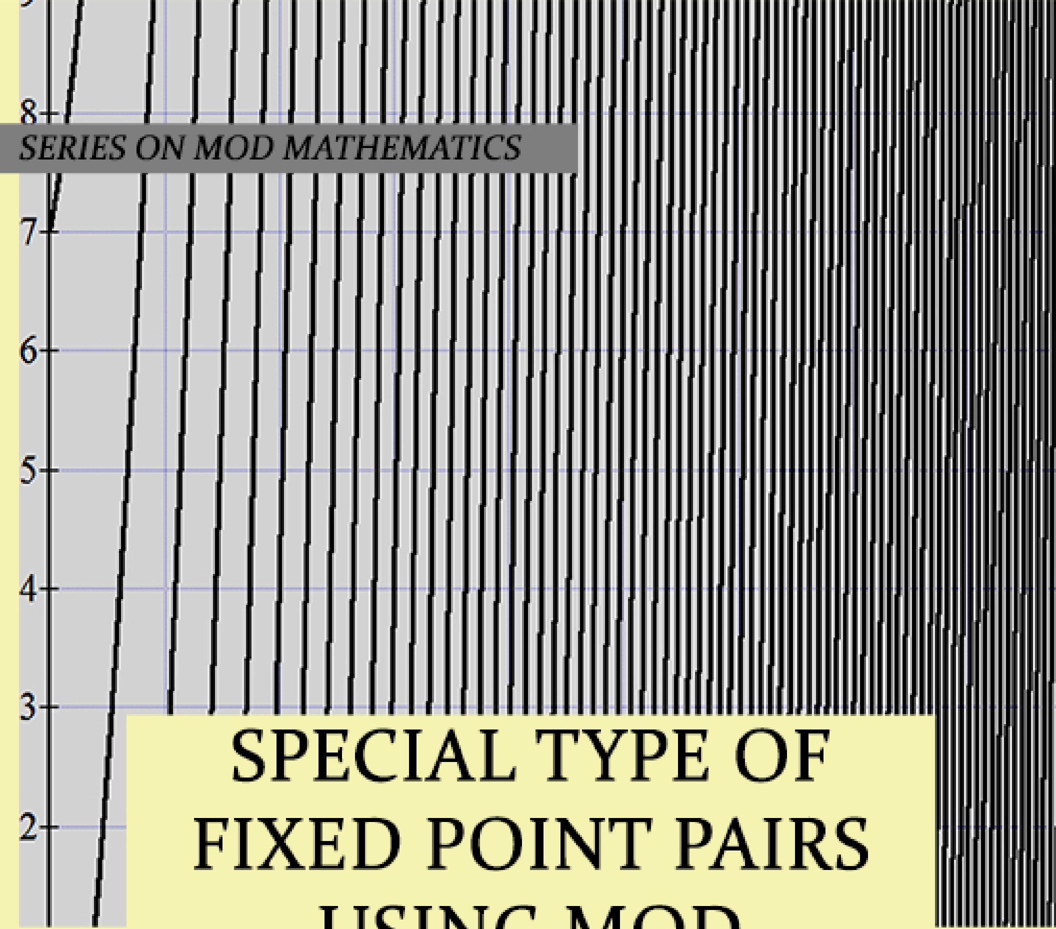


SERIES ON MOD MATHEMATICS

# SPECIAL TYPE OF FIXED POINT PAIRS USING MOD RECTANGULAR MATRIX OPERATORS

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# CONTENTS

# PREFACE

In this book authors for the first time define a special type of fixed points using MOD rectangular matrices as operators. In this case the special fixed points or limit cycles are pairs which is arrived after a finite number of iterations. Such study is both new and innovative for it can find lots of applications in mathematical modeling.

Since all these $Z_n$ or $Z_n^I$ or $\langle Z_n \cup g\rangle$ or $\langle Z_n \cup g\rangle_I$ or $C(Z_n)$ or $C^I(Z_n)$ are all of finite order we are sure to arrive at a MOD fixed point pair or a MOD limit cycle pair after a finite number of iterations.

It is important to mention for each on state of the node the MOD resultant will give node values from $Z_n$ or $Z_n^I$ or $C(Z_n)$ or $C^I(Z_n)$ and so on where the MOD directed bipartite graph takes its values from $Z_n$ or $Z_n^I$ or $C(Z_n)$ or $C^I(Z_n)$ and so on. Such study is new innovative and certainly lead to several special important applications in the field of science, engineering, technology medicine and social issues. These models are discussed and defined in the forth coming book. This book gives lots of illustrative examples to make it easy for a non mathematician.

Several problems are suggested about MOD special fixed point pairs and MOD limit cycle pairs for the interested reader.

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## Chapter One

# INTRODUCTION

In this chapter authors define, develop and describe a special type of fixed point theory in case MOD-matrix operators. Throughout this book by MOD-matrix operators we mean square matrices which take entries from $Z_n$ or $\langle Z_n \cup I\rangle$, $I^2 = I$ or $\langle Z_n \cup g\rangle$ $g^2 = 0$ or $C(Z_n)$; $i_F^2 = n - 1$; or $\langle Z_n \cup h\rangle$, $h^2 = h$ or $\langle Z_n \cup k\rangle$; $k^2 = (n - 1)k$.

These MOD matrix operators, operates on row vectors or column vectors and after a finite number of iterations gives a fixed point pair (fixed row vector or column vector) or a limit cycle pair.

Thus using these MOD matrix operators we develop a new view to the fixed point theory in general. We see in our study also the domain and range space are the same viz. they are either collection of all row vectors or collection of all column vectors with entries from $Z_n$, $\langle Z_n \cup I\rangle$ and so on.

Such study is new and innovative. We are sure to arrive at a realized fixed point pair or a limit cycle pair as the number of elements is finite. That is why modulo numbers are used. For MOD theory refer [20-6].

However if $Z_n$ is replaced by Z or R or C or Q we see in general such study of arriving realized a fixed point pair or a limit cycle pair is NP hard problem.

For more about the notion of neutrosophic element I refer [3-4].

To arrive more properties about $C(Z_n)$ refer [11]. For the notion of dual numbers and their properties refer [12].

For the concept of special dual like number refer [12].

The notion of special quasi dual numbers can be had from [14].

Finally we get using these MOD rectangular matrix operators always arrive at a classical fixed point pair or a realized fixed point pair or a realized fixed cycle.

Several interesting open conjectures are proposed for the reader.

Questions like the behavior of a MOD operator on two elements and their sum.

Further functioning of two distinct MOD operators on a single element and study for problem relations is left as an open problem.

Further the study of interrelation between the number of iterations needed and the type of MOD-matrix operator is also a open conjecture.

Finally the study of the row vector x and $x^t$ the column vectors behavior relative a MOD-matrix problem.

Thus this books has many innovative ideas and a new perspective to the notion of fixed point theory.

## Chapter Two

# MOD FIXED POINT PAIRS OF MOD-RECTANGULAR REAL MATRIX OPERATORS

In this chapter for the first time authors define, develop and describe the new notion of MOD rectangular real matrix operators built using $Z_n$, $2 \leq n < \infty$. These operators are not studied in general. However one can realize the parity check matrix and the generator matrix of an algebraic code to be a MOD - rectangular real matrix operator. Further they generate or give code words.

We cannot define using them the notion of fixed point in the classical sense or a realized fixed point or a realized limit cycle.

The venture of studying the above said fixed points happens to be innovative and challenging. Since we are in the finite collection of state row vectors we are guaranteed that after a finite set of iterations we will arrive at a realized fixed point pair or a realized limit cycle.

We will illustrate these situations by some examples for the better understanding of this notion.

***Example 2.1:*** Let A = $\begin{bmatrix} 3 & 1 & 2 & 0 & 5 \\ 0 & 4 & 0 & 3 & 0 \\ 4 & 0 & 1 & 0 & 2 \\ 0 & 1 & 0 & 3 & 0 \end{bmatrix}$ with entries from $Z_6$.

We call these matrices as MOD rectangular real matrix operator.

Let D = $\{(x_1, x_2, x_3, x_4) \mid x_i \in Z_6, 1 \leq i \leq 4\}$ and

R = $\{(y_1, y_2, y_3, y_4, y_5) \mid y_i \in Z_6, 1 \leq i \leq 5\}$ the row state vectors. These are defined as the related pair of state vectors.

Clearly if $x \in D$ then $xA = y \in R$ and

$Ay^t = x' \in D$ or we may use $yA^t \in D$ we make it clear at this juncture that both the techniques will explained and yield a same fixed point pair.

***Example 2.2:*** Let

$$S = \begin{bmatrix} 0 & 2 & 1 & 5 \\ 3 & 0 & 4 & 0 \\ 0 & 8 & 0 & 6 \\ 1 & 0 & 7 & 0 \\ 0 & 7 & 0 & 2 \\ 2 & 0 & 5 & 0 \\ 0 & 4 & 0 & 1 \end{bmatrix}$$

be the MOD real rectangular matrix operator with entries from $Z_9$.

$$D = \{(x_1, x_2, \ldots, x_7) \mid x_i \in Z_9, 1 \leq i \leq 7\} \text{ and}$$

$R = \{(y_1, y_2, y_3, y_4) \mid y_i \in Z_9, 1 \leq i \leq 4\}$ be the domain and range state vectors respectively associated with S the MOD rectangular operator matrix.

Now we proceed onto make the definition.

**DEFINITION 2.1:** *Let*

$$S = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

*be the MOD-rectangular $m \times n$ real matrix operator with entries from $Z_s$, $2 \leq s < \infty$.*

*Let $D = \{(a_1, a_2, \ldots, a_m) \mid a_i \in Z_m, 1 \leq i \leq m\}$ and*

*$R = \{(b_1, b_2, \ldots, y_n) \mid b_i \in Z_s, 1 \leq i \leq n\}$ be the associated pair of Domain and Range state vectors with entries from S.*

*Many a times S will also be called as MOD-rectangular dynamical system or in short MOD-dynamical system or MOD-matrix operator.*

*When $x \in D$ and $x \times S$ is obtained it will also be termed as effect of x on S.*

The working of this MOD dynamical rectangular system will be explained by an example or two.

***Example 2.3:*** Let

$$S = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 1 & 0 \\ 5 & 0 & 3 \\ 0 & 3 & 0 \\ 2 & 0 & 4 \\ 0 & 4 & 0 \\ 4 & 0 & 2 \end{bmatrix}$$

be the MOD rectangular real matrix operator with entries from $Z_6$.

Let D = $\{(x_1, x_2, \ldots, x_7) \mid x_i \in Z_6, 1 \leq i \leq 7\}$ be the domain state vector associated with S.

R = $\{(x_1, x_2, x_3) \mid x_i \in Z_6, 1 \leq i \leq 3\}$ be the range state vectors associated with S.

Let x = (0, 1, 0, 2, 0, 0, 4) $\in$ D.

The effect of x on the MOD-dynamical operator S.

$xS = (4, 0, 2) = y_1$; $\quad y_1S^t = (2, 0, 2, 0, 4, 0, 2) = x_1$;

$x_1S = (2, 4, 4) = y_2$; $\quad y_2S^t = (0, 4, 4, 0, 2, 4, 4) = x_2$;

$x_2S = (4, 2, 4) = y_3$; $\quad y_3S^t = (2, 2, 2, 0, 0, 2, 0) = x_3$;

$x_3S = (4, 2, 0) = y_4$; $\quad y_4S^t = (4, 2, 2, 0, 2, 2, 4) = x_4$;

$x_4S = (0, 0, 2) = y_5$; $\quad y_5S^t = (2, 0, 0, 0, 2, 0, 4) = x_5$;

$x_5S = (2, 4, 0) = y_6$; $\quad y_6S^t = (2, 4, 4, 0, 4, 4, 2) = x_6$;

$x_6S = (0, 0, 4) = y_7$; $\quad y_7S^t = (4, 0, 0, 0, 4, 0, 2) = x_7$;

$x_7S = (4, 2, 0) = y_8$ ($= y_4$) ;

$y_4S^t = (4, 2, 2, 0, 2, 2, 4) = x_8$ ($=x_4$).

Thus we see the effect of x on the MOD matrix operator is a realized limit cycle pair given by {(4, 2, 2, 0, 2, 2, 4), (4, 2, 0)}.

Let $y = (0, 1, 0) \in R$.

To find the effect of y on S.

$yS^t = (2, 1, 0, 3, 0, 4, 0) = x_1$; $x_1S = (0, 0, 2) = y_1$;

$y_1S^t = (2, 0, 0, 0, 2, 0, 4) = x_2$; $x_2S = (0, 4, 0) = y_2$;

$y_2S^t = (2, 4, 0, 0, 0, 2, 0) = x_3$; $x_3S = (0, 4, 2) = y_3$;

$y_3S^t = (4, 4, 0, 0, 2, 4, 4) = x_4$; $x_4S = (2, 4, 2) = y_4$;

$y_4S^t = (4, 4, 4, 0, 0, 4, 0) = x_5$; $x_5S = (2, 4, 4) = y_5$;

$y_5S^t = (0, 4, 4, 0, 2, 4, 4) = x_6$;

$x_6S = (4, 2, 4) = y_6$ ($=y_3$) ;

$y_6S = (2, 2, 2, 0, 0, 2, 0) = x_7$ ($=x_3$).

Thus the resultant of y on the MOD matrix operator S is a realized limit cycle pair given by {(2, 2, 2, 0, 0, 2, 0), (4, 2, 4)}.

This is the way operations are performed using the MOD rectangular matrix operator S.

Let $x_0 = (1, 0, 0, 0, 0, 0, 0) \in D$.

To find the effect of $x_0$ on S.

$x_0S = (3, 2, 1) = y_1$

$y_1S^t = (2, 2, 0, 0, 4, 2, 2) = x_1$; $x_1S = (4, 2, 4) = y_2$;

$y_2S^t = (2, 2, 2, 0, 0, 2, 0) = x_2$; $x_2S = (4, 2, 2) = y_3$;

$y_3S^t = (0, 2, 2, 0, 4, 2, 2) = x_3$; $x_3S = (2, 0, 2) = y_4$;

$y_4S^t = (2, 0, 4, 0, 0, 0, 0) = x_4$; $x_4S = (2, 4, 2) = y_5$;

$y_5S = (4, 4, 4, 4, 0, 4, 0) = x_5$; $x_5S = (2, 4, 4) = y_6$;

$y_6S = (0, 4, 4, 0, 2, 4, 4) = x_6$;

$x_6S = (4, 2, 4) = y_7$ ( $=y_2$);

$y_7S = (2, 2, 2, 0, 0, 2, 0)$.

Thus the resultant in this case is also a realized limit cycle pair.

Now we leave it as an exercise for the reader to find those x in D or y in R which are classical fixed point pair other than the pair {(0, 0, 0, 0, 0, 0, 0), (0, 0, 0)}.

Now we find will the sum of the resultant pair yield the resultant sum.

Consider

$x = (0, 1, 0, 2, 0, 0, 4)$ and

$x_0 = (1, 0, 0, 0, 0, 0, 0) \in D$.

$x + x_0 = (1, 1, 0, 2, 0, 0, 4) \in D$.

We first find the effect of $x + x_0$ on S.

$(x + x_0) S = (1, 3, 3) = y_1$;

$y_1S^t = (0, 3, 2, 0, 3, 2, 0, 4) = x_1$; $x_1S = (0, 0, 2) = y_2$;

$y_2S^t = (2, 0, 0, 0, 2, 0, 4) = x_2$; $x_2S = (2, 4, 0) = y_3$;

$y_3S^t = (2, 4, 4, 4, 4, 4, 2) = x_3$; $x_3S = (0, 0, 4) = y_4$;

$y_4S^t = (4, 0, 0, 0, 4, 0, 2) = x_4$; $x_4S = (4, 2, 4) = y_5$;

$y_5S^t = (2, 2, 2, 0, 0, 2, 0) = x_5$; $x_5S = (4, 2, 2) = y_6$;

$y_6S^t = (0, 2, 2, 0, 4, 2, 4) = x_6$; $x_6S = (4, 4, 0) = y_7$;

$y_7S^t = (2, 4, 2, 0, 2, 4, 4) = x_7$;

$x_7S = (0, 0, 4) = y_8\ (=y_4)$ ;

$y_8S^t = (4, 0, 0, 0, 4, 0, 2) = x_8\ (=x_4)$.

Thus the resultant of $x+x_0$ on the MOD rectangular matrix operator is a realized limit point pair given by

$\{(4, 0, 0, 0, 4, 0, 2), (0, 0, 4)\}$.

The realized limit point pair of x is
$\{(4, 2, 2, 0, 2, 2, 4), (4, 2, 0)\}$.

The realized limit point pair of $x_0$ is
$\{(2, 2, 2, 0, 0, 2, 0), (4, 2, 4)\}$.

The sum of xS and $x_0S$ is $\{(0, 4, 4, 0, 2, 4, 4), (2, 4, 4)\}$.

Sum of the resultant of $x_0 + x$ is not the resultant sum of $x_0$ and that of x.

***Example 2.4:*** Let

$$M = \begin{bmatrix} 6 & 0 & 2 & 6 & 4 & 6 & 4 \\ 0 & 4 & 0 & 6 & 0 & 4 & 0 \end{bmatrix}$$

be the MOD real rectangular matrix operator with entries from $Z_{10}$.

Let D = {(a, b) | a, b ∈ $Z_{10}$} be the domain of state vectors associated with M.

R = {($a_1$, $a_2$, …, $a_7$) | $a_i \in Z_{10}$; $1 \leq i \leq 7$} be the range of state vectors associated with M.

Let x = (5, 3) ∈ D.

To find the effect of x on M.

$xM = (0, 2, 0, 8, 0, 2, 0) = y_1$; $y_1M^t = (0, 4) = x_1$;

$x_1M = (0, 6, 0, 4, 0, 6, 0) = y_2$; $y_2M^t = (0, 0) = x_2$;

$x_2M = (0, 0, 0, 0, 0, 0, 0) = y_3$;

$y_3M^t = (0, 0) = x_3\ (=x_2)$.

Thus the resultant of x = (5, 3) ∈ D is a realized fixed point pair given by

{(0, 0), (0, 0, 0, 0, 0, 0, 0}.

Consider y = (1, 2, 3, 0, 5, 0, 7) ∈ R.

To find the effect of y on $M^t$.

$yM^t = (0, 8) = x_1$;

$x_1M = (0, 2, 0, 0, 8, 0, 2, 0) = y_1$; $y_1M^t = (0, 4) = x_2$;

$x_2M = (0, 6, 0, 4, 0, 6, 0) = y_2$; $y_2M^t = (0, 0) = x_3$;

$x_3M = (0, 0, 0, 0, 0, 0, 0) = y_3$;

$y_3M^t = (0, 0) = x_4\ (=x_3)$.

Thus we see the resultant of y on the MOD rectangular operator M is the realized fixed point pair
{(0, 0), (0, 0, 0, 0, 0, 0, 0)}.

Now consider another MOD rectangular matrix operator N of order $2 \times 7$ with entries from $Z_{10}$.

$N = \begin{bmatrix} 0 & 2 & 4 & 0 & 6 & 0 & 8 \\ 2 & 0 & 0 & 4 & 0 & 6 & 0 \end{bmatrix}$ be the MOD rectangular

matrix operator with entries from $Z_{10}$.

Now we find the effect of the same

$x = (5, 3) \in D$ on N.

$xN = (6, 0, 0, 2, 0, 8, 0) = y_1$; $y_1N^t = (0, 8) = x_1$;

$x_1N = (6, 0, 0, 2, 0, 8, 0) = y_2$; $y_2N^t = (0, 8) = x_2$ $(=x_1)$.

Thus the resultant of x on N is a realized fixed point pair given by
{(0, 8), (6, 0, 0, 2, 0, 8, 0)}.

Let us find the effect of the same $y = (1, 2, 3, 0, 5, 0, 7) \in R$ on the MOD rectangular matrix operator $N^t$.

$yN^t = (2, 0) = x_1$;

$x_1N = (0, 4, 8, 0, 2, 0, 6) = y_1$; $y_1N^t = (0, 0) = x_2$;

$x_2N = (0, 0, 0, 0, 0, 0, 0) = y_2$; $y_2N^t = (0, 0) = x_3$ $(=x_2)$.

Thus the resultant of y on $N^t$ is a realized fixed point pair given by {(0, 0), (0, 0, 0, 0, 0, 0, 0)}.

Now we find M + N the sum of the MOD rectangular matrix operators.

M + N

$$= \begin{bmatrix} 6 & 0 & 2 & 6 & 4 & 6 & 4 \\ 0 & 4 & 0 & 6 & 0 & 4 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2 & 4 & 0 & 6 & 0 & 8 \\ 2 & 0 & 0 & 4 & 0 & 6 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 2 & 6 & 6 & 0 & 6 & 2 \\ 2 & 4 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} = W.$$

To find the effect of the same $x = (5, 3) \in D$ on the MOD rectangular matrix operator W.

$xW = (6, 2, 0, 0, 0, 0, 0) = y_1;$ $\quad y_1W^t = (0, 0) = x_2;$

$x_2W = (0, 0, 0, 0, 0, 0, 0) = y_2;$ $\quad y_2W^t = (0, 0) = x_3\ (=x_2).$

Thus the resultant of x on the MOD rectangular matrix operator is a realized fixed point pair given by
$\{(0, 0), (0, 0, 0, 0, 0, 0, 0)\}$.

We find the effect of the same $y = (1, 2, 3, 0, 5, 0, 7) \in R$ on the MOD rectangular matrix operator W which is the sum of the MOD rectangular matrix operator M and N.

$yW^t = (2, 0) = x_1;$

$x_1W = (2, 4, 2, 2, 0, 2, 4) = y_1;$ $\quad y_1W^t = (4, 0) = x_2;$

$x_2W = (4, 8, 4, 4, 0, 4, 8) = y_2;$ $\quad y_2W^t = (8, 0) = x_2;$

$x_2W = (8, 6, 8, 8, 0, 8, 6) = y_3;$ $\quad y_3W^t = (6, 0) = x_3;$

$x_3W = (6, 2, 6, 6, 0, 6, 2) = y_4;$

$y_4W^t = (2, 0) = x_4\ (=x_1).$

$x_4W = (2, 4, 2, 2, 0, 2, 4) = y_5$ $(=y_1)$.

Thus the resultant of y on W is a realized limit cycle pair given by

{(2, 0), (2, 4, 2, 2, 0, 2, 4)} I

The resultant of x on M is {(0, 0), (0, 0, 0, 0, 0, 0, 0)} II

The resultant of y on M is {(0, 0), (0, 0, 0, 0, 0, 0, 0)} III

The resultant of x = (5, 3) on N is
{(0, 8), (6, 0, 0, 2, 0, 8, 0)} IV

The resultant of y = (1, 2, 3, 0, 5, 0, 7) on N is
{(0, 0), (0, 0, 0, 0, 0, 0, 0)} V

Resultant of x = (5, 3) on W = M + N is
{(0, 0), (0, 0, 0, 0, 0, 0, 0)}.

The resultant of y = (1, 2, 3, 0, 5, 0, 7) on M + N = W is I

But sum of resultant of x on M and N is
{(0, 8), (6, 0, 0, 2, 0, 8, 0)}.

The sum of resultant of y on M and N is
{(0, 0), (0, 0, 0, 0, 0, 0, 0)}.

But resultant of on W is
{(2, 0), (2, 4, 2, 2, 0, 2, 4)}.

From these examples we enlist some of the result as theorems.

Now one more examples to this effect is given.

***Example 2.5:*** Let

$$M = \begin{bmatrix} 3 & 0 & 4 \\ 0 & 1 & 0 \\ 2 & 0 & 3 \\ 0 & 4 & 0 \end{bmatrix}$$

be the MOD rectangular real matrix operator with entires from $Z_5$.

Let D = $\{(x_1, x_2, x_3, x_4) \mid x_i \in Z_5, 1 \leq i \leq 4\}$ be the domain state row vectors associated with M.

Let R = $\{(y_1, y_2, y_3) \mid y_i \in Z_5, 1 \leq i \leq 3\}$ be the range space of state vectors associated with the MOD rectangular matrix rectangular matrix operator M.

Let x = (3, 0, 2, 0) $\in$ D.

To find the effect of x on M is as follows:

$xM = (3, 0, 3) = y_1$; $\quad y_1M^t = (1, 0, 0, 0) = x_1$;

$x_1M^t = (3, 0, 4) = y_2$; $\quad y_2M^t = (0, 0, 3, 0) = x_2$;

$x_2M = (1, 0, 4) = y_3$; $\quad y_3M^t = (4, 0, 4, 0) = x_3$;

$x_3M = (0, 0, 3) = y_4$; $\quad y_4M^t = (2, 0, 4, 0) = x_4$;

$x_4M = (4, 0, 0) = y_5$; $\quad y_5M^t = (2, 0, 3, 0) = x_5$;

$x_5M = (2, 0, 2) = y_6$; $\quad y_6M^t = (4, 0, 0, 0) = x_6$;

$x_6M = (2, 0, 1) = y_7$; $\quad y_7M^t = (0, 0, 2, 0) = x_7$;

$x_7M = (4, 0, 1) = y_8$; $\quad y_8M^t = (1, 0, 1, 0) = x_8$;

$x_8M = (0, 0, 2) = y_9$; $\quad y_9M^t = (3, 0, 1, 0) = x_9$;

$x_9M = (1, 0, 0) = y_{10}$; $y_{10}M^t = (3, 0, 2, 0) = x_{10}$ (=x).

Thus the resultant is a realized limit cycle pair given by
{(3, 0, 2, 0) (1, 0, 0)}. I

Let $y = (1, 1, 1) \in R$.

To find the effect of y on the MOD-rectangular matrix operator on $M^t$.

$yM^t = (2, 1, 0, 4) = x_1$; $x_1M^t = (1, 2, 3) = y_1$;

$y_1M^t = (0, 2, 1, 3) = x_2$; $x_2M = (2, 4, 3) = y_2$;

$y_2M^t = (3, 4, 3, 1) = x_3$; $x_3M = (0, 3, 1) = y_3$;

$y_3M^t = (4, 3, 3, 2) = x_4$; $x_4M = (3, 1, 0) = y_4$;

$y_4M^t = (4, 1, 1, 4) = x_5$; $x_5M = (4, 2, 4) = y_5$;

$y_5M^t = (3, 2, 0, 3) = x_6$; $x_6M = (4, 4, 2) = y_6$;

$y_6M^t = (0, 4, 4, 1) = x_7$; $x_7M = (4, 3, 2) = y_7$;

$y_7M^t = (0, 3, 4, 2) = x_8$; $x_8M = (3, 1, 2) = y_8$;

$y_8M^t = (2, 1, 2, 4) = x_9$; $x_9M = (0, 2, 4) = y_9$;

$y_9M^t = (1, 2, 2, 3) = x_{10}$; $x_{10}M = (2, 4, 0) = y_{10}$;

$y_{10}M^t = (1, 4, 4, 1) = x_{11}$; $x_{11}M = (1, 3, 2) = y_{11}$;

$y_{11}M^t = (1, 3, 3, 2) = x_{12}$; $x_{12}M = (3, 1, 3) = y_{12}$;

$y_{12}M^t = (1, 1, 0, 4) = x_{13}$; $x_{13}M = (3, 2, 4) = y_{13}$;

$y_{13}M^t = (0, 2, 3, 3) = x_{14}$; $x_{14}M = (1, 4, 4) = y_{14}$;

$y_{14}M^t = (4, 4, 4, 1) = x_{15}$; $x_{15}M = (0, 3, 3) = y_{15}$;

$y_{15}M^t = (2, 3, 4, 2) = x_{16}$; $x_{16}M = (4, 1, 0) = y_{16}$;

$y_{16}M^t = (2, 1, 3, 4) = x_{17}$; $x_{17}M = (2, 2, 2) = y_{17}$;

$y_{17}M^t = (4, 2, 0, 3) = x_{18}$; $x_{18}M = (2, 4, 1) = y_{19}$;

$y_{19}M^t = (0, 4, 2, 1) = x_{19}$; $x_{19}M = (4, 3, 1) = y_{20}$;

$y_{20}M^t = (1, 3, 1, 2) = x_{20}$; $x_{20}M = (0, 1, 2) = y_{21}$;

$y_{21}M^t = (3, 1, 1, 4) = x_{21}$; $x_{21}M = (1, 2, 0) = y_{22}$;

$y_{22}M^t = (3, 2, 2, 3) = x_{22}$; $x_{22}M = (3, 4, 3) = y_{23}$;

$y_{23}M^t = (1, 4, 0, 1) = x_{23}$; $x_{23}M = (3, 3, 4) = y_{24}$;

$y_{24}M^t = (0, 3, 3, 2) = x_{24}$ and so on.

However as the number of elements in D and R is finite we will arrive at a realized fixed point or a realized limit cycle afer a finite number of iterations.

In view of these we have the following results.

**THEOREM 2.1:** *Let*

$$M = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

*be the Mod rectangular matrix real operator with entries from $Z_s$; $2 \leq s < \infty$.*

*Let $D = \{(a_1, a_2, \ldots, a_n) \mid a_i \in Z_s;\ 1 \leq i \leq n\}$ be the domain state vector associated with M.*

*Let $R = \{(b_1, b_2, \ldots, b_m) \mid b_i \in Z_s;\ 1 \leq i \leq m\}$ be the collection of range of state vectors associated with the MOD rectangular matrix M.*

*(i) For $x_1, x_2 \in D$ if $\{a, b_1\}$ and $\{a_2, b_2\}$ are the resultant pair of $x_1$ and $x_2$ respectively using M.*

*If $x_1 + x_2 \in D$ and if $\{c, d\}$ be the resultant pair using M.*

*In general $\{a_1, b_1\}$ and $\{a_2, b_2\}$ are not in any way related with $\{c, d\}$.*

*(ii) $y_1, y_2 \in D$ and $\{c_1, d_1\}$ and $\{c_2, d_2\}$ are the resultant pairs of $y_1$ and $y_2$ respectively using M.*

*If $\{x, y\}$ be the resultant of $y_1 + y_2$ using M. Then in general $\{c_1, d_1\}$ and $\{c_2, d_2\}$ are not related with $\{x, y\}$.*

Proof is direct from the examples given in this chapter.

**THEOREM 2.2:** *Let*

$$M = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

*be the $m \times n$ MOD real rectangular matrix operator with entries from $Z_s$, $2 \leq s > \infty$.*

Let

$$N = \begin{bmatrix} b_{11} & b_{12} & \ldots & b_{1n} \\ b_{21} & b_{22} & \ldots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \ldots & b_{mn} \end{bmatrix}$$

*be the $m \times n$ MOD real rectangular matrix operator with entries from $Z_S$, $2 \leq s < \infty$.*

*Let $D = \{(x_1, x_2, \ldots, x_m) \mid x_i \in Z_s, 1 \leq i \leq m\}$ be the domain row vector associated M and N.*

*Let $R = \{(y_1, \ldots, y_n) \mid y_i \in Z_s, 1 \leq i \leq n\}$ be the range space associated with M and N.*

*If $x \in D$ has $\{a, b\}$ to be the resultant pair on M, $\{c, d\}$ be the resultant pair on N and $\{t, u\}$ be the resultant pair of x using the MOD rectangular matrix operator M + N. Then in general $\{a, b\}$ $\{c, d\}$ and $\{t, u\}$ are in no way related.*

*Similar result is true for $y \in R$.*

Proof is evident from the examples proved.

Next we proceed onto define the develop some more restricted properties on D and R and the mode of operating on the MOD-rectangular real matrix operators.

***Example 2.6:*** Let

$$M = \begin{bmatrix} 3 & 2 & 0 & 4 \\ 0 & 1 & 2 & 0 \\ 1 & 0 & 0 & 3 \\ 0 & 4 & 3 & 1 \\ 2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 4 \end{bmatrix}$$

be a $6 \times 4$ MOD-rectangular real matrix operator with entries from $Z_5$.

Let $D_S = \{(x_1, x_2, \ldots, x_6) \mid x_i \in \{0, 1\}, 1 \leq i \leq 6; x_i = 0$ implies the off state and $x_i = 1$ denotes the on state$\}$ and

$R_S = \{(y_1, y_2, y_3, y_4, y_5) \mid y_i \in \{0, 1\}, 1 \leq i \leq 4; y_i = 0$ implies the off state and $y_i = 1$ the on state$\}$ be the domain and range space of state vectors associated with Mod matrix operator M.

We will show how the elements of $D_S$ and $R_S$ function on M in the following.

Let $x = (1, 0, 0, 0, 0, 0) \in D_S$.

To find the effect of x on M.

$xM = (3, 2, 0, 4)$ after updating and thresholding xM we get $xM = (3, 2, 0, 4) \to (1, 1, 0, 1) = y_1$ say;

$y_1M^t = (4, 1, 4, 0, 2, 2) \to (1, 1, 1, 0, 1, 1) = x_1$;

$x_1M = (0, 1, 2, 1) \to (0, 1, 1, 1) = y_2$;

$y_2M^t = (1, 3, 3, 3, 0, 2) \to (1, 1, 1, 1, 0, 1) = x_2$;

$x_2M = (4, 0, 0, 2) \to (1, 0, 0, 1) = y_3$;

$y_3M^t = (2, 0, 4, 1, 2, 4) \to (1, 0, 1, 1, 1, 1) = x_3$;

$x_3M = (1, 4, 3, 2) \to (1, 1, 1, 1) = y_4$;

$y_4M^t = (4, 3, 4, 3, 2, 2) \to (1, 1, 1, 1, 1, 1) = x_4$;

$x_4M = (1, 0, 0, 2) \to (1, 0, 0, 1) = y_5\ (=y_3)$ ;

$y_3M^t = (2, 0, 4, 1, 2, 4) \to (1, 0, 1, 1, 1, 1) = x_5\ (=x_3)$.

Thus a new type of situation is faced by us there are two realized limit cycles pairs given by

{(1, 0, 1, 1, 1, 1), (1, 0, 0, 1)}.

{(1, 1, 1, 1, 1), (1, 1, 1, 1)} is the MOD realized limit cycle.

Let $y = (0, 0, 0, 1) \in R_S$.

To find the resultant of y on the MOD operator $M^t$.

$yM^t = (4, 0, 3, 1, 0, 4) \to (1, 0, 1, 1, 0, 1) = x_1$;

$x_1M = (4, 4, 3, 2) \to (1, 1, 1, 1) = y_1$;

$y_1M^t = (4, 3, 4, 3, 2, 2) \to (1, 1, 1, 1, 1, 1) = x_2$;

$x_2M = (1, 0, 0, 2) \to (1, 0, 0, 1) = y_2$;

$y_2M^t = (2, 0, 4, 1, 2, 4) \to (1, 0, 1, 1, 1, 1) = x_3$;

$x_3M = (1, 4, 3, 2) \to (1, 1, 1, 1) = y_3$ $(=y_1)$.

Next consider $x = (0, 0, 1, 1, 0, 0) \in D_S$.

To find the effect of x on M.

$xM = (1, 4, 3, 4) \to (1, 1, 1, 1) = y_1$;

$y_1M^t = (4, 3, 4, 3, 2, 2) \to (1, 1, 1, 1, 1, 1) = x_2$;

$x_2M = (1, 0, 0, 2) \to (1, 0, 0, 1) = y_2$;

$y_2M^t = (2, 0, 4, 1, 2, 4) \to (1, 0, 1, 1, 1, 1) = x_3$;

$x_3M = (1, 4, 3, 2) \to (1, 1, 1, 1) = y_3$ $(=y_1)$.

Thus we will in this case also go on getting the same type of realized limit cycle as resultant.

Hence we leave it as an exercise to the reader the work of finding any $x \in D_S$ and $y \in R_S$ which may yield realized fixed point.

However we consider another MOD $6 \times 4$ rectangular matrix operator N with entries from $Z_5$.

$$N = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 3 & 4 & 3 & 4 \\ 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \\ 4 & 3 & 2 & 4 \\ 1 & 0 & 4 & 0 \end{bmatrix}.$$

We find the effect of only those state vectors for which we have already found the resultant using M now using the MOD operator N.

Let $x = (1, 0, 0, 0, 0, 0) \in D_S$.

Let $xN = (1, 2, 1, 2) \to (1, 1, 1, 1) = y_1$;

$y_1N^t = (2, 4, 1, 1, 3, 0) \to (1, 1, 1, 1, 1, 0) = x_1$;

$x_1N = (4, 2, 1, 3) \to (1, 1, 1, 1) = y_2\ (=y_1)$ ;

$y_2N^t = (2, 4, 1, 1, 3, 0) \to (1, 1, 1, 1, 1, 0) = x_2\ (= x_1)$.

Thus we using this MOD -real rectangular matrix operator obtain a realized fixed point just after one pair of iterations given by

$\{(1, 1, 1, 1, 1, 0)\ (1, 1, 1, 1)\}$.

Let $y = (0, 0, 0, 1) \in R_S$ to find the effect of y on the MOD real rectangular matrix operator N

$yN^t = (2, 4, 3, 0, 4, 0) \to (1, 1, 1, 0, 1, 0) = x_1$;

$x_1N = (3, 0, 3, 3) \to (1, 0, 1, 1) = y_1$;

$y_1N^t = (4, 0, 0, 4, 0, 0) \to (1, 0, 0, 1, 0, 0) = x_2$;

$x_2N = (2, 4, 4, 2) \to (1, 1, 1, 1) = y_2$;

$y_2N^t = (1, 4, 1, 1, 3, 0) \to (1, 1, 1, 1, 1, 0) = x_3$;

$x_3N = (4, 2, 1, 3) \to (1, 1, 1, 1) = y_3$ $(=y_2)$.

Thus the resultant of y = (0, 0, 0, 1) on N is a realized limit cycle pair after three pair of iterations given by
{(1, 1, 1, 1, 1, 0), (1, 1, 1, 1)}.

Let $x = (0, 0, 1, 1, 0, 0) \in D_S$.

To find the effect of x on N.

$xN = (1, 3, 0, 3) \to (1, 1, 0, 1) = y_1$;

$y_1N^t = (0, 1, 4, 3, 1, 1) \to (0, 1, 1, 1, 1, 1) = x_1$;

$x_1N = (4, 0, 4, 0) \to (1, 0, 1, 0) = y_2$;

$y_2N^t = (2, 1, 2, 4, 1, 0) \to (1, 1, 1, 1, 1, 0) = x_2$;

$x_2N = (4, 2, 1, 3) \to (1, 1, 1, 1) = y_3$;

$y_3N^t = (1, 0, 1, 1, 3, 0) \to (1, 0, 1, 1, 1, 0) = x_3$;

$x_3N = (1, 3, 3, 4) \to (1, 1, 1, 1) = y_4$ $(= y_3)$.

Thus the resultant of $x = (0, 0, 1, 1, 0, 0) \in D_S$ is a realized fixed point pair given by {(1, 0, 1, 1, 1, 0), (1, 1, 1, 1)}.

We realize the resultant happens to depends mainly on the MOD real rectangular matrix operator. This is very much evident from the above example.

Next we find some other type of MOD-real rectangular operators using the modulo integers $Z_n$.

***Example 2.7:*** Let

$$M = \begin{bmatrix} 5 & 2 & 1 & 0 & 3 \\ 0 & 4 & 3 & 2 & 1 \\ 1 & 0 & 2 & 3 & 4 \\ 5 & 6 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 1 \\ 0 & 2 & 2 & 2 & 2 \\ 3 & 3 & 3 & 0 & 3 \\ 0 & 5 & 0 & 5 & 5 \end{bmatrix}$$

be MOD real rectangular matrix with entries from $Z_8$.

Let $D = \{(x_1, x_2, \ldots, x_8) \mid x_i \in Z_8, 1 \leq i \leq 8\}$ and

$R = \{(y_1, y_2, y_3, y_4, y_5) \mid y_i \in Z_8; 1 \leq i \leq 5\}$ be the domain and range space of state vectors associated with M.

Let $D_S = \{(x_1, x_2, \ldots, x_8) \mid x_i \in \{0, 1\}, 1 \leq i \leq 8\}$ and

$R_S = \{(y_1, y_2, y_3, y_4, y_5) \mid y_i \in \{0, 1\}; 1 \leq i \leq 5\}$ be the special domain and range state vectors which depict the on or off state of the elements in the state vector spaces $D_S$ and $R_S$.

Now we will work with these four types of state vectors. Clearly $D_S \subseteq D$ and $R_S \subseteq R$ but when operations are performed on x in $D_S$ it is distinctly different from the operations performed with x is in D (for the same x is in $D_S$).

This will be illustrated here by this example.

Let x = (1, 1, 0, 0, 0, 0, 1, 1) $\in$ D.

To find the effect of x on M.

xM = (0, 6, 7, 7, 4) = $y_1 \in$ R;

$y_1M^t$ = (7, 7, 3, 3, 1, 0, 3, 5) = $x_1$;

$x_1M$ = (6, 7, 2, 3, 3) = $y_2$;

$y_2M^t$ = (7, 3, 7, 3, 2, 6, 6, 1) = $x_2$;

$x_2M$ = (5, 1, 6, 7, 1) = $y_3$;

$y_3M^t$ = (4, 5, 2, 6, 5, 6, 7, 5) = $x_3$;

$x_3M$ = (6, 7, 5, 5, 0) = $y_4$;

$y_4M^t$ = (1, 5, 7, 6, 2, 2, 6, 4) = $x_4$;

$x_4M$ = (4, 6, 6, 5, 0) = $y_5$;

$y_5M^t$ = (6, 4, 7, 5, 0, 2, 0, 7) = $x_5$;

$x_5M$ = (6, 1, 4, 1, 1) = $y_6$;

$y_6M^t$ = (7, 3, 5, 5, 4, 6, 4, 7) = $x_6$;

$x_6M$ = (5, 5, 6, 1, 3) = $y_7$;

$y_7M^t$ = (2, 3, 0, 0, 3, 6, 1, 5) = $x_6$ and so on.

Even after 7 pair of iterations we are yet to arrive at a realized fixed point pair or a realized limit cycle pair. However we are sure with $8^8 - 1$ iterations we arrive at the result.

Now we find the effect of x = (1, 1, 0, 0, 0, 0, 1, 1) as a point of $D_S$.

$xM = (0, 6, 7, 7, 4) \to (0, 1, 1, 1, 1) = y_1;$

$y_1M^t = (6, 0, 1, 7, 3, 0, 4, 7) \to (1, 1, 1, 1, 1, 0, 1, 1) = x_1;$

$x_1M = (7, 5, 2, 3, 1) \to (1, 1, 1, 1, 1) = y_2;$

$y_2M^t = (3, 2, 2, 5, 4, 0, 4, 7) \to (1, 1, 1, 1, 1, 0, 1, 1) = x_2$
$(= x_1);$

$x_2M = (7, 5, 2, 3, 1) \to (1, 1, 1, 1, 1) = y_3 \; (= y_2).$

Thus with in a pair of two iteration we arrive at a realized fixed point pair given by
$\{(1, 1, 1, 1, 1, 0, 1, 1), (1, 1, 1, 1, 1)\}.$

Now it is important to note the following facts and keep them on record. In the first place if D and R are used we can only after a maximum number of $8^8 - 1$ iterations arrive at a resultant however in case of $D_S$ and $R_S$ we arrive at a resultant after $8^8 - 1$ iterations.

Let $y = (1, 0, 0, 0, 0) \in R$.

To find the effect of y on M.

