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Special Issue

Neutrosophic Information Theory and Applications

Special Issue Editors: Prof. Dr. Florentin Smarandache and Prof. Jun Ye

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Neutrosophic Information Theory and Applications

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Dr. Shu-Kun Lin
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Editorial

Summary of the Special Issue "Neutrosophic Information Theory and Applications" at "Information" Journal

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Abstract: Over a period of seven months (August 2017–February 2018), the Special Issue dedicated to "Neutrosophic Information Theory and Applications" by the "Information" journal (ISSN 2078-2489), located in Basel, Switzerland, was a success. The Guest Editors, Prof. Dr. Florentin Smarandache from the University of New Mexico (USA) and Prof. Dr. Jun Ye from the Shaoxing University (China), were happy to select—helped by a team of neutrosophic reviewers from around the world, and by the "Information" journal editors themselves—and publish twelve important neutrosophic papers, authored by 27 authors and coauthors. There were a variety of neutrosophic topics studied and used by the authors and coauthors in Multi-Criteria (or Multi-Attribute and/or Group) Decision-Making, including Cross Entropy-Based MAGDM, Neutrosophic Hesitant Fuzzy Prioritized Aggregation Operators, Biparametric Distance Measures, Pattern Recognition and Medical Diagnosis, Intuitionistic Neutrosophic Graph, NC-TODIM-Based MAGDM, Neutrosophic Cubic Set, VIKOR Method, Neutrosophic Multiple Attribute Group Decision-Making, Competition Graphs, Intuitionistic Neutrosophic Environment, Neutrosophic Commutative *N*-Ideals, Neutrosophic *N*-Structures Applied to BCK/BCI-Algebras, Neutrosophic Similarity Score, Weighted Histogram, Robust Mean-Shift Tracking, and Linguistic Neutrosophic Cubic Numbers.

Neutrosophic logic, symbolic logic, set, probability, statistics, etc., are, respectively, generalizations of fuzzy and intuitionistic fuzzy logic and set, classical and imprecise probability, classical statistics, and so on. Neutrosophic logic, symbol logic, and set are gaining significant attention in solving many real-life problems that involve uncertainty, impreciseness, vagueness, incompleteness, inconsistency, and indeterminacy. A number of new neutrosophic theories have been proposed and have been applied in computational intelligence, multiple-attribute decision making, image processing, medical diagnosis, fault diagnosis, optimization design, etc. This *Special Issue* gathers original research papers that report on the state of the art, as well as on recent advancements in neutrosophic information theory in soft computing, artificial intelligence, big and small data mining, decision-making problems, pattern recognition, information processing, image processing, and many other practical achievements.

In the first chapter (*NS-Cross Entropy-Based MAGDM under Single-Valued Neutrosophic Set Environment*), the authors Surapati Pramanik, Shyamal Dalapati, Shariful Alam, Florentin Smarandache, Tapan Kumar Roy propose a new cross entropy measure under a single-valued neutrosophic set (SVNS) environment, namely NS-cross entropy, and prove its basic properties. Additionally, they define the weighted NS-cross entropy measure, investigate its basic properties, and develop a novel multi-attribute group decision-making (MAGDM) strategy that is free from the drawbacks of asymmetrical behavior and undefined phenomena. It is capable of dealing with an unknown weight of attributes and an unknown weight of decision-makers. Finally, a numerical example of multi-attribute group decision-making problem of investment potential is solved to show the feasibility, validity and efficiency of the proposed decision-making strategy.

Single-valued neutrosophic hesitant fuzzy set (SVNHFS) is a combination of a single-valued neutrosophic set and a hesitant fuzzy set, and its aggregation tools play an important role in the multiple criteria decision-making (MCDM) process. The second paper (*Generalized Single-Valued Neutrosophic Hesitant Fuzzy Prioritized Aggregation Operators and Their Applications to Multiple Criteria Decision-Making*) investigates MCDM problems in which the criteria under SVNHF environment are in different priority levels. First, the generalized single-valued neutrosophic hesitant fuzzy prioritized weighted average operator and generalized single-valued neutrosophic hesitant fuzzy prioritized weighted geometric operator are developed based on the prioritized average operator. Second, some desirable properties and special cases of the proposed operators are discussed in detail. Third, an approach combining the proposed operators and the score function of single-valued neutrosophic hesitant fuzzy element is constructed to solve MCDM problems. Finally, the authors Rui Wang, Yanlai Li provide the example of investment selection to illustrate the validity and rationality of the proposed method.

Single-valued neutrosophic sets (SVNSs) handling the uncertainties characterized by truth, indeterminacy, and falsity membership degrees are a more flexible way of capturing uncertainty. In the third paper (*Some New Biparametric Distance Measures on Single-Valued Neutrosophic Sets with Applications to Pattern Recognition and Medical Diagnosis*), the authors Harish, Garg and Nancy propose some new types of distance measures, overcoming the shortcomings of the existing measures, for SVNSs with two parameters along with their proofs. The various desirable relations between the proposed measures are also derived. A comparison between the proposed and existing measures is performed in terms of counter-intuitive cases for showing its validity. The proposed measures are illustrated with case studies of pattern recognition, as well as medical diagnoses, along with the effect of the different parameters on the ordering of the objects.

A graph structure is a generalization of simple graphs. Graph structures are very useful tools for the study of different domains of computational intelligence and computer science. In the fourth research paper, *Certain Concepts in Intuitionistic Neutrosophic Graph Structures*, the authors Muhammad Akram and Muzzamal Sitara introduce certain notions of intuitionistic neutrosophic graph structures, illustrating these notions with several examples. They investigate some related properties of intuitionistic neutrosophic graph structures, and also present an application of intuitionistic neutrosophic graph structures.

A neutrosophic cubic set is the hybridization of the concept of a neutrosophic set and an interval neutrosophic set. A neutrosophic cubic set has the capacity to express the hybrid information of both the interval neutrosophic set and the single valued neutrosophic set simultaneously. Since the neutroaophic cubic sets have only recently been defined, not much research on the operations and applications of neutrosophic cubic sets is currently available in the literature. In the fifth paper, *NC-TODIM-Based MAGDM under a Neutrosophic Cubic Set Environment*, the authors Surapati Pramanik, Shyamal Dalapati, Shariful Alam and Tapan Kumar Roy propose score and accuracy functions for neutrosophic cubic sets and prove their basic properties. They also develop a strategy for ranking of neutrosophic cubic numbers based on the score and accuracy functions. The authors firstly develop a TODIM (Tomada de decisao interativa e multicritévio) in the neutrosophic cubic set (NC) environment, which is called the NC-TODIM. They establish a new NC-TODIM strategy for solving multi-attribute group decision-making (MAGDM) problems in neutrosophic cubic set environments. They illustrate the proposed NC-TODIM strategy for solving a multi-attribute group decision-making problem to show the applicability and effectiveness of the developed strategy. They also conduct sensitivity analysis to show the impact of the ranking order of the alternatives on the different values of the attenuation factor of losses for multi-attribute group decision-making strategies.

In the sixth paper, *VIKOR Method for Interval Neutrosophic Multiple Attribute Group Decision-Making*, the authors Yu-Han Huang, Gui-Wu Wei and Cun Wei extend the VIKOR method to multiple-attribute group decision-making (MAGDM) with interval neutrosophic numbers (INNs). Firstly, the basic concepts of INNs are briefly presented. The method first aggregates all individual decision-makers' assessment information based on an interval neutrosophic weighted averaging (INWA) operator, and then employs the extended classical VIKOR method to solve MAGDM problems with INNs. The validity and stability of this method are verified by example analysis and sensitivity analysis, and its superiority is illustrated by a comparison with the existing methods.

The concept of intuitionistic neutrosophic sets provides an additional possibility for representing imprecise, uncertain, inconsistent and incomplete information that exists in real situations. The seventh research article (*Certain Competition Graphs Based on Intuitionistic Neutrosophic Environment*) presents the notion of intuitionistic neutrosophic competition graphs. Then, the authors Muhammad Akram and Maryam Nasir discuss p-competition intuitionistic neutrosophic graphs and m-step intuitionistic neutrosophic competition graphs. Further, applications of intuitionistic neutrosophic competition graphs in ecosystem and career competition are described.

The notion of a neutrosophic commutative N-ideal in BCK-algebras is introduced in the eighth paper (*Neutrosophic Commutative N-Ideals in BCK-Algebras*), and several properties are investigated. Relations between a neutrosophic N-ideal and a neutrosophic commutative N-ideal are discussed by the authors Seok-Zun Song, Florentin Smarandache, and Young Bae Jun. Characterizations of a neutrosophic commutative N-ideal are considered.

Neutrosophic N-Structures Applied to BCK/BCI-Algebras is the title of the ninth paper. The notions of a neutrosophic N-subalgebra and a (closed) neutrosophic N-ideal in a BCK/BCI-algebra are introduced by authors Young Bae Jun, Florentin Smarandache and Hashem Bordbar, and several related properties are investigated. Characterizations of a neutrosophic N-subalgebra and a neutrosophic N-ideal are considered, and relations between a neutrosophic N-subalgebra and a neutrosophic N-ideal are stated. The conditions for a neutrosophic N-ideal being a closed neutrosophic N-ideal are provided.

Recently, TODIM has been used to solve multiple attribute decision making (MADM) problems. Single-valued neutrosophic sets (SVNSs) are useful tools for depicting the uncertainty of the MADM. In the tenth paper, *TODIM Method for Single-Valued Neutrosophic Multiple Attribute Decision Making*, Dong-Sheng Xu, Cun Wei and Gui-Wu Wei extend the TODIM method to the MADM with the single-valued neutrosophic numbers (SVNNs). Firstly, the definition, comparison, and distance of SVNNs are briefly presented, and the steps of the classical TODIM method for MADM problems are introduced. Then, an extended classical TODIM method is proposed for dealing with MADM problems with SVNNs, its significant characteristic being that it can fully consider the decision makers' bounded rationality, which is a real factor in decision-making. Furthermore, the authors extend the proposed model to interval neutrosophic sets (INSs). Finally, a numerical example is proposed.

Visual object tracking is a critical task in computer vision. Challenging things always exist when an object needs to be tracked. For instance, background clutter is one of the most challenging problems. The mean-shift tracker is quite popular because of its efficiency and performance under a range of conditions. However, the challenge of background clutter also disturbs its performance. In the eleventh article, *Neutrosophic Similarity Score Based Weighted Histogram for Robust Mean-Shift Tracking*, the authors Keli Hu, En Fan, Jun Ye, Changxing Fan, Shigen Shen and Yuzhang Gu propose a novel weighted histogram based on neutrosophic similarity score to help the mean-shift tracker discriminate the target from the background. The authors utilize the single-valued neutrosophic set (SVNS), which is a subclass of NS, to improve the mean-shift tracker. First, two kinds of criteria are considered—object feature similarity and background feature similarity—and each bin of the weight histogram is represented in the SVNS domain via three membership functions: T(Truth), I(indeterminacy), and F(Falsity). Second, the neutrosophic similarity score function is introduced to fuse those two criteria and to build the final weighted histogram. Finally, a novel neutrosophic weighted mean-shift tracker is proposed. The proposed tracker is compared with several mean-shift-based trackers on a dataset of 61 public sequences. The results reveal that this method outperforms other trackers, especially when confronting background clutter.

To describe both certain linguistic neutrosophic information and uncertain linguistic neutrosophic information simultaneously in the real world, Jun Ye proposes in the twelfth paper (*Linguistic Neutrosophic Cubic Numbers and Their Multiple Attribute Decision-Making Method*) the concept of a linguistic neutrosophic cubic number (LNCN), including an internal LNCN and external LNCN. In LNCN, its uncertain linguistic neutrosophic number consists of the truth, indeterminacy, and falsity uncertain linguistic variables, and its linguistic neutrosophic number consists of the truth, indeterminacy, and falsity linguistic variables to express their hybrid information. Then, the author presents the operational laws of LNCNs and the score, accuracy, and certain functions of LNCN for comparing/ranking LNCNs. Next, the author proposes a LNCN weighted arithmetic averaging (LNCNWAA) operator and a LNCN weighted geometric averaging (LNCNWGA) operator to aggregate linguistic neutrosophic cubic information and discuss their properties. Further, a multiple attribute decision-making method based on the LNCNWAA or LNCNWGA operator is developed under a linguistic neutrosophic cubic environment. Finally, an illustrative example is provided to indicate the application of the developed method.

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Article

NS-Cross Entropy-Based MAGDM under Single-Valued Neutrosophic Set Environment

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Abstract: A single-valued neutrosophic set has king power to express uncertainty characterized by indeterminacy, inconsistency and incompleteness. Most of the existing single-valued neutrosophic cross entropy bears an asymmetrical behavior and produces an undefined phenomenon in some situations. In order to deal with these disadvantages, we propose a new cross entropy measure under a single-valued neutrosophic set (SVNS) environment, namely NS-cross entropy, and prove its basic properties. Also we define weighted NS-cross entropy measure and investigate its basic properties. We develop a novel multi-attribute group decision-making (MAGDM) strategy that is free from the drawback of asymmetrical behavior and undefined phenomena. It is capable of dealing with an unknown weight of attributes and an unknown weight of decision-makers. Finally, a numerical example of multi-attribute group decision-making problem of investment potential is solved to show the feasibility, validity and efficiency of the proposed decision-making strategy.

Keywords: neutrosophic set; single-valued neutrosophic set; NS-cross entropy measure; multi-attribute group decision-making

1. Introduction

To tackle the uncertainty and modeling of real and scientific problems, Zadeh [1] first introduced the fuzzy set by defining membership measure in 1965. Bellman and Zadeh [2] contributed important research on fuzzy decision-making using max and min operators. Atanassov [3] established the intuitionistic fuzzy set (IFS) in 1986 by adding non-membership measure as an independent component to the fuzzy set. Theoretical and practical applications of IFSs in multi-criteria decision-making (MCDM) have been reported in the literature [4–12]. Zadeh [13] introduced entropy measure in the fuzzy environment. Burillo and Bustince [14] proposed distance measure between IFSs and offered an axiomatic definition of entropy measure. In the IFS environment, Szmidt and Kacprzyk [15] proposed a new entropy measure based on geometric interpretation of IFS. Wei et al. [16] developed an entropy measure for interval-valued intuitionistic fuzzy set (IVIFS) and presented its applications in pattern recognition and MCDM. Li [17] presented a new multi-attribute decision-making (MADM) strategy combining entropy and Technique for Order Preference by Similarity to Ideal Solution (TOPSIS) in an IVIFS environment. Shang and Jiang [18] introduced the cross entropy in the fuzzy environment. Vlachos and Sergiadis [19] presented intuitionistic fuzzy cross entropy by extending fuzzy cross entropy [18]. Ye [20] defined a new cross entropy under an IVIFS environment and presented an optimal decision-making strategy. Xia and Xu [21] put forward a new entropy and a cross entropy and employed them for multi-attribute criteria group decision-making (MAGDM) strategy under an IFS environment. Tong and Yu [22] defined cross entropy under an IVIFS environment and applied it to MADM problems.

The study of uncertainty took a new direction after the publication of the neutrosophic set (NS) [23] and single-valued neutrosophic set (SVNS) [24]. SVNS appeals more to researchers for its applicability in decision-making [25–54], conflict resolution [55], educational problems [56,57], image processing [58–60], cluster analysis [61,62], social problems [63,64], etc. The research on SVNS gained momentum after the inception of the international journal "Neutrosophic Sets and Systems". Combining with the neutrosophic set, a number of hybrid neutrosophic sets such as the neutrosophic soft set [65–72], the neutrosophic soft expert set [73–75], the neutrosophic complex set [76], the rough neutrosophic set [77–86], the rough neutrosophic tri complex set [87], the neutrosophic rough hyper complex set [88], the neutrosophic hesitant fuzzy sets/multi-valued neutrosophic set [89–97], the bipolar neutrosophic set [98–103], the rough bipolar neutrosophic set [104], the neutrosophic cubic set [105–113], and the neutrosophic cubic soft set [114,115] has been reported in the literature. Wang et al. [116] defined the interval neutrosophic set (INS). Different interval neutrosophic hybrid sets and their theoretical development and applications have been reported in the literature, such as the interval-valued neutrosophic soft set [117], the interval neutrosophic complex set [118], the interval neutrosophic rough set [119–121], and the interval neutrosophic hesitant fuzzy set [122]. Other extensions of neutrosophic sets, such as trapezoidal neutrosophic sets [123,124], normal neutrosophic sets [125], single-valued neutrosophic linguistic sets [126], interval neutrosophic linguistic sets [127,128], simplified neutrosophic linguistic sets [129], single-valued neutrosophic trapezoid linguistic sets [130], interval neutrosophic uncertain linguistic sets [131–133], neutrosophic refined sets [134–139], linguistic refined neutrosophic sets [140] bipolar neutrosophic refined sets [141], and dynamic single-valued neutrosophic multi-sets [142] have been proposed to enrich the study of neutrosophics. So the field of neutrosophic study has been steadily developing.

Majumdar and Samanta [143] defined an entropy measure and presented an MCDM strategy under SVNS environment. Ye [144] proposed cross entropy measure under the single-valued neutrosophic set environment, which is not symmetric straight forward and bears undefined phenomena. To overcome the asymmetrical behavior of the cross entropy measure, Ye [144] used a symmetric discrimination information measure for single-valued neutrosophic sets. Ye [145] defined cross entropy measures for SVNSs to overcome the drawback of undefined phenomena of the cross entropy measure [144] and proposed a MCDM strategy.

The aforementioned applications of cross entropy [144,145] can be effective in dealing with neutrosophic MADM problems. However, they also bear some limitations, which are outlined below:

i. The strategies [144,145] are capable of solving neutrosophic MADM problems that require the criterion weights to be completely known. However, it can be difficult and subjective to offer exact criterion weight information due to neutrosophic nature of decision-making situations.
ii. The strategies [144,145] have a single decision-making structure, and not enough attention is paid to improving robustness when processing the assessment information.
iii. The strategies [144,145] cannot deal with the unknown weight of the decision-makers.

Research gap:

MAGDM strategy based on cross entropy measure with unknown weight of attributes and unknown weight of decision-makers.

This study answers the following research questions:

i. Is it possible to define a new cross entropy measure that is free from asymmetrical phenomena and undefined behavior?

ii. Is it possible to define a new weighted cross entropy measure that is free from the asymmetrical phenomena and undefined behavior?
iii. Is it possible to develop a new MAGDM strategy based on the proposed cross entropy measure in single-valued neutrosophic set environment, which is free from the asymmetrical phenomena and undefined behavior?
iv. Is it possible to develop a new MAGDM strategy based on the proposed weighted cross entropy measure in the single-valued neutrosophic set environment that is free from the asymmetrical phenomena and undefined behavior?
v. How do we assign unknown weight of attributes?
vi. How do we assign unknown weight of decision-makers?

Motivation:

The above-mentioned analysis describes the motivation behind proposing a comprehensive NS-cross entropy-based strategy for tackling MAGDM under the neutrosophic environment. This study develops a novel NS-cross entropy-based MAGDM strategy that can deal with multiple decision-makers and unknown weight of attributes and unknown weight of decision-makers and free from the drawbacks that exist in [144,145].

The objectives of the paper are:

1. To define a new cross entropy measure and prove its basic properties, which are free from asymmetrical phenomena and undefined behavior.
2. To define a new weighted cross measure and prove its basic properties, which are free from asymmetrical phenomena and undefined behavior.
3. To develop a new MAGDM strategy based on weighted cross entropy measure under single-valued neutrosophic set environment.
4. To develop a technique to incorporate unknown weight of attributes and unknown weight of decision-makers in the proposed NS-cross entropy-based MAGDM under single-valued neutrosophic environment.

To fill the research gap, we propose NS-cross entropy-based MAGDM, which is capable of dealing with multiple decision-makers with unknown weight of the decision-makers and unknown weight of the attributes.

The main contributions of this paper are summarized below:

1. We define a new NS-cross entropy measure and prove its basic properties. It is straightforward symmetric and it has no undefined behavior.
2. We define a new weighted NS-cross entropy measure in the single-valued neutrosophic set environment and prove its basic properties. It is straightforward symmetric and it has no undefined behavior.
3. In this paper, we develop a new MAGDM strategy based on weighted NS cross entropy to solve MAGDM problems with unknown weight of the attributes and unknown weight of decision-makers.
4. Techniques to determine unknown weight of attributes and unknown weight of decisions makers are proposed in the study.

The rest of the paper is presented as follows: Section 2 describes some concepts of SVNS. In Section 3 we propose a new cross entropy measure between two SVNS and investigate its properties. In Section 4, we develop a novel MAGDM strategy based on the proposed NS-cross entropy with SVNS information. In Section 5 an illustrative example is solved to demonstrate the applicability and efficiency of the developed MAGDM strategy under SVNS environment. In Section 6 we present comparative study and discussion. Section 7 offers conclusions and the future scope of research.

2. Preliminaries

This section presents a short list of mostly known definitions pertaining to this paper.

Definition 1 [23] NS. *Let U be a space of points (objects) with a generic element in U denoted by u, i.e., $u \in U$. A neutrosophic set A in U is characterized by truth-membership measure $T_A(u)$, indeterminacy-membership measure $I_A(u)$ and falsity-membership measure $F_A(u)$, where $T_A(u)$, $I_A(u)$, $F_A(u)$ are the measures from U to $]^-0, 1^+$ [i.e., $T_A(u)$, $I_A(u)$, $F_A(u)$:$U \rightarrow]^-0, 1^+[$ NS can be expressed as A = {<u; ($T_A(u)$, $I_A(u)$, $F_A(u)$)>: $\forall$ u $\in$U}. Since $T_A(u)$, $I_A(u)$, $F_A(u)$ are the subsets of $]^-0, 1^+$ [there the sum ($T_A(u) + I_A(u) + F_A(u)$) lies between $^-0$ and 3^+.*

Example 1. *Suppose that U = {$u_1, u_2, u_3, \ldots$} be the universal set. Let R_1 be any neutrosophic set in U. Then R_1 expressed as R_1 = {<u_1; (0.6, 0.3, 0.4)>: $u_1 \in U$}.*

Definition 2 [24] SVNS. *Assume that U be a space of points (objects) with generic elements $u \in U$. A SVNS H in U is characterized by a truth-membership measure $T_H(u)$, an indeterminacy-membership measure $I_H(u)$, and a falsity-membership measure $F_H(u)$, where $T_H(u)$, $I_H(u)$, $F_H(u) \in [0, 1]$ for each point u in U. Therefore, a SVNS A can be expressed as H = {u, (T_H (u), I_H (u), F_H (u)) ∣ $\forall u \in U$}, whereas, the sum of $T_H(u)$, $I_H(u)$ and $F_H(u)$ satisfy the condition $0 \leq T_H(u) + I_H(u) + F_H(u) \leq 3$ and H(u) = <(T_H (u), I_H (u), F_H (u)> call a single-valued neutrosophic number (SVNN).*

Example 2. *Suppose that U = {$u_1, u_2, u_3, \ldots$} be the universal set. A SVNS H in U can be expressed as: H = {u_1, (0.7, 0.3, 0.5) ∣ $u_1 \in U$} and SVNN presented H = <0.7, 0.3, 0.5>.*

Definition 3 [24] Inclusion of SVNSs. *The inclusion of any two SVNS sets H_1 and H_2 in U is denoted by $H_1 \subseteq H_2$ and defined as follows:*

$$H_1 \subseteq H_2,\ T_{H_1}(u)\ \leq\ T_{H_2}(u),\ I_{H_1}(u) \geq I_{H_2}(u),\ F_{H_1}(u) \geq F_{H_2}(u)\ iff\ for\ all\ u \in U.$$

Example 3. *Let H_1 and H_2 be any two SVNNs in U presented as follows: H_1 = <(0.7, 0.3, 0.5)> and H_2 = <(0.8, 0.2, 0.4)> for all $u \in U$. Using the property of inclusion of two SVNNs, we conclude that $H_1 \subseteq H_2$.*

Definition 4 [24] Equality of two SVNSs. *The equality of any two SVNS H_1 and H_2 in U denoted by $H_1 = H_2$ and defined as follows:*

$$T_{H_1}(u) = T_{H_2}(u),\ I_{H_1}(u) = I_{H_2}(u)\ and\ F_{H_1}(u) = F_{H_2}(u)\ for\ all\ u \in U.$$

Definition 5 Complement of any SVNSs. *The complement of any SVNS H in U denoted by H^c and defined as follows:*

$$H^c\ =\ \{u,\ 1 - T_H, 1 - I_H,\ 1 - F_H \mid u \in U\}.$$

Example 4. *Let H be any SVNN in U presented as follows: H = < (0.7, 0.3, 0.5) >. Then compliment of H is obtained as H^c = <(0.3, 0.7, 0.5)>.*

Definition 6 [24] Union. *The union of two single-valued neutrosophic sets H_1 and H_2 is a neutrosophic set H_3 (say) written as*

$H_3 = H_1 \cup H_2$.
$T_{H_3}(u)$ = max {$T_{H_1}(u)$, $T_{H_2}(u)$}, $I_{HJ_3}(u)$ = min {$I_{H_1}(u)$, $I_{H_2}(u)$}, F_{H_3} (u) = min {$F_{H_1}(u)$, $F_{H_2}(u)$}, $\forall$ u $\in$ U.

Example 5. *Let H_1 and H_2 be two SVNSs in U presented as follows:*

$$H_1 = <(0.6, 0.3, 0.4)> \text{ and } H_2 = <(0.7, 0.3, 0.6)>. \text{ Then union of them is presented as:}$$

$$H_1 \cup H_2 = < (0.7, 0.3, 0.4) > .$$

Definition 7 [24] Intersection. *The intersection of two single-valued neutrosophic sets H_1 and H_2 denoted by H_4 and defined as*

$H_4 = H_1 \cap H_2$
$T_{H_4}(u) = min \{T_{H_1}(u), T_{H_2}(u)\}$, $I_{H_4}(u) = max\{I_{H_1}(u), I_{H_2}(u)\}$
$F_{H_4}(u) = max \{F_{H_1}(u), F_{H_2}(u)\}$, $\forall\, u \in U$.

Example 6. *Let H_1 and H_2 be two SVNSs in U presented as follows:*

$$H_1 = <(0.6, 0.3, 0.4)> \text{ and } H_2 = <(0.7, 0.3, 0.6)>.$$

Then intersection of H_1 and H_2 is presented as follows:

$$H_1 \cap H_2 = <(0.6, 0.3, 0.6)>$$

3. NS-Cross Entropy Measure

In this section, we define a new single-valued neutrosophic cross-entropy measure for measuring the deviation of single-valued neutrosophic variables from an a priori one.

Definition 8 NS-cross entropy measure. *Let H_1 and H_2 be any two SVNSs in $U = \{u_1, u_2, u_3, \ldots, u_n\}$. Then, the single-valued cross-entropy of H_1 and H_2 is denoted by CE_{NS} (H_1, H_2) and defined as follows:*

$$\begin{aligned}
\mathrm{CE_{NS}}\,(\mathrm{H_1}, \mathrm{H_2}) = \frac{1}{2}\Bigg\{ \sum_{i=1}^{n} \Bigg\langle \Bigg[& \frac{2\,|T_{H_1}(u_i) - T_{H_2}(u_i)|}{\sqrt{1+|T_{H_1}(u_i)|^2} + \sqrt{1+|T_{H_2}(u_i)|^2}} + \frac{2\big|(1-T_{H_1}(u_i)) - (1-T_{H_2}(u_i))\big|}{\sqrt{1+\big|(1-T_{H_1}(u_i))\big|^2} + \sqrt{1+\big|(1-T_{H_2}(u_i))\big|^2}} \Bigg] + \\
& \Bigg[\frac{2|I_{H_1}(u_i) - I_{H_2}(u_i)|}{\sqrt{1+|I_{H_1}(u_i)|^2} + \sqrt{1+|I_{H_2}(u)|^2}} + \frac{2\big|(1-I_{H_1}(u_i)) - (1-I_{H_2}(u_i))\big|}{\sqrt{1+\big|(1-I_{H_1}(u_i))\big|^2} + \sqrt{1+\big|(1-I_{H_2}(u_i))\big|^2}} \Bigg] + \\
& \Bigg[\frac{2|F_{H_1}(u_i) - F_{H_2}(u_i)|}{\sqrt{1+|F_{H_1}(u_i)|^2} + \sqrt{1+|F_{H_2}(u_i)|^2}} + \frac{2\big|(1-F_{H_1}(u_i)) - (1-F_{H_2}(u_i))\big|}{\sqrt{1+\big|(1-F_{H_1}(u_i))\big|^2} + \sqrt{1+\big|(1-F_{H_2}(u_i))\big|^2}} \Bigg] \Bigg\rangle \Bigg\}
\end{aligned} \tag{1}$$

Example 7. *Let H_1 and H_2 be two SVNSs in U, which are given by $H_1 = \{u, (0.7, 0.3, 0.4) \mid u \in U\}$ and $H_2 = \{u, (0.6, 0.4, 0.2) \mid u \in U\}$. Using Equation (1), the cross entropy value of H_1 and H_2 is obtained as $\mathrm{CE_{NS}}(\mathrm{H_1}, \mathrm{H_2}) = 0.707$.*

Theorem 1. *Single-valued neutrosophic cross entropy $\mathrm{CE_{NS}}(\mathrm{H_1}, \mathrm{H_2})$ for any two SVNSs $\mathrm{H_1}, \mathrm{H_2}$, satisfies the following properties:*

i. $\mathrm{CE_{NS}}(\mathrm{H_1}, \mathrm{H_2}) \geq 0$.
ii. $\mathrm{CE_{NS}}(\mathrm{H_1}, \mathrm{H_2}) = 0$ if and only if $T_{H_1}(u_i) = T_{H_2}(u_i)$, $I_{H_1}(u_i) = I_{H_2}(u_i)$, $F_{H_1}(u_i) = F_{H_2}(u_i)$, $\forall\, u_i \in U$.
iii. $\mathrm{CE_{NS}}(\mathrm{H_1}, \mathrm{H_2}) = \mathrm{CE_{NS}}\,(\mathrm{H_1^c}, \mathrm{H_2^c})$
iv. $\mathrm{CE_{NS}}\,(\mathrm{H_1}, \mathrm{H_2}) = \mathrm{CE_{NS}}\,(\mathrm{H_2}, \mathrm{H_1})$

Proof. (i) For all values of $u_i \in U$, $\left|T_{H_1}(u_i)\right| \geq 0$, $\left|T_{H_2}(u_i)\right| \geq 0$, $\left|T_{H_1}(u_i) - T_{H_2}(u_i)\right| \geq 0$, $\sqrt{1+\left|T_{H_1}(u_i)\right|^2} \geq 0$, $\sqrt{1+\left|T_{H_2}(u_i)\right|^2} \geq 0$, $\left|(1-T_{H_1}(u_i))\right| \geq 0$, $\left|(1-T_{H_2}(u_i))\right| \geq 0$, $\left|(1-T_{H_1}(u_i)) - (1-T_{H_2}(u_i))\right| \geq 0$, $\sqrt{1+\left|(1-T_{H_1}(u_i))\right|^2} \geq 0$, $\sqrt{1+\left|(1-T_{H_2}(u_i))\right|^2} \geq 0$.

Then, $\left[\frac{2\left|T_{H_1}(u_i)-T_{H_2}(u_i)\right|}{\sqrt{1+\left|T_{H_1}(u_i)\right|^2}+\sqrt{1+\left|T_{H_2}(u_i)\right|^2}} + \frac{2\left|(1-T_{H_1}(u_i))-(1-T_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-T_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{H_2}(u_i))\right|^2}}\right] \geq 0.$

Similarly, $\left[\frac{2\left|I_{H_1}(u_i)-I_{H_2}(u_i)\right|}{\sqrt{1+\left|I_{H_1}(u_i)\right|^2}+\sqrt{1+\left|I_{H_2}(u)\right|^2}} + \frac{2\left|(1-I_{H_1}(u_i))-(1-I_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-I_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{H_2}(u_i))\right|^2}}\right] \geq 0$, and $\left[\frac{2\left|F_{H_1}(u_i)-F_{H_2}(u_i)\right|}{\sqrt{1+\left|F_{H_1}(u_i)\right|^2}+\sqrt{1+\left|F_{H_2}(u_i)\right|^2}} + \frac{2\left|(1-F_{H_1}(u_i))-(1-F_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-F_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{H_2}(u_i))\right|^2}}\right] \geq 0.$

Therefore, $CE_{NS}(H_1, H_2) \geq 0$.

Hence complete the proof.

(ii) $\left[\frac{2\left|T_{H_1}(u_i)-T_{H_2}(u_i)\right|}{\sqrt{1+\left|T_{H_1}(u_i)\right|^2}+\sqrt{1+\left|T_{H_2}(u_i)\right|^2}} + \frac{2\left|(1-T_{H_1}(u_i))-(1-T_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-T_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{H_2}(u_i))\right|^2}}\right] = 0, \Leftrightarrow T_{H_1}(u_i) = T_{H_2}(u_i)$, $\left[\frac{2\left|I_{H_1}(u_i)-I_{H_2}(u_i)\right|}{\sqrt{1+\left|I_{H_1}(u_i)\right|^2}+\sqrt{1+\left|I_{H_2}(u)\right|^2}} + \frac{2\left|(1-I_{H_1}(u_i))-(1-I_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-I_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{H_2}(u_i))\right|^2}}\right] = 0 \Leftrightarrow I_{H_1}(u_i) = I_{H_2}(u_i)$, and, $\left[\frac{2\left|F_{H_1}(u_i)-F_{H_2}(u_i)\right|}{\sqrt{1+\left|F_{H_1}(u_i)\right|^2}+\sqrt{1+\left|F_{H_2}(u_i)\right|^2}} + \frac{2\left|(1-F_{H_1}(u_i))-(1-F_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-F_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{H_2}(u_i))\right|^2}}\right] = 0, \Leftrightarrow F_{H_1}(u_i) = F_{H_2}(u_i)$

Therefore, $CE_{NS}(H_1, H_2) = 0$, iff $T_{H_1}(u_i) = T_{H_2}(u_i)$, $I_{H_1}(u_i) = I_{H_2}(u_i)$, $F_{H_1}(u_i) = F_{H_2}(u_i)$, $\forall\, u_i \in U$.

Hence complete the proof.

(iii) Using Definition 5, we obtain the following expression

$$
\begin{aligned}
CE_{NS}(H_1^c, H_2^c) = \frac{1}{2}\Bigg\{\sum_{i=1}^{n}\Bigg\langle & \left[\frac{2\left|(1-T_{H_1}(u_i))-(1-T_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-T_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{H_2}(u_i))\right|^2}} + \frac{2\left|T_{H_1}(u_i)-T_{H_2}(u_i)\right|}{\sqrt{1+\left|T_{H_1}(u_i)\right|^2}+\sqrt{1+\left|T_{H_2}(u_i)\right|^2}}\right] + \\
& \left[\frac{2\left|(1-I_{H_1}(u_i))-(1-I_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-I_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{H_2}(u_i))\right|^2}} + \frac{2\left|I_{H_1}(u_i)-I_{H_2}(u_i)\right|}{\sqrt{1+\left|I_{H_1}(u_i)\right|^2}+\sqrt{1+\left|I_{H_2}(u)\right|^2}}\right] + \\
& \left[\frac{2\left|(1-F_{H_1}(u_i))-(1-F_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-F_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{H_2}(u_i))\right|^2}} + \frac{2\left|F_{H_1}(u_i)-F_{H_2}(u_i)\right|}{\sqrt{1+\left|F_{H_1}(u_i)\right|^2}+\sqrt{1+\left|F_{H_2}(u_i)\right|^2}}\right]\Bigg\rangle\Bigg\} \\
= \frac{1}{2}\Bigg\{\sum_{i=1}^{n}\Bigg\langle & \left[\frac{2\left|T_{H_1}(u_i)-T_{H_2}(u_i)\right|}{\sqrt{1+\left|T_{H_1}(u_i)\right|^2}+\sqrt{1+\left|T_{H_2}(u_i)\right|^2}} + \frac{2\left|(1-T_{H_1}(u_i))-(1-T_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-T_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{H_2}(u_i))\right|^2}}\right] + \\
& \left[\frac{2\left|I_{H_1}(u_i)-I_{H_2}(u_i)\right|}{\sqrt{1+\left|I_{H_1}(u_i)\right|^2}+\sqrt{1+\left|I_{H_2}(u)\right|^2}} + \frac{2\left|(1-I_{H_1}(u_i))-(1-I_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-I_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{H_2}(u_i))\right|^2}}\right] + \\
& \left[\frac{2\left|F_{H_1}(u_i)-F_{H_2}(u_i)\right|}{\sqrt{1+\left|F_{H_1}(u_i)\right|^2}+\sqrt{1+\left|F_{H_2}(u_i)\right|^2}} + \frac{2\left|(1-F_{H_1}(u_i))-(1-F_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-F_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{H_2}(u_i))\right|^2}}\right]\Bigg\rangle\Bigg\} = CE_{SN}(H_1, H_2)
\end{aligned}
$$

Therefore, $CE_{NS}(H_1, H_2) = CE_{NS}(H_1^c, H_2^c)$.

Hence complete the proof.

(iv) Since, $\left|T_{H_1}(u_i) - T_{H_2}(u_i)\right| = \left|T_{H_2}(u_i) - T_{H_1}(u_i)\right|$, $\left|I_{H_1}(u_i) - I_{H_2}(u_i)\right| = \left|I_{H_2}(u_i) - I_{H_1}(u_i)\right|$, $\left|F_{H_1}(u_i) - F_{H_2}(u_i)\right| = \left|F_{H_2}(u_i) - F_{H_1}(u_i)\right|$, $\left|(1-T_{H_1}(u_i)) - (1-T_{H_2}(u_i))\right| = \left|(1-T_{H_2}(u_i)) - (1-T_{H_1}(u_i))\right|$, $\left|(1-I_{H_1}(u_i)) - (1-I_{H_2}(u_i))\right| = \left|(1-I_{H_2}(u_i)) - (1-I_{H_1}(u_i))\right|$,

$\left|(1-F_{H_1}(u_i))-(1-F_{H_2}(u_i))\right| = \left|(1-F_{H_2}(u_i))-(1-F_{H_1}(u_i))\right|$, then, $\sqrt{1+|T_{H_1}(u_i)|^2}+\sqrt{1+|T_{H_2}(u_i)|^2}=\sqrt{1+|T_{H_2}(u_i)|^2}+\sqrt{1+|T_{H_1}(u_i)|^2}$, $\sqrt{1+|I_{H_1}(u_i)|^2}+\sqrt{1+|I_{H_2}(u_i)|^2}=\sqrt{1+|I_{H_2}(u_i)|^2}+\sqrt{1+|I_{H_1}(u_i)|^2}$, $\sqrt{1+|F_{H_1}(u_i)|^2}+\sqrt{1+|F_{H_2}(u_i)|^2}=\sqrt{1+|F_{H_2}(u_i)|^2}+\sqrt{1+|F_{H_1}(u_i)|^2}$, $\sqrt{1+\left|(1-T_{H_1}(u_i))\right|^2}+\sqrt{1+|(1-T_{H_2}(u_i))|^2}=\sqrt{1+|(-T_{H_2}(u_i)\ )|^2}+\sqrt{1+|(1-T_{H_1}(u_i))|^2}$, $\sqrt{1+|(1-I_{H_1}(u_i))|^2}+\sqrt{1+|(1-I_{H_2}(u_i))|^2}=\sqrt{1+|(1-I_{H_2}(u_i))|^2}+\sqrt{1+|(1-I_{H_1}(u_i))|^2}$, $\sqrt{1+\left|(1-F_{H_1}(u_i))\right|^2}+\sqrt{1+|(1-F_{H_2}(u_i))|^2}=\sqrt{1+|(1-F_{H_2}(u_i))|^2}+\sqrt{1+|(1-F_{H_1}(u_i))|^2}$, $\forall\, u_i \in U$.

Therefore, $\mathrm{CE_{NS}}(\mathrm{H_1},\ \mathrm{H_2}) = \mathrm{CE_{NS}}\ (\mathrm{H_2},\ \mathrm{H_1})$.

Hence complete the proof. □

Definition 9 Weighted NS-cross entropy measure. *We consider the weight w_i (i = 1, 2, ..., n) for the element u_i (i = 1, 2, .., n) with the conditions* $\mathrm{w_i} \in [0,\ 1]$ *and* $\sum_{i=1}^{n} w_i = 1$.

Then the weighted cross entropy between SVNSs H_1 and H_2 can be defined as follows:

$$\mathrm{CE_{NS}^{w}}\ (\mathrm{H_1},\ \mathrm{H_2}) = \frac{1}{2}\left\langle \sum_{i\ =\ 1}^{n} w_i\left\{\left[\frac{2\left|T_{H_1}(u_i)-T_{H_2}(u_i)\right|}{\sqrt{1+\left|T_{H_1}(u_i)\right|^2}+\sqrt{1+\left|T_{H_2}(u_i)\right|^2}}+\frac{2\left|(1-T_{H_1}(u_i))-(1-T_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-T_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{H_2}(u_i))\right|^2}}\right]+\right.\right.$$
$$\left.\left.\left[\frac{2\left|I_{H_1}(u_i)-I_{H_2}(u_i)\right|}{\sqrt{1+\left|I_{H_1}(u_i)\ \right|^2}+\sqrt{1+\left|I_{H_2}(u)\right|^2}}+\frac{2\left|(1-I_{H_1}(u_i))\ -(1-I_{H_2}(u_i))\right|}{\sqrt{1+\left|(1\ -I_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{H_2}(u_i))\right|^2}}\right]+\left[\frac{2\left|F_{H_1}(u_i)-F_{H_2}(u_i)\right|}{\sqrt{1+\left|F_{H_1}(u_i)\right|^2}+\sqrt{1+\left|F_{H_2}(u_i)\right|^2}}+\frac{2\left|(1-F_{H_1}(u_i))-(1-F_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-F_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{H_2}(u_i))\right|^2}}\right]\right\}\right\rangle \tag{2}$$

Theorem 2. *Single-valued neutrosophic weighted NS-cross-entropy (defined in Equation (2)) satisfies the following properties:*

i. $\mathrm{CE_{NS}^{w}}\ (\mathrm{H_1},\ \mathrm{H_2}) \geq 0$.
ii. $\mathrm{CE_{NS}^{w}}\ (\mathrm{H_1},\ \mathrm{H_2}) = 0$, if and only if $T_{H_1}(u_i) = T_{H_2}(u_i)\ I_{H_1}(u_i) = I_{H_2}(u_i)$, $F_{H_1}(u_i) = F_{H_2}(u_i)$, $\forall\, u_i \in U$.
iii. $\mathrm{CE_{NS}^{w}}\ (\mathrm{H_1},\ \mathrm{H_2}) = \mathrm{CE_{NS}^{w}}\ (\mathrm{H_1^c},\ \mathrm{H_2^c})$
iv. $\mathrm{CE_{NS}^{w}}\ (\mathrm{H_1},\ \mathrm{H_2}) = \mathrm{CE_{NS}^{w}}\ (\ \mathrm{H_2},\ \mathrm{H_1})$

Proof. (i). For all values of $u_i \in U$, $|T_{H_1}(u_i)\ | \geq 0\ |T_{H_2}(u_i)\ | \geq 0$, $|T_{H_1}(u_i)-T_{H_2}(u_i)| \geq 0$, $\sqrt{1+|T_{H_1}(u_i)|^2} \geq 0$, $\sqrt{1+|T_{H_2}(u_i)|^2} \geq 0$, $\left|(1-T_{H_1}(u_i))\right| \geq 0$, $|(1-T_{H_2}(u_i))\ | \geq 0$, $\left|(1-T_{H_1}(u_i))-(1-T_{H_2}(u_i))\right| \geq 0$, $\sqrt{1+\left|(1-T_{H_1}(u_i))\right|^2} \geq 0$, $\sqrt{1+\left|(1-\ T_{H_2}(u_i))\ \right|^2} \geq 0$, then,

$$\left[\frac{2\left|T_{H_1}(u_i)-T_{H_2}(u_i)\right|}{\sqrt{1+\left|T_{H_1}(u_i)\right|^2}+\sqrt{1+\left|T_{H_2}(u_i)\right|^2}}+\frac{2\left|(1-T_{H_1}(u_i))-(1-T_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-T_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{H_2}(u_i))\right|^2}}\right] \geq 0.$$

Similarly, $\left[\frac{2\left|I_{H_1}(u_i)-I_{H_2}(u_i)\right|}{\sqrt{1+\left|I_{H_1}(u_i)\right|^2}+\sqrt{1+\left|I_{H_2}(u)\right|^2}}+\frac{2\left|(1-I_{H_1}(u_i))-(1-I_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-I_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{H_2}(u_i))\right|^2}}\right] \geq 0$,

and $\left[\frac{2\left|F_{H_1}(u_i)-F_{H_2}(u_i)\right|}{\sqrt{1+\left|F_{H_1}(u_i)\right|^2}+\sqrt{1+\left|F_{H_2}(u_i)\right|^2}}+\frac{2\left|(1-F_{H_1}(u_i))-(1-F_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-F_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{H_2}(u_i))\right|^2}}\right] \geq 0$.

Since $w_i \in [0,\ 1]$ and $\sum_{i=1}^{n} w_i = 1$, therefore, $\mathrm{CE_{NS}^{w}}\ (\mathrm{H_1},\ \mathrm{H_2}) \geq 0$.

Hence complete the proof.

(ii) Since, $\left[\frac{2\left|T_{H_1}(u_i)-T_{H_2}(u_i)\right|}{\sqrt{1+\left|T_{H_1}(u_i)\right|^2}+\sqrt{1+\left|T_{H_2}(u_i)\right|^2}}+\frac{2\left|(1-T_{H_1}(u_i))-(1-T_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-T_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{H_2}(u_i))\right|^2}}\right]=0, \Leftrightarrow T_{H_1}(u_i)=T_{H_2}(u_i)$,
$\left[\frac{2\left|I_{H_1}(u_i)-I_{H_2}(u_i)\right|}{\sqrt{1+\left|I_{H_1}(u_i)\right|^2}+\sqrt{1+\left|I_{H_2}(u)\right|^2}}+\frac{2\left|(1-I_{H_1}(u_i))-(1-I_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-I_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{H_2}(u_i))\right|^2}}\right]=0, \Leftrightarrow I_{H_1}(u_i)=I_{H_2}(u_i)$,
$\left[\frac{2\left|F_{H_1}(u_i)-F_{H_2}(u_i)\right|}{\sqrt{1+\left|F_{H_1}(u_i)\right|^2}+\sqrt{1+\left|F_{H_2}(u_i)\right|^2}}+\frac{2\left|(1-F_{H_1}(u_i))-(1-F_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-F_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{H_2}(u_i))\right|^2}}\right]=0, \Leftrightarrow F_{H_1}(u_i)=F_{H_2}(u_i)$
and $w_i \in [0,\ 1]$, $\sum_{i=1}^{n} w_i = 1$, $w_i \geq 0$. Therefore, $\mathrm{CE}_{\mathrm{NS}}^{\mathrm{w}}(\mathrm{H}_1, \mathrm{H}_2)=0$ iff $T_{H_1}(u_i)=T_{H_2}(u_i)$, $I_{H_1}(u_i)=I_{H_2}(u_i)$, $F_{H_1}(u_i)=F_{H_2}(u_i)$, $\forall\, u_i \in U$.

Hence complete the proof.

(iii) Using Definition 5, we obtain the following expression

$$\begin{aligned}
&\mathrm{CE}_{\mathrm{NS}}^{\mathrm{w}}(\mathrm{H}_1^{\mathrm{c}}, \mathrm{H}_2^{\mathrm{c}})=\frac{1}{2}\left\{\sum_{i=1}^{n} w_i\left\langle\left[\frac{2\left|(1-T_{H_1}(u_i))-(1-T_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-T_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{H_2}(u_i))\right|^2}}+\frac{2\left|T_{H_1}(u_i)-T_{H_2}(u_i)\right|}{\sqrt{1+\left|T_{H_1}(u_i)\right|^2}+\sqrt{1+\left|T_{H_2}(u_i)\right|^2}}\right]+\right.\right.\\
&\left[\frac{2\left|(1-I_{H_1}(u_i))-(1-I_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-I_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{H_2}(u_i))\right|^2}}+\frac{2\left|I_{H_1}(u_i)-I_{H_2}(u_i)\right|}{\sqrt{1+\left|I_{H_1}(u_i)\right|^2}+\sqrt{1+\left|I_{H_2}(u)\right|^2}}\right]+\\
&\left.\left.\left[\frac{2\left|(1-F_{H_1}(u_i))-(1-F_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-F_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{H_2}(u_i))\right|^2}}+\frac{2\left|F_{H_1}(u_i)-F_{H_2}(u_i)\right|}{\sqrt{1+\left|F_{H_1}(u_i)\right|^2}+\sqrt{1+\left|F_{H_2}(u_i)\right|^2}}\right]\right\rangle\right\}\\
&=\frac{1}{2}\left\{\sum_{i=1}^{n} w_i\left\langle\left[\frac{2\left|T_{H_1}(u_i)-T_{H_2}(u_i)\right|}{\sqrt{1+\left|T_{H_1}(u_i)\right|^2}+\sqrt{1+\left|T_{H_2}(u_i)\right|^2}}+\frac{2\left|(1-T_{H_1}(u_i))-(1-T_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-T_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{H_2}(u_i))\right|^2}}\right]+\right.\right.\\
&\left[\frac{2\left|I_{H_1}(u_i)-I_{H_2}(u_i)\right|}{\sqrt{1+\left|I_{H_1}(u_i)\right|^2}+\sqrt{1+\left|I_{H_2}(u)\right|^2}}+\frac{2\left|(1-I_{H_1}(u_i))-(1-I_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-I_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{H_2}(u_i))\right|^2}}\right]+\\
&\left.\left.\left[\frac{2\left|F_{H_1}(u_i)-F_{H_2}(u_i)\right|}{\sqrt{1+\left|F_{H_1}(u_i)\right|^2}+\sqrt{1+\left|F_{H_2}(u_i)\right|^2}}+\frac{2\left|(1-F_{H_1}(u_i))-(1-F_{H_2}(u_i))\right|}{\sqrt{1+\left|(1-F_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{H_2}(u_i))\right|^2}}\right]\right\rangle\right\}=\mathrm{CE}_{\mathrm{NS}}^{\mathrm{w}}(\mathrm{H}_1, \mathrm{H}_2)
\end{aligned}$$

Therefore, $\mathrm{CE}_{\mathrm{NS}}^{\mathrm{w}}(\mathrm{H}_1, \mathrm{H}_2)=\mathrm{CE}_{\mathrm{NS}}^{\mathrm{w}}(\mathrm{H}_1^{\mathrm{c}}, \mathrm{H}_2^{\mathrm{c}})$.

Hence complete the proof.

(iv) Since $\left|T_{H_1}(u_i)-T_{H_2}(u_i)\right|=\left|T_{H_2}(u_i)-T_{H_1}(u_i)\right|$, $\left|I_{H_1}(u_i)-I_{H_2}(u_i)\right|=\left|I_{H_2}(u_i)-I_{H_1}(u_i)\right|$, $\left|F_{H_1}(u_i)-F_{H_2}(u_i)\right|=\left|F_{H_2}(u_i)-F_{H_1}(u_i)\right|$, $\left|(1-T_{H_1}(u_i))-(1-T_{H_2}(u_i))\right|=\left|(1-T_{H_2}(u_i))-(1-T_{H_1}(u_i))\right|$, $\left|(1-I_{H_1}(u_i))-(1-I_{H_2}(u_i))\right|=\left|(1-I_{H_2}(u_i))-(1-I_{H_1}(u_i))\right|$, $\left|(1-F_{H_1}(u_i))-(1-F_{H_2}(u_i))\right|=\left|(1-F_{H_2}(u_i))-(1-F_{H_1}(u_i))\right|$, we obtain $\sqrt{1+\left|T_{H_1}(u_i)\right|^2}+\sqrt{1+\left|T_{H_2}(u_i)\right|^2}=\sqrt{1+\left|T_{H_2}(u_i)\right|^2}+\sqrt{1+\left|T_{H_1}(u_i)\right|^2}$, $\sqrt{1+\left|I_{H_1}(u_i)\right|^2}+\sqrt{1+\left|I_{H_2}(u_i)\right|^2}=\sqrt{1+\left|I_{H_2}(u_i)\right|^2}+\sqrt{1+\left|I_{H_1}(u_i)\right|^2}$, $\sqrt{1+\left|F_{H_1}(u_i)\right|^2}+\sqrt{1+\left|F_{H_2}(u_i)\right|^2}=\sqrt{1+\left|F_{H_2}(u_i)\right|^2}+\sqrt{1+\left|F_{H_1}(u_i)\right|^2}$, $\sqrt{1+\left|(1-T_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{H_2}(u_i))\right|^2}=\sqrt{1+\left|(-T_{H_2}(u_i))\right|^2}+\sqrt{1+\left|(1-T_{H_1}(u_i))\right|^2}$, $\sqrt{1+\left|(1-I_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{H_2}(u_i))\right|^2}=\sqrt{1+\left|(1-I_{H_2}(u_i))\right|^2}+\sqrt{1+\left|(1-I_{H_1}(u_i))\right|^2}$, $\sqrt{1+\left|(1-F_{H_1}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{H_2}(u_i))\right|^2}=\sqrt{1+\left|(1-F_{H_2}(u_i))\right|^2}+\sqrt{1+\left|(1-F_{H_1}(u_i))\right|^2}$, $\forall\, u_i \in U$ and $\mathrm{w_i} \in [0,\ 1]$, $\sum_{i=1}^{n} w_i = 1$.

Therefore, $\mathrm{CE}_{\mathrm{NS}}^{\mathrm{w}}(\mathrm{H}_1, \mathrm{H}_2)=\mathrm{CE}_{\mathrm{NS}}^{\mathrm{w}}(\mathrm{H}_2, \mathrm{H}_1)$.

Hence complete the proof. □

4. MAGDM Strategy Using Proposed Ns-Cross Entropy Measure under SVNS Environment

In this section, we develop a new MAGDM strategy using the proposed NS-cross entropy measure.

Description of the MAGDM Problem

Assume that $A = \{A_1, A_2, A_3, \ldots, A_m\}$ and $G = \{G_1, G_2, G_3, \ldots, G_n\}$ be the discrete set of alternatives and attributes respectively and $W = \{w_1, w_2, w_3, \ldots, w_n\}$ be the weight vector of attributes $G_j (j = 1, 2, 3, \ldots, n)$, where $w_j \geq 0$ and $\sum_{j=1}^{n} w_j = 1$. Assume that $E = \{E_1, E_2, E_3, \ldots, E_\rho\}$ be the set of decision-makers who are employed to evaluate the alternatives. The weight vector of the decision-makers E_k $(k = 1, 2, 3, \ldots, \rho)$ is $\lambda = \{\lambda_1, \lambda_2, \lambda_3, \ldots, \lambda_\rho\}$ (where, $\lambda_k \geq 0$ and $\sum_{k=1}^{\rho} \lambda_k = 1$), which can be determined according to the decision-makers' expertise, judgment quality and domain knowledge.

Now, we describe the steps of the proposed MAGDM strategy (see Figure 1) using NS-cross entropy measure.

MAGDM Strategy Using Ns-Cross Entropy Measure

Step 1. Formulate the decision matrices

For MAGDM with SVNSs information, the rating values of the alternatives A_i $(i = 1, 2, 3, \ldots, m)$ based on the attribute G_j $(j = 1, 2, 3, \ldots, n)$ provided by the k-th decision-maker can be expressed in terms of SVNN as $a_{ij}^k = < T_{ij}^k, I_{ij}^k, F_{ij}^k >$ $(i = 1, 2, 3, \ldots, m; j = 1, 2, 3, \ldots, n; k = 1, 2, 3, \ldots, \rho)$. We present these rating values of alternatives provided by the decision-makers in matrix form as follows:

$$M^k = \begin{pmatrix} & G_1 & G_2 & \ldots. & G_n \\ A_1 & a_{11}^k & a_{12}^k & \ldots & a_{1n}^k \\ A_2 & a_{21}^k & & a_{2n}^k & a_{22}^k \\ . & . & \ldots & . & \\ A_m & a_{m1}^k & a_{m2}^k & \ldots & a_{mn}^k \end{pmatrix} \tag{3}$$

Step 2. Formulate priori/ideal decision matrix

In the MAGDM, the a priori decision matrix has been used to select the best alternatives among the set of collected feasible alternatives. In the decision-making situation, we use the following decision matrix as a priori decision matrix.

$$P = \begin{pmatrix} & G_1 & G_2 & \ldots. & G_n \\ A_1 & a_{11}^* & a_{12}^* & \ldots & a_{1n}^* \\ A_2 & a_{21}^* & a_{22}^* & & a_{2n}^* \\ . & . & \ldots & . & \\ A_m & a_{m1}^* & a_{m2}^* & \ldots & a_{mn}^* \end{pmatrix} \tag{4}$$

where, $a_{ij}^* = < \max_i (T_{ij}^k), \min_i (I_{ij}^k), \min_i (F_{ij}^k) >$) corresponding to benefit attributes and $a_{ij}^* = < \min_i (T_{ij}^k), \max_i (I_{ij}^k), \max_i (F_{ij}^k) >$ corresponding to cost attributes, and $(i = 1, 2, 3, \ldots, m; j = 1, 2, 3, \ldots, n; k = 1, 2, 3, \ldots, \rho)$.

Step 3. Determinate the weights of decision-makers

To find the decision-makers' weights we introduce a model based on the NS-cross entropy measure. The collective NS-cross entropy measure between M^k and P (Ideal matrix) is defined as follows:

$$CE_{NS}^c(M^k, P) = \frac{1}{m} \sum_{i=1}^{m} CE_{NS}\left(M^k(A_i), P(A_i)\right) \tag{5}$$

where, $CE_{NS}\left(M^k(A_i),\ P(A_i)\right) = \sum_{j=1}^{n} CE_{NS}(M^k(A_i(G_j)),\ P(A_i(G_j)))$.

Thus, we can introduce the following weight model of the decision-makers:

$$\lambda_K = \frac{\left(1 \div CE^c_{NS}(M^k,\ P)\right)}{\sum_{k=1}^{\rho}\left(1 \div CE^c_{NS}(M^k,\ P)\right)} \tag{6}$$

where, $0 \le \lambda_K \le 1$ and $\sum_{k=1}^{\rho} \lambda_K = 1$ for $k = 1, 2, 3, \ldots, \rho$.

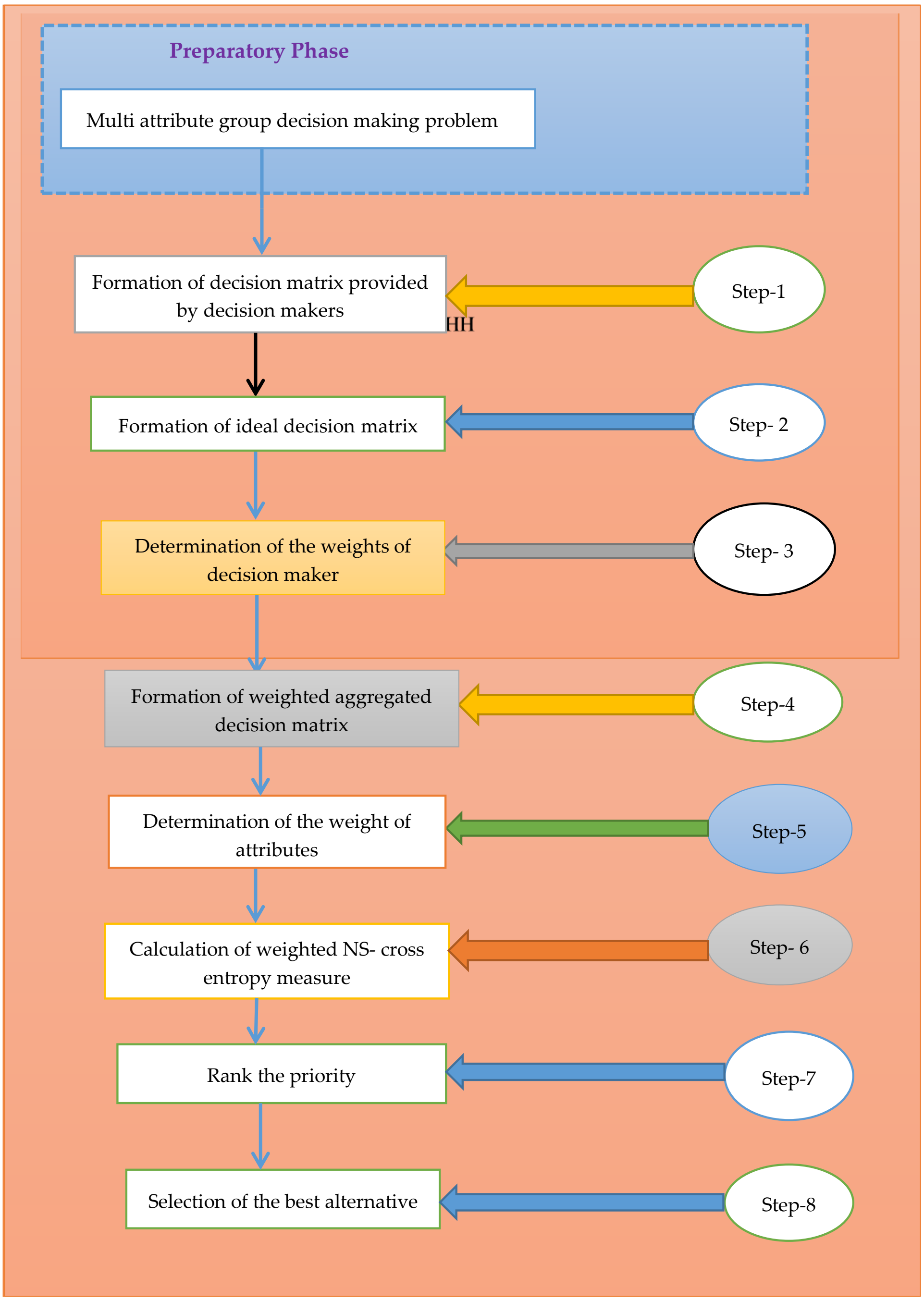

Figure 1. Decision-making procedure of the proposed MAGDM strategy.

Step 4. Formulate the weighted aggregated decision matrix

For obtaining one group decision, we aggregate all the individual decision matrices (M^k) to an aggregated decision matrix (M) using single valued neutrosophic weighted averaging (SVNWA) operator ([51]) as follows:

$$\begin{gathered} a_{ij} = SVNSWA_{\lambda}(\, a_{ij}^1,\ a_{ij}^2,\ a_{ij}^3,\ \ldots,\ a_{ij}^{\rho}) = (\lambda_1 a_{ij}^1 \oplus \lambda_2 a_{ij}^2 \oplus \lambda_3 a_{ij}^3 \oplus \ldots \oplus \lambda_{\rho} a_{ij}^{\rho}) = \\ <\ 1 - \prod_{k=1}^{\rho} (1\ - T_{ij}^k)^{\lambda_k},\ \prod_{k=1}^{\rho} (I_{ij}^k)^{\lambda_k},\ \prod_{k=1}^{\rho} (F_{ij}^k)^{\lambda_k} > \end{gathered} \tag{7}$$

Therefore, the aggregated decision matrix is defined as follows:

$$M = \begin{pmatrix} & G_1 & G_2 & \ldots. & G_n \\ A_1 & a_{11} & a_{12} & \ldots & a_{1n} \\ A_2 & a_{21} & a_{22} & & a_{2n} \\ . & . & \ldots & . & \\ A_m & a_{m1} & a_{m2} & \ldots & a_{mn} \end{pmatrix} \tag{8}$$

where, $a_{ij} =< T_{ij},\ I_{ij},\ F_{ij}\ >$, ($i = 1, 2, 3, \ldots, m$; $j = 1, 2, 3, \ldots, n$; $k = 1, 2, 3, \ldots, \rho$).

Step 5. Determinate the weight of attributes

To find the attributes weight we introduce a model based on the NS-cross entropy measure. The collective NS-cross entropy measure between M (Weighted aggregated decision matrix) and P (Ideal matrix) for each attribute is defined by

$$CE_{NS}^{j}(M,\ P) = \frac{1}{m}\sum_{i=1}^{m} CE_{NS}\ \left(M(A_i(G_j)),\ P(A_i(G_j))\right) \tag{9}$$

where, $i = 1, 2, 3, \ldots, m$; $j = 1, 2, 3, \ldots, n$.

Thus, we defined a weight model for attributes as follows:

$$w_j = \frac{\left(1 \div CE_{NS}^{j}(M,\ P)\right)}{\sum\limits_{J=1}^{n}\left(1 \div CE_{NS}^{j}(M,\ P)\right)} \tag{10}$$

where, $0 \leq w_j \leq 1$ and $\sum\limits_{j=1}^{n} w_j = \ 1$ for $j = 1, 2, 3, \ldots, n$.

Step 6. Calculate the weighted NS-cross entropy measure

Using Equation (2), we calculate weighted cross entropy value between weighted aggregated matrix and priori matrix. The cross entropy values can be presented in matrix form as follows:

$$^{NS}M_{CE}^{w} = \begin{pmatrix} \mathrm{CE}_{\mathrm{NS}}^{\mathrm{w}}\ (A_1) \\ \mathrm{CE}_{\mathrm{NS}}^{\mathrm{w}}\ (A_2) \\ \ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots \\ \mathrm{CE}_{\mathrm{NS}}^{\mathrm{w}}\ (A_m) \end{pmatrix} \tag{11}$$

Step 7. Rank the priority

Smaller value of the cross entropy reflects that an alternative is closer to the ideal alternative. Therefore, the preference priority order of all the alternatives can be determined according to the increasing order of the cross entropy values $CE^{w}_{NS}(A_i)$ (i = 1, 2, 3, ... , m). Smallest cross entropy value indicates the best alternative and greatest cross entropy value indicates the worst alternative.

Step 8. Select the best alternative

From the preference rank order (from step 7), we select the best alternative.

5. Illustrative Example

In this section, we solve an illustrative example adapted from [12] of MAGDM problems to reflect the feasibility, applicability and efficiency of the proposed strategy under the SVNS environment.

Now, we use the example [12] for cultivation and analysis. A venture capital firm intends to make evaluation and selection of five enterprises with the investment potential:

(1) Automobile company (A_1)
(2) Military manufacturing enterprise (A_2)
(3) TV media company (A_3)
(4) Food enterprises (A_4)
(5) Computer software company (A_5)

On the basis of four attributes namely:

(1) Social and political factor (G_1)
(2) The environmental factor (G_2)
(3) Investment risk factor (G_3)
(4) The enterprise growth factor (G_4).

The investment firm makes a panel of three decision-makers.

The steps of decision-making strategy (4.1.1.) to rank alternatives are presented as follows:

Step: 1. Formulate the decision matrices

We represent the rating values of alternatives A_i (i = 1, 2, 3, 4, 5) with respects to the attributes G_j (j = 1, 2, 3, 4) provided by the decision-makers E_k (k = 1, 2, 3) in matrix form as follows:

Decision matrix for E_1 decision-maker

$$M^1 = \begin{pmatrix} & G_1 & G_2 & G_3 & G_4 \\ A_1 & (0.9, 0.5, 0.4) & (0.7, 0.4, 0.4) & (0.7, 0.3, 0.4) & (0.5, 0.4, 0.9) \\ A_2 & (0.7, 0.2, 0.3) & (0.8, 0.4, 0.3) & (0.9, 0.6, 0.5) & (0.9, 0.1, 0.3) \\ A_3 & (0.8, 0.4, 0.4) & (0.7, 0.4, 0.2) & (0.9, 0.7, 0.6) & (0.7, 0.3, 0.3) \\ A_4 & (0.5, 0.8, 0.7) & (0.6, 0.3, 0.4) & (0.7, 0.2, 0.5) & (0.5, 0.4, 0.7) \\ A_5 & (0.8, 0.4, 0.3) & (0.5, 0.4, 0.5) & (0.6, 0.4, 0.4) & (0.9, 0.7, 0.5) \end{pmatrix} \quad (12)$$

Decision matrix for E_2 decision-maker

$$M^2 = \begin{pmatrix} & G_1 & G_2 & G_3 & G_4 \\ A_1 & (0.7, 0.2, 0.3) & (0.5, 0.4, 0.5) & (0.9, 0.4, 0.5) & (0.6, 0.5, 0.3) \\ A_2 & (0.7, 0.4, 0.4) & (0.7, 0.3, 0.4) & (0.7, 0.3, 0.4) & (0.6, 0.4, 0.3) \\ A_3 & (0.6, 0.4, 0.4) & (0.5, 0.3, 0.5) & (0.9, 0.5, 0.4) & (0.6, 0.5, 0.6) \\ A_4 & (0.7, 0.5, 0.3) & (0.6, 0.3, 0.6) & (0.7, 0.4, 0.4) & (0.8, 0.5, 0.4) \\ A_5 & (0.9, 0.4, 0.3) & (0.6, 0.4, 0.5) & (0.8, 0.5, 0.6) & (0.5, 0.4, 0.5) \end{pmatrix} \quad (13)$$

Decision matrix for E_3 decision-maker

$$M^{3} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{pmatrix} G_1 & G_2 & G_3 & G_4 \\ (0.7,\ 0.2,\ 0.5) & (0.6,\ 0.4,\ 0.4) & (0.7,\ 0.4,\ 0.5) & (0.9,\ 0.4,\ 0.3) \\ (0.6,\ 0.5,\ 0.5) & (0.9,\ 0.3,\ 0.4) & (0.7,\ 0.4,\ 0.3) & (0.8,\ 0.4,\ 0.5) \\ (0.8,\ 0.3,\ 0.5) & (0.9,\ 0.3,\ 0.4) & (0.8,\ 0.3,\ 0.4) & (0.7,\ 0.3,\ 0.4) \\ (0.9,\ 0.3,\ 0.4) & (0.6,\ 0.3,\ 0.4) & (0.5,\ 0.2,\ 0.4) & (0.7,\ 0.3,\ 0.5) \\ (0.8,\ 0.3,\ 0.3) & (0.6,\ 0.4,\ 0.3) & (0.6,\ 0.3,\ 0.4) & (0.7,\ 0.3,\ 0.5) \end{pmatrix} \tag{14}$$

Step: 2. Formulate priori/ideal decision matrix

A priori/ideal decision matrix Please provide a sharper picture

$$P = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{pmatrix} G_1 & G_2 & G_3 & G_4 \\ (0.9,\ 0.2,\ 0.3) & (0.7,\ 0.4,\ 0.4) & (0.9,\ 0.3,\ 0.4) & (0.9,\ 0.4,\ 0.3) \\ (0.7,\ 0.2,\ 0.3) & (0.9,\ 0.3,\ 0.3) & (0.9,\ 0.3,\ 0.3) & (0.9,\ 0.1,\ 0.3) \\ (0.8,\ 0.3,\ 0.4) & (0.9,\ 0.3,\ 0.2) & (0.9,\ 0.3,\ 0.4) & (0.7,\ 0.3,\ 0.3) \\ (0.9,\ 0.3,\ 0.3) & (0.6,\ 0.3,\ 0.4) & (0.7,\ 0.2,\ 0.4) & (0.7,\ 0.3,\ 0.4) \\ (0.9,\ 0.3,\ 0.3) & (0.6,\ 0.4,\ 0.3) & (0.8,\ 0.3,\ 0.4) & (0.9,\ 0.3,\ 0.5) \end{pmatrix} \tag{15}$$

Step: 3. Determine the weight of decision-makers

By using Equations (5) and (6), we determine the weights of the three decision-makers as follows:

$$\lambda_1 = \frac{(1 \div\ 0.9)}{3.37} \approx 0.33,\ \lambda_2 = \frac{(1 \div\ 1.2)}{3.37} \approx 0.25,\ \lambda_1 = \frac{(1 \div\ .07)}{3.37} \approx 0.42.$$

Step: 4. Formulate the weighted aggregated decision matrix

Using Equation (7) the weighted aggregated decision matrix is presented as follows:
Weighted Aggregated decision matrix

$$M = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{pmatrix} G_1 & G_2 & G_3 & G_4 \\ (0.8, 0.3,\ 0.4) & (0.6,\ 0.4,\ 0.4) & (0.8, 0.4,\ 0.4) & (0.7, 0.4,\ 0.5) \\ (0.7,\ 0.3, 0\ .4) & (0.8,\ 0.3,\ 0.4) & (0.8,\ 0.4,\ 0.4) & (0.8,\ 0.2,\ 0.3) \\ (0.8,\ 0.4,\ 0.4) & (0.8, 0.3,\ 0.3) & (0.9,\ 0.5,\ 0.5) & (0.7,\ 0.3,\ 0.4) \\ (0.7, 0.5,\ 0.5) & (0.6,\ 0.3,\ 0.4) & (0.6,\ 0.2, 0.4) & (0.7,\ 0.4,\ 0.5) \\ (0.8,\ 0.4,\ 0.4) & (0.6,\ 0.4,\ 0.4) & (0.7,\ 0.4,\ 0.4) & (0.8,\ 0.5,\ 0.5) \end{pmatrix} \tag{16}$$

Step: 5. Determinate the weight of the attributes

By using Equations (9) and (10), we determine the weights of the four attribute as follows:

$$w_1 = \frac{(1 \div 0.26)}{25} \approx 0.16,\ w_2 = \frac{(1 \div 0.11)}{25} \approx 0.37,\ w_3 = \frac{(1 \div 0.20)}{25} \approx 0.20,\ w_4 = \frac{(1 \div 0.15)}{25} \approx 0.27.$$

Step: 6. Calculate the weighted SVNS cross entropy matrix

Using Equation (2) and weights of attributes, we calculate the weighted NS-cross entropy values between ideal matrix and weighted aggregated decision matrix.

$$
{}^{NS}M^{w}_{CE} = \begin{pmatrix} 0.195 \\ 0.198 \\ 0.168 \\ 0.151 \\ 0.184 \end{pmatrix} \tag{17}
$$

Step: 7. Rank the priority

The cross entropy values of alternatives are arranged in increasing order as follows:

$$0.151 < 0.168 < 0.184 < 0.195 < 0.198.$$

Alternatives are then preference ranked as follows:

$$A_4 > A_3 > A_5 > A_1 > A_2.$$

Step: 8. Select the best alternative

From step 7, we identify A_4 is the best alternative. Hence, Food enterprises (A_4) is the best alternative for investment.

In Figure 2, we draw a bar diagram to represent the cross entropy values of alternatives which shows that A_4 is the best alternative according our proposed strategy.

In Figure 3, we represent the relation between cross entropy values and acceptance values of alternatives. The range of acceptance level for five alternatives is taken by five points. The high acceptance level of alternatives indicates the best alternative for acceptance and low acceptance level of alternative indicates the poor acceptance alternative.

We see from Figure 3 that alternative A_4 has the smallest cross entropy value and the highest acceptance level. Therefore A_4 is the best alternative for acceptance. Figure 3 indicates that alternative A_2 has highest cross entropy value and lowest acceptance value that means A_2 is the worst alternative. Finally, we conclude that the relation between cross entropy values and acceptance value of alternatives is opposite in nature.

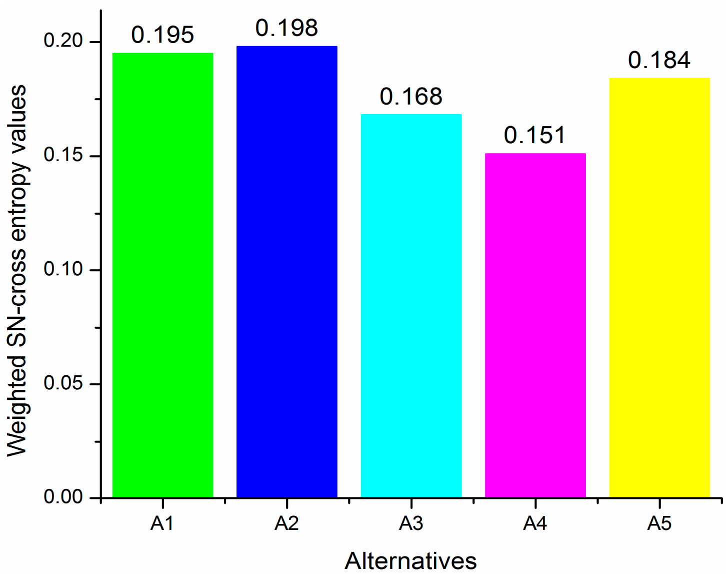

Figure 2. Bar diagram of alternatives versus weighted NS-cross entropy values of alternatives.

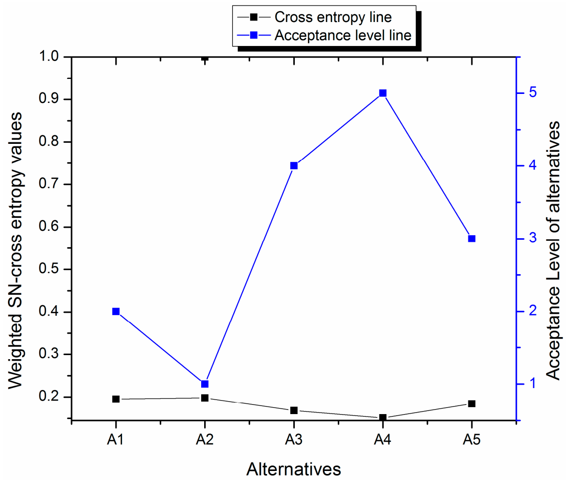

Figure 3. Relation between weighted NS-cross entropy values and acceptance level line of alternatives.

6. Comparative Study and Discussion

In literature only two MADM strategies [144,145] have been proposed. No MADGM strategy is available. So the proposed MAGDM is novel and non-comparable with the existing cross entropy under SVNS for numerical example.

i. The MADM strategies [144,145] are not applicable for MAGDM problems. The proposed MAGDM strategy is free from such drawbacks.
ii. Ye [144] proposed cross entropy that does not satisfy the symmetrical property straightforward and is undefined for some situations but the proposed strategy satisfies symmetric property and is free from undefined phenomenon.
iii. The strategies [144,145] cannot deal with the unknown weight of the attributes whereas the proposed MADGM strategy can deal with the unknown weight of the attributes
iv. The strategies [144,145] are not suitable for dealing with the unknown weight of decision-makers, whereas the essence of the proposed NS-cross entropy-based MAGDM is that it is capable of dealing with the unknown weight of the decision-makers.

7. Conclusions

In this paper, we have defined a novel cross entropy measure in SVNS environment. The proposed cross entropy measure in SVNS environment is free from the drawbacks of asymmetrical behavior and undefined phenomena. It is capable of dealing with the unknown weight of attributes and the unknown weight of decision-makers. We have proved the basic properties of the NS-cross entropy measure. We also defined weighted NS-cross entropy measure and proved its basic properties. Based on the weighted NS-cross entropy measure, we have developed a novel MAGDM strategy to solve neutrosophic multi-attribute group decision-making problems. We have at first proposed a novel MAGDM strategy based on NS-cross entropy measure with technique to determine the unknown weight of attributes and the unknown weight of decision-makers. Other existing cross entropy measures [144,145] can deal only with the MADM problem with single decision-maker and known weight of the attributes. So in general, our proposed NS-cross entropy-based MAGDM strategy is not comparable with the existing cross-entropy-based MADM strategies [144,145] under the single-valued neutrosophic environment. Finally, we solve a MAGDM problem to show the feasibility, applicability and efficiency of the proposed MAGDM strategy. The proposed NS-cross entropy-based MAGDM can be applied in teacher selection, pattern recognition, weaver selection, medical treatment selection options, and other practical problems. In future study, the proposed NS-cross entropy-based MAGDM strategy can be also extended to the interval neutrosophic set environment.

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Generalized Single-Valued Neutrosophic Hesitant Fuzzy Prioritized Aggregation Operators and Their Applications to Multiple Criteria Decision-Making

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Abstract: Single-valued neutrosophic hesitant fuzzy set (SVNHFS) is a combination of single-valued neutrosophic set and hesitant fuzzy set, and its aggregation tools play an important role in the multiple criteria decision-making (MCDM) process. This paper investigates the MCDM problems in which the criteria under SVNHF environment are in different priority levels. First, the generalized single-valued neutrosophic hesitant fuzzy prioritized weighted average operator and generalized single-valued neutrosophic hesitant fuzzy prioritized weighted geometric operator are developed based on the prioritized average operator. Second, some desirable properties and special cases of the proposed operators are discussed in detail. Third, an approach combined with the proposed operators and the score function of single-valued neutrosophic hesitant fuzzy element is constructed to solve MCDM problems. Finally, an example of investment selection is provided to illustrate the validity and rationality of the proposed method.

Keywords: multiple criteria decision-making (MCDM); single-valued neutrosophic hesitant fuzzy set (SVNHFS); generalized single-valued neutrosophic hesitant fuzzy prioritized weighted average operator; generalized single-valued neutrosophic hesitant fuzzy prioritized weighted geometric operator

1. Introduction

In daily life, MCDM problems happen in many fields; decision makers determine the best one from several alternatives through evaluating them with respect to the corresponding criteria. Due to the high complexity of the social environment, the evaluation information given by decision makers is often uncertain, incomplete, and inconsistent. With the demand for accuracy of decision-making results is getting higher and higher, much research in recent years has focused on the MCDM problems under fuzzy environment [1]. In 1965, Zadeh [2] developed the fuzzy set (FS) theory, which is a powerful tool to express the fuzzy information. However, there are several obvious limitations of FS theory in expressing uncertain information, which are attracting widespread interest in improving FS theory.

Atanassov [3] introduced the non-membership function to extend FS theory and proposed the intuitionistic fuzzy set (IFS) theory. IFS can express the membership and non-membership information simultaneously; the property can deal with some applications effectively, which FS cannot. For example, ten decision makers vote for an affair, four present agreement, three suggest different opinions, and the others choose to give up. The example above can be characterized by IFS, i.e., the value of membership is 0.4, and the value of non-membership is 0.3. However, expressing the voting information by FS is impossible. To describe the fuzziness of evaluation information more effective, Atanassov and Gargov [4] utilized the interval number to extend the membership and non-membership functions and put forward the interval-valued intuitionistic fuzzy set (IVIFS) theory. Nevertheless, in the real decision-making process, only considering the membership and non-membership information is not comprehensive sometimes. For instance, a decision maker gives her/his evaluation on a viewpoint, she/he may think the positive probability is 0.5, the false probability is 0.6, and the indeterminacy probability is 0.2 [5]. Obviously, IFS and IVIFS theory cannot deal with this situation. Therefore, Smarandache [6] defined the neutrosophic set (NS), which can be regarded as a generalization of FS and IFS [7]. NS consists of three independent membership functions, namely, truth-membership, indeterminacy-membership, and falsity-membership functions. Whereas, NS theory was originally proposed from a philosophical point of view, and it is difficult to apply NS theory in the field of science and engineering. To solve this problem, Wang [8,9] defined the concepts of interval neutrosophic set (INS) and single-valued neutrosophic set (SVNS), which are specific cases of NS.

Another drawback of FS is that its membership value is single; while determining the exact value of membership may be difficult for decision makers due to doubt. To deal with this situation, Torra and Narukawa [10] and Torra [11] extended the FS theory to hesitant fuzzy set (HFS) theory through allowing decision makers to give several different values of membership. Furthermore, Chen [12] defined the concept of interval-valued hesitant fuzzy set (IVHFS), in which the possible membership values can be expressed by interval numbers. Considering the complex information given by decision makers, Zhu [13] introduced the non-membership hesitancy function to propose the dual hesitant fuzzy set (DHFS) theory. According to the aforementioned analysis of improved FS theory from two directions, Ye [14] developed the single-valued neutrosophic hesitant fuzzy set (SVNHFS) combined with NS and HFS theory, in addition, Liu and Shi [7] extended the SVNHFS to interval neutrosophic hesitant fuzzy set (INHFS). Consequently, SVNHFS and INHFS not only can characterize the inconsistent and indeterminate information but also allow decision makers to give several possible values of truth-membership, indeterminacy-membership, and falsity-membership functions.

Besides the evaluation information, aggregation tools also are important parts of MCDM process. Ye [14] developed the operational laws and cosine measure of single-valued neutrosophic hesitant fuzzy elements (SVNHFEs), and proposed the single-valued neutrosophic hesitant fuzzy weighted average (SVNHFWA) operator and single-valued neutrosophic hesitant fuzzy weighted geometric (SVNHFWG) operator to aggregate SVNHFEs. Şahin and Liu [15] constructed the decision-making approach based on the correlation coefficient and weighted correlation coefficient of SVNHFEs. Biswas et al. [16] put forward several approaches for decision-making under SVNHF environment by using distance measures of SVNHFEs. Liu and Luo [17] proposed the single-valued neutrosophic hesitant fuzzy ordered weighted average (SVNHFOWA) operator and single-valued neutrosophic hesitant fuzzy hybrid weighted average (SVNHFHWA) operator, and applied them into MCDM process. Liu and Zhang [18] developed the single-valued neutrosophic hesitant fuzzy Heronian mean aggregation operators to deal with MCDM problems. Liu and Shi [7] defined the operational laws of INHFSs and proposed interval neutrosophic hesitant fuzzy generalized weighted average (INHFGWA) operator, interval neutrosophic hesitant fuzzy generalized ordered weighted average (INHFGOWA) operator, and interval neutrosophic hesitant fuzzy generalized hybrid weighted average (INHFGHWA) operator. Ye [19] determined the ranking of alternatives combined with the correlation coefficient of INHFSs.

The aforementioned decision-making methods are applied to the situation of the aggregated arguments and are in the same priority; whereas, in many real situations, criteria always have different priorities. For example, a mother chooses the dried milk for her baby, the criteria she considers are price and safety. Obviously, a prioritization ordering exists between the criteria, i.e., safety is much more important than price [20]. To deal with this situation, Yager [21] proposed the prioritized average (PA) operator to aggregate the evaluation information concerning the criteria of different priorities. Since the PA operator was presented, many scholars have focused on extending the PA operator into the fuzzy environment. For instance, Yu [20] proposed the intuitionistic fuzzy prioritized weighted average (IFPWA) operator and intuitionistic fuzzy prioritized weighted geometric (IFPWG) operator, and investigated their properties. Yu et al. [22] extended the PA operator into IVIF environment and developed the interval-valued intuitionistic fuzzy prioritized weighted average (IVIFPWA) operator and interval-valued intuitionistic fuzzy prioritized weighted geometric (IVIFPWG) operator. Liu and Wang [23] studied the aggregation operator under IN environment and put forward the interval neutrosophic prioritized ordered weighted average (INPOWA) operator. Furthermore, Wei [24] extended the PA operator into hesitant fuzzy MCDM problems. Jin et al. [25] developed interval-valued hesitant fuzzy Einstein prioritized weighted average (IVHFEPWA) operator and the interval-valued hesitant fuzzy Einstein prioritized weighted geometric (IVHFEPWG) operator through improving the operations of IVHFSs. However, to our best knowledge, little attention has been paid to the prioritized aggregation operators under SVNHF environment.

This paper proposes the aggregation operators for SVNHFEs, in which the aggregation arguments have different priority levels, and develops an approach for decision-making. To do this, the rest of this paper is organized as follows. Section 2 briefly introduces some basic concepts of SVNS, HFS, SVNHFS, and the PA operator. Section 3 develops the generalized single-valued neutrosophic hesitant fuzzy prioritized weighted average (GSVNHFPWA) operator and generalized single-valued neutrosophic hesitant fuzzy prioritized weighted geometric (GSVNHFPWG) operator, and investigates some desirable properties and special cases of the proposed operators. Section 4 constructs an approach for decision-making based on the proposed operators. Section 5 provides a numerical example to illustrate the applications and advantages of the proposed method. Section 6 summarizes the conclusions of this research.

2. Preliminaries

In this section, we briefly introduce some basic concepts, including the definitions of NS, SVNS, HFS, and SVNHFS. The operations of SVNHFEs and the PA operator are also presented, which are used in the subsequent discussion.

2.1. The Single-Valued Neutrosophic Set

Definition 1. *Ref. [6] Let X be a universe of discourse, with a generic element in X denoted by x. An NS A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership function $I_A(x)$, and a falsity-membership function $F_A(x)$. The functions $T_A(x)$, $I_A(x)$, and $F_A(x)$ are real standard or non-standard subsets of $]^-0,1^+[$, i.e., $T_A(x) : X \to]^-0,1^+[$, $I_A(x) : X \to]^-0,1^+[$, and $F_A(x) : X \to]^-0,1^+[$. Thus, the sum of three aforementioned functions satisfies the condition of $^-0 \le \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \le 3^+$.*

NS theory was originally proposed from the angle of philosophy and can be regarded as a generalization of FS, IFS, and IVIFS. However, the NS is not easily used for real scientific and engineering decision-making problems. To solve this limitation, Wang [8] defined the concept of SVNS, which is a special case of NS.

Definition 2. *Ref. [8] Let X be a universe of discourse, with a generic element in X denoted by x. An SVNS A is given by*

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle | x \in X\}, \tag{1}$$

where $T_A(x)$ is the truth-membership function, $I_A(x)$ is the indeterminacy-membership function, and $F_A(x)$ is the falsity-membership function. For each point x in X, the functions $T_A(x)$, $I_A(x)$, and $F_A(x)$ satisfy the conditions of $T_A(x), I_A(x), F_A(x) \in [0,1]$ and $0 \le T_A(x) + I_A(x) + F_A(x) \le 3$.

2.2. The Hesitant Fuzzy Set

During the decision-making process, decision makers sometimes may be confused when determining the exact membership value of an element to the set because of the existing several possible membership values. Considering this situation, Torra and Narukawa [10] defined the concept of HFS.

Definition 3. *Ref. [10] Let X be a non-empty and finite set, an HFS A on X is defined by a function $h_A(x)$ that when applied to X returns a finite subset of* $[0,1]$, *which can be expressed as*

$$A = \{\langle x, h_A(x)\rangle | x \in X\}, \tag{2}$$

where $h_A(x)$ is a set of some different values in $[0,1]$, indicating the possible membership degrees of the element $x \in X$ to A.

2.3. The Single-Valued Neutrosophic Hesitant Fuzzy Set

Based on the combination of SVNS and HFS, Ye [14] proposed the concept of SVNHFS.

Definition 4. *Ref. [14] Let X be a non-empty and finite set, an SVNHFS N on X is expressed as*

$$N = \left\{\left\langle x, \widetilde{t}(x), \widetilde{i}(x), \widetilde{f}(x)\right\rangle | x \in X\right\}, \tag{3}$$

where $\widetilde{t}(x) = \{\gamma|\gamma \in \widetilde{t}(x)\}$, $\widetilde{i}(x) = \left\{\delta\middle|\delta \in \widetilde{i}(x)\right\}$, and $\widetilde{f}(x) = \left\{\eta\middle|\eta \in \widetilde{f}(x)\right\}$ are three sets of some different values in $[0,1]$, denoting the possible truth-membership hesitant, possible indeterminacy-membership hesitant, and possible falsity-membership hesitant degrees of the element $x \in X$ to N. And they satisfy the conditions of γ, δ, $\eta \subseteq [0,1]$ and $0 \le \sup\gamma^+ + \sup\delta^+ + \sup\eta^+ \le 3$, where $\gamma^+ = \cup_{\gamma\in\widetilde{t}(x)}\max\{\gamma\}$, $\delta^+ = \cup_{\delta\in\widetilde{i}(x)}\max\{\delta\}$, and $\eta^+ = \cup_{\eta\in\widetilde{f}(x)}\max\{\eta\}$ for $x \in X$. For convenience, we call $\widetilde{n} = \left\{\widetilde{t}(x), \widetilde{i}(x), \widetilde{f}(x)\right\}$ is an SVNHFE, denoted by $\widetilde{n} = \left\{\widetilde{t}, \widetilde{i}, \widetilde{f}\right\}$.

Definition 5. *Ref. [14] Let $\widetilde{n} = \left\{\widetilde{t}, \widetilde{i}, \widetilde{f}\right\}$, $\widetilde{n}_1 = \left\{\widetilde{t}_1, \widetilde{i}_1, \widetilde{f}_1\right\}$ and $\widetilde{n}_2 = \left\{\widetilde{t}_2, \widetilde{i}_2, \widetilde{f}_2\right\}$ be three SVNHFEs, $\lambda > 0$, then the basic operations of SVNHFEs are defined as*

$$\widetilde{n}_1 \oplus \widetilde{n}_2 = \left\{\widetilde{t}_1 \oplus \widetilde{t}_2, \widetilde{i}_1 \otimes \widetilde{i}_2, \widetilde{f}_1 \otimes \widetilde{f}_2\right\} = \bigcup_{\gamma_1\in\widetilde{t}_1,\delta_1\in\widetilde{i}_1,\eta_1\in\widetilde{f}_1,\gamma_2\in\widetilde{t}_2,\delta_2\in\widetilde{i}_2,\eta_2\in\widetilde{f}_2} \{\{\gamma_1+\gamma_2-\gamma_1\gamma_2\}, \{\delta_1\delta_2\}, \{\eta_1\eta_2\}\}; \tag{4}$$

$$\widetilde{n}_1 \otimes \widetilde{n}_2 = \left\{\widetilde{t}_1 \otimes \widetilde{t}_2, \widetilde{i}_1 \oplus \widetilde{i}_2, \widetilde{f}_1 \oplus \widetilde{f}_2\right\} = \bigcup_{\gamma_1\in\widetilde{t}_1,\delta_1\in\widetilde{i}_1,\eta_1\in\widetilde{f}_1,\gamma_2\in\widetilde{t}_2,\delta_2\in\widetilde{i}_2,\eta_2\in\widetilde{f}_2} \{\{\gamma_1\gamma_2\}, \{\delta_1+\delta_2-\delta_1\delta_2\}, \{\eta_1+\eta_2-\eta_1\eta_2\}\}; \tag{5}$$

$$\lambda\widetilde{n} = \bigcup_{\gamma\in\widetilde{t},\delta\in\widetilde{i},\eta\in\widetilde{f}} \left\{\left\{1-(1-\gamma)^\lambda\right\}, \left\{\delta^\lambda\right\}, \left\{\eta^\lambda\right\}\right\}; \tag{6}$$

$$\widetilde{n}^\lambda = \bigcup_{\gamma\in\widetilde{t},\delta\in\widetilde{i},\eta\in\widetilde{f}} \left\{\left\{\gamma^\lambda\right\}, \left\{1-(1-\delta)^\lambda\right\}, \left\{1-(1-\eta)^\lambda\right\}\right\}. \tag{7}$$

Definition 6. *Ref. [18] Let $\widetilde{n} = \left\{\widetilde{t}, \widetilde{i}, \widetilde{f}\right\}$ be an SVNHFE, then the score function $s(\widetilde{n})$ of $\widetilde{n}$ is given by*

$$s(\widetilde{n}) = \left[\frac{1}{l}\sum_{i=1}^{l}\gamma_i + \frac{1}{p}\sum_{i=1}^{p}(1-\delta_i) + \frac{1}{q}\sum_{i=1}^{q}(1-\eta_i)\right] \Big/ 3, \tag{8}$$

where l, p, q are the numbers of values in $\widetilde{t}, \widetilde{i}, \widetilde{f}$, respectively. Obviously, the range of $s(\widetilde{n})$ is limited to $[0,1]$.

