

# On Algebraic Multi-Ring Spaces

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**Abstract:** A Smarandache multi-space is a union of  $n$  spaces  $A_1, A_2, \dots, A_n$  with some additional conditions holding. Combining Smarandache multi-spaces with rings in classical ring theory, the conception of multi-ring spaces is introduced. Some characteristics of a multi-ring space are obtained in this paper

**Key words:** ring, multi-space, multi-ring space, ideal subspace chain.

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## 1. Introduction

The notion of multi-spaces is introduced by Smarandache in [6] under his idea of hybrid mathematics: *combining different fields into a unifying field*([7]), which is defined as follows.

**Definition 1.1** For any integer  $i, 1 \leq i \leq n$  let  $A_i$  be a set with ensemble of law  $L_i$ , and the intersection of  $k$  sets  $A_{i_1}, A_{i_2}, \dots, A_{i_k}$  of them constrains the law  $I(A_{i_1}, A_{i_2}, \dots, A_{i_k})$ . Then the union of  $A_i, 1 \leq i \leq n$

$$\tilde{A} = \bigcup_{i=1}^n A_i$$

is called a *multi-space*.

As we known, a set  $R$  with two binary operation  $+$  and  $\circ$ , denoted by  $(R; +, \circ)$ , is said to be a *ring* if for  $\forall x, y \in R, x + y \in R, x \circ y \in R$ , the following conditions hold.

- (i)  $(R; +)$  is an abelian group;
- (ii)  $(R; \circ)$  is a semigroup;
- (iii) For  $\forall x, y, z \in R, x \circ (y + z) = x \circ y + x \circ z$  and  $(x + y) \circ z = x \circ z + y \circ z$ .

By combining Smarandache multi-spaces with rings, a new kind of algebraic structure called multi-ring space is found, which is defined in the following.

**Definition 1.2** Let  $\tilde{R} = \bigcup_{i=1}^m R_i$  be a complete multi-space with double binary operation set  $O(\tilde{R}) = \{(+_i, \times_i), 1 \leq i \leq m\}$ . If for any integers  $i, j, i \neq j, 1 \leq i, j \leq m$ ,  $(R_i; +_i, \times_i)$  is a ring and for  $\forall x, y, z \in \tilde{R}$ ,

$$(x +_i y) +_j z = x +_i (y +_j z), \quad (x \times_i y) \times_j z = x \times_i (y \times_j z)$$

and

$$x \times_i (y +_j z) = x \times_i y +_j x \times_i z, \quad (y +_j z) \times_i x = y \times_i x +_j z \times_i x$$

if all their operation results exist, then  $\tilde{R}$  is called a multi-ring space. If for any integer  $1 \leq i \leq m$ ,  $(R; +_i, \times_i)$  is a filed, then  $\tilde{R}$  is called a multi-filed space.

For a multi-ring space  $\tilde{R} = \bigcup_{i=1}^m R_i$ , let  $\tilde{S} \subset \tilde{R}$  and  $O(\tilde{S}) \subset O(\tilde{R})$ , if  $\tilde{S}$  is also a multi-ring space with double binary operation set  $O(\tilde{S})$ , then call  $\tilde{S}$  a multi-ring subspace of  $\tilde{R}$ . We have the following criterions for the multi-ring subspaces.

The subject of this paper is to find some characteristics of a multi-ring space. For terminology and notation not defined here can be seen in [1], [5], [12] for algebraic terminologies and in [2], [6] – [11] for multi-spaces and logics.

## 2. Characteristics of a multi-ring space

First, we have the following result for multi-ring subspace of a multi-ring space.

**Theorem 2.1** For a multi-ring space  $\tilde{R} = \bigcup_{i=1}^m R_i$ , a subset  $\tilde{S} \subset \tilde{R}$  with  $O(\tilde{S}) \subset O(\tilde{R})$  is a multi-ring subspace of  $\tilde{R}$  if and only if for any integer  $k, 1 \leq k \leq m$ ,  $(\tilde{S} \cap R_k; +_k, \times_k)$  is a subring of  $(R_k; +_k, \times_k)$  or  $\tilde{S} \cap R_k = \emptyset$ .

*Proof* For any integer  $k, 1 \leq k \leq m$ , if  $(\tilde{S} \cap R_k; +_k, \times_k)$  is a subring of  $(R_k; +_k, \times_k)$  or  $\tilde{S} \cap R_k = \emptyset$ , then since  $\tilde{S} = \bigcup_{i=1}^m (\tilde{S} \cap R_i)$ , we know that  $\tilde{S}$  is a multi-ring subspace by definition of a multi-ring space.

