P_3$ and draw the diagram and compare them with squares in (i) and (ii).

7. Let $W = \{(-3, 5i), (0, 5i), (-3, 2i), (0, 2i)\}$ and $V = \{(5, -8i), (8, -8i), (5, -5i), (8, -5i)\}$ be any two complex square of type I.

   (i) Find $W + V$, $W - V$ and $V - W$.
   (ii) Find the graph of $W + V$, $W - V$ and $V - W$ in the complex plane.

8. Let $V = \{(3, 4i), (5, 4i), (3, 6i), (5, 6i)\}$ and $W = \{(-4, -8i), (-1, -8i), (-4, -5i), (-1, -5i)\}$ be the two complex squares of type I.

   (i) Find $V - W$, $W - V$ and $V + W$.
   (ii) Draw the graphs of $V - W$, $W - V$ and $V + W$ in the complex plane.

9. Let $M_1 = \{(3, 0), (5, 0), (3, 2i), (5, 2i)\}$ be the complex square of type I.

   (i) Find $M_1 + M_1$, $M_1 + M_1 + M_1$ and $M_1 + M_1 + M_1 + M_1$ and represent them in the complex plane.

10. Let $B = \{(0, 0), (6, 0), (0, 6i), (6, 6i)\}$ be the complex square of type I.
    Take $P_1 = \{(0, 2i), (0, 2i), (0, 2i), (0, 2i)\}$ and $P_2 = \{(2, 0), (2, 0), (2, 0), (2, 0)\}$ and point squares.

    (i) Find $B + P_1$, $B + P_2$, $B - P_1$, $B - P_2$, $P_1 - B$ and $P_2 - B$ and represent them in the complex plane.

11. Can C_I enjoy any other properties different from N_I and S_I?

12. Let C_{II} be the collection of all complex squares of type II.

    (i) What is the main difference between squares in C_I and

C_{II}?

(ii) Is C_{II} under product operation compatible?

13. Show $(C_{II}, +)$ is an abelian group of infinite order.

14. Prove addition of a point square with any complex square A of type II only makes the complex square A shift its position to a different place in the same plane.

15. Is $P = \{(-5, 4i), (-15, 4i), (-10, 2i), (-10, 6i)\}$ a complex square of type II.

    Justify your claim.

16. Let $M = \{(a, 0), (0, ai), (-a, 0), (0, -ai)\}$ $a \in R \setminus \{0\}$ be the complex square of type II as well as lines $\{(a^2, 0), (0, a^2)\}$.

    (i) Is $(M, \times)$ a group?
    (ii) Can M have subgroups?
    (iii) Can M have subsemigroups?
    (iv) Does every $A \in M$ generate a subsemigroup?

17. Can $\{M, +\}$ be a group?

18. Will $\{M, +\}$ be a semigroup?

19. Let $B = \{(5, 0), (0, 5i), (-5, 0), (0, -5i)\}$ be a complex square of type II.

    If $P_1 = \{(3, -2i), (3, -2i), (3, -2i), (3, -2i)\}$ be a point square.

    Find   (i) $B + P_1$
            (ii) $B - P_1$ and
            (iii) $P_1 - B$
    (iv) Will the squares in (i), (ii) and (iii) be of the same type of the square as that of B?

20. Let A = {(10, 0), (–10, 0), (0, 10i), (0, –10i)} be a complex square of type II.

    Let P = {(0, 2i), (0, 2i), (0, 2i), (0, 2i)} be the point square.

    Study questions (i) to (iv) of problem (19) for this A.

21. Can type I and type II complex square be related by some means?

22. Let C_{II} be the collection of all complex squares of type II.

    Obtain any special property associated with C_{II}.

23. $(C_{II}, \times)$ is not a group or semigroup in general – prove or disprove!

24. Obtain some special features enjoyed by dual number squares of type I.

25. What are the distinct features enjoyed by complex squares from the real Euclid squares and neutrosophic squares of both type I and type II.

26. Let S = {(9, 0), (–9, 0), (0, 5i), (0, –5i)}.

    Is S a complex square of type II?

