A Generalization of Seifert-Van Kampen Theorem for Fundamental Groups

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Abstract: As we known, the *Seifert-Van Kampen theorem* handles fundamental groups of those topological spaces $X = U \cup V$ for open subsets $U,\ V \subset X$ such that $U \cap V$ is arcwise connected. In this paper, this theorem is generalized to such a case of maybe not arcwise-connected, i.e., there are $C_1,\ C_2, \cdots,\ C_m$ arcwise-connected components in $U \cap V$ for an integer $m \geq 1$, which enables one to find fundamental groups of combinatorial spaces by that of spaces with theirs underlying topological graphs, particularly, that of compact manifolds by their underlying graphs of charts.

Key Words: Fundamental group, Seifert-Van Kampen theorem, topological space, combinatorial manifold, topological graph.

AMS(2010): 51H20.

§1. Introduction

All spaces X considered in this paper are arcwise-connected, graphs are connected topological graph, maybe with loops or multiple edges and interior of an arc $a : (0,1) \to X$ is opened. For terminologies and notations not defined here, we follow the reference [1]-[3] for topology and [4]-[5] for topological graphs.

Let X be a topological space. A *fundamental group* $\pi_1(X, x_0)$ of X based at a point $x_0 \in X$ is formed by homotopy arc classes in X based at $x_0 \in X$. For an arcwise-connected space X, it is known that $\pi_1(X, x_0)$ is independent on the base point x_0, that is, for $\forall x_0, y_0 \in X$,

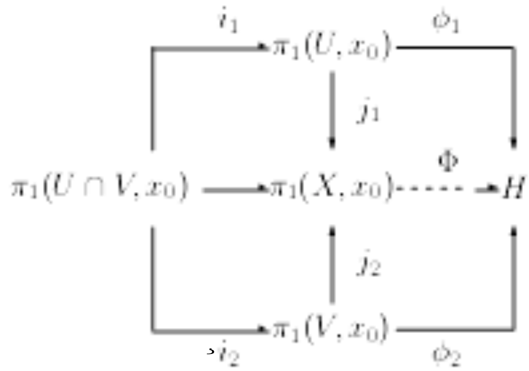
$$\pi_1(X, x_0) \cong \pi_1(X, y_0).$$

Find the fundamental group of a space X is a difficult task in general. Until today, the basic tool is still the *Seifert-Van Kampen theorem* following.

Theorem 1.1(Seifert and Van-Kampen) *Let $X = U \cup V$ with $U,\ V$ open subsets and let $X,\ U,\ V,\ U \cap V$ be non-empty arcwise-connected with $x_0 \in U \cap V$ and H a group. If there are homomorphisms*

$$\phi_1 : \pi_1(U, x_0) \to H \quad and \quad \phi_2 : \pi_1(V, x_0) \to H$$

and



with $\phi_1 \cdot i_1 = \phi_2 \cdot i_2$, *where* $i_1 : \pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$, $i_2 : \pi_1(U \cap V, x_0) \to \pi_1(V, x_0)$, $j_1 : \pi_1(U, x_0) \to \pi_1(X, x_0)$ *and* $j_2 : \pi_1(V, x_0) \to \pi_1(X, x_0)$ *are homomorphisms induced by inclusion mappings, then there exists a unique homomorphism* $\Phi :\ \pi_1(X, x_0) \to H$ *such that* $\Phi \cdot j_1 = \phi_1$ *and* $\Phi \cdot j_2 = \phi_2$.

Applying Theorem 1.1, it is easily to determine the fundamental group of such spaces $X = U \cup V$ with $U \cap V$ an arcwise-connected following.

Theorem 1.2(Seifert and Van-Kampen theorem, classical version) *Let spaces* X, U, V *and* x_0 *be in Theorem* 1.1. *If*

$$j : \pi_1(U, x_0) * \pi_1(V, x_0) \to \pi_1(X, x_0)$$

is an extension homomorphism of j_1 *and* j_2, *then* j *is an epimorphism with kernel* $\mathrm{Ker} j$ *generated by* $i_1^{-1}(g)i_2(g)$, $g \in \pi_1(U \cap V, x_0)$, *i.e.,*

$$\pi_1(X, x_0) \cong \frac{\pi_1(U, x_0) * \pi_1(V, x_0)}{\left[i_1^{-1}(g) \cdot i_2(g)|\ g \in \pi_1(U \cap V, x_0)\right]},$$

where $[A]$ denotes the minimal normal subgroup of a group $\mathscr{G}$ included $A \subset \mathscr{G}$.

Now we use the following convention.

Convention 1.3 *Assume that*

(1) X *is an arcwise-connected spaces,* $x_0 \in X$;

(2) $\{U_\lambda : \lambda \in \Lambda\}$ *is a covering of* X *by arcwise-connected open sets such that* $x_0 \in U_\lambda$ *for* $\forall \lambda \in \Lambda$;

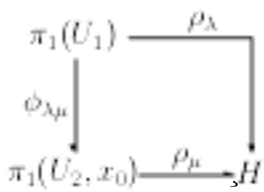
(3) *For any two indices* $\lambda_1, \lambda_2 \in \Lambda$ *there exists an index* $\lambda \in \Lambda$ *such that* $U_{\lambda_1} \cap U_{\lambda_2} = U_\lambda$

If $U_\lambda \subset U_\mu \subset X$, then the notation

$$\phi_{\lambda\mu} : \pi_1(U_\lambda, x_0) \to \pi_1(U_\mu, x_0) \ \text{ and } \ \phi_\lambda : \pi_1(U_\lambda, x_0) \to \pi_1(X, x_0)$$

denote homomorphisms induced by the inclusion mapping $U_\lambda \to U_\mu$ and $U_\lambda \to X$, respectively. It should be noted that the Seifert-Van Kampen theorem has been generalized in [2] under Convention 1.3 by any number of open subsets instead of just by two open subsets following.