$yM^t = (5, 0, 1, 5, 1, 0, 3, 0) = x_1;$

$x_1M = (5, 7, 1, 0, 5) = y_1;$

$y_1M^t = (7, 4, 3, 3, 2, 4, 6, 4) = x_2;$

$x_2M = (1, 2, 5, 3, 5) = y_2;$

$y_2M^t = (5, 2, 0, 4, 5, 6, 7, 2) = x_3;$

$x_3M = (1, 1, 1, 6, 1) = y_3;$

$y_3 M^t = (3, 4, 1, 1, 4, 4, 4, 0) = x_4;$

$x_4M = (5, 4, 1, 2, 1) = y_4$;

$y_4M^t = (5, 0, 1, 3, 3, 0, 1, 3) = x_5$;

$x_5M = (7, 1, 5, 6, 1) = y_5$;

$y_5M^t = (5, 0, 1, 7, 6, 2, 7, 0) = x_6$;

$x_6M = (0, 4, 6, 6, 2) = y_6$;

$y_6M^t = (4, 0, 6, 6, 4, 4, 4, 4) = x_7$;

$x_7M = (0, 0, 0, 4, 0) = y_7$;

$y_7M^t = (0, 0, 4, 4, 0, 0, 0, 4) = x_8$;

$x_8M = (0, 4, 0, 4, 4) = y_8$;

$y_8M^t = (4, 4, 4, 4, 0, 0, 0, 4) = x_9$;

$x_9M = (4, 4, 0, 4, 4) = y_9$;

$y_9M^t = (0, 4, 0, 0, 4, 0, 4, 4) = x_{10}$;

$x_{10}M = (0, 4, 4, 4, 0) = y_{10}$;

$y_{10}M^t = (4, 4, 4, 4, 0, 0, 0, 0) = x_{11}$;

$x_{11}M = (4, 0, 0, 0, 0) = y_{11}$;

$y_{11}M^t = (4, 0, 4, 4, 4, 0, 4, 0) = x_{12}$;

$x_{12}M = (4, 0, 4, 0, 4) = y_{12}$;

$y_{12}M^t = (4, 0, 4, 4, 4, 0, 4, 4) = x_{13}$;

$x_{13}M = (4, 0, 4, 4, 0) = y_{13}$;

$y_{13}M^t = (0, 4, 0, 0, 0, 0, 0, 4) = x_{14}$;

$x_{14}M = (0, 4, 4, 4, 0) = y_{14}$;

$y_{14}M^t = (4, 4, 4, 4, 4, 0, 0, 0) = x_{15}$;

$x_{15}M = (0, 4, 4, 0, 4) = y_{15}$;

$y_{15}M^t = (0, 0, 0, 0, 4, 0, 4, 0) = x_{16}$;

$x_{16}M = (0, 0, 0, 0, 0) = y_{16}$;

$y_{16}M^t = (0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) = x_{17}$;

$x_{17}M = (0\ 0\ 0\ 0\ 0) = y_{17}\ (= y_{16})$.

Thus the resultant is a realized fixed point pair given by
$\{(0, 0, 0, 0, 0, 0, 0, 0), (0, 0, 0, 0, 0)\}$ --- I

Let $y = (1, 0, 0, 0, 0) \in R_S$

To find the resultant of on M.

$yM^t = (5, 0, 1, 5, 1, 0, 3, 0) \to (1, 0, 1, 1, 1, 0, 1, 0) = x_1$;

$x_1M = (7, 4, 7, 4, 3) \to (1, 1, 1, 1, 1) = y_1$;

$y_1M^t = (3, 2, 2, 4, 4, 0, 4, 7) \to (1, 1, 1, 1, 1, 0, 1, 1) = x_2$;

$x_2M = (7, 5, 2, 4, 1) \to (1, 1, 1, 1, 1) = y_2$;

$y_2M^t = (3, 2, 2, 4, 4, 0, 4, 7) \to (1, 1, 1, 1, 1, 0, 1, 1) = x_3$ $(=x_2)$.

Thus the resultant after a pair of iterations is a realized fixed point pair given by
$\{(1, 1, 1, 1, 1, 0, 1, 1), (1, 1, 1, 1)\}$ --- II

From I and II it is clear that after 16 pair of iterations we arrive as a resultant the pair zero state vectors where as from II we see in case of (1, 0, 0, 0, 0) in $R_S$ the resultant is a pair which has all elements to be ones in both the state vectors except in one place.

Consider x = (0, 0, 0, 0, 0, 0, 0, 1) ∈ D. To find the effect of x on M.

$$xM = (0, 5, 0, 5, 5) = y_1;$$

$$y_1M^t = (1, 1, 3, 3, 2, 6, 6, 3) = x_1;$$

$$x_1M = (3, 7, 2, 0, 7) = y_2;$$

$$y_2M^t = (4, 1, 6, 3, 3, 0, 1, 6) = x_2;$$

$$x_2M = (7, 2, 1, 5, 1) = y_3;$$

$$y_3M^t = (3, 6, 3, 1, 3, 2, 1, 0) = x_3;$$

$$x_3M = (7, 6, 5, 2, 5) = y_3;$$

$$y_3M^t = (3, 0, 3, 1, 7, 4, 5, 1) = x_4;$$

$$x_4M = (5, 7, 7, 7, 0) = y_4;$$

$$y_4M^t = (6, 7, 1, 2, 3, 2, 1, 6) = x_5;$$

$$x_5M = (7, 4, 7, 5, 5) = y_5;$$

$$y_5M^t = (1, 4, 0, 0, 7, 2, 5, 6) = x_6;$$

$$x_6M = (3, 2, 7, 4, 7) = y_6;$$

$$y_6M^t = (7, 4, 1, 7, 3, 0, 1, 1) = x_7;$$

$$x_7M = (5, 3, 3, 1, 0) = y_7;$$

$$y_7M^t = (2, 7, 6, 4, 3, 6, 1, 2) = x_8;$$

$x_8M = (2, 4, 5, 2, 7) = y_8;$

$y_8M^t = (4, 2, 6, 4, 3, 6, 1, 2) = x_9$

and so on. However after $8^8 - 1$ number of iterations certainly we will arrive at a realized fixed point pair or a realized limit cycle pair. Now we work with $y = (1, 0, 0, 0, 0) \in R_S$ on M.

$yM^t = (5, 0, 1, 5, 1, 0, 3, 0) \to (1, 0, 1, 1, 1, 0, 1, 0) = x_1;$

$x_1M = (7, 4, 7, 4, 3) \to (1, 1, 1, 1, 1) = y_1;$

$y_1M^t = (3, 2, 2, 4, 4, 0, 4, 7) \to (1, 1, 1, 1, 1, 0, 1, 1) = x_2;$

$x_2M = (7, 5, 2, 3, 1) \to (1, 1, 1, 1, 1) = y_2;$

$y_2M = (3, 2, 2, 4, 4, 0, 4, 7) \to (1, 1, 1, 1, 1, 0, 1, 1) = x_3 \quad (= x_2).$

Thus after just two iterations we arrive at a realized fixed point pair {(1, 1, 1, 1, 1, 0, 1, 1), (1, 1, 1, 1, 1)}.

Let x = (0, 0, 0, 1, 0, 0, 0, 1) $\in$ D. The effect of x on M is as follows:

$xM = (5,3, 0, 6, 5) = y_1$; $y_1M^t = (6, 5, 3, 1, 5, 4, 7, 6) = x_1;$

$x_1M = (0,6, 5, 2,3) = y_2$; $y_2M^t = (2, 6, 4, 6, 6, 0, 0, 6) = x_2;$

$x_2M = (2,4, 4,4,0) = y_3$; $y_3M^t = (6, 4, 4, 6, 2, 0, 6, 0) = x_3;$

$x_3M = (4,4, 6,2,4) = y_4$; $y_4M^t = (6, 2, 6, 6, 2, 0, 6, 2) = x_4;$

$x_4M = (6, 6, 4,6,2) = y_5$; $y_5M^t = (4, 2, 0, 0, 2, 4, 6, 4) = x_5;$

$x_5M = (0, 0, 6, 0, 6) = y_6$; $y_6M^t = (0, 0, 4, 0, 4, 0, 4, 6) = x_6;$

$x_6M = (4, 6, 0, 2, 6) = y_7$; $y_7M^t = (2, 2, 2, 2, 0, 4, 0, 6) = x_7;$

$x_7M = (6, 6, 0, 6, 6) = y_8;$

$y_8M^t = (4, 2, 0, 0, 2, 4, 6, 2) = x_8;$

$x_8M = (0, 6, 6, 6, 6) = y_9;$

$y_9M^t = (4, 4, 6, 2, 2, 0, 6, 2) = x_9;$

$x_9M = (0, 4, 0, 4, 6) = y_{10};$

$y_{10}M^t = (2, 6, 4, 4, 2, 4, 6, 6) = x_{10};$

$x_{10}M = (6, 6, 6, 2, 6) = y_{11};$

$y_{11}M^t = (2, 4, 0, 4, 0, 0, 0, 6) = x_{11};$

$x_{11}m = (6, 2, 6, 2, 0) = y_{12};$

$y_{12}m^t = (0, 6, 0, 4, 6, 4, 2, 4) = x_{12};$

$x_{12}m = (0, 4, 6, 4, 2) = y_{13};$

$y_{13}m^t = (4, 4, 0, 4, 4, 0, 4, 2) = x_{13};$

$x_{14}m = (0, 4, 0, 6, 2) = y_{14};$

$y_{14}m^t = (6, 6, 2, 6, 6, 0, 2, 4) = x_{14};$

$x_{14}m = (2, 0, 0, 4, 0) = y_{15};$

$y_{15}m^t = (4, 0, 6, 6, 2, 0, 6, 2) = x_{15};$

$x_{15}m = (6, 2, 4, 2, 2) = y_{16};$

$y_{16}m^t = (4, 2, 4, 4, 6, 4, 0, 6) = x_{16};$

$x_{16}m = (2, 4, 0, 2, 2) = y_{17}$

and so on certainly after a finite number of iterations we will arrive at a realized fixed point pair or a realized limit cycle pair.

Now take the same

$x = (0, 0, 0, 1, 0, 0, 0, 1)$ as an element of $D_S$.

$xM = (5, 3, 0, 6, 5) \to (1, 1, 0, 1, 1) = y_1$;

$y_1M^t = (2, 7, 0, 4, 3, 6, 1, 7) \to (1, 1, 0, 1, 1, 1, 1, 1) = x_1$;

$x_1M = (6, 7, 2, 2, 7) \to (1, 1, 1, 1, 1) = y_2$;

$y_2M^t = (1, 2, 2, 4, 4, 0, 4, 7) \to (1, 1, 1, 1, 1, 0, 1, 1) = x_2$;

$x_2 M = (7, 5, 2, 3, 1) \to (1, 1, 1, 1, 1) = y_3 (= y_2)$ ;

$y_2M^t = (1, 2, 2, 4, 4, 0, 4, 7) \to (1, 1, 1, 1, 1, 0, 1, 1) = x_3$ $(= x_2)$.

Thus the resultant is a realized limit point pair given by $\{(1, 1, 1, 1, 1, 1, 0, 1, 1), (1, 1, 1, 1, 1)\}$.

Hence the same state vector has different resultants on M.

***Example 2.8:*** Let

$$M = \begin{bmatrix} 3 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 6 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 7 & 0 & 0 \\ 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 9 \\ 0 & 4 & 0 & 0 & 0 & 0 \end{bmatrix}$$

be the MOD rectangular real matrix operator with entries from $Z_{10}$.

Let D = $\{(a_1, a_2, \ldots, a_9) \mid a_i \in Z_{10}, 1 \leq i \leq 9\}$ be the domain space of state vectors.

Let R = $\{(b_1, b_2, \ldots, b_6) \mid b_i \in Z_{10}, 1 \leq i \leq 6\}$ be the range space of state vectors.

$D_S = \{(a_1, a_2, \ldots, a_9) \mid a_i \in \{0, 1\}, 1 \leq i \leq 9\}$ be the domain space of special state vectors.

$R_S = \{(b_1, b_2, \ldots, b_6) \mid b_i \in \{0, 1\}, 1 \leq i \leq 6\}$ be the range space of special state vectors.

We study of effect of x = (1, 1, 0, 0, 0, 0, 0, 0, 0) $\in$ D on the MOD rectangular operator M.

$xM = (3, 0, 0, 0, 0, 6) = y_1;$

$y_1M^t = (9, 8, 0, 0, 0, 0, 4, 4, 0) = x_1;$

$x_1M = (8, 0, 0, 0, 0, 4) = y_2;$

$y_2M^t = (4, 4, 0, 0, 0, 0, 4, 6, 0) = x_2;$

$x_2M = (4, 0, 0, 0, 0, 8) = y_3;$

$y_3M^t = (2, 8, 0, 0, 0, 0, 2, 6, 0) = x_3;$

$x_3M = (2, 0, 0, 0, 0, 2) = y_4;$

$y_4M^t = (6, 2, 0, 0, 0, 0, 6, 8, 0) = x_4;$

$x_4M = (6, 0, 0, 0, 0, 4) = y_5;$

$y_5M^t = (8, 4, 0, 0, 0, 0, 8, 6, 0) = x_5;$

$x_5M = (8, 0, 0, 0, 0, 8) = y_6;$

$y_6M^t = (4, 8, 0, 0, 0, 0, 4, 2, 0) = x_6;$

$x_6M = (4, 0, 0, 0, 0, 6) = y_7;$

$y_7M^t = (2, 6, 0, 0, 0, 0, 2, 4, 0) = x_7;$

$x_7M = (2, 0, 0, 0, 0, 2) = y_8\ (= y_4)\ ;$

$y_8M^t = y_4M = (6, 2, 0, 0, 0, 0, 6, 8, 0) = x_8\ \ (= x_4).$

Thus the resultant of x = (1, 1, 0, 0, 0, 0, 0, 0, 0) is a realized limit cycle pair given by
{(6, 2, 0, 0, 0, 0, 6, 8, 0), (2, 0, 0, 0, 0, 2)}.

Consider the same x = (1, 1, 0, 0, 0, 0, 0, 0, 0) to be an element of $D_S$.

The effect of x on M.

$xM = (3, 0, 0, 0, 0, 6) \to (1, 0, 0, 0, 0, 1) = y_1;$

$y_1M^t = (3, 6, 0, 0, 0, 0, 8, 9, 0) \to (1, 1, 0, 0, 0, 0, 1, 1, 0) = x_1;$

$x_1M = (1, 0, 0, 0, 0, 5) \to (1, 0, 0, 0, 0, 1) = \ y_2\ (= y_1).$

Thus the resultant is a realized fixed point pair just after one iteration given by {(1, 1, 0, 0, 0, 0, 1, 1, 0), (1, 0, 0, 0, 0, 1)}.

Hence we see that in general a state vector in $D \cap D_S$ realized as a state vector of D gives a realized limit cycle pair where as the same state vector in $D_S \cap D$ realized as a state vector in $D_S$ results in a realized fixed point pair just after one iterations.

So a researcher can work with the element in D or as in $D_S$ as per need.

This is one of the flexibilities enjoyed in the MOD-real rectangular matrix operators. Further this cannot be achieved in general if we take a usual matrix operators from the integers Z or rationals Q or reals R or C the complex numbers.

Thus using MOD real rectangular matrix operators happens to be an interesting and innovative work which can be used in mathematical models by researchers.

Let y = (1, 0, 0, 1, 0, 0) $\in$ R $\cap$ $R_S$ to find the effect of y on M realized as a state vector from R.

$yM^t = (3, 0, 0, 0, 0, 7, 8, 0, 0) = x_1;$

$x_1M = (3, 0, 0, 9, 0, 0) = y_1;$

$y_1M^t = (9, 0, 0, 0, 0, 3, 4, 0, 0) = x_2;$

$x_2M = (9, 0, 0, 1, 0, 0) = y_2;$

$y_2M^t = (7, 0, 0, 0, 0, 7, 8, 0, 0) = x_3;$

$x_3M = (5, 0, 0, 9, 0, 0) = y_2;$

$y_2M^t = (5, 0, 0, 0, 0, 3, 0, 0, 0) = x_4;$

$x_4M = (5, 0, 0, 1, 0, 0) = y_3;$

$y_3M^t = (5, 0, 0, 0, 0, 7, 0, 0, 0) = x_5;$

$x_5M = (5, 0, 0, 9, 0, 0) = y_4;$

$y_4M^t = (5, 0, 0, 0, 0, 3, 0, 0, 0) = x_6;$

$x_6M = (5, 0, 0, 1, 0, 0) = y_5;$

$y_5M^t = (5, 0, 0, 0, 0, 7, 0, 0, 0) = x_7 \ (=x_5).$

Thus the resultant a pair is a realized limit cycle pair given by

$\{(5, 0, 0, 0, 0, 7, 0, 0, 0), (5, 0, 0, 9, 0, 0)\}$ and

$\{(5, 0, 0, 0, 0, 3, 0, 0, 0), (5, 0, 0, 1, 0, 0)\}$.

Now we find the effect of $y = (1, 0, 0, 1, 0, 0)$ realized as a vector from $R_S$ on M.

$$yM^t = (3, 0, 0, 0, 0, 7, 8, 0, 0) \to (1, 0, 0, 0, 0, 1, 1, 0, 0) = x_1;$$

$$x_1M = (1, 0, 0, 1, 0, 0) = y_1 \ (= y).$$

Thus the resultant is a classical fixed pair. Hence the difference between the states vectors behavior when realized as element of $D_S$ and $R_S$ and that of elements in R and S is distinctly different.

We give some more examples before we proceed work in a different angle.

***Example 2.9:*** Let

$$M = \begin{bmatrix} 3 & 2 & 0 & 0 & 0 & 0 \\ 1 & 6 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 5 & 0 & 0 \\ 0 & 0 & 2 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

be the MOD matrix rectangular matrix with entries from $Z_8$.

$R = \{(x_1, x_2, x_3, x_4, x_5, x_6)$ where $x_i \in Z_8, 1 \leq i \leq 6\}$ and

$D = \{(y_1, y_2, y_3, y_4, y_5) \mid y_i \in Z_8, 1 \leq i \leq 5\}$

$R_S = \{(a_1, a_2, \ldots, a_6) \mid a_i \in \{0, 1\}, 1 \leq i \leq 6\}$ and

$D_S = \{(b_1, b_2, \ldots, b_5) \mid b_i \in \{0, 1\}, 1 \leq i \leq 5\}$ be the collection of all state vectors associated with M.

Let x = (3, 1, 2, 0, 0) ∈ D.

To find the effect of x on M.

$xM = (2, 4, 2, 2, 0, 0) = y_1$; $y_1M^t = (6, 2, 4, 2, 0) = x_1$;

$x_1M = (4, 0, 0, 2, 0, 0) = y_2$; $y_2M^t = (4, 4, 4, 6, 0) = x_2$;

$x_2M = (0, 0, 0, 6, 0, 0) = y_3$; $y_3M^t = (0, 0, 6, 2, 0) = x_3$;

$x_3M = (0, 0, 2, 2, 0, 0) = y_4$; $y_4 M^t = (0, 0, 4, 4, 0) = x_4$;

$x_4M = (0, 0, 4, 0, 0, 0) = y_5$; $y_5M^t = (0, 0, 4, 0, 0) = x_5$;

$x_5M = (0, 0, 4, 4, 0, 0) = y_6$; $y_6M^t = (0, 0, 0, 4, 0) = x_6$;

$x_6M = (0, 0, 0, 4, 0, 0) = y_7$; $y_7M^t = (0, 0, 2, 4, 0) = x_7$;

$x_7M = (0, 0, 2, 6, 0, 0) = y_8$; $y_8M^t = (0, 0, 0, 2, 0) = x_8$;

$x_8 M = (0, 0, 4, 6, 0, 0 ) = y_9$; $y_9M^t = (0, 0, 2, 2, 0) = x_9$;

$x_9M = (0, 0, 6, 0, 0, 0) = y_{10}$; $y_{10}M^t = (0, 0, 4, 2, 0) = x_{10}$;

$x_{10}M = (0, 0, 0, 2, 0, 0) = y_{11}$; $y_{11}M^t = (0, 0, 2, 6, 0) = x_{11}$;

$x_{11}M = (0, 0, 6, 4, 0, 0) = y_{12}$; $y_{12}M^t = (0, 0, 6, 0, 0) = x_{12}$;

$x_{12}M = (0, 0, 4, 6, 0, 0) = y_{13} = y_9$;

$y_9M^t = (0, 0, 2, 2, 0) = x_{13}$ $(= x_9)$.

Thus we see the resultant is a realized limit cycle pair given by {(0, 0, 2, 2, 0), (0, 0, 6, 0, 0, 0)}.

Next we consider the effect of $y = (0, 1, 2, 0, 0, 0) \in R$ on M.

$yM^t = (2, 6, 2, 4, 0) = x_1$; $x_1M = (4, 0, 2, 6, 0, 0) = y_1$;

$y_1M^t = (4, 4, 0, 6, 0) = x_2$; $x_2M = (0, 0, 4, 2, 0, 0) = y_2$;

$y_2M^t = (0, 0, 6, 6, 0) = x_3$; $x_3M = (0, 0, 4, 0, 0, 0) = y_3$;

$y_3M^t = (0, 0, 4, 6, 0) = x_4$; $x_4M = (0, 0, 0, 6, 0, 0) = y_4$;

$y_4M^t = (0, 0, 4, 4, 0) = x_5$;

$x_5M = (0, 0, 4, 0, 0, 0) = y_5$ $(= y_3)$.

Hence the resultant is a realized limit cycle pair given by $\{(0, 0, 4, 6, 0), (0, 0, 4, 0, 0, 0)\}$.

Now we find the effect of $x = (1, 1, 1, 0, 0) \in D_S$ on the MOD rectangular matrix operator M.

$xM = (4, 0, 1, 5, 0, 0) \to (1, 0, 1, 1, 0, 0) = y_1$;

$y_1M^t = (3, 1, 6, 5, 0) \to (1, 1, 1, 1, 0) = x_1$;

$x_1\ M = (4, 0, 3, 0, 0, 0) \to (1, 0, 1, 0, 0, 0) = y_2$;

$y_2M^t = (3, 1, 1, 2, 0) \to (1, 1, 1, 1, 0) = x_2$ $(= x_1)$.

Thus the resultant is a realized fixed point pair given by $\{(1, 1, 1, 1, 0)\ (1, 0, 1, 0, 0, 0)\}$.

Consider $x = (1, 1, 0, 0, 0) \in D_S$

To find the effect of x on M is as follows:

$xM = (4, 0, 0, 0, 0, 0) \to (1, 0, 0, 0, 0, 0) = y_1$;

$y_1M^t = (3, 1, 0, 0, 0) \to (1, 1, 0, 0, 0) = x_1$;

$x_1M = (4, 0, 0, 0, 0, 0) \rightarrow (1, 0, 0, 0, 0, 0)$

$y_2M^t = (3, 1, 0, 0, 0) \rightarrow (1, 1, 0, 0, 0) = (x_1)$.

Thus the resultant is a realized fixed point pair given by {(1, 1, 0, 0, 0) (1, 0, 0, 0, 0, 0)}.

Let $y = (0, 0, 1, 0, 0, 0) \in R_S$.

To find the effect of y on M is as follows:

$yM^t = (0, 0, 1, 2, 0) \rightarrow (0, 0, 1, 1, 0) = x_1$;

$x_1M = (0, 0, 3, 0, 0, 0) \rightarrow (0, 0, 1, 0, 0, 0) = y_1 ( = y)$.

Thus the resultant is the classical fixed point given by
{(0, 0, 1, 0, 0, 0), (0, 0, 1, 1, 0)} or
{(0, 0, 1, 1, 0), (0, 0, 1, 0, 0, 0)}.

Now having seen various types of MOD rectangular matrix operators we see we can have different type of operations on them which will developed and described in the following.

All matrices used in this chapter are MOD real matrix operators.

In order to develop the notion of face value ordering on MOD-integers we need the following.

We know face value ordering notion is used in pack of cards. We give the face value in case of modulo integers {0, 1, 2, 3, 4, …, n–1} = $Z_n$. The face value ordering is carried out in this manner. Let $2, 3 \in Z_n$ we know 3 is bigger than 2. Thus the face value of 3 is bigger than that of 2.

So we can for any $x, y \in Z_n$ we can say bigger of {x, y} and similarly smaller of {x,y}.

However in case of $Z_5 = \{0, 1, 2, 3, 4\}$ the face value ordering of $Z_5$ is $0 < 1 < 2 < 3 < 4$.

But certainly this face value ordering is not compatible with respect to the algebraic operations of + and ×.

So the ordering cannot be done hence we are forced to give the face value ordering as in case of cards.

Now to make the face value ordering more technical we can define the bigger of two elements by max and smaller of two elements by min. At this juncture we wish to keep on record that the face value order in $Z_n$ can never be compatible with respect to the usual operations of + and ×.

Further it is not the usual order. We can call this face value order also as label value order, so in this book we call it is face value order or label value order synonymously.

Further the term bigger as maximum and smaller as minimum respectively.

We will illustrate this by some examples.

***Example 2.10:*** Let $Z_{12} = \{0, 1, 2, …, 11\}$ be the modulo 12 integers.

bigger $\{7, 3\} = \max \{7, 3\} = 7$,
smaller $\{7, 3\} = \min \{7, 3\} = 3$,
bigger $\{4, 4\} = \max \{4, 4\} = 4$,
and smaller $\{4, 4\} = \min \{4, 4\} = 4$.

This under face value ordering is denoted by $7 > 3$, 7 is bigger than 3.

$3 < 7$, 3 is less than 7
$4 = 4$ is equal to 4.

So bigger $\{4, 4\} = 4$ and smaller $\{4, 4\} = 4$.

Certainly the face value ordering is not compatible with respect to + and ×. For 7 > 3 is the face value ordering.

Add 5 to both 7 + 5 > 3 + 5 = 12 > 8 that is 0 > 8 which will imply bigger of 0 and 8 is zero an absurdity.

Hence we see the face value ordering is not in general compatible with respect to +.

Consider 7 > 3 multiply by two $7 \times 2 > 3 \times 2$ that is 2 > 6 ($7 \times 2 \equiv 2 \pmod{12}$). This is not possible.

So in general the face value order is not compatible with respect to × also.

That is why we made it very clear that we gave a new type of ordering called face value ordering on MOD integers.

***Example 2.11:*** Let $Z_{19}$ be the integers modulo 19. $Z_{19} = \{0, 1, 2, \ldots, 18\}$ $18 > 17 > 16 > \ldots. > 2 > 1 > 0$ and $0 < 1 < 2 < 3 < \ldots < 17 < 18$ . Thus we have only face ordering on $Z_{19}$.

Clearly one cannot have usual ordering.

Let $x, y \in Z_{19}$. $x < y$ ($y > x$).

Take $\{10, 6\} \in Z_{19}$; $10 > 6$ or $6 < 10$.
Clearly is $10 + 10 > 6 + 6$
$20 \pmod{19} > 12$

1 > 12 which is an impossibility so even addition of a > b with itself is not compatible.

Consider for a > b
$a \times a > b \times b$ gives
$10 \times 10 > 6 \times 6$
$100 \pmod{19} > 36 \pmod{19}$;

5 > 17 which is again an impossibility.

Thus both + and × are in general not compatible with face value ordering.

We will be using the notion of face value ordering define max (bigger) and min (smaller) operations on $Z_n$.

Suppose we have two MOD rectangular real matrices MOD n with entries from $Z_n$; where M is a MOD m × t matrix and N is a MOD t × n matrix we get the resultant MOD m × n real matrix by the following method.

Further we can get MOD m × n real rectangular matrix.

***Example 2.12:*** Let

$$A = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 5 & 7 \\ 0 & 2 & 3 \\ 4 & 0 & 6 \\ 5 & 3 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 3 & 0 & 1 & 7 & 1 & 4 \\ 7 & 1 & 2 & 0 & 2 & 0 \\ 3 & 2 & 0 & 6 & 3 & 1 \end{bmatrix}$$

be any two MOD rectangular real matrices with entries from Z8.

$$A \times B = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 5 & 7 \\ 0 & 2 & 3 \\ 4 & 0 & 6 \\ 5 & 3 & 0 \end{bmatrix}_{5\times3} \times \begin{bmatrix} 3 & 0 & 1 & 7 & 1 & 4 \\ 7 & 1 & 2 & 0 & 2 & 0 \\ 3 & 2 & 0 & 6 & 3 & 1 \end{bmatrix}_{3\times6}$$

$$= \begin{bmatrix} 7 & 2 & 7 & 5 & 7 & 4 \\ 5 & 3 & 3 & 1 & 0 & 3 \\ 7 & 0 & 4 & 2 & 5 & 3 \\ 6 & 4 & 4 & 0 & 6 & 6 \\ 4 & 3 & 3 & 3 & 3 & 2 \end{bmatrix}_{5\times 6}$$

is again a MOD rectangular real matrix with entries from $Z_8$.

Thus we can use this technique in the application of mathematical models.

Further we can define for two compatible MOD real rectangular matrices the notion of (max-min) composition.

We define P o Q.

$$= \begin{bmatrix} 3 & 2 & 0 \\ 1 & 5 & 7 \\ 0 & 2 & 3 \\ 4 & 0 & 6 \\ 5 & 3 & 0 \end{bmatrix} \text{ o } \begin{bmatrix} 3 & 0 & 1 & 7 & 1 & 4 \\ 7 & 1 & 2 & 0 & 2 & 0 \\ 3 & 2 & 0 & 6 & 3 & 1 \end{bmatrix}$$

= max min $\{p_{ij} \text{ o } q_{jk}]$ by the following way.

$$P \text{ o } Q = \begin{bmatrix} 3 & 1 & 2 & 3 & 2 & 3 \\ 5 & 2 & 2 & 6 & 3 & 1 \\ 3 & 2 & 2 & 3 & 0 & 1 \\ 3 & 2 & 1 & 6 & 3 & 4 \\ 3 & 1 & 2 & 5 & 3 & 4 \end{bmatrix}$$

$= R \neq P \times Q$ where $P = A$ and $Q = B$.

On similar lines we can find $[r_{ik}] = \min\{\max(p_{ij}, q_{jk})\}$.

We find $\min\{\max(p_{ij}, q_{jk})\}$ for this P and Q.

$$P \text{ o } Q = \begin{bmatrix} 3 & 2 & 0 \\ 1 & 5 & 7 \\ 0 & 2 & 3 \\ 4 & 0 & 6 \\ 5 & 3 & 0 \end{bmatrix} \text{ o } \begin{bmatrix} 3 & 0 & 1 & 7 & 1 & 4 \\ 7 & 1 & 2 & 0 & 2 & 0 \\ 3 & 2 & 0 & 6 & 3 & 1 \end{bmatrix}$$

$$= \min\{\max(p_{ij}, q_{jk})\} = [r_{ik}]$$

$$P \text{ o } Q = \begin{bmatrix} 3 & 2 & 0 & 2 & 2 & 1 \\ 3 & 1 & 1 & 5 & 1 & 4 \\ 3 & 0 & 1 & 2 & 1 & 2 \\ 4 & 1 & 2 & 0 & 2 & 0 \\ 3 & 2 & 0 & 3 & 3 & 1 \end{bmatrix}.$$

Clearly this P o Q operator using min-max is different from max-min and the usual product.

We see all the three operations yield different values.

However here we cannot talk of fixed points our only motivation is to show that such type operations on MOD real matrix operator can yield different MOD matrix operators.

We will illustrate this situation by more examples.

***Example 2.13:*** Let

$$M = \begin{bmatrix} 3 & 2 & 1 & 0 & 5 & 6 \\ 4 & 0 & 9 & 2 & 0 & 1 \\ 3 & 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 5 & 0 & 7 & 1 \\ 2 & 0 & 0 & 4 & 0 & 0 \end{bmatrix} \text{ and } N = \begin{bmatrix} 2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 1 \\ 4 & 0 & 9 & 8 \\ 5 & 7 & 0 & 1 \\ 6 & 0 & 2 & 0 \\ 7 & 1 & 0 & 5 \end{bmatrix}$$

be two MOD rectangular matrix operators with entries from $Z_{10}$.

To find $S = M \times N$, min-max of M o N and max-min M o N.

Consider

$$S = \begin{bmatrix} 3 & 2 & 1 & 0 & 5 & 6 \\ 4 & 0 & 9 & 2 & 0 & 1 \\ 3 & 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 5 & 0 & 7 & 1 \\ 2 & 0 & 0 & 4 & 0 & 0 \end{bmatrix} \begin{bmatrix} 2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 1 \\ 4 & 0 & 9 & 8 \\ 5 & 7 & 0 & 1 \\ 6 & 0 & 2 & 0 \\ 7 & 1 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 8 & 8 & 2 & 6 \\ 1 & 5 & 5 & 7 \\ 6 & 8 & 3 & 8 \\ 2 & 2 & 9 & 1 \\ 4 & 8 & 2 & 6 \end{bmatrix}_{5\times 4}$$

is a $5 \times 4$ MOD-rectangular real matrix operator with entries from $Z_{10}$.

Now we use min-max operation on M and N is as following.

$$M \text{ o } N = \begin{bmatrix} 3 & 2 & 1 & 0 & 5 & 6 \\ 4 & 0 & 9 & 2 & 0 & 1 \\ 3 & 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 5 & 0 & 7 & 1 \\ 2 & 0 & 0 & 4 & 0 & 0 \end{bmatrix} \text{ o } \begin{bmatrix} 2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 1 \\ 4 & 0 & 9 & 8 \\ 5 & 7 & 0 & 1 \\ 6 & 0 & 2 & 0 \\ 7 & 1 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 1 & 0 & 1 \\ 3 & 0 & 0 & 1 \\ 3 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \\ 2 & 0 & 0 & 1 \end{bmatrix}_{5\times4}$$

is a MOD matrix rectangular operator with entries from $Z_{10}$.

Let us now use the max-min operation to find M o N.

$$M \text{ o } N = \begin{bmatrix} 3 & 2 & 1 & 0 & 5 & 6 \\ 4 & 0 & 9 & 2 & 0 & 1 \\ 3 & 1 & 0 & 1 & 2 & 0 \\ 0 & 1 & 5 & 0 & 7 & 1 \\ 2 & 0 & 0 & 4 & 0 & 0 \end{bmatrix} \text{ o } \begin{bmatrix} 2 & 0 & 1 & 2 \\ 3 & 1 & 0 & 1 \\ 4 & 0 & 9 & 8 \\ 5 & 7 & 0 & 1 \\ 6 & 0 & 2 & 0 \\ 7 & 1 & 0 & 5 \end{bmatrix}$$

$$= \begin{bmatrix} 6 & 1 & 2 & 5 \\ 4 & 2 & 9 & 8 \\ 2 & 1 & 2 & 2 \\ 6 & 5 & 5 & 5 \\ 4 & 4 & 1 & 2 \end{bmatrix}$$

is the MOD rectangular matrix operator. All these three MOD matrices are distinct.

Now we instead of using two MOD matrices we make use of column state vector and the MOD-rectangular matrices.

Next we proceed onto give operations on MOD-rectangular matrices in a different way.

Let

$$M = \begin{bmatrix} m_{11} & \dots & m_{1n} \\ m_{21} & \dots & m_{2n} \\ \vdots & & \vdots \\ m_{m1} & \dots & m_{mn} \end{bmatrix}$$

be a m × n rectangular MOD real matrix operator with entries from $Z_8$; $2 \leq s < \infty$.

$$\text{Let } D = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} \middle| \; a_i \in Z_S; 1 \leq i \leq n \right\} \text{ and}$$

$$R = \left\{ \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix} \middle| \; b_j \in Z_S; 1 \leq j \leq m \right\}$$

be the associated column vectors of the MOD operator matrix M.

Let $X \in D$ then $MX = Y \in R$.

Consider $M^tY = X_1 \in D$

Find $MX_1 = Y_1$ and so on.

We proceed onto work till we arrive at a realized fixed point or a limit cycle.

Certainly we will arrive at a realized fixed point or a realized limit cycle after a finite number of iterations as both the sets R and D are of finite cardinality.

We will describe this situation by some examples.

***Example 2.14:*** Let

$$M = \begin{bmatrix} 2 & 1 & 0 & 3 \\ 0 & 3 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 2 & 0 \end{bmatrix}$$

be the MOD matrix operator with entries from $Z_4$.

$$R = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \middle| \; a_i \in Z_4; 1 \leq i \leq 6 \right\} \text{ and}$$

$$D = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \middle| \; a_i \in Z_4; 1 \leq i \leq 4 \right\}$$

be the range vectors associated with M.

Let

$$X = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix} \in D.$$

To find the effect of X on M.

$$MX = \begin{bmatrix} 2 & 1 & 0 & 3 \\ 0 & 3 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 2 & 0 & 3 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 2 & 0 \end{bmatrix}_{6\times 4} \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \end{bmatrix}_{4\times 1} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \\ 0 \\ 3 \end{bmatrix} = y_1 \in R.$$

We find

$$M^t y_1 = \begin{bmatrix} 2 & 0 & 1 & 2 & 0 & 3 \\ 1 & 3 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 3 & 0 & 2 \\ 3 & 0 & 1 & 0 & 2 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \\ 1 \\ 0 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} = X_1 \in D.$$

$$Mx_1 = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 2 \\ 2 \\ 0 \end{bmatrix} = y_2 \in R; \qquad M^t\, y_2 = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 2 \end{bmatrix} = x_2 \in D$$

$$Mx_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ 0 \\ 1 \end{bmatrix} = y_3 \in R; \qquad M^t y_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = X_3;$$

$$MX_3 = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 0 \\ 3 \\ 0 \end{bmatrix} = y_4; \qquad M^t y_4 = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 0 \end{bmatrix} = X_4;$$

$$MX_4 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} = y_5; \qquad M^t y_5 = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 2 \end{bmatrix} = X_5;$$

$$MX_5 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \\ 2 \\ 2 \end{bmatrix} = y_6; \qquad M^t y_6 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 2 \end{bmatrix} = X_6;$$

$$MX_6 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 3 \\ 0 \\ 0 \end{bmatrix} = y_7; \qquad M^t y_7 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix} = X_7;$$

$$MX_7 = \begin{bmatrix} 3 \\ 3 \\ 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} = y_8; \qquad M^t y_8 = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 0 \end{bmatrix} = X_8;$$

$$MX_8 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} = y_9; \qquad M^t y_9 = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 2 \end{bmatrix} = X_9;$$

$$MX_9 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 3 \\ 0 \end{bmatrix} = y_{10}; \qquad M^t y_{10} = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 2 \end{bmatrix} = X_{10};$$

$$MX_{10} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 3 \end{bmatrix} = y_{11}; \qquad M^t y_{11} = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 3 \end{bmatrix} = X_{11};$$

$$MX_{11} = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 2 \\ 2 \\ 0 \end{bmatrix} = y_{12}; \qquad M^t y_{12} = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 1 \end{bmatrix} = X_{12};$$

$$MX_{12} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 2 \\ 3 \\ 3 \end{bmatrix} = y_{13}; \qquad M^t y_{13} = \begin{bmatrix} 2 \\ 3 \\ 3 \\ 1 \end{bmatrix} = X_{13};$$

$$MX_{13} = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 1 \\ 3 \\ 0 \end{bmatrix} = y_{14}; \qquad M^t y_{14} = \begin{bmatrix} 0 \\ 3 \\ 3 \\ 2 \end{bmatrix} = X_{14};$$

$$MX_{14} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 3 \\ 2 \end{bmatrix} = y_{15}.$$

We are sure after a finite number of iterations certainly we will arrive at a realized fixed pont or a realized limit cycle.

Let $Y = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in R$; to find the effect of Y on M.

$$M^tY = \begin{bmatrix} 0 \\ 3 \\ 1 \\ 0 \end{bmatrix} = x_1 \in D; \qquad Mx_1 = \begin{bmatrix} 3 \\ 2 \\ 3 \\ 3 \\ 3 \\ 2 \end{bmatrix} = Y_1;$$

$$M^tY_1 = \begin{bmatrix} 1 \\ 3 \\ 3 \\ 2 \end{bmatrix} = x_2; \qquad Mx_{12} = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 3 \\ 3 \\ 1 \end{bmatrix} = Y_2;$$

$$M^tY_2 = \begin{bmatrix} 1 \\ 0 \\ 3 \\ 3 \end{bmatrix} = x_3; \qquad Mx_3 = \begin{bmatrix} 3 \\ 3 \\ 0 \\ 0 \\ 2 \\ 1 \end{bmatrix} = Y_3;$$

$$M^tY_3 = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 1 \end{bmatrix} = x_4; \qquad Mx_4 = \begin{bmatrix} 3 \\ 3 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = Y_4;$$

$$M^tY_4 = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 1 \end{bmatrix} = x_4; \qquad Mx_4 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \\ 2 \\ 1 \end{bmatrix} = Y_5;$$

$$M^tY_5 = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 3 \end{bmatrix} = x_5; \qquad Mx_5 = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 2 \\ 1 \\ 3 \end{bmatrix} = Y_6;$$

$$M^tY_6 = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 1 \end{bmatrix} = x_6; \qquad Mx_6 = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 2 \\ 1 \\ 2 \end{bmatrix} = Y_7;$$

$$M^tY_7 = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 2 \end{bmatrix} = x_7; \qquad Mx_7 = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 1 \\ 1 \\ 2 \end{bmatrix} = Y_8;$$

$$M^tY_8 = \begin{bmatrix} 3 \\ 2 \\ 1 \\ 1 \end{bmatrix} = x_8; \qquad Mx_8 = \begin{bmatrix} 3 \\ 3 \\ 2 \\ 1 \\ 0 \\ 3 \end{bmatrix} = Y_9;$$

$$M^tY_9 = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 3 \end{bmatrix} = x_9; \qquad Mx_9 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} = Y_{10};$$

$$M^tY_{10} = \begin{bmatrix} 3 \\ 2 \\ 2 \\ 0 \end{bmatrix} = x_{10}; \qquad Mx_{10} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 2 \\ 1 \end{bmatrix} = Y_{11};$$

$$M^tY_{11} = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 1 \end{bmatrix} = x_{11}; \qquad Mx_{11} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} = Y_{12};$$

$$M^tY_{12} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = x_{12}; \qquad Mx_{12} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \\ 0 \\ 2 \end{bmatrix} = Y_{13};$$

$$M^tY_{13} = \begin{bmatrix} 0 \\ 3 \\ 2 \\ 0 \end{bmatrix} = x_{13}; \qquad Mx_{13} = \begin{bmatrix} 3 \\ 3 \\ 3 \\ 2 \\ 3 \\ 0 \end{bmatrix} = Y_{14};$$

$$M^tY_{14} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \end{bmatrix} = x_{14}; \qquad Mx_{14} = \begin{bmatrix} 2 \\ 3 \\ 1 \\ 1 \\ 2 \\ 1 \end{bmatrix} = Y_{15};$$

$$M^tY_{15} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 3 \end{bmatrix} = x_{15}; \qquad Mx_{15} = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 0 \\ 3 \\ 2 \end{bmatrix} = Y_{16};$$

$$M^tY_{16} = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 2 \end{bmatrix} = x_{16}; \qquad Mx_{16} = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 1 \\ 0 \\ 2 \end{bmatrix} = Y_{17};$$

$$M^tY_{17} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} = x_{17}; \qquad Mx_{17} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} = Y_{18};$$

$$M^tY_{18} = \begin{bmatrix} 3 \\ 2 \\ 3 \\ 2 \end{bmatrix} = x_{18}; \qquad Mx_{18} = \begin{bmatrix} 2 \\ 1 \\ 3 \\ 3 \\ 2 \\ 3 \end{bmatrix} = Y_{19};$$

$$M^tY_{19} = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 1 \end{bmatrix} = x_{19}; \qquad Mx_{19} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 0 \\ 0 \\ 2 \end{bmatrix} = Y_{20};$$

$$M^tY_{20} = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} = x_{20}; \qquad Mx_{20} = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 0 \\ 0 \\ 3 \end{bmatrix} = Y_{21};$$

$$M^tY_{21} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 3 \end{bmatrix} = x_{21}; \qquad Mx_{21} = \begin{bmatrix} 2 \\ 3 \\ 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} = Y_{22};$$

$$M^tY_{22} = \begin{bmatrix} 0 \\ 2 \\ 3 \\ 0 \end{bmatrix} = x_{22}; \qquad Mx_{22} = \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \\ 2 \\ 2 \end{bmatrix} = Y_{23};$$

$$M^tY_{23} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = x_{23}; \qquad Mx_{23} = \begin{bmatrix} 1 \\ 3 \\ 3 \\ 0 \\ 1 \\ 2 \end{bmatrix} = Y_{24};$$

$$M^tY_{24} = \begin{bmatrix} 3 \\ 2 \\ 3 \\ 0 \end{bmatrix} = x_{24}; \qquad Mx_{24} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 2 \\ 3 \end{bmatrix} = Y_{25};$$

$$M^tY_{25} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 1 \end{bmatrix} = x_{25}; \qquad Mx_{25} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \\ 0 \\ 2 \end{bmatrix} = Y_{26};$$

$$M^tY_{26} = \begin{bmatrix} 1 \\ 2 \\ 2 \\ 0 \end{bmatrix} = x_{26}; \qquad Mx_{26} = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \\ 2 \\ 3 \end{bmatrix} = Y_{27};$$

$$M^tY_{27} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix} = x_{27}; \qquad Mx_{27} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \\ 3 \\ 0 \end{bmatrix} = Y_{28};$$

$$M^tY_{28} = \begin{bmatrix} 0 \\ 0 \\ 3 \\ 0 \end{bmatrix} = x_{28}; \qquad Mx_{28} = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \\ 0 \\ 2 \end{bmatrix} = Y_{29};$$

$$M^tY_{29} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} = x_{29} \; (=x_{17}).$$

Thus we see the resultant is a realized limit cycle point is a pair given by

$$\left\{ \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \\ 1 \\ 0 \end{bmatrix} \right\}.$$

Thus we see the resultant of $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in$ R is a realized limit cycle pair.