Definition 7. *Ref. [18] Let $\widetilde{n}_1 = \left\{\widetilde{t}_1, \widetilde{i}_1, \widetilde{f}_1\right\}$ and $\widetilde{n}_2 = \left\{\widetilde{t}_2, \widetilde{i}_2, \widetilde{f}_2\right\}$ be two SVNHFEs, then the comparison method of them is expressed by*

(1) If $s(\widetilde{n}_1) > s(\widetilde{n}_2)$, then $\widetilde{n}_1 > \widetilde{n}_2$;
(2) If $s(\widetilde{n}_1) < s(\widetilde{n}_2)$, then $\widetilde{n}_1 < \widetilde{n}_2$;
(3) If $s(\widetilde{n}_1) = s(\widetilde{n}_2)$, then $\widetilde{n}_1 = \widetilde{n}_2$.

2.4. The Prioritized Average Operator

Aggregation operators play an important role in group decision-making to fusion the evaluation information. In view of priority relations between the criteria, Yager [21] developed the PA operator to solve this problem.

Definition 8. *Ref. [21] Let $C = \{C_1, C_2, \ldots, C_n\}$ be a collection of criteria, and priority relations between the criteria exist which can be expressed by the ordering of $C_1 \succ C_2 \succ C_3 \succ \cdots \succ C_n$. That means criteria C_j has a higher priority level than criteria C_k if $j < k$. The value $C_j(x)$ is the evaluation information of alternative x with respect to criteria C_j. Thus, if*

$$\mathrm{PA}\left(C_j(x)\right) = \sum_{j=1}^{n} w_j C_j(x), \tag{9}$$

then the function PA is called the prioritized average (PA) operator, where $w_j = T_j / \sum_{j=1}^{n} T_j, T_j = \prod_{k=1}^{j-1} C_k(x)$, $T_1 = 1$.

3. Generalized Single-Valued Neutrosophic Hesitant Fuzzy Prioritized Aggregation Operators

The PA operator can effectively solve the decision-making problems that the criteria have different priorities; however, it can only be used in the situation where the aggregated arguments are exact values. Combined with the PA operator and the generalized mean operators [26], we extend the PA operator to deal with the decision-making problems under SVNHF environment. In this section, the GSVNHFPWA operator and GSVNHFPWG operator are proposed, and their properties are presented simultaneously. Besides, several special cases of the GSVNHFPWA operator and GSVNHFPWG operator are also discussed through changing the values of the parameter λ.

3.1. Generalized Single-Valued Neutrosophic Hesitant Fuzzy Prioritized Average Operator

Definition 9. *Let $\widetilde{n}_j = \left\{\widetilde{t}_j, \widetilde{i}_j, \widetilde{f}_j\right\} (j = 1, 2, \ldots, n)$ be a collection of SVNHFEs, and let $\mathrm{GSVNHFPWA} : \Omega^n \to \Omega$, if*

$$\mathrm{GSVNHFPWA}_\lambda(\widetilde{n}_1, \widetilde{n}_2, \ldots, \widetilde{n}_n) = \left(\frac{T_1}{\sum_{j=1}^{n} T_j} \widetilde{n}_1^\lambda \oplus \frac{T_2}{\sum_{j=1}^{n} T_j} \widetilde{n}_2^\lambda \oplus \cdots \oplus \frac{T_1}{\sum_{j=1}^{n} T_j} \widetilde{n}_n^\lambda \right)^{1/\lambda}, \tag{10}$$

then the function GSVNHFPWA is called the GSVNHFPWA operator. Where $T_j = \prod_{k=1}^{j-1} s(\widetilde{n}_k) (j = 2, \ldots, n)$, $T_1 = 1$, and $s(\widetilde{n}_k)$ is the score function value of SVNHFE $\widetilde{n}_k$.

According to the operational laws of SVHFEs in Definition 5, we can obtain the theorem as follows.

Theorem 1. *Let $\widetilde{n}_j = \left\{\widetilde{t}_j, \widetilde{i}_j, \widetilde{f}_j\right\} (j = 1, 2, \ldots, n)$ be a collection of SVNHFEs, then their aggregated value by using the GSVNHFPWA operator is also an SVNHFE, and*

$$\begin{aligned} &\mathrm{GSVNHFPWA}_\lambda(\widetilde{n}_1, \widetilde{n}_2, \ldots, \widetilde{n}_n) = \left(\frac{T_1}{\sum_{j=1}^{n} T_j} \widetilde{n}_1^\lambda \oplus \frac{T_2}{\sum_{j=1}^{n} T_j} \widetilde{n}_2^\lambda \oplus \cdots \oplus \frac{T_n}{\sum_{j=1}^{n} T_j} \widetilde{n}_n^\lambda \right)^{1/\lambda} \\ &= \bigcup_{\widetilde{\gamma}_1 \in \widetilde{t}_1, \widetilde{\gamma}_2 \in \widetilde{t}_2, \ldots, \widetilde{\gamma}_n \in \widetilde{t}_n, \widetilde{\delta}_1 \in \widetilde{i}_1, \widetilde{\delta}_2 \in \widetilde{i}_2, \ldots, \widetilde{\delta}_n \in \widetilde{i}_n, \widetilde{\eta}_1 \in \widetilde{f}_1, \widetilde{\eta}_2 \in \widetilde{f}_2, \ldots, \widetilde{\eta}_n \in \widetilde{f}_n} \left\{ \left\{ \left(1 - \prod_{j=1}^{n} \left(1 - \gamma_j^\lambda\right)^{\frac{T_j}{\sum_{j=1}^{n} T_j}} \right)^{1/\lambda} \right\}, \left\{ 1 - \left(1 - \prod_{j=1}^{n} \left(1 - (1 - \delta_j)^\lambda\right)^{\frac{T_j}{\sum_{j=1}^{n} T_j}} \right)^{1/\lambda} \right\}, \right. \\ &\left. \left\{ 1 - \left(1 - \prod_{j=1}^{n} \left(1 - (1 - \eta_j)^\lambda\right)^{\frac{T_j}{\sum_{j=1}^{n} T_j}} \right)^{1/\lambda} \right\} \right\}. \end{aligned} \tag{11}$$

where $T_j = \prod_{k=1}^{j-1} s(\widetilde{n}_k) (j = 2, \ldots, n)$, $T_1 = 1$, and $s(\widetilde{n}_k)$ is the score function value of SVNHFE $\widetilde{n}_k$.

Proof. We can use mathematical induction to prove the Theorem 1:

(a) For $n = 1$, since

$$\text{GSVNHFPWA}_\lambda(\tilde{n}_1) = \left(\frac{T_1}{\sum_{j=1}^n T_j}\tilde{n}_1^\lambda\right)^{1/\lambda} = \left(\frac{T_1}{T_1}\tilde{n}_1^\lambda\right)^{1/\lambda} = \tilde{n}_1.$$

Obviously, Equation (11) holds for $n = 1$.

(b) For $n = 2$, since

$$\tilde{n}_1^\lambda = \bigcup_{\gamma_1\in\tilde{t}_1,\delta_1\in\tilde{i}_1,\eta_1\in\tilde{f}_1}\left\{\left\{\gamma_1^\lambda\right\},\left\{1-(1-\delta_1)^\lambda\right\},\left\{1-(1-\eta_1)^\lambda\right\}\right\},$$

$$\tilde{n}_2^\lambda = \bigcup_{\gamma_2\in\tilde{t}_2,\delta_2\in\tilde{i}_2,\eta_2\in\tilde{f}_2}\left\{\left\{\gamma_2^\lambda\right\},\left\{1-(1-\delta_2)^\lambda\right\},\left\{1-(1-\eta_2)^\lambda\right\}\right\},$$

Then

$$\frac{T_1}{\sum_{j=1}^n T_j}\tilde{n}_1^\lambda = \bigcup_{\gamma_1\in\tilde{t}_1,\delta_1\in\tilde{i}_1,\eta_1\in\tilde{f}_1}\left\{\left\{1-\left(1-\gamma_1^\lambda\right)^{\frac{T_1}{\sum_{j=1}^n T_j}}\right\},\left\{\left(1-(1-\delta_1)^\lambda\right)^{\frac{T_1}{\sum_{j=1}^n T_j}}\right\},\left\{\left(1-(1-\eta_1)^\lambda\right)^{\frac{T_1}{\sum_{j=1}^n T_j}}\right\}\right\},$$

$$\frac{T_2}{\sum_{j=1}^n T_j}\tilde{n}_2^\lambda = \bigcup_{\gamma_2\in\tilde{t}_2,\delta_2\in\tilde{i}_2,\eta_2\in\tilde{f}_2}\left\{\left\{1-\left(1-\gamma_2^\lambda\right)^{\frac{T_2}{\sum_{j=1}^n T_j}}\right\},\left\{\left(1-(1-\delta_2)^\lambda\right)^{\frac{T_2}{\sum_{j=1}^n T_j}}\right\},\left\{\left(1-(1-\eta_2)^\lambda\right)^{\frac{T_2}{\sum_{j=1}^n T_j}}\right\}\right\}.$$

We have

$$\frac{T_1}{\sum_{j=1}^n T_j}\tilde{n}_1^\lambda \oplus \frac{T_2}{\sum_{j=1}^n T_j}\tilde{n}_2^\lambda =$$

$$\bigcup_{\gamma_1\in\tilde{t}_1,\delta_1\in\tilde{i}_1,\eta_1\in\tilde{f}_1,\gamma_2\in\tilde{t}_2,\delta_2\in\tilde{i}_2,\eta_2\in\tilde{f}_2}\left\{\left\{1-(1-\gamma_1^\lambda)^{\frac{T_1}{\sum_{j=1}^n T_j}}+1-(1-\gamma_2^\lambda)^{\frac{T_2}{\sum_{j=1}^n T_j}}-\left(1-(1-\gamma_1^\lambda)^{\frac{T_1}{\sum_{j=1}^n T_j}}\right)\left(1-(1-\gamma_2^\lambda)^{\frac{T_2}{\sum_{j=1}^n T_j}}\right)\right\},\right.$$

$$\left.\left\{\left(\left(1-(1-\delta_1)^\lambda\right)^{\frac{T_1}{\sum_{j=1}^n T_j}}\right)\left(\left(1-(1-\delta_2)^\lambda\right)^{\frac{T_2}{\sum_{j=1}^n T_j}}\right)\right\},\left\{\left(\left(1-(1-\eta_1)^\lambda\right)^{\frac{T_1}{\sum_{j=1}^n T_j}}\right)\left(\left(1-(1-\eta_2)^\lambda\right)^{\frac{T_2}{\sum_{j=1}^n T_j}}\right)\right\}\right\}.$$

$$= \bigcup_{\gamma_1\in\tilde{t}_1,\delta_1\in\tilde{i}_1,\eta_1\in\tilde{f}_1,\gamma_2\in\tilde{t}_2,\delta_2\in\tilde{i}_2,\eta_2\in\tilde{f}_2}\left\{\left\{1-(1-\gamma_1^\lambda)^{\frac{T_1}{\sum_{j=1}^n T_j}}(1-\gamma_2^\lambda)^{\frac{T_2}{\sum_{j=1}^n T_j}}\right\},\right.$$

$$\left.\left\{\left(\left(1-(1-\delta_1)^\lambda\right)^{\frac{T_1}{\sum_{j=1}^n T_j}}\right)\left(\left(1-(1-\delta_2)^\lambda\right)^{\frac{T_2}{\sum_{j=1}^n T_j}}\right)\right\},\left\{\left(\left(1-(1-\eta_1)^\lambda\right)^{\frac{T_1}{\sum_{j=1}^n T_j}}\right)\left(\left(1-(1-\eta_2)^\lambda\right)^{\frac{T_2}{\sum_{j=1}^n T_j}}\right)\right\}\right\}.$$

Thus

$$\text{GSVNHFPWA}_\lambda(\tilde{n}_1,\tilde{n}_1) = \left(\frac{T_1}{\sum_{j=1}^n T_j}\tilde{n}_1^\lambda \oplus \frac{T_2}{\sum_{j=1}^n T_j}\tilde{n}_2^\lambda\right)^{1/\lambda} =$$

$$\bigcup_{\gamma_1\in\tilde{t}_1,\delta_1\in\tilde{i}_1,\eta_1\in\tilde{f}_1,\gamma_2\in\tilde{t}_2,\delta_2\in\tilde{i}_2,\eta_2\in\tilde{f}_2}\left\{\left\{\left(1-(1-\gamma_1^\lambda)^{\frac{T_1}{\sum_{j=1}^n T_j}}(1-\gamma_2^\lambda)^{\frac{T_2}{\sum_{j=1}^n T_j}}\right)^{1/\lambda}\right\},\right.$$

$$\left.\left\{1-\left(1-\left(\left(1-(1-\delta_1)^\lambda\right)^{\frac{T_1}{\sum_{j=1}^n T_j}}\right)\left(\left(1-(1-\delta_2)^\lambda\right)^{\frac{T_2}{\sum_{j=1}^n T_j}}\right)\right)^{1/\lambda}\right\},\left\{1-\left(1-\left(\left(1-(1-\eta_1)^\lambda\right)^{\frac{T_1}{\sum_{j=1}^n T_j}}\right)\left(\left(1-(1-\eta_2)^\lambda\right)^{\frac{T_2}{\sum_{j=1}^n T_j}}\right)\right)^{1/\lambda}\right\}\right\}.$$

i.e., Equation (11) holds for $n = 2$.

(c) If Equation (11) holds for $n = k$, we have

$$\mathrm{GSVNHFPWA}_{\lambda}(\widetilde{n}_1, \widetilde{n}_2, \ldots, \widetilde{n}_k) = \left(\frac{T_1}{\sum_{j=1}^{n} T_j} \widetilde{n}_1^{\lambda} \oplus \frac{T_2}{\sum_{j=1}^{n} T_j} \widetilde{n}_2^{\lambda} \oplus \cdots \oplus \frac{T_k}{\sum_{j=1}^{n} T_j} \widetilde{n}_k^{\lambda} \right)^{1/\lambda}$$

$$= \bigcup_{\widetilde{\gamma}_1 \in \widetilde{t}_1, \widetilde{\gamma}_2 \in \widetilde{t}_2, \ldots, \widetilde{\gamma}_k \in \widetilde{t}_k, \widetilde{\delta}_1 \in \widetilde{i}_1, \widetilde{\delta}_2 \in \widetilde{i}_2, \ldots, \widetilde{\delta}_k \in \widetilde{i}_k, \widetilde{\eta}_1 \in \widetilde{f}_1, \widetilde{\eta}_2 \in \widetilde{f}_2, \ldots, \widetilde{\eta}_k \in \widetilde{f}_k} \left\{ \left\{ \left(1 - \prod_{j=1}^{k} \left(1 - \gamma_j^{\lambda}\right)^{\frac{T_j}{\sum_{j=1}^{n} T_j}} \right)^{1/\lambda} \right\}, \left\{ 1 - \left(1 - \prod_{j=1}^{k} \left(1 - (1 - \delta_j)^{\lambda}\right)^{\frac{T_j}{\sum_{j=1}^{n} T_j}} \right)^{1/\lambda} \right\}, \right.$$
$$\left. \left\{ 1 - \left(1 - \prod_{j=1}^{k} \left(1 - (1 - \eta_j)^{\lambda}\right)^{\frac{T_j}{\sum_{j=1}^{n} T_j}} \right)^{1/\lambda} \right\} \right\}.$$

When $n = k + 1$, combined with the operations of SVNHFE in Definition 5, we have

$$\frac{T_1}{\sum_{j=1}^{n} T_j} \widetilde{n}_1^{\lambda} \oplus \frac{T_2}{\sum_{j=1}^{n} T_j} \widetilde{n}_2^{\lambda} \oplus \cdots \oplus \frac{T_k}{\sum_{j=1}^{n} T_j} \widetilde{n}_k^{\lambda} \oplus \frac{T_{k+1}}{\sum_{j=1}^{n} T_j} \widetilde{n}_{k+1}^{\lambda} =$$

$$= \bigcup_{\widetilde{\gamma}_1 \in \widetilde{t}_1, \widetilde{\gamma}_2 \in \widetilde{t}_2, \ldots, \widetilde{\gamma}_k \in \widetilde{t}_k, \widetilde{\delta}_1 \in \widetilde{i}_1, \widetilde{\delta}_2 \in \widetilde{i}_2, \ldots, \widetilde{\delta}_k \in \widetilde{i}_k, \widetilde{\eta}_1 \in \widetilde{f}_1, \widetilde{\eta}_2 \in \widetilde{f}_2, \ldots, \widetilde{\eta}_k \in \widetilde{f}_k} \left\{ \left\{ 1 - \prod_{j=1}^{k} \left(1 - \gamma_j^{\lambda}\right)^{\frac{T_j}{\sum_{j=1}^{n} T_j}} \right\}, \left\{ \prod_{j=1}^{k} \left(1 - (1 - \delta_j)^{\lambda}\right)^{\frac{T_j}{\sum_{j=1}^{n} T_j}} \right\}, \left\{ \prod_{j=1}^{k} \left(1 - (1 - \eta_j)^{\lambda}\right)^{\frac{T_j}{\sum_{j=1}^{n} T_j}} \right\} \right\} \oplus$$
$$\bigcup_{\gamma_{k+1} \in \widetilde{t}_{k+1}, \delta_{k+1} \in \widetilde{i}_{k+1}, \eta_{k+1} \in \widetilde{f}_{k+1}} \left\{ \left\{ 1 - \left(1 - \gamma_{k+1}^{\lambda}\right)^{\frac{T_{k+1}}{\sum_{j=1}^{n} T_j}} \right\}, \left\{ \left(1 - (1 - \delta_{k+1})^{\lambda}\right)^{\frac{T_{k+1}}{\sum_{j=1}^{n} T_j}} \right\}, \left\{ \left(1 - (1 - \eta_{k+1})^{\lambda}\right)^{\frac{T_{k+1}}{\sum_{j=1}^{n} T_j}} \right\} \right\}$$
$$= \bigcup_{\widetilde{\gamma}_1 \in \widetilde{t}_1, \widetilde{\gamma}_2 \in \widetilde{t}_2, \ldots, \widetilde{\gamma}_{k+1} \in \widetilde{t}_{k+1}, \widetilde{\delta}_1 \in \widetilde{i}_1, \widetilde{\delta}_2 \in \widetilde{i}_2, \ldots, \widetilde{\delta}_{k+1} \in \widetilde{i}_{k+1}, \widetilde{\eta}_1 \in \widetilde{f}_1, \widetilde{\eta}_2 \in \widetilde{f}_2, \ldots, \widetilde{\eta}_{k+1} \in \widetilde{f}_{k+1}} \left\{ \left\{ 1 - \prod_{j=1}^{k+1} \left(1 - \gamma_j^{\lambda}\right)^{\frac{T_j}{\sum_{j=1}^{n} T_j}} \right\}, \left\{ \prod_{j=1}^{k+1} \left(1 - (1 - \delta_j)^{\lambda}\right)^{\frac{T_j}{\sum_{j=1}^{n} T_j}} \right\}, \left\{ \prod_{j=1}^{k+1} \left(1 - (1 - \eta_j)^{\lambda}\right)^{\frac{T_j}{\sum_{j=1}^{n} T_j}} \right\} \right\}.$$

Then

$$\mathrm{GSVNHFPWA}_{\lambda}(\widetilde{n}_1, \widetilde{n}_2, \ldots, \widetilde{n}_{k+1}) = \left(\frac{T_1}{\sum_{j=1}^{n} T_j} \widetilde{n}_1^{\lambda} \oplus \frac{T_2}{\sum_{j=1}^{n} T_j} \widetilde{n}_2^{\lambda} \oplus \cdots \oplus \frac{T_{k+1}}{\sum_{j=1}^{n} T_j} \widetilde{n}_{k+1}^{\lambda} \right)^{1/\lambda}$$

$$= \bigcup_{\widetilde{\gamma}_1 \in \widetilde{t}_1, \widetilde{\gamma}_2 \in \widetilde{t}_2, \ldots, \widetilde{\gamma}_{k+1} \in \widetilde{t}_{k+1}, \widetilde{\delta}_1 \in \widetilde{i}_1, \widetilde{\delta}_2 \in \widetilde{i}_2, \ldots, \widetilde{\delta}_{k+1} \in \widetilde{i}_{k+1}, \widetilde{\eta}_1 \in \widetilde{f}_1, \widetilde{\eta}_2 \in \widetilde{f}_2, \ldots, \widetilde{\eta}_{k+1} \in \widetilde{f}_{k+1}} \left\{ \left\{ \left(1 - \prod_{j=1}^{k+1} \left(1 - \gamma_j^{\lambda}\right)^{\frac{T_j}{\sum_{j=1}^{n} T_j}} \right)^{1/\lambda} \right\}, \left\{ 1 - \left(1 - \prod_{j=1}^{k+1} \left(1 - (1 - \delta_j)^{\lambda}\right)^{\frac{T_j}{\sum_{j=1}^{n} T_j}} \right)^{1/\lambda} \right\}, \right.$$
$$\left. \left\{ 1 - \left(1 - \prod_{j=1}^{k+1} \left(1 - (1 - \eta_j)^{\lambda}\right)^{\frac{T_j}{\sum_{j=1}^{n} T_j}} \right)^{1/\lambda} \right\} \right\}.$$

i.e., Equation (11) holds for $n = k + 1$, thus we can confirm Equation (11) holds for all n. The proof of Theorem 1 is completed. □

Some desirable properties of the GSVNHFPWA operator are presented as below.

Theorem 2. *(Idempotency) Let* $\widetilde{n}_j = \left\{ \widetilde{t}_j, \widetilde{i}_j, \widetilde{f}_j \right\} (j = 1, 2, \ldots, n)$ *be a collection of SVNHFEs, where* $T_j = \prod_{k=1}^{j-1} s(\widetilde{n}_k) (j = 2, \ldots, n), T_1 = 1$, *and* $s(\widetilde{n}_k)$ *is the score function value of SVNHFE* $\widetilde{n}_k$. *If all* $\widetilde{n}_j = \left\{ \widetilde{t}_j, \widetilde{i}_j, \widetilde{f}_j \right\} (j = 1, 2, \ldots, n)$ *are equal, i.e.,* $\widetilde{n}_j = \widetilde{n} = \left\{ \widetilde{t}, \widetilde{i}, \widetilde{f} \right\}$, $\widetilde{t} = \gamma$, $\widetilde{i} = \delta$, *and* $\widetilde{f} = \eta$, *then*

$$\mathrm{GSVNHFPWA}_{\lambda}(\widetilde{n}_1, \widetilde{n}_2, \ldots, \widetilde{n}_n) = \widetilde{n} = \left\{ \widetilde{t}, \widetilde{i}, \widetilde{f} \right\}. \tag{12}$$

Proof. Since $\widetilde{n}_j = \widetilde{n} = \left\{\widetilde{t}, \widetilde{i}, \widetilde{f}\right\}$, by Theorem 1, we have

$$\mathrm{GSVNHFPWA}_\lambda(\widetilde{n}_1, \widetilde{n}_2, \ldots, \widetilde{n}_n) = \left(\frac{T_1}{\sum_{j=1}^n T_j}\widetilde{n}^\lambda \oplus \frac{T_2}{\sum_{j=1}^n T_j}\widetilde{n}^\lambda \oplus \cdots \oplus \frac{T_n}{\sum_{j=1}^n T_j}\widetilde{n}^\lambda\right)^{1/\lambda}$$

$$= \bigcup_{\widetilde{\gamma}\in\widetilde{t}, \widetilde{\delta}\in\widetilde{i}, \widetilde{\eta}\in\widetilde{f}} \left\{\left\{\left(1 - \prod_{j=1}^n \left(1-\gamma^\lambda\right)^{\frac{T_j}{\sum_{j=1}^n T_j}}\right)^{1/\lambda}\right\}, \left\{1 - \left(1 - \prod_{j=1}^n \left(1 - (1-\delta)^\lambda\right)^{\frac{T_j}{\sum_{j=1}^n T_j}}\right)^{1/\lambda}\right\}, \left\{1 - \left(1 - \prod_{j=1}^n \left(1 - (1-\eta)^\lambda\right)^{\frac{T_j}{\sum_{j=1}^n T_j}}\right)^{1/\lambda}\right\}\right\}$$

$$= \bigcup_{\widetilde{\gamma}\in\widetilde{t}, \widetilde{\delta}\in\widetilde{i}, \widetilde{\eta}\in\widetilde{f}} \left\{\left\{\left(1 - \left(1-\gamma^\lambda\right)^{\frac{\sum_{j=1}^n T_j}{\sum_{j=1}^n T_j}}\right)^{1/\lambda}\right\}, \left\{1 - \left(1 - \left(1 - (1-\delta)^\lambda\right)^{\frac{\sum_{j=1}^n T_j}{\sum_{j=1}^n T_j}}\right)^{1/\lambda}\right\}, \left\{1 - \left(1 - \left(1 - (1-\eta)^\lambda\right)^{\frac{\sum_{j=1}^n T_j}{\sum_{j=1}^n T_j}}\right)^{1/\lambda}\right\}\right\}$$

$$= \bigcup_{\widetilde{\gamma}\in\widetilde{t}, \widetilde{\delta}\in\widetilde{i}, \widetilde{\eta}\in\widetilde{f}} \left\{\left\{\left(\gamma^\lambda\right)^{1/\lambda}\right\}, \left\{1 - \left((1-\delta)^\lambda\right)^{1/\lambda}\right\}, \left\{1 - \left((1-\eta)^\lambda\right)^{1/\lambda}\right\}\right\} = \bigcup_{\widetilde{\gamma}\in\widetilde{t}, \widetilde{\delta}\in\widetilde{i}, \widetilde{\eta}\in\widetilde{f}} \{\{\gamma\}, \{\delta\}, \{\eta\}\} = \widetilde{n} = \left\{\widetilde{t}, \widetilde{i}, \widetilde{f}\right\}.$$

Then, the proof of Theorem 2 is completed. □

Theorem 3. *(Boundedness) Let* $\widetilde{n}_j = \left\{\widetilde{t}_j, \widetilde{i}_j, \widetilde{f}_j\right\}(j = 1, 2, \ldots, n)$ *be a collection of SVNHFEs, where* $T_j = \prod_{k=1}^{j-1} s(\widetilde{n}_k)(j = 2, \ldots, n), T_1 = 1,$ *and* $s(\widetilde{n}_k)$ *is the score function value of SVNHFE* $\widetilde{n}_k$*. And let* $\widetilde{n}^- = \{\{\gamma^-\}, \{\delta^+\}, \{\eta^+\}\}$ *and* $\widetilde{n}^+ = \{\{\gamma^+\}, \{\delta^-\}, \{\eta^-\}\}$*, where* $\gamma^+ = \cup_{\gamma_j\in\widetilde{t}_j}\max\{\gamma_j\}$*,* $\delta^+ = \cup_{\delta_j\in\widetilde{i}_j}\max\{\delta_j\}$*,* $\eta^+ = \cup_{\eta_j\in\widetilde{f}_j}\max\{\eta_j\}$*,* $\gamma^- = \cup_{\gamma_j\in\widetilde{t}_j}\min\{\gamma_j\}$*,* $\delta^- = \cup_{\delta_j\in\widetilde{i}_j}\min\{\delta_j\}$*, and* $\eta^- = \cup_{\eta_j\in\widetilde{f}_j}\min\{\eta_j\}$*. Then*

$$\widetilde{n}^- \leq \mathrm{GSVNHFPWA}_\lambda(\widetilde{n}_1, \widetilde{n}_2, \ldots, \widetilde{n}_n) \leq \widetilde{n}^+. \tag{13}$$

Proof. Since $\gamma^- \leq \gamma_j \leq \gamma^+$, $\delta^- \leq \delta_j \leq \delta^+$, and $\eta^- \leq \eta_j \leq \eta^+$. First, when $\lambda \in (0, \infty)$, then

$$\gamma_j^\lambda \geq \left(\gamma^-\right)^\lambda, 1 - \gamma_j^\lambda \leq 1 - \left(\gamma^-\right)^\lambda, \left(1 - \gamma_j^\lambda\right)^{\frac{T_j}{\sum_{j=1}^n T_j}} \leq \left(1 - \left(\gamma^-\right)^\lambda\right)^{\frac{T_j}{\sum_{j=1}^n T_j}},$$

$$\prod_{j=1}^n \left(1 - \gamma_j^\lambda\right)^{\frac{T_j}{\sum_{j=1}^n T_j}} \leq \prod_{j=1}^n \left(1 - \left(\gamma^-\right)^\lambda\right)^{\frac{T_j}{\sum_{j=1}^n T_j}},$$

$$1 - \prod_{j=1}^n \left(1 - \gamma_j^\lambda\right)^{\frac{T_j}{\sum_{j=1}^n T_j}} \geq 1 - \prod_{j=1}^n \left(1 - \left(\gamma^-\right)^\lambda\right)^{\frac{T_j}{\sum_{j=1}^n T_j}},$$

$$\left(1 - \prod_{j=1}^n \left(1 - \gamma_j^\lambda\right)^{\frac{T_j}{\sum_{j=1}^n T_j}}\right)^{1/\lambda} \geq \left(1 - \prod_{j=1}^n \left(1 - \left(\gamma^-\right)^\lambda\right)^{\frac{T_j}{\sum_{j=1}^n T_j}}\right)^{1/\lambda} = \gamma^-.$$

Similarly, we have

$$\left(1 - \prod_{j=1}^n \left(1 - \gamma_j^\lambda\right)^{\frac{T_j}{\sum_{j=1}^n T_j}}\right)^{1/\lambda} \leq \left(1 - \prod_{j=1}^n \left(1 - \left(\gamma^+\right)^\lambda\right)^{\frac{T_j}{\sum_{j=1}^n T_j}}\right)^{1/\lambda} = \gamma^+.$$

And as $\delta^- \leq \delta_j \leq \delta^+$, then

$$1 - \delta_j \leq 1 - \delta^-, \left(1 - \delta_j\right)^\lambda \leq \left(1 - \delta^-\right)^\lambda, 1 - \left(1 - \delta_j\right)^\lambda \geq 1 - \left(1 - \delta^-\right)^\lambda,$$

$$\left(1 - \left(1 - \delta_j\right)^\lambda\right)^{\frac{T_j}{\sum_{j=1}^n T_j}} \geq \left(1 - \left(1 - \delta^-\right)^\lambda\right)^{\frac{T_j}{\sum_{j=1}^n T_j}},$$

$$\prod_{j=1}^{n}\left(1-\left(1-\delta_j\right)^{\lambda}\right)^{\frac{T_j}{\sum_{j=1}^{n}T_j}} \geq \prod_{j=1}^{n}\left(1-\left(1-\delta^{-}\right)^{\lambda}\right)^{\frac{T_j}{\sum_{j=1}^{n}T_j}},$$

$$1-\prod_{j=1}^{n}\left(1-\left(1-\delta_j\right)^{\lambda}\right)^{\frac{T_j}{\sum_{j=1}^{n}T_j}} \leq 1-\prod_{j=1}^{n}\left(1-\left(1-\delta^{-}\right)^{\lambda}\right)^{\frac{T_j}{\sum_{j=1}^{n}T_j}},$$

$$\left(1-\prod_{j=1}^{n}\left(1-\left(1-\delta_j\right)^{\lambda}\right)^{\frac{T_j}{\sum_{j=1}^{n}T_j}}\right)^{1/\lambda} \leq \left(1-\prod_{j=1}^{n}\left(1-\left(1-\delta^{-}\right)^{\lambda}\right)^{\frac{T_j}{\sum_{j=1}^{n}T_j}}\right)^{1/\lambda},$$

$$1-\left(1-\prod_{j=1}^{n}\left(1-\left(1-\delta_j\right)^{\lambda}\right)^{\frac{T_j}{\sum_{j=1}^{n}T_j}}\right)^{1/\lambda} \geq 1-\left(1-\prod_{j=1}^{n}\left(1-\left(1-\delta^{-}\right)^{\lambda}\right)^{\frac{T_j}{\sum_{j=1}^{n}T_j}}\right)^{1/\lambda} = \delta^{-}.$$

Similarly, we have

$$1-\left(1-\prod_{j=1}^{n}\left(1-\left(1-\delta_j\right)^{\lambda}\right)^{\frac{T_j}{\sum_{j=1}^{n}T_j}}\right)^{1/\lambda} \leq 1-\left(1-\prod_{j=1}^{n}\left(1-\left(1-\delta^{+}\right)^{\lambda}\right)^{\frac{T_j}{\sum_{j=1}^{n}T_j}}\right)^{1/\lambda} = \delta^{+}.$$

On the other hand,

$$\eta^{-} \leq 1-\left(1-\prod_{j=1}^{n}\left(1-\left(1-\delta_j\right)^{\lambda}\right)^{\frac{T_j}{\sum_{j=1}^{n}T_j}}\right)^{1/\lambda} \leq \eta^{+}.$$

Let $\text{GSVNHFPWA}_{\lambda}(\widetilde{n}_1, \widetilde{n}_2, \ldots, \widetilde{n}_n) = \widetilde{n} = \{\{\gamma\}, \{\delta\}, \{\eta\}\}$, then

$$s(\widetilde{n}) = \frac{\frac{1}{l}\sum_{i=1}^{l}\gamma_i + \frac{1}{p}\sum_{i=1}^{p}(1-\delta_i) + \frac{1}{q}\sum_{i=1}^{q}(1-\eta_i)}{3} \geq \frac{\frac{1}{l^{-}}\sum_{i=1}^{l^{-}}\gamma_i^{-} + \frac{1}{p^{-}}\sum_{i=1}^{p^{-}}(1-\delta_i^{+}) + \frac{1}{q^{-}}\sum_{i=1}^{q^{-}}(1-\eta_i^{+})}{3} = s(\widetilde{n}^{-}),$$

And

$$s(\widetilde{n}) = \frac{\frac{1}{l}\sum_{i=1}^{l}\gamma_i + \frac{1}{p}\sum_{i=1}^{p}(1-\delta_i) + \frac{1}{q}\sum_{i=1}^{q}(1-\eta_i)}{3} \leq \frac{\frac{1}{l^{+}}\sum_{i=1}^{l^{+}}\gamma_i^{+} + \frac{1}{p^{+}}\sum_{i=1}^{p^{+}}(1-\delta_i^{-}) + \frac{1}{q^{+}}\sum_{i=1}^{q^{+}}(1-\eta_i^{-})}{3} = s(\widetilde{n}^{+}).$$

If $s(\widetilde{n}^{-}) < s(\widetilde{n}) < s(\widetilde{n}^{+})$, we have

$$\widetilde{n}^{-} < \text{GSVNHFPWA}_{\lambda}(\widetilde{n}_1, \widetilde{n}_2, \ldots, \widetilde{n}_n) < \widetilde{n}^{+}.$$

If $s(\widetilde{n}) = s(\widetilde{n}^{-})$, i.e.,

$$\frac{\frac{1}{l}\sum_{i=1}^{l}\gamma_i + \frac{1}{p}\sum_{i=1}^{p}(1-\delta_i) + \frac{1}{q}\sum_{i=1}^{q}(1-\eta_i)}{3} = \frac{\frac{1}{l^{-}}\sum_{i=1}^{l^{-}}\gamma_i^{-} + \frac{1}{p^{-}}\sum_{i=1}^{p^{-}}(1-\delta_i^{+}) + \frac{1}{q^{-}}\sum_{i=1}^{q^{-}}(1-\eta_i^{+})}{3},$$

Then

$$\text{GSVNHFPWA}_{\lambda}(\widetilde{n}_1, \widetilde{n}_2, \ldots, \widetilde{n}_n) = \widetilde{n}^{-}.$$

If $s(\widetilde{n}) = s(\widetilde{n}^{-})$, i.e.,

$$\frac{\frac{1}{l}\sum_{i=1}^{l}\gamma_i + \frac{1}{p}\sum_{i=1}^{p}(1-\delta_i) + \frac{1}{q}\sum_{i=1}^{q}(1-\eta_i)}{3} \leq \frac{\frac{1}{l^{+}}\sum_{i=1}^{l^{+}}\gamma_i^{+} + \frac{1}{p^{+}}\sum_{i=1}^{p^{+}}(1-\delta_i^{-}) + \frac{1}{q^{+}}\sum_{i=1}^{q^{+}}(1-\eta_i^{-})}{3},$$

Then

$$\text{GSVNHFPWA}_{\lambda}(\widetilde{n}_1, \widetilde{n}_2, \ldots, \widetilde{n}_n) = \widetilde{n}^{+}.$$

Based on analysis above, we have

$$\widetilde{n}^{-} \leq \text{GSVNHFPWA}_{\lambda}(\widetilde{n}_1, \widetilde{n}_2, \ldots, \widetilde{n}_n) \leq \widetilde{n}^{+}\lambda \in (0, \infty).$$

Similarly, we can obtain

$$\widetilde{n}^{-} \leq \text{GSVNHFPWA}_{\lambda}(\widetilde{n}_1, \widetilde{n}_2, \ldots, \widetilde{n}_n) \leq \widetilde{n}^{+}\lambda \in (-\infty, 0).$$

The proof of Theorem 3 is completed. □

Theorem 4. *(Monotonicity) Let* $\widetilde{n}_j = \left\{\widetilde{t}_j, \widetilde{i}_j, \widetilde{f}_j\right\}(j = 1, 2, \ldots, n)$ *and* $\widetilde{n}_j^{*} = \left\{\widetilde{t}_j^{*}, \widetilde{i}_j^{*}, \widetilde{f}_j^{*}\right\}(j = 1, 2, \ldots, n)$ *be two collections of SVNHFEs, where* $T_j = \prod_{k=1}^{j-1} s(\widetilde{n}_k)(j = 2, \ldots, n), T_j^{*} = \prod_{k=1}^{j-1} s(\widetilde{n}_k^{*})(j = 2, \ldots, n)$, $T_1 = T_1^{*} = 1$, $s(\widetilde{n}_k)$ *and* $s(\widetilde{n}_k^{*})$ *are the score values of SVNHFE* $\widetilde{n}_k$ *and* $\widetilde{n}_k^{*}$, *respectively. If* $\widetilde{n}_j \leq \widetilde{n}_j^{*}$ $(j = 1, 2, \ldots, n)$, *then*

$$\text{GSVNHFPWA}_{\lambda}(\widetilde{n}_1, \widetilde{n}_2, \ldots, \widetilde{n}_n) \leq \text{GSVNHFPWA}_{\lambda}(\widetilde{n}_1^{*}, \widetilde{n}_2^{*}, \ldots, \widetilde{n}_n^{*}). \tag{14}$$

Proof. It directly follows from Theorem 3. □

Special cases of the GSVNHFPWA operator are shown as follows.

(1) If $\lambda = 1$, then the GSVNHFPWA operator is reduced to the single-valued neutrosophic hesitant fuzzy prioritized weighted average (SVNHFPWA) operator:

$$\text{SVNHFPWA}(\widetilde{n}_1, \widetilde{n}_2, \ldots, \widetilde{n}_n) = \left(\frac{T_1}{\sum_{j=1}^{n} T_j}\widetilde{n}_1 \oplus \frac{T_2}{\sum_{j=1}^{n} T_j}\widetilde{n}_2 \oplus \cdots \oplus \frac{T_n}{\sum_{j=1}^{n} T_j}\widetilde{n}_n\right). \tag{15}$$

(2) If $\lambda \to 0$, then the GSVNHFPWA operator is reduced to the single-valued neutrosophic hesitant fuzzy prioritized weighted geometric (SVNHFPWG) operator:

$$\text{SVNHFPWG}(\widetilde{n}_1, \widetilde{n}_2, \ldots, \widetilde{n}_n) = \left((\widetilde{n}_1)^{\frac{T_1}{\sum_{j=1}^{n} T_j}} \otimes (\widetilde{n}_2)^{\frac{T_2}{\sum_{j=1}^{n} T_j}} \otimes \cdots \otimes (\widetilde{n}_n)^{\frac{T_n}{\sum_{j=1}^{n} T_j}}\right). \tag{16}$$

(3) If $\lambda = 2$, then the GSVNHFPWA operator is reduced to the single-valued neutrosophic hesitant fuzzy prioritized weighted quadratic average (SVNHFPWQA) operator:

$$\text{SVNHFPWQA}(\widetilde{n}_1, \widetilde{n}_2, \ldots, \widetilde{n}_n) = \left(\frac{T_1}{\sum_{j=1}^{n} T_j}\widetilde{n}_1^{2} \oplus \frac{T_2}{\sum_{j=1}^{n} T_j}\widetilde{n}_2^{2} \oplus \cdots \oplus \frac{T_n}{\sum_{j=1}^{n} T_j}\widetilde{n}_n^{2}\right)^{1/2}. \tag{17}$$

(4) If $\lambda = 3$, then the GSVNHFPWA operator is reduced to the single-valued neutrosophic hesitant fuzzy prioritized weighted cubic average (SVNHFPWCA) operator:

$$\text{SVNHFPWCA}(\widetilde{n}_1, \widetilde{n}_2, \ldots, \widetilde{n}_n) = \left(\frac{T_1}{\sum_{j=1}^{n} T_j}\widetilde{n}_1^{3} \oplus \frac{T_2}{\sum_{j=1}^{n} T_j}\widetilde{n}_2^{3} \oplus \cdots \oplus \frac{T_n}{\sum_{j=1}^{n} T_j}\widetilde{n}_n^{3}\right)^{1/3}. \tag{18}$$

(5) If $\lambda = 1$ and the aggregated arguments are in the same priority level, then the GSVNHFPWA operator is reduced to the single-valued neutrosophic hesitant fuzzy weighted average (SVNHFWA) operator [14]:

$$\text{SVNHFWA}(\widetilde{n}_1, \widetilde{n}_2, \ldots, \widetilde{n}_n) = (w_1\widetilde{n}_1 \oplus w_2\widetilde{n}_2 \oplus \cdots \oplus w_n\widetilde{n}_n). \tag{19}$$

(6) If $\lambda \to 0$ and the aggregated arguments are in the same priority level, then the GSVNHFPWA operator is reduced to the single-valued neutrosophic hesitant fuzzy weighted geometric (SVNHFWG) operator [14]:

$$\text{SVNHFWG}(\widetilde{n}_1, \widetilde{n}_2, \ldots, \widetilde{n}_n) = (\widetilde{n}_1^{w_1} \otimes \widetilde{n}_2^{w_2} \otimes \cdots \otimes \widetilde{n}_n^{w_n}). \tag{20}$$

(7) If $w = (1/n, 1/n, \ldots, 1/n)^T$, $\lambda = 1$, and the aggregated arguments are in the same priority level, then the GSVNHFPWA operator is reduced to the single-valued neutrosophic hesitant fuzzy arithmetic average (SVNHFAA) operator:

$$\text{SVNHFAA}(\widetilde{n}_1, \widetilde{n}_2, \ldots, \widetilde{n}_n) = \frac{1}{n}(\widetilde{n}_1 \oplus \widetilde{n}_2 \oplus \cdots \oplus \widetilde{n}_n). \tag{21}$$

(8) If $w = (1/n, 1/n, \ldots, 1/n)^T$, $\lambda \to 0$, and the aggregated arguments are in the same priority level, then the GSVNHFPWA operator is reduced to the single-valued neutrosophic hesitant fuzzy geometric average (SVNHFGA) operator:

$$\text{SVNHFGA}(\widetilde{n}_1, \widetilde{n}_2, \ldots, \widetilde{n}_n) = (\widetilde{n}_1 \otimes \widetilde{n}_2 \otimes \cdots \otimes \widetilde{n}_n)^{1/n}. \tag{22}$$

3.2. Generalized Single-Valued Neutrosophic Hesitant Fuzzy Prioritized Geometric Operator

Based on the GSVNHFPWA operator investigated above, we develop the GSVNHFPWG operator as the following.

Definition 10. *Let* $\widetilde{n}_j = \left\{\widetilde{t}_j, \widetilde{i}_j, \widetilde{f}_j\right\} (j = 1, 2, \ldots, n)$ *be a collection of SVNHFEs, and let* $\text{GSVNHFPWG}: \Omega^n \to \Omega$, *if*

$$\text{GSVNHFPWG}_\lambda(\widetilde{n}_1, \widetilde{n}_2, \ldots, \widetilde{n}_n) = \frac{1}{\lambda}\left((\lambda\widetilde{n}_1)^{\frac{T_1}{\sum_{j=1}^n T_j}} \otimes (\lambda\widetilde{n}_2)^{\frac{T_2}{\sum_{j=1}^n T_j}} \otimes \cdots \otimes (\lambda\widetilde{n}_n)^{\frac{T_n}{\sum_{j=1}^n T_j}}\right), \tag{23}$$

then the function GSVNHFPWG is called the GSVNHFPWG operator. Where $T_j = \prod_{k=1}^{j-1} s(\widetilde{n}_k) (j = 2, \ldots, n)$, $T_1 = 1$, *and* $s(\widetilde{n}_k)$ *is the score function value of SVNHFE* $\widetilde{n}_k$.

Similarly, according to the operations of SVHFEs in Definition 5, the theorem is obtained as below.

Theorem 5. *Let* $\widetilde{n}_j = \left\{\widetilde{t}_j, \widetilde{i}_j, \widetilde{f}_j\right\} (j = 1, 2, \ldots, n)$ *be a collection of SVNHFEs, then their aggregated value by using the GSVNHFPWG operator is also an SVNHFE, and*

$$\begin{aligned}
&\text{GSVNHFPWG}_\lambda(\widetilde{n}_1, \widetilde{n}_2, \ldots, \widetilde{n}_n) = \frac{1}{\lambda}\left((\lambda\widetilde{n}_1)^{\frac{T_1}{\sum_{j=1}^n T_j}} \otimes (\lambda\widetilde{n}_2)^{\frac{T_2}{\sum_{j=1}^n T_j}} \otimes \cdots \otimes (\lambda\widetilde{n}_n)^{\frac{T_n}{\sum_{j=1}^n T_j}}\right) \\
&= \bigcup_{\widetilde{\gamma}_1 \in \widetilde{t}_1, \widetilde{\gamma}_2 \in \widetilde{t}_2, \ldots, \widetilde{\gamma}_n \in \widetilde{t}_n, \widetilde{\delta}_1 \in \widetilde{i}_1, \widetilde{\delta}_2 \in \widetilde{i}_2, \ldots, \widetilde{\delta}_n \in \widetilde{i}_n, \widetilde{\eta}_1 \in \widetilde{f}_1, \widetilde{\eta}_2 \in \widetilde{f}_2, \ldots, \widetilde{\eta}_n \in \widetilde{f}_n} \left\{\left\{1 - \left(1 - \prod_{j=1}^n \left(1 - (1-\gamma_j)^\lambda\right)^{\frac{T_j}{\sum_{j=1}^n T_j}}\right)^{1/\lambda}\right\}, \left\{\left(1 - \prod_{j=1}^n \left(1 - \delta_j^\lambda\right)^{\frac{T_j}{\sum_{j=1}^n T_j}}\right)^{1/\lambda}\right\},\right. \\
&\left.\left\{\left(1 - \prod_{j=1}^n \left(1 - \eta_j^\lambda\right)^{\frac{T_j}{\sum_{j=1}^n T_j}}\right)^{1/\lambda}\right\}\right\}.
\end{aligned} \tag{24}$$

where $T_j = \prod_{k=1}^{j-1} s(\widetilde{n}_k) (j = 2, \ldots, n)$, $T_1 = 1$, *and* $s(\widetilde{n}_k)$ *is the score function value of SVNHFE* $\widetilde{n}_k$.

Proof. The proof procedure of Theorem 5 is similar to Theorem 1. □

Some desirable properties of the GSVNHFPWG operator are presented as below.

Theorem 6. *(Idempotency) Let* $\widetilde{n}_j = \left\{\widetilde{t}_j, \widetilde{i}_j, \widetilde{f}_j\right\} (j = 1, 2, \ldots, n)$ *be a collection of SVNHFEs, where* $T_j = \prod_{k=1}^{j-1} s(\widetilde{n}_k) (j = 2, \ldots, n), T_1 = 1$, *and* $s(\widetilde{n}_k)$ *is the score function value of SVNHFE* $\widetilde{n}_k$. *If all* $\widetilde{n}_j = \left\{\widetilde{t}_j, \widetilde{i}_j, \widetilde{f}_j\right\} (j = 1, 2, \ldots, n)$ *are equal, i.e.,* $\widetilde{n}_j = \widetilde{n} = \left\{\widetilde{t}, \widetilde{i}, \widetilde{f}\right\}$, $\widetilde{t} = \gamma$, $\widetilde{i} = \delta$, *and* $\widetilde{f} = \eta$, *then*

$$\mathrm{GSVNHFPWG}_\lambda(\widetilde{n}_1, \widetilde{n}_2, \ldots, \widetilde{n}_n) = \widetilde{n} = \left\{\widetilde{t}, \widetilde{i}, \widetilde{f}\right\}. \tag{25}$$

Proof. The proof procedure of Theorem 6 is similar to Theorem 2. □

Theorem 7. *(Boundedness) Let* $\widetilde{n}_j = \left\{\widetilde{t}_j, \widetilde{i}_j, \widetilde{f}_j\right\} (j = 1, 2, \ldots, n)$ *be a collection of SVNHFEs, where* $T_j = \prod_{k=1}^{j-1} s(\widetilde{n}_k) (j = 2, \ldots, n), T_1 = 1$, *and* $s(\widetilde{n}_k)$ *is the score value of SVNHFE* $\widetilde{n}_k$. *And let* $\widetilde{n}^- = \{\{\gamma^-\}, \{\delta^+\}, \{\eta^+\}\}$ *and* $\widetilde{n}^+ = \{\{\gamma^+\}, \{\delta^-\}, \{\eta^-\}\}$, *where* $\gamma^+ = \cup_{\gamma_j \in \widetilde{t}_j} \max\{\gamma_j\}$, $\delta^+ = \cup_{\delta_j \in \widetilde{i}_j} \max\{\delta_j\}$, $\eta^+ = \cup_{\eta_j \in \widetilde{f}_j} \max\{\eta_j\}$, $\gamma^- = \cup_{\gamma_j \in \widetilde{t}_j} \min\{\gamma_j\}$, $\delta^- = \cup_{\delta_j \in \widetilde{i}_j} \min\{\delta_j\}$, *and* $\eta^- = \cup_{\eta_j \in \widetilde{f}_j} \min\{\eta_j\}$. *Then*

$$\widetilde{n}^- \leq \mathrm{GSVNHFPWG}_\lambda(\widetilde{n}_1, \widetilde{n}_2, \ldots, \widetilde{n}_n) \leq \widetilde{n}^+. \tag{26}$$

Proof. The proof procedure of Theorem 7 is similar to Theorem 3. □

Theorem 8. *(Monotonicity) Let* $\widetilde{n}_j = \left\{\widetilde{t}_j, \widetilde{i}_j, \widetilde{f}_j\right\} (j = 1, 2, \ldots, n)$ *and* $\widetilde{n}_j^* = \left\{\widetilde{t}_j^*, \widetilde{i}_j^*, \widetilde{f}_j^*\right\} (j = 1, 2, \ldots, n)$ *be two collections of SVNHFEs, where* $T_j = \prod_{k=1}^{j-1} s(\widetilde{n}_k) (j = 2, \ldots, n)$, $T_j^* = \prod_{k=1}^{j-1} s(\widetilde{n}_k^*) (j = 2, \ldots, n)$, $T_1 = T_1^* = 1$, $s(\widetilde{n}_k)$ *and* $s(\widetilde{n}_k^*)$ *are the score function values of SVNHFE* $\widetilde{n}_k$ *and* $\widetilde{n}_k^*$, *respectively. If* $\widetilde{n}_j \leq \widetilde{n}_j^* (j = 1, 2, \ldots, n)$, *then*

$$\mathrm{GSVNHFPWG}_\lambda(\widetilde{n}_1, \widetilde{n}_2, \ldots, \widetilde{n}_n) \leq \mathrm{GSVNHFPWG}_\lambda(\widetilde{n}_1^*, \widetilde{n}_2^*, \ldots, \widetilde{n}_n^*). \tag{27}$$

Proof. It directly follows from Theorem 7. □

Special cases of the GSVNHFPWG operator are shown as follows:

(1) If $\lambda = 1$, then the GSVNHFPWG operator is reduced to the single-valued neutrosophic hesitant fuzzy prioritized weighted geometric (SVNHFPWG) operator:

$$\mathrm{SVNHFPWG}(\widetilde{n}_1, \widetilde{n}_2, \ldots, \widetilde{n}_n) = \left((\widetilde{n}_1)^{\frac{T_1}{\sum_{j=1}^n T_j}} \otimes (\widetilde{n}_2)^{\frac{T_2}{\sum_{j=1}^n T_j}} \otimes \cdots \otimes (\widetilde{n}_n)^{\frac{T_n}{\sum_{j=1}^n T_j}} \right). \tag{28}$$

(2) If $\lambda = 1$ and the aggregated arguments are in the same priority level, then the GSVNHFPWG operator is reduced to the SVNHFWG operator [14]:

$$\mathrm{SVNHFWG}(\widetilde{n}_1, \widetilde{n}_2, \ldots, \widetilde{n}_n) = \left((\widetilde{n}_1)^{w_1} \otimes (\widetilde{n}_2)^{w_2} \otimes \cdots \otimes (\widetilde{n}_n)^{w_n} \right). \tag{29}$$

(3) If $w = (1/n, 1/n, \ldots, 1/n)^T$, $\lambda = 1$, and the aggregated arguments are in the same priority level, then the GSVNHFPWG operator is reduced to the SVNHFGA operator:

$$\mathrm{SVNHFGA}(\widetilde{n}_1, \widetilde{n}_2, \ldots, \widetilde{n}_n) = (\widetilde{n}_1 \otimes \widetilde{n}_2 \otimes \cdots \otimes \widetilde{n}_n)^{1/n}. \tag{30}$$

4. An Approach for Decision-Making under Single-Valued Neutrosophic Hesitant Fuzzy Environment

In this section, we utilize the GSVNHFPWA operator and GSVNHFPWG operator to solve the MCDM problems under SVNHF environment, respectively. For a MCDM problem, let $A = \{A_1, A_2, \ldots A_m\}$ be a set of m alternatives to be evaluated, $C = \{C_1, C_2, \ldots, C_n\}$ be a collection of criteria that prioritizations between the criteria expressed by the linear ordering $C_1 \succ C_2 \succ \cdots \succ C_n$ exist, i.e., criteria C_j has a higher priority level than the criteria C_k if $j < k$. Decision makers evaluates the alternatives over the criteria by using SVNHFEs, let $N = \left(\tilde{n}_{ij}\right)_{m\times n}(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$ be an SVNHF decision matrix, and $\tilde{n}_{ij} = \left\{\tilde{t}_{ij}, \tilde{i}_{ij}, \tilde{f}_{ij}\right\}$ is the evaluation information given by decision maker. Where $\tilde{t}_{ij} = \{\gamma_{ij} | \gamma_{ij} \in \tilde{t}_{ij}\}$ represents the possible degrees that the alternative A_i satisfies the criteria C_j provided by decision maker, $\tilde{i}_{ij} = \left\{\delta_{ij} \middle| \delta_{ij} \in \tilde{i}_{ij}\right\}$ represents the possible indeterminacy degrees that decision maker judges whether the alternative A_i satisfies the criteria C_j, and $\tilde{f}_{ij} = \left\{\eta_{ij} \middle| \eta_{ij} \in \tilde{f}_{ij}\right\}$ represents the possible degrees that the alternative A_i does not satisfy the criteria C_j provided by decision maker.

Based on the assumptions above, we use the GSVNHFPWA operator or GSVNHFPWG operator to construct an approach for decision-making under SVNHF environment. The main steps are presented below.

Step 1. Calculate the values of $T_{ij}(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n)$ by the equations as follows.

$$T_{ij} = \prod\nolimits_{k=1}^{j-1} s(\tilde{n}_{ik})(i = 1, 2, \ldots, m; j = 1, 2, \ldots, n), T_{i1} = 1. \tag{31}$$

Step 2. Utilize the GSVNHFPWA operator:

$$\begin{aligned}
\tilde{n}_i &= \mathrm{GSVNHFPWA}_\lambda(\tilde{n}_{i1}, \tilde{n}_{i2}, \ldots, \tilde{n}_{in}) = \left(\frac{T_{i1}}{\sum_{j=1}^n T_{ij}}(\tilde{n}_{i1})^\lambda \oplus \frac{T_{i2}}{\sum_{j=1}^n T_{ij}}(\tilde{n}_{i2})^\lambda \oplus \cdots \oplus \frac{T_{in}}{\sum_{j=1}^n T_{ij}}(\tilde{n}_{in})^\lambda\right)^{1/\lambda} \\
&= \bigcup_{\tilde{\gamma}_{i1}\in\tilde{t}_{i1}, \tilde{\gamma}_{i2}\in\tilde{t}_{i2}, \ldots, \tilde{\gamma}_{in}\in\tilde{t}_{in}, \tilde{\delta}_{i1}\in\tilde{i}_{i1}, \tilde{\delta}_{i2}\in\tilde{i}_{i2}, \ldots, \tilde{\delta}_{in}\in\tilde{i}_{in}, \tilde{\eta}_{i1}\in\tilde{f}_{i1}, \tilde{\eta}_{i2}\in\tilde{f}_{i2}, \ldots, \tilde{\eta}_{in}\in\tilde{f}_{in}} \left\{\left\{\left(1 - \prod_{j=1}^n \left(1 - (\gamma_{ij})^\lambda\right)^{\frac{T_{ij}}{\sum_{j=1}^n T_{ij}}}\right)^{1/\lambda}\right\},\right. \\
&\left.\left\{1 - \left(1 - \prod_{j=1}^n \left(1 - (1 - \delta_{ij})^\lambda\right)^{\frac{T_{ij}}{\sum_{j=1}^n T_{ij}}}\right)^{1/\lambda}\right\}, \left\{1 - \left(1 - \prod_{j=1}^n \left(1 - (1 - \eta_{ij})^\lambda\right)^{\frac{T_{ij}}{\sum_{j=1}^n T_{ij}}}\right)^{1/\lambda}\right\}\right\}.
\end{aligned} \tag{32}$$

or the GSVNHFPWG operator:

$$\begin{aligned}
\tilde{n}_i &= \mathrm{GSVNHFPWG}_\lambda(\tilde{n}_{i1}, \tilde{n}_{i2}, \ldots, \tilde{n}_{in}) = \frac{1}{\lambda}\left((\lambda\tilde{n}_{i1})^{\frac{T_{i1}}{\sum_{j=1}^n T_{ij}}} \otimes (\lambda\tilde{n}_{i2})^{\frac{T_{i2}}{\sum_{j=1}^n T_{ij}}} \otimes \cdots \otimes (\lambda\tilde{n}_{in})^{\frac{T_{in}}{\sum_{j=1}^n T_{ij}}}\right) \\
&= \bigcup_{\tilde{\gamma}_{i1}\in\tilde{t}_{i1}, \tilde{\gamma}_{i2}\in\tilde{t}_{i2}, \ldots, \tilde{\gamma}_{in}\in\tilde{t}_{in}, \tilde{\delta}_{i1}\in\tilde{i}_{i1}, \tilde{\delta}_{i2}\in\tilde{i}_{i2}, \ldots, \tilde{\delta}_{in}\in\tilde{i}_{in}, \tilde{\eta}_{i1}\in\tilde{f}_{i1}, \tilde{\eta}_{i2}\in\tilde{f}_{i2}, \ldots, \tilde{\eta}_{in}\in\tilde{f}_{in}} \left\{\left\{1 - \left(1 - \prod_{j=1}^n \left(1 - (1 - \gamma_{ij})^\lambda\right)^{\frac{T_{ij}}{\sum_{j=1}^n T_{ij}}}\right)^{1/\lambda}\right\},\right. \\
&\left.\left\{\left(1 - \prod_{j=1}^n \left(1 - (\delta_{ij})^\lambda\right)^{\frac{T_{ij}}{\sum_{j=1}^n T_{ij}}}\right)^{1/\lambda}\right\}, \left\{\left(1 - \prod_{j=1}^n \left(1 - (\eta_{ij})^\lambda\right)^{\frac{T_{ij}}{\sum_{j=1}^n T_{ij}}}\right)^{1/\lambda}\right\}\right\}.
\end{aligned} \tag{33}$$

to aggregate the SVNHF decision matrix $N = \left(\tilde{n}_{ij}\right)_{m\times n}$ into the SVNHFE $\tilde{n}_i = \left\{\tilde{t}_i, \tilde{i}_i, \tilde{f}_i\right\}$ of each alternative.

Step 3. Rank all the alternatives by calculating the score function value of the SVNHFE $\widetilde{n}_i = \left\{\widetilde{t}_i, \widetilde{i}_i, \widetilde{f}_i\right\}$ combined with Definition 6.

$$s(\widetilde{n}_i) = \left[\frac{1}{l_i}\sum\nolimits_{\gamma_i \in \widetilde{t}_i} \gamma_i + \frac{1}{p_i}\sum\nolimits_{\delta_i \in \widetilde{i}_i} (1-\delta_i) + \frac{1}{q_i}\sum\nolimits_{\eta_i \in \widetilde{f}_i} (1-\eta_i)\right] \Big/ 3. \tag{34}$$

Then the bigger the score function value $s(\widetilde{n}_i)$, the higher the ranking of alternative x_i will be.

5. Numerical Example

In this section, we apply a numerical example of MCDM problem under SVNHF environment to illustrate the applications and advantages of the proposed method [14].

5.1. Implementation

Suppose that an investment company wants to invest a sum of money in a target company. After a market survey, four alternative companies are identified to be chosen from, namely, a car company (A_1), a food company (A_2), a computer company (A_3), and an arms company (A_4). To evaluate the investment potential of a company needs to consider many aspects, such as the growth prospects of the company, risk degree of the investment, and the impact of the company on the environment. Therefore, the investment company shall evaluate the four alternative companies above with respect to three criteria, namely, the environmental impact (C_1), the risk (C_2), and the growth (C_3). In the real decision-making process, compared with determining the weights of criteria, identifying the priority level of criteria is more feasible and accurate. Then, according to the weight vector of three criteria $\boldsymbol{w} = (0.40, 0.35, 0.25)^T$ [14], we set up the criteria C_1 with the first priority level, followed by criteria C_2 and C_3. Decision makers from the investment company express the evaluation information combined with SVNHFEs, and the SVNHF decision matrix $N = \left(\widetilde{n}_{ij}\right)_{m \times n}$ is obtained shown in Table 1 [14].

Table 1. SVNHF decision matrix.

Alternatives	C_1	C_2	C_3
A_1	{{0.2, 0.3}, {0.1, 0.2}, {0.5, 0.6}}	{{0.3, 0.4, 0.5}, {0.1}, {0.3, 0.4}}	{{0.5, 0.6}, {0.2, 0.3}, {0.3, 0.4}}
A_2	{{0.6, 0.7}, {0.1, 0.2}, {0.1, 0.2}}	{{0.6, 0.7}, {0.1, 0.2}, {0.2, 0.3}}	{{0.6, 0.7}, {0.1}, {0.3}}
A_3	{{0.5, 0.6}, {0.1}, {0.3}}	{{0.5, 0.6}, {0.4}, {0.2, 0.3}}	{{0.6}, {0.3}, {0.4}}
A_4	{{0.3, 0.5}, {0.2}, {0.1, 0.2, 0.3}}	{{0.7, 0.8}, {0.1}, {0.1, 0.2}}	{{0.6, 0.7}, {0.1}, {0.2}}

Then, we use the proposed method to determine the ranking result of the four alternative companies, which are presented as follows.

Step 1. Calculate the values of $T_{ij}(i = 1,2,3,4; j = 1,2,3)$ according to Equation (31) as follows:

$$T_{ij} = \begin{bmatrix} 1.000 & 0.5167 & 0.3358 \\ 1.000 & 0.7833 & 0.5875 \\ 1.000 & 0.7167 & 0.4539 \\ 1.000 & 0.6667 & 0.5556 \end{bmatrix}.$$

Step 2. Utilize the GSVNHFPWA operator (which the parameter $\lambda = 1$) to aggregate the SVNHF decision matrix $N = \left(\widetilde{n}_{ij}\right)_{m \times n}(i = 1,2,3,4; j = 1,2,3)$ into the SVNHFE $\widetilde{n}_i = \left\{\widetilde{t}_i, \widetilde{i}_i, \widetilde{f}_i\right\}(i = 1,2,3,4)$ of each alternative company. Take the alternative company A_1 for instance, we have

$$\widetilde{n}_1 = \mathrm{GSVNHFPWA}_1(\widetilde{n}_{11}, \widetilde{n}_{12}, \widetilde{n}_{13}) = \left(\frac{T_{11}}{\sum_{j=1}^{3} T_{1j}} (\widetilde{n}_{11})^1 \oplus \frac{T_{12}}{\sum_{j=1}^{3} T_{1j}} (\widetilde{n}_{12})^1 \oplus \frac{T_{13}}{\sum_{j=1}^{3} T_{1j}} (\widetilde{n}_{13})^1 \right)^{1/1}$$

$$= \bigcup_{\widetilde{\gamma}_{11} \in \widetilde{t}_{11}, \widetilde{\gamma}_{12} \in \widetilde{t}_{12}, \widetilde{\gamma}_{13} \in \widetilde{t}_{13}, \widetilde{\delta}_{11} \in \widetilde{i}_{11}, \widetilde{\delta}_{12} \in \widetilde{i}_{12}, \widetilde{\delta}_{13} \in \widetilde{i}_{13}, \widetilde{\eta}_{11} \in \widetilde{f}_{11}, \widetilde{\eta}_{12} \in \widetilde{f}_{12}, \widetilde{\eta}_{13} \in \widetilde{f}_{13}} \left\{ \left\{ \left(1 - \prod_{j=1}^{3} \left(1 - (\gamma_{1j})^1 \right)^{\frac{T_{1j}}{\sum_{j=1}^{3} T_{1j}}} \right)^{1/1} \right\}, \left\{ 1 - \left(1 - \prod_{j=1}^{3} \left(1 - (1 - \delta_{1j})^1 \right)^{\frac{T_{1j}}{\sum_{j=1}^{3} T_{1j}}} \right)^{1/1} \right\}, \right.$$

$$\left. \left\{ 1 - \left(1 - \prod_{j=1}^{3} \left(1 - (1 - \eta_{1j})^1 \right)^{\frac{T_{1j}}{\sum_{j=1}^{3} T_{1j}}} \right)^{1/1} \right\} \right\}$$

$$\begin{aligned}
= \Big\{ \Big\{ & 1-(1-0.2)^{0.54}(1-0.3)^{0.28}(1-0.5)^{0.18}, 1-(1-0.2)^{0.54}(1-0.3)^{0.28}(1-0.6)^{0.18}, 1-(1-0.2)^{0.54}(1-0.4)^{0.28}(1-0.5)^{0.18}, \\
& 1-(1-0.2)^{0.54}(1-0.4)^{0.28}(1-0.6)^{0.18}, 1-(1-0.2)^{0.54}(1-0.5)^{0.28}(1-0.5)^{0.18}, 1-(1-0.2)^{0.54}(1-0.5)^{0.28}(1-0.6)^{0.18}, \\
& 1-(1-0.3)^{0.54}(1-0.3)^{0.28}(1-0.5)^{0.18}, 1-(1-0.2)^{0.54}(1-0.3)^{0.28}(1-0.6)^{0.18}, 1-(1-0.2)^{0.54}(1-0.4)^{0.28}(1-0.5)^{0.18}, \\
& 1-(1-0.3)^{0.54}(1-0.4)^{0.28}(1-0.6)^{0.18}, 1-(1-0.2)^{0.54}(1-0.5)^{0.28}(1-0.5)^{0.18}, 1-(1-0.2)^{0.54}(1-0.5)^{0.28}(1-0.6)^{0.18} \Big\}, \\
& \Big\{ 1-\Big(1-(1-(1-0.1))^{0.54}(1-(1-0.1))^{0.28}(1-(1-0.2))^{0.18}\Big), 1-\Big(1-(1-(1-0.1))^{0.54}(1-(1-0.1))^{0.28}(1-(1-0.3))^{0.18}\Big), \\
& 1-\Big(1-(1-(1-0.2))^{0.54}(1-(1-0.1))^{0.28}(1-(1-0.2))^{0.18}\Big), 1-\Big(1-(1-(1-0.2))^{0.54}(1-(1-0.1))^{0.28}(1-(1-0.3))^{0.18}\Big) \Big\}, \\
& \Big\{ 1-\Big(1-(1-(1-0.5))^{0.54}(1-(1-0.3))^{0.28}(1-(1-0.3))^{0.18}\Big), 1-\Big(1-(1-(1-0.5))^{0.54}(1-(1-0.3))^{0.28}(1-(1-0.4))^{0.18}\Big), \\
& 1-\Big(1-(1-(1-0.5))^{0.54}(1-(1-0.4))^{0.28}(1-(1-0.3))^{0.18}\Big), 1-\Big(1-(1-(1-0.5))^{0.54}(1-(1-0.4))^{0.28}(1-(1-0.4))^{0.18}\Big), \\
& 1-\Big(1-(1-(1-0.6))^{0.54}(1-(1-0.3))^{0.28}(1-(1-0.3))^{0.18}\Big), 1-\Big(1-(1-(1-0.6))^{0.54}(1-(1-0.3))^{0.28}(1-(1-0.4))^{0.18}\Big), \\
& 1-\Big(1-(1-(1-0.6))^{0.54}(1-(1-0.4))^{0.28}(1-(1-0.3))^{0.18}\Big), 1-\Big(1-(1-(1-0.6))^{0.54}(1-(1-0.4))^{0.28}(1-(1-0.4))^{0.18}\Big) \Big\}.
\end{aligned}$$

and obtain the SVNHFE $\widetilde{n}_1$ as the following.

$$\widetilde{n}_1 = \{\{0.2922, 0.3203, 0.3220, 0.3414, 0.3489, 0.3556, 0.3675, 0.3691, 0.3811, 0.3941, 0.4004, 0.4242\}, \{0.1134, 0.1220, 0.1648, 0.1774\}, \{0.3953, 0.4164, 0.4283, 0.4512, 0.4361, 0.4595, 0.4726, 0.4979\}\}.$$

Similarly, the SVNHFEs of other alternative companies can be computed as follows:

$$\widetilde{n}_2 = \{\{0.6000, 0.6275, 0.6363, 0.6613, 0.6457, 0.6701, 0.6778, 0.7000\}, \{0.1000, 0.1257, 0.1340, 0.1684\}, \{0.1651, 0.1887, 0.2211, 0.2528\}\};$$

$$\widetilde{n}_3 = \{\{0.5228, 0.5567, 0.5694, 0.6000\}, \{0.1989\}, \{0.2787, 0.3186\}\};$$

$$\widetilde{n}_4 = \{\{0.5280, 0.5608, 0.5821, 0.6111, 0.5943, 0.6225, 0.6408, 0.6657\}, \{0.1366\}, \{0.1189, 0.1464, 0.1625, 0.2000, 0.1950, 0.2400\}\}.$$

Step 3. Calculate the score function value of the SVNHFE $\widetilde{n}_i$ by using Equation (34):

$$s(\widetilde{n}_1) = 0.5902, s(\widetilde{n}_2) = 0.7711, s(\widetilde{n}_3) = 0.6882, s(\widetilde{n}_4) = 0.7623.$$

Then, we can obtain the ranking order of four alternative companies is $A_2 \succ A_4 \succ A_3 \succ A_1$, the food company A_2 is the best alternative.

If we replace the GSVNHFPWA operator in the aforementioned procedures with the GSVNHFPWG operator, the decision-making steps of the proposed method can be described as follows.

Step 1′. See **Step 1.**

Step 2′. Utilize the GSVNHFPWG operator (which the parameter $\lambda = 1$) to aggregate the SVNHF decision matrix $N = (\widetilde{n}_{ij})_{m \times n} (i = 1, 2, 3, 4; j = 1, 2, 3)$ into the SVNHFE $\widetilde{n}_i = \left\{ \widetilde{t}_i, \widetilde{i}_i, \widetilde{f}_i \right\} (i = 1, 2, 3, 4)$ of each alternative company. Take an alternative company A_1 for example, we have

$$\widetilde{n}_1 = \mathrm{GSVNHFPWG}_1(\widetilde{n}_{11}, \widetilde{n}_{12}, \widetilde{n}_{13}) = \left((\widetilde{n}_{11})^{\frac{T_{11}}{\sum_{j=1}^{3} T_{1j}}} \otimes (\widetilde{n}_{12})^{\frac{T_{12}}{\sum_{j=1}^{3} T_{1j}}} \otimes (\widetilde{n}_{13})^{\frac{T_{13}}{\sum_{j=1}^{3} T_{1j}}} \right)$$

$$= \bigcup_{\widetilde{\gamma}_{11}\in\widetilde{t}_{11},\widetilde{\gamma}_{12}\in\widetilde{t}_{12},\widetilde{\gamma}_{13}\in\widetilde{t}_{13},\widetilde{\delta}_{11}\in\widetilde{i}_{11},\widetilde{\delta}_{12}\in\widetilde{i}_{12},\widetilde{\delta}_{13}\in\widetilde{i}_{13},\widetilde{\eta}_{11}\in\widetilde{f}_{11},\widetilde{\eta}_{12}\in\widetilde{f}_{12},\widetilde{\eta}_{13}\in\widetilde{f}_{13}} \left\{ \left\{ 1 - \left(1 - \prod_{j=1}^{3} \left(1 - (1-\gamma_{1j})^1 \right)^{\frac{T_{1j}}{\sum_{j=1}^{3} T_{1j}}} \right)^{1/1} \right\}, \left\{ \left(1 - \prod_{j=1}^{3} \left(1 - (\delta_{1j})^1 \right)^{\frac{T_{1j}}{\sum_{j=1}^{3} T_{1j}}} \right)^{1/1} \right\}, \right.$$

$$\left. \left\{ \left(1 - \prod_{j=1}^{3} \left(1 - (\eta_{1j})^1 \right)^{\frac{T_{1j}}{\sum_{j=1}^{3} T_{1j}}} \right)^{1/1} \right\} \right\}$$

$$\begin{aligned}
= \Big\{ \Big\{ & 1 - \left(1-(1-(1-0.2))^{0.54}(1-(1-0.3))^{0.28}(1-(1-0.5))^{0.18}\right), 1 - \left(1-(1-(1-0.2))^{0.54}(1-(1-0.3))^{0.28}(1-(1-0.6))^{0.18}\right), \\
& 1 - \left(1-(1-(1-0.2))^{0.54}(1-(1-0.4))^{0.28}(1-(1-0.5))^{0.18}\right), 1 - \left(1-(1-(1-0.2))^{0.54}(1-(1-0.4))^{0.28}(1-(1-0.6))^{0.18}\right), \\
& 1 - \left(1-(1-(1-0.2))^{0.54}(1-(1-0.5))^{0.28}(1-(1-0.5))^{0.18}\right), 1 - \left(1-(1-(1-0.2))^{0.54}(1-(1-0.5))^{0.28}(1-(1-0.6))^{0.18}\right), \\
& 1 - \left(1-(1-(1-0.3))^{0.54}(1-(1-0.3))^{0.28}(1-(1-0.5))^{0.18}\right), 1 - \left(1-(1-(1-0.3))^{0.54}(1-(1-0.3))^{0.28}(1-(1-0.6))^{0.18}\right), \\
& 1 - \left(1-(1-(1-0.3))^{0.54}(1-(1-0.4))^{0.28}(1-(1-0.5))^{0.18}\right), 1 - \left(1-(1-(1-0.3))^{0.54}(1-(1-0.4))^{0.28}(1-(1-0.6))^{0.18}\right), \\
& 1 - \left(1-(1-(1-0.2))^{0.54}(1-(1-0.5))^{0.28}(1-(1-0.5))^{0.18}\right), 1 - \left(1-(1-(1-0.2))^{0.54}(1-(1-0.5))^{0.28}(1-(1-0.6))^{0.18}\right) \Big\}, \\
& \Big\{ 1-(1-0.1)^{0.54}(1-0.1)^{0.28}(1-0.2)^{0.18}, 1-(1-0.1)^{0.54}(1-0.1)^{0.28}(1-0.3)^{0.18}, 1-(1-0.2)^{0.54}(1-0.1)^{0.28}(1-0.2)^{0.18}, \\
& 1-(1-0.2)^{0.54}(1-0.1)^{0.28}(1-0.3)^{0.18} \Big\}, \Big\{ 1-(1-0.5)^{0.54}(1-0.3)^{0.28}(1-0.3)^{0.18}, 1-(1-0.5)^{0.54}(1-0.3)^{0.28}(1-0.4)^{0.18}, \\
& 1-(1-0.5)^{0.54}(1-0.4)^{0.28}(1-0.3)^{0.18}, 1-(1-0.5)^{0.54}(1-0.4)^{0.28}(1-0.4)^{0.18}, 1-(1-0.6)^{0.54}(1-0.3)^{0.28}(1-0.3)^{0.18}, \\
& 1-(1-0.6)^{0.54}(1-0.3)^{0.28}(1-0.4)^{0.18}, 1-(1-0.6)^{0.54}(1-0.4)^{0.28}(1-0.3)^{0.18}, 1-(1-0.6)^{0.54}(1-0.4)^{0.28}(1-0.4)^{0.18} \Big\}.
\end{aligned}$$

and obtain the SVNHFE $\widetilde{n}_1$ as the following:

$$\widetilde{n}_1 = \{\{0.2644, 0.2733, 0.2865, 0.2961, 0.3049, 0.3151, 0.3291, 0.3402, 0.3566, 0.3686, 0.3795, 0.3923\}, \{0.1190, 0.1401, 0.1733, 0.1931\}, \{0.4163, 0.4324, 0.4408, 0.4562, 0.4825, 0.4968, 0.5043, 0.5179\}\}.$$

Similarly, the SVNHFEs of other alternative companies can be computed as follows:

$$\widetilde{n}_2 = \{\{0.6000, 0.6234, 0.6314, 0.6559, 0.6403, 0.6652, 0.6738, 0.7000\}, \{0.1000, 0.1344, 0.1436, 0.1763\}, \{0.1866, 0.2217, 0.2260, 0.2594\}\};$$

$$\widetilde{n}_3 = \{\{0.5194, 0.5517, 0.5649, 0.6000\}, \{0.2531\}, \{0.2917, 0.3222\}\};$$

$$\widetilde{n}_4 = \{\{0.4600, 0.4781, 0.4788, 0.4976, 0.5789, 0.6016, 0.6026, 0.6262\}, \{0.1465\}, \{0.1261, 0.1565, 0.1712, 0.2000, 0.2196, 0.2467\}\}.$$

Step 3′. Calculate the score function value of the SVNHFE $\widetilde{n}_i$ by using Equation (34):

$$s(\widetilde{n}_1) = 0.5669, s(\widetilde{n}_2) = 0.7622, s(\widetilde{n}_3) = 0.6663, s(\widetilde{n}_4) = 0.7358.$$

Then, we can obtain the ranking order of four alternative companies is $A_2 \succ A_4 \succ A_3 \succ A_1$, and the food company A_2 is also the best alternative.

In real life, decision makers may determine the value of the parameter λ according to the decision-making problem itself or their preference. To analyze the influence of the parameter λ on the final ranking result, we change the parameter λ of the GSVNHFPWA operator and GSVNHFPWG operator in the numerical example above. Different values of the parameter λ are provided, such as 0.001, 0.5, 1, 2, 3, 5, 10, 20, and 50, which is determined by decision makers in decision-making process. Combined with the proposed method, we can obtain the score function values of four alternative companies, then the ranking results are determined as shown in Tables 2 and 3. Tables 2 and 3 show that when the GSVNHFPWA operator is used to aggregate arguments, the best alternative is the food company A_2 for $0 < \lambda \leq 3$, but the best alternative is the arms company A_4 for $5 \leq \lambda \leq 50$. Besides, when the GSVNHFPWG operator is used to aggregate arguments, the best alternative is always the food company A_2 for $0 < \lambda \leq 50$, however, there are some differences in specific ranking for $\lambda = 50$. Thus, the different ranking results indicate that the parameter λ plays a very important role in the aggregation process; decision makers should be cautious to determine the value of λ in real decision-making process.

Table 2. Score function values obtained by the GSVNHFPWA operator and the rankings of alternatives for different values of λ.

The Value of λ	$s(\tilde{n}_1)$	$s(\tilde{n}_2)$	$s(\tilde{n}_3)$	$s(\tilde{n}_4)$	Ranking
$\lambda = 0.001$	0.5834	0.7702	0.6856	0.7571	$A_2 \succ A_4 \succ A_3 \succ A_1$
$\lambda = 0.5$	0.5866	0.7706	0.6869	0.7596	$A_2 \succ A_4 \succ A_3 \succ A_1$
$\lambda = 1$	0.5902	0.7711	0.6882	0.7623	$A_2 \succ A_4 \succ A_3 \succ A_1$
$\lambda = 2$	0.5984	0.7721	0.6910	0.7676	$A_2 \succ A_4 \succ A_3 \succ A_1$
$\lambda = 3$	0.6071	0.7732	0.6937	0.7727	$A_2 \succ A_4 \succ A_3 \succ A_1$
$\lambda = 5$	0.6232	0.7753	0.6991	0.7811	$A_4 \succ A_2 \succ A_3 \succ A_1$
$\lambda = 10$	0.6500	0.7810	0.7109	0.7954	$A_4 \succ A_2 \succ A_3 \succ A_1$
$\lambda = 20$	0.6734	0.7902	0.7253	0.8104	$A_4 \succ A_2 \succ A_3 \succ A_1$
$\lambda = 50$	0.6927	0.8023	0.7394	0.8261	$A_4 \succ A_2 \succ A_3 \succ A_1$

Table 3. Score function values obtained by the GSVNHFPWG operator and the rankings of alternatives for different values of λ.

The Value of λ	$s(\tilde{n}_1)$	$s(\tilde{n}_2)$	$s(\tilde{n}_3)$	$s(\tilde{n}_4)$	Ranking
$\lambda = 0.01$	0.5735	0.7667	0.6766	0.7454	$A_2 \succ A_4 \succ A_3 \succ A_1$
$\lambda = 0.5$	0.5704	0.7647	0.6718	0.7408	$A_2 \succ A_4 \succ A_3 \succ A_1$
$\lambda = 1$	0.5669	0.7622	0.6663	0.7358	$A_2 \succ A_4 \succ A_3 \succ A_1$
$\lambda = 2$	0.5592	0.7569	0.6553	0.7251	$A_2 \succ A_4 \succ A_3 \succ A_1$
$\lambda = 3$	0.5512	0.7518	0.6459	0.7152	$A_2 \succ A_4 \succ A_3 \succ A_1$
$\lambda = 5$	0.5372	0.7435	0.6324	0.6998	$A_2 \succ A_4 \succ A_3 \succ A_1$
$\lambda = 10$	0.5166	0.7317	0.6132	0.6806	$A_2 \succ A_4 \succ A_3 \succ A_1$
$\lambda = 20$	0.5013	0.7311	0.5964	0.6686	$A_2 \succ A_4 \succ A_3 \succ A_1$
$\lambda = 50$	0.5718	1.0000	0.8765	0.8030	$A_2 \succ A_3 \succ A_4 \succ A_1$

5.2. Comparison and Discussion

To further verify the effectiveness of the proposed method, we compare the aforementioned ranking order with the results of other decision-making methods for analyzing the same numerical example as shown in Table 4; these methods include the SVNHFWA operator and SVNHFWG operator [14], correlation coefficient of DHFSs [27], correlation coefficient of SVNEs [28], and correlation coefficient of SVNHFEs [15]. From Table 4, we can see that the ranking order of four alternatives obtained by the SVNHFWA operator is $A_4 \succ A_2 \succ A_3 \succ A_1$ due to the feature of emphasizing group major points; besides, the ranking order of four alternatives in other methods are always $A_2 \succ A_4 \succ A_3 \succ A_1$, which is consistent with our proposed method.

Table 4. Comparison result of different decision-making methods.

Decision-Making Method	Ranking
The GSVNHFPWA operator ($\lambda = 1$)	$A_2 \succ A_4 \succ A_3 \succ A_1$
The GSVNHFPWG operator ($\lambda = 1$)	$A_2 \succ A_4 \succ A_3 \succ A_1$
The SVNHFWA operator	$A_4 \succ A_2 \succ A_3 \succ A_1$
The SVNHFWG operator	$A_2 \succ A_4 \succ A_3 \succ A_1$
Correlation coefficient of DHFSs	$A_2 \succ A_4 \succ A_3 \succ A_1$
Correlation coefficient of SVNEs	$A_2 \succ A_4 \succ A_3 \succ A_1$
Correlation coefficient of SVNHFEs	$A_2 \succ A_4 \succ A_3 \succ A_1$

With regard to the existing five decision-making methods above, the methods based on the correlation coefficient of DHFSs and correlation coefficient of SVNEs are only applicable to the DHF and SVN environment, respectively, while DHFS and SVNS are the specific cases of SVNHFS. On the other hand, the other three methods can only solve the decision-making problems that the criteria are in the same priority level. Therefore, the comparison result indicates that the proposed method, not only can deal with the decision-making problems effectively but, also has several advantages as follows: (1) decision makers evaluate the alternatives by using SVNHFEs, which contains truth-membership, indeterminacy-membership, and falsity-membership degrees, and SVNHFS is also a generalization of HFS, DHFS, and SVNS; thus, SVNHFEs can express more reliable evaluation information of decision makers; (2) the GSVNHFPWA operator and GSVNHFPWG operator can solve the decision-making problems that the criteria are in different priority levels, which is not considered in other decision-making methods under SVNHF environment; and (3) the GSVNHFPWA operator and GSVNHFPWG operator can be reduced to several aggregation operators through adjusting the value of the parameter λ, including the SVNHFWA operator and SVNHFWG operator [14]. Decision makers can determine the exact value of the parameter λ to respond to the possible situations in real life.

6. Conclusions

This paper studies the MCDM problems under SVNHF environment, while the criteria are in different priority levels. Motivated by the idea of the PA operator, we develop the GSVNHFPWA operator and GSVNHFPWG operator for aggregating SVNHFEs based on the related researches of SVNS and HFS theory. Some desirable properties of the proposed operators are investigated in detail, such as idempotency, boundedness, and monotonicity. Furthermore, we obtained several special cases that reduced from the proposed operators by changing the value of the parameter λ. Then, an approach for MCDM in which the criteria have different priorities is constructed combined with these operators. Finally, a numerical example is provided to illustrate the applications of the proposed method, and several advantages are reflected by the comparison between the proposed method and several existing decision-making methods. In the future, we shall investigate the SVNHF prioritized aggregation operators according to the different t-norm and t-conorm operational laws, and develop more aggregation operators for SVNHFSs.

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Some New Biparametric Distance Measures on Single-Valued Neutrosophic Sets with Applications to Pattern Recognition and Medical Diagnosis

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Abstract: Single-valued neutrosophic sets (SVNSs) handling the uncertainties characterized by truth, indeterminacy, and falsity membership degrees, are a more flexible way to capture uncertainty. In this paper, some new types of distance measures, overcoming the shortcomings of the existing measures, for SVNSs with two parameters are proposed along with their proofs. The various desirable relations between the proposed measures have also been derived. A comparison between the proposed and the existing measures has been performed in terms of counter-intuitive cases for showing its validity. The proposed measures have been illustrated with case studies of pattern recognition as well as medical diagnoses, along with the effect of the different parameters on the ordering of the objects.

Keywords: decision-making; single-valued neutrosophic sets; distance measure; pattern recognition; uncertainties

1. Introduction

The classical measure theory has been widely used to represent uncertainties in data. However, these measures are valid only for precise data, and hence they may be unable to give accurate judgments for data uncertain and imprecise in nature. To handle this, fuzzy set (FS) theory, developed by Zadeh [1], has received much attention over the last decades because of its capability of handling uncertainties. After this, Atanassov [2] proposed the concept of an intuitionistic fuzzy set (IFS), which extends the theory of FSs with the addition of a degree of non-membership. As IFS theory has widely been used by researchers [3–16] in different disciplines for handling the uncertainties in data, hence its corresponding analysis is more meaningful than FSs' crisp analysis. Nevertheless, neither the FS nor IFS theory are able to deal with indeterminate and inconsistent information. For instance, we take a person giving their opinion about an object with 0.5 being the possibility that the statement is true, 0.7 being the possibility that the statement is false and 0.2 being the possibility that he or she is not sure. To resolve this, Smarandache [17] introduced a new component called the "indeterminacy-membership function" and added the "truth membership function" and "falsity membership function", all which are independent components lying in $]0^-, 1^+[$, and hence the corresponding set is known as a neutrosophic set (NS), which is the generalization of the IFS and FS. However, without specification, NSs are difficult to apply to real-life problems. Thus, a particular case of the NS called a single-valued NS (SVNS) has been proposed by Smarandache [17], Wang et al. [18].

After this pioneering work, researchers have been engaged in extensions and applications to different disciplines. However, the most important task for the decision-maker is to rank the objects so as to obtain the desired object(s). For this, researchers have made efforts to enrich the concept of information measures in neutrosophic environments. Broumi and Smarandache [19] introduced the Hausdorff distance, while Majumdar [20] presented the Hamming and Euclidean distance for comparing the SVNSs. Ye [21] presented the concept of correlation for single-valued neutrosophic numbers (SVNNs). Additionally, Ye [22] improved the concept of cosine similarity for SVNSs, which was firstly introduced by Kong et al. [23] in a neutrosophic environment. Nancy and Garg [24] presented an improved score function for ranking the SVNNs and applied them to solve the decision-making problem. Garg and Nancy [25] presented the entropy measure of order α and applied them to solve decision-making problems. Recently, Garg and Nancy [26] presented a technique for order preference by similarity to ideal solution (TOPSIS) method under an interval NS environment to solve decision-making problems. Aside from these, various authors have incorporated the idea of NS theory into the similarity measures [27,28], distance measures [29,30], the cosine similarity measure [19,22,31], and aggregation operators [22,31–40].

Thus, on the basis of the above observations, it has been observed that distance or similarity measures are of key importance in a number of theoretical and applied statistical inference and data processing problems. It has been deduced from studies that similarity, entropy and divergence measures could be induced by the normalized distance measure on the basis of their axiomatic definitions. On the other hand, SVNSs are one of the most successful theories to handle the uncertainties and certainties in the system, but little systematic research has explored these problems. The gap in the research motivates us to develop some families of the distance measures of the SVNS to solve the decision-making problem, for which preferences related to different alternatives are taken in the form of neutrosophic numbers. The main contributions of this work are summarized as follows: (i) to highlight the shortcomings of the various existing distance measures under the single-valued neutrosophic information through illustrative examples; (ii) to overcome the shortcomings of the existing measures, this paper defines some new series of biparametric distance measures between SVNSs, which depend on two parameters, namely, p and t, where p is the L_p norm and t identifies the level of uncertainty. The various desirable relations between these have been investigated in detail. Then, we utilized these measures to solve the problem of pattern recognition as well as medical diagnosis and compared their performance with that of some of the existing approaches.

The rest of this paper is organized as follows. Section 2 briefly describes the concepts of NSs, SVNSs and their corresponding existing distance measures. Section 3 presents a family of the normalized and weighted normalized distance measures between two SVNSs. Some of their desirable properties have also been investigated in detail, while generalized distance measures have been proposed in Section 4. The defined measures are illustrated, by an example in Section 5, using the field of pattern recognition and medical diagnosis for demonstrating the effectiveness and stability of the proposed measures. Finally, a concrete conclusion has been drawn in Section 6.

2. Preliminaries

An overview of NSs and SVNSs is addressed here on the universal set X.

2.1. Basic Definitions

Definition 1 ([17,41])**.** *A neutrosophic set (NS) A in X is defined by its truth membership function* $(T_A(x))$*, an indeterminacy-membership function* $(I_A(x))$ *and a falsity membership function* $(F_A(x))$*, where all are subsets of* $]0^-, 1^+[$*. There is no restriction on the sum of* $T_A(x)$*,* $I_A(x)$ *and* $F_A(x)$*; thus* $0^- \leq \sup T_A(x) + \sup I_A(x) + \sup F_A(x) \leq 3^+$ *for all* $x \in X$*. Here,* sup *represents the supremum of the set.*

Wang et al. [18], Smarandache [41] defined the SVNS, which is an instance of a NS.

Definition 2 ([18,41])**.** *A single-valued neutrosophic set (SVNS) A is defined as*

$$A = \{\langle x, T_A(x), I_A(x), F_A(x)\rangle \mid x \in X\}$$

where $T_A : X \to [0,1]$, $I_A : X \to [0,1]$ and $F_A : X \to [0,1]$ with $T_A(x) + I_A(x) + F_A(x) \leq 3$ for all $x \in X$. The values $T_A(x)$, $I_A(x)$ and $F_A(x)$ denote the truth-membership degree, the indeterminacy-membership degree and the falsity-membership degree of x to A, respectively. The pairs of these are called single-valued neutrosophic numbers (SVNNs), which are denoted by $\alpha = \langle \mu_A, \rho_A, \nu_A \rangle$, and class of SVNSs is denoted by $\Phi(X)$.

Definition 3. *Let $A = \langle \mu_A, \rho_A, \nu_A \rangle$ and $B = \langle \mu_B, \rho_B, \nu_B \rangle$ be two single-valued neutrosophic sets (SVNSs). Then the following expressions are defined by [18]:*

(i) $A \subseteq B$ *if and only if (iff)* $\mu_A(x) \leq \mu_B(x), \rho_A(x) \geq \rho_B(x)$ *and* $\nu_A(x) \geq \nu_B(x)$ *for all x in X;*
(ii) $A = B$ *iff* $A \subseteq B$ *and* $B \subseteq A$;
(iii) $A^c = \{\langle \nu_A(x), 1 - \rho_A(x), \mu_A(x) \mid x \in X \rangle\}$;
(iv) $A \cap B = \langle \min(\mu_A(x), \mu_B(x)), \max(\rho_A(x), \rho_B(x)), \max(\nu_A(x), \nu_B(x)) \rangle$;
(v) $A \cup B = \langle \max(\mu_A(x), \mu_B(x)), \min(\rho_A(x), \rho_B(x)), \min(\nu_A(x), \nu_B(x)) \rangle$.

2.2. Existing Distance Measures

Definition 4. *A real function $d : \Phi(X) \times \Phi(X) \to [0,1]$ is called a distance measure [19], where d satisfies the following axioms for $A, B, C \in \Phi(X)$:*

(P1) $0 \leq d(A,B) \leq 1$;
(P2) $d(A,B) = 0$ *iff* $A = B$;
(P3) $d(A,B) = d(B,A)$;
(P4) *If* $A \subseteq B \subseteq C$, *then* $d(A,C) \geq d(A,B)$ *and* $d(A,C) \geq d(B,C)$.

On the basis of this, several researchers have addressed the various types of distance and similarity measures between two SVNSs $A = \langle x_i, \mu_A(x_i), \rho_A(x_i), \nu_A(x_i) | x_i \in X \rangle$ and $B = \langle x_i, \mu_B(x_i), \rho_B(x_i), \nu_B(x_i) | x_i \in X \rangle$, $i = 1, 2, ..., n$, which are given as follows:

(i) The extended Hausdorff distance [19]:

$$D_H(A,B) = \frac{1}{n} \sum_{i=1}^{n} \max \Big\{ |\mu_A(x_i) - \mu_B(x_i)|, |\rho_A(x_i) - \rho_B(x_i)|, |\nu_A(x_i) - \nu_B(x_i)| \Big\} \tag{1}$$

(ii) The normalized Hamming distance [20]:

$$D_{NH}(A,B) = \frac{1}{3n} \sum_{i=1}^{n} \Big\{ |\mu_A(x_i) - \mu_B(x_i)| + |\rho_A(x_i) - \rho_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| \Big\} \tag{2}$$

(iii) The normalized Euclidean distance [20]:

$$D_{NE}(A,B) = \left(\frac{1}{3n} \sum_{i=1}^{n} \Big\{ (\mu_A(x_i) - \mu_B(x_i))^2 + (\rho_A(x_i) - \rho_B(x_i))^2 + (\nu_A(x_i) - \nu_B(x_i))^2 \Big\} \right)^{1/2} \tag{3}$$

(iv) The cosine similarities [22]:

$$S_{CS1}(A,B) = \frac{1}{n} \sum_{i=1}^{n} \cos \left[\frac{\pi \big(|\mu_A(x_i) - \mu_B(x_i)| \vee |\rho_A(x_i) - \rho_B(x_i)| \vee |\nu_A(x_i) - \nu_B(x_i)| \big)}{2} \right] \tag{4}$$

and

$$S_{CS2}(A,B) = \frac{1}{n} \sum_{i=1}^{n} \cos \left[\frac{\pi \big(|\mu_A(x_i) - \mu_B(x_i)| + |\rho_A(x_i) - \rho_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)| \big)}{6} \right] \tag{5}$$

and their corresponding distances denoted by $D_{CS1} = 1 - S_{CS1}$ and $D_{CS2} = 1 - S_{CS2}$.