Now if  $\tilde{S} = \bigcup_{j=1}^s S_j$  is a multi-ring subspace of  $\tilde{R}$  with double binary operation set  $O(\tilde{S}) = \{(+_{i_j}, \times_{i_j}), 1 \leq j \leq s\}$ , then  $(S_{i_j}; +_{i_j}, \times_{i_j})$  is a subring of  $(R_{i_j}; +_{i_j}, \times_{i_j})$ . Therefore, for any integer  $j, 1 \leq j \leq s$ ,  $S_{i_j} = R_{i_j} \cap \tilde{S}$ . But for other integer  $l \in \{i; 1 \leq i \leq m\} \setminus \{i_j; 1 \leq j \leq s\}$ ,  $\tilde{S} \cap S_l = \emptyset$ .  $\spadesuit$

Applying the criterions for subrings of a ring, we get the following result.

**Theorem 2.2** For a multi-ring space  $\tilde{R} = \bigcup_{i=1}^m R_i$ , a subset  $\tilde{S} \subset \tilde{R}$  with  $O(\tilde{S}) \subset O(\tilde{R})$  is a multi-ring subspace of  $\tilde{R}$  if and only if for any double binary operations  $(+_j, \times_j) \in O(\tilde{S})$ ,  $(\tilde{S} \cap R_j; +_j) \prec (R_j; +_j)$  and  $(\tilde{S}; \times_j)$  is complete.

*Proof* According to Theorem 2.1, we know that  $\tilde{S}$  is a multi-ring subspace if and only if for any integer  $i, 1 \leq i \leq m$ ,  $(\tilde{S} \cap R_i; +_i, \times_i)$  is a subring of  $(R_i; +_i, \times_i)$  or  $\tilde{S} \cap R_i = \emptyset$ . By a well known criterions for subrings of a ring (see also [5]), we know

that  $(\tilde{S} \cap R_i; +_i, \times_i)$  is a subring of  $(R_i; +_i, \times_i)$  if and only if for any double binary operations  $(+_j, \times_j) \in O(\tilde{S})$ ,  $(\tilde{S} \cap R_j; +_j) \prec (R_j; +_j)$  and  $(\tilde{S}; \times_j)$  is a complete set. This completes the proof.  $\spadesuit$

We use the *ideal subspace chain* of a multi-ring space to characteristic its structure properties. An *ideal subspace*  $\tilde{I}$  of a multi-ring space  $\tilde{R} = \bigcup_{i=1}^m R_i$  with double binary operation set  $O(\tilde{R})$  is a multi-ring subspace of  $\tilde{R}$  satisfying the following conditions:

- (i)  $\tilde{I}$  is a multi-group subspace with operation set  $\{+ | (+, \times) \in O(\tilde{I})\}$ ;
- (ii) for any  $r \in \tilde{R}, a \in \tilde{I}$  and  $(+, \times) \in O(\tilde{I})$ ,  $r \times a \in \tilde{I}$  and  $a \times r \in \tilde{I}$  if their operation results exist.

**Theorem 2.3** *A subset  $\tilde{I}$  with  $O(\tilde{I}), O(\tilde{I}) \subset O(\tilde{R})$  of a multi-ring space  $\tilde{R} = \bigcup_{i=1}^m R_i$  with double binary operation set  $O(\tilde{R}) = \{(+_i, \times_i) | 1 \leq i \leq m\}$  is an ideal subspace if and only if for any integer  $i, 1 \leq i \leq m$ ,  $(\tilde{I} \cap R_i, +_i, \times_i)$  is an ideal of the ring  $(R_i, +_i, \times_i)$  or  $\tilde{I} \cap R_i = \emptyset$ .*

*Proof* By definition of an ideal subspace, the necessity of the condition is obvious.

For the sufficiency, denote by  $\tilde{R}(+, \times)$  the set of elements in  $\tilde{R}$  with binary operations  $+$  and  $\times$ . If there exists an integer  $i$  such that  $\tilde{I} \cap R_i \neq \emptyset$  and  $(\tilde{I} \cap R_i, +_i, \times_i)$  is an ideal of  $(R_i, +_i, \times_i)$ , then for  $\forall a \in \tilde{I} \cap R_i, \forall r_i \in R_i$ , we know that

$$r_i \times_i a \in \tilde{I} \cap R_i; \quad a \times_i r_i \in \tilde{I} \cap R_i.$$

