Introduction to Neutrosophic Nearrings

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Abstract

The objective of this paper is to introduce the concept of neutrosophic nearrings. The concept of neutrosophic N-group of a neutrosophic nearring is introduced. We studied neutrosophic subnearrings of neutrosophic nearrings and also neutrosophic N-subgroups of neutrosophic N- groups. The notions of neutrosophic ideals in neutrosophic nearrings and neutrosophic N-groups are introduced and their elementary properties are presented. In addition, we introduced the concepts of neutrosophic homomorphisms of neutrosophic nearrings and neutrosophic N-homomorphisms of neutrosophic N-groups and also, we presented neutrosophic quotient nearrings and quotient N-groups.
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1 Introduction

Definition 1.1. Let $(N, +, .)$ be a set with two binary operations $+$ and $.$. N is called a right nearring if the following conditions hold:

(1) $(N, +)$ is a group, not necessarily abelian.

(2) $(N, .)$ is a semigroup.

(3) For all $x, y, z \in N$, $(x + y)z = xz + yz$.

A left nearring is similarly defined. If $(N, +)$ is an abelian group, then $(N, +, .)$ is called an abelian nearring. Also, if $(N, .)$ is commutative, then $(N, +, .)$ is called a commutative nearring.

$(N, +, .)$ is a nearfield if $(N \setminus \{0\}, .)$ is a group.

Neutrosophy is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. Neutrosophic set and neutrosophic logic were introduced in 1995 by Smarandache as generalizations of classical sets, conventional fuzzy set, intuitionistic fuzzy set, interval valued fuzzy set and respectively fuzzy logic and intuitionistic fuzzy logic. In neutrosophic logic, each proposition is approximated to have the percentage of truth in a subset (T), the percentage of indeterminacy in a subset (I), and the percentage of falsity in a subset (F), where T, I, F are standard or non-standard subsets of the non-standard unit interval $]^-0, 1^+[$, (see [8, 9, 10]). In technical applications, T, I and F are only standard subsets of the standard unit interval $[0, 1]$ with $^-0 \leq \sup(T) + \sup(I) + \sup(F) \leq 3^+$ where $\sup(X)$ means the superior of the subset X.

Neutrosophic logic has wide applications in science, engineering, Information Technology, law, politics, economics, finance, econometrics, operations research, optimization theory, game theory and simulation etc.

The notion of neutrosophic algebraic structures was introduced by Kandasamy and Smarandache in 2006. The indeterminate element I was combined with the elements of a given algebraic structure $(X, *)$, and, the new algebraic structure $(X(I), *) =< X, I >$ generated by X and I is called a neutrosophic algebraic structure. Some of the neutrosophic algebraic structures developed by Kandasamy and Smarandache include neutrosophic groupoids, neutrosophic semigroups, neutrosophic groups, neutrosophic loops, neutrosophic rings, neutrosophic fields, neutrosophic vector spaces, neutrosophic modules, neutrosophic bigroupoids, neutrosophic bisemigroups, neutrosophic bigroups, neutrosophic biloops, neutrosophic N-groups, neutrosophic N-semigroups, neutrosophic N-loops and so on (see [17, 18]). Recently, Agboola and Davvaz introduced neutrosophic hypergroups, neutrosophic canonical hypergroups, neutrosophic hyperrings and neutrosophic BCI/BCK-Algebras (see [4],[5],[6]). Many researchers have also developed and studied several neutrosophic algebraic structures (see [12],[13] [14], [16]).

Let X be a nonempty set and let I be an indeterminate. The set

$$X(I) =< X, I >= \{(x, yI) : x, y \in X\} \tag{1}$$

is called a neutrosophic set generated by X and I. If $+$ and $.$ are ordinary addition and multiplication, I has the following properties:

(1) $I + I + \cdots + I = nI$.

(2) $I + (-I) = 0$.

(3) $I.I.\cdots.I = I^n = I$ for all positive integer n.

(4) $0.I = 0$.

(5) I^{-1} is undefined and therefore does not exist.

If $* : X(I) \times X(I) \to X(I)$ is a binary operation defined on $X(I)$, then the couple $(X(I), *)$ is called a neutrosophic algebraic structure and it is named according the axioms satisfied by $*$. If $(X(I), *)$ and and $(Y(I), *')$ are two neutrosophic algebraic structures, the mapping $\phi : (X(I), *) \to (Y(I), *')$ is called a neutrosophic homomorphism if the following conditions hold:

(1) $\phi((w,xI)*(y,zI)) = \phi((w,xI)) *' \phi((y,zI))$.

(2) $\phi(I) = I \ \ \forall (w,xI),(y,zI) \in X(I)$.

Definition 1.2. Let $(G,*)$ be any group. The couple $(G(I),*)$ is called a neutrosophic group generated by G and I. $(G(I),*)$ is said to be commutative if for all $x,y \in G(I)$, we have $x*y = y*x$. $(\mathbb{Z}(I),+)$ and $(\mathbb{R}(I),.)$ are commutative neutrosophic groups.