(v) The tangent similarities [42]:

$$S_{T1}(A,B) = 1 - \frac{1}{n}\sum_{i=1}^{n} \tan\left[\frac{\pi\left(|\mu_A(x_i) - \mu_B(x_i)| \vee |\rho_A(x_i) - \rho_B(x_i)| \vee |\nu_A(x_i) - \nu_B(x_i)|\right)}{4}\right] \tag{6}$$

and

$$S_{T2}(A,B) = 1 - \frac{1}{n}\sum_{i=1}^{n} \tan\left[\frac{\pi\left(|\mu_A(x_i) - \mu_B(x_i)| + |\rho_A(x_i) - \rho_B(x_i)| + |\nu_A(x_i) - \nu_B(x_i)|\right)}{12}\right] \tag{7}$$

and their corresponding distances denoted by $D_{T1} = 1 - S_{T1}$ and $D_{T2} = 1 - S_{T2}$.

2.3. Shortcomings of the Existing Measures

The above measures have been widely used; however, simultaneously they have some drawbacks, which are illustrated with the numerical example that follows.

Example 1. *Consider two known patterns A and B, which are represented by SVNSs in a universe X given by* $A = \langle x, 0.5, 0.0, 0.0 \mid x \in X\rangle$, $B = \langle x, 0.0, 0.5, 0.0 \mid x \in X\rangle$. *Consider an unknown pattern* $C \in SVNSs(X)$, *which is recognized where* $C = \langle x, 0.0, 0.0, 0.5 \mid x \in X\rangle$; *then the target of this problem is to classify the pattern C in one of the classes A or B. If we apply the existing measures [19,20,22,42] defined in Equations (1)–(7) above, then we obtain the following:*

Pair	D_H	D_{NH}	D_{NE}	D_{CS1}	D_{CS2}	D_{T1}	D_{T1}
(A,C)	0.5	0.3333	0.4048	0.2929	0.1340	0.4142	0.2679
(B,C)	0.5	0.3333	0.4048	0.2929	0.1340	0.4142	0.2679

Thus, from this, we conclude that these existing measures are unable to classify the pattern C with A and B. Hence these measures are inconsistent and unable to perform ranking.

Example 2. *Consider two SVNSs defined on the universal set X given by* $A = \langle x, 0.3, 0.2, 0.3 \mid x \in X\rangle$ *and* $B = \langle x, 0.4, 0.2, 0.4 \mid x \in X\rangle$. *If we replace the degree of falsity membership of A (0.3) with 0.4, and that of B (0.4) with 0.3, then we obtain new SVNSs as* $C = \langle x, 0.3, 0.2, 0.4 \mid x \in X\rangle$ *and* $D = \langle x, 0.4, 0.2, 0.3 \mid x \in X\rangle$. *Now, by using the distance measures defined in Equations (1)–(7), we obtain their corresponding values as follows:*

Pair	D_H	D_{NH}	D_{NE}	D_{CS1}	D_{CS2}	D_{T1}	D_{T1}
(A,B)	0.1	0.066	0.077	0.013	0.006	0.078	0.052
(C,D)	0.1	0.066	0.077	0.013	0.006	0.078	0.052

Thus, it has been concluded that by changing the falsity degree of SVNSs and keeping the other degrees unchanged, the values of their corresponding measures remain the same. Thus, there is no effect of the degree of falsity membership on the distance measures. Similarly, we can observe the same for the degree of the truth membership functions.

This seems to be worthless to calculate distance using the measures mentioned above. Thus, there is a need to build up a new distance measure that overcomes the shortcomings of the existing measures.

3. Some New Distance Measures between SVNSs

In this section, we present the Hamming and the Euclidean distances between SVNSs, which can be used in real scientific and engineering applications.

Letting $\Phi(X)$ be the class of SVNSs over the universal set X, then we define the distances for SVNSs, $A = \langle \mu_A(x_i), \rho_A(x_i), \nu_A(x_i) \mid x_i \in X\rangle$ and $B = \langle \mu_B(x_i), \rho_B(x_i), \nu_B(x_i) \mid x_i \in X\rangle$, by considering the uncertainty parameter t, as follows:

(i) Hamming distance:

$$d_1(A,B) = \frac{1}{3(2+t)} \sum_{i=1}^{n} \begin{pmatrix} |-t(\mu_A(x_i)-\mu_B(x_i)) + (\rho_A(x_i)-\rho_B(x_i)) + (\nu_A(x_i)-\nu_B(x_i))| \\ +|-t(\rho_A(x_i)-\rho_B(x_i)) - (\nu_A(x_i)-\nu_B(x_i)) + (\mu_A(x_i)-\mu_B(x_i))| \\ +|-t(\nu_A(x_i)-\nu_B(x_i)) - (\rho_A(x_i)-\rho_B(x_i)) + (\mu_A(x_i)-\mu_B(x_i))| \end{pmatrix} \tag{8}$$

(ii) Normalized Hamming distance:

$$d_2(A,B) = \frac{1}{3n(2+t)} \sum_{i=1}^{n} \begin{pmatrix} |-t(\mu_A(x_i)-\mu_B(x_i)) + (\rho_A(x_i)-\rho_B(x_i)) + (\nu_A(x_i)-\nu_B(x_i))| \\ +|-t(\rho_A(x_i)-\rho_B(x_i)) - (\nu_A(x_i)-\nu_B(x_i)) + (\mu_A(x_i)-\mu_B(x_i))| \\ +|-t(\nu_A(x_i)-\nu_B(x_i)) - (\rho_A(x_i)-\rho_B(x_i)) + (\mu_A(x_i)-\mu_B(x_i))| \end{pmatrix} \tag{9}$$

(iii) Euclidean distance:

$$d_3(A,B) = \left(\frac{1}{3(2+t)^2} \sum_{i=1}^{n} \begin{pmatrix} |-t(\mu_A(x_i)-\mu_B(x_i)) + (\rho_A(x_i)-\rho_B(x_i)) + (\nu_A(x_i)-\nu_B(x_i))|^2 \\ +|-t(\rho_A(x_i)-\rho_B(x_i)) - (\nu_A(x_i)-\nu_B(x_i)) + (\mu_A(x_i)-\mu_B(x_i))|^2 \\ +|-t(\nu_A(x_i)-\nu_B(x_i)) - (\rho_A(x_i)-\rho_B(x_i)) + (\mu_A(x_i)-\mu_B(x_i))|^2 \end{pmatrix} \right)^{1/2} \tag{10}$$

(iv) Normalized Euclidean distance:

$$d_4(A,B) = \left(\frac{1}{3n(2+t)^2} \sum_{i=1}^{n} \begin{pmatrix} |-t(\mu_A(x_i)-\mu_B(x_i)) + (\rho_A(x_i)-\rho_B(x_i)) + (\nu_A(x_i)-\nu_B(x_i))|^2 \\ +|-t(\rho_A(x_i)-\rho_B(x_i)) - (\nu_A(x_i)-\nu_B(x_i)) + (\mu_A(x_i)-\mu_B(x_i))|^2 \\ +|-t(\nu_A(x_i)-\nu_B(x_i)) - (\rho_A(x_i)-\rho_B(x_i)) + (\mu_A(x_i)-\mu_B(x_i))|^2 \end{pmatrix} \right)^{1/2} \tag{11}$$

where $t \geq 3$ is a parameter.

Then, on the basis of the distance properties as defined in Definition 4, we can obtain the following properties:

Proposition 1. *The above-defined distance $d_2(A,B)$, between two SVNSs A and B, satisfies the following properties (P1)–(P4):*

(P1) $0 \leq d_2(A,B) \leq 1, \forall A,B \in \Phi(X)$;
(P2) $d_2(A,B) = 0$ *iff* $A = B$;
(P3) $d_2(A,B) = d_2(B,A)$;
(P4) *If* $A \subseteq B \subseteq C$, *then* $d_2(A,C) \geq d_2(A,B)$ *and* $d_2(A,C) \geq d_2(B,C)$.

Proof. For two SVNSs A and B, we have

(P1) $0 \leq \mu_A(x_i), \mu_B(x_i) \leq 1$, $0 \leq \rho_A(x_i), \rho_B(x_i) \leq 1$ and $0 \leq \nu_A(x_i), \nu_B(x_i) \leq 1$. Thus, $|\mu_A(x_i) - \mu_B(x_i)| \leq 1$, $|\rho_A(x_i) - \rho_B(x_i)| \leq 1$, $|\nu_A(x_i) - \nu_B(x_i)| \leq 1$ and $|t(\mu_A(x_i) - \mu_B(x_i))| \leq t$. Therefore,

$$\begin{aligned} &|(t\mu_A(x_i) - \nu_A(x_i) - \rho_A(x_i)) - (t\mu_B(x_i) - \nu_B(x_i) - \rho_B(x_i))| \leq (2+t) \\ &|(t\rho_A(x_i) + \nu_A(x_i) - \mu_A(x_i)) - (t\rho_B(x_i) + \nu_B(x_i) - \mu_B(x_i))| \leq (2+t) \\ &|(t\nu_A(x_i) + \rho_A(x_i) - \mu_A(x_i)) - (t\nu_B(x_i) + \rho_B(x_i) - \mu_B(x_i))| \leq (2+t) \end{aligned}$$

Hence, by the definition of d_2, we obtain $0 \leq d_2(A,B) \leq 1$.

(P2) Firstly, we assume that $A = B$, which implies that $\mu_A(x_i) = \mu_B(x_i)$, $\rho_A(x_i) = \rho_B(x_i)$, and $\nu_A(x_i) = \nu_B(x_i)$ for $i = 1,2,..,n$. Thus, by the definition of d_2, we obtain $d_2(A,B) = 0$. Conversely, assuming that $d_2(A,B) = 0$ for two SVNSs A and B, this implies that

$$\left.\begin{array}{ll} & |-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))| \\ + & |-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))| \\ + & |-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))| \end{array}\right\} = 0$$

or

$$\begin{array}{rcl} |-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))| &=& 0 \\ |-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))| &=& 0 \\ |-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))| &=& 0 \end{array}$$

After solving, we obtain $\mu_A(x_i)-\mu_B(x_i)=0$, $\rho_A(x_i)-\rho_B(x_i)=0$ and $\nu_A(x_i)-\nu_B(x_i)=0$, which implies $\mu_A(x_i)=\mu_B(x_i)$, $\rho_A(x_i)=\rho_B(x_i)$ and $\nu_A(x_i)=\nu_B(x_i)$. Therefore, $A=B$. Hence $d_2(A,B)=0$ iff $A=B$.

(P3) This is straightforward from the definition of d_2.

(P4) If $A\subseteq B\subseteq C$, then $\mu_A(x_i)\leq\mu_B(x_i)\leq\mu_C(x_i)$, $\rho_A(x_i)\geq\rho_B(x_i)\geq\rho_C(x_i)$ and $\nu_A(x_i)\geq\nu_B(x_i)\geq\nu_C(x_i)$, which implies that $\mu_A(x_i)-\mu_B(x_i)\geq\mu_A(x_i)-\mu_C(x_i)$, $\nu_A(x_i)-\nu_B(x_i)\leq\nu_A(x_i)-\nu_C(x_i)$, and $\rho_A(x_i)-\rho_B(x_i)\leq\rho_A(x_i)-\rho_C(x_i)$.

Therefore,

$$\begin{array}{l} |-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))| \\ \qquad\leq|-t(\mu_A(x_i)-\mu_c(x_i))+(\rho_A(x_i)-\rho_C(x_i))+(\nu_A(x_i)-\nu_C(x_i))| \\ |-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))| \\ \qquad\leq|-t(\rho_A(x_i)-\rho_C(x_i))-(\nu_A(x_i)-\nu_C(x_i))+(\mu_A(x_i)-\mu_C(x_i))| \\ |-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))| \\ \qquad\leq|-t(\nu_A(x_i)-\nu_C(x_i))-(\rho_A(x_i)-\rho_C(x_i))+(\mu_A(x_i)-\mu_C(x_i))| \end{array}$$

By adding, we obtain $d_2(A,B)\leq d_2(A,C)$. Similarly, we obtain $d_2(B,C)\leq d_2(A,C)$.

□

Proposition 2. *Distance d_4 as defined in Equation (11) is also a valid measure.*

Proof. For two SVNSs A and B, we have

(P1) $0\leq\mu_A(x_i),\mu_B(x_i)\leq1$, $0\leq\rho_A(x_i),\rho_B(x_i)\leq1$ and $0\leq\nu_A(x_i),\nu_B(x_i)\leq1$. Thus, $|\mu_A(x_i)-\mu_B(x_i)|\leq1$, $|\rho_A(x_i)-\rho_B(x_i)|\leq1$, $|\nu_A(x_i)-\nu_B(x_i)|\leq1$ and $|t(\mu_A(x_i)-\mu_B(x_i))|\leq t$. Therefore,

$$\begin{array}{l} |-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))|^2\leq(2+t)^2 \\ |-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|^2\leq(2+t)^2 \\ |-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|^2\leq(2+t)^2 \end{array}$$

Hence, by the definition of d_4, we obtain $0\leq d_4(A,B)\leq1$.

(P2) Assuming that $A=B$ implies that $\mu_A(x_i)=\mu_B(x_i)$, $\rho_A(x_i)=\rho_B(x_i)$ and $\nu_A(x_i)=\nu_B(x_i)$ for $i=1,2,\ldots,n$, and hence using Equation (11), we obtain $d_4(A,B)=0$. Conversely, assuming that $d_4(A,B)=0$ implies

$$\begin{array}{l} |-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))|^2=0 \\ |-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|^2=0 \\ |-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|^2=0 \end{array}$$

After solving these, we obtain $\mu_A(x_i) - \mu_B(x_i) = 0$, $\rho_A(x_i) - \rho_B(x_i) = 0$ and $\nu_A(x_i) - \nu_B(x_i) = 0$; that is, $\mu_A(x_i) = \mu_B(x_i)$, $\rho_A(x_i) = \rho_B(x_i)$ and $\nu_A(x_i) = \nu_B(x_i)$ for $t \geq 3$. Hence $A = B$. Therefore, $d_4(A, B) = 0$ iff $A = B$.

(P3) This is straightforward from the definition of d_4.

(P4) If $A \subseteq B \subseteq C$, then $\mu_A(x_i) \leq \mu_B(x_i) \leq \mu_C(x_i)$, $\rho_A(x_i) \geq \rho_B(x_i) \geq \rho_C(x_i)$, and $\nu_A(x_i) \geq \nu_B(x_i) \geq \nu_C(x_i)$. Therefore

$$\begin{aligned}
&|-t(\mu_A(x_i) - \mu_B(x_i)) + (\rho_A(x_i) - \rho_B(x_i)) + (\nu_A(x_i) - \nu_B(x_i))|^2 \\
&\qquad \leq |-t(\mu_A(x_i) - \mu_c(x_i)) + (\rho_A(x_i) - \rho_C(x_i)) + (\nu_A(x_i) - \nu_C(x_i))|^2 \\
&|-t(\rho_A(x_i) - \rho_B(x_i)) - (\nu_A(x_i) - \nu_B(x_i)) + (\mu_A(x_i) - \mu_B(x_i))|^2 \\
&\qquad \leq |-t(\rho_A(x_i) - \rho_C(x_i)) - (\nu_A(x_i) - \nu_C(x_i)) + (\mu_A(x_i) - \mu_C(x_i))|^2 \\
&|-t(\nu_A(x_i) - \nu_B(x_i)) - (\rho_A(x_i) - \rho_B(x_i)) + (\mu_A(x_i) - \mu_B(x_i))|^2 \\
&\qquad \leq |-t(\nu_A(x_i) - \nu_C(x_i)) - (\rho_A(x_i) - \rho_C(x_i)) + (\mu_A(x_i) - \mu_C(x_i))|^2
\end{aligned}$$

Hence by the definition of d_4, we obtain $d_4(A, B) \leq d_4(A, C)$. Similarly, we obtain $d_4(B, C) \leq d_4(A, C)$.

□

Now, on the basis of these proposed distance measures, we conclude that this successfully overcomes the shortcomings of the existing measures as described above.

Example 3. *If we apply the proposed distance measures d_2 and d_4 on the data considered in Example 1 to classify the pattern C, then corresponding to the parameter $t = 3$, we obtain $d_2(A, C) = 0.3333$, $d_2(B, C) = 0.1333$, $d_4(A, C) = 0.3464$ and $d_4(B, C) = 0.1633$. Thus, the pattern C is classified with the pattern B and hence is able to identify the best pattern.*

Example 4. *If we utilize the proposed distances d_2 and d_4 for the above-considered Example 2, then their corresponding values are $d_2(A, B) = 0.0267$, $d_2(C, D) = 0.0667$, $d_4(A, B) = 0.0327$ and $d_4(C, D) = 0.6930$. Therefore, there is a significant effect of the change in the falsity membership on the measure values and hence consequently on the ranking values.*

Proposition 3. *Measures d_1 and d_3 satisfy the following properties:*

(i) $0 \leq d_1 \leq n$;

(ii) $0 \leq d_3 \leq n^{1/2}$.

Proof. We can easily obtain that $d_1(A, B) = nd_2(A, B)$, and thus by Proposition 1, we obtain $0 \leq d_1(A, B) \leq n$. Similarly, we can obtain $0 \leq d_3(A, B) \leq n^{1/2}$. □

However, in many practical situations, the different sets may have taken different weights, and thus weight $\omega_i (i = 1, 2, \ldots, n)$ of the element $x_i \in X$ should be taken into account. In the following, we develop a weighted Hamming distance and the normalized weighted Euclidean distance between SVNSs.

(i) The normalized weighted Hamming distance:

$$\begin{aligned}
&d_5(A, B) \\
&= \frac{1}{3n(2+t)} \sum_{i=1}^{n} \omega_i \begin{pmatrix} |-t(\mu_A(x_i) - \mu_B(x_i)) + (\rho_A(x_i) - \rho_B(x_i)) + (\nu_A(x_i) - \nu_B(x_i))| \\ + |-t(\rho_A(x_i) - \rho_B(x_i)) - (\nu_A(x_i) - \nu_B(x_i)) + (\mu_A(x_i) - \mu_B(x_i))| \\ + |-t(\nu_A(x_i) - \nu_B(x_i)) - (\rho_A(x_i) - \rho_B(x_i)) + (\mu_A(x_i) - \mu_B(x_i))| \end{pmatrix} \qquad (12)
\end{aligned}$$

(ii) The normalized weighted Euclidean distance:

$$d_6(A,B) = \left\{ \frac{1}{3n(2+t)^2} \sum_{i=1}^{n} \omega_i \left(\begin{array}{l} |-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))|^2 \\ +|-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|^2 \\ +|-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|^2 \end{array} \right) \right\}^{1/2} \quad (13)$$

where $t \geq 3$ is a parameter.

It is straightforward to check that the normalized weighted distance $d_k(A,B)(k=5,6)$ between SVNSs A and B also satisfies the above properties (P1)–(P4).

Proposition 4. *Distance measures d_2 and d_5 satisfy the relation $d_5 \leq d_2$.*

Proof. Because $\omega_i \geq 0, \sum_{i=1}^{n} \omega_i = 1$, then for any two SVNSs A and B, we have $d_5(A,B) = \frac{1}{3n(2+t)} \sum_{i=1}^{n} \omega_i \Big\{ \Big(|-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))| + |-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))| + |-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))| \Big) \Big\} \leq \frac{1}{3n(2+t)} \sum_{i=1}^{n} \Big(|-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))| + |-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))| + |-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))| \Big)$; that is, $d_5(A,B) \leq d_2(A,B)$. □

Proposition 5. *Let A and B be two SVNSs in X; then d_5 and d_6 are the distance measures.*

Proof. Because $\omega_i \in [0,1]$ and $\sum\limits_{i=1}^{n} \omega_i = 1$ then we can easily obtain $0 \leq d_5(A,B) \leq d_2(A,B)$. Thus, $d_5(A,B)$ satisfies (P1). The proofs of (P2)–(P4) are similar to those of Proposition 1. Similar is true for d_6. □

Proposition 6. *The distance measures d_4 and d_6 satisfy the relation $d_6 \leq d_4$.*

Proof. The proof follows from Proposition 4. □

Proposition 7. *The distance measures d_2 and d_4 satisfy the inequality $d_4 \leq \sqrt{d_2}$.*

Proof. For two SVNSs A and B, we have

$$\begin{array}{l} |-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))|^2 \leq (2+t)^2 \\ |-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|^2 \leq (2+t)^2 \\ |-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|^2 \leq (2+t)^2 \end{array}$$

which implies that

$$\begin{array}{l} \left|\dfrac{-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))}{2+t}\right|^2 \leq 1 \\ \left|\dfrac{-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))}{2+t}\right|^2 \leq 1 \\ \left|\dfrac{-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))}{2+t}\right|^2 \leq 1 \end{array}$$

For any $a \in [0,1]$, we have $a^2 \le a$. Therefore,

$$\left|\frac{-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))}{2+t}\right|^2 \le \left|\frac{-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))}{2+t}\right|$$

$$\left|\frac{-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))}{2+t}\right|^2 \le \left|\frac{-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))}{2+t}\right|$$

and

$$\left|\frac{-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))}{2+t}\right|^2 \le \left|\frac{-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))}{2+t}\right|$$

By adding these inequalities and by the definition of d_4, we have

$$\begin{aligned} d_4(A,B) &= \\ &\left[\frac{1}{3n(2+t)^2}\sum_{i=1}^{n}\begin{pmatrix} |-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))|^2 \\ +|-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|^2 \\ +|-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|^2 \end{pmatrix}\right]^{1/2} \\ &\le \left[\frac{1}{3n(2+t)}\sum_{i=1}^{n}\begin{pmatrix} |-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))| \\ +|-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))| \\ +|-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))| \end{pmatrix}\right]^{1/2} \\ &\le (d_2(A,B))^{1/2} \end{aligned}$$

As A and B are arbitrary SVNSs, thus we obtain $d_4 \le \sqrt{d_2}$. □

Proposition 8. *Measures d_6 and d_5 satisfy the inequality $d_6 \le \sqrt{d_5}$.*

Proof. The proof follows from Proposition 7. □

The Hausdroff distance between two non-empty closed and bounded sets is a measure of the resemblance between them. For example, we consider $A = [x_1, x_2]$ and $B = [y_1, y_2]$ in the Euclidean domain R; the Hausdroff distance in the additive set environment is given by the following [8]:

$$H(A,B) = \max\{\,|\,x_1 - y_1\,|, |\,x_2 - y_2\,|\,\}$$

Now, for any two SVNSs A and B over $X = \{x_1, x_2, \ldots, x_n\}$, we propose the following utmost distance measures:

- Utmost normalized Hamming distance:

$$d_1^H(A,B) = \frac{1}{3n(2+t)}\sum_{i=1}^{n}\max_i\begin{pmatrix} |-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))|, \\ |-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|, \\ |-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))| \end{pmatrix} \quad (14)$$

- Utmost normalized weighted Hamming distance:

$$
\begin{aligned}
& d_2^H(A,B) \\
= \ & \frac{1}{3n(2+t)} \sum_{i=1}^{n} \omega_i \max_i \begin{pmatrix} |-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))|, \\ |-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|, \\ |-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))| \end{pmatrix}
\end{aligned} \tag{15}
$$

- Utmost normalized Euclidean distance:

$$
\begin{aligned}
& d_3^H(A,B) \\
= \ & \left\{ \frac{1}{3n(2+t)^2} \sum_{i=1}^{n} \max_i \begin{pmatrix} |-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))|^2, \\ |-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|^2, \\ |-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|^2 \end{pmatrix} \right\}^{1/2}
\end{aligned} \tag{16}
$$

- Utmost normalized weighted Euclidean distance:

$$
\begin{aligned}
& d_4^H(A,B) \\
= \ & \left\{ \frac{1}{3n(2+t)^2} \sum_{i=1}^{n} \omega_i \max_i \begin{pmatrix} |-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))|^2, \\ |-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|^2, \\ |-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|^2 \end{pmatrix} \right\}^{1/2}
\end{aligned} \tag{17}
$$

Proposition 9. *The distance $d_1^H(A,B)$ defined in Equation (14) for two SVNSs A and B is a valid distance measure.*

Proof. The above measure satisfies the following properties:

(P1) As A and B are SVNSs, so $|\ \mu_A(x_i)-\mu_B(x_i)\ | \leq 1$, $|\ \rho_A(x_i)-\rho_B(x_i)\ | \leq 1$ and $|\ \nu_A(x_i)-\nu_B(x_i)\ | \leq 1$. Thus,

$$
\begin{aligned}
&|\ (t\mu_A(x_i)-\nu_A(x_i)-\rho_A(x_i))-(t\mu_B(x_i)-\nu_B(x_i)-\rho_B(x_i))\ | \leq (2+t) \\
&|\ (t\rho_A(x_i)+\nu_A(x_i)-\mu_A(x_i))-(t\rho_B(x_i)+\nu_B(x_i)-\mu_B(x_i))\ | \leq (2+t) \\
&|\ (t\nu_A(x_i)+\rho_A(x_i)-\mu_A(x_i))-(t\nu_B(x_i)+\rho_B(x_i)-\mu_B(x_i))\ | \leq (2+t)
\end{aligned}
$$

Hence, by the definition of d_1^H, we obtain $0 \leq d_1^H(A,B) \leq 1$.

(P2) Similar to the proof of Proposition 1.

(P3) This is clear from Equation (14).

(P4) Let $A \subseteq B \subseteq C$, which implies $\mu_A(x_i) \leq \mu_B(x_i) \leq \mu_C(x_i)$, $\rho_A(x_i) \geq \rho_B(x_i) \geq \rho_C(x_i)$ and $\nu_A(x_i) \geq \nu_B(x_i) \geq \nu_C(x_i)$. Therefore, $|-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))| \leq |-t(\mu_A(x_i)-\mu_c(x_i))+(\rho_A(x_i)-\rho_C(x_i))+(\nu_A(x_i)-\nu_C(x_i))|$, $|-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))| \leq |-t(\rho_A(x_i)-\rho_C(x_i))-(\nu_A(x_i)-\nu_C(x_i))+(\mu_A(x_i)-\mu_C(x_i))|$ and $|-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))| \leq |-t(\nu_A(x_i)-\nu_C(x_i))-(\rho_A(x_i)-\rho_C(x_i))+(\mu_A(x_i)-\mu_C(x_i))|$, which implies that $\max_i \big(|-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))|, |-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|, |-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|\big)$ $\leq \max_i \big(|-t(\mu_A(x_i)-\mu_c(x_i))+(\rho_A(x_i)-\rho_C(x_i))+(\nu_A(x_i)-\nu_C(x_i))|, |-t(\rho_A(x_i)-\rho_C(x_i))-(\nu_A(x_i)-\nu_C(x_i))+(\mu_A(x_i)-\mu_C(x_i))|$ and $|-t(\nu_A(x_i)-\nu_C(x_i))-(\rho_A(x_i)-\rho_C(x_i))+(\mu_A(x_i)-\mu_C(x_i))|\big)$. Hence $d_1^H(A,B) \leq d_1^H(A,C)$. Similarly, we obtain $d_1^H(B,C) \leq d_1^H(A,C)$.

□

Proposition 10. *For $A,B \in \Phi(X)$, d_2^H, d_3^H and d_4^H are the distance measures.*

Proof. The proof follows from the above proposition. □

Proposition 11. *The measures d_2^H and d_1^H satisfy the following inequality: $d_2^H \leq d_1^H$.*

Proof. Because $w_i \in [0,1]$, therefore

$$\begin{aligned} d_2^H(A,B) &= \frac{1}{3n(2+t)} \sum_{i=1}^{n} w_i \Bigg(\max_i \Big(|-t(\mu_A(x_i)-\mu_B(x_i)) + (\rho_A(x_i)-\rho_B(x_i)) + (\nu_A(x_i)-\nu_B(x_i))|, \\ &\quad |-t(\rho_A(x_i)-\rho_B(x_i)) - (\nu_A(x_i)-\nu_B(x_i)) + (\mu_A(x_i)-\mu_B(x_i))|, |-t(\nu_A(x_i) \\ &\quad -\nu_B(x_i)) - (\rho_A(x_i)-\rho_B(x_i)) + (\mu_A(x_i)-\mu_B(x_i))| \Big) \Bigg) \\ &\leq \frac{1}{3n(2+t)} \sum_{i=1}^{n} \max_i \Big(|-t(\mu_A(x_i)-\mu_B(x_i)) + (\rho_A(x_i)-\rho_B(x_i)) + (\nu_A(x_i)-\nu_B(x_i))|, \\ &\quad |-t(\rho_A(x_i)-\rho_B(x_i)) - (\nu_A(x_i)-\nu_B(x_i)) + (\mu_A(x_i)-\mu_B(x_i))|, |-t(\nu_A(x_i) \\ &\quad -\nu_B(x_i)) - (\rho_A(x_i)-\rho_B(x_i)) + (\mu_A(x_i)-\mu_B(x_i))| \Big) \\ &= d_1^H(A,B) \end{aligned}$$

Hence, $d_2^H \leq d_1^H$. □

Proposition 12. *The measures d_3^H and d_4^H satisfy the inequality $d_4^H(A,B) \leq d_3^H(A,B)$.*

Proof. The proof follows from Proposition 11. □

Proposition 13. *The measures d_3^H and d_1^H satisfy the inequality $d_3^H \leq \sqrt{d_1^H}$.*

Proof. Because for any $a \in [0,1]$, $a^2 \leq a \leq a^{1/2}$, the remaining proof follows from Proposition 7. □

Proposition 14. *The measures d_4^H and d_2^H satisfy the inequality $d_4^H \leq \sqrt{d_2^H}$.*

Proof. The proof follows from Proposition 13. □

Proposition 15. *The measures d_1^H and d_2 satisfy the following inequality:*

$$d_1^H \leq d_2$$

Proof. For positive numbers a_i, $i = 1,2,...,n$, we have $\max_i\{a_i\} \leq \sum_{i=1}^{n} a_i$. Thus, for any two SVNSs A and B, we have $d_1^H(A,B) = \frac{1}{3n(2+t)} \sum_{i=1}^{n} \max_i \Big(|-t(\mu_A(x_i)-\mu_B(x_i)) + (\rho_A(x_i) - \rho_B(x_i)) + (\nu_A(x_i)-\nu_B(x_i))|, |-t(\rho_A(x_i)-\rho_B(x_i)) - (\nu_A(x_i)-\nu_B(x_i)) + (\mu_A(x_i)-\mu_B(x_i))|, |-t(\nu_A(x_i) - \nu_B(x_i)) - (\rho_A(x_i)-\rho_B(x_i)) + (\mu_A(x_i)-\mu_B(x_i))| \Big) \leq \frac{1}{3n(2+t)} \sum_{i=1}^{n} |-t(\mu_A(x_i)-\mu_B(x_i)) + (\rho_A(x_i) - \rho_B(x_i)) + (\nu_A(x_i)-\nu_B(x_i))| + |-t(\rho_A(x_i)-\rho_B(x_i)) - (\nu_A(x_i)-\nu_B(x_i)) + (\mu_A(x_i)-\mu_B(x_i))| + |-t(\nu_A(x_i)-\nu_B(x_i)) - (\rho_A(x_i)-\rho_B(x_i)) + (\mu_A(x_i)-\mu_B(x_i))| = d_2(A,B)$. Hence $d_1^H \leq d_2$. □

Proposition 16. *The measures d_3^H and d_4 satisfy the following inequality:*

$$d_3^H \leq d_4$$

Proof. The proof follows from Proposition 15. □

Proposition 17. *The measures d_2, d_5 and d_1^H satisfy the following inequalities:*

(i) $d_2 \geq \frac{d_5+d_1^H}{2}$;
(ii) $d_2 \geq \sqrt{d_5 \cdot d_1^H}$.

Proof. Because $d_2 \geq d_5$ and $d_2 \geq d_1^H$, by adding these inequalities, we obtain $d_2 \geq \frac{d_5 + d_1^H}{2}$. On the other hand, by multiplying these, we obtain $d_2 \geq \sqrt{d_5 \cdot d_1^H}$. $\square$

4. Generalized Distance Measure

The above-defined Hamming and Euclidean distance measures are generalized for the two SVNSs A and B on the universal set X as follows:

$$\begin{aligned} d^p(A,B) \; = \; & \Bigg\{ \frac{1}{3n(2+t)^p} \sum_{i=1}^{n} \Big(| - t(\mu_A(x_i) - \mu_B(x_i)) + (\rho_A(x_i) - \rho_B(x_i)) + (\nu_A(x_i) - \nu_B(x_i))|^p \\ & + | - t(\rho_A(x_i) - \rho_B(x_i)) - (\nu_A(x_i) - \nu_B(x_i)) + (\mu_A(x_i) - \mu_B(x_i))|^p \\ & + | - t(\nu_A(x_i) - \nu_B(x_i)) - (\rho_A(x_i) - \rho_B(x_i)) + (\mu_A(x_i) - \mu_B(x_i))|^p \Big) \Bigg\}^{1/p} \end{aligned} \tag{18}$$

where $p \geq 1$ is an L_p norm and $t \geq 3$ represents the uncertainty index parameters.

In particular, if $p = 1$ and $p = 2$, then the above measure, given in Equation (18), reduces to measures d_2 and d_4 defined in Equations (9) and (11), respectively.

Proposition 18. *The above-defined distance $d^p(A,B)$, between SVNSs A and B, satisfies the following properties (P1)–(P4):*

(P1) $0 \leq d^p(A,B) \leq 1, \forall A,B \in \Phi(X)$;
(P2) $d^p(A,B) = 0$, *iff* $A = B$;
(P3) $d^p(A,B) = d^p(B,A)$;
(P4) *If* $A \subseteq B \subseteq C$, *then* $d^p(A,C) \geq d^p(A,B)$ *and* $d^p(A,C) \geq d^p(B,C)$.

Proof. For $p \geq 1$ and $t \geq 3$, we have the following:

(P1) For SVNSs, $| \mu_A(x_i) - \mu_B(x_i) | \leq 1$, $| \rho_A(x_i) - \rho_B(x_i) | \leq 1$ and $| \nu_A(x_i) - \nu_B(x_i) | \leq 1$. Thus, we obtain

$$\begin{aligned} -(2+t) \leq t(\mu_A(x_i) - \mu_B(x_i)) - (\nu_A(x_i) - \nu_B(x_i)) - (\rho_A(x_i) - \rho_B(x_i)) \leq (2+t) \\ -(2+t) \leq -t(\rho_A(x_i) - \rho_B(x_i)) - (\nu_A(x_i) - \nu_B(x_i)) + (\mu_A(x_i) - \mu_B(x_i)) \leq (2+t) \\ -(2+t) \leq -t(\nu_A(x_i) - \nu_B(x_i)) - (\rho_A(x_i) - \rho_B(x_i)) + (\mu_A(x_i)) - \mu_B(x_i) \leq (2+t) \end{aligned}$$

which implies that

$$\begin{aligned} 0 \leq \left| t(\mu_A(x_i) - \mu_B(x_i)) - (\nu_A(x_i) - \nu_B(x_i)) - (\rho_A(x_i) - \rho_B(x_i)) \right|^p \leq (2+t)^p \\ 0 \leq \left| - t(\rho_A(x_i) - \rho_B(x_i)) - (\nu_A(x_i) - \nu_B(x_i)) + (\mu_A(x_i) - \mu_B(x_i)) \right|^p \leq (2+t)^p \\ 0 \leq \left| - t(\nu_A(x_i) - \nu_B(x_i)) - (\rho_A(x_i) - \rho_B(x_i)) + (\mu_A(x_i)) - \mu_B(x_i) \right|^p \leq (2+t)^p \end{aligned}$$

Thus, by adding these inequalities, we obtain $0 \leq d^p(A,B) \leq 1$.

(P2) Assuming that $A = B \Leftrightarrow \mu_A(x) = \mu_B(x_i)$, $\rho_A(x_i) = \rho_B(x_i)$, and $\nu_A(x) = \nu_B(x_i)$, thus, $d^p(A,B) = 0$.

Conversely, assuming that $d^p(A,B)=0$ implies that

$$
\begin{aligned}
&|-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))|=0\\
&|-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|=0\\
&|-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|=0
\end{aligned}
$$

and hence, after solving, we obtain $\mu_A(x_i)=\mu_B(x_i)$, $\rho_A(x_i)=\rho_B(x_i)$ and $\nu_A(x_i)=\nu_B(x_i)$. Thus, $A=B$.

(P3) This is straightforward.

(P4) Let $A\subseteq B\subseteq C$; then $\mu_A(x_i)\le\mu_B(x_i)\le\mu_C(x_i)$, $\rho_A(x_i)\ge\rho_B(x_i)\ge\rho_C(x_i)$ and $\nu_A(x_i)\ge\nu_B(x_i)\ge\nu_C(x_i)$. Thus, $\mu_A(x_i)-\mu_B(x_i)\ge\mu_A(x_i)-\mu_C(x_i)$, $\rho_A(x_i)-\rho_B(x_i)\le\rho_A(x_i)-\rho_C(x_i)$ and $\nu_A(x_i)-\nu_B(x_i)\le\nu_A(x_i)-\nu_C(x_i)$. Hence, we obtain

$$
\begin{aligned}
&|-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))|^p\\
&\qquad\le|-t(\mu_A(x_i)-\mu_c(x_i))+(\rho_A(x_i)-\rho_C(x_i))+(\nu_A(x_i)-\nu_C(x_i))|^p\\
&|-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|^p\\
&\qquad\le|-t(\rho_A(x_i)-\rho_C(x_i))-(\nu_A(x_i)-\nu_C(x_i))+(\mu_A(x_i)-\mu_C(x_i))|^p\\
\text{and}\quad&|-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|^p\\
&\qquad\le|-t(\nu_A(x_i)-\nu_C(x_i))-(\rho_A(x_i)-\rho_C(x_i))+(\mu_A(x_i)-\mu_C(x_i))|^p
\end{aligned}
$$

Thus, we obtain $d^p(A,B)\le d^p(A,C)$. Similarly, $d^p(B,C)\le d^p(A,C)$.

□

If the weight vector $\omega_i,(i=1,2,\ldots,n)$ of each element is considered such that $\omega_i\in[0,1]$ and $\sum_i\omega_i=1$, then a generalized parametric distance measure between SVNSs A and B takes the following form:

$$
\begin{aligned}
d_w^p(A,B)\;=\;&\left(\frac{1}{3n(2+t)^p}\sum_{i=1}^{n}\omega_i\Big\{\Big(|-t(\mu_A(x_i)-\mu_B(x_i))+(\rho_A(x_i)-\rho_B(x_i))+(\nu_A(x_i)-\nu_B(x_i))|^p\right.\\
&+|-t(\rho_A(x_i)-\rho_B(x_i))-(\nu_A(x_i)-\nu_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|^p\\
&\left.+|-t(\nu_A(x_i)-\nu_B(x_i))-(\rho_A(x_i)-\rho_B(x_i))+(\mu_A(x_i)-\mu_B(x_i))|^p\Big)\Big\}\right)^{1/p}
\end{aligned}
\tag{19}
$$

In particular, if $p=1$ and $p=2$, Equation (19) reduces to Equations (12) and (13), respectively.

Proposition 19. *Let $\omega=(\omega_1,\omega_2,\ldots,\omega_n)^T$ be the weight vector of $x_i,(i=1,2,\ldots,n)$ with $\omega_i\ge0$ and $\sum_{i=1}^{n}\omega_i=1$; then the generalized parametric distance measure between the SVNSs A and B defined by Equation (19) satisfies the following:*

(P1) $0\le d_w^p(A,B)\le1,\forall A,B\in\Phi(X)$;
(P2) $d_w^p(A,B)=0$ *iff* $A=B$;
(P3) $d_w^p(A,B)=d_w^p(B,A)$;
(P4) $A\subseteq B\subseteq C$ *then* $d_w^p(A,C)\ge d_w^p(A,B)$ *and* $d_w^p(A,C)\ge d_w^p(B,C)$.

Proof. The proof follows from Proposition 18. □

5. Illustrative Examples

In order to illustrate the performance and validity of the above-proposed distance measures, two examples from the fields of pattern recognition and medical diagnosis have been taken into account.

5.1. Example 1: Application of Distance Measure in Pattern Recognition

Consider three known patterns A_1, A_2 and A_3, which are represented by the following SVNSs in a given universe $X = \{x_1, x_2, x_3, x_4\}$:

$$A_1 = \{\langle x_1, 0.7, 0.0, 0.1\rangle, \langle x_2, 0.6, 0.1, 0.2\rangle, \langle x_3, 0.8, 0.7, 0.6\rangle, \langle x_4, 0.5, 0.2, 0.3\rangle\}$$
$$A_2 = \{\langle x_1, 0.4, 0.2, 0.3\rangle, \langle x_2, 0.7, 0.1, 0.0\rangle, \langle x_3, 0.1, 0.1, 0.6\rangle, \langle x_4, 0.5, 0.3, 0.6\rangle\}$$
$$A_3 = \{\langle x_1, 0.5, 0.2, 0.2\rangle, \langle x_2, 0.4, 0.1, 0.2\rangle, \langle x_3, 0.1, 0.1, 0.4\rangle, \langle x_4, 0.4, 0.1, 0.2\rangle\}$$

Consider an unknown pattern $B \in SVNS(X)$, which will be recognized where

$$B = \{\langle x_1, 0.4, 0.1, 0.4\rangle, \langle x_2, 0.6, 0.1, 0.1\rangle, \langle x_3, 0.1, 0.0, 0.4\rangle, \langle x_4, 0.4, 0.4, 0.7\rangle\}$$

Then the target of this problem is to classify the pattern B in one of the classes A_1, A_2 or A_3. For this, proposed distance measures, $d_1, d_2, d_3, d_4, d_1^H$ and d_3^H, have been computed from B to $A_k (k = 1, 2, 3)$ corresponding to $t = 3$, and the results are given as follows:

$$\begin{aligned}
d_1(A_1, B) &= 0.5600; & d_1(A_2, B) &= 0.2932; & d_1(A_3, B) &= 0.4668\\
d_2(A_1, B) &= 0.1400; & d_2(A_2, B) &= 0.0733; & d_2(A_3, B) &= 0.1167\\
d_3(A_1, B) &= 0.3499; & d_3(A_2, B) &= 0.1641; & d_3(A_3, B) &= 0.3120\\
d_4(A_1, B) &= 0.1749; & d_4(A_2, B) &= 0.0821; & d_4(A_3, B) &= 0.1560\\
d_1^H(A_1, B) &= 0.0633; & d_1^H(A_2, B) &= 0.0300; & d_1^H(A_3, B) &= 0.0567\\
d_3^H(A_1, B) &= 0.1252; & d_3^H(A_2, B) &= 0.0560; & d_3^H(A_3, B) &= 0.1180
\end{aligned}$$

Thus, from these distance measures, we conclude that the pattern B belongs to the pattern A_2. On the other hand, if we assume that the weights of x_1, x_2, x_3 and x_4 are 0.3, 0.4, 0.2 and 0.1, respectively, then we utilize the distance measures d_5, d_6, d_2^H and d_4^H for obtaining the most suitable pattern as follows:

$$\begin{aligned}
d_5(A_1, B) &= 0.0338; & d_5(A_2, B) &= 0.0162; & d_5(A_3, B) &= 0.0233\\
d_6(A_1, B) &= 0.0861; & d_6(A_2, B) &= 0.0369; & d_6(A_3, B) &= 0.0604\\
d_2^H(A_1, B) &= 0.0148; & d_2^H(A_2, B) &= 0.0068; & d_2^H(A_3, B) &= 0.0117\\
d_4^H(A_1, B) &= 0.0603; & d_4^H(A_2, B) &= 0.0258; & d_4^H(A_3, B) &= 0.0464
\end{aligned}$$

Thus, the ranking order of the three patterns is A_2, A_3 and A_1, and hence A_2 is the most desirable pattern to be classified with B. Furthermore, it can be easily verified that these results validate the above-proposed propositions on the distance measures.

Comparison of Example 1 Results with Existing Measures

The above-mentioned measures have been compared with some existing measures under a NS environment for showing the validity of the approach whose results are summarized in Table 1. From these results, it has been shown that the final ordering of the pattern coincides with the proposed measures, and hence it shows the conservative nature of the measures.

Table 1. Ordering value of Example 1.

Methods	Measure Value of B from A_1	A_2	A_3	Ranking Order
D_H (defined in Equation (1)) [19]	0.3250	0.1250	0.2500	$A_2 \succ A_3 \succ A_1$
Correlation coefficient [19]	0.7883	0.9675	0.8615	$A_2 \succ A_3 \succ A_1$
D_{NE} (defined in Equation (3)) [20]	0.5251	0.7674	0.6098	$A_1 \succ A_3 \succ A_2$
S_{CS1} (defined in Equation (4)) [22]	0.8209	0.9785	0.8992	$A_2 \succ A_3 \succ A_1$
S_{CS2} (defined in Equation (5)) [22]	0.8949	0.9911	0.9695	$A_2 \succ A_3 \succ A_1$
S_{T1} (defined in Equation (6)) [42]	0.7275	0.9014	0.7976	$A_2 \succ A_3 \succ A_1$
S_{T2} (defined in Equation (7)) [42]	0.9143	0.9673	0.9343	$A_2 \succ A_3 \succ A_1$

5.2. *Example 2: Application of Distance Measure in Medical Diagnosis*

Consider a set of diseases $Q = \{Q_1(\text{Viral fever}), Q_2(\text{Malaria}),\ Q_3(\text{Typhoid}),\ Q_4(\text{Stomach Problem}),$ Q_5 (Chest problem)$\}$ and a set of symptoms $S = \{s_1$ (Temperature), s_2 (HeadAche), s_3 (Stomach Pain), s_4 (Cough), s_5 (Chest pain)$\}$. Suppose a patient, with respect to all the symptoms, can be represented by the following SVNS:

$$P(\text{Patient}) = \{(s_1, 0.8, 0.2, 0.1), (s_2, 0.6, 0.3, 0.1), (s_3, 0.2, 0.1, 0.8), (s_4, 0.6, 0.5, 0.1), (s_5, 0.1, 0.4, 0.6)\}$$

and each diseases $Q_k (k = 1, 2, 3, 4, 5)$ is as follows:

$$Q_1(\text{Viral fever}) = \{(s_1, 0.4, 0.6, 0.0), (s_2, 0.3, 0.2, 0.5), (s_3, 0.1, 0.3, 0.7), (s_4, 0.4, 0.3, 0.3), (s_5, 0.1, 0.2, 0.7)\}$$
$$Q_2(\text{Malaria}) = \{(s_1, 0.7, 0.3, 0.0), (s_2, 0.2, 0.2, 0.6), (s_3, 0.0, 0.1, 0.9), (s_4, 0.7, 0.3, 0.0), (s_5, 0.1, 0.1, 0.8)\}$$
$$Q_3(\text{Typhoid}) = \{(s_1, 0.3, 0.4, 0.3), (s_2, 0.6, 0.3, 0.1), (s_3, 0.2, 0.1, 0.7), (s_4, 0.2, 0.2, 0.6), (s_5, 0.1, 0.0, 0.9)\}$$
$$Q_4(\text{Stomach problem}) = \{(s_1, 0.1, 0.2, 0.7), (s_2, 0.2, 0.4, 0.4), (s_3, 0.8, 0.2, 0.0), (s_4, 0.2, 0.1, 0.7), (s_5, 0.2, 0.1, 0.7)\}$$
$$Q_5(\text{Chest problem}) = \{(s_1, 0.1, 0.1, 0.8), (s_2, 0.0, 0.2, 0.8), (s_3, 0.2, 0.0, 0.8), (s_4, 0.2, 0.0, 0.8), (s_5, 0.8, 0.1, 0.1)\}$$

Now, the target is to diagnose the disease of patient P among Q_1, Q_2, Q_3, Q_4 and Q_5. For this, proposed distance measures, $d_1, d_2, d_3, d_4, d_1^H$ and d_3^H, have been computed from P to $Q_k (k = 1, 2, \ldots, 5)$ and are given as follows:

$$d_1(Q_1, P) = 0.6400; \quad d_1(Q_2, P) = 0.9067; \quad d_1(Q_3, P) = 0.6333; \quad d_1(Q_4, P) = 1.4600; \quad d_1(Q_5, P) = 1.6200$$
$$d_2(Q_1, P) = 0.1280; \quad d_2(Q_2, P) = 0.1813; \quad d_2(Q_3, P) = 0.1267; \quad d_2(Q_4, P) = 0.2920; \quad d_2(Q_5, P) = 0.3240$$
$$d_3(Q_1, P) = 0.3626; \quad d_3(Q_2, P) = 0.4977; \quad d_3(Q_3, P) = 0.4113; \quad d_3(Q_4, P) = 0.7566; \quad d_3(Q_5, P) = 0.8533$$
$$d_4(Q_1, P) = 0.1622; \quad d_4(Q_2, P) = 0.2226; \quad d_4(Q_3, P) = 0.1840; \quad d_4(Q_4, P) = 0.3383; \quad d_4(Q_5, P) = 0.3816$$
$$d_1^H(Q_1, P) = 0.0613; \quad d_1^H(Q_2, P) = 0.0880; \quad d_1^H(Q_3, P) = 0.0627; \quad d_1^H(Q_4, P) = 0.1320; \quad d_1^H(Q_5, P) = 0.1400$$
$$d_3^H(Q_1, P) = 0.1175; \quad d_3^H(Q_2, P) = 0.1760; \quad d_3^H(Q_3, P) = 0.1373; \quad d_3^H(Q_4, P) = 0.2439; \quad d_3^H(Q_5, P) = 0.2661$$

Thus, from these distance measures, we conclude that the patient P suffers from the disease Q_3.

On the other hand, if we assign weights 0.3, 0.2, 0.2, 0.1 and 0.2 corresponding to $Q_k (k = 1, 2, \ldots, 5)$, respectively, then we utilize the distance measures d_5, d_6, d_2^H and d_4^H for obtaining the most suitable pattern as

$$d_5(Q_1, P) = 0.0284; \quad d_5(Q_2, P) = 0.0403; \quad d_5(Q_3, P) = 0.0273; \quad d_5(Q_4, P) = 0.0625; \quad d_5(Q_5, P) = 0.0684$$
$$d_6(Q_1, P) = 0.0795; \quad d_6(Q_2, P) = 0.1101; \quad d_6(Q_3, P) = 0.0862; \quad d_6(Q_4, P) = 0.1599; \quad d_6(Q_5, P) = 0.1781$$
$$d_2^H(Q_1, P) = 0.0135; \quad d_2^H(Q_2, P) = 0.0200; \quad d_2^H(Q_3, P) = 0.0129; \quad d_2^H(Q_4, P) = 0.0276; \quad d_2^H(Q_5, P) = 0.0289$$
$$d_4^H(Q_1, P) = 0.0572; \quad d_4^H(Q_2, P) = 0.0885; \quad d_4^H(Q_3, P) = 0.0636; \quad d_4^H(Q_4, P) = 0.1139; \quad d_4^H(Q_5, P) = 0.1226$$

Thus, on the basis of the ranking order, we conclude that the patient P suffers from the disease Q_3.

Comparison of Example 2 Results with Existing Approaches

In order to verify the feasibility of the proposed decision-making approach based on the distance measure, we conducted a comparison analysis based on the same illustrative example. For this, various measures as presented in Equations (1)–(7) were taken, and their corresponding results are summarized in Table 2, which shows that the patient P suffers from the disease Q_1.

Table 2. Comparison of diagnosis result using existing measures.

Approach	Ranking Order
D_H (defined in Equation (1)) [19]	$Q_1 \succ Q_3 \succ Q_2 \succ Q_4 \succ Q_5$
Correlation [19]	$Q_1 \succ Q_2 \succ Q_3 \succ Q_4 \succ Q_5$
Distance measure [27]	
$p = 1$	$Q_3 \succ Q_1 \succ Q_2 \succ Q_4 \succ Q_5$
$p = 2$	$Q_1 \succ Q_3 \succ Q_2 \succ Q_4 \succ Q_5$
$p = 3$	$Q_1 \succ Q_3 \succ Q_2 \succ Q_4 \succ Q_5$
$p = 5$	$Q_1 \succ Q_3 \succ Q_2 \succ Q_4 \succ Q_5$
D_{NH} (defined in Equation (2)) [20]	$Q_3 \succ Q_1 \succ Q_2 \succ Q_4 \succ Q_5$
D_{NH} (defined in Equation (3)) [20]	$Q_1 \succ Q_3 \succ Q_2 \succ Q_4 \succ Q_5$
S_{CS1} (defined in Equation (4)) [22]	$Q_1 \succ Q_3 \succ Q_2 \succ Q_4 \succ Q_5$
S_{CS1} (defined in Equation (5)) [22]	$Q_1 \succ Q_2 \succ Q_3 \succ Q_4 \succ Q_5$
S_{T1} (defined in Equation (6)) [42]	$Q_1 \succ Q_3 \succ Q_2 \succ Q_4 \succ Q_5$
S_{T1} (defined in Equation (7)) [42]	$Q_1 \succ Q_3 \succ Q_2 \succ Q_4 \succ Q_5$

5.3. *Effect of the Parameters p and t on the Ordering*

However, in order to analyze the effect of the parameters t and p on the measure values, an experiment was performed by taking different values of p ($p = 1, 1.5, 2, 3, 5, 10$) corresponding to a different value of the uncertainty parameter t ($t = 3, 5, 7$). On the basis of these different pairs of parameters, distance measures were computed, and their results are summarized in Tables 3 and 4, respectively, for Examples 1 and 2 corresponding to different criterion weights.

From these, the following have been computed:

(i) For a fixed value of p, it has been observed that the measure values corresponding to each alternative increase with the increase in the value of t. On the other hand, by varying the value of t from 3 to 7, corresponding to a fixed value of p, this implies that values of the distance measures of each diagnosis from the patient P increase.

(ii) It has also been observed from this table that when the weight vector has been assigned to each criterion weight, then the measure values are less than that of an equal weighting case.

(iii) Finally, it is seen from the table that the measured values corresponding to each alternative $Q_k (k = 1, 2, 3, 4, 5)$ are conservative in nature.

For each pair, the measure values lie between 0 and 1, and hence, on the basis of this, we conclude that the patient P suffers from the Q_1 disease. The ranking order for the decision-maker is shown in the table as (13245), which indicates that the order of the different attributes is of the form $Q_1 \succ Q_3 \succ Q_2 \succ Q_4 \succ Q_5$. Hence Q_1 is the most desirable, while Q_5 is the least desirable for different values of t and p.

Table 3. Results of classification of given sample using proposed distance measure.

		When Equal Importance Is given to Each Criteria				When Weight Vector $(0.3, 0.4, 0.2, 0.1)^T$ Is Taken			
p	t	$d^p(A_1, B)$	$d^p(A_2, B)$	$d^p(A_3, B)$	Ranking	$d_w^p(A_1, B)$	$d_w^p(A_2, B)$	$d_w^p(A_3, B)$	Ranking
1	3	0.1400	0.0733	0.1167	$A_2 \succ A_3 \succ A_1$	0.0338	0.0162	0.0233	$A_2 \succ A_3 \succ A_1$
	5	0.1667	0.0762	0.1214	$A_2 \succ A_3 \succ A_1$	0.0387	0.0170	0.0248	$A_2 \succ A_3 \succ A_1$
	7	0.1815	0.0778	0.1241	$A_2 \succ A_3 \succ A_1$	0.0414	0.0175	0.0256	$A_2 \succ A_3 \succ A_1$
1.5	3	0.1598	0.0783	0.1374	$A_2 \succ A_3 \succ A_1$	0.0620	0.0277	0.0426	$A_2 \succ A_3 \succ A_1$
	5	0.1924	0.0817	0.1437	$A_2 \succ A_3 \succ A_1$	0.0723	0.0293	0.0452	$A_2 \succ A_3 \succ A_1$
	7	0.2116	0.0838	0.1480	$A_2 \succ A_3 \succ A_1$	0.0784	0.0304	0.0469	$A_2 \succ A_3 \succ A_1$
2	3	0.1749	0.0821	0.1560	$A_2 \succ A_3 \succ A_1$	0.0861	0.0369	0.0604	$A_2 \succ A_3 \succ A_1$
	5	0.2137	0.0859	0.1646	$A_2 \succ A_3 \succ A_1$	0.1021	0.0392	0.0644	$A_2 \succ A_3 \succ A_1$
	7	0.2374	0.0885	0.1705	$A_2 \succ A_3 \succ A_1$	0.1120	0.0408	0.0671	$A_2 \succ A_3 \succ A_1$
3	3	0.1970	0.0880	0.1875	$A_2 \succ A_3 \succ A_1$	0.1229	0.0507	0.0927	$A_2 \succ A_3 \succ A_1$
	5	0.2469	0.0929	0.02012	$A_2 \succ A_3 \succ A_1$	0.1497	0.0543	0.1000	$A_2 \succ A_3 \succ A_1$
	7	0.2785	0.0962	0.2098	$A_2 \succ A_3 \succ A_1$	0.1672	0.0566	0.1046	$A_2 \succ A_3 \succ A_1$
5	3	0.2240	0.0967	0.2314	$A_2 \succ A_1 \succ A_3$	0.1680	0.0689	0.1469	$A_2 \succ A_3 \succ A_1$
	5	0.2902	0.1041	0.2526	$A_2 \succ A_3 \succ A_1$	0.2128	0.0749	0.1605	$A_2 \succ A_3 \succ A_1$
	7	0.3326	0.1087	0.2650	$A_2 \succ A_3 \succ A_1$	0.2426	0.0786	0.1685	$A_2 \succ A_3 \succ A_1$
10	3	0.2564	0.1107	0.2830	$A_2 \succ A_1 \succ A_3$	0.2203	0.0939	0.2248	$A_2 \succ A_1 \succ A_3$
	5	0.3421	0.1231	0.3131	$A_2 \succ A_3 \succ A_1$	0.2915	0.1047	0.2487	$A_2 \succ A_3 \succ A_1$
	7	0.3942	0.1304	0.3301	$A_2 \succ A_3 \succ A_1$	0.3356	0.1109	0.2622	$A_2 \succ A_3 \succ A_1$

Table 4. Diagnosis result on basis of proposed distance measure.

		When Equal Importance Is Given to Each Criteria					When Weight Vector $(0.3, 0.2, 0.2, 0.1, 0.2)^T$ is Taken				
p	t	$d^p(Q_1, P)$	$d^p(Q_2, P)$	$d^p(Q_3, P)$	$d^p(Q_4, P)$	$d^p(Q_5, P)$	$d_w^p(Q_1, P)$	$d_w^p(Q_2, P)$	$d_w^p(Q_3, P)$	$d_w^p(Q_4, P)$	$d_w^p(Q_5, P)$
1	3	0.1280	0.1813	0.1267	0.2920	0.3240	0.0284	0.0403	0.0273	0.0625	0.0684
	5	0.1410	0.1867	0.1457	0.3076	0.3400	0.0304	0.0413	0.0300	0.0643	0.0700
	7	0.1481	0.1896	0.1563	0.3178	0.3489	0.0315	0.0419	0.0315	0.0656	0.070
1.5	3	0.1465	0.2023	0.1600	0.3175	0.3574	0.0553	0.0768	0.0579	0.1154	0.1282
	5	0.1612	0.2131	0.1794	0.3364	0.3778	0.0598	0.0808	0.0628	0.1202	0.1334
	7	0.1711	0.2205	0.1916	0.3492	0.3913	0.0630	0.0836	0.0658	0.1237	0.1369
2	3	0.1622	0.2226	0.1840	0.3383	0.3816	0.0795	0.1101	0.0862	0.1599	0.1781
	5	0.1787	0.2391	0.2038	0.3609	0.4052	0.0867	0.1183	0.0928	0.1686	0.1872
	7	0.1895	0.2501	0.2168	0.3760	0.4211	0.0914	0.1238	0.0972	0.1744	0.1933
3	3	0.1870	0.2601	0.2163	0.3715	0.4142	0.1182	0.1662	0.1312	0.2276	0.2509
	5	0.2061	0.2876	0.2376	0.4004	0.4421	0.1297	0.1842	0.1409	0.2436	0.2666
	7	0.2175	0.3047	0.2516	0.4185	0.4601	0.1365	0.1954	0.1475	0.2535	0.2765
5	3	0.2185	0.3187	0.2531	0.4170	0.4504	0.1675	0.2471	0.1892	0.3127	0.3354
	5	0.2405	0.3625	0.2782	0.4531	0.4826	0.1841	0.2817	0.2045	0.3384	0.3588
	7	0.2529	0.3877	0.2940	0.4740	0.5023	0.1934	0.3016	0.2145	0.3532	0.3729
10	3	0.2519	0.3980	0.2969	0.4731	0.4896	0.2215	0.3524	0.2599	0.4095	0.4235
	5	0.2771	0.4586	0.3271	0.5170	0.5252	0.2434	0.4063	0.2840	0.4464	0.4547
	7	0.2912	0.4624	0.3451	0.5420	0.5466	0.2556	0.4363	0.2981	0.4675	0.4730

5.4. Advantages of the Proposed Method

According to the above comparison analysis, the proposed method for addressing decision-making problems has the following advantages:

(i) The distance measure under the IFS environment can only handle situations in which the degree of membership and non-membership is provided to the decision-maker. This kind of measure is unable to deal with indeterminacy, which commonly occurs in real-life applications. Because SVNSs are a successful tool in handling indeterminacy, the proposed distance measure in the neutrosophic domain can effectively be used in many real applications in decision-making.

(ii) The proposed distance measure depends upon two parameters p and t, which help in adjusting the hesitation margin in computing data. The effect of hesitation will be diminished or almost neglected if the value of t is taken very large, and for smaller values of t, the effect of hesitation will rise. Thus, according to requirements, the decision-maker can adjust the parameter to handle incomplete as well as indeterminate information. Therefore, this proposed approach is more suitable for engineering, industrial and scientific applications.

(iii) As has been observed from existing studies, various existing measures under NS environments have been proposed by researchers, but there are some situations that cannot be distinguished by these existing measures; hence their corresponding algorithm may give an irrelevant result. The proposed measure has the ability to overcome these flaws; thus it is a more suitable measure to tackle problems.

6. Conclusions

SVNSs are applied to problems with imprecise, uncertain, incomplete and inconsistent information existing in the real world. Although several measures already exist to deal with such kinds of information systems, they have several flaws, as described in the manuscript. Here in this article, we overcome these flaws by proposing an alternative way to define new generalized distance measures between the two SVNNs. Further, a family of normalized and weighted normalized Hamming and Euclidean distance measures have been proposed for the SVNSs. Some desirable properties and their relations have been studied in detail. Finally, a decision-making method has been proposed on the basis of these distance measures. To demonstrate the efficiency of the proposed coefficients, numerical examples of pattern recognition as well as medical diagnosis have been taken. A comparative study, as well as the effect of the parameters on the ranking of the alternative, will support the theory and hence demonstrate that the proposed measures are an alternative way to solve the decision-making problems. In the future, we will extend the proposed approach to the soft set environment [43–45], the multiplicative environment [46–48], and other uncertain and fuzzy environments [7,49–53].

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information

Article

Certain Concepts in Intuitionistic Neutrosophic Graph Structures

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Abstract: A graph structure is a generalization of simple graphs. Graph structures are very useful tools for the study of different domains of computational intelligence and computer science. In this research paper, we introduce certain notions of intuitionistic neutrosophic graph structures. We illustrate these notions by several examples. We investigate some related properties of intuitionistic neutrosophic graph structures. We also present an application of intuitionistic neutrosophic graph structures.

Keywords: graph structure; intuitionistic neutrosophic graph structure; ψ-complement

MSC: 03E72; 05C72; 05C78; 05C99

1. Introduction

Fuzzy graph models are advantageous mathematical tools for dealing with combinatorial problems of various domains including operations research, optimization, social science, algebra, computer science, environmental science and topology. Fuzzy graphical models are obviously better than graphical models due to natural existence of vagueness and ambiguity. Initially, we needed fuzzy set theory to cope with many complex phenomenons having incomplete information. Fuzzy set theory [1] is a very strong mathematical tool for solving approximate reasoning related problems. These notions describe complex phenomenons very well, which are not properly described using classical mathematics. Atanassov [2] generalized the fuzzy set theory by introducing the notion of intuitionistic fuzzy sets. The intuitionistic fuzzy sets have more describing possibilities as compared to fuzzy sets. An intuitionistic fuzzy set is inventive and more useful due to the existence of non-membership degree. In many situations like information fusion, indeterminacy is explicitly quantified. Smarandache [3] introduced the concept of neutrosophic sets, and he combined the tricomponent logic, non-standard analysis, and philosophy. It is a branch of philosophy which studies the origin, nature and scope of neutralities as well as their interactions with different ideational spectra. Three independent components of neutrosophic set are: truth value, indeterminacy value and falsity value [3]. For convenient use of neutrosophic sets in real-life phenomena, Wang et al. [4] proposed single valued neutrosophic sets, which is a generalization of intuitionistic fuzzy sets [2] and has three independent components having values in a standard unit interval [0, 1]. Ye [5–8] proposed several multi criteria decision-making methods based on neutrosophic sets. Bhowmik and Pal [9,10] introduced the notion of intuitionistic neutrosophic sets.

Kauffman [11] introduced fuzzy graphs on the basis of Zadeh's fuzzy relations [12]. Rosenfeld [13] discussed fuzzy analogue of many graph-theoretic notions. Later on, Bhattacharya [14] gave some remarks on fuzzy graphs. The complement of a fuzzy graph was defined by Sunitha and Vijayakumar [15]. Bhutani and Rosenfeld studied the notion of M-strong fuzzy graphs and their properties in [16]. Parvathi et al. defined operations on intuitionistic fuzzy graphs in [17]. Akram and Shahzadi [18] introduced neutrosophic soft graphs with applications. Dinesh and Ramakrishnan [19] introduced the notion of fuzzy graph structures and discussed some related properties. Akram and Akmal [20] introduced the concept of bipolar fuzzy graph structures. Recently, Akram and Sitara [21] introduced the concept of intuitionistic neutrosophic graph structures. Several notions' graph structures have been studied by the same authors in [22–27]. In this research paper, we introduce certain notions of intuitionistic neutrosophic graph structures and illustrate these notions by examples. We also present an application of intuitionistic neutrosophic graph structures in decision-making. For other notations and applications, readers are referred to [28–45] .

2. Intuitionistic Neutrosophic Graph Structures

Sampathkumar [46] introduced the graph structure, which is a generalization of an undirected graph and is quite useful in studying some structures like graphs, signed graphs, labeled graphs and edge colored graphs.

Definition 1. *[46] A graph structure $G = (V, R_1, \ldots, R_r)$ consists of a non-empty set V together with relations $R_1, R_2, \ldots, R_r$ on V, which are mutually disjoint such that each R_h, $1 \leq h \leq r$ is symmetric and irreflexive.*

One can represent a graph structure $G = (V, R_1, ..., R_r)$ in the plane, just like a graph where each edge is labeled as R_h, $1 \leq h \leq r$.

Definition 2. *[3] An ordered triple $< T_N, I_N, F_N >$ in $]0^-, 1^+[$ in the universe of discourse V is called neutrosophic set, where T_N, I_N, F_N: $V \rightarrow]0^-, 1^+[$, and their sum is without any restriction.*

Definition 3. *[4] An ordered triple $< T_N, I_N, F_N >$ in $[0, 1]$ in a universe of discourse V is called single-valued neutrosophic set, where T_N, I_N, F_N: $V \rightarrow [0, 1]$, and their sum is restricted between 0 and 3.*

Definition 4. *[47] Let V be a fixed set. A generalized intuitionistic fuzzy set I of V is an object having the form $I = \{(u, \mu_I(u), \nu_I(u)) | u \in V\}$, where the functions $\mu_I(u) :\rightarrow [0, 1]$ and $\nu_I(u) :\rightarrow [0, 1]$ define the degree of membership and degree of nonmembership of an element $u \in V$, respectively, such that*

$$\min\{\mu_I(u), \nu_I(u)\} \leq 0.5, \textit{for all } u \in V.$$

Definition 5. *[9,10] An intuitionistic neutrosophic set can be stated as a set having the form $I = \{T_I(u), I_I(u), F_I(u) : u \in V\}$, where*

$$\min\{T_I(u), I_I(u)\} \leq 0.5,$$
$$\min\{F_I(u), I_I(u)\} \leq 0.5,$$
$$\min\{T_I(u), F_I(u)\} \leq 0.5,$$

and $0 \leq T_I(u) + I_I(u) + F_I(u) \leq 2$.

Definition 6. *Let $\check{G} = (P, P_1, P_2, \ldots, P_r)$ be a graph structure(GS), and then $\check{G}_i = (O, O_1, O_2 and \ldots, O_r)$ is called an intuitionistic neutrosophic graph structure (INGS), if $O = < k, T(k), I(k), F(k) >$ and $O_h = < (k, l), T_h(k, l), I_h(k, l), F_h(k, l) >$ are intuitionistic neutrosophic sets on P and P_h, respectively, such that*

1. $T_h(k, l) \leq T(k) \wedge T(l)$, $I_h(k, l) \leq I(k) \wedge I(l)$, $F_h(k, l) \leq F(k) \vee F(l)$;
2. $T_h(k, l) \wedge I_h(k, l) \leq 0.5$, $T_h(k, l) \wedge F_h(k, l) \leq 0.5$, $I_h(k, l) \wedge F_h(k, l) \leq 0.5$;

3. $0 \leq T_h(k,l) + I_h(k,l) + F_h(k,l) \leq 2, \quad \forall\ (k,l) \in O_h,\ h = 1,2,\ldots,r,$

where O is an underlying vertex set of $\check{G}_i$ *and* O_h *(h = 1, 2, …, r) are underlying h-edge sets of* $\check{G}_i$.

Example 1. *Consider a GS* $\check{G} = (P, P_1, P_2)$ *such that* O, O_1,O_2 *are IN subsets of* P, P_1, P_2*, respectively, where*

$$\begin{aligned} P &= \{k_1, k_2, k_3, k_4, k_5, k_6, k_7, k_8\}, \\ P_1 &= \{k_1k_2, k_3k_4, k_5k_6, k_3k_7, k_6k_8\}, \\ P_2 &= \{k_2k_3, k_4k_5, k_1k_6, k_5k_7, k_2k_8\}. \end{aligned}$$

Through direct calculations, it is easy to show that $\check{G}_i = (O, O_1, O_2)$ *is an INGS of* $\check{G}$ *as represented in Figure 1.*

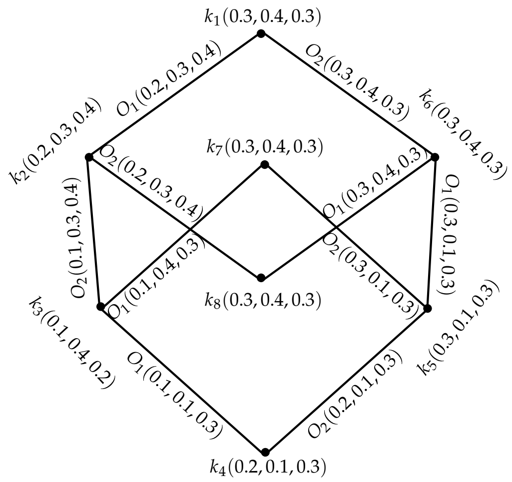

Figure 1. An intuitionistic neutrosophic graph structure.

Definition 7. *Let* $\check{G}_i = (O, O_1, O_2, \ldots, O_r)$ *be an INGS of* $\check{G}$*. If* $\check{H}_i = (O', O'_1, O'_2, \ldots, O'_r)$ *is an INGS of* $\check{G}$ *such that*

$$T'(k) \leq T(k), I'(k) \leq I(k), F'(k) \geq F(k)\ \forall k \in P,$$

$$T'_h(k,l) \leq T_h(k,l),\ I'_h(k,l) \leq I_h(k,l),\ F'_h(k,l) \geq F_h(k,l), \forall (k,l) \in P_h,\ h = 1,2,...,r.$$

Then, $\check{H}_i$ *is said to be an intuitionistic neutrosophic (IN) subgraph structure of INGS* $\check{G}_i$.

Example 2. *Consider an INGS* $\check{H}_i = (O', O'_1, O'_2)$ *of GS* $\check{G} = (P, P_1, P_2)$ *as represented in Figure 2. Through routine calculations, it can be easily shown that* $\check{H}_i$ *is an IN subgraph structure of INGS* $\check{G}_i$.

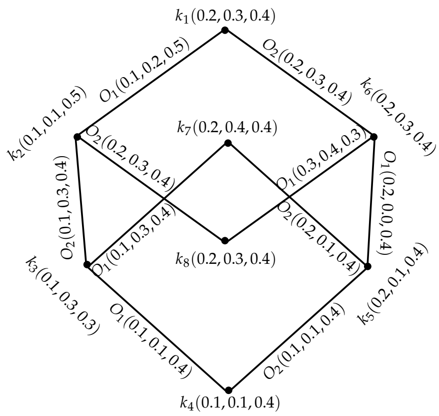

Figure 2. IN subgraph structure.

Definition 8. *An INGS* $\check{H}_i = (O', O'_1, O'_2, \ldots, O'_r)$ *is called an IN induced-subgraph structure of* $\check{G}_i$ *by* $Q \subseteq P$ *if*

$$T'(k) = T(k), I'(k) = I(k), F'(k) = F(k), \forall k \in Q,$$

$$T'_h(k,l) = T_h(k,l),\ I'_h(k,l) = I_h(k,l),\ F'_h(k,l) = F_h(k,l), \forall k, l \in Q, h = 1, 2, \ldots, r.$$

Example 3. *The INGS in the given Figure 3 is an IN induced-subgraph structure of an INGS in Figure 1.*

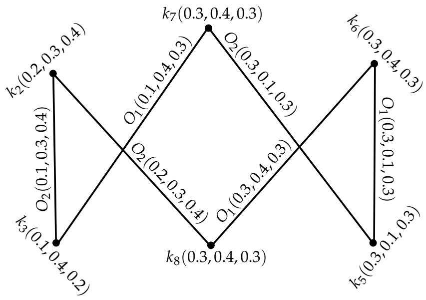

Figure 3. An IN induced-subgraph structure.

Definition 9. *An INGS* $\check{H}_i = (O', O'_1, O'_2, \ldots, O'_r)$ *is said to be a IN spanning-subgraph structure of* $\check{G}_i$ *if* $O' = O$ *and*

$$T'_h(k,l) \le T_h(k,l),\ I'_h(k,l) \le I_h(k,l),\ F'_h(k,l) \ge F_h(k,l), h = 1, 2, \ldots, r.$$

Example 4. *An INGS shown in Figure 4 is an IN spanning-subgraph structure of an INGS in Figure 1.*

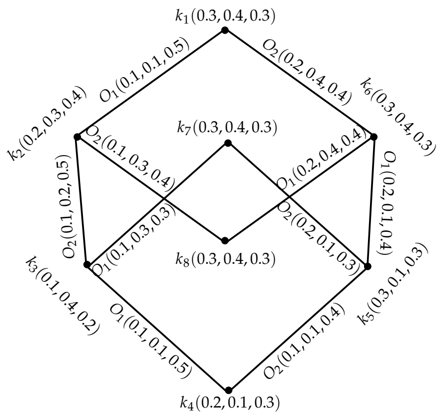

Figure 4. An IN spanning-subgraph structure.

Definition 10. *Let $\check{G}_i = (O, O_1, O_2, \dots, O_r)$ be an INGS. Then, $kl \in P_h$ is named as a IN O_h-edge or shortly O_h-edge, if $T_h(k,l) > 0$ or $I_h(k,l) > 0$ or $F_h(k,l) > 0$ or all these conditions are satisfied. As a result, support of O_h is:*

$$supp(O_h) = \{kl \in O_h : T_h(k,l) > 0\} \cup \{kl \in O_h : I_h(k,l) > 0\} \cup \{kl \in O_h : F_h(k,l) > 0\},$$

$h = 1, 2, ..., r$.

Definition 11. *O_h-path in an INGS $\check{G}_i = (O, O_1, O_2, \dots, O_r)$ is a sequence $k_1, k_2, ..., k_r$ of distinct vertices (except $k_r = k_1$) in P, such that $k_{h-1}k_h$ is an IN O_h-edge $\forall h = 2, ..., r$.*

Definition 12. *An INGS $\check{G}_i = (O, O_1, O_2, \dots, O_r)$ is O_h-strong for any $h \in \{1, 2, ..., r\}$ if*

$$T_h(k,l) = \min\{T(k), T(l)\},\ I_h(k,l) = \min\{I(k), I(l)\},\ F_h(k,l) = \max\{F(k), F(l)\},$$

$\forall kl \in supp(O_h)$. If $\check{G}_i$ is O_h-strong for all $h \in \{1, 2, \dots, r\}$, then $\check{G}_i$ is a strong INGS.

Example 5. *Consider an INGS $\check{G}_i = (O, O_1, O_2)$ as represented in Figure 5. Then, $\check{G}_i$ is strong INGS, as it is O_1- and $O_2-strong$.*

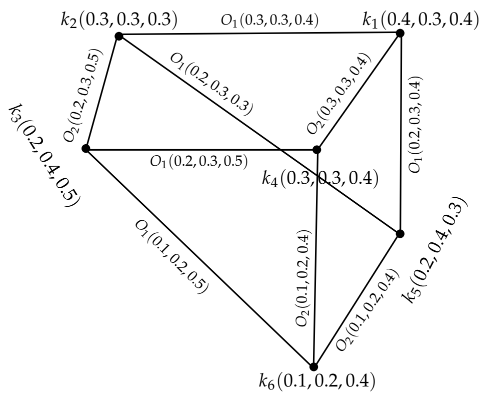

Figure 5. A strong INGS.

Definition 13. *An INGS $\check{G}_i = (O, O_1, O_2, \dots, O_r)$ is a complete INGS, if*

1. *$\check{G}_i$ is strong INGS.*

2. $supp(O_h) \neq \emptyset$, *for all h = 1, 2, ..., r.*
3. *For all* $k, l \in P$, *kl is a* $O_h - edge$ *for some h.*

Example 6. *Let* $\check{G}_i = (O, O_1, O_2)$ *be an INGS of GS* $\check{G} = (P, P_1, P_2)$, *such that*

$$P = \{k_1, k_2, k_3, k_4, k_5, k_6\},$$
$$P_1 = \{k_1k_6, k_1k_2, k_2k_4, k_2k_5, k_2k_6, k_1k_6\},$$
$$P_2 = \{k_2k_6, k_4k_3, k_5k_6, k_1k_4\},$$
$$P_3 = \{k_1k_5, k_5k_3, k_2k_3, k_1k_3, k_4k_6\}.$$

By means of direct calculations, it is easy to show that $\check{G}_i$ *is strong INGS.*
Moreover, $supp(O_1) \neq \emptyset$, $supp(O_2) \neq \emptyset$, $supp(O_3) \neq \emptyset$, *and every pair* k_hk_q *of vertices of P, is* O_1*-edge or* O_2*-edge or an* O_3*-edge. Hence,* $\check{G}_i$ *is a complete INGS, that is,* $O_1O_2O_3$*-complete INGS.*

Definition 14. *Let* $\check{G}_i = (O, O_1, O_2, \ldots, O_r)$ *be an INGS. The truth strength* $T.P_{O_h}$, *falsity strength* $F.P_{O_H}$, *and indeterminacy strength* $I.P_{O_h}$ *of an* O_h*-path,* $P_{O_h} = k_1, k_2, \ldots, k_n$ *is defined as:*

$$T.P_{O_h} = \bigwedge_{i=2}^{n} [T^P_{O_h}(k_{i-1}k_i)],$$
$$I.P_{O_h} = \bigwedge_{i=2}^{n} [I^P_{O_h}(k_{i-1}k_i)],$$
$$F.P_{O_h} = \bigvee_{i=2}^{n} [F^P_{O_h}(k_{i-1}k_i)].$$

Example 7. *Consider an INGS* $\check{G}_i = (O, O_1, O_2, O_3)$ *as in Figure 6. We found an* O_1*-path* $P_{O_1} = k_2, k_1, k_6$. *So,* $T.P_{O_1}$*= 0.2,* $I.P_{O_1}$*= 0.1 and* $F.P_{O_2}$*= 0.5.*

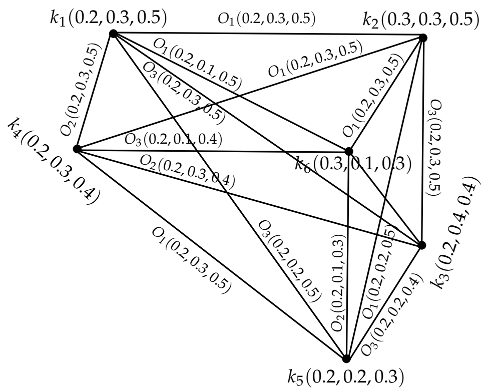

Figure 6. A complete INGS.

Definition 15. *Let* $\check{G}_i = (O, O_1, O_2, \ldots, O_r)$ *be an INGS. Then,*

- O_h*-strength of connectedness of truth between k and l is defined as:* $T^{\infty}_{O_h}(kl) = \bigvee_{i \geq 1} \{T^i_{O_h}(kl)\}$, *such that* $T^i_{O_h}(kl) = (T^{i-1}_{O_h} \circ T^1_{O_h})(kl)$ *for* $i \geq 2$ *and* $T^2_{O_h}(kl) = (T^1_{O_h} \circ T^1_{O_h})(kl) = \bigvee_y (T^1_{O_h}(ky) \wedge T^1_{O_h})(yl)$.
- O_h*-strength of connectedness of indeterminacy between k and l is defined as:* $I^{\infty}_{O_h}(kl) = \bigvee_{i \geq 1} \{I^i_{O_h}(kl)\}$, *such that* $I^i_{O_h}(kl) = (I^{i-1}_{O_h} \circ I^1_{O_h})(kl)$ *for* $i \geq 2$ *and* $I^2_{O_h}(kl) = (I^1_{O_h} \circ I^1_{O_h})(kl) = \bigvee_y (I^1_{O_h}(ky) \wedge I^1_{O_i})(yl)$.

- O_h-*strength of connectedness of falsity between k and l is defined as:* $F^{\infty}_{O_h}(kl) = \bigwedge_{i\geq 1}\{F^{i}_{O_h}(kl)\}$, *such that* $F^{i}_{O_h}(kl) = (F^{i-1}_{O_h} \circ F^{1}_{O_h})(kl)$ *for* $i \geq 2$ *and* $F^{2}_{Q_h}(kl) = (F^{1}_{O_h} \circ F^{1}_{O_h})(kl) = \bigwedge_{y}(F^{1}_{O_h}(ky) \vee F^{1}_{O_h})(yl)$.

Definition 16. *An INGS* $\check{G}_i = (O, O_1, O_2, \ldots, O_r)$ *is called an* O_h-*cycle if* $(supp(O), supp(O_1), supp(O_2), \ldots, supp(O_r))$ *is an* $O_h - cycle$.

Definition 17. *An INGS* $\check{G}_i = (O, O_1, O_2, \ldots, O_r)$ *is an IN fuzzy* O_h-*cycle (for any h) if*

1. $\check{G}_i$ *is an* O_h-*cycle.*
2. *There exists no unique* O_h-*edge kl in* $\check{G}_i$ *such that* $T_{O_h}(kl) = \min\{T_{O_h}(yz) : yz \in P_h = supp(O_h)\}$ *or* $I_{O_h}(kl) = \min\{I_{O_h}(yz) : yz \in P_h = supp(O_h)\}$ *or* $F_{O_h}(kl) = \max\{F_{O_h}(yz) : yz \in P_h = supp(O_h)\}$.

Example 8. *Consider an INGS* $\check{G}_i = (O, O_1, O_2)$ *as in Figure 6. Then,* $\check{G}_i$ *is an* O_1-*cycle and IN fuzzy* $O_1 -$ *cycle, since* $(supp(O),\ supp(O_1),\ supp(O_2))$ *is an* O_1-*cycle and no unique* O_1-*edge kl satisfies the condition:* $T_{O_h}(kl) = \min\{T_{O_h}(yz) : yz \in P_h = supp(O_h)\}$ *or* $I_{O_h}(kl) = \min\{I_{O_h}(yz) : yz \in P_h = supp(O_h)\}$ *or* $F_{O_h}(kl) = \max\{F_{O_h}(yz) : yz \in P_h = supp(O_h)\}$.

Definition 18. *Let* $\check{G}_i = (O, O_1, O_2, \ldots, O_r)$ *be an INGS and k a vertex in* $\check{G}_i$. *Let* $(O', O'_1, O'_2, \ldots, O'_r)$ *be an IN subgraph structure of* $\check{G}_i$ *induced by* $P \setminus \{k\}$ *such that* $\forall y \neq k, z \neq k$.

$T_{O'}(k) = 0 = I_{O'}(k) = F_{O'}(k)$, $T_{O'_h}(ky) = 0 = I_{O'_h}(ky) = F_{O'_h}(ky)$ $\forall$ *edges* $ky \in \check{G}_i$; $T_{O'}(y) = T_O(y)$, $I_{O'}(y) = I_O(y)$, $F_{O'}(y) = F_O(y)$, $\forall y \neq k$; $T_{O'_h}(yz) = T_{O_h}(yz)$, $I_{O'_h}(yz) = I_{O_h}(yz)$, $F_{O'_h}(yz) = F_{O_h}(yz)$.

Then, k is IN fuzzy O_h *cut-vertex, for some h, if*

$$T^{\infty}_{O_h}(yz) > T^{\infty}_{O'_h}(yz),\ I^{\infty}_{O_h}(yz) > I^{\infty}_{O'_h}(yz)$$

and

$$F^{\infty}_{O_h}(yz) > F^{\infty}_{O'_h}(yz), \text{for some } y, z \in P \setminus \{k\}.$$

Note that k is an IN fuzzy $O_h - T$ *cut-vertex, if* $T^{\infty}_{O_h}(yz) > T^{\infty}_{O'_h}(yz)$, *IN fuzzy* $O_h - I$ *cut-vertex, if* $I^{\infty}_{O_h}(yz) > I^{\infty}_{O'_h}(yz)$ *and IN fuzzy* $O_h - F$ *cut-vertex, if* $F^{\infty}_{O_h}(yz) > F^{\infty}_{O'_h}(yz)$.

Example 9. *Consider an INGS* $\check{G}_i = (O, O_1, O_2)$ *as represented in Figure 7 and* $\check{G}'_h = (O', O'_1, O'_2)$ *is an IN subgraph structure of an INGS* $\check{G}_i$, *and we found it by deleting the vertex* k_2. *The vertex* k_2 *is an IN fuzzy* O_1-*I cut-vertex, since* $I^{\infty}_{O'_1}(k_2k_5) = 0 < 0.5 = I^{\infty}_{O_1}(k_2k_5)$, $I^{\infty}_{O'_1}(k_4k_3) = 0.7 = I^{\infty}_{O_1}(k_4k_3)$ *and* $I^{\infty}_{O'_1}(k_3k_5) = 0.3 < 0.4 = I^{\infty}_{O_1}(k_3k_5)$.

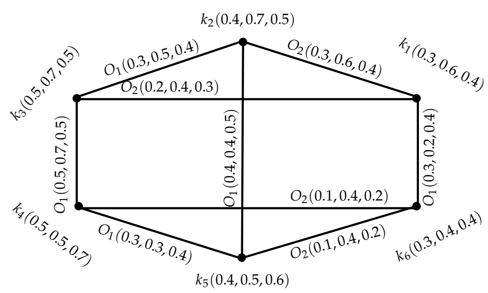

Figure 7. An INGS $\check{G}_i = (O, O_1, O_2)$.

Definition 19. *Let $\check{G}_i = (O, O_1, O_2, \ldots, O_r)$ be an INGS and kl an $O_h -$ edge.*

Let $(O', O'_1, O'_2, \ldots, O'_r)$ be an IN fuzzy spanning-subgraph structure of $\check{G}_i$, such that $T_{O'_h}(kl) = 0 = I_{O'_h}(kl) = F_{O'_h}(kl)$, $T_{O'_h}(qt) = T_{O_h}(qt)$, $I_{O'_h}(qt) = I_{O_h}(qt)$, $F_{O'_h}(qt) = F_{O_h}(qt)$, $\forall$ edges $qt \neq kl$.

Then, kl is an IN fuzzy O_h-bridge if

$T^{\infty}_{O_h}(yz) > T^{\infty}_{O'_h}(yz)$, $I^{\infty}_{O_h}(yz) > I^{\infty}_{O'_h}(yz)$ *and* $F^{\infty}_{O_h}(yz) > F^{\infty}_{O'_h}(yz)$, *for some* $y, z \in P$.

Note that kl is an IN fuzzy $O_h - T$ bridge if $T^{\infty}_{O_h}(yz) > T^{\infty}_{O'_h}(yz)$, IN fuzzy $O_h - I$ bridge if $I^{\infty}_{O_h}(yz) > I^{\infty}_{O'_h}(yz)$ and IN fuzzy $O_h - F$ bridge if $F^{\infty}_{O_h}(yz) > F^{\infty}_{O'_h}(yz)$.

Example 10. *Consider an INGS $\check{G}_i = (O, O_1, O_2)$ as shown in Figure 7 and $\check{G}'_H = (O'', O''_1, O''_2)$ is IN spanning-subgraph structure of an INGS $\check{G}_i$ found by the deletion of O_1-edge (k_2k_5). Edge (k_2k_5) is an IN fuzzy O_1-bridge. As* $T^{\infty}_{O''_1}(k_2k_5) = 0.3 < 0.4 = T^{\infty}_{O_1}(k_2k_5)$, $I^{\infty}_{O''_1}(k_2k_5) = 0.3 < 0.4 = I^{\infty}_{O_1}(k_2k_5)$, $F^{\infty}_{O''_1}(k_2k_5) = 0.4 < 0.5 = F^{\infty}_{O_1}(k_2k_5)$.

Definition 20. *An INGS $\check{G}_i = (O, O_1, O_2, \ldots, O_r)$ is an O_h-tree, if $(supp(O), supp(O_1), supp(O_2), \ldots, supp(O_r))$ is an $O_h -$ tree. Alternatively, $\check{G}_i$ is an O_h-tree, if there is a subgraph of $\check{G}_i$ induced by $supp(O_h)$, which forms a tree.*

Definition 21. *An INGS $\check{G}_i = (O, O_1, O_2, \ldots, O_r)$ is an IN fuzzy O_h-tree if $\check{G}_i$ has an IN fuzzy spanning-subgraph structure $\check{H}_i = (O'', O''_1, O''_2, \ldots, O''_r)$, such that, for all O_h-edges kl not in $\check{H}_i$, $\check{H}_i$ is an O''_h-tree, and $T_{O_h}(kl) < T^{\infty}_{O''_h}(kl)$, $I_{O_h}(kl) < I^{\infty}_{O''_h}(kl)$, $F_{O_h}(kl) < F^{\infty}_{O''_h}(kl)$.*
In particular, $\check{G}_i$ is an IN fuzzy O_h-T tree if $T_{O_h}(kl) < T^{\infty}_{O''_h}(kl)$, an IN fuzzy O_h-I tree if $I_{O_h}(kl) < I^{\infty}_{O''_h}(kl)$, and an IN fuzzy O_h-F tree if $F_{O_h}(kl) > F^{\infty}_{O''_h}(kl)$.

Example 11. *Consider an INGS $\check{G}_i = (O, O_1, O_2)$ as shown in Figure 8. It is an O_2-tree, not an O_1-tree but it is IN fuzzy O_1-tree because it has an IN fuzzy-spanning subgraph (O', O'_1, O'_2) as an O'_1-tree, which is found by the deletion of O_1-edge k_2k_5 from $\check{G}_i$. Moreover,* $T^{\infty}_{O'_1}(k_2k_5) = 0.3 > 0.2 = T_{O_1}(k_2k_5)$, $I^{\infty}_{O'_1}(k_2k_5) = 0.3 > 0.1 = I_{O_1}(k_2k_5)$ *and* $F^{\infty}_{O'_1}(k_2k_5) = 0.4 < 0.5 = F_{O_1}(k_2k_5)$.

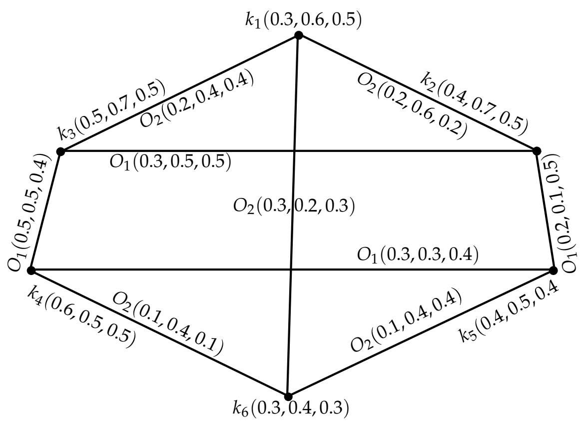

Figure 8. An IN fuzzy O_1-tree.

Definition 22. *An INGS* $\check{G}_{i1} = (O_1, O_{11}, O_{12}, \ldots, O_{1r})$ *of graph structure* $\check{G}_1 = (P_1, P_{11}, P_{12}, \ldots, P_{1r})$ *is said to be isomorphic to an INGS* $\check{G}_{i2} = (O_2, O_{21}, O_{22}, \ldots, O_{2r})$ *of the graph structure* $\check{G}_2 = (P_2, P_{21}, P_{22}, ..., P_{2r})$*, if there is a pair* (g, ψ)*, where* $g : P_1 \to P_2$ *is a bijective mapping and* ψ *is any permutation on this set* $\{1, 2, \ldots, r\}$ *such that;*

$$T_{O_1}(k) = T_{O_2}(g(k)),\ I_{O_1}(k) = I_{O_2}(g(k)),\ F_{O_1}(k) = F_{O_2}(g(k)),\ \forall k \in P_1,$$

$$T_{O_{1h}}(kl) = T_{O_{2\phi(h)}}(g(k)g(l)),\ I_{O_{1h}}(kl) = I_{O_{2\phi(h)}}(g(k)g(l),\ F_{Q_{1h}}(kl) = F_{O_{2\phi(h)}}(g(k)g(l)),$$

$\forall kl \in P_{1h}$, $h = 1,2,\ldots,r$.