Theorem 1.4([2]) *Let X, U_λ, $\lambda \in \Lambda$ be arcwise-connected space with Convention 1.3 satisfies the following universal mappping condition: Let H be a group and let $\rho_\lambda : \pi_1(U_\lambda, x_0) \to H$ be any collection of homomorphisms defined for all $\lambda \in \Lambda$ such that the following diagram is commutative for $U_\lambda \subset U_\mu$:*



Then there exists a unique homomorphism $\Phi : \pi_1(X, x_0) \to H$ such that for any $\lambda \in \Lambda$ the following diagram is commutative:

Moreover, this universal mapping condition characterizes $\pi_1(X, x_0)$ up to a unique isomorphism.

Theorem 1.4 is useful for determining the fundamental groups of CW-complexes, particularly, the adjunction of n-dimensional cells to a space for $n \geq 2$. Notice that the essence in Theorems 1.2 and 1.4 is that $\cap_{\lambda \in \Lambda} U_\lambda$ is arcwise-connected, which limits the application of such kind of results. The main object of this paper is to generalize the Seifert-Van Kampen theorem to such an intersection maybe non-arcwise connected, i.e., there are $C_1, C_2, \cdots, C_m$ arcwise-connected components in $U \cap V$ for an integer $m \geq 1$. This enables one to determine the fundamental group of topological spaces, particularly, combinatorial manifolds introduced in [6]-[8] following which is a special case of Smarandache multi-space ([9]-[10]).

Definition 1.4 *A combinatorial Euclidean space $\mathscr{E}_G(n_\nu; \nu \in \Lambda)$ underlying a connected graph G is a topological spaces consisting of $\mathbf{R}^{n_\nu}$, $\nu \in \Lambda$ for an index set Λ such that*

$V(G) = \{\mathbf{R}^{n_\nu} | \nu \in \Lambda\}$;

$E(G) = \{ (\mathbf{R}^{n_\mu}, \mathbf{R}^{n_\nu}) | \mathbf{R}^{n_\mu} \cap \mathbf{R}^{n_\nu} \neq \emptyset, \mu, \nu \in \Lambda\}$.

A *combinatorial fan-space* $\widetilde{\mathbf{R}}(n_\nu; \nu \in \Lambda)$ is a combinatorial Euclidean space

$\mathscr{E}_{K_{|\Lambda|}}(n_\nu;\nu\in\Lambda)$ of $\mathbf{R}^{n_\nu},\ \nu\in\Lambda$ such that for any integers $\mu,\nu\in\Lambda,\ \mu\neq\nu$,

$$\mathbf{R}^{n_\mu}\bigcap\mathbf{R}^{n_\nu}=\bigcap_{\lambda\in\Lambda}\mathbf{R}^{n_\lambda},$$

which enables us to generalize the conception of manifold to combinatorial manifold, also a locally combinatorial Euclidean space following.

Definition 1.5 *For a given integer sequence* $0<n_1<n_2<\cdots<n_m,\ m\geq 1$, *a topological combinatorial manifold* $\widetilde{M}$ *is a Hausdorff space such that for any point* $p\in\widetilde{M}$, *there is a local chart* (U_p,φ_p) *of* p, *i.e., an open neighborhood* U_p *of* p *in* $\widetilde{M}$ *and a homoeomorphism* $\varphi_p:U_p\to\widetilde{\mathbf{R}}(n_1(p),n_2(p),\cdots,n_{s(p)}(p))=\bigcup\limits_{i=1}^{s(p)}\mathbf{R}^{n_i(p)}$ *with* $\{n_1(p),n_2(p),\cdots,n_{s(p)}(p)\}\subseteq\{n_1,n_2,\cdots,n_m\}$ *and* $\bigcup\limits_{p\in\widetilde{M}}\{n_1(p),n_2(p),\cdots,n_{s(p)}(p)\}=\{n_1,n_2,\cdots,n_m\}$, *denoted by* $\widetilde{M}(n_1,n_2,\cdots,n_m)$ *or* $\widetilde{M}$ *on the context and*

$$\widetilde{\mathcal{A}}=\{(U_p,\varphi_p)|p\in\widetilde{M}(n_1,n_2,\cdots,n_m))\}$$

an atlas on $\widetilde{M}(n_1,n_2,\cdots,n_m)$.

A topological combinatorial manifold $\widetilde{M}(n_1,n_2,\cdots,n_m)$ *is finite if it is just combined by finite manifolds without one manifold contained in the union of others.*

If these manifolds $M_i,\ 1\leq i\leq m$ in $\widetilde{M}(n_1,n_2,\cdots,n_m)$ are Euclidean spaces $\mathbf{R}^{n_i},\ 1\leq i\leq m$, then $\widetilde{M}(n_1,n_2,\cdots,n_m)$ is nothing but a combinatorial Euclidean space $\mathscr{E}_G(n_\nu;\nu\in\Lambda)$ with $\Lambda=\{1,2,\cdots,m\}$. Furthermore, If $m=1$ and $n_1=n$, or $n_\nu=n$ for $\nu\in\Lambda$, then $\widetilde{M}(n_1,n_2,\cdots,n_m)$ or $\mathscr{E}_G(n_\nu;\nu\in\Lambda)$ is exactly a manifold M^n by definition.

§2. Topological Space Attached Graphs

A topological graph G is itself a topological space formally defined as follows.

Definition 2.1 *A topological graph* G *is a pair* (S,S^0) *of a Hausdorff space* S *with its a subset* S^0 *such that*

(1) S^0 *is discrete, closed subspaces of* S;

(2) $S-S^0$ *is a disjoint union of open subsets* $e_1,e_2,\cdots,e_m$, *each of which is homeomorphic to an open interval* $(0,1)$;

(3) *the boundary* $\overline{e}_i-e_i$ *of* e_i *consists of one or two points. If* $\overline{e}_i-e_i$ *consists of two points, then* $(\overline{e}_i,e_i)$ *is homeomorphic to the pair* $([0,1],(0,1))$; *if* $\overline{e}_i-e_i$ *consists of one point, then* $(\overline{e}_i,e_i)$ *is homeomorphic to the pair* $(S^1,S^1-\{1\})$;

(4) *a subset* $A\subset G$ *is open if and only if* $A\cap\overline{e}_i$ *is open for* $1\leq i\leq m$.