Let $x = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \in D$. To find the effect of x on M.

$Mx = \begin{bmatrix} 3 \\ 0 \\ 1 \\ 0 \\ 2 \\ 0 \end{bmatrix} = y_1 \in R$; $\qquad M^t y_1 = \begin{bmatrix} 3 \\ 2 \\ 0 \\ 2 \end{bmatrix} = x_1$;

$Mx_1 = \begin{bmatrix} 2 \\ 2 \\ 3 \\ 2 \\ 2 \\ 1 \end{bmatrix} = y_2$; $\qquad M^t y_2 = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 1 \end{bmatrix} = x_2$;

$$Mx_2 = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 2 \\ 1 \\ 2 \end{bmatrix} = y_3; \qquad M^t y_3 = \begin{bmatrix} 0 \\ 2 \\ 1 \\ 2 \end{bmatrix} = x_3;$$

$$Mx_3 = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 3 \\ 3 \\ 2 \end{bmatrix} = y_3; \qquad M^t y_3 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} = x_4;$$

$$Mx_4 = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} = y_4; \qquad M^t y_4 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \end{bmatrix} = x_5;$$

$$Mx_5 = \begin{bmatrix} 0 \\ 2 \\ 2 \\ 2 \\ 0 \\ 2 \end{bmatrix} = y_5; \qquad M^t y_5 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} = x_6;$$

$$Mx_6 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = y_6; \qquad M^t y_6 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = x_7;$$

$$Mx_7 = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} = y_7; \qquad M^t y_7 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \end{bmatrix} = x_8;$$

$$Mx_8 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} = y_8; \qquad M^t y_8 = \begin{bmatrix} 0 \\ 0 \\ 2 \\ 0 \end{bmatrix} = x_9;$$

$$Mx_9 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} = y_9; \qquad M^t y_9 = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} = x_{10} \ (= x_4) .$$

Thus the resultant is a realized limit cycle pair given by

$$\left\{\begin{bmatrix}0\\2\\0\\0\end{bmatrix}, \begin{bmatrix}2\\2\\2\\0\\2\\0\end{bmatrix}\right\}.$$

Now having seen the example of column state vectors associated with MOD real rectangular matrices using the usual product ×.

We use the min-max and max-min operations and arrive at a result.

We will illustrate this situation by some examples.

***Example 2.15:*** Let

$$M = \begin{bmatrix} 3 & 2 & 0 & 1 & 4 & 5 & 1 \\ 0 & 1 & 2 & 0 & 3 & 0 & 7 \\ 1 & 0 & 6 & 2 & 0 & 1 & 0 \\ 0 & 5 & 0 & 1 & 7 & 0 & 3 \end{bmatrix}$$

be the MOD rectangular real matrix operator with entries.

$$D = \left\{\begin{bmatrix}a_1\\a_2\\\vdots\\a_7\end{bmatrix} \middle| a_i \in Z_8;\ 1 \le i \le 7\right\} \text{ and}$$

$$R = \left\{ \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \middle| \; b_i \in Z_8;\ 1 \leq i \leq 4 \right\}$$

be the domain and column matrix with entries from $Z_8$ associated with M.

$$\text{Let } x = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in D.$$

To find the effect of x on M.

$$Mx = \begin{bmatrix} 6 \\ 0 \\ 2 \\ 0 \end{bmatrix} = y_1; \qquad M^t y_1 = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 2 \\ 0 \\ 0 \\ 6 \end{bmatrix} = x_1;$$

$$Mx_1 = \begin{bmatrix} 4 \\ 6 \\ 0 \\ 0 \end{bmatrix} = y_1; \qquad M^t y_1 = \begin{bmatrix} 4 \\ 6 \\ 4 \\ 4 \\ 0 \\ 4 \\ 6 \end{bmatrix} = x_2$$

$$Mx_2 = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 4 \end{bmatrix} = y_3; \qquad M^t y_3 = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 2 \\ 4 \\ 6 \\ 6 \end{bmatrix} = x_3;$$

$$Mx_3 = \begin{bmatrix} 0 \\ 6 \\ 0 \\ 0 \end{bmatrix} = y_4; \qquad M^t y_4 = \begin{bmatrix} 0 \\ 6 \\ 4 \\ 0 \\ 4 \\ 0 \\ 4 \end{bmatrix} = x_4;$$

$$Mx_4 = \begin{bmatrix} 0 \\ 6 \\ 0 \\ 6 \end{bmatrix} = y_5; \qquad M^t y_5 = \begin{bmatrix} 0 \\ 4 \\ 4 \\ 6 \\ 4 \\ 0 \\ 4 \end{bmatrix} = x_5;$$

$$Mx_5 = \begin{bmatrix} 2 \\ 4 \\ 4 \\ 2 \end{bmatrix} = y_6; \qquad M^t y_6 = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 4 \\ 2 \\ 2 \\ 4 \end{bmatrix} = x_6;$$

$$Mx_6 = \begin{bmatrix} 4 \\ 4 \\ 4 \\ 0 \end{bmatrix} = y_7; \qquad M^t y_7 = \begin{bmatrix} 0 \\ 4 \\ 0 \\ 4 \\ 4 \\ 0 \\ 0 \end{bmatrix} = x_7;$$

$$Mx_7 = \begin{bmatrix} 4 \\ 0 \\ 0 \\ 4 \end{bmatrix} = y_8; \qquad M^t y_8 = \begin{bmatrix} 4 \\ 4 \\ 0 \\ 0 \\ 4 \\ 4 \\ 0 \end{bmatrix} = x_8;$$

$$Mx_8 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = y_9; \qquad M^t y_9 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = x_9.$$

Thus the resultant of $X = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is the realized fixed point

pair given by

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Let y = $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$ $\in$ R. To find the effect of y on M is as follows.

$M^t y = \begin{bmatrix} 1 \\ 7 \\ 0 \\ 3 \end{bmatrix} = x_1$;

$Mx_1 = \begin{bmatrix} 3 \\ 1 \\ 6 \\ 4 \\ 6 \\ 5 \\ 3 \end{bmatrix} = y_1$;

$M^t y_1 = \begin{bmatrix} 3 \\ 4 \\ 4 \\ 4 \end{bmatrix} = x_2$;

$Mx_2 = \begin{bmatrix} 5 \\ 6 \\ 0 \\ 7 \\ 4 \\ 3 \\ 3 \end{bmatrix} = y_2$;

$M^t y_2 = \begin{bmatrix} 5 \\ 7 \\ 6 \\ 2 \end{bmatrix} = x_3$.

$Mx_3$ and so on. Certain the resultant will be a realized fixed point pair or a realized limit cycle pair.

Now we for the same x find the realized limit cycle pair or a realized fixed point pair using the operation min-max and max-min respectively.

Let $x = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in D$. We find using the min-max operation.

$$M \text{ o } x = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}; \qquad M^t y_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = x_1;$$

$$\text{and } Mx_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Thus the resultant is a realized fixed point pair given by

$$\left\{\begin{bmatrix}0\\0\\0\\0\\0\\0\\0\end{bmatrix},\begin{bmatrix}0\\0\\0\\0\end{bmatrix}\right\}.$$

Now we use the max-min operation for the same x = $\begin{bmatrix}2\\0\\0\\0\\0\\0\\0\end{bmatrix}$

$$M \text{ o } x = \begin{bmatrix}2\\0\\1\\0\end{bmatrix} = y_1; \qquad M^t \text{ o } y_1 = \begin{bmatrix}2\\2\\1\\1\\2\\2\\1\end{bmatrix} = x_1;$$

$$M \text{ o } x_1 = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} = y_2; \qquad M^t \text{ o } y_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} = x_2;$$

$$M \text{ o } x_2 = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} = y_3 \ (=y_2).$$

Thus the resultant of $x = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in D$

is a realized fixed point pair given by $\left\{ \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 2 \\ 2 \\ 2 \end{bmatrix} \right\}$.

We see all the three resultant for the same $x = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in D$ is distinctly different.

Let $y = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \in R$.

To find the resultant of y using X, max-min and min-max operation on M.

$$M^t y = \begin{bmatrix} 3 & 2 & 0 & 1 & 4 & 5 & 1 \\ 0 & 1 & 2 & 0 & 3 & 0 & 7 \\ 1 & 0 & 6 & 2 & 0 & 1 & 0 \\ 0 & 5 & 0 & 1 & 7 & 0 & 3 \end{bmatrix}^t \times \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 0 & 1 & 0 \\ 2 & 1 & 0 & 5 \\ 0 & 2 & 6 & 0 \\ 1 & 0 & 2 & 1 \\ 4 & 3 & 0 & 7 \\ 5 & 0 & 1 & 0 \\ 1 & 7 & 0 & 3 \end{bmatrix} \times \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 6 \\ 2 \\ 0 \\ 1 \\ 0 \end{bmatrix} = x_1$$

$$Mx_1 = \begin{bmatrix} 2 \\ 4 \\ 2 \\ 2 \end{bmatrix} = y_1; \qquad M^t y_1 = \begin{bmatrix} 0 \\ 2 \\ 4 \\ 6 \\ 2 \\ 4 \\ 6 \end{bmatrix} = x_2;$$

$$Mx_2 = \begin{bmatrix} 4 \\ 2 \\ 0 \\ 0 \end{bmatrix} = y_2; \qquad M^t y_2 = \begin{bmatrix} 4 \\ 2 \\ 4 \\ 4 \\ 6 \\ 4 \\ 2 \end{bmatrix} = x_3;$$

$$Mx_3 = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 6 \end{bmatrix} = y_3; \qquad M^t y_3 = \begin{bmatrix} 6 \\ 4 \\ 4 \\ 2 \\ 0 \\ 2 \\ 2 \end{bmatrix} = x_4;$$

$$Mx_4 = \begin{bmatrix} 0 \\ 2 \\ 4 \\ 4 \end{bmatrix} = y_4; \qquad M^t y_4 = \begin{bmatrix} 4 \\ 6 \\ 4 \\ 4 \\ 2 \\ 4 \\ 2 \end{bmatrix} = x_5;$$

$$Mx_5 = \begin{bmatrix} 2 \\ 2 \\ 0 \\ 6 \end{bmatrix} = y_5; \qquad M^t y_5 = \begin{bmatrix} 6 \\ 6 \\ 4 \\ 0 \\ 0 \\ 2 \\ 2 \end{bmatrix} = x_6;$$

$$Mx_6 = \begin{bmatrix} 2 \\ 4 \\ 0 \\ 4 \end{bmatrix} = y_6; \qquad M^t y_6 = \begin{bmatrix} 6 \\ 4 \\ 0 \\ 6 \\ 0 \\ 4 \\ 2 \end{bmatrix} = x_7;$$

$$Mx_7 = \begin{bmatrix} 0 \\ 2 \\ 6 \\ 6 \end{bmatrix} = y_7; \qquad M^t y_7 = \begin{bmatrix} 6 \\ 0 \\ 0 \\ 2 \\ 0 \\ 6 \\ 0 \end{bmatrix} = x_8;$$

$$Mx_8 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 2 \end{bmatrix} = y_8; \qquad M^t y_8 = \begin{bmatrix} 6 \\ 4 \\ 0 \\ 4 \\ 6 \\ 2 \\ 0 \end{bmatrix} = x_9;$$

$$Mx_9 = \begin{bmatrix} 4 \\ 6 \\ 0 \\ 2 \end{bmatrix} = y_9; \qquad M^t y_9 = \begin{bmatrix} 4 \\ 0 \\ 4 \\ 6 \\ 0 \\ 4 \\ 4 \end{bmatrix} = x_{10};$$

$$Mx_{10} = \begin{bmatrix} 2 \\ 4 \\ 4 \\ 2 \end{bmatrix} = y_{10}$$

and so on.

We are not able to get the resultant even after 10 iterations. However we will arrive at a realized fixed point pair or a realized limit cycle pair after a finite number of steps. The number of steps is less than $8^4 - 1$.

Next we work using min-max operation for the same

$$y = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} \in R.$$

$$M^t \text{ o } y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = x_1; \qquad M \text{ o } x_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = y_1;$$

$$M^t \text{ o } y_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = x_2 \ (= x_1).$$

Thus the min-max operation results in a realized fixed pair given by

$$\left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \right\}.$$

Now we use the max-min operation for finding the resultant of y on M.

$$M^t \text{ o } y = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = x_1 \text{ ; } \qquad M \text{ o } x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = y_1 \text{ ;}$$

$$M^t \text{ o } y_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = x_2 \text{ ; } \qquad M \text{ o } x_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = y_2 \;\; (=y_1).$$

Once again the resultant is a realized fixed pair given by

$$\left\{\begin{bmatrix}1\\1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\1\\1\\1\\1\\1\\1\end{bmatrix}\right\}.$$

Thus we see the resultant of y = $\begin{bmatrix}0\\0\\1\\0\end{bmatrix}$ on M for the three different operations yields three different resultants.

Hence the researcher can choose any of the operations as per need and work with the model.

Let us now work with MOD-matrix operators using the domain and range vectors as on and off state vectors.

We will illustrate this by some examples.

***Example 2.16:*** Let

$$M = \begin{bmatrix}3 & 0 & 1 & 2 & 5 & 1\\0 & 3 & 0 & 1 & 0 & 3\\2 & 0 & 3 & 0 & 4 & 0\\0 & 1 & 2 & 3 & 0 & 1\\1 & 0 & 0 & 6 & 0 & 3\\3 & 1 & 0 & 7 & 1 & 2\\1 & 0 & 6 & 0 & 1 & 0\\0 & 6 & 1 & 4 & 0 & 7\end{bmatrix}$$

be the MOD real rectangular matrix operator with entries from $Z_{10}$.

$$\text{Let } D = \left\{ \left. \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \end{bmatrix} \right| a_i \in \{0, 1\}; 1 \le i \le 6 \right\}$$

be the special domain space of on and off state vectors.

$$\text{Let } R_S = \left\{ \left. \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_8 \end{bmatrix} \right| b_i \in \{0, 1\}; 1 \le i \le 8 \right\}$$

be the special range space of state vectors.

Let us find the effect of

$$x = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in D_S.$$

$$Mx = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = y_1$$

( $\rightarrow$ the state vector has been updated and thresholded).

$$M^t y_1 = \begin{bmatrix} 3 \\ 1 \\ 3 \\ 5 \\ 1 \\ 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = x_1; \qquad Mx_1 = \begin{bmatrix} 2 \\ 7 \\ 9 \\ 7 \\ 0 \\ 4 \\ 8 \\ 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = y_2\,;$$

$$M^t y_2 = \begin{bmatrix} 0 \\ 1 \\ 3 \\ 3 \\ 1 \\ 7 \end{bmatrix} \rightarrow \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = x_2; \qquad Mx_2 = \begin{bmatrix} 9 \\ 7 \\ 7 \\ 7 \\ 9 \\ 1 \\ 7 \\ 8 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = y_3 = (y_2).$$

Thus we see the resultant is a realized fixed point pair given by

$$\left\{\begin{bmatrix}0\\1\\1\\1\\1\\1\end{bmatrix},\begin{bmatrix}1\\1\\1\\1\\1\\1\\1\\1\end{bmatrix}\right\}.$$

Next we find the effect of $y = \begin{bmatrix}0\\0\\0\\0\\0\\1\\0\\0\end{bmatrix} \in R_S$.

$$M^t y = \begin{bmatrix}3\\1\\0\\7\\1\\2\end{bmatrix} \to \begin{bmatrix}1\\1\\0\\1\\1\\1\end{bmatrix} = x_1; \qquad Mx_1 = \begin{bmatrix}1\\7\\6\\5\\0\\4\\2\\7\end{bmatrix} \to \begin{bmatrix}1\\1\\1\\1\\0\\1\\1\\1\end{bmatrix} = y_1;$$

$$M^t y_1 = \begin{bmatrix} 9 \\ 1 \\ 3 \\ 7 \\ 1 \\ 4 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = x_2; \qquad Mx_2 = \begin{bmatrix} 2 \\ 7 \\ 9 \\ 7 \\ 0 \\ 4 \\ 8 \\ 8 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = y_2 = (y_1).$$

Thus the resultant is again a realized fixed point pair given by

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

This is the way resultant are calculated.

Now if the same $x = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ is taken as element of D we see

$$Mx = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 6 \end{bmatrix} = y_1; \qquad M^t y_1 = \begin{bmatrix} 3 \\ 7 \\ 8 \\ 7 \\ 1 \\ 4 \end{bmatrix} = x_1;$$

$$Mx_1 = \begin{bmatrix} 0 \\ 0 \\ 4 \\ 8 \\ 5 \\ 4 \\ 8 \\ 6 \end{bmatrix} = y_2; \qquad M^t\, y_2 = \begin{bmatrix} 3 \\ 8 \\ 2 \\ 6 \\ 8 \\ 3 \end{bmatrix} = x_2;$$

$$Mx_2 = \begin{bmatrix} 6 \\ 9 \\ 4 \\ 3 \\ 8 \\ 7 \\ 3 \\ 5 \end{bmatrix} = y_3; \qquad M^t y_3 = \begin{bmatrix} 8 \\ 7 \\ 7 \\ 7 \\ 6 \\ 6 \end{bmatrix} = x_3 \text{ and so on.}$$

However we are guranteed of a realized fixed point pair or a realized limit point pair with in the maximum number of iterations $10^6$.

$$\text{Let } y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} \in R_S.$$

To find the effect of y on M.

$$M^t y = \begin{bmatrix} 3 \\ 1 \\ 0 \\ 7 \\ 1 \\ 2 \end{bmatrix} = x_1; \qquad Mx_1 = \begin{bmatrix} 0 \\ 6 \\ 0 \\ 4 \\ 0 \\ 4 \\ 4 \\ 8 \end{bmatrix} = y_1;$$

$$M^t y_1 = \begin{bmatrix} 6 \\ 4 \\ 0 \\ 8 \\ 8 \\ 6 \end{bmatrix} = x_1 \text{ and so on.}$$

Thus we have to work with finite number of iterations say $10^6$ times.

***Example 2.17:*** Let

$$M = \begin{bmatrix} 5 & 0 & 2 & 1 & 4 & 3 \\ 0 & 3 & 0 & 2 & 0 & 1 \\ 4 & 0 & 3 & 0 & 5 & 0 \\ 0 & 2 & 0 & 1 & 0 & 2 \\ 1 & 0 & 4 & 0 & 3 & 0 \end{bmatrix}$$

be the MOD rectangular real matrix operator with entries from $Z_6$.

$$\text{Let } R_S = \left\{ \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| \; a_i \in \{0, 1\}; 1 \leq i \leq 5 \right\}$$

be the range space of state vectors.

$$D_S = \left\{ \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_6 \end{bmatrix} \middle| \; b_i \in \{0, 1\}; 1 \leq i \leq 6 \right\}$$

be the domain space of state vectors associated with M.

$$R = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_5 \end{bmatrix} \middle| \; x_i \in Z_6; 1 \leq i \leq 5 \right\}$$

be the range state vectors associated with M.

$$D = \left\{ \left. \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_6 \end{bmatrix} \right| x_i \in Z_6;\ 1 \leq i \leq 6 \right\}$$

be the domain of state vectors associated with M.

Now we find the effect of $x = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$ on M as element of $D_S$.

$$M \times x = \begin{bmatrix} 5 \\ 0 \\ 4 \\ 0 \\ 1 \end{bmatrix} \to \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} = y_1; \qquad M^t y_1 = \begin{bmatrix} 4 \\ 0 \\ 3 \\ 1 \\ 2 \\ 3 \end{bmatrix} \to \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = x_1;$$

$$Mx_1 = \begin{bmatrix} 3 \\ 3 \\ 0 \\ 3 \\ 2 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \\ 1 \end{bmatrix} = y_2; \qquad M^t y_2 = \begin{bmatrix} 0 \\ 5 \\ 0 \\ 4 \\ 1 \\ 0 \end{bmatrix} \to \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 1 \\ 0 \end{bmatrix} = x_2;$$

$$Mx_2 = \begin{bmatrix} 5 \\ 5 \\ 5 \\ 3 \\ 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = y_3; \qquad M^t y_3 = \begin{bmatrix} 4 \\ 5 \\ 3 \\ 4 \\ 0 \\ 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = x_4;$$

$$Mx_4 = \begin{bmatrix} 2 \\ 5 \\ 7 \\ 3 \\ 5 \end{bmatrix} \rightarrow \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = y_4 \ (=y_3).$$

Thus the resultant of x is a realized fixed point pair given by

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

$$\text{Let } y = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \in R_S;$$

To find the effect of y on M.

$$M^t y = \begin{bmatrix} 1 \\ 0 \\ 4 \\ 0 \\ 3 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} = x_1; \qquad Mx_1 = \begin{bmatrix} 5 \\ 0 \\ 0 \\ 0 \\ 2 \end{bmatrix} \to \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = y_1;$$

$$M^t y_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 3 \end{bmatrix} \to \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \end{bmatrix} = x_2; \qquad Mx_1 = \begin{bmatrix} 2 \\ 3 \\ 5 \\ 3 \\ 3 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = y_2;$$

$$M^t y_2 = \begin{bmatrix} 4 \\ 5 \\ 3 \\ 4 \\ 0 \\ 0 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = x_3; \qquad Mx_3 = \begin{bmatrix} 2 \\ 5 \\ 1 \\ 3 \\ 5 \end{bmatrix} \to \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = y_3.$$

Thus we see the effect of $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

is a realized fixed point pair given by

$$\left\{\begin{bmatrix}1\\1\\1\\1\\0\\0\end{bmatrix},\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}\right\}.$$

We can find the effect of any $x \in D$ (or $y \in R$) any one of the operations $\times$ or max-min or min-max on the MOD matrix operator M.

Clearly it can be easily verified that the resultant of $x \in D$ (or $y \in R$) can be realized fixed point pair or a realized limit cycle pair depending on that x and M after a finite number of iterations.

**Problems**

1. Obtain all special features associated with MOD real rectangular matrices M with entries from $Z_n$ with associated domain state vectors D and range state vectors R.

2. Find all essential and important properties enjoyed by MOD real rectangular matrices M with associated from $Z_n$ but with associated domain space of state vectors $D_S$ and range space of state vectors $R_S$.

3. Compare for any M problems (1) and (2).

4. Let M = $\begin{bmatrix} 3 & 0 & 2 & 1 & 5 \\ 0 & 6 & 0 & 8 & 11 \\ 16 & 0 & 17 & 0 & 5 \\ 6 & 11 & 0 & 10 & 1 \\ 15 & 0 & 14 & 0 & 12 \\ 0 & 1 & 0 & 5 & 0 \\ 7 & 0 & 2 & 0 & 4 \\ 10 & 16 & 14 & 11 & 12 \end{bmatrix}$ be the MOD rectangular real matrix operator with entries from $Z_{19}$.

Let D = $\{(x_1, x_2, \ldots, x_8) \mid x_i \in Z_{19},\ 1 \leq i \leq 8\}$ and R = $\{(y_1, y_2, y_3, y_4, y_5) \mid y_i \in Z_{19}, 1 \leq i \leq 5\}$ be the domain and range space of state vectors associated with M respectively.

Let $D_S = \{(x_1, x_2, \ldots, x_8) \mid x_i \in \{0, 1\}, 1 \leq i \leq 8\}$ and $R_S = \{(y_1, \ldots, y_5) \mid y_i \in \{0, 1\}, 1 \leq i \leq 5\}$ be the special domain state vectors and special range state vectors associated with M respectively.

(i) Find all classical fixed points of M for D.
(ii) Find all classical fixed points of R associated with M.
(iii) Find all realized fixed point pairs of R and D associated with M.
(iv) Find all realized limit cycle pairs of R and D associated with M.
(v) Find the realized fixed point or the realized limit cycle point pair which has maximum number of iterations.
(vi) Enumerate all realized fixed point pair and realized limit cycle pair which has minimum number of iterations.
(vii) Can we say state vectors B and D mentioned in questions (i) and (ii) are inter related.
(viii) Study questions (i) to (vii) for $D_S$ and $R_S$.

5. Let $M_1 = \begin{bmatrix} a_1 & a_2 & \ldots & a_{10} \\ a_{11} & a_{12} & \ldots & a_{20} \\ a_{21} & a_{22} & \ldots & a_{30} \\ a_{31} & a_{32} & \ldots & a_{40} \end{bmatrix}$ $a_i \in Z_6$, $1 \leq i \leq 40$

   be the MOD rectangular real matrix operator with entries from $Z_6$.

   Let D, R, $D_S$ and $R_S$ be as in problem 4 with appropriate changes.

   (i) Study questions (i) to (viii) of problem 4 for this $M_1$.
   (ii) Can we say the structure of $M_1$ as a matrix contributes to questions (i) to (viii) of problem 4.

6. Let $P = \begin{bmatrix} a_1 & a_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ a_3 & a_4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_5 & a_6 & 0 & 0 & 0 & 0 \\ 0 & 0 & a_7 & a_8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a_9 & a_{10} & 0 & 0 \\ 0 & 0 & 0 & 0 & a_{11} & a_{12} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{13} & a_{14} \\ 0 & 0 & 0 & 0 & 0 & 0 & a_{15} & a_{16} \\ a_{17} & a_{18} & 0 & 0 & 0 & 0 & 0 & 0 \\ a_{19} & a_{20} & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$

   be the MOD rectangular real matrix operator with entries from $a_i \in Z_{143}$, $1 \leq i \leq 20$.

   (i) Study questions (i) to (vii) of problem 4 for this P with appropriate changes.
   (ii) Enumerate all the special features enjoyed by P.

(iii) If $P_1$ and $P_2$ are two distinct matrices of the form P. Can we say if x is a realized fixed point pair of $P_1$ and $P_2$ so will be x on the MOD operator $P_1 + P_2$?

7. Let M = $\begin{bmatrix} 3 & 0 & 1 & 2 & 4 & 0 & 5 & 7 & 1 \\ 0 & 7 & 0 & 1 & 0 & 6 & 0 & 5 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 4 & 4 & 4 \\ 0 & 0 & 0 & 7 & 6 & 5 & 0 & 0 & 0 \end{bmatrix}$

be the MOD rectangular real matrix operator with entries from $Z_8$.

(i) Study questions (i) to (viii) of problem 4 for this M.
(ii) If $x_1 = (0\ 1\ 2\ 3)$ and $x_2 = (1\ 0\ 1\ 0) \in D$. Find $x_1M$ and $x_2M$. If $x_1M$ and $x_2M$ related with $(x_1 + x_2)M$?

8. Let B = $\begin{bmatrix} 3 & 7 & 0 & 1 \\ 0 & 5 & 0 & 6 \\ 7 & 1 & 1 & 0 \\ 6 & 0 & 0 & 2 \\ 1 & 2 & 3 & 4 \\ 5 & 0 & 6 & 3 \\ 3 & 0 & 1 & 0 \\ 4 & 1 & 0 & 7 \\ 1 & 2 & 3 & 4 \end{bmatrix}$ be the MOD rectangular matrix with

entries from $Z_{10}$.

Study questions (i) to (viii) of problem 4 for this B.

Let C = $\begin{bmatrix} 1 & 2 & 6 & 0 \\ 0 & 1 & 1 & 0 \\ 2 & 0 & 0 & 2 \\ 0 & 3 & 3 & 0 \\ 4 & 0 & 0 & 4 \\ 0 & 5 & 5 & 0 \\ 6 & 0 & 0 & 6 \\ 0 & 7 & 7 & 0 \\ 9 & 0 & 0 & 9 \end{bmatrix}$ be another MOD real matrix with

entries from $Z_{10}$.

(i) If x = (1 2 3 0 0 0 0 0 1) ∈ D find the resultant of xB and xC.
(ii) Compare it with the resultant of x(B+C).
(iii) Study questions (i) to (viii) of problem 4 for this B, C and B + C.

9. Let $S_1$ = $\begin{bmatrix} 3 & 6 & 4 & 0 \\ 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 6 \\ 2 & 0 & 0 & 3 \\ 5 & 0 & 2 & 0 \\ 6 & 0 & 1 & 1 \\ 7 & 6 & 3 & 0 \end{bmatrix}$ and $S_2$ = $\begin{bmatrix} 0 & 0 & 0 & 7 \\ 0 & 7 & 3 & 0 \\ 6 & 3 & 0 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 6 & 0 & 3 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 0 & 6 \end{bmatrix}$ and

be two MOD rectangular matrix operation with entries from $Z_{12}$.

(i) Study questions (i) to (viii) of problem 4 for this $S_1$ and $S_2$.
(ii) Compare the resultants of $S_1$ and $S_2$ with that of $S_1 + S_2$.

(iii) If x and y are in D.

(a) Find the resultant of $xS_1$, $yS_2$, $x(S_1 + S_2)$, $y(S_1 + S_2)$, $(x+y)S_1$, $(x+y)S_2$ and $(x+y)\ S_1+S_2$.

(b) Are these resultants in any way related?

10. Does there exist a MOD restangular real matrix operator M which yields for every $x \in D$ and every $y \in R$ only classical fixed point pairs.

11. Is it possible to find MOD real rectangular matrix operators M which are such that all resultants of $x \in R$ and $y \in D$ are only realized fixed point pairs?

12. Is it possible to find MOD find rectangular matrix operators M which are such that all $x \in D$ and $y \in R$ give the resultant only as realized limit cycle pairs?

13. Find those MOD real rectangular matrix operators M for which y, $x \in D$ and xD, yD are resultants then the resultant sum is the sum of the resultants of x+y on D.

14. Characterize all those MOD rectangular matrices for which problem 13 is not true.

15. Let M = $\begin{bmatrix} 3 & 4 & 0 & 1 & 2 \\ 0 & 1 & 6 & 0 & 5 \\ 1 & 0 & 7 & 2 & 0 \\ 4 & 4 & 0 & 0 & 3 \\ 0 & 0 & 1 & 1 & 0 \\ 5 & 0 & 6 & 0 & 7 \\ 0 & 7 & 0 & 8 & 0 \end{bmatrix}$

be a MOD rectangular matrix operator with entries from $Z_{10}$.

Let D = $\left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{bmatrix} \middle| a_i \in Z_{10}, 1 \leq i \leq 5\right\}$ be the domain state vectors associated with M and R = $\left\{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ \vdots \\ b_7 \end{bmatrix} \middle| b_i \in Z_{10}, 1 \leq i \leq 7\right\}$ be the range state vectors associated with M.

(i) Find all realized fixed point pairs of M.

(ii) If x = $\begin{bmatrix} 3 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} \in$ D, find the resultant of x on M, that $M \times x$.

(iii) Can M have classical fixed point pairs associated

with it other than $\left\{\begin{bmatrix}0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\0\\0\\0\end{bmatrix}\right\}$?

(iv) Obtain any other special feature associated with this type of MOD rectangular matrix operator.

(v) Find the resultant of x on M if 'x' operation is replaced by max-min.

(vi) Find the resultant of x on M if 'x' operation is replaced by min-max.

(vii) Compare the resultants in all the three cases.

(viii) Suppose we take y = $\begin{bmatrix}1\\1\\0\\0\\0\\0\\1\end{bmatrix} \in R$. Find the resultant

of $M^t \times y$, $M^t$ o y under max-min operation and min-max operations.

Compare the resultants.

16. Let M = $\begin{bmatrix} 3 & 2 & 1 & 0 & 5 & 2 & 3 & 4 \\ 0 & 4 & 0 & 6 & 0 & 7 & 0 & 2 \\ 6 & 0 & 1 & 0 & 5 & 0 & 1 & 0 \\ 0 & 7 & 0 & 3 & 0 & 4 & 0 & 5 \end{bmatrix}$

be the MOD rectangular real matrix operator with entries from $Z_{11}$.

$$\text{Let } x = \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \in D;$$

find M × x, min-max and max-min using M.

(i) Is this a realized fixed point pair or a realized limit cycle pair.

(ii) Let y = $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 2 \end{bmatrix}$ ∈ R, find $M^t o y$ with max-min and min-max. Find all the resultant $M^t \times y$.

(iii) Characterize all realized limit cycle pair associated with M.

(iv) Characterize all those realized fixed point pairs associated with M.

(v) Can M have classical fixed point pair?

17. Let W = $\begin{bmatrix} 2 & 11 & 1 \\ 0 & 13 & 0 \\ 4 & 5 & 9 \\ 10 & 17 & 0 \\ 4 & 8 & 16 \\ 5 & 17 & 0 \\ 0 & 19 & 12 \\ 14 & 0 & 0 \end{bmatrix}$ be the MOD rectangular matrix operator with entries from $Z_{20}$.

Let D = $\left\{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \middle|\ x_i \in Z_{20}; 1 \leq i \leq 3\right\}$ and

$D_S = \left\{\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \middle|\ a_i \in \{0,1\}; 1 \leq i \leq 3\right\}$ be the domain space of state vectors and special domain space of state vectors respectively.

Let R = $\left\{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_8 \end{bmatrix} \middle|\ y_i \in Z_{20}; 1 \leq i \leq 8\right\}$ and

$R_S = \left\{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_8 \end{bmatrix} \middle|\ b_i \in Z_{20}; 1 \leq i \leq 8\right\}$ be the range space of state vectors and special range space of state vectors respectively.

(i) Let x = $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\in$ D find the resultant of x on W.

(ii) Let x = $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ $\in$ $D_S$ find the special resultant of x on W.

(iii) Let y = $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ $\in$ R, find the resultant of y on W.

(iv) Let y = $\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$ $\in$ $R_S$ find the special resultant of y on W.

(v) Compare the resultants given in questions (i) and (ii).

(vi) Compare the resultants given in questions (iii) and (iv).

(vii) Can there be a $x \in D_S \subseteq D$ which gives same resultant over W?

(viii) Does there exist a $y \in D_S \subseteq D$ which gives same resultant over W realized as an element of $D_S$ and that of D?

(ix) Study questions (i) to (viii) if X operation is replaced by min-max and max-min.

18. Give any applications of the MOD real rectangular operators built using $Z_n$.

19. Let $X = \begin{bmatrix} 3 & 0 & 1 & 2 & 3 \\ 0 & 1 & 0 & 1 & 0 \\ 4 & 0 & 1 & 0 & 2 \\ 0 & 2 & 0 & 2 & 0 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 4 & 0 & 4 & 0 \\ 3 & 0 & 3 & 0 & 3 \end{bmatrix}$

be the MOD matrix real operator with entries from $Z_5$.

Let $D = \left\{ \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_5 \end{bmatrix} \middle| \; x_i \in Z_5; 1 \leq i \leq 5 \right\}$

be the domain space associated with X.

Let $R = \left\{\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_7 \end{bmatrix} \middle|\, y_i \in Z_5;\ 1 \leq i \leq 7\right\}$

be the range space associated with X.

(i) Find all realized fixed point pairs associated with X.

(ii) If X is realized as a MOD rectangular real matrix $W_1$ with entries from $Z_6$ compare them as MOD operators.

(iii) If $a = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} \in R$ find $W^t$ o a and $W^t \times a$ and $W_1^t$ o a and $W_1^t \times a$. Compare the resultant.

(iv) What are the advantages or disadvantages of using the field $Z_5$ instead of the ring $Z_6$?

(v) Let $b = \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} \in D$ find $W \times b$, W o b, $W_1$ o b and $W_1 \times b$.
Compare the four resultant appropriately.

(vi) Obtain any other special feature associated with W and $W_1$.

20. Let W = $\begin{bmatrix} 3 & 0 \\ 6 & 1 \\ 1 & 0 \\ 0 & 3 \\ 2 & 0 \\ 0 & 1 \\ 4 & 1 \\ 5 & 0 \\ 2 & 5 \end{bmatrix}$ be the MOD real rectangular matrix

operator with entries.

(i) Study the special features associated with W.

(ii) If $Z_7$ is replaced by $Z_8$ can we say the number of iterations in case of $Z_7$ is more than that of iterations resulting by using $Z_8$.

(iii) For x = $\begin{bmatrix} 3 \\ 4 \end{bmatrix}$ $\in$ D find the resultant on W.

(iv) For y = $\begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $\in$ R find the resultant on W.

(v) For $x_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \in D$ find the resultant of $x_1$ on W.

(vi) Find the resultant of $x + x_1$ on W.

(vii) Is resultants in (iii), (iv) and (v) related?

21. Find all $n \times m$ $(n \neq m)$ MOD matrix operators with entries from $Z_2$ which are such that all element x in D and y in D are classical fixed point pairs.

22. Let $M = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ be a MOD rectangular matrix

operator with entries from $Z_2$.

(i) Let $x = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in D$. Find the resultant of x on M.

(ii) Let y = $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$ $\in$ R, find the resultant of y on M.

(iii) Can M have state vectors in D or R which are classical fixed point pairs?

(iv) Let $x_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \in$ D find the resultant of $x_1$ on M.

(v) Find the resultant of $x_1 + x_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ on M.

(vi) Does there exist a relation between (i), (iv) and (v)?

23. Give all the probable applications of MOD rectangular real matrix operators.

24. Characterize all those MOD matrix operators M which have every x $\in$ D and y $\in$ R have only realized fixed point pairs.

25. Let $P = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 \end{bmatrix}$ be the MOD real matrix operator with entries from $Z_2$.

Let $D = \{(a_1, \ldots, a_8) \mid a_i \in Z_2; 1 \leq i \leq 8\} = D_S$ and

$R = \{(b_1, b_2, \ldots, b_5) \mid b_i \in Z_2; 1 \leq i \leq 5\} = R_S$ be the special domain state vectors and special range state vectors.

If $x = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \in D$; find the realized resultant.

If $x \in \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \in D_S$ find the realized resultant.

Compare the resultants.

## Chapter Three

# MOD FIXED POINT PAIRS USING MOD RECTANGULAR MATRIX OPERATORS WITH ENTRIES FROM $\langle Z_n \cup I\rangle$ OR $\langle Z_n \cup g\rangle$ OR $\langle Z_n \cup h\rangle$ OR $\langle Z_n \cup k\rangle$ OR $C(Z_n)$.

In this chapter we study MOD rectangular matrix operators with entries from various fields or rings $\langle Z_n \cup I\rangle$ the neutrosophic ring with $I^2 = I$ or from dual number MOD ring $\langle Z_n \cup g\rangle$, $g^2 = 0$ and so on.

All these situations will be represented by examples and their properties will be enumerated.

***Example 3.1:*** Let S = $\begin{bmatrix} 3+I & 2I & 1 \\ 0 & 2+3I & I \\ 2I & 0 & 4I+1 \\ 0 & 2 & 0 \\ 3I+1 & 0 & 2+I \\ 0 & 4I+1 & 2 \end{bmatrix}$

be the MOD neutrosophic matrix operator with entries from $\langle Z_5 \cup I\rangle = \{a + bI / a, b \in Z_5, I^2 = I\}$.

$D = \{(a_1, a_2, \ldots, a_6) \mid a_i \in \langle Z_5 \cup I\rangle\ 1 \leq i \leq 6\}$ and

$R = \{(b_1, b_2, b_3) \mid b_i \in \langle Z_5 \cup I\rangle;\ 1 \leq i \leq 3\}$ be the collection of domain state of vectors and range space of state vectors respectively.

Let $x = (2, 1+I, 0, 0, 0, 0) \in D$.

To find the effect of x on S.

$xS = (1 + 2I, 2 + 2I, 2 + 2I) = y_1;$

$y_1S^t = (4I, 4, 2 + 2I, 4 + 4I, 4I, 2I + 1) = x_1;$

$x_1S = (3I, 2 + 2I, 2I + 4) = y_2;$

$y_2S^t = (2I + 4, 4 + 2I, 2I + 4, 4 + 4I, 2I + 3, 2) = x_2;$

$x_2S = (I, 3, 3) = y_3;$

$y_3S^t = (3, 1 + 2I, 4I + 3, 1, 1 + 2I, 4 + 2I) = x_3$ and so on.

However after a finite number of iterations we arrive at a MOD realized fixed point pair or a MOD realized limit cycle pair.

Let $y = (1, 0, 0) \in R$.

To find the effect of y on M;

$yM^t = (3 + I, 0, 2I, 0, 3I + 1, 0) = x_1;$

$x_1M = (I, 3I, 3I + 2) = y_1;$

$y_1M^t = (3I + 2, 0, 2, I, 4, I + 4) = x_2;$

$x_2M = (0, 4 + 3I, 2I) = y_2;$

$y_2M^t = (I, 3 + 4I, 0, 3 + I, I, 4) = x_3$ and so on.

However after a finite number of iterations we will arrive at a realized fixed point pair or a realized limit cycle.

Let $x = (I, 0, 0, 0, 0, 0) \in D$.

To find the effect of x on M;

$xM = (4I, 2I, I) = y_1$;

$y_1M^t = (I, I, 3I, 4I, 4I, 2I) = x_1$;

$x_1M^t = (I, 4I, 3I) = y_2$;

$y_2M = (0, 3I\ 2I, 3I, 3I, I) = x_2$ and so on.

Certainly after a finite number of iterations we will arrive at a realized fixed point pair or a realized limit cycle pair, however the pair of state vectors will only be a pure neutrosophic pair of row vectors.

***Example 3.2:*** Let

$$M = \begin{bmatrix} 0 & I & 0 & 3+I & 4 & 3I \\ 1 & 0 & 2 & 0 & 1+I & 0 \\ 1+I & 2 & 0 & 4+I & 0 & 4 \\ 0 & 2+I & I & 0 & 1+2I & 2 \end{bmatrix}$$

be the MOD neutrosophic matrix operator with entries from
$\langle Z_6 \cup I\rangle = \{(a + bI \mid I^2 = I, a, b \in Z_6\}$.

Let $D = \{(a_1, a_2, a_3, a_4) \mid a_i \in \langle Z_6 \cup I\rangle\ 1 \leq i \leq 4\}$ and

$R = \{(b_1, b_2, \ldots, b_6) \mid b_i \in \langle Z_6 \cup I\rangle; 1 \leq i \leq 6\}$ be the domain and range state vectors respectively associated with MOD operator.

Let $x = (1, 0, 0, 1) \in D$.

To find the resultant of x on M.

$xM = (0, 2 + 2I, I, 3 + I, 5 + 2I, 2 + 3I) = y_1;$

$y_1M^t = (5 + 4I, 5 + 5I, 0, 1) = x_1;$

$x_1M = (5 + 5I, 2 + 4I, 4 + 5I, 3 + 3I, 2 + 3I, 3I + 2) = y_2$

we can find $y_2M^t$ and so on.

Certainly as the cardinality of R and D are finite we are sure to arrive at a resultant as a realized limit cycle pair or a realized fixed point pair after a finite number of iterations.

Let $y = (I, 0, 0, 0, 0, 0) \in R$.

To find the effect of y on M.

$yM^t = (0, 1, 1 + I, 0) = x_1;$

$x_1M = (2 + I, 2 + 2I, 2, 4, 1 + I, 4 + 4I) = y_1;$

$y_1M^t = (4, 1 + 4I, 2 + 4I, 5 + 4I) = x_2;$

$x_2M = (3 + 2I, 2 + 5I, 2 + 5I, 2 + 2I, 4 + I, 0) = y_2.$

However we are guaranteed of arriving at a realized limit point pair or a realized limit cycle pair after a finite number of iterations as a resultant of y on M.

Next we proceed on to give another example.

***Example 3.3:*** Let

$$S = \begin{bmatrix} 3 & 1+I \\ 0 & 2I \\ 5I+1 & 0 \\ I & 3 \\ 0 & 4I+2 \\ 6I & I \\ 0 & 6I+6 \\ 4 & 4I \\ 2+2I & 0 \end{bmatrix}$$

be the MOD neutrosophic rectangular matrix operator with entries from $\langle Z_8 \cup I\rangle$.

Let $D = \{(a_1, a_2, \ldots, a_9) \mid a_i \in \langle Z_8 \cup I\rangle; 1 \leq i \leq 9\}$ and

$R = \{(a, b) \mid a, b \in \langle Z_8 \cup I\rangle\}$ be the domain and range space of state vectors respectively associated with S.

Let $x = (I, 0, 0, 0, 0, 0, 0, 0, 1) \in D$.

To find the effect of x on S;

$xS = (2 + 5I, 1 + I) = y_1 \in R$;

$y_1S^t = (7 + 2I, 4I, 2, 3+2I, 2 + 2I, 4I, 2I+6, 4I, 4) = x_1$;

$x_1S = (7 + 5I, 6I) = y_2$;

$y_2S^t = (5 + 3I, 4I, 7 + I, 6I, 4I, 6I, 0, 4 + 4I, 6 + 4I) = x_2$ and so on.