    If P = {(0, i), (0, i), (0, i), (0, i)} ∈ C_{II} find PS and SP.

27. Will all complex squares of type II with (0, 0) as the point of intersection of diagonals together with point squares be a group under product?

    Justify your claim.

28. Let W = {(–a, 0), (a, 0), (0, ai), (0, –ai)} be a collection of complex squares of type II.

(i) Is W a semigroup under +?

(ii) Can W be closed under product?

(iii) If X = {(b, ci) | b, c (R} be a complex point square. Find X + W, XW, W–X and X–W.

(iv) Discuss the algebraic structure enjoyed by X + W, XW, X–W and W – X.

29. Let B = {(–7, 0), (7, 0), (0, 7i), (0, –7i)} be a complex square of type II.

Let P = {(3, –2i), (3, –2i), (3, –2i), (3, –2i)} be the complex point square.

Find B + P, B – P and P – B.

Give the graph of all the five squares in the complex plane.

30. For B and P given in problem 28 find B^2 and PB and give their respective graphs.

31. Let M = {(4, 5i), (3, 2i), (2, 3i), (3, 4i)} be the complex square of type II.

P_1 = {(5, 3i), (5, 3i), (5, 3i), (5, 3i)} be the complex point square.

(i) Find M + P_1, P_1 – M and M – P_1
(ii) Find P_1M.
(iii) Draw the figure of M, P_1, P_1M, M + P_1, P_1 – M and M – P_1 on the complex plane.

32. Let W = {(1, i), (10, i), (10, 10i), (1, 10i)} be a complex square of type I.

(i) Find W_2, W_3, ….

    (ii) Does W generate a semigroup under (?

    (iii) Find W + W, W+W+W, W+W+W+W, …

    (iv) Does W generate a semigroup under the operation addition?

33. Let V = {(2, 3i), (3, 4i), (4, 3i), (3, 2i)} be a complex square of type II.

    Study questions (i) to (iv) of problem 31 for this V.

34. Let P = {(–2, 3i), (–1, 2i), (–2, i), (–3, 2i)} be a complex square of type II.

    Study questions (i) to (iv) of problem 31 for this P.

35. Using V of problem 32 and P of problem 33 find V + P, P–V and V–P and draw their graphs in the complex plane.

36. Let D_I denote the collection of all dual number squares of type I.

    Obtain any of the special and interesting feature enjoyed by D_I.

37. Let A = {(5, 2g), (7, 2g), (5, 4g), (7, 4g)} be a dual number square of a type I.

        (i) Can A generate group under '+'?

        (ii) Can A generate a semigroup under ×?

        (iii) If P = {(9, 2g), (9, 2g), (9, 2g), (9, 2g)} be a point squares will A+P, A–P and P–A be dual number squares of type I?

        (iv) Find A × P is A × P a dual number square of type I?

        (v) Does $A^2 \in D_I$?

38. Let M = {(–3, 4g), (0, 4g), (–3, 7g), (0, 7g)} and N = {(4, 5g), (1, 5g), (4, 2g), (1, 2g)} be two dual number squares of type I in D_I.

(i)   Find M+N, M–N and N–M and represent them graphically.

(ii) Find M_2, N_2 and NM. Do they belong to D_I?

(iii) Find –M and –N.
Is $-M \times -N = M \times N$?

(iv) Find M+N+M and N+M+N and represent them graphically.

39. Let M and N be as in problem 37.

Let P_1 = {(–4, –2g), (–4, –2g), (–4, –2g) and

P_2 = {(–1, –5g), (–1, –5g), (–1, –5g), (–1, –5g)} be two point squares.

(i)   Find $M + P_1$, $M – P_1$, $P_1 – M$ and represent them graphically.

(ii) Find $N + P_1$, $N – P_1$ and $P_1 – N$ and represent graphically in the dual number plane.

(iii) Find MP_1 and NP_1; do they belong to D_I.

(iv) Find $M – P_2$, $M + P_2$ and $P_2 – M$ and represent it in the dual number plane.