Example 12. *Let* $\check{G}_{i1} = (O, O_1, O_2)$ *and* $\check{G}_{i2} = (O', O'_1, O'_2)$ *be two INGSs as shown in the Figure 9.* $\check{G}_{i1}$ *and* $\check{G}_{i2}$ *are isomorphic under* (g, ψ)*, where* $g : P \to P'$ *is a bijective mapping and* ψ *is the permutation on* $\{1, 2\}$*, which is defined as* $\psi(1) = 2$*,* $\psi(2) = 1$*, and the following conditions hold:*

$$\begin{aligned} T_O(k_h) &= T_{O'}(g(k_h)), \\ I_O(k_h) &= I_{O'}(g(k_h)), \\ F_O(k_h) &= F_{O'}(g(k_h)), \end{aligned}$$

$\forall k_h \in P$ *and*

$$\begin{aligned} T_{O_h}(k_h k_q) &= T_{O'_{\psi(h)}}(g(k_h)g(k_q)), \\ I_{O_h}(k_h k_q) &= I_{O'_{\psi(h)}}(g(k_h)g(k_q)), \\ F_{O_h}(k_h k_q) &= F_{O'_{\psi(h)}}(g(k_h)g(k_q)), \end{aligned}$$

$\forall k_h k_q \in P_h$, $h = 1, 2$.

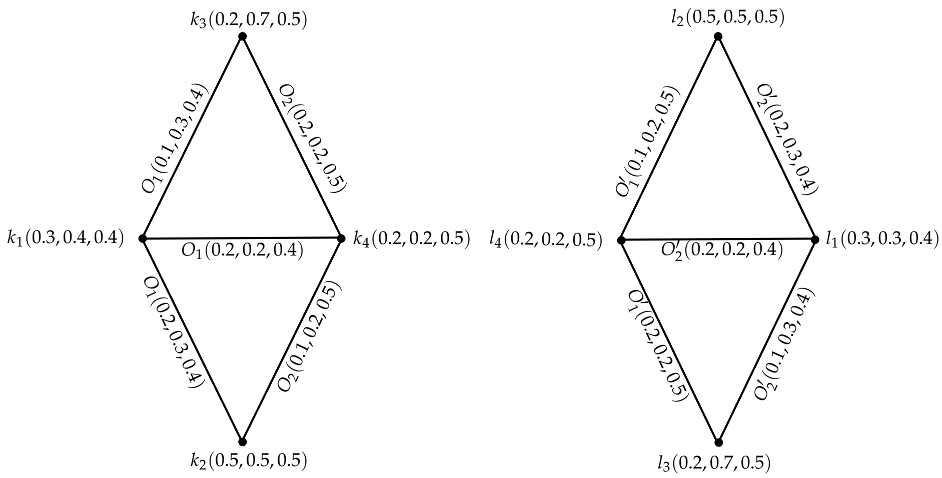

Figure 9. Two isomorphic INGSs.

Definition 23. *An INGS* $\check{G}_{i1} = (O_1, O_{11}, O_{12}, \ldots, O_{1r})$ *of the graph structure* $\check{G}_1 = (P_1, P_{11}, P_{12}, ..., P_{1r})$ *is identical with an INGS* $\check{G}_{i2} = (O_2, O_{21}, O_{22}, ..., O_{2r})$ *of the graph structure* $\check{G}_2 = (P_2, P_{21}, P_{22}, ..., P_{2r})$ *if* $g : P_1 \to P_2$ *is a bijective mapping such that*

$$T_{O_1}(k) = T_{O_2}(g(k)),\ I_{O_1}(k) = I_{O_2}(g(k)),\ F_{O_1}(k) = F_{O_2}(g(k)),\ \forall k \in P_1,$$

$$T_{O_{1h}}(kl) = T_{O_{2h}}(g(k)g(l)),\ I_{O_{1h}}(kl) = I_{O_{2h}}(g(k)g(l)),\ F_{O_{1h}}(kl) = F_{O_{2(h)}}(g(k)g(l)),$$

$\forall kl \in P_{1h}$, $h = 1, 2, \ldots, r$.

Example 13. *Let $\check{G}_{i1} = (O, O_1, O_2)$ and $\check{G}_{i2} = (O', O'_1, O'_2)$ be two INGSs of the GSs $\check{G}_1 = (P, P_1, P_2)$, $\check{G}_2 = (P', P'_1, P'_2)$, respectively, as they are shown in Figures 10 and 11.*
SVINGSs $\check{G}_{i1}$ and $\check{G}_{i2}$ are identical under $g : P \to P'$ is defined as :

$g(k_1) = l_2$, $g(k_2) = l_1$, $g(k_3) = l_4$, $g(k_4) = l_3$, $g(k_5) = l_5$, $g(k_6) = l_8$, $g(k_7) = l_7$, $g(k_8) = l_6$.

Moreover, $T_O(k_h) = T_{O'}((k_h))$, $I_O(k_h) = I_{O'}(g(k_h))$, $F_O(k_h) = F_{O'}(g(k_h))$, $\forall k_h \in P$ and $T_{O_h}(k_h k_q) = T_{O'_h}(g(k_h)g(k_q))$, $I_{O_h}(k_h k_q) = I_{O'_h}(g(k_h)g(k_q))$, $F_{O_h}(k_h k_q) = F_{O'_h}(g(k_h)g(k_q))$, $\forall k_h k_q \in P_h$, $h = 1, 2$.

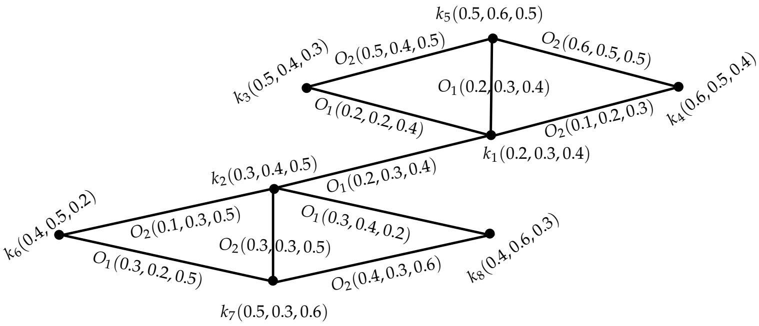

Figure 10. An INGS $\check{G}_{i1}$.

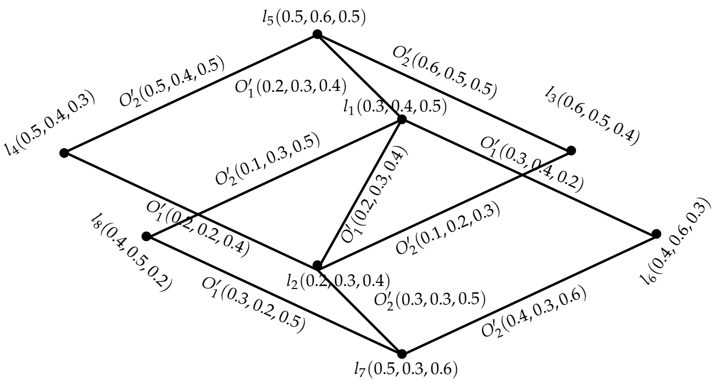

Figure 11. An INGS $\check{G}_{i2}$.

Definition 24. *Let $\check{G}_i = (O, O_1, O_2, ..., O_r)$ be an INGS and ψ is any permutation on $\{O_1, O_2, ..., O_r\}$ and on set $\{1, 2, ..., r\}$, that is, $\psi(O_h) = O_q$ if and only if $\psi(h) = q$ $\forall h$. If $kl \in O_h$, for any h and*

$T_{O_h^\psi}(kl) = T_O(k) \wedge T_O(l) - \bigvee_{q \neq h} T_{\psi(O_q)}(kl)$, $I_{O_h^\psi}(kl) = I_O(k) \wedge I_O(l) - \bigvee_{q \neq h} I_{\psi(O_q)}(kl)$,
$F_{O_h^\psi}(kl) = F_O(k) \vee F_O(l) - \bigwedge_{q \neq h} T_{\psi(O_q)}(kl)$, $h = 1, 2, ..., r$, *then,* $kl \in O_t^\psi$, *where t is chosen such that*
$T_{O_t^\psi}(kl) \geq T_{O_h^\psi}(kl)$, $I_{O_t^\psi}(kl) \geq I_{O_h^\psi}(kl)$, $F_{O_t^\psi}(kl) \geq F_{O_h^\psi}(kl)$ $\forall h$. *In addition, INGS $(O, O_1^\psi, O_2^\psi, \ldots, O_r^\psi)$ is called a ψ-complement of an INGS $\check{G}_i$, and it is symbolized as $\check{G}_i^{\psi c}$.*

Example 14. *Let $O = \{(k_1, 0.3, 0.4, 0.7), (k_2, 0.5, 0.6, 0.4), (k_3, 0.7, 0.5, 0.3)\}$, $O_1 = \{(k_1k_3, 0.3, 0.4, 0.3)\}$, $O_2 = \{(k_2k_3, 0.5, 0.4, 0.3)\}$, $O_3 = \{(k_1k_2, 0.3, 0.3, 0.4)\}$ be IN subsets of P, P_1, P_2, P_3, respectively.*

Thus, $\check{G}_i = (O, O_1, O_2, O_3)$ is an INGS of GS $\check{G} = (P, P_1, P_2, P_3)$. Let $\psi(O_1) = O_2$, $\psi(O_2) = O_3$, $\psi(O_3) = O_1$, where ψ is permutation on $\{O_1, O_2, O_3\}$. Now, for $k_1k_3, k_2k_3, k_1k_2 \in O_1, O_2, O_3$, respectively:

$T_{O_1^\psi}(k_1k_3) = 0$, $I_{O_1^\psi}(k_1k_3) = 0$, $F_{O_1^\psi}(k_1k_3) = 0.7$, $T_{O_2^\psi}(k_1k_3) = 0$, $I_{O_2^\psi}(k_1k_3) = 0$, $F_{O_2^\psi}(k_1k_3) = 0.7$, $T_{O_3^\psi}(k_1k_3) = 0.3$, $I_{O_3^\psi}(k_1k_3) = 0.4$, $F_{O_3^\psi}(k_1k_3) = 0.7$. *So* $k_1k_3 \in O_3^\psi$,

$T_{O_1^\psi}(k_2k_3) = 0.5$, $I_{O_1^\psi}(k_2k_3) = 0.5$, $F_{O_1^\psi}(k_2k_3) = 0.4$, $T_{O_2^\psi}(k_2k_3) = 0$, $I_{O_2^\psi}(k_2k_3) = 0.1$, $F_{O_2^\psi}(k_2k_3) = 0.4$, $T_{O_3^\psi}(k_2k_3) = 0$, $I_{O_3^\psi}(k_2k_3) = 0.1$, $F_{O_3^\psi}(k_2k_3) = 0.4$. *So* $k_2k_3 \in O_1^\psi$,

$T_{O_1^\psi}(k_1k_2) = 0$, $I_{O_1^\psi}(k_1k_2) = 0.1$, $F_{O_1^\psi}(k_1k_2) = 0.7$, $T_{O_2^\psi}(k_1k_2) = 0.3$, $I_{O_2^\psi}(k_1k_2) = 0.4$, $F_{O_2^\psi}(k_1k_2) = 0.7$, $T_{O_3^\psi}(k_1k_2) = 0$, $I_{O_3^\psi}(k_1k_2) = 0.1$, $F_{O_3^\psi}(k_1k_2) = 0.7$. *This shows* $k_1k_2 \in O_2^\psi$.

Hence, $\check{G}_i^{\psi c} = (O, O_1^\psi, O_2^\psi, O_3^\psi)$ is a ψ-complement of an INGS $\check{G}_i$ as presented in Figure 12.

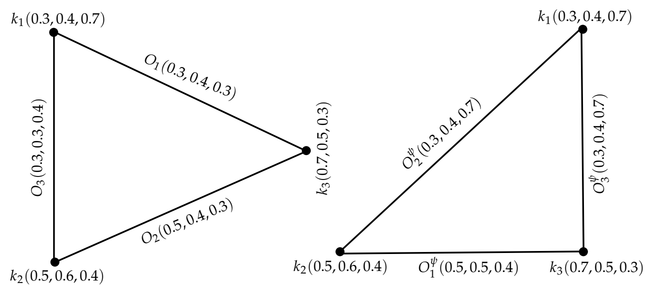

Figure 12. INGSs $\check{G}_i$, $\check{G}_i^{\psi c}$.

Proposition 1. *A ψ-complement of an INGS $\check{G}_i = (O, O_1, O_2, \ldots, O_r)$ is a strong INGS. Moreover, if $\psi(h) = t$, where $h, t \in \{1, 2, ..., r\}$; then, all O_t-edges in an INGS $(O, O_1, O_2, \ldots, O_r)$ become O_h^ψ-edges in $(O, O_1^\psi, O_2^\psi, ..., O_r^\psi)$.*

Proof. By definition of ψ-complement,

$$T_{O_h^\psi}(kl) = T_O(k) \wedge T_O(l) - \bigvee_{q \neq h} T_{\psi(O_q)}(kl), \quad (1)$$

$$I_{O_h^\psi}(kl) = I_O(k) \wedge I_O(l) - \bigvee_{q \neq h} I_{\psi(O_q)}(kl), \quad (2)$$

$$F_{O_h^\psi}(kl) = F_O(k) \vee F_O(l) - \bigwedge_{q \neq h} F_{\psi(O_q)}(kl), \quad (3)$$

for $h \in \{1, 2, ..., r\}$. For Expression 1.
As $T_O(k) \wedge T_O(l) \geq 0$, $\bigvee_{q \neq h} T_{\psi(O_q)}(kl) \geq 0$ and $T_{O_h}(kl) \leq T_O(k) \wedge T_O(l)$ $\forall O_h$.
$\Rightarrow \bigvee_{q \neq h} T_{\psi(O_q)}(kl) \leq T_O(k) \wedge T_O(l) \Rightarrow T_O(k) \wedge T_O(l) - \bigvee_{q \neq h} T_{\psi(O_q)}(kl) \geq 0$.
Hence, $T_{O_h^\psi}(kl) \geq 0$ $\forall h$.
Furthermore, $T_{O_h^\psi}(kl)$ gets a maximum value, when $\bigvee_{q \neq h} T_{\psi(O_q)}(kl)$ is zero. Clearly, when $\psi(O_h) = O_t$ and kl is an O_t-edge, then $\bigvee_{q \neq h} T_{\psi(O_q)}(kl)$ attains zero value. Hence,

$$T_{O_h^\psi}(kl) = T_O(k) \wedge T_O(l), \; for \; (kl) \in O_t, \; \psi(O_h) = O_t. \quad (4)$$

Similarly, for I, the results are:
Since $I_O(k) \wedge I_O(l) \geq 0$, $\bigvee_{q \neq h} I_{\psi(O_q)}(kl) \geq 0$ and $I_{O_h}(kl) \leq I_O(k) \wedge I_O(l)\ \forall O_h$.
$\Rightarrow \bigvee_{q \neq h} I_{\psi(O_q)}(kl) \leq I_O(k) \wedge I_O(l) \Rightarrow I_O(k) \wedge I_O(l) - \bigvee_{q \neq h} I_{\psi(O_q)}(kl) \geq 0$.
Therefore, $I_{O_h^{\psi}}(kl) \geq 0\ \forall\ i$.
Value of the $I_{O_h^{\psi}}(kl)$ is maximum when $\bigvee_{q \neq h} I_{\psi(O_q)}(kl)$ gets zero value. Clearly, when $\psi(O_h) = O_t$ and kl is an O_t-edge, then $\bigvee_{q \neq h} I_{\psi(O_q)}(kl)$ is zero. Thus,

$$I_{O_h^{\psi}}(kl) \quad = \quad I_O(k) \wedge I_O(l),\ for\ (kl) \in O_t,\ \psi(O_h) = O_t. \tag{5}$$

On a similar basis for F in ψ-complement, the results are:

Since $F_O(k) \vee F_O(l) \geq 0$, $\bigwedge_{q \neq h} F_{\psi(O_q)}(kl) \geq 0$ and $F_{O_h}(kl) \leq F_O(k) \vee F_O(l)\ \forall O_h$.
$\Rightarrow \bigwedge_{q \neq h} F_{\psi(O_q)}(kl) \leq F_O(k) \vee F_O(l) \Rightarrow F_O(k) \vee F_O(l) - \bigwedge_{q \neq h} F_{\psi(O_q)}(kl) \geq 0$.
Hence, $F_{O_h^{\psi}}(kl) \geq 0\ \forall h$.
Furthermore, $F_{O_h^{\psi}}(kl)$ is maximum, when $\bigwedge_{q \neq h} F_{\psi(O_q)}(kl)$ is zero. Definitely, when $\psi(O_h) = O_t$ and kl is an O_t-edge, then $\bigwedge_{q \neq h} F_{\psi(O_q)}(kl)$ is zero. Hence,

$$F_{O_h^{\psi}}(kl) \quad = \quad F_O(k) \vee F_O(l),\ for\ (kl) \in O_t,\ \psi(O_h) = O_t. \tag{6}$$

Expressions (4)–(6) give the required proof. □

Definition 25. *Let $\check{G}_i = (O, O_1, O_2, ..., O_r)$ be an INGS and ψ be any permutation on $\{1, 2, ..., r\}$. Then,*

(i) *$\check{G}_i$ is a self-complementary INGS if $\check{G}_i$ is isomorphic to $\check{G}_i^{\psi c}$;*
(ii) *$\check{G}_i$ is a strong self-complementary INGS if $\check{G}_i$ is identical to $\check{G}_i^{\psi c}$.*

Definition 26. *Let $\check{G}_i = (O, O_1, O_2, \ldots, O_r)$ be an INGS. Then,*

(i) *$\check{G}_i$ is a totally self-complementary INGS if $\check{G}_i$ is isomorphic to $\check{G}_i^{\psi c}$, $\forall$ permutations ψ on $\{1, 2, \ldots, r\}$;*
(ii) *$\check{G}_i$ is a totally-strong self-complementary INGS if $\check{G}_i$ is identical to $\check{G}_i^{\psi c}$, $\forall$ permutations ψ on $\{1, 2, \ldots, r\}$.*

Example 15. *INGS $\check{G}_i = (O, O_1, O_2, O_3)$ in Figure 13 is totally-strong self-complementary INGS.*

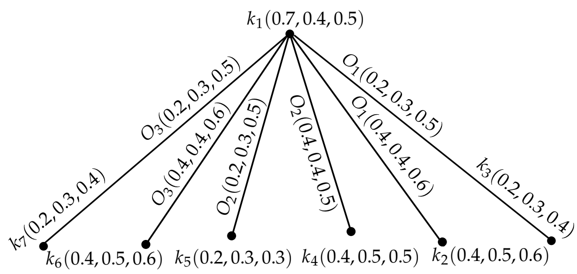

Figure 13. Totally-strong self-complementary INGS.

Theorem 1. *A strong INGS is a totally self-complementary INGS and vice versa.*

Proof. Consider any strong INGS $\check{G}_i$ and Permutation ψ on {1,2, ..., r}. By proposition 1, ψ-complement of an INGS $\check{G}_i = (O, O_1, O_2, \ldots, O_r)$ is a strong INGS. Moreover, if $\psi^{-1}(t) = h$, where $h, t \in \{1, 2, ..., r\}$, then all O_t-edges in an INGS $(O, O_1, O_2, ..., O_r)$ become O_h^ψ-edges in $(O, O_1^\psi, O_2^\psi, ..., O_r^\psi)$, this leads

$$T_{O_t}(kl) = T_O(k) \wedge T_O(l) = T_{O_h^\psi}(kl), I_{O_t}(kl) = I_O(k) \wedge I_O(l) = I_{O_h^\psi}(kl),$$
$$F_{O_t}(kl) = F_O(k) \vee F_O(l) = F_{O_h^\psi}(kl).$$

Therefore, under $g : P \to P$ (identity mapping), $\check{G}_i$ and $\check{G}_i^\psi$ are isomorphic, such that

$$T_O(k) = T_O(g(k)), I_O(k) = I_O(g(k)), F_O(k) = F_O(g(k))$$

and

$$T_{O_t}(kl) = T_{O_h^\psi}(g(k)g(l)) = T_{O_h^\psi}(kl),$$
$$I_{O_t}(kl) = I_{O_h^\psi}(g(k)g(l)) = I_{O_h^\psi}(kl) ,$$
$$F_{O_t}(kl) = F_{O_h^\psi}(g(k)g(l)) = F_{O_h^\psi}(kl),$$

$\forall kl \in P_t$, for $\psi^{-1}(t) = h$; h,t = $1, 2, \ldots, r$.
For each permutation ψ on $\{1, 2, ..., r\}$, this holds. Hence, $\check{G}_i$ is a totally self-complementary INGS.
Conversely, let $\check{G}_i$ is isomorphic to $\check{G}_i^\psi$ for each permutation ψ on $\{1, 2, ..., r\}$. Then, by definitions of ψ-complement of INGS and isomorphism of INGS, we have

$$T_{O_t}(kl) = T_{O_h^\psi}(g(k)g(l)) = T_O(g(k)) \wedge T_O(g(l)) = T_O(k) \wedge T_O(l),$$
$$I_{O_t}(kl) = I_{O_h^\psi}(g(k)g(l)) = I_O(g(k)) \wedge I_O(g(l)) = T_O(k) \wedge I_O(l),$$
$$F_{O_t}(kl) = F_{O_h^\psi}(g(k)g(l)) = F_O(g(k)) \vee F_O(g(l)) = F_O(k) \vee F_O(l),$$

$\forall kl \in P_t$, t = 1,2,...,r. Hence, $\check{G}_i$ is strong INGS. □

Remark 1. *Each self-complementary INGS is a totally self-complementary INGS.*

Theorem 2. *If $\check{G} = (P, P_1, P_2, \ldots, P_r)$ is a totally strong self-complementary GS and $O = (T_O, I_O, F_O)$ is an IN subset of P, where T_O, I_O, F_O are the constant functions, then any strong INGS of $\check{G}$ with IN vertex set O is necessarily totally-strong self-complementary INGS.*

Proof. Let $u \in [0, 1]$, $v \in [0, 1]$ and $w \in [0, 1]$ be three constants, and

$$T_O(k) = u, I_O(k) = v, F_O(k) = w \ \forall k \in P.$$

Since $\check{G}$ is a totally strong self-complementary GS, so, for each permutation ψ^{-1} on $\{1, 2, \ldots, r\}$, there exists a bijective mapping $g : P \to P$, such that, for each P_t-edge (kl), (g(k)g(l)) [a P_h-edge in $\check{G}$] is a P_t-edge in $\check{G}^{\psi^{-1}c}$. Thus, for every O_t-edge (kl), (g(k)g(l)) [an O_h-edge in $\check{G}_i$] is an O_t^ψ-edge in ${\check{G}_i}^{\psi^{-1}c}$.
Moreover, $\check{G}_i$ is a strong INGS, so

$$T_O(k) = u = T_O(g(k)), I_O(k) = v = I_O(g(k)), F_O(k) = w = F_O(g(k)) \ \forall k \in P$$

and

$$T_{O_t}(kl) = T_O(k) \wedge T_O(l) = T_O(g(k)) \wedge T_O(g(l)) = T_{O_h^{\psi}}(g(k)g(l)),$$
$$I_{O_t}(kl) = I_O(k) \wedge I_O(l) = I_O(g(k)) \wedge I_O(g(l)) = I_{O_h^{\psi}}(g(k)g(l)),$$
$$F_{O_t}(kl) = F_O(k) \vee I_O(l) = F_O(g(k)) \vee F_O(g(l)) = F_{O_h^{\psi}}(g(k)g(l)),$$

$\forall kl \in P_h, h = 1, 2, \ldots, r.$
This shows that $\check{G}_i$ is a strong self-complementary INGS. This exists for each permutation ψ and ψ^{-1} on set $\{1, 2, \ldots, r\}$, thus $\check{G}_i$ is a totally strong self-complementary INGS. Hence, required proof is obtained. □

Remark 2. *Converse of the Theorem 2 may or may not true, as an INGS shown in Figure 2 is totally strong self-complementary INGS, and it is also a strong INGS with a totally strong self-complementary underlying GS but T_O, I_O, F_O are not the constant-valued functions.*

3. Application

First, we explain the general procedure of this application by the following algorithm.
Algorithm: Crucial interdependence relations

Step 1. Input vertex set $P = \{B_1, B_2, \ldots, B_n\}$ and IN set O defined on P.
Step 2. Input IN set of interdependence relations of any vertex with all other vertices and calculate T, F, and I of every pair of vertices by using, $T(B_iB_j) \leq \min(T(B_i), T(B_j))$, $F(B_iB_j) \leq \max(F(B_i), F(B_j))$, $I(B_iB_j) \leq \min(I(B_i), I(B_j))$.
Step 3. Repeat the Step 2 for every vertex in P.
Step 4. Define relations $P_1, P_2, \ldots, P_n$ on set P such that $(P, P_1, P_2, \ldots, P_n)$ is a GS.
Step 5. Consider an element of that relation, for which its value of T is comparatively high, and its values of F and I are lower than other relations.
Step 6. Write down all elements in relations with T, F and I values, corresponding relations $O_1, O_2, \ldots, O_n$ are IN sets on $P_1, P_2, P_3, \ldots, P_n$, respectively, and $(O, O_1, O_2, \ldots, O_n)$ is an INGS.

Human beings, the main creatures in the world, depend on many things for their survival. Interdependence is a very important relationship in the world. It is a natural phenomenon that nobody can be 100% independent, and the whole world is relying on interdependent relationships. Provinces or states of any country, especially of a progressive country, can not be totally independent, more or less they have to depend on each other. They depend on each other for many things, that is, there are many interdependent relationships among provinces or states of a progressive country—for example, education, natural energy resources, agricultural items, industrial products, and water resources, etc. However, all of these interdependent relationships are not of equal importance. Some are very important to run the system of a progressive country. Between any two provinces, all interdependent relationships do not have the same strength. Some interdependent relationships are like the backbone for the country. We can make an INGS of provinces or states of a progressive country, and can highlight those interdependent relationships, due to which the system of the country is running properly. This INGS can guide the government as to which interdependent relationships are very crucial, and they must try to make them strong and overcome the factors destroying or weakening them.

We consider a set P of provinces and states of Pakistan:

P = {Punjab, Sindh, Khyber Pakhtunkhawa(KPK), Balochistan, Gilgit-Baltistan, Azad Jammu and Kashmir(AJK) }. Let O be the IN set on P, as defined in Table 1.

Table 1. IN set O of provinces of Pakistan.

Provinces or States	T	I	F
Punjab	0.5	0.3	0.3
Sindh	0.5	0.4	0.4
Khyber Pakhtunkhawa(KPK)	0.4	0.4	0.4
Balochistan	0.3	0.4	0.4
Gilgit-Baltistan	0.3	0.4	0.4
Azad Jammu and Kashmir	0.3	0.4	0.3

In Table 1, symbol T demonstrates the positive role of that province or state for the strength of the Federal Government, and symbol F indicates its negative role, whereas I denotes the percentage of ambiguity of its role for the strength of the Federal Government. Let us use the following alphabets for the provinces' names:

PU = Punjab, SI = Sindh, KPK = Khyber Pakhtunkhwa, BA = Balochistan, GB = Gilgit-Baltistan, AJK = Azad Jammu and Kashmir. For every pair of provinces of Pakistan in set P, different interdependent relationships with their T, I and F values are demonstrated in Tables 2–6.

Table 2. IN set of interdependent relations between Punjab and other provinces.

Type of Interdependent Relationships	(PU, SI)	(PU, KPK)	(PU, BA)
Education	(0.5, 0.1, 0.1)	(0.4, 0.3, 0.2)	(0.3, 0.2, 0.2)
Natural energy resources	(0.3, 0.2, 0.3)	(0.4, 0.2, 0.2)	(0.3, 0.2, 0.1)
Agricultural items	(0.3, 0.2, 0.2)	(0.4, 0.2, 0.1)	(0.3, 0.2, 0.1)
Industrial products	(0.4, 0.2, 0.1)	(0.4, 0.1, 0.1)	(0.3, 0.1, 0.1)
Water resources	(0.3, 0.1, 0.1)	(0.4, 0.3, 0.2)	(0.2, 0.2, 0.2)

Table 3. IN set of interdependent relationships between Sindh and other provinces.

Type of Interdependent Relationships	(SI, KPK)	(SI, BA)	(SI, GB)
Education	(0.3, 0.2, 0.1)	(0.3, 0.2, 0.3)	(0.3, 0.2, 0.4)
Natural energy resources	(0.3, 0.2, 0.3)	(0.3, 0.1, 0.0)	(0.2, 0.2, 0.4)
Agricultural items	(0.4, 0.1, 0.1)	(0.3, 0.1, 0.2)	(0.3, 0.1, 0.1)
Industrial products	(0.4, 0.2, 0.1)	(0.3, 0.2, 0.2)	(0.3, 0.2, 0.2)
Water resources	(0.3, 0.2, 0.2)	(0.2, 0.3, 0.2)	(0.2, 0.2, 0.3)

Table 4. IN set of interdependent relationships between KPK and other provinces.

Type of Interdependent Relationships	(KPK, BA)	(KPK, GB)	(KPK, AJK)
Education	(0.1, 0.4, 0.3)	(0.1, 0.4, 0.3)	(0.1, 0.4, 0.4)
Natural energy resources	(0.3, 0.2, 0.1)	(0.3, 0.2, 0.2)	(0.3, 0.3, 0.2)
Agricultural items	(0.1, 0.2, 0.4)	(0.1, 0.4, 0.4)	(0.1, 0.3, 0.3)
Industrial products	(0.1, 0.3, 0.4)	(0.1, 0.4, 0.3)	(0.1, 0.2, 0.2)
Water resources	(0.3, 0.2, 0.2)	(0.3, 0.3, 0.2)	(0.3, 0.2, 0.2)

Table 5. IN set of interdependent relationships between AJK and other provinces.

Type of Interdependent Relationships	(AJK, PU)	(AJK, SI)	(AJK, BA)
Education	(0.3, 0.1, 0.1)	(0.1, 0.4, 0.3)	(0.1, 0.3, 0.4)
Natural energy resources	(0.1, 0.2, 0.3)	(0.2, 0.4, 0.3)	(0.3, 0.3, 0.3)
Agricultural items	(0.3, 0.2, 0.1)	(0.3, 0.3, 0.2)	(0.3, 0.2, 0.2)
Industrial products	(0.3, 0.2, 0.2)	(0.3, 0.2, 0.2)	(0.3, 0.2, 0.3)
Water resources	(0.3, 0.2, 0.1)	(0.3, 0.3, 0.2)	(0.3, 0.0, 0.1)

Table 6. IN set of interdependent relationships of Gilgit-Baltistan with other provinces.

Type of Interdependent Relationships	(GB, PU)	(GB, BA)	(GB, AJK)
Education	(0.3, 0.2, 0.1)	(0.1, 0.4, 0.4)	(0.2, 0.1, 0.4)
Natural energy resources	(0.1, 0.3, 0.4)	(0.3, 0.1, 0.0)	(0.2, 0.2, 0.4)
Agricultural items	(0.3, 0.2, 0.2)	(0.1, 0.3, 0.3)	(0.1, 0.4, 0.4)
Industrial products	(0.3, 0.3, 0.2)	(0.2, 0.4, 0.4)	(0.1, 0.4, 0.2)
Water resources	(0.2, 0.3, 0.3)	(0.2, 0.3, 0.2)	(0.3, 0.1, 0.1)

Many relations can be defined on the set P, we define following relations on set P as:

P_1 = Education, P_2 = Natural energy resources , P_3 = Agricultural items, P_4 = Industrial products, P_5 = Water resources, such that $(P, P_1, P_2, P_3, P_4, P_5)$ is a GS. Any element of a relation demonstrates a particular interdependent relationship between these two provinces. As $(P, P_1, P_2, P_3, P_4, P_5)$ is GS; this is why any element can appear in only one relation. Therefore, any element will be considered in that relationship, whose value of T is high, and values of I, F are comparatively low, using the data of above tables.

Write down T, I and F values of the elements in relations according to the above data, such that O_1, O_2, O_3, O_4, O_5 are IN sets on relations P_1, P_2, P_3, P_4, P_5, respectively.

Let $P_1 = \{(\text{Punjab}, \text{Sindh}), (\text{Gilgit} - \text{Baltistan}, \text{Punjab}), (\text{AzadJammuandKashmir}, \text{Punjab})\}$;
P_2 = {(Sindh, Balochistan), (Khyber Pakhtunkhawa, Balochistan), (Balochistan, Gilgit-Baltistan), (Khyber Pakhtunkhawa, Gilgit-Baltistan)};
P_3 = {(Sindh, Khyber Pakhtunkhwa), (Gilgit-Baltistan, Sindh) };
$P_4 = \{(\text{Punjab}, \text{KhyberPakhtunkhwa}), (\text{Sindh}, \text{AzadJammuandKashmir}), (\text{Balochistan}, \text{Punjab})\}$;
$P_5 = \{(\text{KheberPakhtunkhwa}, \text{AzadJammuandKashmir}), (\text{Balochistan}, \text{AzadJammuandKashmir}), (\text{Gilgit} - \text{Baltistan}, \text{ Azad Jammu } and \text{ Kashmir})\}$.

Let $O_1 = \{((PU, SI), 0.5, 0.1, 0.1), ((GB, PU), 0.3, 0.2, 0.1), ((AJK, PU), 0.3, 0.1, 0.1)\}$,
$O_2 = \{((SI, BA), 0.3, 0.1, 0.0), ((KPK, BA), 0.3, 0.2, 0.1), ((BA, GB), 0.3, 0.1, 0.0), ((KPK, GB), 0.3, 0.2, 0.2)\}$,
$O_3 = \{((SI, KPK), 0.4, 0.1, 0.1), ((GB, SI), 0.3, 0.1, 0.1), \}$,
$O_4 = \{((PU, KPK), 0.4, 0.1, 0.1), ((SI, AJK), 0.3, 0.2, 0.2), ((BA, PU), 0.3, 0.1, 0.1)\}$,
$O_5 = \{((KPK, AJK), 0.3, 0.2, 0.2), ((BA, AJK), 0.3, 0.0, 0.1), ((GB, AJK), 0.3, 0.1, 0.1)\}$.

Obviously, (O, O_1, O_2, O_3, O_4, O_5) is an INGS as shown in Figure 14.

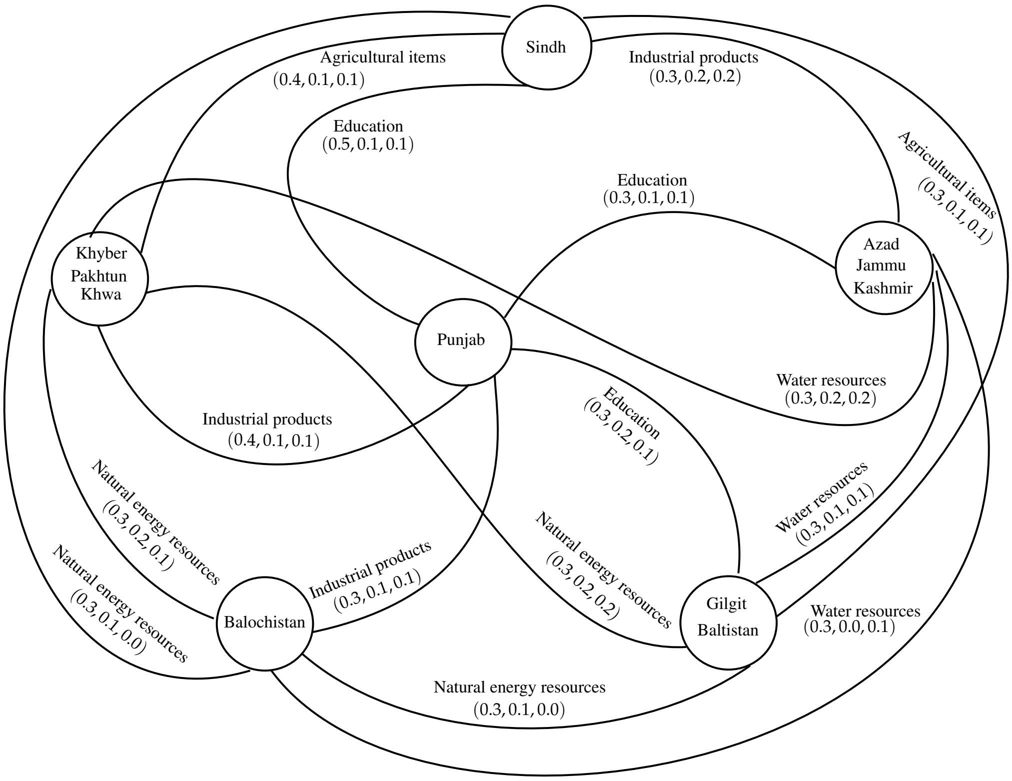

Figure 14. INGS identifying crucial interdependence relation between any two provinces.

Every edge of this INGS demonstrates the most dominating interdependent relationship between those two provinces—for example, the most dominating interdependent relationship between Punjab and Gilgit-Baltistan is education, and its T, F and I values are 0.3, 0.2 and 0.1, respectively. It shows that education is the strongest connection bond between Punjab and Gilgit-Baltistan; it is 30% stable, 10% unstable, and 20% unpredictable or uncertain. Using INGS, we can also elaborate the strength of any province, e.g., Punjab has the highest vertex degree for interdependent relationship education, and Balochistan has the highest vertex degree for the interdependent relationship natural energy resources. This shows that the strength of Punjab is education, and the strength of Balochistan is the natural energy resources. This INGS can be very helpful for Provincial Governments, and they can easily estimate which kind of interdependent relationships they have with other provinces, and what is the percentage of its stability and instability. It can also guide the Federal Government in regards to, between any two provinces, which relationships are crucial and what is their status. The Federal Government should be conscious of making decisions such that the most crucial interdependent relationships of its provinces are not disturbed and need to overcome the counter forces that are trying to destroy them.

4. Conclusions

Graph theory is a useful tool for solving combinatorial problems of different fields, including optimization, algebra, computer science, topology and operations research. An intuitionistic neutrosophic set constitutes a generalization of an intuitionistic fuzzy set. In this research paper, we have introduced the notion of intuitionistic neutrosophic graph structure. We have discussed a real-life application of intuitionistic neutrosophic graph structure in decision-making. Our aim is to extend our research work to (1) fuzzy rough graph structures; (2) rough fuzzy graph structures; (3) soft rough graph structures; and (4) roughness in graph structures.

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NC-TODIM-Based MAGDM under a Neutrosophic Cubic Set Environment

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Abstract: A neutrosophic cubic set is the hybridization of the concept of a neutrosophic set and an interval neutrosophic set. A neutrosophic cubic set has the capacity to express the hybrid information of both the interval neutrosophic set and the single valued neutrosophic set simultaneously. As newly defined, little research on the operations and applications of neutrosophic cubic sets has been reported in the current literature. In the present paper, we propose the score and accuracy functions for neutrosophic cubic sets and prove their basic properties. We also develop a strategy for ranking of neutrosophic cubic numbers based on the score and accuracy functions. We firstly develop a TODIM (Tomada de decisao interativa e multicritévio) in the neutrosophic cubic set (NC) environment, which we call the NC-TODIM. We establish a new NC-TODIM strategy for solving multi attribute group decision making (MAGDM) in neutrosophic cubic set environment. We illustrate the proposed NC-TODIM strategy for solving a multi attribute group decision making problem to show the applicability and effectiveness of the developed strategy. We also conduct sensitivity analysis to show the impact of ranking order of the alternatives for different values of the attenuation factor of losses for multi-attribute group decision making strategies.

Keywords: neutrosophic cubic set; single valued neutrosophic set; interval neutrosophic set; multi attribute group decision making; TODIM strategy; NC-TODIM

1. Introduction

While modelling multi attribute decision making (MADM) and multi attribute group decision making (MAGDM), it is often observed that the parameters of the problem are not precisely known. The parameters often involve uncertainty. To deal with uncertainty, Zadeh [1] left an important mark to represent and compute with imperfect information by introducing the fuzzy set. The fuzzy set fostered a broad research community, and its impact has also been clearly felt at the application level in MADM [2–4] and MAGDM [5–9].

Atanassov [10] incorporated the non-membership function as an independent component and defined the intuitionistic fuzzy set (IFS) at first to express uncertainty in a more meaningful way. IFSs have been applied in many MADM problems [11–13]. Smarandache [14] proposed the notion of the neutrosophic set (NS) by introducing indeterminacy as an independent component. Wang et al. [15] grounded the concept of the single valued neutrosophic set (SVNS), an instance of the neutrosophic set, to deal with incomplete, inconsistent, and indeterminate information in a realistic way. Wang et al. [16] proposed the interval neutrosophic set (INS) as a subclass of neutrosophic sets in which the values of truth, indeterminacy, and falsity membership degrees are interval numbers. Theoretical development and applications of SVNSs and INSs are found in [17–37] for MADM or MAGDM. Some studies on MADM in single valued neutrosophic hesitant fuzzy set environments are found in [38–41].

NS and INS are both capable of handling uncertainty and incomplete information. By fusing NS and INS, Ali et al. [42] proposed the neutrosophic cubic set (NCS) and defined external and internal neutrosophic cubic sets, and established some of their properties. In the same study, Ali et al. [42] proposed an adjustable strategy to NCS-based decision making. Jun et al. [43] also defined NCS by combining NS and INS. In decision making process, the advantage of NCSs is that the decision makers can employ the hybrid information comprising of INSs and SVNSs for evaluating and rating of the alternatives with respect to their predefined attributes. However, there are only a few studies in the literature to deal with MADM and MAGDM in the NCS environment. Banerjee et al. [44] established grey relational analysis (GRA) [45–47] based on the new MADM strategy in the NCS environment. In the same study, Banerjee et al. [44] proposed the Hamming distances for weighted grey relational coefficients and ideal grey relational coefficients, and offered the concept of relative closeness coefficients for presenting the ranking order of the alternatives based on the descending order of their relative closeness coefficients.

Similarity measure is an important mathematical tool in decision-making problems. Pramanik et al. [48] at first defined similarity measure for NCSs and proved its basic properties. In the same study, Pramanik et al. [48] developed a new MAGDM strategy in the NCS environment. Lu and Ye [49] proposed cosine measures between NCSs and established their basic properties. In the same study, Lu and Ye [49] proposed three new cosine measures-based MADM strategies under a NCS environment.

Due to little research on the operations and application of NCSs, Pramanik et al. [50] proposed several operational rules on NCSs, and defined Euclidean distances and arithmetic average operators of NCSs. In the same study, Pramanik et al. [50] also employed the information entropy scheme to calculate the unknown weights of the attributes, and developed a new extended TOPSIS strategy for MADM under the NCS environment. Zhan et al. [51] proposed a new algorithm for multi-criteria decision making (MCDM) in an NCS environment based on a weighted average operator and a weighted geometric operator. Ye [52] established the concept of a linguistic neutrosophic cubic number (LNCN). In the same study, Ye [52] developed a new MADM strategy based on a LNCN weighted arithmetic averaging (LNCNWAA) operator and a LNCN weighted geometric averaging (LNCNWGA) operator under a linguistic NCS environment.

In the literature, there are only six strategies [44,48–52] for MADM and MAGDM in NCS environment. However, we say that none of them is generally superior to all others. So, new strategies for MADM and MAGDM should be explored under the NCS environment for the development of neutrosophic studies.

TODIM (an acronym in Portuguese for interactive multi-criteria decision making strategy named Tomada de decisao interativa e multicritévio) is an important MADM strategy, since it considers the decision makers' bounded rationality. Firstly, Gomes and Lima [53] introduced the TODIM strategy based on prospect theory [54]. Krohling and Souza [55] defined the fuzzy TODIM strategy to solve MCDM problems. Several researchers applied the TODIM strategy in various fuzzy MADM or MAGDM problems [56–58]. Fan et al. [59] introduced the extended TODIM strategy to deal with the hybrid MADM problems. Krohling et al. [60] extended the TODIM strategy from fuzzy environment to intuitionistic fuzzy environment to process the intuitionistic fuzzy information. Wang [61] introduced TODIM strategy into multi-valued neutrosophic set environment. Zhang et al. [62] proposed the TODIM strategy for MAGDM problems under a neutrosophic number environment. Ji et al. [63] proposed the TODIM strategy under a multi-valued neutrosophic environment and employed it to solve personal selection problems. In 2017, Xu et al. [64] developed the TODIM strategy in a single valued neutrosophic setting and extended it into interval neutrosophic setting. Neutrosophic TODIM [64] is capable of dealing with only single-valued neutrosophic information or interval neutrosophic information.

NCS can be used to express the interval neutrosophic information and neutrosophic information in the process of MADGM. It seems that TODIM in NCSs has an enormous chance of success to deal with group decision making problems. In the NCS environment, the TODIM strategy is yet to appear. Motivated by these, we initiated the study of TODIM in the NCS environment, which we call NC-TODIM.

However, NCSs comprise of hybrid information of INSs and SVNSs simultaneously, which are more flexible and elegant for expressing neutrosophic cubic information. To apply NCSs to MADGM problems, we introduce some basic operations of neutrosophic cubic (NC) numbers and the score, and accuracy functions of NC numbers, and the ranking strategy of NC numbers.

In this paper we develop a TODIM strategy (for short, NC-TODIM strategy) for MAGDM in the NCS environment. The proposed NC-TODIM strategy was proven to be capable of successfully dealing with MAGDM problems by solving an illustrative example. What is more, a comparative analysis ensured the feasibility of the proposed NC-TODIM strategy.

The remainder of the paper is divided into seven sections that are organized as follows: Section 2 presents some basic definitions of NS, SVNS, INS, and NCS. Section 3 presents comparison strategy of two NC-numbers. Section 4 is devoted to present the proposed NC-TODIM strategy. Section 5 presents an illustrative numerical example of MAGDM in the NCS environment. Section 6 is devoted to analyzing the ranking order with different values of attenuation factors of losses. Section 7 presents a comparative analysis between the developed strategy and other existing strategies in the NCS environment. Section 8 presents the conclusion and the future scope of research.

2. Preliminaries

In this section, we review some basic definitions which are important to develop the paper.

Definition 1. *[14]* NS. *Let* U *be a space of points (objects) with a generic element in* U *denoted by u, i.e.,* $u \in U$. *A neutrosophic set* R *in* U *is characterized by truth-membership function* t_R, *indeterminacy-membership function* i_R, *and falsity-membership function* f_R, *where* t_R, i_R, f_R *are the functions from* U *to* $]^-0, 1^+[$ *i.e.,* t_R, i_R, f_R*:* $U \rightarrow]^-0, 1^+[$ *that means* $t_R(u)$, $i_R(u)$, $f_R(u)$ are the real standard or non-standard subset of $]^-0, 1^+[$. *The neutrosophic set can be expressed as* $R = \{<u; (t_R(u), i_R(u), f_R(u))>: \forall u \in U\}$. *Since* $t_R(u)$, $i_R(u)$, $f_R(u)$ *are the subset of* $]^-0, 1^+[$, *there the sum* $(t_R(u) + i_R(u) + f_R(u))$ *lies between* $^-0$ *and* 3^+, *where* $^-0 = 0 - \varepsilon$ *and* $3^+ = 3 + \varepsilon$, $\varepsilon > 0$.

Example 1. *Suppose that* $U = \{u_1, u_2, u_3, \ldots\}$ *is the universal set. Let* R_1 *be any neutrosophic set in* U. *Then* R_1 *expressed as* $R_1 = \{<u_1; (0.6, 0.3, 0.4)>: u_1 \in U\}$.

Definition 2. *[15] SVNS. Let* U *be a space of points (objects) with a generic element in* U *denoted by* u. *A single valued neutrosophic set* H *in* U *is expressed by* $H = \{<u, (t_H(u), i_H(u), f_H(u))>; u \in U\}$, *where* $t_H(u)$, $i_H(u)$, $f_H(u) \in [0, 1]$. *Therefore for each* $u \in U$, $t_H(u)$, $i_H(u)$, $f_H(u) \in [0, 1]$ *and* $0 \leq t_H(u) + i_H(u) + f_H(u) \leq 3$.

Definition 3. *[16] INS. Let* G *be a non-empty set. An interval neutrosophic set* $\widetilde{G}$ *in* G *is characterized by truth-membership function* $t_{\widetilde{G}}$, *the indeterminacy membership function* $i_{\widetilde{G}}$ *and falsity membership function* $f_{\widetilde{G}}$. *For each* $g \in G$, $t_{\widetilde{G}}(g)$, $i_{\widetilde{G}}(g)$, $f_{\widetilde{G}}(g) \subseteq [0, 1]$ *and* $\widetilde{G}$ *defined as*

$$\widetilde{G} = \{< g; [t^-_{\widetilde{G}}(g), t^+_{\widetilde{G}}(g)], [i^-_{\widetilde{G}}(g), i^+_{\widetilde{G}}(g),], [f^-_{\widetilde{G}}(g), f^+_{\widetilde{G}}(g)] : \forall g \in G\}.$$

Here,

$$t^-_{\widetilde{G}}(g), t^+_{\widetilde{G}}(g), i^-_{\widetilde{G}}(g), i^+_{\widetilde{G}}(g), f^-_{\widetilde{G}}(g), f^+_{\widetilde{G}}(g) : G \rightarrow]^-0, 1^+[,$$

and

$$^-0 \leq \sup t^+_{\widetilde{G}}(g) + \sup i^+_{\widetilde{G}}(g) + \sup f^+_{\widetilde{G}}(g) \leq 3^+.$$

In real problems it is difficult to express the truth-memberships function, indeterminacy-membership function and falsity-membership function in the form of

$$t^{-}_{\tilde{G}}(g), t^{+}_{\tilde{G}}(g), i^{-}_{\tilde{G}}(g), i^{+}_{\tilde{G}}(g), f^{-}_{\tilde{G}}(g), f^{+}_{\tilde{G}}(g) : G \to]^{-}0, 1^{+}[.$$

Here,

$$t^{-}_{\tilde{G}}(g), t^{+}_{\tilde{G}}(g), i^{-}_{\tilde{G}}(g), i^{+}_{\tilde{G}}(g), f^{-}_{\tilde{G}}(g), f^{+}_{\tilde{G}}(g) : G \to [0, 1].$$

Example 2. *Suppose that* $G = \{g_1, g_2, g_3, \ldots, g_n\}$ *is a non-empty set. Let* $\tilde{G}_1$ *be an INS. Then* $\tilde{G}_1$ *is expressed as*

$$\tilde{G}_1 = \{< g_1; [0.39, 0.47], [0.17, 0.43], [0.18, 0.36] : g_1 \in G\}.$$

Definition 4. *[42,43] NCS. A NCS in a non-empty set* G *is defined as* © = {<g; $\tilde{G}$(g), R(g)>: ∀g∈G}, *where* $\tilde{G}$ *and R are the INS and NS in* G *respectively. NCS can be presented as an order pair* © = <$\tilde{G}$, R>, *then we call it as a neutrosophic cubic (NC) number.*

Example 3. *Suppose that* $G = \{g_1, g_2, g_3, \ldots, g_n\}$ *is a non-empty set. Let* $©_1$ *be any NC-number. Then* $©_1$ *can be expressed as* $©_1$ = {<g_1; [0.39, 0.47], [0.17, 0.43], [0.18, 0.36], (0.6, 0.3, 0.4)>: g_1∈G}.

Some operations of NC-numbers:

i. Union of any two NC-numbers

Let $©_1 =< \tilde{G}_1, R_1 >$ and $©_2 =< \tilde{G}_2, R_2 >$ be any two NC-numbers in a non-empty set G. Then the union of $©_1$ and $©_2$denoted by $©_1 \cup ©_2$ and defined as

$$©_1 \cup ©_2 =< \tilde{G}_1(g) \cup \tilde{G}_1(g), R_1(g) \cup R_2(g)\, \forall g \in G >,$$

where $\tilde{G}_1(g) \cup \tilde{G}_1(g)$ = {<g, [max {$t^{-}_{\tilde{G}_1}(g), t^{-}_{\tilde{G}_2}(g)$},max {$t^{+}_{\tilde{G}1}(g), t^{+}_{\tilde{G}_2}(g)$}], [max {$i^{-}_{\tilde{G}_1}(g), i^{-}_{\tilde{G}_2}(g)$}, max {$i^{+}_{\tilde{G}_1}(g), i^{+}_{\tilde{G}_2}(g)$}], [min {$f^{-}_{\tilde{G}_1}(g), f^{-}_{\tilde{G}_2}(g)$}, min {$f^{+}_{\tilde{G}_1}(g), f^{+}_{\tilde{G}_2}(g)$}]>: g∈G} and $R_1(g) \cup R_2(g)$ = {<g, max {$t_{R_1}(g), t_{R_2}(g)$}, max {$i_{R_1}(g), i_{R_2}(g)$}, min {$f_{R_1}(g), f_{R_2}(g)$}>:∀g∈U}.

Example 4. *Let* $©_1$ *and* $©_2$ *be two NC-numbers in* G *presented as follows:*

$$©_1 =< [0.39, 0.47], [0.17, 0.43], [0.18, 0.36], (0.6, 0.3, 0.4) >$$

and

$$©_2 =< [0.56, 0.70], [0.27, 0.42], [0.15, 0.26], (0.7, 0.3, 0.6) > .$$

Then

$$©_1 \cup ©_2 =< [0.56, 0.70], [0.27, 0.43], [0.15, 0.26], (0.7, 0.3, 0.4) > .$$

ii. Intersection of any two NC-numbers

Intersection of two NC-numbers denoted and defined as follows:

$$©_1 \cap ©_2 =<\tilde{G}_1(g) \cap \tilde{G}_1(g), R_1(g) \cap R_2(g)\, \forall g \in G >,$$

where $\tilde{G}_1(g) \cap \tilde{G}_1(g)$ = {<g, [min {$t^{-}_{\tilde{G}_1}(g), t^{-}_{\tilde{G}_2}(g)$},min {$t^{+}_{\tilde{G}1}(g), t^{+}_{\tilde{G}_2}(g)$}], [min {$i^{-}_{\tilde{G}_1}(g), i^{-}_{\tilde{G}_2}(g)$}, min {$i^{+}_{\tilde{G}_1}(g), i^{+}_{\tilde{G}_2}(g)$}], [max {$f^{-}_{\tilde{G}_1}(g), f^{-}_{\tilde{G}_2}(g)$}, max {$f^{+}_{\tilde{G}_1}(g), f^{+}_{\tilde{G}_2}(g)$}]>: g∈G} and $R_1(g) \cap R_2(g)$= {<g, min {$t_{R_1}(g), t_{R_2}(g)$}, min {$i_{R_1}(g)$,$i_{R_2}(g)$}, max {$f_{R_1}(g), f_{R_2}(g)$}>:∀g∈U}.

Example 5. *Let* $©_1$ *and* $©_2$ *be any two NC-numbers in* G *presented as follows:*

$$©_1 =< [0.45,\ 0.57],\ [0.27,\ 0.33],\ [0.18,\ 0.46],\ (0.7,\ 0.3,\ 0.5) >$$

and

$$©_2 =< [0.67,\ 0.75],\ [0.22,\ 0.44],\ [0.17,\ 0.21],\ (0.8,\ 0.4,\ 0.4) > .$$

Then

$$©_1 \cap ©_2 =< [0.45,\ 0.57],\ [0.22,\ 0.33],\ [0.18,\ 0.46],\ (0.7,\ 0.3,\ 0.4) > .$$

iii. Compliment of a NC-number

Let $©_1 =< \widetilde{G}_1,\ R_1 >$ be a NCS in G. Then, the compliment of $©_1 =< \widetilde{G}_1,\ R_1 >$ denoted by $©_1^c$ = {<g, $\widetilde{G}_1^c$(g), R_1^c(g)>: ∀g∈G}.

Here, $\widetilde{G}_1{}^c$ = {<g, $[t^-_{\widetilde{G}_1{}^c}(g),\ t^+_{\widetilde{G}_1{}^c}(g)]$, $[i^-_{\widetilde{G}_1{}^c}(g),\ i^+_{\widetilde{G}_1{}^c}(g)]$, $[f^-_{\widetilde{G}_1{}^c}(g),\ f^+_{\widetilde{G}_1{}^c}(g)]$>: ∀g∈G}, where, $t^-_{\widetilde{G}_1{}^c}(g) = f^-_{\widetilde{G}_1}(g)$, $t^+_{\widetilde{G}_1{}^c}(g) = f^+_{\widetilde{G}_1}(g)$, $i^-_{\widetilde{G}_1{}^c}(g) = \{1\} - i^-_{\widetilde{G}_1}(g)$, $i^+_{\widetilde{G}_1{}^c}(g) = \{1\} - i^+_{\widetilde{G}_1}(g)$, $f^-_{\widetilde{G}_1{}^c}(g) = t^-_{\widetilde{G}_1}(g)$, $f^+_{\widetilde{G}_1{}^c}(g) = f^+_{\widetilde{G}_1}(g)$ and $t_{R_1^c}(g) = f_{R1}(g)$, $i_{R'^c_1}(g) = \{1^+\} - i_{R_1}(g)$, $f_{R_1^c}(g) = t_{R_1}(g)$.

Example 6. *Assume that* $©_1$ *be any NC-number in* G *in the form:*

$$©_1 =< [0.45,\ 0.57],\ [0.27,\ 0.33],\ [0.18,\ 0.46],\ (0.7,\ 0.3,\ 0.5) >$$

Then compliment of $©_1$ *is obtained as*

$$©_1^c =< [0.18,\ 0.46],\ [0.73,\ 0.67],\ [0.45,\ 0.57],\ (0.5,\ 0.7,\ 0.7) > .$$

Definition 5. *Score function. Let* $©_1$ *be a NC-number in a non-empty set* G. *Then, a score function of* $©_1$, *denoted by* Sc $(©_1)$ *is defined as:*

$$\mathrm{Sc}\,(©_1) = \frac{1}{2}\left[\left(\frac{2 + a_1 + a_2 - 2b_1 - 2b_2 - c_1 - c_2}{4}\right) + \left(\frac{1 + a - 2b - c}{2}\right)\right] \tag{1}$$

where, $©_1$ = <[a_1, a_2], [b_1, b_2], [c_1, c_2], (a, b, c) > *and* Sc $(©_1)$∈[–1, 1].

Proposition 1. *Score function of two NC-numbers lies between* −1 *to* 1.

Proof. Using the definition of INS and NS, we have all a_1, a_2, b_1, b_2, c_1, c_2, a, b, and c [0, 1].

Since,

$$\begin{gathered} 0 \le a_1 \le 1, 0 \le a_2 \le 1 \\ 0 \le a_1 + a_2 \le 2, \\ \Rightarrow 2 \le 2 + a_1 + a_2 \le 4 \end{gathered} \tag{2}$$

$$\begin{gathered} 0 \le b_1 \le 1 \Rightarrow 0 \le 2b_1 \le 2 \text{ and } 0 \le b_2 \le 1 \Rightarrow 0 \le 2b_2 \le 2 \\ \Rightarrow -2 \le -2b_1 \le 0 \\ \Rightarrow -2 \le -2b_2 \le 0 \\ \Rightarrow -4 \le -2b_1 - 2b_2 \le 0 \end{gathered} \tag{3}$$

$$\begin{gathered} 0 \le c_1 \le 1 \Rightarrow -1 \le -c_1 \le 0 \\ 0 \le c_2 \le 1 \Rightarrow -1 \le -c_2 \le 0 \\ \Rightarrow -2 \le -c_1 - c_2 \le 0 \end{gathered} \tag{4}$$

Adding Equations (2)–(4), we obtain

$$\begin{aligned} &\Rightarrow -4 \leq 2 + a_1 + a_2 - 2b_1 - 2b_2 - c_1 - c_2 \leq 4, \\ &\Rightarrow -1 \leq \tfrac{2 + a_1 + a_2 - 2b_1 - 2b_2 - c_1 - c_2}{4} \leq 1 \end{aligned} \tag{5}$$

Again,

$$0 \leq a \leq 1 \Rightarrow 1 \leq 1 + a \leq 2, \tag{6}$$

$$\begin{gathered} 0 \leq b \leq 1 \Rightarrow 0 \leq 2b \leq 2, \\ 0 \leq c \leq 1, \\ \Rightarrow 0 \leq 2b + c \leq 3, \\ \Rightarrow -3 \leq -2b - c \leq 0 \end{gathered} \tag{7}$$

Adding (6) and (7), we obtain

$$\begin{gathered} -2 \leq 1 + a - 2b - c \leq 2, \\ \Rightarrow -1 \leq \tfrac{1 + a - 2b - c}{2} \leq 1 \end{gathered} \tag{8}$$

Adding (5) and (8) and dividing by 2, we obtain

$$-1 \leq \frac{1}{2}[(\frac{2 + a_1 + a_2 - 2b_1 - 2b_2 - c_1 - c_2}{4}) + (\frac{1 + a - 2b - c}{2})] \leq 1$$

$$Sc\,(©_1) \in [-1, 1].$$

Hence the proof is complete. □

Example 7. *Let* $©_1$ *and* $©_2$ *be two NC-numbers in* G*, presented as follows:*

$$©_1 = < [0.39, 0.47], [0.17, 0.43], [0.18, 0.36], (0.6, 0.3, 0.4) >$$

and

$$©_2 = < [0.56, 0.70], [0.27, 0.42], [0.15, 0.26], (0.7, 0.3, 0.6) > .$$

Then, by applying Definition 5, we obtain $Sc\,(©_1) = -0.01$ *and* $Sc\,(©_2) = 0.07$*, In this case, we can say that* $©_2 > ©_1$.

Definition 6. *Accuracy function. Let* $©_1$ *be a NC-number in a non-empty set* G*, an accuracy function of* $©_1$ *is defined as:*

$$Ac(©_1) = 1/2\,[1/2(a_1 + a_2 - b_2(1 - a_2) - b_1(1 - a_1) - c_2(1 - b_1) - c_1(1 - b_2) + a - b(1 - a) - c(1 - b)] \tag{9}$$

Here, $Ac\,(©_1) \in [-1, 1]$.

When the value of $Ac\,(©_1)$ *increases, we say that the degree of accuracy of the NC-number* $©_1$ *increases.*

Proposition 2. *Accuracy function of two NC-numbers lies between* -1 *to* 1.

Proof. The values of accuracy function depend upon

$$\{\frac{1}{2}(a_1 + a_2 - b_2(1 - a_2) - b_1(1 - a_1) - c_2(1 - b_1) - c_1(1 - b_2)) \text{ and } \{ a - b(1 - a) - c(1 - b)\}$$

The values of

$$\{\frac{1}{2}(a_1 + a_2 - b_2(1 - a_2) - b_1(1 - a_1) - c_2(1 - b_1) - c_1(1 - b_2))\}$$

and

$$\{a - b(1 - a) - c(1 - b)\}$$

lie between -1 to 1 from [37].

Thus, $-1 \leq \mathrm{Ac}(©_1) \leq 1$.

Hence the proof is completed. □

Example 8. *Let* $©_1$ *and* $©_2$ *be two NC-numbers in* G *presented as follows:*

$$©_1 =< [0.41,\ 0.52],\ [0.10,\ 0.18],\ [0.06,\ 0.17],\ (0.48,\ 0.11,\ 0.11) >$$

and

$$©_2 =< [0.40,\ 0.51],\ [0.10,\ 0.20],\ [0.10,\ 0.19],\ (0.50,\ 0.11,\ 0.11) > .$$

Then, by applying Definition 6, we obtain $\mathrm{Ac}(©_1) = 0.14$ *and* $\mathrm{Ac}(©_2) = 0.30$. In this case, we can say that alternative $©_2$ *is better than* $©_1$.

With respect to the score function Sc and the accuracy function Ac, a strategy for comparing NC-numbers can be defined as follows:

3. Comparison Strategy of Two NC-Numbers

Let $©_1$ and $©_2$ be any two NC-numbers. Then we define comparison strategy as follows:

i. If

$$\mathrm{Sc}(©_1) > \mathrm{Sc}(©_2), \text{ then } ©_1 > ©_2. \tag{10}$$

ii. If

$$\mathrm{Sc}(©_1) = \mathrm{Sc}(©_2) \text{ and } \mathrm{Ac}(©_1) > \mathrm{Ac}(©_2), \text{ then } ©_1 > ©_2. \tag{11}$$

iii. If

$$\mathrm{Sc}(©_1) = \mathrm{Sc}(©_2) \text{ and } \mathrm{Ac}(©_1) = \mathrm{Ac}(©_2), \text{ then } ©_1 = ©_2. \tag{12}$$

Example 9. *Let* $©_1$ *and* $©_2$ *be two NC-numbers in* G*, presented as follows:*

$$©_1 =< [0.23,\ 0.29],\ [0.37,\ 0.46],\ [0.34,\ 0.42],\ (0.26,\ 0.26,\ 0.26) >$$

and

$$©_2 =< [0.25,\ 0.31],\ [0.35,\ 0.44],\ [0.35,\ 0.44],\ (0.28,\ 0.28,\ 0.28) > .$$

Then, applying Definition 5, we obtain $\mathrm{Sc}(©_1) = 0.13$ *and* $\mathrm{Sc}(©_2) = 0.13$. *Applying Definition 6, we obtain* $\mathrm{Ac}(©_1) = -0.20$ and $\mathrm{Ac}(©_2) = -0.18$. *In this case, we say that alternative* $©_2 > ©_1$. *(Score values and Accuracy values taking correct up to two decimal places).*

Definition 7. *Let* $©_1$ *and* $©_2$ *be any two NC-numbers, then the distance between them is defined by*

$$\partial(©_1,\ ©_2) = \tfrac{1}{9}[|a_1 - d_1| + |a_2 - d_2| + |b_1 - e_1| + |b_2 - e_2| + |c_1 - f_1| + |c_2 - f_2| + |a - d| + |b - e| + |c - f|] \tag{13}$$

where, $©_1 = <[a_1, a_2], [b_1, b_2], [c_1, c_2], (a, b, c)>$ and $©_2 = <[d_1, d_2], [e_1, e_2], [f_1, f_2], (d, e, f)>$.

Example 10. *Let* $©_1$ *and* $©_2$ *be two NC-numbers in* G *presented as follows:*

$$©_1 =< [0.66,\ 0.75],\ [0.25,\ 0.32],\ [0.17,\ 0.34],\ (0.53,\ 0.17,\ 0.22) >$$

and

$$©_2 = < [0.35, 0.55], [0.12, 0.25], [0.12, 0.20], (0.60, 0.23, 0.43) >$$

Then, applying Definition 7, we obtain $\partial\,(©_1, ©_2) = 0.12$.

Definition 8. *Let* $©_{ij} = \{< [t_{ij}^{-}, t_{ij}^{+}], [i_{ij}^{-}, i_{ij}^{+}], [f_{ij}^{-}, f_{ij}^{+}], (t, i, f) >\}$ *be any neutrosophic cubic value.* $©_{ij}$ *used to evaluate i-th alternative with respect to j-th criterion. The normalized form of* $©_{ij}$ *is defined as follows:*

$$\begin{aligned} ©_{ij}^{\otimes} = \{< &\left[\frac{t_{ij}^{-}}{\left(\sum_{i=1}^{m}(t_{ij}^{-})^2+(t_{ij}^{+})^2\right)^{\frac{1}{2}}}, \frac{t_{ij}^{+}}{\left(\sum_{i=1}^{m}(t_{ij}^{-})^2+(t_{ij}^{+})^2\right)^{\frac{1}{2}}}\right], \\ &\left[\frac{i_{ij}^{-}}{\left(\sum_{i=1}^{m}(i_{ij}^{-})^2+(i_{ij}^{+})^2\right)^{\frac{1}{2}}}, \frac{i_{ij}^{+}}{\left(\sum_{i=1}^{m}(i_{ij}^{-})^2+(i_{ij}^{+})^2\right)^{\frac{1}{2}}}\right], \\ &\left[\frac{f_{ij}^{-}}{\left(\sum_{i=1}^{m}(f_{ij}^{-})^2+(f_{ij}^{+})^2\right)^{\frac{1}{2}}}, \frac{f_{ij}^{+}}{\left(\sum_{i=1}^{m}(f_{ij}^{-})^2+(f_{ij}^{+})^2\right)^{\frac{1}{2}}}\right] \\ &\left[\frac{t_{ij}}{\left(\sum_{i=1}^{m}(t_{ij})^2+(i_{ij})^2+(f_{ij})^2\right)^{\frac{1}{2}}}, \frac{i_{ij}}{\left(\sum_{i=1}^{m}(t_{ij})^2+(i_{ij})^2+(f_{ij})^2\right)^{\frac{1}{2}}}, \frac{f_{ij}}{\left(\sum_{i=1}^{m}(t_{ij})^2+(i_{ij})^2+(f_{ij})^2\right)^{\frac{1}{2}}}\right] >\}. \end{aligned} \tag{14}$$

A conceptual model of the evolution of the neutrosophic cubic set is shown in Figure 1.

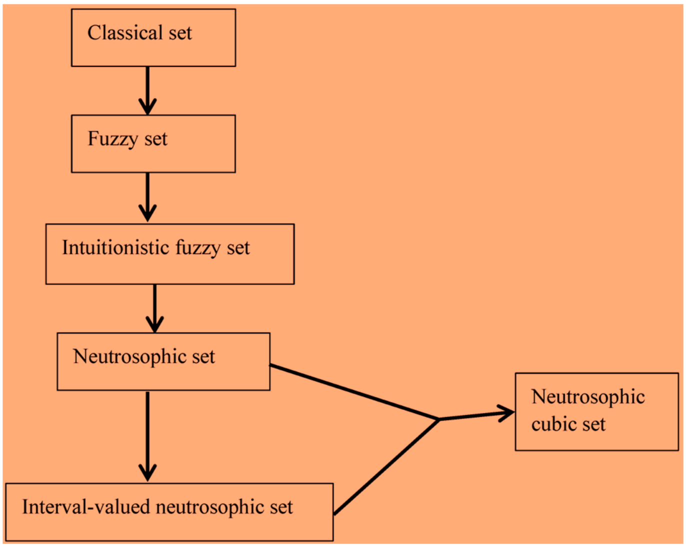

Figure 1. Evolution of the neutrosophic cubic set.

4. NC-TODIM Based MAGDM under a NCS Environment

Assume that $A = \{A_1, A_2, \ldots, A_m\}$ $(m \geq 2)$ and $C = \{C_1, C_2, \ldots, C_n\}$ $(n \geq 2)$ are the discrete set of alternatives and attributes respectively. $W = \{W_1, W_2, \ldots, W_n\}$ is the weight vector of attributes C_j $(j = 1, 2, \ldots, n)$, where $W_j > 0$ and $\sum_{j=1}^{n} W_j = 1$. Let $E = \{E_1, E_2, \ldots, E_r\}$ be the set of decision makers and $\gamma = \{\gamma_1, \gamma_2, \ldots, \gamma_r\}$ be the weight vector of decision makers, where $\gamma_k > 0$ and $\sum_{k=1}^{r} \gamma_k = 1$.

NC-TODIM Strategy

Now, we describe the NC-TODIM strategy to solve the MAGDM problems with NC-numbers. The NC-TODIM strategy consists of the following steps:

Step 1. Formulate the decision matrix

Assume that $M^k = (©_{ij}^k)_{m \times n}$ be the decision matrix, where $©_{ij}^k$ = <$\widetilde{G}_{ij}^k$, R_{ij}^k> is the rating value provided by the k-th (E_k) decision maker for alternative A_i, with respect to attribute C_j. The matrix form of M^k is presented as:

$$M^k = \begin{pmatrix} & C_1 & C_2 & \ldots & C_n \\ A_1 & ©_{11}^k & ©_{12}^k & \ldots & ©_{1n}^k \\ A_2 & ©_{21}^k & ©_{22}^k & & ©_{2n}^k \\ . & . & \ldots & . & \\ A_m & ©_{m1}^k & ©_{m2}^k & \ldots & ©_{mnj}^k \end{pmatrix} \tag{15}$$

Step 2. Normalize the decision matrix

The MAGDM problem generally consists of cost criteria and benefit criteria. So, the decision matrix needs to be normalized. For cost criterion C_j, we use the Definition 8 to normalize the decision matrix (Equation (15)) provided by the decision makers. For benefit criterion C_j we don't need to normalize the decision matrix. When C_j is a cost criterion, the normalized form of decision matrix (see Equation (15)) is presented below.

$$M^{\otimes k} = \begin{pmatrix} & C_1 & C_2 & \ldots & C_n \\ A_1 & ©_{11}^{\otimes k} & ©_{12}^{\otimes k} & \ldots & ©_{1n}^{\otimes k} \\ A_2 & ©_{21}^{\otimes k} & ©_{22}^{\otimes k} & & ©_{2n}^{\otimes k} \\ . & . & \ldots & . & \\ A_m & ©_{m1}^{\otimes k} & ©_{m2}^{\otimes k} & \ldots & ©_{mnj}^{\otimes k} \end{pmatrix} \tag{16}$$

Here $©_{ij}^{\otimes k}$ is the normalized form of the NC-number.

Step 3. Determine the relative weight of each criterion

The relative weight W_{ch} of each criterion is obtained by the following equation.

$$W_{ch} = \frac{W_C}{W_h} \tag{17}$$

where, W_h = max {W_1, W_2, ..., W_n}.

Step 4. Calculate score values

Using Equation (1), calculate the score value Sc ($©_{ij}^{\otimes k}$) (i = 1, 2, ..., m; j = 1, 2, ..., n) of $©_{ij}^{\otimes k}$ if C_j is a cost criterion. Using Equation (1), calculate the score value Sc $((c)_{ij}^k)$ (i = 1, 2, ..., m; j = 1, 2, ..., n) of $©_{ij}^k$ if C_j is a benefit criterion.

Step 5. Calculate accuracy values

Using Equation (9), calculate the accuracy value Ac ($©_{ij}^{\otimes k}$) (I = 1, 2, ..., m; j= 1, 2, ..., n) of $©_{ij}^{\otimes k}$ if C_j is a cost criterion. Using Equation (9), calculate the accuracy value Ac ($©_{ij}^k$) (I = 1, 2, ..., m; j = 1, 2, ..., n) of $©_{ij}^k$ if C_j is a benefit criterion.

Step 6. Formulate the dominance matrix

Calculate the dominance of each alternative A_i over each alternative A_j with respect to the criteria C (C_1, C_2, ..., C_n), of the k-th decision maker E_k by the following Equations (18) and (19).

(For cost criteria)

$$\left.\begin{array}{ll}\Psi_c^k(A_i,\ A_j) & = \sqrt{\left(\frac{W_{Ch}}{\sum_{c=1}^{n} W_{ch}} \partial(\copyright_{ic}^{\otimes k},\ \copyright_{jc}^{\otimes k}\right)},\ \text{if}\ \copyright_{ic}^{\otimes k} > \copyright_{jc}^{\otimes k} \\ & = 0, \qquad\qquad \text{if}\ \copyright_{ic}^{\otimes k} = \copyright_{jc}^{\otimes k} \\ & = -\frac{1}{\alpha}\sqrt{\left(\frac{\sum_{c=1}^{n} W_{ch}}{W_{Ch}} \partial(\copyright_{ic}^{\otimes k},\ \copyright_{jc}^{\otimes k}\right)},\ \text{if}\ \copyright_{ic}^{\otimes k} < \copyright_{jc}^{\otimes k}\end{array}\right\} \tag{18}$$

(For benefit criteria)

$$\left.\begin{array}{ll}\Psi_c^k(A_i,\ A_j) & = \sqrt{\left(\frac{W_{Ch}}{\sum_{c=1}^{n} W_{ch}} \partial(\copyright_{ic}^{k},\ \copyright_{jc}^{k}\right)},\ \text{if}\ \copyright_{ic}^{k} > \copyright_{jc}^{k} \\ & = 0, \qquad\qquad \text{if}\ \copyright_{ic}^{k} = \copyright_{jc}^{k} \\ & = -\frac{1}{\alpha}\sqrt{\left(\frac{\sum_{c=1}^{n} W_{ch}}{W_{Ch}} \partial(\copyright_{ic}^{k},\ ,\copyright_{jc}^{k}\right)},\ \text{if}\ \copyright_{ic}^{k} < \copyright_{jc}^{k}\end{array}\right\} \tag{19}$$

where, parameter α represents the attenuation factor of losses and α must be positive.

Step 7. Formulate the individual overall dominance matrix

Using Equation (20), calculate the individual total dominance matrix of each alternative A_i over each alternative A_j under the criterion C_j.

$$\phi^k = (A_i,\ A_j) = \sum_{c=1}^{n} \Psi_c^k(A_i,\ A_j) \tag{20}$$

Step 8. Aggregate the dominance matrix

Using Equation (21), calculate the collective overall dominance of alternative A_i over each alternative A_j.

$$\phi(A_i,\ A_j) = \sum_{k=1}^{m} \gamma_k \lambda^k(A_i,\ A_j) \tag{21}$$

Step 9. Calculate global values

We present the global value of each alternative as follows:

$$\Omega_i = \frac{\sum_{j=1}^{n} \phi\,(A_i, A_j) - \min_{1\le i\le m} \left(\sum_{j=1}^{n} \phi\,(A_i, A_j)\right)}{\max_{1\le i\le m} \left(\sum_{j=1}^{n} \phi\,(A_i, A_j)\right) - \min_{1\le i\le m} \left(\sum_{j=1}^{n} \phi\,(A_i, A_j)\right)} \tag{22}$$

Step 10. Rank the priority

Sorting the values of Ω_i provides the rank of each alternative. A set of alternatives can be preference-ranked according to the descending order of Ω_i. The highest global value corresponds to the best alternative.

A conceptual model of the NC-TODIM strategy is shown in Figure 2.

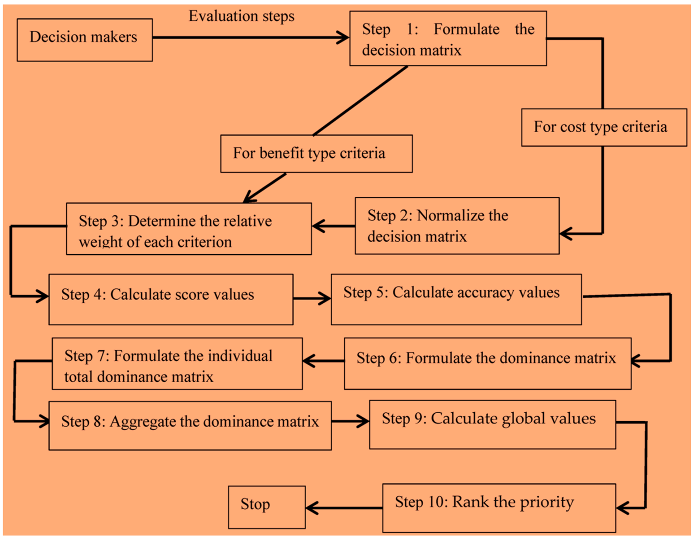

Figure 2. A flow chart of the proposed neutrosophic cubic set (NC)-TODIM strategy.

5. Illustrative Example

In this section, a MAGDM problem is adapted from the study [18] under the NCS environment. An investment company wants to select the best alternative among the set of feasible alternatives. The feasible alternatives are

1. Car company (A_1)
2. Food company (A_2)
3. Computer company (A_3)
4. Arms company (A_4).

The best alternative is selected based on the following criteria:

1. Risk analysis (C_1)
2. Growth analysis (C_2)
3. Environmental impact analysis (C_3).

An investment company forms a panel of three decision makers $\{E_1, E_2, E_3\}$ who evaluate four alternatives in decision making process. The weight vector of attributes and decision makers are considered as $W = (0.4, 0.35, 0.25)^T$ $\gamma = (0.32, 0.33, 0.35)^T$ respectively.

The proposed strategy is presented using the following steps:

Step 1. Formulate the decision matrix

Formulate the decision matrices M^k $(k = 1, 2, 3)$ using the rating values of alternatives with respect to three criteria provided by the three decision makers in terms of NC-numbers. Assume that the NC-numbers $©^k_{ij} = <\widetilde{G}^k_{ij}, R^k_{ij}>$ present the rating value provided by the decision maker E_k for alternative A_i with respect to attribute C_j. Using these rating values $©^k_{ij}$ $(k = 1, 2, 3; i = 1, 2, 3, 4; j = 1, 2, 3)$, three decision matrices $M^k = (©^k_{ij})_{4\times3}$ $(k = 1, 2, 3)$ are constructed (see Equations (23)–(25)).

Decision matrix for E_1

$$M^1 = \begin{array}{cc} & \begin{array}{ccc} \qquad\qquad C_1 \qquad\qquad & \qquad\qquad C_2 \qquad\qquad & \qquad\qquad C_3 \qquad\qquad \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{pmatrix} \langle [0.41, 0.52], [0.10, 0.18], [0.06, 0.17], (0.48, 0.11, 0.11) \rangle & \langle [0.40, 0.51], [0.10, 0.20], [0.10, 0.19], (0.50, 0.11, 0.11) \rangle & \langle [0.22, 0.27], [0.41, 0.52], [0.41, 0.52], (0.31, 0.31, 0.31) \rangle \\ \langle [0.35, 0.46], [0.18, 0.27], [0.17, 0.34], (0.43, 0.16, 0.21) \rangle & \langle [0.22, 0.28], [0.40, 0.50], [0.39, 0.48], (0.28, 0.28, 0.28) \rangle & \langle [0.38, 0.49], [0.10, 0.21], [0.10, 0.21], (0.57, 0.12, 0.12) \rangle \\ \langle [0.23, 0.29], [0.36, 0.45], [0.34, 0.42], (0.26, 0.26, 0.26) \rangle & \langle [0.34, 0.45], [0.20, 0.30], [0.19, 0.39], (0.44, 0.16, 0.22) \rangle & \langle [0.22, 0.27], [0.41, 0.52], [0.41, 0.52], (0.31, 0.31, 0.31) \rangle \\ \langle [0.17, 0.23], [0.45, 0.55], [0.42, 0.59], (0.21, 0.32, 0.37) \rangle & \langle [0.22, 0.28], [0.40, 0.50], [0.39, 0.48], (0.28, 0.28, 0.28) \rangle & \langle [0.38, 0.49], [0.10, 0.21], [0.10, 0.21], (0.57, 0.12, 0.12) \rangle \end{pmatrix} \end{array} \quad (23)$$

Decision matrix for E_2

$$M^2 = \begin{array}{cc} & \begin{array}{ccc} \qquad\qquad C_1 \qquad\qquad & \qquad\qquad C_2 \qquad\qquad & \qquad\qquad C_3 \qquad\qquad \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{pmatrix} \langle [0.17, 0.23], [0.46, 0.55], [0.42, 0.59], (0.21, 0.32, 0.37) \rangle & \langle [0.25, 0.31], [0.35, 0.44], [0.35, 0.44], (0.28, 0.28, 0.28) \rangle & \langle [0.34, 0.43], [0.13, 0.27], [0.13, 0.27], (0.49, 0.11, 0.11) \rangle \\ \langle [0.23, 0.29], [0.37, 0.46], [0.34, 0.42], (0.26, 0.26, 0.26) \rangle & \langle [0.25, 0.31], [0.35, 0.44], [0.35, 0.44], (0.28, 0.28, 0.28) \rangle & \langle [0.34, 0.43], [0.13, 0.27], [0.13, 0.27], (0.49, 0.11, 0.11) \rangle \\ \langle [0.41, 0.52], [0.10, 0.18], [0.10, 0.17], (0.48, 0.11, 0.11) \rangle & \langle [0.44, 0.57], [0.10, 0.17], [0.10, 0.17], (0.51, 0.11, 0.11) \rangle & \langle [0.19, 0.24], [0.53, 0.67], [0.53, 0.67], (0.27, 0.27, 0.27) \rangle \\ \langle [0.35, 0.46], [0.20, 0.28], [0.17, 0.34], (0.42, 0.16, 0.21) \rangle & \langle [0.25, 0.31], [0.35, 0.44], [0.35, 0.44], (0.28, 0.28, 0.28) \rangle & \langle [0.34, 0.43], [0.13, 0.27], [0.13, 0.27], (0.49, 0.11, 0.11) \rangle \end{pmatrix} \end{array} \quad (24)$$

Decision matrix for E_3

$$M^3 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} \begin{array}{ccc} C_1 & C_2 & C_3 \end{array}$$

$$M^3 = \begin{pmatrix} < [0.22, 0.27], [0.42, 0.52], [0.42, 0.52], (0.28, 0.28, 0.28) > & < [0.22, 0.28], [0.40, 0.50], [0.39, 0.48], (0.28, 0.28, 0.28) > & < [0.41, 0.52], [0.10, 0.18], [0.10, 0.17], (0.48, 0.11, 0.11) > \\ < [0.22, 0.27], [0.42, 0.52], [0.42, 0.52], (0.28, 0.28, 0.28) > & < [0.40, 0.51], [0.10, 0.20], [0.10, 0.19], (0.50, 0.11, 0.11) > & < [0.23, 0.29], [0.36, 0.45], [0.34, 0.42], (0.26, 0.26, 0.26) > \\ < [0.38, 0.49], [0.10, 0.21], [0.10, 0.21], (0.50, 0.11, 0.11) > & < [0.34, 0.45], [0.20, 0.30], [0.19, 0.39], (0.44, 0.16, 0.22) > & < [0.38, 0.49], [0.10, 0.21], [0.10, 0.21], (0.50, 0.11, 0.11) > \\ < [0.38, 0.49], [0.10, 0.21], [0.10, 0.21], (0.50, 0.11, 0.11) > & < [0.22, 0.28], [0.40, 0.50], [0.39, 0.48], (0.28, 0.28, 0.28) > & < [0.17, 0.23], [0.45, 0.54], [0.42, 0.59], (0.21, 0.32, 0.37) > \end{pmatrix} \quad (25)$$

Step 2. Normalize the decision matrix

Since all the criteria are benefit type, we do not need to normalize the decision matrix.

Step 3. Determine the relative weight of each criterion

Using Equation (17), we obtain the relative weight vector W_{ch} of criteria as follows:

$$W_{ch} = (1, 0.875, 0.625)^T.$$

Step 4. Calculate score values

The score values of each alternative relative to each criterion obtained by Equation (1) are presented in the Tables 1–3.

Table 1. Score values for M^1.

	C_1	C_2	C_2
A_1	0.56	0.54	0.06
A_2	0.40	0.09	0.54
A_3	0.50	0.38	0.06
A_4	−0.03	0.09	0.54

Table 2. Score values for M^2.

	C_1	C_2	C_2
A_1	−0.03	0.13	0.49
A_2	0.13	0.13	0.49
A_3	0.56	0.60	−0.04
A_4	0.39	0.13	0.49

Table 3. Score values for M^3.

	C_1	C_2	C_2
A_1	0.07	0.09	0.56
A_2	0.07	0.52	0.13
A_3	0.51	0.37	0.39
A_4	0.51	0.09	−0.03

Step 5. Calculate accuracy values

The accuracy values of each alternative relative to each criterion obtained by Equation (9) are presented in Tables 4–6.

Table 4. Accuracy values for M^1.

	C_1	C_2	C_2
A_1	0.14	0.30	−0.24
A_2	0.12	−0.23	0.32
A_3	−0.20	0.09	−0.24
A_4	−0.38	−0.23	0.32

Table 5. Accuracy values for M^2.

	C_1	C_2	C_2
A_1	−0.38	−0.18	0.21
A_2	−0.20	−0.18	0.21
A_3	0.14	0.36	−0.21
A_4	0.12	−0.18	0.21

Table 6. Accuracy values for M^3.

	C_1	C_2	C_2
A_1	−0.24	−0.23	0.41
A_2	−0.24	0.30	−0.20
A_3	0.26	0.09	0.12
A_4	0.26	−0.23	−0.38

Step 6. Formulate the dominance matrix

Using Equation (19), we construct dominance matrix for $\alpha = 1$. The dominance matrices are represented in matrix form (See Equations (26)–(34)).

The dominance matrix Ψ_1^1, the dominance matrix Ψ_2^1

$$\Psi_1^1 = \begin{array}{cc} & \begin{matrix} A_1 & A_2 & A_3 & A_4 \end{matrix} \\ \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} & \begin{pmatrix} 0 & 0.18 & 0.30 & 0.35 \\ -0.46 & 0 & -0.58 & 0.30 \\ -0.74 & 0.23 & 0 & 0.19 \\ -0.88 & -0.74 & -0.47 & 0 \end{pmatrix} \end{array} \tag{26}$$

$$\Psi_2^1 = \begin{array}{cc} & \begin{matrix} A_1 & A_2 & A_3 & A_4 \end{matrix} \\ \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} & \begin{pmatrix} 0 & 0.29 & 0.18 & 0.28 \\ -0.82 & 0 & -0.69 & 0 \\ -0.51 & 0.24 & 0 & 0.29 \\ -0.81 & 0 & -0.65 & 0 \end{pmatrix} \end{array} \tag{27}$$

The dominance matrix Ψ_3^1, the dominance matrix Ψ_1^2

$$\Psi_3^1 = \begin{array}{cc} & \begin{matrix} A_1 & A_2 & A_3 & A_4 \end{matrix} \\ \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} & \begin{pmatrix} 0 & -1 & 0 & -1 \\ 0.25 & 0 & 0.26 & 0 \\ 0 & -1 & 0 & -1 \\ 0.25 & 0 & 0.26 & 0 \end{pmatrix} \end{array} \tag{28}$$

$$\Psi_1^2 = \begin{array}{cc} & \begin{matrix} A_1 & A_2 & A_3 & A_4 \end{matrix} \\ \begin{matrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{matrix} & \begin{pmatrix} 0 & -0.46 & -0.88 & -0.74 \\ 0.18 & 0 & -0.75 & -0.58 \\ 0.35 & 0.09 & 0 & 0.04 \\ 0.30 & 0.23 & 0.19 & 0 \end{pmatrix} \end{array} \tag{29}$$

The dominance matrix Ψ_2^2, the dominance matrix Ψ_3^2

$$\Psi_2^2 = \begin{array}{c@{}c} & \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{pmatrix} 0 & 0 & -0.84 & 0 \\ 0 & 0 & 0.84 & 0 \\ 0.29 & 0.29 & 0 & 0.29 \\ 0 & 0 & -0.84 & 0 \end{pmatrix} \end{array} \quad (30)$$

$$\Psi_3^2 = \begin{array}{c@{}c} & \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{pmatrix} 0 & 0 & 0.26 & 0 \\ 0 & 0 & 0.26 & 0 \\ -1 & -1 & 0 & -1 \\ 0 & 0 & 0.26 & 0 \end{pmatrix} \end{array} \quad (31)$$

The dominance matrix Ψ_1^3, the dominance matrix Ψ_2^3

$$\Psi_1^3 = \begin{array}{c@{}c} & \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{pmatrix} 0 & 0 & 0.78 & 0.78 \\ 0 & 0 & 0.78 & 0.78 \\ 0.31 & 0.31 & 0 & 0 \\ 0.31 & 0.31 & 0 & 0 \end{pmatrix} \end{array} \quad (32)$$

$$\Psi_2^3 = \begin{array}{c@{}c} & \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{pmatrix} 0 & -0.83 & 0.65 & 0 \\ 0.29 & 0 & 0.18 & 0.29 \\ 0.23 & -0.51 & 0 & 0.23 \\ 0 & -0.83 & -0.65 & 0 \end{pmatrix} \end{array} \quad (33)$$

The dominance matrix Ψ_3^3

$$\Psi_3^3 = \begin{array}{c@{}c} & \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{pmatrix} 0 & -0.94 & 0.59 & -1.1 \\ 0.23 & 0 & -0.73 & 0.15 \\ 0.59 & 0.18 & 0 & 0.23 \\ -1.1 & 0.58 & -0.94 & 0 \end{pmatrix} \end{array} \quad (34)$$

Step 7. Formulate the individual overall dominance matrix

The individual overall dominance matrix is calculated by the Equation (20) and the dominance matrices are represented in matrix form (see Equations (35)–(37)).

First decision maker's overall dominance matrix ϕ^1

$$\phi^1 = \begin{array}{c@{}c} & \begin{array}{cccc} A_1 & A_2 & A_3 & A_4 \end{array} \\ \begin{array}{c} A_1 \\ A_2 \\ A_3 \\ A_4 \end{array} & \begin{pmatrix} 0 & -0.53 & 0.47 & -0.37 \\ -1 & 0 & -1 & 0.30 \\ -1.3 & 0.53 & 0 & 0.52 \\ -1.5 & 0.74 & -0.86 & 0 \end{pmatrix} \end{array} \quad (35)$$

Second decision maker's overall dominance matrix ϕ^2

$$\phi^2 = \begin{pmatrix} & A_1 & A_2 & A_3 & A_4 \\ A_1 & 0 & -0.46 & -1.5 & -0.74 \\ A_2 & 0.18 & 0 & -1.3 & -0.58 \\ A_3 & -0.36 & -0.62 & 0 & 0.67 \\ A_4 & 0.30 & 0.23 & -0.39 & 0 \end{pmatrix} \quad (36)$$

Third decision maker's overall dominance matrix ϕ^3

$$\phi^3 = \begin{pmatrix} & A_1 & A_2 & A_3 & A_4 \\ A_1 & 0 & -1.8 & -2 & 1.9 \\ A_2 & 0.52 & 0 & -1.3 & -0.34 \\ A_3 & -0.05 & -0.02 & 0 & 0.46 \\ A_4 & -0.79 & -1.1 & -1.6 & 0 \end{pmatrix} \quad (37)$$

Step 8. Aggregate the dominance matrix

Using Equation (21), the aggregate dominance matrix ϕ is constructed (see Equation (38)) as follows:

$$\phi = \begin{pmatrix} & A_1 & A_2 & A_3 & A_4 \\ A_1 & 0 & -0.94 & -1.1 & -0.53 \\ A_2 & -0.10 & 0 & -1.23 & -0.22 \\ A_3 & -0.54 & -0.38 & 0 & -0.23 \\ A_4 & -0.64 & -0.55 & -0.96 & 0 \end{pmatrix} \quad (38)$$

Step 9. Calculate global values

Using Equation (22), we calculate the values of Ω_i (i = 1, 2, 3, 4) and represented in Table 7.

Table 7. Global values of alternatives.

A_i	A_1	A_2	A_3	A_4
Ω_i	0.49	0.61	1	0

Step 10. Rank the priority

Since $\Omega_3 > \Omega_2 > \Omega_1 > \Omega_4$, alternatives are then preference ranked as follows: $A_3 > A_2 > A_1 > A_4$. Hence A_3 is the best alternative.

From the illustrative example, we see that the proposed NC-TODIM strategy is more suitable for real scientific and engineering applications because it can handle hybrid information consisting of INS and SVNS information simultaneously to cope with indeterminate and inconsistent information. Thus, NC-TODIM extends the existing decision-making strategies and provides a sophisticated mathematical tool for decision makers.

6. Rank of Alternatives with Different Values of α

Table 8 shows that the ranking order of alternatives depends on the values of the attenuation factor, which reflects the importance of the attenuation factor in the NC-TODIM strategy.

Table 8. Global values and ranking of alternatives for different values of α.

Values of α	Global Values of Alternative (Ω_i)	Rank Order of A_i
0.5	$\Omega_1 = 0, \Omega_2 = 0.89, \Omega_3 = 1, \Omega_4 = 0.46$ $\Omega_3 > \Omega_2 > \Omega_4 > \Omega_1$	$A_3 > A_2 > A_4 > A_1$
1	$\Omega_1 = 0.49, \Omega_2 = 0.61, \Omega_3 = 1, \Omega_4 = 0$ $\Omega_3 > \Omega_2 > \Omega_1 > \Omega_4$	$A_3 > A_2 > A_1 > A_4$
1.5	$\Omega_1 = 0, \Omega_2 = 0.72, \Omega_3 = 1, \Omega_4 = 0.44$ $\Omega_3 > \Omega_2 > \Omega_4 > \Omega_1$	$A_3 > A_2 > A_4 > A_1$
2	$\Omega_1 = 0, \Omega_2 = 1, \Omega_3 = 0.81, \Omega_4 = 0.38$ $\Omega_2 > \Omega_3 > \Omega_4 > \Omega_1$	$A_2 > A_3 > A_4 > A_1$
3	$\Omega_1 = 0, \Omega_2 = 0.56, \Omega_3 = 1, \Omega_4 = 0.45$ $\Omega_3 > \Omega_2 > \Omega_4 > \Omega_1$	$A_3 > A_2 > A_4 > A_1$

Analysis on Influence of the Parameter α to Ranking Order

The impact of parameter α on ranking order is examined by comparing the ranking orders taken with varying the different values of α. When α = 0.5, 1, 1.5, 2, 3, ranking order are presented in Table 8. We draw Figures 3 and 4 to compare the ranking order for different values of α. When α = 0.5, α = 1.5 and α = 3, the ranking order is unchanged and A_3 is the best alternative, while A_1 is the worst alternative. When α = 1, the ranking order is changed and A_3 is the best alternative and A_4 is the worst alternative. For α = 2, the ranking order is changed and A_2 is the best alternative and A_1 is the worst alternative. From Table 8, we see that A_3 is the best alternative in four cases and A_1 is the worst for four cases. We can say that ranking order depends on parameter α.