We will arrive after a finite number of iterations the resultant of x to be a realized fixed point pair or a realized limit cycle pair.

Let y = (0, 1). To find the effect of y on S.

$yS^t = (1 + I, 2I, 0, 3, 4I + 2, I, 6I + 6, 4I, 0) = x_1$;

$x_1S = (3 + 4I, 2 + 4I) = y_1$;

$y_1S^t = (3 + 6I, 4I, 7I + 3, 6 + 3I, 4, 0, 4 + 4I, 4, 6 + 6I) = x_2$.

We can proceed on to find the resultant. Certainly after a finite number of iterations we will arrive at a resultant.

Next we give some examples of special type.

***Example 3.4:*** Let

$$P = \begin{bmatrix} 3 & 0 & 1 & 2 & 5 \\ 1 & 1 & 2 & 0 & 1 \\ 0 & 4 & 0 & 3 & 0 \\ 1 & 5 & 0 & 1 & 1 \end{bmatrix}$$

be the MOD neutrosophic rectangular operator with entries from $\langle Z_6 \cup I\rangle$.

Let D = $\{(a_1, a_2, a_3, a_4) \mid a_i \in \langle Z_6 \cup I\rangle; 1 \leq i \leq 4\}$ and

R = $\{(b_1, b_2, b_3, b_4, b_5) \mid b_i \in \langle Z_6 \cup I\rangle; 1 \leq i \leq 5\}$ be the neutrosophic initial state vectors.

Let x = (3, 0, 1, 0) ∈ D.

To find the effect of x on P.

$xP = (3, 4, 3, 3, 3) = y_1$; $\quad y_1P^t = (3, 4, 1, 5) = x_1$;

$x_1P = (0, 3, 5, 2, 0) = y_2$; $\quad y_2P^t = (3, 1, 0, 5) = x_2$;

$x_2P = (3,4, 5, 5, 3) = y_3;$ $y_3P^t = (3, 2, 1, 1) = x_3;$

$x_3P = (0, 5, 1, 3, 0) = y_4;$ $y_4P^t = (1, 1, 5, 4) = x_4;$

$x_4P = (2, 5, 3, 3, 4) = y_5;$ $y_5P^t = (5, 5, 5, 4) = x_5;$

$x_5P = (0, 3, 3, 5, 4) = y_6;$ $y_6P^t = (3, 1, 3, 0) = x_6;$

$x_6P = (4, 1, 5, 3, 4) = y_7;$ $y_7P^t = (1, 1, 1, 4) = x_7;$

$x_7P = (2, 1, 3, 3, 4) = y_8;$ $y_8P^t = (5, 1, 1, 2) = x_8;$

$x_8P = (0, 3, 1, 1, 4) = y_9;$ $y_9P^t = (5, 3, 3, 2) = x_9;$

$x_9P = (2, 1, 5, 3, 0) = y_{10};$ $y_{10}P^t = (5, 1, 1, 4) = x_{10};$

$x_{10}P = (2, 1, 1, 5, 0) = y_{11};$ $y_{11}P^t = (5, 5, 1, 0) = x_{11};$

$x_{11}P = (2, 3, 3, 1, 0) = y_{12};$ $y_{12}P^t = (5, 5, 3, 0) = x_{12};$

$x_{12}P = (2, 5, 3, 1, 0) = y_{13};$ $y_{13}P^t = (5, 1, 5, 4) = x_{13};$

$x_{13}P = (2, 5, 1, 5, 0) = y_{14};$ $y_{14}P^t = (5, 3, 5, 2) = x_{14};$

$$x_{14}P = (2, 3, 5, 3, 0) = y_{15};$$

$$y_{15}P^t = (5, 3, 3, 2) = x_{15}\ (= x_9)\ ;$$

$$x_{15}P = (2, 1, 5, 3, 0) = y_{16}\ (=y_{10}).$$

Thus we see the resultant is achieved after 14 pairs of iterations and is a realized limit cycle pair given by
{(5, 3, 3, 2), (2, 1, 5, 3, 0)}.

It is important to observe that as the MOD-matrix operator had only real entries and $x \in D$ with which we worked with also happened to be real.

This is an important observation as far as the MOD neutrosophic operator matrix is considered.

Let y = (1, 0, 0, 1, 0) $\in$ R; to find the result of y on the MOD neutrosophic matrix operator P.

$yP^t = (5, 1, 3, 2) = x_1$; $x_1P = (0, 5, 1, 3, 4) = y_1$;

$y_1P^t = (3, 5, 5, 2) = x_2$; $x_2P = (4, 5, 1, 5, 4) = y_2$;

$y_2P^t = (1, 3, 5, 2) = x_3$; $x_3P = (2, 3, 1, 1, 4) = y_3$;

$y_3P^t = (5, 5, 3, 4) = x_4$; $x_4P = (0, 1, 3, 5, 4) = y_4$;

$y_4P^t = (3, 5, 1, 2) = x_5$; $x_5P = (4, 1, 1, 5, 4) = y_5$;

$y_5P^t = (3, 5, 1, 0) = x_6$; $x_6P = (2, 3, 1, 3, 2) = y_6$;

$y_6P^t = (5, 3, 3, 4) = x_7$; $x_7P = (4, 5, 5, 5, 2) = y_7$;

$y_7P^t = (1, 3, 5, 0) = x_8$; $x_8P = (0, 5, 3, 5, 2) = y_8$;

$y_8P^t = (5, 1, 5, 2) = x_9$; $x_9P = (0, 1, 1, 3, 4) = y_9$;

$y_9P^t = (3, 1, 1, 0) = x_{10}$; $x_{10}P = (4, 5, 5, 3, 4) = y_{10}$;

$y_{10}P^t = (1, 5, 5, 0) = x_{11}$; $x_{11}P = (2, 1, 5, 5, 4) = y_{11}$;

$y_{11}P^t = (5, 5, 1, 4) = x_{12}$; $x_{12}P = (0, 5, 3, 5, 4) = y_{12}$;

$y_{12}P^t = (3, 3, 5, 4) = x_{13}$; $x_{13}P = (4, 1, 3, 1, 4) = y_{13}$;

$y_{13}P^t = (1, 3, 1, 2) = x_{14}$; $x_{14}P = (2, 1, 1, 1, 4) = y_{14}$;

$y_{14}P^t = (5, 3, 1, 0) = x_{15}$; $x_{15}P = (0, 1, 5, 5, 4) = y_{15}$;

$y_{15}P^t = (5, 1, 1, 2) = x_{16}$; $x_{16}P = (0, 3, 1, 1, 4) = y_{16}$;

$y_{16}P^t = (5, 3, 3, 2) = x_{17}$; $x_{17}P = (2, 1, 5, 3, 0) = y_{17}$;

$y_{17}P^t = (5, 1, 1, 0) = x_{18};$ $x_{18}P = (4, 5, 1, 1, 2) = y_{18};$

$y_{18}P^t = (1, 1, 5, 2) = x_{19};$ $x_{19}P = (0, 1, 3, 1, 2) = y_{19};$

$y_{19}P^t = (3, 3, 1, 2) = x_{20};$ $x_{20}P = (2, 5, 3, 5, 2) = y_{20};$

$y_{20}P^t = (5, 3, 5, 4) = x_{21};$ $x_{21}P = (4, 1, 5, 5, 2) = y_{21};$

$y_{21}P^t = (1, 5, 1, 4) = x_{22};$ $x_{22}P = (0, 5, 5, 0, 2) = y_{22};$

and so on.

However we are sure that after a finite number of iterations we will arrive at a realized limit cycle pair or a realized fixed point pair.

***Example 3.5:*** Let

$$B = \begin{bmatrix} 3 & 0 & 1 & 2 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 3 & 1 \\ 0 & 2 & 0 & 1 \\ 2 & 0 & 1 & 0 \\ 0 & 3 & 0 & 2 \end{bmatrix}$$

be the MOD rectangular neutrosophic matrix operator with entries from $\langle Z_4 \cup I \rangle$.

Let D = $\{(a_1, a_2, \ldots, a_6) \mid a_i \in \langle Z_4 \cup I \rangle; 1 \leq i \leq 6\}$ be the domain space of neutrosophic state vectors associated with B.

R = $\{(b_1, b_2, b_3, b_4) \mid \langle Z_4 \cup I \rangle; 1 \leq i \leq 4\}$ be the range space of neutrosophic state vectors associated with B.

Let x = $(0, 1, 0, 0, 0, 0) \in$ D; to find the effect of x on B.

$xB = (0, 1, 0, 0) = y_1$; $y_1B^t = (0, 1, 2, 2, 0, 3) = x_1$;

$x_1B = (2, 2, 2, 2) = y_2$; $y_2B^t = (0, 2, 2, 2, 2, 2) = x_2$;

$x_2B = (2, 0, 0, 0) = y_3$; $y_3B^t = (2, 0, 2, 0, 0, 0) = x_3$;

$x_3B = (0, 0, 0, 2) = y_4$; $y_4B^t = (0, 0, 2, 2, 0, 0) = x_4$;

$x_4B = (2, 0, 2, 0) = y_5$; $y_5B^t = (0, 0, 0, 0, 2, 0) = x_5$;

$x_5B = (0, 0, 2, 0) = y_6$; $y_6B^t = (2, 0, 2, 0, 2, 0) = x_6$;

$x_6B = (0, 0, 2, 2) = y_7$; $y_7B^t = (2, 0, 0, 2, 2, 0) = x_7$;

$x_7B = (2, 0, 0, 2) = y_8$; $y_8B^t = (2, 0, 0, 2, 0, 0) = x_8$;

$x_8B = (2, 0, 2, 2) = y_9$; $y_9B^t = (0, 0, 2, 2, 2, 0) = x_9$;

$x_9B = (2, 0, 0, 0) = y_{10}$ $(=y_4)$ ;

$y_{10}B^t = (2, 0, 2, 0, 0, 0) = x_{10}$ $(=x_3)$.

Thus the resultant of x = (0, 1, 0, 0, 0, 0) is a realized limit cycle pair given by {(2, 0, 2, 0, 0, 0), (2, 0, 0, 0)}.

Let $y = (0, 0, 1, 0) \in R$.

To find the resultant of y on B.

$yB^t = (1, 0, 3, 0, 1, 0) = x_1$; $x_1B = (0, 2, 3, 1) = y_1$;

$y_1B^t = (1, 2, 2, 2, 3, 0) = x_2$; $x_2B = (3, 2, 2, 2) = y_2$;

$y_2B^t = (3, 2, 3, 2, 0\ 2) = x_3$; $x_3B = (0, 2, 0, 3) = y_3$;

$y_3B^t = (2, 2, 3, 3, 0, 0) = x_4$; $x_4B = (1, 2\ 3\ 2) = y_4$;

$y_4B^t = (2, 2, 0, 2, 1, 2) = x_5$; $x_5B = (0, 0, 3, 2) = y_5$;

$y_5B^t = (3, 0, 3, 2, 3, 0) = x_6$; $x_6B = (2, 2, 3, 3) = y_7$;

$y_7B^t = (3, 2, 2, 3, 2, 0) = x_7$; $x_7B = (3, 0, 3, 3) = y_8$;

$y_8B^t = (2, 0, 3, 3, 1, 2) = x_8$; $x_8B = (3,2, 0, 2) = y_9$;

$y_9B^t = (1, 2, 1, 2, 2, 2) = x_9$; $x_9B = (0, 2, 2, 3) = y_{10}$;

$y_{10}B^t = (0, 2, 1, 1, 2, 0) = x_{10}$; $x_{10}B = (1, 2, 1, 2) = y_{11}$;

$y_{11}B^t = (0, 2, 2, 2, 3\ 2) = x_{11}$; $x_{11}B = (0, 0, 1, 0) = y_2$ (=y).

Thus the resultant of y is a realized limit cycle or a pseudo fixed point as it is y and (1, 0, 3, 0, 1, 0) that is
{(1, 0, 3, 0, 1, 0), (0, 0, 1, 0)}.

Hence from these examples we arrive at the following important fact.

**THEOREM 3.1:** *Let*

$$M = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ a_{21} & \cdots & a_{2n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix} \text{ where } a_{ij} \in Z_s \subseteq \langle Z_s \cup I\rangle;\ 2 \leq s < \infty;$$

$$1 \leq i \leq m \text{ and } 1 \leq j \leq n$$

*be the MOD neutrosophic rectangular matrix operator.*

*$D = \{(a_1, a_2, \ldots, a_m) \mid a_i \in \langle Z_s \cup I\rangle = \{a + bI \mid a, b \in Z_s.\ I^2 = I\};\ 1 \leq i \leq m\}$ be the domain space of neutrosophic state vectors and*

*$R = \{(b_1, b_2, \ldots, b_n) \mid b_i \in \langle Z_s \cup I\rangle;\ 1 \leq i \leq n\}$ be the range space of neutrosophic state vectors associated with M.*

*If $x = (a_1, \ldots, a_m) \in D$ is such that $a_i \in Z_s$; $1 \leq i \leq m$ then the MOD resultant is a MOD realized limit cycle pair or a MOD realized fixed point pair is always a real pair.*

*Similarly if $y = (b_1, \ldots, b_n)$ such that $b_i \in Z_s$; $1 \leq i \leq n$ then the MOD resultant is always a MOD realized limit cycle real pair or a MOD realized fixed point real pair.*

Proof follows from the fact as M has all its entries to be reals so when the $x \in D$ has only real entries the resultant is real as it is impossible to get any neutrosophic term. Similar argument is true in case of y a real range state vector of R.

Next we give some more special examples.

***Example 3.6:*** Let

$$S = \begin{bmatrix} 3I & 0 & I \\ 2I & 2I & 3I \\ 0 & 5I & 0 \\ I & 2I & I \\ 3I & 0 & 4I \\ 5I & 0 & 3I \\ I & 2I & 3I \\ 0 & I & 0 \end{bmatrix}$$

be a MOD neutrosophic rectangular matrix operator. The entries of S are from the pure neutrosophic modulo integers

$$Z_6 I \subseteq \langle Z_6 \cup I \rangle.$$

Let $D = \{(a_1, a_2, \ldots, a_8) \mid a_i \in \langle Z_6 \cup I \rangle = \{a + bI \mid a, b \in Z_6\}, 1 \leq i \leq 8\}$ and

$R = \{(b_1, b_2, b_3) \mid b_i \in \langle Z_6 \cup I \rangle, 1 \leq i \leq 3\}$ be the neutrosophic domain and range space of state vectors respectively.

Let x = (1, 0, 0, 0, 0, 0, 0, 0) $\in$ D.

To find effect of x on S.

$xS = (3I, 0\ I) = y_1$;

$y_1S^t = (4I, 5I, 5I, 6I, 3I, 0, 2I, 0) = x_1$;

$x_1S = (3I, 3I, I) = y_2$;

$y_2S^t = (4I, 3I, 3I, 4I, I, 0, 0, 3I) = x_2$ and so on.

We see the resultant when x is a pure real number is a neutrosophic pair.

Let y = (0, 0, 2) $\in$ R be the true real range space state vector.

To find the effect of y on S.

$yS^t = (2I, 0, 0, 2I, 2I, 0, 0, 0) = x_1$;

$x_1S = (2I, 4I, 0) = y_1$;

$y_1S^t = (0, 0, 2I, 4I, 0, 0, 4I, 4I) = x_2$;

$x_2S = (2I, 0, 4I) = y_2$;

$y_2S^t = (4I, 4I, 0, 0, 4I, 4I, 2I, 0) = x_3$;

$x_3S = (0, 0, 2I) = y_3$;

$y_3S^t = (2I, 0, 0, 2I, 2I, 0, 0, 0) = x_2$;

$x_4S = (2I, 4I, 0) = y_4\ (= y_1)$.

Thus the MOD resultant is a realized limit cycle pair given by {(0, 0, 2I, 4I, 0, 0, 4I, 4I), (2I, 4I, 0)}.

y = (0, 0, 2) is real however the resultant is a pure neutrosophic pair.

Let x = (0, 2 + I, 0, 0, 0, 0, 0, 0) $\in$ D.

To find the effect of x on S;

$xS = (0, 0, 3I) = y_1$;

$y_1S^t = (3I, 3I, 0, 3I, 0, 3I, 3I, 0) = x_1$;

$x_1S = (0, 0, 3I) = y_2$ $(=y_1)$.

Thus the MOD resultant is a fixed point pair given by {(3I, 3I, 0, 3I, 0, 3I, 3I, 0), (0, 0, 3I)} which is also pure neutrosophic. However x is D is not pure neutrosophic.

Let y = (1 + 2I, 0, 0) $\in$ R.

To find the effect of y on S;

$yS^t = (3I, 0, 0, 3I, 3I, 3I, 3I, 0) = x_1$;

$x_1S = (3I, 0, 0) = y_1$;

$y_1S^t = (3I, 0, 0, 3I, 3I, 3I, 3I, 0) = x_2$ $(=x_1)$.

Thus the resultant is a realized fixed point pair given by {(3I, 0, 0, 3I, 3I, 3I, 3I, 0), (3I, 0, 0)} which is a pure neutrosophic pair though y = (1 + 2I, 0, 0) $\in$ R is not pure neutrosophic state vector.

Let x = (I, 0, 0, 0, 0, 0, 0, 0) $\in$ D.

To find the effect of this pure neutrosophic vector x on S.

$$xS = (3I, 0, I) = y_1;$$

$$y_1S^t = (4I, 3I, 0, 4I, I, 0, 0, 0) = x_1;$$

$x_1S = (I, 2I, 3I) = y_2;$

$y_2S^t = (0, 3I, 4I, 2I, 3I, 2I, 2I, 2I) = x_2;$

$x_2S = (5I, 0, 5I) = y_3;$

$y_3S^t = (2I, I, 0, 4I, 5I, 4I, 4I, 0) = x_3;$

$x_3S = (3I, 0, 5I) = y_4;$

$y_4S^t = (2I, I, 0, 2I, 5I, 0, 0, 0) = x_4;$

$x_4S = (I, 0, 3I) = y_5;$

$y_5S^t = (0, 5I, 0, 4I, 3I, 2I, 4I, 0) = x_5;$

$x_5S = (I, 2I, I) = y_6;$

$y_6S^t = (4I, 3I, 4I, 0, I, 2I, 2I, 2I) = x_6;$

$x_6S = (3I, 4I, 4I) = y_7;$

$y_7S^t = (I, 2I, 2I, 3I, I, 3I, 5I, 4I) = x_7;$

$x_7S = (3I, 4I, 2I) = y_8;$

$y_8S^t = (5I, 2I, 2I, I, 5I, 3I, 5I, 4I) = x_8;$

$x_8S = (I, 0, 2I) = y_9;$

$y_9S^t = (5I, 5I, 0, 3I, 5I, 5I, I, 0) = x_9;$

$x_9S = (3I, 0, I) = y_{10}\ (=y_1).$

Thus the resultant is a realized limit cycle pair given by $\{(4I, 3I, 0, 4I, I, 0, 0, 0), (3I, 0, I)\}$.

We see x = (I, 0, 0, 0, 0, 0, 0, 0) is pure neutrosophic and the resultant is also pure neutrosophic.

Let y = (0, 0, 3I) $\in$ R.

To find the effect of y on S.

$$yS^t = (3I, 3I, 0, 3I, 0, 3I, 3I, 0) = x_1;$$

$$x_1S = (0, 0, 3I) = y_1 \ (=y).$$

Thus the effect of y = (0, 0, 3I) is the classical fixed point pair given by
{(3I, 3I, 0, 3I, 0, 3I, 3I, 0), (0, 0, 3I)}.

We give one more example.

***Example 3.7:*** Let

$$M = \begin{bmatrix} 2I & 6I & 0 & I & 2I \\ 4I & 0 & 4I & 0 & 6I \end{bmatrix}$$

be the MOD neutrosophic matrix operator with entries from $\langle Z_8 \cup I \rangle = \{a + bI \mid a, b \in Z_8, I^2 = I\}$.

Let D = {(a, b) | a, b $\in \langle Z_8 \cup I \rangle$} be the domain space of neutrosophic state vectors

R = {($a_1, a_2, a_3, a_4, a_5$) | $a_i \in \langle Z_8 \cup I \rangle$, $1 \leq i \leq 4$} be the range space of neutrosophic state vectors.

Let x = (2, 0) $\in$ D. To find the effect of x on M.

$$xM = (4I, 4I, 0, 2I, 4I) = y_1;$$

$$y_1M^t = (2I, 0) = x_1;$$

$$x_1M = (4I, 4I, 0, 2I, 4I) = y_2 \ (=y_1).$$

Thus the resultant is a realized fixed point pair given by $\{(2I, 0), (4I, 4I, 0, 2I, 4I)\}$.

We see the resultant is a pure neutrosophic fixed point pair though $x = (2, 0)$ is just real.

Consider $y = (0, 0, 1, 0, 0)$ be the range space of state vectors.

To find the effect of y on M is as follows.

$yM^t = (0, 4I) = x_1$; $\quad x_1M = (0, 0, 0, 0, 0) = y_1$;

$y_1M^t = (0, 0) = x_2$; $\quad x_2M = (0, 0, 0, 0, 0) = y_2\ (=y_1)$.

Thus the resultant is a realized fixed point pair given by $\{(0, 0), (0, 0, 0, 0, 0)\}$.

Let $x = (2 + 3I, 0) \in D$, to find the effect of x on M is as follows.

$xM = (2I, 6I, 0, 5I, 2I) = y_1$; $\quad yM^t = (I, 4I) = x_1$;

$x_1M = (2I, 6I, 0, I, 2I) = y_2$; $\quad y_2M^t = (5I, 0) = x_2$;

$x_2M = (2I, 6I, 0, 5I, 4I) = y_3$; $\quad y_3M^t = (5I, 0) = x_3\ ( = x_2)$.

Thus the resultant is a realized fixed point pair given by $\{(5I, 0), (2I, 6I, 0, 5I, 4I)\}$.

Let $y = (0, 0, 0, 0, 2I + 1) \in R$ be the state vector in the range space.

To find the effect of y on M.

$yM^t = (6I, 2I) = x_1$; $\quad x_1M = (4I, 4I, 0, 6I, 0) = y_1$;

$y_1M = (6I, 0) = x_2$; $\quad x_2M = (4I, 4I, 0, 6I, 4I) = y_2$;

$y_2M = (6I, 0) = x_3$ ($=x_2$).

Thus we see the resultant is a realized fixed point given by the pair {(6I, 0), (4I, 4I, 0, 6I, 4I)}.

From these it is observed that using this pure neutrosophic matrix as the MOD operator we get resultant of every vector from D or R gives only a pure neutrosophic pair.

This result is given by the following theorem.

**THEOREM 3.2:** *Let*

$$M = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

*be the pure neutrosophic matrix with entries* $a_{ij} \in Z_sI \subseteq \langle Z_s \cup I \rangle = \{a + bI / a, b \in Z_s; I^2 = I\}$; $(2 \leq s < \infty)$, $1 \leq i \leq m$ *and* $1 \leq j \leq n$.

$D = \{(a_1, \ldots, a_m) \mid a_i \in \langle Z_s \cup I \rangle;\ i = 1, 2, \ldots, m\}$ *be the domain space of state vectors.*

$R = \{(b_1, \ldots, b_n) \mid b_i \in \langle Z_s \cup I \rangle;\ i = 1, 2, \ldots, n\}$ *be the range space of state vectors.*

*Every* $x \in D$ *(or* $y \in R$*) gives the resultant only as a pure neutrosophic pair.*

**Proof:** Follows from the fact that if M the MOD neutrosophic matrix operator has only pure neutrosophic entries then we see $x \in D$ (or $y \in R$) happens to be only a pure neutrosophic pair.

We will give examples by working with on or off of indeterminate state of domain state vectors and range state vectors.

We will illustrate this situation by some examples.

***Example 3.8:*** Let

$$\begin{bmatrix} 3+I & 5+2I & 0 \\ 0 & 7I & 3I+1 \\ 2I & 0 & 4 \\ 5+2I & 2 & 0 \\ 4+I & 0 & 2I+4 \end{bmatrix}$$

be the MOD neutrosophic rectangular matrix operator with entries from $\langle Z_{10} \cup I\rangle = \{a + bI / a, b \in Z_{10}, I^2 = I\}$.

Let D = $\{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in \{0, 1, I, 1 + I\}; 1 \leq i \leq 5\}$ be the domain space special state vectors which has entries from off state on state or I the indeterminate state.

R = $\{(b_1, b_2, b_3) \mid b_i \in \{0, 1, I, 1 + I); 1 \leq i \leq 3\}$ be the range of space of special state vectors.

We follow the following rules.

$a + bI = I$ if $a < b$; $a + bI = 1$ if $a > 1$;

in $a + bI = 1 + I$ if $a = b$

while finding the effect on any state vector $x \in D$ (or $y \in R$).

Let $x = (1, 0, 0, 0, 0) \in D$;

$xM = (3 + I, 5 + 2I, 0) \to (1, 1, 0) = y_1$;

$y_1M^t = (8 + 3I, 7I, 2I, 7 + 2I, 4 + I) \to (1, I, I, 1, 1) = x_1$;

$x_1M = (2 + 6I, 7 + 9I, 4 + I) \to (I, I, 1) = y_2$;

$y_2M^t = (I, 1, 2I + 4, 9I, 7I + 4) \to (I, 1, 1, I, I) = x_2$;

$x_2M = (8I, 2I, 9I) \to (I, I, 1) = y_3$ $(=y_2)$.

Thus the resultant is a fixed point pair given by
$\{(I, 1, 1, I, I), (I, I, 1)\}$.

Consider $y = (0, 0, 1) \in R$;

To find the effect of y on M.

$yM^t = (0, 3I + 1, 4, 0, 2I + 4) \to (0, I, 1, 0, 1) = x_1$;

$x_1M = (4 + 3I, 9I + 5\ 6I + 8) \to (1, I, 1) = y_1$;

$y_1M^t = (3 + 8I, 1, 4 + 2I, 7 + 2I, 3I + 8) \to$
$(I, 1, 1, 1, 1) = x_2$;

$x_2M = (9 + 9I, 7 + 9I, 5I + 5) = (1 + I, I, 1 + I) = y_2$;

$y_2M^t = (3 + 2I, 1 + 4I, 4 + 8I, 9 + 7I, 8 + 4I)$
$\to (1, I, I, 1, 1) = x_3$;

$x_3M = (2 + 6I, 7 + 9I, 6I + 4) \to (I, I, I) = y_3$;

$y_3M^t = (I, I, 6I\ 9I\ I) \to (I, I, I, I, I) = x_4$;

$x_4M = (2I, 6I\ 0) \to (I, I, 0) = y_4$;

$y_4M^t = (I, 7I, 2I, 9I, 5I) \to (I, I, I, I, I) = x_5$ $(=x_4)$.

Clearly the resultant is realized fixed point pair given by
$\{(I, I, I, I, I), (I, I, 0)\}$.

Let $x = (0, 0, 0, 0, I) \in D$ to find the effect of x on M.

$xM = (5I, 0, 6I) \to (I, 0, I) = y_1$;

$y_1M^t = (4I, 4I, 6I\ 7I\ I) \to (I\ I\ I\ I\ I) = x_1$;

$x_1M = (8I\ 6I\ 4I) \to (I, I, I) = y_2;$

$y_2M^t = (I, I, 6I, 9I, I) \to (I, I, I, I, I) = x_2\ (=x_1).$

Thus the effect of x = (0, 0, 0, 0, I) on M gives a fixed point pair given by

{(I, I, I, I, I), (I, I, I)}.

Let y = (0, I, 0) ∈ R; to find the effect of y on M.

$yM^t = (7I, 7I, 0, 2I, 0) \to (I, I, 0, I, 0) = x_1;$

$x_1M = (I, 6I, 0) \to (I, I, 0) = y_1;$

$y_1M^t = (I, 7I, 2I, 9I, 5I) \to (I, I, I, I, I) = x_2;$

$x_2M = (8I, 6I, 0) \to (I, I, 0) = y_2;$

$y_2M^t = (I, 7I, 2I, 9I, 5I) \to (I, I, I, I, I) = x_3.$

Thus the resultant is a fixed point pair given by
{(I, I, I, I, I), (I, I, 0)}

Let x = (0, 1 + I, 0, 0, 0) ∈ D.

$xM = (0, 8I\ 7I + 1) \to (0, I, I) = y_1;$

$y_1M^t = (7I, I, 4I, 2I, 6I) \to (I, I, I, I, I) = x_1;$

$x_1M = (6I\ 6I\ 4I) \to (I, I, I) = y_2;$

$y_2M^t = (I, I, 6I, 9I, I) \to (I, I, I, I, I) = x_2.$

Thus the resultant is a realized fixed point pair given by
{(I, I, I, I, I), (I, I, I)}.

***Example 3.9:*** Let

$$M = \begin{bmatrix} 5 & 0 & 1 & 2 \\ 0 & 1 & 0 & 5 \\ 1 & 0 & 2 & 2 \\ 4 & 1 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix}$$

be the neutrosophic rectangular MOD matrix operator with elements from $\langle Z_6 \cup I\rangle$.

Let D = $\{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in \{0, 1, I, 1 + I\}; 1 \leq i \leq 5\}$ be the domain special space of state vectors.

R = $\{(b_1, b_2, b_3, b_4) \mid b_i \in \{1, 0, I, 1 + I\}; 1 \leq i \leq 4\}$ be the range space of special state of vectors.

Let x = (1, 0, 0, 0, 1) $\in$ D.

$xM = (2, 0, 1, 3) \to (1, 0, 1, 1) = y_1;$

$y_1M^t = (2, 5, 5, 5, 4) \to (1, 1, 1, 1, 1) = x_1;$

$x_1M = (1, 2, 4, 4) \to (1, 1, 1, 1) = y_2;$

$y_2M^t = (2, 0, 5, 0, 4) \to (1, 0, 1, 0\ 1) = x_2;$

$x_2M = (3, 0, 3, 5) \to (1\ 0\ 1\ 1) = y_3\ (= y_1).$

Thus the resultant is a realized fixed point pair given by
{(1, 1, 1, 1, 1), (1, 0, 1, 1)}.

Let y = (0 1 1 0) $\in$ R.

To find the effect of y on M.

$yM^t = (1, 1, 2, 2, 0) \to (1, 1, 1, 1, 0) = x_1;$

$x_1M = (4, 2, 4, 9) \rightarrow (1, 1, 1, 1) = y_1;$

$y_1M^t = (2, 0, 5, 0, 4) \rightarrow (1, 0, 1, 0, 1) = x_2;$

$x_2M = (3, 0, 3, 5) \rightarrow (1, 0, 1, 1) = y_2;$

$y_2M^t = (2, 5, 5, 5, 4) \rightarrow (1, 1, 1, 1, 1) = x_3;$

$x_3M = (1\ 2\ 4\ 4) \rightarrow (1\ 1\ 1\ 1) = y_3\ (= y_1).$

Thus the resultant is a realized fixed point pair given by $\{(1, 0, 1, 0, 1), (1, 1, 1, 1)\}$.

Let $x = (I, 1, 0\ 0\ 0) \in D$.

To find the effect of x on M.

$xM = (5I, 1, I, 2I + 5) \rightarrow (I, 1, I, 1) = y_1;$

$y_1M^t = (6I+2, 6, 3I+2, 5I+1, 3I+1) \rightarrow (I, 1, I, I, I) = x_1;$

$x_1M = (I , 1 + I, 4I, 5 + 5I) \rightarrow (I, 1 + I, I, 1 + I) = y_2;$

$y_2M^t = (2 + 2I, 0, 5I + 2, 1, 4I + 1) \rightarrow (1 + I, 0, I, 1, I) = x_2;$

$x_2M = (3 + 3I, 1, 2 + 3I, 5I + 2) \rightarrow (1 + I, 1, I, I) = y_3;$

$$y_3M^t = (5 + 2I, 1 + 5I, 1 + 3I, 5I + 5, 2I + 1) \rightarrow (1, I, I, 1 + I, I) = x_3;$$

$x_3M = (3 + 2I, 1 + 2I, 2 + 3I, 2 + I) \rightarrow (1, I, I, 1) = y_4;$

$$y_4M^t = (1 + I, 5 +I, 3 + 2I, 4 + 2I, 4) \rightarrow (1 + I, 1, 1, 1, 1) = x_4;$$

$x_4M = (5I + 2, 2, 3+ I, 3 + 2I) \rightarrow (I, 1, 1, 1) = y_5;$

$y_5M^t = (5I + 3, 0, 4 + I, 4I + 2, 3I + 1) \to (I, 0, 1, I, I) = x_5$;

$x_5M = (1, I, 1 + 2I, 2 + 3I) \to (1, I, I, I) = y_6$;

$y_6M^t = (5 + 3I, 0, 4I + 1, 4 + 2I, 3 + I) \to (1, 0, I, 1, 1) = x_6$;

$x_6M = (0 + I, 1, 2 + 2I, 3 + 2I) \to (I, 1, 1 + I, 1) = y_7$;

$y_7M^t = (3, 0, 4 + 3I, 2 + 5I, 3I + 1) \to (1, 0, 1, I, I) = x_7$;

$x_7M = (I, I, 2 + I, 4 + I) \to (I, I, 1, 1) = y_8$;

$y_8M^t \to (I, 1, 1, I, I) = x_8$;

$x_8M \to (1, 1 + I, 1 + I, I)$ and so on.

However after a finite number of iterations certainly we will arrive at a realized fixed point pair or a realized limit cycle pair which will be mixed neutrosophic as we have started with a neutrosophic element. It may be possible we get pure real or pure neutrosophic or zero also.

***Example 3.10:*** Let

$$M = \begin{bmatrix} 3 & 1 & 0 \\ 0 & 6 & 6 \\ 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 5 \\ 4 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$$

be the MOD neutrosophic matrix operator with entries from $Z_7 \subseteq \langle Z_7 \cup I\rangle = \{a + bI \mid a, b \in Z_7, I^2 = I\}$.

$D = \{(x_1, x_2, \ldots, x_7) \mid x_i \in \{0, 1, I, 1 + I\}, 1 \leq i \leq 7\}$ and

R = {(a, b, c) | a, b, c ∈ {0, 1, I, 1 + I}} be the domain and range space of special state vectors respectively.

Let x = (1, 0, 0, 0, 0, 0, 0) be a real state vectors in D.

To find the effect of x on M;

$xM = (3, 1, 0) \to (1, 1, 0) = y_1$;

$y_1M^t = (4, 6, 1, 2, 1, 5, 1) \to (1, 1, 1, 1, 1, 1, 1) = x_1$ ;

$x_1M = (2, 4, 6) \to (1, 1, 1) = y_2$;

$y_2M^t = (4, 5, 2, 2, 6, 5, 2) \to (1, 1, 1, 1, 1, 1, 1) = x_2 \ (=x_1)$.

Thus the resultant is a realized fixed point pair given by {(1, 1, 1, 1, 1, 1, 1), (1, 1, 1)}.

Let y = (0, 0, 1) ∈ R be the real initial state vector. The effect of y on M is as follows.

$yM^t = (0, 6, 1, 0, 5, 0, 1) \to (0, 1, 1, 0, 1, 0, 1) = x_1$;

$x_1M = (2, 0, 6) \to (1, 0, 1) = y_1$;

$y_1M^t = (3, 6, 2, 0, 6, 4, 1) \to (1, 1, 1, 0, 1, 1, 1) = x_2$;

$x_2M = (2, 2, 6) \to (1, 1, 1) = y_2$;

$y_2M^t = (4, 5, 2, 2, 6, 5, 2) \to (1, 1, 1, 1, 1, 1, 1) = x_3$;

$x_3M = (2, 4, 6) \to (1, 1, 1) = y_3 \ (=y_2)$.

Thus the resultant is a realized fixed point pair given by
{(1, 1, 1, 1, 1, 1, 1), (1, 1, 1)}.

Let x = (0, 0, 0, 0, 0, I, 0) ∈ D.

A pure neutrosophic number. To find the effect of x on M;

$xM = (4I, I, 0) \rightarrow (I, I, 0) = y_1;$

$y_1M^t = (4I, 6I\ I, 2I\ I, 5I, I) \rightarrow (I, I, I, I, I, I, I) = x_1;$

$x_1M = (2I, 4I, 6I) \rightarrow (I, I, I) = y_2;$

$y_2M^t = (4I, 5I, 2I, 2I, 6I, 5I, 2I) \rightarrow (I, I, I, I, I, I, I)$
$= x_2\ (=x_1).$

The resultant is a realized fixed point pair given by {(I, I, I, I, I, I, I), (I, I, I)} which is pure neutrosophic.

Let $x = (0, 1 + I, 0, 0, 0, 0, 0) \in D$.

Clearly x is mixed neutrosophic.

To find the effect of x on M.

$xM = (0\ 6 + 6I, 6 + 6I) \rightarrow (0, 1 + I, 1 + I) = y_1;$

$y_1M^t = (1 + I, 5 + 5I, I + 1, 2 + 2I, 6 + 6I, 5 + 5I, 2 + 2I) \rightarrow$

$(1 + I, 1 + I, 1 + I, 1 + I, 1 + I, 1 + I, 1 + I) = x_1;$

$x_1M = (2 + 2I, 4 + 4I, 6 + 6I) \rightarrow (1 + I, 1 + I, 1 + I) = y_2;$

$y_2M^t = (4 + 4I, 5 + 5I, 2 + 2I, 2 + 2I, 6 + 6I, 5 + 5I, 2 + 2I)$

$\rightarrow (1 + I, 1 + I, 1 + I, 1 + I, 1 + I, 1 + I, 1+ I).$

Thus the resultant is a realized fixed point pair given by {(1 + I, 1 + I, 1 + I, 1 + I, 1 + I, 1 + I), (1 + I, 1 + I, 1 + I)}.

We see if M the MOD neutrosophic operator is real then real domain state vector gives real resultant pair, pure neutrosophic state vector gives pure neutrosophic resultant pair and a mixed domain state vector results in mixed resultant pair.

This is true in case of range space state vectors also. In view of this we prove the following theorem.

**THEOREM 3.3:** *Let*

$$M = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} a_{ij}\ Z_s \subseteq \langle Z_s \cup I \rangle$$

$$= \{a + bI \,/\, a,b \in Z_s,\ I^2 = I,\ 2 \leq s < \infty\};\ 1 \leq i \leq M \text{ and } 1 \leq j \leq n$$

*be the MOD neutrosophic real matrix operator with entries from* $\langle Z_s \cup I \rangle$.

*Let* $D_s = \{(a_1, a_2, \ldots, a_m) \,/\, a_i \in \{0, 1, I, 1 + I\};\ 1 \leq i \leq m\}$ *and*

$R_s = \{(b_1,\ b_2, \ldots, b_n) \,/\, b_j \in \{0, 1, I, 1 + I\};\ 1 \leq j \leq n\}$ *be the space initial state vectors associated with M.*

*i) If $x \in D$ (or $y \in R$) is real state vector then the resultant pair is also real.*

*ii) If $x \in D$ (or $y \in R$) is a pure neutrosophic state vector then the resultant pair is also pure neutrosophic.*

*iii) If $x \in D$ (or $y \in R$) is a mixed initial state vector then so is the resultant pair.*

Proof is direct and hence left as an exercise to the reader.

It is important to record that M can also be equal to n and we can use the MOD-neutrosophic matrix operator with domain and range space.

To this effect we give some examples.

***Example 3.11:*** Let

$$S = \begin{bmatrix} 3 & 0 & 2+I & 0 & 2+I \\ 0 & 1+I & 0 & 4+I & 0 \\ 1 & 0 & 3I & 0 & 1+I \\ I & 1 & 0 & 3+I & 0 \\ 2+3I & 0 & 4 & 0 & I \end{bmatrix}$$

be the MOD neutrosophic matrix operator with entries from $\langle Z_6 \cup I\rangle = \{a + bI \mid a, b \in Z_6, I^2 = I\}$.

$D = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in \langle Z_6 \cup I\rangle, 1 \leq i \leq 5\}$ and

$R = \{(b_1, b_2, \ldots, b_5) \mid b_i \in \langle Z_6 \cup I\rangle; 1 \leq i \leq 5\}$ be the domain and range space of state vectors respectively associated with S.

Let $x = (1, 0, 0, 0, 0) \in D$ to find the effect of x on S;

$xS = (3, 0, 1, I, 2 + 3I) = y_1$;

$$y_1S^t = (3, 0, 1, I, 2 + 3I) \begin{bmatrix} 3 & 0 & 2+I & 0 & 2+I \\ 0 & 1+I & 0 & 4+I & 0 \\ 1 & 0 & 3I & 0 & 1+I \\ I & 1 & 0 & 3+I & 0 \\ 2+3I & 0 & 4 & 0 & I \end{bmatrix}$$

$= (2 + 4I, I, 2, 4I, 1 + 3I) = x_1$;

$x_1S = (0, 4I, 3+5I, 4 + 3I, 4 + 4I) = y_1$;

$y_1S^t = (5 + 2I, 4 + 5I, 4I +4, 0, 3 + 3I) = x_1$;

$x_1S = (5 + 4I, 4 + 3I, 3 + 2I, 4, 5I + 2) = y_1$.

We work with $y_1S^t$ and so on.

However after a finite number of steps we arrive at a realized fixed point pair or a realized limit cycle point pair.

Let y = (1, 0, 0, 0, 0) ∈ R.

To find the effect of y on S.

$yS^t = (3, 0, 2 + I, 0, 2 + I) = x_1;$

$x_1S = (4I + 5, 0, 5, 3I, I + 2) = y_1;$

$y_1S^t = (2I, 3I, 0, 0, 3 + 5I) = x_2;$

$x_2S = (0, 0, 3 + 3I, 5I, 0) = y_1$ and so on we arrive at a realized fixed point pair or a realized limit cycle pair.

Let us give one more example before we make conclusions.

***Example 3.12:*** Let

$$P = \begin{bmatrix} 3+I & 2I & 0 & 4+3I \\ 0 & 1+I & 1 & 0 \\ 0 & 0 & 4I+1 & 3 \\ 0 & 2 & 0 & 1+I \end{bmatrix}$$

be the MOD neutrosophic matrix operator with entries from $\langle Z_6 \cup I\rangle$.

Let D = R = $\{(a_1, a_2, a_3, a_4) \mid a_i \in \langle Z_6 \cup I\rangle = \{a + bI;\ a, b \in Z_6;\ I^2 = I\}\ 1 \leq i \leq 4\}$ be the state vectors.

$D_S = R_S = \{(a_1, a_2, a_3, a_4) \mid a_i \in \{0, 1, I, 1 + I\};\ 1 \leq i \leq 4\}$ be the special class of state vectors.

Let x = (1, 0, 0, 0) ∈ D.

$$xP = (3 + I, 2I, 0, 4 + 3I) = y_1;$$

$$y_1P^t = (1 + 2I, 4I, 2, 4 + 2I) = x_1;$$

$$x_1P = (3 + 3I, 2, 2, I + 2) = y_2;$$

$$y_2P^t = (2I + 5, 4 + 2I, 2 + 5I, 4I) = x_2.$$

and so on; we will get a realized fixed point pair or a realized limit cycle pair.

Let us consider $x = (1, 0\ 0\ 0) \in D_S$;

$xP = (3 + I, 2I, 0, 4 + 3I) = (1, I, 0, 1) = y_1$;

$y_1P^t \to (1, I, 1, I) = x_1$; $\quad$ $x_1P \to (1, I, I, 1) = y_2$;

$y_2P^t \to (1, I, I, I) = x_2$; $\quad$ $x_2P \to (1, I, I, 1) = y_3\ (=y_2)$.

Thus the resultant is a realized fixed point pair or a realized limit cycle pair.

Clearly there is a difference between the resultant of $x = (1, 0, 0, 0)$ realized as an element of D and as an element of $D_S$.

So there is a difference in the resultant for the same x.

Let $y = (1, 0, 0, 0) \in R$. The effect of y on P.

$yP^t = (3 + I, 0, 0, 0) = x_1$; $\quad$ $x_1P = (4 + I, 2I, 0, 4I) = y_1$;

$y_1P^t = (0, 4I, 0, 0) = x_2$; $\quad$ $x_2P = (0, 2I, 2I, 4I) = x_3$;

$x_3P = (4I, 2I, 0, 0) = y_2$ and so on.