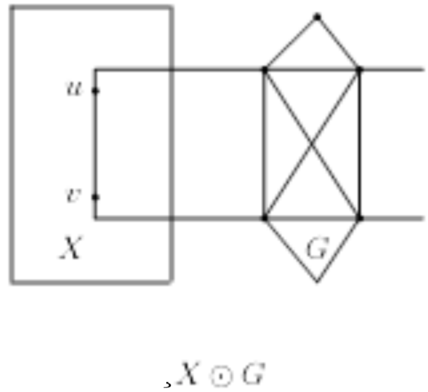


Fig.2.1

Notice that a topological graph maybe with semi-edges, i.e., those edges $e^+ \in E(G)$ with $e^+ : [0,1)$ or $(0,1] \to S$. A topological space X attached with a graph G is such a space $X \odot G$ such that

$$X \bigcap G \neq \emptyset, \quad G \not\subset X$$

and there are semi-edges $e^+ \in (X \bigcap G) \setminus G$, $e^+ \in G \setminus X$. An example for $X \odot G$ can be found in Fig.2.1. In this section, we characterize the fundamental groups of such topological spaces attached with graphs.

Theorem 2.2 *Let X be arc-connected space, G a graph and H the subgraph $X \cap G$ in $X \odot G$. Then for $x_0 \in X \cap G$,*

$$\pi_1(X \odot G, x_0) \cong \frac{\pi_1(X, x_0) * \pi_1(G, x_0)}{\left[i_1^{-1}(\alpha_{e_\lambda}) i_2(\alpha_{e_\lambda}) |\ e_\lambda \in E(H) \setminus T_{span})\right]},$$

where $i_1 : \pi_1(H, x_0) \to X$, $i_2 : \pi_1(H, x_0) \to G$ are homomorphisms induced by inclusion mappings, T_{span} is a spanning tree in H, $\alpha_\lambda = A_\lambda e_\lambda B_\lambda$ is a loop associated with an edge $e_\lambda = a_\lambda b_\lambda \in H \setminus T_{span}$, $x_0 \in G$ and A_λ, B_λ are unique paths from x_0 to a_λ or from b_λ to x_0 in T_{span}.

Proof This result is an immediately conclusion of Seifert-Van Kampen theorem. Let $U = X$ and $V = G$. Then $X \odot G = X \cup G$ and $X \cap G = H$. By definition, there are both semi-edges in G and H. Whence, they are opened. Applying the Seifert-Van Kampen theorem, we get that

$$\pi_1(X \odot G, x_0) \cong \frac{\pi_1(X, x_0) * \pi_1(G, x_0)}{\left[i_1^{-1}(g) i_2(g) |\ g \in \pi_1(X \cap G, x_0)\right]},$$

Notice that the fundamental group of a graph H is completely determined by those of its cycles ([2]), i.e.,

$$\pi_1(H, x_0) = \langle \alpha_\lambda | e_\lambda \in E(H) \setminus T_{span} \rangle,$$

where T_{span} is a spanning tree in H, $\alpha_\lambda = A_\lambda e_\lambda B_\lambda$ is a loop associated with an edge $e_\lambda = a_\lambda b_\lambda \in H \setminus T_{span}$, $x_0 \in G$ and A_λ, B_λ are unique paths from x_0 to a_λ or from b_λ to x_0 in T_{span}. We finally get the following conclusion,

$$\pi_1(X \odot G, x_0) \cong \frac{\pi_1(X, x_0) * \pi_1(G, x_0)}{\left[i_1^{-1}(\alpha_{e_\lambda}) i_2(\alpha_{e_\lambda}) |\ e_\lambda \in E(H) \setminus T_{span})\right]} \qquad \Box$$

Corollary 2.3 *Let X be arc-connected space, G a graph. If $X \cap G$ in $X \odot G$ is a tree, then*

$$\pi_1(X \odot G, x_0) \cong \pi_1(X, x_0) * \pi_1(G, x_0).$$

Particularly, if G is graphs shown in Fig.2.2 following

Fig.2.2

and $X \cap G = K_{1,m}$, Then

$$\pi_1(X \odot B_m^T, x_0) \cong \pi_1(X, x_0) * \langle L_i | 1 \leq i \leq m \rangle,$$

where L_i is the loop of parallel edges (x_0, x_i) in B_m^T for $1 \leq i \leq m-1$ and

$$\pi_1(X \odot S_m^T, x_0) \cong \pi_1(X, x_0).$$

Theorem 2.4 *Let $\mathscr{X}_m \odot G$ be a topological space consisting of m arcwise-connected spaces $X_1, X_2, \cdots, X_m$, $X_i \cap X_j = \emptyset$ for $1 \leq i, j \leq m$ attached with a graph G, $V(G) = \{x_0, x_1, \cdots, x_{l-1}\}$, $m \leq l$ such that $X_i \cap G = \{x_i\}$ for $0 \leq i \leq l-1$. Then*

$$\begin{aligned}\pi_1(\mathscr{X}_m \odot G, x_0) &\cong \left(\prod_{i=1}^{m} \pi_1(X_i^*, x_0)\right) * \pi_1(G, x_0) \\ &\cong \left(\prod_{i=1}^{m} \pi_1(X_i, x_i)\right) * \pi_1(G, x_0),\end{aligned}$$

where $X_i^* = X_i \bigcup (x_0, x_i)$ with $X_i \cap (x_0, x_i) = \{x_i\}$ for $(x_0, x_i) \in E(G)$, integers $1 \leq i \leq m$.

Proof The proof is by induction on m. If $m = 1$, the result is hold by Corollary 2.3.

Now assume the result on $\mathscr{X}_m \odot G$ is hold for $m \leq k < l-1$. Consider $m = k+1 \leq l$. Let $U = \mathscr{X}_k \odot G$ and $V = X_{k+1}$. Then we know that $\mathscr{X}_{k+1} \odot G = U \cup V$ and $U \cap V = \{x_{k+1}\}$.