It should be noted that generally, a neutrosophic group $(G(I),*)$ is not a group. However, every additive neutrosophic group $(G(I),+)$ is a group.

Definition 1.3. Let $(R,+,.)$ be any ring. The triple $(R(I),+,.)$ is called a neutrosophic ring generated by R and I. $R(I)$ is said to be commutative if for all $x,y \in R(I)$, we have $x.y = y.x$. $(\mathbb{Z}(I),+,.)$ and $(\mathbb{R}(I),+,.)$ are commutative neutrosophic rings of integers and real numbers respectively.

Generally, every neutrosophic ring $(R(I),+,.)$ is a ring but not the converse.

Definition 1.4. Let $(F,+,.)$ be any field. The triple $(F(I),+,.)$ is called a neutrosophic field generated by F and I. $(\mathbb{Q}(I),+,.)$ and $(\mathbb{R}(I),+,.)$ are neutrosophic fields.

2 Main Results

Definition 2.1. Let $(N,+,.)$ be any right nearring. The triple $(N(I),+,.)$ is called a right neutrosophic nearring. For all $x=(a,bI), y=(c,dI) \in N(I)$ with $a,b,c,d \in N$, we define:

$$x+y = (a,bI)+(c,dI) = (a+c,(b+d)I). \quad (2)$$

$$-x = -(a,bI) = (-a,-bI). \quad (3)$$

$$x.y = (a,bI).(c,dI) = (ac,(ad+bc+bd)I). \quad (4)$$

The zero element in $(N,+)$ is represented by $(0,0)$ in $(N(I),+)$. Any element $x \in N$ is represented by $(x,0)$ in $N(I)$. I in $N(I)$ is sometimes represented by $(0,I)$ in $N(I)$.

Definition 2.2. Let $(N(I),+,.)$ be a right neutrosophic nearring.

(1) $N(I)$ is called abelian if

$$(a,bI)+(c,dI) = (c,dI)+(a,bI) \ \forall (a,bI),(c,dI) \in N(I).$$

(2) $N(I)$ is called commutative if

$$(a,bI).(c,dI) = (c,dI).(a,bI) \ \forall (a,bI),(c,dI) \in N(I).$$

(3) $N(I)$ is said to be distributive if $N(I) = N_d(I)$ where

$$N_d(I) = \{d \in N(I) : d(m+n) = dm+dn \ \forall m,n \in N(I)\}.$$

(4) $N(I)$ is said to be zero-symmetric if $N(I) = N_0(I)$ where

$$N_0(I) = \{n \in N(I) : n0 = 0\}.$$

The following should be noted:

(1) $N(I)$ is abelian only if $(N, +)$ is abelian.

(2) $N(I)$ is commutative only if $(N, .)$ is commutative.

(3) $N(I)$ is distributive only if N is distributive.

(4) $N(I)$ is zero-symmetric only if N is zero-symmetric.

Except otherwise stated, all nearrings in this paper will be right nearrings and all neutrosophic nearrings will be right neutrosophic nearrings.

EXAMPLE 1. Let $(X(I), +)$ be a neutrosophic group and let $M^{X(I)}$ be a neutrosophic set defined by

$$M^{X(I)} = \{\phi : X(I) \to X(I)\}.$$

For all $\phi, \psi \in M^{X(I)}$, define:

$$\begin{aligned} (\phi + \psi)((x, yI)) &= \phi((x, yI)) + \psi((x, yI)) \\ \phi \circ \psi((x, yI)) &= \phi(\psi((x, yI)) \ \ \forall \ (x, yI) \in X(I). \end{aligned}$$

Then $(M^{X(I)}, +, \circ)$ is a neutrosophic nearring.

EXAMPLE 2. Let

$$\begin{aligned} N(I) &= \mathbb{Z}_{12}(I) \\ &= \{(0,0), (1,0), \ldots, (11,0), (0,I), (0,2I), \ldots, \\ &\quad (0,11I), (1,I), (2,I), \ldots, (11,11I)\}. \end{aligned}$$

For all $(x, yI) \in N(I)$ with $x, y \in \mathbb{Z}_{12}$, let $x.y = x$. Then $(N(I), +, .)$ is a neutrosophic nearring.

Theorem 2.3. *Let $(N(I), +, .)$ be a neutrosophic nearring. Then $N(I)$ is a nearring.*

Theorem 2.4. *Let $\{N_i(I)\}_{i=1}^n$ be a family of neutrosophic nearrings. Then $\Pi_{i=1}^n N_i(I)$ is a neutrosophic nearring.*

Definition 2.5. let $(N, +, .)$ be any nearring and let $(\Gamma(I), +)$ be any neutrosophic group. Suppose that $\mu : N \times \Gamma(I) \to \Gamma(I)$ is an action of N on $\Gamma(I)$ defined by juxtaposition. $\Gamma(I)$ is called a neutrosophic N-group if for all $m, n \in N$, $x \in \Gamma(I)$, the following conditions hold:

(1) $m.I = mI$.