Now let $y = (1, 0, 0, 0) \in R_S$;

$yP^t \to (1, 0, 0, 0) = x_1$; $\quad$ $x_1P \to (1, I, 0, 1) = y_1$;

$y_1P^t \to (1, I, 1, I) = x_2;$ $\quad$ $x_2P \to (1, I, I, I) = y_2;$

$y_2P^t \to (I, I, I, I) = x_3;$ $\quad$ $x_3P \to (I, 0, 0 ,0) = y_3;$

$y_3P^t \to (I, 0, 0, 0) = x_4;$ $\quad$ $x_4P \to (I, I, 0, I) = y_4;$

$y_4P^t \to (I, I, I, I) = x_5;$ $\quad$ $x_5P \to (I, 0, 0, 0) = y_5\ (=y_3).$

Thus we see the resultant is a realized limit cycle pair given by

$$\{(I, 0, 0, 0), (I, 0, 0, 0)\}.$$

We see the resultant of y in R is different from the resultant of y in $R_S$.

Now we proceed onto describe the MOD finite complex integer matrix operators.

$$\text{Let } M = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} a_{ij} \in C(Z_s),$$

$$2 \leq s < \infty,\ 1 \leq i \leq m,\ 1 \leq j \leq n;$$

be the MOD-finite complex modulo integer matrix operator where

$$C(Z_s) = \{a + bi_F \mid i_F^2 = (s-1);\ a, b \in Z_s\}.$$

There is no problem even if m = n happens still we call it the rectangular matrix MOD operators only.

We will first illustrate this situation by some examples.

***Example 3.13:*** Let

$$S = \begin{bmatrix} 3i_F & 2+i_F & 0 \\ 0 & 2i_F & 1 \\ 1+4i_F & 0 & 1+i_F \\ 0 & 3 & 4i_F \end{bmatrix}$$

be the MOD complex modulo integer rectangular matrix operator with entries from $C(Z_5) = \{(a + bi_F \mid i_F^2 = 4; a, b \in Z_5\}$.

Let $D = \{(a_1, a_2, a_3, a_4) \mid a_i \in C(Z_5); 1 \leq i \leq 4\}$ and

$R = \{(b_1, b_2, b_3) \mid b_i \in C(Z_s); 1 \leq i \leq 3\}$ be the domain and range space of state vectors respectively associated with the MOD complex number rectangular operator S.

Let $x = (3i_F, 0, 1, 0) \in D$;

To find the effect of x on S.

$xS = (2 + 4i_F, i_F + 2, 1 + i_F) = y_1$;

$y_1S^t = (1, 3i_F + 4, 1 + 2i_F, 2 + 2i_F) = x_1$;

$x_1S = (3 + 4i_F, 2, 4i_F) = y_2$.

We work for the resultant and after a finite number of iterations we arrive at a realized fixed point pair or a realized limit cycle pair.

Let $y = (0, 0, 1) \in R$.

To find the effect of y on S;

$yS^t = (0, 1, 1 + i_F, 4i_F) = x_1$; $\qquad x_1S = (2, 4i_F, 2i_F) = y_1$;

$y_1S^t = (1 + 4i_F, 1, 0, 2 + 2i_F) = x_2$.

Here also we are guaranteed that after a finite number of iterations we will arrive at a realized limit cycle pair or a realized fixed point pair.

We are sure for every $x \in D$ (or R) we will arrive at a realized fixed point pair or a realized limit cycle pair after a finite number of iterations.

Consider yet another example.

***Example 3.14:*** Let

$$M = \begin{bmatrix} 3 & 1 \\ 0 & 2 \\ 4 & 5 \\ 0 & 1 \\ 1 & 0 \\ 2 & 3 \\ 1 & 0 \end{bmatrix}$$

be the MOD complex integer rectangular operator with entries from $C(Z_6) = \{a + bi_F) \mid i_F^2 = 5, a, b \in Z_6\}$.

$D = \{(x_1, x_2, \ldots, x_7) \mid x_i \in C(Z_6); 1 \leq i \leq 7\}$ be the domain space of state vectors associated with M.

$R = \{(x, y) \mid x, y \in C(Z_6)\}$ be the range space of state vectors associated with M.

Let $x = (1, 0, 0, 0, 0, 0, 0) \in D$.

To find the effect of x on M.

$xM = (3, 1) = y_1$; $\quad y_1M^t = (4, 2, 5, 1, 3, 3, 3) = x_1$;

$x_1M = (2, 1) = y_2$; $\quad y_2M^t = (1, 2, 1, 1, 2, 1, 2) = x_2$;

$x_2M = (1, 2) = y_3$; $\quad y_3M = (5, 4, 2, 2, 1, 2, 1) = x_3$;

$x_3M = (5, 1) = y_4$; $\quad y_4M^t = (4, 2, 1, 1, 5, 1, 5) = x_4$;

$x_4M = (4, 5) = y_5$; $\quad y_5M^t = (5, 4, 5, 5, 4, 5, 4) = x_5$;

$x_5M = (5,0) = y_6$; $\quad y_6M^t = (3, 0, 2, 0, 5, 4, 5) = x_6$;

$x_6M = (5,1) = y_7$; $\quad y_7M^t = (4, 1, 1, 1, 5, 1, 5) = x_7$;

$x_7M = (4, 3) = y_8$; $\quad y_8M^t = (3, 0, 1, 3, 4, 5, 4) = x_8$;

$x_8M = (1, 2) = y_9$; $\quad y_9M^t = (5, 4, 2, 2, 1, 2, 1) = x_9$;

and so on.

It is important to note that if the entries of the MOD finite complex number matrix operator has only real and the state vectors taken from the domain space is also real the resultant is a realized fixed point pair or a limit cycle pair which will always be a real pair of vectors.

Consider $y= (0, 1) \in R$ to find the effect of y on M.

$yM^t = (1, 2, 5, 1, 0, 3, 0) = x_1$; $\quad x_1M = (5, 4) = y_1$;

$y_1M = (1, 2, 4, 4, 5, 4, 5) = x_2$; $\quad x_2M = (1, 5) = y_2$;

$y_2M^t = (2, 4, 5, 5, 1, 5, 1) = x_3$; $\quad x_3M = (2, 2) = y_3$;

$y_3M^t = (2, 4, 0, 2, 2, 4, 2) = x_4$; $\quad x_4M = (0, 0) = y_4$;

$y_4M^t = (0, 0, 0, 0, 0, 0, 0) = x_5$; $\quad x_5M = (0, 0) = y_5 (=y_4)$.

Thus the resultant is a realized fixed point pair given by

$\{(0, 0, 0, 0, 0, 0, 0), (0, 0)\}$.

Let $x = (0, 0, 0, i_F, 0, 0, 0) \in D$ find the effect of on M.

$xM = (0, i_F) = y_1$;

$y_1M^t = (i_F, 2i_F, 5i_F, i_F, 0, 3i_F, 0) = x_1$;

$x_1M = (5i_F, 4i_F) = y_2$;

$y_2M^t = (i_F, 2i_F, 4i_F, 4i_F, 5i_F, 2i_F, 5i_F) = x_2$;

$x_2M = (3i_F, 5i_F) = y_3$;

$y_3M^t = (2i_F, 4i_F, i_F, 5i_F, 3i_F, 3i_F, 3i_F) = x_3$;

$x_3M = (5i_F, 5i_F) = y_4$;

$y_4M^t = (2i_F, 4i_F, 3i_F, 5i_F, 5i_F, i_F, 5i_F) = x_4$;

$x_2M = (0, 3i_F) = y_5$;

$y_5M^t = (3i_F, 0, 3i_F, 3i_F, 0, 3i_F, 0) = x_5$;

$x_5M = (3i_F, 0) = y_6$;

$y_6M^t = (3i_F, 0, 0, 0, 3i_F, 0, 3i_F) = x_6$;

$x_6M = (3i_F, 3i_F) = y_7$;

$y_7M^t = (0, 0, 3i_F, 3i_F, 3i_F, 3i_F, 3i_F) = x_7$;

$x_7M = (0, 3i_F) = y_8\ (=y_5)$.

Thus we see the resultant is a realized limit cycle pair given by $\{(3i_F, 0, 3i_F, 3i_F, 0, 3i_F, 0), (0, 3i_F)\}$.

Let $y = (i_F, 0) \in R$. To find the effect of y on M.

$yM^t = (3i_F, 0, 4i_F, 0, i_F, 2i_F, i_F) = x_1$;

$x_1M = (i_F, 5i_F) = y_1$ ;

$y_1M^t = (2i_F, 4i_F, 5i_F, 5i_F, i_F, 5i_F, i_F) = x_2$;

$x_2M = (2i_F, i_F) = y_2$;

$y_2M^t = (i_F, 2i_F, i_F, i_F, 2i_F, i_F, 2i_F) = x_3$;

$x_3M = (i_F, i_F) = y_3$;

$y_3M^t = (4i_F, 2i_F, 3i_F, i_F, 5i_F, i_F) = x_4$;

$x_4M = (0, 3i_F) = y_4$;

$y_4M^t = (3i_F, 0, 3i_F, 3i_F, 0, 3i_F, 0) = x_5$;

$x_5M = (3i_F, 0) = y_5$;

$y_5M^t = (3i_F, 0, 0, 0, 3i_F, 0, 3i_F) = x_6$;

$x_6M = (3i_F, 3i_F) = y_6$;

$y_6M^t = (0, 0, 3i_F, 3i_F, 3i_F, 3i_F, 3i_F) = x_7$;

$x_7M = (0, 3i_F) = y_7$ $(=y_4)$.

Thus the resultant is a realized limit cycle pair given by
$\{(3i_F, 0, 3i_F, 3i_F, 0, 3i_F, 0), (0, 3i_F)\}$.

Thus it is important to keep on record that if we use pure complex number as the state vectors from domain space or range space the resultant will only be a pure complex number provided the matrix operator is a real matrix operator.

In view of this the following result is proved.

**THEOREM 3.4**: *Let*

$$M = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \cdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$$

*be the MOD complex rectangular matrix operator with entries from $Z_s \subseteq C(Z_s)$ $(2 \le s < \infty)$; $i_F^2 = s - 1$.*

*Let $D = \{(a_1, a_2, \ldots, a_m) \mid a_i \in C(Z_s); 1 \le i < m\}$ and*

*$R = \{(b_1, b_2, \ldots, b_n) \mid b_i \in C(Z_s) ; 1 \le i \le n\}$ be the domain and range spaces of state vectors respectively associated with M the MOD-complex operator.*

*Every $x = (x_1, \ldots, x_m)$ and $y = (y_1, y_2, \ldots, y_n)$; where $x_i, y_j \in Z_s i_F = \{a i_F / a \in Z_s\}$, $1 \le i \le m$; $1 \le j \le n$ have the resultant to be always a pure complex number pair.*

Proof follows from the fact that since M the MOD complex operator is real when a pure complex number state vector acts on it the resultant is also a pure complex number. Hence the claim.

***Example 3.15:*** Let

$$M = \begin{bmatrix} 3 & 0 & 1 & 2 & 5 & 0 \\ 0 & 1 & 0 & 1 & 0 & 2 \\ 6 & 0 & 7 & 0 & 1 & 0 \end{bmatrix}$$

be MOD complex matrix operator with entries from

$$C(Z_8) = \{a + bi_F \mid a, b \in Z_8,\ i_F^2 = 7\}.$$

Let $D = \{(a_1, a_2, a_3) \mid a_i \in C(Z_8); 1 \le i \le 3\}$ and

$R = \{(b_1, b_2, \ldots, b_6) \mid b_i \in C(Z_8); 1 \leq i \leq 6\}$ be the domain and range space of state vectors respectively associated with M respectively.

Let $x = (1 + i_F, 0, 0) \in D$.

To find the effect of x on M.

$xM = (3 + 3i_F, 0, 1 + i_F, 2 + 2i_F, 5 + 5i_F, 0) = y_1;$

$y_1M^t = (7 + 7i_F, 2 + 2i_{F,} 6 + 6i_F) = x_2;$

$x_2M = (1 + i_F, 2 + 2i_F, 1 + i_F, 0, 1 + i_F, 4 + 4i_F) = y_2;$

$y_2M^t = (1 + i_F, 2 + 2i_F, 6 + 6i_F) = x_3;$

$x_3M^t = (7 + 7i_F, 2 + 2i_F\ 3 + 3i_F, 4 + 4i_F, 3 + 3i_F, 4 + 4i_F) = y_3;$

$y_3M^t = (7 + 7i_F, 2 + 2i_F, 2 + 2i_F) = x_4;$

$x_4M = (1 + i_F, 2 + 2i_F, 5 + 3i_F, 0, 5 + 5i_F, 4 + 4i_F) = y_4;$

$y_4M^t = (1 + i_F, 2 + 2i_f, 6 + 6i_F) = x_5\ (=x_3).$

Thus the resultant is a realized limit cycle pair given by $\{(1+ i_F, 2 + 2i_F, 6 + 6i_F), (7 + 7i_F, 2 + 2i_F, 3 + 3i_{F,} 4 + 4i_F, 3 + 3i_F, 4 + 4i_F)\}$.

Thus the resultant of a mixed complex number is also a mixed complex number.

Let $y = (0, 0, 0, 0, 0, 1 + i_F) \in R$. To find the effect of y on M.

$yM^t = (0, 2 + 2i_F, 0) = x_1;$

$x_1M = (0, 2 + 2i_F, 0, 2 + 2i_F, 0, 4 + 4i_F) = y_1;$

$y_1M^t = (4+4i_F, 4+4i_F, 0) = x_2;$

$x_2M = (4+4i_F, 4 + 4i_F, 4 + 4i_F, 4 + 4i_F, 4 + 4i_F, 0) = y_2;$

$y_2M^t = (4+4i_F, 0, 0) = x_3;$

$x_3M = (4+4i_F, 0, 4 + 4i_F, 0, 4 + 4i_F, 0) = y_3;$

$y_3M^t = (4 + 4i_F, 0, 0) = x_4 (=x_3).$

Thus we see the resultant is a realized fixed point pair given by
$\{(4 + 4i_F, 0, 0), (4 + 4i_F, 0, 4 + 4i_F, 0, 4 + 4i_F, 0)\}.$

We see the resultant is also a mixed complex number state vector pair.

Next we give an example of a pure complex MOD matrix operator and show how it functions.

***Example 3.16:*** Let

$$M = \begin{bmatrix} 3i_F & i_F & 0 \\ 0 & 2i_F & i_F \\ i_F & 0 & 2i_F \\ 0 & 2i_F & 0 \\ 2i_F & 0 & i_F \end{bmatrix}$$

be the pure complex number MOD matrix operator with entries from $C(Z_4) = \{a + bi_F \mid a, b \in Z_4, i_F^2 = 3\}$.

Let $D = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in C(Z_4); 1 \leq i \leq 5\}$ be the domain state vector associated with M.

$R = \{(a_1, a_2, a_3) \mid a_i \in C(Z_4); 1 \leq i \leq 3\}$ be the range state of vectors associated with M.

Let $x = (1, 0, 0, 0, 0) \in D$.

To find the effect of x on M.

$xM = (3i_F, i_F, 0) = y$; $yM^t = (2, 2, 1, 2, 2) = x_1$;

$x_1M = (3i_F, 2i_F, 0) = y_1$; $y_1M^t = (1, 0, 1, 0, 2) = x_2$;

$x_2M = (0, i_F, 0) = y_2$; $y_2M^t = (3, 2, 0, 2, 0) = x_3$;

$x_3M = (i_F, 3i_F, 2i_F) = y_3$; $y_3M^t = (2, 0, 3, 2, 0) = x_4$;

$x_4M = (i_F, 2i_F, 2i_F) = y_4$; $y_4M^t = (3, 2, 3, 0, 0) = x_5$;

$x_5M = (0, 3i_F, 0) = y_5$; $y_5M^t = (1, 2, 0, 2, 0) = x_6$;

$x_6M = (3i_F, i_F, 2i_F) = y_6$; $y_6M^t = (2, 0, 1, 2, 0) = x_7$;

$x_7M = (3i_F, 2i_F, 2i_F) = y_7$; $y_7M^t = (1, 2, 1, 0, 0) = x_8$;

$$x_8M = (0, i_F, 0) = y_8 \ (= y_2).$$

The resultant of x = (1, 0, 0, 0, 0) ∈ D is a realized limit cycle point pair given by {(3, 2, 0, 2, 0), $(0, i_F, 0)$} we see one of the state vector is real where as other is a pure complex.

It is a very unique property enjoyed by MOD complex matrix operators enjoyed by $C(Z_s)$; $2 \leq s < \infty$.

Let y = (0, 1, 0) ∈ R. To find the effect of y on M.

$yM^t = (i_F, 2i_F, 0, 2i_F, 0) = x_1$;

$x_1M = (1, 3, 2) = y_1$; $y_1M^t = (2i_F, 0, i_F, 2i_F, 0) = x_2$;

$x_2M = (1, 2, 2) = y_2$; $y_2M^t = (i_F, 2i_F, i_F, 0, 0) = x_3$;

$x_3M = (0, 3, 0) = y_3$; $y_3M^t = (3i_F, 2i_F, 0, 2i_F, 0) = x_3$;

$x_3M = (3, 1, 2) = y_4$; $y_4M^t = (2i_F, 0, 3i_F, 2i_F, 0) = x_4$;

$x_4M = (3, 2, 2) = y_5$; $\quad$ $y_5M^t = (3i_F, 2i_F, 3i_F, 0, 0) = x_5$;

$x_5M = (0, 1, 0) = y_6$ (= y).

Thus the resultant is a realized limit cycle point pair given by

$\{(i_F, 2i_F, 0, 2i_F, 0), (0, 1, 0)\}$.

Clearly the range is a real state vector so is the resultant, however the domain space state vector is a pure complex number.

Thus if real state vector from range space is taken; it gives the domain state vector resultant to be a pure complex state vector.

Now if $x = (0, 2 + i_F, 0, 0, 0) \in D$.

To find the effect of x on M.

$xM = (0, 2, 2i_F + 3) = y_1$;

$y_1M^t = (2i_F, 3i_F + 2, 2i_F, 0, 2 + 3i_F) = x_1$;

$x_1M = (2, 0, 2) = y_2$;

$y_2M^t = (2i_F, 2i_F, 2i_F, 0, 2i_F) = x_2$;

$x_2M = (0, 2, 0) = y_3$;

$y_3M^t = (2i_F, 0, 0, 0, 0) = x_3$;

$x_3M = (2, 2, 0) = y_4$;

$y_4M^t = (0, 0, 2i_F, 0, 0) = x_4$;

$x_4M = (2, 0, 0) = y_4$;

$y_4M^t = (2i_F, 0, 2i_F, 0, 0) = x_5$;

$x_5M = (0, 2, 0) = y_5 ( = y_3)$.

Thus the resultant is a realized limit cycle pair given by $\{(2i_F, 0, 0, 0, 0), (0, 2, 0)\}$.

Even if x is a mixed complex number we can get the resultant y to be real.

Let $y = (0, 0, 2 + i_F) \in R$.

To find the effect of y on M.

$yM^t = (0, 0, 2i_F + 3, 2, 0, 2i_F + 3) = x_1$;

$x_1M = (i_F + 2, 0, 2 + i_F) = y_1$;

$y_1M^t = (1 + 2i_F, 2i_F + 3, 2i_F + 1, 0, 2i_F + 1) = x_2$;

$x_2M = (2i_F, 2+3i_F, 2i_F) = y_2$;

$y_2M^t = (2i_F, 0, 2, 2, 0) = x_3$;

$x_3M = (2 + 2i_F, 2, 2i_F) = y_3$;

$y_3M^t = (2, 2, 2 + 2i_F, 0, 2) = x_4$;

$x_4M = (2, 2i_F, 0) = y_3$;

$y_3M^t = (2i_F, 0, 2i_F, 0, 0) = x_5$;

$x_5M = (0, 2, 0) = y_4$;

$y_4M^t= (2i_F, 0, 0, 0, 0) = x_6$;

$x_6M = (2, 2, 0) = y_5$;

$y_5M^t = (0, 0, 2i_F, 0, 0) = x_7$;

$x_7M = (2, 0, 0) = y_6$;

$y_6M^t = (2i_F, 0, 2i_F, 0, 0) = x_7$ ( $= x_5$).

Thus the resultant is a realized fixed point pair given by $\{(2i_F, 0, 2i_F, 0, 0), (0, 2, 0)\}$.

Though we have taken the range state vector to be a mixed complex number yet the resultant of the range space is real where as that of the domain space is pure complex. Thus they behave in a very different way.

Next we proceed onto describe special type of domain space of state vectors and range space of state vectors for the same MOD finite complex modulo number matrix operator by examples.

***Example 3.17:*** Let

$$P = \begin{bmatrix} 3+i_F & 2 & 1+2i_F \\ 0 & 1+5i_F & 0 \\ 4i_F & 0 & 5 \\ 1+i_F & 3i_F & 0 \\ 0 & 5+2i_F & i_F \\ 1 & 0 & 3+4i_F \end{bmatrix}$$

be the MOD finite complex modulo integer matrix operator with entries from $C(Z_6) = \{(a + bi_F$ where $a, b \in Z_6$; $i_F^2 = 5\}$.

Let $D_S = \{(a_1, a_2, \ldots, a_6) \mid a_i \in \{1, 0, i_F, 1 + i_F), 1 \leq i \leq 6$; $i_F^2 = 5\}$ and $R_S = \{(a_1, a_2, a_3) \mid a_i \in \{0, 1, i_F, 1 + i_F\}$; $i_F^2 = 5$; $1 \leq i \leq 3\}$ be the collection of special state vectors of domain and range spaces respectively.

Clearly $D_S \subseteq D$ and $R_S \subseteq R$.

Now we work with elements of $D_S$ and $R_S$ for effect on P.

Let $x = (1, 0, 0, 0, 0, 0) \in D_S$. To find the effect of x on P.

$xP = (3 + i_F, 2, 1 + 2i_F) \to (1, 1, i_F) = y_1;$

$y_1P^t = (3 + 2i_F, 1 + 5i_F, 3i_F, 1 + 4i_F, 4 + 2i_F, 4 + 4i_F) \to (1, i_F, i_F, i_F, 1, 1+ i_F) = x_1;$

$x_1P = (1 + 3i_F, 5 + 3i_F, 3i_F) \to (i_F, 1, i_F) = y_2;$

$y_2P^t = (5 + 4i_F, 1 + 5i_F, 2 + 5i_F, 5 + 4i_F, 4 + 2i_F, 2 + 4i_F) \to (1, i_F, i_F, 1, 1, i_F) = x_2;$

$x_2P = (3i_F, 2, 3 + 5i_F) \to (i_F, 1, i_F) = y_3 \ (= y_2).$

Thus the resultant is a realized fixed point pair given by $\{(1, i_F, i_F, 1, 1, i_F), (i_F, 1, i_F)\}$.

Let us find the effect of x = (1, 0, 0, 0, 0, 0) realized as a vector of D on P.

$xP = (3 + i_F, 2, 1 + 2i_F) = y_1;$

$y_1P^t = (3 + 4i_F, 2 + 4i_F, 1 + 4i_F, 2 + 4i_F, 2 + 5i_F, 4 + 5i_F) = x_1;$

$x_1P = (3, 3i_F, i_F + 5) = y_2;$

$y_2P^t = (2i_F, 3i_F + 3, 5i_F + 1, 3i_F, 5 + 2i_F, 2 + 5i_F) = x_2;$

$x_2P = (1, 0, 3 + i_F) = y_3;$

$y_3P^t = (4 + 2i_F, 0, 3 + 3i_F, 1 + i_F, 3i_F + 5, 3i_F) = x_3;$

$x_3P = (4 + 3i_F, 2i_F, 3i_F) = y_4;$

$y_4P^t = (2i_F + 3, 2 + 2i_F, i_F, 1 + i_F, 5 + 4i_F, 4) = x_4$ and so on.

We will arrive at a resultant which will be a realized fixed point or a realized limit cycle point pair.

Now we see if $x \in D_S \subseteq D$; the effect of x in $D_S$ can give a realized limit cycle or realized fixed point with lesser number of terms and if the same x ∈ D then the number of iterations is more when x is realized as an element of D but when the same x is in $D_S$ the realized limit cycle or realized fixed point is reached by a finite number of iterations.

Let y = (0, 0, 1) ∈ $R_S$ we find the effect of y on P in the following

$yP^t = (1 + 2i_F, 0, 5, 0, i_F, 3 + 4i_F) \to (i_F, 0, 1, 0, i_F, i_F) = x_1;$

$x_1P = (3i_F + 4, i_F + 4, 4i_F + 4) = y_1;$

$y_1P^t = (1 + 3i_F, 3i_F, 2, 4 + i_F, 2 + 5i_F, 4 + 3i_F) = x_2$ and so on.

We see the effect of y = (0, 0, 1) in R has several iterations to go how ever it will finite.

We will arrive at a realized fixed point pair or the realized limit cycle pair only.

Now we will take y = (0, 0, 1) as an element of $R_S$ and find the resultant on P.

$yP^t = (1 + 2i_F, 0, 5, 0, i_F, 3+4i_F) \to (i_F, 0, 1, 0, i_F, i_F) = x_1;$

$x_1P= (3i_F + 4, i_F + 4, 4i_F + 4) \to (1, 1, 1 + i_F) = y_1;$

$y_1P^t = (4 + 4i_F, 1 + 5i_F\ \ 5 + 3i_F, 1 + 4i_F\ 4 + 3i_F, i_F) \to$
$(1 + i_F, i_F, 1, i_F, 1, i_F) = x_2;$

$x_2P = (1 + 4i_F, 5 + 5i_F, i_F) \to (i_F, 1 + i_F, i_F) = y_2;$

$y_2P^t = (5, 2, 4 + 5i_F, 4i_F + 2, 2 + i_F, 4i_F + 2) \to$
$(1, 1, i_F, i_F, 1, i_F) = x_3;$

$x_3P = (4 + 3i_F, 5 + i_F, 3 + 5i_F) \to (1, 1, i_F) = y_3;$

$y_3P^t = (3 + 2i_F, 1 + 5i_F, 3i_F, 1 + 4i_F, 4 + 2i_F, 3 + 3i_F) \to$
$1, i_F, i_F, i_F, 1, 1 + i_F) = x_4;$

$x_4P = (5 + 3i_F, 3 + 5i_F, 3i_F) \to (1, i_F, i_F) = y_4;$

$y_4P^t = (1 + 4i_F, 1 + i_F, 3i_F, 4 + i_F, 3 + 5i_F, 3 + 3i_F) \to$
$(i_F, 1 + i_F, i_F, 1, i_F, 1 + i_F) = x_5;$

$x_5P = (3 + 5i_F, 4i_F, 2 + 3i_F) \to (i_F, i_F, i_F) = y_5;$

$y_5P^t = (3, i_F + 1, 2 + 5i_F, i_F + 2, 5i_F + 3, 4i_F + 2) \to$
$(1, 1 + i_F, i_F, 1, i_F, 1_F) = x_6;$

$x_6P = (3i_F, 4i_F, 2 + 4i_F) \to (i_F, i_F, 1_F) = y_6\ ( = y_5).$

Thus the resultant is a realized fixed point pair given by $\{1, 1 + i_F, i_F, 1, I_F, 1_F), (i_F, i_F, i_F)\}$.

However when we work with elements of $D_S$ or $R_S$ we arrive at a resultant with a fewer number of iterations than when the elements are from D and R.

***Example 3.18:*** Let

$$S = \begin{bmatrix} 2 & 0 & 1 & 4 & 0 \\ 0 & 5 & 1 & 0 & 2 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 2 & 2 & 2 \\ 3 & 3 & 0 & 1 & 1 \\ 5 & 0 & 4 & 0 & 3 \end{bmatrix}$$

be the MOD finite complex modulo integer operator with entries from $C(Z_6) = \{a + bi_F \mid a, b \in Z_6,\ i_F^2 = 5\}$.

Let $D = \{(a_1, a_2, \ldots, a_6) \mid a_i \in C(Z_6); 1 \le i \le 6\}$ and

$R = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in C(Z_6); 1 \le i \le 5\}$ be the domain and range space of state vectors respectively associated with S.

Let $x = (5 + 2i_F, 0, 0, 0, 0, 0) \in D$.

To find the effect of x on S.

$xS = (4 + 4i_F, 0, 5 \; 2i_F, 2 + 3i_F, 0) = y_1;$

$y_1St = (3 + 4i_F, 5 + 2i_F, 3, 2 + 4i_F, 2 + 3i_F, 0) = x_1;$

$x_1S = (3 + 5i_F, 4 + i_F, 5 + 2i_F, 0, 4 + 3i_F) = y_2$ and so on.

From the trend of results we realized the resultant would be a realized limit cycle pair or a realized fixed point pair which are also mixed finite complex numbers.

Consider $y = (0, 0, 0, 0, 2 + i_F) \in R$.

To find the effect of y on S;

$yS^t = (0, 4 + 2i_F, 0, 4 + 2i_F, 2 + i_F, 3i_F) = x_1;$

$x_1S = (0, 2 + i_F, 0, 4 + 5i_F, 0) = y_1;$

$y_1S^t = (4 + 2i_F, 4 + 5i_F, 2 + i_F, 2 + 4i_F, 4 + 2i_F, 0) = x_2;$

$x_2S = (4 + 5i_F, 4 + 2i_F, 2 + 2i_F, 0, 2 + 4i_F) = y_2;$

$y_2S^t = (4, 2 + 2i_F, 2 + 3i_F, 2, \; 3i_F, 4 + 3i_F) = x_3;$

$x_3S = (3i_F, 4i_F, 4 + 5i_F, 4 + 3i_F, 2 + 4i_F) = y_3;$

$y_3S^t = (2 + 5i_F, 3i_F + 2, 4, 2, 4i_F, 4 + 3i_F) = x_4;$

$x_4S = (4 + i_F, 3i_F + 2, 2i_F, 0, 4 + 3i_F) = y_4;$

$y_4S^t = (2 + 4i_F, 2 + i_F, 0, 2 + 4i_F, 1, 4i_F + 2) = x_5;$

$x_5S = (5 + 4i_F, 1 + 5i_F, 5 + i_F, 1, 2 + 4i_F) = y_5;$

$y_5S^t = (4 + 4i_F, 4 + 5i_F, 2 + i_F, 2 + 4i_F, 4 + 2i_F, 0) = x_6;$

$x_6S = (4 + 3i_F, 4 + 2i_F, 2i_F, 2i_F, 4 + 2i_F) = y_6;$

$y_6S^t = (4 + 4i_F, 4 + 4i_F, 2 + i_F, 4, 4 + i_F, 4 + i_F) = x_7;$

$x_7S = (5i_F, 4, 4 + i_F, 4 + 2i_F, 2) = y_7;$

$y_7S^t = (2 + i_F, 4 + i_F, 2, 2, 5i_F, 5i_F + 4) = x_8;$

$x_8S = (2, 4 + 4i_F, 4 + 4i_F, 3i_F, 4i_F) = y_8;$

$y_8S^t = (2 + 4i_F, 2i_F, 4 + 2i_F, 4 + 2i_F, i_F, 2 + 4i_F) = x_9;$

$x_9S = (2 + 3i_F, 4 + 3i_F, 4 + 4i_F, 0, 2 + 3i_F) = y_9;$

$y_9S^t = (2 + 4i_F, 4 + i_F, 4 + 4i_F, 2i_F, 2 + 3i_F, 2 + 4i_F) = x_{10};$

we find $x_{10}S$ and so on until we arrive at a realized fixed point pair or a realized limit cycle pair.

Certainly this will be achieved after a finite number of iterations.

Let $x = (1, 0, 0, 0, 0, 0) \in D_S$ to find the effect of x on S.

$xS = (2, 0, 1, 4, 0) \to (1, 0, 1, 1, 0) = y_1;$

$y_1S^t = (1, 1, 2, 4, 4, 3) \to (1, 1, 1, 1, 1, 1) = x_1;$

$x_1S = (5, 3, 3, 1, 2) \to (1, 1, 1, 1, 1) = y_2;$

$y_2S^t = (1, 2, 3, 0, 2, 0) \to (1, 1, 1, 0, 1, 0) = x_2;$

$x_2S = (0, 3, 3, 5, 3) \to (0, 1, 1, 1, 1) = y_3;$

$y_3S^t = (5, 2, 2, 0, 5, 1) \to (1, 1, 1, 0, 1, 1) = x_3;$

$x_3S = (5, 3, 1, 5, 0) \to (1, 1, 1, 1, 0) = y_4;$

$y_4S^t = (1, 0, 3, 4, 1, 3) \to (1, 0, 1, 1, 1, 1) = x_4;$

$x_4S = (5, 4, 2, 1, 0) \to (1, 1, 1, 1, 0) = y_5$ (= $y_4$).

Thus the resultant is a realized fixed point pair given by {(1, 0, 1, 1, 1, 1), (1, 1, 1, 1, 0)}.

We see x is real and the resultant is also a real pair.

Let y = (0, 1, 0, 0, 0) $\in R_S$.

To find the effect of y on S;

$yS^t = (0, 5, 1, 0, 3, 0) \to (0, 1, 1, 0, 1, 0) = x_1$;

$x_1S = (4, 3, 2, 1, 3) \to (1, 1, 1, 1, 1) = y_1$;

$y_1S^t = (1, 2, 3, 0, 2, 0) \to (1, 1, 1, 0, 1, 0) = x_2$;

$x_2S = (0, 3, 3, 5, 3) \to (0, 1, 1, 1, 1) = y_2$;

$y_2S^t = (5, 2, 2, 0, 5, 1) \to (1, 1, 1, 0, 1, 1) = x_3$;

$x_3S = (5, 3, 1, 5, 0) \to (1, 1, 1, 1, 0) = y_3$;

$y_3S^t = (1, 0, 3, 4, 1, 3) \to (1, 0, 1, 1, 1, 1) = x_4$;

$x_4S = (5, 4, 2, 1, 0) \to (1, 1, 1, 1, 0) = y_4$ (=$y_3$).

Thus the resultant of y is a realized fixed pair of real vectors given by

{(1, 0, 1, 1, 1, 1), (1 , 1, 1, 1, 0)}.

Let x = (0, $i_F$, 0 0 0 0) $\in D_S$. To find the effect of x on S.

$xS = (0, 5i_F, i_F, 0, 2i_F) \to (0, i_F, i_F, 0, i_F) = y_1$;

$y_1S^t = (i_F, 2i_F, 2i_F, 4i_F, 4i_F, i_F) \to (i_F, i_F, i_F, i_F, i_F, i_F) = x_1$;

$x_1S = (5i_F, 3i_F, 3i_F, i_F, 2i_F) \to (i_F, i_F, i_F, i_F, i_F) = y_2$;

$y_2S^t = (i_F, 2i_F, 3i_F, 0, 2i_F, 0) \to (i_F, i_F, i_F, 0, i_F, 0) = x_2;$

$x_2S = (0, 3i_F, 3i_F, 5i_F, 3i_F) \to (0, i_F, i_F, i_F, i_F) = y_3;$

$y_3S^t = (5i_F, 2i_F, 2i_F, 0, 5i, i_F) \to (i_F, i_F, i_F, 0, i_F, i_F) = x_3;$

$x_3S = (5i_F, 3i_F, i_F, 5i_F, 0) \to (i_F, i_F, i_F, i_F, 0) = y_4;$

$y_4S^t = (i_F, 0, 3i_F, 4i_F, i_F, 3i_F) \to (i_F, i_F, i_F, i_F, i_F, i_F) = x_4 (= x_1).$

Thus the resultant is a realized limit cycle pair given by $\{(i_F, i_F, i_F, i_F, i_F, i_F), (i_F, i_F, i_F, i_F, i_F)\}$.

Thus if $x \in D_S$ is a pure complex number so as the resultant pair.

Let $y = (0, 0, i_F, 0, 0) \in R_S$.

To find the effect of y on S;

$yS^t = (i_F, i_F, i_F, 2i_F, 0, 4i_F) \to (i_F, i_F, i_F, i_F, 0, i_F) = x_1;$

$x_1S = (2i_F, 0, 3i_F, 0, i_F) \to (i_F, 0, i_F, 0, i_F) = y_1;$

$y_1S^t = (3i_F, 0, 3i_F, 2i_F, i_F, 5i_F) \to (i_F, 0, i_F, i_F, i_F, i_F) = x_2;$

$x_2S = (5i_F, 4i_F, 2i_F, i_F, 0) \to (i_F, i_F, i_F, i_F, 0) = y_2;$

$y_2S^t = (i_F, 0, 3i_F, 4i_F, i_F, 3i_F) \to (i_F, 0, i_F, i_F, i_F, i_F) = x_3;$

$x_3S = (5i_F, 4i_F, 2i_F, i_F, 0) \to (i_F, i_F, i_F, i_F, 0) = y_3 (=y_2).$

The resultant of y is a realized fixed point pair given by $\{(i_F, 0, i_F, i_F, i_F, i_F), (i_F, i_F, i_F, i_F, 0)\}$.

This is also a pure complex pair.

Let us consider $x = (0, 1 + i_F, 0, 0, 0, 0) \in D_S$.

To find the effect of x on S.

$xS = (0, 5 + 5i_F, 1 + i_F, 0, 2 + 2i_F) \to$
$(0, 1 + i_F, 1 + i_F, 0, 1 + i_F) = y_1;$

$y_1S^t = (1 + i_F, 2 + 2i_F, 2 + 2i_F, 4 + 4i_F\ 4 + 4i_F, i_F + 1) \to$
$(1 + i_F, 1 + i_F, 1 + i_F, 1 + i_F, 1 + i_F, 1 + i_F) = x_1;$

$x_1S = (5 + 5i_F, 3i_F + 3, 3 + 3i_F, i_F + 1, 2 + 2i_F) \to$
$(1 + i_F, 1 + i_F, 1 + i_F, 1 + i_F, 1 + i_F) = y_2;$

$y_2S^t = (i_F + 1, 2 + 2i_F, 3 + 3i_F, 0, 2 + 2i_F, 0) \to$
$(1 + i_F, 1 + i_F, 1 + i_F, 0, 1 + i_F, 0) = x_2;$

$x_2S = (0, 3 + 3i_F, 3 + 3i_F, 5 + 5\ i_F, 3 + 3i_F) \to$
$(0, 1 + i_F, 1 + i_F, 1 + 1i_F, 1 + i_F) = y_3;$

$y_3S^t = (5 + 5i_F, 2 + 2i_F, 2 + 2i_F, 0, 5 + 5i_F, 1 + i_F) \to$
$(1 + i_F, 1 + i_F, 1 + i_F, 0, 1 + i_F, 1 + i_F) = x_3;$

$x_3S = (4 + 4i_F, 3 + 3i_F, 4 + 4i_F, 5 + 5i_F, 0) \to$
$(1 + i_F, 1 = i_F, 1 + i_F, 1 + i_F, 0) = y_4;$

$y_4S^t = (1 + i_F, 0, 3 + 3i_F, 4 + 4i_F, 1 + i_F, 3 + 3i_F) \to$
$(1 + i_F, 0, 1 + i_F, 1 + i_F, 1 + i_F, 1 + i_F) = x_4;$

$x_4S = (5 + 5i_F, 4 + 4i_F, 2 + 2i_F, 1 + i_F, 0) \to$
$(1 + i_F, 1 + i_F, 1 + i_F, 1 + i_F, 0) = y_5\ (=y_4).$

The resultant of x on S is a realized fixed point pair given by $\{(1 + i_F, 0, 1 + i_F, 1 + i_F, 1 + i_F, 1 + i_F), (1 + i_F, 1 + i_F, 1 + i_F, 1 + i_F, 0)\}$.

The resultant is also mixed as x is mixed.

Next we proceed onto study the MOD dual number operator using $\langle Z_n \cup g\rangle = \{a + bg \mid a, b \in Z_n, g^2 = 0\}$.

We will illustrate them by examples.

***Example 3.19:*** Let

$$S = \begin{bmatrix} 3+g & 1+g \\ 2g & 0 \\ 3g+2 & 4+g \\ 0 & 3+2g \\ 4+2g & 3g \end{bmatrix}$$

be the MOD dual number matrix operator with entries from
$\langle Z_5 \cup g \rangle = \{a + bg \mid a, b \in Z_5, g^2 = 0\}$.

Let $D = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in \langle Z_5 \cup g \rangle; 1 \leq i \leq 5\}$ and

$R = \{(b_1, b_2) \mid b_1, b_2 \in \langle Z_5 \cup g \rangle\}$ be the domain and range state vector spaces respectively related with S.

Now let $x = (0, 1, 0, 0, 0) \in D$ to find the effect of x on S.

$xS = (2g, 0) = y_1;$ $\quad y_1S^t = (g, 0, 4g, 0, 3g) = x_1;$

$x_1S = (3g, 2g) = y_2;$ $\quad y_2S^t = (g, 0, 4g, g, 2g) = x_2;$

$x_2S = (4g, 0) = y_3;$ $\quad y_3S^t = (2g, 0, 3g, 0, g) = x_3;$

$x_3S = (g, 3g) = y_4;$ $\quad y_4S^t = (g, 0, 4g, 4g, 4g) = x_4;$

$x_4S = (2g, 4g) = y_5;$ $\quad y_5S^t = (0, 0, 3g, 2g, 3g) = x_5;$

$x_5S = (3g, 3g) = y_6;$ $\quad y_6S^t = (2g, 0, 3g, 4g, 2g) = x_6;$

$x_6S = (0, g) = y_7;$ $\quad y_7S^t = (g, 0, 4g, 3g, 0) = x_7;$

$x_7S = (g, g) = y_8;$ $\quad y_gS^t = (4g, 0, g, 3g, 4g) = x_8;$

$x_8S = (0, 2g) = y_9$; $\quad y_9S^t = (2g, 0, 3g, g, 0) = x_9$;

$x_9S = (2g, 2g) = y_{10}$; $\quad y_{10}S^t = (3g, 0, 2g, g, 3g) = x_{10}$;

$x_{10}S = (0, 4g) = y_{11}$; $\quad y_{11}S^t = (4g, 0, g, 2g, 0) = x_{11}$;

$x_{11}S = (4g, 4g) = y_{12}$; $\quad y_{12}S^t = (g, 0, 4g, 2g, g) = x_{12}$;

$x_{12}S = (0, 3g) = y_{13}$; $\quad y_{13}S^t = (3g, 0, 2g, 4g, 0) = x_{13}$;

$x_{13}S = (3g, 3g) = y_{14}$ $(=y_6)$.

Thus the resultant of $x = (0, 1, 0, 0, 0)$ is a realized limit point pair given by $\{(2g, 0, 3g, 4g, 2g), (3g, 3g)\}$.

Though x is real the resultant pair is a pure dual number

Let $y = (g, 0) \in R$.

To find the effect of y on S.

$yS^t = (3g, 0, 2g, 0, 4g) = x_1$; $\quad x_1S = (4g, g) = y_1$;

$y_1S^t = (3g, 0, 2g, 3g, g) = x_2$; $\quad x_2S = (2g, 0) = y_2$;

$y_2S^t = (g, 0, 4g, 0, 3g) = x_3$; $\quad x_3S = (3g, 2g) = y_3$;

$y_3S^t = (g, 0, 4g, g, 2g) = x_4$; $\quad x_4S = (4g, 3g) = y_4$;

$y_4S^t = (0, 0, 0, 4g, 4g) = x_5$; $\quad x_5S = (4g, 2g) = y_5$;

$y_5S^t = (4g, 0, g, g, g) = x_6$; $\quad x_6S = (3g, g) = y_6$;

$y_6S^t = (0, 0, 0, 3g, 2g) = x_7$; $\quad x_7S = (3g, 4g) = y_7$;

$y_7S^t = (3g, 0, 2g, 2g, 2g) = x_8$; $\quad x_8S = (g, 2g) = y_8$;

$y_8S^t = (0, 0, 0, g, 4g) = x_9$; $\quad x_9S = (g, 3g) = y_9$;

$y_9S^t = (g, 0, 4g, 4g, 4g) = x_{10}$; $\quad$ $x_{10}S = (2g, 4g) = y_{10}$;

$y_{10}S^t = (0, 0, 0, 2g, 3g) = x_{11}$;

$x_{11}S = (3g, 4g) = y_{11}$ $(= y_7)$.

Thus the resultant of y = (g, 0) is realized limit cycle pair given by {(3g, 0, 2g, 2g, 2g), (3g, 4g)} which is also a pure dual number.

Let y = (0, 1) to find the effect of y on S.

$yS^t = (1 + g, 0, 4 + g, 3 + 2g, 3g) = x_1$;

$x_1S = (1, 1 + 2g) = y_1$;

$y_1S^t = (4 + 4g, 2g, 2g + 1, 3 + 3g, 4) = x_2$;

$x_2S = (g, 2g + 2) = y_2$;

$y_2S^t = (2g + 2, 0, 3 + 2g, 1, 0)) = x_3$;

$x_3S = (g + 2, 2 + 2g) = y_3$

and so on we may arrive at a resultant which may be a realized limit cycle pair or a realized fixed point pair with mixed dual numbers in general.