Applying the Seifert-Van Kampen theorem, we find that

$$\begin{aligned}
\pi_1(\mathscr{X}_{k+1} \odot G, x_{k+1}) &\cong \frac{\pi_1(U, x_{k+1}) * \pi_1(V, x_{k+1})}{\left[i_1^{-1}(g)i_2(g)|\ g \in \pi_1(U \cap V, x_{k+1})\right]} \\
&\cong \frac{\pi_1(\mathscr{X}_k \odot G, x_0) * \pi_1(X_{k+1}, x_{k+1})}{\left[i_1^{-1}(g)i_2(g)|\ g \in \{\mathbf{e}_{x_{k+1}}\}\right]} \\
&\cong \left(\left(\prod_{i=1}^{k} \pi_1(X_i^*, x_0)\right) * \pi_1(G, x_0)\right) * \pi_1(X_{k+1}, x_{k+1}) \\
&\cong \left(\prod_{i=1}^{k+1} \pi_1(X_i^*, x_0)\right) * \pi_1(G, x_0) \\
&\cong \left(\prod_{i=1}^{m} \pi_1(X_i, x_i)\right) * \pi_1(G, x_0),
\end{aligned}$$

by the induction assumption. □

Particularly, for the graph B_m^T or star S_m^T in Fig.2.2, we get the following conclusion.

Corollary 2.5 *Let G be the graph B_m^T or star S_m^T. Then*

$$\begin{aligned}
\pi_1(\mathscr{X}_m \odot B_m^T, x_0) &\cong \left(\prod_{i=1}^{m} \pi_1(X_i^*, x_0)\right) * \pi_1(B_m^T, x_0) \\
&\cong \left(\prod_{i=1}^{m} \pi_1(X_i, x_{i-1})\right) * \langle L_i | 1 \leq i \leq m \rangle,
\end{aligned}$$

where L_i is the loop of parallel edges (x_0, x_i) in B_m^T for integers $1 \leq i \leq m-1$ and

$$\pi_1(\mathscr{X}_m \odot S_m^T, x_0) \cong \prod_{i=1}^{m} \pi_1(X_i^*, x_0) \cong \prod_{i=1}^{m} \pi_1(X_i, x_{i-1}).$$

Corollary 2.6 Let $X = \mathscr{X}_m \odot G$ be a topological space with simply-connected spaces X_i for integers $1 \leq i \leq m$ and $x_0 \in X \cap G$. Then we know that

$$\pi_1(X, x_0) \cong \pi_1(G, x_0).$$

§3. A Generalization of Seifert-Van Kampen Theorem

These results and graph B_m^T shown in Section 2 enables one to generalize the Seifert-Van Kampen theorem to the case of $U \cap V$ maybe not arcwise-connected.

Theorem 3.1 *Let $X = U \cup V$, $U, V \subset X$ be open subsets, X, U, V arcwise-connected and let $C_1, C_2, \cdots, C_m$ be arcwise-connected components in $U \cap V$ for an integer $m \geq 1$, $x_{i-1} \in C_i$, $b(x_0, x_{i-1}) \subset V$ an arc $: I \to X$ with $b(0) = x_0, b(1) = x_{i-1}$ and $b(x_0, x_{i-1}) \cap U = \{x_0, x_{i-1}\}$, $C_i^E = C_i \bigcup b(x_0, x_{i-1})$ for any integer $i,\ 1 \leq i \leq m$, H a group and there are homomorphisms*

$$\phi_1^i : \pi_1(U \bigcup b(x_0, x_{i-1}), x_0) \to H, \quad \phi_2^i : \pi_1(V, x_0) \to H$$

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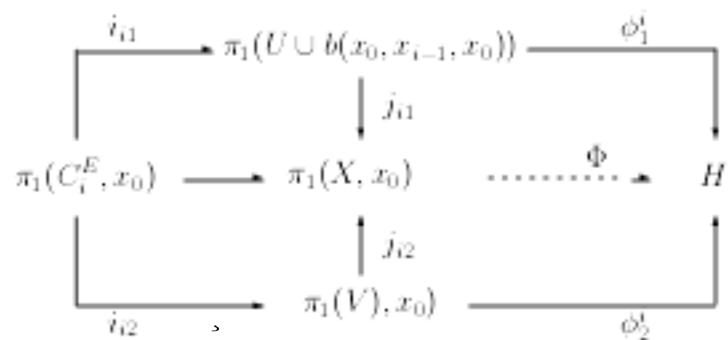


with $\phi_1^i \cdot i_{i1} = \phi_2^i \cdot i_{i2}$, where $i_{i1} : \pi_1(C_i^E, x_0) \to \pi_1(U \cup b(x_0, x_{i-1}), x_0)$, $i_{i2} : \pi_1(C_i^E, x_0) \to \pi_1(V, x_0)$ and $j_{i1} : \pi_1(U \cup b(x_0, x_{i-1}, x_0)) \to \pi_1(X, x_0)$, $j_{i2} : \pi_1(V, x_0)) \to \pi_1(X, x_0)$ are homomorphisms induced by inclusion mappings, then there exists a unique homomorphism $\Phi :\ \pi_1(X, x_0) \to H$ such that $\Phi \cdot j_{i1} = \phi_1^i$ and $\Phi \cdot j_{i2} = \phi_2^i$ for integers $1 \leq i \leq m$.

Proof Define $U^E = U \bigcup \{\ b(x_0, x_i) \mid 1 \leq i \leq m-1\}$. Then we get that $X = U^E \cup V$, $U^E, V \subset X$ are still opened with an arcwise-connected intersection $U^E \cap V = \mathscr{X}_m \odot S_m^T$, where S_m^T is a graph formed by arcs $b(x_0, x_{i-1})$, $1 \leq i \leq m$.

Notice that $\mathscr{X}_m \odot Sm^T = \bigcup\limits_{i=1}^{m} C_i^E$ and $C_i^E \bigcap C_j^E = \{x_0\}$ for $1 \leq i, j \leq m,\ i \neq j$. Therefore, we get that

$$\pi_1(\mathscr{X}_m \odot S_m^T, x_0) = \bigotimes_{i=1}^{m} \pi_1(C_i^E, x_0).$$

This fact enables us knowing that there is a unique m-tuple $(h_1, h_2, \cdots, h_m)$, $h_i \in \pi_1(C_i^E, x_{i-1}),\ 1 \leq i \leq m$ such that

$$\mathscr{I} = \prod_{i=1}^{m} h_i$$

for $\forall \mathscr{I} \in \pi_1(\mathscr{X}_m \odot S_m^T, x_0)$.