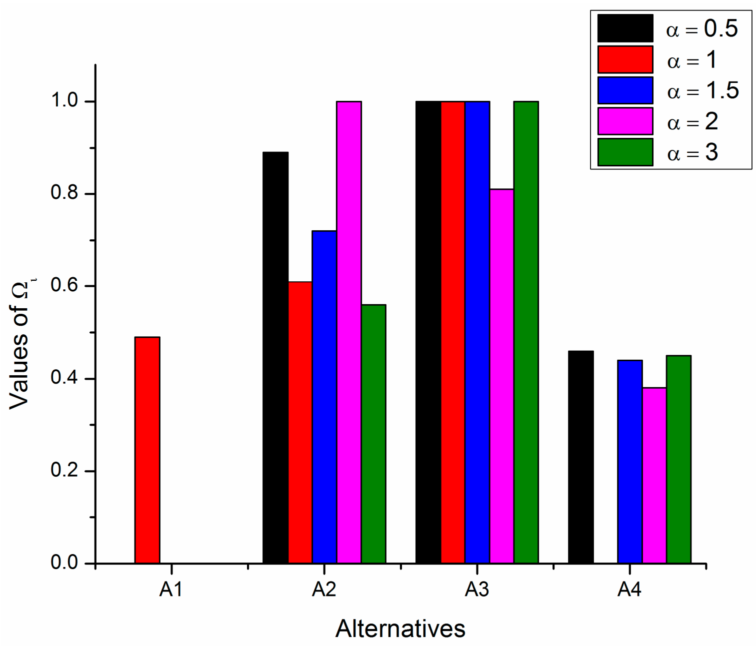

Figure 3. Global values of the alternatives for different values of attenuation factor α = 0.5, 1, 1.5, 2, 3.

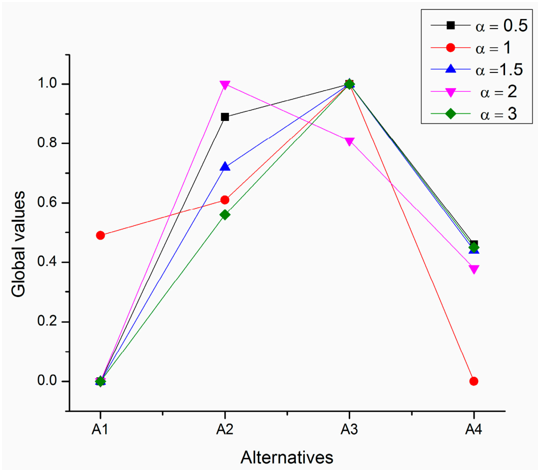

Figure 4. Ranking of the alternatives for $\alpha = 0.5, 1, 1.5, 2, 3$.

7. Comparative Analysis and Discussion

On comparing with the existing neutrosophic decision making strategies [26–29,33–35,64–69], we see that the decision information used in the proposed NC-TODIM strategy is NC numbers, which comprises of interval neutrosophic information and single-valued neutrosophic information simultaneously; whereas the decision information in the existing literature is either SVNSs or INSs. Since NC numbers comprises of much more information, the NC numbers based on the TODIM strategy proposed in this paper is more elegant, typical and more general in applications, while the existing neutrosophic decision-making strategies cannot deal with the NC number decision-making problem developed in this paper.

The first decision making paper in NCS environment was studied by Banerjee et al. [44]. On comparison with existing GRA-based NCS decision making strategies [44], we observe that the proposed NC-TODIM strategy uses the score, and accuracy functions, while the decision making-strategy in [44] uses Hamming distances for weighted grey relational coefficients and standard (ideal) grey relational coefficients, and ranks the alternatives based on the relative closeness coefficients. Hence, the proposed NC-TODIM strategy is relatively simple in the decision making process.

On comparing with cosine measures of NCSs [49], we observe that the proposed NC-TODIM involves multiple decision makers, while in [49] only a single decision maker is involved. This shows that [49] cannot deal with group decision making, while the proposed NC-TODIM strategy is more sophisticated as it can deal with single as well as group decision making in the NCS environment.

On comparison with extended TOPSIS [50] with neutrosophic cubic information, we observe that nine components are present in NCSs. Therefore, by calculation of a weighted decision matrix, a neutrosophic cubic positive ideal solution (NCPIS), and a neutrosophic cubic negative ideal solution, the distance measures of alternatives from NCPIS and NCNIS (NCNIS,) and entropy weight, and use of an aggregation operator are lengthy, time consuming, and hence expensive. The proposed NC-TODIM strategy is free from different kinds of typical aggregation operators. The calculations required for the proposed strategy are relatively straightforward and time-saving. Therefore, the final ranking obtained by the proposed strategy is more conclusive than those produced by the other strategies, and it is evident that the proposed strategy is accurate and reliable.

On comparison with the strategy proposed by Zhan et al. [51], we see that they employ score, accuracy, and certainty functions, and a weighted average operator and weighted geometric operator of NCSs for decision making problem involving only a single decision maker. This reflects that the strategy introduced by Zhan et al. [51] is only applicable for decision making problems involving single decision maker. However, our proposed NC-TODIM strategy is more general as it is capable of dealing with group decision-making problems.

A comparative study is conducted with the existing strategy [48] for group decision making under a NCS environment (See Table 9). Since the philosophy of two strategies are different, the obtained results (ranking order) are different. At a glance, it cannot be said which strategy is superior to the other. However, on comparison with similarity measure-based strategies studied in [48], we observed that ideal solutions are needed for ranking of alternatives but in a real world ideal solution, this is an imaginary case, which means that an indeterminacy arises automatically, whereas in our proposed NC-TODIM strategy we can calculate the rank of the alternatives based on global values of alternatives. So, the proposed NC-TODIM strategy is relatively easy to implement and apply for solving MAGDM problems.

Table 9. Ranking order of alternatives using three different decision making strategies in the neutrosophic cubic set (NCS) environment.

Proposed NC-TODIM Strategy	Similarity Measure [48]
$\Omega_1 = 0$, $\Omega_2 = 0.89$, $\Omega_3 = 1$, $\Omega_4 = 0.46$ Ranking order: $A_3 > A_2 > A_4 > A_1$	$\rho_1 = 0.20$, $\rho_2 = 0.80$, $\rho_3 = 0.22$, $\rho_4 = 0.19$ Ranking order: $A_2 > A_3 > A_1 > A_4$

8. Conclusions

NCSs can better describe hybrid information comprising of INSs and NSs. In this study, we proposed a score function and an accuracy function, and established their properties. We developed a NC-TODIM strategy, which is capable for tackling MAGDM problems affected by uncertainty and indeterminacy represented by NC numbers. The standard TODIM, in its original formulation, is only applicable to a crisp environment. Existing neutrosophic TODIM strategies deal with single valued neutrosophic information or interval neutrosophic information. Therefore, proposed NC-TODIM strategy demonstrates the advantages of presenting and manipulating MAGDM problems with NCSs comprising of the hybrid information of INSs and NSs. Furthermore, NC-TODIM strategy that considers the risk preferences of decision makers, is significant to solve MAGDM problems. The proposed NC-TODIM strategy was verified to be applicable, feasible, and effective by solving an illustrative example regarding the selection problem of investment alternatives. In addition, we investigated the influence of attenuation factor of losses α on ranking the order of alternatives.

The contribution of this study can be concluded as follows. First, this study utilized NCSs to present the interval neutrosophic information and neutrosophic information in the MAGDM process. Second, the NC-TODIM strategy established in this paper is simpler and easier than the existing strategy proposed by Pramanik et al. [48] for group decision making with neutrosophic cubic information based on similarity measure and demonstrates the main advantage of its simple and easy group decision making process. Third, TODIM strategy was extended to the NCS environment. Fourth, we defined the NC number. Fifth, we defined the score and accuracy functions and proved their basic properties. Sixth, we developed the ranking of NC numbers using score and accuracy functions. Therefore, two functions namely, score function, accuracy function, and proofs of their basic properties, ranking of NC numbers, and NC-TODIM strategy for MAGDM are the main contributions of the paper.

Several directions for future research are generated from this study. First, this study employs the NC-TODIM strategy to deal with MAGDM. In addition to MAGDM, MAGDM problems in a variety of other fields can be solved using the NC-TODIM strategy, including logistics center selection, personnel selection, teacher selection, renewable energy selection, medical diagnosis, image processing, fault diagnosis, etc. Second, this study considers the risk preferences of decision makers i.e., the essence of TODIM, while the interrelationship between criteria are ignored. In future research, the NC-TODIM strategy will be improved to address this deficiency. Third, the proposed strategy can only deal with crisp weights of attributes and decision makers, rather than NCS, which reflects its main limitation. This limitation will be effectively addressed in our future research. Fourth, in our illustrative example, three criteria are considered as an example. However, in real world group decision making problems, many other criteria should be included. A comprehensive framework for MAGDM problem comprising of all relevant criteria should be designed based on prior studies and the proposed NC-TODIM strategy in future research. Finally, we conclude that the developed NC-TODIM strategy offers a novel and effective strategy for decision makers under the NCS environment, and will open up a new avenue of research into the neutrosophic hybrid environment.

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VIKOR Method for Interval Neutrosophic Multiple Attribute Group Decision-Making

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Abstract: In this paper, we will extend the VIKOR (VIsekriterijumska optimizacija i KOmpromisno Resenje) method to multiple attribute group decision-making (MAGDM) with interval neutrosophic numbers (INNs). Firstly, the basic concepts of INNs are briefly presented. The method first aggregates all individual decision-makers' assessment information based on an interval neutrosophic weighted averaging (INWA) operator, and then employs the extended classical VIKOR method to solve MAGDM problems with INNs. The validity and stability of this method are verified by example analysis and sensitivity analysis, and its superiority is illustrated by a comparison with the existing methods.

Keywords: MAGDM; INNs; VIKOR method

1. Introduction

Multiple attribute group decision-making (MAGDM), which has been increasingly investigated and considered by all kinds of researchers and scholars, is one of the most influential parts of decision theory. It aims to provide a comprehensive solution by evaluating and ranking alternatives based on conflicting attributes with respect to decision-makers' (DMs) preferences, and has widely been utilized in engineering, economics, and management. Several traditional MAGDM methods have been developed by scholars in literature, such as the TOPSIS (Technique for Order Preference by Similarity to an Ideal Solution) method [1,2], the VIKOR (VIsekriterijumska optimizacija i KOmpromisno Resenje) method [3–5], the PROMETHEE (Preference Ranking Organization Method for Enrichment Evaluations) method [6], the ELECTRE (ELimination Et Choix Traduisant la Realité) method [7], the GRA (Grey Relational Analysis) method [8–10], and the MULTIMOORA (Multiobjective Optimization by Ratio Analysis plus Full Multiplicative Form) method [11,12].

Due to the fuzziness and uncertainty of the alternatives in different attributes, attribute values in MAGDM are not always represented as real numbers, and they can be described as fuzzy numbers in more suitable occasions [13–15]. Since fuzzy set (FS) was first defined by Zadeh [16], is has been used as a better tool to solve MAGDM [17,18]. Smarandache [19,20] proposed a neutrosophic set (NS). Furthermore, the concepts of single-valued neutrosophic sets (SVNSs) [21] and interval neutrosophic sets(INSs) [22] were presented for actual applications. Ye [23] proposed a simplified neutrosophic set (SNS). Broumi and Smarandache [24] defined the correlation coefficient of INS. Zhang et al. [25] gave the correlation coefficient of interval neutrosophic numbers (INNs) in MAGDM. Zhang et al. [26] gave an outranking approach for INN MAGDM. Tian et al. [27] defined a cross-entropy in INN MAGDM. Zhang et al. [28] proposed some INN aggregating. Some other INN operators are proposed in References [29–32]. Ye [33] proposed two similarity measures between INNs. The SVNS and INS have received more and more attention since their appearance [34–42].

Opricovic [3] proposed the VIKOR method for a MAGDM problem with conflicting attributes [43–45]. Some scholars proposed fuzzy VIKOR models [46], intuitionistic fuzzy VIKOR models [47–49], the linguistic VIKOR method [50], the interval type-2 fuzzy VIKOR model [51], the hesitant fuzzy linguistic VIKOR method [52], the dual hesitant fuzzy VIKOR method [53], the linguistic intuitionistic fuzzy [54], and the single-valued neutrosophic number (SVNN) VIKOR method [38]. However, there has not yet been an academic investigation of the VIKOR method for MAGDM problems with INNs. Therefore, it is necessary to pay great attention to this novel and worthy research issue. The purpose of our paper is to use the VIKOR idea to solve MAGDM with INNs, to fill this vacancy of knowledge. In Section 2, we give the definition of INNs. We propose the VIKOR method for INN MAGDM. In Section 3, an example is provided, and the comparative analysis is proposed in Section 4. We finish with our conclusions in Section 5.

2. Preliminaries

The concepts of SVNSs and INSs are introduced.

SVNSs and INSs

NSs [19,20] are not easy to apply to real applications. Wang et al. [21] developed SNSs. Furthermore, Wang et al. [22] defined INSs.

Definition 1 [21]. *Let X be a space of points (objects), a SVNSs A in X is characterized as following:*

$$A = \{(x, \xi_A(x), \psi_A(x), \zeta_A(x)) | x \in X\} \tag{1}$$

where the truth-membership function $\xi_A(x)$, *indeterminacy-membership* $\psi_A(x)$ *and falsity-membership function* $\zeta_A(x)$, $\xi_A(x) \to [0,1], \psi_A(x) \to [0,1]$ *and* $\zeta_A(x) \to [0,1]$, *with the condition* $0 \le \xi_A(x) + \psi_A(x) + \zeta_A(x) \le 3$.

Definition 2 [22]. *Let X be a space of points (objects) with a generic element in fixed set X, denoted by x, where an INS* $\widetilde{A}$ *in X is characterized as follows:*

$$\widetilde{A} = \left\{ (x, \xi_{\widetilde{A}}(x), \psi_{\widetilde{A}}(x), \zeta_{\widetilde{A}}(x)) | x \in X \right\} \tag{2}$$

where truth-membership function $\xi_{\widetilde{A}}(x)$, *indeterminacy-membership* $\psi_{\widetilde{A}}(x)$, *and falsity-membership function* $\zeta_{\widetilde{A}}(x)$ *are interval values,* $\xi_{\widetilde{A}}(x) \subseteq [0,1], \psi_{\widetilde{A}}(x) \subseteq [0,1]$ *and* $\zeta_{\widetilde{A}}(x) \subseteq [0,1]$, *and* $0 \le sup(\xi_{\widetilde{A}}(x)) + sup(\psi_{\widetilde{A}}(x)) + sup(\zeta_{\widetilde{A}}(x)) \le 3$.

An INN can be expressed as $\widetilde{A} = (\xi_{\widetilde{A}}, \psi_{\widetilde{A}}, \zeta_{\widetilde{A}}) = \left(\left[\xi^L_{\widetilde{A}}, \xi^R_{\widetilde{A}}\right], \left[\psi^L_{\widetilde{A}}, \psi^R_{\widetilde{A}}\right], \left[\zeta^L_{\widetilde{A}}, \zeta^R_{\widetilde{A}}\right] \right)$, *where* $\left[\xi^L_{\widetilde{A}}, \xi^R_{\widetilde{A}}\right] \subseteq [0,1]$, $\left[\psi^L_{\widetilde{A}}, \psi^R_{\widetilde{A}}\right] \subseteq [0,1]$, $\left[\zeta^L_{\widetilde{A}}, \zeta^R_{\widetilde{A}}\right] \subseteq [0,1]$, *and* $0 \le \xi^R_{\widetilde{A}} + \psi^R_{\widetilde{A}} + \zeta^R_{\widetilde{A}} \le 3$.

Definition 3 [45]. *Let* $\widetilde{A} = \left(\left[\xi^L_{\widetilde{A}}, \xi^R_{\widetilde{A}}\right], \left[\psi^L_{\widetilde{A}}, \psi^R_{\widetilde{A}}\right], \left[\zeta^L_{\widetilde{A}}, \zeta^R_{\widetilde{A}}\right] \right)$ *be an INN, then a score function, SF, is:*

$$SF\left(\widetilde{A}\right) = \frac{\left(2 + \xi^L_{\widetilde{A}} - \psi^L_{\widetilde{A}} - \zeta^L_{\widetilde{A}}\right) + \left(2 + \xi^R_{\widetilde{A}} - \psi^R_{\widetilde{A}} - \zeta^R_{\widetilde{A}}\right)}{6}, SF\left(\widetilde{A}\right) \in [0,1] \tag{3}$$

Definition 4 [45]. *Let* $\widetilde{A} = \left(\left[\xi^L_{\widetilde{A}}, \xi^R_{\widetilde{A}}\right], \left[\psi^L_{\widetilde{A}}, \psi^R_{\widetilde{A}}\right], \left[\zeta^L_{\widetilde{A}}, \zeta^R_{\widetilde{A}}\right] \right)$ *be an INN, then an accuracy function,* $AF\left(\widetilde{A}\right)$, *is defined as*:

$$AF\left(\widetilde{A}\right) = \frac{\left(\xi^L_{\widetilde{A}} + \xi^R_{\widetilde{A}}\right) - \left(\zeta^L_{\widetilde{A}} + \zeta^R_{\widetilde{A}}\right)}{2}, AF\left(\widetilde{A}\right) \in [-1,1] \tag{4}$$

Definition 5 [45]. *Let* $\widetilde{A} = \left(\left[\xi^L_{\widetilde{A}}, \xi^R_{\widetilde{A}}\right], \left[\psi^L_{\widetilde{A}}, \psi^R_{\widetilde{A}}\right], \left[\zeta^L_{\widetilde{A}}, \zeta^R_{\widetilde{A}}\right]\right)$ *and* $\widetilde{B} = \left(\left[\xi^L_{\widetilde{B}}, \xi^R_{\widetilde{B}}\right], \left[\psi^L_{\widetilde{B}}, \psi^R_{\widetilde{B}}\right], \left[\zeta^L_{\widetilde{B}}, \zeta^R_{\widetilde{B}}\right]\right)$ *be two INNs,* $SF\left(\widetilde{A}\right) = \frac{\left(2+\xi^L_{\widetilde{A}}-\psi^L_{\widetilde{A}}-\zeta^L_{\widetilde{A}}\right)+\left(2+\xi^R_{\widetilde{A}}-\psi^R_{\widetilde{A}}-\zeta^R_{\widetilde{A}}\right)}{6}$ *and* $SF\left(\widetilde{B}\right) = \frac{\left(2+\xi^L_{\widetilde{B}}-\psi^L_{\widetilde{B}}-\zeta^L_{\widetilde{B}}\right)+\left(2+\xi^R_{\widetilde{B}}-\psi^R_{\widetilde{B}}-\zeta^R_{\widetilde{B}}\right)}{6}$ *be the score functions, and* $AF\left(\widetilde{A}\right) = \frac{\left(\xi^L_{\widetilde{A}}+\xi^R_{\widetilde{A}}\right)-\left(\zeta^L_{\widetilde{A}}+\zeta^R_{\widetilde{A}}\right)}{2}$ *and* $AF\left(\widetilde{B}\right) = \frac{\left(\xi^L_{\widetilde{B}}+\xi^R_{\widetilde{B}}\right)-\left(\zeta^L_{\widetilde{B}}+\zeta^R_{\widetilde{B}}\right)}{2}$ *be the accuracy functions, then if* $SF\left(\widetilde{A}\right) < SF\left(\widetilde{B}\right)$*, then* $\widetilde{A} < \widetilde{B}$*; if* $SF\left(\widetilde{A}\right) = SF\left(\widetilde{B}\right)$*, then (1) if* $AF\left(\widetilde{A}\right) = AF\left(\widetilde{B}\right)$*, then* $\widetilde{A} = \widetilde{B}$*; (2) if* $AF\left(\widetilde{A}\right) < AF\left(\widetilde{B}\right)$*, then* $\widetilde{A} < \widetilde{B}$.

Definition 6 [22,33]. *Let* $\widetilde{A} = \left(\left[\xi^L_{\widetilde{A}}, \xi^R_{\widetilde{A}}\right], \left[\psi^L_{\widetilde{A}}, \psi^R_{\widetilde{A}}\right], \left[\zeta^L_{\widetilde{A}}, \zeta^R_{\widetilde{A}}\right]\right)$ *and* $\widetilde{B} = \left(\left[\xi^L_{\widetilde{B}}, \xi^R_{\widetilde{B}}\right], \left[\psi^L_{\widetilde{B}}, \psi^R_{\widetilde{B}}\right], \left[\zeta^L_{\widetilde{B}}, \zeta^R_{\widetilde{B}}\right]\right)$ *be two INNs, then*:

$$(1)\ \widetilde{A} \oplus \widetilde{B} = \left([\xi^L_A + \xi^L_B - \xi^L_A\xi^L_B, \xi^R_A + \xi^R_B - \xi^R_A\xi^R_B], [\psi^L_A\psi^L_B, \psi^R_A\psi^R_B], [\zeta^L_A\zeta^L_B, \zeta^R_A\zeta^R_B]\right);$$

$$(2)\ \widetilde{A} \otimes \widetilde{B} = \left(\begin{array}{c} [\xi^L_A\xi^L_B, \xi^R_A\xi^R_B], [\psi^L_A + \psi^L_B - \psi^L_A\psi^L_B, \psi^R_A + \psi^R_B - \psi^R_A\psi^R_B], \\ [\zeta^L_A + \zeta^L_B - \zeta^L_A\zeta^L_B, \zeta^R_A + \zeta^R_B - \zeta^R_A\zeta^R_B] \end{array}\right);$$

$$(3)\ \lambda\widetilde{A} = \left(\left[1-(1-\xi^L_A)^\lambda, 1-(1-\xi^R_A)^\lambda\right], \left[(\psi^L_A)^\lambda, (\psi^R_A)^\lambda\right], \left[(\zeta^L_A)^\lambda, (\zeta^R_A)^\lambda\right]\right), \lambda > 0;$$

$$(4)\ \left(\widetilde{A}\right)^\lambda = \left(\left[(\xi^L_A)^\lambda, (\xi^R_A)^\lambda\right], \left[(\psi^L_A)^\lambda, (\psi^R_A)^\lambda\right], \left[1-(1-\zeta^L_A)^\lambda, 1-(1-\zeta^R_A)^\lambda\right]\right), \lambda > 0.$$

Definition 7 [45]. *Let* $\widetilde{A}$ *and* $\widetilde{B}$ *be two INNs, then the normalized Hamming distance between* $\widetilde{A}$ *and* $\widetilde{B}$ *is defined as follows*:

$$d\left(\widetilde{A}, \widetilde{B}\right) = \frac{1}{6}\left(\begin{array}{c} |\xi^L_A - \xi^L_B| + |\xi^R_A - \xi^R_B| + |\psi^L_A - I^L_B| \\ +|\psi^R_A - \psi^R_B| + |\zeta^L_A - \zeta^L_B| + |\zeta^R_A - \zeta^R_B| \end{array}\right) \tag{5}$$

3. VIKOR Method for INN MAGDM Problems

Let $\phi = \{\phi_1, \phi_2, \cdots, \phi_m\}$ be alternatives and $\varphi = \{\varphi_1, \varphi_2, \cdots, \varphi_n\}$ be attributes. Let $\tau = (\tau_1, \tau_2, \cdots, \tau_n)$ be the weight of φ_j, $0 \le \tau_j \le 1$, $\sum_{j=1}^{n} \tau_j = 1$. Let $D = \{D_1, D_2, \cdots, D_t\}$ be the set of DMs, $\sigma = (\sigma_1, \sigma_2, \cdots, \sigma_t)$ be the weighting of DMs, with $0 \le \sigma_k \le 1$, $\sum_{k=1}^{t} \sigma_k = 1$. Suppose that $\widetilde{R}_k = \left(\widetilde{r}^{(k)}_{ij}\right)_{m\times n} = \left(\left[\xi^{L(k)}_{ij}, \xi^{R(k)}_{ij}\right], \left[\psi^{L(k)}_{ij}, \psi^{R(k)}_{ij}\right], \left[\zeta^{L(k)}_{ij}, \zeta^{R(k)}_{ij}\right]\right)_{m\times n}$ is the INN decision matrix $\left[\xi^{L(k)}_{ij}, \xi^{R(k)}_{ij}\right] \subseteq [0,1]$, $\left[\psi^{L(k)}_{ij}, \psi^{R(k)}_{ij}\right] \subseteq [0,1]$, $\left[\zeta^{L(k)}_{ij}, \zeta^{R(k)}_{ij}\right] \subseteq [0,1]$, $0 \le \xi^{R(k)}_{ij} + \psi^{R(k)}_{ij} + \zeta^{R(k)}_{ij} \le 3$, $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$, $k = 1, 2, \cdots, t$.

To cope with the MAGDM with INNs, we develop the INN VIKOR model.

Step 1. Utilize the $\widetilde{R}_k$ and the interval neutrosophic number weighted averaging (INNWA) operator

$$\begin{array}{l} \widetilde{r}_{ij} = \left(\left[\xi^L_{ij}, \xi^R_{ij}\right], \left[\psi^L_{ij}, \psi^R_{ij}\right], \left[\zeta^L_{ij}, \zeta^R_{ij}\right]\right) = \mathrm{INNWA}_\sigma\left(\widetilde{r}^{(1)}_{ij}, \widetilde{r}^{(2)}_{ij}, \cdots, \widetilde{r}^{(t)}_{ij}\right) \\ i = 1, 2, \cdots, m, j = 1, 2, \cdots, n \end{array} \tag{6}$$

to get $\widetilde{R} = (\widetilde{r}_{ij})_{m\times n}$.

Step 2. Define the positive ideal solutions $\widetilde{R}^+$ and negative ideal solutions $\widetilde{R}^-$.

$$\widetilde{R}^+ = \left(\left[\xi^{L+}_j, \xi^{R+}_j\right], \left[\psi^{L+}_j, \psi^{R+}_j\right], \left[\zeta^{L+}_j, \zeta^{R+}_j\right]\right) \tag{7}$$

$$\widetilde{R}^- = \left(\left[\xi^{L-}_j, \xi^{R-}_j\right], \left[\psi^{L-}_j, \psi^{R-}_j\right], \left[\zeta^{L-}_j, \zeta^{R-}_j\right]\right) \tag{8}$$

For the benefit attribute:

$$\begin{aligned}&\left(\left[\xi_j^{L+},\xi_j^{R+}\right],\left[\psi_j^{L+},\psi_j^{R+}\right],\left[\zeta_j^{L+},\zeta_j^{R+}\right]\right)\\&=\left(\left[\max_i \xi_{ij}^L,\max_i \xi_{ij}^R\right],\left[\min_i \psi_{ij}^L,\min_i \psi_{ij}^R\right],\left[\min_i \zeta_{ij}^L,\min_i \zeta_{ij}^R\right]\right)\end{aligned} \tag{9}$$

$$\begin{aligned}&\left(\left[\xi_j^{L-},\xi_j^{R-}\right],\left[\psi_j^{L-},\psi_j^{R-}\right],\left[\zeta_j^{L-},\zeta_j^{R-}\right]\right)\\&=\left(\left[\min_i \xi_{ij}^L,\min_i \xi_{ij}^R\right],\left[\max_i \psi_{ij}^L,\max_i \psi_{ij}^R\right],\left[\max_i \zeta_{ij}^L,\max_i \zeta_{ij}^R\right]\right)\end{aligned} \tag{10}$$

For the cost attribute:

$$\begin{aligned}&\left(\left[\xi_j^{L+},\xi_j^{R+}\right],\left[\psi_j^{L+},\psi_j^{R+}\right],\left[\zeta_j^{L+},\zeta_j^{R+}\right]\right)\\&=\left(\left[\min_i \xi_{ij}^L,\min_i \xi_{ij}^R\right],\left[\max_i \psi_{ij}^L,\max_i \psi_{ij}^R\right],\left[\max_i \zeta_{ij}^L,\max_i \zeta_{ij}^R\right]\right)\end{aligned} \tag{11}$$

$$\begin{aligned}&\left(\left[\xi_j^{L-},\xi_j^{R-}\right],\left[\psi_j^{L-},\psi_j^{R-}\right],\left[\zeta_j^{L-},\zeta_j^{R-}\right]\right)\\&=\left(\left[\max_i \xi_{ij}^L,\max_i \xi_{ij}^R\right],\left[\min_i \psi_{ij}^L,\min_i \psi_{ij}^R\right],\left[\min_i \zeta_{ij}^L,\min_i \zeta_{ij}^R\right]\right)\end{aligned} \tag{12}$$

Step 3. Compute the Γ_i and Z_i.

$$\Gamma_i=\sum_{j=1}^{n}\frac{\tau_j\times d\left(\begin{array}{c}\left(\left[\xi_j^{L+},\xi_j^{R+}\right],\left[\psi_j^{L+},\psi_j^{R+}\right],\left[\zeta_j^{L+},\zeta_j^{R+}\right]\right),\\\left(\left[\xi_{ij}^{L},\xi_{ij}^{R}\right],\left[\psi_{ij}^{L},\psi_{ij}^{R}\right],\left[\zeta_{ij}^{L},\zeta_{ij}^{R}\right]\right)\end{array}\right)}{d\left(\begin{array}{c}\left(\left[\xi_j^{L+},\xi_j^{R+}\right],\left[\psi_j^{L+},\psi_j^{R+}\right],\left[\zeta_j^{L+},\zeta_j^{R+}\right]\right),\\\left(\left[\xi_j^{L-},\xi_j^{R-}\right],\left[\psi_j^{L-},\psi_j^{R-}\right],\left[\zeta_j^{L-},\zeta_j^{R-}\right]\right)\end{array}\right)} \tag{13}$$

$$Z_i=\max_j\left\{\frac{\tau_j\times d\left(\begin{array}{c}\left(\left[\xi_j^{L+},\xi_j^{R+}\right],\left[\psi_j^{L+},\psi_j^{R+}\right],\left[\zeta_j^{L+},\zeta_j^{R+}\right]\right),\\\left(\left[\xi_{ij}^{L},\xi_{ij}^{R}\right],\left[\psi_{ij}^{L},\psi_{ij}^{R}\right],\left[\zeta_{ij}^{L},\zeta_{ij}^{R}\right]\right)\end{array}\right)}{d\left(\begin{array}{c}\left(\left[\xi_j^{L+},\xi_j^{R+}\right],\left[\psi_j^{L+},\psi_j^{R+}\right],\left[\zeta_j^{L+},\zeta_j^{R+}\right]\right),\\\left(\left[\xi_j^{L-},\xi_j^{R-}\right],\left[\psi_j^{L-},\psi_j^{R-}\right],\left[\zeta_j^{L-},\zeta_j^{R-}\right]\right)\end{array}\right)}\right\} \tag{14}$$

where τ_j is weight of φ_j.

Step 4. Compute the Θ_i by the following formula:

$$\Theta_i=\theta\frac{(\Gamma_i-\Gamma_i^*)}{(\Gamma_i^- -\Gamma_i^*)}+(1-\theta)\frac{(Z_i-Z_i^*)}{(Z_i^- -Z_i^*)} \tag{15}$$

where

$$\Gamma_i^*=\min_i \Gamma_i,\ \ \Gamma_i^-=\max_i \Gamma_i \tag{16}$$

$$Z_i^*=\min_i Z_i,\ \ Z_i^-=\max_i \Gamma_i \tag{17}$$

where θ depicts the decision-making mechanism coefficient. If $\theta>0.5$, it is for "the maximum group utility"; If $\theta<0.5$, it is "the minimum regret"; and it is both if $\theta=0.5$.

Step 5. Rank the alternatives by Θ_i, Γ_i and Z_i according to the selection rule of the traditional VIKOR method.

4. Numerical Example

4.1. Numerical Example

In this section, a numerical example is given with INNs. Five possible emerging technology enterprises (ETEs) $\phi_i(i = 1,2,3,4,5)$ are selected. Four attributes are selected to evaluate the five possible ETEs: ① φ_1 is the employment creation; ② φ_2 is the development of science and technology; ③ φ_3 is the technical advancement; ④ φ_4 is the industrialization infrastructure. The five ETEs are to be evaluated by using INNs under the attributes ($\tau = (0.2, 0.1, 0.3, 0.4)^T$) by the DMs ($\sigma = (0.2, 0.5, 0.3)^T$), as listed in Tables 1–3.

Table 1. The decision matrix $\widetilde{R}_1$.

	φ_1	φ_2
ϕ_1	([0.3, 0.4], [0.6, 0.7], [0.3, 0.5])	([0.4, 0.5], [0.2, 0.3], [0.1, 0.2])
ϕ_2	([0.5, 0.7], [0.6, 0.8], [0.2, 0.4])	([0.5, 0.6], [0.3, 0.5], [0.2, 0.3])
ϕ_3	([0.4, 0.5], [0.5, 0.6], [0.2, 0.3])	([0.3, 0.4], [0.5, 0.6], [0.1, 0.2])
ϕ_4	([0.6, 0.7], [0.2, 0.3], [0.1, 0.2])	([0.4, 0.5], [0.1, 0.2], [0.2, 0.3])
ϕ_5	([0.4, 0.5], [0.2, 0.3], [0.2, 0.3])	([0.2, 0.3], [0.6, 0.7], [0.2, 0.3])
	φ_3	φ_4
ϕ_1	([0.1, 0.2], [0.4, 0.5], [0.1, 0.2])	([0.3, 0.4], [0.5, 0.6], [0.2, 0.3])
ϕ_2	([0.5, 0.7], [0.4, 0.6], [0.2, 0.3])	([0.6, 0.7], [0.3, 0.4], [0.2, 0.3])
ϕ_3	([0.3, 0.4], [0.1, 0.2], [0.2, 0.3])	([0.4, 0.5], [0.1, 0.2], [0.3, 0.4])
ϕ_4	([0.4, 0.5], [0.2, 0.3], [0.1, 0.2])	([0.3, 0.4], [0.4, 0.5], [0.2, 0.3])
ϕ_5	([0.5, 0.6], [0.4, 0.5], [0.2, 0.3])	([0.3, 0.4], [0.6, 0.7], [0.3, 0.4])

Table 2. The decision matrix $\widetilde{R}_2$.

	φ_1	φ_2
ϕ_1	([0.4, 0.6], [0.5, 0.7], [0.3, 0.4])	([0.6, 0.7], [0.5, 0.6], [0.5, 0.6])
ϕ_2	([0.6, 0.9], [0.4, 0.5], [0.3, 0.4])	([0.7, 0.8], [0.6, 0.7], [0.4, 0.5])
ϕ_3	([0.8, 0.9], [0.8, 0.9], [0.4, 0.5])	([0.7, 0.8], [0.5, 0.6], [0.5, 0.6])
ϕ_4	([0.6, 0.7], [0.3, 0.4], [0.5, 0.6])	([0.8, 0.9], [0.5, 0.6], [0.6, 0.7])
ϕ_5	([0.4, 0.5], [0.6, 0.7], [0.6, 0.7])	([0.6, 0.7], [0.3, 0.4], [0.3, 0.4])
	φ_3	φ_4
ϕ_1	([0.5, 0.6], [0.4, 0.5], [0.3, 0.4])	([0.6, 0.7], [0.4, 0.5], [0.3, 0.4])
ϕ_2	([0.7, 0.8], [0.3, 0.4], [0.3, 0.4])	([0.8, 0.9], [0.4, 0.5], [0.3, 0.4])
ϕ_3	([0.7, 0.8], [0.1, 0.2], [0.3, 0.4])	([0.8, 0.9], [0.5, 0.6], [0.2, 0.3])
ϕ_4	([0.5, 0.6], [0.2, 0.3], [0.4, 0.5])	([0.5, 0.6], [0.7, 0.9], [0.3, 0.4])
ϕ_5	([0.9, 1.0], [0.4, 0.5], [0.3, 0.4])	([0.7, 0.8], [0.8, 0.9], [0.1, 0.2])

Table 3. The decision matrix $\widetilde{R}_3$.

	φ_1	φ_2
ϕ_1	([0.7, 0.8], [0.4, 0.5], [0.4, 0.5])	([0.7, 0.8], [0.3, 0.4], [0.6, 0.7])
ϕ_2	([0.6, 0.7], [0.5, 0.6], [0.4, 0.5])	([0.7, 0.8], [0.6, 0.7], [0.5, 0.6])
ϕ_3	([0.7, 0.8], [0.3, 0.4], [0.5, 0.6])	([0.8, 0.9], [0.2, 0.4], [0.6, 0.7])
ϕ_4	([0.7, 0.8], [0.4, 0.5], [0.6, 0.7])	([0.6, 0.9], [0.1, 0.2], [0.7, 0.8])
ϕ_5	([0.6, 0.7], [0.7, 0.8], [0.2, 0.3])	([0.7, 0.8], [0.3, 0.5], [0.4, 0.5])
	φ_3	φ_4
ϕ_1	([0.6, 0.7], [0.3, 0.4], [0.4, 0.5])	([0.5, 0.6], [0.4, 0.5], [0.4, 0.5])
ϕ_2	([0.8, 0.9], [0.2, 0.3], [0.7, 0.8])	([0.6, 0.7], [0.3, 0.4], [0.4, 0.6])
ϕ_3	([0.8, 0.9], [0.2, 0.4], [0.4, 0.5])	([0.9, 1.0], [0.1, 0.2], [0.5, 0.6])
ϕ_4	([0.6, 0.7], [0.1, 0.2], [0.5, 0.6])	([0.6, 0.7], [0.3, 0.4], [0.4, 0.5])
ϕ_5	([0.7, 0.9], [0.3, 0.4], [0.4 0.5])	([0.8, 0.9], [0.5, 0.6], [0.5, 0.6])

Then, we use the proposed model to select the best ETE.

Step 1. Utilize $\widetilde{R}_k(k=1,2,3)$ and the INNWA operator, in order to obtain matrix $\widetilde{R}=\left(\widetilde{r}_{ij}\right)_{5\times 4}$ by Equation (6) which is listed in Table 4.

Table 4. The decision matrix $\widetilde{R}$.

	φ_1	φ_2
ϕ_1	([0.4974, 0.6477], [0.4850, 0.6328], [0.3270, 0.4472])	([0.6021, 0.7058], [0.3571, 0.4625], [0.3828, 0.5044])
ϕ_2	([0.5817, 0.8268], [0.4638, 0.5802], [0.3016, 0.4277])	([0.6677, 0.7703], [0.5223, 0.6544], [0.3723, 0.4768])
ϕ_3	([0.7186, 0.8301], [0.5426, 0.6507], [0.3723, 0.4768])	([0.6853, 0.7976], [0.3798, 0.5313], [0.3828, 0.5044])
ϕ_4	([0.6331, 0.7344], [0.3016, 0.4038], [0.3828, 0.5044])	([0.6933, 0.8620], [0.2236, 0.3464], [0.5044, 0.6150])
ϕ_5	([0.4687, 0.5710], [0.5044, 0.6150], [0.3464, 0.4583])	([0.5785, 0.6853], [0.3446, 0.4783], [0.3016, 0.4083])
	φ_3	φ_4
ϕ_1	([0.4740, 0.5785], [0.3669, 0.4676], [0.2625, 0.3723])	([0.5127, 0.6243], [0.4183, 0.5186], [0.3016, 0.4038])
ϕ_2	([0.7058, 0.8238], [0.2814, 0.3979], [0.3567, 0.4649])	([0.7172, 0.8268], [0.3464, 0.4472], [0.3016, 0.4265])
ϕ_3	([0.6853, 0.7976], [0.1231, 0.2462], [0.3016, 0.4038])	([0.7976, 1.0000], [0.2236, 0.3464], [0.2855, 0.3912])
ϕ_4	([0.5150, 0.6163], [0.1625, 0.2656], [0.3241, 0.4397])	([0.4998, 0.6021], [0.4854, 0.6274], [0.3016, 0.4038])
ϕ_5	([0.8082, 1.0000], [0.3669, 0.4676], [0.3016, 0.4038])	([0.6853, 0.7976], [0.6559, 0.7579], [0.2019, 0.3194])

Step 2. Define the $\widetilde{R}^{+}$ and $\widetilde{R}^{-}$ by Equations (7) and (8).

$$\widetilde{R}^{+}=\left\{\begin{array}{c}([0.7186,0.8301],[0.3016,0.4038],[0.3016,0.4277]),\\ ([0.6933,0.8620],[0.2236,0.3464],[0.3016,0.4038]),\\ ([0.8082,1.0000],[0.1231,0.2462],[0.2625,0.3723]),\\ ([0.7976,1.1000],[0.2236,0.3464],[0.2019,0.3194])\end{array}\right\}$$

$$\widetilde{R}^{-}=\left\{\begin{array}{c}([0.4687,0.5710],[0.5426,0.6507],[0.3828,0.5044]),\\ ([0.5785,0.6853],[0.5223,0.6544],[0.5044,0.6150]),\\ ([0.4740,0.5785],[0.3669,0.4676],[0.3567,0.4649]),\\ ([0.4998,0.6021],[0.6559,0.7579],[0.3016,0.4265])\end{array}\right\}$$

Step 3. Compute the Γ_i and Z_i by Equation (14).

$$\Gamma_1=0.6507,\Gamma_2=0.4182,\Gamma_3=0.2416,\Gamma_4=0.5261,\Gamma_5=0.5195$$
$$Z_1=0.2386,Z_2=0.1515,Z_3=0.0921,Z_4=0.2765,Z_5=0.2252$$

Step 4. Compute the Θ_i (let $\theta=0.5$) by Equation (15).

$$\Theta_1=0.8974,\Theta_2=0.3772,\Theta_3=0.0000,\Theta_4=0.8477,\Theta_5=0.7006$$

Step 5. The order of ETEs is determined by Θ_i $(i=1,2,3,4,5)$: $\phi_3\succ\phi_2\succ\phi_5\succ\phi_4\succ\phi_1$, and thus the most desirable ETE is ϕ_3.

4.2. Comparative Analysis

In what follows, we compare with the interval neutrosophic number weighted averaging (INNWA) operator and interval neutrosophic number weighted geometric (INNWG) operator [28], INN similarity [33], and INN VIKOR [55]. The results are shown in Table 5.

From the above analysis, it can be seen that the five methods have the same best emerging technology enterprise ϕ_3, and the ranking results of Method 1 and Method 2 are slightly different. The proposed INN VIKOR method can reasonably focus a MAGDM problem with INNs. At the same time, compared with Method 5 based on the INN VIKOR method in Reference [55], our proposed method avoids the interval numbers' comparison.

Table 5. The orders by utilizing five methods.

Methods	Ranking Orders	Best Alternatives
Method 1 with INNWA operator in [28]	$\phi_3 \succ \phi_5 \succ \phi_2 \succ \phi_4 \succ \phi_1$	ϕ_3
Method 2 with INNWG operator in [28]	$\phi_3 \succ \phi_2 \succ \phi_5 \succ \phi_4 \succ \phi_1$	ϕ_3
Method 3 based on similarity in [33]	$\phi_3 \succ \phi_2 \succ \phi_5 \succ \phi_4 \succ \phi_1$	ϕ_3
Method 4 based on similarity in [33]	$\phi_3 \succ \phi_2 \succ \phi_5 \succ \phi_4 \succ \phi_1$	ϕ_3
Method 5 based on INN VIKOR in [55]	$\phi_3 \succ \phi_2 \succ \phi_5 \succ \phi_4 \succ \phi_1$	ϕ_3
The proposed method	$\phi_3 \succ \phi_2 \succ \phi_5 \succ \phi_4 \succ \phi_1$	ϕ_3

5. Conclusions

The VIKOR method for a MAGDM presents some conflicting attributes. We extended the VIKOR method to MAGDM with INNs. Firstly, the basic concepts of INNs were briefly presented. The method first aggregates all individual decision-makers' assessment information based on an INNWA operator, and then employs the extended classical VIKOR method for MAGDM problems with INNs. The validity and stability of this method were verified by example analysis and comparative analysis, and its superiority was illustrated by a comparison with the existing methods. In the future, many other methods of INSs need to be explored in for MAGDM, risk analysis, and many other uncertain and fuzzy environments [56–78].

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Certain Competition Graphs Based on Intuitionistic Neutrosophic Environment

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Abstract: The concept of intuitionistic neutrosophic sets provides an additional possibility to represent imprecise, uncertain, inconsistent and incomplete information, which exists in real situations. This research article first presents the notion of intuitionistic neutrosophic competition graphs. Then, p-competition intuitionistic neutrosophic graphs and m-step intuitionistic neutrosophic competition graphs are discussed. Further, applications of intuitionistic neutrosophic competition graphs in ecosystem and career competition are described.

Keywords: intuitionistic neutrosophic competition graphs; intuitionistic neutrosophic open-neighborhood graphs; p-competition intuitionistic neutrosophic graphs; m-step intuitionistic neutrosophic competition graphs

MSC: 03E72; 68R10; 68R05

1. Introduction

Euler [1] introduced the concept of graph theory in 1736, which has applications in various fields, including image capturing, data mining, clustering and computer science [2–5]. A graph is also used to develop an interconnection between objects in a known set of objects. Every object can be illustrated by a vertex, and interconnection between them can be illustrated by an edge. The notion of competition graphs was developed by Cohen [6] in 1968, depending on a problem in ecology. The competition graphs have many utilizations in solving daily life problems, including channel assignment, modeling of complex economic, phytogenetic tree reconstruction, coding and energy systems.

Fuzzy set theory and intuitionistic fuzzy sets theory are useful models for dealing with uncertainty and incomplete information. However, they may not be sufficient in modeling of indeterminate and inconsistent information encountered in the real world. In order to cope with this issue, neutrosophic set theory was proposed by Smarandache [7] as a generalization of fuzzy sets and intuitionistic fuzzy sets. However, since neutrosophic sets are identified by three functions called truth-membership (t), indeterminacy-membership (i) and falsity-membership (f), whose values are the real standard or non-standard subset of unit interval $]0^-, 1^+[$. There are some difficulties in modeling of some problems in engineering and sciences. To overcome these difficulties, Smarandache in 1998 [8] and Wang et al. [9] in 2010 defined the concept of single-valued neutrosophic sets and their operations as a generalization of intuitionistic fuzzy sets. Yang et al. [10] introduced the concept of the single-valued neutrosophic relation based on the single-valued neutrosophic set. They also developed kernels and closures of a single-valued neutrosophic set. The concept of the single-valued intuitionistic neutrosophic set was proposed by Bhowmik and Pal [11,12].

The valuable contribution of fuzzy graph and generalized structures has been studied by several researchers [13–22]. Smarandache [23] proposed the notion of the neutrosophic graph and separated them into four main categories. Wu [24] discussed fuzzy digraphs. Fuzzy m-competition and p-competition graphs were introduced by Samanta and Pal [25]. Samanta et al. [26] introduced m-step fuzzy competition graphs. Dhavaseelan et al. [27] defined strong neutrosophic graphs. Akram and Shahzadi [28] introduced the notion of a single-valued neutrosophic graph and studied some of its operations. They also discussed the properties of single-valued neutrosophic graphs by level graphs. Akram and Shahzadi [29] introduced the concept of neutrosophic soft graphs with applications. Broumi et al. [30] proposed single-valued neutrosophic graphs and discussed some properties. Ye [31–33] has presented several novel concepts of neutrosophic sets with applications. In this paper, we first introduce the concept of intuitionistic neutrosophic competition graphs. We then discuss m-step intuitionistic neutrosophic competition graphs. Further, we describe applications of intuitionistic neutrosophic competition graphs in ecosystem and career competition. Finally, we present our developed methods by algorithms.

Our paper is divided into the following sections: In Section 2, we introduce certain competition graphs using the intuitionistic neutrosophic environment. In Section 3, we present applications of intuitionistic neutrosophic competition graphs in ecosystem and career competition. Finally, Section 4 provides conclusions and future research directions.

2. Intuitionistic Neutrosophic Competition Graphs

We have used standard definitions and terminologies in this paper. For other notations, terminologies and applications not mentioned in the paper, the readers are referred to [34–44].

Definition 1. *[38] Let X be a fixed set. A generalized intuitionistic fuzzy set I of X is an object having the form* $I=\{(u,\mu_I(u),\nu_I(u))|u\in U\}$, *where the functions* $\mu_I(u):\to[0,1]$ *and* $\nu_I(u):\to[0,1]$ *define the degree of membership and degree of non-membership of an element* $u\in X$, *respectively, such that:*

$$\min\{\mu_I(u),\nu_I(u)\}\leq 0.5, \text{for all } u\in X.$$

This condition is called the generalized intuitionistic condition.

Definition 2. *[11] An intuitionistic neutrosophic set (IN-set) is defined as* $\check{A}=(w,t_{\check{A}}(w),i_{\check{A}}(w),f_{\check{A}}(w))$, *where:*

$$t_{\check{A}}(w)\wedge f_{\check{A}}(w)\leq 0.5,$$
$$t_{\check{A}}(w)\wedge i_{\check{A}}(w)\leq 0.5,$$
$$i_{\check{A}}(w)\wedge f_{\check{A}}(w)\leq 0.5,$$

for all, $w\in X$, *such that:*

$$0\leq t_{\check{A}}(w)+i_{\check{A}}(w)+f_{\check{A}}(w)\leq 2.$$

Definition 3. *[12] An intuitionistic neutrosophic relation (IN-relation) is defined as an intuitionistic neutrosophic subset of* $X\times Y$, *which has of the form:*

$$R=\{((w,z),t_R(w,z),i_R(w,z),f_R(w,z)):w\in X,z\in Y\},$$

where t_R, i_R *and* f_R *are intuitionistic neutrosophic subsets of* $X\times Y$ *satisfying the conditions:*

1. *one of these* $t_R(w,z)$, $i_R(w,z)$ *and* $f_R(w,z)$ *is greater than or equal to* 0.5,
2. $0\leq t_R(w,z)+i_R(w,z)+f_R(w,z)\leq 2$.

Definition 4. *An intuitionistic neutrosophic graph (IN-graph)* $\mathfrak{G}=(X,h,k)$ *(in short* $\mathfrak{G}$*) on X (vertex set) is a triplet such that:*

1. $t_k(w,z) \leq t_h(w) \wedge t_h(z)$, $\quad i_k(w,z) \leq i_h(w) \wedge i_h(z)$, $\quad f_k(w,z) \leq f_h(w) \vee f_h(z)$,
2. $t_k(w,z) \wedge i_k(w,z) \leq 0.5$, $\quad t_k(w,z) \wedge f_k(w,z) \leq 0.5$, $\quad i_k(w,z) \wedge f_k(w,z) \leq 0.5$,
3. $0 \leq t_k(w,z) + i_k(w,z) + f_k(w,z) \leq 2$, *for all* $w, z \in X$,

where,

$$t_h,\ i_h \text{ and } f_h \to [0,1]$$

denote the truth-membership, indeterminacy-membership and falsity-membership of an element $w \in X$ *and:*

$$t_k,\ i_k \text{ and } f_k \to [0,1]$$

denote the truth-membership, indeterminacy-membership and falsity-membership of an element $(w,\ z) \in E$ *(edge set).*

We now illustrate this with an example.

Example 1. *Consider IN-graph* $\mathfrak{G}$ *on non-empty set X, as shown in Figure 1.*

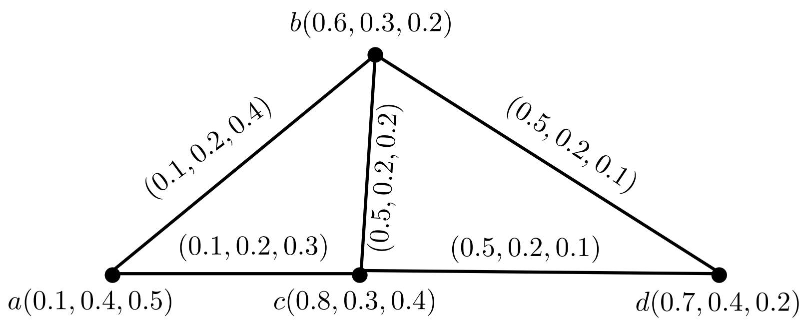

Figure 1. Intuitionistic neutrosophic graph (IN-graph).

Definition 5. *Let* $\overrightarrow{\mathfrak{G}}$ *be an intuitionistic neutrosophic digraph (IN-digraph), then intuitionistic neutrosophic out-neighborhoods (IN-out-neighborhoods) of a vertex w are an IN-set:*

$$\mathbb{N}^+(w) = (X_w^+, t_w^+, i_w^+, f_w^+),$$

where,

$$X_w^+ = \{z | k_1(\overrightarrow{w,z}) > 0, k_2(\overrightarrow{w,z}) > 0, k_3(\overrightarrow{w,z}) > 0\},$$

such that $t_w^+ : X_w^+ \to [0,1]$ *defined by* $t_w^+(z) = k_1(\overrightarrow{w,z})$, $i_w^+ : X_w^+ \to [0,1]$ *defined by* $i_w^+(z) = k_2(\overrightarrow{w,z})$ *and* $f_z^+ : X_z^+ \to [0,1]$ *defined by* $f_w^+(z) = k_3(\overrightarrow{w,z})$.

Definition 6. *Let* $\overrightarrow{\mathfrak{G}}$ *be an IN-digraph, then the intuitionistic neutrosophic in-neighborhoods (IN-in-neighborhoods) of a vertex w are an IN-set:*

$$\mathbb{N}^-(w) = (X_w^-, t_w^-, i_w^-, f_w^-),$$

where,

$$X_w^- = \{z | k_1(\overrightarrow{z,w}) > 0, k_2(\overrightarrow{z,w}) > 0, k_3(\overrightarrow{z,w}) > 0\},$$

such that $t_w^- : X_w^- \to [0,1]$ *defined by* $t_w^-(z) = k_1(\overrightarrow{z,w})$, $i_w^- : X_w^- \to [0,1]$ *defined by* $i_w^-(z) = k_2(\overrightarrow{z,w})$ *and* $f_w^- : X_w^- \to [0,1]$ *defined by* $f_w^-(z) = k_3(\overrightarrow{z,w})$.

Example 2. *Consider* $\overrightarrow{\mathfrak{G}} = (X, h, k)$ *to be an IN-digraph, such that,* $X = \{a, b, c, d, e\}$, $h = \{(a, 0.5, 0.3, 0.1), (b, 0.6, 0.4, 0.2), (c, 0.8, 0.3, 0.1), (d, 0.1, 0.9, 0.4), (e, 0.4, 0.3, 0.6)\}$ *and* $k = \{(\overrightarrow{ab}, 0.3, 0.3, 0.1), (\overrightarrow{ae}, 0.3, 0.2, 0.4), (\overrightarrow{bc}, 0.5, 0.2, 0.1), (\overrightarrow{ed}, 0.1, 0.2, 0.5), (\overrightarrow{dc}, 0.1, 0.2, 0.3), (\overrightarrow{bd}, 0.1, 0.3, 0.3)\}$, *as shown in Figure 2.*

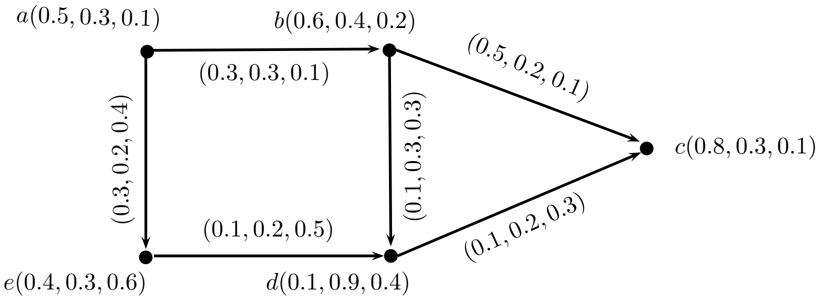

Figure 2. IN-digraph.

Then, $\mathbb{N}^+(a) = \{(b, 0.3, 0.3, 0.1), (e, 0.3, 0.2, 0.4)\}$, $\mathbb{N}^+(c) = \emptyset$, $\mathbb{N}^+(d) = \{(c, 0.1, 0.2, 0.3)\}$, *and* $\mathbb{N}^-(b) = \{(a, 0.3, 0.3, 0.1)\}$, $\mathbb{N}^-(c) = \{(b, 0.5, 0.2, 0.1), (d, 0.1, 0.2, 0.3)\}$. *Similarly, we can calculate IN-out and in-neighborhoods of the remaining vertices.*

Definition 7. *The height of an IN-set* $\breve{A} = (w, t_{\breve{A}}, i_{\breve{A}}, f_{\breve{A}})$ *is defined as:*

$$H(\breve{A}) = (\sup_{w\in X} t_{\breve{A}}(w), \sup_{w\in X} i_{\breve{A}}(w), \inf_{w\in X} f_{\breve{A}}(w)) = (H_1(\breve{A}), H_2(\breve{A}), H_3(\breve{A})).$$

For example, the height of an IN-set $\breve{A} = \{(a, 0.5, 0.7, 0.2), (b, 0.1, 0.2, 1), (c, 0.3, 0.5, 0.3)\}$ *in* $X = \{a, b, c\}$ *is* $H(\breve{A}) = (0.5, 0.7, 0.2)$.

Definition 8. *An intuitionistic neutrosophic competition graph (INC-graph)* $\mathbb{C}(\overrightarrow{\mathfrak{G}})$ *of an IN-digraph* $\overrightarrow{\mathfrak{G}} = (X, h, k)$ *is an undirected IN-graph* $\mathfrak{G} = (X, h, k)$, *which has the same intuitionistic neutrosophic set of vertices as in* $\overrightarrow{\mathfrak{G}}$ *and has an intuitionistic neutrosophic edge between two vertices* $w, z \in X$ *in* $\mathbb{C}(\overrightarrow{\mathfrak{G}})$ *if and only if* $\mathbb{N}^+(w) \cap \mathbb{N}^+(z)$ *is a non-empty IN-set in* $\overrightarrow{\mathfrak{G}}$. *The truth-membership, indeterminacy-membership and falsity-membership values of edge* (w, z) *in* $\mathbb{C}(\overrightarrow{\mathfrak{G}})$ *are:*

$$\begin{aligned}
t_k(w,z) &= (t_h(w) \wedge t_h(z))H(\mathbb{N}^+(w) \cap \mathbb{N}^+(z)),\\
i_k(w,z) &= (i_h(w) \wedge i_h(z))H(\mathbb{N}^+(w) \cap \mathbb{N}^+(z)),\\
f_k(w,z) &= (f_h(w) \vee f_h(z))H(\mathbb{N}^+(w) \cap \mathbb{N}^+(z)), \quad \textit{respectively.}
\end{aligned}$$

Example 3. *Consider* $\overrightarrow{\mathfrak{G}} = (X, h, k)$ *to be an IN-digraph, such that,* $X = \{a, b, c, d\}$, $h = \{(a, 0.1, 0.4, 0.5), (b, 0.6, 0.3, 0.2), (c, 0.8, 0.3, 0.4), (d, 0.7, 0.4, 0.2)\}$ *and* $k = \{(\overrightarrow{ab}, 0.1, 0.2, 0.4), (\overrightarrow{ac}, 0.1, 0.2, 0.3), (\overrightarrow{bc}, 0.5, 0.2, 0.2), (\overrightarrow{bd}, 0.5, 0.2, 0.1), (\overrightarrow{cd}, 0.5, 0.2, 0.1)\}$, *as shown in Figure 3.*

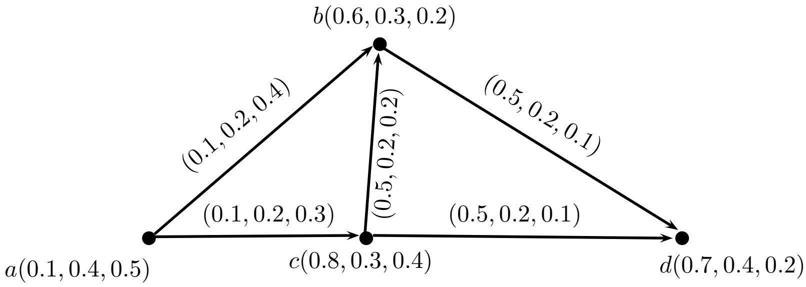

Figure 3. IN-digraph.

By direct calculations, we have Tables 1 and 2 representing IN-out and in-neighborhoods, respectively.

Table 1. IN-out-neighborhoods.

w	$\mathbb{N}^+(w)$
a	{(b, 0.1, 0.2, 0.4), (c, 0.1, 0.2, 0.3)}
b	{(d, 0.5, 0.2, 0.1)}
c	{(b, 0.5, 0.2, 0.2), (d, 0.5, 0.2, 0.1)}
d	$\emptyset$

Table 2. IN-in-neighborhoods.

w	$\mathbb{N}^-(w)$
a	$\emptyset$
b	{(a, 0.1, 0.2, 0.4), (c, 0.1, 0.2, 0.3)}
c	{(a, 0.1, 0.2, 0.3)}
d	{(b, 0.5, 0.2, 0.1), (c, 0.5, 0.2, 0.1)}

The INC-graph of Figure 3 is shown in Figure 4.

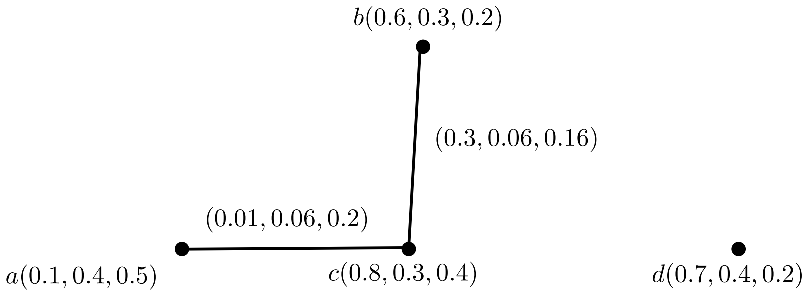

Figure 4. Intuitionistic neutrosophic competition graph (INC-graph).

Therefore, there is an edge between two vertices in INC-graph $\mathbb{C}(\overrightarrow{\mathfrak{G}})$, whose truth-membership, indeterminacy-membership and falsity-membership values are given by the above formula.

Definition 9. *For an IN-graph $\mathfrak{G} = (X, h, k)$, where $h = (h_1, h_2, h_3)$ and $k = (k_1, k_2, k_3)$, then an edge (w, z), $w, z \in X$ is called independent strong if:*

$$\frac{1}{2}[h_1(w) \wedge h_1(z)] < k_1(w,z),$$
$$\frac{1}{2}[h_2(w) \wedge h_2(z)] > k_2(w,z),$$
$$\frac{1}{2}[h_3(w) \vee h_3(z)] > k_3(w,z).$$

Otherwise, it is called weak.

Theorem 1. *Suppose $\overrightarrow{\mathfrak{G}}$ is an IN-digraph. If $\mathbb{N}^+(w) \cap \mathbb{N}^+(z)$ contains only one element of $\overrightarrow{\mathfrak{G}}$, then the edge (w, z) of $\mathbb{C}(\overrightarrow{\mathfrak{G}})$ is independent strong if and only if:*

$$|[\mathbb{N}^+(w) \cap \mathbb{N}^+(z)]|_t > 0.5,$$
$$|[\mathbb{N}^+(w) \cap \mathbb{N}^+(z)]|_i < 0.5,$$
$$|[\mathbb{N}^+(w) \cap \mathbb{N}^+(z)]|_f < 0.5.$$

Proof. Suppose, $\overrightarrow{\mathfrak{G}}$ is an IN-digraph. Suppose $\mathbb{N}^+(w) \cap \mathbb{N}^+(z) = (a, \check{p}, q, r)$, where $\check{p}$, q and r are the truth-membership, indeterminacy-membership and falsity-membership values of either the edge (w, a) or the edge (z, a), respectively. Here,

$$|[\mathbb{N}^+(w) \cap \mathbb{N}^+(z)]|_t = \check{p} = H_1(\mathbb{N}^+(w) \cap \mathbb{N}^+(z)),$$
$$|[\mathbb{N}^+(w) \cap \mathbb{N}^+(z)]|_i = q = H_2(\mathbb{N}^+(w) \cap \mathbb{N}^+(z)),$$
$$|[\mathbb{N}^+(w) \cap \mathbb{N}^+(z)]|_f = r = H_3(\mathbb{N}^+(w) \cap \mathbb{N}^+(z)).$$

Then,

$$k_1(w,z) = \check{p} \times [h_1(w) \wedge h_1(z)],$$
$$k_2(w,z) = q \times [h_2(w) \wedge h_2(z)],$$
$$k_3(w,z) = r \times [h_3(w) \vee h_3(z)].$$

Therefore, the edge (w, z) in $\mathbb{C}(\overrightarrow{\mathfrak{G}})$ is independent strong if and only if $\check{p} > 0.5$, $q < 0.5$ and $r < 0.5$. Hence, the edge (w, z) of $\mathbb{C}(\overrightarrow{\mathfrak{G}})$ is independent strong if and only if:

$$|[\mathbb{N}^+(w) \cap \mathbb{N}^+(z)]|_t > 0.5,$$
$$|[\mathbb{N}^+(w) \cap \mathbb{N}^+(z)]|_i < 0.5,$$
$$|[\mathbb{N}^+(w) \cap \mathbb{N}^+(z)]|_f < 0.5.$$

□

We illustrate the theorem with an example as shown in Figure 5.

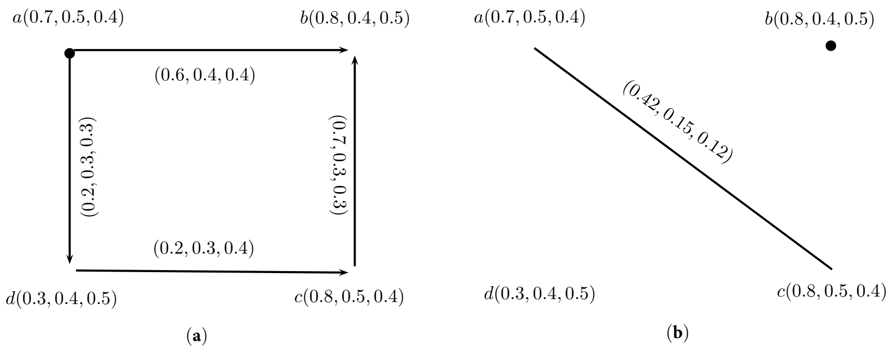

Figure 5. INC-graph. (**a**) IN-digraph; (**b**) corresponding INC-graph.

Theorem 2. *If all the edges of an IN-digraph $\overrightarrow{\mathfrak{G}}$ are independent strong, then:*

$$\frac{k_1(w,z)}{(h_1(w)\wedge h_1(z))^2} > 0.5,$$
$$\frac{k_2(w,z)}{(h_2(w)\wedge h_2(z))^2} < 0.5,$$
$$\frac{k_3(w,z)}{(h_3(w)\vee f_3(z))^2} < 0.5$$

for all edges (w, z) in $\mathbb{C}(\overrightarrow{\mathfrak{G}})$.

Proof. Suppose all the edges of IN-digraph $\overrightarrow{\mathfrak{G}}$ are independent strong. Then:

$$\frac{1}{2}[h_1(w)\wedge h_1(z)] < k_1\overrightarrow{(w,z)},$$
$$\frac{1}{2}[h_2(w)\wedge h_2(z)] > k_2\overrightarrow{(w,z)},$$
$$\frac{1}{2}[h_3(w)\vee h_3(z)] > k_3\overrightarrow{(w,z)},$$

for all the edges (w, z) in $\overrightarrow{\mathfrak{G}}$. Let the corresponding INC-graph be $\mathbb{C}(\overrightarrow{\mathfrak{G}})$.

Case (1): When $\mathbb{N}^+(w)\cap\mathbb{N}^+(z)=\emptyset$ for all $w, z\in X$, then there does not exist any edge in $\mathbb{C}(\overrightarrow{\mathfrak{G}})$ between w and z. Thus, we have nothing to prove in this case.

Case (2): When $\mathbb{N}^+(w)\cap\mathbb{N}^+(z)\neq\emptyset$, let $\mathbb{N}^+(w)\cap\mathbb{N}^+(z)=\{(a_1, m_1, n_1, \check{p}_1), (a_2, m_2, n_2, \check{p}_2), \ldots, (a_l, m_l, n_l, \check{p}_l)\}$, where m_i, n_i and $\check{p}_i$ are the truth-membership, indeterminacy-membership and falsity-membership values of either $\overrightarrow{(w, a_i)}$ or $\overrightarrow{(z, a_i)}$ for $i = 1, 2, \ldots, l$, respectively. Therefore,

$$m_i = [k_1\overrightarrow{(w,a_i)}\wedge k_1\overrightarrow{(z,a_i)}],$$
$$n_i = [k_2\overrightarrow{(w,a_i)}\wedge k_2\overrightarrow{(z,a_i)}],$$
$$\check{p}_i = [k_3\overrightarrow{(w,a_i)}\vee k_3\overrightarrow{(z,a_i)}], \quad for \quad i=1,2,\ldots,l.$$

Suppose,

$$H_1(\mathbb{N}^+(w)\cap\mathbb{N}^+(z)) = \max\{m_i, \quad i=1,2,\ldots,l\} = m_{\max},$$
$$H_2(\mathbb{N}^+(w)\cap\mathbb{N}^+(z)) = \max\{n_i, \quad i=1,2,\ldots,l\} = n_{\max},$$
$$H_3(\mathbb{N}^+(w)\cap\mathbb{N}^+(z)) = \min\{\check{p}_i, \quad i=1,2,\ldots,l\} = \check{p}_{\min}.$$

Obviously, $m_{\max} > k_1\overrightarrow{(w,z)}$ and $n_{\max} < k_2\overrightarrow{(w,z)}$ and $\check{p}_{\min} < k_3\overrightarrow{(w,z)}$ for all edges $\overrightarrow{(w,z)}$ show that:

$$\frac{m_{\max}}{h_1(w) \wedge h_1(z)} > \frac{k_1\overrightarrow{(w,z)}}{h_1(w) \wedge h_1(z)} > 0.5,$$
$$\frac{n_{\max}}{h_2(w) \wedge h_2(z)} < \frac{k_2\overrightarrow{(w,z)}}{h_2(w) \wedge h_2(z)} < 0.5,$$
$$\frac{\check{p}_{\min}}{h_3(w) \vee h_3(z)} < \frac{k_3\overrightarrow{(w,z)}}{h_3(w) \wedge h_3(z)} < 0.5,$$

therefore,

$$\begin{aligned} k_1(w,z) &= (h_1(w) \wedge h_1(z))H_1(\mathbb{N}^+(w) \cap \mathbb{N}^+(z)), \\ k_1(w,z) &= [h_1(w) \wedge h_1(z)] \times m_{\max}, \\ \frac{k_1(w,z)}{(h_1(w) \wedge h_1(z))} &= m_{\max}, \\ \frac{k_1(w,z)}{(h_1(w) \wedge h_1(z))^2} &= \frac{m_{\max}}{(h_1(w) \wedge h_1(z))} > 0.5, \end{aligned}$$

$$\begin{aligned} k_2(w,z) &= (h_2(w) \wedge h_2(z))H_2(\mathbb{N}^+(w) \cap \mathbb{N}^+(z)), \\ k_2(w,z) &= [h_2(w) \wedge h_2(z)] \times n_{\max}, \\ \frac{k_2(w,z)}{(h_2(w) \wedge h_2(z))} &= n_{\max}, \\ \frac{k_2(w,z)}{(h_2(w) \wedge h_2(z))^2} &= \frac{n_{\max}}{(h_2(w) \wedge h_2(z))} < 0.5, \end{aligned}$$

and:

$$\begin{aligned} k_3(w,z) &= (h_3(w) \vee h_3(z))H_3(\mathbb{N}^+(w) \cap \mathbb{N}^+(z)), \\ k_3(w,z) &= [h_3(w) \vee h_3(z)] \times \check{p}_{\min}, \\ \frac{k_3(w,z)}{(h_3(w) \vee h_3(z))} &= \check{p}_{\min}, \\ \frac{k_3(w,z)}{(h_3(w) \vee h_3(z))^2} &= \frac{\check{p}_{\min}}{(h_3(w) \vee h_3(z))} < 0.5. \end{aligned}$$

Hence, $\frac{k_1(w,z)}{(h_1(w) \wedge h_1(z))^2} > 0.5$, $\frac{k_2(w,z)}{(h_2(w) \wedge h_2(z))^2} < 0.5$, and $\frac{k_3(w,z)}{(h_3(w) \vee h_3(z))^2} < 0.5$ for all edges (w, z) in $\mathbb{C}(\overrightarrow{\mathfrak{G}})$. $\square$

Theorem 3. *Let $\mathbb{C}(\overrightarrow{\mathfrak{G}_1}) = (h_1, k_1)$ and $\mathbb{C}(\overrightarrow{\mathfrak{G}_2}) = (h_2, k_2)$ be two INC-graph of IN-digraphs $\overrightarrow{\mathfrak{G}_1} = (h_1, \overrightarrow{l_1})$ and $\overrightarrow{\mathfrak{G}_2} = (h_2, \overrightarrow{l_2})$, respectively. Then, $\mathbb{C}(\overrightarrow{\mathfrak{G}_1} \square \overrightarrow{\mathfrak{G}_2}) = \mathfrak{G}_{\mathbb{C}(\overrightarrow{\mathfrak{G}_1})^* \square \mathbb{C}(\overrightarrow{\mathfrak{G}_2})^*} \cup \mathfrak{G}^{\square}$ where, $\mathfrak{G}_{\mathbb{C}(\overrightarrow{\mathfrak{G}_1})^* \square \mathbb{C}(\overrightarrow{\mathfrak{G}_2})^*}$ is an IN-graph on the crisp graph $(X_1 \times X_2, E_{\mathbb{C}(\overrightarrow{\mathfrak{G}_1})^*} \square E_{\mathbb{C}(\overrightarrow{\mathfrak{G}_2})^*})$, $\mathbb{C}(\overrightarrow{\mathfrak{G}_1})^*$ and $\mathbb{C}(\overrightarrow{\mathfrak{G}_2})^*$ are the crisp competition graphs of $\overrightarrow{\mathfrak{G}_1}$ and $\overrightarrow{\mathfrak{G}_2}$, respectively. $\mathfrak{D}^{\square}$ is an IN-graph on $(X_1 \times X_2, E^{\square})$ such that:*

1. $E^{\square} = \{(w_1, w_2)(z_1, z_2) : z_1 \in \mathbb{N}^-(w_1)^*, z_2 \in \mathbb{N}^+(w_2)^*\}$
   $E_{\mathbb{C}(\overrightarrow{\mathfrak{G}_1})^*} \square E_{\mathbb{C}(\overrightarrow{\mathfrak{G}_2})^*} = \{(w_1, w_2)(w_1, z_2) : w_1 \in X_1, w_2 z_2 \in E_{\mathbb{C}(\overrightarrow{\mathfrak{G}_2})^*}\} \cup \{(w_1, w_2)(z_1, w_2) : w_2 \in X_2, w_1 z_1 \in E_{\mathbb{C}(\overrightarrow{\mathfrak{G}_1})^*}\}$.

2. $t_{h_1\Box h_2} = t_{h_1}(w_1) \wedge t_{h_2}(w_2), \quad i_{h_1\Box h_2} = i_{h_1}(w_1) \wedge i_{h_2}(w_2), \quad f_{h_1\Box h_2} = f_{h_1}(w_1) \vee f_{h_2}(w_2).$
3. $t_k((w_1,w_2)(w_1,z_2)) = [t_{h_1}(w_1) \wedge t_{h_2}(w_2) \wedge t_{h_2}(z_2)] \times \vee_{x_2}\{t_{h_1}(w_1) \wedge t_{\overrightarrow{l_2}}(w_2x_2) \wedge t_{\overrightarrow{l_2}}(z_2x_2)\}$,
$(w_1,w_2)(w_1,z_2) \in E_{\mathbb{C}(\overrightarrow{\mathfrak{G}_1})^*}\Box E_{\mathbb{C}(\overrightarrow{\mathfrak{G}_2})^*}, \quad x_2 \in (\mathbb{N}^+(w_2) \cap \mathbb{N}^+(z_2))^*$.
4. $i_k((w_1,w_2)(w_1,z_2)) = [i_{h_1}(w_1) \wedge i_{h_2}(w_2) \wedge i_{h_2}(z_2)] \times \vee_{x_2}\{i_{h_1}(w_1) \wedge i_{\overrightarrow{l_2}}(w_2x_2) \wedge i_{\overrightarrow{l_2}}(z_2x_2)\}$,
$(w_1,w_2)(w_1,z_2) \in E_{\mathbb{C}(\overrightarrow{\mathfrak{G}_1})^*}\Box E_{\mathbb{C}(\overrightarrow{\mathfrak{G}_2})^*}, \quad x_2 \in (\mathbb{N}^+(w_2) \cap \mathbb{N}^+(z_2))^*$.
5. $f_k((w_1,w_2)(w_1,z_2)) = [f_{h_1}(w_1) \vee f_{h_2}(w_2) \vee f_{h_2}(z_2)] \times \vee_{x_2}\{f_{h_1}(w_1) \vee f_{\overrightarrow{l_2}}(w_2x_2) \vee f_{\overrightarrow{l_2}}(z_2x_2)\}$,
$(w_1,w_2)(w_1,z_2) \in E_{\mathbb{C}(\overrightarrow{\mathfrak{G}_1})^*}\Box E_{\mathbb{C}(\overrightarrow{\mathfrak{G}_2})^*}, \quad x_2 \in (\mathbb{N}^+(w_2) \cap \mathbb{N}^+(z_2))^*$.
6. $t_k((w_1,w_2)(z_1,w_2)) = [t_{h_1}(w_1) \wedge t_{h_1}(z_1) \wedge t_{h_2}(w_2)] \times \vee_{x_1}\{t_{h_2}(w_2) \wedge t_{\overrightarrow{l_1}}(w_1x_1) \wedge t_{\overrightarrow{l_1}}(z_1x_1)\}$,
$(w_1,w_2)(z_1,w_2) \in E_{\mathbb{C}(\overrightarrow{\mathfrak{G}_1})^*}\Box E_{\mathbb{C}(\overrightarrow{\mathfrak{G}_2})^*}, \quad x_1 \in (\mathbb{N}^+(w_1) \cap \mathbb{N}^+(z_1))^*$.
7. $i_k((w_1,w_2)(z_1,w_2)) = [i_{h_1}(w_1) \wedge i_{h_1}(z_1) \wedge i_{h_2}(w_2)] \times \vee_{x_1}\{i_{h_2}(w_2) \wedge i_{\overrightarrow{l_1}}(w_1x_1) \wedge i_{\overrightarrow{l_1}}(z_1x_1)\}$,
$(w_1,w_2)(z_1,w_2) \in E_{\mathbb{C}(\overrightarrow{\mathfrak{G}_1})^*}\Box E_{\mathbb{C}(\overrightarrow{\mathfrak{G}_2})^*}, \quad x_1 \in (\mathbb{N}^+(w_1) \cap \mathbb{N}^+(z_1))^*$.
8. $f_k((w_1,w_2)(z_1,w_2)) = [f_{h_1}(w_1) \vee f_{h_1}(z_1) \vee f_{h_2}(w_2)] \times \vee_{x_1}\{f_{h_2}(w_2) \vee f_{\overrightarrow{l_1}}(w_1x_1) \vee t_{\overrightarrow{l_1}}(z_1x_1)\}$,
$(w_1,w_2)(z_1,w_2) \in E_{\mathbb{C}(\overrightarrow{\mathfrak{G}_1})^*}\Box E_{\mathbb{C}(\overrightarrow{\mathfrak{G}_2})^*}, \quad x_1 \in (\mathbb{N}^+(w_1) \cap \mathbb{N}^+(z_1))^*$.
9. $t_k((w_1,w_2)(z_1,z_2)) = [t_{h_1}(w_1) \wedge t_{h_1}(z_1) \wedge t_{h_2}(w_2) \wedge t_{h_2}(z_2)] \times [t_{h_1}(w_1) \wedge t_{\overrightarrow{l_1}}(z_1w_1) \wedge t_{h_2}(z_2) \wedge t_{\overrightarrow{l_2}}(w_2z_2)]$,
$(w_1,z_1)(w_2,z_2) \in E^{\Box}$.
10. $i_k((w_1,w_2)(z_1,z_2)) = [i_{h_1}(w_1) \wedge i_{h_1}(z_1) \wedge i_{h_2}(w_2) \wedge i_{h_2}(z_2)] \times [i_{h_1}(w_1) \wedge i_{\overrightarrow{l_1}}(z_1w_1) \wedge i_{h_2}(z_2) \wedge i_{\overrightarrow{l_2}}(w_2z_2)]$,
$(w_1,z_1)(w_2,z_2) \in E^{\Box}$.
11. $f_k((w_1,w_2)(z_1,z_2)) = [f_{h_1}(w_1) \vee f_{h_1}(z_1) \vee f_{h_2}(w_2) \vee f_{h_2}(z_2)] \times [f_{h_1}(w_1) \vee f_{\overrightarrow{l_1}}(z_1w_1) \vee f_{h_2}(z_2) \vee f_{\overrightarrow{l_2}}(w_2z_2)]$,
$(w_1,z_1)(w_2,z_2) \in E^{\Box}$.

Proof. Using similar arguments as in Theorem 2.1. [39], it can be proven. □

Example 4. *Consider $\overrightarrow{\mathfrak{G}_1} = (X_1, h_1, l_1)$ and $\overrightarrow{\mathfrak{G}_2} = (X_2, h_2, l_2)$ to be two IN-digraphs, respectively, as shown in Figure 6. The intuitionistic neutrosophic out and in-neighborhoods of $\overrightarrow{\mathfrak{G}_1}$ and $\overrightarrow{\mathfrak{G}_2}$ are given in Tables 3 and 4. The INC-graphs $\mathbb{C}(\overrightarrow{\mathfrak{G}_1})$ and $\mathbb{C}(\overrightarrow{\mathfrak{G}_2})$ are given in Figure 7.*

Table 3. IN-out and in-neighborhoods of $\overrightarrow{\mathfrak{G}_1}$.

$w \in X_1$	$\mathbb{N}^+(w)$	$\mathbb{N}^-(w)$
w_1	$\{w_2(0.2,0.2,0.3)\}$	$\emptyset$
w_2	$\emptyset$	$\{w_1(0.2,0.2,0.3), w_3(0.3,0.1,0.1)\}$
w_3	$\{w_2(0.3,0.2,0.1)\}$	$\{w_4(0.3,0.1,0.1)\}$
w_4	$\{w_3(0.3,0.1,0.1)\}$	$\emptyset$

Table 4. IN-out and in-neighborhoods of $\overrightarrow{\mathfrak{G}_2}$.

$w \in X_2$	$\mathbb{N}^+(w)$	$\mathbb{N}^-(w)$
z_1	$\{z_3(0.3,0.2,0.2)\}$	$\emptyset$
z_2	$\{z_3(0.3,0.1,0.1)\}$	$\emptyset$
z_3	$\emptyset$	$\{z_1(0.3,0.2,0.2), z_2(0.3,0.1,0.1)\}$

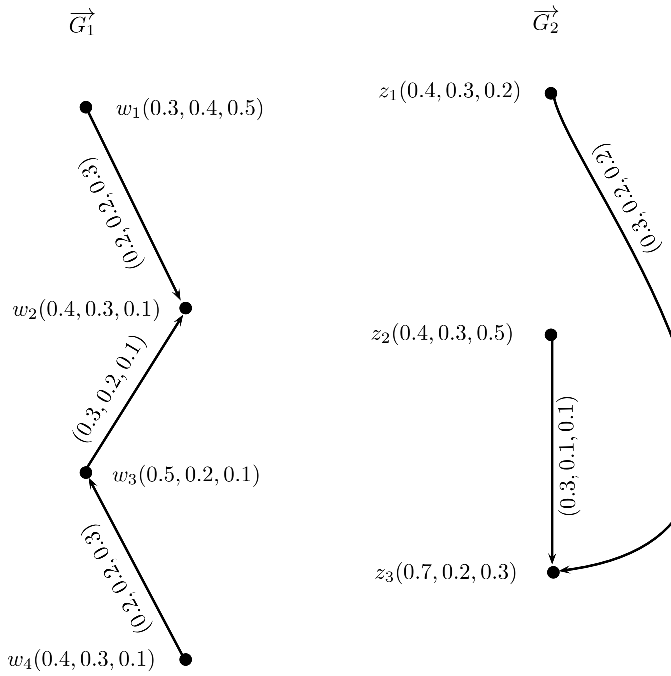

Figure 6. IN-digraphs.

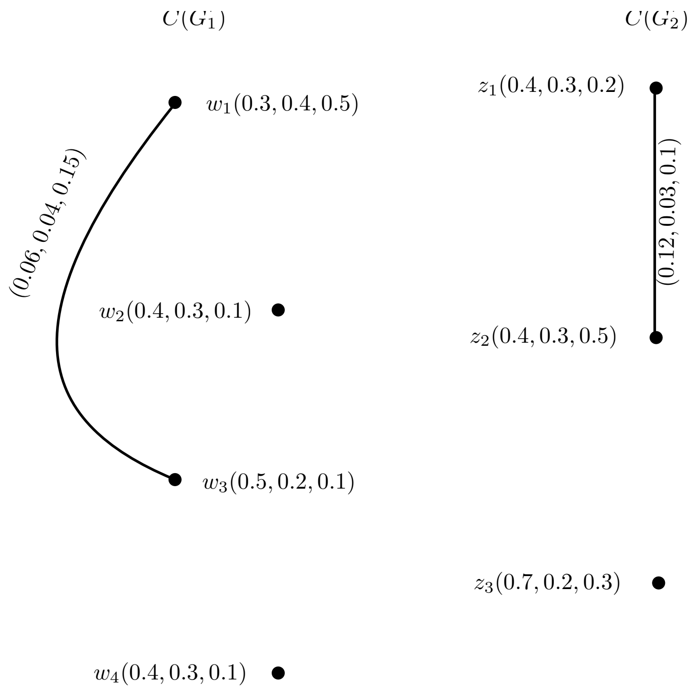

Figure 7. INC-graphs of $\overrightarrow{\mathfrak{G}_1}$ and $\overrightarrow{\mathfrak{G}_2}$.

We now construct the INC-graph $\mathfrak{G}_{\mathbb{C}(\overrightarrow{\mathfrak{G}_1})^* \square \mathbb{C}(\overrightarrow{\mathfrak{G}_2})^*} \cup \mathfrak{G}^{\square} = (w, k)$*, where* $w = (t_w, i_w, f_w)$ *and* $k = (t_k, i_k, f_k)$*, from* $\mathbb{C}(\overrightarrow{\mathfrak{G}_1})^*$ *and* $\mathbb{C}(\overrightarrow{\mathfrak{G}_2})^*$ *using Theorem* 2.14*. We obtained two sets of edges by using Condition* (1)*.*

$$
\begin{aligned}
E_{\mathbb{C}(\overrightarrow{\mathfrak{G}_1})^*} \Box E_{\mathbb{C}(\overrightarrow{\mathfrak{G}_2})^*} = & \{(w_1,z_1)(w_1,z_2),(w_2,z_1)(w_2,z_2),(w_3,z_1)(w_3,z_2),\\
& (w_4,z_1)(w_4,z_2),(w_1,z_1)(w_3,z_1),\\
& (w_1,z_2)(w_3,z_2),(w_1,z_3)(w_3,z_3)\},\\
E^{\Box} = & \{(w_2,z_1)(w_1,z_3),(w_2,z_1)(w_3,z_3),(w_2,z_2)(w_1,z_3)\\
& (w_2,z_2)(w_3,z_3),(w_3,z_1)(w_4,z_3),(w_3,z_2)(w_4,z_3)\}.
\end{aligned}
$$

The truth-membership, indeterminacy-membership and falsity-membership of edges can be calculated by using Conditions (3) *to* (11) *as,*

$$
\begin{aligned}
k((w_1,z_1)(w_1,z_2)) = & (t_{h_1}(w_1) \wedge t_{h_2}(z_1) \wedge t_{h_2}(z_2), \quad i_{h_1}(w_1) \wedge i_{h_2}(z_1) \wedge i_{h_2}(z_2), \quad f_{h_1}(w_1) \vee f_{h_2}(z_1) \vee f_{h_2}(z_2))\\
& \times (t_{h_1}(w_1) \wedge t_{l_2}(z_1z_3) \wedge t_{l_2}(z_2z_3), \quad i_{h_1}(w_1) \wedge i_{l_2}(z_1z_3) \wedge i_{l_2}(z_2z_3),\\
& f_{h_1}(w_1) \vee f_{l_2}(z_1z_3) \vee f_{l_2}(z_2z_3)\\
= & (0.3,0.3,0.5) \times (0.3,0.1,0.5)\\
= & (0.09,0.03,0.25),\\
k((w_2,z_1)(w_1,z_3)) = & (t_{h_1}(w_2) \wedge t_{h_2}(z_1) \wedge t_{h_1}(w_1) \wedge t_{h_2}(z_3), \quad i_{h_1}(w_2) \wedge i_{h_2}(z_1) \wedge i_{h_1}(w_1) \wedge i_{h_2}(z_3),\\
& f_{h_1}(w_2) \vee f_{h_2}(z_1) \vee f_{h_1}(w_1) \vee f_{h_2}(z_3))\\
& \times (t_{h_1}(w_2) \wedge t_{l_1}(w_1w_2) \wedge t_{l_2}(z_3) \wedge t_{l_2}(z_1z_3), \quad i_{h_1}(w_2) \wedge i_{l_1}(w_1w_2) \wedge i_{l_2}(z_3) \wedge i_{l_2}(z_1z_3),\\
& f_{h_1}(w_2) \vee f_{l_1}(w_1w_2) \vee f_{l_2}(z_3) \vee f_{l_2}(z_1z_3))\\
= & (0.3,0.2,0.5) \times (0.2,0.2,0.3)\\
= & (0.06,0.04,0.15).
\end{aligned}
$$

All the truth-membership, indeterminacy-membership and falsity-membership degrees of adjacent edges of $\mathfrak{G}_{\mathbb{C}(\overrightarrow{\mathfrak{G}_1})^* \Box \mathbb{C}(\overrightarrow{\mathfrak{G}_2})^*}$ *and* $\mathfrak{G}^{\Box}$ *are given in Table 5.*

Table 5. Adjacent edges of $\mathfrak{G}_{\mathbb{C}(\overrightarrow{\mathfrak{G}_1})^* \Box \mathbb{C}(\overrightarrow{\mathfrak{G}_2})^*} \cup \mathfrak{G}^{\Box}$.

$(w, \acute{w})\ (z, \acute{z})$	$k\ (w, \acute{w})\ (z, \acute{z})$
$(w_1,z_1)(w_1,z_2)$	(0.09, 0.03, 0.25)
$(w_2,z_1)(w_2,z_2)$	(0.12, 0.03, 0.1)
$(w_3,z_1)(w_3,z_2)$	(0.12, 0.02, 0.1)
$(w_4,z_1)(w_4,z_2)$	(0.12, 0.03, 0.1)
$(w_1,z_1)(w_3,z_1)$	(0.06, 0.04, 0.15)
$(w_1,z_3)(w_3,z_3)$	(0.06, 0.04, 0.15)
$(w_2,z_1)(w_1,z_3)$	(0.06, 0.04, 0.15)
$(w_2,z_1)(w_3,z_3)$	(0.12, 0.04, 0.09)
$(w_2,z_2)(w_1,z_3)$	(0.06, 0.02, 0.15)
$(w_2,z_2)(w_3,z_3)$	(0.12, 0.02, 0.15)
$(w_3,z_1)(w_4,z_3)$	(0.12, 0.02, 0.09)
$(w_3,z_2)(w_4,z_3)$	(0.12, 0.02, 0.15)
$(w_1,z_2)(w_3,z_2)$	(0.06, 0.04, 0.25)

The INC-graph obtained by using this method is given in Figure 8 where solid lines indicate part of INC-graph obtained from $\mathfrak{G}_{\mathbb{C}(\overrightarrow{\mathfrak{G}_1})^* \Box \mathbb{C}(\overrightarrow{\mathfrak{G}_2})^*}$*, and the dotted lines indicate the part of* $\mathfrak{G}^{\Box}$*.*

The Cartesian product $\overrightarrow{\mathfrak{G}_1} \Box \overrightarrow{\mathfrak{G}_2}$ *of IN-digraphs* $\overrightarrow{\mathfrak{G}_1}$ *and* $\overrightarrow{\mathfrak{G}_2}$ *is shown in Figure 9. The IN-out-neighborhoods of* $\overrightarrow{\mathfrak{G}_1} \Box \overrightarrow{\mathfrak{G}_2}$ *are calculated in Table 6. The INC-graphs of* $\overrightarrow{\mathfrak{G}_1} \Box \overrightarrow{\mathfrak{G}_2}$ *are shown in Figure 10.*

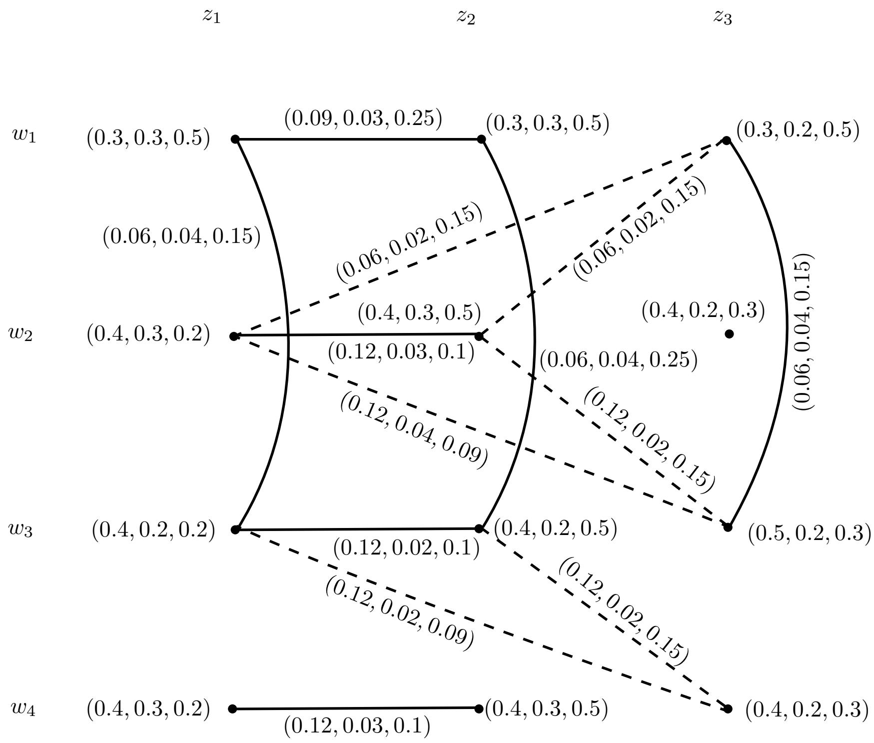

Figure 8. $\mathfrak{G}_{\mathbb{C}(\overrightarrow{\mathfrak{G}_1})^*\square\mathbb{C}(\overrightarrow{\mathfrak{G}_2})^*} \cup \mathfrak{G}^{\square}$.