***Example 3.20:*** Let

$$B = \begin{bmatrix} 2 & 1 & 0 & 1 & 4 & 5 \\ 0 & 3 & 2 & 0 & 1 & 0 \\ 4 & 0 & 3 & 1 & 0 & 2 \\ 5 & 1 & 0 & 0 & 2 & 0 \end{bmatrix}$$

be the MOD dual number matrix operator with entries from

$$\langle Z_6 \cup g\rangle = \{a + bg \mid a, b \in Z_6, g^2 = 0\}.$$

$D = \{(a_1, a_2, a_3, a_4) \mid a_i \in \langle Z_6 \cup g\rangle; 1 \leq i \leq 4\}$ and

$R = \{(b_1, b_2, \ldots, b_6) \mid b_i \in \langle Z_6 \cup g\rangle; 1 \leq i \leq 6\}$ be the domain and range space of state vectors of dual numbers respectively associated with B.

Let $x = (0, 1, 0, 0) \in D$.

To find the effect of x on B.

$xB = (0, 3, 2, 0, 1, 0) = y$; $\quad yB^t = (1, 2, 0, 5) = x_1$;

$x_1B = (3, 1, 4, 1, 4, 5) = y_1$; $\quad y_1B^t = (1, 3, 4\ 0) = x_2$;

$x_2B = (0, 4, 0, 5, 1, 1) = y_2$; $\quad y_2B^t = (0, 1, 1, 0) = x_3$;

$x_3B = (4, 3, 5, 1, 1, 2) = y_3$; $\quad y_3B^t = (2, 2, 0\ 1) = x_4$;

$x_4B = (1, 3, 4, 2, 0, 4) = y_4$; $\quad y_4B^t = (3, 5, 2, 2) = x_5$;

$x_5B = (0, 4, 4, 3, 3, 1) = y_5$; $\quad y_5B^t = (0, 5, 5, 4) = x_6$;

$x_6B = (4, 1, 1, 5, 1, 4) = y_6$; $\quad y_6B^t = (2, 0, 0, 4) = x_7$;

$x_7B = (0, 0, 0, 2, 4, 4) = y_7$; $\quad y_7B^t = (4, 4, 4, 2) = x_8$;

$x_8B = (4, 0, 2, 2, 0, 4) = y_8$; $\quad y_8B^t = (0, 4, 0, 2) = x_9$;

$x_9B = (4, 2, 2, 0, 2, 0) = y_9$ and so on.

However from the trend it clearly follows that the resultant of a real state vector from the domain or range space always yields a real realized resultant pair.

Let $y = (0, 0, 0, 0, 1, 0) \in R$. To find the effect of y on B.

$yB^t = (4, 1, 0, 2) = x_1$; $\quad x_1B = (0, 3, 2, 4, 3, 4) = y_1$;

$y_1B^t = (3, 4, 0, 3) = x_2$; $x_2B = (3, 0, 2, 3, 0, 3) = y_2$;

$y_2B^t = (0, 4, 3, 3) = x_3$; $x_3B = (3, 3, 5, 3, 4 0) = y_3$;

$y_3B^t = (4, 5, 0, 2) = x_4$; $x_4B = (0, 3, 4, 4, 1, 2) = y_4$;

$y_4B^t = (3, 0, 2, 5) = x_5$; $x_5B = (3, 2, 0, 5, 4, 1) = y_5$;

$y_5B^t = (4, 4, 1, 3) = x_6$; $x_6B = (3, 1, 5, 5, 2, 4) = y_6$;

$y_6B^t = (4, 3, 4, 2) = x_7$ and so on.

The resultant will be only a real pair of vectors.

Let $x = (0, g, 0, 0) \in D$.

To find the effect of x on B.

$xB = (0, 3g, 2g, 0, g, 0) = y_1$; $y_1B^t = (g, 2g, 0, 5g) = x_1$;

$x_1B = (3g, 0, 4g, g, 4g, 5g) = y_2$; $y_2B^t = (4g, 0, 5g, 5g) = x_2$

and so on.

It is clear from the trend we get only a pure dual number as a resultant pair if we started to work with a pure dual number state vector either from the range space of a domain space.

However if we take a mixed dual number state vector say $x = (3 + 2g, 0, 0, 0) \in D$; to find its effect on B.

$xB = (4g, 3 + 2g, 0, 3 + 2g, 2g, 3 + 4g) = y_1$;

$y_1B^t = (3 + 2g, 1 + 2g, 3 + 2g, 3 + 2g) = x_2$;

$x_2B$ = will also be a mixed dual number.

Infact we see the resultant pair will be a realized fixed pair or a realized limit cycle pair given by the mixed dual number pair.

In view of all these we have the following theorem.

**THEOREM 3.5:** *Let*

$$M = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

*be the MOD dual number matrix operator with entries from* $Z_s \subseteq \langle Z_s \cup g \rangle = \{a + bg \mid a, b \in Z_s;\ g^2 = 0\}$; $2 \leq s < \infty$ $(a_{ij} \in Z_s;\ 1 \leq i \leq m,\ 1 \leq j \leq n)$.

$D = \{(a_1, a_2, \ldots, a_m) \mid a_i \in \langle Z_s \cup g \rangle;\ 1 \leq i \leq m\}$ *and*

$R = \{(b_1, b_2, \ldots, b_n) \mid b_i \in \langle Z_s \cup g \rangle;\ 1 \leq i \leq n\}$ *be the domain and range space state vectors respectively associated with M.*

i) *If* $x = (a_1, \ldots, a_m)$ *where all* $a_i$*'s are real* $a_i \in Z_s$; $1 \leq i \leq m$ *then the resultant of x on M is a realized pair of state vectors which are all real.*

*ii)* *If* $y = (b_1, \ldots, b_n)$; $b_i \in Z_s$; $1 \leq i \leq n$ *are real then the effect of y on M is also a real pair of state vectors.*

*iii)* *If* $x = (a, g, \ldots, a_m g)$ *where* $a_i \in Z_s$; $1 \leq i \leq m$ *(or* $y = (b_1 g, b_2 g, \ldots, b_n g)$ *where* $b_I \in Z_s$; $1 \leq i \leq n$*) be the pure dual number state vectors then the resultant is also a pure dual number pair.*

*iv)* *If* $x \in D$ *(or* $y \in R$*) are mixed dual number state vectors so is the resultant pair is general.*

**Proof**. Follows from the fact that all entries of the MOD dual number operator matrix M are real so if a real vector is multiplied the resultant pair will be real so (i) and (ii) are true.

Further if the initial state vector x is a pure dual number then when it is multiplied b y a real number the resultant will be only a pure dual number pair.

Hence (iii) is true.

Finally if x is a mixed dual number state vector then the resultant will be a mixed dual number as the MOD matrix operator has only real entries, hence (iv) is true.

Now we give different type of examples.

***Example 3.21:*** Let

$$B = \begin{bmatrix} 2g & 4g & 6g \\ 0 & 6g & 0 \\ 3g & 0 & 5g \\ 0 & g & 2g \\ 4g & 5g & 0 \end{bmatrix}$$

be the MOD dual number matrix operator with entries from $\langle Z_8 \cup g \rangle$. All entries of B are pure dual numbers.

Let $D = \{(a_1, a_2, \ldots, a_5) \mid a_i \in \langle Z_8 \cup g \rangle = \{a + bg \mid a, b \in Z_8, g^2 = 0\}; 1 \leq i \leq 5\}$ and

$R = \{(b_1, b_2, b_3) \mid b_i \in \langle Z_8 \cup g \rangle; 1 \leq i \leq 3\}$ be the domain and range space of state vectors respectively associated with B.

Let $x = (1, 0, 0, 0, 0) \in D$.

To find the effect of x on B.

$xB = (2g, 4g, 6g) = y$; $\quad yB^t = (0, 0, 0, 0, 0) = x_1$;

$$x_1B = (0, 0, 0).$$

Thus the resultant pair is a realized fixed point pair given by
{(0, 0, 0, 0, 0), (0, 0, 0)}.

Let y = (0, 0, 3) ∈ R.

To find the effect of y on B;

$yB^t = (2g, 0, 7g, 6g, 0) = x_1$; $x_1B = (0, 0, 0) = y_1$;

$$y_1B^t = (0, 0, 0, 0, 0) = x_2.$$

Thus the realized resultant is a fixed point pair given by
{(0, 0, 0, 0, 0), (0, 0, 0)}

Let us consider x = (2g, 0, 0, g, 0) ∈ D.

To find the effect of x on B.

$xB = (0, 0, 0) = y_1$;

$y_1B^t = (0, 0, 0, 0, 0) = x_7$.

Thus the resultant is a realized fixed point pair given by
{(0, 0, 0, 0, 0), (0, 0, 0)},

Let y = (0, 2g, 3g) ∈ R.

To find the effect of y on B.

$yB^t = (0, 0, 0, 0, 0) = x_1$;

$x_1B = (0, 0, 0) = y_1$.

Thus the resultant is a realized fixed point pair given by
{(0, 0, 0, 0, 0), (0, 0, 0)}.

From this it is observed if the MOD matrix operator is a pure dual number matrix then the resultant of pure dual number state vectors and real number state vectors are only pair of zero vectors.

Let $x = (2 + g, 3 + 2g, 0, 0, 0) \in D$.

To find the effect of x on B.

$xB = (5g, 2g, 4g) = y_1$; $\quad y_1B^t = (0, 0, 0, 0, 0) = x_1$;

$x_1B = (0, 0, 0)$.

Thus again the resultant pair is a zero pair.

In view of this we make the following conditions.

If we take or consider those MOD dual number matrices if the entries are pure dual numbers than we see the resultant is always the zero pair given by $\{(0, 0,\ldots, 0), (0, 0, \ldots, 0)\}$.

Thus there is no use of working with MOD dual number matrix operators which has pure dual number entries. However if we have MOD dual number matrix operator to have real or mixed dual number entries certainly we will have non zero resultants.

To this end we will give one more example of a MOD dual number matrix operator.

***Example 3.22:*** Let

$$M = \begin{bmatrix} 3g+1 & 2g+3 & 0 & 2g & 1 \\ 6+5g & 0 & 4+g & 0 & g \\ 2g+4 & 4g+1 & 2g & 4 & 2 \end{bmatrix}$$

be the MOD dual number matrix operator with entries from $\langle Z_{10} \cup g\rangle = \{a + bg \mid g^2 = 0, a, b \in Z_{10}\}$.

Let $D = \{(a_1, a_2, a_3) \mid a_i \in \langle Z_{10} \cup g\rangle, 1 \leq i \leq 3\}$ and

$R = \{(b_1, b_2, b_3, b_4, b_5, b_6) \mid b_i \in \langle Z_{10} \cup g\rangle; 1 \leq i \leq 6\}$ be the state vectors of the domain and range spaces respectively.

Let $D_S = \{(a_1, a_2, a_3) \mid a_i \in \{0, 1, g, 1 + g)\}; 1 \leq i \leq 3\}$ and $R_S = \{(b_1, b_2, \ldots, b_6) \mid b_i \in \{0, 1, g, 1 + g\}; 1 \leq i \leq 6\}$ be the special state vectors of domain and range space respectively.

Let $x = (1, 0\ 0) \in D$. We find the effect of x on M.

$xM = (3g + 1, 2g + 3, 0, 2g, 1) = y;$

$yM^t = (8g + 1, 4g + 6, 6g + 9) = x_1;$

$x_1M = (3 + 7g, 8g + 2, 8 + 2g, 6 + 6g, 6g + 9) = y_1;$

$y_1M^t$ and so on we will after a finite number of iterations arrive at a realized fixed point pair or a limit cycle pair.

Suppose the same $x = (1, 0, 0)$ is taken as an element of $D_S$ we find the effect of x on M as a special state vector of $D_S$.

$xM \to (g, 1, 0, g, 1) = y_1;$ $\qquad y_1M^t \to (1, g, 1) = x_1;$

$x_1M \to (1, g, g, 1, 1) = y_2;$ $\qquad y_2M^t \to (g, 1, g) = x_2;$

$x_2M \to (1, g, 1, g, g) = y_3;$ $\qquad y_3M^t \to (g, g, 1) = x_4;$

$x_4M \to (g, g, g, 1, 1) = y_4;$ $\qquad y_{4M}{}^t \to (g, g, g) = x_5;$

$x_5M \to (g, g, g, g, g) = y_5;$ $\qquad y_5M^t \to (g, 0, g) = x_6;$

$x_6M \to (g, g, 0, g, g) = y_7;$ $\quad y_7M^t \to (g, 0, g) = x_7\ (= x_6).$

Thus the resultant as an element of $D_S$ is a realized fixed point pair given by

$$\{(g, 0, g), (g, g, 0, g, g)\}.$$

Next let us consider $y = (0, 0, 0, 1, 0) \in R$.

To find the effect of y on M.

$yM^t = (0, 4 + g, 2g) = x_1$;

$x_1M = (4, 2g, 6 + 8g, 8g, 8g) = y_1$;

$y_1M^t = (4 + 6g, 8 + 8g, 6) = x_2$;

$x_2M = (6 + g, 8, 2 + 2g, 4 + 8g, 6 + 4g) = y_2$;

$y_2M^t = (6 + 4g, 4 + 4g, 0) = x_3$;

$x_3M = (6g, 2 + 4g, 6, 2g, 6 + 8g) = y_3$ and so on.

Thus we will after a finite number of steps or iterations we will arrive at a realized limit cycle pair or a realized fixed point pair.

Now we will work with the same y = (0, 0, 0, 1, 0) realized as a special state vector of $R_S$.

$yM^t \to (g, 0, 1) = x_1$; $x_1M \to (1, g, g, 1, 1) = y_1$;

$y_1M^t \to (g, 1, g) = x_2$; $x_2M \to (1, g, 1, g, g) = y_2$;

$y_2M \to (g, g, 1) = x_3$; $x_3M \to (g, g, g, 1, 1) = y_3$;

$y_3M^t \to (g, g, 1) = x_4 (= x_3)$.

Thus the resultant is a realized fixed point pair given by
$\{(g, g, 1), (g, g, g, 1, 1)\}$.

Let x = (0, g, 0) ∈ D. To find the effect of x on M.

$xM = (6g, 0, 4g, 0, 0) = y_1$; $y_1M^t = (8g, 6g, 8g) = x_1$;

$x_1M = (6g, 2g, 4g, 2g, 4g) = y_2$; $y_2M^t = (6g, 2g, 2g) = x_2$;

$x_2M = (6g, 8g, 8g, 8g, 0) = y_3$; $y_3M^t = (0, 8g, 4g) = x_3$;

$x_3M = (4g, 4g, 2g, 6g, 8g) = y_4$; $y_4M^t = (4g, 2g, 0) = x_4$;

$x_4M = (6g, 2g, 8g, 0, 4g) = y_5$; $y_5M^t = (6g, 8g, 6g) = x_5$;

$x_5M = (8g, 4g, 2g, 4g, 8g) = y_6$;

$y_6M^t = (8g, 6g, 8g) = x_6 (= x_1)$.

Thus the resultant is a realized limit cycle pair given by {(8g, 6g, 8g), (6g, 2g, 4g, 2g, 4g)}.

Now we study the effect of x = (0, g, 0) as a special state vector of $D_S$ on M.

$xM = (6g, 0, 4g, 0, 0) \to (g, 0, g, 0, 0) = y_1$;

$y_1M^t \to (g, 0, g) = x_1$; $x_1M \to (g, g, 0, g, g) = y_2$;

$y_2M^t \to (g, 0, g) = x_2 (= x_1)$.

Thus we see the resultant is a realized fixed point pair given after just one iteration {(g, 0, g), (g, g, 0, g, g)}.

Now we see special state vectors in $D_S$ and $R_S$ behave in a very different way from that of the state vectors in D and R.

Next we proceed onto work with the MOD special dual like number matrix operators.

***Example 3.23:*** Let

$$M = \begin{bmatrix} h & 0 & 3h+2 \\ 0 & 4h+1 & 0 \\ 4h+1 & 0 & 3h \\ 0 & 2h & h+1 \\ 3h+3 & 0 & 2 \\ 4 & 3 & 0 \end{bmatrix}$$

be the MOD special dual like number matrix operator with entries from $\langle Z_5 \cup h\rangle = \{a + bh \mid a, b \in Z_5, h^2 = h\}$.

$D = \{(a_1, a_2, a_3, a_4, a_5, a_6) \mid a_i \in \langle Z_5 \cup h\rangle; 1 \leq i \leq 6\}$ and

$R = \{(b_1, b_2, b_3) \mid b_i \in \langle Z_5 \cup h\rangle\ 1 \leq i \leq 3\}$ be the state vector related with the domain and range spaces respectively associated with the MOD operator M.

Let $x = (1, 0, 0, 0, 0, 0) \in D$.

To find the effect of x on M.

$xM = (h, 0, 3h + 2) = y_1$;

$y_1M^t = (4 + 2h, 0, 0, 3h + 2, 4 + 2h, 4h) = x_1$;

$x_1M = (2, 2h, 4h + 3) = y_2$;

$y_2M^t = (1 + h, 0, 4h + 2, 3, 2 + 4h, h + 3) = x_2$;

$x_2M = (4h, 4h+4, 2h + 4) = y_3$;

$y_3M^t = (h + 3, 4 + h, 3h, 4 + 4h, h + 3\ 3h + 2)$ and so on.

We see the resultant of x will be a realized limit cycle pair or a realized fixed point pair.

Let $y = (0, 0, 1) \in R$, to find the effect of y on M.

$yM^t = (3h + 2, 0, 3h, h + 1, 2, 0) = x$;

$xM = (1 + h, 4h, 3h + 4) = y_1$;

$y_1M^t = (4h + 3, 0, 1, 3h + 4\ 1, 4 + h) = x_1$

and so on.

Thus we see the resultant is a realized limit cycle pair or a realized fixed point pair.

Let $x = (0, 0, h, 0, 0) \in D$ to find the effect of x on M.

$xM = (0, 0, 3h) = y_1$; $y_1M^t = (0, 0, 4h, h, h, 0) = x_1$;

$x_1M = (h, 2h, h) = y_2$; $y_2M^t = (h, 0, 3h, h, 3h, 0) = x_2$;

$x_2M = (4h, 2h, 2h) = y_3$; $y_3M^t = (4h,0,h,3h, 3h, 2h) = x_3$;

$x_3M = (0, 2h, 0) = y_4$; $y_4M^t = (0, 0, 0, 4h, 0\ h) = x_4$;

$x_4M = (4h, h, 3h) = y_5$; $y_5M^t = (4h,0,4h,3h,0,4h) = x_5$;

$x_5M = (0, 3h, 3h) = y_6$; $y_6M^t = (0,0,4h,2h,h,4h) = x_6$;

$x_6M = (2h, 0, 3h) = y_7$; $y_7M^t = (2h, 0, 4h, h, 3h,3h) = x_7$;

$x_7M = (2h, h, 0) = y_8$; $y_8M^t = (2h, 0, 0, 2h, 4h, h) = x_8$;

$x_8M = (0, 2h, 2h) = y_9$; $y_9M^t = (0, 0, h, 3h, 4h, h) = x_9$;

$x_9M = (3h, 4h, 4h) = y_{10}$;

$y_{10}M^t = (3h, 0, 2h, h, h, 4h) = x_{10}$;

$x_{10}M = (0, 0, 0) = y_{11}$;

$y_{11}M^t = (0, 0, 0, 0, 0, 0)$.

Thus the resultant is a realized fixed point pair given by
$\{(0, 0, 0, 0, 0, 0), (0, 0, 0)\}$.

Thus the resultant can be a zero pair.

***Example 3.24:*** Let

$$M = \begin{bmatrix} 3h & h & 0 \\ 0 & 2h & h \\ 2h & 0 & 3h \\ 0 & 3h & 0 \\ h & 0 & 2h \\ 0 & h & 0 \end{bmatrix}$$

be the MOD special dual like number matrix operator with entries from $\langle Z_4 \cup h\rangle = \{a + bh \mid h^2 = h, a, b \in Z_4\}$.

$D = \{(a_1, a_2, \ldots, a_6) \mid a_i \in \langle Z_4 \cup h\rangle; 1 \leq i \leq 6\}$ and

$R = \{(b_1, b_2, b_3) \mid b_i \in \langle Z_4 \cup h\rangle, 1 \leq i \leq 3\}$ be the domain space of state vector and range space of state vectors respectively related with the MOD matrix operator M.

Let x = (0, 0, 0, 1, 0, 0) $\in$ D; to find the effect of x on M.

$xM = (0, 3h, 0) = y_1;$ $\quad y_1M^t = (3h, 2h, 0, h, 0\ 3h) = x_1;$

$x_1M = (h, 3h, 2h) = y_2;$ $\quad y_2M^t = (2h, 0, 0, h, h, 3h) = x_2;$

$x_2M = (3h, 0, 2h) = y_3;$ $\quad y_3M^t = (h, 2h, 0, 0, 3h, 0) = x_3;$

$x_3M = (2h, h, 0) = y_4;$ $\quad y_4M^t = (3h, 2h, 0, 3h, 2h, h) = x_4;$

$x_4M = (3h, h, 2h) = y_5;$ $\quad y_5M^t = (2h, 0, 0, 3h, 3h, h) = x_5;$

$x_5M = (h, 0, 2h) = y_6;$ $\quad y_6M^t = (3h, 2h, 0, 0, h, 0) = x_6;$

$x_6M = (2h, 3h, 0) = y_7;$ $\quad y_7M^t = (h, 2h, 0, h, 2h, 3h) = x_7;$

$x_7M = (h, 3h, 2h) = y_8\ (=y_2).$

Thus the resultant is a realized limit cycle pair given by $\{(2h, 0, 0, h, h, 3h), (h, 3h, 2h)\}$.

However the $x \in D$ was real.

Let $y = (0, 0, 1) \in R$, to find the effect of y on M.

$yM^t = (0, h, 3h, 0, 2h, 0) = x_1$; $x_1M = (0, 2h, 2h) = y_1$;

$y_1M^t = (2h, 2h, 2h, 2h, 0, 2h) = x_2$; $x_2M = (2h, 2h, 0) = y_2$;

$y_2M^t = (0, 0, 0, 2h, 2h, 2h) = x_3$; $x_3M = (2h, 0, 0) = y_3$;

$y_3M^t = (2h, 0, 0, 0, 2h, 0)= x_4$; $x_4M = (0, 2h, 0) = y_4$;

$y_4M^t = (2h, 0, 0, 2h, 0, 2h) = x_5$; $x_5M=(2h,2h,0)=y_5(=y_2)$

Thus the resultant is a realized limit cycle pair given by
$\{(0, 0, 0, 2h, 2h, 2h), (2h, 2h, 0)\}$.

Let $x = (2 + h, 0, 0, 0, 0, 0) \in D$.

To find the effect of x on M.

$xM = (h, 3h, 0) = y_1$; $y_1M^t = (2h, 2h, 2h, 2h, 2h, 3h) = x_1$;

$x_1M = (0, 3h, 0) = y_2$; $y_2M^t = (3h, 2h, 0, h, 0\ 3h) = x_2$;

$x_2M = (h, h, 2h) = y_3$; $y_3M^t = (0, 0, 0, 3h, h, h) = x_3$;

$x_3M = (h, 2h, 2h) = y_4$; $y_4M^t = (h, 2h, 0, 2h, h, 2h) = x_4$;

$x_4M = (0, h, 0) = y_5$; $y_5M^t = (h\ 2h\ 0\ 3h, 0, h) = x_5$;

$x_5M^t = (3h, 3h, 2h) = y_6$; $y_6M^t = (0, 0, 0, h, 3h, 3h) = x_6$;

$x_6M = (3h, 2h, 2h) = y_7$; $y_7M^t = (3h, 2h, 0, 2h, 3h, 2h) = x_7$;

$x_7M = (0, 3h, 0) = y_8 (= y_2)$.

Thus the resultant is a realized limit cycle pair given by

$$\{(3h, 2h, 0, h, 0, 3h), (0, 3h, 0)\}.$$

Hence from this study we make the following observations. If the MOD special dual like number matrix operator M has all its entries from $Z_nh = \{ah \mid a \in Z_n, h^2 = h\}$ then certainly the resultant of every state vector from the domain space of state vectors as well as from the range space of state vectors is always a pair $\{(a_1, \ldots, a_k), (b_1, \ldots, b_t)\}$ where $a_i$, $b_j \in Z_nh$, $1 \leq i \leq k$ and $1 \leq j \leq t$ for any $k \times t$ MOD special dual like number matrix operator with entries from $Z_nh$.

In view of this we have the following theorem.

**THEOREM 3.6:** *Let*

$$M = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

*be the MOD special quasi dual number matrix operator with entries from $Z_sh = \{ah \mid a \in Z_s, h^2 = h\}$.*

*$D = \{(a_1, a_2, \ldots, a_m) \mid a_i \in \langle Z_s \cup h\rangle = \{a + bh \mid a, b \in Z_s, h^2 = h\}; 1 \leq i \leq m\}$ and $R = \{(b_1, \ldots, b_n) \mid b_i \in \langle Z_s \cup h\rangle; 1 \leq i \leq n\}$ be the domain and range space of state vectors associated with M.*

*Every $x \in D$ (or R) is such that its resultant is always a realized fixed point pair or a realized limit cycle pair which have only pure special dual number entries from $Z_sh$.*

Proof is direct and hence left as an exercise to the reader.

***Example 3.25:*** Let

$$S = \begin{bmatrix} 2 & 3 & 5 & 8 \\ 1 & 0 & 6 & 0 \\ 4 & 6 & 0 & 7 \\ 2 & 1 & 5 & 0 \\ 0 & 1 & 0 & 2 \\ 3 & 0 & 4 & 0 \end{bmatrix}$$

be the MOD special dual like number matrix operator with entries from $Z_{10} \subseteq \langle Z_{10} \cup g\rangle = \{a + bg \mid g^2 = g, a, b \in Z_{10}\}$.

Let $D = \{(a_1, a_2, \ldots, a_6) \mid a_i \in \langle Z_{10} \cup g\rangle; 1 \leq i \leq 6\}$ and

$T = \{(b_1, b_2, b_3, b_4) \mid b_i \in \langle Z_{10} \cup g\rangle; 1 \leq i \leq 4\}$ be the domain and range space of state vectors respectively associated with the MOD matrix operator S.

Let $x = (0, 0, 1, 0, 0, 0) \in D$ to find the effect of x on S.

$xS = (4, 6, 0, 7) = y_1$; $\quad y_1S^t = (4, 4, 9, 4, 0, 2) = x_1$;

$x_1S = (2, 0, 2, 5) = y_2$; $\quad y_2S^t = (4, 4, 3, 4, 0, 4) = x_2$;

$x_2M = (4, 4, 0, 3) = y_3$ ; $\quad y_3M^t = (4, 4, 1, 2, 0, 2) = x_3$;

$x_3M = (6, 0, 2, 9) = y_4$;

$y_4M^t = (4, 8, 7, 2, 8, 6) = x_4$ and so on.

From the trend it is very clear after a finite number of iterations we will arrive at a realized limit cycle or a realized fixed point pair which will have its entries to be only from $Z_{10}$ that is they will be real pair of state vectors if the state vector we took was also real.

Let $y = (0, 0, 0, 1) \in R$.

To find the effect of y on S.

$yS^t = (8, 0, 7, 0, 2, 0) = x_1$; $x_1S = (4, 8, 0, 7) = y_1$;

$y_1S^t = (8, 4, 3, 6, 2, 2) = x_2$; $x_2S = (0, 0, 2, 9) = y_2$;

$y_2S^t = (2, 2, 3, 0, 8, 8) = x_3$ and so on.

We are sure to arrive after a finite number of iterations to arrive at a realized limit cycle pair or a realized fixed point pair which will be only real entries from $Z_{10}$.

Let $x = (2g, 0, 0, 0, 0, 0) \in D$.

To find the effect of x on S.

$xS = (4g, 6g, 0, 6g) = y_1$;

$y_1S^t = (4g, 4g, 4g, 4g, 6g, 2g) = x_1$;

$x_1S = (2g, 6g, 2g, 2g) = y_2$;

$y_2S^t = (8g, 4g, 8g, 0, 0, 4g) = x_2$;

$x_2S = (4g, 2g, 0, 0) = y_3$;

$y_3S^t = (4g, 4g, 8g, 0, 2g, 2g) = x_3$;

$x_3S = (2g, 2g, 2g, 2g) = y_4$;

$y_4S^t = (4g, 4g, 4g, 6g, 6g, 4g) = x_4$;

$x_4S = (2g, 8g, 0, 2g) = y_5$;

$y_5S^t = (4g, 2g, 0, 2g, 2g, 6g) = x_5$;

$x_5S = (2g, 6g, 6g, 6g) = y_6$;

$y_6S^t = (0, 8g, 6g, 0, 8g, 0) = x_6$;

$x_6S = (2g, 4g, 8g, 8g) = y_7$;

$y_7S^t = (0, 0, 8g, 8g, 4g, 8g) = x_7$;

$x_7S = (2g, 0, 2g, 4g) = y_8$;

$y_8S^t = (6g, 4g, 6g, 4g, 8g, 4g) = x_8$;

$x_8S = (0, 6g, 0, 6g) = y_9$;

$y_9S^t = (6g, 0, 8g, 6g, 8g, 0) = x_9$ and so on.

We see of the initial state vector is a pure special dual like number so is the resultant pair when the MOD matrix used is real.

Let $y = (0, 2 + 3g, 0, 0) \in R$.

To find the effect of y on S.

$yS^t = (5g, 2 + 8g, 0, 5g, 0, 8 + 2g) = x_1$;

$x_1S = (6 + 4g, 5g, 4 + 6g, 0) = y_1$.

We see if the $y \in R$ (or $x \in D$) happens to be a mixed special dual like number so will be the resultant pair.

In view of all these we have the following theorem to be true.

**THEOREM 3.7:** *Let*

$$M = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

*be the MOD special dual like number matrix operator with real entries from $Z_s \not\subseteq \langle Z_s \cup h\rangle = \{a + bh \mid h^2 = h, a, b \in Z_s\}$ $(2 \leq s < \infty)$.*

*Let $D = \{(a_1, a_2, \ldots, a_m) \mid a_i \in \langle Z_s \cup h\rangle; 1 \leq i \leq m\}$ and*

*$R = \{(b_1, b_2, \ldots, b_n) \mid b_i \in \langle Z_s \cup h\rangle, 1 \leq i \leq n\}$ be the domain and range space of state vectors respectively associated with M.*

i) *Every $x \in D$ (or $y \in R$) such that all entries from them are from $Z_s$ then certainly in the resultant pair all entries are from $Z_s$.*

ii) *If $x \in D$ (or $y \in R$) take its entries from $Z_sh$ then the resultant pair will have its entries from $Z_sh$.*

iii) *If $x \in D$ (or $y \in R$) takes its entries from $\{\langle Z_s \cup h\rangle \mid \{Z_s \cup Z_sh\}$ then its entries are from $\langle Z_s \cup h\rangle$.*

Proof is direct and hence left as an exercise to the reader.

Next we work with special state vector model.

***Example 3.26:*** Let

$$M = \begin{bmatrix} 3h & 0 & 4h+1 & 2h \\ 0 & 2h+1 & 0 & 3 \\ 4 & 0 & 2h+3 & 0 \\ 5h+1 & 2 & 0 & 5+h \\ 2h & 0 & 2+h & 0 \end{bmatrix}$$

be the MOD special dual like number matrix operator with entries from $\langle Z_6 \cup h\rangle = \{a + bh \mid a, b \in Z_6, h^2 = h\}$.

Let $D = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in \langle Z_6 \cup h\rangle, \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in \langle Z_6 \cup h\rangle, 1 \leq i \leq 5\}$ and

$R = \{(a_1, a_2, a_3, a_4) \mid a_i \in \langle Z_6 \cup h\rangle\ 1 \leq i \leq 4\}$ be the state vectors of the domain and range space associated with M respectively.

Let $D_S = \{(a_1, a_2,\ldots, a_5) \mid a_i \in \{0, 1, h, 1 + h\};\ 1 \leq i \leq 5\}$ and $R_S = \{(a_1, a_2, a_3, a_4) \mid a_i \in \{0, 1, h, 1 + h\}\ 1 \leq i \leq 4\}$ be the special state vectors of the domain and range space respectively associated with M.

Let $x = (1, 0, 0, 0, 0) \in D$.

The effect of x on M is as follows.

$xM = (3h, 0, 4h + 1, 2h) = y_1;$

$yM^t = (h + 1, 0, 3 + 4h, 0, h + 2) = x_1;$

$x_1M = (2h, 3, 2 + 4h, 4h) = y_1;$

$y_1M^t = (2, 3, 2h, 0, 4) = x_2;$

$x_2M = (4h, 3, 4h, 3 + 4h) = y_2;$

$y_2M^t = (4h, 0, 0, 3 + 3h, 2h) = x_3;$

$x_3M = (h, 0, 2h, 3 + 5h) = y_3$ and so on.

Clearly the resultant of $x = (1, 0, 0, 0, 0) \in D$ will be a pair which is a mixed special dual number pair.

Let $x = (1, 0, 0, 0, 0) \in D_S$.

We find the resultant of x as a special state vector of $D_S$

$xM = (3h, 0, 4h + 1, 2h) \to (h, 0, h, h) = y_1;$

$y_1M^t \to (1, h, h, h, h) = x_1;$

$x_1M \to (h, h, 1, h) = y_2;$

$y_2M^t \to (1, h, h, h, h) = x_2 \; (= x_1)$.

Thus the resultant is a realized fixed pair given by

$$\{(1, h, h, h, h), (h, h, 1, h)\}.$$

Let us consider $a = (0, h, 0, 0, 0) \in D$.

Clearly the resultant will be a realized pure special dual like number pair.

However a realized as a special state vector of $D_S$ will also yield only a resultant pair which will be a special dual like number pair

Now let $y = (0, 0, 1, 0) \in R$; to find the effect of y on M.

$yM^t = (4h + 1, 0, 2h + 3, 0, 2 + h) = x_1$;

$x_1M = (5h, 0, 2 + 3h, 4h) = y_1$;

$y_1M^t = (4h + 2, 0, 3h, 0, 4 + 3h) = x_2$ and so on.

Thus the resultant in this case will also be a mixed special dual like number pair.

Let $y = (0, 0, 1, 0) \in R_S$, that is y is realized as a special state vector from $R_S$.

The effect of y on $R_S$.

$yM^t = (4h + 1, 0, 2h + 3, 0, 2 + h) \to (h, 0, 1, 0, 1) = x_1$;

$x_1M \to (h, 1, 1, h) = y_1$; $\qquad y_1M^t \to (h, h, 1, 1, h) = x_2$;

$x_2M \to (1, h, 1, 1) = y_2$; $\qquad y_2M^t \to (h, h + 1, h, h, h) = x_3$

and so on.

We will after a finite number of iterations arrive at a realized fixed point pair or a realized limit cycle.

In view of all these we have the following theorem.

**THEOREM 3.8:** *Let*

$$M = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ldots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

*be the MOD special dual like number matrix operator with entries from* $\langle Z_s \cup h\rangle = \{a + bh / h^2 = h, a, b \in Z_s\}$*;* $2 \leq s < \infty$ *and*

$D = \{(a_1, a_2, \ldots, a_m) \mid a_i \in \langle Z_s \cup h\rangle; 1 \leq i \leq m\}$ *and*

$R = \{(b_1, b_2, \ldots, b_n) \mid b_j \in \langle Z_s \cup h\rangle; 1 \leq j \leq n\}$ *be the domain and range space of vectors.*

*If x ∈ R (or D) is a pure special dual like number state vector then so will be the resultant of x on M.*

Proof is direct and hence left as an exercise to the reader.

Next we proceed on to study the MOD special quasi dual number matrix operators.

Let $\langle Z_s \cup k\rangle = \{a + bk \mid a, b \in Z_s, k^2 = (s – 1) k\}$ be the ring of special quasi dual numbers.

We will use $\langle Z_s \cup k\rangle$ to build the matrix which we call as the MOD special quasi dual number matrix operators.

***Example 3.27:*** Let

$$M = \begin{bmatrix} 3k+2 & 1+k \\ 0 & 2k \\ 1+2k & 0 \\ 2 & 3+k \\ 1+3k & 2 \end{bmatrix}$$

be the MOD special quasi dual number matrix operator with entries from $\langle Z_4 \cup k\rangle = \{a + bk \mid a, b \in Z_4, k^2 = 3k\}$.

Let $D = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in \langle Z_4 \cup k\rangle\ 1 \leq i \leq 4\}$ and

$R = \{(b_1, b_2) \mid b_1, b_2 \in \langle Z_4 \cup k\rangle\}$ be the special quasi dual number state vectors respectively associated with M.

Let $x = (1, 0, 0, 0, 0) \in D$. To find the effect of x on M.

$xM = (3k + 2, 1 + k) = y_1;$

$y_1M^t = (1, 0, k + 2, 3 + k, 2) = x_1;$

$x_1M = (0, 2 + 2k) = y_2;$

$y_2M^t = (2 + 2k\ 0, 0, 2 + 2k, 0) = x_2;$

$x_2M = (0, 3 + 3k) = y_2;$

$y_2M^t = (3 + 3k, 0, 0, 1 + 3k, 2 + 2k) = x_3;$

$x_3M = (2 + 2k, 2 + 2k) = y_3;$

$y_3M^t = (0, 0, 2 + 2k, 2 + 2k, 0) = x_4;$

$x_4M = (2 + 2k, 2 + 2k) = y_4\ (= y_3).$

Thus the resultant is a realized fixed point pair given by $\{(0, 0, 2 + 2k, 2 + 2k, 0), (2 + 2k, 2 + 2k)\}$.

Let $y = (0, 1) \in R$, to find the effect of y on M.

$yM^t = (1 + k, 2k, 0, 3 + k, 2) = x_1$;

$x_1M = (2 + 2k, 2 + 2k) = y_1$;

$y_1M^t = (0, 0, 2 + 2k, 2 + 2k, 0) = x_2$;

$x_2M = (2 + 2k, 2 + 2k) = y_2$ $(=y_1)$.

Thus the resultant is a realized fixed point pair given by

$$\{(0, 0, 2 + 2k, 2 + 2k, 0), (2 + 2k, 2 + 2k)\}.$$

Let $x = (0, k, 0, 0, 0) \in D$.

$xM = (0, 2k) = y_1$; $y_1M^t = (0, 0, 0, 0, 0) = x_1$;

$x_1M = (0, 0) = y_2$.

Thus the resultant is a fixed point pair given by

$$\{(0, 0, 0, 0, 0), (0, 0)\}.$$

Let $y = (2k, 0) \in R$.

To find the effect of y on M.

$yM^t = (2k, 0, 2k, 0, 0) = x_1$; $x_1M = (0, 0) = y_1$;

$y_1M^t = (0, 0, 0, 0, 0) = x_2$.

Thus the resultant is once again a realized fixed point pair given by

$$\{(0, 0, 0, 0, 0), (0, 0)\}.$$

***Example 3.28:*** Let

$$M = \begin{bmatrix} 2k+3 & 3k & 0 & 1 & 4+k \\ 0 & 1+2k & 2k & 3k & 0 \\ k+2 & 0 & 1+k & 0 & 2+k \end{bmatrix}$$

be the MOD special quasi dual number matrix operator with entries from $\langle Z_6 \cup k\rangle = \{a + bk \mid a, b \in Z_6, k^2 = 5k\}$.

Let $D = \{(a_1, a_2, a_3) \mid a_i \in \langle Z_6 \cup k\rangle, i = 1, 2, 3\}$ and

$R = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in \langle Z_6 \cup k\rangle\ 1 \leq i \leq 5\}$ be the special quasi dual number initial state vectors of domain and range space respectively associated with M.

Let $x = (1, 0, 0) \in D$, to find the effect of x on M is as follows.

$$xM = (2k + 3, 3k, 0, 1, 4 + k) = y_1;$$

$$y_1M^t = (2, 0, 0) = x_1;$$

$$x_1M = (4k, 0, 0, 2, 2 + 2k) = y_2;$$

$$y_2M^t = (4, 0, 4 + 4k) = x_2;$$

$$x_2M = (4k + 2, 0, 4 + 4k, 4, 0) = y_3;$$

$$y_3M^t = ( 2+ 4k, 0, 2 + 2k) = x_3;$$

$$x_3M = (4, 0, 2 + 2k, 2 + 4k, 0) = y_4;$$

$$y_4M^t = (2, 0, 4) = x_4;$$

$$x_4M = (2 + 2k, 0, 4 + 4k, 2, 4) = y_5;$$

$$y_5M^t = (4k, 0, 4k) = x_5;$$

$x_5M = (2k, 2k, 0, 4k, 4k) = y_6;$

$y_6M^t = (0, 4k, 0) = x_6;$

$x_6M = (0, 4k, 4k, 0, 0) = y_7;$

$y_7M^t = (0, 0, 0) = x_7;$

$x_7M = (0, 0, 0, 0, 0) = y_8.$

Thus the resultant is a realized fixed point pair given by
$\{(0, 0, 0), (0, 0, 0, 0, 0)\}$.

Let $y = (0, 0, 0, 1, 0) \in R$, to find the effect of y on M.

$yM^t = (1, 3k, 0) = x_1;$

$x_1M = (2k + 3, 0, 0, 1 + 3k, 3k) = y_1;$

$y_1M^t = (3 + 2k, 0, 2k) = x_2;$

$x_2M = (4k + 3, 3k, 0, 3 + 2k, 2k) = y_2;$

$y_2M^t = (3 + 3k, 0, 3k) = x_3;$

$x_3M = (3, 0, 0, 3 + 3k, 3k) = y_3;$

$y_3M^t = (0, 3 + 3k, 0) = x_4;$

$x_4M = (0, 3 + 3k, 0, 3 + 3k, 0) = y_4;$

$y_4M^t = (3 + 3k, 3 + 3k, 0) = x_5;$

$x_5M = (3 + 3k, 0, 0, 3 + 3k, 0) = y_5;$

$y_5M^t = (0, 0, 0) = x_6;$

$x_6M = (0, 0, 0, 0, 0).$

Thus the resultant is a realized fixed point pair given by
$\{(0, 0, 0), (0, 0, 0, 0, 0)\}$.

***Example 3.29:*** Let

$$M = \begin{bmatrix} 5 & 2 & 3 \\ 4 & 0 & 6 \\ 0 & 8 & 0 \\ 1 & 2 & 3 \\ 0 & 1 & 0 \end{bmatrix}$$

be the MOD special quasi dual number matrix operator with real entries from $Z_{10} \subseteq \langle Z_{10} \cup k\rangle = \{a + bk \mid a, b \in Z_{10}, k^2 = 9k\}$.

Let $D = \{(a_1, a_2, \ldots, a_5) \mid a_i \in \langle Z_{10} \cup k\rangle; 1 \leq i \leq 5\}$ and

$R = \{(a_1, a_2, a_3) \mid a_i \in \langle Z_{10} \cup k\rangle; 1 \leq i \leq 3\}$ be the domain and range space of state vectors respectively associated with the MOD special quasi dual number matrix operator M.

Let $x = (0, 1, 0, 0, 0) \in D$.

To find the effect of x on M.

$xM = (4, 0, 6) = y_1;$ $y_1M^t = 8, 2, 0, 2, 0) = x_1;$

$x_1M = (0, 0, 2) = y_2;$ $y_2M^t = (6, 2, 0, 6, 0) = x_2;$

$x_2M = (4, 4,8) = y_3;$ $y_3M^t = (2, 4, 2, 6, 4) = x_3;$

$x_3M^t = (2, 6, 8) = y_4;$ $y_4M^t = (2, 6, 8, 8, 6) = x_4;$

$x_4M = (2, 6, 6) = y_5;$ $y_5M^t = (0, 4, 8, 2, 6) = x_5;$

$x_5M = (8, 4, 0) = y_6;$ $y_6M^t = (8, 2, 2, 6, 4) = x_5;$

$x_5M = (4, 8, 8) = y_7;$ $y_7M^t = (0, 4, 4, 4, 8) = x_6;$

$x_6M = (0\ 8\ 6) = y_8$; $y_8M^t = (4, 6, 4, 4, 8) = x_7$;

$x_7M = (8, 6, 0) = y_9$; $y_9M^t = (2, 2, 8, 0, 6) = x_8$;

$x_8M = (8, 4, 8) = y_{10}$; $y_{10}M^t = (2, 0, 2, 0, 4) = x_9$;

$x_9M = (0, 4, 6) = y_{11}$; $y_{11}M^t = (6, 6, 2, 6, 4) = x_{10}$;

$x_{10}M = (0, 4, 2) = y_{12}$; $y_{12}M^t = (4, 2, 2, 4, 4) = x_{11}$;

$x_{11}M = (2, 6, 6) = y_{13}$; $y_{13}M^t = (0, 4, 8, 2, 6) = x_{12}$;

$x_{12}M = (8, 4, 0) = y_{14}$ $(=y_6)$.

