Combinatorial Speculations and the Combinatorial Conjecture for Mathematics

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Abstract: Combinatorics is a powerful tool for dealing with relations among objectives mushroomed in the past century. However, an more important work for mathematician is to apply combinatorics to other mathematics and other sciences not merely to find combinatorial behavior for objectives. Recently, such research works appeared on journals for mathematics and theoretical physics on cosmos. The main purpose of this paper is to survey these thinking and ideas for mathematics and cosmological physics, such as those of multi-spaces, map geometries and combinatorial cosmoses, also the combinatorial conjecture for mathematics proposed by myself in 2005. Some open problems are included for the 21th mathematics by a combinatorial speculation.

Key words: combinatorial speculation, combinatorial conjecture for mathematics, Smarandache multi-space, M-theory, combinatorial cosmos.

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1. The role of classical combinatorics in mathematics

Modern science has so advanced that to find a universal genus in the society of sciences is nearly impossible. Thereby a scientist can only give his or her contribution in one or several fields. The same thing also happens for researchers in combinatorics. Generally, combinatorics deals with twofold questions:

Question 1.1. *determine or find structures or properties of configurations, such as those structure results appeared in graph theory, combinatorial maps and design theory,..., etc..*

Question 1.2. *enumerate configurations, such as those appeared in the enumeration of graphs, labelled graphs, rooted maps, unrooted maps and combinatorial designs,...,etc..*

Consider the contribution of a question to science. We can separate mathematical

questions into three grades:

Grade 1 *they contribute to all sciences.*

Grade 2 *they contribute to all or several branches of mathematics.*

Grade 3 *they contribute only to one branch of mathematics, for instance, just to the graph theory or combinatorial theory.*

Classical combinatorics is just a *grade* 3 *mathematics* by this view. This conclusion is gloomy for researchers in combinatorics, also for me 4 years ago. *Whether can combinatorics be applied to other mathematics or other sciences? Whether can it contribute to human's lives, not just in games?*

Although become a universal genus in science is nearly impossible, *our world is a combinatorial world.* A combinatorician should stand on all mathematics and all sciences, not just on classical combinatorics and then with a real combinatorial notion, i.e., *combining different fields into a unifying field* ([25]-[28]), such as combine different or even anti branches in mathematics or science into a unifying science for its freedom of research ([24]). This combination also requires us answering three questions for solving a combinatorial question before. *What is this question working for? What is its objective? What is its contribution to science or human's society?* After these works be well done, modern combinatorics can be applied to all sciences and all sciences are combinatorization.

2. The combinatorics metrization and mathematics combinatorization

There is a prerequisite for the application of combinatorics to other branch mathematics and other sciences, i.e, to introduce various metrics into combinatorics, ignored by the classical combinatorics since they are the fundamental of scientific realization for our world. This speculation is firstly appeared in the beginning of Chapter 5 of my book [16]:

$\cdots$ *our world is full of measures. For applying combinatorics to other branch of mathematics, a good idea is pullback measures on combinatorial objects again, ignored by the classical combinatorics and reconstructed or make combinatorial generalization for the classical mathematics, such as those of algebra, differential geometry, Riemann geometry, Smarandache geometries, $\cdots$ and the mechanics, theoretical physics, $\cdots$.*

The combinatorial conjecture for mathematics, abbreviated to *CCM* is stated in the following.

Conjecture 2.1(CCM Conjecture) *Mathematics can be reconstructed from or turned into combinatorization.*

Remark 2.1 We need some further clarifications for this conjecture.

(i) This conjecture assumes that one can selects finite combinatorial rulers and axioms to reconstruct or make generalization for classical mathematics.

(ii) Classical mathematics is a particular case in the combinatorization of mathematics, i.e., the later is a combinatorial generalization of the former.

(iii) We can make combinatorizations of different branches in mathematics into one and find new theorems after then.

Therefore, a branch in mathematics can not be ended if it has not been combinatorization and all mathematics can not be ended if its combinatorization has not completed. There is an assumption in one's realization of our world, i.e., *every science can be turned into mathematization*. Whence, we similarly get the combinatorial conjecture for sciences.

Conjecture 2.2(CCS Conjecture) *Sciences can be reconstructed from or turned into combinatorization.*

A typical example for the combinatorization of classical mathematics is the *combinatorial map theory*, i.e., a combinatorial theory for surfaces([14]-[15]). Combinatorially, a surface is topological equivalent to a polygon with even number of edges by identifying each pairs of edges along a given direction on it. If label each pair of edges by a letter $e, e \in \mathcal{E}$, a surface S is also identifying with a cyclic permutation such that each edge $e, e \in \mathcal{E}$ just appears two times in S, one is e and another is e^{-1}. Let $a, b, c, \cdots$ denote the letters in $\mathcal{E}$ and $A, B, C, \cdots$ the sections of successive letters in a linear order on a surface S (or a string of letters on S). Then, a surface can be represented as follows:

$$S = (\cdots, A, a, B, a^{-1}, C, \cdots),$$

where, $a \in \mathcal{E}$,A, B, C denote a string of letters. Define three elementary transformations as follows:

(O_1) $\quad (A, a, a^{-1}, B) \Leftrightarrow (A, B)$;

(O_2) $\quad (i) \quad (A, a, b, B, b^{-1}, a^{-1}) \Leftrightarrow (A, c, B, c^{-1})$;
$\qquad\quad (ii) \quad (A, a, b, B, a, b) \Leftrightarrow (A, c, B, c)$;

(O_3) $\quad (i) \quad (A, a, B, C, a^{-1}, D) \Leftrightarrow (B, a, A, D, a^{-1}, C)$;
$\qquad\quad (ii) \quad (A, a, B, C, a, D) \Leftrightarrow (B, a, A, C^{-1}, a, D^{-1})$.