(v) Find $N – P_2$, $N + P_2$ and $P_2 – N$ and represent it in the dual number plane.

(vi) Does MP_2 and $NP_2 \times D_I$?

40. Let N = {(0, 5g), (6, 5g), (0, 11g), (6, 11g)} and
P_1 = {(0, –2g), (0, –2g), (0, –2g), (0, –2g)} be the dual number square of type I and point square.

(i) Find $N + P_1$, $N - P_1$ and $P_1 - N$ and show this is a translation of the square N by some distance.

(ii) Find $N \times N$ and $N \times P_1$ and show these two are not elements of D_I.

41. Let $M_1 = \{(2, -5g), (6, -5g), (2, -g), (6, -g)\}$,
$M_2 = \{(-5, 2g), (-1, 2g), (-5, 6g), (-1, 6g)\}$ be any two dual number squares of type I.

Find $M_1 + M_2$, $M_1 - M_2$, $M_2 - M_1$ and $M_1 \times M_2$ and draw their graphs in dual number plane.

42. What is the special features associated with dual number squares D_{II} of type II?

43. Distinguish between neutrosophic square of type I with dual number squares of type I.

44. Study the differences between complex squares of type II with that of dual number squares of type II.

45. Is type II squares always squares or just regular nice figures whose parallel sides are equal and diagonals parallel to the axis of the planes?

46. Let $W = \{(5, -4g), (-5, -4g), (0, 0), (0, -8g)\}$ be a dual number square of type II.

Let $P_1 = \{(6, 0), (6, 0), (6, 0), (6, 0)\}$

$P_2 = \{(-3, 0), (-3, 0), (-3, 0), (-3, 0)\}$

$P_3 = \{(0, 2g), (0, 2g), (0, 2g), (0, 2g)\}$

$P_4 = \{(0, -5g), (0, -5g), (0, -5g), (0, -5g)\}$

$P_5 = \{(4, 2g), (4, 2g), (4, 2g), (4, 2g)\}$

$P_6 = \{(-2, 6g), (-2, 6g), (-2, 6g), (-2, 6g)\}$

$P_7 = \{(3, -4g), (3, -4g), (3, -4g), (3, -4g)\}$

$P_8 = \{(-1, -3g), (-1, -3g), (-1, -3g), (-1, -3g)\}$ be eight distinct point squares from the dual number plane.

(i)   Find $W + P_i$, i = 1, 2, …, 8 and sketch their graphs.

(ii) Find $W - P_i$, i = 1, 2, 3, …, 8 and sketch their graphs.

(iii) Find $P_i - W$; i = 1, 2, 3, …, 8 and sketch their graphs.

(iv) Compare the squares in (i), (ii) and (iii) among themselves.

(v) Find $W \times P_i$, i = 1, 2, 3, …, 8 and give the graphs of $W \times Pi$.

(vi) Does $W \times P_i \in D_{II}$?

47. Let $A = \{(a, 0),\ (-a, 0),\ (0, ag),\ (0, -ag)\} \in D_{II}$.
    (i)   Does $A \times A \in D_{II}$?
    (ii) Find $A + B$?
    (iii) Find $A \times P_1$ where
          $P_1 = \{(5, 8g),\ (5, 8g),\ (5, 8g),\ (5, 8g)\}$.
    (iv) Find $A + P_1$.
          Does $A + P_1 \in D_{II}$?

48. Let $B = \{$Collection all squares of type II of the form $\{(a, 0), (0, ag), (-a, 0), (0, -ag)\}$ where $a \in R\}$.
    (i)   Is B a group under product?
    (ii) Is B a semigroup under product?
    (iii) Is B closed under product?
    (iv) If $Y \in R$ what is $Y \times Y$?

49. Compare squares in D_{II} with squares in N_{II}.

50. Compare squares in D_{II} with squares in R_{II}.

51. Compare squares in C_{II} with that of squares in D_{II}.

52. Let be the special dual number squares of type I.
    What are the special features enjoyed by D_I^1 ?

53. Compare with D_I^1.

54. Compare N_I with D_I^1.

55. Compare R_I with D_I^1.

56. What is the special feature enjoyed by squares in D_I^1 as
    $g^2 = g$?

57. Let be the special dual like number squares of type II.
    Enumerate all the special properties associated with D_{II}^1.