By definition,

$$i_{i1} : \pi_1(C_i^E, x_0) \to \pi_1(U \cap b(x_0, x_{i-1}), x_0),$$

$$i_{i2}:\pi_1(C_i^E,x_0)\to\pi_1(V,x_0)$$

are homomorphisms induced by inclusion mappings. We know that there are homomorphisms

$$i_1^E:\pi_1(\mathscr{X}_m\odot S_m^T,x_0)\to\pi_1(U^E,x_0),$$
$$i_2^E:\pi_1(\mathscr{X}_m\odot S_m^T,x_0)\to\pi_1(V,x_0)$$

with $i_1^E|_{\pi_1(C_i^E,x_0)}=i_{i1}$, $i_2^E|_{\pi_1(C_i^E,x_0)}=i_{i2}$ for integers $1\leq i\leq m$.

Similarly, because of

$$\pi_1(U^E,x_0)=\bigcup_{i=1}^m\pi_1(U\cup b(x_0,x_{i-1},x_0))$$

and

$$j_{i1}:\pi_1(U\cup b(x_0,x_{i-1},x_0))\to\pi_1(X,x_0),$$
$$j_{i2}:\pi_1(V\to\pi_1(X,x_0)$$

being homomorphisms induced by inclusion mappings, there are homomorphisms

$$j_1^E:\pi_1(U^E,x_0)\to\pi_1(X,x_0),\quad j_2^E:\pi_1(V,x_0)\to\pi_1(X,x_0)$$

induced by inclusion mappings with $j_1^E|_{\pi_1(U\cup b(x_0,x_{i-1},x_0))}=j_{i1}$, $j_2^E|_{\pi_1(V,x_0)}=j_{i2}$ for integers $1\leq i\leq m$ also.

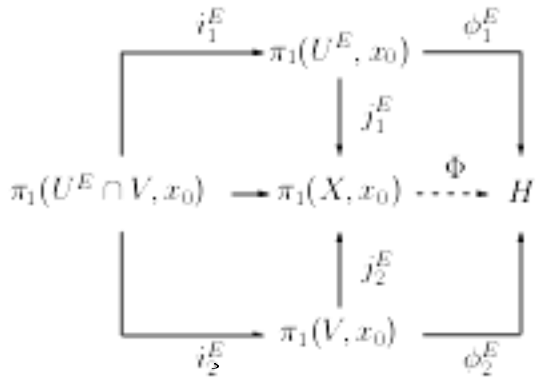
Define ϕ_1^E and ϕ_2^E by

$$\phi_1^E(\mathscr{I})=\prod_{i=1}^m\phi_1^i(i_{i1}(h_i)),\quad \phi_2^E(\mathscr{I})=\prod_{i=1}^m\phi_2^i(i_{i2}(h_i)).$$

Then they are naturally homomorphic extensions of homomorphisms ϕ_1^i, ϕ_2^i for integers $1\leq i\leq m$. Notice that $\phi_1^i\cdot i_{i1}=\phi_2^i\cdot i_{i2}$ for integers $1\leq i\leq m$, we get that

$$\begin{aligned}\phi_1^E\cdot i_1^E(\mathscr{I}) &= \phi_1^E\cdot i_1^E\left(\prod_{i=1}^m h_i\right)\\ &= \prod_{i=1}^m\left(\phi_1^i\cdot i_{i1}(h_i)\right)=\prod_{i=1}^m\left(\phi_2^i\cdot i_{i2}(h_i)\right)\\ &= \phi_2^E\cdot i_2^E\left(\prod_{i=1}^m h_i\right)=\phi_2^E\cdot i_2^E(\mathscr{I}),\end{aligned}$$

i.e., the following diagram



is commutative with $\phi_1^E \cdot i_1^E = \phi_2^E \cdot i_2^E$. Applying Theorem 1.1, we know that there exists a unique homomorphism $\Phi : \pi_1(X, x_0) \to H$ such that $\Phi \cdot j_1^E = \phi_1^E$ and $\Phi \cdot j_2^E = \phi_2^E$. Whence, $\Phi \cdot j_{i1} = \phi_1^i$ and $\Phi \cdot j_{i2} = \phi_2^i$ for integers $1 \leq i \leq m$. $\square$

The following result is a generalization of the classical Seifert-Van Kampen theorem to the case of maybe non-arcwise connected.

Theorem 3.2 *Let X, U, V, C_i^E, $b(x_0, x_{i-1})$ be arcwise-connected spaces for any integer i, $1 \leq i \leq m$ as in Theorem 3.1, $U^E = U \bigcup \{\ b(x_0, x_i) \mid 1 \leq i \leq m-1\}$ and B_m^T a graph formed by arcs $a(x_0, x_{i-1})$, $b(x_0, x_{i-1})$, $1 \leq i \leq m$, where $a(x_0, x_{i-1}) \subset U$ is an arc : $I \to X$ with $a(0) = x_0, a(1) = x_{i-1}$ and $a(x_0, x_{i-1}) \cap V = \{x_0, x_{i-1}\}$. Then*

$$\pi_1(X, x_0) \cong \frac{\pi_1(U, x_0) * \pi_1(V, x_0) * \pi_1(B_m^T, x_0)}{\left[(i_1^E)^{-1}(g) \cdot i_2(g)|\ g \in \prod_{i=1}^{m} \pi_1(C_i^E, x_0)\right]},$$

where $i_1^E : \pi_1(U^E \cap V, x_0) \to \pi_1(U^E, x_0)$ and $i_2^E : \pi_1(U^E \cap V, x_0) \to \pi_1(V, x_0)$ are homomorphisms induced by inclusion mappings.