Notice that  $\tilde{R}(+_i, \times_i) = R_i$ . Therefore, we get that for  $\forall r \in \tilde{R}$ ,

$$r \times_i a \in \tilde{I} \cap R_i; \quad \text{and} \quad a \times_i r \in \tilde{I} \cap R_i,$$

if their operation result exist. Whence,  $\tilde{I}$  is an ideal subspace of  $\tilde{R}$ .  $\spadesuit$

An ideal subspace  $\tilde{I}$  of a multi-ring space  $\tilde{R}$  is said *maximal* if for any ideal subspace  $\tilde{I}'$ , if  $\tilde{R} \supseteq \tilde{I}' \supseteq \tilde{I}$ , then  $\tilde{I}' = \tilde{R}$  or  $\tilde{I}' = \tilde{I}$ . For any order of the double binary operation set  $O(\tilde{R})$  of a multi-ring space  $\tilde{R} = \bigcup_{i=1}^m R_i$ , not loss of generality, assume it being  $(+_1, \times_1) \succ (+_2, \times_2) \succ \cdots \succ (+_m, \times_m)$ , we can define an *ideal subspace chain* of  $\tilde{R}$  by the following programming.

- (i) Construct the ideal subspace chain

$$\tilde{R} \supset \tilde{R}_{11} \supset \tilde{R}_{12} \supset \cdots \supset \tilde{R}_{1s_1}$$

under the double binary operation  $(+_1, \times_1)$ , where  $\tilde{R}_{11}$  is a maximal ideal subspace of  $\tilde{R}$  and in general, for any integer  $i, 1 \leq i \leq m - 1$ ,  $\tilde{R}_{1(i+1)}$  is a maximal ideal subspace of  $\tilde{R}_{1i}$ .

- (ii) If the ideal subspace

$$\tilde{R} \supset \tilde{R}_{11} \supset \tilde{R}_{12} \supset \cdots \supset \tilde{R}_{1s_1} \supset \cdots \supset \tilde{R}_{i1} \supset \cdots \supset \tilde{R}_{is_i}$$

has been constructed for  $(+_1, \times_1) \succ (+_2, \times_2) \succ \cdots \succ (+_i, \times_i)$ ,  $1 \leq i \leq m-1$ , then construct an ideal subspace chain of  $\tilde{R}_{is_i}$

$$\tilde{R}_{is_i} \supset \tilde{R}_{(i+1)1} \supset \tilde{R}_{(i+1)2} \supset \cdots \supset \tilde{R}_{(i+1)s_i}$$

under the operations  $(+_{i+1}, \times_{i+1})$ , where  $\tilde{R}_{(i+1)1}$  is a maximal ideal subspace of  $\tilde{R}_{is_i}$  and in general,  $\tilde{R}_{(i+1)(i+1)}$  is a maximal ideal subspace of  $\tilde{R}_{(i+1)j}$  for any integer  $j$ ,  $1 \leq j \leq s_i - 1$ . Define the ideal subspace chain of  $\tilde{R}$  under  $(+_1, \times_1) \succ (+_2, \times_2) \succ \cdots \succ (+_{i+1}, \times_{i+1})$  being

$$\tilde{R} \supset \tilde{R}_{11} \supset \cdots \supset \tilde{R}_{1s_1} \supset \cdots \supset \tilde{R}_{i1} \supset \cdots \supset \tilde{R}_{is_i} \supset \tilde{R}_{(i+1)1} \supset \cdots \supset \tilde{R}_{(i+1)s_{i+1}}.$$

Similar to a multi-group space([3]), we have the following results for the ideal subspace chain of a multi-ring space.

**Theorem 2.4** For a multi-ring space  $\tilde{R} = \bigcup_{i=1}^m R_i$ , its ideal subspace chain only has finite terms if and only if for any integer  $i$ ,  $1 \leq i \leq m$ , the ideal chain of the ring  $(R_i; +_i, \times_i)$  has finite terms, i.e., each ring  $(R_i; +_i, \times_i)$  is an Artin ring.