58. Let V = {(5, 3g), (2, 3g), (5, 0), (2, 0)} be a special dual like
    number squares in D_I^1.

    Let P = {(6, –8g), (6, –8g), (6, –8g), (6, –8g)} be the point
    square.

    (i) Find P+V, P–V and V–P and draw the graphs
         associated with them.

    (ii) Does $V \times V \in D_I^1$?

    (iii) Find $P \times V$ and give the graph in the special dual like
          number plane.

59. Let A = {(5, 3g), (8, 3g), (5, 6g), (8, 6g)} and

B = {(–3, 4g), (–1, 4g), (–3, 6g), (–1, 6g)} ∈ D_I^1 .

Find the graphs of A+B, A–B, B–A and AB. Compare them.

60. Let M = {(5, –3g), (–4, –3g), (5, –12g), (–4, –12g)} and N = {(2, 5g), (6, 5g), (2, 9g), (6, 9g)} ∈ D_I^1.

    (i) Find M+N, M–N, M × N and N–M and give their graphic representations in the special dual like number plane.

61. Let M = {(7, 0), (–7, 0), (0, –7g), (0, 7g)} be a special dual like number square of type II; M ∈ D_I^1.

    If A = {(5, 7g), (5, 7g), (5, 7g), (5, 7g)} is a point square.

    Find M+A, M–A, A–M and AM and give their graphic representation.

62. Obtain all the special feature enjoyed by D_{II}^1 .

63. Compare D_{II} with D_{II}^1 .

64. Compare N_{II} with D_{II}^1 .

65. Distinguish between D_{II}^1 and C_{II}.

66. Differentiate R_{II} and D_{II}^1 .

67. Let V = {(5, 3g), (2, 3g), (5, g), (2, 3g)} where $g^2 = –g$ be a special quasi dual number square of type I.

    If X = {(3, 2g), (3, 2g), (3, 2g), (3, 2g)} find V+X, X–V, V–X and VX and give their graphic representation in the special quasi dual number planes.

68. Let W = {(5, –2g), (8 –2g), (5, g), (8, g)} and
    V = {(–6, 5g), (–9, 5g), (–6, 2g), (–9, 2g)} be two special
    quasi dual squares of type I.

    (i) Find W+V, W–V, V–W and VW and give their graphic
        representation in the special quasi dual plane.

69. Let P_1 = {(8, 3g), (8, 3g), (8, 3g), (8, 3g)} be a point square.

    Find W+P_1, W–P_1, P_1–W, W × P_1 and give their graph in
    special quasi dual planes.

70. Let W = {(–3, 4g), (2, 4g), (–3, 9g), (2, 9g)} and
    B = {(5, 2g), (2, 2g), (5, –g), (2, –g)} be elements.

    (i) Draw the graph of W+B, W–B, B–W and W × B in the
        special quasi dual planes.

71. Let M = {(5, 0), (–5, 0), (0, 5g), (0, –5g)} be special quasi
    dual number squares of type II. Let M ∈ D_{II}^1 .

    (i) Does M generate a semigroup under ×?

    (ii) Let P_1 = {(3, 4g), (3, 4g), (3, 4g), (3, 4g)} be a point
    square. Find M+P_1, P_1–M, M–P_1 and MP_1 and represent
    them in the special quasi dual number plane.

72. Let M_1 = {(3, 0), (–3, 0), (0, 3g), (0, –3g)} and
    N_1 = {(8, 0), (–8, 0), (0, –8g), (0, 8g)} be the elements in .

    Find M_1 + N_1, M_1–N_1, N_1–M_1 and M_1 × N_1 and give the
    graph in special quasi dual number plane.

73. Let B = ⟨{(a, 0), (–a, 0), (0, ag), (0, –ag)} | a ∈ R; g^2 = –g⟩.

    (i) Is B closed under product?