Proof Similarly, $X = U^E \cup V$, $U^E, V \subset X$ are opened with $U^E \cap V = \mathscr{X}_m \odot S_m^T$. By the proof of Theorem 3.1 we have known that there are homomorphisms ϕ_1^E and ϕ_2^E such that $\phi_1^E \cdot i_1^E = \phi_2^E \cdot i_2^E$. Applying Theorem 1.2, we get that

$$\pi_1(X, x_0) \cong \frac{\pi_1(U^E, x_0) * \pi_1(V, x_0)}{[(i_1^E)^{-1}(\mathscr{I}) \cdot i_2^E(\mathscr{I})|\mathscr{I} \in \pi_1(U^E \cap V, x_0)]}.$$

Notice that $U^E \cap V^E = \mathscr{X}_m \odot S_m^T$. We have known that

$$\pi_1(U^E, x_0) \cong \pi_1(U, x_0) * \pi_1(B_m^T, x_0)$$

by Corollary 2.3. As we have shown in the proof of Theorem 3.1, an element $\mathscr{I}$ in

$\pi_1(\mathscr{X}_m \odot S_m^T, x_0)$ can be uniquely represented by

$$\mathscr{I} = \prod_{i=1}^{m} h_i,$$

where $h_i \in \pi_1(C_i^E, x_0),\ 1 \leq i \leq m$. We finally get that

$$\pi_1(X, x_0) \cong \frac{\pi_1(U, x_0) * \pi_1(V, x_0) * \pi_1(B_m^T, x_0)}{\left[(i_1^E)^{-1}(g) \cdot i_2^E(g) |\ g \in \prod_{i=1}^{m} \pi_1(C_i^E, x_0)\ \right]}. \qquad \square$$

The form of elements in $\pi_1(\mathscr{X}_m \odot S_m^T, x_0)$ appeared in Corollary 2.5 enables one to obtain another generalization of classical Seifert-Van Kampen theorem following.

Theorem 3.3 *Let X, U, V, $C_1, C_2, \cdots, C_m$ be arcwise-connected spaces, $b(x_0, x_{i-1})$ arcs for any integer i, $1 \leq i \leq m$ as in Theorem 3.1, $U^E = U \bigcup \{\ b(x_0, x_{i-1})\ |\ 1 \leq i \leq m\}$ and B_m^T a graph formed by arcs $a(x_0, x_{i-1})$, $b(x_0, x_{i-1})$, $1 \leq i \leq m$. Then*

$$\pi_1(X, x_0) \cong \frac{\pi_1(U, x_0) * \pi_1(V, x_0) * \pi_1(B_m^T, x_0)}{\left[(i_1^E)^{-1}(g) \cdot i_2^E(g) |\ g \in \prod_{i=1}^{m} \pi_1(C_i, x_{i-1})\right]},$$

where $i_1^E : \pi_1(U^E \cap V, x_0) \to \pi_1(U^E, x_0)$ and $i_2^E : \pi_1(U^E \cap V, x_0) \to \pi_1(V, x_0)$ are homomorphisms induced by inclusion mappings.

Proof Notice that $U^E \cap V = \mathscr{X}_m \odot S_m^T$. Applying Corollary 2.5, replacing

$$\pi_1(\mathscr{X}_m \odot S_m^T, x_0) = \left[(i_1^E)^{-1}(g) \cdot i_2^E(g) |\ g \in \prod_{i=1}^{m} \pi_1(C_i^E, x_0)\right]$$

by

$$\pi_1(\mathscr{X}_m \odot S_m^T, x_0) = \left[(i_1^E)^{-1}(g) \cdot i_2^E(g) |\ g \in \prod_{i=1}^{m} \pi_1(C_i, x_{i-1})\right]$$

in the proof of Theorem 3.2. We get this conclusion. $\square$

Particularly, we get corollaries following by Theorems 3.1, 3.2 and 3.3.

Corollary 3.4 *Let $X = U \cup V$, $U, V \subset X$ be open subsets and X, U, V and $U \cap V$ arcwise-connected. Then*

$$\pi_1(X, x_0) \cong \frac{\pi_1(U, x_0) * \pi_1(V, x_0)}{\left[i_1^{-1}(g) \cdot i_2(g) |\ g \in \pi_1(U \cap V, x_0)\right]},$$

where $i_1 : \pi_1(U \cap V, x_0) \to \pi_1(U, x_0)$ and $i_2 : \pi_1(U \cap V, x_0) \to \pi_1(V, x_0)$ are homomorphisms induced by inclusion mappings.

Corollary 3.5 *Let X, U, V, C_i, $a(x_0, x_i)$, $b(x_0, x_i)$ for integers i, $1 \le i \le m$ be as in Theorem* 3.1. *If each C_i is simply-connected, then*

$$\pi_1(X, x_0) \cong \pi_1(U, x_0) * \pi_1(V, x_0) * \pi_1(B_m^T, x_0).$$

Proof Notice that $C_1^E, C_2^E, \cdots, C_m^E$ are all simply-connected by assumption. Applying Theorem 3.3, we easily get this conclusion. □

Corollary 3.6 *Let X, U, V, C_i, $a(x_0, x_i)$, $b(x_0, x_i)$ for integers i, $1 \le i \le m$ be as in Theorem* 3.1. *If V is simply-connected, then*

$$\pi_1(X, x_0) \cong \frac{\pi_1(U, x_0) * \pi_1(B_m^T, x_0)}{\left[(i_1^E)^{-1}(g) \cdot i_2^E(g) | \ g \in \prod\limits_{i=1}^{m} \pi_1(C_i^E, x_0) \right]},$$

where $i_1^E : \pi_1(U^E \cap V, x_0) \to \pi_1(U^E, x_0)$ and $i_2^E : \pi_1(U^E \cap V, x_0) \to \pi_1(V, x_0)$ are homomorphisms induced by inclusion mappings.

§4. Fundamental Groups of Combinatorial Spaces

4.1 Fundamental groups of combinatorial manifolds

By definition, a combinatorial manifold $\widetilde{M}$ is arcwise-connected. So we can apply Theorems 3.2 and 3.3 to find its fundamental group $\pi_1(\widetilde{M})$ up to isomorphism in this section.