Thus the resultant is a realized limit cycle pair given by {(8, 2, 2, 6, 4), (8, 4, 0)}.

Let y = (0, 0, 1) ∈ R. To find the effect of y on M.

$yM^t = (3, 6, 0, 3, 0) = x_1$; $x_1M = (2, 2, 4) = y_1$;

$y_1M^t = (6, 4, 6, 8, 2) = x_2$; $x_2M = (4, 8,6) = y_2$;

$y_2M^t = (4, 2, 4, 8, 8) = x_3$; $x_3M = (6, 4, 8) = y_3$;

$y_3M^t = (2, 2, 2, 8, 4) =x_4$; $x_4M = (6, 0, 2) = y_4$;

$y_4M^r = (6, 6, 0\ 2\ 0) = x_5$; $x_5M = (6, 6, 0) = y_5$;

$y_5M^r = (2, 4, 8, 6\ 6) = x_6$; $x_6M = (2, 6, 8) = y_6$;

$y_6M^r = (6, 6, 8, 8, 6) = x_7$; $x_7M = (2, 8, 8) = y_7$;

$y_7M^r = (0, 4, 4, 2, 8) = x_8$; $x_8M = (8, 2, 0) = y_8$;

$y_8M^r = (4\ 2, 6, 2, 2) = x_9$; $x_9M = (0, 2, 0) = y_9$;

$y_9M^r = (4, 0, 6, 4, 2) = x_{10}$; $x_{10}M = (4, 6, 4) = y_{10}$;

$y_{10}M^r = (4, 0, 8, 8, 6) = x_{11}$; $x_{11}M = (8, 4, 6) = y_{11}$;

$y_{11}M^r = (6, 8, 2, 0, 4) = x_{12}$; $x_{12}M = (2, 2, 6) = y_{12}$;

$y_{12}M^r = (2, 4, 6, 4, 2) = x_{13}$; $x_{13}M = (0, 2, 2) = y_{13}$;

$y_{13}M^r = (0, 2, 6, 0, 2) = x_{14}$; $x_{14}M = (8, 0, 2) = y_{14}$;

$y_{14}M^r = (6, 4, 0, 4, 0) = x_{15}$; $x_{15}M = (0, 0, 4) = y_{15}$;

$y_{15}M^r = (2, 4, 0, 6, 0) = x_{16}$; $x_{16}M = (0, 6, 8) = y_{16}$;

$y_{16}M^r = (6, 8, 8, 0, 6) = x_{17}$; $x_{17}M = (2, 2, 6) = y_{17}$ $(=y_{12})$.

Thus the resultant is a realized limit cycle given by the pair $\{(2, 4, 6, 4, 2), (2, 2, 6)\}$.

We see the resultant is real if the MOD special quasi dual number matrix is real.

Let $x = (0, k, 0, 0, 0) \in D$.

To find the effect of x on M.

$xM = (4k, 0, 6k) = y_1$; $y_1M^t = (8k, 2k, 0, 2k, 0) = x_1$;

$x_1M = (0, 0, 6k) = y_2$; $y_2M^t = (8k, 6k, 0, 8k, 0) = x_2$;

$x_2M = (2k, 2k, 4k) = y_3$; $y_3M^t =(6k, 2k, 6k, 8k, 2k) = x_3$;

$x_3M = (6k, 8k, 4k) = y_4$; $y_4M^t = (8k, 8k, 4k, 4k,8k) = x_4$;

$x_4M = (6k, 4k, 4k) = y_5$; $y_5M^t = (0, 8k, 2k, 6k, 4k) = x_5$;

$x_5M = (8k, 2k, 6k) = y_6$; $y_6M^t = (8k, 8k, 6k, 0, 2k) = x_6$;

$x_6M = (2k, 6k, 2k) = y_7$; $y_7M^t = (8k, 0, 8k, 4k, 6k) = x_7$;

$x_7M = (4k, 4k, 6k) = y_8$; $\quad y_8M^t = (6k, 2k, 2k, 0, 4k) = x_8$;

$x_8M = (8k, 2k, 0) = y_9$; $\quad y_9M^t= (4k, 2k, 6k, 2k, 2k)=x_{10}$;

$x_{10}M = (0, 2k, 0) = y_{10}$; $\quad y_{10}M^t = (4k, 0, 6k, 4k, 2k)=x_{11}$;

$x_{11}M = (4k, 6k, 4k) = y_{12}$; $\quad y_{12}M^t = (4k, 0, 8k, 8k, 6k)=x_{12}$;

$x_{12}M = (8k, 4k, 6k) = y_{13}$; $\quad y_{13}M^t = (6k,8k,2k,4k,4k)= x_{13}$;

$x_{13}M = (6k, 0, 8k) = y_{14}$; $\quad y_{14}M^t = (4k, 2k, 0, 0, 0) = x_{14}$;

$x_{14}M = (8k, 8k, 4k) = y_{15}$; $\quad y_{15}M^t = (8k,8k,4k,6k,8k) = x_{15}$

and so on.

However after a finite number of iterations we will arrive at a realized fixed point pair or a realized limit cycle pair which will certainly be a pure special quasi dual number state vectors.

In view of this we prove the following theorem.

**THEOREM 3.9:** *Let*

$$M = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

*be a MOD special quasi dual number matrix operator with real entries from $Z_s \subseteq \langle Z_s \cup k\rangle = \{a + bk \,/\, a, b \in Z_s, k^2 = (s-1)k\}$.*

*$D = \{(a_1, a_2, \dots, a_m) \mid a_i \in \langle Z_s \cup k\rangle\ 1 \leq i \leq m\}$ and*

*$R = \{(a_1\, a_2, \dots, a_n) \mid a_i \in \langle Z_s \cup k\rangle;\ 1 \leq i \leq n\}$ be the domain and range state vectors respectively associated with M.*

i) *Every $x \in D$ (or $y \in R$) where all entries of x (or y) are real is such that the resultant pair of x on M is also a real pair.*

ii) *If $x \in D$ (or $y \in R$) is such that the entries of x are in $Z_s k$ that is pure special quasi dual numbers then the resultant pair given by x on M is also a pure special quasi dual number pair.*

Proof is direct and hence left as an exercise to the reader.

We now work with pure special quasi dual number MOD matrix operator M.

***Example 3.30:*** Let

$$M = \begin{bmatrix} 3k & 2k & k \\ 0 & k & 2k \\ 2k & 0 & k \end{bmatrix}$$

be the MOD special quasi dual number matrix operator with entries from $Z_4k \subseteq \langle Z_4 \cup k\rangle = \{a + bk \mid a, b \in Z_4, k^2 = 3k\}$.

Let $D = \{(x_1, x_2, x_3) \mid x_i \in \langle Z_4 \cup k\rangle; 1 \leq i \leq 3\}$ and

$R = \{(y_1, y_2, y_3) \mid y_i \in \langle Z_4 \cup k\rangle; 1 \leq i \leq 3\}$ be the domain and range space of state vectors respectively associated with M.

Let $x = (1, 0, 0) \in D$.

To find the effect of x on M.

$xM = (3k, 2k, k) = y_1$; $\quad y_1M^t = (2k, 0, k) = x_1$;

$x_1M = (0, 0, k) = y_2$; $\quad y_2M^t = (3k, 2k, 3k) = x_2$;

$x_2M = (k, 0, 2k) = y_3$; $\quad y_3M^t = (3k, 0, 0) = x_3$;

$x_3M = (3k, 2k, k) = y_4\ (= y_1)$.

Thus the resultant is a realized limit cycle pair given by {(2k,0, k), (3k, 2k, k)} though the initial state vector

$$x = (1, 0, 0) \in D.$$

Now let y = (1, 0, 0) ∈ R to find the effect of y on M.

$yM^t = (3k, 0, 2k) = x_1$; $x_1M = (3k, 2k, k) = y_1$;

$y_1M^t = (2k, 0, k) = x_2$; $x_2M = (0, 0, k) = y_2$;

$y_2M^t = (3k, 2k, 3k) = x_3$; $x_3M = (k, 0, 2k) = y_3$;

$y_3M^t = (3k, 0, 0) = x_4$; $x_4M = (3k, 2k, k) = y_4(=y_1)$.

Thus the resultant is a realize limit cycle pair given by {(2k, 0, k), (3k, 2k, k)}.

If x = (0, 0, k) ∈ D to find the effect of x on M.

$xM = (2k, 0, 3k) = y_1$; $y_1M^t = (3k, 2k, k) = x_1$;

$x_1M = (k, 0, 0) = y_2$; $y_2M^t = (k, 0, 2k) = x_2$;

$x_2M = (k, 2k, k) = y_3$; $y_3M^t = (0, 0, k) = x_3(=x_1)$.

Thus the resultant is a realized limit cycle pair given by {(0, 0, k), (2k, 0, 3k)}.

Let y = (0, 0, k) ∈ R. To find the effect of y on M.

$yM^t = (3k, 2k, 3k) = x_1$; $x_1M = (k, 0, 2k) = y_1$;

$y_1M^t = (3k, 0, 0) = x_2$; $x_2M = (3k, 2k, k) = y_2$;

$y_2M^t = (2k, 0, k) = x_3$; $xM = (0, 0, k) = y_5 (=y_1)$.

Thus the resultant is a realized limit cycle pair given by {(3k, 2k, 3k), (0, 0, k)}.

In view of this we make the following observation.

**THEOREM 3.10:** *Let*

$$M = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

*be the MOD special quasi dual number matrix operator with entries from $Z_sk = \{ak / a \in Z_s, k^2 = (s-1)k\} \subseteq \langle Z_s \cup k\rangle = \{a + bk / a, b \in Z_s, k^2 = (s-1)k\}; 2 \leq s < \infty$.*

*Let $D = \{(a_1, a_2, \dots a_m) \mid a_i \in \langle Z_s \cup k\rangle; 1 \leq i \leq m\}$ and*

*$R = \{(b_1, b_2, \dots, b_n) \mid b_i \in \langle Z_s \cup k\rangle; 1 \leq i \leq m\}$ be the domain and range space of state vectors respectively associated with M.*

*Every $x \in D$ (or $y \in R$) has its resultant pair to be a pure special quasi dual number that is every element of the resultant is from $Z_sk$.*

Proof is direct and hence left as an exercise to the reader.

***Example 3.31:*** Let

$$M = \begin{bmatrix} 2k+2 & 6k \\ 0 & 4+2k \\ 6+2k & 0 \\ 2k & 4+6k \\ 0 & 4k \\ 6k & 0 \end{bmatrix}$$

be the MOD special quasi dual number matrix operator with entries from $\langle Z_8 \cup k\rangle = \{a + bk \mid a, b \in Z_8, k^2 = 7k\}$.

$D = \{(a_1, a_2, \ldots, a_6) / a_i \in \langle Z_8 \cup k\rangle; 1 \leq i \leq 6\}$ and

$R = \{(b_1, b_2) / b_i \in \langle Z_8 \cup k\rangle, 1 \leq i \leq 2\}$ be the domain and range space of state vectors respectively associated with M.

Let $D_s = \{(a_1, a_2, \ldots, a_6) \mid a_i \in \{0, 1, k, 1 + k\}\ 1 \leq i \leq 6\}$ and

$R_S = \{(b_1, b_2) \mid b_1, b_2 \in \{0, 1, k, 1 + k\}\}$ be the domain and range space of special state vectors associated with M.

Let $x = (1, 0, 0, 0, 0, 0) \in D$.

To find the effect of x on M;

$xM = (2k + 2, 6k) = y_1;$

$y_1M^t = (4, 4k\ 4+ 4k, 4k, 0, 0) = x_1;$

$x_1M = (0, 0) = y_2;$

$y_2M^t = (0, 0, 0, 0, 0, 0).$

Thus the resultant is a realized fixed point pair given by

$$\{(0, 0, 0, 0, 0, 0), (0, 0)\}.$$

Let $x = (1, 0, 0, 0, 0, 0) \in D_S$ be realized as a special state vector from $D_S$.

To find the effect of x on M.

$xM \to (k + 1, k) = y_1;$

$y_1M^t \to (k, k, 1 + k, k, k, 0) = x_1;$

$x_1M \to (1, 1 + k) = y_2;$

$y_2M^t \to (1 + k, 1 + k, 1, k, 0, k) = x_2;$

$x_2M \to (k, 1) = y_3;$

$y_3M^t \to (k, 1, k, 1 + k, k, k) = x_3;$

$x_3M = (6k, 4k) \to (k, k)$ ;

$y_4M^t \to (k, k, k, k, k, k) = x_4;$

$x_4M \to (k, k).$

Thus the resultant is a realized fixed point pair given by {(k, k, k, k, k, k), (k, k)}. When x = (1, 0, 0, 0, 0, 0) is realized as a special state vector of $D_S$.

Let y = (0, 1) ∈ $R_S$ to find the effect of y on M.

$yM^t \to (k, 1, 0, k, k, 0) = x_1;$ $\qquad$ $x_1M \to (k, k) = y_1;$

$y_1M^t \to (k, k, k, k, k, k) = x_2;$ $\qquad$ $x_2M \to (k, k).$

Thus y = (0, 1) realized as a special state vector yields in a realized fixed point pair given by {(k, k, k, k, k, k), (k, k)}.

Let y = (0, 1) ∈ R. To find the effect of y on M.

$yM^t = (6k, 4 + 2k, 0, 4 + 6k, 4k, 0) = x_1;$

$x_1M = (0, 4k) = y_2;$

$y_2M^t = (0, 0, 0, 0, 0, 0) = x_2;$

$x_2M = (0, 0) = y_3.$

Thus the resultant is a realized fixed point pair resulting in {(0, 0, 0, 0, 0, 0), (0, 0)}.

We see there is a different between the resultant of y = (0, 1) realized as a state vector of R and secondly when realized initial special sate vector of $R_S$.

Now we make the following conclusions for this chapter. The MOD operators are matrices with entries from $\langle Z_n \cup I\rangle$ or $\langle Z_n \cup g\rangle$ ($g^2 = 0$) or $\langle Z_n \cup h\rangle$ ($h^2 = h$) or $\langle Z_n \cup k\rangle$ ($k^2 = (n - 1)k$ or $C(Z_n)$; $i_F^2 = n - 1$. It is clearly seen when the elements are from different MOD operators the resultants are also different. Our main motivation at this juncture is to show that there are special MOD fixed point pairs which are termed as realized fixed points. Here it is pertinent to keep on record that these realized fixed points are not classical fixed points like the ones appearing from projection of a space to a subspace and so on but fixed point in the natural sense for these points after a few iterations give a particular fixed value.

Clearly matrix operator operates from (D, R) → (D, R) with appropriate dimensions and notations given earlier.

This MOD matrix operator acts

$[\ \ ]_{m\times n} : ( [\ ]_{1\times m}, [\ ]_{1\times m}) \to ([\ ]_{1\times m}, [\ ]_{1\times n})$ as a special type of map functioning in a very different way. Special properties related with them are discussed in this chapter.

Further it is important to observe always the resultant may not be a realized fixed point pair but it may be a realized limit cycle pair.

However as the entries are always from a finite set $Z_n$; $2 \leq n < \infty$ we are always guaranteed of a realized fixed point pair or a realized limit cycle pair.

If $Z_n$ is replaced by Z or C or Q or R or $\langle R \cup I\rangle$ or $\langle R \cup g\rangle$ or $\langle R \cup k\rangle$ or $\langle R \cup h\rangle$ the resultant or to be more specific arriving at a resultant is a NP hard problem.

Further we do another technique of working using this MOD operator, namely this operator works on the special state vectors as the domain and range space, on this situation also as we use $Z_n$ we will arrive at a realized fixed point pair.

But we wish to keep on record the most important fact we want to express is that we can take any matrix as a operator with entries from Z or Q or R or C or $\langle Q \cup g\rangle$ or $\langle R \cup I\rangle$ or so on.

But for us to arrive a realized fixed point or a limit cycle is possible provided we use the state vectors of domain and range space from {0, 1} or {0, I, I + 1, 1} or {1, g, 1 + g, 0}, {0, I, 1 + I, 1} and so on.

We will illustrate these situations by some examples.

***Example 3.32:*** Let

$$M = \begin{bmatrix} 3 & 1 & 0 \\ 7 & 0 & 9 \\ 12 & 10 & 0 \\ 0 & 15 & 0 \\ 21 & 0 & 18 \end{bmatrix}$$

matrix operator from Z.

Let D = $\{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in \{0, 1\}; 1 \leq i \leq 5\}$ and

R = $\{(a_1, a_2, a_3) \mid a_i \in \{0, 1\}; 1 \leq i \leq 3\}$ be the state vectors associated with M.

We use the operations of thresholding and updating the state vectors at each stage of working.

Let x = $(1, 0, 0, 0, 0) \in D$;

To find the resultant of x on M;

$xM = (3, 1, 0) \to (1, 1, 0) = y_1$;

$y_1M^t \to (1, 1, 1, 1, 1) = x_1$;

$x_1M \to (1, 1, 1) = y_2$;

$y_2M \to (1, 1, 1, 1, 1) = x_2 \ (= x_1)$.

So the resultant is a fixed point pair given by
$\{(1, 1, 1, 1, 1), (1, 1, 1)\}$

***Example 3.33:*** Let

$$M = \begin{bmatrix} 5 & -2 & 0 \\ -3 & 0 & -1 \\ -2 & 0 & 1 \\ 0 & 1 & 0 \\ -4 & 0 & 0 \end{bmatrix}$$

be the matrix operator.

Let $D = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in \{0, 1\}; 1 \leq i \leq 5\}$ and

$R = \{(a_1, a_2, a_3) \mid a_i \in \{0, 1\}; 1 \leq i \leq 3\}$ be the domain and range space of state vectors respectively associated with M.

Let $x = (0, 0, 1, 0, 0) \in D$;

To find the resultant of x on M;

$xM \to (0, 0\ 1) = y_1$; $\qquad y_1M^t \to (0, 0, 1, 0, 0) = x_1 \ (=x)$.

So the resultant is a realized classical fixed point given by
$\{(0, 0, 1, 0, 0)\ (0, 0, 1)\}$.

Thus we will arrive at a realized fixed point or a classical fixed point or a limit cycle pair.

However this topic is not within the purview of this book but it is pertinent to keep on record realized fixed points and limit cycle can exist for any matrix operator.

***Example 3.34:*** Let

$$M = \begin{bmatrix} 3 & -3 & 2 & 0 \\ -5 & 0 & -3 & 0 \\ 2 & -1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

be the matrix operator.

$S = \{(a_1, a_2, a_3, a_4) \mid a_i \in \{0, 1\}; 1 \leq i \leq 4\}$ be the initial state of vectors related with the matrix operator M.

Let $x \in S$; to find the effect of $x = (1, 0, 0, 0)$ on M.

$xM = (3, –3, 2, 0) \to (1, 0, 1, 0) = x_1$;

$x_1M \to (1, 0, 1, 1) = x_2$;

$x_2M \to (1, 0, 1, 0) = x_3 = (x_1)$.

Thus the resultant is a limit cycle $\{(1, 0, 1, 0), (1, 0, 1, 1)\}$.

Let $x = (0, 0, 0, 1) \in S$ to find the effect of x on M.

$xM \to (0, 0, 1, 1) = x_1$; $\quad$ $x_1M \to (1, 0, 1, 1) = x_2$;

$x_2M \to (1, 0, 1, 1) = x_3 \; (=x_2)$.

Thus the resultant is a realized fixed point given by $\{(1, 0, 1, 1)\}$.

We suggest a set of problems for the reader as an exercise.

**Problems:**

1. Obtain all special features enjoyed by MOD neutrosophic rectangular operators.

2. Let M = $\begin{bmatrix} 3+2I & 5I+1 & 6I & 2 \\ 0 & 4I+9 & 2+4I & 0 \\ 7I+3I & 0 & 0 & 4I+5 \\ 0 & 9I+1 & 8I+3 & 0 \\ 3I & 0 & 2 & 5+7I \\ 0 & 8I & 0 & 3I \end{bmatrix}$

be the MOD neutrosophic matrix operator with entries from $\langle Z_{10} \cup I\rangle = \{a + bI \mid a, b \in Z_{10}; I^2 = I\}$.

$D = \{(a_1, a_2, \ldots, a_6) \mid a_i \in \langle Z_{10} \cup I\rangle; 1 \leq i \leq 6\}$ and

$R = \{(b_1, b_2, b_3, b_4) \mid b_i \in \langle Z_{10} \cup I\rangle, 1 \leq i \leq 4\}$ be the domain and range space of state vectors associated with M.

i. Can M have classical fixed point?

ii. If $x \in \{(a_1, a_2, \ldots, a_6) / a_i \in Z_{10}\}; 1 \leq i \leq 6 \subseteq D$; what is the special feature associated with x?

iii. If $y \in \{(d_1, d_2, d_3, d_4) / d_i \in Z_{10}I; 1 \leq i \leq 4\} \subseteq R$; what is the special feature enjoyed by y?

iv. Let $x_1 = (3I + 1, 0\ 0\ 0\ 0\ 0)$ and $x_2 = (0, 0\ 0, 2I + 7, 0, 0) \in D$.

(a) Find the resultant of $x_1$ and $x_2$ on M.
(b) Let $x = x_1 + x_2 \in D$, find the resultant of x on M.
(c) Compare (b) with sum of the resultants of $x_1$ and $x_2$ on M mentioned in (a).

v. Characterize all those state vectors in D and R which give only realized fixed points.

vi. Characterize all those state vectors in R and D which yield as resultant realized limit cycle pair.

vii. Obtain any other interesting feature enjoyed by this MOD neutrosophic operators.

3. Study problem (2) using the same MOD matrix but entries instead taken from $\langle Z_{11} \cup I\rangle$.

What are the differences between the resultants when $\langle Z_{11} \cup I\rangle$ is used instead of $\langle Z_{10} \cup I\rangle$?

4. Let S = $\begin{bmatrix} 3I+2 & 4I+7 & 3I \\ 0 & 8I+10 & 2 \\ 4I+7 & 0 & 5I+1 \\ 0 & 9I & 8I+1 \\ 7I+8 & 3 & 4 \\ 0 & 3+2I & 0 \\ 4+9I & 0 & 0 \end{bmatrix}$ MOD neutrosophic

matrix operator with entries from $\langle Z_{12} \cup I\rangle = \{a + bI$ where $a, b \in Z_{12}, I^2 = I\}$.

Let D and R the appropriate state vectors associated with S.

Study questions (i) to (vii) of problem 2 for this S with appropriate changes.

5. Let S = $\begin{bmatrix} 3+i_F & 0 & 7i_F+1 & 2+3i_F & 4 \\ 0 & 5i_F & 0 & 8 & 3+8i_F \\ 4+3i_F & 6i_F+1 & 7+3i_F & 0 & 0 \\ 0 & 0 & 4 & 2+i_F & 1+i_F \end{bmatrix}$

be the MOD finite complex modulo integer matrix operator with entries from $C(Z_9)$.

D = $\{(a_1, a_2, a_3, a_4) \mid a_i \in C(Z_9) = \{a + bi_F$ where $a, b \in Z_9,\ i_F^2 = 8\};\ 1 \leq i \leq 4\}$ and

R = $\{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in C(Z_9);\ 1 \leq i \leq 5\}$ be the domain and range space of state vectors respectively associated with S.

i) Characterize all classical fixed points of S.

ii) Characterize all realized fixed point pairs of S.

iii) Characterize all realized limit cycle pair of S.
iv) Let x = $(3 + 4i_F, 0, 0, 2)$ and
    y = $(4i_F + 1, 0\ 3i_F, 0) \in$ D.
    a) Find the resultant of x on S.
    b) Find the resultant of y on S.
    c) Find the resultant of x + y on S.
    d) Does there exist any relation between the resultants in (a), (b) and (c)?
    e) Obtain all special features enjoyed by MOD complex number matrix operators.
    f) Compare MOD complex number matrix operators with MOD neutrosophic number matrix operators.

6. Let M =

$$\begin{bmatrix} 2+i_F & 4+3i_F & 0 & 7+5I & 4 & 2i_F+3 \\ 0 & 1+i_F & 8 & 0 & 4+3i_F & 0 \\ 7+5i_F & 0 & 3+5i_F & 4i_F & 0 & 3 \end{bmatrix}$$

be the MOD complex integer matrix operator with entries from $C(Z_{10})$.

i) Study questions (i) to (iv) of problem (5) for this M.

ii) If $D_S = \{(a_1, a_2, a_3) \mid a_i \in \{0, 1, i_F, 1 + i_F\}\ 1 \leq i \leq 3\}$ and $R_S = \{(b_1, b_2, \ldots, b_6)\}\ b_i \in \{0, 1, i_F, 1 + i_F\}, 1 \leq i \leq 6\}$ be the domain and range space of special state vectors respectively.

Study questions (i) to (iv) of problem 5 for this $D_S$ and $R_{S.}$

iii) Compare the resultants of those common in D and R with $D_S$ and $R_S$ using both type of operations.

7. Let N = $\begin{bmatrix} 2 & 3 & 0 & 4 & 5 & 8 & 1 \\ 0 & 1 & 2 & 0 & 3 & 0 & 4 \\ 5 & 0 & 7 & 9 & 0 & 1 & 2 \\ 1 & 5 & 0 & 6 & 1 & 0 & 0 \\ 6 & 0 & 3 & 0 & 0 & 4 & 0 \end{bmatrix}$

be the MOD neutrosophic matrix operators with entries from $Z_{10} \subseteq \langle Z_{10} \cup I\rangle = \{a + bI\} \mid a, b \in Z_{10}, I^2 = I\}$.

Study all the special features associated with N.

8. Let $Z = \begin{bmatrix} 3I & 2I & 0 & 4I \\ 0 & 5I & 6I & 0 \\ 7I & 0 & I & 2I \\ 0 & 4I & 0 & 3I \\ I & 0 & 2I & 0 \\ 3I & I & 0 & 2I \\ 0 & 6I & I & 0 \\ 4I & 0 & 0 & 3I \end{bmatrix}$ be the MOD pure neutrosophic matrix operator with entries from $Z_8I \subseteq \langle Z_8 \cup I\rangle = \{a + bI \mid a, b \in Z_8, I^2 = I\}$.

i. Study all special features enjoyed by Z.

ii. Compare problems (7) and (8).

9. Let $M = \begin{bmatrix} 3 & 1 & 2 & 0 & 5 & 6 & 7 & 8 \\ 0 & 3 & 1 & 2 & 0 & 5 & 6 & 7 \\ 8 & 0 & 3 & 1 & 2 & 0 & 5 & 6 \\ 7 & 8 & 0 & 1 & 2 & 3 & 0 & 0 \end{bmatrix}$ be the MOD complex integer matrix operator with real entries from $Z_{12} \subseteq C(Z_{12}) = \{a + bi_F \mid a, b \in Z_{12}; i_F^2 = 11\}$.

Study all special features enjoyed by the MOD operator. M and their resultants.

10. Let $P = \begin{bmatrix} 2i_F & 0 & 4i_F & i_F & 3i_F & 0 & 2i_F \\ 0 & 3i_F & 0 & 5i_F & 0 & i_F & 0 \\ 4i_F & 0 & 2i_F & 0 & 3i_F & 0 & 4i_F \end{bmatrix}$ be the

MOD pure complex modulo integer matrix operator with entries from $Z_6 i_F = \{ai_F / a \in 6;\ i_F^2 = 5\} \subseteq C(Z_6) = \{a + bi_F / a, b \in Z_6,\ i_F^2 = 5\}$.

Study all special features enjoy by the resultant state vectors associated with P.

11. Let $M = \begin{bmatrix} 3+g & 2g+1 & 0 & 5g+2 & 6g \\ 0 & 7g & 3g+1 & 0 & 1+g \\ 2+2g & 0 & 4+g & 3g & 0 \\ g & 4 & 0 & 3+2g & 7 \end{bmatrix}$ be

the MOD dual number matrix operator with entries from $\langle Z_8 \cup g\rangle = \{a + bg / g^2 = 0, a, b \in Z_8\}$.

$D = \{(a_1, a_2, a_3\ a_4) / a_i \in \langle Z_8 \cup g\rangle\ \ 1 \leq i \leq 4\}$ and

$R = \{(b_1, b_2, b_3, b_4, b_5) / b_i \in \langle Z_8 \cup g\rangle\ 1 \leq i \leq 5\}$ be the domain and range space of state vectors respectively associated with M.

Let $D_S = \{(a_1, a_2, a_3, a_4) / a_i \in \{0, 1, g, 1 + g; 1 \leq i \leq 4\}$

and $R_S = \{(b_1, b_2, b_3, b_4, b_5) / b_i \in \{0, 1, g, 1 + g\}; 1 \leq i \leq 5\}$ be the domain and range space special state vectors associated with M.

i) Obtain all classical fixed points of M associated with D and R.

ii) Obtain all classical fixed points of M associated with $D_S$ and $R_S$.

iii) Compare the resultants in (i) and (ii).

iv) Characterize all realized fixed points of M associated with R and D.

v) Characterize all realized limit cycles of M associated with D and R.

vi) Study questions (iv) and (v) in case of $D_S$ and $R_S$.

vii) Let x = (1, 0, 0, 0) $\in$ D find its resultant.

viii) For same x = (1, 0, 0, 0) $\in$ $D_S$ find its resultant.

ix) Compare the resultants in (vii) and (viii).

x) Characterize those x in $D_S \subseteq D$ and y on $R_S \subseteq R$ have same resultants when treated as elements as $D_S$ as well as element of D.

12. Let $M = \begin{bmatrix} 3 & 1 & 2 & 0 & 4 & 5 & 6 & 0 & 7 & 8 \\ 0 & 5 & 0 & 6 & 0 & 4 & 0 & 8 & 0 & 1 \\ 1 & 0 & 2 & 0 & 3 & 0 & 4 & 0 & 5 & 0 \\ 6 & 6 & 0 & 1 & 0 & 3 & 0 & 1 & 0 & 7 \end{bmatrix}$ be the

MOD dual number matrix operator with entries from $Z_9 \subseteq \langle Z_9 \cup g\rangle = \{a + bg / a, b \in Z_9, g^2 = g\}$.

i) Show all $x \in \{(a_1, a_2, a_3, a_4)$ with $a_i \in Z_9$, $1 \leq i \leq 4\}$ and all $y \in \{(x_1, x_2, \ldots, x_{10}) / x_i \in Z_9; 1 \leq i \leq 10\}$ give the resultant pairs which are real, that is then entries are in $Z_9$.

ii) If $x \in \{(a_1g, a_2g, a_3g, a_4g) \mid a_i \in Z_9;\ 1 \leq i \leq 4\}$ and $y = \{(a_1g, a_2g, \ldots, a_{10}g) \mid a_i \in Z_9;\ 1 \leq i \leq 10\}$ then all their resultants pairs are also pure dual number state vectors with entries from $Z_9g = \{ag \mid a \in Z_9, g^2 = 0\}$.

iii) Obtain all special features enjoyed by this M.

13. Let $M = \begin{bmatrix} 3g & 4g & 0 & 5g \\ 0 & 6g & 7g & 0 \\ 9g & 0 & 11g & 12g \\ 0 & 15g & 0 & g \\ 19g & 0 & 8g & 0 \\ 0 & 14g & 0 & 13g \\ g & 0 & 7g & 0 \end{bmatrix}$ be the MOD pure

dual number matrix operator with entries from

$Z_{20}g = \{ag \mid a \in Z_{20}, g^2 = 0\} \subseteq \langle Z_{20} \cup g \rangle$
$= \{a + bg \mid a, b \in Z_{20}, g^2 = 0\}$.

i) Study all special features associated with resultant of $M_1$.

ii) Compare the MOD operator M of problem (12) with this $M_1$.

14. Let S =

$$\begin{bmatrix} 3h+1 & 2h+5 & 3h & 0 & 4+2h & 6 \\ 0 & 4h & 5h+1 & 6h & 0 & 7h \\ 6h+5 & 0 & 6 & 3+4h & 6h & 1+h \\ 0 & 5 & 0 & 3 & 0 & 8h+1 \\ 4 & 3+2h & 7+h & 0 & 1+h & 0 \\ h & 0 & 1+h & 5h & 0 & 7+h \\ 3+4h & 7+h & 0 & 7+2h & h & 0 \end{bmatrix}$$

be the MOD special dual like number matrix operator with entries from $\langle Z_9 \cup h\rangle = \{a + bh / a, b \in Z_9, h^2 = h\}$.

Let D = $\{(a_1, a_2, \ldots, a_7) / a_i \in \langle Z_9 \cup h \rangle; 1 \leq i \leq 7\}$ and R = $\{(b_1, b_2, \ldots, b_6) / b_i \in \{0, \{0, 1, h, 1 + h\}; 1 \leq i \leq 7\}$ be the domain and range space of specials state of vectors respectively associated with S.

i) Obtain all special features associated with S.

ii) Let x = (1, 0, …, 0) $\in$ D. Find its resultant on S.

iii) Let x = (1, 0, …, 0) $\in D_S$. Find its resultant on S.

iv) Compare the resultants given in (ii) and (iii).

v) Characterize all those state vectors in D or $D_S$ or R or $R_S$ which are classical fixed points.

vi) Characterize those state vectors in D, $D_S$, R and $R_S$ which give (a) realized fixed point pair. (b) realized limit cycle pair.

15. Let $V = \begin{bmatrix} 2 & 3 & 4 & 0 & 5 & 6 & 7 & 8 \\ 0 & 1 & 0 & 3 & 0 & 5 & 0 & 2 \\ 7 & 0 & 1 & 0 & 6 & 0 & 9 & 0 \\ 5 & 7 & 0 & 2 & 0 & 4 & 0 & 6 \\ 0 & 2 & 4 & 0 & 6 & 0 & 8 & 0 \\ 3 & 0 & 5 & 0 & 7 & 2 & 0 & 1 \end{bmatrix}$

be the MOD special dual like number matrix operator with entries from $Z_{10} \subseteq \langle Z_{10} \cup h\rangle = \{a + bh \mid a, b \in Z_{10}\ h^2 = h\}$.

Study all special features associated with V.

16. Let $W = \begin{bmatrix} 3h & 5h & 4h & 0 & 2h & h & 7h \\ 0 & 2h & 0 & 6h & 0 & 5h & 0 \\ 4h & 6h & 8h & 0 & 4h & 0 & 6h \\ 0 & 3h & 0 & 5h & 0 & 7h & 0 \\ 9h & 0 & 11h & 0 & 3h & 0 & h \\ 0 & h & 0 & 3h & 0 & 2h & 0 \\ h & 0 & 2h & 0 & 3h & 0 & 4h \\ 0 & 5h & 0 & 6h & 0 & 7h & 0 \\ 8h & 0 & 9h & 0 & 10h & 0 & 11h \\ 0 & h & 0 & 2h & 0 & 3h & 0 \\ 4h & 0 & 5h & 0 & 6h & 0 & 7h \end{bmatrix}$

be the MOD special dual like number from $Z_{12}h = \{ah \mid a \in Z_{12},\ h^2 = h\} \subseteq \langle Z_{12} \cup h\rangle = \{a + bh \,/\, a, b \in Z_{12},\ h^2 = h\}$.

i) Study all special features enjoyed by the resultant vectors acting on W.

ii) Compare the two operators V of problem (15) with this W.

17. Let

$$M = \begin{bmatrix} 3+k & 2+4k & 0 & 5K= & 3 & 1+k \\ 0 & k & 1+5k & 0 & 1+2k & 0 \\ 6+7k & 0 & k & 8 & 9 & 4+3k \\ 0 & 5+3k & 3 & 1+k & 0 & 2 \\ 4+2k & 0 & 1+3k & 0 & 7+k & 0 \end{bmatrix}$$

be the MOD special quasi dual number matrix operator with entries from $\langle Z_{10} \cup k\rangle = \{a + bk / a, b \in Z_{10}, k^2 = 9k\}$.

Let $D = \{(a_1, a_2, \ldots, a_5) \mid a_i \in \langle Z_{10} \cup k\rangle, 1 \leq i \leq 5\}$ and

$R = \{(b_1, b_2, \ldots, b_6) \mid b_j \in \langle Z_{10} \cup k\rangle; 1 \leq j \leq 6\}$ be the domain and range state of vectors respectively associated with the MOD special quasi dual number matrix operator M.

Let $D_S = \{(a_1, a_2, \ldots, a_5) / a_i \in \{0, 1, k, 1 + k\}, 1 \leq i \leq 5\}$ and $R_S = \{(b_1, b_2, \ldots, b_6) / b_j \in \{1, k, 0, 1 + k\}; 1 \leq j \leq 6\}$ be the domain and range space of special state vectors associated with M.

i) Obtain all classical fixed points contributed by $D_S$, D, $R_S$ and R by acting on M.

ii) Characterize all realized fixed point pairs associated with M.

iii) Characterize all those realized limit cycle pairs associated with M.

iv) Let x = (0, 0, 1, 0 0) $\in$ D find the resultant of x on M.

v) Let x = (0, 0, 1, 0, 0) $\in D_S$ find the resultant of x on M.

vi) Compare the resultants of x on (iv) and (v).

vii) Let x = (0, 0, 0, 0, k, 0) $\in$ R find the resultant of x on M.

viii) Let x = (0, 0, 0, 0, k, 0) $\in R_S$; find the resultant of x on M.

ix) Compare the resultants of x in (vii) and (viii).

x) Find all vectors in D and $D_S$ (R and $R_S$) which yield the same resultant on M.

xi) Study all special features associated with this M.

18. Let $B = \begin{bmatrix} 9 & 2 & 0 & 6 & 1 & 8 & 9 & 7 \\ 0 & 1 & 2 & 0 & 4 & 0 & 6 & 0 \\ 6 & 0 & 4 & 2 & 0 & 5 & 0 & 8 \\ 1 & 2 & 3 & 0 & 0 & 4 & 5 & 6 \\ 0 & 0 & 7 & 8 & 0 & 0 & 9 & 5 \\ 11 & 0 & 2 & 0 & 10 & 7 & 0 & 11 \\ 0 & 1 & 2 & 3 & 4 & 5 & 6 & 0 \\ 7 & 8 & 9 & 10 & 0 & 11 & 0 & 1 \\ 1 & 1 & 1 & 2 & 1 & 3 & 1 & 0 \\ 0 & 5 & 0 & 3 & 0 & 7 & 0 & 5 \end{bmatrix}$

be the MOD special quasi dual number matrix operator with entries from

$Z_{14} \subseteq \langle Z_{14} \cup k\rangle = \{a + bk \mid a, b \in Z_{14}, k^2 = 13k\}$.

i) Study all the special features associated with this B.

ii) Enumerate any property which is distinctly different from other MOD operators.

19. Let $W = \begin{bmatrix} 3k & 2k & 0 & 4k & 0 & 5k \\ 0 & k & 7k & 0 & 6k & 0 \\ k & 0 & 2k & 4k & 0 & 3k \\ 0 & 5k & 0 & 0 & 9k & 0 \\ k & 0 & 2k & k & 0 & k \\ k & 2k & 3k & 4k & 5k & 6k \\ 0 & 3k & 0 & 6k & 0 & 9k \end{bmatrix}$

be the MOD special quasi dual number matrix operator with entries from $Z_{10}k = \{ak \mid a \in Z_{10}\} \subseteq \langle Z_{10} \cup k\rangle = \{a + bk \mid a, b \in Z_{10};\ k^2 = 9k\}$.

i) Study all special features associated with this W.

ii) Compare the state vectors resultants in case of W and B.

20. Specify any applications of these MOD matrix operators.

21. Compare the MOD dual number matrix operator with MOD special quasi dual number matrix operators.

22. Which of the MOD matrix operators will be the power tool to give classical fixed point pairs?

23. Does there exist a MOD dual which gives all state vectors in R and D as classical fixed points?

24. Characterize all those MOD complex number matrix operators which has no classical fixed point pair associated with it.

25. Study any other special feature associated with MOD neutrosophic matrix operator.

26. Does there exist two MOD matrix dual number operators A and B of same order with entries from same set $\langle Z_n \cup g\rangle$ such that the resultant sum of x on A and B is the same resultant of x on the sum A + B?

27. Let M be the finite complex MOD matrix operator M characterize those M such that the sum of the resultant of x and y on M is the same as the resultant of x + y on M.

28. Study the special features associated with MOD special quasi dual number matrix operators M which are such that they have all resultants to be only realized limit cycle pairs.

29. Let M
$$\begin{bmatrix} 8 & 3 & 6 & 0 & 0 & 0 \\ 4 & 2 & 1 & 0 & 0 & 0 \\ 5 & 6 & 7 & 0 & 0 & 0 \\ 0 & 0 & 0 & k & 2k & 7k \\ 0 & 0 & 0 & 5k & 4k & k \\ 0 & 0 & 0 & 2k & 3k & 6k \\ 1+k & 2+3k & 1+4k & 0 & 0 & 0 \\ 2+k & 3+k & 4+k & 0 & 0 & 0 \\ 5+k & 5k+2 & 3k+1 & 0 & 0 & 0 \end{bmatrix}$$

be the MOD special quasi dual number matrix operator with entries from
$\langle Z_9 \cup k\rangle = \{a + bk \mid a, b \in Z_9, k^2 = 8k\}$.

Study the special features enjoyed by this M.

30. Let $V = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ g & 2g & 3g \\ 4g & 5g & 6g \\ 7g & g & 2g \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$ be the MOD dual number matrix operator with entries from $\langle Z_{10} \cup g \rangle = \{a + bg \mid a, b \in Z_{10}, g^2 = 0\}$.

Study all special features enjoyed by this V.

## Chapter Four

# MOD MATRIX OPERATORS USING MOD INTERVAL [0, n)

In this chapter we for the first time give face value ordering to the interval [0, n). Using this we can use min or max operations on interval MOD matrix operators or MOD matrix operators built on using the MOD interval [0, n); $2 \leq n < \infty$.

We also show these MOD matrix operators can yield realized fixed point pair or realized limit cycle pair.

We will describe all these situations by some examples.

***Example 4.1:*** Let

$$M = \begin{bmatrix} 0.3 & 2.9 & 4 \\ 1.5 & 0 & 2.1 \\ 0 & 4.2 & 0 \\ 2.1 & 5.1 & 0.1 \end{bmatrix}$$

be the MOD interval matrix operator with entries from [0,6).

Let $D = \{(a_1, a_2, a_3, a_4) \mid a \in \{0, 1\}, 1 \leq i \leq 4\}$ and

$R = \{(b_1, b_2, b_3) \mid b_i \in \{0, 1\}; 1 \leq j \leq 3\}$ be the domain and range space of state vectors respectively associated with M.

Let $x = (1, 0, 0, 0) \in D$;

$xM = (0.3, 2.9, 4) = (x_1, x_2, x_3) \to (0, 1, 1) = y_1$;

if $0 < x_1 < 1$ replace it by 0 if $x_i \geq 1$ replace it by 1.

$y_1M^t \to (1, 1, 1, 1) = x_1$; $x_1M \to (1, 0, 0) = y_2$;

$y_2M^t \to (1, 1, 0, 1) = x_2$; $x_2M \to (1, 1, 0) = y_3$;

$y_3M^t \to (1, 1, 1, 1) = x_3 (= x_1)$.

Thus the resultant is a realized limit cycle pair given by
$\{(1, 1, 1, 1), \{1, 0, 0)\}$.

Let $y = (0, 1, 0) \in R$.

To find the effect of y on M.

$yM^t \to (1, 0, 1, 1) = x_1$; $x_1M \to (1, 1, 1) = y_1$;

$y_1M \to (1, 1, 1, 1) = x_2$; $x_2M \to (1, 1, 0) = y_2$;

$y_2M^t \to (1, 1, 1, 1) = x_3$.

Thus the resultant is again a realized limit cycle given by
$\{(1, 1, 1, 1), (1, 1, 0)\}$.

We will certainly arrive at a realized fixed point pair or a realized limit cycle pair as D and R are of finite cardinality.

***Example 4.2:*** Let

$$P = \begin{bmatrix} 0.03 & 0.1 & 0.11 \\ 3.1 & 3.5 & 0 \\ 0 & 3.0 & 1.01 \\ 2.1 & 0 & 2.1 \\ 2.01 & 0 & 0 \\ 3.02 & 0.11 & 2.01 \end{bmatrix}$$

be the MOD interval matrix operator with entries from [0, 4).