If a surface S can be obtained from S_0 by these elementary transformations O_1-O_3, we say that S is elementary equivalent with S_0, denoted by $S \sim_{El} S_0$. Then we can get the classification theorem of compact surfaces as follows([29]):

Any compact surface is homeomorphic to one of the following standard surfaces:
(P_0) the sphere: aa^{-1};
(P_n) the connected sum of $n, n \geq 1$ tori:

$$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1} \cdots a_n b_n a_n^{-1} b_n^{-1};$$

3

(Q_n) *the connected sum of* $n, n \geq 1$ *projective planes:*

$$a_1 a_1 a_2 a_2 \cdots a_n a_n.$$

A *map* M is a connected topological graph cellularly embedded in a surface S. In 1973, Tutte suggested an algebraic representation for an embedding graph on a locally orientable surface ([16]):

A *combinatorial map* $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ is defined to be a basic permutation $\mathcal{P}$, i.e, for any $x \in \mathcal{X}_{\alpha,\beta}$, no integer k exists such that $\mathcal{P}^k x = \alpha x$, acting on $\mathcal{X}_{\alpha,\beta}$, the disjoint union of *quadricells* Kx of $x \in X$ (the base set), where $K = \{1, \alpha, \beta, \alpha\beta\}$ is the *Klein group* satisfying the following two conditions:

(i) $\alpha \mathcal{P} = \mathcal{P}^{-1}\alpha$;

(ii) *the group* $\Psi_J = < \alpha, \beta, \mathcal{P} >$ *is transitive on* $\mathcal{X}_{\alpha,\beta}$.

For a given map $M = (\mathcal{X}_{\alpha,\beta}, \mathcal{P})$, it can be shown that $M^* = (\mathcal{X}_{\beta,\alpha}, \mathcal{P}\alpha\beta)$ is also a map, call it the *dual* of the map M. The vertices of M are defined as the pairs of conjugatcy orbits of $\mathcal{P}$ action on $\mathcal{X}_{\alpha,\beta}$ by the condition (i) and edges the orbits of K on $\mathcal{X}_{\alpha,\beta}$, for example, for $\forall x \in \mathcal{X}_{\alpha,\beta}$, $\{x, \alpha x, \beta x, \alpha\beta x\}$ is an edge of the map M. Define the faces of M to be the vertices in the dual map M^*. Then the Euler characteristic $\chi(M)$ of the map M is

$$\chi(M) = \nu(M) - \varepsilon(M) + \phi(M)$$

where, $\nu(M), \varepsilon(M), \phi(M)$ are the number of vertices, edges and faces of the map M, respectively. For each vertex of a map M, its valency is defined to be the length of the orbits of $\mathcal{P}$ action on a quadricell incident with u.

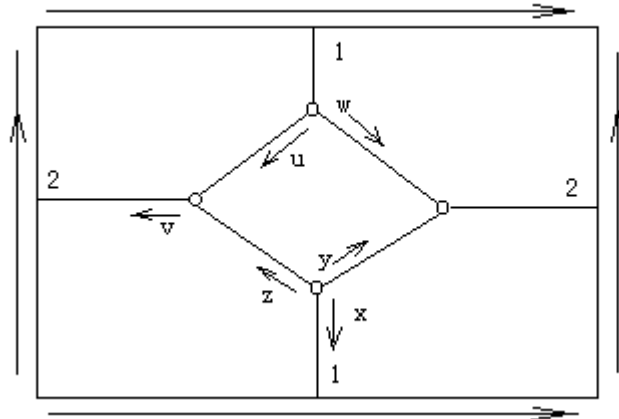For example, the graph K_4 on the tours with one face length 4 and another 8 shown in Fig.2.1

Fig.2.1

can be algebraically represented by $(\mathcal{X}_{\alpha,\beta}, \mathcal{P})$ with $\mathcal{X}_{\alpha,\beta} = \{x, y, z, u, v, w, \alpha x, \alpha y, \alpha z,$
$\alpha u, \alpha v, \alpha w, \beta x, \beta y, \beta z, \beta u, \beta v, \beta w, \alpha\beta x, \alpha\beta y, \alpha\beta z, \alpha\beta u, \alpha\beta v, \alpha\beta w\}$ and

$$
\begin{aligned}
\mathcal{P} &= (x, y, z)(\alpha\beta x, u, w)(\alpha\beta z, \alpha\beta u, v)(\alpha\beta y, \alpha\beta v, \alpha\beta w) \\
&\times (\alpha x, \alpha z, \alpha y)(\beta x, \alpha w, \alpha u)(\beta z, \alpha v, \beta u)(\beta y, \beta w, \beta v)
\end{aligned}
$$

with 4 vertices, 6 edges and 2 faces on an orientable surface of genus 1.

By the view of combinatorial maps, these standard surfaces P_0, P_n, Q_n for $n \geq 1$ is nothing but the bouquet B_n on a locally orientable surface with just one face. Therefore, combinatorial maps are the combinatorization of surfaces.

Many open problems are motivated by the *CCM Conjecture*. For example, a *Gauss mapping* among surfaces is defined as follows.

Let $\mathcal{S} \subset R^3$ be a surface with an orientation $\mathbf{N}$. The mapping $\mathbf{N}:\mathcal{S} \to R^3$ takes its value in the unit sphere

$$
S^2 = \{(x, y, z) \in R^3 | x^2 + y^2 + z^2 = 1\}
$$

along the orientation $\mathbf{N}$. The map $\mathbf{N}:\mathcal{S} \to S^2$, thus defined, is called the Gauss mapping.

we know that for a point $P \in \mathcal{S}$ such that the Gaussian curvature $K(P) \neq 0$ and V a connected neighborhood of P with K does not change sign,

$$
K(P) = \lim_{A \to 0} \frac{N(A)}{A},
$$

where A is the area of a region $B \subset V$ and $N(A)$ is the area of the image of B by the Gauss mapping $N : \mathcal{S} \to S^2$([2],[4]). Questions for the Gauss mapping are

(i) what is its combinatorial meaning of the Gauss mapping? How to realizes it by combinatorial maps?

(ii) how can we define various curvatures for maps and rebuilt the results in the classical differential geometry?

Let $\mathcal{S}$ be a compact orientable surface. Then the *Gauss-Bonnet theorem* asserts that

$$
\int\int_{\mathcal{S}} K d\sigma = 2\pi\chi(\mathcal{S}),
$$

where K is the Gaussian curvature of $\mathcal{S}$.