(2) $(mn)x = m(nx)$.

(3) $(m+n)x = mx + nx$.

Theorem 2.6. *Every neutrosophic N-group is an N-group.*

Proof. Suppose that $\Gamma(I)$ is a neutrosophic N-group. Then $(\Gamma(I), +)$ is a group. The required result follows. □

Definition 2.7. Let $N(I)$ be a neutrosophic nearring and let $A(I)$ be a nonempty subset of $N(I)$. $A(I)$ is called a neutrosophic subnearring of $N(I)$ if the following conditions hold:

(1) $A(I)$ is a neutrosophic subgroup of $N(I)$ that is, $xy \in A(I)$ for all $x, y \in A(I)$.

(2) $A(I)$ contains a proper subset which is a subnearring of $N(I)$.

$A(I)$ is called a pseudo neutrosophic subnearring of $N(I)$ if it does not contain a proper subset which is a subnearring of $N(I)$.

Definition 2.8. Let $\Gamma(I)$ be a neutrosophic N-group and let $B(I)$ be a nonempty subset of $\Gamma(I)$. $B(I)$ is called a neutrosophic N-subgroup of $\Gamma(I)$ if the following conditions hold:

(1) $B(I)$ is a neutrosophic subgroup of $\Gamma(I)$ that is, $nx \in B(I)$ for all $n \in N(I), x \in B(I)$.

(2) $B(I)$ contains a proper subset which is an N-subgroup of $\Gamma(I)$.

$B(I)$ is called a pseudo neutrosophic N-subgroup of $\Gamma(I)$ if it does not contain a proper subset which is an N-subgroup of $\Gamma(I)$.

Example 3. Let $N(I) = \mathbb{Z}(I)$ and let $A(I) = 2\mathbb{Z}(I)$. Then $A(I)$ is a neutrosophic subnearring of $N(I)$.

Example 4. Let $\Gamma(I) = \mathbb{R}(I)$ and let $B(I) = \mathbb{Q}(I)$. Then $B(I)$ is a neutrosophic N-subgroup of $\Gamma(I)$.

Definition 2.9. Let $(N(I), +, .)$ be a neutrosophic nearring and let $A(I)$ be a normal neutrosophic subgroup of $(N(I), +)$.

(1) $A(I)$ is called a right neutrosophic ideal of $N(I)$ if for all $x \in A(I)$, $m \in N(I)$, we have $xm \in A(I)$.

(2) $A(I)$ is called a left neutrosophic ideal of $N(I)$ if for all $x \in A(I)$, $m, n \in N(I)$, we have $m(n+x) - mn \in A(I)$.

$A(I)$ will be called a neutrosophic ideal of $N(I)$ if it is both a left and right neutrosophic ideal of $N(I)$.

Definition 2.10. Let $(N(I), +, .)$ be a neutrosophic nearring, $\Gamma(I)$ a neutrosophic N-group and let $B(I)$ be a normal neutrosophic N-subgroup of $\Gamma(I)$. $B(I)$ is called a neutrosophic ideal of $\Gamma(I)$ if for all $x \in B(I)$, $y \in \Gamma(I)$, $m \in N(I)$, we have $m(y+x) - my \in B(I)$.

Example 5. Let

$$\begin{aligned} N(I) &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{Z}(I) \right\} \text{ be a neutrosophic nearring,} \\ A_r(I) &= \left\{ \begin{bmatrix} x & y \\ 0 & 0 \end{bmatrix} : x,y \in \mathbb{Z}(I) \right\}, \\ A_l(I) &= \left\{ \begin{bmatrix} x & 0 \\ y & 0 \end{bmatrix} : x,y \in \mathbb{Z}(I) \right\}. \end{aligned}$$

Then $A_r(I)$ is a right neutrosophic ideal of $N(I)$ and $A_l(I)$ is a left neutrosophic ideal of $N(I)$.

Example 6. Let

$$\begin{aligned} N(I) &= \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} : a,b,c,d \in \mathbb{R}(I) \right\} \text{ be a neutrosophic nearring,} \\ \Gamma(I) &= \left\{ \begin{bmatrix} w & x \\ y & z \end{bmatrix} : w,x,y,z \in \mathbb{Q}(I) \right\} \text{ a neutrosophic } N\text{-group and let} \\ B(I) &= \left\{ \begin{bmatrix} p & 0 \\ q & 0 \end{bmatrix} : p,q \in \mathbb{Q}(I) \right\}. \end{aligned}$$

Then $B(I)$ is a neutrosophic ideal of $\Gamma(I)$.