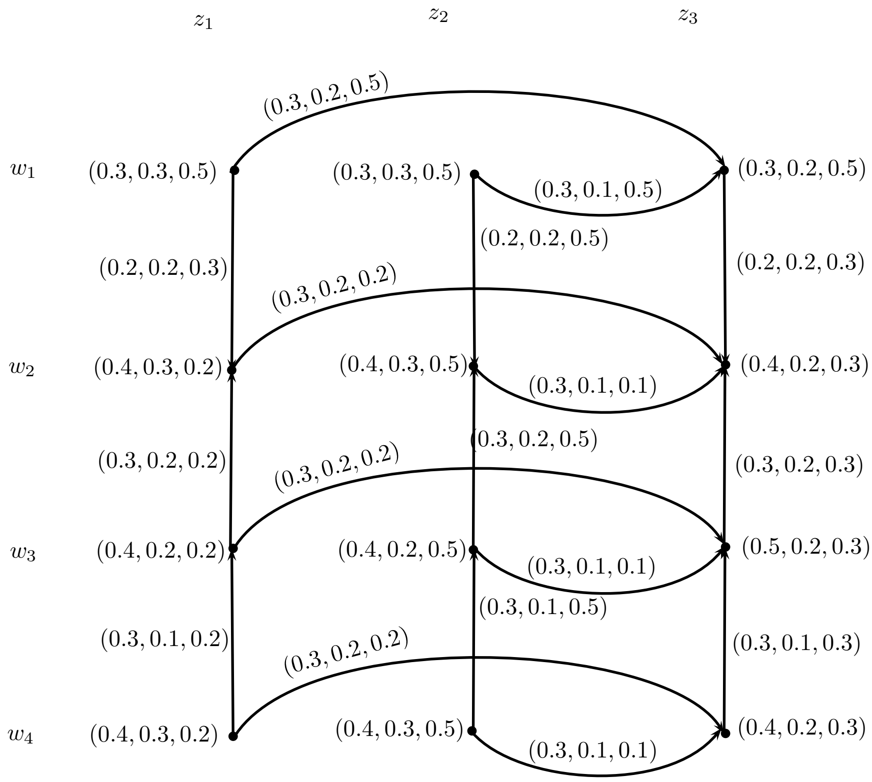

Figure 9. $\overrightarrow{\mathfrak{G}_1}\square\overrightarrow{\mathfrak{G}_2}$.

Table 6. IN-out-neighborhoods of $\overrightarrow{\mathfrak{G}_1}\Box\overrightarrow{\mathfrak{G}_2}$.

(w, z)	$\mathbb{N}^+(w, z)$
(w_1, z_1)	$\{((w_2, z_1), 0.2, 0.2, 0.3), ((w_1, z_3), 0.3, 0.2, 0.5)\}$
(w_1, z_2)	$\{((w_1, z_3), 0.3, 0.1, 0.5), ((w_2, z_2), 0.2, 0.2, 0.5)\}$
(w_1, z_3)	$\{((w_2, z_3),0.2, 0.2, 0.3)\}$
(w_2, z_1)	$\{((w_2, z_3),0.3, 0.2, 0.2)\}$
(w_2, z_2)	$\{((w_2, z_3),0.3, 0.1, 0.1)\}$
(w_2, z_3)	$\emptyset$
(w_3, z_1)	$\{((w_3, z_3), 0.3, 0.2, 0.2), ((w_2, z_1), 0.3, 0.2, 0.2)\}$
(w_3, z_2)	$\{((w_2, z_2), 0.3, 0.2, 0.5), ((w_3, z_3), 0.3, 0.1, 0.1)\}$
(w_3, z_3)	$\{((w_2, z_3),0.3, 0.2, 0.3)\}$
(w_4, z_1)	$\{((w_4, z_3), 0.3, 0.2, 0.2), ((w_3, z_1), 0.3, 0.1, 0.2)\}$
(w_4, z_2)	$\{((w_4, z_3), 0.3, 0.1, 0.1), ((w_3, z_2), 0.3, 0.1, 0.5)\}$
(w_4, z_2)	$\{((w_3, z_3),0.3, 0.1, 0.3)\}$

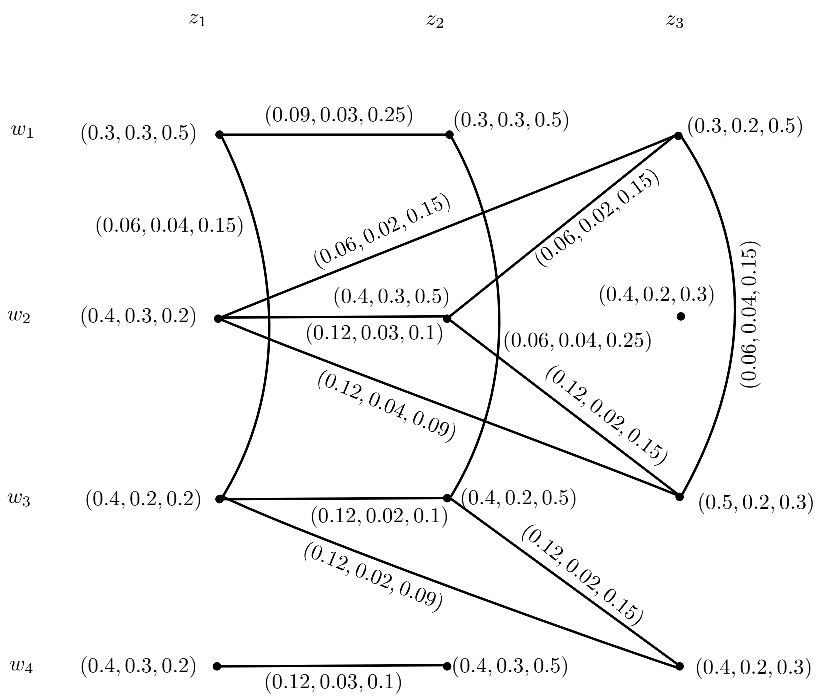

Figure 10. $\mathbb{C}(\overrightarrow{\mathfrak{G}_1}\Box\overrightarrow{\mathfrak{G}_2})$.

It can be seen that $\mathbb{C}(\overrightarrow{\mathfrak{G}_1}\Box\overrightarrow{\mathfrak{G}_2}) \cong \mathfrak{G}_{\mathbb{C}(\overrightarrow{\mathfrak{G}_1})^*\Box\mathbb{C}(\overrightarrow{\mathfrak{G}_2})^*} \cup \mathfrak{G}^{\Box}$ *from Figures 8 and 10.*

Definition 10. *The intuitionistic neutrosophic open-neighborhood of a vertex w of an IN-graph $\mathfrak{G} = (X, h, k)$ is IN-set $\mathbb{N}(w) = (X_w, t_w, i_w, f_w)$, where,*

$$X_w = \{z | k_1(w,z) > 0, k_2(w,z) > 0, k_3(w,z) > 0\},$$

and $t_w : X_w \to [0,1]$ defined by $t_w(z) = k_1(w, z)$, $i_w : X_w \to [0,1]$ defined by $i_w(z) = k_2(w, z)$ and $f_z : X_w \to [0,1]$ defined by $f_w(z) = k_3(w, z)$. For every vertex $w \in X$, the intuitionistic neutrosophic singleton set, $A_w = (w, h'_1, h'_2, h'_3)$, such that: $h'_1 : \{w\} \to [0,1]$, $h'_2 : \{w\} \to [0,1]$, $h'_3 : \{w\} \to [0,1]$ defined by $h'_1(w) = h_1(w)$, $h'_2(w) = h_2(w)$ and $h'_3(w) = h_3(w)$, respectively. The intuitionistic neutrosophic closed-neighborhood of a vertex w is $\mathbb{N}[w] = \mathbb{N}(w) \cup A_w$.

Definition 11. *Suppose* $\mathfrak{G} = (X, h, k)$ *is an IN-graph. The single-valued intuitionistic neutrosophic open-neighborhood graph of* $\mathfrak{G}$ *is an IN-graph* $\mathbb{N}(\mathfrak{G}) = (X, h, k')$*, which has the same intuitionistic neutrosophic set of vertices in* $\mathfrak{G}$ *and has an intuitionistic neutrosophic edge between two vertices* $w, z \in X$ *in* $\mathbb{N}(\mathfrak{G})$ *if and only if* $\mathbb{N}(w) \cap \mathbb{N}(z)$ *is a non-empty IN-set in* $\mathfrak{G}$*. The truth-membership, indeterminacy-membership and falsity-membership values of the edge* (w, z) *are given by:*

$$\begin{aligned}
k_1'(w,z) &= [h_1(w) \wedge h_1(z)]H_1(\mathbb{N}(w) \cap \mathbb{N}(z)),\\
k_2'(w,z) &= [h_2(w) \wedge h_2(z)]H_2(\mathbb{N}(w) \cap \mathbb{N}(z)),\\
k_3'(w,z) &= [h_3(w) \vee h_3(z)]H_3(\mathbb{N}(w) \cap \mathbb{N}(z)), \quad \textit{respectively}.
\end{aligned}$$

Definition 12. *Suppose* $\mathfrak{G} = (X, h, k)$ *is an IN-graph. The single-valued intuitionistic neutrosophic closed-neighborhood graph of* $\mathfrak{G}$ *is an IN-graph* $\mathbb{N}(\mathfrak{G}) = (X, h, k')$*, which has the same intuitionistic neutrosophic set of vertices in* $\mathfrak{G}$ *and has an intuitionistic neutrosophic edge between two vertices* $w, z \in X$ *in* $\mathbb{N}[\mathfrak{G}]$ *if and only if* $\mathbb{N}[w] \cap \mathbb{N}[z]$ *is a non-empty IN-set in* $\mathfrak{G}$*. The truth-membership, indeterminacy-membership and falsity-membership values of the edge* (w, z) *are given by:*

$$\begin{aligned}
k_1'(w,z) &= [h_1(w) \wedge h_1(z)]H_1(\mathbb{N}[w] \cap \mathbb{N}[z]),\\
k_2'(w,z) &= [h_2(w) \wedge h_2(z)]H_2(\mathbb{N}[w] \cap \mathbb{N}[z]),\\
k_3'(w,z) &= [h_3(w) \vee h_3(z)]H_3(\mathbb{N}[w] \cap \mathbb{N}[z]), \quad \textit{respectively}.
\end{aligned}$$

Example 5. *Consider* $G = (X, h, k)$ *to be an IN-graph, such that* $X = \{a, b, c, d\}$, $h = \{(a, 0.5, 0.4, 0.3), (b, 0.6, 0.3, 0.1), (c, 0.7, 0.3, 0.1), (d, 0.5, 0.6, 0.3)\}$*, and* $k = \{(ab, 0.3, 0.2, 0.2), (ad, 0.4, 0.3, 0.2), (bc, 0.5, 0.2, 0.1), (cd, 0.4, 0.2, 0.2)\}$*, as shown in Figure 11. Then, corresponding intuitionistic neutrosophic open and closed-neighborhood graphs are shown in Figure 12.*

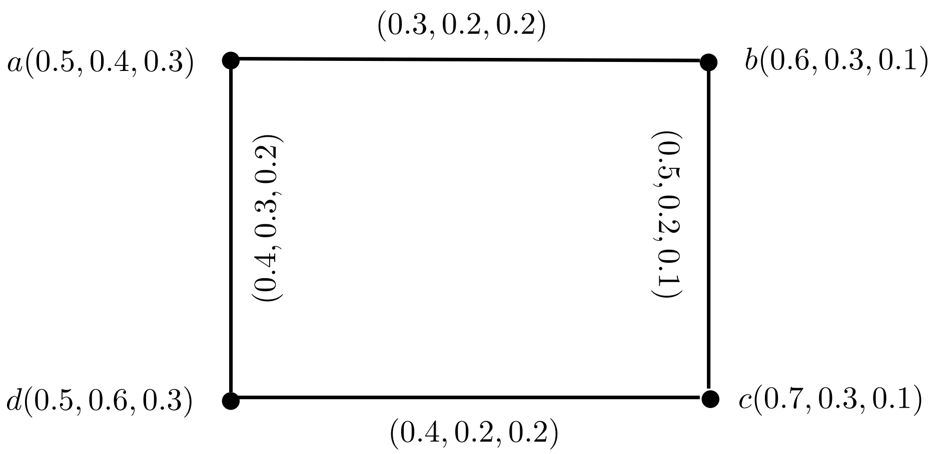

Figure 11. IN-digraph.

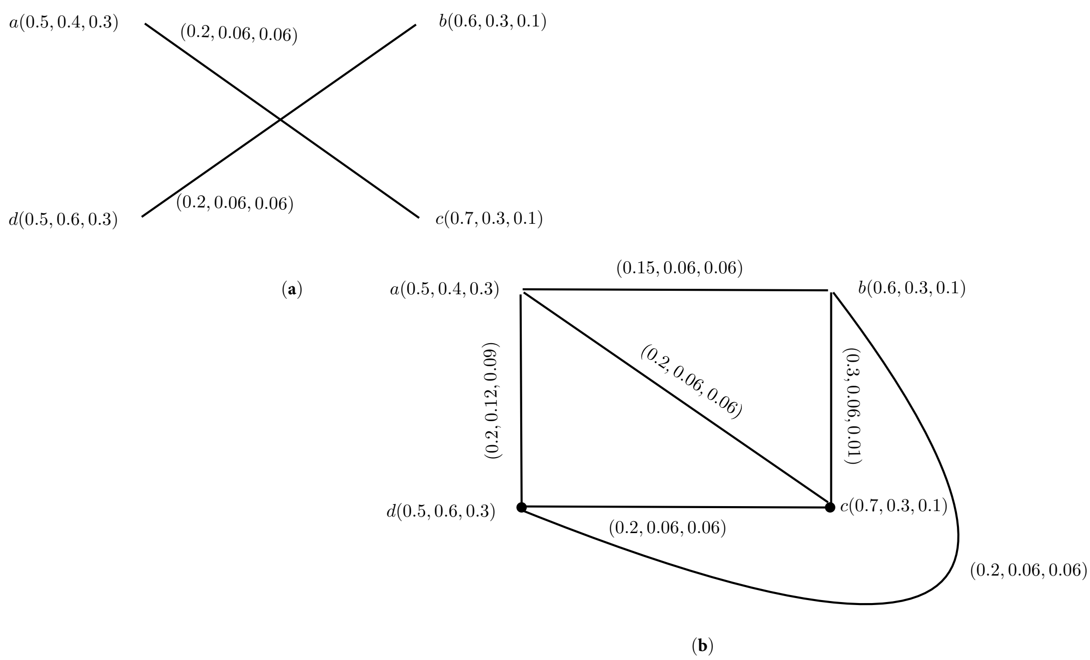

Figure 12. (**a**) $\mathbb{N}(\mathfrak{G})$; (**b**) $\mathbb{N}[\mathfrak{G}]$.

Theorem 4. *For each edge of an IN-graph $\mathfrak{G}$, there exists an edge in $\mathbb{N}[\mathfrak{G}]$.*

Proof. Suppose (w, z) is an edge of an IN-graph $\mathfrak{G} = (V, h, k)$. Suppose $\mathbb{N}[\mathfrak{G}] = (V, h, k')$ is the corresponding closed neighborhood of an IN-graph. Suppose $w, z \in \mathbb{N}[w]$ and $w, z \in \mathbb{N}[z]$. Then, w, $z \in \mathbb{N}[w] \cap \mathbb{N}[z]$. Hence,

$$H_1(\mathbb{N}[w] \cap \mathbb{N}[z]) \neq 0,$$
$$H_2(\mathbb{N}[w] \cap \mathbb{N}[z]) \neq 0,$$
$$H_3(\mathbb{N}[w] \cap \mathbb{N}[z]) \neq 0.$$

Then,

$$k'_1(w,z) = [h_1(w) \wedge h_1(z)]H_1(\mathbb{N}[w] \cap \mathbb{N}[z]) \neq 0,$$
$$k'_2(w,z) = [h_2(w) \wedge h_2(z)]H_2(\mathbb{N}[w] \cap \mathbb{N}[z]) \neq 0,$$
$$k'_3(w,z) = [h_3(w) \vee h_3(z)]H_3(\mathbb{N}[w] \cap \mathbb{N}[z]) \neq 0.$$

Thus, for each edge (w, z) in IN-graph $\mathfrak{G}$, there exists an edge (w, z) in $\mathbb{N}[\mathfrak{G}]$. □

Definition 13. *The support of an IN-set $\check{A} = (w, t_{\check{A}}, i_{\check{A}}, f_{\check{A}})$ in X is the subset $\hat{A}$ of X defined by:*

$$\hat{A} = \{w \in X : t_{\check{A}}(w) \neq 0, i_{\check{A}}(w) \neq 0, f_{\check{A}}(w) \neq 1\}$$

and $|supp(\hat{A})|$ is the number of elements in the set.

We now discuss p-competition intuitionistic neutrosophic graphs.

Suppose p is a positive integer. Then, p-competition IN-graph $\mathbb{C}^p(\overrightarrow{\mathfrak{G}})$ of the IN-digraph $\overrightarrow{\mathfrak{G}} = (X, h, k)$ is an undirected IN-graph $\mathfrak{G} = (X, h, k)$, which has the same intuitionistic neutrosophic set of vertices as in $\overrightarrow{\mathfrak{G}}$ and has an intuitionistic neutrosophic edge between two vertices $w, z \in X$ in

$\mathbb{C}^p(\overrightarrow{\mathfrak{G}})$ if and only if $|supp(\mathbb{N}^+(w) \cap \mathbb{N}^+(z))| \geq p$. The truth-membership value of edge (w, z) in $\mathbb{C}^p(\overrightarrow{\mathfrak{G}})$ is $t(w, z) = \frac{(i-p)+1}{i}[h_1(w) \wedge h_1(z)]H_1(\mathbb{N}^+(w) \cap \mathbb{N}^+(z))$; the indeterminacy-membership value of edge (w, z) in $\mathbb{C}^p(\overrightarrow{\mathfrak{G}})$ is $i(w, z) = \frac{(i-p)+1}{i}[h_2(w) \wedge h_2(z)]H_2(\mathbb{N}^+(w) \cap \mathbb{N}^+(z))$; and the falsity-membership value of edge (w, z) in $\mathbb{C}^p(\overrightarrow{\mathfrak{G}})$ is $f(w, z) = \frac{(i-p)+1}{i}[h_3(w) \vee h_3(z)]H_3(\mathbb{N}^+(w) \cap \mathbb{N}^+(z))$ where $i = |supp(\mathbb{N}^+(w) \cap \mathbb{N}^+(z))|$.

The three-competition IN-graph is illustrated by the following example.

Example 6. *Consider* $\overrightarrow{\mathfrak{G}} = (X, h, k)$ *to be an IN-digraph, such that* $X = \{w_1, w_2, w_3, z_1, z_2, z_3\}$, $h = \{(w_1, 0.5, 0.1, 0.2), (w_2, 0.1, 0.6, 0.3), (w_3, 0.1, 0.2, 0.5), (z_1, 0.7, 0.2, 0.1), (z_2, 0.5, 0.2, 0.3), (z_3, 0.3, 0.7, 0.2)\}$ *and* $k = \{(\overrightarrow{(w_1, z_1)}, 0.4, 0.1, 0.1), (\overrightarrow{(w_1, z_2)}, 0.5, 0.1, 0.3), (\overrightarrow{(w_1, z_3)}, 0.2, 0.1, 0.1), (\overrightarrow{(w_2, z_1)}, 0.1, 0.1, 0.2), (\overrightarrow{(w_2, z_2)}, 0.1, 0.1, 0.2), (\overrightarrow{(w_2, z_3)}, 0.1, 0.5, 0.2), (\overrightarrow{(w_3, z_1)}, 0.1, 0.1, 0.1)\ (\overrightarrow{(w_3, z_2)}, 0.1, 0.1, 0.2)\}$, *as shown in Figure 13. Then,* $\mathbb{N}^+(w_1) = \{(z_1, 0.4, 0.1, 0.1), (z_2, 0.5, 0.1, 0.3), (z_3, 0.2, 0.1, 0.1)\}$, $\mathbb{N}^+(w_2) = \{(z_1, 0.1, 0.1, 0.2), (z_2, 0.1, 0.1, 0.2), (z_3, 0.1, 0.5, 0.2)\}$ *and* $\mathbb{N}^+(w_3) = \{(z_1, 0.1, 0.1, 0.1), (z_2, 0.1, 0.1, 0.2)\}$. *Therefore,* $\mathbb{N}^+(w_1) \cap \mathbb{N}^+(w_2) = \{(z_1, 0.1, 0.1, 0.2), (z_2, 0.1, 0.1, 0.3), (z_3, 0.1, 0.1, 0.2)\}$, $\mathbb{N}^+(w_1) \cap \mathbb{N}^+(w_3) = \{(z_1, 0.1, 0.1, 0.1), (z_2, 0.1, 0.1, 0.3)\}$ *and* $\mathbb{N}^+(w_2) \cap \mathbb{N}^+(w_3) = \{(z_1, 0.1, 0.1, 0.2), (z_2, 0.1, 0.1, 0.2)\}$.

Now, $i = |supp(\mathbb{N}^+(w_1) \cap \mathbb{N}^+(w_2))| = 3$. *For* $p = 3$, $t(w_1, w_2) = 0.003$, $i(w_1, w_2) = 0.003$ *and* $f(w_1, w_2) = 0.02$. *As shown in Figure 14.*

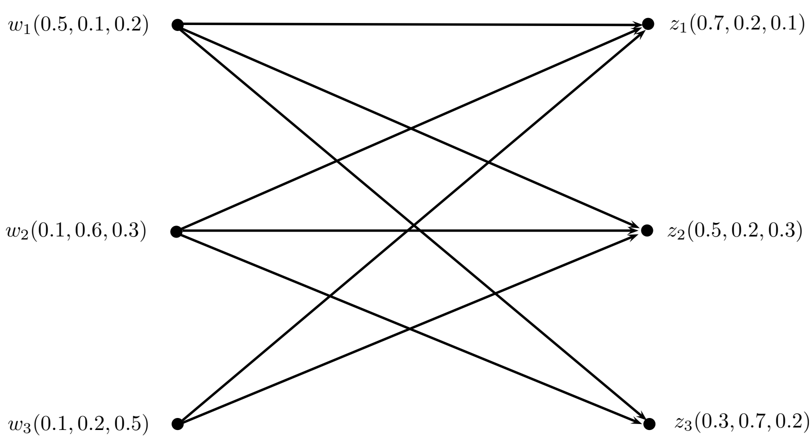

Figure 13. IN-digraph.

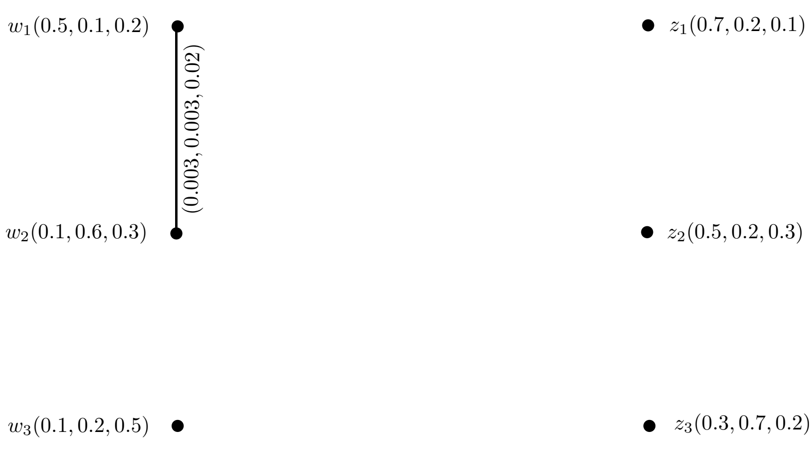

Figure 14. Three-competition IN-graph.

We now define another extension of INC-graph known as the m-step INC-graph.
$\overrightarrow{P}^{m}_{z,w}$: a directed intuitionistic neutrosophic path of length m from z to w.
$\mathbb{N}^{+}_{m}(z)$: single-valued intuitionistic neutrosophic m-step out-neighborhood of vertex z.
$\mathbb{N}^{-}_{m}(z)$: single-valued intuitionistic neutrosophic m-step in-neighborhood of vertex z.
$\mathbb{C}_m(\overrightarrow{\mathfrak{G}})$: m-step INC-graph of the IN-digraph $\overrightarrow{\mathfrak{G}}$.

Definition 14. *Suppose $\overrightarrow{\mathfrak{G}} = (X, h, k)$ is an IN-digraph. The m-step IN-digraph of $\overrightarrow{\mathfrak{G}}$ is denoted by $\overrightarrow{G}_m = (X, h, k)$ where the intuitionistic neutrosophic set of vertices of $\overrightarrow{\mathfrak{G}}$ is the same as the intuitionistic neutrosophic set of vertices of $\overrightarrow{G}_m$ and has an edge between z and w in $\overrightarrow{G}_m$ if and only if there exists an intuitionistic neutrosophic directed path $\overrightarrow{P}^{m}_{z,w}$ in $\overrightarrow{\mathfrak{G}}$.*

Definition 15. *The intuitionistic neutrosophic m-step out-neighborhood of vertex z of an IN-digraph $\overrightarrow{\mathfrak{G}} = (X, h, k)$ is IN-set:*

$$\mathbb{N}^{+}_{m}(z) = (X^{+}_{z}, t^{+}_{z}, i^{+}_{z}, f^{+}_{z}), \quad \textit{where}$$

$X^{+}_{z} = \{w|$ there exists a directed intuitionistic neutrosophic path of length m from z to w, $\overrightarrow{P}^{m}_{z,w}\}$, $t^{+}_{z} : X^{+}_{z} \to [0, 1]$, $i^{+}_{z} : X^{+}_{z} \to [0, 1]$ and $f^{+}_{z} : X^{+}_{z} \to [0, 1]$ are defined by $t^{+}_{z} = min\{t(\overrightarrow{w_1, w_2}), (w_1, w_2)$ is an edge of $\overrightarrow{P}^{m}_{z,w}\}$, $i^{+}_{z} = min\{i(\overrightarrow{w_1, w_2}), (w_1, w_2)$ is an edge of $\overrightarrow{P}^{m}_{z,w}\}$ and $f^{+}_{z} = max\{f(\overrightarrow{w_1, w_2}), (w_1, w_2)$ is an edge of $\overrightarrow{P}^{m}_{z,w}\}$, respectively.

Definition 16. *The intuitionistic neutrosophic m-step in-neighborhood of vertex z of an IN-digraph $\overrightarrow{\mathfrak{G}} = (X, h, k)$ is IN-set:*

$$\mathbb{N}^{-}_{m}(z) = (X^{-}_{z}, t^{-}_{z}, i^{-}_{z}, f^{-}_{z}), \quad \textit{where}$$

$X^{-}_{z} = \{w|$ there exists a directed intuitionistic neutrosophic path of length m from w to z, $\overrightarrow{P}^{m}_{w,z}\}$, $t^{-}_{z} : X^{-}_{z} \to [0, 1]$, $i^{-}_{z} : X^{-}_{z} \to [0, 1]$ and $f^{-}_{z} : X^{-}_{z} \to [0, 1]$ are defined by $t^{-}_{z} = min\{t(\overrightarrow{w_1, w_2}), (w_1, w_2)$ is an edge of $\overrightarrow{P}^{m}_{w,z}\}$, $i^{-}_{z} = min\{i(\overrightarrow{w_1, w_2}), (w_1, w_2)$ is an edge of $\overrightarrow{P}^{m}_{w,z}\}$ and $f^{-}_{z} = max\{f(\overrightarrow{w_1, w_2}), (w_1, w_2)$ is an edge of $\overrightarrow{P}^{m}_{w,z}\}$, respectively.

Definition 17. *Suppose $\overrightarrow{\mathfrak{G}} = (X, h, k)$ is an IN-digraph. The m-step INC-graph of IN-digraph $\overrightarrow{\mathfrak{G}}$ is denoted by $\mathbb{C}_m(\overrightarrow{\mathfrak{G}}) = (X, h, k)$, which has the same intuitionistic neutrosophic set of vertices as in $\overrightarrow{\mathfrak{G}}$ and has an edge between two vertices $w, z \in X$ in $\mathbb{C}_m(\overrightarrow{\mathfrak{G}})$ if and only if $(\mathbb{N}^{+}_{m}(w) \cap \mathbb{N}^{+}_{m}(z))$ is a non-empty IN-set in $\overrightarrow{\mathfrak{G}}$. The truth-membership value of edge (w, z) in $\mathbb{C}_m(\overrightarrow{\mathfrak{G}})$ is $t(w, z) = [h_1(w) \wedge h_1(z)]H_1(\mathbb{N}^{+}_{m}(w) \cap \mathbb{N}^{+}_{m}(z))$; the indeterminacy-membership value of edge (w, z) in $\mathbb{C}_m(\overrightarrow{\mathfrak{G}})$ is $i(w, z) = [h_2(w) \wedge h_2(z)]H_2(\mathbb{N}^{+}_{m}(w) \cap \mathbb{N}^{+}_{m}(z))$; and the falsity-membership value of edge (w, z) in $\mathbb{C}_m(\overrightarrow{\mathfrak{G}})$ is $f(w, z) = [h_3(w) \vee h_3(z)]H_3(\mathbb{N}^{+}_{m}(w) \cap \mathbb{N}^{+}_{m}(z))$.*

The two-step INC-graph is illustrated by the following example.

Example 7. *Consider $\overrightarrow{\mathfrak{G}} = (X, h, k)$ is an IN-digraph, such that, $X = \{w_1, w_2, z_1, z_2, z_3\}$, $h = \{(w_1,$ 0.3, 0.4, 0.6), $(w_2$, 0.2, 0.5, 0.3), $(z_1$, 0.4, 0.2, 0.3), $(z_2$, 0.7, 0.2, 0.1), $(z_3$, 0.5, 0.1, 0.2), $(z_4$, 0.6, 0.3, 0.2)$\}$, and $k = \{((\overrightarrow{w_1, z_1})$, 0.2, 0.1, 0.2), $((\overrightarrow{w_2, z_4})$, 0.1, 0.2, 0.3), $((\overrightarrow{z_1, z_3})$, 0.3, 0.1, 0.2), $((\overrightarrow{z_1, z_2})$, 0.3, 0.1, 0.2), $((\overrightarrow{z_4, z_2})$, 0.2, 0.1, 0.1), and $((\overrightarrow{z_4, z_3})$, 0.4, 0.1, 0.4)$\}$, as shown in Figure 15.*

Then, $\mathbb{N}^{+}_{2}(w_1) = \{(z_2$, 0.2, 0.1, 0.2), $(z_3$, 0.2, 0.1, 0.2)$\}$ and $\mathbb{N}^{+}_{2}(w_2) = \{(z_2$, 0.1, 0.1, 0.3), $(z_3$, 0.1, 0.1, 0.4)$\}$. Therefore, $\mathbb{N}^{+}_{2}(w_1) \cap \mathbb{N}^{+}_{2}(w_2) = \{(z_2$, 0.1, 0.1, 0.3), $(z_3$, 0.1, 0.1, 0.4)$\}$. Thus, $t(w_1, w_2) = 0.02$, $i(w_1, w_2) = 0.04$ and $f(w_1, w_2) = 0.18$. This is shown in Figure 16.

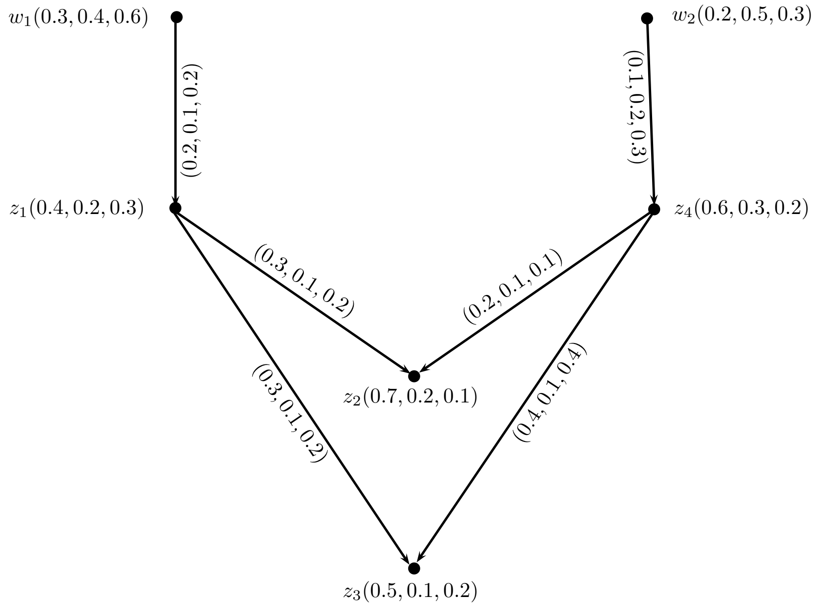

Figure 15. IN-digraph.

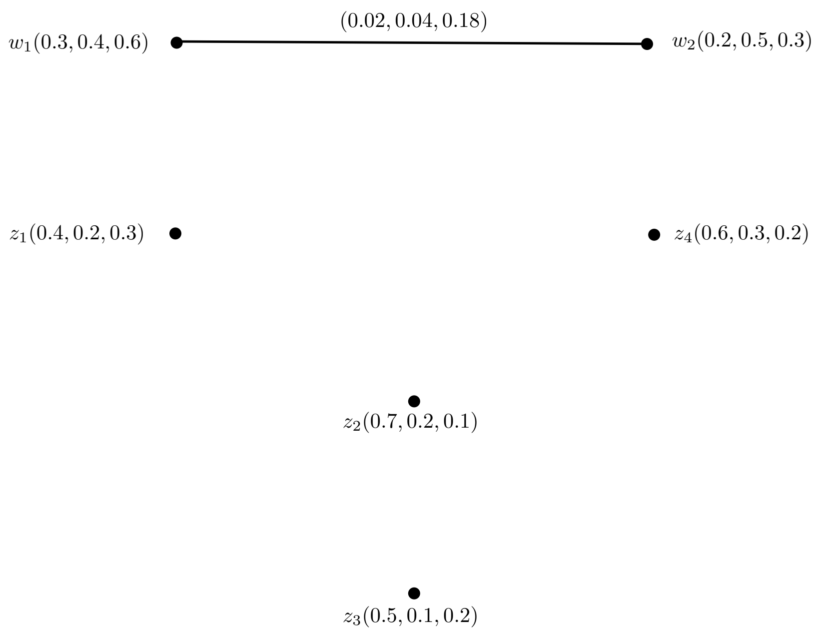

Figure 16. Two-step INC-graph.

Definition 18. *The intuitionistic neutrosophic m-step out-neighborhood of vertex z of an IN-digraph* $\overrightarrow{\mathfrak{G}} = (X, h, k)$ *is IN-set:*

$$\mathbb{N}_m(z) = (X_z, t_z, i_z, f_z), \quad where$$

$X_z = \{w|$ *there exists a directed intuitionistic neutrosophic path of length m from z to w,* $P^m_{z,w}\}$, $t_z : X_z \to [0, 1]$, $i_z : X_z \to [0, 1]$ *and* $f_z : X_z \to [0, 1]$ *are defined by* $t_z = min\{t(w_1, w_2), (w_1, w_2)$ *is an edge of* $P^m_{z,w}\}$, $i_z = min\{i(w_1, w_2), (w_1, w_2)$ *is an edge of* $P^m_{z,w}\}$ *and* $f_z = max\{f(w_1, w_2), (w_1, w_2)$ *is an edge of* $P^m_{z,w}\}$*, respectively.*

Definition 19. *Suppose $\mathfrak{G} = (X, h, k)$ is an IN-graph. Then, the m-step intuitionistic neutrosophic neighborhood graph (IN-neighborhood-graph) $\mathbb{N}_m(\mathfrak{G})$ is defined by $\mathbb{N}_m(\mathfrak{G}) = (X, h, \kappa)$, where $h = (h_1, h_2, h_3)$, $\kappa = (\kappa_1, \kappa_2, \kappa_3)$, $\kappa_1 : X \times X \to [0, 1]$, $\kappa_2 : X \times X \to [0, 1]$ and $\kappa_3 : X \times X \to [0, 1]$ are such that:*

$$\begin{aligned}
\kappa_1(w,z) &= h_1(w) \wedge h_1(z) H_1(\mathbb{N}_m(w) \cap \mathbb{N}_m(z)), \\
\kappa_2(w,z) &= h_2(w) \wedge h_2(z) H_2(\mathbb{N}_m(w) \cap \mathbb{N}_m(z)), \\
\kappa_3(w,z) &= h_3(w) \vee h_3(z) H_3(\mathbb{N}_m(w) \cap \mathbb{N}_m(z)), \quad \textit{respectively.}
\end{aligned}$$

Theorem 5. *If all the edges of IN-digraph $\overrightarrow{\mathfrak{G}} = (X, h, k)$ are independent strong, then all the edges of $\mathbb{C}_m(\overrightarrow{\mathfrak{G}})$ are independent strong.*

Proof. Suppose $\overrightarrow{\mathfrak{G}} = (X, h, k)$ is an IN-digraph and $\mathbb{C}_m(\overrightarrow{\mathfrak{G}}) = (X, h, k)$ is the corresponding m-step INC-graph. Since all the edges of $\overrightarrow{\mathfrak{G}}$ are independent strong, then $H_1(\mathbb{N}^+_m(w) \cap \mathbb{N}^+_m(z)) > 0.5$, $H_2(\mathbb{N}^+_m(w) \cap \mathbb{N}^+_m(z)) < 0.5$ and $H_3(\mathbb{N}^+_m(w) \cap \mathbb{N}^+_m(z)) < 0.5$. Then, $t(w,z) = (h_1(w) \wedge h_1(z))H_1(\mathbb{N}^+_m(w) \cap \mathbb{N}^+_m(z))$, or $t(w,z) > 0.5(h_1(w) \wedge h_1(z))$, or $\frac{t(w,z)}{(h_1(w) \wedge h_1(z))} > 0.5$, $i(w,z) = (h_2(w) \wedge h_2(z))H_2(\mathbb{N}^+_m(w) \cap \mathbb{N}^+_m(z))$, or $i(w,z) < 0.5(h_2(w) \wedge h_2(z))$, or $\frac{i(w,z)}{(h_2(w) \wedge h_2(z))} < 0.5$ and $f(w,z) = (h_3(w) \vee h_3(z))H_3(\mathbb{N}^+_m(w) \cap \mathbb{N}^+_m(z))$, or $f(w,z) < 0.5(h_3(w) \vee h_3(z))$, or $\frac{f(w,z)}{(h_3(w) \vee h_3(z))} < 0.5$.

Hence, the edge (w, z) is independent strong in $\mathbb{C}_m(\overrightarrow{\mathfrak{G}})$. Since, (w, z) is taken to be the arbitrary edge of $\mathbb{C}_m(\overrightarrow{\mathfrak{G}})$, thus all the edges of $\mathbb{C}_m(\overrightarrow{\mathfrak{G}})$ are independent strong. □

3. Applications

Competition graphs are very important to represent the competition between objects. However, still, these representations are unsuccessful to deal with all the competitions of world; for that purpose, INC-graphs are introduced. Now, we discuss the applications of INC-graphs to study the competition along with algorithms. The INC-graphs have many utilizations in different areas.

3.1. Ecosystem

Consider a small ecosystem: human eats trout; bald eagle eats trout and salamander; trout eats phytoplankton, mayfly and dragonfly; salamander eats dragonfly and mayfly; snake eats salamander and frog; frog eats dragonfly and mayfly; mayfly eats phytoplankton; dragonfly eats phytoplankton. These nine species human, bald eagle, salamander, snake, frog, dragonfly, trout, mayfly and phytoplankton are taken as vertices. Let the degree of existence in the ecosystem of human be 60%, the degree of indeterminacy of existence be 30% and the degree of false-existence be 10%, i.e., the truth-membership, indeterminacy-membership and falsity-membership values of the vertex human are (0.6, 0.3, 0.1). Similarly, we assume the truth-membership, indeterminacy-membership and falsity-membership values of other vertices as (0.7, 0.3, 0.2), (0.4, 0.3, 0.5), (0.3, 0.5, 0.1), (0.3, 0.4, 0.5), (0.3, 0.5, 0.2), (0.7, 0.3, 0.2), (0.6, 0.4, 0.2) and (0.3, 0.5, 0.2). Suppose that human likes to eat trout 20%, indeterminate to eat 10% and dislike to eat, say 10%. The likeness, indeterminacy and dislikeness of preys for predators are shown in Table 7.

It is clear that if trout is removed from the food cycle, then human must be lifeless, and in such a situation bald eagle, phytoplankton, dragonfly and mayfly grow in an undisciplined manner. Thus, we can evaluate the food cycle with the help of INC-graphs.

Table 7. Likeness, indeterminacy and dislikeness of preys and predators.

Name of Predator	Name of Prey	Like to Eat	Indeterminate to Eat	Dislike to Eat
Human	*Trout*	20	10	10
Bald eagle	*Trout*	20	20	20
Bald eagle	*Salamander*	30	20	30
Snake	*Salamander*	20	20	10
Snake	*Frog*	30	20	40
Salamander	*Dragonfly*	20	20	20
Salamander	*Mayfly*	20	20	40
Frog	*Dragonfly*	30	30	30
Trout	*Dragonfly*	20	40	10
Trout	*Mayfly*	30	10	10
Trout	*Phytoplankton*	20	10	10
Dragonfly	*Phytoplankton*	10	10	10
Mayfly	*Phytoplankton*	30	30	20
Frog	*Mayfly*	10	10	10

For this food web Figure 17, we have the following Table 8 of IN-out-neighborhoods.

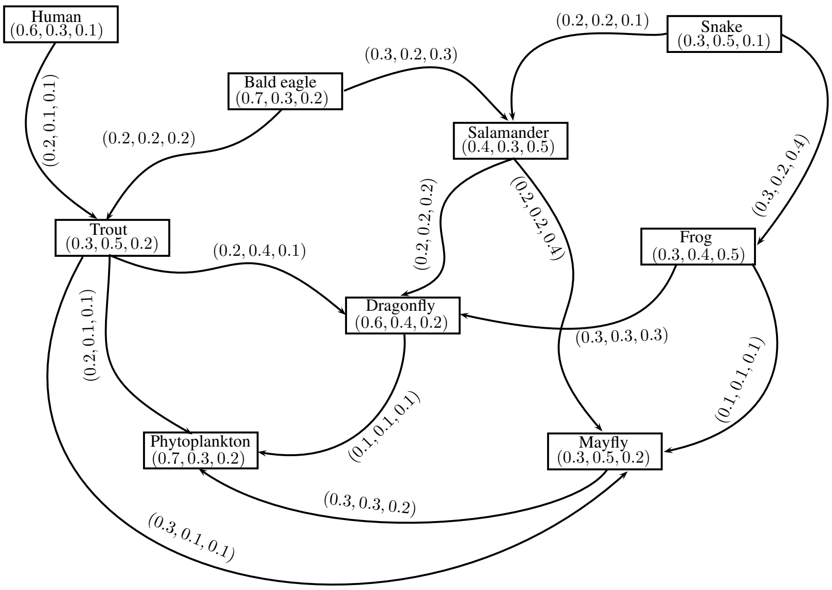

Figure 17. IN-food web.

Table 8. IN-out-neighborhoods.

$w \in X$	$\mathbb{N}^+(w)$
Human	$\{(Trout, 0.2, 0.1, 0.1)\}$
Bald eagle	$\{(Trout, 0.2, 0.2, 0.2), (Salamander, 0.3, 0.2, 0.3)\}$
Salamander	$\{(Dragonfly, 0.2, 0.2, 0.2), (Mayfly, 0.2, 0.2, 0.4)\}$
Snake	$\{(Salamander, 0.2, 0.2, 0.1), (Frog, 0.3, 0.2, 0.4)\}$
Frog	$\{(Dragonfly, 0.3, 0.3, 0.3), (Mayfly, 0.1, 0.1, 0.1)\}$
Mayfly	$\{(Phytoplankton, 0.3, 0.3, 0.2)\}$
Phytoplankton	$\emptyset$
Dragonfly	$\{(Phytoplankton, 0.1, 0.1, 0.1)\}$
Trout	$\{(Phytoplankton, 0.2, 0.1, 0.1), (Mayfly, 0.3, 0.1, 0.1), (Dragonfly, 0.2, 0.4, 0.1)\}$

Therefore, $\mathbb{N}^+(Human \cap Bald\ \ eagle) = \{(Trout,\ 0.2,\ 0.1,\ 0.2)\}$, $\mathbb{N}^+(Bald\ \ eagle \cap Snake) = \{(Salamander,\ 0.2,\ 0.2,\ 0.3)\}$, $\mathbb{N}^+(Salamander \cap Frog) = \{(Dragonfly,\ 0.2,\ 0.2,\ 0.3),\ (Mayfly,\ 0.1,\ 0.1,\ 0.4)\}$, $\mathbb{N}^+(Salamander \cap Trout) = \{(Dragonfly, 0.2, 0.2, 0.2), (Mayfly, 0.2, 0.1, 0.4)\}$, $\mathbb{N}^+(Trout \cap Frog) = \{(Dragonfly,\ 0.2,\ 0.3,\ 0.3),\ (Mayfly,\ 0.1,\ 0.1,\ 0.1)\}$, $\mathbb{N}^+(Mayfly \cap Trout) = \{(Phytoplankton,\ 0.2,\ 0.1,\ 0.2)\}$, $\mathbb{N}^+(Mayfly \cap Dragonfly) = \{(Phytoplankton,\ 0.1,\ 0.1,\ 0.2)\}$ and $\mathbb{N}^+(Dragonfly \cap Trout) = \{(Phytoplankton, 0.1, 0.1, 0.1)\}$.

Now, there is an edge between human and bald eagle; snake and bald eagle; salamander and trout; salamander and frog; trout and frog; trout and dragonfly; trout and mayfly; dragonfly and mayfly in the INC-graph, which highlights the competition between them; and for the other pair of species, there is no edge, which indicates that there is no competition in the INC-graph Figure 18. For example, there is an edge between human and bald eagle indicating a 12% degree of likeness to prey on the same species, a 3% degree of indeterminacy and a 4% degree of non-likeness between them.

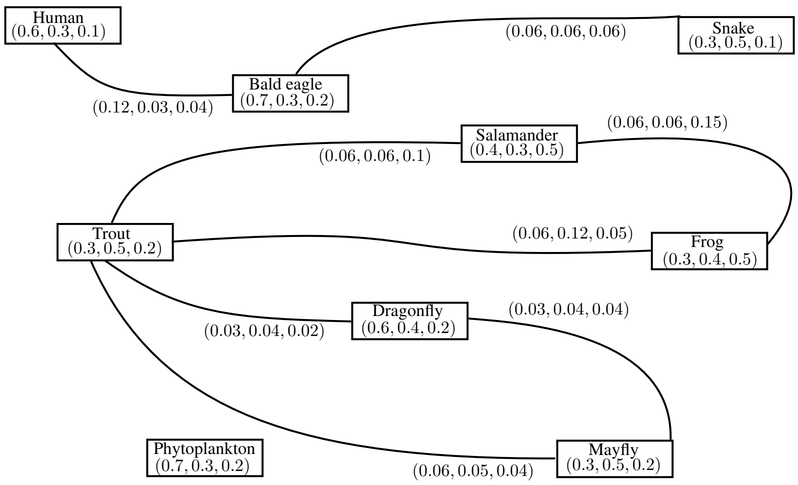

Figure 18. Corresponding INC-graph

We present our method, which is used in our ecosystem application in Algorithm 1.

Algorithm 1: Ecosystem.

Step 1. Input the truth-membership, indeterminacy-membership and falsity-membership values for set of n species.

Step 2. If for any two distinct vertices w_i and w_j, $t(w_iw_j) > 0$, $i(w_iw_j) > 0$, $f(w_iw_j) > 0$, then

$$(w_j, t(w_iw_j), i(w_iw_j), f(w_iw_j)) \in \mathbb{N}^+(w_i).$$

Step 3. Repeat Step 2 for all vertices w_i and w_j to calculate IN-out-neighborhoods $\mathbb{N}^+(w_i)$.

Step 4. Calculate $\mathbb{N}^+(w_i) \cap \mathbb{N}^+(w_j)$ for each pair of distinct vertices w_i and w_j.

Step 5. Calculate $H[\mathbb{N}^+(w_i) \cap \mathbb{N}^+(w_j)]$.

Step 6. If $\mathbb{N}^+(w_i) \cap \mathbb{N}^+(w_j) \neq \emptyset$, then draw an edge w_iw_j.

Step 7. Repeat Step 6 for all pairs of distinct vertices.

Step 8. Assign membership values to each edge w_iw_j using the conditions:

$$\begin{aligned} t(w_iw_j) &= (w_i \wedge w_j)H_1[\mathbb{N}^+(w_i) \cap \mathbb{N}^+(w_j)] \\ i(w_iw_j) &= (w_i \wedge w_j)H_2[\mathbb{N}^+(w_i) \cap \mathbb{N}^+(w_j)] \\ f(w_iw_j) &= (w_i \vee w_j)H_3[\mathbb{N}^+(w_i) \cap \mathbb{N}^+(w_j)]. \end{aligned}$$

3.2. Career Competition

Consider the IN-digraph Figure 19 representing the competition between applicants for a career. Let $\{Rosaleen, Nazneen, Abner, Amara, Casper\}$ be the set of applicants for the particular careers $\{Medicine, Pharmacy, Anatomy, Surgery\}$. The truth-membership value of each applicant represents the degree of loyalty quality; the indeterminacy-value represents the indeterminate state of loyalty; and the false-membership value represents the disloyalty of each applicant towards their careers. Let the degree of truth-membership of Nazneen of her loyalty towards her career be 30%: degree of indeterminacy is 50%, and degree of disloyalty is 10%, i.e., the truth-membership, indeterminacy and falsity-membership values of the vertex Nazneen are $(0.3, 0.5, 0.1)$. The truth-membership value of each directed edge between an applicant and a career represents the eligibility for that career; the indeterminacy-value represents the indeterminate state of that career; and the false-membership value represents non-eligibility for that particular career.

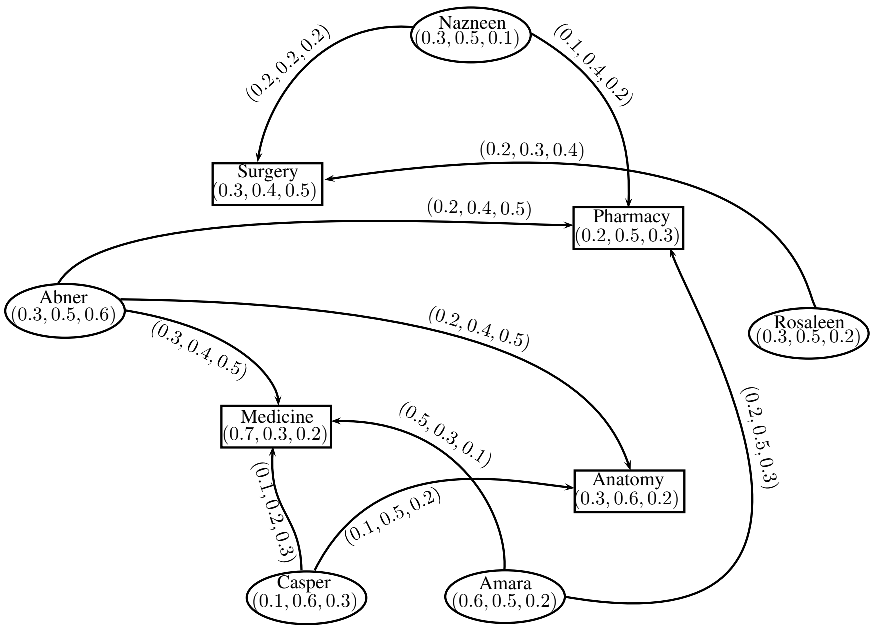

Figure 19. IN-digraph.

Thus, in Table 9, $\mathbb{N}^+(Nazneen) \cap \mathbb{N}^+(Rosaleen) = \{(Surgery, 0.2, 0.2, 0.4)\}$, $\mathbb{N}^+(Nazneen) \cap \mathbb{N}^+(Amara) = \{(Pharmacy, 0.1, 0.4, 0.3)\}$, $\mathbb{N}^+(Nazneen) \cap \mathbb{N}^+(Abner) = \{(Pharmacy, 0.1, 0.4, 0.5)\}$, $\mathbb{N}^+(Nazneen) \cap \mathbb{N}^+(Casper) = \emptyset$, $\mathbb{N}^+(Rosaleen) \cap \mathbb{N}^+(Amara) = \emptyset$, $\mathbb{N}^+(Rosaleen) \cap \mathbb{N}^+(Casper) = \emptyset$, $\mathbb{N}^+(Rosaleen) \cap \mathbb{N}^+(Abner) = \emptyset$, $\mathbb{N}^+(Amara) \cap \mathbb{N}^+(Casper) = \{(Medicine, 0.1, 0.2, 0.3)\}$, $\mathbb{N}^+(Amara) \cap \mathbb{N}^+(Abner) = \{(Medicine, 0.3, 0.3, 0.5), (Pharmacy, 0.2, 0.4, 0.5)\}$ and $\mathbb{N}^+(Casper) \cap \mathbb{N}^+(Abner) = \{(Medicine, 0.1, 0.2, 0.5), (Anatomy, 0.1, 0.4, 0.5)\}$.

Table 9. IN-out-neighborhoods.

$w \in X$	$\mathbb{N}^+(w)$
Nazneen	$\{(Surgery, 0.2, 0.2, 0.2), (Pharmacy, 0.1, 0.4, 0.2)\}$
Rosaleen	$\{(Surgery, 0.2, 0.3, 0.4)\}$
Amara	$\{(Medicine, 0.5, 0.3, 0.1), (Pharmacy, 0.2, 0.5, 0.3)\}$
Casper	$\{(Medicine, 0.1, 0.2, 0.3), (Anatomy, 0.1, 0.5, 0.2)\}$
Abner	$\{(Medicine, 0.3, 0.4, 0.5), (Anatomy, 0.2, 0.4, 0.5), (Pharmacy, 0.2, 0.4, 0.5)\}$

The INC-graph is shown in Figure 20. The solids lines indicate the strength of competition between two applicants, and dashed lines indicate the applicant competing for the particular career. For example, Nazneen and Rosaleen both are competing for the career, surgery, and the strength of competition between them is $(0.06, 0.1, 0.08)$. In Table 10, $W(z, c)$ represents the competition of applicant z for career c with respect to loyalty quality, indeterminacy and disloyalty to compete with the others. The strength to compete with the other applicants with respect to a particular career is calculated in Table 10.

From Table 10, Nazneen and Rosaleen have equal strength to compete with the other for the career, surgery. Abner and Casper have equal strength of competition for the career, anatomy. Amara competes with the others for the career, pharmacy and medicine.

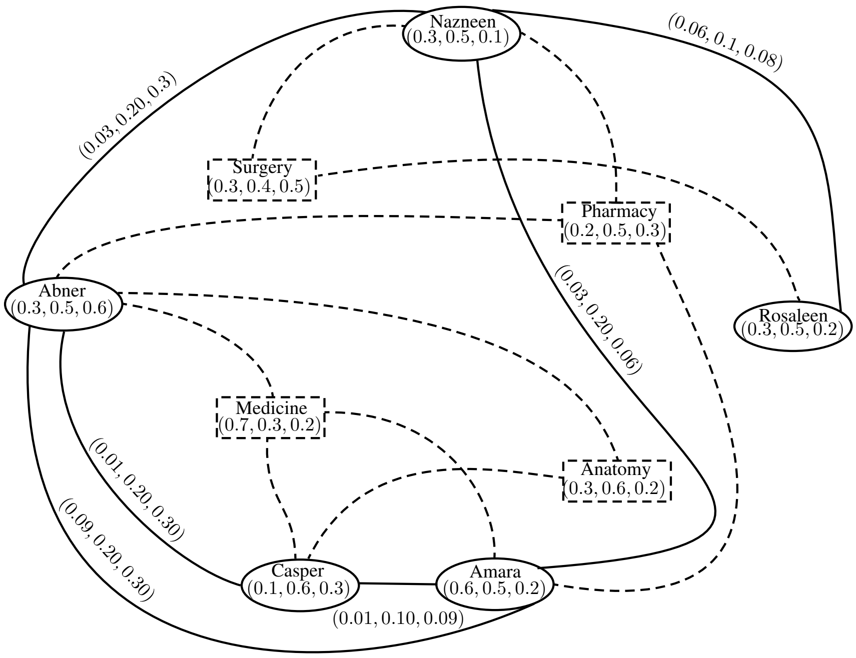

Figure 20. Corresponding INC-graph.

Table 10. Strength of competition of the applicant for a particular career.

(Applicant, Career)	*In Competition*	*W(Applicant, Career)*	*S(Applicant, Career)*
$(Nazneen, Surgery)$	*Rosaleen*	(0.06, 0.1, 0.08)	0.88
$(Rosaleen, Surgery)$	*Nazneen*	(0.06, 0.1, 0.08)	0.88
$(Abner, Anatomy)$	*Casper*	(0.01, 0.20, 0.30)	0.51
$(Casper, Anatomy)$	*Abner*	(0.01, 0.20, 0.30)	0.51
$(Nazneen, Pharmacy)$	*Abner, Amara*	(0.03, 0.20, 0.18)	0.65
$(Abner, Pharmacy)$	*Amara, Nazneen*	(0.06, 0.20, 0.30)	0.56
$(Amara, Pharmacy)$	*Nazneen, Abner*	(0.06, 0.20, 0.18)	0.68
$(Amara, Medicine)$	*Abner, Casper*	(0.05, 0.15, 0.195)	0.705
$(Casper, Medicine)$	*Abner, Amara*	(0.01, 0.15, 0.195)	0.665
$(Abner, Medicine)$	*Casper, Amara*	(0.05, 0.20, 0.30)	0.55

We present our method, which is used in our career competition application in Algorithm 2.

Algorithm 2: Career Competition

Step 1. Input the truth-membership, indeterminacy-membership and falsity-membership values for set of n applicants.

Step 2. If for any two distinct vertices z_i and z_j, $t(z_iz_j) > 0$, $i(z_iz_j) > 0$, $f(z_iz_j) > 0$, then

$$(z_j, t(z_iz_j), i(z_iz_j), f(z_iz_j)) \in \mathbb{N}^+(z_i).$$

Step 3. Repeat Step 2 for all vertices z_i and z_j to calculate IN-out-neighborhoods $\mathbb{N}^+(z_i)$.

Step 4. Calculate $\mathbb{N}^+(z_i) \cap \mathbb{N}^+(z_j)$ for each pair of distinct vertices z_i and z_j.

Step 5. Calculate $H[\mathbb{N}^+(z_i) \cap \mathbb{N}^+(z_j)]$.

Step 6. If $\mathbb{N}^+(z_i) \cap \mathbb{N}^+(z_j) \neq \emptyset$, then draw an edge z_iz_j.

Step 7. Repeat Step 6 for all pairs of distinct vertices.

Step 8. Assign membership values to each edge z_iz_j using the conditions:

$$\begin{aligned} t(z_iz_j) &= (z_i \wedge z_j)H_1[\mathbb{N}^+(z_i) \cap \mathbb{N}^+(z_j)] \\ i(z_iz_j) &= (z_i \wedge z_j)H_2[\mathbb{N}^+(z_i) \cap \mathbb{N}^+(z_j)] \\ f(z_iz_j) &= (z_i \vee z_j)H_3[\mathbb{N}^+(z_i) \cap \mathbb{N}^+(z_j)]. \end{aligned}$$

Step 9. If $z, r_1, r_2, r_3, \ldots, r_n$ are the applicants competing for career c, then the strength of competition $W(z,c) = (t(z,c), i(z,c), f(z,c))$ of each applicant z for the career c is:

$$W(z,c) = \frac{(t(zr_1)+t(zr_2)+\ldots+t(zr_n), i(zr_1)+i(zr_2)+\ldots+i(zr_n), f(zr_1)+f(zr_2)+\ldots+f(zr_n))}{n}.$$

Step 10. Calculate $S(z,c)$, the strength of competition of each applicant z for career c.

$$S(z,c) = t(z,c) - (i(z,c) + f(z,c)) + 1.$$

4. Conclusions

Graphs serve as mathematical models to analyze many concrete real-world problems successfully. Certain problems in physics, chemistry, communication science, computer technology, sociology and linguistics can be formulated as problems in graph theory. Intuitionistic neutrosophic set theory is a mathematical tool to deal with incomplete and vague information. Intuitionistic neutrosophic set theory deals with the problem of how to understand and manipulate imperfect knowledge. In this research paper, we have described the concept of intuitionistic neutrosophic competition graphs. We have also presented applications of intuitionistic neutrosophic competition graphs in ecosystem and career competition. We aim to extend our research work of fuzzification to (1) fuzzy soft competition graphs, (2) fuzzy rough soft competition graphs, (3) bipolar fuzzy soft competition graphs and (4) the application of fuzzy soft competition graphs in decision support systems.

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information

Article

Neutrosophic Commutative $\mathcal{N}$-Ideals in BCK-Algebras

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Abstract: The notion of a neutrosophic commutative $\mathcal{N}$-ideal in BCK-algebras is introduced, and several properties are investigated. Relations between a neutrosophic $\mathcal{N}$-ideal and a neutrosophic commutative $\mathcal{N}$-ideal are discussed. Characterizations of a neutrosophic commutative $\mathcal{N}$-ideal are considered.

Keywords: neutrosophic $\mathcal{N}$-structure; neutrosophic $\mathcal{N}$-ideal; neutrosophic commutative $\mathcal{N}$-ideal

MSC: 06F35, 03G25, 03B52

1. Introduction

As a generalization of fuzzy sets, Atanassov [1] introduced the degree of nonmembership/falsehood (f) in 1986 and defined the intuitionistic fuzzy set.

Smarandache proposed the term "neutrosophic" because "neutrosophic" etymologically comes from "neutrosophy" [French *neutre* < Latin *neuter*, neutral, and Greek *sophia*, skill/wisdom] which means knowledge of neutral thought, and this third/neutral represents the main distinction between "fuzzy"/"intuitionistic fuzzy" logic/set and "neutrosophic" logic/set, i.e., the *included middle* component (Lupasco–Nicolescu's logic in philosophy), i.e., the neutral/indeterminate/unknown part (besides the "truth"/"membership" and "falsehood"/"non-membership" components that both appear in fuzzy logic/set). Smarandache introduced the degree of indeterminacy/neutrality (i) as an independent component in 1995 (published in 1998) and defined the neutrosophic set on three components

$$(\text{t, i, f}) = (\text{truth, indeterminacy, falsehood}).$$

For more details, refer to the site http://fs.gallup.unm.edu/FlorentinSmarandache.htm.

Jun et al. [2] introduced a new function which is called negative-valued function, and constructed $\mathcal{N}$-structures. Khan et al. [3] introduced the notion of neutrosophic $\mathcal{N}$-structure and applied it to a semigroup. Jun et al. [4] applied the notion of neutrosophic $\mathcal{N}$-structure to BCK/BCI-algebras. They introduced the notions of a neutrosophic $\mathcal{N}$-subalgebra and a (closed) neutrosophic $\mathcal{N}$-ideal in a BCK/BCI-algebra, and investigated related properties. They also considered characterizations of a neutrosophic $\mathcal{N}$-subalgebra and a neutrosophic $\mathcal{N}$-ideal, and discussed relations between a neutrosophic $\mathcal{N}$-subalgebra and a neutrosophic $\mathcal{N}$-ideal. They provided conditions for a neutrosophic $\mathcal{N}$-ideal to be a closed neutrosophic $\mathcal{N}$-ideal. BCK-algebras entered into mathematics in 1966 through the work of Imai and Iséki [5], and have been applied to many branches of mathematics, such as group theory, functional analysis, probability theory and topology. Such algebras generalize Boolean rings as well as Boolean D-posets (= MV-algebras). Also, Iséki introduced the notion of a BCI-algebra which is a generalization of a BCK-algebra (see [6]).

In this paper, we introduce the notion of a neutrosophic commutative $\mathcal{N}$-ideal in BCK-algebras, and investigate several properties. We consider relations between a neutrosophic $\mathcal{N}$-ideal and a neutrosophic commutative $\mathcal{N}$-ideal. We discuss characterizations of a neutrosophic commutative $\mathcal{N}$-ideal.

2. Preliminaries

By a *BCI*-algebra, we mean a system $X := (X, *, 0) \in K(\tau)$ in which the following axioms hold:

(I) $((x * y) * (x * z)) * (z * y) = 0,$
(II) $(x * (x * y)) * y = 0,$
(III) $x * x = 0,$
(IV) $x * y = y * x = 0 \;\Rightarrow\; x = y$

for all $x, y, z \in X$. If a *BCI*-algebra X satisfies $0 * x = 0$ for all $x \in X$, then we say that X is a *BCK*-algebra.

We can define a partial ordering $\preceq$ by

$$(\forall x, y \in X)\,(x \preceq y \;\Rightarrow\; x * y = 0).$$

In a *BCK/BCI*-algebra X, the following hold:

$$(\forall x \in X)\;(x * 0 = x), \tag{1}$$
$$(\forall x, y, z \in X)\;((x * y) * z = (x * z) * y). \tag{2}$$

A *BCK*-algebra X is said to be *commutative* if it satisfies the following equality:

$$(\forall x, y \in X)\,(x * (x * y) = y * (y * x))\,. \tag{3}$$

A subset I of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies

$$0 \in I, \tag{4}$$
$$(\forall x, y \in X)\,(x * y \in I,\; y \in I \;\Rightarrow\; x \in I)\,. \tag{5}$$

A subset I of a *BCK*-algebra X is called a *commutative ideal* of X if it satisfies (4) and

$$(\forall x, y, z \in X)\,((x * y) * z \in I,\; z \in I \;\Rightarrow\; x * (y * (y * x)) \in I)\,. \tag{6}$$

Lemma 1. *An ideal I is commutative if and only if the following assertion is valid.*

$$(\forall x, y \in X)\,(x * y \in I \;\Rightarrow\; x * (y * (y * x)) \in I)\,. \tag{7}$$

We refer the reader to the books [7,8] for further information regarding *BCK/BCI*-algebras.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

$$\bigwedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite,} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise.} \end{cases}$$

Denote by $\mathcal{F}(X, [-1, 0])$ the collection of functions from a set X to $[-1, 0]$. We say that an element of $\mathcal{F}(X, [-1, 0])$ is a *negative-valued function* from X to $[-1, 0]$ (briefly, $\mathcal{N}$*-function* on X). By an $\mathcal{N}$*-structure*, we mean an ordered pair (X, f) of X and an $\mathcal{N}$-function f on X (see [2]). A *neutrosophic* $\mathcal{N}$*-structure* over a nonempty universe of discourse X (see [3]) is defined to be the structure

$$X_{\mathbf{N}} := \frac{X}{(T_N, I_N, F_N)} = \left\{ \frac{x}{(T_N(x), I_N(x), F_N(x))} \mid x \in X \right\} \tag{8}$$

where T_N, I_N and F_N are $\mathcal{N}$-functions on X which are called the *negative truth membership function*, the *negative indeterminacy membership function* and the *negative falsity membership function*, respectively, on X.

Note that every neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over X satisfies the condition:

$$(\forall x \in X)\,(-3 \leq T_N(x) + I_N(x) + F_N(x) \leq 0)\,.$$

3. Neutrosophic Commutative $\mathcal{N}$-Ideals

In what follows, let X denote a BCK-algebra unless otherwise specified.

Definition 1 ([4]). *A neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over X is called a neutrosophic $\mathcal{N}$-ideal of X if the following assertion is valid.*

$$(\forall x, y \in X) \begin{pmatrix} T_N(0) \leq T_N(x) \leq \bigvee\{T_N(x*y), T_N(y)\} \\ I_N(0) \geq I_N(x) \geq \bigwedge\{I_N(x*y), I_N(y)\} \\ F_N(0) \leq F_N(x) \leq \bigvee\{F_N(x*y), F_N(y)\} \end{pmatrix}. \tag{9}$$

Definition 2. *A neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over X is called a neutrosophic commutative $\mathcal{N}$-ideal of X if the following assertions are valid.*

$$(\forall x \in X)\,(T_N(0) \leq T_N(x),\ I_N(0) \geq I_N(x),\ F_N(0) \leq F_N(x))\,, \tag{10}$$

$$(\forall x, y, z \in X) \begin{pmatrix} T_N(x*(y*(y*x))) \leq \bigvee\{T_N((x*y)*z), T_N(z)\} \\ I_N(x*(y*(y*x))) \geq \bigwedge\{I_N((x*y)*z), I_N(z)\} \\ F_N(x*(y*(y*x))) \leq \bigvee\{F_N((x*y)*z), F_N(z)\} \end{pmatrix}. \tag{11}$$

Example 1. *Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the Cayley table which is given in Table 1.*

Table 1. Cayley table for the binary operation "*".

*	**0**	**1**	**2**	**3**	**4**
0	0	0	0	0	0
1	1	0	1	1	1
2	2	2	0	2	2
3	3	3	3	0	3
4	4	4	4	4	0

The neutrosophic $\mathcal{N}$-structure

$$X_{\mathbf{N}} = \left\{ \frac{0}{(-0.8,-0.2,-0.9)}, \frac{1}{(-0.3,-0.9,-0.5)}, \frac{2}{(-0.7,-0.7,-0.4)}, \frac{3}{(-0.3,-0.6,-0.7)}, \frac{4}{(-0.5,-0.3,-0.1)} \right\}$$

over X is a neutrosophic commutative $\mathcal{N}$-ideal of X.

Theorem 1. *Every neutrosophic commutative $\mathcal{N}$-ideal is a neutrosophic $\mathcal{N}$-ideal.*

Proof. Let $X_{\mathbf{N}}$ be a neutrosophic commutative $\mathcal{N}$-ideal of X. For every $x, z \in X$, we have

$$T_N(x) = T_N(x * (0 * (0 * x))) \leq \bigvee\{T_N((x * 0) * z), T_N(z)\} = \bigvee\{T_N(x * z), T_N(z)\},$$
$$I_N(x) = I_N(x * (0 * (0 * x))) \geq \bigwedge\{I_N((x * 0) * z), I_N(z)\} = \bigwedge\{I_N(x * z), I_N(z)\},$$
$$F_N(x) = F_N(x * (0 * (0 * x))) \leq \bigvee\{F_N((x * 0) * z), F_N(z)\} = \bigvee\{F_N(x * z), F_N(z)\}$$

by putting $y = 0$ in (11) and using (1). Therefore, $X_{\mathbf{N}}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X. □

The converse of Theorem 1 is not true in general as seen in the following example.

Example 2. *Consider a BCK-algebra $X = \{0, 1, 2, 3, 4\}$ with the Cayley table which is given in Table 2.*

Table 2. Cayley table for the binary operation "$*$"

*	0	1	2	3	4
0	0	0	0	0	0
1	1	0	1	0	0
2	2	2	0	0	0
3	3	3	3	0	0
4	4	4	4	3	0

The neutrosophic $\mathcal{N}$-structure

$$X_{\mathbf{N}} = \left\{ \frac{0}{(-0.8, -0.1, -0.7)}, \frac{1}{(-0.7, -0.6, -0.6)}, \frac{2}{(-0.6, -0.2, -0.4)}, \frac{3}{(-0.3, -0.8, -0.4)}, \frac{4}{(-0.3, -0.8, -0.4)} \right\}$$

over X is a neutrosophic $\mathcal{N}$-ideal of X. But it is not a neutrosophic commutative $\mathcal{N}$-ideal of X since $F_N(2 * (3 * (3 * 2)) = F_N(2) = -0.4 \nleq -0.7 = \bigvee\{F_N((2 * 3) * 0), F_N(0)\}$.

We consider characterizations of a neutrosophic commutative $\mathcal{N}$-ideal.

Theorem 2. *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-ideal of X. Then, $X_{\mathbf{N}}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X if and only if the following assertion is valid.*

$$(\forall x, y \in X) \left(\begin{array}{l} T_N(x * (y * (y * x))) \leq T_N(x * y), \\ I_N(x * (y * (y * x))) \geq I_N(x * y), \\ F_N(x * (y * (y * x))) \leq F_N(x * y) \end{array} \right). \tag{12}$$

Proof. Assume that $X_{\mathbf{N}}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X. The assertion (12) is by taking $z = 0$ in (11) and using (1) and (10).

Conversely, suppose that a neutrosophic $\mathcal{N}$-ideal $X_{\mathbf{N}}$ of X satisfies the condition (12). Then,

$$(\forall x, y \in X) \left(\begin{array}{l} T_N(x * y) \leq \bigvee\{T_N((x * y) * z), T_N(z)\} \\ I_N(x * y) \geq \bigwedge\{I_N((x * y) * z), I_N(z)\} \\ F_N(x * y) \leq \bigvee\{F_N((x * y) * z), F_N(z)\} \end{array} \right). \tag{13}$$

It follows that the condition (11) is induced by (12) and (13). Therefore, $X_{\mathbf{N}}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X. □

Lemma 2 ([4]). *For any neutrosophic $\mathcal{N}$-ideal $X_{\mathbf{N}}$ of X, we have*

$$(\forall x, y, z \in X) \left(x * y \preceq z \Rightarrow \left\{ \begin{array}{l} T_N(x) \leq \bigvee\{T_N(y), T_N(z)\} \\ I_N(x) \geq \bigwedge\{I_N(y), I_N(z)\} \\ F_N(x) \leq \bigvee\{F_N(y), F_N(z)\} \end{array} \right. \right). \tag{14}$$

Theorem 3. *In a commutative BCK-algebra, every neutrosophic $\mathcal{N}$-ideal is a neutrosophic commutative $\mathcal{N}$-ideal.*

Proof. Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-ideal of a commutative BCK-algebra X. For any $x, y, z \in X$, we have

$$\begin{aligned}
&((x * (y * (y * x))) * ((x * y) * z)) * z \\
&= ((x * (y * (y * x))) * z) * ((x * y) * z) \\
&\preceq (x * (y * (y * x))) * (x * y) \\
&= (x * (x * y)) * (y * (y * x)) = 0,
\end{aligned}$$

that is, $(x * (y * (y * x))) * ((x * y) * z) \preceq z$. It follows from Lemma 2 that

$$\begin{aligned}
T_N(x * (y * (y * x))) &\leq \bigvee\{T_N((x * y) * z),\ T_N(z)\}, \\
I_N(x * (y * (y * x))) &\geq \bigwedge\{I_N((x * y) * z),\ I_N(z)\}, \\
F_N(x * (y * (y * x))) &\leq \bigvee\{F_N((x * y) * z),\ F_N(z)\}.
\end{aligned}$$

Therefore, $X_{\mathbf{N}}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X. $\square$

Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. Consider the following sets.

$$\begin{aligned}
T_N^{\alpha} &:= \{x \in X \mid T_N(x) \leq \alpha\}, \\
I_N^{\beta} &:= \{x \in X \mid I_N(x) \geq \beta\}, \\
F_N^{\gamma} &:= \{x \in X \mid F_N(x) \leq \gamma\}.
\end{aligned}$$

The set

$$X_{\mathbf{N}}(\alpha, \beta, \gamma) := \{x \in X \mid T_N(x) \leq \alpha, I_N(x) \geq \beta, F_N(x) \leq \gamma\}$$

is called the (α, β, γ)*-level set* of $X_{\mathbf{N}}$. It is clear that

$$X_{\mathbf{N}}(\alpha, \beta, \gamma) = T_N^{\alpha} \cap I_N^{\beta} \cap F_N^{\gamma}.$$

Theorem 4. *If $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X, then T_N^{α}, I_N^{β} and F_N^{γ} are commutative ideals of X for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$ whenever they are nonempty.*

We call T_N^{α}, I_N^{β} and F_N^{γ} *level commutative ideals* of $X_{\mathbf{N}}$.

Proof. Assume that T_N^{α}, I_N^{β} and F_N^{γ} are nonempty for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Then, $x \in T_N^{\alpha}$, $y \in I_N^{\beta}$ and $z \in F_N^{\gamma}$ for some $x, y, z \in X$. Thus, $T_N(0) \leq T_N(x) \leq \alpha$, $I_N(0) \geq I_N(y) \geq \beta$, and $F_N(0) \leq F_N(z) \leq \gamma$, that is, $0 \in T_N^{\alpha} \cap I_N^{\beta} \cap F_N^{\gamma}$. Let $(x * y) * z \in T_N^{\alpha}$ and $z \in T_N^{\alpha}$. Then, $T_N((x * y) * z) \leq \alpha$ and $T_N(z) \leq \alpha$, which imply that

$$T_N(x * (y * (y * x))) \leq \bigvee\{T_N((x * y) * z), T_N(z)\} \leq \alpha,$$

that is, $x * (y * (y * x)) \in T_N^{\alpha}$. If $(a * b) * c \in I_N^{\beta}$ and $c \in I_N^{\beta}$, then $I_N((a * b) * c) \geq \beta$ and $I_N(c) \geq \beta$. Thus

$$I_N(a * (b * (b * c))) \geq \bigwedge\{I_N((a * b) * c), I_N(c)\} \geq \beta,$$

and so $a * (b * (b * c)) \in I_N^\beta$. Finally, suppose that $(u * v) * w \in F_N^\gamma$ and $w \in F_N^\gamma$. Then, $F_N((u * v) * w) \leq \gamma$ and $F_N(w) \leq \gamma$. Thus,

$$F_N(u * (v * (v * w))) \leq \bigvee\{F_N((u * v) * w), F_N(w)\} \leq \gamma,$$

that is, $u * (v * (v * w)) \in F_N^\gamma$. Therefore, T_N^α, I_N^β and F_N^γ are commutative ideals of X. □

Corollary 1. *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. If $X_{\mathbf{N}}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X, then the nonempty (α, β, γ)-level set of $X_{\mathbf{N}}$ is a commutative ideal of X.*

Proof. Straightforward. □

Lemma 3 ([4]). *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X and assume that T_N^α, I_N^β and F_N^γ are ideals of X for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Then $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X.*

Theorem 5. *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X and assume that T_N^α, I_N^β and F_N^γ are commutative ideals of X for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Then, $X_{\mathbf{N}}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X.*

Proof. If T_N^α, I_N^β and F_N^γ are commutative ideals of X, then they are ideals of X. Hence, $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X by Lemma 3. Let $x, y \in X$ and $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$ such that $T_N(x * y) = \alpha$, $I_N(x * y) = \beta$ and $F_N(x * y) = \gamma$. Then, $x * y \in T_N^\alpha \cap I_N^\beta \cap F_N^\gamma$. Since $T_N^\alpha \cap I_N^\beta \cap F_N^\gamma$ is a commutative ideal of X, it follows from Lemma 1 that $x * (y * (y * x)) \in T_N^\alpha \cap I_N^\beta \cap F_N^\gamma$. Hence

$$\begin{aligned} T_N(x * (y * (y * x))) &\leq \alpha = T_N(x * y), \\ I_N(x * (y * (y * x))) &\geq \beta = I_N(x * y), \\ F_N(x * (y * (y * x))) &\leq \gamma = F_N(x * y). \end{aligned}$$

Therefore, $X_{\mathbf{N}}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X by Theorem 2. □

Theorem 6. *Let $f : X \to X$ be an injective mapping. Given a neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over X, the following are equivalent.*

(1) *$X_{\mathbf{N}}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X, satisfying the following condition.*

$$(\forall x \in X) \begin{pmatrix} T_N(f(x)) = T_N(x) \\ I_N(f(x)) = I_N(x) \\ F_N(f(x)) = F_N(x) \end{pmatrix}. \tag{15}$$

(2) *T_N^α, I_N^β and F_N^γ are commutative ideals of $X_{\mathbf{N}}$, satisfying the following condition.*

$$f(T_N^\alpha) = T_N^\alpha, f(I_N^\beta) = I_N^\beta, f(F_N^\gamma) = F_N^\gamma. \tag{16}$$

Proof. Let $X_{\mathbf{N}}$ be a neutrosophic commutative $\mathcal{N}$-ideal of X, satisfying the condition (15). Then, T_N^α, I_N^β and F_N^γ are commutative ideals of $X_{\mathbf{N}}$ by Theorem 4. Let $\alpha \in \mathrm{Im}(T_N)$, $\beta \in \mathrm{Im}(I_N)$, $\gamma \in \mathrm{Im}(F_N)$ and $x \in T_N^\alpha \cap I_N^\beta \cap F_N^\gamma$. Then $T_N(f(x)) = T_N(x) \leq \alpha$, $I_N(f(x)) = I_N(x) \geq \beta$ and $F_N(f(x)) = F_N(x) \leq \gamma$. Thus, $f(x) \in T_N^\alpha \cap I_N^\beta \cap F_N^\gamma$, which shows that $f(T_N^\alpha) \subseteq T_N^\alpha$, $f(I_N^\beta) \subseteq I_N^\beta$ and $f(F_N^\gamma) \subseteq F_N^\gamma$. Let $y \in X$ be such that $f(y) = x$. Then, $T_N(y) = T_N(f(y)) = T_N(x) \leq \alpha$, $I_N(y) = I_N(f(y)) = I_N(x) \geq \beta$

and $F_N(y) = F_N(f(y)) = F_N(x) \leq \gamma$, which imply that $y \in T_N^\alpha \cap I_N^\beta \cap F_N^\gamma$. Thus, $x = f(y) \in f(T_N^\alpha) \cap f(I_N^\beta) \cap f(F_N^\gamma)$, and so $T_N^\alpha \subseteq f(T_N^\alpha)$, $I_N^\beta \subseteq f(I_N^\beta)$ and $F_N^\gamma \subseteq f(F_N^\gamma)$. Therefore (16) is valid.

Conversely, assume that T_N^α, I_N^β and F_N^γ are commutative ideals of $X_{\mathbf{N}}$, satisfying the condition (16). Then, $X_{\mathbf{N}}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X by Theorem 5. Let $x, y, z \in X$ be such that $T_N(x) = \alpha$, $I_N(y) = \beta$ and $F_N(z) = \gamma$. Note that

$$T_N(x) = \alpha \Longleftrightarrow x \in T_N^\alpha \text{ and } x \notin T_N^{\tilde{\alpha}} \text{ for all } \alpha > \tilde{\alpha},$$
$$I_N(y) = \beta \Longleftrightarrow y \in I_N^\beta \text{ and } y \notin I_N^{\tilde{\beta}} \text{ for all } \beta < \tilde{\beta},$$
$$F_N(z) = \gamma \Longleftrightarrow z \in F_N^\gamma \text{ and } z \notin F_N^{\tilde{\gamma}} \text{ for all } \gamma > \tilde{\gamma}.$$

It follows from (16) that $f(x) \in T_N^\alpha$, $f(y) \in I_N^\beta$ and $f(z) \in F_N^\gamma$. Hence, $T_N(f(x)) \leq \alpha$, $I_N(f(y)) \geq \beta$ and $F_N(f(z)) \leq \gamma$. Let $\tilde{\alpha} = T_N(f(x))$, $\tilde{\beta} = I_N(f(y))$ and $\tilde{\gamma} = F_N(f(z))$. If $\alpha > \tilde{\alpha}$, then $f(x) \in T_N^{\tilde{\alpha}} = f\left(T_N^{\tilde{\alpha}}\right)$, and thus $x \in T_N^{\tilde{\alpha}}$ since f is one to one. This is a contradiction. Hence, $T_N(f(x)) = \alpha = T_N(x)$. If $\beta < \tilde{\beta}$, then $f(y) \in I_N^{\tilde{\beta}} = f\left(I_N^{\tilde{\beta}}\right)$ which implies from the injectivity of f that $y \in I_N^{\tilde{\beta}}$, a contradiction. Hence, $I_N(f(x)) = \beta = I_N(x)$. If $\gamma > \tilde{\gamma}$, then $f(z) \in F_N^{\tilde{\gamma}} = f\left(F_N^{\tilde{\gamma}}\right)$. Since f is one to one, we have $z \in F_N^{\tilde{\gamma}}$ which is a contradiction. Thus, $F_N(f(x)) = \gamma = F_N(x)$. This completes the proof. □

For any elements $\omega_t, \omega_i, \omega_f \in X$, we consider sets:

$$X_{\mathbf{N}}^{\omega_t} := \{x \in X \mid T_N(x) \leq T_N(\omega_t)\},$$
$$X_{\mathbf{N}}^{\omega_i} := \{x \in X \mid I_N(x) \geq I_N(\omega_i)\},$$
$$X_{\mathbf{N}}^{\omega_f} := \left\{x \in X \mid F_N(x) \leq F_N(\omega_f)\right\}.$$

Obviously, $\omega_t \in X_{\mathbf{N}}^{\omega_t}$, $\omega_i \in X_{\mathbf{N}}^{\omega_i}$ and $\omega_f \in X_{\mathbf{N}}^{\omega_f}$.

Lemma 4 ([4])**.** *Let ω_t, ω_i and ω_f be any elements of X. If $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X, then $X_{\mathbf{N}}^{\omega_t}$, $X_{\mathbf{N}}^{\omega_i}$ and $X_{\mathbf{N}}^{\omega_f}$ are ideals of X.*

Theorem 7. *Let ω_t, ω_i and ω_f be any elements of X. If $X_{\mathbf{N}}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X, then $X_{\mathbf{N}}^{\omega_t}$, $X_{\mathbf{N}}^{\omega_i}$ and $X_{\mathbf{N}}^{\omega_f}$ are commutative ideals of X.*

Proof. If $X_{\mathbf{N}}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X, then it is a neutrosophic $\mathcal{N}$-ideal of X and so $X_{\mathbf{N}}^{\omega_t}$, $X_{\mathbf{N}}^{\omega_i}$ and $X_{\mathbf{N}}^{\omega_f}$ are ideals of X by Lemma 4. Let $x * y \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$ for any $x, y \in X$. Then, $T_N(x * y) \leq T_N(\omega_t)$, $I_N(x * y) \geq T_N(\omega_i)$ and $F_N(x * y) \leq F_N(\omega_f)$. It follows from Theorem 2 that

$$T_N(x * (y * (y * x))) \leq T_N(x * y) \leq T_N(\omega_t),$$
$$I_N(x * (y * (y * x))) \geq I_N(x * y) \geq I_N(\omega_i),$$
$$F_N(x * (y * (y * x))) \leq F_N(x * y) \leq F_N(\omega_f).$$

Hence, $x * (y * (y * x)) \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$, and therefore $X_{\mathbf{N}}^{\omega_t}$, $X_{\mathbf{N}}^{\omega_i}$ and $X_{\mathbf{N}}^{\omega_f}$ are commutative ideals of X by Lemma 1. □

Theorem 8. *Any commutative ideal of X can be realized as level commutative ideals of some neutrosophic commutative $\mathcal{N}$-ideal of X.*

Proof. Let A be a commutative ideal of X and let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X in which

$$T_N : X \to [-1,0], \ x \mapsto \begin{cases} \alpha & \text{if } x \in A, \\ 0 & \text{otherwise,} \end{cases}$$
$$I_N : X \to [-1,0], \ x \mapsto \begin{cases} \beta & \text{if } x \in A, \\ -1 & \text{otherwise,} \end{cases}$$
$$F_N : X \to [-1,0], \ x \mapsto \begin{cases} \gamma & \text{if } x \in A, \\ 0 & \text{otherwise} \end{cases}$$

where $\alpha, \gamma \in [-1,0)$ and $\beta \in (-1,0]$. Division into the following cases will verify that $X_{\mathbf{N}}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X.

If $(x*y)*z \in A$ and $z \in A$, then $x*(y*(y*x) \in A$. Thus,

$$\begin{aligned} T_N((x*y)*z) &= T_N(z) = T_N(x*(y*(y*x))) = \alpha, \\ I_N((x*y)*z) &= I_N(z) = I_N(x*(y*(y*x))) = \beta, \\ F_N((x*y)*z) &= F_N(z) = F_N(x*(y*(y*x))) = \gamma, \end{aligned}$$

and so (11) is clearly verified.

If $(x*y)*z \notin A$ and $z \notin A$, then $T_N((x*y)*z) = T_N(z) = 0$, $I_N((x*y)*z) = I_N(z) = -1$ and $F_N((x*y)*z) = F_N(z) = 0$. Hence

$$\begin{aligned} T_N(x*(y*(y*x))) &\leq \bigvee\{T_N((x*y)*z),\ T_N(z)\}, \\ I_N(x*(y*(y*x))) &\geq \bigwedge\{I_N((x*y)*z),\ I_N(z)\}, \\ F_N(x*(y*(y*x))) &\leq \bigvee\{F_N((x*y)*z),\ F_N(z)\}. \end{aligned}$$

If $(x*y)*z \in A$ and $z \notin A$, then $T_N((x*y)*z) = \alpha$, $T_N(z) = 0$, $I_N((x*y)*z) = \beta$, $I_N(z) = -1$, $F_N((x*y)*z) = \gamma$ and $F_N(z) = 0$. Therefore,

$$\begin{aligned} T_N(x*(y*(y*x))) &\leq \bigvee\{T_N((x*y)*z),\ T_N(z)\}, \\ I_N(x*(y*(y*x))) &\geq \bigwedge\{I_N((x*y)*z),\ I_N(z)\}, \\ F_N(x*(y*(y*x))) &\leq \bigvee\{F_N((x*y)*z),\ F_N(z)\}. \end{aligned}$$

Similarly, if $(x*y)*z \notin A$ and $z \in A$, then (11) is verified. Therefore, $X_{\mathbf{N}}$ is a neutrosophic commutative $\mathcal{N}$-ideal of X. Obviously, $T_N^{\alpha} = A$, $I_N^{\beta} = A$ and $F_N^{\gamma} = A$. This completes the proof. □

4. Conclusions

In order to deal with the negative meaning of information, Jun et al. [2] have introduced a new function which is called negative-valued function, and constructed $\mathcal{N}$-structures. The concept of neutrosophic set (NS) has been developed by Smarandache in [9,10] as a more general platform which extends the concepts of the classic set and fuzzy set, intuitionistic fuzzy set and interval valued intuitionistic fuzzy set. In this article, we have introduced the notion of a neutrosophic commutative $\mathcal{N}$-ideal in BCK-algebras, and investigated several properties. We have considered relations between a neutrosophic $\mathcal{N}$-ideal and a neutrosophic commutative $\mathcal{N}$-ideal. We have discussed characterizations of a neutrosophic commutative $\mathcal{N}$-ideal.

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Article

Neutrosophic $\mathcal{N}$-Structures Applied to BCK/BCI-Algebras

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Abstract: Neutrosophic $\mathcal{N}$-structures with applications in BCK/BCI-algebras is discussed. The notions of a neutrosophic $\mathcal{N}$-subalgebra and a (closed) neutrosophic $\mathcal{N}$-ideal in a BCK/BCI-algebra are introduced, and several related properties are investigated. Characterizations of a neutrosophic $\mathcal{N}$-subalgebra and a neutrosophic $\mathcal{N}$-ideal are considered, and relations between a neutrosophic $\mathcal{N}$-subalgebra and a neutrosophic $\mathcal{N}$-ideal are stated. Conditions for a neutrosophic $\mathcal{N}$-ideal to be a closed neutrosophic $\mathcal{N}$-ideal are provided.

Keywords: neutrosophic $\mathcal{N}$-structure; neutrosophic $\mathcal{N}$-subalgebra; (closed) neutrosophic $\mathcal{N}$-ideal

MSC: 06F35, 03G25, 03B52

1. Introduction

BCK-algebras entered into mathematics in 1966 through the work of Imai and Iséki [1], and they have been applied to many branches of mathematics, such as group theory, functional analysis, probability theory and topology. Such algebras generalize Boolean rings as well as Boolean D-posets (MV-algebras). Additionally, Iséki introduced the notion of a BCI-algebra, which is a generalization of a BCK-algebra (see [2]).

A (crisp) set A in a universe X can be defined in the form of its characteristic function $\mu_A : X \to \{0,1\}$ yielding the value 1 for elements belonging to the set A and the value 0 for elements excluded from the set A. So far, most of the generalizations of the crisp set have been conducted on the unit interval $[0,1]$, and they are consistent with the asymmetry observation. In other words, the generalization of the crisp set to fuzzy sets relied on spreading positive information that fit the crisp point $\{1\}$ into the interval $[0,1]$. Because no negative meaning of information is suggested, we now feel a need to deal with negative information. To do so, we also feel a need to supply a mathematical tool. To attain such an object, Jun et al. [3] introduced a new function, called a negative-valued function, and constructed $\mathcal{N}$-structures. Zadeh [4] introduced the degree of membership/truth (t) in 1965 and defined the fuzzy set. As a generalization of fuzzy sets, Atanassov [5] introduced the degree of nonmembership/falsehood (f) in 1986 and defined the intuitionistic fuzzy set. Smarandache introduced the degree of indeterminacy/neutrality (i) as an independent component in 1995 (published in 1998) and defined the neutrosophic set on three components:

$$(\text{t, i, f}) = (\text{truth, indeterminacy, falsehood})$$

For more details, refer to the following site:

http://fs.gallup.unm.edu/FlorentinSmarandache.htm

In this paper, we discuss a neutrosophic $\mathcal{N}$-structure with an application to BCK/BCI-algebras. We introduce the notions of a neutrosophic $\mathcal{N}$-subalgebra and a (closed) neutrosophic $\mathcal{N}$-ideal in a BCK/BCI-algebra, and investigate related properties. We consider characterizations of a neutrosophic $\mathcal{N}$-subalgebra and a neutrosophic $\mathcal{N}$-ideal. We discuss relations between a neutrosophic $\mathcal{N}$-subalgebra and a neutrosophic $\mathcal{N}$-ideal. We provide conditions for a neutrosophic $\mathcal{N}$-ideal to be a closed neutrosophic $\mathcal{N}$-ideal.

2. Preliminaries

We let $K(\tau)$ be the class of all algebras with type $\tau = (2,0)$. A *BCI-algebra* refers to a system $X := (X, *, \theta) \in K(\tau)$ in which the following axioms hold:

(I) $((x * y) * (x * z)) * (z * y) = \theta$,
(II) $(x * (x * y)) * y = \theta$,
(III) $x * x = \theta$,
(IV) $x * y = y * x = \theta \Rightarrow x = y$.

for all $x, y, z \in X$. If a BCI-algebra X satisfies $\theta * x = \theta$ for all $x \in X$, then we say that X is a *BCK-algebra*. We can define a partial ordering $\preceq$ by

$$(\forall x, y \in X)\,(x \preceq y \Rightarrow x * y = \theta)$$

In a BCK/BCI-algebra X, the following hold:

$$(\forall x \in X)\ (x * \theta = x) \tag{1}$$
$$(\forall x, y, z \in X)\ ((x * y) * z = (x * z) * y) \tag{2}$$

A non-empty subset S of a BCK/BCI-algebra X is called a *subalgebra* of X if $x * y \in S$ for all $x, y \in S$.

A subset I of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies the following:

(I1) $0 \in I$,
(I2) $(\forall x, y \in X)(x * y \in I,\ y \in I \Rightarrow x \in I)$.

We refer the reader to the books [6,7] for further information regarding BCK/BCI-algebras.