By the *CCM Conjecture*, the following questions should be considered.

(i) How can we define various metrics for combinatorial maps, such as those of length, distance, angle, area, curvature, $\cdots$?

(ii) Can we rebuilt the Gauss-Bonnet theorem by maps for dimensional 2 or higher dimensional compact manifolds without boundary?

One can see the references [15] and [16] for more open problems for the classical mathematics motivated by this *CCM Conjecture*, also raise new open problems for his or her research works.

3. The contribution of combinatorial speculation to mathematics

3.1. The combinatorization of algebra

By the view of combinatorics, algebra can be also seen as a combinatorial mathematics. The combinatorial speculation can generalize it by means of the combinatorization. For this objective, these Smarandache multi-algebraic systems are needed, defined in the following.

Definition 3.1([17],[18]) *For any integers $n, n \geq 1$ and $i, 1 \leq i \leq n$, let A_i be a set with an operation set $O(A_i)$ such that $(A_i, O(A_i))$ is a complete algebraic system. Then the union*

$$\bigcup_{i=1}^{n} (A_i, O(A_i))$$

is called an n multi-algebra system.

An example of multi-algebra systems can be constructed by a finite additive group. Let n be an integer, $Z_1 = (\{0, 1, 2, \cdots, n-1\}, +)$ an additive group $(modn)$ and $P = (0, 1, 2, \cdots, n-1)$ a permutation. For any integer $i, 0 \leq i \leq n-1$, define

$$Z_{i+1} = P^i(Z_1)$$

such that $P^i(k) +_i P^i(l) = P^i(m)$ in Z_{i+1} if $k + l = m$ in Z_1, where $+_i$ denotes the binary operation $+_i : (P^i(k), P^i(l)) \rightarrow P^i(m)$. Then we know that

$$\bigcup_{i=1}^{n} Z_i$$

is an n multi-algebra system .

The conception of multi-algebra systems can be extensively applied for generalizing conceptions and results in the algebraic structure, such as those of groups, rings, bodies, fields and vector spaces, $\cdots$, etc.. Some of them are explained in the following.

Definition 3.2 *Let $\widetilde{G} = \bigcup_{i=1}^{n} G_i$ be a complete multi-algebra system with a binary operation set $O(\widetilde{G}) = \{\times_i, 1 \leq i \leq n\}$. If for any integer $i, 1 \leq i \leq n$, $(G_i; \times_i)$ is a group and for $\forall x, y, z \in \widetilde{G}$ and any two binary operations $\times$ and $\circ$, $\times \neq \circ$, there is one operation, for example the operation $\times$ satisfying the distribution law to the operation $\circ$ provided their operation results exist , i.e.,*

$$x \times (y \circ z) = (x \times y) \circ (x \times z),$$

$$(y \circ z) \times x = (y \times x) \circ (z \times x),$$

then $\widetilde{G}$ is called a *multi-group*.

For a multi-group $(\widetilde{G}, O(\widetilde{G}))$, $\widetilde{G}_1 \subset \widetilde{G}$ and $O(\widetilde{G}_1) \subset O(\widetilde{G})$, call $(\widetilde{G}_1, O(\widetilde{G}_1))$ a *sub-multi-group* of $(\widetilde{G}, O(\widetilde{G}))$ if $\widetilde{G}_1$ is also a multi-group under the operations in $O(\widetilde{G}_1)$, denoted by $\widetilde{G}_1 \preceq \widetilde{G}$. For two sets A and B, denote the union $A \bigcup B$ by $A \oplus B$ if $A \bigcap B = \emptyset$. Then we get a generalization of the *Lagrange theorem* of finite groups.

Theorem 3.1([18]) *For any sub-multi-group $\widetilde{H}$ of a finite multi-group $\widetilde{G}$, there is a representation set T, $T \subset \widetilde{G}$, such that*

$$\widetilde{G} = \bigoplus_{x \in T} x\widetilde{H}.$$

For a sub-multi-group $\widetilde{H}$ of $\widetilde{G}$, $\times \in O(\widetilde{H})$ and $\forall g \in \widetilde{G}(\times)$, if for $\forall h \in \widetilde{H}$,

$$g \times h \times g^{-1} \in \widetilde{H},$$

then $\widetilde{H}$ is called a *normal sub-multi-group of $\widetilde{G}$*. We call an arrangement of all operations in $O(\widetilde{G})$ in order an *oriented operation sequence*, denote it by $\overrightarrow{O}(\widetilde{G})$. Then a generalization of the *Jordan-Hölder theorem* for finite multi-groups is described in the next result.

Theorem 3.2([18]) *For a finite multi-group $\widetilde{G} = \bigcup\limits_{i=1}^{n} G_i$ and an oriented operation sequence $\overrightarrow{O}(\widetilde{G})$, the length of maximal series of normal sub-multi-groups is a constant, only dependent on $\widetilde{G}$ itself.*

In Definition 2.2, choose $n = 2, G_1 = G_2 = \widetilde{G}$. Then $\widetilde{G}$ is a body. If $(G_1; \times_1)$ and $(G_2; \times_2)$ both are commutative groups, then $\widetilde{G}$ is a field. For multi-algebra systems with two or more operations on one set, we introduce the conception of multi-rings and multi-vector spaces in the following.

Definition 3.3 *Let $\widetilde{R} = \bigcup\limits_{i=1}^{m} R_i$ be a complete multi-algebra system with a double binary operation set $O(\widetilde{R}) = \{(+_i, \times_i), 1 \leq i \leq m\}$. If for any integers i, j, $i \neq j, 1 \leq i, j \leq m$, $(R_i; +_i, \times_i)$ is a ring and for $\forall x, y, z \in \widetilde{R}$,*

$$(x +_i y) +_j z = x +_i (y +_j z), \quad (x \times_i y) \times_j z = x \times_i (y \times_j z)$$

and

$$x \times_i (y +_j z) = x \times_i y +_j x \times_i z, \quad (y +_j z) \times_i x = y \times_i x +_j z \times_i x$$

provided all thpse operation results exist, then $\widetilde{R}$ is called a multi-ring. If for any integer $1 \leq i \leq m$, $(R; +_i, \times_i)$ is a filed, then $\widetilde{R}$ is called a multi-filed.