Theorem 2.11. *Let $N(I)$ be a neutrosophic nearring. If $A(I)$ and $B(I)$ are any two neutrosophic ideals of $N(I)$ and $\{A_i(I)\}_{i=1}^n$ is a family of neutrosophic ideals of $N(I)$, then:*

(1) $A(I) + B(I) = \{a + b : a \in A(I), b \in B(I)\}$ is a neutrosophic ideal of $N(I)$.

(2) $A(I)B(I) = \{\sum_1^n a_i b_i : a_i \in A(I), b_i \in B(I)\}$ is a neutrosophic ideal of $N(I)$.

(3) $\bigcap_1^n A_i(I)$ is a neutrosophic ideal of $N(I)$.

Theorem 2.12. *Let $N(I)$ be a distributive neutrosophic nearring and let $X(I)$ be a nonempty subset of $N(I)$. Then*

$$(O : X(I))_r = \{n \in N(I) : X(I)n = O\}$$

is a neutrosophic right ideal of $N(I)$.

Proof. Let $a = (u, vI), b = (p, qI) \in (O : X(I))_r$ with $u, v, p, q \in N$, let $x = (y, zI) \in X(I)$ with $y, z \in X$ and let $n = (s, tI) \in N(I)$ with $s, t \in N$ be arbitrary elements. Then $xa = (y, zI)(u, vI) = (0, 0)$ from which we obtain $yu = 0$ and $yv + zu + zv = 0$. Also, $xb = (y, zI)(p, qI) = (0, 0)$ which implies that $yp = 0$ and $yq + zp + zq = 0$. Now,

$$\begin{aligned} x(ab) &= (y, zI)(up, (uq + vp + vq)I) \\ &= (y(up), (y(uq) + y(vp) + y(vq) + z(up) + z(uq) + z(vp) + z(vq))I) \\ &= ((yu)p, ((yu)q + (yv)p + (yv)q + (zu)p + (zu)q + (zv)p + (zv)q))I) \\ &= ((yu)p, ((yu + yv + zu + zv)q + (yv + zu + zv)p)I) \\ &= (0p, (0q + 0p)I \\ &= (0, 0). \end{aligned}$$

This shows that $ab \in (O : X(I))_r$. Lastly,

$$\begin{aligned} x(an) &= (y, zI)((u, vI)(s, tI)) \\ &= (y, zI)(us, (ut + vs + vt)I) \\ &= (y(us), (y(ut + vs + vt) + z(us) + z(ut + vs + vt))I) \\ &= ((yu)s, ((yu + yv + zu + zv)t + (yv + zu + zv)s)I) \\ &= (0s, (0t + 0s)I) \\ &= (0, 0). \end{aligned}$$

This shows that $an \in (O : X(I))_r$ and consequently, $(O : X(I))_r$ is a neutrosophic right ideal of $N(I)$. □

Theorem 2.13. *Let $N(I)$ be a zero-symmetric distributive neutrosophic nearring and let $X(I)$ be a nonempty subset of $N(I)$. Then*

$$(O : X(I))_l = \{n \in N(I) : nX(I) = O\}$$

is a neutrosophic left ideal of $N(I)$.

Proof. Let $a = (u, vI), b = (p, qI) \in (O : X(I))_l$ with $u, v, p, q \in N$, let $x = (y, zI) \in X(I)$ with $y, z \in X$ and let $m = (g, hI), n = (s, tI) \in N(I)$ with $g, h, s, t \in N$ be arbitrary elements . Then $ax = (u, vI)(y, zI) = (0, 0)$ from which we obtain $uy = 0$ and $uz + vy + vz = 0$. Also, $bx = (p, qI)(y, zI) = (0, 0)$ which implies that $py = 0$ and $pz + qy + qz = 0$. Now,

$$\begin{aligned} (ab)x &= (up, (uq + vp + vq)I)(y, zI) \\ &= ((up)y, ((u - p)z + (v - q)y + (v - q)z)I) \\ &= ((up)y, ((up)z + (uq)y + (vp)y + +(vq)y + (uq)z + (vp)z + (vq)z)I) \\ &= (u(py), (u(pz) + u(qy) + v(py) + v(qz) + u(qz) + v(pz) + v(qz))I) \\ &= (u(py), (u(pz + qy + qz) + v(py + qy + pz + qz))I) \\ &= (u0, (u0 + v0)I) \\ &= (0, 0) \end{aligned}$$

showing that $ab \in (O : X(I))_l$. Also,

$$\begin{aligned} (m(n + a) - mn)x &= ((g, hI)((s, tI) + (u, vI)) - (g, hI)(s, tI))(y, zI) \\ &= ((g, hI)(s + u, (t + v)I) - (gs, (gt + hs + ht)I)(y, zI) \\ &= ((g(s + u), (g(t + v) + h(s + u) + h(t + v))I) - (gs, (gt + hs + ht)I))(y, zI) \\ &= (gs, (gv + hu + hv)I)(y, zI) \\ &= ((gu)y, ((gu)z + (gv)y + (hu)y + (hv)y + (gv)z + (hu)z + (hv)z)I) \\ &= (g(uy), (g(uz + vy + vz) + h(uy + vy + uz + vz))I) \\ &= (g0, (g0 + h0)I) \\ &= (0, 0). \end{aligned}$$