(ii) If $P = \langle B \rangle \cup \{(a^{2n}, 0), (0, -ag^{2n})$ where $a \in R$, is P a semigroup under $\times$.

74. Let $M = \{(-9, 0), (9, 0), (0, 9g), (0, -9g)\}$ and $N = \{(-6, 8g), (-6, 8g), (-6, 8g), (-6, 8g)\}$ be the point square in the special quasi dual number plane.

Find $M \times N$, $M-N$, $M+N$ and $N-M$ and draw their graphs in the special quasi dual number plane.

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INDEX

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On India's 60th Independence Day, Dr.Vasantha was conferred the Kalpana Chawla Award for Courage and Daring Enterprise by the State Government of Tamil Nadu in recognition of her sustained fight for social justice in the Indian Institute of Technology (IIT) Madras and for her contribution to mathematics. The award, instituted in the memory of Indian-American astronaut Kalpana Chawla who died aboard Space Shuttle Columbia, carried a cash prize of five lakh rupees (the highest prize-money for any Indian award) and a gold medal.

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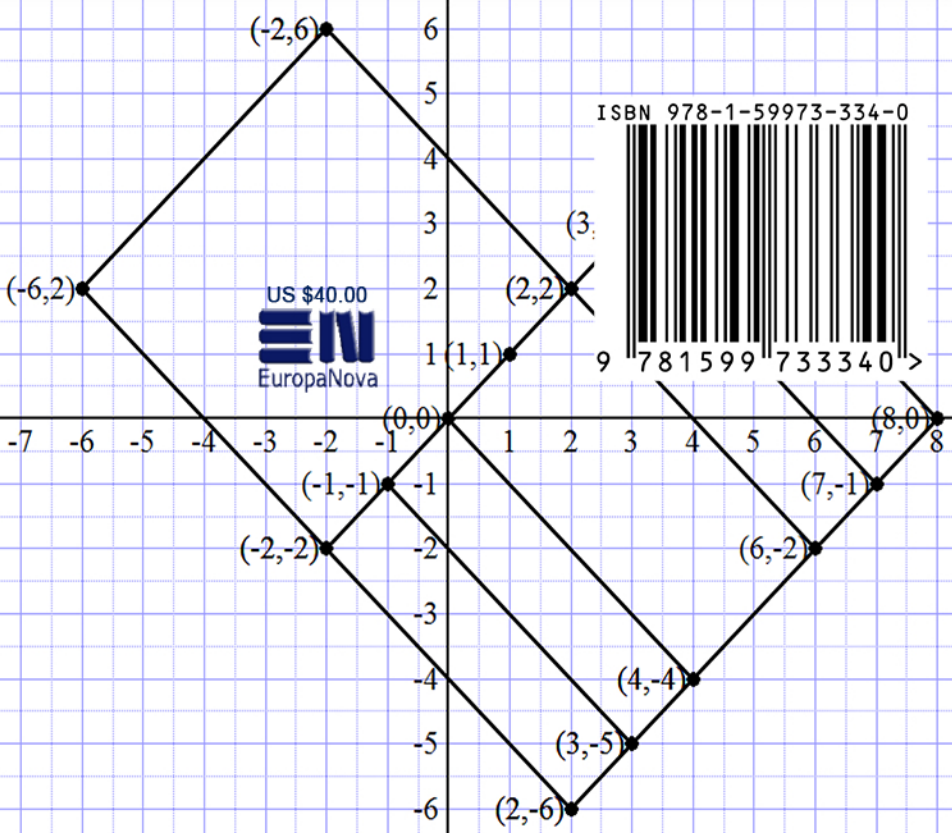Web Site: http://mat.iitm.ac.in/home/wbv/public_html/
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In this book authors for the first time study special types of Euclid squares in the real plane, complex plane, neutrosophic plane, dual number plane and their specializations. This study can be visualized as the blending of algebra, geometry and analysis.

(-2,6)
(-6,2)
(2,2)
(1,1)
(0,0)
(-1,-1)
(-2,-2)
(8,0)
(7,-1)
(6,-2)
(4,-4)
(3,-5)
(2,-6)