*Proof* Let the order of double operations in  $\vec{O}(\tilde{R})$  be

$$(+_1, \times_1) \succ (+_2, \times_2) \succ \cdots \succ (+_m, \times_m)$$

and a maximal ideal chain in the ring  $(R_1; +_1, \times_1)$  is

$$R_1 \succ R_{11} \succ \cdots \succ R_{1t_1}.$$

Calculate

$$\begin{aligned} \tilde{R}_{11} &= \tilde{R} \setminus \{R_1 \setminus R_{11}\} = R_{11} \bigcup_{i=2}^m R_i, \\ \tilde{R}_{12} &= \tilde{R}_{11} \setminus \{R_{11} \setminus R_{12}\} = R_{12} \bigcup_{i=2}^m R_i, \\ &\dots\dots\dots \end{aligned}$$

$$\tilde{R}_{1t_1} = \tilde{R}_{1t_1} \setminus \{R_{1(t_1-1)} \setminus R_{1t_1}\} = R_{1t_1} \bigcup_{i=2}^m R_i.$$

According to Theorem 3.10, we know that

$$\tilde{R} \supset \tilde{R}_{11} \supset \tilde{R}_{12} \supset \cdots \supset \tilde{R}_{1t_1}$$

is a maximal ideal subspace chain of  $\tilde{R}$  under the double binary operation  $(+_1, \times_1)$ . In general, for any integer  $i, 1 \leq i \leq m-1$ , assume

$$R_i \succ R_{i1} \succ \cdots \succ R_{it_i}$$

is a maximal ideal chain in the ring  $(R_{(i-1)t_{i-1}}; +_i, \times_i)$ . Calculate

$$\tilde{R}_{ik} = R_{ik} \bigcup \left( \bigcup_{j=i+1}^m \right) \tilde{R}_{ik} \cap R_i$$

Then we know that

$$\tilde{R}_{(i-1)t_{i-1}} \supset \tilde{R}_{i1} \supset \tilde{R}_{i2} \supset \cdots \supset \tilde{R}_{it_i}$$

is a maximal ideal subspace chain of  $\tilde{R}_{(i-1)t_{i-1}}$  under the double operation  $(+_i, \times_i)$  by Theorem 3.10. Whence, if for any integer  $i, 1 \leq i \leq m$ , the ideal chain of the ring  $(R_i; +_i, \times_i)$  has finite terms, then the ideal subspace chain of the multi-ring space  $\tilde{R}$  only has finite terms and if there exists one integer  $i_0$  such that the ideal chain of the ring  $(R_{i_0}, +_{i_0}, \times_{i_0})$  has infinite terms, then there must be infinite terms in the ideal subspace chain of the multi-ring space  $\tilde{R}$ .  $\spadesuit$ .

A multi-ring space is called an *Artin multi-ring space* if each ideal subspace chain only has finite terms. We have the following corollary by Theorem 3.11.

**Corollary 2.1** *A multi-ring space  $\tilde{R} = \bigcup_{i=1}^m$  with double binary operation set  $O(\tilde{R}) = \{(+_i, \times_i) \mid 1 \leq i \leq m\}$  is an Artin multi-ring space if and only if for any integer  $i, 1 \leq i \leq m$ , the ring  $(R_i; +_i, \times_i)$  is an Artin ring.*

For a multi-ring space  $\tilde{R} = \bigcup_{i=1}^m$  with double binary operation set  $O(\tilde{R}) = \{(+_i, \times_i) \mid 1 \leq i \leq m\}$ , an element  $e$  is an *idempotent* element if  $e_{\times}^2 = e \times e = e$  for a double binary operation  $(+, \times) \in O(\tilde{R})$ . We define the *directed sum*  $\tilde{I}$  of two ideal subspaces  $\tilde{I}_1$  and  $\tilde{I}_2$  as follows:

- (i)  $\tilde{I} = \tilde{I}_1 \cup \tilde{I}_2$ ;
- (ii)  $\tilde{I}_1 \cap \tilde{I}_2 = \{0_+\}$ , or  $\tilde{I}_1 \cap \tilde{I}_2 = \emptyset$ , where  $0_+$  denotes an unit element under the operation  $+$ .

Denote the directed sum of  $\tilde{I}_1$  and  $\tilde{I}_2$  by

$$\tilde{I} = \tilde{I}_1 \oplus \tilde{I}_2.$$

If for any  $\tilde{I}_1, \tilde{I}_2, \tilde{I} = \tilde{I}_1 \oplus \tilde{I}_2$  implies that  $\tilde{I}_1 = \tilde{I}$  or  $\tilde{I}_2 = \tilde{I}$ , then  $\tilde{I}$  is called *non-reducible*. We have the following result for the Artin multi-ring space similar to a well-known result for the Artin ring (see [12]).