Thus $m(n + a) - mn \in (O : X(I))_l$ and hence, $(O : X(I))_l$ is a neutrosophic left ideal of $N(I)$. □

Theorem 2.14. *Let $N(I)$ be a distributive neutrosophic nearring and let $\Gamma(I)$ be a neutrosophic N-group of $N(I)$. If $A(I)$ and $B(I)$ are any two neutrosophic ideals of $\Gamma(I)$, then*

$$(A(I) : B(I)) = \{n \in N(I) : nB(I) \subseteq A(I)\}$$

is a neutrosophic ideal of $\Gamma(I)$.

Proof. Let $a = (u, vI) \in (A(I) : B(I))$, $b = (e, fI) \in B(I)), x = (w, yI) \in \Gamma(I)$ and $m = (n, rI) \in N(I)$ be arbitrary elements. Then $ab = a$ that is $(u, vI)(e, fI) = (u, vI)$ from which we obtain $ue = u$ and $uf + ve + vf = v$. Now,

$$\begin{aligned}
(ma)b &= ((n, rI)(u, vI))(e, fI) \\
&= (nu, (nv + ru + rv)I)(e, fI) \\
&= ((nu)e, ((nu)f + (nv)e + (ru)e + (rv)e + (nv)f + (ru)f + (rv)f)I) \\
&= (n(ue), (n(uf) + n(ve) + r(ue) + r(ve) + n(vf) + r(uf) + r(vf))I) \\
&= (nu, n(uf + ve + vf) + r(u + ve + uf + vf) \\
&= (nu, (nv + r(u + v))I) \\
&= (nu, (nv + ru + rv)I) \\
&= (n, rI)(u, vI) \\
&= ma.
\end{aligned}$$

This shows that $ma \in (A(I) : B(I))$. Lastly,

$$\begin{aligned}
(m(x + a) - mx)b &= [(n, rI)((w, yI) + (u, vI)) - (n, rI)(w, yI)](e, fI) \\
&= [(n, rI)((w + u, (y + v)I) - (nw, (ny + rw + ry)I)](e, fI) \\
&= [(n(w + u), (n(y + v) + r(w + u) + r(y + v))I) - (nw, ny + rw + ry)I)](e, fI) \\
&= ((nw + nu, (ny + nv + rw + ru + ry + rv)I) - (nw, (ny + rw + ry)I)](e, fI) \\
&= (nu, (nv + ru + rv)I)(e, fI) \\
&= ((nu)e, ((nu)f + (nv)e + (ru)e + (rv)e + (nv)f + (ru)f + (rv)f)I) \\
&= (n(ue), (n(uf) + n(ve) + r(ue) + r(ve) + n(vf) + r(uf) + r(vf))I) \\
&= (nu, (n(uf + ve + vf) + r(ue + ve + uf + vf))I) \\
&= (nu, (nv + ru + rv)I) \\
&= (n, rI)(u, vI) \\
&= ma.
\end{aligned}$$

This shows that $m(x + a) - mx \in (A(I) : B(I))$. Accordingly, $(A(I) : B(I))$ is a neutrosophic ideal of $\Gamma(I)$. □

Theorem 2.15. *Let $N(I)$ be a zero-symmetric distributive neutrosophic nearring and let $\Gamma(I)$ be a neutrosophic N-group of $N(I)$. If $A(I)$ is a neutrosophic ideal of $\Gamma(I)$, then*

$$(O : A(I)) = \{n \in N(I) : nA(I) = O\}$$

is a neutrosophic ideal of $\Gamma(I)$.

Proof. Let $a = (u, vI), b = (p, qI) \in (O : A(I))$, $x = (w, yI) \in \Gamma(I)$, $m = (n, rI) \in N(I)$ and $c = (e, fI) \in A(I)$ be arbitrary elements. Then $ac = 0$ and $bc = 0$ that is $(u, vI)(e, fI) = (0, 0)$ and $(p, qI)(e, fI) = (0, 0)$ from which we obtain $ue = 0$, $uf + ve + vf = 0$ and $pe = 0$, $pf + qe + qf = 0$. Now,

$$\begin{aligned}
(ab)c &= ((u, vI)(p, qI))(e, fI) \\
&= (up, (uq + vp + vq)I)(e, fI) \\
&= ((up)e, ((up)f + (uq)e + (vp)e + (vq)e + (uq)f + (vp)f + (vq)f)I) \\
&= ((u(pe), (u(pf) + u(qe) + v(pe) + v(qe) + u(qf) + v(pf) + v(qf))I) \\
&= (u(pe), (u(pf + qe + qf) + (pe + qe + pf + qf))I) \\
&= (u0, (u0 + v0)I) \\
&= (0, 0).
\end{aligned}$$