For any family $\{a_i \mid i \in \Lambda\}$ of real numbers, we define

$$\bigvee\{a_i \mid i \in \Lambda\} := \begin{cases} \max\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite} \\ \sup\{a_i \mid i \in \Lambda\} & \text{otherwise} \end{cases}$$

$$\bigwedge\{a_i \mid i \in \Lambda\} := \begin{cases} \min\{a_i \mid i \in \Lambda\} & \text{if } \Lambda \text{ is finite} \\ \inf\{a_i \mid i \in \Lambda\} & \text{otherwise} \end{cases}$$

We denote by $\mathcal{F}(X, [-1,0])$ the collection of functions from a set X to $[-1,0]$. We say that an element of $\mathcal{F}(X, [-1,0])$ is a *negative-valued function* from X to $[-1,0]$ (briefly, $\mathcal{N}$-*function* on X). An $\mathcal{N}$-*structure* refers to an ordered pair (X, f) of X and an $\mathcal{N}$-function f on X (see [3]). In what follows, we let X denote the nonempty universe of discourse unless otherwise specified.

A *neutrosophic* $\mathcal{N}$-*structure* over X (see [8]) is defined to be the structure:

$$X_{\mathbf{N}} := \frac{X}{(T_N, I_N, F_N)} = \left\{ \frac{x}{(T_N(x), I_N(x), F_N(x))} \mid x \in X \right\} \tag{3}$$

where T_N, I_N and F_N are $\mathcal{N}$-functions on X, which are called the *negative truth membership function*, the *negative indeterminacy membership function* and the *negative falsity membership function*, respectively, on X.

We note that every neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over X satisfies the condition:

$$(\forall x \in X)\,(-3 \leq T_N(x) + I_N(x) + F_N(x) \leq 0)$$

3. Application in *BCK*/*BCI*-Algebras

In this section, we take a BCK/BCI-algebra X as the universe of discourse unless otherwise specified.

Definition 1. *A neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over X is called a neutrosophic $\mathcal{N}$-subalgebra of X if the following condition is valid:*

$$(\forall x, y \in X) \begin{pmatrix} T_N(x * y) \leq \bigvee\{T_N(x), T_N(y)\} \\ I_N(x * y) \geq \bigwedge\{I_N(x), I_N(y)\} \\ F_N(x * y) \leq \bigvee\{F_N(x), F_N(y)\} \end{pmatrix} \tag{4}$$

Example 1. *Consider a BCK-algebra $X = \{\theta, a, b, c\}$ with the following Cayley table.*

$*$	θ	a	b	c
θ	θ	θ	θ	θ
a	a	θ	θ	a
b	b	a	θ	b
c	c	c	c	θ

The neutrosophic $\mathcal{N}$-structure

$$X_{\mathbf{N}} = \left\{ \frac{\theta}{(-0.7,-0.2,-0.6)}, \frac{a}{(-0.5,-0.3,-0.4)}, \frac{b}{(-0.5,-0.3,-0.4)}, \frac{c}{(-0.3,-0.8,-0.5)} \right\}$$

over X is a neutrosophic $\mathcal{N}$-subalgebra of X.

Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. Consider the following sets:

$$\begin{aligned} T_N^{\alpha} &:= \{x \in X \mid T_N(x) \leq \alpha\} \\ I_N^{\beta} &:= \{x \in X \mid I_N(x) \geq \beta\} \\ F_N^{\gamma} &:= \{x \in X \mid F_N(x) \leq \gamma\} \end{aligned}$$

The set

$$X_{\mathbf{N}}(\alpha, \beta, \gamma) := \{x \in X \mid T_N(x) \leq \alpha, I_N(x) \geq \beta, F_N(x) \leq \gamma\}$$

is called the (α, β, γ)-*level set* of $X_{\mathbf{N}}$. Note that

$$X_{\mathbf{N}}(\alpha, \beta, \gamma) = T_N^{\alpha} \cap I_N^{\beta} \cap F_N^{\gamma}$$

Theorem 1. *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. If $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra of X, then the nonempty (α, β, γ)-level set of $X_{\mathbf{N}}$ is a subalgebra of X.*

Proof. Let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$ and $X_{\mathbf{N}}(\alpha, \beta, \gamma) \neq \emptyset$. If $x, y \in X_{\mathbf{N}}(\alpha, \beta, \gamma)$, then $T_N(x) \leq \alpha$, $I_N(x) \geq \beta$, $F_N(x) \leq \gamma$, $T_N(y) \leq \alpha$, $I_N(y) \geq \beta$ and $F_N(y) \leq \gamma$. It follows from Equation (4) that

$T_N(x * y) \leq \bigvee\{T_N(x), T_N(y)\} \leq \alpha$,
$I_N(x * y) \geq \bigwedge\{I_N(x), I_N(y)\} \geq \beta$, and
$F_N(x * y) \leq \bigvee\{F_N(x), F_N(y)\} \leq \gamma$.

Hence, $x * y \in X_{\mathbf{N}}(\alpha, \beta, \gamma)$, and therefore $X_{\mathbf{N}}(\alpha, \beta, \gamma)$ is a subalgebra of X. □

Theorem 2. *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X and assume that T_N^{α}, I_N^{β} and F_N^{γ} are subalgebras of X for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Then $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra of X.*

Proof. Assume that there exist $a, b \in X$ such that $T_N(a * b) > \bigvee\{T_N(a), T_N(b)\}$. Then $T_N(a * b) > t_\alpha \geq \bigvee\{T_N(a), T_N(b)\}$ for some $t_\alpha \in [-1, 0)$. Hence $a, b \in T_N^{t_\alpha}$ but $a * b \notin T_N^{t_\alpha}$, which is a contradiction. Thus

$$T_N(x * y) \leq \bigvee\{T_N(x), T_N(y)\}$$

for all $x, y \in X$. If $I_N(a * b) < \bigwedge\{I_N(a), I_N(b)\}$ for some $a, b \in X$, then

$$I_N(a * b) < t_\beta < \bigwedge\{I_N(a), I_N(b)\}$$

where $t_\beta := \frac{1}{2}\{I_N(a * b) + \bigwedge\{I_N(a), I_N(b)\}\}$. Thus $a, b \in I_N^{t_\beta}$ and $a * b \notin I_N^{t_\beta}$, which is a contradiction. Therefore

$$I_N(x * y) \geq \bigwedge\{I_N(x), I_N(y)\}$$

for all $x, y \in X$. Now, suppose that there exist $a, b \in X$ and $t_\gamma \in [-1, 0)$ such that

$$F_N(a * b) > t_\gamma \geq \bigvee\{F_N(a), F_N(b)\}$$

Then $a, b \in F_N^{t_\gamma}$ and $a * b \notin F_N^{t_\gamma}$, which is a contradiction. Hence

$$F_N(x * y) \leq \bigvee\{F_N(x), F_N(y)\}$$

for all $x, y \in X$. Therefore $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra of X. □

Because $[-1, 0]$ is a completely distributive lattice with respect to the usual ordering, we have the following theorem.

Theorem 3. *If $\{X_{N_i} \mid i \in \mathbb{N}\}$ is a family of neutrosophic $\mathcal{N}$-subalgebras of X, then $(\{X_{N_i} \mid i \in \mathbb{N}\}, \subseteq)$ forms a complete distributive lattice.*

Proposition 1. *If a neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over X is a neutrosophic $\mathcal{N}$-subalgebra of X, then $T_N(\theta) \leq T_N(x)$, $I_N(\theta) \geq I_N(x)$ and $F_N(\theta) \leq F_N(x)$ for all $x \in X$.*

Proof. Straightforward. □

Theorem 4. *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-subalgebra of X. If there exists a sequence $\{a_n\}$ in X such that $\lim_{n\to\infty} T_N(a_n) = -1$, $\lim_{n\to\infty} I_N(a_n) = 0$ and $\lim_{n\to\infty} F_N(a_n) = -1$, then $T_N(\theta) = -1$, $I_N(\theta) = 0$ and $F_N(\theta) = -1$.*

Proof. By Proposition 1, we have $T_N(\theta) \leq T_N(x)$, $I_N(\theta) \geq I_N(x)$ and $F_N(\theta) \leq F_N(x)$ for all $x \in X$. Hence $T_N(\theta) \leq T_N(a_n)$, $I_N(a_n) \leq I_N(\theta)$ and $F_N(\theta) \leq F_N(a_n)$ for every positive integer n. It follows that

$$
\begin{aligned}
-1 &\leq T_N(\theta) \leq \lim_{n\to\infty} T_N(a_n) = -1 \\
0 &\geq I_N(\theta) \geq \lim_{n\to\infty} I_N(a_n) = 0 \\
-1 &\leq F_N(\theta) \leq \lim_{n\to\infty} F_N(a_n) = -1
\end{aligned}
$$

Hence $T_N(\theta) = -1$, $I_N(\theta) = 0$ and $F_N(\theta) = -1$. □

Proposition 2. *If every neutrosophic $\mathcal{N}$-subalgebra $X_{\mathbf{N}}$ of X satisfies:*

$$T_N(x*y) \leq T_N(y), I_N(x*y) \geq I_N(y), F_N(x*y) \leq F_N(y) \tag{5}$$

for all $x, y \in X$, then $X_{\mathbf{N}}$ is constant.

Proof. Using Equations (1) and (5), we have $T_N(x) = T_N(x*\theta) \leq T_N(\theta)$, $I_N(x) = I_N(x*\theta) \geq I_N(\theta)$ and $F_N(x) = F_N(x*\theta) \leq F_N(\theta)$ for all $x \in X$. It follows from Proposition 1 that $T_N(x) = T_N(\theta)$, $I_N(x) = I_N(\theta)$ and $F_N(x) = F_N(\theta)$ for all $x \in X$. Therefore $X_{\mathbf{N}}$ is constant. □

Definition 2. *A neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over X is called a neutrosophic $\mathcal{N}$-ideal of X if the following assertion is valid:*

$$(\forall x, y \in X) \begin{pmatrix} T_N(\theta) \leq T_N(x) \leq \bigvee\{T_N(x*y), T_N(y)\} \\ I_N(\theta) \geq I_N(x) \geq \bigwedge\{I_N(x*y), I_N(y)\} \\ F_N(\theta) \leq F_N(x) \leq \bigvee\{F_N(x*y), F_N(y)\} \end{pmatrix} \tag{6}$$

Example 2. *The neutrosophic $\mathcal{N}$-structure $X_{\mathbf{N}}$ over X in Example 1 is a neutrosophic $\mathcal{N}$-ideal of X.*

Example 3. *Consider a BCI-algebra $X := Y \times \mathbb{Z}$ where $(Y, *, \theta)$ is a BCI-algebra and $(\mathbb{Z}, -, 0)$ is the adjoint BCI-algebra of the additive group $(\mathbb{Z}, +, 0)$ of integers (see [6]). Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X given by*

$$X_{\mathbf{N}} = \left\{ \tfrac{x}{(\alpha,0,\gamma)} \mid x \in Y \times (\mathbb{N} \cup \{0\}) \right\} \cup \left\{ \tfrac{x}{(0,\beta,0)} \mid x \notin Y \times (\mathbb{N} \cup \{0\}) \right\}$$

where $\alpha, \gamma \in [-1, 0)$ and $\beta \in (-1, 0]$. Then $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X.

Proposition 3. *Every neutrosophic $\mathcal{N}$-ideal $X_{\mathbf{N}}$ of X satisfies the following assertions:*

$$(x, y \in X)\,(x \preceq y \Rightarrow T_N(x) \leq T_N(y), I_N(x) \geq I_N(y), F_N(x) \leq F_N(y)) \tag{7}$$

Proof. Let $x, y \in X$ be such that $x \preceq y$. Then $x * y = \theta$, and so

$$
\begin{aligned}
&T_N(x) \leq \bigvee\{T_N(x*y), T_N(y)\} = \bigvee\{T_N(\theta), T_N(y)\} = T_N(y) \\
&I_N(x) \geq \bigwedge\{I_N(x*y), I_N(y)\} = \bigwedge\{I_N(\theta), I_N(y)\} = I_N(y) \\
&F_N(x) \leq \bigvee\{F_N(x*y), F_N(y)\} = \bigvee\{F_N(\theta), F_N(y)\} = F_N(y)
\end{aligned}
$$

This completes the proof. □

Proposition 4. *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-ideal of X. Then*

(1) $T_N(x*y) \leq T_N((x*y)*y) \Leftrightarrow T_N((x*z)*(y*z)) \leq T_N((x*y)*z)$
(2) $I_N(x*y) \geq I_N((x*y)*y) \Leftrightarrow I_N((x*z)*(y*z)) \geq I_N((x*y)*z)$
(3) $F_N(x*y) \leq F_N((x*y)*y) \Leftrightarrow F_N((x*z)*(y*z)) \leq F_N((x*y)*z)$

for all $x, y, z \in X$.

Proof. Note that
$$((x * (y * z)) * z) * z \preceq (x * y) * z \tag{8}$$
for all $x, y, z \in X$. Assume that $T_N(x * y) \leq T_N((x * y) * y)$, $I_N(x * y) \geq I_N((x * y) * y)$ and $F_N(x * y) \leq F_N((x * y) * y)$ for all $x, y \in X$. It follows from Equation (2) and Proposition 3 that

$$\begin{aligned} T_N((x * z) * (y * z)) &= T_N((x * (y * z)) * z) \\ &\leq T_N(((x * (y * z)) * z) * z) \\ &\leq T_N((x * y) * z) \end{aligned}$$

$$\begin{aligned} I_N((x * z) * (y * z)) &= I_N((x * (y * z)) * z) \\ &\geq I_N(((x * (y * z)) * z) * z) \\ &\geq I_N((x * y) * z) \end{aligned}$$

and

$$\begin{aligned} F_N((x * z) * (y * z)) &= F_N((x * (y * z)) * z) \\ &\leq F_N(((x * (y * z)) * z) * z) \\ &\leq F_N((x * y) * z) \end{aligned}$$

for all $x, y \in X$.

Conversely, suppose

$$\begin{aligned} T_N((x * z) * (y * z)) &\leq T_N((x * y) * z) \\ I_N((x * z) * (y * z)) &\geq I_N((x * y) * z) \\ F_N((x * z) * (y * z)) &\leq F_N((x * y) * z) \end{aligned} \tag{9}$$

for all $x, y, z \in X$. If we substitute z for y in Equation (9), then

$$\begin{aligned} T_N(x * z) &= T_N((x * z) * \theta) = T_N((x * z) * (z * z)) \leq T_N((x * z) * z) \\ I_N(x * z) &= I_N((x * z) * \theta) = I_N((x * z) * (z * z)) \geq I_N((x * z) * z) \\ F_N(x * z) &= F_N((x * z) * \theta) = F_N((x * z) * (z * z)) \leq F_N((x * z) * z) \end{aligned}$$

for all $x, z \in X$ by using (III) and Equation (1). □

Theorem 5. *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X and let $\alpha, \beta, \gamma \in [-1, 0]$ be such that $-3 \leq \alpha + \beta + \gamma \leq 0$. If $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X, then the nonempty (α, β, γ)-level set of $X_{\mathbf{N}}$ is an ideal of X.*

Proof. Assume that $X_{\mathbf{N}}(\alpha, \beta, \gamma) \neq \emptyset$ for $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Clearly, $\theta \in X_{\mathbf{N}}(\alpha, \beta, \gamma)$. Let $x, y \in X$ be such that $x * y \in X_{\mathbf{N}}(\alpha, \beta, \gamma)$ and $y \in X_{\mathbf{N}}(\alpha, \beta, \gamma)$. Then $T_N(x * y) \leq \alpha$, $I_N(x * y) \geq \beta$, $F_N(x * y) \leq \gamma$, $T_N(y) \leq \alpha$, $I_N(y) \geq \beta$ and $F_N(y) \leq \gamma$. It follows from Equation (6) that

$$\begin{aligned} T_N(x) &\leq \bigvee\{T_N(x * y), T_N(y)\} \leq \alpha \\ I_N(x) &\geq \bigwedge\{I_N(x * y), I_N(y)\} \geq \beta \\ F_N(x) &\leq \bigvee\{F_N(x * y), F_N(y)\} \leq \gamma \end{aligned}$$

so that $x \in X_{\mathbf{N}}(\alpha, \beta, \gamma)$. Therefore $X_{\mathbf{N}}(\alpha, \beta, \gamma)$ is an ideal of X. □

Theorem 6. *Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X and assume that T_N^{α}, I_N^{β} and F_N^{γ} are ideals of X for all $\alpha, \beta, \gamma \in [-1, 0]$ with $-3 \leq \alpha + \beta + \gamma \leq 0$. Then $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X.*

Proof. If there exist $a, b, c \in X$ such that $T_N(\theta) > T_N(a)$, $I_N(\theta) < I_N(b)$ and $F_N(\theta) > F_N(c)$, respectively, then $T_N(\theta) > a_t \geq T_N(a)$, $I_N(\theta) < b_i \leq I_N(b)$ and $F_N(\theta) > c_f \geq F_N(c)$ for some $a_t, c_f \in [-1, 0)$ and $b_i \in (-1, 0]$. Then $\theta \notin T_N^{a_t}$, $\theta \notin I_N^{b_i}$ and $\theta \notin F_N^{c_f}$. This is a contradiction. Hence, $T_N(\theta) \leq T_N(x)$, $I_N(\theta) \geq I_N(x)$ and $F_N(\theta) \leq F_N(x)$ for all $x \in X$. Assume that there exist $a_t, b_t, a_i, b_i, a_f, b_f \in X$ such that $T_N(a_t) > \bigvee\{T_N(a_t * b_t), T_N(b_t)\}$, $I_N(a_i) < \bigwedge\{I_N(a_i * b_i), I_N(b_i)\}$ and $F_N(a_f) > \bigvee\{F_N(a_f * b_f), F_N(b_f)\}$. Then there exist $s_t, s_f \in [-1, 0)$ and $s_i \in (-1, 0]$ such that

$$\begin{aligned}
T_N(a_t) &> s_t \geq \bigvee\{T_N(a_t * b_t), T_N(b_t)\} \\
I_N(a_i) &< s_i \leq \bigwedge\{I_N(a_i * b_i), I_N(b_i)\} \\
F_N(a_f) &> s_f \geq \bigvee\{F_N(a_f * b_f), F_N(b_f)\}
\end{aligned}$$

It follows that $a_t * b_t \in T_N^{s_t}$, $b_t \in T_N^{s_t}$, $a_i * b_i \in I_N^{s_i}$, $b_i \in I_N^{s_i}$, $a_f * b_f \in F_N^{s_f}$ and $b_f \in F_N^{s_f}$. However, $a_t \notin T_N^{s_t}$, $a_i \notin I_N^{s_i}$ and $a_f \notin F_N^{s_f}$. This is a contradiction, and so

$$\begin{aligned}
T_N(x) &\leq \bigvee\{T_N(x * y), T_N(y)\} \\
I_N(x) &\geq \bigwedge\{I_N(x * y), I_N(y)\} \\
F_N(x) &\leq \bigvee\{F_N(x * y), F_N(y)\}
\end{aligned}$$

for all $x, y \in X$. Therefore $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X. □

Proposition 5. *For any neutrosophic $\mathcal{N}$-ideal $X_{\mathbf{N}}$ of X, we have*

$$(\forall x, y, z \in X) \left(x * y \preceq z \;\Rightarrow \begin{cases} T_N(x) \leq \bigvee\{T_N(y), T_N(z)\} \\ I_N(x) \geq \bigwedge\{I_N(y), I_N(z)\} \\ F_N(x) \leq \bigvee\{F_N(y), F_N(z)\} \end{cases} \right) \tag{10}$$

Proof. Let $x, y, z \in X$ be such that $x * y \preceq z$. Then $(x * y) * z = \theta$, and so

$$\begin{aligned}
T_N(x * y) &\leq \bigvee\{T_N((x * y) * z), T_N(z)\} = \bigvee\{T_N(\theta), T_N(z)\} = T_N(z) \\
I_N(x * y) &\geq \bigwedge\{I_N((x * y) * z), I_N(z)\} = \bigwedge\{I_N(\theta), I_N(z)\} = I_N(z) \\
F_N(x * y) &\leq \bigvee\{F_N((x * y) * z), F_N(z)\} = \bigvee\{F_N(\theta), F_N(z)\} = F_N(z)
\end{aligned}$$

It follows that

$$\begin{aligned}
T_N(x) &\leq \bigvee\{T_N(x * y), T_N(y)\} \leq \bigvee\{T_N(y), T_N(z)\} \\
I_N(x) &\geq \bigwedge\{I_N(x * y), I_N(y)\} \geq \bigwedge\{I_N(y), I_N(z)\} \\
F_N(x) &\leq \bigvee\{F_N(x * y), F_N(y)\} \leq \bigvee\{F_N(y), F_N(z)\}
\end{aligned}$$

This completes the proof. □

Theorem 7. *In a BCK-algebra, every neutrosophic $\mathcal{N}$-ideal is a neutrosophic $\mathcal{N}$-subalgebra.*

Proof. Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-ideal of a BCK-algebra X. For any $x, y \in X$, we have

$$
\begin{aligned}
T_N(x*y) &\leq \bigvee\{T_N((x*y)*x), T_N(x)\} = \bigvee\{T_N((x*x)*y), T_N(x)\} \\
&= \bigvee\{T_N(\theta*y), T_N(x)\} = \bigvee\{T_N(\theta), T_N(x)\} \\
&\leq \bigvee\{T_N(x), T_N(y)\}
\end{aligned}
$$

$$
\begin{aligned}
I_N(x*y) &\geq \bigwedge\{I_N((x*y)*x), I_N(x)\} = \bigwedge\{I_N((x*x)*y), I_N(x)\} \\
&= \bigwedge\{I_N(\theta*y), I_N(x)\} = \bigwedge\{I_N(\theta), I_N(x)\} \\
&\geq \bigwedge\{I_N(y), I_N(x)\}
\end{aligned}
$$

and

$$
\begin{aligned}
F_N(x*y) &\leq \bigvee\{F_N((x*y)*x), F_N(x)\} = \bigvee\{F_N((x*x)*y), F_N(x)\} \\
&= \bigvee\{F_N(\theta*y), F_N(x)\} = \bigvee\{F_N(\theta), F_N(x)\} \\
&\leq \bigvee\{F_N(x), F_N(y)\}
\end{aligned}
$$

Hence $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra of a BCK-algebra X. $\square$

The converse of Theorem 7 may not be true in general, as seen in the following example.

Example 4. *Consider a BCK-algebra $X = \{\theta, 1, 2, 3, 4\}$ with the following Cayley table.*

$*$	$\boldsymbol{\theta}$	**1**	**2**	**3**	**4**
θ	θ	θ	θ	θ	θ
1	1	θ	θ	θ	θ
2	2	1	θ	1	θ
3	3	3	3	θ	θ
4	4	4	4	3	θ

Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X, which is given as follows:

$$
\begin{aligned}
X_{\mathbf{N}} = \Big\{ &\tfrac{\theta}{(-0.8,0,-1)}, \tfrac{1}{(-0.8,-0.2,-0.9)}, \\
&\tfrac{2}{(-0.2,-0.6,-0.5)}, \tfrac{3}{(-0.7,-0.4,-0.7)}, \tfrac{4}{(-0.4,-0.8,-0.3)} \Big\}
\end{aligned}
$$

Then $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra of X, but it is not a neutrosophic $\mathcal{N}$-ideal of X as $T_N(2) = -0.2 > -0.7 = \bigvee\{T_N(2*3), T_N(3)\}$, $I_N(4) = -0.8 < -0.4 = \bigwedge\{I_N(4*3), I_N(3)\}$, or $F_N(4) = -0.3 > -0.7 = \bigvee\{F_N(4*3), F_N(3)\}$.

Theorem 7 is not valid in a BCI-algebra; that is, if X is a BCI-algebra, then there is a neutrosophic $\mathcal{N}$-ideal that is not a neutrosophic $\mathcal{N}$-subalgebra, as seen in the following example.

Example 5. *Consider the neutrosophic $\mathcal{N}$-ideal $X_{\mathbf{N}}$ of X in Example 3. If we take $x := (\theta, 0)$ and $y := (\theta, 1)$ in $Y \times (\mathbb{N} \cup \{0\})$, then $x * y = (\theta, 0) * (\theta, 1) = (\theta, -1) \notin Y \times (\mathbb{N} \cup \{0\})$. Hence*

$$
\begin{aligned}
T_N(x*y) &= 0 > \alpha = \bigvee\{T_N(x), T_N(y)\} \\
I_N(x*y) &= \beta < 0 = \bigwedge\{I_N(x), I_N(y)\} \text{ or} \\
F_N(x*y) &= 0 > \gamma = \bigvee\{F_N(x), F_N(y)\}
\end{aligned}
$$

Therefore $X_{\mathbf{N}}$ is not a neutrosophic $\mathcal{N}$-subalgebra of X.

For any elements $\omega_t, \omega_i, \omega_f \in X$, we consider sets:

$$\begin{aligned} X_{\mathbf{N}}^{\omega_t} &:= \{x \in X \mid T_N(x) \leq T_N(\omega_t)\} \\ X_{\mathbf{N}}^{\omega_i} &:= \{x \in X \mid I_N(x) \geq I_N(\omega_i)\} \\ X_{\mathbf{N}}^{\omega_f} &:= \left\{x \in X \mid F_N(x) \leq F_N(\omega_f)\right\} \end{aligned}$$

Clearly, $\omega_t \in X_{\mathbf{N}}^{\omega_t}$, $\omega_i \in X_{\mathbf{N}}^{\omega_i}$ and $\omega_f \in X_{\mathbf{N}}^{\omega_f}$.

Theorem 8. *Let ω_t, ω_i and ω_f be any elements of X. If $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X, then $X_{\mathbf{N}}^{\omega_t}$, $X_{\mathbf{N}}^{\omega_i}$ and $X_{\mathbf{N}}^{\omega_f}$ are ideals of X.*

Proof. Clearly, $\theta \in X_{\mathbf{N}}^{\omega_t}$, $\theta \in X_{\mathbf{N}}^{\omega_i}$ and $\theta \in X_{\mathbf{N}}^{\omega_f}$. Let $x, y \in X$ be such that $x * y \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$ and $y \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$. Then

$$\begin{aligned} &T_N(x * y) \leq T_N(\omega_t),\ T_N(y) \leq T_N(\omega_t) \\ &I_N(x * y) \geq I_N(\omega_i),\ I_N(y) \geq I_N(\omega_i) \\ &F_N(x * y) \leq F_N(\omega_f),\ F_N(y) \leq F_N(\omega_f) \end{aligned}$$

It follows from Equation (6) that

$$\begin{aligned} &T_N(x) \leq \bigvee\{T_N(x * y), T_N(y)\} \leq T_N(\omega_t) \\ &I_N(x) \geq \bigwedge\{I_N(x * y), I_N(y)\} \geq I_N(\omega_i) \\ &F_N(x) \leq \bigvee\{F_N(x * y), F_N(y)\} \leq F_N(\omega_f) \end{aligned}$$

Hence $x \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$, and therefore $X_{\mathbf{N}}^{\omega_t}$, $X_{\mathbf{N}}^{\omega_i}$ and $X_{\mathbf{N}}^{\omega_f}$ are ideals of X. □

Theorem 9. *Let $\omega_t, \omega_i, \omega_f \in X$ and let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X. Then*

(1) *If $X_{\mathbf{N}}^{\omega_t}$, $X_{\mathbf{N}}^{\omega_i}$ and $X_{\mathbf{N}}^{\omega_f}$ are ideals of X, then the following assertion is valid:*

$$(\forall x, y, z \in X) \left(\begin{array}{l} T_N(x) \geq \bigvee\{T_N(y * z), T_N(z)\} \ \Rightarrow\ T_N(x) \geq T_N(y) \\ I_N(x) \leq \bigwedge\{I_N(y * z), I_N(z)\} \ \Rightarrow\ I_N(x) \leq I_N(y) \\ F_N(x) \geq \bigvee\{F_N(y * z), F_N(z)\} \ \Rightarrow\ F_N(x) \geq F_N(y) \end{array} \right) \tag{11}$$

(2) *If $X_{\mathbf{N}}$ satisfies Equation (11) and*

$$(\forall x \in X)\ (T_N(\theta) \leq T_N(x), I_N(\theta) \geq I_N(x), F_N(\theta) \leq F_N(x)) \tag{12}$$

then $X_{\mathbf{N}}^{\omega_t}$, $X_{\mathbf{N}}^{\omega_i}$ and $X_{\mathbf{N}}^{\omega_f}$ are ideals of X for all $\omega_t \in \mathrm{Im}(T_N)$, $\omega_i \in \mathrm{Im}(I_N)$ and $\omega_f \in \mathrm{Im}(F_N)$.

Proof. (1) Assume that $X_{\mathbf{N}}^{\omega_t}$, $X_{\mathbf{N}}^{\omega_i}$ and $X_{\mathbf{N}}^{\omega_f}$ are ideals of X for $\omega_t, \omega_i, \omega_f \in X$. Let $x, y, z \in X$ be such that $T_N(x) \geq \bigvee\{T_N(y * z), T_N(z)\}$, $I_N(x) \leq \bigwedge\{I_N(y * z), I_N(z)\}$ and $F_N(x) \geq \bigvee\{F_N(y * z), F_N(z)\}$. Then $y * z \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$ and $z \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$, where $\omega_t = \omega_i = \omega_f = x$. It follows from (I2) that $y \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$ for $\omega_t = \omega_i = \omega_f = x$. Hence $T_N(y) \leq T_N(\omega_t) = T_N(x)$, $I_N(y) \geq I_N(\omega_i) = I_N(x)$ and $F_N(y) \leq F_N(\omega_f) = F_N(x)$.

(2) Let $\omega_t \in \mathrm{Im}(T_N)$, $\omega_i \in \mathrm{Im}(I_N)$ and $\omega_f \in \mathrm{Im}(F_N)$ and suppose that $X_{\mathbf{N}}$ satisfies Equations (11) and (12). Clearly, $\theta \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$ by Equation (12). Let $x, y \in X$ be such that $x * y \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$ and $y \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$. Then

$$\begin{aligned} &T_N(x*y) \le T_N(\omega_t),\ T_N(y) \le T_N(\omega_t) \\ &I_N(x*y) \ge I_N(\omega_i),\ I_N(y) \ge I_N(\omega_i) \\ &F_N(x*y) \le F_N(\omega_f),\ F_N(y) \le F_N(\omega_f) \end{aligned}$$

which implies that $\bigvee\{T_N(x*y), T_N(y)\} \le T_N(\omega_t)$, $\bigwedge\{I_N(x*y), I_N(y)\} \ge I_N(\omega_i)$, and $\bigvee\{F_N(x*y), F_N(y)\} \le F_N(\omega_f)$. It follows from Equation (11) that $T_N(\omega_t) \ge T_N(x)$, $I_N(\omega_i) \le I_N(x)$ and $F_N(\omega_f) \ge F_N(x)$. Thus, $x \in X_{\mathbf{N}}^{\omega_t} \cap X_{\mathbf{N}}^{\omega_i} \cap X_{\mathbf{N}}^{\omega_f}$, and therefore $X_{\mathbf{N}}^{\omega_t}$, $X_{\mathbf{N}}^{\omega_i}$ and $X_{\mathbf{N}}^{\omega_f}$ are ideals of X. □

Definition 3. *A neutrosophic $\mathcal{N}$-ideal $X_{\mathbf{N}}$ of X is said to be closed if it is a neutrosophic $\mathcal{N}$-subalgebra of X.*

Example 6. *Consider a BCI-algebra $X = \{\theta, 1, a, b, c\}$ with the following Cayley table.*

$*$	θ	1	a	b	c
θ	θ	θ	a	b	c
1	1	θ	a	b	c
a	a	a	θ	c	b
b	b	b	c	θ	a
c	c	c	b	a	θ

Let $X_{\mathbf{N}}$ be a neutrosophic $\mathcal{N}$-structure over X which is given as follows:

$$X_{\mathbf{N}} = \left\{ \frac{\theta}{(-0.9,-0.3,-0.8)}, \frac{1}{(-0.7,-0.4,-0.7)}, \frac{a}{(-0.6,-0.8,-0.3)}, \frac{b}{(-0.2,-0.6,-0.3)}, \frac{c}{(-0.2,-0.8,-0.5)} \right\}$$

Then $X_{\mathbf{N}}$ is a closed neutrosophic $\mathcal{N}$-ideal of X.

Theorem 10. *Let X be a BCI-algebra, For any $\alpha_1, \alpha_2, \gamma_1, \gamma_2 \in [-1, 0)$ and $\beta_1, \beta_2 \in (-1, 0]$ with $\alpha_1 < \alpha_2$, $\gamma_1 < \gamma_2$ and $\beta_1 > \beta_2$, let $X_{\mathbf{N}} := \frac{X}{(T_N, I_N, F_N)}$ be a neutrosophic $\mathcal{N}$-structure over X given as follows:*

$$T_N : X \to [-1,0],\ \ x \mapsto \begin{cases} \alpha_1 & \text{if } x \in X_+ \\ \alpha_2 & \text{otherwise} \end{cases}$$

$$I_N : X \to [-1,0],\ \ x \mapsto \begin{cases} \beta_1 & \text{if } x \in X_+ \\ \beta_2 & \text{otherwise} \end{cases}$$

$$F_N : X \to [-1,0],\ \ x \mapsto \begin{cases} \gamma_1 & \text{if } x \in X_+ \\ \gamma_2 & \text{otherwise} \end{cases}$$

where $X_+ = \{x \in X \mid \theta \preceq x\}$. Then $X_{\mathbf{N}}$ is a closed neutrosophic $\mathcal{N}$-ideal of X.

Proof. Because $\theta \in X_+$, we have $T_N(\theta) = \alpha_1 \le T_N(x)$, $I_N(\theta) = \beta_1 \ge I_N(x)$ and $F_N(\theta) = \gamma_1 \le F_N(x)$ for all $x \in X$. Let $x, y \in X$. If $x \in X_+$, then

$$\begin{aligned} &T_N(x) = \alpha_1 \le \bigvee\{T_N(x*y), T_N(y)\} \\ &I_N(x) = \beta_1 \ge \bigwedge\{I_N(x*y), I_N(y)\} \\ &F_N(x) = \gamma_1 \le \bigvee\{F_N(x*y), F_N(y)\} \end{aligned}$$

Suppose that $x \notin X_+$. If $x * y \in X_+$ then $y \notin X_+$, and if $y \in X_+$ then $x * y \notin X_+$. In either case, we have

$$\begin{aligned} T_N(x) &= \alpha_2 = \bigvee\{T_N(x*y), T_N(y)\} \\ I_N(x) &= \beta_2 = \bigwedge\{I_N(x*y), I_N(y)\} \\ F_N(x) &= \gamma_2 = \bigvee\{F_N(x*y), F_N(y)\} \end{aligned}$$

For any $x, y \in X$, if any one of x and y does not belong to X_+, then

$$\begin{aligned} T_N(x*y) &\leq \alpha_2 = \bigvee\{T_N(x), T_N(y)\} \\ I_N(x*y) &\geq \beta_2 = \bigwedge\{I_N(x), I_N(y)\} \\ F_N(x*y) &\leq \gamma_2 = \bigvee\{F_N(x), F_N(y)\} \end{aligned}$$

If $x, y \in X_+$, then $x * y \in X_+$. Hence

$$\begin{aligned} T_N(x*y) &= \alpha_1 = \bigvee\{T_N(x), T_N(y)\} \\ I_N(x*y) &= \beta_1 = \bigwedge\{I_N(x), I_N(y)\} \\ F_N(x*y) &= \gamma_1 = \bigvee\{F_N(x), F_N(y)\} \end{aligned}$$

Therefore $X_{\mathbf{N}}$ is a closed neutrosophic $\mathcal{N}$-ideal of X. □

Proposition 6. *Every closed neutrosophic $\mathcal{N}$-ideal $X_{\mathbf{N}}$ of a BCI-algebra X satisfies the following condition:*

$$(\forall x \in X)\ (T_N(\theta * x) \leq T_N(x),\ I_N(\theta * x) \geq I_N(x),\ F_N(\theta * x) \leq F_N(x)) \tag{13}$$

Proof. Straightforward. □

We provide conditions for a neutrosophic $\mathcal{N}$-ideal to be closed.

Theorem 11. *Let X be a BCI-algebra. If $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-ideal of X that satisfies the condition of Equation (13), then $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra and hence is a closed neutrosophic $\mathcal{N}$-ideal of X.*

Proof. Note that $(x * y) * x \preceq \theta * y$ for all $x, y \in X$. Using Equations (10) and (13), we have

$$\begin{aligned} T_N(x*y) &\leq \bigvee\{T_N(x), T_N(\theta*y)\} \leq \bigvee\{T_N(x), T_N(y)\} \\ I_N(x*y) &\geq \bigwedge\{I_N(x), I_N(\theta*y)\} \geq \bigwedge\{I_N(x), I_N(y)\} \\ F_N(x*y) &\leq \bigvee\{F_N(x), F_N(\theta*y)\} \leq \bigvee\{F_N(x), F_N(y)\} \end{aligned}$$

Hence $X_{\mathbf{N}}$ is a neutrosophic $\mathcal{N}$-subalgebra and is therefore a closed neutrosophic $\mathcal{N}$-ideal of X. □

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TODIM Method for Single-Valued Neutrosophic Multiple Attribute Decision Making

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Abstract: Recently, the TODIM has been used to solve multiple attribute decision making (MADM) problems. The single-valued neutrosophic sets (SVNSs) are useful tools to depict the uncertainty of the MADM. In this paper, we will extend the TODIM method to the MADM with the single-valued neutrosophic numbers (SVNNs). Firstly, the definition, comparison, and distance of SVNNs are briefly presented, and the steps of the classical TODIM method for MADM problems are introduced. Then, the extended classical TODIM method is proposed to deal with MADM problems with the SVNNs, and its significant characteristic is that it can fully consider the decision makers' bounded rationality which is a real action in decision making. Furthermore, we extend the proposed model to interval neutrosophic sets (INSs). Finally, a numerical example is proposed.

Keywords: multiple attribute decision making (MADM); single-valued neutrosophic numbers; interval neutrosophic numbers; TODIM method; prospect theory

1. Introduction

Multiple attribute decision making (MADM) is a hot research area of the decision theory domain, which has had wide applications in many fields, and attracted increasing attention [1,2]. Due to the fuzziness and uncertainty of the alternatives in different attributes, attribute values in decision making problems are not always represented as real numbers, and they can be described as fuzzy numbers in more suitable occasions, such as interval-valued numbers [3,4], triangular fuzzy variables [5–8], linguistic variables [9–13] or uncertain linguistic variables [14–21], intuitionistic fuzzy numbers (IFSs) [22–27] or interval-valued intuitionistic fuzzy numbers (IVIFSs) [28–31], and SVNSs [32] or INSs [33]. Since Fuzzy set (FS), which is a very useful tool to process fuzzy information, was firstly proposed by Zadeh [34], it has been regarded as an useful tool to solve MADM [35,36], fuzzy logic [37], and patterns recognition [38]. Atanassov [22] introduced IFSs with the membership degree and non-membership degree, which were extended to IVIFSs [28]. Smarandache [39,40] proposed a neutrosophic set (NS) with truth-membership function, indeterminacy-membership function, and falsity-membership function. Furthermore, the concepts of a SVNS [32] and an INS [33] were presented for actual applications. Ye [41] proposed a simplified neutrosophic set (SNS), including the SVNS and INS. Recently, SNSs (INSs, and SVNSs) have been utilized to solve many MADM problems [42–67].

In order to depict the increasing complexity in the actual world, the DMs' risk attitudes should be taken into consideration to deal with MADM [68–70]. Based on the prospect theory, Gomes and Lima [71] established TODIM (an acronym in Portuguese for Interactive Multi-Criteria Decision Making) method to solve the MADM problems with the DMs' psychological behaviors are considered. Some scholars have paid attention to depict the DMs' attitudinal characters in the MADM [72–74]. Also, some scholars proposed fuzzy TODIM models [75,76], intuitionistic fuzzy

TODIM models [77,78], the Pythagorean fuzzy TODIM approach [68], the multi-hesitant fuzzy linguistic TODIM approach [79,80], the interval type-2 fuzzy TODIM model [81], the intuitionistic linguistic TODIM method [82], and the 2-dimension uncertain linguistic TODIM method [83]. However, there is no scholar to investigate the TODIM model with SVNNS. Therefore, it is very necessary to pay abundant attention to this novel and worthy issue. The aim of this paper is to extend the TODIM idea to solve the MADM with the SVNNs, to fill up this vacancy. In Section 2, we give the basic concepts of SVNSs and the classical TODIM method for MADM problems. In Section 3, we propose the TODIM method for SVN MADM problems. In Section 4, we extend the proposed SVN TODIM method to INNs. In Section 5, an illustrative example is pointed out and some comparative analysis is conducted. We give a conclusion in Section 6.

2. Preliminaries

Some basic concepts and definitions of NSs and SVNSs are introduced.

2.1. NSs and SVNSs

Definition 1 [39,40]. *Let X be a space of points (objects) with a generic element in fix set X, denoted by x. NSs A in X is characterized by a truth-membership function $T_A(x)$, an indeterminacy-membership $I_A(x)$ and a falsity-membership function $F_A(x)$, where $T_A(x) : X \to]^-0, 1^+[, I_A(x) : X \to]^-0, 1^+[$ and $F_A(x) : X \to]^-0, 1^+[$ and $0^- \leq supT_A(x) + supI_A(x) + supF_A(x) \leq 3^+$.*

The NSs was difficult to apply to real applications. Wang [32] develop the SNSs.

Definition 2 [32]. *Let X be a space of points (objects); a SVNSs A in X is characterized as the following:*

$$A = \{(x, T_A(x), I_A(x), F_A(x))|x \in X\} \tag{1}$$

where the truth-membership function $T_A(x)$, indeterminacy-membership $I_A(x)$ and falsity-membership function $F_A(x)$, $T_A(x) : X \to [0, 1], I_A(x) : X \to [0, 1]$ and $F_A(x) : X \to [0, 1]$, with the condition $0 \leq T_A(x) + I_A(x) + F_A(x) \leq 3$.

For convenience, a SVNN can be expressed to be $A = (T_A, I_A, F_A)$, $T_A \in [0, 1], I_A \in [0, 1], F_A \in [0, 1]$, and $0 \leq T_A + I_A + F_A \leq 3$.

Definition 3 [50]. *Let $A = (T_A, I_A, F_A)$ be a SVNN, a score function $S(A)$ is defined:*

$$S(A) = \frac{(2 + T_A - I_A - F_A)}{3}, S(A) \in [0, 1]. \tag{2}$$

Definition 4 [50]. *Let $A = (T_A, I_A, F_A)$ be a SVNN, an accuracy function $H(A)$ of a SVNN is defined:*

$$H(A) = T_A - F_A, H(A) \in [-1, 1]. \tag{3}$$

to evaluate the degree of accuracy of the SVNN $A = (T_A, I_A, F_A)$, where $H(A) \in [-1, 1]$. The larger the value of $H(A)$ is, the higher the degree of accuracy of the SVNN A.

Zhang et al. [50] gave an order relation between two SVNNs, which is defined as follows:

Definition 5 [50]. *Let $A = (T_A, I_A, F_A)$ and $B = (T_B, I_B, F_B)$ be two SVNNs, if $S(A) < S(B)$, then $A < B$; if $S(A) = S(B)$, then*

(1) *if $H(A) = H(B)$, then $A = B$;*
(2) *if $H(A) < H(B)$, then $A < B$.*

Definition 6 [32]. *Let A and B be two SVNNs, the basic operations of SVNNs are:*

(1) $A \oplus B = (T_A + T_B - T_A T_B, I_A I_B, F_A F_B)$;
(2) $A \otimes B = (T_A T_B, I_A + I_B - I_A I_B, F_A + F_B - F_A F_B)$;
(3) $\lambda A = \left(1 - (1 - T_A)^{\lambda}, (I_A)^{\lambda}, (F_A)^{\lambda}\right), \lambda > 0$;
(4) $(A)^{\lambda} = \left((T_A)^{\lambda}, (I_A)^{\lambda}, 1 - (1 - F_A)^{\lambda}\right), \lambda > 0$.

Definition 7 [42]. *Let A and B be two SVNNs, then the normalized Hamming distance between A and B is:*

$$d(A, B) = \frac{1}{3}(|T_A - T_B| + |I_A - I_B| + |F_A - F_B|) \tag{4}$$

2.2. The TODIM Approach

The TODIM approach [71], developed to consider the DM's psychological behavior, can effectively solve the MADM problems. Based on the prospect theory, this approach depicts the dominance of each alternative over others by constructing a function of multi-attribute values [69].

Let $G = \{G_1, G_2, \cdots, G_n\}$ be the attributes, $w = (w_1, w_2, \cdots, w_n)$ be the weight of G_j, $0 \le w_j \le 1$, and $\sum\limits_{j=1}^{n} w_j = 1$. $A = \{A_1, A_2, \cdots, A_m\}$ are alternatives. Let $A = (a_{ij})_{m\times n}$ be a decision matrix, where a_{ij} is given for the alternative A_i under the G_j, $i = 1, 2, \cdots, m$, and $j = 1, 2, \cdots, n$. We set $w_{jr} = w_j / w_r (j, r = 1, 2, \cdots, n)$ are relative weight of G_j to G_r, and $w_r = \max\{w_j | j = 1, 2, \cdots, n\}$, and $0 \le w_{jr} \le 1$.

Then the traditional TODIM model concludes the following computing steps:

Step 1. Normalizing $A = (a_{ij})_{m\times n}$ into $B = (b_{ij})_{m\times n}$.

Step 2. Computing the dominance degree of A_i over every alternative A_t under attribute G_j:

$$\delta(A_i, A_t) = \sum_{j=1}^{n} \phi_j(A_i, A_t), \ (i, t = 1, 2, \cdots, m) \tag{5}$$

where

$$\phi_j(A_i, A_t) = \begin{cases} \sqrt{w_{jr}(b_{ij} - b_{tj}) / \sum\limits_{j=1}^{n} w_{jr}}, & if\ b_{ij} - b_{tj} > 0 \\ 0, & if\ b_{ij} - b_{tj} = 0 \\ -\frac{1}{\theta}\sqrt{\left(\sum\limits_{j=1}^{n} w_{jr}\right)(b_{tj} - b_{ij}) / w_{jr}}, & if\ b_{ij} - b_{tj} < 0 \end{cases} \tag{6}$$

and the parameter θ shows the attenuation factor of the losses. If $b_{ij} - b_{tj} > 0$, then $\phi_j(A_i, A_t)$ represents a gain; if $b_{ij} - b_{tj} < 0$, then $\phi_j(A_i, A_t)$ signifies a loss.

Step 3. Deriving the overall dominance value of A_i by the Equation (7):

$$\phi(A_i) = \frac{\sum\limits_{t=1}^{m} \delta(A_i, A_t) - \min\limits_{i}\left\{\sum\limits_{t=1}^{m} \delta(A_i, A_t)\right\}}{\max\limits_{i}\left\{\sum\limits_{t=1}^{m} \delta(A_i, A_t)\right\} - \min\limits_{i}\left\{\sum\limits_{t=1}^{m} \delta(A_i, A_t)\right\}}, \ i = 1, 2, \cdots, m. \tag{7}$$

Step 4. Ranking all alternatives and selecting the most desirable alternative in accordance with $\phi(A_i)$. The alternative with minimum value is the worst. Inversely, the maximum value is the best one.

3. TODIM Method for SVN MADM Problems

Let $A = \{A_1, A_2, \cdots, A_m\}$ be alternatives, and $G = \{G_1, G_2, \cdots, G_n\}$ be attributes. Let $w = (w_1, w_2, \cdots, w_n)$ be the weight of attributes, where $w_j \in [0,1]$, $\sum_{j=1}^{n} w_j = 1$. Suppose that $R = (r_{ij})_{m\times n} = (T_{ij}, I_{ij}, F_{ij})_{m\times n}$ be a SVN matrix, where $\tilde{r}_{ij} = (T_{ij}, I_{ij}, F_{ij})$, which is an attribute value, given by an expert, for the alternative A_i under G_j, $T_{ij} \in [0,1], I_{ij} \in [0,1], F_{ij} \in [0,1]$, $0 \le T_{ij} + I_{ij} + F_{ij} \le 3$, $i = 1,2,\cdots,m$, $j = 1,2,\cdots,n$.

To solve the MADM problem with single-valued neutrosophic information, we try to present a single-valued neutrosophic TODIM model based on the prospect theory and can depict the DMs' behaviors under risk.

Firstly, we calculate the relative weight of each attribute G_j as:

$$w_{jr} = w_j / w_r,\ j, r = 1,2,\cdots,n. \tag{8}$$

where w_j is the weight of the attribute of G_j, $w_r = \max\{w_j | j = 1,2,\cdots,n\}$, and $0 \le w_{jr} \le 1$.

Based on the Equation (8), we can derive the dominance degree of A_i over each alternative A_t with respect to the attribute G_j:

$$\phi_j(A_i, A_t) = \begin{cases} \sqrt{w_{jr} d(r_{ij}, r_{tj}) / \sum_{j=1}^{n} w_{jr}}, & if\ r_{ij} > r_{tj} \\ 0, & if\ r_{ij} = r_{tj} \\ -\frac{1}{\theta}\sqrt{\left(\sum_{j=1}^{n} w_{jr}\right) d(r_{ij}, r_{tj}) / w_{jr}}, & if\ r_{ij} < r_{tj} \end{cases} \tag{9}$$

$$d(r_{ij}, r_{tj}) = \frac{1}{3}\left(|T_{ij} - T_{tj}| + |I_{ij} - I_{tj}| + |F_{ij} - F_{tj}|\right). \tag{10}$$

where the parameter θ shows the attenuation factor of the losses, and $d(r_{ij}, r_{tj})$ is to measure the distances between the SVNNs r_{ij} and r_{tj} by Definition 7. If $r_{ij} > r_{tj}$, then $\phi_j(A_i, A_t)$ represents a gain; if $r_{ij} < r_{tj}$, then $\phi_j(A_i, A_t)$ signifies a loss.

For indicating functions $\phi_j(A_i, A_t)$ clearly, a dominance degree matrix $\phi_j = [\phi_j(A_i, A_t)]_{m\times m}$ under G_j is expressed as:

$$\phi_j = [\phi_j(A_i, A_t)]_{m\times m} = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & \cdots & A_m \end{array} \\ \begin{bmatrix} o & \phi_j(A_1, A_2) & \cdots & \phi_j(A_1, A_m) \\ \phi_j(A_2, A_1) & 0 & \cdots & \phi_j(A_2, A_m) \\ \vdots & \vdots & \cdots & \vdots \\ \phi_j(A_m, A_1) & \phi_j(A_m, A_2) & \cdots & 0 \end{bmatrix} \end{array}, \ j = 1,2,\cdots,n\ . \tag{11}$$

On the basis of Equation (11), the overall dominance degree $\delta(A_i, A_t)$ of the A_i over each A_t can be calculated:

$$\delta(A_i, A_t) = \sum_{j=1}^{n} \phi_j(A_i, A_t), \quad (i, t = 1,2,\cdots,m). \tag{12}$$

Thus, the overall dominance degree matrix $\delta = [\delta(A_i, A_t)]_{m\times m}$ can be derived by Equation (12):

$$\delta = [\delta(A_i, A_t)]_{m\times m} = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & \cdots & A_m \end{array} \\ \begin{bmatrix} o & \delta(A_1, A_2) & \cdots & \delta(A_1, A_m) \\ \delta(A_2, A_1) & 0 & \cdots & \delta(A_2, A_m) \\ \vdots & \vdots & \cdots & \vdots \\ \delta(A_m, A_1) & \delta(A_m, A_2) & \cdots & 0 \end{bmatrix} \end{array}. \tag{13}$$

Then, the overall value of each A_i can be calculated Equation (14):

$$\delta(A_i) = \frac{\sum_{t=1}^{m} \delta(A_i, A_t) - \min_i \left\{ \sum_{t=1}^{m} \delta(A_i, A_t) \right\}}{\max_i \left\{ \sum_{t=1}^{m} \delta(A_i, A_t) \right\} - \min_i \left\{ \sum_{t=1}^{m} \delta(A_i, A_t) \right\}}, \quad i = 1, 2, \cdots, m. \tag{14}$$

Also the greater the overall value $\delta(A_i)$, the better the alternative A_i.

In general, single-valued neutrosophic TODIM model includes the computing steps:

(Procedure one)

Step 1. Identifying the single-valued neutrosophic matrix $R = (r_{ij})_{m\times n} = (T_{ij}, I_{ij}, F_{ij})_{m\times n}$ in the MADM, where r_{ij} is a SVNN.

Step 2. Calculating the relative weight of G_j by using Equation (8).

Step 3. Calculating the dominance degree $\phi_j(A_i, A_t)$ of A_i over each alternative A_t under attribute G_j by Equation (9).

Step 4. Calculating the overall dominance degree $\delta(A_i, A_t)$ of A_i over each alternative A_t by using Equation (12).

Step 5. Deriving the overall value $\delta(A_i)$ of each alternative A_i using Equation (14).

Step 6. Determining the order of the alternatives in accordance with $\delta(A_i)(i = 1, 2, \cdots, m)$.

4. TODIM Method for Interval Neutrosophic MADM Problems

Furthermore, Wang et al. [33] defined INSs.

Definition 8 [33]. *Let X be a space of points (objects) with a generic element in fix set X, an INSs $\widetilde{A}$ in X is characterized as follows:*

$$\widetilde{A} = \left\{ \left(x, T_{\widetilde{A}}(x), I_{\widetilde{A}}(x), F_{\widetilde{A}}(x) \right) | x \in X \right\} \tag{15}$$

where truth-membership function $T_{\widetilde{A}}(x)$, indeterminacy-membership $I_{\widetilde{A}}(x)$ and falsity-membership function $F_{\widetilde{A}}(x)$ are interval values, $T_A(x) \subseteq [0,1], I_A(x) \subseteq [0,1]$ and $F_A(x) \subseteq [0,1]$, and $0 \le \sup\left(T_{\widetilde{A}}(x)\right) + \sup\left(I_{\widetilde{A}}(x)\right) + \sup\left(F_{\widetilde{A}}(x)\right) \le 3$.

An interval neutrosophic number (INN) can be expressed as $\widetilde{A} = \left(T_{\widetilde{A}}, I_{\widetilde{A}}, F_{\widetilde{A}}\right) = \left(\left[T^L_{\widetilde{A}}, T^R_{\widetilde{A}}\right], \left[I^L_{\widetilde{A}}, I^R_{\widetilde{A}}\right], \left[F^L_{\widetilde{A}}, F^R_{\widetilde{A}}\right]\right)$, where $\left[T^L_{\widetilde{A}}, T^R_{\widetilde{A}}\right] \subseteq [0,1], \left[I^L_{\widetilde{A}}, I^R_{\widetilde{A}}\right] \subseteq [0,1], \left[F^L_{\widetilde{A}}, F^R_{\widetilde{A}}\right] \subseteq [0,1]$, and $0 \le T^R_{\widetilde{A}} + I^R_{\widetilde{A}} + F^R_{\widetilde{A}} \le 3$.

Definition 9 [84]. *Let $\widetilde{A} = \left(\left[T^L_{\widetilde{A}}, T^R_{\widetilde{A}}\right], \left[I^L_{\widetilde{A}}, I^R_{\widetilde{A}}\right], \left[F^L_{\widetilde{A}}, F^R_{\widetilde{A}}\right]\right)$ be an INN, a score function S of an INN can be represented as follows:*

$$S\left(\widetilde{A}\right) = \frac{\left(2 + T^L_{\widetilde{A}} - I^L_{\widetilde{A}} - F^L_{\widetilde{A}}\right) + \left(2 + T^R_{\widetilde{A}} - I^R_{\widetilde{A}} - F^R_{\widetilde{A}}\right)}{6}, S\left(\widetilde{A}\right) \in [0,1]. \tag{16}$$

Definition 10 [84]. *Let $\widetilde{A} = \left(\left[T^L_{\widetilde{A}}, T^R_{\widetilde{A}}\right], \left[I^L_{\widetilde{A}}, I^R_{\widetilde{A}}\right], \left[F^L_{\widetilde{A}}, F^R_{\widetilde{A}}\right]\right)$ be an INN, an accuracy function $H\left(\widetilde{A}\right)$ is defined:*

$$H\left(\widetilde{A}\right) = \frac{\left(T^L_{\widetilde{A}} + T^R_{\widetilde{A}}\right) - \left(F^L_{\widetilde{A}} + F^R_{\widetilde{A}}\right)}{2}, H\left(\widetilde{A}\right) \in [-1,1]. \tag{17}$$

Tang [84] defined an order relation between two INNs.

Definition 11 [84]**.** *Let* $\widetilde{A} = \left(\left[T^L_{\widetilde{A}}, T^R_{\widetilde{A}}\right], \left[I^L_{\widetilde{A}}, I^R_{\widetilde{A}}\right], \left[F^L_{\widetilde{A}}, F^R_{\widetilde{A}}\right]\right)$ *and* $\widetilde{B} = \left(\left[T^L_{\widetilde{B}}, T^R_{\widetilde{B}}\right], \left[I^L_{\widetilde{B}}, I^R_{\widetilde{B}}\right], \left[F^L_{\widetilde{B}}, F^R_{\widetilde{B}}\right]\right)$ *be two INNs,* $S\left(\widetilde{A}\right) = \frac{\left(2+T^L_{\widetilde{A}}-I^L_{\widetilde{A}}-F^L_{\widetilde{A}}\right)+\left(2+T^R_{\widetilde{A}}-I^R_{\widetilde{A}}-F^R_{\widetilde{A}}\right)}{6}$ *and* $S\left(\widetilde{B}\right) = \frac{\left(2+T^L_{\widetilde{B}}-I^L_{\widetilde{B}}-F^L_{\widetilde{B}}\right)+\left(2+T^R_{\widetilde{B}}-I^R_{\widetilde{B}}-F^R_{\widetilde{B}}\right)}{6}$ *be the scores, and* $H\left(\widetilde{A}\right) = \frac{\left(T^L_{\widetilde{A}}+T^R_{\widetilde{A}}\right)-\left(F^L_{\widetilde{A}}+F^R_{\widetilde{A}}\right)}{2}$ *and* $H\left(\widetilde{B}\right) = \frac{\left(T^L_{\widetilde{B}}+T^R_{\widetilde{B}}\right)-\left(F^L_{\widetilde{B}}+F^R_{\widetilde{B}}\right)}{2}$ *be the accuracy function, then if* $S\left(\widetilde{A}\right) < S\left(\widetilde{B}\right)$*, then* $\widetilde{A} < \widetilde{B}$ *; if* $S\left(\widetilde{A}\right) = S\left(\widetilde{B}\right)$*, then*

(1) *if* $H\left(\widetilde{A}\right) = H\left(\widetilde{B}\right)$*, then* $\widetilde{A} = \widetilde{B}$ *;*

(2) *if* $H\left(\widetilde{A}\right) < H\left(\widetilde{B}\right)$*,* $\widetilde{A} < \widetilde{B}$*.*

Definition 12 [33,61]**.** *Let* $\widetilde{A}_1 = ([T^L_1, T^R_1], [I^L_1, I^R_1], [F^L_1, F^R_1])$ *and* $\widetilde{A}_2 = ([T^L_2, T^R_2], [I^L_2, I^R_2], [F^L_2, F^R_2])$ *be two INNs, and some basic operations on them are defined as follows:*

(1) $\widetilde{A}_1 \oplus \widetilde{A}_2 = ([T^L_1 + T^L_1 - T^L_1 T^L_1, T^R_1 + T^R_1 - T^R_1 T^R_1], [I^L_1 I^L_2, I^R_1 I^R_2], [F^L_1 F^L_2, F^R_1 F^R_2])$;

(2) $\widetilde{A}_1 \otimes \widetilde{A}_2 = \begin{pmatrix} [T^L_1 T^L_2, T^R_1 T^R_2], [I^L_1 + I^L_1 - I^L_1 I^L_1, I^R_1 + I^R_1 - I^R_1 I^R_1], \\ [F^L_1 + F^L_1 - F^L_1 F^L_1, F^R_1 + F^R_1 - F^R_1 F^R_1] \end{pmatrix}$;

(3) $\lambda \widetilde{A}_1 = \left(\left[1 - (1 - T^L_1)^\lambda, 1 - (1 - T^R_1)^\lambda\right], \left[(I^L_1)^\lambda, (I^R_1)^\lambda\right], \left[(F^L_1)^\lambda, (F^R_1)^\lambda\right]\right), \lambda > 0$;

(4) $\left(\widetilde{A}_1\right)^\lambda = \left(\left[(T^L_1)^\lambda, (T^R_1)^\lambda\right], \left[(I^L_1)^\lambda, (I^R_1)^\lambda\right], \left[1 - (1 - F^L_1)^\lambda, 1 - (1 - F^R_1)^\lambda\right]\right), \lambda > 0.$

Definition 13 [84]**.** *Let* $\widetilde{A}_1 = ([T^L_1, T^R_1], [I^L_1, I^R_1], [F^L_1, F^R_1])$ *and* $\widetilde{A}_2 = ([T^L_2, T^R_2], [I^L_2, I^R_2], [F^L_2, F^R_2])$ *be two INNs, then the normalized Hamming distance between* $\widetilde{A}_1 = ([T^L_1, T^R_1], [I^L_1, I^R_1], [F^L_1, F^R_1])$ *and* $\widetilde{A}_2 = ([T^L_2, T^R_2], [I^L_2, I^R_2], [F^L_2, F^R_2])$ *is defined as follows:*

$$d\left(\widetilde{A}_1, \widetilde{A}_2\right) = \frac{1}{6}\left(\left|T^L_1 - T^L_2\right| + \left|T^R_1 - T^R_2\right| + \left|I^L_1 - I^L_2\right| + \left|I^R_1 - I^R_2\right| + \left|F^L_1 - F^L_2\right| + \left|F^R_1 - F^R_2\right|\right) \tag{18}$$

Let A*,* G *and* w *be presented as in Section* 3*. Suppose that* $\widetilde{R} = (\widetilde{r}_{ij})_{m \times n} = \left(\left[T^L_{ij}, T^R_{ij}\right], \left[I^L_{ij}, I^R_{ij}\right], \left[F^L_{ij}, F^R_{ij}\right]\right)_{m \times n}$ *is the interval neutrosophic decision matrix, where* $\left[T^L_{ij}, T^R_{ij}\right], \left[I^L_{ij}, I^R_{ij}\right], \left[F^L_{ij}, F^R_{ij}\right]$ *is truth-membership function, indeterminacy-membership function and falsity-membership function,* $\left[T^L_{ij}, T^R_{ij}\right] \subseteq [0,1]$, $\left[I^L_{ij}, I^R_{ij}\right] \subseteq [0,1]$, $\left[F^L_{ij}, F^R_{ij}\right] \subseteq [0,1]$, $0 \leq T^R_{ij} + I^R_{ij} + F^R_{ij} \leq 3$, $i = 1, 2, \cdots, m$, $j = 1, 2, \cdots, n$.

To cope with the MADM with INNs, we develop interval neutrosophic TODIM model.

Firstly, we calculate the relative weight of each attribute G_j as:

$$w_{jr} = w_j / w_r, j, r = 1, 2, \cdots, n \tag{19}$$

where w_j is the weight of the attribute of G_j, $w_r = \max\{w_j | j = 1, 2, \cdots, n\}$, and $0 \leq w_{jr} \leq 1$.

Based on the Equation (20), we can derive the dominance degree of A_i over each alternative A_t with respect to the attribute G_j:

$$\phi_j(A_i, A_t) = \begin{cases} \sqrt{w_{jr} d(\widetilde{r}_{ij}, \widetilde{r}_{tj}) / \sum\limits_{j=1}^{n} w_{jr}}, & if\ \widetilde{r}_{ij} > \widetilde{r}_{tj} \\ 0, & if\ \widetilde{r}_{ij} = \widetilde{r}_{tj} \\ -\frac{1}{\theta}\sqrt{\left(\sum\limits_{j=1}^{n} w_{jr}\right) d(\widetilde{r}_{ij}, \widetilde{r}_{tj}) / w_{jr}}, & if\ \widetilde{r}_{ij} < \widetilde{r}_{tj} \end{cases} \tag{20}$$

$$d(\widetilde{r}_{ij},\widetilde{r}_{tj}) = \frac{1}{6}\left(\left|T_{ij}^L - T_{tj}^L\right| + \left|T_{ij}^R - T_{tj}^R\right| + \left|I_{ij}^L - I_{tj}^L\right| + \left|I_{ij}^R - I_{tj}^R\right| + \left|F_{ij}^L - F_{tj}^L\right| + \left|F_{ij}^R - F_{tj}^R\right|\right). \quad (21)$$

where the parameter θ shows the attenuation factor of the losses, and $d(\widetilde{r}_{ij},\widetilde{r}_{tj})$ is to measure the distances between the INNs $\widetilde{r}_{ij}$ and $\widetilde{r}_{tj}$ by Definition 13. If $\widetilde{r}_{ij} > \widetilde{r}_{tj}$, then $\phi_j(A_i, A_t)$ represents a gain; if $\widetilde{r}_{ij} < \widetilde{r}_{tj}$, then $\phi_j(A_i, A_t)$ signifies a loss.

For indicating functions $\phi_j(A_i, A_t)$ clearly, a dominance degree matrix $\phi_j = [\phi_j(A_i, A_t)]_{m\times m}$ under G_j is expressed as:

$$\phi_j = [\phi_j(A_i, A_t)]_{m\times m} = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & \cdots & A_m \end{array} \\ \begin{bmatrix} o & \phi_j(A_1,A_2) & \cdots & \phi_j(A_1,A_m) \\ \phi_j(A_2,A_1) & 0 & \cdots & \phi_j(A_2,A_m) \\ \vdots & \vdots & \cdots & \vdots \\ \phi_j(A_m,A_1) & \phi_j(A_m,A_2) & \cdots & 0 \end{bmatrix} \end{array}, j = 1,2,\cdots,n \quad (22)$$

On the basis of Equation (22), the overall dominance degree $\delta(A_i, A_t)$ of the A_i over each A_t can be calculated:

$$\delta(A_i, A_t) = \sum_{j=1}^{n} \phi_j(A_i, A_t), \quad (i,t = 1,2,\cdots,m) \quad (23)$$

Thus, the overall dominance degree matrix $\delta = [\delta(A_i, A_t)]_{m\times m}$ can be derived by Equation (23):

$$\delta = [\delta(A_i, A_t)]_{m\times m} = \begin{array}{c} \\ A_1 \\ A_2 \\ \vdots \\ A_m \end{array} \begin{array}{c} \begin{array}{cccc} A_1 & A_2 & \cdots & A_m \end{array} \\ \begin{bmatrix} o & \delta(A_1,A_2) & \cdots & \delta(A_1,A_m) \\ \delta(A_2,A_1) & 0 & \cdots & \delta(A_2,A_m) \\ \vdots & \vdots & \cdots & \vdots \\ \delta(A_m,A_1) & \delta(A_m,A_2) & \cdots & 0 \end{bmatrix} \end{array} \quad (24)$$

Then, the overall value of each A_i can be calculated Equation (25):

$$\delta(A_i) = \frac{\sum_{t=1}^{m} \delta(A_i, A_t) - \min_i \left\{ \sum_{t=1}^{m} \delta(A_i, A_t) \right\}}{\max_i \left\{ \sum_{t=1}^{m} \delta(A_i, A_t) \right\} - \min_i \left\{ \sum_{t=1}^{m} \delta(A_i, A_t) \right\}}, \; i = 1,2,\cdots,m. \quad (25)$$

Also the greater the overall value $\delta(A_i)$, the better the alternative A_i.

In general, interval neutrosophic TODIM model includes the computing steps:

(Procedure two)

Step 1. Identifying the interval neutrosophic matrix $\widetilde{R} = (\widetilde{r}_{ij})_{m\times n} = \left(\left[T_{ij}^L, T_{ij}^R\right], \left[I_{ij}^L, I_{ij}^R\right], \left[F_{ij}^L, F_{ij}^R\right]\right)_{m\times n}$ in the MADM, where $\widetilde{r}_{ij}$ is an INN.

Step 2. Calculating the relative weight of G_j by using Equation (19).

Step 3. Calculating the dominance degree $\phi_j(A_i, A_t)$ of A_i over each alternative A_t under attribute G_j by Equation (20).

Step 4. Calculating the overall dominance degree $\delta(A_i, A_t)$ of A_i over each alternative A_t by using Equation (23).

Step 5. Deriving the overall value $\delta(A_i)$ of each alternative A_i using Equation (25).

Step 6. Determining the order of the alternatives in accordance with $\delta(A_i)(i = 1,2,\cdots,m)$.

5. Numerical Example and Comparative Analysis

5.1. Numerical Example 1

In this part, a numerical example is given to show potential evaluation of emerging technology commercialization with SVNNs. Five possible emerging technology enterprises (ETEs) $A_i(i = 1, 2, 3, 4, 5)$ are to be evaluated and selected. Four attributes are selected to evaluate the five possible ETEs: ① G_1 is the employment creation; ② G_2 is the development of science and technology; ③ G_3 is the technical advancement; and ④ G_4 is the industrialization infrastructure. The five ETEs $A_i(i = 1, 2, 3, 4, 5)$ are to be evaluated by using the SVNNs under the above four attributes (whose weighting vector $\omega = (0.2, 0.1, 0.3, 0.4)^T$), as listed in the following matrix.

$$\widetilde{R} = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \begin{array}{cccc} G_1 & G_2 & G_3 & G_4 \end{array} \\ \begin{bmatrix} (0.5, 0.8, 0.1) & (0.6, 0.3, 0.3) & (0.3, 0.6, 0.1) & (0.5, 0.7, 0.2) \\ (0.7, 0.2, 0.1) & (0.7, 0.2, 0.2) & (0.7, 0.2, 0.4) & (0.8, 0.2, 0.1) \\ (0.6, 0.7, 0.2) & (0.5, 0.7, 0.3) & (0.5, 0.3, 0.1) & (0.6, 0.3, 0.2) \\ (0.8, 0.1, 0.3) & (0.6, 0.3, 0.4) & (0.3, 0.4, 0.2) & (0.5, 0.6, 0.1) \\ (0.6, 0.4, 0.4) & (0.4, 0.8, 0.1) & (0.7, 0.6, 0.1) & (0.5, 0.8, 0.2) \end{bmatrix} \end{array}$$

Then, we use **Procedure One** to select the best ETE.

Firstly, since $w_4 = \max\{w_1, w_2, w_3, w_4\}$, then G_4 is the reference attribute and the reference attribute's weight is $w_r = 0.4$. Then, we can calculate the relative weights of the attributes $G_j(j = 1, 2, 3, 4)$ as $w_{1r} = 0.50, w_{2r} = 0.25, w_{3r} = 0.75$ and $w_{4r} = 1.00$. Let $\theta = 2.5$, then the dominance degree matrix $\phi_j(A_i, A_t)(j = 1, 2, 3, 4)$ with respect to G_j can be calculated:

$$\phi_1 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \begin{array}{ccccc} A_1 & A_2 & A_3 & A_4 & A_5 \end{array} \\ \begin{bmatrix} 0.0000 & -0.4619 & -0.2828 & -0.5657 & -0.4619 \\ 0.2309 & 0.0000 & 0.2160 & 0.1633 & 0.2000 \\ 0.1414 & -0.4320 & 0.0000 & -0.4899 & -0.3651 \\ 0.2828 & -0.3266 & 0.2449 & 0.0000 & 0.2000 \\ 0.2309 & -0.4000 & 0.1826 & -0.4000 & 0.0000 \end{bmatrix} \end{array}$$

$$\phi_2 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \begin{array}{ccccc} A_1 & A_2 & A_3 & A_4 & A_5 \end{array} \\ \begin{bmatrix} 0.0000 & -0.4000 & 0.1291 & 0.0577 & 0.1732 \\ 0.1000 & 0.0000 & 0.1633 & 0.1155 & 0.1826 \\ -0.5164 & -0.6532 & 0.0000 & -0.5657 & -0.4619 \\ -0.2309 & -0.4619 & 0.1414 & 0.0000 & 0.1826 \\ -0.6928 & -0.7303 & 0.1155 & -0.7303 & 0.0000 \end{bmatrix} \end{array}$$

$$\phi_3 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \begin{array}{ccccc} A_1 & A_2 & A_3 & A_4 & A_5 \end{array} \\ \begin{bmatrix} 0.0000 & -0.4422 & -0.2981 & -0.2309 & -0.2667 \\ 0.3317 & 0.0000 & -0.3266 & 0.2828 & 0.2646 \\ 0.2236 & 0.2449 & 0.0000 & 0.2000 & 0.2236 \\ 0.1732 & -0.3771 & -0.2667 & 0.0000 & -0.3528 \\ 0.2000 & -0.3528 & -0.2981 & 0.2646 & 0.0000 \end{bmatrix} \end{array}$$

$$\phi_4 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \begin{array}{ccccc} A_1 & A_2 & A_3 & A_4 & A_5 \end{array} \\ \begin{bmatrix} 0.0000 & -0.3464 & -0.2582 & -0.1633 & 0.1155 \\ 0.3464 & 0.0000 & 0.2309 & 0.3055 & 0.3651 \\ 0.2582 & -0.2309 & 0.0000 & 0.2582 & 0.2828 \\ 0.1633 & -0.3055 & -0.2582 & 0.0000 & 0.2000 \\ -0.1155 & -0.3651 & -0.2828 & -0.2000 & 0.0000 \end{bmatrix} \end{array}$$

The overall dominance degree $\delta(A_i, A_t)$ of the candidate A_i over each candidate A_t can be derived by Equation (13):

$$\delta = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \begin{array}{ccccc} A_1 & A_2 & A_3 & A_4 & A_5 \end{array} \\ \begin{bmatrix} 0.0000 & -1.6505 & -0.7100 & -0.9022 & -0.4399 \\ 1.0090 & 0.0000 & 0.2836 & 0.8671 & 1.01234 \\ 0.1068 & -1.0712 & 0.0000 & -0.5974 & -0.3206 \\ 0.3884 & -1.4711 & -0.1386 & 0.0000 & 0.2298 \\ -0.3774 & -1.8482 & -0.2828 & -1.0657 & 0.0000 \end{bmatrix} \end{array}$$

Then, we get the overall value $\delta(A_i)(i = 1, 2, 3, 4, 5)$ by using Equation (14):

$$\begin{gathered} \delta(A_1) = 0.0000, \delta(A_2) = 1.0000, \delta(A_3) = 0.2648 \\ \delta(A_4) = 0.3944, \delta(A_5) = 0.0187 \end{gathered}$$

Finally, we get order of ETEs by $\delta(A_i)(i = 1, 2, 3, 4, 5)$: $A_2 \succ A_4 \succ A_3 \succ A_5 \succ A_1$, and thus the most desirable ETE is A_2.

5.2. Comparative Analysis 1

In what follows, we compare our proposed method with other existing methods including the SVNWA operator and SVNWG operator proposed by Sahin [85] as follows:

Definition 14 [85]. *Let $A_j = (T_j, I_j, F_j)$ $(j = 1, 2, \cdots, n)$ be a collection of SVNNs, $w = (w_1, w_2, \cdots, w_n)^T$ be the weight of $A_j(j = 1, 2, \cdots, n)$, and $w_j > 0$, $\sum_{j=1}^{n} w_j = 1$. Then*

$$\begin{aligned} r_i &= (T_i, I_i, F_i) \\ &= \text{SVNWA}_w(r_{i1}, r_{i2}, \cdots, r_{in}) = \mathop{\oplus}_{j=1}^{n} (w_j r_{ij}) \\ &= \left(1 - \prod_{j=1}^{n} (1 - T_{ij})^{w_j}, \prod_{j=1}^{n} (I_{ij})^{w_j}, \prod_{j=1}^{n} (F_{ij})^{w_j}\right) \end{aligned} \tag{26}$$

$$\begin{aligned} r_i &= (T_i, I_i, F_i) \\ &= \text{SVNWG}_\omega(r_{i1}, r_{i2}, \cdots, r_{in}) = \mathop{\otimes}_{j=1}^{n} (r_{ij})^{w_j} \\ &= \left(\prod_{j=1}^{n} (T_{ij})^{w_j}, 1 - \prod_{j=1}^{n} (1 - I_{ij})^{w_j}, 1 - \prod_{j=1}^{n} (1 - F_{ij})^{w_j}\right) \end{aligned} \tag{27}$$

By utilizing the $\widetilde{R}$, as well as the SVNWA and SVNWG operators, the aggregating values are derived in Table 1.

Table 1. The aggregating values of the emerging technology enterprises by the SVNWA (SVNWG) operators.

	SVNWA	**SVNWG**
A_1	(0.4591, 0.6307, 0.1473)	(0.4369, 0.6718, 0.1627)
A_2	(0.7449, 0.2000, 0.1625)	(0.7384, 0.2000, 0.2124)
A_3	(0.5627, 0.3868, 0.1692)	(0.5578, 0.4571, 0.1822)
A_4	(0.5497, 0.3464, 0.1762)	(0.4799, 0.4381, 0.2067)
A_5	(0.5822, 0.6389, 0.1741)	(0.5610, 0.6933, 0.2083)

According to the aggregating results in Table 1, the score functions are listed in Table 2.

Table 2. The score functions of the emerging technology enterprises.

	SVNWA	SVNWG
A_1	0.5604	0.5341
A_2	0.7942	0.7753
A_3	0.6689	0.6398
A_4	0.6757	0.6117
A_5	0.5898	0.5531

According to the score functions shown in Table 2, the order of the emerging technology enterprises are in Table 3.

Table 3. Order of the emerging technology enterprises.

	Order
SVNWA	$A_2 > A_4 > A_3 > A_5 > A_1$
SVNWG	$A_2 > A_3 > A_4 > A_5 > A_1$

From the above analysis, it can be seen that two operators have the same best emerging technology enterprise A_2 and two methods' ranking results are slightly different. However, the SVN TODIM approach can reasonably depict the DMs' psychological behaviors under risk, and thus, it may deal with the above issue effectively. This verifies the method we proposed is reasonable and effective in this paper.

5.3. Numerical Example 2

If the five possible emerging technology enterprises $A_i (i = 1, 2, 3, 4, 5)$ are to be evaluated by using the INNS under the above four attributes (whose weighting vector $\omega = (0.2, 0.1, 0.3, 0.4)^T$), as listed in the matrix $\widetilde{R}$, then:

$$\widetilde{R} = \begin{bmatrix} ([0.5,0.6],[0.8,0.9],[0.1,0.2]) & ([0.6,0.7],[0.3,0.4],[0.3,0.4]) & ([0.3,0.4],[0.6,0.7],[0.1,0.2]) & ([0.5,0.6],[0.7,0.8],[0.1,0.2]) \\ ([0.7,0.9],[0.2,0.3],[0.1,0.2]) & ([0.7,0.8],[0.1,0.2],[0.2,0.3]) & ([0.7,0.9],[0.2,0.3],[0.4,0.5]) & ([0.8,0.9],[0.2,0.3],[0.1,0.2]) \\ ([0.6,0.7],[0.7,0.8],[0.2,0.3]) & ([0.5,0.6],[0.7,0.8],[0.3,0.4]) & ([0.5,0.6],[0.3,0.4],[0.1,0.2]) & ([0.6,0.7],[0.3,0.4],[0.2,0.3]) \\ ([0.8,0.9],[0.1,0.2],[0.3,0.4]) & ([0.6,0.7],[0.3,0.4],[0.4,0.5]) & ([0.3,0.4],[0.4,0.5],[0.2,0.3]) & ([0.5,0.6],[0.6,0.7],[0.1,0.2]) \\ ([0.6,0.7],[0.4,0.5],[0.4,0.5]) & ([0.4,0.5],[0.8,0.9],[0.1,0.2]) & ([0.7,0.8],[0.6,0.7],[0.1,0.2]) & ([0.5,0.6],[0.8,0.9],[0.2,0.3]) \end{bmatrix}$$

Then, we use **Procedure Two** to select the best ETE.

Firstly, since $w_4 = \max\{w_1, w_2, w_3, w_4\}$, then G_4 is the reference attribute and the reference attribute's weight is $w_r = 0.4$. Then, we can calculate the relative weights of the attributes

$G_j (j = 1, 2, 3, 4)$ as: $w_{1r} = 0.50, w_{2r} = 0.25, w_{3r} = 0.75$ and $w_{4r} = 1.00$. Let $\theta = 2.5$, then the dominance degree matrix $\phi_j(A_i, A_t)(j = 1, 2, 3, 4)$ with respect to G_j can be calculated:

$$\phi_1 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \begin{array}{ccccc} A_1 & A_2 & A_3 & A_4 & A_5 \end{array} \\ \begin{bmatrix} 0.0000 & -0.4761 & -0.2828 & -0.5657 & -0.4619 \\ 0.2380 & 0.0000 & 0.2236 & 0.1528 & 0.2082 \\ 0.1414 & -0.4472 & 0.0000 & -0.4899 & -0.3651 \\ 0.2828 & -0.3055 & 0.2449 & 0.0000 & 0.2000 \\ 0.2309 & -0.4163 & 0.1826 & -0.4000 & 0.0000 \end{bmatrix} \end{array}$$

$$\phi_2 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \begin{array}{ccccc} A_1 & A_2 & A_3 & A_4 & A_5 \end{array} \\ \begin{bmatrix} 0.0000 & -0.4619 & 0.1291 & 0.0577 & 0.1732 \\ 0.1155 & 0.0000 & 0.1732 & 0.1291 & 0.1915 \\ -0.5164 & -0.6928 & 0.0000 & -0.5657 & -0.4619 \\ -0.2309 & -0.5164 & 0.1414 & 0.0000 & 0.1826 \\ -0.6928 & -0.7659 & 0.1155 & -0.7303 & 0.0000 \end{bmatrix} \end{array}$$

$$\phi_3 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \begin{array}{ccccc} A_1 & A_2 & A_3 & A_4 & A_5 \end{array} \\ \begin{bmatrix} 0.0000 & -0.4522 & -0.2981 & -0.2309 & -0.2667 \\ 0.3391 & 0.0000 & 0.2550 & 0.2915 & 0.2739 \\ 0.2236 & -0.3399 & 0.0000 & 0.2000 & 0.2236 \\ 0.1732 & -0.3887 & -0.2667 & 0.0000 & -0.3528 \\ 0.2000 & -0.3651 & -0.2981 & 0.2646 & 0.0000 \end{bmatrix} \end{array}$$

$$\phi_4 = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \begin{array}{ccccc} A_1 & A_2 & A_3 & A_4 & A_5 \end{array} \\ \begin{bmatrix} 0.0000 & -0.3266 & -0.2828 & -0.1155 & 0.1633 \\ 0.3266 & 0.0000 & 0.2309 & 0.3055 & 0.3651 \\ 0.2828 & -0.2309 & 0.0000 & 0.2582 & 0.2828 \\ 0.1155 & -0.3055 & -0.2582 & 0.0000 & 0.2000 \\ -0.1633 & -0.3651 & -0.2828 & -0.2000 & 0.0000 \end{bmatrix} \end{array}$$

The overall dominance degree $\delta(A_i, A_t)$ of the candidate A_i over each candidate A_t can be derived by Equation (24):

$$\delta = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \end{array} \begin{array}{c} \begin{array}{ccccc} A_1 & A_2 & A_3 & A_4 & A_5 \end{array} \\ \begin{bmatrix} 0.0000 & -1.7168 & -0.7346 & -0.7506 & 0.0698 \\ 1.0192 & 0.0000 & 0.3727 & 0.3513 & 0.8305 \\ 0.1314 & -1.0310 & 0.0000 & -0.4726 & 0.0445 \\ 0.3406 & -1.5161 & -0.1386 & 0.2000 & 0.0298 \\ -0.4252 & -1.9124 & -0.8654 & -0.6657 & 0.0000 \end{bmatrix} \end{array}$$

Then, we get the overall value $\delta(A_i)(i = 1, 2, 3, 4, 5)$ by using Equation (25):

$$\delta(A_1) = 0.1143, \delta(A_2) = 1.0000, \delta(A_3) = 0.3944$$
$$\delta(A_4) = 0.4322, \delta(A_5) = 0.0000$$

Finally, we get order of ETEs by $\delta(A_i)(i = 1, 2, 3, 4, 5)$: $A_2 \succ A_4 \succ A_3 \succ A_1 \succ A_5$, and thus the most desirable ETE is A_2.

5.4. Comparative Analysis 2

In what follows, we compare our proposed method with other existing methods including the INWA operator and INWG operator proposed by Zhang et al. [50] as follows:

Definition 15 [50]. *Let* $\widetilde{A}_j = \left(\left[T_j^L, T_j^R\right], \left[I_j^L, I_j^R\right], \left[F_j^L, F_j^R\right]\right)(j = 1, 2, \cdots, n)$ *be a collection of INNs,* $w = (w_1, w_2, \cdots, w_n)^T$ *be the weight of* $A_j(j = 1, 2, \cdots, n)$*, and* $w_j > 0$, $\sum_{j=1}^{n} w_j = 1$ *. Then*

$$\begin{aligned}
&\widetilde{r}_i = \left(\left[T_i^L, T_i^R\right], \left[I_i^L, I_i^R\right], \left[F_i^L, F_i^R\right]\right) \\
&= \mathrm{INWA}_w(\widetilde{r}_{i1}, \widetilde{r}_{i2}, \cdots, \widetilde{r}_{in}) = \bigoplus_{j=1}^{n} (w_j \widetilde{r}_{ij}) \\
&= \left(\begin{array}{l} \left[1 - \prod_{j=1}^{n}\left(1 - T_{ij}^L\right)^{w_j}, 1 - \prod_{j=1}^{n}\left(1 - T_{ij}^R\right)^{w_j}\right], \\ \left[\prod_{j=1}^{n}\left(I_{ij}^L\right)^{w_j}, \prod_{j=1}^{n}\left(I_{ij}^R\right)^{w_j}\right], \left[\prod_{j=1}^{n}\left(F_{ij}^L\right)^{w_j}, \prod_{j=1}^{n}\left(F_{ij}^R\right)^{w_j}\right] \end{array} \right)
\end{aligned} \tag{28}$$

$$\begin{aligned}
&\widetilde{r}_i = \left(\left[T_i^L, T_i^R\right], \left[I_i^L, I_i^R\right], \left[F_i^L, F_i^R\right]\right) \\
&= \mathrm{INWG}_w(\widetilde{r}_{i1}, \widetilde{r}_{i2}, \cdots, \widetilde{r}_{in}) = \bigotimes_{j=1}^{n} (\widetilde{r}_{ij})^{w_j} \\
&= \left(\begin{array}{l} \left[\prod_{j=1}^{n}\left(T_{ij}^L\right)^{w_j}, \prod_{j=1}^{n}\left(T_{ij}^R\right)^{w_j}\right], \left[1 - \prod_{j=1}^{n}\left(1 - I_{ij}^L\right)^{w_j}, 1 - \prod_{j=1}^{n}\left(1 - I_{ij}^R\right)^{w_j}\right], \\ \left[1 - \prod_{j=1}^{n}\left(1 - F_{ij}^L\right)^{w_j}, 1 - \prod_{j=1}^{n}\left(1 - F_{ij}^R\right)^{w_j}\right] \end{array} \right)
\end{aligned} \tag{29}$$

By utilizing the decision matrix $\widetilde{R}$, and the INWA and INWG operators, the aggregating values are in Table 4.

Table 4. The aggregating values of the emerging technology enterprises by the INWA and INWG operators.

	INWA
A_1	([0.4591, 0.5611], [0.6307, 0.7342], [0.1116, 0.2144])
A_2	([0.7449, 0.8928], [0.1866, 0.2881], [0.1625, 0.2742])
A_3	([0.5627, 0.6634], [0.3868, 0.4925], [0.1692, 0.2734])
A_4	([0.5497, 0.6674], [0.3464, 0.4657], [0.1762, 0.2844])
A_5	([0.5822, 0.6863], [0.6389, 0.7421], [0.1741, 0.2825])
	INWG
A_1	([0.4369, 0.5395], [0.6718, 0.7805], [0.1223, 0.2227])
A_2	([0.7384, 0.8895], [0.1905, 0.2906], [0.2124, 0.3144])
A_3	([0.5578, 0.6581], [0.4571, 0.5685], [0.1822, 0.2825])
A_4	([0.4799, 0.5851], [0.4381, 0.5440], [0.2067, 0.3077])
A_5	([0.5610, 0.6624], [0.6933, 0.8082], [0.2083, 0.3097])

According to the aggregating values in Table 4, the score functions are in Table 5.

Table 5. The score functions of the emerging technology enterprises.

	INWA	**INWG**
A_1	0.5549	0.5298
A_2	0.7877	0.7700
A_3	0.6507	0.6209
A_4	0.6574	0.5948
A_5	0.5718	0.5340

According to the score functions shown in Table 5, the order of the emerging technology enterprises are in Table 6.

Table 6. Order of the emerging technology enterprises.

	Ordering
INWA	$A_2 > A_4 > A_3 > A_5 > A_1$
INWG	$A_2 > A_3 > A_4 > A_5 > A_1$

From the above analysis, it can be seen that two operators have the same best emerging technology enterprise A_2 and two methods' ranking results are slightly different. However, the interval neutrosophic TODIM approach can reasonably depict the DMs' psychological behaviors under risk, and thus, it may deal with the above issue effectively. This verifies the method we proposed is reasonable and effective.

6. Conclusions

In this paper, we will extend the TODIM method to the MADM with the single-valued neutrosophic numbers (SVNNs). Firstly, the definition, comparison and distance of SVNNs are briefly presented, and the steps of the classical TODIM method for MADM problems are introduced. Then, the extended classical TODIM method is proposed to deal with MADM problems with the SVNNs, and its significant characteristic is that it can fully consider the decision makers' bounded rationality which is a real action in decision making. Furthermore, we extend the proposed model to interval neutrosophic sets (INSs). Finally, a numerical example is proposed to verify the developed approach.

In the future, the application of the proposed models and methods of SVNSs and INSs needs to be explored in the decision making [86–99], risk analysis and many other uncertain and fuzzy environment [100–112].

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Neutrosophic Similarity Score Based Weighted Histogram for Robust Mean-Shift Tracking

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Abstract: Visual object tracking is a critical task in computer vision. Challenging things always exist when an object needs to be tracked. For instance, background clutter is one of the most challenging problems. The mean-shift tracker is quite popular because of its efficiency and performance in a range of conditions. However, the challenge of background clutter also disturbs its performance. In this article, we propose a novel weighted histogram based on neutrosophic similarity score to help the mean-shift tracker discriminate the target from the background. Neutrosophic set (NS) is a new branch of philosophy for dealing with incomplete, indeterminate, and inconsistent information. In this paper, we utilize the single valued neutrosophic set (SVNS), which is a subclass of NS to improve the mean-shift tracker. First, two kinds of criteria are considered as the object feature similarity and the background feature similarity, and each bin of the weight histogram is represented in the SVNS domain via three membership functions T(*Truth*), I(*indeterminacy*), and F(*Falsity*). Second, the neutrosophic similarity score function is introduced to fuse those two criteria and to build the final weight histogram. Finally, a novel neutrosophic weighted mean-shift tracker is proposed. The proposed tracker is compared with several mean-shift based trackers on a dataset of 61 public sequences. The results revealed that our method outperforms other trackers, especially when confronting background clutter.

Keywords: tracking; mean-shift; neutrosophic set; single valued neutrosophic set; neutrosophic similarity score

1. Introduction

Currently, applications in the computer vision field such as surveillance, video indexing, traffic monitoring, and auto-driving have come into our life. However, most of the key algorithms still lack the performance of those applications. One of the most important tasks is visual object tracking, and it is still a challenging problem [1–3].

Challenges like illumination variation, scale variation, motion blur, background clutters, etc. may happen when dealing with the task of visual object tracking [2]. A specific classifier is always considered for tackling such kinds of challenging problems. Boosting [4] and semi-supervised boosting [5] were employed for building a robust classifier; multiple instance learning [6] was introduced into the classifier training procedure due to the interference of the inexact training instance; compressive sensing theory [7] was applied for developing effective and efficient appearance models for robust object tracking, due to factors such as pose variation, illumination change, occlusion, and motion blur.

The mean-shift procedure was first introduced into visual object tracking by Comaniciu et al. [8,9]. The color histogram was employed as the tracking feature. The location of the target in each frame was decided by minimizing the distance between two probability density functions, which are represented by a target histogram and a target candidate histogram. By utilizing the color histogram feature and the efficient seeking method, such a mean-shift tracker demonstrates high efficiency and good performance, even when confronting motion blur and deformation problems. On the other hand, the color histogram feature cannot help the tracker discriminate the target from the background effectively, especially when background clutter exists. Several new metrics or features were considered to deal with such a problem. For instance, Cross-Bin metric [10], SIFT (Scale-invariant feature transform) [11], and texture feature [12] were introduced into the mean shift based tracker, and the proposed trackers all earn a better performance than the traditional one. Besides, Tomas et al. [13] exploited the background to discriminate the target and proposed the background ratio weighting method. In addition, since estimating an adequate scale is essential for robust tracking, a more robust method for estimating the scale of the searching bounding box was proposed through the forward–backward consistency check. This mean-shift based tracker [13] outperforms several state-of-the-art algorithms. Robert et al. [14] also proposed a scale selecting scheme by utilizing the Lindeberg's theory [15] based on the local maxima of differential scale-space filters. Although so many kinds of visual trackers have been proposed, the visual tracking is still an open problem, due to the challenging conditions in the real tracking tasks. All in all, the mean-shift tracker demonstrates high efficiency and may earn an even better performance if a more effective method can be found to discriminate the target from the background. Thus, finding a suitable way to represent the information presented by the background, as well as the target, is of high relevance.

Neutrosophic set (NS) [16] is a new branch of philosophy to deal with the origin, nature, and scope of neutralities. It has an inherent ability to handle the indeterminate information like the noise included in images [17–21] and video sequences. Until now, NS has been successfully applied in many areas [22]. For the computer vision research fields, the NS theory is widely utilized in image segmentation [17–21], skeleton extraction [23] and object tracking [24], etc. Before calculating the segmentation result for an image, a specific neutrosophic image was usually computed via several criteria in NS domain [17–21]. For object tracking, in order to improve the traditional color based CAMShift tracker, the single valued neutrosophic cross-entropy was employed for fusing color and depth information [24]. In addition, the NS theory is also utilized for improving the clustering algorithms, such as c-means [25]. While several criteria are always proposed to handle a specific image processing problem, an appropriate way for fusing information is needed. Decision-making [26–30] can be regarded as a problem-solving activity terminated by a solution deemed to be satisfactory, and it has been frequently employed for dealing with such an information fusion problem. The similarity measurement [30] using the correlation coefficient under single valued neutrosophic environment was successfully applied into the issue of image thresholding [21]. A single valued neutrosophic set (SVNS) [31] is an instance of a neutrosophic set and provides an additional possibility to represent uncertainty, imprecise, incomplete, and inconsistent information, which exists in the real world. The correlation coefficient of SVNS was proposed by the authors of [30] and was successfully applied for handling the multicriteria decision making problem. For the mean-shift tracker, the color histogram is employed for representing the tracked target. Due to the challenging conditions during the tracking procedure, indeterminate information always exists. For instance, object feature may changes due to object pose or external environment changes between frames. It is difficult to localize the object exactly during the tracking procedure. Thus, there exists indeterminate information when you try to utilize the uncertain bounding box to extract object feature. All in all, how to utilize the information of the object and the corresponding background to help the tracker discriminate the object is also an indeterminate problem.

In this work, we propose a novel mean-shift tracker based on the neutrosophic similarity score [21,30] under the SVNS environment. We build a neutrosophic weight histogram, which jointly considered the indeterminate attributes of the object and the background information. First, we propose two criteria of the object feature similarity and the background feature similarity, where each one is represented as its bin of the histogram corresponding to three membership functions for the *T(Truth), I(indeterminacy), and F(Falsity)* element of the neutrosophic set. Second, the neutrosophic similarity score function is introduced to fuse those two criteria and build the final weighted histogram. Finally, the weight of each bin of the histogram is applied to modify the traditional mean-shift tracker, and a novel neutrosophic weighted mean-shift tracker is proposed. To our own knowledge, it is the first time to introduce the NS theory into the mean-shift procedure. Experiments results revealed that the proposed neutrosophic weighted mean-shift tracker outperforms several kinds of mean-shift based trackers [9,13,14].