Definition 3.4 *Let $\widetilde{V} = \bigcup\limits_{i=1}^{k} V_i$ be a complete multi-algebra system with a binary*

operation set $O(\widetilde{V}) = \{(\dot{+}_i, \cdot_i) \mid 1 \leq i \leq m\}$ and $\widetilde{F} = \bigcup\limits_{i=1}^{k} F_i$ a multi-filed with a

double binary operation set $O(\widetilde{F}) = \{(+_i, \times_i) \mid 1 \leq i \leq k\}$. If for any integers i, j, $1 \leq i, j \leq k$ and $\forall \mathbf{a}, \mathbf{b}, \mathbf{c} \in \widetilde{V}$, $k_1, k_2 \in \widetilde{F}$,
    (i) $(V_i; \dot{+}_i, \cdot_i)$ is a vector space on F_i with vector additive $\dot{+}_i$ and scalar multiplication $\cdot_i$;
    (ii) $(\mathbf{a}\dot{+}_i\mathbf{b})\dot{+}_j\mathbf{c} = \mathbf{a}\dot{+}_i(\mathbf{b}\dot{+}_j\mathbf{c})$;
    (iii) $(k_1 +_i k_2) \cdot_j \mathbf{a} = k_1 +_i (k_2 \cdot_j \mathbf{a})$;
provided all those operation results exist, then $\widetilde{V}$ is called a multi-vector space on the multi-filed $\widetilde{F}$ with a binary operation set $O(\widetilde{V})$, denoted by $(\widetilde{V}; \widetilde{F})$.

Similar to multi-groups, results can be also obtained for multi-rings or multi-vector spaces by generalizing classical results in rings or linear spaces. Notice that in the references [17] and [18], some such results have been gotten.

3.2. The combinatorization of geometries

First, we generalize classical metric spaces by the combinatorial speculation.

Definition 3.5 *A multi-metric space is a union $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$ such that each M_i is a*

space with metric ρ_i for $\forall i, 1 \leq i \leq m$.

Two well-known results in metric spaces are generalized.

Theorem 3.3([19]) *Let $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$ be a completed multi-metric space. For an ϵ-disk sequence $\{B(\epsilon_n, x_n)\}$, where $\epsilon_n > 0$ for $n = 1, 2, 3, \cdots$, the following conditions hold:*
    (i) $B(\epsilon_1, x_1) \supset B(\epsilon_2, x_2) \supset B(\epsilon_3, x_3) \supset \cdots \supset B(\epsilon_n, x_n) \supset \cdots$;
    (ii) $\lim\limits_{n \to +\infty} \epsilon_n = 0$.
Then $\bigcap\limits_{n=1}^{+\infty} B(\epsilon_n, x_n)$ only has one point.

Theorem 3.4([19]) *Let $\widetilde{M} = \bigcup\limits_{i=1}^{m} M_i$ be a completed multi-metric space and T a contraction on $\widetilde{M}$. Then*

$$1 \leq^{\#} \Phi(T) \leq m,$$

where $^\#\Phi(T)$ denotes the number of fixed points of T.

Particularly, let $m = 1$. We get the *Banach fixed-point theorem* again.

**Corollary** 3.1(Banach) *Let M be a metric space and T a contraction on M. Then T has just one fixed point.*

Smarandache geometries were proposed by Smarandache in [25] which are generalization of classical geometries, i.e., these *Euclid, Lobachevshy-Bolyai-Gauss* and *Riemann geometries* may be united altogether in a same space, by some Smarandache geometries under the combinatorial speculation. These geometries can be either partially Euclidean and partially Non-Euclidean, or Non-Euclidean. In general, Smarandache geometries are defined by the next definition.

Definition 3.6 *An axiom is said to be Smarandachely denied if the axiom behaves in at least two different ways within the same space, i.e., validated and invalided, or only invalided but in multiple distinct ways.*

A Smarandache geometry is a geometry which has at least one Smarandachely denied axiom(1969).

For example, let us consider an euclidean plane $\mathbf{R}^2$ and three non-collinear points A, B and C. Define s-points as all usual euclidean points on $\mathbf{R}^2$ and s-lines as any euclidean line that passes through one and only one of points A, B and C. Then this geometry is a Smarandache geometry because two axioms are Smarandachely denied comparing with an Euclid geometry:

(i) The axiom (A5) that through a point exterior to a given line there is only one parallel passing through it is now replaced by two statements: *one parallel* and *no parallel*. Let L be an s-line passing through C and is parallel in the euclidean sense to AB. Notice that through any s-point not lying on AB there is one s-line parallel to L and through any other s-point lying on AB there are no s-lines parallel to L such as those shown in Fig.3.1(a).
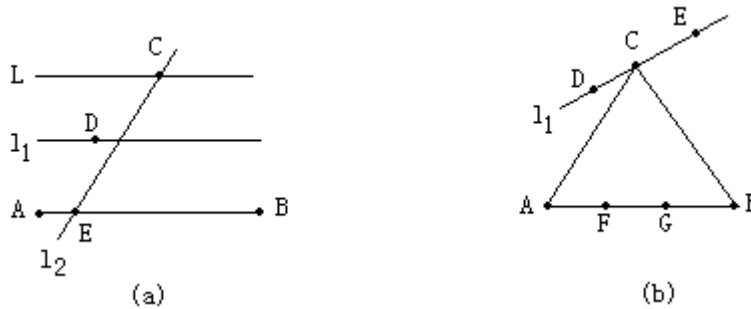
Fig.3.1

(ii) The axiom that through any two distinct points there exists one line passing through them is now replaced by; *one s-line* and *no s-line*. Notice that through any two distinct s-points D, E collinear with one of A, B and C, there is one s-line passing through them and through any two distinct s-points F, G lying on AB or

non-collinear with one of A, B and C, there is no s-line passing through them such as those shown in Fig.3.1(b).