Therefore, $ab \in (O : A(I))$. Lastly,

$$\begin{aligned}
(m(x + a) - mx)c &= [(n, rI)((w, yI) + (u, vI)) - (n, rI)(w, yI))](e, fI) \\
&= [(n, rI)((w + u, (y + v)I) - (nw, (ny + rw + ry)I)](e, fI) \\
&= [(n(w + u), (n(y + v) + r(w + u) + r(y + v))I) - (nw, (ny + rw + ry)I)](e, fI) \\
&= [(nw + nu, (ny + nv + rw + ru + ry + rv)I) - (nw, (ny + rw + ry)I)](e, fI) \\
&= [(nu, (nv + ru + rv)I)](e, fI) \\
&= (n(ue), (n(uf) + n(ve) + r(ue) + r(ve) + n(vf) + r(uf) + r(vf))I) \\
&= (n(ue), (n(uf + ve + vf) + r(ue + ve + uf + vf))I) \\
&= (n0, (n0 + r0)I) \\
&= (0, 0).
\end{aligned}$$

This shows that $m(x + a) - mx \in (O : A(I))$. Accordingly, $(O : A(I))$ is a neutrosophic ideal of $\Gamma(I)$. □

If $N(I)$ is a neutrosophic nearring, let $x \in N(I)$ and consider $< x >_l$, $< x >_r$ the subsets of $N(I)$ given by

$$\begin{aligned}
< x >_l &= \{nx : n \in N(I)\}, \\
< x >_r &= \{xn : n \in N(I)\}.
\end{aligned}$$

Provided that $N(I)$ is distributive, it can be shown that $< x >_l$ is a left neutrosophic ideal of $N(I)$ generated by x and $< x >_r$ is a right neutrosophic ideal of $N(I)$ generated by x.

Definition 2.16. Let $N(I)$ be a distributive neutrosophic nearring. Then:

(1) $A(I) =< x >_l$ is called a left neutrosophic principal ideal of $N(I)$ generated by $x \in N(I)$.

(2) $A(I) =< x >_r$ is called a right neutrosophic principal ideal of $N(I)$ generated by $x \in N(I)$.

(3) $A(I) =< x >$ is called a neutrosophic principal ideal of $N(I)$ generated by $x \in N(I)$ if it is both left and right neutrosophic principal ideals of $N(I)$.

EXAMPLE 7. Let $N(I) = \mathbb{Z}_3(I) = \{(0,0), (1,0), (2,0), (0,I), (0,2I), (1,I), (2,I), (1,2I), (2,2I)$ be a neutrosophic neearing. Then $A(I) =< (0,I) >=< (0,2I) >= \{(0,0), (0,I), (0,2I)\}$ is a neutrosophic principal ideal of $N(I)$.

Definition 2.17. Let $N(I)$ be a neutrosophic nearring and let $x \in N(I)$ be any element of $N(I)$.

(1) x is said to be idempotent if $x^2 = x$.

(2) x is said to be nilpotent if there exists a positive integer $n > 0$ such that $x^n = (0,0)$.

(3) $x \neq (0,0)$ is called left zero divizor of $(0,0) \neq y = (c,dI) \in N(I)$ if $xy = (0,0)$.

(4) $x \neq (0,0)$ is called right zero divizor of $(0,0) \neq y = (c,dI) \in N(I)$ if $yx = (0,0)$.

(5) $x \neq (0,0)$ is called zero divizor of $(0,0) \neq y = (c,dI) \in N(I)$ if $xy = yx = (0,0)$.

EXAMPLE 8. Let $N(I) = \mathbb{Z}_4(I) = \{(0,0), (1,0), (2,0), (3,0), (0,I), (0,2I), (0,3I), (1,I), (2,I), (3,I), (1,2I), (2,2I), (3,2I), (1,3I), (2,3I), (3,3I)\}$ be a neutrosophic neearing. Then $E(I) = \{(2,0), (0,2I), (2,2I), (2,3I)\}$ is a neutrosophic set of idempotents of $N(I)$. Also, $A(I) = \{(0,0), (2,0), (2,2I)\}$ is a neutrosophic set of nilpotents of $N(I)$ which is a neutrosophic ideal of $N(I)$.