**Theorem 2.5** Any Artin multi-ring space  $\tilde{R} = \bigcup_{i=1}^m R_i$  with double binary operation set  $O(\tilde{R}) = \{(+_i, \times_i) \mid 1 \leq i \leq m\}$  is a directed sum of finite non-reducible ideal subspaces, and if for any integer  $i, 1 \leq i \leq m$ ,  $(R_i; +_i, \times_i)$  has unit  $1_{\times_i}$ , then

$$\tilde{R} = \bigoplus_{i=1}^m \left( \bigoplus_{j=1}^{s_i} (R_i \times_i e_{ij}) \bigcup (e_{ij} \times_i R_i) \right),$$

where  $e_{ij}, 1 \leq j \leq s_i$  are orthogonal idempotent elements of the ring  $R_i$ .

*Proof* Denote by  $\tilde{M}$  the set of ideal subspaces which can not be represented by a directed sum of finite ideal subspaces in  $\tilde{R}$ . According to Theorem 3.11, there is a minimal ideal subspace  $\tilde{I}_0$  in  $\tilde{M}$ . It is obvious that  $\tilde{I}_0$  is reducible.

Assume that  $\tilde{I}_0 = \tilde{I}_1 + \tilde{I}_2$ . Then  $\tilde{I}_1 \notin \tilde{M}$  and  $\tilde{I}_2 \notin \tilde{M}$ . Therefore,  $\tilde{I}_1$  and  $\tilde{I}_2$  can be represented by directed sums of finite ideal subspaces. Whence,  $\tilde{I}_0$  can be also represented by a directed sum of finite ideal subspaces. Contradicts that  $\tilde{I}_0 \in \tilde{M}$ .

Now let

$$\tilde{R} = \bigoplus_{i=1}^s \tilde{I}_i,$$

where each  $\tilde{I}_i, 1 \leq i \leq s$ , is non-reducible. Notice that for a double operation  $(+, \times)$ , each non-reducible ideal subspace of  $\tilde{R}$  has the form

$$(e \times R(\times)) \bigcup (R(\times) \times e), \quad e \in R(\times).$$

Whence, we know that there is a set  $T \subset \tilde{R}$  such that

$$\tilde{R} = \bigoplus_{e \in T, \times \in O(\tilde{R})} (e \times R(\times)) \bigcup (R(\times) \times e).$$

For any operation  $\times \in O(\tilde{R})$  and the unit  $1_{\times}$ , assume that

$$1_{\times} = e_1 \oplus e_2 \oplus \cdots \oplus e_l, \quad e_i \in T, \quad 1 \leq i \leq l.$$

Then

$$e_i \times 1_{\times} = (e_i \times e_1) \oplus (e_i \times e_2) \oplus \cdots \oplus (e_i \times e_l).$$

Therefore, we get that

$$e_i = e_i \times e_i = e_i^2 \quad \text{and} \quad e_i \times e_j = 0_i \quad \text{for} \quad i \neq j.$$

That is,  $e_i, 1 \leq i \leq l$ , are orthogonal idempotent elements of  $\tilde{R}(\times)$ . Notice that  $\tilde{R}(\times) = R_h$  for some integer  $h$ . We know that  $e_i, 1 \leq i \leq l$  are orthogonal idempotent elements of the ring  $(R_h, +_h, \times_h)$ . Denoted by  $e_{hj}$  for  $e_j, 1 \leq j \leq l$ . Consider all units in  $\tilde{R}$ , we get that

$$\tilde{R} = \bigoplus_{i=1}^m \left( \bigoplus_{j=1}^{s_i} (R_i \times_i e_{ij}) \cup (e_{ij} \times_i R_i) \right).$$

This completes the proof.  $\spadesuit$

**Corollary 2.2**([12]) *Any Artin ring  $(R ; +, \times)$  is a directed sum of finite ideals, and if  $(R ; +, \times)$  has unit  $1_\times$ , then*

$$R = \bigoplus_{i=1}^s R_i e_i,$$

where  $e_i, 1 \leq i \leq s$  are orthogonal idempotent elements of the ring  $(R; +, \times)$ .

### 3. Open problems for a multi-ring space

Similar to Artin multi-ring space, we can also define *Noether multi-ring spaces, simple multi-ring spaces, half-simple multi-ring spaces,  $\dots$ , etc.* The open problems for these new algebraic structure are as follows.

**Problem 3.1** Call a ring  $R$  a Noether ring if its every ideal chain only has finite terms. Similarly, for a multi-ring space  $\tilde{R}$ , if its every ideal multi-ring subspace chain only has finite terms, it is called a Noether multi-ring space. *Whether can we find its structures similar to Corollary 2.2 and Theorem 2.5?*

**Problem 3.2** *Similar to ring theory, define a Jacobson or Brown-McCoy radical for multi-ring spaces and determine their contribution to multi-ring spaces.*

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