A *Smarandache n-manifold* is an n-dimensional manifold that support a Smarandache geometry. Now there are many approaches for constructing Smarandache manifolds in the case of $n = 2$. A general way is by the so called *map geometries* without or with boundary underlying orientable or non-orientable maps proposed in references [14] and [15] firstly.

Definition 3.7 *For a combinatorial map M with each vertex valency≥ 3, endow each vertex $u, u \in V(M)$ a real number $\mu(u), 0 < \mu(u) < \frac{4\pi}{\rho_M(u)}$. Call (M, μ) a map geometry without boundary, $\mu : V(M) \rightarrow R$ an angle function on M.*

Definition 3.8 *For a map geometry (M, μ) without boundary and faces $f_1, f_2, \cdots, f_l \in F(M), 1 \leq l \leq \phi(M) - 1$, if $S(M) \setminus \{f_1, f_2, \cdots, f_l\}$ is connected, then call $(M, \mu)^{-l} = (S(M) \setminus \{f_1, f_2, \cdots, f_l\}, \mu)$ a map geometry with boundary $f_1, f_2, \cdots, f_l$, where $S(M)$ denotes the locally orientable surface underlying M.*

A realization for vertices $u, v, w \in V(M)$ in a space $\mathbf{R}^3$ is shown in Fig.3.2, where $\rho_M(u)\mu(u) < 2\pi$, $\rho_M(v)\mu(v) = 2\pi$ and $\rho_M(w)\mu(w) > 2\pi$, are called to be elliptic, euclidean or hyperbolic, respectively.

$$\rho_M(u)\mu(u) < 2\pi \qquad \rho_M(u)\mu(u) = 2\pi \qquad \rho_M(u)\mu(u) > 2\pi$$
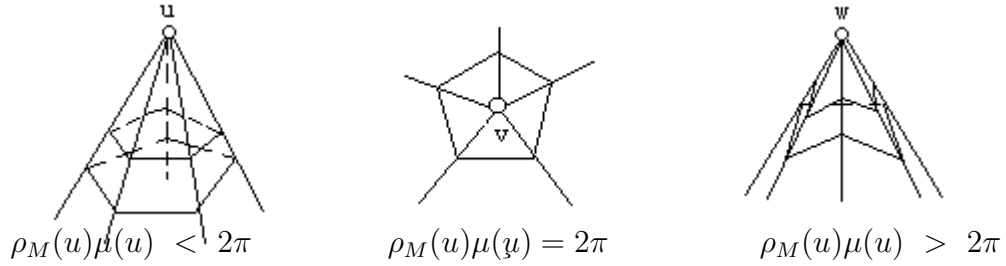
Fig.3.2

On an Euclid plane $\mathbf{R}^2$, a straight line passing through an elliptic or a hyperbolic point is shown in Fig.3.3.
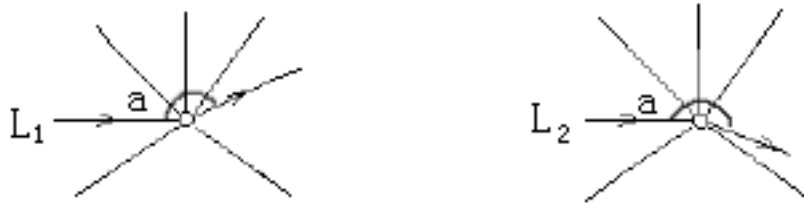
Fig.3.3

Theorem 3.5([17]) *There are Smarandache geometries, including paradoxist ge-*

ometries, non-geometries and anti-geometries in map geometries without or with boundary.

Generally, we can generalize the ideas in Definitions 3.7 and 3.8 to metric spaces further and find new geometries.

Definition 3.9 *Let U and W be two metric spaces with metric ρ, $W \subseteq U$. For $\forall u \in U$, if there is a continuous mapping $\omega : u \to \omega(u)$, where $\omega(u) \in \mathbf{R}^n$ for an integer $n, n \geq 1$ such that for any number $\epsilon > 0$, there exist a number $\delta > 0$ and a point $v \in W$, $\rho(u - v) < \delta$ such that $\rho(\omega(u) - \omega(v)) < \epsilon$, then U is called a metric pseudo-space if $U = W$ or a bounded metric pseudo-space if there is a number $N > 0$ such that $\forall w \in W$, $\rho(w) \leq N$, denoted by (U, ω) or (U^-, ω), respectively.*

For the case $n = 1$, we can also explain $\omega(u)$ being an angle function with $0 < \omega(u) \leq 4\pi$ as the case in map geometries without or with boundary, i.e.,

$$\omega(u) = \begin{cases} \omega(u)(mod 4\pi), & \text{if } u \in W, \\ 2\pi, & \text{if } u \in U \setminus W \end{cases}$$

and get some interesting metric pseudo-space geometries. For example, let $U = W = $ Euclid plane $= \sum$, then we obtained some interesting results for pseudo-plane geometries $(\sum, \omega)$ as shown in the following([17]).

Theorem 3.6 *In a pseudo-plane $(\sum, \omega)$, if there are no euclidean points, then all points of $(\sum, \omega)$ is either elliptic or hyperbolic.*

Theorem 3.7 *There are no saddle points and stable knots in a pseudo-plane $(\sum, \omega)$.*

Theorem 3.8 *For two constants ρ_0, θ_0, $\rho_0 > 0$ and $\theta_0 \neq 0$, there is a pseudo-plane $(\sum, \omega)$ with*

$$\omega(\rho, \theta) = 2(\pi - \frac{\rho_0}{\theta_0 \rho}) \ or \ \omega(\rho, \theta) = 2(\pi + \frac{\rho_0}{\theta_0 \rho})$$

such that

$$\rho = \rho_0$$

is a limiting ring in $(\sum, \omega)$.