Theorem 2.18. *Let $F(I)$ be a neutrosophic nearfield. Then $F(I)$ has nontrivial zero divisors.*

Proof. Suppose that $F(I)$ is a neutrosophic nearfield and suppose that $x = (0, \alpha I)$ and $y = (\beta, -\beta I)$ are nonzero elements of $F(I)$. Then

$$\begin{aligned} xy &= (0,\alpha I)(\beta, -\beta I) \\ &= (0, (0 + \alpha\beta - \alpha\beta)I) \\ &= (0,0). \end{aligned}$$

This shows that x and y are nontrivial zero divisors. □

Corollary 2.19. A neutrosophic nearring $N(I)$ is not an integral domain even if the nearring N is an integral domain or a field.

Definition 2.20. Let $N_1(I)$ and $N_2(I)$ be two neutrosophic nearrings. $\phi : N_1(I) \to N_2(I)$ is called a neutrosophic nearring homomorphism if the following conditions hold:

(1) ϕ is a nearring homomorphism.

(2) $\phi(I) = I$.

$Ker\phi = \{x \in N_1(I) : \phi(x) = 0\}$ and $Im\phi = \{y \in N_2(I) : y = \phi(x), x \in N_1(I)\}$.

EXAMPLE 9. Let $N_1(I)$ and $N_2(I)$ be two neutrosophic nearrings. Let $\pi : N_1(I) \times N_2(I) \to N_1(I)$ be a projection defined by $\pi((a, bI), (c, dI)) = (a, bI)$ for all $(a, bI) \in N_1(I)$ and $(c, dI) \in N_2(I)$. Then π is a neutrosophic nearring homomorphism.

Definition 2.21. Let $\Gamma_1(I)$ and $\Gamma_2(I)$ be two neutrosophic N-groups. $\phi : \Gamma_1(I) \to \Gamma_2(I)$ is called a neutrosophic N-homomorphism if the following conditions hold:

(1) ϕ is an N-homomorphism.

(2) $\phi(I) = I$.

EXAMPLE 10. Let $\Gamma(I)$ be a neutrosophic N-group and let $\phi : \Gamma(I) \to \Gamma(I)$ be a mapping defined by $\phi((x, yI)) = (x, yI)$ for all $(x, yI) \in \Gamma(I)$. Then ϕ is a neutrosophic N-homomorphism.

Theorem 2.22. *Let $\phi : M(I) \to N(I)$ be a neutrosophic nearring homomorphism. Then*

(2) $Ker\phi$ is a subnearring of M.

(2) $Ker\phi$ is not a neutrosophic ideal of $M(I)$.

(3) $Im\phi$ is a neutrosophic subnearring of $N(I)$.

Proof. (1) Let $(a, bI) \in M(I)$ be arbitrary where $a, b \in M$. Since $\phi(I) = I$, it follows that $(a, bI) \in Ker\phi$ if and only if $b \neq 0$ that is only elements of the form $(a, 0) \in M$ can be in the kernel of ϕ. Hence $Ker\phi$ is a subnearring of M.
(2) Since $Ker\phi$ is a subnearring of M and not a neutrosophic subnearring of $M(I)$, it follows that $Ker\phi$ cannot be a neutrosophic ideal of $M(I)$.
(3) The same as the classical case. □

EXAMPLE 11. Consider $\mathbb{Z}_2(I) = \{(0, 0), (1, 0), (0, I), (1, I)\}$ and let $\phi : \mathbb{Z}_2(I) \times \mathbb{Z}_2(I) \to \mathbb{Z}_2(I)$ be a neutrosophic nearring homomorphism defined by $\phi((a, bI), (c, dI)) = (a, bI)$ for all $(a, bI), (c, dI) \in \mathbb{Z}_2(I)$. Then

$$\begin{aligned} Ker\phi &= \{((0,0),(0,0)),((0,0),(1,0)),((0,0),(0,I)),((0,0),(1,I))\}. \\ Im\phi &= \{(0,0),(1,0),(0,I),(1,I)\} = \mathbb{Z}_2(I). \end{aligned}$$

Theorem 2.23. *Let $M(I)$ and $N(I)$ be neutrosophic distributive nearrings and let $\phi : M(I) \to N(I)$ be a neutrosophic nearring homomorphism. If $A(I)$ is a neutrosophic ideal of $M(I)$, then $\phi(A(I))$ is a neutrosophic ideal of $M(I)$.*

Proof. Suppose that $x, y \in \phi(A(I))$. Then there exist $(a, bI), (c, dI) \in A(A)$ with $a, b, c, d \in A$ such that $x = \phi((a, bI))$ and $y = \phi((c, dI))$. Now,

$$\begin{aligned} xy &= \phi((a, bI))\phi((c, dI)) \\ &= \phi((a, bI)(c, dI)) \\ &= \phi((ac, (ad + bc + bd)I) \\ &\in \phi((A(I)). \end{aligned}$$

Next, Suppose that $m, n \in N(I)$. Then there exist $(e, fI), (g, hI) \in M(I)$ with $e, f, g, h \in M$ such that $m = \phi((e, fI))$ and $n = \phi((g, hI))$. now,

$$\begin{aligned} xm &= \phi((a, bI))\phi((e, fI)) \\ &= \phi((a, bI)(e, fI)) \\ &= \phi((ae, (af + be + bf)I)) \\ &\in \phi(A(I)). \end{aligned}$$