Definition 4.1 *Let $\widetilde{M}$ be a combinatorial manifold underlying a graph $G[\widetilde{M}]$. An edge-induced graph $G^\theta[\widetilde{M}]$ is defined by*

$$V(G^\theta[\widetilde{M}]) = \{x_M, x_{M'}, x_1, x_2, \cdots, x_{\mu(M,M')} | \ for \ \forall (M, M') \in E(G[\widetilde{M}])\},$$

$$E(G^\theta[\widetilde{M}]) = \{(x_M, x_{M'}), (x_M, x_i), (x_{M'}, x_i) | \ 1 \le i \le \mu(M, M')\},$$

where $\mu(M, M')$ is called the edge-index of (M, M') with $\mu(M, M') + 1$ equal to the number of arcwise-connected components in $M \cap M'$.

By the definition of edge-induced graph, we finally get $G^\theta[\widetilde{M}]$ of a combinatorial manifold $\widetilde{M}$ if we replace each edge (M, M') in $G[\widetilde{M}]$ by a subgraph $TB^T_{\mu(M,M')}$ shown in Fig.4.1 with $x_M = M$ and $x_{M'} = M'$.

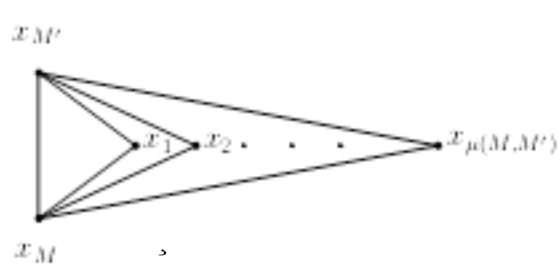


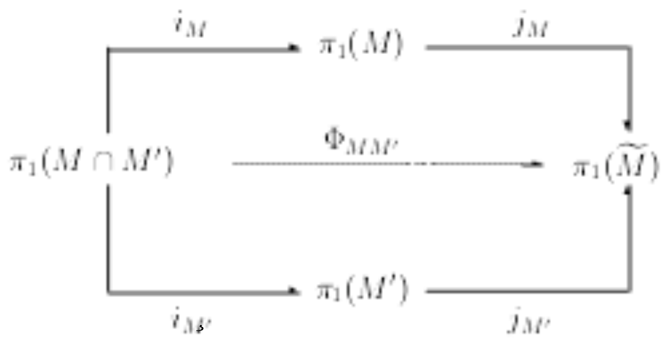
Fig.4.1

Then we have the following result.

Theorem 4.2 *Let $\widetilde{M}$ be a finitely combinatorial manifold. Then*

$$\pi_1(\widetilde{M}) \cong \frac{\left(\prod\limits_{M\in V(G[\widetilde{M}])} \pi_1(M)\right) * \pi_1(G^\theta[\widetilde{M}])}{\left[(i_1^E)^{-1}(g)\cdot i_2^E(g)|\ g \in \prod\limits_{(M_1,M_2)\in E(G[\widetilde{M}])} \pi_1(M_1 \bigcap M_2)\right]},$$

where i_1^E and i_2^E are homomorphisms induced by inclusion mappings $i_M : \pi_1(M \cap M') \to \pi_1(M)$, $i_{M'} : \pi_1(M \cap M') \to \pi_1(M')$ such as those shown in the following diagram:



for $\forall (M, M') \in E(G[\widetilde{M}])$.

Proof This result is obvious for $|G[\widetilde{M}]| = 1$. Notice that $G^\theta[\widetilde{M}] = B^T_{\mu(M,M')+1}$ if $V(G[\widetilde{M}]) = \{M,\ M'\}$. Whence, it is an immediately conclusion of Theorem 3.2 for $|G[\widetilde{M}]| = 2$.

Now let $k \geq 3$ be an integer. If this result is true for $|G[\widetilde{M}]| \leq k$, we prove it hold for $|G[\widetilde{M}]| = k$. It should be noted that for an arcwise-connected graph H we can always find a vertex $v \in V(H)$ such that $H - v$ is also arcwise-connected. Otherwise, each vertex v of H is a cut vertex. There must be $|H| = 1$, a contradiction. Applying this fact to $G[\widetilde{M}]$, we choose a manifold $M \in V(G[\widetilde{M}])$ such that $\widetilde{M} - M$ is arcwise-connected, which is also a finitely combinatorial manifold.

Let $U = \widetilde{M} \setminus (M \setminus \widetilde{M})$ and $V = M$. By definition, they are both opened. Applying Theorem 3.2, we get that

$$\pi_1(\widetilde{M}) \cong \frac{\pi_1(\widetilde{M} - M) * \pi_1(M) * \pi_1(B^T_m)}{\left[(i_1^E)^{-1}(g)\cdot i_2^E(g)|\ g \in \prod\limits_{i=1}^{m} \pi_1(C_i)\right]},$$

where C_i is an arcwise-connected component in $M \cap (\widetilde{M} - M)$ and

$$m = \sum_{(M,M')\in E(G[\widetilde{M}])} \mu(M, M').$$

Notice that

$$\pi_1(B_m^T) \cong \prod_{(M,M')\in E(G[\widetilde{M}]} \pi_1(TB_{\mu(M,M')}).$$

By the induction assumption, we know that

$$\pi_1(\widetilde{M} - M) \cong \frac{\left(\prod_{M\in V(G[\widetilde{M}-M])} \pi_1(M)\right) * \pi_1(G^\theta[\widetilde{M} - M])}{\left[(i_1^E)^{-1}(g) \cdot i_2^E(g) |\ g \in \prod_{(M_1,M_2)\in E(G[\widetilde{M}-M])} \pi_1(M_1 \cap M_2)\right]},$$

where i_1^E and i_2^E are homomorphisms induced by inclusion mappings $i_{M_1} : \pi_1(M_1 \cap M_2) \to \pi_1(M_1)$, $i_{M_2} : \pi_1(M_1 \cap M_2) \to \pi_1(M_2)$ for $\forall (M_1, M_2) \in E(G[\widetilde{M} - M])$. Therefore, we finally get that