Now for an m-manifold M^m and $\forall u \in M^m$, choose $U = W = M^m$ in Definition 3.9 for $n = 1$ and $\omega(u)$ a smooth function. We get pseudo-manifold geometries (M^m, ω) on M^m. By the reference [2], a *Minkowski norm* on M^m is a function $F : M^m \to [0, +\infty)$ such that

(*i*) F is smooth on $M^m \setminus \{0\}$;
(*ii*) F is 1-homogeneous, i.e., $F(\lambda \overline{u}) = \lambda F(\overline{u})$ for $\overline{u} \in M^m$ and $\lambda > 0$;
(*iii*) for $\forall y \in M^m \setminus \{0\}$, the symmetric bilinear form $g_y : M^m \times M^m \to R$ with

$$g_y(\overline{u}, \overline{v}) = \frac{1}{2} \frac{\partial^2 F^2(y + s\overline{u} + t\overline{v})}{\partial s \partial t}\Big|_{t=s=0}$$

is positive definite and a *Finsler manifold* is a manifold M^m endowed with a function $F : TM^m \to [0, +\infty)$ such that

(i) F is smooth on $TM^m \setminus \{0\} = \bigcup\{T_{\overline{x}}M^m \setminus \{0\} : \overline{x} \in M^m\}$;

(ii) $F|_{T_{\overline{x}}M^m} \to [0, +\infty)$ is a Minkowski norm for $\forall \overline{x} \in M^m$.

As a special case, we choose $\omega(\overline{x}) = F(\overline{x})$ for $\overline{x} \in M^m$, then (M^m, ω) is a *Finsler manifold*. Particularly, if $\omega(\overline{x}) = g_{\overline{x}}(y, y) = F^2(x, y)$, then (M^m, ω) is a *Riemann manifold*. Therefore, we get a relation for Smarandache geometries with Finsler or Riemann geometry.

Theorem 3.9 *There is an inclusion for Smarandache, pseudo-manifold, Finsler and Riemann geometries as shown in the following:*

$$\{Smarandache\ geometries\} \supset \{pseudo-manifold\ geometries\}$$
$$\supset \{Finsler\ geometry\}$$
$$\supset \{Riemann\ geometry\}.$$

4. The contribution of combinatorial speculation to theoretical physics

The progress of theoretical physics in last twenty years of the 20th century enables human beings to probe the mystic cosmos: *where are we came from? where are we going to?* Today, these problems still confuse eyes of human beings. Accompanying with research in cosmos, new puzzling problems also arose: *Whether are there finite or infinite cosmoses? Is just one? What is the dimension of our cosmos? We do not even know what the right degree of freedom in the universe is*, as Witten said([3]).

We are used to the idea that our living space has three dimensions: *length, breadth* and *height*, with time providing the fourth dimension of spacetime by Einstein. Applying his *principle of general relativity*, i.e. *all the laws of physics take the same form in any reference system* and the *equivalence principle*, i.e., *there are no difference for physical effects of the inertial force and the gravitation in a field small enough*, Einstein got the *equation of gravitational field*

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \lambda g_{\mu\nu} = -8\pi G T_{\mu\nu}.$$

where $R_{\mu\nu} = R_{\nu\mu} = R^{\alpha}_{\mu i \nu}$,

$$R^{\alpha}_{\mu i \nu} = \frac{\partial \Gamma^i_{\mu i}}{\partial x^\nu} - \frac{\partial \Gamma^i_{\mu\nu}}{\partial x^i} + \Gamma^{\alpha}_{\mu i}\Gamma^i_{\alpha\nu} - \Gamma^{\alpha}_{\mu\nu}\Gamma^i_{\alpha i},$$

$$\Gamma^g_{mn} = \frac{1}{2}g^{pq}\left(\frac{\partial g_{mp}}{\partial u^n} + \frac{\partial g_{np}}{\partial u^m} - \frac{\partial g_{mn}}{\partial u^p}\right)$$

and $R = g^{\nu\mu}R_{\nu\mu}$.

Combining the Einstein's equation of gravitational field with the *cosmological principle*, i.e., *there are no difference at different points and different orientations at a point of a cosmos on the metric* $10^4 l.y.$, Friedmann got a standard model of cosmos. The metrics of the standard cosmos are

$$ds^2 = -c^2 dt^2 + a^2(t)\left[\frac{dr^2}{1 - Kr^2} + r^2(d\theta^2 + \sin^2\theta d\varphi^2)\right]$$

and

$$g_{tt} = 1, \ g_{rr} = -\frac{R^2(t)}{1 - Kr^2}, g_{\phi\phi} = -r^2 R^2(t)\sin^2\theta.$$

The standard model of cosmos enables the birth of big bang model for our cosmos in thirties of the 20th century. The following diagram describes the developing process of our cosmos in different periods after the big bang.
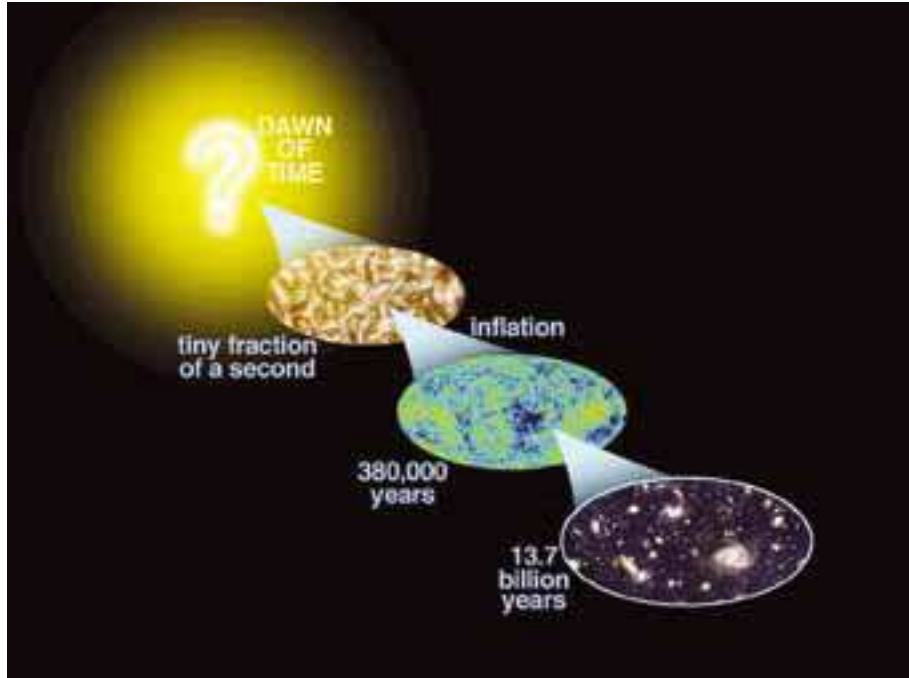