This shows that $\phi(A(I))$ is neutrosophic right ideal of $M(I)$. Lastly,

$$\begin{aligned} m(n + x) - mn &= \phi((e, fI))[\phi((g, hI)) + \phi((a, bI))] - \phi((e, fI))\phi((g, hI)) \\ &= \phi((e, fI)(g + a, (h + b)I)) - \phi((e, fI)(g, hI)) \\ &= \phi((e, fI)(g + a, (h + b)I) - (e, fI)(g, hI)) \\ &= \phi((e(g + a), (e(h + b) + f(g + a) + f(h + b))I) - (eg, (eh + fg + fh)I)) \\ &= \phi((eg + ea, (eh + eb + fg + fa + fh + fb)I) - (eg, (eh + fg + fh)I)) \\ &= \phi((ea, (eb + fa + fb)I) \\ &= \phi((e, fI)(a, bI)) \\ &\in \phi(A(I)). \end{aligned}$$

This shows that $\phi(A(I))$ is neutrosophic left ideal of $M(I)$ and therefore, $\phi(A(I))$ is an ideal of $M(I)$. □

Lemma 2.24. *Let $N(I)$ be a neutrosophic nearring and let $A(I)$ be a neutrosophic ideal of $N(I)$. Then*

(1) $A(I)A(I) = A(I)$.

(2) $(a, bI) + A(I) = A(I)\ \forall\ (a, bI) \in A(I)$.

If $A(I)$ is a neutrosophic ideal of a neutrosophic nearring $N(I)$, let $N(I)/A(I)$ be a neutrosophic set defined by

$$N(I)/A(I) = \{(m, nI) + A(I) : (m, nI) \in N(I)\}.$$

For all $(m, nI)+A(I), (p, qI)+A(I) \in N(I)/A(I)$, we define addition and multiplication in $N(I)/A(I)$ as follows:

$$\begin{aligned} ((m, nI) + A(I)) \oplus ((p, qI) + A(I)) &= (m + p, (n + q)I) + A(I), \\ ((m, nI) + A(I)) \odot ((p, qI) + A(I)) &= (mp, (mq + np + nq)I) + A(I). \end{aligned}$$

It can be shown that $\oplus$ and $\odot$ are well-defined on $N(I)/A(I)$ and the triple $(N(I)/A(I), \oplus, \odot)$ is a neutrosophic nearring called neutrosophic quotient nearring or neutrosophic factored nearring. The zero element of $(N(I)/A(I), \oplus)$ is simply $A(I)$.

Similarly, if $\Gamma(I)$ is a neutrosophic N-group of $N(I)$ and $B(I)$ is a neutrosophic ideal of $\Gamma(I)$, we can define the neutrosophic quotient or factored N-group $(\Gamma(I)/B(I), \oplus, \odot)$.

EXAMPLE 12. Let $N(I) = \mathbb{Z}(I)$. For all $(x, yI) \in N(I)$ with $x, y \in \mathbb{Z}$, define $x.y = x$. Then $(N(I), +, .)$ is a neutrosophic nearring. Suppose that

$$A(I) = 3\mathbb{Z}(I) = \{(0,0), (3,0), (6,0), \ldots, (0,3I), (0,6I), \ldots, (3,3I), (3,6I), (6,3I), (6,6I), \ldots\}.$$

Then $A(I)$ is a neutrosophic ideal of $N(I)$ and the neutrosophic quotient nearring $N(I)/A(I)$ is obtained as

$$\begin{aligned} N(I)/A(I) &= \{(0,0) + A(I), (1,0) + A(I), (2,0) + A(I), (0,I) + A(I), \\ &\quad (0,2I) + A(I), (1,I) + A(I), (1,2I) + A(I), (2,I) + A(I), (2,2I) + A(I)\} \end{aligned}$$

which is a neutrosophic nearring.

Theorem 2.25. *Let $A(I)$ be a neutrosophic ideal of a neutrosophic nearring $N(I)$. Then the mapping $\phi : N(I) \to N(I)/A(I)$ defined for all $(x, yI) \in N(I)$ by $\phi((x, yI)) = (x, yI) + A(I)$ is a nearring homomorphism and not a neutrosophic nearring homomorphism.*

Proof. Let $(w, xI), (y, zI) \in N(I)$ be arbitrary. It can be shown that $\phi((w, xI) + (y, zI)) = \phi((w, xI)) + \phi((y, zI))$ and $\phi((w, xI)(y, zI)) = \phi((w, xI))\phi((y, zI))$. However, $\phi(0, I) = (0, I) + A(I) = A(I) \neq (0, I)$. The proof is complete. □

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