The remainder of this paper is organized as follows: in Section 2, the traditional mean-shift procedure for visual object tracking and the definition of the neutrosophic similarity score are first given. Then the details of the method for calculating the neutrosophic weight histogram are presented, and the main steps of the proposed mean-shift tracker are illustrated in the following subsection. Experimental evaluations and discussions are presented in Section 3, and Section 4 has the conclusions.

2. Problem Formulation

In this section, we present the algorithmic details of this paper.

For the visual tracking problem, the initial location of the target will be given in the first frame, and the location is always represented by a rectangle bounding box [1–3]. Then the critical task for a visual tracker is to calculate the displacement of the bounding box in the following frame corresponding to the previous one.

2.1. Traditional Mean-Shift Tracker

The main steps of the traditional mean-shift visual tracker are summarized in this subsection.

The kernel-based histogram is employed by the traditional mean-shift tracker. At the beginning, the feature model of the target is calculated by

$$\hat{q}_u = C\sum_{i=1}^{n} k\left(\|\mathbf{x}_i^*\|^2\right)\delta[b(\mathbf{x}_i^*) - u] \tag{1}$$

where $\hat{q}$ is the target model, $\hat{q} = \{\hat{q}_u\}_{u=1\ldots m}$; $\hat{q}_u$ is the u-th bin of the target model satisfying $\sum\limits_{u=1}^{m} \hat{q}_u = 1$; $\mathbf{x}_i^*$ is the normalized pixel location which located in the initial bounding box; and n is the number of pixels belonging to the target. In order to reduce the interference of the background clutters, the kernel $k(x)$ is utilized. $k(x)$ is an isotropic, convex, and monotonic decreasing kernel. The kernel assigns smaller weights to pixels farther than the center. In this work, $k(x)$ is defined as $k(x) = \begin{cases} \frac{2}{\pi}(1-x) & if\ x < 1 \\ 0 & else \end{cases}$. The function $b(\mathbf{x})$: $R^2 \to 1\ldots m$ associates to the pixel at location x the index $b(\mathbf{x})$ of the histogram bin corresponding to the color of that pixel. Then, C is the normalization constant, which is denoted by

$$C = \frac{1}{\sum_{i=1}^{n} k\left(\|\mathbf{x}_i^*\|^2\right)} \tag{2}$$

The function $\delta(x)$ is the Kronecker delta function. Let $\mathbf{y}$ be the center of the target candidate and $\{\mathbf{x}_i\}_{i\,=1,\ldots,nh}$ be the pixel locations in the bounding box of the target candidate. Here, nh is the total number of the pixels falling in the bounding box. Then when using the same kernel profile $k(x)$, the probability of the feature in the target candidate is given by

$$\hat{p}_u = C_h\sum_{i=1}^{n_h} k\left(\|\frac{\mathbf{y}-\mathbf{x}_i}{h}\|^2\right)\delta[b(\mathbf{x}_i - u)] \tag{3}$$

where h is the bandwidth and C_h is the normalization constant derived by imposing the condition $\sum_{u=1}^{m} \hat{p}_u = 1$.

The metric based on Bhattacharyya coefficient is proposed to evaluate the similarity between the probability distributions of the target and the candidate target. Let $\rho[\mathbf{p}(\hat{\mathbf{y}}), \hat{\mathbf{q}}]$ be the similarity probability, then it can be calculated by

$$\rho[\mathbf{p}(\hat{\mathbf{y}}), \hat{\mathbf{q}}] = \sum_{u=1}^{m} \sqrt{\hat{p}(y)\hat{q}_u} \tag{4}$$

For the mean-shift tracker, the location $\hat{\mathbf{y}}_0$ in the previous frame is employed as the starting location for searching the new target location in the current frame. The estimate of a new target location is then obtained by maximizing the Bhattacharyya coefficient $\rho[\mathbf{p}(\hat{\mathbf{y}}), \hat{\mathbf{q}}]$ using a Taylor series expansion, see [8,9] for further details. To reach the maximum of the Bhattacharyya coefficient, the kernel is repeatedly moved from the current location $\hat{\mathbf{y}}_0$ to the new location

$$\hat{\mathbf{y}}_1 = \frac{\sum_{i=1}^{n_h} \mathbf{x}_i w_i g\left(\| \frac{\hat{\mathbf{y}}_0 - \mathbf{x}_i}{h} \|^2 \right)}{\sum_{i=1}^{n_h} w_i g\left(\| \frac{\hat{\mathbf{y}}_0 - \mathbf{x}_i}{h} \|^2 \right)} \tag{5}$$

where $g(x)$ is the negative derivative of the kernel $k(x)$, i.e., $g(x) = -k'(x)$. Furthermore, it is assumed that $g(x)$ exists for all $x \in [0, \infty)$ except for a finite set of points. The parameter w_i in Equation (5) is denoted by

$$w_i = \sum_{u=1}^{m} \delta[b(\mathbf{x}_i) - u] \sqrt{\frac{\hat{q}_u}{\hat{p}_u(\hat{y}_0)}} \tag{6}$$

2.2. Neutrosophic Similarity Score

A neutrosophic set with multiple criteria can be expressed as follows:

Let $A = \{A_1, A_2, \ldots, A_m\}$ be a set of alternatives and $C = \{C_1, C_2, \ldots, C_n\}$ be a set of criteria. Then the character of the alternative A_i ($i = 1, 2, \ldots, m$) can be represented by the following information:

$$A_i = \left\{ \left\langle C_j, T_{C_j}(A_i), I_{C_j}(A_i), F_{C_j}(A_i) \right\rangle \middle| C_j \in C \right\} \; i = 1 \ldots m, j = 1 \ldots n \tag{7}$$

where $T_{C_j}(A_i), I_{C_j}(A_i), F_{C_j}(A_i) \in [0,1]$. Here, $T_{C_j}(A_i)$ denotes the degree to which the alternative A_i satisfies the criterion C_j; $I_{C_j}(A_i)$ indicates the indeterminacy degree to which the alternative A_i satisfies or does not satisfy the criterion C_j; $F_{C_j}(A_i)$ indicates the degree to which the alternative A_i does not satisfy the criterion C_j.

A method for multicriteria decision-making based on the correlation coefficient under single-valued neutrosophic environment is proposed in [30]. The similarity degree between two elements A_i and A_j is defined as:

$$S_{C_k}(A_i, A_j) = \frac{T_{C_k}(A_i)T_{C_k}(A_j) + I_{C_k}(A_i)I_{C_k}(A_j) + F_{C_k}(A_i)F_{C_k}(A_j)}{\sqrt{{T_{C_k}}^2(A_i) + {I_{C_k}}^2(A_i) + {F_{C_k}}^2(A_i)}\sqrt{{T_{C_k}}^2(A_j) + {I_{C_k}}^2(A_j) + {F_{C_k}}^2(A_j)}} \tag{8}$$

Assume the ideal alternative $A^* = \left\{ \left\langle C_j, T_{C_j}(A^*), I_{C_j}(A^*), F_{C_j}(A^*) \right\rangle \middle| C_j \in C \right\}$ $i = 1 \ldots m$, $j = 1 \ldots n$. Then the similarity degree between any alternative A_i and the ideal alternative A^* can be calculated by

$$S_{C_k}(A_i, A^*) = \frac{T_{C_k}(A_i)T_{C_k}(A^*) + I_{C_k}(A_i)I_{C_k}(A^*) + F_{C_k}(A_i)F_{C_k}(A^*)}{\sqrt{{T_{C_k}}^2(A_i) + {I_{C_k}}^2(A_i) + {F_{C_k}}^2(A_i)}\sqrt{{T_{C_k}}^2(A^*) + {I_{C_k}}^2(A^*) + {F_{C_k}}^2(A^*)}} \tag{9}$$

Suppose $w_k \in [0,1]$ is the weight of each criterion C_k and $\sum_{j=1}^{n} w_j = 1$, then the weighted correlation coefficient between an alternative A_i and the ideal alternative A^* is defined by

$$W(A_i, A^*) = \sum_{k=1}^{n} w_k \frac{T_{C_k}(A_i)T_{C_k}(A^*) + I_{C_k}(A_i)I_{C_k}(A^*) + F_{C_k}(A_i)F_{C_k}(A^*)}{\sqrt{{T_{C_k}}^2(A_i) + {I_{C_k}}^2(A_i) + {F_{C_k}}^2(A_i)}\sqrt{{T_{C_k}}^2(A^*) + {I_{C_k}}^2(A^*) + {F_{C_k}}^2(A^*)}} \tag{10}$$

The alternative with high correlation coefficient is considered to be a good choice for the current decision.

2.3. Calculate the Neutrosophic Weight Histogram

Employing the information discriminated from the background is one of the most important issues for robustly tacking a visual object. As shown in Figure 1, the smallest region $\mathbf{G}_O$ inside the red bounding box is the object region and this region corresponds to the location of the object in the corresponding frame. Then $\mathbf{G}_O$ is decided by the tracker and its accuracy depends on the robustness of the tracker. In this work, the surrounding area of $\mathbf{G}_O$ is defined as the background region $\mathbf{G}_B$. In order to eliminate the indeterminacy of the region $\mathbf{G}_O$ to some extent, the region far from $\mathbf{G}_O$ is employed as $\mathbf{G}_B$ and $\mathbf{G}_B = \beta\mathbf{G}_O - \alpha\mathbf{G}_O$.

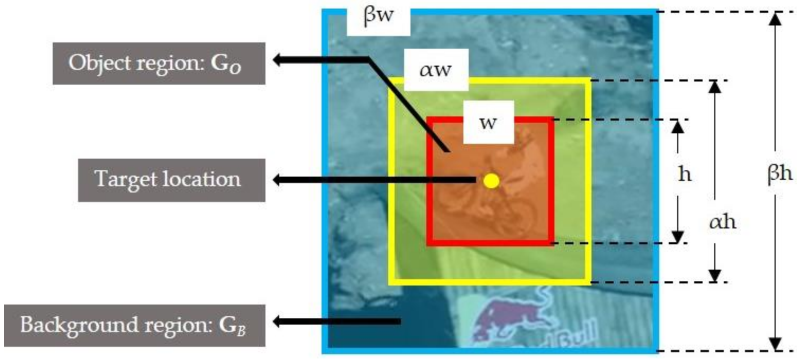

Figure 1. Illustration of the object region.

To enhance the robustness of the traditional mean-shift tracker, a novel weight histogram w^{NS} is defined in the neutrosophic domain. Each bin of the weighted histogram w^{NS} is expressed in the SVNS domain via three membership functions $T(Truth)$, $I(indeterminacy)$, and $F(Falsity)$.

For the proposition of object feature is a discriminative feature, T_{CO}, I_{CO}, and F_{CO} represent the probabilities when a proposition is true, indeterminate and false degrees, respectively. Finding the location of the tracked object in a new frame is the main task for a tracker, and the target model (object feature histogram in the initial frame) is frequently employed as major information to discriminate the object from the background. The region which owns more similarity to the object feature is more likely to be the object region. Using the object feature similarity criterion, we can further give the definitions:

$$T_{CO}(u) = \hat{q}_u \tag{11}$$

$$I_{CO}(u) = |\hat{q}_u - \hat{q}_u(t-1)| \tag{12}$$

$$F_{CO}(u) = 1 - T_{CO}(u) \tag{13}$$

where $\hat{q}_u$ is the u-th bin of the target model corresponding to the object region $\mathbf{G}_O$ in the first frame of the tracking process and it is calculated by using Equation (1).

The indeterminacy degree $I_{CO}\,(u)$ is defined in Equation (12). Then, $\hat{q}_u(t-1)$ is the *u-th* bin of the updated object feature histogram in the previous frame. Suppose $\hat{p}_u(t-1)$ is the feature histogram corresponding to the extracted object region at time $t-1$, then $\hat{q}_u(t-1)$ is calculated by

$$\hat{q}_u(t-1) = (1-\lambda)\hat{q}_u(t-2) + \lambda \hat{p}_u(t-1) \tag{14}$$

where λ is the updating rate for $\lambda \in (0,1)$.

As the tracker may drift from the object due to the similar surroundings, using the object features with high similarity to the background will bring risk to the accuracy of the tracker. The background feature similarity criterion is considered in this work. The corresponding three membership functions T_{CB}, I_{CB} and F_{CB} are defined as follows:

$$T_{CB}(u) = \hat{q}_u \tag{15}$$

$$I_{CB}(u) = \begin{cases} 0 & if\ \hat{b}_u = 0 \\ 1 & if\ \hat{b}_u > \hat{q}_u \\ \hat{b}_u/\hat{q}_u & else \end{cases} \tag{16}$$

$$F_{CB}(u) = \hat{b}_u \tag{17}$$

where $\hat{b}_u$ is the *u-th* bin of the object background feature histogram. This histogram is initialized in the background region $\mathbf{G}_B$ in the first frame, as shown in Figure 1. For $\hat{q}_u$, Equation (1) is also employed to calculate $\hat{b}_u$, and $\hat{b}_u$, which will be updated when the surroundings of the tracked target change dramatically.

By substituting the corresponding T(*Truth*), I(*indeterminacy*), and F(*Falsity*) under the criteria of the object feature similarity and the background feature similarity into Equation (10), the u-th bin of the neutrosophic weight histogram can be calculated by

$$\begin{aligned} w_u^{NS} &= w_{CO}S_{CO}(u, A^*) + w_{CB}S_{CB}(u, A^*) \\ &= w_{CO}\frac{T_{CO}(u)T_{C_k}(A^*)+I_{CO}(u)I_{C_k}(A^*)+F_{CO}(u)F_{C_k}(A^*)}{\sqrt{T_{CO}{}^2(u)+I_{CO}{}^2(u)+F_{CO}{}^2(u)}\sqrt{T_{C_k}{}^2(A^*)+I_{C_k}{}^2(A^*)+F_{C_k}{}^2(A^*)}} \\ &\quad + w_{CB}\frac{T_{CB}(u)T_{C_k}(A^*)+I_{CB}(u)I_{C_k}(A^*)+F_{CB}(u)F_{C_k}(A^*)}{\sqrt{T_{CB}{}^2(u)+I_{CB}{}^2(u)+F_{CB}{}^2(u)}\sqrt{T_{C_k}{}^2(A^*)+I_{C_k}{}^2(A^*)+F_{C_k}{}^2(A^*)}} \end{aligned} \tag{18}$$

where w_{CO}, $w_{CB} \in [0,1]$ are the corresponding weights of criteria and $w_{CO} + w_{CB} = 1$. The ideal alternative under two criteria is the same as $A^* = \langle 1,\ 0,\ 0\rangle$.

2.4. Neutrosophic Weighted Mean-Shift Tracker

In this work, the neutrosophic weighted histogram is introduced into the traditional mean-shift procedure, and this improved mean-shift tracker is called the neutrosophic weighted mean-shift tracker. The basic flow of the proposed tracker is described below:

Initialization

Step 1: Read the first frame and select an object on the image plane as the target to be tracked.

Step 2: Calculate the object feature histogram $\hat{q}$ and object background feature histogram $\hat{b}$ by using Equation (1).

Tracking

Input: (t + 1)-th video frame

Step 3: Employ the location $\hat{\mathbf{y}}_0$ in the previous frame as the starting location for searching the new target location in the current frame.

Step 4: Based on the mean-shift algorithm and neutrosophic weight histogram, derive the new location of the object according to Equation (19) and Equation (5) as follows:

$$w_i = \sum_{u=1}^{m} \delta[b(\mathbf{x}_i) - u] \sqrt{w_u^{NS} \frac{\hat{q}_u}{\hat{p}_u(\hat{y}_0)}} \tag{19}$$

Step 5: If $\|\hat{\mathbf{y}}_1 - \hat{\mathbf{y}}_0\| < \varepsilon_0$, stop. Otherwise, set $\hat{\mathbf{y}}_0 \leftarrow \hat{\mathbf{y}}_1$ and go to Step 4.

Step 6: Derive $\hat{q}_u(t-1)$ according to Equation (14) and then update object background feature histogram $\hat{b} \leftarrow \hat{b}_c$ when the Bhattacharyya coefficient $\rho\left[\hat{b}, \hat{b}_c\right] < \varepsilon_1$, where $\hat{b}_c$ is the corresponding feature histogram in the current background region $\mathbf{G}_B$.

Output: Tracking location.

3. Experiment Results and Analysis

We tested the neutrosophic weighted mean-shift tracker on a challenging benchmark [2]. As mentioned at the outset, background clutter is one of the most challenging problems for the mean-shift tracker. Besides the 50 challenging sequences in this benchmark [2], another 10 sequences with the challenge of background clutter are also selected as testing sequences. The information of those 10 sequences is given in Table 1. The abbreviations of several kinds of challenges included in the testing sequences are shown in the footer of Table 1.

Table 1. An overview of another 10 sequences.

Sequence	Target	Challenges	Frames
Board	board	SV, MB, FM, OPR, OV, BC	698
Bolt2	human	DEF, BC	293
Box	box	IV, SV, OCC, MB, IPR, OPR, OV, BC, LR	1161
ClifBar	book	SV, OCC, MB, FM, IPR, OV, BC	472
Coupon	coupon	OCC, BC	327
Crowds	human	IV, DEF, BC	347
Car2	car	IV, SV, MB, FM, BC	913
Car1	car	IV, SV, MB, FM, BC, LR	1020
Human3	human	SV, OCC, DEF, OPR, BC	1698
Car24	car	IV, SV, BC	3059

Note: IV: Illumination Variation, SV: Scale Variation, OCC: Occlusion, DEF: Deformation, MB: Motion Blur, FM: Fast Motion, IPR: In-Plane Rotation, OPR: Out-of-Plane Rotation, OV: Out-of-View, BC: Background Clutters, and LR: Low Resolution.

To gauge the performance of the proposed tracker, we compare our results to another three mean-shift based trackers including ASMS [13], KMS [9] and SMS [14]. Some experimental results have shown that ASMS [13] outperforms several state-of-the-art algorithms. KMS is the traditional mean-shift tracker. Both SMS and ASMS are scale adaptive. All of the tested algorithms employ the color histogram as object feature.

3.1. Setting Parameters

For the proposed neutrosophic weighted mean-shift tracker, the parameter α and β relate to the background region $\mathbf{G}_B$ are set to 1.2 and 1.48 respectively. The parameter λ in Equation (14) decides the updating rate of the object feature histogram. With the assumption that the appearance of the tracked object will not change dramatically, a low updating rate should be given. In this work, λ is set to 0.05. As seen in the Section 2.4, the accuracy of the result of the mean-shift procedure depends on the parameter ε_0 to some extent, where ε_0 is set to 0.1. A much greater value of ε_0 may lead to failure. The parameter ε_1 is a threshold for updating the object background feature histogram. During the tracking procedure, the surroundings of the object always change. Hence, it is essential to update the object background feature histogram when the similarity between the current surroundings and the object background feature falls to a specific value. If ε_1 is set to 0, the updating process of the background feature will stop. If ε_1 is set to 1, the updating frequency will be too high. Thus, a medium value is chosen as ε_1 = 0.5. The neutrosophic weight histogram plays an essential role in this proposed mean-shift based tracker. In order to emphasize the background information when constructing the neutrosophic weight histogram, the corresponding parameter w_{CB} should be set to a relatively high value. However, if this value is set too high, the effect of the first neutrosophic criteria will reduce, even to zero. In this work, w_{CB} is set to 0.6, and w_{CO} is set to 0.4. Finally, all the values of these parameters are chosen by hand-tuning, and all of them are constant for all experiments.

3.2. Evaluation Criteria

The overlap rate of the bounding box is used as the evaluation criterion, and the overlap rate is defined as

$$s_i = \frac{area\left(\mathrm{ROI}_{T_i} \cap \mathrm{ROI}_{G_i}\right)}{area\left(\mathrm{ROI}_{T_i} \cup \mathrm{ROI}_{G_i}\right)} \tag{20}$$

where ROI_{Ti} is the target bounding box in the i-th frame and ROI_{Gi} is the corresponding ground truth bounding box. For the video datasets applied in this work, the ground truth bounding boxes of the tracked target are manually labeled for each frame. The success ratio is defined as:

$$R = \sum_{i=1}^{N} u_i / N, \ \ u_i = \begin{cases} 1 & if\ s_i > r \\ 0 & otherwise \end{cases} \tag{21}$$

where N is the number of frames and r is the overlap threshold which decides the corresponding tracking result is correct or not. The success ratio is $R \in [0,1]$. When the overlap ratio s_i is greater than r on each frame, R achieves the maximum, and then this means the corresponding tracker performs very well in this sequence. On the contrary, R achieves the minimum when s_i is smaller than r on each frame, and then this means the corresponding tracker performs the worst.

Both the one-pass evaluation (OPE) and temporal robustness evaluation (TRE) are employed as the evaluation metric. For the TRE, each testing sequence is partitioned into 20 segments, and each tracker is tested throughout all of the segments. The results for the OPE evaluation metric are derived by testing the tracker with one time initialization from the ground truth position in the first frame of each testing sequence. Finally, we use the area under curve (AUC) of each success plot to rank the tracking algorithms. For each success plot, the tracker with a greater value of AUC ranks better.

3.3. Tracking Results

Several screen captures for some of the testing sequences are given in Figures 2–5. Success plots of TRE and OPE for the whole testing sequences are shown in Figures 6a and 7a, and the success plots for those sequences including background clutter challenge are shown in Figures 6b and 7b. In the following section, a more detailed discussion of the tracking results is documented.

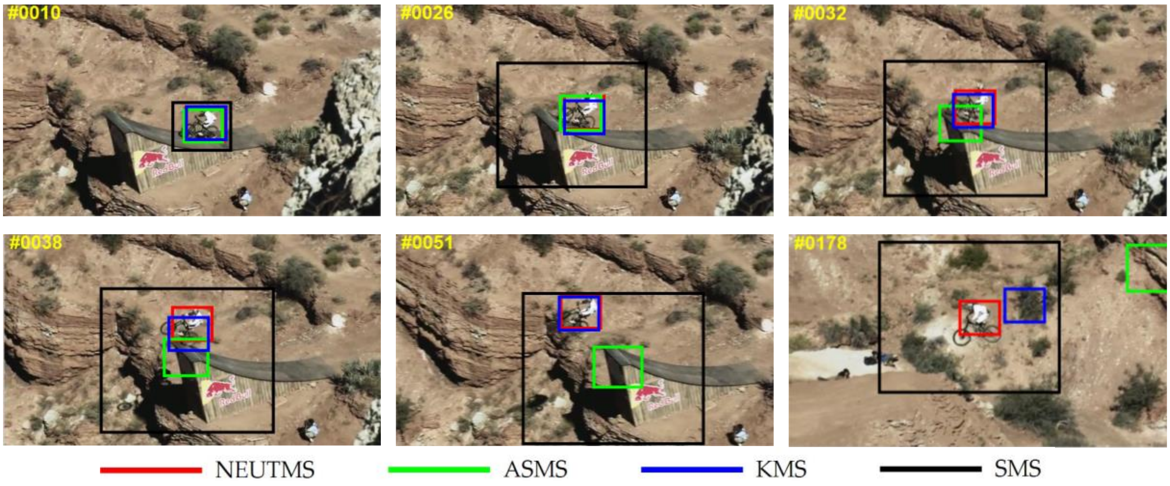

Figure 2. Screenshots of tracking results of the video sequence used for testing (*mountainBike*, target is selected in frame #1).

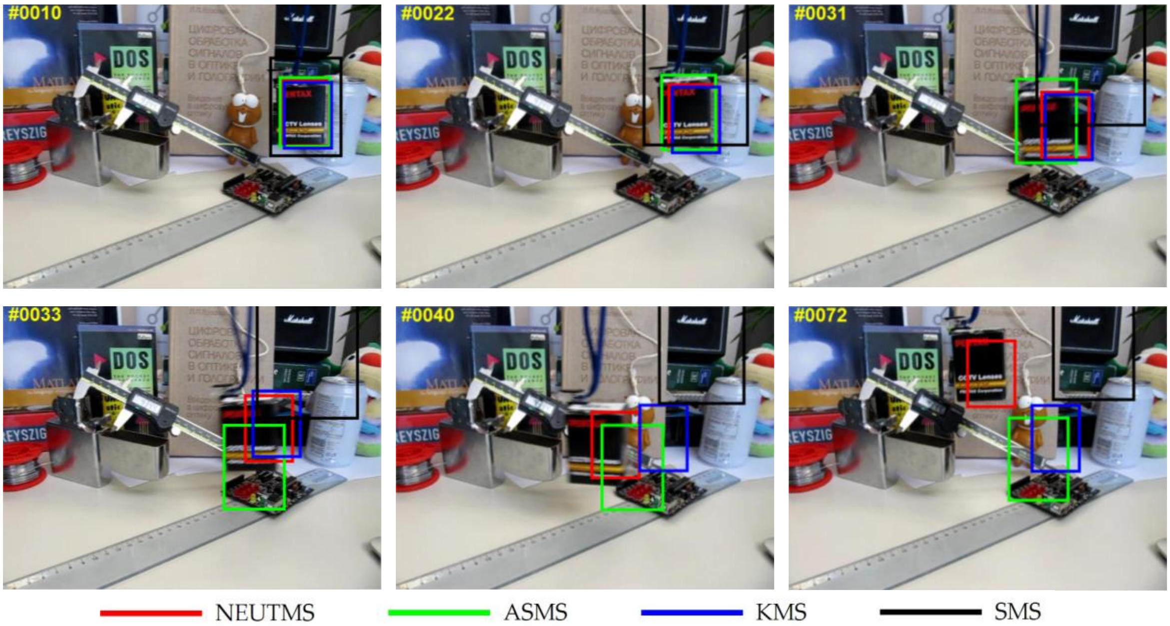

Figure 3. Screenshots of tracking results of the video sequence used for testing (*Box*, target is selected in frame #1).

MountainBike sequence: This sequence highlights the challenges of BC, IPR and OPR. As shown in Figure 2, an improper scale of the bounding box is estimated by the SMS tracker, and the SMS tracker has failed in frame #26. The ASMS tracker, as can be seen in frame #32, has drifted from the tracking target because of the similar color of the surroundings, although an appropriate scale is given. During the first half of the tracking process, both of the KMS and our NEUTMS perform well. However, compared to the NEUTMS, the KMS tracker sometimes drifts a little farer from the biker, as seen in frame #38. When the challenge of background clutter appears, the KMS tracker may also drift from the right location of the target, as seen in frame #178. During the whole tracking process, the NEUTMS tracker performs the best result.

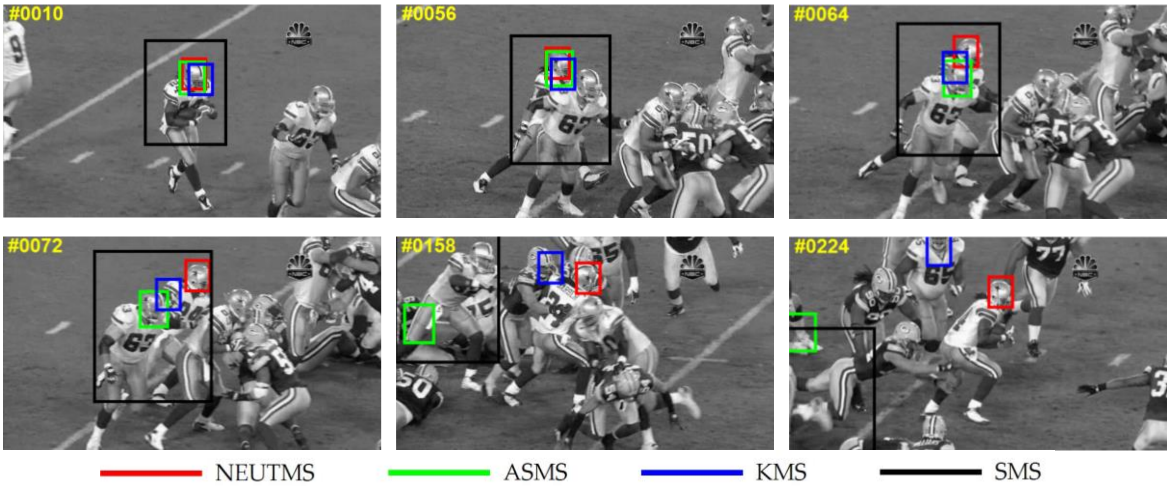

Figure 4. Screenshots of tracking results of the video sequence used for testing (*Football*, target is selected in frame #1).

Box sequence: The challenges included in this sequence can be found in Table 1. This sequence is more challenging than the *MountainBike* sequence. As seen in frame #31 in Figure 3, all the trackers except for the SMS tracker can give a right location of the tracked box, and the ASMS performs the best result so far. Due to the black background upon the box, the SMS tracker fails soon. While the box is passing by the circuit board on the table, both the ASMS and the KMS tracker begin to lose the box. By employing the information of the background region, our NEUTMS tracker has successfully overcome the challenges like BC and MB during this sequence.

Football sequence: Challenges of BC, OCC, IPR and OPR are presented in this sequence. As shown in Figure 4, the SMS tracker has already failed in frame #10. The ASMS and KMS trackers fail when the tracked player getting close to another player on account of the factor of all the players wear the same helmet. However, the NEUTMS tracker performs well even the tracked player runs through some players with similar feature.

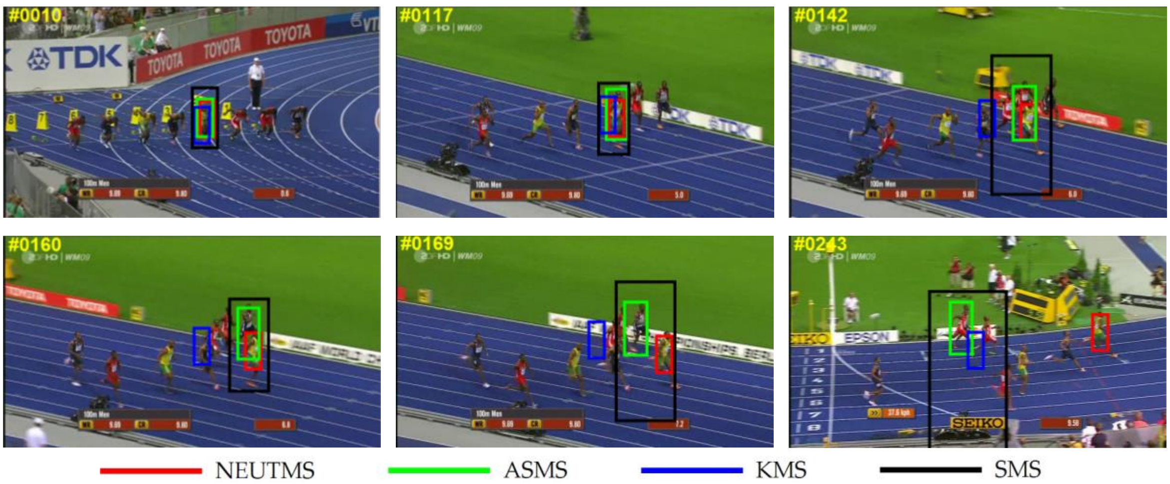

Figure 5. Screenshots of tracking results of the video sequence used for testing (*Bolt*, target is selected in frame #1).

Bolt sequence: This sequence presents the challenges of OCC, DEF, IPR and OPR. As shown in Figure 5, all the trackers perform well till frame #117. Compared to the ASMS and SMS trackers, the KMS and NEUTMS trackers cannot calculate a proper size for the bounding box due to the fixed scale. The KMS tracker has begun to drift form the target on the account of the improper size of the bounding box since frame #117. By fusing the information of the feature of the object and background region, the NEUTMS tracker has successfully tracked the target throughout this sequence even with an inappropriate scale. Though a good scale is estimated by the ASMS tracker, it fails when Bolt passes by some other runners, as seen in frame #142 and #160.

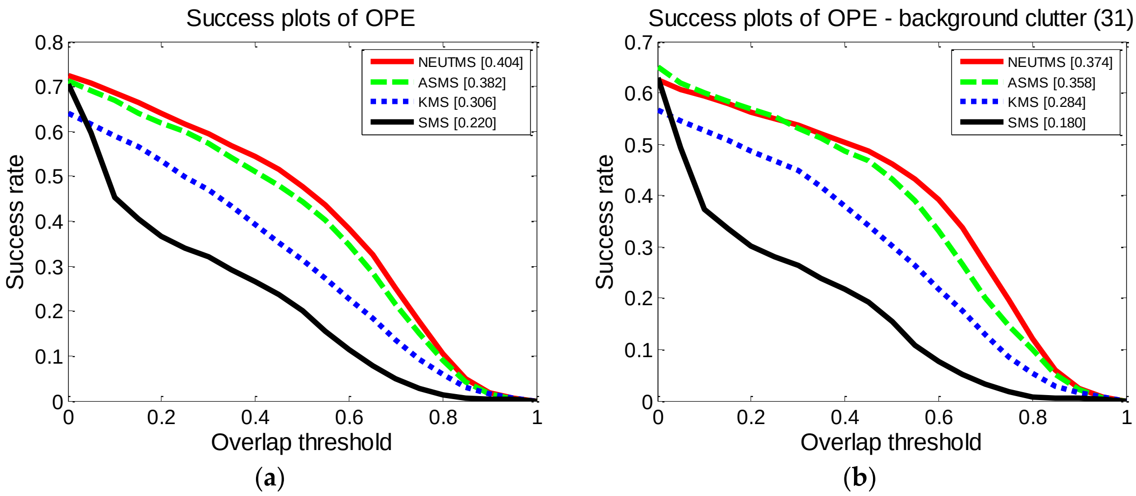

Figure 6. Success plots of one-pass evaluation (OPE): (**a**) Success plots of OPE over all the testing sequences; (**b**) Success plots of OPE over all the 31 testing sequences included the challenge of background clutters (BC). The value shown between the brackets is the area under curve (AUC) value corresponds to the tracker.

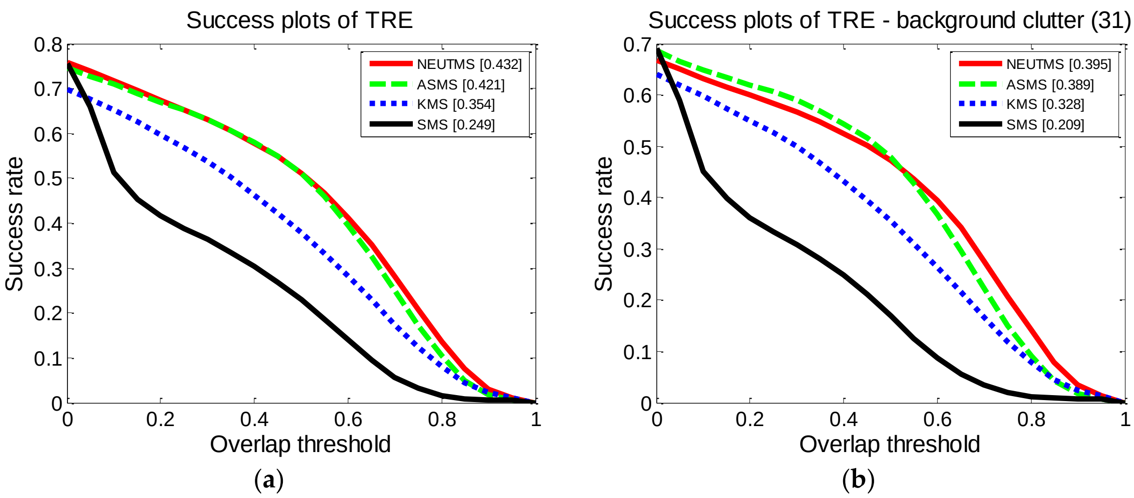

Figure 7. Success plots of temporal robustness evaluation (TRE): (**a**) Success plots of TRE over all the testing sequences; (**b**) Success plots of TRE over all the 31 testing sequences included the challenge of BC. The value shown between the brackets is the AUC value corresponds to the tracker.

We employ all the 61 sequences as the testing sequence dataset. Success plots of OPE and TRE over all the sequences are shown in Figures 6a and 7a respectively, which show our NEUTMS tracker is superior to other trackers. Due to the fact that the focus of our work in this paper is to employ both the object and background feature to enhance the mean-shift tracker's ability of overcoming the problem of similar surroundings, only the success plots for the challenge of BC are given, and then the BC challenge is one of the most challenging problems for the traditional mean-shift tracker [13]. The results of the corresponding success plots are shown in Figures 6b and 7b, which show the robustness of the NEUTMS tracker when handling the challenge of BC.

In order to test the performance of the proposed NEUTMS tracker over other kinds of challenges, all the AUC results for each tracker are given in Tables 2 and 3. The best result is highlighted in red italic type and the second result is highlighted in bold type. As seen in Tables 2 and 3, the NEUTMS tracker performs the best result when tackling the challenge of BC, MB, DEF, IPR, OCC or OPR when the OPE evaluation is considered. For TRE, the NEUTMS tracker performs the best result when confronting the same kind of challenge to OPE except for the challenge of MB. The ASMS tracker wins over SV because a robust scale updating scheme is used. The NEUTMS tracker performs the second best result over FM, IV and OV mainly because some inaccurate background information may be brought into the background feature model. The NEUTMS tracker performs the second best result when confronting the challenge of LR on account of less information can be employed for enhancing the tracker.

Table 2. AUC results of each tracker on sequences with different challenge for OPE.

Challenge	BC	FM	MB	DEF	IV	IPR	LR	OCC	OPR	OV	SV	Total
NEUTMS	*0.374*	**0.409**	*0.408*	*0.444*	**0.306**	*0.365*	**0.235**	*0.413*	*0.422*	**0.380**	**0.340**	*0.404*
ASMS	**0.358**	*0.436*	**0.406**	**0.399**	*0.338*	**0.346**	*0.271*	**0.387**	**0.393**	*0.413*	*0.390*	**0.382**
KMS	0.284	0.325	0.322	0.302	0.292	0.277	0.185	0.315	0.315	0.369	0.290	0.306
SMS	0.180	0.255	0.222	0.219	0.193	0.184	0.131	0.251	0.235	0.274	0.242	0.220

Table 3. AUC results of each tracker on sequences with different challenge for TRE.

Challenge	BC	FM	MB	DEF	IV	IPR	LR	OCC	OPR	OV	SV	Total
NEUTMS	*0.395*	**0.422**	**0.418**	*0.480*	**0.361**	*0.402*	**0.252**	*0.432*	*0.442*	**0.392**	**0.366**	*0.432*
ASMS	**0.389**	*0.442*	*0.434*	**0.453**	*0.392*	**0.401**	*0.271*	**0.416**	**0.437**	*0.418*	*0.387*	**0.421**
KMS	0.328	0.346	0.342	0.371	0.328	0.334	0.237	0.361	0.363	0.357	0.320	0.354
SMS	0.209	0.274	0.243	0.277	0.224	0.220	0.153	0.281	0.268	0.258	0.247	0.249

4. Conclusions

In this paper, a neutrosophic weighted mean-shift tracker is proposed. The experimental results have revealed its robustness. While calculating the neutrosophic weighted histogram, two kinds of criteria are considered as the object feature similarity and the background feature similarity, and each bin of the weight histogram is represented in the SVNS domain via three membership functions T, I and F. Both the feature in the object and the background region are fused by introducing the weighted neutrosophic similarity score function. Finally, the neutrosophic weighted histogram is employed to decide the new location of the object. As discussed in this work, we have not considered the scale variation problem. To further improve the performance of our tracker in the future, our primary mission is to introduce a scale updating scheme into this neutrosophic weighted mean-shift tracker.

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information

Article

Linguistic Neutrosophic Cubic Numbers and Their Multiple Attribute Decision-Making Method

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Abstract: To describe both certain linguistic neutrosophic information and uncertain linguistic neutrosophic information simultaneously in the real world, this paper originally proposes the concept of a linguistic neutrosophic cubic number (LNCN), including an internal LNCN and external LNCN. In LNCN, its uncertain linguistic neutrosophic number consists of the truth, indeterminacy, and falsity uncertain linguistic variables, and its linguistic neutrosophic number consists of the truth, indeterminacy, and falsity linguistic variables to express their hybrid information. Then, we present the operational laws of LNCNs and the score, accuracy, and certain functions of LNCN for comparing/ranking LNCNs. Next, we propose a LNCN weighted arithmetic averaging (LNCNWAA) operator and a LNCN weighted geometric averaging (LNCNWGA) operator to aggregate linguistic neutrosophic cubic information and discuss their properties. Further, a multiple attribute decision-making method based on the LNCNWAA or LNCNWGA operator is developed under a linguistic neutrosophic cubic environment. Finally, an illustrative example is provided to indicate the application of the developed method.

Keywords: linguistic neutrosophic cubic number; score function; accuracy function; certain function; linguistic neutrosophic cubic number weighted arithmetic averaging (LNCNWAA) operator; linguistic neutrosophic cubic number weighted geometric averaging (LNCNWGA) operator; decision-making

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1. Introduction

In terms of complex objective aspects of real life, human preference judgments may use linguistic expression, instead of numerical value expression, in order to be more suitable for people's thinking habits. Hence, Zadeh [1] firstly introduced the concept of a linguistic variable and applied it to fuzzy reasoning. After that, linguistic decision analysis and linguistic aggregation operators have been proposed to solve linguistic decision-making problems [2–5]. Due to the incompleteness and uncertainty of linguistic decision environments, uncertain linguistic variables and their various aggregation operators were developed and applied to uncertain linguistic decision-making problems [6–11]. As to the extension of linguistic variables, the concept of linguistic intuitionistic fuzzy numbers and their linguistic intuitionistic multicriteria group decision-making methods were introduced in the literature [12,13], and then linguistic intuitionistic multicriteria decision-making method was proposed based on the Frank Heronian mean operator [14]. Recently, the concept of a neutrosophic linguistic number, which indicates a changeable uncertain linguistic number corresponding to some specified indeterminate range, and some weighted aggregation operators of neutrosophic linguistic numbers, were presented to solve multiple attribute group decision-making problems with neutrosophic linguistic numbers [15]. Then, the concept of a linguistic neutrosophic number, which is described independently by the truth, indeterminacy, and falsity linguistic variables, and some aggregation operators of linguistic neutrosophic numbers, were proposed to solve multiple attribute group decision-making problems with linguistic neutrosophic numbers [16,17].

To express vagueness and uncertainty in real life, the concept of a (fuzzy) cubic set (including the internal cubic set and external cubic set) was introduced based on the hybrid information of both partial certain and partial uncertain values in [18], where the first component is an interval/uncertain value and the second component is an exact/certain value. After that, the concept of a neutrosophic cubic set (including the internal neutrosophic cubic set and external neutrosophic cubic set), where a neutrosophic cubic number (a basic element in a neutrosophic cubic set) is composed of both the interval neutrosophic number and the single-valued neutrosophic number, and the distance measure of neutrosophic cubic sets were proposed and applied to pattern recognition [19,20]. Then, decision-making methods with neutrosophic cubic information were put forward based on grey relational analysis [21] and cosine measures [22], respectively.

However, all the existing linguistic variables, including: uncertain linguistic variables, linguistic intuitionistic fuzzy numbers (basic elements in a linguistic intuitionistic fuzzy set), neutrosophic linguistic numbers (basic elements in a neutrosophic linguistic set), and linguistic neutrosophic numbers (basic elements in a linguistic neutrosophic set), cannot express the hybrid information of both uncertain linguistic and certain linguistic neutrosophic numbers simultaneously in linguistic decision-making environments. Furthermore, the cubic set and neutrosophic cubic set cannot also express linguistic arguments and handle linguistic decision-making problems under linguistic environments. Hence, it is necessary to extend neutrosophic cubic sets to linguistic neutrosophic arguments. For this purpose, this study presents a new concept of a linguistic neutrosophic cubic number (LNCN), where the uncertain linguistic neutrosophic number corresponding to its first part is composed of the truth, indeterminacy, and falsity uncertain linguistic variables and the linguistic neutrosophic number corresponding to its second part is composed of the truth, indeterminacy, and falsity linguistic variables. Then, we propose the operational laws of LNCNs and the score, accuracy, and certain functions of LNCN for comparing/ranking LNCNs. Further, we present a LNCN weighted arithmetic averaging (LNCNWAA) operator and a LNCN weighted geometric averaging (LNCNWGA) operator. Moreover, we develop a decision-making method based on the LNCNWAA or LNCNWGA operator and the score, accuracy, and certain functions to solve decision-making problems with the hybrid information of both certain linguistic neutrosophic numbers and uncertain linguistic neutrosophic numbers under linguistic environments.

The rest of this paper is structured as follows: Section 2 proposes the concept of LNCN (including the internal LNCN and external LNCN), the operational laws of LNCNs, and the score, accuracy, and certain functions of LNCNs to rank LNCNs. In Section 3, we propose the LNCNWAA and LNCNWGA operators to aggregate LNCNs and discuss their properties. In Section 4, a multiple attribute decision-making method is developed based on the LNCNWAA or LNCNWGA operator under a LNCN environment. In Section 5, an example illustrates the application of the proposed method. Section 6 gives conclusions and future work.

2. Linguistic Neutrosophic Cubic Numbers (LNCNs) and Their Operational Laws

This section proposes the concept of LNCN, which include the internal LNCN and external LNCN, and the operational laws of LNCNs.

Definition 1. *Let a linguistic term set be* $S = \{s_j \mid j \in [0, p]\}$, *where* $p + 1$ *is an odd number/cardinality. A LNCN* h *in* S *is constructed as* $h = (u, c)$, *where* $u = \langle [s_{Ta}, s_{Tb}], [s_{Ia}, s_{Ib}], [s_{Fa}, s_{Fb}] \rangle$ *is an uncertain linguistic neutrosophic number with the truth, indeterminacy, and falsity uncertain linguistic variables* $[s_{Ta}, s_{Tb}]$, $[s_{Ia}, s_{Ib}]$, *and* $[s_{Fa}, s_{Fb}]$ *for* s_{Ta}, s_{Ia}, s_{Fa} s_{Tb}, s_{Ib}, $s_{Fb} \in S$ *and* $Ta \leq Tb$, $Ia \leq Ib$, $Fa \leq Fb$; $c = \langle s_T, s_I, s_F \rangle$ *is a linguistic neutrosophic number with the truth, indeterminacy, and falsity linguistic variables* s_T, s_I, *and* s_F *for* s_T, s_I, $s_F \in S$.

Definition 2. *Let a LNCN be* $h = (\langle [s_{Ta}, s_{Tb}], [s_{Ia}, s_{Ib}], [s_{Fa}, s_{Fb}] \rangle, \langle s_T, s_I, s_F \rangle)$ *for* s_{Ta}, s_{Ia}, s_{Fa} s_{Tb}, s_{Ib}, s_{Fb}, s_T, s_I, $s_F \in S$. *Then, we call*

(1) *h an internal LNCN if* $Ta \leq T \leq Tb$, $Ia \leq I \leq Ib$, $Fa \leq F \leq Fb$;

(2) *h an external LNCN if* $T \notin (Ta, Tb)$, $I \notin (Ia, Ib)$, *and* $F \notin (Fa, Fb)$.

Based on the operational laws of linguistic intuitionistic fuzzy numbers and linguistic neutrosophic numbers introduced in the existing literature [12–14,16,17], we propose the following operational laws of LNCNs.

Definition 3. *Let two LNCNs be* $h_1 = (\langle [s_{Ta1}, s_{Tb1}], [s_{Ia1}, s_{Ib1}], [s_{Fa1}, s_{Fb1}] \rangle, \langle s_{T1}, s_{I1}, s_{F1} \rangle)$ *and* $h_2 = (\langle [s_{Ta2}, s_{Tb2}], [s_{Ia2}, s_{Ib2}], [s_{Fa2}, s_{Fb2}] \rangle, \langle s_{T2}, s_{I2}, s_{F2} \rangle)$. *Then, their operational laws are defined as follows:*

$$h_1 \oplus h_2 = \left(\begin{array}{l} \left\langle \left[s_{Ta1+Ta2-\frac{Ta1\cdot Ta2}{p}}, s_{Tb1+Tb2-\frac{Tb1\cdot Tb2}{p}} \right], \left[s_{\frac{Ia1\cdot Ia2}{p}}, s_{\frac{Ib1\cdot Ib2}{p}} \right], \left[s_{\frac{Fa1\cdot Fa2}{p}}, s_{\frac{Fb1\cdot Fb2}{p}} \right] \right\rangle, \\ \left\langle s_{T1+T2-\frac{T1\cdot T2}{p}}, s_{\frac{I1\cdot I2}{p}}, s_{\frac{F1\cdot F2}{p}} \right\rangle \end{array} \right)$$

$$h_1 \otimes h_2 = \left(\begin{array}{l} \left\langle \left[s_{\frac{Ta1\cdot Ta2}{p}}, s_{\frac{Tb1\cdot Tb2}{p}} \right], \left[s_{Ia1+Ia2-\frac{Ia1\cdot Ia2}{p}}, s_{Ib1+Ib2-\frac{Ib1\cdot Ib2}{p}} \right], \left[s_{Fa1+Fa2-\frac{Fa1\cdot Fa2}{p}}, s_{Fb1+Fb2-\frac{Fb1\cdot Fb2}{p}} \right] \right\rangle, \\ \left\langle s_{\frac{T1\cdot T2}{p}}, s_{I1+I2-\frac{I1\cdot I2}{p}}, s_{F1+F2-\frac{F1\cdot F2}{p}} \right\rangle \end{array} \right)$$

$$\lambda h_1 = \left(\begin{array}{l} \left\langle \left[s_{p-p(1-\frac{Ta1}{p})^{\lambda}}, s_{p-p(1-\frac{Tb1}{p})^{\lambda}} \right], \left[s_{p(\frac{Ia1}{p})^{\lambda}}, s_{p(\frac{Ib1}{p})^{\lambda}} \right], \left[s_{p(\frac{Fa1}{p})^{\lambda}}, s_{p(\frac{Fb1}{p})^{\lambda}} \right] \right\rangle \\ \left\langle s_{p-p(1-\frac{T1}{p})^{\lambda}}, s_{p(\frac{I1}{p})^{\lambda}}, s_{p(\frac{F1}{g})^{\lambda}} \right\rangle \end{array} \right), \lambda > 0$$

$$h_1^{\lambda} = \left(\begin{array}{l} \left\langle \left[s_{p(\frac{Ta1}{p})^{\lambda}}, s_{p(\frac{Tb1}{p})^{\lambda}} \right], \left[s_{p-p(1-\frac{Ia1}{p})^{\lambda}}, s_{p-p(1-\frac{Ib1}{p})^{\lambda}} \right], \left[s_{p-p(1-\frac{Fa1}{p})^{\lambda}}, s_{p-p(1-\frac{Fb1}{p})^{\lambda}} \right] \right\rangle, \\ \left\langle s_{p(\frac{T1}{p})^{\lambda}}, s_{p-p(1-\frac{I1}{p})^{\lambda}}, s_{p-p(1-\frac{F1}{p})^{\lambda}} \right\rangle \end{array} \right), \lambda > 0$$

Then, the above operational results are still LNCNs.

Based on the score and accuracy functions of a linguistic neutrosophic number in the literature [16], we present the score, accuracy, and certain functions of LNCN to compare/rank LNCNs.

Definition 4. *Let a LNCN be* $h = (\langle [s_{Ta}, s_{Tb}], [s_{Ia}, s_{Ib}], [s_{Fa}, s_{Fb}] \rangle, \langle s_T, s_I, s_F \rangle)$ *for* s_{Ta}, s_{Ia}, s_{Fa} s_{Tb}, s_{Ib}, s_{Fb}, s_T, s_I, $s_F \in S$. *Then, its score, accuracy, and certain functions are defined as follows:*

$$S(h) = \frac{1}{9p}[(4p + Ta + Tb - Ia - Ib - Fa - Fb) + (2p + T - I - F)], \text{ for } S(h) \in [0, 1] \quad (1)$$

$$H(h) = \frac{1}{3p}[(Ta + Tb - Fa - Fb) + (T - F)], \text{ for } H(h) \in [-1,\ 1] \quad (2)$$

$$C(h) = \frac{Ta + Tb + T}{3p}, \text{for } C(h) \in [0,\ 1] \quad (3)$$

Then, we introduce a ranking method based on the values of the score, accuracy, and certain functions.

Definition 5. *Let two LNCNs be* $h_1 = (\langle [s_{Ta1}, s_{Tb1}], [s_{Ia1}, s_{Ib1}], [s_{Fa1}, s_{Fb1}] \rangle, \langle s_{T1}, s_{I1}, s_{F1} \rangle)$ *and* $h_2 = (\langle [s_{Ta2}, s_{Tb2}], [s_{Ia2}, s_{Ib2}], [s_{Fa2}, s_{Fb2}] \rangle, \langle s_{T2}, s_{I2}, s_{F2} \rangle)$. *Then, their ranking method based on their score, accuracy, and certain functions are defined as follows:*

(1) *If* $S(h_1) > S(h_2)$, *then* $h_1 \succ h_2$;

(2) *If* $S(h_1) = S(h_2)$ *and* $H(h_1) > H(h_2)$, *then* $h_1 \succ h_2$;

(3) *If $S(h_1) = S(h_2)$, $H(h_1) = H(h_2)$, and $C(h_1) > C(h_2)$, then $h_1 \succ h_2$;*
(4) *$S(h_1) = S(h_2)$, $H(h_1) = H(h_2)$, and $C(h_1) = C(h_2)$, then $h_1 \sim h_2$*

Example 1. *Let h_1 = (<[s_4, s_6], [s_1, s_2], [s_1, s_3]>, <s_5, s_1, s_2>), h_2 = (<[s_4, s_5], [s_1, s_2], [s_1, s_2]>, <s_4, s_1, s_1>), and h_3 = (<[s_6, s_7], [s_2, s_3], [s_1, s_3]>, <s_6, s_2, s_3>) in the linguistic term set $S = \{s_j \mid j \in [0, 8]\}$ are three LNCNs. Then, we need to compare them.*

By using Equations (1) to (3), the values of their score, accuracy, and certain functions are as follows:

$S(h_1)$ = [32 + 4 + 6 − (1 + 2 + 1 + 3) + 16 + 5 − (1 + 2)]/72 = 0.7361, $S(h_2)$ = [32 + 4 + 5 − (1 + 2 + 1 + 2) + 16 + 4− (1 + 1)]/72 = 0.7361, and $S(h_3)$ = [32 + 6 + 7 − (2 + 3 + 1 + 3) + 16 + 6 − (2 + 3)]/72 = 0.7361; $H(h_1)$ = [4 + 6 − (1 + 3) + 5 − 2]/24 = 0.375, $H(h_2)$ = [4 + 5 − (1 + 2) + 4 − 1]/24 = 0.375, and $H(h_3)$ = [6 + 7 − (1 + 3) + 6 − 3]/24 = 0.5; and $C(h_1)$ = (4 + 6 + 5)/24 = 0.625 and $C(h_2)$ = (4 + 5 + 4)/24 = 0.5417.

According to the ranking method of Definition 5, their ranking order is $h_3 \succ h_1 \succ h_2$.

3. Two Weighted Aggregation Operators of LNCNs

3.1. Linguistic Neutrosophic Cubic Number Weighted Arithmetic Averaging (LNCNWAA) Operator

Definition 6. *Let $h_j = \left(\left\langle [s_{Taj}, s_{Tbj}], [s_{Iaj}, s_{Ibj}], [s_{Faj}, s_{Fbj}] \right\rangle, \left\langle s_{Tj}, s_{Ij}, s_{Fj} \right\rangle\right)$ (j =1, 2, ..., n) be a group of LNCNs, then the LNCNWAA operator can be defined as follows:*

$$LNCNWAA(h_1, h_2, \cdots, h_n) = \sum_{j=1}^{n} w_j h_j \tag{4}$$

where w_j is the weight of h_j (j = 1, 2, ..., n) for $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$.

According to Definitions 3 and 6, there is the following theorem.

Theorem 1. *Let $h_j = \left(\left\langle [s_{Taj}, s_{Tbj}], [s_{Iaj}, s_{Ibj}], [s_{Faj}, s_{Fbj}] \right\rangle, \left\langle s_{Tj}, s_{Ij}, s_{Fj} \right\rangle\right)$ (j = 1, 2, ..., n) be a group of LNCNs, then the aggregation result obtained by Equation (4) is still a LNCN, which is calculated by the following aggregation formula:*

$$\begin{aligned} &LNCNWAA(h_1, h_2, \cdots, h_n) = \sum_{j=1}^{n} w_j h_j \\ &= \left(\begin{array}{l} \left\langle \left[s_{p - p\prod_{j=1}^{n}(1-\frac{Taj}{p})^{w_j}}, s_{p - p\prod_{j=1}^{n}(1-\frac{Tbj}{p})^{w_j}} \right], \left[s_{p\prod_{j=1}^{n}(\frac{Iaj}{p})^{w_j}}, s_{p\prod_{j=1}^{n}(\frac{Ibj}{p})^{w_j}} \right], \left[s_{p\prod_{j=1}^{n}(\frac{Faj}{p})^{w_j}}, s_{p\prod_{j=1}^{n}(\frac{Fbj}{p})^{w_j}} \right] \right\rangle, \\ \left\langle s_{p - p\prod_{j=1}^{n}(1-\frac{Tj}{p})^{w_j}}, s_{p\prod_{j=1}^{n}(\frac{Ij}{p})^{w_j}}, s_{p\prod_{j=1}^{n}(\frac{Fj}{p})^{w_j}} \right\rangle \end{array} \right) \end{aligned} \tag{5}$$

where w_j is the weight of h_j (j = 1, 2, ..., n) for $w_j \in [0, 1]$ and $\sum_{j=1}^{n} w_j = 1$.

In the following, the mathematical induction is used to prove Theorem 1.

Proof. (1) Set $n = 2$, according the operational laws of LNCNs, we have the following results:

$$w_1 h_1 = \left(\begin{array}{l} \left\langle \left[s_{p-p(1-\frac{Ta1}{p})^{w_1}}, s_{p-p(1-\frac{Tb1}{p})^{w_1}} \right], \left[s_{p(\frac{Ia1}{p})^{w_1}}, s_{p(\frac{Ib1}{p})^{w_1}} \right], \left[s_{p(\frac{Fa1}{p})^{w_1}}, s_{p(\frac{Fb1}{p})^{w_1}} \right] \right\rangle, \\ \left\langle s_{p-p(1-\frac{T1}{p})^{w_1}}, s_{p(\frac{I1}{p})^{w_1}}, s_{p(\frac{F1}{p})^{w_1}} \right\rangle \end{array} \right)$$

$$w_2h_2 = \left(\begin{array}{l}\left\langle \left[s_{p-p(1-\frac{Ta2}{p})^{w_2}}, s_{p-p(1-\frac{Tb2}{p})^{w_2}}\right], \left[s_{p(\frac{Ia2}{p})^{w_2}}, s_{p(\frac{Ib2}{p})^{w_2}}\right], \left[s_{p(\frac{Fa2}{p})^{w_2}}, s_{p(\frac{Fb2}{p})^{w_2}}\right]\right\rangle, \\ \left\langle s_{p-p(1-\frac{T2}{p})^{w_2}}, s_{p(\frac{I2}{p})^{w_2}}, s_{p(\frac{F2}{p})^{w_2}}\right\rangle\end{array}\right)$$

Then, there exists the following result:

$$\begin{aligned}
&LNCNWAA(h_1,h_2) = w_1h_1 \oplus w_2h_2 \\
&= \left(\begin{array}{l}
\left\langle \begin{array}{l}\left[\begin{array}{l} s_{p-p(1-\frac{Ta1}{p})^{w_1}+p-p(1-\frac{Ta2}{p})^{w_2}-\frac{(p-p(1-\frac{Ta1}{p})^{w_1})(p-p(1-\frac{Ta2}{p})^{w_2})}{p}}, \\ s_{p-p(1-\frac{Tb1}{p})^{w_1}+p-p(1-\frac{Tb2}{p})^{w_2}-\frac{(p-p(1-\frac{Tb1}{p})^{w_1})(p-p(1-\frac{Tb2}{p})^{w_2})}{p}} \end{array}\right], \\
\left[s_{p(\frac{Ia1}{p})^{w_1}(\frac{Ia2}{p})^{w_2}}, s_{p(\frac{Ib1}{p})^{w_1}(\frac{Ib2}{p})^{w_2}}\right], \left[s_{p(\frac{Fa1}{p})^{w_1}(\frac{Fa2}{p})^{w_2}}, s_{p(\frac{Fb1}{p})^{w_1}(\frac{Fb2}{p})^{w_2}}\right]\end{array}\right\rangle, \\
\left\langle s_{p-p(1-\frac{T1}{p})^{w_1}+p-p(1-\frac{T2}{p})^{w_2}-\frac{(p-p(1-\frac{T1}{p})^{w_1})(p-p(1-\frac{T2}{p})^{w_2})}{p}}, s_{p(\frac{I1}{p})^{w_1}(\frac{I2}{p})^{w_2}}, s_{p(\frac{F1}{p})^{w_1}(\frac{F2}{p})^{w_2}} \right\rangle
\end{array}\right) \\
&= \left(\begin{array}{l}
\left\langle \begin{array}{l}\left[\begin{array}{l} s_{p-p(1-\frac{Ta1}{p})^{w_1}+p-p(1-\frac{Ta2}{p})^{w_2}-(p-p(1-\frac{Ta1}{p})^{w_1}-p(1-\frac{Ta2}{p})^{w_2}+p(1-\frac{Ta1}{p})^{w_1}(1-\frac{Ta2}{p})^{w_2})}, \\ s_{p-p(1-\frac{Tb1}{p})^{w_1}+p-p(1-\frac{Tb2}{p})^{w_2}-(p-p(1-\frac{Tb1}{p})^{w_1}-p(1-\frac{Tb2}{p})^{w_2}+p(1-\frac{Tb1}{p})^{w_1}(1-\frac{Tb2}{p})^{w_2})} \end{array}\right], \\
\left[s_{p\prod_{j=1}^{2}(\frac{Iaj}{p})^{w_j}}, s_{p\prod_{j=1}^{2}(\frac{Ibj}{p})^{w_j}}\right], \left[s_{p\prod_{j=1}^{2}(\frac{Faj}{p})^{w_j}}, s_{p\prod_{j=1}^{2}(\frac{Fbj}{p})^{w_j}}\right]\end{array}\right\rangle, \\
\left\langle s_{p-p(1-\frac{T1}{p})^{w_1}+p-p(1-\frac{T2}{p})^{w_2}-(p-p(1-\frac{T1}{p})^{w_1}-p(1-\frac{T2}{p})^{w_2}+p(1-\frac{T1}{p})^{w_1}(1-\frac{T2}{p})^{w_2})}, s_{p\prod_{j=1}^{2}(\frac{Ij}{p})^{w_j}}, s_{p\prod_{j=1}^{2}(\frac{Fj}{p})^{w_j}} \right\rangle
\end{array}\right) \\
&= \left(\begin{array}{l}
\left\langle \begin{array}{l}\left[s_{p-p(1-\frac{Ta1}{p})^{w_1}(1-\frac{Ta2}{p})^{w_2}}, s_{p-p(1-\frac{Tb1}{p})^{w_1}(1-\frac{Tb2}{p})^{w_2}}\right], \\
\left[s_{p\prod_{j=1}^{2}(\frac{Iaj}{p})^{w_j}}, s_{p\prod_{j=1}^{2}(\frac{Ibj}{p})^{w_j}}\right], \left[s_{p\prod_{j=1}^{2}(\frac{Faj}{p})^{w_j}}, s_{p\prod_{j=1}^{2}(\frac{Fbj}{p})^{w_j}}\right]\end{array}\right\rangle, \\
\left\langle s_{p-p(1-\frac{T1}{p})^{w_1}(1-\frac{T2}{p})^{w_2}}, s_{p\prod_{j=1}^{2}(\frac{Ij}{p})^{w_j}}, s_{p\prod_{j=1}^{2}(\frac{Fj}{p})^{w_j}} \right\rangle
\end{array}\right) \\
&= \left(\begin{array}{l}
\left\langle \left[s_{p-p\prod_{j=1}^{2}(1-\frac{Taj}{p})^{w_j}}, s_{p-p\prod_{j=1}^{2}(1-\frac{Tbj}{p})^{w_j}}\right], \left[s_{p\prod_{j=1}^{2}(\frac{Iaj}{p})^{w_j}}, s_{p\prod_{j=1}^{2}(\frac{Ibj}{p})^{w_j}}\right], \left[s_{p\prod_{j=1}^{2}(\frac{Faj}{p})^{w_j}}, s_{p\prod_{j=1}^{2}(\frac{Fbj}{p})^{w_j}}\right]\right\rangle, \\
\left\langle s_{p-p\prod_{j=1}^{2}(1-\frac{Tj}{p})^{w_j}}, s_{p\prod_{j=1}^{2}(\frac{Ij}{p})^{w_j}}, s_{p\prod_{j=1}^{2}(\frac{Fj}{p})^{w_j}} \right\rangle
\end{array}\right).
\end{aligned} \tag{6}$$

(2) Set $n = k$, by Equation (5) we obtain

$$\begin{aligned}
&LNCNWAA(h_1, h_2, \cdots, h_k) = \sum_{j=1}^{k} w_jh_j \\
&= \left(\begin{array}{l}
\left\langle \left[s_{p-p\prod_{j=1}^{k}(1-\frac{Taj}{p})^{w_j}}, s_{p-p\prod_{j=1}^{k}(1-\frac{Tbj}{p})^{w_j}}\right], \left[s_{p\prod_{j=1}^{k}(\frac{Iaj}{p})^{w_j}}, s_{p\prod_{j=1}^{k}(\frac{Ibj}{p})^{w_j}}\right], \left[s_{p\prod_{j=1}^{k}(\frac{Faj}{p})^{w_j}}, s_{p\prod_{j=1}^{k}(\frac{Fbj}{p})^{w_j}}\right]\right\rangle, \\
\left\langle s_{p-p\prod_{j=1}^{k}(1-\frac{Tj}{p})^{w_j}}, s_{p\prod_{j=1}^{k}(\frac{Ij}{p})^{w_j}}, s_{p\prod_{j=1}^{k}(\frac{Fj}{p})^{w_j}} \right\rangle
\end{array}\right)
\end{aligned} \tag{7}$$

(3) Set $n = k + 1$, based on Equations (6) and (7), we can obtain the following result:

$$LNCNWAA(h_1, h_2, \cdots, h_{k+1}) = \sum_{j=1}^{k+1} w_j h_j$$

$$= \left(\begin{array}{l} \left\langle \begin{array}{l} \left[\begin{array}{l} s_{p-p\prod_{j=1}^{k}(1-\frac{Taj}{p})^{w_j}+p-p(1-\frac{Tak+j}{p})^{w_{k+1}}-\frac{(p-p\prod_{j=1}^{k}(1-\frac{Taj}{p})^{wj})(p-p(1-\frac{Tak+1}{p})^{w_{k+1}})}{p}}, \\ s_{p-p\prod_{j=1}^{k}(1-\frac{Tbj}{p})^{w_j}+p-p(1-\frac{Tbk+j}{p})^{w_{k+1}}-\frac{(p-p\prod_{j=1}^{k}(1-\frac{Tbj}{p})^{wj})(p-p(1-\frac{Tbk+1}{p})^{w_{k+1}})}{p}} \end{array} \right], \\ \left[s_{p\prod_{j=1}^{k}(\frac{Iaj}{p})^{w_j}(\frac{Iak+j}{p})^{w_{k+1}}}, s_{p\prod_{j=1}^{k}(\frac{Ibj}{p})^{w_j}(\frac{Ibk+j}{p})^{w_{k+1}}} \right], \left[s_{p\prod_{j=1}^{k}(\frac{Faj}{p})^{w_j}(\frac{Fak+j}{p})^{w_{k+1}}}, s_{p\prod_{j=1}^{k}(\frac{Fbj}{p})^{w_j}(\frac{Fbk+j}{p})^{w_{k+1}}} \right] \end{array} \right\rangle, \\ \left\langle s_{p-p\prod_{j=1}^{k}(1-\frac{Tj}{p})^{w_j}+p-p(1-\frac{Tk+j}{p})^{w_{k+1}}-\frac{(p-p\prod_{j=1}^{k}(1-\frac{Tj}{p})^{wj}))(p-p(1-\frac{Tk+1}{p})^{w_{k+1}})}{p}}, s_{p\prod_{j=1}^{k}(\frac{Ij}{p})^{w_j}(\frac{Ik+j}{p})^{w_{k+1}}}, s_{p\prod_{j=1}^{k}(\frac{Fj}{p})^{w_j}(\frac{Fk+j}{p})^{w_{k+1}}} \right\rangle \end{array} \right)$$

$$= \left(\begin{array}{l} \left\langle \begin{array}{l} \left[\begin{array}{l} s_{p-p\prod_{j=1}^{k}(1-\frac{Taj}{p})^{w_j}+p-p(1-\frac{Tak+1}{p})^{w_{k+1}}-(p-p\prod_{j=1}^{k}(1-\frac{Taj}{p})^{w_j}-p(1-\frac{Tak+1}{p})^{w_{k+1}}+p\prod_{j=1}^{k}(1-\frac{Taj}{p})^{w_j}(1-\frac{Tak+1}{p})^{w_{k+1}})}, \\ s_{p-p\prod_{j=1}^{k}(1-\frac{Tbj}{p})^{w_j}+p-p(1-\frac{Tbk+1}{p})^{w_{k+1}}-(p-p\prod_{j=1}^{k}(1-\frac{Tbj}{p})^{w_j}-p(1-\frac{Tbk+1}{p})^{w_{k+1}}+p\prod_{j=1}^{k}(1-\frac{Tbj}{p})^{w_j}(1-\frac{Tbk+1}{p})^{w_{k+1}})} \end{array} \right], \\ \left[s_{p\prod_{j=1}^{k}(\frac{Iaj}{p})^{w_j}(\frac{Iak+1}{p})^{w_{k+1}}}, s_{p\prod_{j=1}^{k}(\frac{Ibj}{p})^{w_j}(\frac{Ibk+1}{p})^{w_{k+1}}} \right], \left[s_{p\prod_{j=1}^{k}(\frac{Faj}{p})^{w_j}(\frac{Fak+1}{p})^{w_{k+1}}}, s_{p\prod_{j=1}^{k}(\frac{Fbj}{p})^{w_j}(\frac{Fbk+1}{p})^{w_{k+1}}} \right] \end{array} \right\rangle, \\ \left\langle s_{p-p\prod_{j=1}^{k}(1-\frac{Tj}{p})^{w_j}+p-p(1-\frac{Tk+1}{p})^{w_{k+1}}-(p-p\prod_{j=1}^{k}(1-\frac{Tj}{p})^{w_j}-p(1-\frac{Tk+1}{p})^{w_{k+1}}+p\prod_{j=1}^{k}(1-\frac{Tj}{p})^{w_j}(1-\frac{Tk+1}{p})^{w_{k+1}})}, s_{p\prod_{j=1}^{k}(\frac{Ij}{p})^{w_j}(\frac{Ik+1}{p})^{w_{k+1}}}, s_{p\prod_{j=1}^{k}(\frac{Fj}{p})^{w_j}(\frac{Fk+1}{p})^{w_{k+1}}} \right\rangle \end{array} \right)$$

$$= \left(\begin{array}{l} \left\langle \begin{array}{l} \left[s_{p-p\prod_{j=1}^{k}(1-\frac{Taj}{p})^{w_j}(1-\frac{Tak+1}{p})^{w_{k+1}}}, s_{p-p\prod_{j=1}^{k}(1-\frac{Tbj}{p})^{w_j}(1-\frac{Tbk+1}{p})^{w_{k+1}}} \right], \\ \left[s_{p\prod_{j=1}^{k}(\frac{Iaj}{p})^{w_j}(\frac{Iak+1}{p})^{w_{k+1}}}, s_{p\prod_{j=1}^{k}(\frac{Ibj}{p})^{w_j}(\frac{Ibk+1}{p})^{w_{k+1}}} \right], \left[s_{p\prod_{j=1}^{k}(\frac{Faj}{p})^{w_j}(\frac{Fak+1}{p})^{w_{k+1}}}, s_{p\prod_{j=1}^{k}(\frac{Fbj}{p})^{w_j}(\frac{Fbk+1}{p})^{w_{k+1}}} \right] \end{array} \right\rangle, \\ \left\langle s_{p-p\prod_{j=1}^{k}(1-\frac{Tj}{p})^{w_j}(1-\frac{Tk+1}{p})^{w_{k+1}}}, s_{p\prod_{j=1}^{k}(\frac{Ij}{p})^{w_j}(\frac{Ik+1}{p})^{w_{k+1}}}, s_{p\prod_{j=1}^{k}(\frac{Fj}{p})^{w_j}(\frac{Fk+1}{p})^{w_{k+1}}} \right\rangle \end{array} \right)$$

$$= \left(\begin{array}{l} \left\langle \left[s_{p-p\prod_{j=1}^{k+1}(1-\frac{Taj}{p})^{w_j}}, s_{p-p\prod_{j=1}^{k+1}(1-\frac{Tbj}{p})^{w_j}} \right], \left[s_{p\prod_{j=1}^{k+1}(\frac{Iaj}{p})^{w_j}}, s_{p\prod_{j=1}^{k+1}(\frac{Ibj}{p})^{w_j}} \right], \left[s_{p\prod_{j=1}^{k+1}(\frac{Faj}{p})^{w_j}}, s_{p\prod_{j=1}^{k+1}(\frac{Fbj}{p})^{w_j}} \right] \right\rangle, \\ \left\langle s_{p-p\prod_{j=1}^{k+1}(1-\frac{Tj}{p})^{w_j}}, s_{p\prod_{j=1}^{k+1}(\frac{Ij}{p})^{w_j}}, s_{p\prod_{j=1}^{k+1}(\frac{Fj}{p})^{w_j}} \right\rangle \end{array} \right).$$

Based on the above results, Equation (5) can hold for any n. The proof is finished.

Clearly, the LNCNWAA operator contains the following properties:

(1) Idempotency: Let h_j (j = 1, 2, ... , n) be a group of LNCNs. When $h_j = h$ for j = 1, 2, ... , n, there is $LNCNWAA(h_1, h_2, \cdots, h_n) = h$.

(2) Boundedness: Let h_j (j = 1, 2, ... , n) be a group of LNCNs and the minimum and maximum LNCNs be $h^- = \left(\left\langle \left[\min_j(s_{Taj}), \min_j(s_{Tbj}) \right], \left[\max_j(s_{Iaj}), \max_j(s_{Ibj}) \right], \left[\max_j(s_{Faj}), \max_j(s_{Fbj}) \right] \right\rangle, \left\langle \min_j(s_{Tj}), \max_j(s_{Ij}), \max_j(s_{Fj}) \right\rangle \right)$ and $h^+ = \left(\left\langle \left[\max_j(s_{Taj}), \max_j(s_{Tbj}) \right], \left[\min_j(s_{Iaj}), \min_j(s_{Ibj}) \right], \left[\min_j(s_{Faj}), \min_j(s_{Fbj}) \right] \right\rangle, \left\langle \max_j(s_{Tj}), \min_j(s_{Ij}), \min_j(s_{Fj}) \right\rangle \right)$ respectively. Then, there exists $h^- \le LNCNWAA(h_1, h_2, \cdots, h_n) \le h^+$.

(3) Monotonicity: Let h_j (j = 1, 2, ... , n) be a group of LNCNs. When $h_j \le h_j^*$ for j = 1, 2, ... , n, then there exists $LNCNWAA(h_1, h_2, \cdots, h_n) \le LNCNWAA\left(h_1^*, h_2^*, \cdots, h_n^*\right)$.

Proof. (1) For $h_j = h$ (j = 1, 2, ..., n), we have the following result:

$$
\begin{aligned}
&LNCNWAA(h_1, h_2, \cdots, h_n) = \sum_{j=1}^{n} w_j h_j \\
&= \left(\begin{array}{l} \left\langle \left[s_{p-p\prod\limits_{j=1}^{n}(1-\frac{Taj}{p})^{w_j}}, s_{p-p\prod\limits_{j=1}^{n}(1-\frac{Tbj}{p})^{w_j}} \right], \left[s_{p\prod\limits_{j=1}^{n}(\frac{Iaj}{p})^{w_j}}, s_{p\prod\limits_{j=1}^{n}(\frac{Ibj}{p})^{w_j}} \right], \left[s_{p\prod\limits_{j=1}^{n}(\frac{Faj}{p})^{w_j}}, s_{p\prod\limits_{j=1}^{n}(\frac{Fbj}{p})^{w_j}} \right] \right\rangle, \\ \left\langle s_{p-p\prod\limits_{j=1}^{n}(1-\frac{Tj}{p})^{w_j}}, s_{p\prod\limits_{j=1}^{n}(\frac{Ij}{p})^{w_j}}, s_{p\prod\limits_{j=1}^{n}(\frac{Fj}{p})^{w_j}} \right\rangle \end{array} \right) \\
&= \left(\begin{array}{l} \left\langle \left[s_{p-p(1-\frac{Ta}{p})^{\sum_{j=1}^{n} w_j}}, s_{p-p(1-\frac{Tb}{p})^{\sum_{j=1}^{n} w_j}} \right], \left[s_{p(\frac{Ia}{p})^{\sum_{j=1}^{n} w_j}}, s_{p(\frac{Ib}{p})^{\sum_{j=1}^{n} w_j}} \right], \left[s_{p(\frac{Fa}{p})^{\sum_{j=1}^{n} w_j}}, s_{p(\frac{Fb}{p})^{\sum_{j=1}^{n} w_j}} \right] \right\rangle, \\ \left\langle s_{p-p(1-\frac{T}{p})^{\sum_{j=1}^{n} w_j}}, s_{p(\frac{I}{p})^{\sum_{j=1}^{n} w_j}}, s_{p(\frac{F}{p})^{\sum_{j=1}^{n} w_j}} \right\rangle \end{array} \right) \\
&= \left(\left\langle \left[s_{p-p(1-\frac{Ta}{p})}, s_{p-p(1-\frac{Tb}{p})} \right], \left[s_{p(\frac{Ia}{p})}, s_{p(\frac{Ib}{p})} \right], \left[s_{p(\frac{Fa}{p})}, s_{p(\frac{Fb}{p})} \right] \right\rangle, \left\langle s_{p-p(1-\frac{T}{p})}, s_{p(\frac{I}{p})}, s_{p(\frac{F}{p})} \right\rangle \right) \\
&= (\langle [s_{Ta}, s_{Tb}], [s_{Ia}, s_{Ib}], [s_{Fa}, s_{Fb}] \rangle, \langle s_T, s_I, s_F \rangle) = h.
\end{aligned}
$$

(2) Since the minimum LNCN is h^- and the maximum LNCN is h^+, there is $h^- \leq h_j \leq h^+$. Thus, there exists $\sum_{j=1}^{n} w_j h^- \leq \sum_{j=1}^{n} w_j h_j \leq \sum_{j=1}^{n} w_j h^+$. According to the above property (1), there exists $h^- \leq \sum_{j=1}^{n} w_j h_j \leq h^+$. Then, $h^- \leq LNCNWAA(h_1, h_2, \cdots, h_n) \leq h^+$ can hold.

(3) For $h_j \leq h_j^*$ (j = 1, 2, ..., n), there exists $\sum_{j=1}^{n} w_j h_j \leq \sum_{j=1}^{n} w_j h_j^*$. Then, $LNCNWAA(h_1, h_2, \cdots, h_n) \leq LNCNWAA(h_1^*, h_2^*, \cdots, h_n^*)$ can hold.

Hence, we complete the proofs of these properties.

Obviously, when $w_j = 1/n$ for j = 1, 2, ..., n, the LNCNWAA operator is reduced to the LNCN arithmetic averaging operator.

3.2. LNCNWGA Operator

Definition 7. *Let* $h_j = \left(\left\langle [s_{Taj}, s_{Tbj}], [s_{Iaj}, s_{Ibj}], [s_{Faj}, s_{Fbj}] \right\rangle, \langle s_{Tj}, s_{Ij}, s_{Fj} \rangle \right)$ *(j =1, 2, ..., n) be a group of LNCNs, then the LNCNWGA operator is defined as follows:*

$$LNCNWGA(h_1, h_2, \cdots, h_n) = \prod_{j=1}^{n} h_j^{w_j} \tag{8}$$

where w_j *is the weight of* h_j *(j =1, 2, ..., n) for* $w_j \in [0, 1]$ *and* $\sum_{j=1}^{n} w_j = 1$.

According to Definitions 3 and 7, we can introduce the following theorem.

Theorem 2. *Let* $h_j = \left(\left\langle [s_{Taj}, s_{Tbj}], [s_{Iaj}, s_{Ibj}], [s_{Faj}, s_{Fbj}] \right\rangle, \langle s_{Tj}, s_{Ij}, s_{Fj} \rangle \right)$ *(j = 1, 2, . . . , n) be a group of LNCNs. Then, the aggregation result of Equation (8) is still a LNCN, which is calculated by the following aggregation equation:*

$$LNCNWGA(h_1, h_2, \cdots, h_n) = \prod_{j=1}^{n} h_j^{w_j}$$
$$= \left(\begin{array}{c} \left\langle \left[s_{p\prod_{j=1}^{n}(\frac{Taj}{p})^{w_j}}, s_{p\prod_{j=1}^{n}(\frac{Tbj}{p})^{w_j}} \right], \left[s_{p-p\prod_{j=1}^{n}(1-\frac{Iaj}{p})^{w_j}}, s_{p-p\prod_{j=1}^{n}(1-\frac{Ibj}{p})^{w_j}} \right], \left[s_{p-p\prod_{j=1}^{n}(1-\frac{Faj}{p})^{w_j}}, s_{p-p\prod_{j=1}^{n}(1-\frac{Fbj}{p})^{w_j}} \right] \right\rangle, \\ \left\langle s_{p\prod_{j=1}^{n}(\frac{Tj}{p})^{w_j}}, s_{p-p\prod_{j=1}^{n}(1-\frac{Ij}{p})^{w_j}}, s_{p-p\prod_{j=1}^{n}(1-\frac{Fj}{p})^{w_j}} \right\rangle \end{array} \right) \quad (9)$$

where w_j *is the weight of* h_j *(j = 1, 2, . . . , n) for* $w_j \in [0, 1]$ *and* $\sum_{j=1}^{n} w_j = 1$. *Obviously, when* w_j *= 1/n for j = 1, 2, ..., n, the LNCNWGA operator is reduced to the LNCN geometric averaging operator.*

Based on the similar proof manner of Theorem 1, we can prove Theorem 2. Hence, it is omitted here.

Obviously, the LNCNWGA operator also contains the following properties:

(1) Idempotency: Let h_j (j = 1, 2, . . . , n) be a group of LNCNs. When h_j = h for j = 1, 2, . . . , n, there exists $LNCNWGA(h_1, h_2, \cdots, h_n) = h$.

(2) Boundedness: Let h_j (j = 1, 2, . . . , n) be a group of LNCNs and the minimum and maximum LNCNs be $h^- = \left(\left\langle \left[\min_j(s_{Taj}), \min_j(s_{Tbj}) \right], \left[\max_j(s_{Iaj}), \max_j(s_{Ibj}) \right], \left[\max_j(s_{Faj}), \max_j(s_{Fbj}) \right] \right\rangle, \left\langle \min_j(s_{Tj}), \max_j(s_{Ij}), \max_j(s_{Fj}) \right\rangle \right)$ and $h^+ = \left(\left\langle \left[\max_j(s_{Taj}), \max_j(s_{Tbj}) \right], \left[\min_j(s_{Iaj}), \min_j(s_{Ibj}) \right], \left[\min_j(s_{Faj}), \min_j(s_{Fbj}) \right] \right\rangle, \left\langle \max_j(s_{Tj}), \min_j(s_{Ij}), \min_j(s_{Fj}) \right\rangle \right)$ respectively. Then, there exists $h^- \leq LNCNWGA(h_1, h_2, \cdots, h_n) \leq h^+$.

(3) Monotonicity: Let h_j (j = 1, 2, . . . , n) be a group of LNCNs. When $h_j \leq h_j^*$ for j = 1, 2, . . . , n, there exists $LNCNWGA(h_1, h_2, \cdots, h_n) \leq LNCNWGA(h_1^*, h_2^*, \cdots, h_n^*)$.

Based on the similar proofs of the properties corresponding to the LNCNWAA operator, we can also prove these properties of the LNCNWGA operator. Hence, these proofs are omitted here.

4. Decision-Making Method Based on the LNCNWAA or Linguistic Neutrosophic Cubic Number Weighted Geometric Averaging (LNCNWGA) Operator

This section proposes a decision-making method based the LNCNWAA or LNCNWGA operator to solve multiple attribute decision-making problems with LNCN information.

If there is a multiple attribute decision-making problem, we consider Q = {Q_1, Q_2, . . . , Q_m} as a set of alternatives and R = {R_1, R_2, . . . , R_n} as a set of attributes. The weigh vector of the attributes R_j (j = 1, 2, . . . , n) is specified as $\boldsymbol{w}$ = (w_1, w_2, . . . , w_n). Then, decision-makers are invited to evaluate the alternatives Q_i (i = 1, 2, . . . , m) over the attributes R_j (j = 1, 2, . . . , n) by LNCNs from the predefined linguistic term set S = {s_j | $j \in [0, p]$}, where p + 1 is an odd number/cardinality. Based on the linguistic term set, the decision-makers can assign the uncertain linguistic arguments corresponding to the truth, indeterminacy, and falsity linguistic terms and the certain linguistic arguments corresponding to the truth, indeterminacy, and falsity linguistic terms in each LNCN as the linguistic evaluation of each attribute R_j (j = 1, 2, . . . , n) on each alternative Q_i (i = 1, 2, . . . , m) in the evaluation process. Thus, all the LNCNs can be constructed as a LNCN decision matrix $D = (h_{ij})_{m \times n}$, where $h_{ij} = \left(\left\langle [s_{Taij}, s_{Tbij}], [s_{Iaij}, s_{Ibij}], [s_{Faij}, s_{Fbij}] \right\rangle, \langle s_{Tij}, s_{Iij}, s_{Fij} \rangle \right)$ (i = 1, 2, . . . , m; j = 1, 2, . . . , n) is a LNCN.

Thus, the decision-making method based on the LNCNWAA or LNCNWGA operator is described by the following decision steps:

Step 1 Calculate h_i = LNCNWAA(h_{i1}, h_{i2}, ..., h_{in}) or h_i = LNCNWGA(h_{i1}, h_{i2}, ..., h_{in}) (i = 1, 2, . . . , m) by using Equation (5) or Equation (9) and obtain the collective overall LNCN h_i for Q_i (i = 1, 2, . . . , m).

Step 2 Calculate the values of $S(h_i)$ ($H(h_i)$ and/or $C(h_i)$ if necessary) (i = 1, 2, ..., m) for each collective overall LNCN h_i (i = 1, 2, ..., m) by Equation (1) (Equation (2) and/or Equation (3)).

Step 3 Rank the alternatives corresponding to the ranking method of Definition 5, and then select the best one.

Step 4 End.

5. Illustrative Example

This section provides an illustrative example in order to demonstrate the application of the proposed decision-making method under a linguistic neutrosophic cubic environment.

A manufacturing company needs to hire a mechanical designer. After all applicants are chosen preliminarily by the human resources department, four potential candidates Q_1, Q_2, Q_3, and Q_4 need to be further evaluated according to the three requirements/attributes: (1) R_1 is the innovation skill; (2) R_2 is the design experience; (3) R_3 is the self-confidence. A group of experts is required to conduct the interview and to choose the most suitable candidate. Then, the weigh vector w = (0.45, 0.35, 0.2) is considered as the importance of the three attributes. Herewith, the experts (decision-makers) need to evaluate the four potential candidates/alternatives Q_i (i = 1, 2, 3, 4) corresponding to the three attributes R_j (j = 1, 2, 3) by the form of LNCNs based on the given linguistic term set $S = \{s_j \mid j \in [0, p]\}$, where S = {s_0 = extremely poor, s_1 = very poor, s_2 = poor, s_3 = slightly poor, s_4 = fair, s_5 = slightly good, s_6 = good, s_7 = very good, s_8 = extremely good} for p = 8. Thus, all the LNCNs are given by the experts and constructed as the following LNCN decision matrix $D(h_{ij})_{4\times 3}$:

$$D(h_{ij})_{4\times 3} = \begin{bmatrix} (\langle [s_4,s_6],[s_1,s_2],[s_1,s_3]\rangle,\langle s_5,s_1,s_2\rangle) & (\langle [s_4,s_6],[s_1,s_3],[s_1,s_3]\rangle,\langle s_5,s_2,s_2\rangle) & (\langle [s_4,s_7],[s_1,s_3],[s_2,s_3]\rangle,\langle s_6,s_2,s_3\rangle) \\ (\langle [s_3,s_5],[s_1,s_2],[s_1,s_2]\rangle,\langle s_4,s_1,s_1\rangle) & (\langle [s_5,s_7],[s_1,s_2],[s_1,s_2]\rangle,\langle s_6,s_1,s_2\rangle) & (\langle [s_4,s_6],[s_2,s_3],[s_1,s_2]\rangle,\langle s_5,s_3,s_1\rangle) \\ (\langle [s_4,s_7],[s_1,s_2],[s_2,s_3]\rangle,\langle s_5,s_2,s_3\rangle) & (\langle [s_6,s_7],[s_1,s_3],[s_1,s_3]\rangle,\langle s_7,s_2,s_2\rangle) & (\langle [s_5,s_7],[s_1,s_3],[s_2,s_3]\rangle,\langle s_5,s_2,s_3\rangle) \\ (\langle [s_6,s_7],[s_2,s_3],[s_2,s_3]\rangle,\langle s_7,s_3,s_3\rangle) & (\langle [s_5,s_7],[s_1,s_2],[s_1,s_2]\rangle,\langle s_6,s_1,s_2\rangle) & (\langle [s_4,s_6],[s_1,s_2],[s_1,s_2]\rangle,\langle s_5,s_1,s_1\rangle) \end{bmatrix}$$

Thus, the proposed decision-making method can be applied to the decision-making problem with LNCN information.

On the one hand, we can use the decision-making method based on the LNCNWAA operator, which is described by the following decision steps:

Step 1 By using Equation (5), the collective overall LNCNs of h_i for Q_i (i = 1, 2, 3, 4) can be given as follows:

$h_1 = (<[s_4, s_{6.2589}], [s_1, s_{2.4997}], [s_{1.1487}, s_3]>, <s_{5.2337}, s_{1.4641}, s_{2.1689}>)$, $h_2 = ([s_{4.0011}, s_{6.1167}], [s_{1.1487}, s_{2.1689}], [s_1, s_2]>, <s_{5.0371}, s_{1.2457}, s_{1.2746}>)$, $h_3 = ([s_{5.0371}, s_7], [s_1, s_{2.4997}], [s_{1.5692}, s_3]>, <s_{5.9577}, s_2, s_{2.6031}>)$, and $h_4 = ([s_{5.3523}, s_{6.8513}], [s_{1.366}, s_{2.4003}], [s_{1.366}, s_{2.4003}]>, <s_{6.4122}, s_{1.6395}, s_{2.0896}>)$.

Step 2 Calculate the score values of $S(h_i)$ (i = 1, 2, 3, 4) by Equation (1):

$S(h_1)$ = 0.7252, $S(h_2)$ = 0.7544, $S(h_3)$ = 0.7406, and $S(h_4)$ = 0.7688.

Step 3 The ranking order of the four alternatives is $Q_4 \succ Q_2 \succ Q_3 \succ Q_1$ based on the score values. Thus, the candidate Q_4 is the best choice among the four candidates.

On the other hand, we can also use the decision-making method based on the LNCNWGA operator, which is described by the following decision procedures:

Step 1′ By using Equation (9), the collective overall LNCNs of h_i for Q_i (i = 1, 2, 3, 4) are given as follows:

$h_1 = (<[s_4, s_{6.1879}], [s_1, s_{2.5725}], [s_{1.2125}, s_3]>, <s_{5.1857}, s_{1.569}, s_{2.2148}>)$, $h_2 = (<[s_{3.7998}, s_{5.8338}], [s_{1.2125}, s_{2.2148}], [s_1, s_2]>, <s_{4.8203}, s_{1.4556}, s_{1.3677}>)$, $h_3 = (<[s_{4.8203}, s_7], [s_1, s_{2.5725}], [s_{1.6674}, s_3]>, <s_{5.6249}, s_2, s_{2.6705}>)$, and $h_4 = (<[s_{5.1906}, s_{6.7875}], [s_{1.4691}, s_{2.4726}], [s_{1.4691}, s_{2.4726}]>, <s_{6.2007}, s_{1.9835}, s_{2.2996}>)$.

Step 2′ By using Equation (1), we calculate the score values of $S(h_i)$ (i = 1, 2, 3, 4) as follows:

$S(h_1)$ = 0.7195, $S(h_2)$ = 0.7389, $S(h_3)$ = 0.7296, and $S(h_4)$ = 0.7502.

Step 3' The ranking order of the four candidates is $Q_4 \succ Q_2 \succ Q_3 \succ Q_1$. Thus, the candidate Q_4 is still the best choice among the four candidates.

Obviously, the above two ranking orders based on the LNCNWAA and LNCNWGA operators and the best candidate are identical.

Compared with existing currant linguistic neutrosophic decision-making methods [16,17], the decision information in this study is LNCNs, while the decision information used in [16,17] is linguistic neutrosophic numbers. As mentioned above, since LNCN is composed of its uncertain neutrosophic number and its linguistic neutrosophic number, LNCN contains more information than the unique linguistic neutrosophic number in [16,17]. However, existing linguistic neutrosophic decision-making methods in [16,17] cannot handle such a decision-making problem with linguistic neutrosophic cubic information in this paper. Therefore, the decision-making method proposed in this paper can solve decision-making problems with both certain linguistic and uncertain linguistic neutrosophic information. It can also provide a new way for hybrid linguistic decision-making problems under certain and uncertain linguistic environments.

Due to no related studies in the existing literature, this is the first study to propose a new concept of LNCN and a new linguistic neutrosophic cubic decision-making method. However, decision-makers can select one of two weighted aggregation operators of LNCNs to solve linguistic neutrosophic cubic decision-making problems according to their preference and actual requirements.

6. Conclusions

This paper originally proposed the concept of LNCN, including the internal LNCN and external LNCN, and the operational laws of LNCNs, and introduced the score, accuracy, and certain functions of LNCNs for comparing/ranking LNCNs. Then, we proposed the LNCNWAA and LNCNWGA operators to aggregate LNCNs and discussed their properties. Next, we developed a multiple attribute decision-making method based on the LNCNWAA or LNCNWGA operator for solving multiple attribute decision-making problems with LNCN information. Finally, an example illustrated the application of the developed method under a LNCN environment. The proposed decision-making method can solve decision-making problems with determinate and uncertain linguistic neutrosophic arguments.

Obviously, the main advantages of this study are summarized as follows:

(1) The LNCN expression is superior to existing linguistic expressions in the certain and uncertain linguistic environment.
(2) The developed linguistic neutrosophic cubic decision-making method extends existing ones to deal with linguistic neutrosophic cubic decision-making problems with the hybrid information of both uncertain linguistic neutrosophic arguments and certain linguistic neutrosophic arguments.
(3) The developed new method enriches linguistic neutrosophic expressions and linguistic neutrosophic decision-making methods.

In the future work, we shall further introduce new aggregation operators of LNCNs and applications in group decision-making, pattern recognition, and medical diagnoses.

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