Fig.4.1

4.1. The M-theory

The M-theory was established by Witten in 1995 for the unity of those five already known string theories and superstring theories, which postulates that all matter and energy can be reduced to *branes* of energy vibrating in an 11 dimensional space, and

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then in a higher dimensional space solve the Einstein's equation of gravitational field under some physical conditions ([1],[3],[22]-[23]). Here, a *brane* is an object or subspace which can have various spatial dimensions. For any integer $p \geq 0$, a *p-brane* has length in p dimensions. For example, a 0-*brane* is just a point or particle; a 1-*brane* is a string and a 2-*brane* is a surface or membrane, $\cdots$.

One mainly discuss line elements in differential forms in Riemann geometry. By a geometrical view, these p-branes in M-theory can be seen as these volume elements in spaces in the counterpart. Whence, we can construct a graph model for p-branes in a space and combinatorially research graphs in spaces.

Definition 4.1 *For each m-brane* **B** *of a space* $\mathbf{R}^m$, *let* $(n_1(\mathbf{B}), n_2(\mathbf{B}), \cdots, n_p(\mathbf{B}))$ *be its unit vibrating normal vector along these p directions and* $q : \mathbf{R}^m \to \mathbf{R}^4$ *a continuous mapping. Now construct a graph phase* $(\mathcal{G}, \omega, \Lambda)$ *by*

$$V(\mathcal{G}) = \{p - branes \ q(\mathbf{B})\},$$

$$E(\mathcal{G}) = \{(q(\mathbf{B}_1), q(\mathbf{B}_2)) | there \ is \ an \ action \ between \ \mathbf{B}_1 \ and \ \mathbf{B}_2\},$$

$$\omega(q(\mathbf{B})) = (n_1(\mathbf{B}), n_2(\mathbf{B}), \cdots, n_p(\mathbf{B})),$$

and

$$\Lambda(q(\mathbf{B}_1), q(\mathbf{B}_2)) = \ forces \ between \ \mathbf{B}_1 \ and \ \mathbf{B}_2.$$

Then we get a graph phase $(\mathcal{G}, \omega, \Lambda)$ *in* $\mathbf{R}^4$. *Similarly, if* $m = 11$, *it is a graph phase for the M-theory.*

As an example for applying M-theory to find an accelerating expansion cosmos of 4-dimensional cosmos from supergravity compactification on hyperbolic spaces is the *Townsend-Wohlfarth type metric* in which the line element is

$$ds^2 = e^{-m\phi(t)}(-S^6 dt^2 + S^2 dx_3^2) + r_C^2 e^{2\phi(t)} ds_{H_m}^2,$$

where

$$\phi(t) = \frac{1}{m-1}(\ln K(t) - 3\lambda_0 t), \quad S^2 = K^{\frac{m}{m-1}} e^{-\frac{m+2}{m-1}\lambda_0 t}$$

and

$$K(t) = \frac{\lambda_0 \zeta r_c}{(m-1)\sin[\lambda_0 \zeta |t + t_1|]}$$

with $\zeta = \sqrt{3 + 6/m}$. This solution is obtainable from space-like brane solution and if the proper time ς is defined by $d\varsigma = S^3(t)dt$, then the conditions for expansion

and acceleration are $\frac{dS}{d\varsigma} > 0$ and $\frac{d^2S}{d\varsigma^2} > 0$. For example, the expansion factor is 3.04 if $m = 7$, i.e., a really expanding cosmos.

According to M-theory, the evolution picture of our cosmos started as a perfect 11 dimensional space. However, this 11 dimensional space was unstable. The original 11 dimensional space finally cracked into two pieces, a 4 and a 7 dimensional cosmos. The cosmos made the 7 of the 11 dimensions curled into a tiny ball, allowing the remaining 4 dimensional cosmos to inflate at enormous rates.

4.2. The combinatorial cosmos

The combinatorial speculation enables us to introduce the conception of combinatorial cosmoss([17]).

Definition 4.2 *A combinatorial cosmos is constructed by a triple (Ω, Δ, T), where*

$$\Omega = \bigcup_{i \geq 0} \Omega_i, \quad \Delta = \bigcup_{i \geq 0} O_i$$

and $T = \{t_i; i \geq 0\}$ are respectively called the cosmos, the operation or the time set with the following conditions hold.

(1) (Ω, Δ) is a Smarandache multi-space dependent on T, i.e., the cosmos (Ω_i, O_i) is dependent on time parameters t_i for any integer $i, i \geq 0$.

(2) For any integer $i, i \geq 0$, there is a sub-cosmos sequence

$$(S): \ \Omega_i \supset \cdots \supset \Omega_{i1} \supset \Omega_{i0}$$

in the cosmos (Ω_i, O_i) and for two sub-cosmoses (Ω_{ij}, O_i) and (Ω_{il}, O_i), if $\Omega_{ij} \supset \Omega_{il}$, then there is a homomorphism $\rho_{\Omega_{ij}, \Omega_{il}} : (\Omega_{ij}, O_i) \to (\Omega_{il}, O_i)$ such that

(i) for $\forall (\Omega_{i1}, O_i), (\Omega_{i2}, O_i), (\Omega_{i3}, O_i) \in (S)$, if $\Omega_{i1} \supset \Omega_{i2} \supset \Omega_{i3}$, then

$$\rho_{\Omega_{i1}, \Omega_{i3}} = \rho_{\Omega_{i1}, \Omega_{i2}} \circ \rho_{\Omega_{i2}, \Omega_{i3}},$$

where $\circ$ denotes the composition operation on homomorphisms.

(ii) for $\forall g, h \in \Omega_i$, if for any integer i, $\rho_{\Omega, \Omega_i}(g) = \rho_{\Omega, \Omega_i}(h)$, then $g = h$.