$$\begin{aligned}
\pi_1(\widetilde{M}) &\cong \frac{\pi_1(\widetilde{M} - M) * \pi_1(M) * \pi_1(B_m^T)}{\left[(i_1^E)^{-1}(g) \cdot i_2^E(g) |\ g \in \prod_{i=1}^{m} \pi_1(C_i)\ \right]} \\
&\cong \frac{\dfrac{\left(\prod_{M\in V(G[\widetilde{M}-M])} \pi_1(M)\right) * \pi_1(G^\theta[\widetilde{M} - M])}{\left[(i_1^E)^{-1}(g) \cdot i_2^E(g) |\ g \in \prod_{(M_1,M_2)\in E(G[\widetilde{M}-M])} \pi_1(M_1 \cap M_2)\right]}}{\left[(i_1^E)^{-1}(g) \cdot i_2(g) |\ g \in \prod_{i=1}^{m} \pi_1(C_i)\ \right]} \\
&* \frac{\pi_1(M) * \prod_{(M,M')\in E(G[\widetilde{M}]} \pi_1(TB_{\mu(M,M')})}{\left[(i_1^E)^{-1}(g) \cdot i_2(g) |\ g \in \prod_{i=1}^{m} \pi_1(C_i)\ \right]} \\
&\cong \frac{\left(\prod_{M\in V(G[\widetilde{M}])} \pi_1(M)\right) * \pi_1(G^\theta[\widetilde{M}])}{\left[(i_1^E)^{-1}(g) \cdot i_2^E(g) |\ g \in \prod_{(M_1,M_2)\in E(G[\widetilde{M}])} \pi_1(M_1 \bigcap M_2)\right]}
\end{aligned}$$

by facts

$$(\mathscr{G}/\mathscr{H}) * H \cong \mathscr{G} * H/\mathscr{H}$$

for groups $\mathscr{G}$, $\mathscr{H}$, G and

$$G^\theta[\widetilde{M}] = G^\theta[\widetilde{M} - M] \bigcup_{(M,M')\in E(G[\widetilde{M}]} TB_{\mu(M,M')},$$

$$\pi_1(G^\theta[\widetilde{M}]) = \pi_1(G^\theta[\widetilde{M} - M]) * \prod_{(M,M')\in E(G[\widetilde{M}]} \pi_1(TB_{\mu(M,M')}),$$

$$\prod_{M\in V(G[\widetilde{M}])} \pi_1(M) = \left(\prod_{M\in V(G[\widetilde{M}-M])} \pi_1(M)\right) * \pi_1(M),$$

where i_1^E and i_2^E are homomorphisms induced by inclusion mappings $i_M : \pi_1(M \cap M') \to \pi_1(M)$, $i_{M'} : \pi_1(M \cap M') \to \pi_1(M')$ for $\forall (M, M') \in E(G[\widetilde{M}])$. This completes the proof. □

Applying Corollary 3.5, we get a result known in [8] by noted that $G^\theta[\widetilde{M}] = G[\widetilde{M}]$ if $\forall (M_1, M_2) \in E(G^L[\widetilde{M}])$, $M_1 \cap M_2$ is simply connected.

Corollary 4.3([8]) *Let $\widetilde{M}$ be a finitely combinatorial manifold. If for $\forall (M_1, M_2) \in E(G^L[\widetilde{M}])$, $M_1 \cap M_2$ is simply connected, then*

$$\pi_1(\widetilde{M}) \cong \left(\bigoplus_{M\in V(G[\widetilde{M}])} \pi_1(M)\right) \bigoplus \pi_1(G[\widetilde{M}]).$$

4.2 Fundamental groups of manifolds

Notice that $\pi_1(\mathbf{R}^n) = identity$ for any integer $n \geq 1$. If we choose $M \in V(G[\widetilde{M}])$ to be a chart $(U_\lambda, \varphi_\lambda)$ with $\varphi_\lambda : U_\lambda \to \mathbf{R}^n$ for $\lambda \in \Lambda$ in Theorem 4.2, i.e., an n-manifold, we get the fundamental group of n-manifold following.

Theorem 4.4 *Let M be a compact n-manifold with charts $\{(U_\lambda, \varphi_\lambda)| \ \varphi_\lambda : U_\lambda \to \mathbf{R}^n, \lambda \in \Lambda)\}$. Then*

$$\pi_1(M) \cong \frac{\pi_1(G^\theta[M])}{\left[(i_1^E)^{-1}(g) \cdot i_2^E(g)| \ g \in \prod_{(U_\mu,U_\nu)\in E(G[M])} \pi_1(U_\mu \cap U_\nu)\right]},$$

where i_1^E and i_2^E are homomorphisms induced by inclusion mappings $i_{U_\mu} : \pi_1(U_\mu \cap U_\nu) \to \pi_1(U_\mu)$, $i_{U_\nu} : \pi_1(U_\mu \cap U_\nu) \to \pi_1(U_\nu)$, $\mu, \nu \in \Lambda$.

Corollary 4.5 *Let M be a simply connected manifold with charts $\{(U_\lambda, \varphi_\lambda)| \ \varphi_\lambda : U_\lambda \to \mathbf{R}^n, \lambda \in \Lambda)\}$, where $|\Lambda| < +\infty$. Then $G^\theta[M] = G[M]$ is a tree.*

Particularly, if $U_\mu \cap U_\nu$ is simply connected for $\forall \mu, \nu \in \Lambda$, then we obtain an interesting result following.

Corollary 4.6 *Let M be a compact n-manifold with charts $\{(U_\lambda, \varphi_\lambda)|\ \varphi_\lambda : U_\lambda \to \mathbf{R}^n, \lambda \in \Lambda)\}$. If $U_\mu \cap U_\nu$ is simply connected for $\forall \mu, \nu \in \Lambda$, then*

$$\pi_1(M) \cong \pi_1(G[M]).$$

Therefore, by Theorem 4.4 we know that the fundamental group of a manifold M is a subgroup of that of its edge-induced graph $G^\theta[M]$. Particularly, if $G^\theta[M] = G[\widetilde{M}]$, i.e., $U_\mu \cap U_\nu$ is simply connected for $\forall \mu, \nu \in \Lambda$, then it is nothing but the fundamental group of $G[\widetilde{M}]$.

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