Let D = $\{(a_1, a_2, \ldots, a_6) \mid a_i \in \{0, 1\}; 1 \leq i \leq 6\}$ and

R = $\{(b_1, b_2, b_3) \mid b_j \in \{0, 1\}; 1 \leq j \leq 3\}$ be the domain and range space of special state vectors respectively.

Let x = (0, 0, 0, 0, 0, 1) ∈ D.

To find the effect of x on P.

$xP = (3.02, 0.11, 2.01) \to (1, 0, 1) = y_1$;

$y_1P^t \to (0, 1, 0, 1, 1, 1) = x_1$; $x_1P \to (0, 1, 1) = y_2$;

$y_2P^t \to (0, 1, 1, 1, 0, 1) = x_2$; $x_2P \to (0, 0, 1) = y_3$;

$y_3P^t \to (0, 0, 0, 1, 0, 1) = x_3$; $x_3P \to (1, 0, 1) = y_4$;

$y_4P^t \to (0, 1, 0, 1, 1, 1) = x_4$ $(=x_1)$.

Thus the resultant is a realized limit cycle pair given by
{(0, 1, 0, 1, 1, 1), (1, 0, 1)}.

Let y = (0, 1, 0) ∈ R; to find the effect of y on P.

$yP^t \to (0\ 1, 1, 0, 0, 0) = x_1$; $x_1P \to (1, 1, 0) = y_1$;

$y_1P^t \rightarrow (0, 0, 1, 1, 1, 1) = x_2$; $x_2P \rightarrow (1, 1, 1) = y_2$;

$y_2P^t \rightarrow (0, 0, 1, 1, 1, 1) = x_3$ (=$x_2$).

Thus the resultant is a realized limit cycle pair given by
{(0, 0, 1, 1, 1, 1), (1, 1, 1)}.

Let x = (0, 0, 1, 0, 0, 0) ∈ E.

To find x on P.

$xP \rightarrow (0, 1, 0) = y_1$; $y_1P^t \rightarrow (0, 1, 1, 0, 0, 0) = x_1$;

$x_1P \rightarrow (1, 0, 0) = y_2$; $y_2P^t \rightarrow (0, 1, 0, 1, 1, 1) = x_2$;

$x_2P \rightarrow (1, 1, 1) = y_3$; $y_3P^t \rightarrow (0, 1, 1, 1, 0, 1) = x_3$;

$x_3P \rightarrow (1, 0, 1) = y_4$; $y_4P^t \rightarrow (0, 1, 0, 1, 1, 1) = x_4$;

$x_4P \rightarrow (0, 1, 1) = y_5$ (=$y_3$).

Thus the resultant is a realized limit cycle point pair
{(0, 1, 0, 1, 1, 1), (0, 1, 1)}.

Let x = (0, 1, 0, 0, 0, 0) ∈ D.

$xP \rightarrow (1, 1, 0) = y_1$; $y_1P^t \rightarrow (0, 1, 1, 1, 1, 1) = x_1$;

$x_1P \rightarrow (1, 0, 1) = y_2$; $y_2P^t \rightarrow (0, 1, 1, 1, 1, 1) = x_2$
(=$x_1$).

The resultant is a realized limit cycle given by
{(0, 1, 1, 1, 1, 1), (1, 0, 1)}.

***Example 4.3:*** Let

$$M = \begin{bmatrix} 0.3 & 0.6 & 1.2 & 0.2 & 1.4 \\ 5.1 & 1.2 & 0 & 0 & 4.2 \\ 1.2 & 0 & 5.1 & 0.2 & 2.1 \\ 0.1 & 0.3 & 0.1 & 0.2 & 1.1 \end{bmatrix}$$

be the MOD rectangular matrix operator with entries from [0, 6).

Let D = $\{(x_1, x_2, x_3, x_4) \mid x_i \in \{0, 1\}; 1 \leq i \leq 4\}$ and

R = $\{(y_1, y_2, \ldots, y_5) \mid y_i \in \{0, 1\}; 1 \leq i \leq 5\}$ be the domain and range space of special row vectors associated with M.

Let x = (1, 0, 0, 0) $\in$ D, to find the effect of x on M.

xP = (1, 0, 0, 0) $\in$ D; to find the effect of x on M.

$xM \to (0, 0, 1, 0, 1) = y_1$; $\quad y_1M^t \to (1, 1, 1, 1) = x_1$;

$x_1M \to (0, 1, 0, 0, 1) = y_2$; $\quad y_2M^t \to (1, 1, 1, 1) = x_2$;

$x_2M \to (0, 1, 0, 0, 1) = y_3\ (= y_2)$.

Thus the resultant of x on M is a realized limit cycle pair given by {(1, 1, 1, 1), (0, 1, 0, 0, 1)}

Let x = (0, 0, 1, 0 0) $\in$ R, to find the effect of x on M.

$xM^t \to (1, 0, 1, 0) = y_1$; $\quad y_1M \to (1, 0\ 1, 0, 1) = x_1$;

$x_1M^t \to (1, 1, 1, 1) = y_2$; $\quad y_2M \to (0, 1, 1, 0, 1) = x_2$;

$x_2M^t \to (1, 1, 1, 1) = y_3$;

$y_3M \to (0, 1, 1, 0, 1) = x_3\ (=x_2)$.

Thus the resultant is a realized limit cycle given by
{(1, 1, 1, 1), (0, 1, 1, 0, 1)}.

***Example 4.4:*** Let

$$P = \begin{bmatrix} 0.3 & 1 & 4 & 0.01 & 0.02 \\ 0.01 & 0 & 0.1 & 0.1 & 0.01 \\ 0.03 & 4 & 1 & 0.05 & 4.1 \\ 0 & 0 & 0 & 0.03 & 1.1 \\ 3.1 & 0.01 & 0 & 0 & 0.01 \\ 0 & 0.04 & 0.04 & 0.01 & 0 \\ 2.0 & 0 & 0.16 & 0 & 0 \end{bmatrix}$$

be the MOD interval matrix operator with entries from [0, 5)

$D = \{(a_1, a_2, \ldots, a_7) \mid a_i \in \{0, 1\}, 1 \leq i \leq 7\}$ and

$R = \{(b_1, b_2, \ldots, b_5)$ where $b_i \in \{0, 1\}; 1 \leq i \leq 5\}$ be the special state vectors of domain and range space respectively associated with P.

Let $x = (0, 1, \ldots, 0) \in D$. To find the effect of x on P.

$xP \to (0, 1, 1, 0, 0) = y_1;$

$y_1P^t \to (1, 0, 0, 0, 0, 0, 0) = x_1 (=x).$

Thus the resultant of x is a classical fixed point pair given by {(1, 0, 0, 0, 0, 0, 0), (0, 1, 1, 0, 0)}.

Let $x = (0, 0, 0, 1, 0, 0, 0) = D$.

To find the effect of x on P.

$x \to (0, 0, 0, 0, 0) = y_1;$

$y_1P^t \rightarrow (0, 0, 0, 1, 0, 0, 0) = x_1$ (=x).

Thus the resultant of x is once again a classical fixed point pair given by {(0, 0, 0, 1, 0, 0, 0), (0, 0, 0, 0, 0)}.

Let $x = (0, 0, 0, 0, 0, 0, 1) \in D$.

$xP \rightarrow (1, 0, 0, 0, 0) = y_1$;

$y_1P^t \rightarrow (0, 0, 0, 0, 1, 0, 1)) = x_1$;

$x_1P \rightarrow (0, 0, 0, 0, 0) = y_2$;

$y_2P \rightarrow (0, 0, 0, 0, 0, 0, 1)$.

Thus the resultant is a realized limit cycle pair given by
{(0, 0, 0, 0, 1, 0, 1), (0, 0, 0, 0, 0)} or
{(0, 0, 0, 0, 0, 0, 1), (1, 0, 0, 0, 0)}.

Thus we get these four sets of pairs given in the following form

$(0, 0, 0, 0, 0, 0, 1) \rightarrow (1, 0, 0, 0, 0) \rightarrow (0, 0, 0, 0, 1, 0, 1)$

$\rightarrow (0, 0, 0, 0, 0) \rightarrow (0, 0, 0, 0, 0, 0, 1)$ and so on.

Let $x = (1, 0, 1, 0, 0, 0, 1) \in D$.

$xP \rightarrow (1, 0, 0, 0\ 1) = y_1$; $\quad y_1P^t \rightarrow (1, 0\ 1, 1, 1, 0, 1) = x_1$;

$x_1P \rightarrow (0, 0, 0, 0, 0) = y_2$; $\quad y_2P^t \rightarrow (1, 0, 1, 0, 0, 0, 1)$

and so on leading to a realized limit cycle pairs.

***Example 4.5:*** Let

$$Z = \begin{bmatrix} 3 & 1 & 0 & 1 & 2 \\ 0 & 5 & 1 & 2 & 0 \\ 1 & 0 & 4 & 2 & 0 \\ 3 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

be the MOD interval matrix operator with entries from [0, 6).

Let D = $\{(x_1, x_2, \ldots, x_6) \mid x_i \in \{0, 1\}; 1 \leq i \leq 6\}$ and

R = $\{(y_1, y_2, \ldots, y_5) \mid y_1 \in \{0, 1\}, 1 \leq i \leq 6\}$ be the special domain and range space of state vectors respectively associated with Z.

Let x = (1, 0, 0, 0, 0 0) $\in$ D.

To find the effect of x on Z.

$xZ \to (1, 1, 0, 1, 1) = y_1$; $\quad y_1Z^t \to (1, 1, 1, 1, 1, 1) = x_1$;

$x_1Z \to (1, 0, 0, 0, 0) = y_2$; $\quad y_2Z^t \to (1, 0, 1, 1, 0, 0) = x_2$ ;

$x_2Z \to (1, 1, 1, 1, 1) = y_3$; $\quad y_3Z^t \to (1, 1, 1, 1, 1, 1) = x_3$;

$x_3Z \to (1, 0, 0, 0, 0) = y_4$;

$y_4Z^t \to (1, 0, 1, 1, 0, 0) = x_4$ $(= x_2)$;

$x_2Z \to (1, 1, 1, 1, 1) = y_5$ $(=y_3)$ ;

$y_5Z^t \to (1, 1, 1, 1, 1, 1) = x_5$ $(=x_3)$ and so on.

The resultant of x is a realized limit cycles.

Let y = (0, 0, 1, 0, 0) $\in$ R,

$yZ^t \to (0, 1, 1, 0, 1, 0) = x_1$; $x_1Z \to (1, 1, 0, 1, 1) = y_1$;

$y_1Z^t \to (1, 1, 1, 1, 1, 1) = x_2$; $x_2Z \to (1, 0, 0, 0, 0) = y_2$;

$y_2Z^t \to (1, 0, 1, 1, 0, 0) = x_3$; $x_3Z \to (1, 1, 1, 1, 1) = y_3$;

$y_3Z^t \to (1, 1, 1, 1, 1, 1) = x_4$ ($=x_2$) and so on.

Thus the resultant is a realized limit cycle pairs.

Now having seen examples of MOD interval real matrix operator we proceed on to describe MOD interval neutrosophic matrix by some examples.

***Example 4.6:*** Let

$$M = \begin{bmatrix} 3I & 0.7I & I \\ 0 & 0.02I & 0.3I \\ 4I & 0 & 5I \\ 0.3I & 0.0I & I \\ 0.002I & 0.002I & 0 \end{bmatrix}$$

be the MOD interval neutrosophic matrix operator with entries from [0, 7)I.

Let D = $\{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in \{0, I\}; 1 \le i \le 5\}$ and

R = $\{(y_1, y_2, y_3) \mid y_i \in \{0, I\}; 1 \le i \le 3\}$ be the special set of domain and range space state vectors respectively related with M.

Let x = (I, 0, 0, 0, 0) $\in$ D.

$xM \to (I, 0, I) = y_1$; $y_1M^t \to (I, 0, I, I, 0) = x_1$;

$x_1M \to (0, 0, 0) = y_2$; $\quad$ $y_2M^t \to (I, 0, 0, 0, 0)$;

so the resultant is a realized limit cycle pair.

Let $y = (0, 0, I) \in R$.

$yM^t \to (I, 0, I, I, 0) = x_1$; $\quad$ $x_1M \to (0, 0, I) = y_1$ $(=y_1)$.

Thus the resultant of this y is a classical fixed point pair given by $\{(I, 0, I, I, 0), (0, 0, I)\}$.

This is the way MOD interval neutrosophic matrix operators functions. Interested reader can construct more such models.

Now we proceed onto describe the MOD interval complex matrix operator built using $[0, n)\, i_F$, $i_F^2 = n - 1$.

***Example 4.7:*** Let

$$M = \begin{bmatrix} 3i_F & 0.2i_F & 2i_F \\ 0 & 1.2i_F & 0 \\ 2i_F & 3.8i_F & i_F \\ 0 & 0 & i_F \\ 0.03i_F & 0.12i_F & i_F \end{bmatrix}$$

be the MOD interval complex matrix operator with entries from $[0,5)i_F = \{ai_F \mid a \in [0,5); = i_F^2 = 4\}$.

Let $D = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in \{0, i_F\}.\ 1 \le i \le 5\}$ and

$R = \{(b_1, b_2, b_3) \mid b_i \in \{0, i_F\};\ 1 \le i \le 3\}$ be the special state vectors of the domain and range space respectively associated with M realized maximum using the condition if $ai_F = i_F$ if $a \ge 1$ and $ai_F = 0$ if $a < 1$.

Let $x = (i_F, 0, 0, 0, 0)$ ;

$$x \circ M \to (i_F, i_F, i_F) = y_1;$$

$$y_1M^t \to (i_F, i_F, i_F, i_F, i_F) = x_1;$$

$x_1M$ and so on.

However it is important to note that working with [0, m)I or [0, m)$i_F$ or [0, m)g, or [0, m)h or [0, m)k does not yield interesting results so we study only using $R_n^I(m)$ (m) or $R_n^g(m)$ or $R_n^h(m)$ or $R_n^k(m)$ or $C_n(m)$ and not intervals and accordingly the special state vectors would be taking values from {0, I, 1, 1 + I) or {0, g, 1 + g, 1} or {0, h, 1, 1 + h} or {0, k, 1, 1 + k} or {0, $i_F$, 1, 1 + $i_F$} respectively.

We give one example of each and leave the rest of the work to the reader as it is a matter of routine.

***Example 4.8:*** Let

$$M = \begin{bmatrix} 2+I & 0.3I & 0.I \\ 0.3 & 0 & 2+I \\ 0.33I & 0.1+0.2I & 0 \\ 2I+1 & 0.31+0.004I & 1+2I \end{bmatrix}$$

be the MOD neutrosophic matrix operator with entries from $R_n^I(3) = \{a + bI \mid a, b \in [0,3), I^2 = I\}$.

Let D = $\{(a_1, a_2, a_3, a_4) \mid a_i \in \{0, 1, I, 1 + I\}; 1 \leq i \leq 4\}$ and

R = $\{(b_1, b_2, b_3) \mid b_i \in \{0, 1, I, 1 + I\}; 1 \leq i \leq 3\}$ be the domain and range space of special row vectors respectively associated with the MOD neutrosophic matrix operator M.

Let x = (1, 0, 0, 0) $\in$ D.

To find the effect of x on M.

$xM \to (1, 0, 0) = y_1$; $\qquad$ $y_1M^t \to (1, 0, 0, I) = x_1$;

$x_1M \to (1, 0, 0) = y_2$; $\qquad$ $y_2M^t \to (1, 0, 0, I) = x_2\ (= x_1)$.

Thus the resultant is a realized fixed point pair given by $\{(1, 0, 0, I), (1, 0, 0)\}$.

Let $x = (0, 0, 0\ I) \in D$.

To find the effect of x on M.

$xM \to (0, 0, 0) = y_1$; $\qquad$ $y_1M^t \to (0, 0, 0, I)$.

Thus $x = (0, 0, 0\ I)$ is a classical fixed point pair given by
$\{(0, 0, 0, I), (0, 0, 0)\}$.

Let $y = (0, I, 0) \in R$.

To find the effect of y on M.

$yM^t \to (0, 0, 0, 0) = x_1$; $\qquad$ $x_1M \to (0, I, 0) = y_1\ (=y)$.

Thus y is a classical fixed point pair given by
$\{(0, 0, 0, 0), (0, I, 0)\}$.

Let $y = (0, 0, I) \in R$.

To find the effect of y on M ;

$yM^t \to (0, 1, 0, I) = x_1$; $\qquad$ $x_1M \to (0, 0, 1) = y_1\ (= y)$.

The resultant is again a classical fixed point pair given by $\{(0, 1, 0, I), (0, 0, 1)\}$.

Next we give an example of MOD interval complex matrix operator).

***Example 4.9:*** Let

$$P = \begin{bmatrix} 3+i_F & 0 & 0.4i_F \\ 0 & 1+3i_F & 0.5 \\ 1+3i_F & 0 & 3+i_F \\ 0 & 3+i_F & 0 \\ 0.3i_F & 0 & 0.5i_F \\ 0.08 & 0.11 & 0.32 \end{bmatrix}$$

be the MOD complex matrix operator with entries from

$C_n(4)= \{a + bi_F / a, b \in [0,4), i_F^2 = 3\}$.

Let D = $\{(a_1, a_2, \ldots, a_6) \mid a_i \in \{0, 1, i_F, 1 + i_F\}; 1 \leq i \leq 6\}$ and

R = $\{(b_1, b_2, b_3) \mid b_i \in \{0, 1, i_F, 1 + i_F\}; 1 \leq i \leq 3\}$ be the special state of domain and range space vector respectively associated with P.

Let x = (1, 0, 0, 0, 0, 0) $\in$ D. To find the effect of x on P.

$xP \to (1, 0, 0) = y_1$; $\quad y_1P^t \to (1, 0, i_F, 0, 0, 0) = x_1$;

$x_1P \to (i_F, 0, 1 + i_F) = y_2$; $\quad y_2P^t \to (1, 1, 1, 0, 1, 0) = x_2$;

$x_2P \to (0, i_F, 1) = y_3$; $\quad y_3P^t \to (1, 1, 1, 1+i_F, 1, 0) = x_3$;

$x_3P \to (0, 1 + i_F, 1) = y_4$; $\quad y_4P^t \to (1, 1, 1, 1, 0, 0) = x_4$;

$x_4M \to (0, 0, 1) = y_5$; $\quad y_5M^t \to (1, 0, 1, 0, 0, 0) = x_5$;

$x_5M \to (0, 0, 1) = y_6$;

$y_6M^t \to (1, 0, 1, 0, 0\ 0) = x_6\ (=x_5)$.

Thus the resultant is a realized fixed point pair given by

$\{(1, 0, 1, 0, 0, 0), (0, 0, 1)\}$.

Next we proceed onto describe MOD dual number matrix operator on $R_n^g$ (m).

***Example 4.10:*** Let

$$P = \begin{bmatrix} 2+g & 0.3+2g & 0.6g \\ 0.7 & 2+2g & 0.3g+0.7 \\ 2+3g & 0.2g & 0.05g \\ 0.8g & 0 & 0.003g \\ 0.04 & 0.03g & 0 \end{bmatrix}$$

be the MOD interval dual number matrix operator with entries from $R_n^g(4)$.

Let D = $\{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in \{0, 1, g, 1+g\} ; 1 \leq i \leq 5\}$ and

R = $\{(b_1, b_2, b_3) \mid b_i \in \{0, 1, g, 1+g\}; 1 \leq i \leq 3\}$ be the special state row vectors of the domain and range spaces associated with P.

Let x = (1, 0, 0, 0, 0) $\in$ D. To find the effect of x on P.

$xP \to (1, g, 0) = y_1$; $\quad y_1P^t \to (1, g, g, 0, 0) = x_1$;

$x_1P \to (0, 0, 0) = y_2$; $\quad y_2P^t \to (1, 0, 0, 0, 0) = x_2$.

Thus the resultant is a realized fixed point pair given by
{(1, 0, 0, 0, 0), (0, 0, 0)}.

In fact this pair is also nothing but the classical point pair.

Let x = (0, 0, g, 1, 0) $\in$ D.

To find the effect of x on P.

$xP \to (g, 0, 0) = y_1$; $\quad y_1P^t \to (g, 0, g, 1, 0) = x_1$;

$x_1P \to (0, 0, 0) = y_2$; $\quad$ $y_2P^t \to (0, 0, g, 1, 0) = x_2(=x)$.

Thus once again the resultant is a classical fixed point pair given by {(0, 0, g, 1, 0), (0, 0, 0)}.

Let $x = (1, 1, 0, 0, 1) \in D$.

To find the effect of x on P.

$xP \to (1, 1, 0) = y_1$; $\quad$ $y_1P^t \to (1, 1, g, 0, 1) = x_1$;

$x_1P \to (g, 1, 0) = y_2$; $\quad$ $y_2P^t \to (1, 1, g, 0, 1) = x_2 (=x_1)$.

Thus the resultant is again a realized fixed point pair given by {(1, 1, g, 0, 1), (g, 1, 0)}.

Let $y = (0, 0, 1) \in R$; to find the effect of y on P.

$$yP^t \to (0, 0, 0, 0, 0).$$

Thus the resultant of y is the classical fixed point pair {(0, 0, 0, 0, 0), (0, 0, 1)}.

Let $y = (0, g, 0) \in R$. To find the effect of y on P.

$yP^t \to (0, g, 0, 0, 0) = x_1$; $\quad$ $x_1P \to (0, g, 0) = y_1 (=y)$.

Thus once again the resultant is the classical fixed point pair given by {(0, g, 0, 0, 0), (0, g, 0)}.

The reader is requested to work with more models.

Now we proceed onto give an example of a MOD special dual like number matrix operator built using

$$R_n^h(m) = \{a + bh / a, b \in [0, m); h^2 = h\}.$$

***Example 4.11:*** Let

$$M = \begin{bmatrix} 2+h & 0.3 & 0.52h \\ 0.07h & 0 & 1+4h \\ 0 & 1+2h & 0 \\ 0.005 & 0 & 4+0.02 \\ 0 & 4+h & 0 \\ 3+4h & 2h & h+0.03 \\ 0 & 0.007h & 0 \\ 0.52h & 0 & 0.3h+0.1 \end{bmatrix}$$

be the MOD special dual like number matrix operator with entries from $R_n^h(5) = \{a + bh \mid a, b \in [0,5), h^2 = h\}$.

Let $D = \{(a_1, a_2, \ldots, a_8) / a_i \in \{0, 1, h, 1 + h\}; 1 \leq i \leq 8\}$ and $R = \{(b_1, b_2, b_3) / b_i \in \{0, h, 1, 1 + h\} ; 1 \leq i \leq 3\}$ be the domain and range space of special vectors respectively associated with MOD matrix operator M.

Let $x = (1, 0, 0, 0, 0, 1, 0, 0) \in D$.

To find the effect of x on M.

$xM \to (0, h, h) = y_1$; $\quad y_1M^t \to (1, 0, h, h, h, 1, 0, 0) = x_1$;

$x_1M \to (0, 0, h) = y_2$; $\quad y_2M^t \to (1, 0, 0, h, 0, 1, 0, 0) = x_2$;

$x_2M \to (0, h, 0) = y_3$; $\quad y_3M^t \to (1, 0, h, 0, 0, 1, 0, 0) = x_3$;

$x_3M \to (0, 0, h) = y_4 (=y_2)$.

Thus the resultant is a realized limit cycle pair given by

$$\{(1, 0, 0, 0, 0, 1, 0, 0), (0, 0, h)\}.$$

Let $y = (0, 1, 0) \in R$. To find the effect of y on M.

$yM^t \to (0, 0, h, 0, 1, h, 0\ 0) = x_1$; $\quad x_1M \to (h, 1, h) = y_1$;

$y_1M^t \to (h, 0, h, h, 1, 0, 0, 0) = x_2$; $\quad x_2M \to (h, 1, 1) = y_2$;

$y_2M^t \to (h, h, h, 1, 1, 0, 0, 0) = x_3$;

$x_3M \to (h, 1, 1) = y_3\ (y_2)$.

Thus the resultant is a realized limit cycle pair given by {(h, h, h, 1, 1, 0, 0, 0), (h, 1, 1)}.

Next we proceed onto describe by examples the MOD special quasi dual number matrix operators built using

$$R_n^k(m) = \{a + bk \mid a, b \in [0, m), k^2 = (m - 1)\, k\}.$$

***Example 4.12:*** Let

$$B = \begin{bmatrix} 3k+0.21 & 0 & 4k+3 & 0.003k & 0 & 4+0.03k \\ 0 & 1+4k & 0.033k & 3+2k & k+3 & 0.002k \\ 4k+0.31 & 3k+6 & 3k+4 & 4+5k & 6k+4 & 3+0.2k \end{bmatrix}$$

be the 3 × 6 MOD special quasi dual number matrix operator with entries from $R_n^k(7)$.

Let $x = (1, 0, 0) \in D$. To find the effect of x on B.

$xB \to (k, 0, k, 0, 0, 1) = y_1$; $\quad y_1B^t \to (1, 0, k) = x_1$;

$x_1B \to (k, k, k, k, k, k) = y_2$; $\quad y_2B^t \to (1, 0, k) = x_2\ (= x_1)$.

Thus the resultant is a realized fixed pair given by

{(1, 0, k), (k, k, k, k, k, 0)}.

Let $y = (1, 0, 0, 0, 0, 0) \in R$.

To find the effect of y on B.

$yB^t \to (k, 0, k) = x_1;$

$x_1B \to (1, k, k, k, k, 0) = y_1;$

$y_1B^t \to (k, 0, k) = x_2.$

Thus the resultant of y is a realized fixed point pair given by $\{(k, 0, k), (1, k, k, k, k, 0)\}$.

We leave it as an exercise for the reader to work with more problems using $R_n^k(m)$.

We suggest some problems for the reader.

**Problems**

1. Show face value ordering can be done on [0, m), $2 \leq m < \infty$.

2. Show by examples that by implementing face value ordering in $R_n(m)$ we can use them in MOD matrix operators.

3. Obtain any other special feature that can be associated with face value ordering on [0, m).

4. Let $M = \begin{bmatrix} 3.001 & 2.01 & 0.523 & 1.112 & 0.72 \\ 0.73 & 5.02 & 0 & 5.315 & 7.08 \\ 1.142 & 0.11 & 4.02 & 0 & 1.009 \end{bmatrix}$ be the

   MOD matrix interval operator with entries from [0,8).

   Let $D = \{(a, b, c) \mid a, b, c \in \{0, 1\}\}$ and $R = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in \{0, 1\}; 1 \leq i \leq 5\}$ be the special state vectors of the domain and range space respectively associated in M.

i) Characterized all classical fixed point pairs of M.
ii) Characterize all those realized fixed point pairs of M.
iii) Characterize all those realized limit cycle pairs of M.
iv) Let x = (0, 1, 0) $\in$ D find the resultant of x on M.
v) Let y = (0, 1, 0, 1, 0) $\in$ R find the resultant of y on M.

5. Let $P = \begin{bmatrix} 0.312 & 6.03 & 1.23 & 0 & 1.52 & 0 \\ 1 & 3.02 & 0 & 1.631 & 0 & 3.12 \\ 0 & 1.23 & 1.5 & 0 & 4.31 & 0 \\ 6.312 & 0 & 2.5 & 1.5 & 0 & 1.77 \\ 7.77 & 1.22 & 0 & 0 & 0.55 & 0 \\ 0 & 0 & 2.7 & 4.3 & 0.3 & 1 \\ 4.31 & 3.77 & 0 & 0 & 4.31 & 0 \\ 0 & 0 & 1.53 & 1.42 & 0 & 3.11 \end{bmatrix}$ be the

MOD real interval matrix operator built using elements from [0, 9).

i) Does this MOD matrix operator have special state vectors from D (or R) which are classical fixed points?
ii) Find all realized fixed point pairs associated with P.
iii) Let x = (1, 0, 0, 0, 0, 1, 0, 0) $\in$ D; find the resultant of x on P.
iv) Let y = (0, 0, 1, 0, 0, 1) $\in$ R find the resultant of y on P.
v) Let $x_1$ = (0, 0, 0, 0, 0, 0, 0, 1) and $x_2$ = (0, 1, 0, 0, 0, 0, 0, 0) $\in$ D.

a. Find the effect of $x_1$ on P.
b. Find the effect of $x_2$ on P.
c. Find the resultant of $x_1 + x_2$ on P.

d. Are these three resultants given in (a), (b) and (c) related in way?

vi) Let $y_1 = (1, 0, 0, 0, 0, 1) \in R$ and $y_2 = (0, 0, 1, 1, 0, 0) \in R$.

a. Find the effect of $y_1$ on P.
b. Find the effect of $y_2$ on P.
c. Find the effect of $y_1 + y_2$ on P.
d. Are these resultants in (a), (b) and (c) are in any way related?

vii) Find the maximum number of iterations that is needed to get the resultant.

viii) Will $x \in D$ in general has more number of iterations to arrive at the resultant or will $y \in R$ in general has more number of iterations to arrive at a resultant? Justice your answer!

6. Prove all MOD interval real matrix operators built on [0, n) includes all real matrix operators with elements from $Z_n$.

7. Let $M = \begin{bmatrix} 0.3+2i_F & 2+0.03i_F & 4i_F+3 \\ 4+0.02i_F & 0 & i_F+0.004 \\ 0 & 2i_F+0.007 & 0 \\ 1+2i_F & 0 & 2i_F+0.03 \\ 1+2i_F & 4+4i_F & 3+0.033i_F \end{bmatrix}$ be the

MOD interval complex matrix operator built using elements from $C_n(6) = \{a + bi_F / a, b \in Z_6;\ i_F^2 = 5\}$.

$D = \{(a_1, a_2, a_3, a_4, a_5) \mid a_i \in \{0, 1, i_F, 1 + i_F\};\ 1 \leq i \leq 5\}$ and $R = \{(b_1, b_2, b_3) \mid b_i \in \{0, 1, i_F, 1 + i_F\},\ 1 \leq i \leq 3\}$ be the special domain and range space of state vectors respectively associated with M.

i) Let I = (1, 0, 0, 0 $i_F$) ∈ D; find the resultant of x on M.
ii) Let y = (0 0 $i_F$) ∈ R find the resultant of y on M.
iii) Find all classical fixed point pairs associated in M.
iv) Characterize all realized fixed point pair of D and R on M.
v) Characterize all realized limit point pairs of D and R on M.
vi) Find all special features enjoyed by M.

8. Let P =

$$\begin{bmatrix} 0.31i_F & 5i_F + 0.003 & 0 & 0.4005 \\ 0 & 4 + 0.03i_F & 0.432 & 0 \\ 3 + 4i_F & 0 & 0 & 0.007i_F \\ 2 + 2i_F & 2i_F + 0.003 & 0.00i_F & 0 \\ 2 + i_F & 3 + 0.0002i_F & 0 & 0.003 + 0.0006i_F \end{bmatrix} \text{ and}$$

Q =

$$\begin{bmatrix} 0.04 & 0.03 + 0.005i_F & 3i_F & 2 + 4i_F \\ 0.6 + 3i_F & 0 & 2 + 0.5i_F & 2 + 3i_F \\ 0 & 0.7 + 3i_F & 3i_F + 1 & 0 \\ 1 + 4i_F & 0 & 2 + i_F & 1 + 0.06i_F \\ 6 + 0.0003i_F & 0 & 3 + 0.00021 & 0.03 \end{bmatrix}$$

be two MOD interval complex matrix operators both getting then entries from $C_n(7)$.

i) Let x = (1, 0, 0, 0, 0) ∈ D
 a. Find the resultant of x using P.
 b. Find the resultant of x using Q.
 c. Find the resultant of x using P + Q.
 d. Compare the resultants in (a), (b) and (c).

ii) Find all classical fixed points of P and Q.

a. Will they be different or the same?
b. If x is a classical fixed point of P and y the classical fixed point of Q will y + x be the classical fixed point of P + Q.

iii) Let $x_1 = (0\ 0\ 0\ 1\ 0) \in D$ and $x_2 = (0\ \ 0\ 0\ 0\ i_F) \in D$.

a. Find the resultant of $x_1$ and $x_2$ on P and Q.
b. Find the resultant of $x_1$ and $x_2$ on P + Q.
c. Find the resultant of $x_1 + x_2$ on P, Q and P + Q.
d. Compare (a), (b) and (c) with each other. Are they related in any manner.

9. Enumerate all special features enjoyed MOD complex modulo integer matrix operators.

10. Let $W = \begin{bmatrix} 3+0.0I & 4I+0.3 & 7I+5 & 0 \\ 4I+0.32 & 2I+3 & 0 & 4I+2 \\ 0 & 0 & 0.3I & 0.7I \\ 2I+3 & 0.3I+4 & 2+0.314I & 0 \\ 0 & 0 & 0 & 4+3I \\ 4+0.315I & 0.6I+3 & 3+2I & 0 \\ 3I+0.310 & 2I+0.31 & 1+I & 3I \\ 0.732I & 4.312 & 0.734 & 4+2I \\ 0.73I+0.42 & 3I+2.3 & 0 & 0 \end{bmatrix}$ be

the MOD neutrosophic interval matrix operator with entries from $R_n^I(10) = \{a + bI / a, b \in [0, 10), I^2 = I\}$.

Let $D = \{(a_1, a_2, \ldots, a_9) / a_i \in \{0, 1, I, 1 + I\}, 1 \leq i \leq 9\}$

and $R = \{(a_1, a_2, a_3, a_4) / a_i \in \{0, 1, I, 1 + I\}; 1 \leq i \leq 4\}$ be the special state vectors of the domain and range space respectively associated with the MOD neutrosophic interval matrix operator W.

i) Let x = (0 1 0 0 0 0 0 1 0) ∈ D.
Find the resultant of x on W.

ii) Let $x_1$, $x_2$ ∈ D find the resultants of $x_1$ on W, $x_2$ on W and $x_1 + x_2$ on W.
Compare the 3 resultants?

iii) Let $y_1$, $y_2$ ∈ R; find the resultants of $y_1$ on W, $y_2$ on W and $y_1 + y_2$ on W and compare them.

iv) If x ∈ D is a classical fixed point pair given by (a, b) should be b ∈ R be a zero state vector of R?

v) Characterize all classical fixed point pairs of W.

vi) Characterize all realized limit cycles of W.

vii) If the resultant of $x_1$ is a realized fixed point pair and that of $x_2$ is the realized limit cycle pair what will be the resultant of $x_1 + x_2$ on W?

viii) Let $y_1$, $y_2$ ∈ R are if the resultant of $y_1$ and $y_2$ are realized fixed point pairs; will the resultant of $y_1 + y_2$ ∈ R be also a realized fixed point pair?
Justify you claim!

ix) Obtain any other special and important features enjoyed by the MOD interval neutrosophic matrix operators.

11. Let $S = \begin{bmatrix} 0.3I & 2I & 4I & 0.3112I & 7.32I \\ 2 & 1 & 3 & 0 & 4.002 \\ 4.5I & 0 & 1.23I & 6.002I & 1.3I \\ 4.02 & 3.15 & 0 & 7 & 5.12 \\ 0 & 4I & 3.2I & 0 & 4.7I \\ 7.02 & 0 & 0.331 & 4.005 & 0 \\ 6I & 2.5I & 0 & 3.712I & 0.35I \\ 4.52 & 0 & 7.23 & 0 & 6.31 \end{bmatrix}$ be the

MOD interval neutrosophic matrix operator with entries from $R_n^I(11) = \{a + bI / a, b \in [0, 11), I^2 = I\}$

Let D = $\{(a_1, a_2, \ldots, a_8) / a_i \in \{0, 1, 1 + I, I\}; 1 \leq i \leq 8\}$ and R = $\{(b_1, b_2, \ldots, b_5) / b_j \in \{0, 1, 1 + I, I\}; 1 \leq j \leq 5\}$ be the special set of domain and range state vectors respectively associated with MOD neutrosophic operator S.

i) Study questions (ii) to (ix) of problem 10 for this MOD interval neutrosophic matrix operator S.

ii) Compare S with a MOD interval complex modulo integer operator.

12. Let $V = \begin{bmatrix} 0.3g & 2g+5 & 0.725 & 0 & 4.3+g \\ 0 & 0.335g & 0 & 1+3.2g & 0 \\ 1.5g+4 & 0 & 4+3.2g & 0 & 7+4g \\ 0 & 2+5g & 0 & 4+3g & 0 \\ 3+2.5g & 0 & 7+2.5g & 0 & 0.33g \\ 0 & 0.3315 & 0 & 0.775g & 0 \\ 8+3g & 11.4g & 12.5+g & 0 & 3g \\ 10.311g & 0 & 12.3777 & 5g & 0 \\ 2+10g & 3+5g & 0 & 6+4g & g \end{bmatrix}$

be the MOD interval dual number matrix operator with entries from $R_n^g(13) = \{a + bg / a, b \in [0, 13); g^2 = 0\}$,

Let $D = \{(a_1, a_2, \ldots, a_9) / a_i \in \{0, g, 1, 1+g\}\ 1 \leq i \leq 9\}$ and $R = \{(b_1, b_2, b_3, b_4, b_5) / b_i \in \{0, 1, g, 1+g\}; 1 \leq i \leq 5\}$ be the special state vectors of the domain and range space respectively associated with the MOD matrix operator V.

i) Study questions (i) to (ix) of problem (10) for this V.
ii) Compare this MOD interval dual number matrix operator with MOD interval neutrosophic matrix operator.

iii) Compare this MOD interval dual number matrix operator with MOD interval complex modulo integer matrix operator.

13. Let L =

$$\begin{bmatrix} 3g+1 & 0.332g & 4g+3 & 2g+1 & 0 \\ 0.331 & 6+2g & 0.5321g & 0 & 0.37g \\ 0 & 5+2.002g & 0 & 3.88g & 0 \\ 4.32g+2 & 0 & 7+0.3g & 0 & 4+3.2g \end{bmatrix} \text{be}$$

the MOD interval dual number matrix operator with entries from $R_n^g(8) = \{a + bg / a, b \in [0, 8); g^2 = 0\}$.

i) Study questions (ii) to (ix) of problem (10) for this L

ii) Find all special features associated with L.

14. Let B = $\begin{bmatrix} 3h+2 & 0.334h & 0.752 & 0 \\ 0 & 4+5h & 0 & 3+h \\ 0.7742 & 0 & 5h+1 & 0.53 \\ 0 & 4.2315g & 0 & 0.92g \\ 1+3.75h & 0 & 3+0.74h & 0 \\ 0 & 2+0.33h & 1+4.05h & 2h \\ 7.21h & 0 & 4.23 & 0 \end{bmatrix}$ be the

MOD interval special dual like number matrix operator with entries from $R_n^h(9) = \{a + bh / a, b \in [0, 9), h^2 = h\}$.

Let D = $\{(a_1, a_2, \ldots, a_7) \mid a_i \in \{0, 1, h, 1 + h\}; 1 \leq i \leq 7\}$

and R = $\{(b_1, b_2, b_3, b_4) \mid b_j \in \{0, 1, h, 1 + h\}\ 1 \leq j \leq 4\}$ be the special state vectors of the domain and range space respectively associated with the MOD interval matrix operator B.

i) Study questions (ii) to (ix) problem (10) for this B.

ii) Compare the MOD interval matrix special dual like number operators with the MOD interval dual number matrix operators.

iii) Compare the MOD interval matrix special dual like number operator with the MOD interval complex finite modulo integer matrix operator.

15. Let $A = \begin{bmatrix} 3.72h & 0 & 4.325h \\ 0.775 & 4.2 & 0.7531 \\ 2h & 6h & 0.1115h \\ 0 & 7.23 & 4 \\ 6.375h & 0 & 4.367h \\ 8.32 & 5.77 & 0 \\ 7.23h & 6.2h & 10.3h \\ 0 & 0 & 7.24 \\ 4.372h & 3.7h & 5.5h \end{bmatrix}$ be the MOD interval special dual like number matrix operator with entries from $R_n^h(11) = \{a + bh / a, b \in [0, 11], h^2 = h\}$.

i) Study questions (i) to (ix) of problem (10) for this A.

ii) Obtain any other special use of these MOD operators.

16. Let $D_1 = \begin{bmatrix} 0.33k + 0.5 & 0.789 + 6.2k \\ 0 & 8 + 2k \\ 19 + 4k & 14 + 21.3k \\ 22.43k & 16.749 \\ 0 & 10 + 5k \\ 4 + 12.312k & 19.3115k \\ 21.0315k & 21.0125 \end{bmatrix}$ be the MOD interval special quasi dual number matrix operator with entries from $R_n^k(23) = [a + bk / a, b \in [0,23), k^2 = 22k\}$.

Let $D = \{(a_1, a_2, \ldots, a_7) \mid a_i \in \{0, k, 1, 1 + k\}< 1 \leq i \leq 7\}$ be the special domain space of state vectors associated with $D_1$.

Let R = {(a, b) | a, b ∈ {0, 1, k, 1 + k} be the special range space of state vectors associated with the MOD matrix operator $D_1$.

i) Study questions (ii) to (ix) of problem (10) for this $D_1$.

ii) What the distinct features between the MOD interval dual number matrix operators and the MOD interval special quasi dual number matrix operators?

iii) Bring out the similarities and differences between the MOD interval special dual like number matrix operators and MOD interval special quasi dual number matrix operators.

iv) Compare and contrast the MOD interval complex modulo integer matrix operators with MOD interval special quasi dual number matrix operators.

17. Let Q = $$\begin{bmatrix} 3.4k+2 & 3.75 & 0 & 0 \\ 5.32k & 1+4.2k & 0 & 0 \\ 0 & 0 & 3.7k & 4.3 \\ 0 & 0 & 1+5k & 2+3k \\ 7.52k & 1.53k & 0 & 0 \\ 7+3.2k & 6+3.4k & 0 & 0 \\ 0 & 0 & 4k & 2k \\ 0 & 0 & 4.34 & 5.2 \\ 7.3 & 4.5 & 0 & 0 \\ 0.3k & 6.8k & 0 & 0 \\ 0 & 0 & 6 & 2 \\ 0 & 0 & 3.7 & 4.2 \\ 7k & 2k & 0 & 0 \\ 4.31k & 5.2k & 0 & 0 \end{bmatrix}$$ be the

MOD interval special quasi dual number matrix operator with entries from $R_n^k(10)$.

i) Study question (ii) to (ix) of problem (10) for this Q.

ii) Study questions (ii) to (iv) of problem 16 for this Q.

iii) Can we find any special feature enjoyed by this Q as it elements are blocks of zeros and so on?

18. Can we have 2 MOD interval neutrosophic matrix operators A and B of same order with elements from the same set $R_n^I(24)$ such that A ≠ B but they have same set of classical fixed point pairs.

    i) Will there be changes for 2A and 2B.

    ii) Find all realized fixed points pairs of A and B in D and R.

    iii) Can a resultant of x which is realized fixed point pair related to A be a realized limit cycle pair of B and vice versa for all x ∈ D [x ∈ R) for any two MOD interval matrix operator in $R_n^I$ (24)?

19. Show these have lots of applications in mathematical modeling.

20. Let $A = \begin{bmatrix} 5.31 & 2.7 & 5.45 & 0 \\ 0 & 1 & 2 & 3.45 \\ 2.11 & 0 & 3.12 & 0 \\ 0 & 2.5 & 0 & 1.2 \\ 1.4 & 2 & 3 & 0 \end{bmatrix}$ be a MOD interval real matrix operator with entries from $R_n(16)$.

Let $D = R = \{(a_1, a_2, a_3, a_4) / a_i \in [0, 6); 1 \leq i \leq 4\}$.

i) Find all $x \in D$ and $x \in R$ which give the same resultant.

ii) Characterize all these $x \in D$ and $x \in R$ which gives different sets of resultants.

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# INDEX

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On India's 60th Independence Day, Dr.Vasantha was conferred the Kalpana Chawla Award for Courage and Daring Enterprise by the State Government of Tamil Nadu in recognition of her sustained fight for social justice in the Indian Institute of Technology (IIT) Madras and for her contribution to mathematics. The award, instituted in the memory of Indian-American astronaut Kalpana Chawla who died aboard Space Shuttle Columbia, carried a cash prize of five lakh rupees (the highest prize-money for any Indian award) and a gold medal.

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In this book authors for the first time define a special type of fixed points using MOD rectangular matrices as operators. In this case the special fixed points or limit cycles are pairs which is arrived after a finite number of iterations. Such study is both new and innovative for it can find lots of applications in mathematical modeling.

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