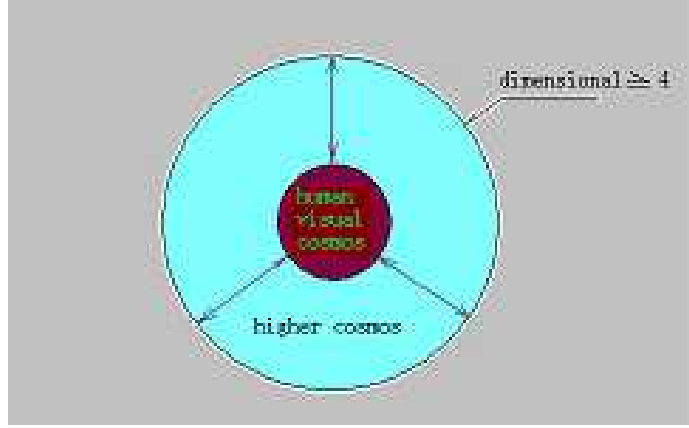*(iii) for $\forall i$, if there is an $f_i \in \Omega_i$ with*

$$\rho_{\Omega_i, \Omega_i \bigcap \Omega_j}(f_i) = \rho_{\Omega_j, \Omega_i \bigcap \Omega_j}(f_j)$$

for integers $i, j, \Omega_i \bigcap \Omega_j \neq \emptyset$, then there exists an $f \in \Omega$ such that $\rho_{\Omega, \Omega_i}(f) = f_i$ for any integer i.

By this definition, there is just one cosmos Ω and the sub-cosmos sequence is

$$\mathbf{R}^4 \supset \mathbf{R}^3 \supset \mathbf{R}^2 \supset \mathbf{R}^1 \supset \mathbf{R}^0 = \{P\} \supset \mathbf{R}_7^- \supset \cdots \supset \mathbf{R}_1^- \supset \mathbf{R}_0^- = \{Q\}.$$

in the string/M-theory. In Fig.4.1, we show a combinatorial cosmos with a higher dimensional cosmos outside our visual cosmos.

Fig.4.2

If the dimensional of this cosmos outside our visual cosmos is 5 or 6, there has been established a dynamical theory by this combinatorial speculation([20][21]). In this dynamics, we look for a solution in the Einstein equation of gravitational field in 6-dimensional spacetime with a metric of the form

$$ds^2 = -n^2(t,y,z)dt^2 + a^2(t,y,z)d\sum_k^2 + b^2(t,y,z)dy^2 + d^2(t,y,z)dz^2$$

where $d\sum_k^2$ represents the 3-dimensional spatial sections metric with $k = -1, 0, 1$ respective corresponding to the hyperbolic, flat and elliptic spaces. For 5-dimensional spacetime, deletes the undefinite z in this metric form. Now consider a 4-brane moving in a 6-dimensional *Schwarzschild-ADS spacetime*, the metric can be written as

$$ds^2 = -h(z)dt^2 + \frac{z^2}{l^2}d\sum_k^2 + h^{-1}(z)dz^2,$$

where

$$d\sum_k^2 = \frac{dr^2}{1-kr^2} + r^2d\Omega_{(2)}^2 + (1-kr^2)dy^2$$

and

$$h(z) = k + \frac{z^2}{l^2} - \frac{M}{z^3}.$$

Then the equation of a 4-dimensional cosmos moving in a 6-spacetime is

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$$2\frac{\ddot{R}}{R} + 3(\frac{\dot{R}}{R})^2 = -3\frac{\kappa^4_{(6)}}{64}\rho^2 - \frac{\kappa^4_{(6)}}{8}\rho p - 3\frac{\kappa}{R^2} - \frac{5}{l^2}$$

by applying the *Darmois-Israel conditions* for a moving brane. Similarly, for the case of $a(z) \neq b(z)$, the equations of motion of the brane are

$$\frac{d^2\dot{d}\dot{R} - d\ddot{R}}{\sqrt{1+d^2\dot{R}^2}} - \frac{\sqrt{1+d^2\dot{R}^2}}{n}(d\dot{n}\dot{R} + \frac{\partial_z n}{d} - (d\partial_z n - n\partial_z d)\dot{R}^2) = -\frac{\kappa^4_{(6)}}{8}(3(p+\rho)+\hat{p}),$$

$$\frac{\partial_z a}{ad}\sqrt{1+d^2\dot{R}^2} = -\frac{\kappa^4_{(6)}}{8}(\rho+p-\hat{p}),$$

$$\frac{\partial_z b}{bd}\sqrt{1+d^2\dot{R}^2} = -\frac{\kappa^4_{(6)}}{8}(\rho-3(p-\hat{p})),$$

where the energy-momentum tensor on the brane is

$$\hat{T}_{\mu\nu} = h_{\nu\alpha}T^\alpha_\mu - \frac{1}{4}Th_{\mu\nu}$$

with $T^\alpha_\mu = diag(-\rho, p, p, p, \hat{p})$ and the *Darmois-Israel conditions*

$$[K_{\mu\nu}] = -\kappa^2_{(6)}\hat{T}_{\mu\nu},$$

where $K_{\mu\nu}$ is the extrinsic curvature tensor.

The idea of combinatorial cosmoses also presents new questions to combinatorics, such as:

(i)   to embed a graph into spaces with dimensional≥ 4;
(ii) to research the phase space of a graph embedded in a space;
(iii) to establish graph dynamics in a space with dimensional$\geq 4, \cdots$, etc..

For example, we have gotten the following result for graphs in spaces in [17].

Theorem 4.1   *A graph G has a nontrivial including multi-embedding on spheres $P_1 \supset P_2 \supset \cdots \supset P_s$ if and only if there is a block decomposition $G = \overset{s}{\underset{i=1}{\uplus}} G_i$ of G such that for any integer $i, 1 < i < s$,*
   (i) G_i is planar;
   (ii) for $\forall v \in V(G_i), N_G(x) \subseteq (\overset{i+1}{\underset{j=i-1}{\cup}} V(G_j))$.

Further research of these combinatorial cosmoses will richen the knowledge of combinatorics and cosmology, also get the combinatorization for cosmology.

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